Spectral Algebraic Geometry (Under Construction!)

February 3, 2018

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Chapter 0

Introduction

Let $X, Y \subseteq \mathbf{CP}^2$ be smooth algebraic curves of degrees m and n in the complex projective plane $\mathbf{CP}^2$. If X and Y meet transversely, then the classical theorem of Bezout (see for example [71]) asserts that the intersection $X \cap Y$ has precisely mn points. This statement has a natural formulation in the language of cohomology. The curves X and Y have fundamental classes $[X], [Y] \in \mathrm{H}^2(\mathbf{CP}^2; \mathbf{Z})$. If C and C' meet transversely, then we have the formula

$$[X] \cup [Y] = [X \cap Y],$$

where the fundamental class $[X \cap Y] \in \mathrm{H}^4(\mathbf{CP}^2; \mathbf{Z}) \simeq \mathbf{Z}$ of the intersection $X \cap Y$ simply counts the number of points where X and Y meet. Of course, this should not be surprising: the cup product on cohomology classes is defined so as to encode the operation of intersection. However, it would be a mistake to regard the equation $[X] \cup [Y] = [X \cap Y]$ as obvious, because it is not always true. For example, if the curves X and Y meet nontransversely (but still in a finite number of points), then we always have a strict inequality

$$[X] \cup [Y] > [X \cap Y]$$

if the right hand side is again interpreted as counting the number of points in the set-theoretic intersection of X and Y.

If we want a formula which is valid for non-transverse intersections, then we must alter the definition of $[X \cap Y]$ so that it counts each intersection point with the appropriate multiplicity. In the situation described above, the multiplicity of an intersection point $p \in X \cap Y$ can be defined as the dimension of the tensor product

$$\mathscr{O}_{X,p} \otimes_{\mathscr{O}_{\mathbf{CP}^2,p}} \mathscr{O}_{Y,p}.$$

as a vector space over the complex numbers. This tensor product has a natural algebro-geometric interpretation: it is the local ring of the *scheme-theoretic* intersection $X \times_{\mathbf{CP}^2} Y$

at the point p. Consequently, the equation $[X] \cup [Y] = [X \cap Y]$ remains valid if the right hand side is properly interpreted: we must define the fundamental class of the intersection $[X \cap Y]$ in a way which takes into account the structure of $X \cap Y = X \times_{\mathbf{CP}^2} Y$ as a scheme.

In more complicated situations, the appropriate intersection multiplicities cannot always be determined from the scheme-theoretic intersection alone. Suppose that X and Y are singular subvarieties of a smooth algebraic variety Z having complementary dimension and intersecting in a finite number of points. In this case, the appropriate intersection multiplicity at a point $p \in X \cap Y$ is not always given by the complex dimension of the local ring

$$\mathscr{O}_{X \cap Y, p} = \mathscr{O}_{X,p} \otimes_{\mathscr{O}_{Z,p}} \mathscr{O}_{Y,p} .$$

The reason for this is easy to understand from the point of view of homological algebra. Since the tensor product functor $\otimes_{\mathscr{O}_{Z,p}}$ is not exact, it does not have good properties when considered alone. According to Serre's intersection formula, the correct intersection multiplicity is instead the Euler characteristic

$$\sum (-1)^m \dim_{\mathbf{C}} \operatorname{Tor}_m^{\mathscr{O}_{Z,p}}(\mathscr{O}_{X,p}, \mathscr{O}_{Y,p}).$$

This Euler characteristic contains the dimension of the local ring of the scheme-theoretic intersection as its leading term, but also higher-order corrections. We refer the reader to [188] for further discussion of this formula for the intersection multiplicity.

If we would like the equation $[X] \cup [Y] = [X \cap Y]$ to remain valid in the more complicated situations described above, then we need to interpret the right hand side in a more sophisticated way. It is not enough to contemplate the intersection $X \cap Y$ as a set or even as a scheme: we need to to remember *all* of the Tor-groups $\operatorname{Tor}_m^{\mathscr{O}_{Z,p}}(\mathscr{O}_{X,p}, \mathscr{O}_{Y,p})$, rather than simply the tensor product $\mathscr{O}_{X \cap Y, p} = \mathscr{O}_{X,p} \otimes_{\mathscr{O}_{Z,p}} \mathscr{O}_{Y,p} = \operatorname{Tor}_0^{\mathscr{O}_{Z,p}}(\mathscr{O}_{X,p}, \mathscr{O}_{Y,p})$.

Let us begin by recalling how these invariants are defined. Suppose that R is a commutative ring and that we are given R-modules A and B. We can then choose a projective resolution of A as an R-module: that is, an exact sequence of R-modules

$$\cdots \to P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \to A \to 0$$

where each P_m is projective. By definition, the groups $\operatorname{Tor}_m^R(A, B)$ are given by the homology groups of the chain complex

$$\cdots \to P_2 \otimes_R B \xrightarrow{d'} P_1 \otimes_R B \xrightarrow{d'} P_0 \otimes_R B,$$

whose differential d' is given by tensoring d with the identity map id_B.

In the situation of interest to us, A and B are not simply R-modules: they are commutative algebras over R. In this case, one can arrange that resolution (P_*, d) is compatible with the algebra structure on A in the following sense:

(*i*) There exist multiplication maps $P_m \otimes_R P_n \to P_{m+n}$ which endow the direct sum $\bigoplus_{n \geqslant 0} P_n$ with the structure of a graded ring which is commutative in the graded sense: that is, we have $xy = (-1)^{mn} yx$ for $x \in P_m$, $y \in P_n$.

(*ii*) The differential $d : P_* \to P_{*-1}$ satisfies the (graded) Leibniz rule $d(xy) = (dx)y + (-1)^m x(dy)$ for $x \in P_m$.

(*iii*) The surjection $P_0 \to A$ is a ring homomorphism.

Properties (*i*) and (*ii*) can be summarized by saying that (P_*, d) is a *commutative differential graded algebra* over R. If B is any commutative R-algebra, then the tensor product complex

$$\cdots \to P_2 \otimes_R B \xrightarrow{d'} P_1 \otimes_R B \xrightarrow{d'} P_0 \otimes_R B$$

inherits the structure of a commutative differential graded algebra over R (or even over B). We will denote this differential graded algebra by $A \otimes_R^L B$ and refer to it as the *derived tensor product of A and B over R.*

Warning 0.0.0.1. The definition of $A \otimes_R^L B$ depends on a choice of projective resolution of A by a differential graded algebra (P_*, d). However, the resulting commutative differential graded algebra turns out to be independent of (P_*, d) *up to quasi-isomorphism.* In particular, the homology groups of $A \otimes_R^L B$ are independent of the resolution chosen: these are simply the Tor-groups $\mathrm{Tor}_n^R(A, B)$.

Let R be a commutative ring. Then every commutative R-algebra R' can be regarded as a commutative differential graded R-algebra by identifying it with a chain complex which is concentrated in degree zero. We can therefore think of a commutative differential graded algebra as a generalized of ordinary commutative rings. In particular, the derived tensor product $A \otimes_R^L B$ bundles the information contained in the Tor-groups $\mathrm{Tor}_n^R(A, B)$ together into a single package which behaves, in some sense, like a commutative ring. The central idea of this book is that this heuristic can be taken seriously: objects like commutative differential graded algebras are, for many purposes, just as good as commutative rings and can be used equally well as the basic building blocks of algebraic geometry.

To fix ideas, let us introduce the a preliminary definition:

Definition 0.0.0.2. Let X be a topological space and let $\mathscr{O}_X$ be a sheaf of commutative differential graded $\mathbf{C}$-algebras on X. For each integer n, we let $\mathrm{H}_n(\mathscr{O}_X)$ denote sheaf of vector spaces given by the nth homology of $\mathscr{O}_X$, so that $\mathrm{H}_0(\mathscr{O}_X)$ is a sheaf of commutative rings on X and each $\mathrm{H}_n(\mathscr{O}_X)$ is a sheaf of $\mathrm{H}_0(\mathscr{O}_X)$-modules. We will say that $(X, \mathscr{O}_X)$ is a *differential graded* $\mathbf{C}$*-scheme* if the following conditions are satisfied:

(*a*) The pair $(X, \mathrm{H}_0(\mathscr{O}_X))$ is a scheme.

(*b*) Each $\mathrm{H}_n(\mathscr{O}_X)$ is a quasi-coherent sheaf on the scheme $(X, \mathrm{H}_0(\mathscr{O}_X))$.

(*c*) The sheaves $\mathrm{H}_n(\mathscr{O}_X)$ vanish for $n < 0$.

Warning 0.0.0.3. The notion of differential graded scheme has been studied by many authors (see [121], [41], [42], [19], [20]) using definitions which are different from (but closely related to) Definition 0.0.0.2.

Warning 0.0.0.4. Definition 0.0.0.2 captures the spirit of the kinds of objects that we will be studying in this book, at least when work over the field **C**. However, it does not really capture the spirit of *how* we will work with them. If $(X, \mathscr{O}_X)$ is a differential graded **C**-scheme, then one should think of the structure sheaf $\mathscr{O}_X$ as something that is well-defined only up to quasi-isomorphism, rather than "on the nose." This idea needs to be incorporated systematically into every aspect of the theory, beginning with the notion of morphism between differential graded **C**-scheme.

The theory of differential graded **C**-schemes has the following features:

- Every ordinary **C**-scheme $(X, \mathscr{O}_X)$ can be regarded as a differential graded **C**-scheme: we can simply regard the structure sheaf $\mathscr{O}_X$ as a sheaf of commutative differential graded algebras which is concentrated in degree zero.

- Every differential graded **C**-scheme $(X, \mathscr{O}_X)$ determines a **C**-scheme $(X, \mathrm{H}_0(\mathscr{O}_X))$, which we will refer to as the *underlying scheme* of $(X, \mathscr{O}_X)$.

- If $(X, \mathscr{O}_X)$ is a differential graded **C**-scheme, then the difference between $(X, \mathscr{O}_X)$ and its underlying scheme $(X, \mathrm{H}_0(\mathscr{O}_X))$ is measured by the quasi-coherent sheaves $\{\mathrm{H}_n(\mathscr{O}_X)\}_{n>0}$: if these sheaves vanish, then one should regard $(X, \mathscr{O}_X)$ and $(X, \mathrm{H}_0(\mathscr{O}_X))$ as equivalent data (see Warning 0.0.0.4).

- The theory of differential graded **C**-schemes has a good notion of fiber product. However, the inclusion of ordinary **C**-schemes into differential graded **C**-schemes does not preserve fiber products. In the setting affine schemes, the usual fiber product $\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B$ is given by the spectrum of the tensor product $A \otimes_R B$. However, the same fiber product in the setting of differential graded **C**-schemes can be described as the spectrum (in a sense we will define later) of the *derived* tensor product $A \otimes_R^L B$.

Recall that a scheme $(X, \mathscr{O}_X)$ is said to be *reduced* if the structure sheaf $\mathscr{O}_X$ has no nonzero nilpotent sections. The relationship between differential graded **C**-schemes and ordinary **C**-schemes is analogous to the relationship between schemes and reduced schemes. Every scheme $(X, \mathscr{O}_X)$ determines a reduced scheme $(X, \mathscr{O}_X^{\mathrm{red}})$, where $\mathscr{O}_X^{\mathrm{red}}$ is the quotient of the structure sheaf $\mathscr{O}_X$ by the ideal sheaf of locally nilpotent sections. However, the passage from $(X, \mathscr{O}_X)$ to $(X, \mathscr{O}_X^{\mathrm{red}})$ loses information. Moreover, the lost information could be useful even if one is primarily interested in smooth algebraic varieties: recall that if $X, Y \subseteq \mathbf{CP}^2$ are smooth algebraic curves, then the (possibly non-reduced) scheme-theoretic intersection $X \times_{\mathbf{CP}^2} Y$ retains information about the multiplicity of each point $p \in X \cap Y$, but this information is lost by passing to the reduced scheme $(X \times_{\mathbf{CP}^2} Y)^{\mathrm{red}}$ (which remembers only the set-theoretic intersection of X and Y).

The situation for differential graded $\mathbf{C}$-schemes is similar: if X and Y are (possibly singular) subvarieties of a smooth algebraic variety Z which have complementary dimension and meet in a finite number of points, then Serre's formula for the intersection multiplicity of X and Y at a point p can be written $\sum (-1)^n \dim_{\mathbf{C}} \mathrm{H}_n(\mathscr{O}_{X \cap Y})_p$, where $\mathscr{O}_{X \cap Y}$ denotes structure sheaf of the fiber product $X \times_Z Y$ *in the setting of differential graded* $\mathbf{C}$*-schemes*. By passing to the underlying scheme of this fiber product $X \times_Z Y$, we lose information about all but the leading term of Serre's formula.

Remark 0.0.0.5. To get a feeling for the sort of information which is encoded by the fiber product $X \times_Z Y$ in the setting of differential graded $\mathbf{C}$-schemes, it is instructive to consider the case where $Z = \operatorname{Spec} R$ is an affine scheme and $X = \operatorname{Spec} R/I$, $Y = \operatorname{Spec} R/J$ are closed subschemes given by the vanishing loci of ideals $I, J \subseteq R$. In this case, the usual (scheme-theoretic) intersection of X and Y is the affine scheme $\operatorname{Spec} R/(I + J)$. The difference between $X \times_Z Y$ and this scheme-theoretic intersection is controlled by the groups $\{\mathrm{H}_n(R/I \otimes_R^L R/J) = \mathrm{Tor}_n^R(R/I, R/J)\}_{n>0}$. The group $\mathrm{Tor}_1^R(R/I, R/J)$ can be described concretely as the quotient $(I \cap J)/IJ$. Any element $f \in R$ which belongs to the intersection $I \cap J$ can be viewed as a regular function on Z which vanishes on *both* of the closed subschemes X and Y. Heuristically, such a function f might be said to vanish on the intersection $X \cap Y$ for *two* reasons, and we have $f \notin IJ$ if these reasons are "different" in some essential way. Consequently, the quotient $\mathrm{Tor}_1^R(R/I, R/J) = (I \cap J)/IJ$ is a measure of the redundancy of the equations defining the subschemes X and Y. Forming the fiber product $X \times_Z Y$ in differential graded $\mathbf{C}$-schemes retains information about this sort of redundancy: it remembers not only *which* functions vanish on the intersection of X and Y, but also *why* they vanish.

If X and Y are *smooth* subvarieties of a smooth complex algebraic variety Z, then some simplifications occur. As long as the intersection $X \cap Y$ has the "expected" dimension $\dim X + \dim Y - \dim Z$, the Tor-groups $\mathrm{Tor}_n^{\mathscr{O}_{Z,p}}(\mathscr{O}_{X,p}, \mathscr{O}_{Y,p})$ automatically vanish for each $p \in X \cap Y$. This means that the fiber product $X \times_Z Y$ in the setting of differential graded $\mathbf{C}$-schemes agrees with the usual scheme-theoretic intersection, so the theory of differential graded schemes has nothing new to tell us. However, the theory can be quite useful in the case where the intersection $X \cap Y$ does *not* have the expected dimension. We will say that a differential graded $\mathbf{C}$-scheme $(W, \mathscr{O}_W)$ is *quasi-smooth* if it is locally of the form $X \times_Z Y$, where X and Y are smooth subvarieties of a smooth complex algebraic variety Z. Then:

- Every quasi-smooth differential graded **C**-scheme $(W, \mathscr{O}_W)$ has a well-defined *virtual dimension* $\mathrm{vdim}_w(W) \in \mathbf{Z}$ at each point $w \in W$, which is a locally constant as a function of w.

- The integer $\mathrm{vdim}_w(W)$ can be thought of as the "expected dimension of W." When W is given as a fiber product $X \times_Z Y$ as above, its virtual dimension is given by the formula $\mathrm{vdim}_w(W) = \dim_w(X) + \dim_w(Y) - \dim_w(Z)$. Roughly speaking, the virtual dimension can be described as "the number of variables minus the number of equations."

- If $(W, \mathscr{O}_W)$ is quasi-smooth, then we always have $\mathrm{vdim}_w(W) \leqslant \dim_w(W)$, with equality if and only if $(W, \mathscr{O}_W)$ is an ordinary **C**-scheme in a neighborhood of w. Beware that unlike the actual dimension $\dim_w(W)$, the virtual dimension $\mathrm{vdim}_w(W)$ can be negative.

- Let $(W, \mathscr{O}_W)$ be a quasi-smooth differential graded **C**-scheme of virtual dimension d and let $W(\mathbf{C})$ denote the set of closed points of W, equipped with the complex-analytic topology. If the space $W(\mathbf{C})$ is compact, then there is a canonical element $[W] \in \mathrm{H}_{2d}(W(\mathbf{C}); \mathbf{Z})$ called the *virtual fundamental class* of $(W, \mathscr{O}_W)$ (this element can also be defined when W is not compact, in which case it lies in the Borel-Moore homology of $W(\mathbf{C})$). In the special case where W is a smooth ordinary **C**-scheme, the space $W(\mathbf{C})$ is a compact complex manifold of dimension d and $[W]$ is its usual fundamental class.

- Let Z be a smooth projective variety of dimension n over the complex numbers. Given a quasi-smooth differential graded **C**-scheme X of virtual dimension d and a map $X \hookrightarrow Z$ which is a closed embedding at the level of topological spaces, let us abuse notation by identifying the virtual fundamental class $[X] \in \mathrm{H}_{2d}(X(\mathbf{C}); \mathbf{Z})$ with its image under the canonical map

$$\mathrm{H}_{2d}(X(\mathbf{C}); \mathbf{Z}) \to \mathrm{H}_{2d}(Z(\mathbf{C}); \mathbf{Z}) \simeq \mathrm{H}^{2n-2d}(Z(\mathbf{C}); \mathbf{Z}),$$

where the isomorphism is provided by Poincare duality (note that when X is a smooth subvariety of Z, this recovers the usual interpretation of $[X]$ as an element in the cohomology ring $\mathrm{H}^*(Z(\mathbf{C}); \mathbf{Z})$). Then the equation $[X] \cup [Y] = [X \cap Y]$ holds in complete generality, provided that we interpret $X \cap Y$ as the fiber product $X \times_Z Y$ in the setting of differential graded **C**-schemes.

Example 0.0.0.6. Consider the easiest case of Bezout's theorem, where we are given a pair of lines $L, L' \subseteq \mathbf{CP}^2$ in the complex projective plane $\mathbf{CP}^2$. The lines L and L' *always* intersect transversely in exactly one point, except in the trivial case where the lines L and

L' are the same. In this degenerate case, the equation $[L] \cup [L'] = [L \cap L']$ seems to fail dramatically, because the naive intersection $L \cap L'$ (formed either in the set-theoretic or scheme-theoretic sense) does not even have the right dimension. However, the fiber product $(W, \mathscr{O}_W) = L \times_{\mathbf{CP}^2} L'$ in the differential graded setting is not equivalent to L as a differential graded $\mathbf{C}$-scheme: the homology sheaf $\mathrm{H}_1(\mathscr{O}_W)$ is a line bundle of degree -1 on W. This allows us to extract some useful information:

- The virtual dimension of W is 0, which differs from the dimension of the its underlying classical scheme L.
- As a topological space, $W(\mathbf{C})$ is a 2-sphere. However, in addition to its usual fundamental class in $\mathrm{H}_2(W(\mathbf{C}); \mathbf{Z})$, the space $W(\mathbf{C})$ also has a *virtual* fundamental class $[W] \in \mathrm{H}_0(W(\mathbf{C}); \mathbf{Z}) \simeq \mathbf{Z}$. One can show that this virtual fundamental class is given by the formula $[W] = \deg(\mathrm{H}_0(\mathscr{O}_W)) - \deg(\mathrm{H}_1(\mathscr{O}_W)) = 1$.

More informally, the structure sheaf $\mathscr{O}_W$ "knows" both that W is expected to be zero-dimensional and that it is expected to consist of exactly one point.

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0.1 Why Spectral Algebraic Geometry?

Our goal in this book is to study algebro-geometric objects like the differential graded $\mathbf{C}$-schemes of Definition 0.0.0.2. However, this merits a warning: in the setting of Definition 0.0.0.2, we could replace $\mathbf{C}$ by an arbitrary field κ, but the resulting theory is not well-behaved if κ is of positive characteristic. To get a sensible theory in positive and mixed characteristic, we will replace the theory of differential graded commutative algebras by the more sophisticated theory of $\mathbb{E}_\infty$-*rings* (for the reader who is not familiar with the theory of $\mathbb{E}_\infty$-rings, we will give a brief review below; for the moment, it is enough to know that they are mathematical objects that are equivalent to commutative differential graded algebras when working over a field of characteristic zero, but better behaved in general). Among our basic objects of study in this book are *spectral schemes*: pairs $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a sheaf of $\mathbb{E}_\infty$-rings on X which satisfies analogues of the hypotheses which appear in Definition 0.0.0.2. Every spectral scheme $(X, \mathscr{O}_X)$ has an *underlying scheme* which we will denote by $(X, \pi_0 \mathscr{O}_X)$.

A reader wanting to get a sense of the subject might ask the following:

(Q) What is the difference between spectral algebraic geometry and classical algebraic geometry? For example, what is the difference between a spectral scheme $(X, \mathscr{O}_X)$ and its underlying scheme $(X, \pi_0 \mathscr{O}_X)$?

A more skeptical reader might put the question in a more pointed way:

(Q') What use is the theory of spectral algebraic geometry? What can one do with a spectral scheme $(X, \mathscr{O}_X)$ that cannot already be done with the underlying scheme $(X, \pi_0 \mathscr{O}_X)$?

One answer to these questions was already sketched in the introduction: the language of spectral algebraic geometry provides a natural framework in which to understand issues of excess intersection and the theory of virtual fundamental classes. However, let us offer three more:

($A1$) The difference between spectral algebraic geometry and classical algebraic geometry lies in the nature of the structure sheaves considered: the structure sheaf of a spectral scheme $(X, \mathscr{O}_X)$ is a sheaf of $\mathbb{E}_\infty$-rings, while the structure sheaf of an ordinary scheme is a sheaf of commutative rings. Structured ring spectra (such as $\mathbb{E}_\infty$-rings) are ubiquitous in the study of stable homotopy theory and its applications. The language of spectral algebraic geometry provides a novel way of thinking about these objects, just as the language of classical algebraic geometry supplies geometric insights which are valuable in the study of commutative algebra.

($A2$) The difference between spectral algebraic geometry and classical algebraic geometry is analogous to the difference between triangulated categories and abelian categories. To every spectral scheme $(X, \mathscr{O}_X)$, one can assign a triangulated category $\mathrm{hQCoh}(X)$ whose objects we will refer to as *quasi-coherent sheaves on* X (this triangulated category arises as the homotopy category of a more fundamental invariant $\mathrm{QCoh}(X)$, which is a *stable* ∞*-category* rather than a triangulated category). The triangulated category $\mathrm{hQCoh}(X)$ contains, as a full subcategory, the abelian category $\mathcal{A}$ of quasi-coherent sheaves on the underlying scheme $(X, \pi_0 \mathscr{O}_X)$. Roughly speaking, the difference between the spectral scheme $(X, \mathscr{O}_X)$ and the ordinary scheme $(X, \pi_0 \mathscr{O}_X)$ is measured by the failure of $\mathrm{hQCoh}(X)$ to be the derived category of $\mathcal{A}$. In some situations, this failure is a feature rather than a bug: the triangulated category $\mathrm{hQCoh}(X)$ may be better suited to a particular application.

($A3$) The difference between spectral algebraic geometry and classical algebraic geometry can be understood in terms of deformation theory. One can think of a spectral scheme $(X, \mathscr{O}_X)$ as given by an ordinary scheme $X_0 = (X, \pi_0 \mathscr{O}_X)$ together with an "obstruction theory" for X_0 (of a somewhat elaborate type). In many cases, this obstruction theory is more natural and easier to work with than the obstruction theory which is intrinsic to X_0 itself.

We now briefly expand on each of these answers (we will discuss each one in much greater detail in the body of the text; ($A2$) and ($A3$) are, in some sense, the main themes of Parts III and **??**, respectively).

0.1.1 Homotopy Theory and $\mathbb{E}_\infty$-Rings

Algebraic topology can be described as the study of topological spaces by means of algebraic invariants. One of the main goals of the algebraic topologist is to answer questions of the following general form:

Question 0.1.1.1. Let X be an interesting topological space (perhaps a classifying space, a sphere, an Eilenberg-MacLane space, a compact Lie group, ...) and let E be an algebraic invariant of spaces (such as homology, cohomology, K-theory, stable or unstable homotopy, ...). What is $E(X)$?

Before attempting to answer a question of this kind, we would first need to decide what sort of answer we are looking for. For example, suppose that we are given a topological space X and asked to compute the cohomology groups $\mathrm{H}^*(X; \kappa)$ with coefficients in a field κ. These cohomology groups form a graded vector space over κ, so one could interpret Question 0.1.1.1 as follows:

(a) Give a basis for the cohomology $\mathrm{H}^*(X; \kappa)$ as a vector space over κ.

The invariants which arise in algebraic topology often have a very rich structure: for example, the cohomology $\mathrm{H}^*(X; \kappa)$ is not just a graded vector space over κ, it is a graded *algebra* over κ. Consequently, one can formulate Question 0.1.1.1 differently:

(*b*) Give a presentation of the ring $\mathrm{H}^*(X;\kappa)$ by generators and relations.

Remark 0.1.1.2. Problems (*a*) and (*b*) are not really the same. For example, it is possible to solve problem (*a*) without having any idea what the multiplication on $\mathrm{H}^*(X;\kappa)$ looks like. Conversely, the task of extracting a vector space basis from a presentation of $\mathrm{H}^*(X;\kappa)$ by generators and relations is nontrivial (but is at least a purely algebraic problem).

If the field κ is of characteristic 2 (or if the cohomology of X is concentrated in even degrees) then the cohomology ring $\mathrm{H}^*(X;\kappa)$ is *commutative*. In this case, we can reformulate (*b*) using the language of algebraic geometry:

(*c*) Describe the affine scheme $\operatorname{Spec}\mathrm{H}^*(X;\kappa)$ (for example, by specifying its functor of points).

The paradigm of (*c*) has turned out to be a surprisingly useful way of thinking about Question 0.1.1.1: ideas from algebraic geometry can be a powerful tool for organizing and understanding the results of many calculations in algebraic topology.

Example 0.1.1.3 (The Dual Steenrod Algebra)**.** Let $\mathcal{A}^\vee$ denote the dual Steenrod algebra: that is, the graded ring given by the direct limit

$$\mathcal{A}^\vee = \varinjlim \mathrm{H}^{\mathrm{red}}_{*+n}(K(\mathbf{F}_2,n);\mathbf{F}_2).$$

Then $\mathcal{A}^\vee$ is a graded Hopf algebra which is of central importance in algebraic topology by virtue of the fact that it controls the *co-operations* on $\mathbf{F}_2$-homology: for every topological space (or spectrum) X, the homology $\mathrm{H}_*(X;\mathbf{F}_2)$ has the structure of a comodule over $\mathcal{A}^\vee$.

The structure of $\mathcal{A}^\vee$ has a simple description (due to Milnor; see [152]): as an algebra, it is isomorphic to a polynomial ring $\mathbf{F}_2[\zeta_1,\zeta_2,\zeta_3,\ldots]$, where each variable ζ_n is homogeneous of degree 2^n-1 and the comultiplication $\Delta:\mathcal{A}^\vee\to\mathcal{A}^\vee\otimes_{\mathbf{F}_2}\mathcal{A}^\vee$ is given by the formula

$$\Delta(\zeta_n) = 1\otimes\zeta_n + \zeta_1\otimes\zeta_{n-1}^2 + \zeta_2\otimes\zeta_{n-2}^4 + \cdots + \zeta_{n-1}\otimes\zeta_1^{2^{n-1}} + \zeta_n\otimes 1.$$

Let $G=\operatorname{Spec}\mathcal{A}^\vee$ denote the associated affine scheme. For any commutative $\mathbf{F}_2$-algebra R, we can identify the set $G(R)$ of R-valued points of G with the subset of $R[[t]]$ consisting of those formal power series having the form $t+\zeta_1t^2+\zeta_2t^4+\zeta_3t^8+\cdots$; equivalently, we can describe $G(R)$ as the subset of $R[[t]]$ consisting of those power series $f(t)$ satisfying the conditions

$$f(t)\equiv t \mod t^2 \qquad f(t+t')=f(t)+f(t').$$

This supplies a conceptual way of thinking about co-operations on $\mathbf{F}_2$-homology: for any space X, we can regard the vector space $\mathrm{H}_*(X;\mathbf{F}_2)$ as an algebraic representation of the group scheme G (in fact, by taking into account the grading of $\mathrm{H}_*(X;\mathbf{F}_2)$, we can regard it as a representation of the larger group scheme G^+ parametrizing *all* power series f satisfying $f(t+t')=f(t)+f(t')$ which are invertible under composition).

Example 0.1.1.4 (Complex Bordism and Formal Group Laws)**.** For each $n \geqslant 0$, let MU_n denote the group of bordism classes of stably almost-complex manifolds of dimension n. The direct sum $\mathrm{MU}_* = \bigoplus \mathrm{MU}_n$ is a commutative ring, called the *complex bordism ring*. The structure of this ring was determined by Milnor ([153]): it is isomorphic to a polynomial ring $\mathbf{Z}[x_1, x_2, x_3, \ldots]$ where each variable x_i is homogeneous of degree $2i$. This result was refined by Quillen, who showed that there is a *canonical* isomorphism of MU_* with the Lazard ring L classifying 1-dimensional formal group laws ([169]). In other words, if X denotes the affine scheme $\operatorname{Spec} \mathrm{MU}_*$, then for any commutative ring R we can identify the set $X(R)$ of R-valued points of X with the set of power series $f(u,v) \in R[[u,v]]$ which satisfy the identities

$$f(u,0) = u \qquad f(u,v) = f(v,u) \qquad f(u, f(v,w)) = f(f(u,v), w).$$

Quillen's theorem is the starting point for the subject of *chromatic homotopy theory*, which has revealed a surprisingly tight connection between the study of cohomology theories and the study of formal groups and their arithmetic properties.

Examples 0.1.1.3 and 0.1.1.4 are concerned with algebraic structures that one sees at the level of homology and homotopy, respectively. For many applications, it is important to understand algebraic structures at a more primitive level: for example, at the level of chain complexes before passing to homology. To take a simple example, for *any* topological space M and *any* commutative ring R, the cohomology groups $\mathrm{H}^*(M;R)$ form a graded-commutative ring. However, when M is a smooth manifold and $R = \mathbf{R}$ is the field of real numbers, then $\mathrm{H}^*(M;R)$ can be described as the cohomology of the de Rham complex

$$\Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \Omega^2_M \to \cdots .$$

One of the many convenient features of this description is it makes the graded-commutative ring structure on $\mathrm{H}^*(M;\mathbf{R})$ visible at the level of cochains: the de Rham complex (Ω^*_M, d) itself is a commutative differential graded algebra. Motivated by this observation, Sullivan introduced a construction which associates to an arbitrary topological space X a "polynomial de Rham complex" $C^*_{\mathrm{dR}}(X;\mathbf{Q})$, given by a mixture of singular and de Rham complexes. This construction is naturally quasi-isomorphic to the usual singular cochain complex $C^*(X;\mathbf{Q})$ but has the virtue of admitting a ring structure which is commutative at the level of cochains: $C^*_{\mathrm{dR}}(X;\mathbf{Q})$ is a commutative differential graded algebra over $\mathbf{Q}$. The result is a powerful algebraic invariant of X. For example, one has the following result:

Theorem 0.1.1.5 (Sullivan)**.** *Let X be a simply connected topological space whose rational cohomology groups $\mathrm{H}^n(X;\mathbf{Q})$ are finite-dimensional for every n. Then the rational homotopy type of X can be recovered from its polynomial de Rham complex $C^*_{\mathrm{dR}}(X;\mathbf{Q})$. More precisely, if we let $X_{\mathbf{Q}}$ denote the space of maps from $C^*_{\mathrm{dR}}(X;\mathbf{Q})$ into $\mathbf{Q}$ (in the homotopy theory of* *commutative differential graded algebras over* $\mathbf{Q}$*), then there is a canonical map* $X \to X_{\mathbf{Q}}$ *which is an isomorphism on rational cohomology.*

We refer to [?] for a more precise formulation and proof of Theorem 0.1.1.5 (we will discuss a version of Theorem 0.1.1.5 in §**??**).

Remark 0.1.1.6. It follows from Theorem 0.1.1.5 that the ring structure on the polynomial de Rham complex $C^*_{\mathrm{dR}}(X;\mathbf{Q})$ contains much more information than the ring structure on its homology $\mathrm{H}^*(X;\mathbf{Q})$. It is easy to give examples of finite CW complexes X and Y for which the cohomology rings $\mathrm{H}^*(X;\mathbf{Q})$ and $\mathrm{H}^*(Y;\mathbf{Q})$ are isomorphic, but much harder to give examples in which the polynomial de Rham complexes $C^*_{\mathrm{dR}}(X;\mathbf{Q})$ and $C^*_{\mathrm{dR}}(Y;\mathbf{Q})$ are quasi-isomorphic: if X and Y are simply connected, this can happen only if there exist maps of spaces $X \to Z \leftarrow Y$ which induce isomorphisms on rational cohomology (in this case, we say that X and Y are *rationally* homotopy equivalent).

Remark 0.1.1.7. The language of differential graded schemes suggests the possibility of formulating Theorem 0.1.1.5 in an algebro-geometric way. Let X be a topological space and let $C^*_{\mathrm{dR}}(X;\mathbf{Q})$ be its polynomial de Rham complex. One might try to form some sort of spectrum $\widehat{X} = \operatorname{Spec} C^*_{\mathrm{dR}}(X;\mathbf{Q})$ in the setting of differential graded $\mathbf{Q}$-schemes, so that the space $X_{\mathbf{Q}}$ appearing in Theorem 0.1.1.5 can be interpreted as a space of $\mathbf{Q}$-valued points of $\widehat{X}$. The object $\widehat{X}$ can be regarded as an algebro-geometric incarnation of the topological space X (in the terminology of [209], it is the *schematization* of X).

The geometric object $\widehat{X}$ does not quite fit into the framework of differential graded schemes introduced in Definition 0.0.0.2, because commutative differential graded algebra $C^*_{\mathrm{dR}}(X;\mathbf{Q})$ usually has nonzero homology in negative degrees (or, equivalently, nonzero cohomology in positive degrees). However, it is an example of a different sort of algebro-geometric object (a *coaffine stack*) which we will study in Chapter 9.

To define the polynomial de Rham complex $C^*_{\mathrm{dR}}(X;\mathbf{Q})$, it is necessary to work over $\mathbf{Q}$: if κ is a field of positive characteristic, then there is no canonical way to choose quasi-isomorphism of the singular cochain complex $C^*(X;\kappa)$ with a commutative differential graded algebra over κ. However, this should be regarded as a defect not of $C^*(X;\kappa)$, but of the notion of commutative differential graded algebra. The cochain complex $C^*(X;\kappa)$ is an example of an $\mathbb{E}_\infty$-algebra over κ: it can be equipped with a multiplication law

$$m : C^*(X;\kappa) \otimes_\kappa C^*(X;\kappa) \to C^*(X;\kappa)$$

which is "commutative and associative up to coherent homotopy": in other words, it satisfies every reasonable demand that can be formulated in a homotopy-invariant way (for example, m need not be commutative, but it is commutative up to a chain homotopy $h : C^*(X;\kappa) \otimes_\kappa C^*(X;\kappa) \to C^{*-1}(X;\kappa)$).

Remark 0.1.1.8. Any commutative differential graded algebra (A_*, d) over a field κ determines an $\mathbb{E}_\infty$-algebra over κ: a multiplication law which is commutative and associative "on the nose" is, in particular, commutative and associative up to coherent homotopy. If the field κ has characteristic zero, then every $\mathbb{E}_\infty$-algebra arises in this way, up to quasi-isomorphism. Over fields of positive characteristic, this is not true: there exists $\mathbb{E}_\infty$-algebras which are not quasi-isomorphic to commutative differential graded algebras. For example, if A is an $\mathbb{E}_\infty$-algebra over the field $\mathbf{F}_2$, then the homology groups of A (regarded as a cochain complex over $\mathbf{F}_2$) can be equipped with Steenrod operations $\mathrm{Sq}^n : \mathrm{H}^*(A) \to \mathrm{H}^{*+n}(A)$). If A is obtained from a commutative differential graded algebra over $\mathbf{F}_2$, then these operations automatically vanish for $* \neq n$. However, in the case $A = C^*(X; \mathbf{F}_2)$, they are usually nontrivial (and are a useful and important tool for studying the $\mathbf{F}_2$-cohomology of X).

Let X be a topological space. Just as the polynomial de Rham complex $C^*_{\mathrm{dR}}(X; \mathbf{Q})$ is a much more powerful invariant than the rational cohomology ring $\mathrm{H}^*(X; \mathbf{Q})$ (Remark 0.1.1.6), the structure of the cochain complex $C^*(X; \mathbf{F}_p)$ as an $\mathbb{E}_\infty$-algebra is a much more powerful invariant than the $\mathbf{F}_p$-cohomology ring $\mathrm{H}^*(X; \mathbf{F}_p)$. For example, the $\mathbb{E}_\infty$-structure on $C^*(X; \mathbf{F}_p)$ determines not only the ring structure on $\mathrm{H}^*(X; \mathbf{F}_p)$, but also the behavior of Steenrod operations. In fact, from the $\mathbb{E}_\infty$-structure on $C^*(X; \mathbf{F}_p)$ one can recover the entire p-adic homotopy type of X, thanks to the following analogue of Theorem 0.1.1.5:

Theorem 0.1.1.9 (Mandell)**.** *Let X be a simply connected space whose cohomology groups $\mathrm{H}^n(X; \mathbf{F}_p)$ are finite for every n, and let $X_p^\wedge$ denote the space of $\mathbb{E}_\infty$-algebra morphisms from $C^*(X; \mathbf{F}_p)$ to $\overline{\mathbf{F}}_p$ (here $\overline{\mathbf{F}}_p$ denotes an algebraic closure of $\mathbf{F}_p$). Then there is a canonical map $X \to X_p^\wedge$ which induces an isomorphism on $\mathbf{F}_p$-cohomology.*

Remark 0.1.1.10. In the situation of Theorem 0.1.1.9, the space $X_p^\wedge$ is a *p-adic completion of X*. This implies, for example, that each homotopy group $\pi_n X_p^\wedge$ an be identified with the p-adic completion of $\pi_n X$.

Remark 0.1.1.11. As with Theorem 0.1.1.5, it may be useful to think of Theorem 0.1.1.9 in algebro-geometric terms. If we view $C^*(X; \mathbf{F}_p)$ as a generalized commutative ring and form some kind of spectrum $\widehat{X} = C^*(X; \mathbf{F}_p)$, then we can view the p-adic completion $X_p^\wedge$ as the space $\widehat{X}(\overline{\mathbf{F}}_p)$ of $\overline{\mathbf{F}}_p$-valued points of $\widehat{X}$. The geometric perspective is a bit more useful here than in the rational case, because one can give an analogous description of the space $\widehat{X}(R)$ for any commutative $\mathbf{F}_p$-algebra R: it can be identified with the space of maps from the étale homotopy type of $\operatorname{Spec} R$ into $X_p^\wedge$. For more details, we refer the reader to Part IX.

If A is an $\mathbb{E}_\infty$-algebra over a field κ, then the multiplication on A (which is commutative up to homotopy) endows the homology $\mathrm{H}_*(A)$ with the structure of a graded-commutative ring. Many of the graded-commutative rings which arise naturally in algebraic topology (such as the cohomology rings of spaces) can be obtained in this way. However, there are many examples which are similar in spirit to which the formalism of chain complexes does not quite apply. For example, for each $n \geqslant 0$ let MSO_n denote the collection of bordism classes of closed oriented n-dimensional smooth manifolds. The collection $\{\mathrm{MSO}_n\}_{n \geqslant 0}$ forms a graded-commutative ring, where addition is given by the formation of disjoint unions and multiplication is given by the formation of Cartesian products. Heuristically, one can think of this graded ring as given by the homology of a "chain complex"

$$\cdots \xrightarrow{\partial} \Omega_3 \xrightarrow{\partial} \Omega_2 \xrightarrow{\partial} \Omega_1 \xrightarrow{\partial} \Omega_0$$

where Ω_n denotes the "collection" of compact oriented n-manifolds with boundary, and ∂ is given by forming the boundary. We have addition and multiplication operations

$$\coprod : \Omega_n \times \Omega_n \to \Omega_n \qquad \prod : \Omega_m \times \Omega_n \to \Omega_{m+n}$$

which are commutative and associative up to diffeomorphism. For many applications, it is important that we do *not* define Ω_n to simply be the set of diffeomorphism classes of compact oriented n-manifolds with boundary: passing to diffeomorphism classes loses important information about the behavior of addition and multiplication (for example, the natural action of the symmetric group Σ_2 on the manifolds $M \amalg M$ and $M \times M$). One can retain this information by regarding oriented bordism as an example of a more sophisticated object which we will refer to as an $\mathbb{E}_\infty$*-ring*. We will give an informal review of the theory of $\mathbb{E}_\infty$-rings in §0.2.3 (for a more detailed and precise account, see [139]). Let us summarize some of the features of this theory which are relevant to the present discussion:

- Every $\mathbb{E}_\infty$-ring A has an underlying cohomology theory $X \mapsto A^*(X)$. Roughly speaking, one can think of an $\mathbb{E}_\infty$-ring A as a cohomology theory equipped with a multiplicative structure which is commutative not only at the level of cohomology classes, but also at the level of representatives for cohomology classes (at least up to coherent homotopy).

- If A is an $\mathbb{E}_\infty$-ring and $n \in \mathbf{Z}$ is an integer, then we denote the cohomology group $A^{-n}(*)$ by $\pi_n A$ and refer to it as the *nth homotopy group of* A. The direct sum $\bigoplus_{n \in \mathbf{Z}} \pi_n A$ has the structure of a graded-commutative ring. In particular, $\pi_0 A$ is a commutative ring and each $\pi_n A$ has the structure of a module over $\pi_0 A$.

- Every commutative ring R can be regarded as an $\mathbb{E}_\infty$-ring: the corresponding cohomology theory is ordinary cohomology with coefficients in R. The $\mathbb{E}_\infty$-rings A which arise in this way are characterized by the fact that the homotopy groups $\pi_n A$ vanish for $n \neq 0$.

- If R is an $\mathbb{E}_\infty$-ring, we define an $\mathbb{E}_\infty$*-algebra over* R to be an $\mathbb{E}_\infty$-ring A equipped with a morphism of $\mathbb{E}_\infty$-rings $R \to A$. When R is an ordinary commutative ring, this reduces to the notion described informally earlier: one can identify A with a chain complex of R-modules equipped with a multiplication which is commutative and associative up to coherent homotopy. Under this identification, the homotopy groups of A (regarded as an abstract $\mathbb{E}_\infty$-ring) correspond to the homology groups of the associated chain complex.

There are many important examples of $\mathbb{E}_\infty$-rings A which do not arise as $\mathbb{E}_\infty$-algebras over any commutative ring R.

Example 0.1.1.12 (Complex K-Theory)**.** Let X be a finite CW complex. We define $K^0(X)$ to be the Grothendieck group of the commutative monoid

$$\{ \text{ Complex vector bundles on } X \ \}/ \text{ isomorphism } .$$

We refer to $K^0(X)$ as the *complex K-theory of X*. It is a commutative ring whose addition and multiplication arise from the operation of direct sum and tensor product on complex vector bundles, respectively.

One can extend the construction $X \mapsto K^0(X)$ to define invariants $K^n(X)$ for any integer n and any topological space X. These invariants determine a cohomology theory which we refer to as *complex K-theory*. This cohomology theory is represented by an $\mathbb{E}_\infty$-ring roughly speaking, the $\mathbb{E}_\infty$-structure reflects the fact that multiplication of K-theory classes can be arises concretely for the formation of tensor products of complex vector bundles, which is commutative and associative up to canonical isomorphism. Complex K-theory does not admit the structure of an $\mathbb{E}_\infty$-algebra over any commutative ring R.

Example 0.1.1.13 (Complex Bordism)**.** The complex bordism groups $\{\mathrm{MU}_n\}_{n \geqslant 0}$ of Example 0.1.1.4 can be identified with the homotopy groups of an $\mathbb{E}_\infty$-ring MU, called the *complex bordism spectrum*. In this case, the $\mathbb{E}_\infty$-structure reflects the fact the fact that the formation of Cartesian products of (stably almost) complex manifolds is commutative up to isomorphism. As in Example 0.1.1.12, MU is not an $\mathbb{E}_\infty$-algebra over any commutative ring R.

Example 0.1.1.14 (The Dual Steenrod Algebra)**.** The dual Steenrod algebra $\mathcal{A}^\vee$ of Example 0.1.1.3 can be defined by the formula

$$\mathcal{A}^\vee = \pi_*(\mathbf{F}_2 \wedge \mathbf{F}_2),$$

where $\mathbf{F}_2 \wedge \mathbf{F}_2$ denotes the $\mathbb{E}_\infty$-ring given by a coproduct of two copies of the ordinary commutative ring $\mathbf{F}_2$. This $\mathbb{E}_\infty$-ring does arise as an $\mathbb{E}_\infty$-algebra over $\mathbf{F}_2$, but in two *different* ways. In some situations, one might not want to choose between these (for example, if one wants to study the action of the symmetric group Σ_2 on $\mathbf{F}_2 \wedge \mathbf{F}_2$ given by permuting the factors), in which case it is better to view $\mathbf{F}_2 \wedge \mathbf{F}_2$ as an abstract $\mathbb{E}_\infty$-ring.

We can summarize the preceding discussion as follows:

(*i*) Many calculations in algebraic topology yield commutative (or graded-commutative) rings R. In these cases, it is sometimes easier to think about the affine scheme $\operatorname{Spec} R$ (see Examples 0.1.1.3 and 0.1.1.4).

(*ii*) Many of the graded-commutative rings which arise in algebraic topology can be described as $\pi_* A$ for some $\mathbb{E}_\infty$-ring A. Passage from A to $\pi_* A$ loses a lot of potentially useful information (see Theorems 0.1.1.5 and 0.1.1.9).

The theory of spectral algebraic geometry developed in this book can be described by the rough heuristic

$$\text{Spectral algebraic geometry} = \text{Algebraic Geometry} + \mathbb{E}_\infty\text{-Rings}.$$

One of the aims of this theory is to provide a setting which we can make use of insights (*i*) and (*ii*) simultaneously: given an $\mathbb{E}_\infty$-ring A, we might wish to contemplate the spectrum of A itself (regarded as a kind of generalized affine scheme), rather than the spectrum of some ordinary commutative ring extracted from A (such as $\pi_0 A$ or $\pi_* A$).

At this point, the reader might reasonably object that all of the schemes considered in this section are affine. If we are interested only in affine schemes, then the language of algebraic geometry is superfluous: the datum of an affine scheme is equivalent to the datum of a commutative ring, and the datum of an affine spectral scheme is equivalent to the datum of a (connective) $\mathbb{E}_\infty$-ring. However, there are also non-affine algebro-geometric objects which are relevant to algebraic topology. This is particularly true in the study of chromatic homotopy theory, where many non-affine objects arise naturally as parameter spaces for families of formal groups.

Example 0.1.1.15 (Elliptic Cohomology)**.** Let $\mathcal{M}_{1,1}$ denote the moduli stack of elliptic curves. It follows from the work of Goerss, Hopkins, and Miller that there an essentially unique sheaf $\mathscr{O}^+$ of $\mathbb{E}_\infty$-rings on (the étale site of) $\mathcal{M}_{1,1}$ with the following features:

($*$) Let $U = \operatorname{Spec} R$ be an affine scheme, let $\eta : U \to \mathcal{M}_{1,1}$ be an étale map which classifies an elliptic curve E over R, and set $A = \mathscr{O}^+(U)$. Then there is a canonical isomorphism of commutative rings $\pi_0 A \simeq R$ and a canonical isomorphism of the formal R-scheme $\operatorname{Spf} A^0(\mathbf{CP}^\infty)$ with the formal completion of E (compatible with the group structure on E). Moreover, the homotopy groups $\pi_n A$ vanish when n is odd, and the multiplication maps $\pi_2 A \otimes_R \pi_n A \simeq \pi_{n+2} A$ are isomorphisms for all n.

Passing to global sections, the sheaf $\mathscr{O}^+$ determines an $\mathbb{E}_\infty$-ring TMF called the *spectrum of topological modular forms*. The resulting cohomology theory manifests a rich interplay between ideas from algebraic topology, the arithmetic of modular forms, and mathematical physics.

The sheaf $\mathscr{O}^+$ has a natural interpretation in the language of spectral algebraic geometry. In the terminology of Part I, the pair $(\mathcal{M}_{1,1}, \mathscr{O}^+)$ is an example of a *nonconnective spectral Deligne-Mumford stack*, whose underlying classical Deligne-Mumford stack is the usual moduli stack of elliptic curves. Moreover, the pair $(\mathcal{M}_{1,1}, \mathscr{O}^+)$ can itself be interpreted as a moduli stack: it classifies elliptic curves (defined over $\mathbb{E}_\infty$-rings) which are equipped with an additional datum called an *orientation* (for an informal summary, we refer the reader to [140]). To make sense of this picture, it is important to have a theory of spectral algebraic geometry which includes non-affine objects: elliptic curves are not affine, and the moduli stack of elliptic curves is not even a scheme.

0.1.2 Derived Categories

Suppose that we are given some category $\mathcal{C}$ that we wish to understand (for example, the category of complex representations of a finite group G). One basic strategy is to first find some select some particularly simple objects $\{C_x \in \mathcal{C}\}_{x \in X}$ (for example, the collection of irreducible representations of the group G) and hope that an arbitrary object $C \in \mathcal{C}$ can be expressed as a combination or superposition of the objects $\{C_x\}_{x \in X}$. In many cases of interest, the "simple" objects $\{C_x\}$ admit an algebraic classification, meaning that they are parametrized by the points of some algebro-geometric object X.

Example 0.1.2.1. Let G be a finite flat commutative group scheme over a field κ. Then the one-dimensional (algebraic) representations of G can be identified with maps from G to the multiplicative group $\mathbf{G}_m$, which we can identify with the κ-valued points of the *Cartier dual* group scheme $G^\vee$. In this case, the category $\mathrm{Rep}(G)$ of algebraic representations of G can be identified with the category of quasi-coherent sheaves on $G^\vee$. In concrete terms, if we write $G = \operatorname{Spec} H$ for some finite-dimensional Hopf algebra H over κ, then $G^\vee$ is the spectrum of the dual Hopf algebra $H^\vee$, and the desired equivalence is given by

$$\begin{aligned}\{ \text{ Quasi-coherent sheaves on } G^\vee \ \} &\simeq \{ \ H^\vee\text{-modules } \} \\ &\simeq \{ \ H\text{-comodules } \} \\ &\simeq \{ \text{ Representations of } G \ \}.\end{aligned}$$

For every κ-valued point $\iota : \operatorname{Spec} \kappa \to G^\vee$, this equivalence carries the skyscraper sheaf $\iota_* \kappa$ (regarded as a quasi-coherent sheaf on $G^\vee$) to the one-dimensional representation of G classified by ι.

In Example 0.1.2.1, the identification of representations of G with quasi-coherent sheaves $G^\vee$ holds at the level of abelian categories. However, there are many examples in which one can apply essentially the same paradigm, but it only provides an equivalence at the level of *derived* categories. Recall that for any abelian $\mathcal{A}$, the *derived category* $D(\mathcal{A})$ is obtained from the category $K(\mathcal{A})$ of chain complexes with values in $\mathcal{A}$ by formally adjoining inverses to all quasi-isomorphisms (see [**?**]). If $\mathcal{A}$ is the abelian category of quasi-coherent sheaves on a scheme X, we will denote $D(\mathcal{A})$ simply by $D(X)$ and refer to it as the *derived category* of X.

Example 0.1.2.2 (The Fourier-Mukai Transform)**.** Let E be an elliptic curve defined over the field $\mathbf{C}$ of complex numbers. For every (closed) point $x \in E$, let $\mathscr{O}(x)$ denote the line bundle on E whose section are regular away from x and permitted to have a simple pole at the point x. If we fix a base point $e \in E$, then the construction

$$x \mapsto \mathscr{O}(x - e) = \mathscr{O}(x) \otimes \mathscr{O}(e)^{-1}$$

determines a bijection from the set of (closed) points of E to the set of isomorphism classes of line bundles of degree zero on E. The line bundles $\{\mathscr{O}(x-e)\}_{x \in E}$ are the fibers of a line bundle $\mathcal{P}$ on $E \times E$ corresponding to the Cartier divisor $\Delta - (\{e\} \times E) - (E \times \{e\})$, where Δ is the image of the diagonal map $E \to E \times E$. If we let $\pi_0, \pi_1 : E \times E \to E$ denote the projection maps, then the construction $\mathscr{F} \mapsto \pi_{1*}(\mathcal{P} \otimes \pi_0^* \mathscr{F})$ determines a functor from the abelian category of quasi-coherent sheaves on E to itself, which carries the skyscraper sheaf at a closed point $x \in E$ to the line bundle $\mathscr{O}(x-e)$. At the level of abelian categories, this functor is poorly behaved: it is neither exact nor faithful (for example, it annihilates any line bundle of degree $\leqslant 0$ on E). However, if we work instead at the level of derived categories, then the analogous construction $\mathscr{F} \mapsto R\pi_{1*}(\mathcal{P} \otimes \pi_0^* \mathscr{F})$ determines an equivalence from the category $D(E)$ to itself. Moreover, an analogous statement holds if we replace E by an abelian scheme over any commutative ring A (see [158]).

The derived category $D(X)$ of a scheme X is a fundamental invariant of X. In many cases it is even a *complete* invariant: if X is a smooth projective variety over a field κ whose canonical bundle is either ample or anti-ample, then a celebrated result of Bondal and Orlov asserts that X is determined (up to isomorphism in the category of schemes) by the full subcategory $D^b_{\mathrm{coh}}(X) \subseteq D(X)$ spanned by chain complexes with bounded coherent cohomology; see [163]. One of the main objects of study in this book is an extension of the construction $X \mapsto D(X)$ to the case where X is a spectral scheme. The main features of this extension can be summarized as follows:

- To every spectral scheme $(X, \mathscr{O}_X)$, we will associate a triangulated category $\mathrm{hQCoh}(X)$.
- If $(X, \mathscr{O}_X)$ is an ordinary scheme which is quasi-compact and separated, then $\mathrm{hQCoh}(X)$ can be identified with the derived category $D(X)$ (see Corollary 10.3.4.13).

- For a general spectral scheme $(X, \mathscr{O}_X)$, we can think of $\mathrm{hQCoh}(X)$ informally as the "derived category of X." However, this heuristic has the potential to cause confusion: in general, the triangulated category $\mathrm{hQCoh}(X)$ need not be equivalent to the derived category of *any* abelian category (however, it does arise as the homotopy category of a stable ∞-category $\mathrm{QCoh}(X)$, which is our actual object of interest).

- Let $(X, \mathscr{O}_X)$ be a spectral scheme and let $\mathcal{A}$ denote the abelian category of quasi-coherent sheaves on the underlying ordinary scheme $X_0 = (X, \pi_0 \mathscr{O}_X)$. Then the triangulated category $\mathrm{hQCoh}(X)$ contains the abelian category $\mathcal{A}$ as a full subcategory. Moreover, the inclusion $\mathcal{A} \hookrightarrow \mathrm{hQCoh}(X)$ extends to a triangulated functor $D(X_0) \to \mathrm{hQCoh}(X)$. Assuming that X is quasi-compact and separated, this functor is an equivalence if and only if $(X, \mathscr{O}_X)$ is an ordinary scheme (see Corollary 10.3.4.12).

- Let X_0 be a scheme and let $\mathcal{A}$ be the abelian category of quasi-coherent sheaves on X_0. Then the construction

$$\{\text{ spectral schemes with underlying ordinary scheme } X_0 \text{ }\} \downarrow \{\text{ triangulated categories containing } \mathcal{A} \text{ }\}$$

$$X \mapsto \mathrm{hQCoh}(X)$$

is not too far from being an equivalence (see Corollary 9.6.0.2). In other words, we have a rough heuristic

Spectral algebraic geometry = Algebraic Geometry + Triangulated Categories.

Extending the theory of derived categories to the setting of spectral schemes is not an empty theoretical exercise: it is often necessary when we wish to extend the paradigm of Examples 0.1.2.1 and 0.1.2.2 to more complicated situations. Given a triangulated category $\mathcal{C}$ and a family of objects $\{C_x\}_{x \in X_0}$ parametrized (in some sense) by a scheme X_0, there is often a canonical way to realize X_0 as the underlying ordinary scheme of a spectral scheme X in such a way that the construction $x \mapsto C_x$ extends to a triangulated equivalence $\mathrm{hQCoh}(X) \to \mathcal{C}$.

Example 0.1.2.3. Let G be a semisimple algebraic group defined over a field κ of characteristic $p > 0$, let $\mathfrak{g}$ be its Lie algebra, and let $U_0(\mathfrak{g})$ denote the restricted universal enveloping algebra of $\mathfrak{g}$. Let X denote the flag variety parametrizing choices of Borel subgroup $B \subseteq G$, let $X^{(1)}$ denote the pullback of X along the Frobenius map $\varphi : \operatorname{Spec} \kappa \to \operatorname{Spec} \kappa$, and let $\varphi_{\mathrm{geom}} : X \to X^{(1)}$ denote the geometric Frobenius map associated to X. For each κ-valued point $x \in X^{(1)}(\kappa)$, we can write the scheme-theoretic fiber $\varphi_{\mathrm{geom}}^{-1}\{x\}$ as the spectrum $\operatorname{Spec} A_x$, where A_x is a finite-dimensional κ-algebra. The action of G on X determines an action of the restricted universal enveloping algebra $U_0(\mathfrak{g})$ on A_x, and the construction $x \mapsto A_x$ extends to a functor of derived categories $T : D(X^{(1)}) \to D(U_0(\mathfrak{g}))$. The functor T is far from being an equivalence. However, if the prime number p is sufficiently large (with respect to the Dynkin diagram of G), then the functor T can be factored as a composition

$$D(X^{(1)}) = \mathrm{hQCoh}(X^{(1)}) \xrightarrow{\pi_*} \mathrm{hQCoh}(Y^{(1)}) \xrightarrow{T^+} D(A)$$

with the following features:

- The functor T^+ is a fully faithful embedding whose essential image is a direct summand (a "block") of the derived category $D(A)$.
- The spectral scheme $Y^{(1)}$ is the Frobenius pullback of the (spectral) fiber product $Y = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \{0\}$, where $\widetilde{\mathfrak{g}} \subseteq X \times_{\operatorname{Spec} \kappa} \mathfrak{g}$ is the closed subscheme whose κ-valued points are pairs $(x, v) \in X(\kappa) \times \mathfrak{g}$ satisfying $v \in \mathfrak{b}_x$ (here we abuse notation by identifying $\mathfrak{g}$ with the affine scheme $\operatorname{Spec} \operatorname{Sym}^*_\kappa \mathfrak{g}^\vee$).
- The triangulated functor $\pi_* : \mathrm{hQCoh}(X^{(1)}) \to \mathrm{hQCoh}(Y^{(1)})$ is induced by pushforward along a map of spectral schemes $\pi : X^{(1)} \to Y^{(1)}$ which exhibits $X^{(1)}$ as the ordinary scheme underlying $Y^{(1)}$.

We refer the reader to [174] and [**?**] for more details.

Another motivation for the theory of spectral algebraic geometry is that the triangulated categories which appear have better formal properties than their classical counterparts.

Example 0.1.2.4. Suppose we are given a pullback diagram σ :

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of quasi-compact separated schemes. To the morphisms f and f' we can associate (derived) pushforward functors

$$Rf_* : D(X) \to D(Y) \qquad Rf'_* : D(X') \to D(Y'),$$

to the morphisms g and g' we can associate (derived) pullback functors

$$Lg^* : D(Y) \to D(Y') \qquad Lg'^* : D(X) \to D(X'),$$

and there is an associated Beck-Chevalley transformation $\alpha : Lg^* \circ Rf_* \to Rf'_* \circ Lg'^*$. This natural transformation is an equivalence if either f or g is flat, but not in general. One can be more precise: the morphism α is an equivalence precisely when the diagram σ is *also* a pullback square in the setting of spectral schemes.

The situation in spectral algebraic geometry is simpler: for *any* pullback diagram σ :

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of quasi-compact separated spectral schemes, the associated diagram of triangulated categories

$$\begin{array}{ccc} \mathrm{hQCoh}(X') & \xleftarrow{g'^*} & \mathrm{hQCoh}(X) \\ \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{hQCoh}(Y') & \xleftarrow{g^*} & \mathrm{hQCoh}(Y) \end{array}$$

commutes up to canonical isomorphism (see Proposition 2.5.4.5).

Example 0.1.2.4 illustrates another general point: in classical algebraic geometry, the role of flatness hypotheses is often to guarantee that fiber products of ordinary schemes agree with the analogous fiber products in the setting of spectral schemes. If one extends the vocabulary of algebraic geometry to include spectral schemes, then flatness hypotheses are often superfluous (or can be substantially weakened).

Example 0.1.2.5 (Proper Descent)**.** Let $f : X \to S$ be a morphism of schemes which is quasi-compact and faithfully flat. Then f is of effective descent for quasi-coherent sheaves. More precisely, the abelian category $\mathcal{A}_S$ of quasi-coherent sheaves on S can be identified with the (homotopy) inverse limit $\varprojlim \mathcal{A}_{X_\bullet}$, where $X_\bullet$ denotes the simplicial scheme obtained from f (so that $X_n = X \times_S X \times_S \cdots \times_S X$ is the $(n+1)$-fold fiber product of X over S) and $\mathcal{A}_{X_\bullet}$ denotes the associated cosimplicial abelian category.

In the setting of spectral algebraic geometry, one can formulate and prove variants of this assumption without a flatness hypothesis. For example, if $f : X \to S$ is a map of Noetherian schemes which is proper, surjective, and of finite presentation and $X_\bullet$ is defined as above, then there is an associated equivalence $\mathrm{QCoh}(S) \to \varprojlim \mathrm{QCoh}(X_\bullet)$ of stable ∞-categories (see Theorem 5.6.6.1). One can even relax the hypothesis that f is surjective: in general, the inverse limit $\varprojlim \mathrm{QCoh}(X_\bullet)$ can be identified with $\mathrm{QCoh}(\widehat{S})$, where $\widehat{S}$ denotes the formal completion of S along the image of f. For example, suppose that S is a Noetherian scheme and let $\eta : \{s\} \hookrightarrow S$ be the inclusion of a closed point $s \in S$ (regarded as a reduced closed subscheme of S). Then, for any quasi-coherent sheaf $\mathscr{F}$ on S, the restriction of $\mathscr{F}$ to the formal completion $\widehat{S}$ can be recovered from the fiber $\eta^* \mathscr{F}$, together with "derived descent data" whose specification involves the iterated fiber products $X_n = \{s\} \times_S \{s\} \times_S \cdots \times_S \{s\}$.

Of course, it is essential here that these iterated fiber product are formed in the setting of spectral schemes (and therefore depend on the local structure of S near the point s, not only on the residue field of S at s).

0.1.3 Deformation Theory

Let X_0 be a smooth algebraic variety over the field $\mathbf{C}$ of complex numbers. One of the central aims of *deformation theory* is to understand the algebraic varieties which are "close to X_0" in some sense. For example, suppose that X_0 appears as the fiber of a morphism of schemes $f : X \to S$ at some point $s \in S$ (having residue field $\mathbf{C}$). One might hope that there is a close relationship between properties of the fiber X_0 and properties of other fibers $X_{s'} = X \times_S \{s'\}$, or of the scheme X itself. To guarantee this, one typically needs to make global assumptions about the morphism f (for example, that it is flat or proper) and also about base scheme S. For example, if $s, s' \in S$ are points belonging to different connected components of S, then there is no reason to expect any relationship between the fibers $X_0 = X \times_S \{s\}$ and $X_{s'} = X \times_S \{s'\}$. One way to avoid this concern is to restrict attention to the case where the scheme S has only one point: for example, the case where $S = \operatorname{Spec} R$ for some local Artinian ring R. This leads to the subject of *formal* deformation theory: the study of algebraic varieties which are, in some sense, "infinitesimally close" to X_0.

Let us begin by considering the simplest nontrivial case: let S denote the spectrum of the ring $\mathbf{C}[\epsilon]/(\epsilon^2)$ (sometimes referred to as the *ring of dual numbers*). If X_0 is an algebraic variety over $\mathbf{C}$, then a *first order deformation* of X_0 is pullback diagram of schemes

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbf{C} & \longrightarrow & \operatorname{Spec} \mathbf{C}[\epsilon]/(\epsilon^2), \end{array}$$

where the vertical maps are flat. The following proposition summarizes some standard facts about first-order deformations:

Proposition 0.1.3.1. *Let X_0 be a smooth algebraic variety over $\mathbf{C}$. Then:*

(a) For every first-order deformation X of X_0, there is a canonical bijection

$$\{ \text{ Automorphisms of } X \text{ that restrict to the identity on } X_0 \ \} \simeq \mathrm{H}^0(X_0; T_{X_0}).$$

Here T_{X_0} denotes the tangent bundle of the smooth variety X_0.

(b) There is a canonical bijection

$$\{ \text{ Isomorphism classes of first-order deformations of } X_0 \ \} \simeq \mathrm{H}^1(X_0; T_{X_0}).$$

Many of the central ideas of spectral algebraic geometry originated from the desire to extend Proposition 0.1.3.1 to the case where X_0 is not smooth. To understand the issues involved, let us begin by how the tangent bundle is defined in the special case where $X_0 = \operatorname{Spec} B$ is an affine algebraic variety over $\mathbf{C}$. Recall that to any homomorphism of commutative rings $\phi : A \to B$, one can introduce a B-module $\Omega_{B/A}$ called the *module of Kähler differentials of B relative to A*: it is generated as an B-module by symbols $\{db\}_{b \in B}$ which are subject only to the relations

$$d(b+b') = db + db' \qquad d(bb') = bdb' + b'db \qquad d\phi(a) = 0.$$

In the special case where B is a *smooth* A-algebra, the B-module $\Omega_{B/A}$ is locally free of finite rank. In particular, if $X_0 = \operatorname{Spec} B$ is smooth affine algebraic variety over $\mathbf{C}$, then the B-module $\Omega_{B/\mathbf{C}}$ is locally free of finite rank, and therefore corresponds to an algebraic vector bundle $T^*_{X_0}$ on X_0 whose dual is (by definition) the tangent bundle T_{X_0}.

To a pair of ring homomorphisms $\phi : A \to B$ and $\psi : B \to C$, one can associate a short exact sequence of Kähler differentials

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0. \tag{1}$$

This sequence is generally not exact on the left unless the ring homomorphism ψ is smooth. To remedy this defect, André and Quillen (independently) introduced the theory known as *André-Quillen homology* (generalizing earlier work of Lichtenbaum and Schlessinger), which has the following features:

- To every homomorphism of commutative rings $\phi : A \to B$ and every B-module M, one associates a sequence of B-modules $\{D_n(B/A; M)\}_{n \geqslant 0}$, called the *André-Quillen homology groups of B relative to A with coefficients in M*.

- The theory of André-Quillen homology generalizes the theory of Kähler differentials: for every homomorphism of commutative rings $\phi : A \to B$ and every B-module M, there is a canonical isomorphism $D_0(B/A; M) \simeq M \otimes_B \Omega_{B/A}$.

- If $\phi : A \to B$ is a smooth homomorphism of commutative rings, then the André-Quillen homology groups $D_n(B/A; M)$ vanish for $n > 0$ and any B-module M.

- To every composable pair of commutative ring homomorphisms $\phi : A \to B$ and $\psi : B \to C$ and every C-module M, one can associate a long exact sequence

  $$\cdots \to D_{n+1}(C/B; M) \to D_n(B/A; M) \to D_n(C/A; M) \to D_n(C/B; M) \to \cdots,$$

  extending the short exact sequence (1) in the special case $M = C$.

The André-Quillen homology groups $D_*(B/A; M)$ are obtained from a more fundamental invariant $L^{\mathrm{alg}}_{B/A}$, called the *cotangent complex of B over A*, a chain complex of (projective) B-modules which determines André-Quillen homology via the formula $D_*(B/A; M) = \mathrm{H}_*(M \otimes_B L^{\mathrm{alg}}_{B/A})$. This invariant was generalized in the work of Illusie (see [?] and [100]), which associates to every morphism $f : X \to S$ of schemes an object $L^{\mathrm{alg}}_{X/S}$ of the derived category $D(X)$, specializing to the chain complex $L^{\mathrm{alg}}_{B/A}$ in the case where $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$ are affine. With this generalization, Proposition 0.1.3.1 extends as follows:

Proposition 0.1.3.2. *Let X_0 be an algebraic variety over $\mathbf{C}$ (not necessarily smooth). Then:*

(a) *For every first-order deformation X of X_0, there is a canonical bijection*

$$\{\textit{Automorphisms of } X \textit{ restricting to the identity on } X_0\} \simeq \mathrm{Ext}^0_{D(X_0)}(L^{\mathrm{alg}}_{X/\operatorname{Spec}\mathbf{C}}, \mathscr{O}_X).$$

(b) *There is a canonical bijection*

$$\{\textit{Isomorphism classes of first-order deformations of } X_0\} \simeq \mathrm{Ext}^1_{D(X_0)}(L^{\mathrm{alg}}_{X/\operatorname{Spec}\mathbf{C}}, \mathscr{O}_X).$$

Illusie's work on the cotangent complex was an important precursor to the theory of spectral algebraic geometry developed in this book. To understand this point, let us begin by recalling that the module of Kähler differentials $\Omega_{B/A}$ can be characterized by a universal property: for any B-module M, there is a canonical bijection

$$\begin{array}{c}\{\ B\text{-module homomorphisms } \Omega_{B/A} \to M\} \\ \downarrow \theta_0 \\ \{\ A\text{-algebra sections of the projection map } B \oplus M \to B\ \}\end{array}$$

which carries a map $\lambda : \Omega_{B/A} \to M$ to the A-algebra homomorphism $s_\lambda : B \to B \oplus M$ given by $s_\lambda(b) = b + \lambda(db)$. The entire cotangent complex $L^{\mathrm{alg}}_{B/A}$ can be characterized by an analogous universal property. To simplify the discussion, let us assume that A and B are $\mathbf{Q}$-algebras; in this case, we will denote the cotangent complex $L^{\mathrm{alg}}_{B/A}$ simply by $L_{B/A}$ (see Remark 0.1.3.7 below). If M is a chain complex of B-modules, then the direct sum $B \oplus M$ can be regarded as an $\mathbb{E}_\infty$-algebra over B, with homotopy groups given by

$$\pi_*(B \oplus M) = \begin{cases} B \oplus \mathrm{H}_0(M) & \text{if } * = 0 \\ \mathrm{H}_*(M) & \text{if } * \neq 0. \end{cases}$$

We then have a canonical bijection

$$\begin{array}{c}\{\ \text{Maps from } L_{B/A} \text{ into } M \text{ in the derived category } D(B)\ \} \\ \downarrow \theta \\ \{\text{Homotopy classes of sections of the projection } q : B \oplus M \to B\}\end{array}$$

where we interpret q as a map of $\mathbb{E}_\infty$-algebras over A. Note that $L_{B/A}$ is characterized by this universal property: that is, we can *define* $L_{B/A}$ to be an object which corepresents the functor

$$M \mapsto \{\text{Homotopy classes of sections of the projection } q : B \oplus M \to B\}.$$

This supplies a definition of $L_{B/A}$ which makes sense for *any* morphism of $\mathbb{E}_\infty$-rings $\phi : A \to B$. This leads to the theory of *topological* André-Quillen homology (see [145], [15], [16]). Like the theory of ordinary André-Quillen homology, it can be "relativized" to non-affine settings: to every morphism of spectral schemes $f : X \to S$, one can associate an object $L_{X/S}$ of the triangulated category $\mathrm{hQCoh}(X)$ called the *relative cotangent complex of f*. The construction $(f : X \to S) \mapsto L_{X/S}$ has the following features:

(1) Suppose that X and S are ordinary schemes over $\operatorname{Spec} \mathbf{Q}$. Then the relative cotangent complex $L_{X/S}$ (in the setting of spectral schemes) agrees with the cotangent complex $L^{\mathrm{alg}}_{X/S}$ of Illusie (for the case of positive and mixed characteristic, see Remark 0.1.3.7 below).

(2) Let $f : (X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$ be a morphism of spectral schemes, let $X_0 = (X, \pi_0 \mathscr{O}_X)$ and $S_0 = (S, \pi_0 \mathscr{O}_S)$ be their underlying ordinary schemes, and let $\mathcal{A}$ be the abelian category of quasi-coherent sheaves on X_0. Then the "degree zero" part of $L_{X/S}$ agrees, as an object of $\mathcal{A}$, with the sheaf of Kähler differentials Ω_{X_0/S_0} of the underlying map $f_0 : X_0 \to S_0$.

One of the advantages of working in the setting of spectral algebraic geometry is that there is a much larger class of geometric objects for which the cotangent complex is well-behaved:

(i) If X is a spectral $\mathbf{C}$-scheme of finite presentation, then the cotangent complex $L_{X/\operatorname{Spec} \mathbf{C}} \in \mathrm{QCoh}(X)$ is perfect: that is, it is dualizable as an object of $\mathrm{QCoh}(X)$.

(ii) If X is an ordinary $\mathbf{C}$-scheme of finite presentation, then the cotangent complex $L_{X/\operatorname{Spec} \mathbf{C}}$ is a perfect complex if and only if X is a local complete intersection (see [**?**]).

To reconcile (i) and (ii), we remark that if X is an ordinary scheme, then the assumption that X is of finite presentation as a $\mathbf{C}$-scheme does not imply that it is of finite presentation as a spectral $\mathbf{C}$-scheme. However, the converse does hold: more generally, if $(X, \mathscr{O}_X)$ is a spectral scheme of finite presentation over $\mathbf{C}$ (as a spectral scheme), then the underlying ordinary scheme $X_0 = (X, \pi_0 \mathscr{O}_X)$ is of finite presentation over $\mathbf{C}$ (as a scheme). In this case, the cotangent complex $L_{X/\operatorname{Spec} \mathbf{C}}$ is often a much more natural and useful object than $L_{X_0/\operatorname{Spec} \mathbf{C}}$.

Example 0.1.3.3 (Quot Schemes)**.** Let X be a projective algebraic variety over $\mathbf{C}$ and let $\mathscr{F}$ be a quasi-coherent sheaf on X. For every commutative $\mathbf{C}$-algebra R, let X_R denote the fiber product $X \times_{\operatorname{Spec} \mathbf{C}} \operatorname{Spec} R$ and let $\mathscr{F}_R$ denote the pullback of $\mathscr{F}$ to X_R. We let $F(R)$ denote the collection of isomorphism classes of exact sequences

$$0 \to \mathscr{F}' \to \mathscr{F}_R \to \mathscr{F}'' \to 0$$

in the abelian category of quasi-coherent sheaves on X_R for which $\mathscr{F}''$ is flat over R. The construction $R \mapsto F(R)$ is representable by a $\mathbf{C}$-scheme Quot (see [**?**]). Given a $\mathbf{C}$-valued point $\eta \in F(\mathbf{C})$ classifying an exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0,$$

the Zariski cotangent space of Quot at the point η is the dual of the vector space $\operatorname{Hom}_{\mathcal{A}}(\mathscr{F}', \mathscr{F}'')$, where $\mathcal{A}$ is the abelian category of quasi-coherent sheaves on X. This vector space can be described as the 0th homology of the cotangent fiber $\eta^* L_{\operatorname{Quot} / \operatorname{Spec} \mathbf{C}}$. However, the scheme Quot is usually highly singular, so the entire cotangent fiber $\eta^* L_{\operatorname{Quot} / \operatorname{Spec} \mathbf{C}}$ can be difficult to describe.

The situation is better if we work in the setting of spectral algebraic geometry. The definitions of X_R, $\mathscr{F}_R$, and $F(R)$ make sense more generally when R is an $\mathbb{E}_\infty$-algebra over $\mathbf{C}$, and the resulting functor on $\mathbb{E}_\infty$-rings is representable by a spectral scheme Quot^+ having Quot is its underlying scheme (this object was introduced in [41]). The structure of the cotangent complex $L_{\operatorname{Quot}^+ / \operatorname{Spec} \mathbf{C}}$ has an immediate description in terms of the functor F (on $\mathbb{E}_\infty$-algebras). For example, if $\eta \in F(\mathbf{C})$ is as above, then the complex $\eta^* L_{\operatorname{Quot}^+ / \operatorname{Spec} \mathbf{C}}$ is dual to $\operatorname{RHom}(\mathscr{F}', \mathscr{F}'')$: in particular, the nth homology group of $\eta^* L_{\operatorname{Quot}^+ / \operatorname{Spec} \mathbf{C}}$ can be identified with the $\mathbf{C}$-linear dual of $\operatorname{Ext}^n_{\mathcal{A}}(\mathscr{F}', \mathscr{F}'')$.

Example 0.1.3.3 illustrates a general phenomenon: if Z_0 is a scheme representing which represents a functor F on the category of commutative rings, there is often a natural way to extend the definition of $F(R)$ to the case where R is an $\mathbb{E}_\infty$-ring in such a way that extended functor is representable by a spectral scheme Z. Roughly speaking, we can think of the specification of Z as given by "equipping Z_0 with a deformation theory" (we will make this idea precise in Part **??**; see Theorem 18.1.0.1), according to the rough heuristic

$$\text{Spectral algebraic geometry} = \text{Algebraic Geometry} + \text{Deformation Theory}.$$

From the above perspective, the subject of this book is a natural outgrowth of deformation theory, which allows us to think about invariants like the cotangent complex in a more flexible and general setting. However, the ideas of spectral algebraic geometry lead to new and useful ways to think about deformation-theoretic questions even for smooth algebraic varieties. For example, consider the following variant of Proposition 0.1.3.1:

Proposition 0.1.3.4. *Let X_0 be a smooth algebraic variety over $\mathbf{C}$. Then there is a canonical "obstruction class map"*

$$\rho : \{ \textit{ First order deformations of } X_0 \} \to \mathrm{H}^2(X_0; T_{X_0})$$

with the following property: if X is a first-order deformation of X_0, then the obstruction class $\rho(X) \in \mathrm{H}^2(X_0; T_{X_0})$ vanishes if and only if X can be extended to a second-order deformation of X_0: that is, if and only if there exists a pullback square of schemes

$$\begin{array}{ccc} X & \longrightarrow & \overline{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbf{C}[\epsilon]/(\epsilon^2) & \longrightarrow & \operatorname{Spec} \mathbf{C}[\epsilon]/(\epsilon^3) \end{array}$$

in which the vertical maps are flat.

Proposition 0.1.3.4 is very useful: it allows us to convert an *a priori* nonlinear problem (deforming an algebraic variety) into a linear one (checking that a certain cohomology class vanishes). This linear problem is often difficult, but in some cases it is trivial: for example, if X_0 is a smooth curve, then $\mathrm{H}^2(X_0; T_{X_0}) \simeq 0$, so Proposition 0.1.3.4 implies that *any* first-order deformation can be extended to a second-order deformation.

One might object that Proposition 0.1.3.4 is not as satisfying as Proposition 0.1.3.1. Note that Proposition 0.1.3.1 provides concrete geometric interpretations for the cohomology groups $\mathrm{H}^0(X_0; T_{X_0})$ and $\mathrm{H}^1(X_0; T_{X_0})$. Proposition 0.1.3.4 tells us that the cohomology group $\mathrm{H}^2(X_0; T_{X_0})$ is *related* to the problem of extending first-order deformations of X_0 to second-order deformations, but it does not tell us what a general element of $\mathrm{H}^2(X_0; T_{X_0})$ *is*. The language of spectral algebraic geometry provides a remedy for this situation. More precisely, it allows us to formulate a generalization of Proposition 0.1.3.1 which supplies a geometric interpretation for *all* of the cohomology groups $\mathrm{H}^n(X_0; T_{X_0})$ and which has Proposition 0.1.3.4 as a consequence.

Fix a smooth $\mathbf{C}$-scheme X_0 as in Proposition 0.1.3.4. For every commutative $\mathbf{C}$-algebra R equipped with an augmentation $\rho : R \to \mathbf{C}$, let us define a *deformation of X_0 over R* to be a flat R-scheme X_R together with an isomorphism $X_0 \simeq \operatorname{Spec} \mathbf{C} \times_{\operatorname{Spec} R} X_R$. The theory of spectral algebraic geometry allows us to consider a more general notion of deformation: if R is an $\mathbb{E}_\infty$-algebra over $\mathbf{C}$ (still equipped with an augmentation $\rho : R \to \mathbf{C}$), then we can contemplate *spectral schemes* X_R equipped with a flat map $X_R \to \operatorname{Spec} R$ and an equivalence $X_0 \simeq \operatorname{Spec} \mathbf{C} \times_{\operatorname{Spec} R} X_R$ (if R is an ordinary commutative ring, this reduces to the previous definition: the flatness of X_R over $\operatorname{Spec} R$ guarantees that X_R is an ordinary scheme). We then have the following generalization of Proposition 0.1.3.1, which we will discuss in Part V (see Proposition 19.4.3.1 and Corollary 19.4.3.3):

Proposition 0.1.3.5. *Let X_0 be a smooth algebraic variety over the field $\mathbf{C}$ of complex numbers and let R denote the direct sum $\mathbf{C} \oplus M$, where M is the chain complex of complex vector spaces which is concentrated in degree $n \geqslant 0$ and is isomorphic to $\mathbf{C}$ in degree n. Let us regard R as an $\mathbb{E}_\infty$-algebra over $\mathbf{C}$. Then:*

(a) If X_R is any deformation of X_0 over R, then there is a canonical bijection

$$\{\textit{Automorphisms of } X_R \textit{ that are the identity on } X_0\}/\textit{homotopy} \simeq \mathrm{H}^n(X_0; T_{X_0}).$$

(b) There is a canonical bijection

$$\{\textit{Deformations of } X_0 \textit{ over } R\}/\textit{homotopy equivalence} \simeq \mathrm{H}^{n+1}(X_0; T_{X_0}).$$

Remark 0.1.3.6. In the special case $n = 0$, the $\mathbb{E}_\infty$-ring $R = \mathbf{C} \oplus M$ can be identified with the ordinary ring of dual numbers $\mathbf{C}[\epsilon]/(\epsilon^2)$, and Proposition 0.1.3.5 reduces to Proposition 0.1.3.1. Proposition 0.1.3.5 implies that *all* of the cohomology groups $\mathrm{H}^m(X_0; T_{X_0})$ arise naturally when studying deformations of X_0 over "shifted" versions of the ring of dual numbers.

To relate Propositions 0.1.3.4 and 0.1.3.5, we note that the $\mathbf{C}[\epsilon]/(\epsilon^3)$ can be regarded as a an extension of $\mathbf{C}[\epsilon]/(\epsilon^2)$ by the square-zero ideal $(\epsilon^2) \simeq \mathbf{C}$. This implies that there is a pullback diagram of $\mathbb{E}_\infty$-algebras

$$\begin{array}{ccc} \mathbf{C}[\epsilon]/(\epsilon^3) & \longrightarrow & \mathbf{C} \\ \downarrow & & \downarrow \\ \mathbf{C}[\epsilon]/(\epsilon^2) & \xrightarrow{\phi} & R, \end{array}$$

where R is defined as in Proposition 0.1.3.5 in the special case $n = 1$. We can interpret this pullback square geometrically as supplying a pushout diagram of affine spectral schemes

$$\begin{array}{ccc} \operatorname{Spec} \mathbf{C}[\epsilon]/(\epsilon^3) & \longleftarrow & \operatorname{Spec} \mathbf{C} \\ \uparrow & & \uparrow \\ \operatorname{Spec} \mathbf{C}[\epsilon]/(\epsilon^2) & \longleftarrow & \operatorname{Spec} R. \end{array}$$

It follows that if X is a first-order deformation of X_0, then X extends to a second-order deformation if and only if the fiber product $X_R = X \times_{\operatorname{Spec} \mathbf{C}[\epsilon]/(\epsilon^2)} \operatorname{Spec} R$ is a trivial deformation of X_0 over R (that is, if and only if it is equivalent to $X_0 \times_{\operatorname{Spec} \mathbf{C}} \operatorname{Spec} R$). This is equivalent to the vanishing of a certain element $\rho(X) \in \mathrm{H}^2(X_0; T_{X_0})$, where ρ is the "obstruction map" given by the composition

$$\begin{aligned} \{ \text{ Deformations of } X_0 \text{ over } \mathbf{C}[\epsilon]/(\epsilon^2) \ \} \ &\to \ \{ \text{ Deformations of } X_0 \text{ over } R\} \\ &\simeq \ \mathrm{H}^2(X_0; T_{X_0}). \end{aligned}$$

where the first map is given by extension of scalars along ϕ and the second follows from the identification of Proposition 0.1.3.5.

Remark 0.1.3.7. Let $\phi : A \to B$ be a homomorphism of commutative rings, which we can also regard as a morphism of $\mathbb{E}_\infty$-rings. In general, the algebraic André-Quillen homology of ϕ (denoted by $L^{\mathrm{alg}}_{B/A}$ in the above discussion and throughout this book) is different from the topological André-Quillen homology of ϕ (denoted by $L_{B/A}$ in the above discussion and throughout this book), though they are rationally equivalent (and therefore coincide whenever B is a $\mathbf{Q}$-algebra). The deformation theory of spectral schemes is controlled by topological André-Quillen homology. For applications in positive and mixed characteristic, it is often more appropriate to use the theory of *derived algebraic geometry*, in which deformations are controlled by algebraic André-Quillen homology. We will give a detailed exposition of derived algebraic geometry and its relationship to spectral algebraic geometry in Part VII.

0.2 Prerequisites

Throughout this book, we will make extensive use of the language of ∞-categories developed in [138] and [139]. The reader will also need some familiarity with stable homotopy theory and the theory of structured ring spectra, which are developed from the ∞-categorical perspective in [139] (for a different approach to the same material, see [60]). For the reader's convenience, we include a brief (and incomplete) expository account of some of the relevant material below. For a more detailed account (which includes precise definitions and proofs), we refer the reader to [138] and [139]. Since we will need to refer to these texts frequently in this book, we adopt the following conventions:

(HTT) We will indicate references to [138] using the letters HTT.

(HA) We will indicate references to [139] using the letters HA.

For example, Theorem HTT.6.1.0.6 refers to Theorem 6.1.0.6 of [138].

The other main prerequisite for reading this book is some familiarity with classical algebraic geometry. To some extent, this is logically unnecessary: the theory of spectral algebraic geometry is developed "from scratch" in this book, and most of our references to the classical theory are purely for motivation. Moreover, we have made an effort to keep this book as self-contained as possible as far as algebraic geometry and commutative algebra are concerned: we have generally opted to include proofs of standard results (particularly in cases where the use of "derived" methods can shed some additional light) except in a few cases which would take us too far afield. Nevertheless, a reader who is not familiar with the classical theory of schemes will almost surely find this book impenetrable (if he or she has even made it this far).

0.2.1 Homotopy Theory and Simplicial Sets

For every integer $n \geqslant 0$, let $|\Delta^n|$ denote the topological n-simplex, given by

$$|\Delta^n| = \{(x_0, \ldots, x_n) \in \mathbf{R}^{n+1}_{\geqslant 0} : x_0 + x_1 + \cdots + x_n = 1\}.$$

If X is a topological space, we let $\mathrm{Sing}_n(X)$ denote the set of continuous maps from $|\Delta^n|$ into X. These sets play a role in defining many important invariants of X: for example, the singular homology groups of X are obtained from the chain complex of free abelian groups

$$\cdots \to \mathbf{Z}[\mathrm{Sing}_2 X] \xrightarrow{d_0 - d_1 + d_2} \mathbf{Z}[\mathrm{Sing}_1 X] \xrightarrow{d_0 - d_1} \mathbf{Z}[\mathrm{Sing}_0 X]$$

where each $d_k : \mathrm{Sing}_n X \to \mathrm{Sing}_{n-1} X$ is the map which assigns to a simplex the face opposite its kth vertex. To describe the structure given by the sets $\{\mathrm{Sing}_n(X)\}_{n \geqslant 0}$ and the face maps d_k in a more systematic way, it will be useful to introduce a bit of terminology.

Definition 0.2.1.1. For each integer $n \geqslant 0$, we let $[n]$ denote the finite linearly ordered set $\{0 < 1 < \cdots < n\}$. We define a category $\mathbf{\Delta}$ as follows:

- The objects of $\mathbf{\Delta}$ are sets of the form $[n]$ for $n \geqslant 0$.
- A morphism from $[m]$ to $[n]$ in $\mathbf{\Delta}$ consists of a nondecreasing function $\alpha : [m] \to [n]$.

We will refer to $\mathbf{\Delta}$ as the *category of combinatorial simplices*. A *simplicial set* is a functor $S_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, where $\mathcal{S}\mathrm{et}$ denotes the category of sets. In this case, we will denote the value of $S_\bullet$ on the object $[n] \in \mathbf{\Delta}$ by S_n, and refer to it as the *set of n-simplices of S*. We let $\mathcal{S}\mathrm{et}_\Delta$ denote the category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ of simplicial sets.

For each $n \geqslant 0$, it is useful to think of the set $[n] = \{0 < 1 < \cdots < n\}$ as the set of vertices of the topological n-simplex $|\Delta^n|$. Every map of sets $\alpha : [m] \to [n]$ extends uniquely to a linear map $\rho : |\Delta^m| \to |\Delta^n|$, given in coordinates by the construction

$$(x_0, \ldots, x_m) \mapsto (\sum_{\alpha(i)=0} x_i, \ldots, \sum_{\alpha(i)=n} x_i).$$

If X is a topological space, then composition with ρ determines a map $\mathrm{Sing}_n X \to \mathrm{Sing}_m X$. In particular, we can regard the construction $([n] \in \mathbf{\Delta}) \mapsto (\mathrm{Sing}_n X \in \mathcal{S}\mathrm{et})$ as a simplicial set. We refer to this simplicial set as the *singular simplicial set* of X and denote it by $\mathrm{Sing}_\bullet X$.

The construction $X \mapsto \mathrm{Sing}_\bullet X$ determines a functor $\mathrm{Sing}_\bullet$ from the category $\mathcal{T}\mathrm{op}$ of topological spaces to the category $\mathcal{S}\mathrm{et}_\Delta$ of simplicial sets. This functor admits a left adjoint

$$\mathcal{S}\mathrm{et}_\Delta \to \mathcal{T}\mathrm{op}$$

$$S_\bullet \mapsto |S_\bullet|$$

which we refer to as *geometric realization.* If X is a topological space which has the homotopy type of a CW complex, then the counit map $|\operatorname{Sing}_\bullet X| \to X$ is a homotopy equivalence (the assumption that X has the homotopy type of a CW complex is necessary here: for any simplicial set $S_\bullet$, the geometric realization $|S_\bullet|$ is a CW complex). Consequently, from the perspective of homotopy theory, no information is lost by discarding the original space X in favor of the simplicial set $\operatorname{Sing}_\bullet X$ (see Remark 0.2.1.5 below). In fact, it is possible to develop the theory of algebraic topology in entirely combinatorial terms, using simplicial sets as surrogates for topological spaces. Moreover, this approach has many advantages:

- Many of the most important algebraic invariants of a topological space X (such as homotopy groups, homology, and cohomology) are obtained by studying maps from n-simplices (and their boundaries) into X. Consequently, these invariants can be extracted more directly from the singular simplicial set $\operatorname{Sing}_\bullet X$ itself.

- Working with simplicial sets rather than topological spaces avoids many of the technicalities and pathologies of point-set topology.

- When applying homotopy-theoretic methods to areas of mathematics outside of topology (a major theme of this book), the association between homotopy theory and topological spaces can be an unnecessary distraction.

We will assume throughout this book that the reader is familiar with the homotopy theory of simplicial sets. We devote the remainder of this section to giving a quick review of some basic definitions and notations; for a more thorough introduction, we refer the reader to the texts [81] and [151].

Notation 0.2.1.2. For each $n \geqslant 0$, we let $\Delta^n \in \mathcal{S}\mathrm{et}_\Delta = \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ denote the simplicial set which is represented by the object $[n] \in \mathbf{\Delta}$, so that the m-simplices of Δ^n are given by nondecreasing maps $[m] \to [n]$. We will refer to Δ^n as the *standard n-simplex.* For any simplicial set $S_\bullet$, Yoneda's lemma provides a canonical bijection $S_n \simeq \operatorname{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n, S_\bullet)$. We will often abuse notation by using this bijection to identify elements of S_n with the corresponding maps $\sigma : \Delta^n \to S_\bullet$, referring to either datum as an *n-simplex* of $S_\bullet$.

We let $\partial \Delta^n$ denote the simplicial subset of Δ^n whose m-simplices are nondecreasing maps $\alpha : [m] \to [n]$ which are not surjective. For $0 < i < n$, we let Λ^n_i denote the simplicial subset of $\partial \Delta^n \subset \Delta^n$ whose m-simplices are nondecreasing maps $\alpha : [m] \to [n]$ having the property that $\alpha([m]) \cup \{i\} \subsetneq [n]$. We will refer to $\partial \Delta^n$ as the *boundary* of Δ^n and to Λ^n_i as the *ith horn* of Δ^n.

If $S_\bullet$ is a simplicial set, we will refer to the element of S_0 as the *vertices* of S and the elements of S_1 as the *edges* of S. Each vertex $v \in S_0$ can be identified with a map $\Delta^0 \to S_\bullet$. We will generally abuse notation by denoting the domain of this map by $\{v\}$.

Taking $S_\bullet$ to be the standard n-simplex, we see that each element $0 \leqslant i \leqslant n$ determines a map $\{i\} \simeq \Delta^0 \to \Delta^n$, which we will refer to as the *ith vertex of* Δ^n.

Remark 0.2.1.3. For each $n \geqslant 0$, the geometric realization of the standard n-simplex $\Delta^n \in \mathcal{S}\mathrm{et}_\Delta$ can be identified with the topological n-simplex

$$|\Delta^n| = \{(x_0, \dots, x_n) \in \mathbf{R}^{n+1} : x_0 + x_1 + \cdots + x_n = 1\}.$$

Under this identification, the geometric realization of $\partial \Delta^n$ corresponds to the subset of $|\Delta^n|$ consisting of those n-tuples $(x_0, \dots, x_n)$ for which at least one coordinate x_j vanishes. The ith horn Λ^n_i corresponds to the subset of $|\Delta^n|$ consisting of those tuples $(x_0, \dots, x_n)$ satisfying $x_j = 0$ for some $j \neq i$. More informally: $|\partial \Delta^n|$ is obtained from the topological n-simplex $|\Delta^n|$ by deleting its interior, while $|\Lambda^n_i|$ is obtained from $|\Delta^n|$ by deleting its interior together with the face opposite the ith vertex.

Definition 0.2.1.4. Let $S_\bullet$ be a simplicial set. We will say that $S_\bullet$ is a *Kan complex* if, for every pair of integers $0 \leqslant i \leqslant n$, every map $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended to an n-simplex $\sigma : \Delta^n \to S_\bullet$. We let $\mathcal{K}\mathrm{an}$ denote the full subcategory of $\mathcal{S}\mathrm{et}_\Delta$ spanned by the Kan complexes; we will refer to $\mathcal{K}\mathrm{an}$ as the *category of Kan complexes*.

Let $S_\bullet$ and $T_\bullet$ be simplicial sets. Given a pair of maps $f, g : S_\bullet \to T_\bullet$, a *simplicial homotopy* from f to g is a map of simplicial sets $h : S_\bullet \times \Delta^1 \to T_\bullet$ such that $h|_{S_\bullet \times \{0\}} = f$ and $h|_{S_\bullet \times \{1\}} = g$. We will say that f and g are *simplicially homotopic* if there exists a simplicial homotopy from f to g. If $T_\bullet$ is a Kan complex, then this is an equivalence relation. We let $\mathrm{h}\mathcal{K}\mathrm{an}$ denote the category whose objects are Kan complexes, where the morphisms from $S_\bullet$ to $T_\bullet$ in $\mathrm{h}\mathcal{K}\mathrm{an}$ are the simplicial homotopy classes of maps from $S_\bullet$ to $T_\bullet$. We will refer to $\mathrm{h}\mathcal{K}\mathrm{an}$ as the *homotopy category of Kan complexes*.

Remark 0.2.1.5. The homotopy category of Kan complexes is equivalent to the homotopy category of CW complexes. More precisely, one can prove the following:

- For every topological space X, the singular simplicial set $\mathrm{Sing}_\bullet X$ is a Kan complex (this follows from the observation that each horn $|\Lambda^n_i|$ is a retract of the corresponding n-simplex $|\Delta^n|$; see Remark 0.2.1.3). Moreover, if $f, g : X \to Y$ are homotopic maps between topological spaces X and Y, then the induced maps $\mathrm{Sing}_\bullet(f), \mathrm{Sing}_\bullet(g) : \mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(Y)$ are simplicially homotopic. Consequently, construction $X \mapsto \mathrm{Sing}_\bullet X$ determines a functor $\mathrm{h}\mathcal{T}\mathrm{op} \to \mathrm{h}\mathcal{K}\mathrm{an}$, where $\mathrm{h}\mathcal{T}\mathrm{op}$ is the homotopy category of topological spaces.

- For every simplicial set $S_\bullet$, the geometric realization $|S_\bullet|$ is a CW complex. Moreover, if $f, g : S_\bullet \to T_\bullet$ are maps of simplicial sets which are simplicially homotopic, then the induced maps of topological spaces $|f|, |g| : |S_\bullet| \to |T_\bullet|$ are homotopic. Consequently, the construction $S_\bullet \mapsto |S_\bullet|$ determines a functor $\mathrm{h}\mathcal{K}\mathrm{an} \to \mathrm{h}\mathcal{T}\mathrm{op}^{\mathrm{CW}}$, where $\mathrm{h}\mathcal{T}\mathrm{op}^{\mathrm{CW}}$ denotes the homotopy category of CW complexes.

- For any CW complex X, the counit map $v : |\operatorname{Sing}_\bullet X| \to X$ is a homotopy equivalence. For any Kan complex $S_\bullet$, the unit map $S_\bullet \to \operatorname{Sing}_\bullet |S_\bullet|$ is a simplicial homotopy equivalence. Consequently, the functors

$$\mathrm{h}\mathcal{T}\mathrm{op}^{\mathrm{CW}} \underset{\operatorname{Sing}_\bullet}{\overset{|\,|}{\rightleftarrows}} \mathrm{h}\mathcal{K}\mathrm{an}$$

are mutually inverse equivalences of categories.

Remark 0.2.1.6. From a spectral scheme $(X, \mathscr{O}_X)$, one can extract topological spaces of two very different types:

(a) The underlying topological space X. This space will typically be non-Hausdorff and therefore very far from the type of spaces which are usually studied in algebraic topology. Consequently, we will not be interested in the homotopy type of such a space: it is primarily a device that allows us to talk about sheaves.

(b) The underlying spaces of the $\mathbb{E}_\infty$-rings $\mathscr{O}_X(U)$. These spaces should really be regarded as only well-defined up to homotopy equivalence: in other words, *all* that we care about is their homotopy type. For example, we would never want to consider sheaves on such a space (other than locally constant sheaves), because the notion of sheaf is not homotopy invariant.

To distinguish between these possibilities, we will usually regard objects of type (b) as Kan complexes rather than as topological spaces. Unless otherwise specified, we use the term "space" to refer to a Kan complex. When we wish to refer to a set equipped with a topology, we will use the term "topological space."

0.2.2 Higher Category Theory

For a reader trained in classical algebraic geometry, the most exotic feature of spectral algebraic geometry is likely to be that all of its basic objects come equipped with an "internal" homotopy theory. To fix ideas, let us introduce a bit of terminology:

Definition 0.2.2.1. Let $\mathcal{C}$ be a category. For every pair of objects $X, Y \in \mathcal{C}$, let $\operatorname{Map}_{\mathcal{C}}(X, Y)$ denote the set of morphisms from X to Y. We will say that $\mathcal{C}$ is a *topological category* if each of the sets $\operatorname{Map}_{\mathcal{C}}(X, Y)$ has been equipped with a topology for which the composition maps

$$\operatorname{Map}_{\mathcal{C}}(Y, Z) \times \operatorname{Map}_{\mathcal{C}}(X, Y) \to \operatorname{Map}_{\mathcal{C}}(X, Z)$$

are continuous (for every triple of objects $X, Y, Z \in \mathcal{C}$).

The collection of spectral schemes (and other more general geometric objects that we study in this book) can be organized into a topological category SpSch. In particular, if X and Y are spectral schemes, then there is a notion of a *homotopy* from a morphism $f : X \to Y$ to another morphism $g : X \to Y$: namely, a path in the topological space $\mathrm{Map}_{\mathrm{SpSch}}(X, Y)$. Consequently, there is an associated notion of *homotopy equivalence*: we say that spectral schemes X and Y are *equivalent* if there exist morphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity map id_Y and $g \circ f$ is homotopic to the identity map id_X.

Warning 0.2.2.2. Let X be a spectral scheme over the field $\mathbf{C}$ of complex numbers. Then we can associate to X its set $X(\mathbf{C})$ of $\mathbf{C}$-valued points (which is the same as the set of $\mathbf{C}$-valued points of the underlying ordinary scheme of X). The set $X(\mathbf{C})$ inherits the structure of a topological space (from the usual topology on $\mathbf{C}$), which we will refer to as the *complex-analytic topology*. This topology is completely unrelated to the structure of SpSch as a topological category. If $f, g : X \to Y$ are homotopic morphisms between spectral schemes over $\mathbf{C}$, then they induce the same map at the level of underlying schemes, so that the induced maps $X(\mathbf{C}) \to Y(\mathbf{C})$ are the same. In particular, if $f : X \to Y$ is an equivalence of spectral schemes over $\mathbf{C}$, then the induced map on $\mathbf{C}$-valued points $X(\mathbf{C}) \to Y(\mathbf{C})$ is a homeomorphism (not merely a homotopy equivalence). In more informal terms: the "internal" homotopy theory of a spectral scheme X is a purely infinitesimal datum, and is invisible to all classical invariants of X (like the space $X(\mathbf{C})$ with its complex-analytic topology, or the underlying topological space of X with its Zariski topology).

The notion of homotopy plays a central role in the theory in the theory of spectral algebraic geometry: all meaningful properties of spectral schemes are invariant under equivalence, and all meaningful properties of morphisms between spectral schemes are invariant under homotopy. This motivates the following definition:

Definition 0.2.2.3. Let $\mathcal{C}$ be a topological category. The *homotopy category* $\mathrm{h}\mathcal{C}$ is defined as follows:

- The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathrm{h}\mathcal{C}$, the set $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$ of morphisms from X to Y in $\mathrm{h}\mathcal{C}$ is the set of path components $\pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)$ of morphisms from X to Y in $\mathcal{C}$.
- For every triple of objects $X, Y, Z \in \mathcal{C}$, the composition law $\circ : \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Z)$ is the unique map for which the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(Y, Z) \times \mathrm{Map}_{\mathcal{C}}(X, Y) & \xrightarrow{\circ} & \mathrm{Map}_{\mathcal{C}}(X, Z) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Z) \end{array}$$

commutes.

By definition, two morphisms of spectral schemes $f, g : X \to Y$ are homotopic if and only if they induce the same morphism in the homotopy category hSpSch, and a morphism $f : X \to Y$ is an equivalence if and only if it induces an isomorphism in hSpSch. Consequently, one way to enforce the philosophy that all constructions should be homotopy invariant is to restrict attention to constructions which can be described entirely in terms of hSpSch. However, this turns out to be too restrictive:

Example 0.2.2.4. Let $\mathcal{C}$ be a topological category. By definition, commutative diagrams in the homotopy category $\mathrm{h}\mathcal{C}$ correspond to diagrams in $\mathcal{C}$ which commute *up to homotopy*. For example, a diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{f} & Y \end{array}$$

in $\mathrm{h}\mathcal{C}$ is specified by giving maps $\overline{f}, \overline{f}', \overline{g}$, and $\overline{g}'$ (well-defined up to homotopy) for which there exists a path α joining $\overline{g} \circ \overline{f}'$ to $\overline{f} \circ \overline{g}'$ in the topological space $\mathrm{Map}_{\mathcal{C}}(X', Y)$. In practice, we are often interested not only in knowing that α exists, but in *specifying* a particular choice of α. A choice of homotopy is not something that can be described in terms of the homotopy category $\mathrm{h}\mathcal{C}$ alone.

Example 0.2.2.5. Let $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ be morphisms of spectral schemes. To the pair (f_0, f_1), one can canonically associate a new spectral scheme which we will denote by $X_0 \times_X X_1$ and refer to as the *fiber product of* X_0 *with* X_1 *over* X. However, this object is usually *not* a fiber product of X_0 with X_1 over X in the homotopy category hSpSch. In the language of topological categories, it is a *homotopy* fiber product, which can be characterized as follows: there is a diagram of spectral schemes

$$\begin{array}{ccc} X_{01} & \xrightarrow{p_0} & X_0 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f_0} \\ X_1 & \xrightarrow{f_1} & X \end{array}$$

and a path α from $f_0 \circ p_0$ to $f_1 \circ p_1$ in the topological space $\mathrm{Map}_{\mathrm{SpSch}}(X_{01}, X)$ which enjoys the following universal property: for every spectral scheme Y, the induced map from $\mathrm{Map}_{\mathrm{SpSch}}(Y, X_{01})$ to the fiber product

$$\mathrm{Map}_{\mathrm{SpSch}}(Y, X_0) \times_{\mathrm{Map}_{\mathrm{SpSch}}(Y,X)} \mathrm{Map}_{\mathrm{SpSch}}(Y, X)^{[0,1]} \times_{\mathrm{Map}_{\mathrm{SpSch}}(Y,X)} \mathrm{Map}_{\mathrm{SpSch}}(Y, X_1)$$

is a (weak) homotopy equivalence. In other words, the datum of a morphism from Y to X_{01} is equivalent to the data of a pair of morphisms $g_0 : Y \to X_0$ and $g_1 : Y \to X_1$ together with a homotopy β from $g_0 \circ f_0$ to $g_1 \circ f_1$. In general, different choices for the homotopy β give rise to different morphisms $Y \to X_{01}$ (even up to homotopy). Consequently, X_{01} is usually *not* a fiber product of X_0 with X_1 over X in the homotopy category hSpSch (such a fiber product usually does not exist). It enjoys an analogous universal property, but one which cannot be formulated in terms of the homotopy category hSpSch alone.

Examples 0.2.2.4 and 0.2.2.5 illustrate that passage from a topological category $\mathcal{C}$ to its homotopy category $\mathrm{h}\mathcal{C}$ loses essential information about diagrams even of very simple shape (indexed by a square). For more complicated diagrams, the situation is worse. To fix ideas, let us suppose that we are given some index category $\mathcal{J}$ and a functor $F : \mathcal{J} \to \mathrm{h}\mathcal{C}$, which we interpret as a commutative diagram in $\mathrm{h}\mathcal{C}$. In practice, this is often not good enough: many basic constructions require that we promote F to a *homotopy coherent* diagram $\overline{F}$ in $\mathcal{C}$. This involves supplying the following sort of data:

(D_1) For every morphism $v : X \to Y$ in $\mathcal{J}$, a choice of point $\overline{F}(v) \in \mathrm{Map}_{\mathcal{C}}(F(X), F(Y))$ belonging to the path component $F(v) \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(F(X), F(Y)) = \pi_0 \mathrm{Map}_{\mathcal{C}}(F(X), F(Y))$.

(D_2) For every composable pair of morphisms $X \xrightarrow{v} Y \xrightarrow{w} Z$ in $\mathcal{C}$, a choice of path

$$\alpha_{w,v} : [0,1] \to \mathrm{Map}_{\mathcal{C}}(F(X), F(Z))$$

which begins at the point $\overline{F}(w \circ v)$ and ends at the point $\overline{F}(w) \circ \overline{F}(v)$. Note that such a path always exist, since

$$F(w \circ v) = F(w) \circ F(v) \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(F(X), F(Z)) = \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Z).$$

(D_3) For every triple of composable morphisms

$$W \xrightarrow{u} X \xrightarrow{v} Y \xrightarrow{w} Z,$$

a continuous map of topological spaces $[0,1]^2 \to \mathrm{Map}_{\mathcal{C}}(W, Z)$ whose restriction to the boundary of the square $[0,1]^2$ is as indicated in the diagram

$$\begin{array}{ccc} \overline{F}(w \circ v \circ u) & \xrightarrow{\alpha_{w, v \circ u}} & \overline{F}(w) \circ \overline{F}(v \circ u) \\ \downarrow{\scriptstyle \alpha_{w \circ v, u}} & & \downarrow{\scriptstyle \overline{F}(w) \circ \alpha_{v,u}} \\ \overline{F}(w \circ v) \circ \overline{F}(u) & \xrightarrow{\alpha_{w,v} \circ \overline{F}(u)} & \overline{F}(w) \circ \overline{F}(v) \circ \overline{F}(u). \end{array}$$

Beware that the existence of such a map might depend on the choices made in (D_2).

(D_n) An analogous datum for every n-tuple of composable morphisms

$$X_0 \xrightarrow{u_1} X_1 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-1}} X_{n-1} \xrightarrow{u_n} X_n,$$

taking the form of a map of topological spaces $[0,1]^{n-1} \to \mathrm{Map}_{\mathcal{C}}(X_0, X_n)$ whose restriction to the boundary $\partial[0,1]^{n-1}$ is determined by the data (D_m) for $m < n$.

To give a precise and succinct formulation of (D_n), it is useful to introduce a bit of terminology.

Construction 0.2.2.6. Let $n \geqslant 0$ be an integer and let $[n]$ denote the linearly ordered set $\{0 < 1 < \ldots < n\}$. We define a topological category $\mathcal{T}_n$ as follows:

- The objects of $\mathcal{T}_n$ are the elements of $[n]$.
- Given a pair of objects $i, j \in \mathcal{T}_n$, the space of maps from i to j is given by

$$\mathrm{Map}_{\mathcal{T}_n}(i,j) = \begin{cases} \varnothing & \text{if } i > j \\ \{f \in [0,1]^{\{i,i+1,\ldots,j\}} : f(i) = f(j) = 1\} & \text{if } i \leqslant j. \end{cases}$$

- For $0 \leqslant i \leqslant j \leqslant k \leqslant n$, the composition law

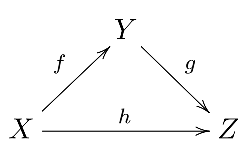
$$\circ : \mathrm{Map}_{\mathcal{T}_n}(j,k) \times \mathrm{Map}_{\mathcal{T}_n}(i,j) \to \mathrm{Map}_{\mathcal{T}_n}(i,k)$$

is given by the formula $(f \circ g)(t) = \begin{cases} f(t) & \text{if } j \leqslant t \leqslant k \\ g(t) & \text{if } i \leqslant t \leqslant j. \end{cases}$

If $\mathcal{C}$ is a topological category, we let $\mathrm{N}(\mathcal{C})_n$ denote the collection of all functors $F : \mathcal{T}_n \to \mathcal{C}$ which are continuous (meaning that the induced map $\mathrm{Map}_{\mathcal{T}_n}(i,j) \to \mathrm{Map}_{\mathcal{C}}(F(i), F(j))$ is continuous for $i, j \in [n]$.

Example 0.2.2.7. Let $\mathcal{C}$ be a topological category. Then:

- An element of $\mathrm{N}(\mathcal{C})_0$ is given by an object X of $\mathcal{C}$.
- An element of $\mathrm{N}(\mathcal{C})_1$ is given by a morphism $f : X \to Y$ in $\mathcal{C}$.
- An element of $\mathrm{N}(\mathcal{C})_2$ is given by a (non-commuting) diagram



in $\mathcal{C}$ together with a path α from h to $g \circ f$ (which "witnesses" that the diagram commutes up to homotopy).

Remark 0.2.2.8. For each $n \geqslant 0$, the topological category $\mathcal{T}_n$ appearing in Construction 0.2.2.6 can be regarded as a "thickened version" of the partially ordered set $[n] = \{0 < 1 < \cdots < n\}$ (regarded as a category): each of the nonempty mapping spaces in $\mathcal{T}_n$ is contractible, and the homotopy category of $\mathcal{T}_n$ can be identified with $[n]$.

Construction 0.2.2.9 (Homotopy Coherent Nerve: [45], [44])**.** Let $m, n \geqslant 0$ be integers and suppose we are given a nondecreasing map $\rho : [m] \to [n]$. Then we can extend ρ to a (continuous) functor $\mathcal{T}_m \to \mathcal{T}_n$, which is given on morphisms by the construction

$$(f \in \mathrm{Map}_{\mathcal{T}_m}(i,j)) \mapsto (\rho(f) \in \mathrm{Map}_{\mathcal{T}_n}(\rho(i), \rho(j)))$$

$$\rho(f)(t) = \sup(\{0\} \cup \{f(\bar{t})\}_{\rho(\bar{t})=t}).$$

If $\mathcal{C}$ is a topological category, then composition with ρ induces a map $\mathrm{N}(\mathcal{C})_n \to \mathrm{N}(\mathcal{C})_m$. Using this construction, we can regard the construction $[n] \mapsto \mathrm{N}(\mathcal{C})_n$ as a simplicial set. We will denote this simplicial set by $\mathrm{N}(\mathcal{C})_\bullet$ and refer to it as the *homotopy coherent nerve* of $\mathcal{C}$.

Example 0.2.2.10. Let $\mathcal{C}$ be any category. Then we can regard $\mathcal{C}$ as a topological category by equipping each of the morphism sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ with the discrete topology. In this case, we will refer to the homotopy coherent nerve $\mathrm{N}(\mathcal{C})_\bullet$ simply as the *nerve* of $\mathcal{C}$. Unwinding the definitions, we see that for each $n \geqslant 0$, the set $\mathrm{N}(\mathcal{C})_n$ can be identified with the set of all diagrams

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_n} X_n$$

consisting of n-tuples of composable morphisms in $\mathcal{C}$.

Using Construction 0.2.2.9, one can formulate the notion of a homotopy coherent diagram in a very simple way:

Definition 0.2.2.11. Let $\mathcal{J}$ and $\mathcal{C}$ be topological categories. A *homotopy coherent diagram in $\mathcal{C}$ indexed by $\mathcal{J}$* is a map of simplicial sets $\mathrm{N}(\mathcal{J})_\bullet \to \mathrm{N}(\mathcal{C})_\bullet$.

Remark 0.2.2.12. When $\mathcal{J}$ is an ordinary category (regarded as a topological category with the discrete topology), Definition 0.2.2.11 supplies a precise formulation of the incomplete definition sketched earlier (note that Definition 0.2.2.11 incorporates some additional requirements regarding identity morphisms that we did not mention earlier: for example, a homotopy coherent diagram should send each identity morphism in $\mathcal{J}$ to an identity morphism in $\mathcal{C}$).

Definition 0.2.2.11 illustrates a general phenomenon. Working with a topological category $\mathcal{C}$ in a homotopy-invariant way often requires us to introduce definitions which seem complicated because they involve a bottomless hierarchy of coherences (conditions which hold up to homotopy, which must be specified and which must satisfy further conditions, but only up to homotopy, which must also be specified, and so forth). However, these definitions can often be expressed in a simple and efficient way in terms of the homotopy coherent nerve $\mathrm{N}(\mathcal{C})_\bullet$, using the language of simplicial sets. For many purposes, it is convenient to discard the topological category $\mathcal{C}$ and work directly with the simplicial set $\mathrm{N}(\mathcal{C})$. It turns out that passage to the homotopy coherent nerve does not lose any essential information. For ordinary categories, it does not lose *any* information:

Example 0.2.2.13. Let $\mathcal{C}$ be an ordinary category. Then $\mathcal{C}$ can be recovered (up to canonical isomorphism) from its nerve $\mathrm{N}(\mathcal{C})_\bullet$. For example, the objects of $\mathcal{C}$ are just the 0-simplices of $\mathrm{N}(\mathcal{C})_\bullet$, and the morphisms of $\mathcal{C}$ are just the 1-simplices of $\mathrm{N}(\mathcal{C})_\bullet$. Given a pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, the composition $g \circ f$ is the unique morphism $h : X \to Z$ for which there exists a 2-simplex in $\mathrm{N}(\mathcal{C})$ whose boundary is given by

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ X & & \xrightarrow{\quad h \quad} & & Z. \end{array}$$

Elaborating on Example 0.2.2.13, one can prove the following:

Proposition 0.2.2.14. *The construction $\mathcal{C} \mapsto \mathrm{N}(\mathcal{C})_\bullet$ determines a fully faithful embedding from the category $\mathcal{C}\mathrm{at}$ of small categories to the category $\mathcal{S}\mathrm{et}_\Delta$ of simplicial sets. The essential image of this embedding consists of those simplicial sets $S_\bullet$ with the following property:*

($*$) *For every pair of integers $0 < i < n$ and every map $\sigma_0 : \Lambda^n_i \to S_\bullet$, there is a unique map $\sigma : \Delta^n \to S_\bullet$ extending σ_0.*

Remark 0.2.2.15. In the special case $i = 1$ and $n = 2$, condition ($*$) of Proposition 0.2.2.14 corresponds to the assertion that every composable pair of morphisms $f : X \to Y$ and $g : Y \to Z$ uniquely determine a commutative diagram

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ X & & \xrightarrow{\quad h \quad} & & Z. \end{array}$$

Note that condition ($*$) of Proposition 0.2.2.14 bears a striking resemblance to the definition of a Kan complex (Definition 0.2.1.4). However, there are two important differences: in the definition of a Kan complex, one requires that horns $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended to simplices $\sigma : \Delta^n \to S_\bullet$ whenever $0 \leqslant i \leqslant n$, not only for the "inner" horns where $0 < i < n$. On the other hand, the definition of a Kan complex only requires the existence of σ, not its uniqueness. Neither of these conditions implies the other: the nerve of a category $\mathcal{C}$ is not a Kan complex if there are non-invertible morphisms in $\mathcal{C}$, and the singular complex of a topological space X is not the nerve of a category if there are nonconstant paths in X. However, these definitions admit a common generalization:

Definition 0.2.2.16. Let $S_\bullet$ be a simplicial set. We will say that $S_\bullet$ is an ∞-*category* if it satisfies the following condition:

($\star$) For every pair of integers $0 < i < n$ and every map $\sigma_0 : \Lambda^n_i \to S_\bullet$, there exists a map $\sigma : \Delta^n \to S_\bullet$ extending σ_0.

Definition 0.2.2.16 was introduced originally by Boardman and Vogt in their work on homotopy invariant algebraic structures (see [?]). They referred to condition $(\star)$ as the *weak Kan condition* and to simplicial sets $S_\bullet$ satisfying $(\star)$ as *weak Kan complexes*. The theory was developed more extensively by Joyal, who refers to simplicial sets satisfying $(\star)$ as *quasicategories* (see, for example, [106], [107], and [108]).

Example 0.2.2.17. Every Kan complex is an ∞-category. In particular, if X is a topological space, then the singular simplicial set $\mathrm{Sing}_\bullet X$ is an ∞-category. More informally, this means any topological space X can be regarded as an ∞-category (by passing to its singular simplicial set), so that the theory of ∞-categories subsumes the classical homotopy theory of spaces.

Example 0.2.2.18. For every ordinary category $\mathcal{C}$, the nerve $\mathrm{N}(\mathcal{C})_\bullet$ is an ∞-category. More informally, this means that any category $\mathcal{C}$ can be regarded as an ∞-category (by passing to its nerve). By virtue of Proposition 0.2.2.14, this involves no loss of information. We will sometimes abuse terminology by not distinguishing between a category $\mathcal{C}$ and the corresponding ∞-category $\mathrm{N}(\mathcal{C})_\bullet$.

We will make extensive use of the theory of ∞-categories in this book. For the reader's convenience, we include a brief account of some of the most important definitions and notations; for a much more extensive discussion (which includes proofs of all of the assertions made in this section), we refer the reader to [138].

Notation 0.2.2.19. We will typically denote ∞-categories by caligraphic letters like $\mathcal{C}$ and $\mathcal{D}$, emphasizing the perspective that an ∞-category is a kind of generalized category. If $\mathcal{C}$ is an ∞-category, we will refer to the 0-simplices of $\mathcal{C}$ as *objects* and to the 1-morphisms of $\mathcal{C}$ as *morphisms*. If $e : \Delta^1 \to \mathcal{C}$ is a 1-simplex of $\mathcal{C}$, then $X = e|_{\{0\}}$ and $Y = e|_{\{1\}}$ are 0-simplices of $\mathcal{C}$, and we will say that *e is a morphism from X to Y* and write $e : X \to Y$. For every object $X \in \mathcal{C}$, we let id_X denote the 1-simplex of $\mathcal{C}$ given by the composition $\Delta^1 \to \Delta^0 \xrightarrow{X} \mathcal{C}$. Then id_X is a morphism from X to itself, which we will refer to as *the identity morphism*.

If $\mathcal{C}$ is an ∞-category, then one can associate to every pair of objects $X, Y \in \mathcal{C}$ a space of map $\mathrm{Map}_{\mathcal{C}}(X, Y)$, which we will regard as a Kan complex. In fact, there are several natural constructions of this space which yield homotopy equivalent (but nonisomorphic) results. Perhaps the easiest is this: one can define $\mathrm{Map}_{\mathcal{C}}(X, Y)$ as the Kan complex whose n-simplices are maps $\Delta^n \times \Delta^1$ to $\mathcal{C}$ which carry $\Delta^n \times \{0\}$ to the vertex X and $\Delta^n \times \{1\}$ to the vertex Y. By definition, a vertex of $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a morphism $f : X \to Y$ in $\mathcal{C}$. We wil say that two morphisms $f, g : X \to Y$ are *homotopic* if the belong to the same path component of $\mathrm{Map}_{\mathcal{C}}(X, Y)$.

Let σ be a 2-simplex in an ∞-category $\mathcal{C}$ whose boundary is given as follows:

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ X & & \xrightarrow{h} & & Z. \end{array}$$

In this case, we will say that σ *exhibits* h *as a composition of* g *with* f and write $h = g \circ f$. This is an abuse of notation: neither the 2-simplex σ nor the morphism h are uniquely determined by f and g. However, one can show that h is determined uniquely up to homotopy (and that the homotopy class of h depends only on the homotopy classes of f and g).

To every ∞-category $\mathcal{C}$ one can associate an ordinary category $\mathrm{h}\mathcal{C}$, called the *homotopy category* of $\mathcal{C}$. The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$, and a morphism from X to Y in $\mathrm{h}\mathcal{C}$ is a homotopy class of morphisms from X to Y in $\mathcal{C}$. There is a canonical map of simplicial sets $\mathcal{C} \to \mathrm{N}(\mathrm{h}\mathcal{C})_\bullet$, and the homotopy category $\mathrm{h}\mathcal{C}$ is universal with respect to this property. We will say that a morphism in $\mathcal{C}$ is an *equivalence* if its image in $\mathrm{h}\mathcal{C}$ is an isomorphism. We say that two objects $X, Y \in \mathcal{C}$ are *equivalent* if there exists an equivalence from X to Y.

If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then a *functor* from $\mathcal{C}$ to $\mathcal{D}$ is a map of simplicial sets $F : \mathcal{C} \to \mathcal{D}$. For every pair of simplicial sets $\mathcal{C}$ and $\mathcal{D}$, we let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the simplicial set parametrizing maps from $\mathcal{C}$ to $\mathcal{D}$: that is, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is characterized by the existence of a canonical bijection $\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(S, \mathrm{Fun}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(S \times \mathcal{C}, \mathcal{D})$. If $\mathcal{D}$ is an ∞-category, then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is also an ∞-category. If $\mathcal{C}$ and $\mathcal{D}$ are both ∞-categories, then we will refer to $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as the *∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$*.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that F is an *equivalence of ∞-categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ is equivalent to $\mathrm{id}_\mathcal{C}$ in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and $F \circ G$ is equivalent to $\mathrm{id}_\mathcal{D}$ in the ∞-category $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$. In this case, the functor G is also an equivalence of ∞-categories, and we will say that the functors F and G are *mutually inverse*.

The construction of homotopy coherent nerves establishes a close connection between the theory of ∞-categories and the theory of topological categories. One can show that the construction $\mathcal{C} \mapsto \mathrm{N}(\mathcal{C})$ admits a left adjoint $S_\bullet \mapsto |\mathfrak{C}(S_\bullet)|$ (modulo a slight technicality: one needs to adjust Definition 0.2.2.1 to work in the setting of compactly generated topological spaces, rather than arbitrary topological spaces). Moreover, one has the following:

(*i*) For every ∞-category $\mathcal{D}$, the unit map $\mathcal{D} \to \mathrm{N}(|\mathfrak{C}(\mathcal{D})|)$ is an equivalence of ∞-categories.

(*ii*) For any topological category $\mathcal{C}$, the counit map $v : |\mathfrak{C}(\mathcal{C})| \to \mathcal{C}$ is a weak equivalence of topological categories. More precisely, it is bijective on objects, and for each pair of maps $X, Y \in \mathcal{C}$ the induced map

$$v_{X,Y} : \mathrm{Map}_{|\mathfrak{C}(\mathcal{C})|}(X, Y) \to \mathrm{Map}_\mathcal{C}(X, Y)$$

is a weak homotopy equivalence (that is, it induces isomorphisms on all homotopy groups). If $\mathrm{Map}_{\mathcal{C}}(X, Y)$ has the homotopy type of a CW complex, then $v_{X,Y}$ is a homotopy equivalence.

Assertions (i) and (ii) imply that the notions of topological category and ∞-category are, in some sense, equivalent. However, the latter theory is often much simpler to work with in practice. For example, if $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors from $\mathcal{C}$ to $\mathcal{D}$ is very easy to define, but the corresponding construction in the setting of topological categories is much more involved.

We close this section by mentioning some of the most important examples of ∞-categories:

Example 0.2.2.20. The collection of all Kan complexes can be organized into an ∞-category, which we will denote by $\mathcal{S}$ and refer to as *the ∞-category of spaces*. It is obtained by applying a variant of Construction 0.2.2.9 to the category $\mathcal{K}\mathrm{an}$ of Kan complexes (which is enriched in simplicial sets). In low degrees, it can be described explicitly as follows:

- A 0-simplex of $\mathcal{S}$ is a Kan complex X.
- A 1-simplex of $\mathcal{S}$ is a map of simplicial sets $f : X \to Y$, where X and Y are Kan complexes.
- A 2-simplex of $\mathcal{S}$ is given by a (non-commuting) diagram of Kan complexes

$$\begin{array}{ccc} & Y & \\ {}^{f}\nearrow & & \searrow^{g} \\ X & \xrightarrow{h} & Z \end{array}$$

  together with a simplicial homotopy from h to $g \circ f$.

One can obtain an equivalent (but nonisomorphic) ∞-category by applying the homotopy coherent nerve to the (topological) category $\mathcal{T}\mathrm{op}^{\mathrm{CW}}$ of CW complexes.

Example 0.2.2.21. The collection of all (small) ∞-categories can be organized into an ∞-category, which we will denote by $\mathcal{C}\mathrm{at}_\infty$ and refer to as *the ∞-category of ∞-categories*. In low degrees, it can be described explicitly as follows:

- A 0-simplex of $\mathcal{C}\mathrm{at}_\infty$ is an ∞-category $\mathcal{C}$.
- A 1-simplex of $\mathcal{C}\mathrm{at}_\infty$ is a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$.
- A 2-simplex of $\mathcal{C}\mathrm{at}_\infty$ is given by a (non-commuting) diagram of ∞-categories

$$\begin{array}{ccc} & \mathcal{D} & \\ {}^{F}\nearrow & & \searrow^{G} \\ \mathcal{C} & \xrightarrow{H} & \mathcal{E} \end{array}$$

  together with an equivalence $u : H \to G \circ F$ in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$.

0.2.3 Stable Homotopy Theory and Structured Ring Spectra

Let X and Y be finite CW complexes equipped with base points $x \in X$ and $y \in Y$. One of the primary aims of algebraic topology is to describe the set $[X, Y]$ of homotopy classes of pointed maps from X to Y. This is generally quite difficult, even in the case where X and Y are relatively simple spaces (such as spheres). A more reasonable (but still difficult) problem is to determine the set $[X, Y]_s$ of *stable* homotopy classes of maps from X to Y, which is defined as the direct limit $\varinjlim [\Sigma^n X, \Sigma^n Y]$; here $\Sigma^n X$ and $\Sigma^n Y$ denote the n-fold suspensions of X and Y, respectively. To study these invariants systematically, it is convenient to introduce the following definition:

Definition 0.2.3.1. The *Spanier-Whitehead category* $\mathcal{SW}$ is defined as follows:

- An object of the category $\mathcal{SW}$ consists of a pair (X, m), where X is a pointed finite CW complex and $m \in \mathbf{Z}$ is an integer.
- Given a pair of objects $(X, m), (Y, n) \in \mathcal{SW}$, the set of morphisms from (X, m) to (Y, n) is given by the direct limit $\varinjlim_k [\Sigma^{m+k} X, \Sigma^{n+k} Y]$ (note that the set $[\Sigma^{m+k} X, \Sigma^{n+k} Y]$ is well-defined as soon as $m + k$ and $n + k$ are both nonnegative).

Given a pointed finite CW complex X and an integer $m \in \mathbf{Z}$, one should think of the object $(X, m) \in \mathcal{SW}$ as playing the role of the suspension $\Sigma^m X$. Note that m is allowed to be negative: the Spanier-Whitehead category enlarges the homotopy category of pointed finite CW complexes by allowing "formal desuspensions."

Example 0.2.3.2. For each integer $n \in \mathbf{Z}$, we let S^n denote the object of the Spanier-Whitehead category given by (S^0, n), where S^0 is the 0-sphere. We will refer to S^n as the *n-sphere*. Note that for $n \geqslant 0$, the object S^n can be identified with the pair $(S^n, 0)$.

Remark 0.2.3.3. Let X and Y be pointed finite CW complexes, and let us abuse notation by identifying X and Y with the objects $(X, 0), (Y, 0) \in \mathcal{SW}$. Then we have $\mathrm{Hom}_{\mathcal{SW}}(X, Y)$ is the set $[X, Y]_s$ of stable homotopy classes of maps from X to Y.

Remark 0.2.3.4. For any pair of objects $(X, m), (Y, n) \in \mathcal{SW}$, it follows from the Freudenthal suspension theorem that the diagram of sets $\{[\Sigma^{m+k} X, \Sigma^{n+k} Y]\}$ is eventually constant: that is, the natural map

$$[\Sigma^{m+k} X, \Sigma^{n+k} Y] \to [\Sigma^{m+k+1} X, \Sigma^{n+k+1} Y]$$

is bijective for $k \gg 0$.

Let $\mathcal{H}$ denote the category whose objects are pointed finite CW complexes and whose morphisms are homotopy classes of pointed maps. Then the construction $X \mapsto \Sigma X$

determines a functor Σ from the category $\mathcal{H}$ to itself. Unwinding the definitions, the Spanier-Whitehead category can be described as the direct limit of the sequence of categories

$$\cdots \to \mathcal{H} \xrightarrow{\Sigma} \mathcal{H} \xrightarrow{\Sigma} \mathcal{H} \xrightarrow{\Sigma} \mathcal{H} \xrightarrow{\Sigma} \mathcal{H} \to \cdots .$$

The category $\mathcal{H}$ arises naturally as the homotopy category of an ∞-category: namely, the ∞-category $\mathcal{S}_*^{\mathrm{fin}}$ whose objects are pointed Kan complexes X for which the geometric realization $|X|$ has the homotopy type of a finite CW complex. Moreover, the suspension functor $\Sigma : \mathcal{H} \to \mathcal{H}$ is obtained from a functor from $\mathcal{S}_*^{\mathrm{fin}}$ to itself, which we will also denote by Σ. It follows that the Spanier-Whitehead category $\mathcal{SW}$ can also be described as the homotopy category of an ∞-category: namely, the direct limit of the sequence

$$\cdots \to \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\mathrm{fin}} \to \cdots .$$

We will denote this direct limit by $\mathrm{Sp}^{\mathrm{fin}}$ and refer to it as the *∞-category of finite spectra*.

The set $[X, Y]_s$ of stable homotopy classes of maps from X to Y is generally easier to compute than the set $[X, Y]$. This is in part because the problem is more structured: for example, the set $[X, Y]_s$ has the structure of an abelian group. In fact, one can say much more: the Spanier-Whitehead category $\mathcal{SW}$ is an example of a *triangulated category* in the sense of Verdier (see [**?**]). The next definition axiomatizes those features of the ∞-category $\mathrm{Sp}^{\mathrm{fin}}$ that are responsible for this phenomenon:

Definition 0.2.3.5. Let $\mathcal{C}$ be an ∞-category. We will say that $\mathcal{C}$ is *stable* if it satisfies the following axioms:

(a) The ∞-category $\mathcal{C}$ admits finite colimits.

(b) The ∞-category $\mathcal{C}$ has an object which is both initial and final (we will refer to such an object as a *zero object* of $\mathcal{C}$ and denote it by $0 \in \mathcal{C}$).

(c) The suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ (given by the formula $\Sigma X = 0 \amalg_X 0$) is an equivalence of ∞-categories.

Example 0.2.3.6. The ∞-category $\mathcal{S}_*^{\mathrm{fin}}$ satisfies axioms (a) and (b) of Definition 0.2.3.5: axiom (b) follows from the fact that we are working with *pointed* spaces (so that the one-point space is both initial and final), and axiom (a) follows from from the observation that the pointed finite spaces are precisely those that can be built from the 0-sphere S^0 by means of finite colimits. However, the ∞-category $\mathcal{S}_*^{\mathrm{fin}}$ does not satisfy (c): for example, the 0-sphere S^0 cannot be obtained as the suspension of another space. The ∞-category $\mathrm{Sp}^{\mathrm{fin}}$ of finite spectra can be regarded as remedy for the fact that $\mathcal{S}_*^{\mathrm{fin}}$ does not satisfy (c): it satisfies property (c) by construction, and inherits properties (a) and (b) from $\mathcal{S}_*^{\mathrm{fin}}$. Consequently, $\mathrm{Sp}^{\mathrm{fin}}$ is a stable ∞-category.

Remark 0.2.3.7. If $\mathcal{C}$ is a stable ∞-category, then its homotopy category $\mathrm{h}\mathcal{C}$ inherits the structure of a triangulated category. Moreover, essentially all of the triangulated categories which arise naturally can be described as the homotopy category of a stable ∞-category.

Remark 0.2.3.8. Suppose we are given a commutative diagram σ :

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ W & \longrightarrow & Z \end{array}$$

in a stable ∞-category $\mathcal{C}$. Then σ is a pullback square if and only if it is a pushout square.

Remark 0.2.3.9. Let $\mathcal{C}$ be an ∞-category with a zero object 0, and suppose we are given a commutative diagram σ :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z \end{array}$$

in $\mathcal{C}$. If σ is a pullback square, we abuse terminology by saying that the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a *fiber sequence* (here we are implicitly referring to the entire diagram σ, which we can think of as supplying the morphisms f and g together with a nullhomotopy of the composition $g \circ f$). Similarly, if σ is a pushout square, then we abuse terminology by saying that the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a cofiber sequence. If the ∞-category $\mathcal{C}$ is stable, then the fiber sequence and cofiber sequences in $\mathcal{C}$ are the same.

For many purposes, the ∞-category $\mathrm{Sp}^{\mathrm{fin}}$ of finite spectra is too small: it admits finite limits and colimits, but does not admit many other categorical constructions such as infinite products. One can remedy this by passing to a larger ∞-category.

Construction 0.2.3.10. Let $\mathcal{C}$ be a small ∞-category. Then one can form a new ∞-category $\mathrm{Ind}(\mathcal{C})$, called *the ∞-category of* Ind-*objects of* $\mathcal{C}$. This ∞-category admits two closely related descriptions:

(a) It is obtained from $\mathcal{C}$ by formally adjoining filtered colimits. In particular, every object of $\mathrm{Ind}(\mathcal{C})$ can be written as the colimit $\varinjlim C_\alpha$ of some filtered diagram $\{C_\alpha\}$ in $\mathcal{C}$, and the mapping spaces in $\mathrm{Ind}(\mathcal{C})$ can be described informally by the formula

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\varinjlim C_\alpha, \varinjlim D_\beta) \simeq \varprojlim_\alpha \varinjlim_\beta \mathrm{Map}_{\mathcal{C}}(C_\alpha, D_\beta).$$

(*b*) If $\mathcal{C}$ admits finite colimits, then $\mathrm{Ind}(\mathcal{C})$ can be described as the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ spanned by those functors which preserve finite limits.

We let Sp denote the ∞-category $\mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}})$. We will refer to Sp as the *∞-category of spectra*. A *spectrum* is an object of the ∞-category Sp.

Remark 0.2.3.11. If $\mathcal{C}$ is a stable ∞-category, then the ∞-category $\mathrm{Ind}(\mathcal{C})$ is also stable. In particular, the ∞-category Sp is stable, so the homotopy category $\mathrm{h}\mathcal{C}$ is triangulated.

Definition 0.2.3.12. For each $n \in \mathbf{Z}$, let $S^n \in \mathrm{CW}$ be defined as in Example 0.2.3.2 and regard S^n as an object of the ∞-category Sp. In the special case $n = 0$, we will denote S^n simply by S and refer to it as the *sphere spectrum*.

Let E be an arbitrary spectrum and let $n \in \mathbf{Z}$ be an integer. Since the homotopy category hSp is additive, the set $\mathrm{Hom}_{\mathrm{hSp}}(S^n, E) = \pi_0 \mathrm{Map}_{\mathrm{Sp}}(S^n, E)$ has the structure of an abelian group. We will denote this group by $\pi_n E$ and refer to it as the *nth homotopy group of E*. We say that a spectrum E is *connective* if the homotopy groups $\pi_n E$ vanish for $n < 0$. We let $\mathrm{Sp}^{\mathrm{cn}}$ denote the full subcategory of Sp spanned by the connective spectra.

There are many different ways of looking at the notion of a spectrum (most of which lead to alternative definitions of the ∞-category Sp). Let us summarize a few of the most useful:

Spectra are infinite loop spaces: Let E be a spectrum. For each integer $n \in \mathbf{Z}$, we let $\Omega^{\infty-n} E$ denote the mapping space $\mathrm{Map}_{\mathrm{Sp}}(S^{-n}, E)$. We refer to $\Omega^{\infty-n}E$ as the *nth space of E*. Note that each $\Omega^{\infty-n}E$ can be identified with the loop space of $\Omega^{\infty-n-1}E$. Consequently, the construction $E \mapsto \{\Omega^{\infty-n}E\}_{n \in \mathbf{Z}}$ determines a functor from the ∞-category Sp to the inverse limit of the tower of ∞-categories

$$\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \cdots .$$

One can show that this functor is an equivalence of ∞-categories. In other words, the data of a spectrum E is equivalent to the data of an *infinite loop space*: that is, a sequence of pointed spaces $\{E(n)\}_{n \in \mathbf{Z}}$ which are equipped with homotopy equivalences $E(n) \simeq \Omega E(n+1)$.

Spectra are cohomology theories: Let E be a spectrum. For every space X, let $E^n(X)$ denote the set $\pi_0 \mathrm{Map}_{\mathcal{S}}(X, \Omega^{\infty-n}E)$ of homotopy classes of (unpointed) maps from X into the nth space of E. We will refer to $E^n(X)$ as the *nth cohomology group of X with coefficients in E*. One can show that the construction $X \mapsto \{E^n(X)\}_{n \in \mathbf{Z}}$ (which extends in a canonical way to an invariant of pairs of spaces $A \subseteq X$) is a *generalized cohomology theory*: that is, it satisfies all of the Eilenberg-Steenrod axioms characterizing singular cohomology, with the exception of the dimension axiom. Moreover, the converse is true as well: according to the Brown representability theorem, every generalized cohomology theory arises in this way. More precisely, this construction yields a bijection

$$\{ \text{ Spectra } \}/\text{ equivalence } \simeq \{ \text{ Cohomology theories } \}/\text{ isomorphism } .$$

Spectra are generalized abelian groups: Let E be a spectrum. Then the 0th space $\Omega^\infty E$ is an example of an $\mathbb{E}_\infty$*-space*: that is, it can be equipped with an addition law

$$+ : \Omega^\infty E \times \Omega^\infty E \to \Omega^\infty E$$

which is unital, commutative, and associative up to *coherent* homotopy. Moreover, the construction $E \mapsto \Omega^\infty E$ restricts to an equivalence from the ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ of connective spectra to the ∞-category $\mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$ of *grouplike* $\mathbb{E}_\infty$-spaces (an $\mathbb{E}_\infty$-space A is said to be *grouplike* if the addition on A exhibits the set of connected components $\pi_0 A$ as an abelian group).

Spectra are the universal stable ∞-category: The ∞-category Sp is stable, admits small colimits, and contains a distinguished object S (the sphere spectrum). Moreover, it is *universal* with respect to these properties: if $\mathcal{C}$ is any stable ∞-category which admits small colimits and $\mathrm{LFun}(\mathrm{Sp}, \mathcal{C})$ denotes the full subcategory of $\mathrm{Fun}(\mathrm{Sp}, \mathcal{C})$ spanned by those functors which preserve small colimits, then the construction $F \mapsto F(S)$ induces an equivalence of ∞-categories $e : \mathrm{LFun}(\mathrm{Sp}, \mathcal{C}) \to \mathcal{C}$. In particular, for each object $C \in \mathcal{C}$, there is an essentially unique functor $F : \mathrm{Sp} \to \mathcal{C}$ which preserves small colimits and satisfies $F(S) = C$.

Let X be a spectrum. We will say that X is *discrete* if the homotopy groups $\pi_n X$ vanish for $n \neq 0$. In this case, X is determined (up to canonical equivalence) by the abelian group $\pi_0 X$. More precisely, the construction $X \mapsto \pi_0 X$ induces an equivalence from the full subcategory $\mathrm{Sp}^\heartsuit \subseteq \mathrm{Sp}$ spanned by the discrete spectra to the ordinary category of abelian groups (which we can regard as an ∞-category by taking its nerve). We can use an inverse of this equivalence to identify the category of abelian groups with the full subcategory $\mathrm{Sp}^\heartsuit \subseteq \mathrm{Sp}$. If A is an abelian group, then the image of A under this identification is called the *Eilenberg-MacLane spectrum of* A. As an infinite loop space, it is given by the sequence $\{K(A,n)\}$; here $K(A,n)$ denotes the Eilenberg-MacLane space characterized by the formula

$$\pi_* K(A,n) = \begin{cases} A & \text{if } * = n \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding cohomology theory is ordinary cohomology with coefficients in A. Throughout this book, we will will often abuse notation by identifying an abelian group A with its corresponding Eilenberg-MacLane spectrum.

It follows from the universal property of the ∞-category Sp that there is an essentially unique functor $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ which preserves small colimits separately in each variable and satisfies $S \otimes S = S$. We will refer to this functor as the *smash product*. Using the universal property of the ∞-category Sp, one can show that the smash product functor endows Sp with the structure of a *symmetric monoidal* ∞-category: that is, the functor $\otimes$ is commutative, associative, and unital up to coherent homotopy.

Definition 0.2.3.13. For any symmetric monoidal ∞-category $\mathcal{C}$, we let $\mathrm{CAlg}(\mathcal{C})$ denote the ∞-category of commutative algebra objects of $\mathcal{C}$. Roughly speaking, an object of $\mathrm{CAlg}(\mathcal{C})$ is given by an object $A \in \mathcal{C}$ equipped with a multiplication $m : A \otimes A \to A$ which is commutative, associative, and unital *up to coherent homotopy*. In the special case where $\mathcal{C}$ is the ∞-category of spectra (equipped with the symmetric monoidal structure given by the smash product), we will denote $\mathrm{CAlg}(\mathcal{C})$ simply by CAlg. We will refer to the objects of CAlg as $\mathbb{E}_\infty$*-rings* and to the ∞-category $\mathcal{C}$ as *the* ∞*-category of* $\mathbb{E}_\infty$*-rings*.

Remark 0.2.3.14. Let E be a spectrum, so that E determines a cohomology theory which assigns to each space X a graded abelian group $E^*(X)$. If E is an $\mathbb{E}_\infty$-ring, then the associated cohomology theory is *multiplicative*: that is, it assigns to each space X a graded ring $E^*(X)$ which is commutative in the graded sense (meaning that $xy = (-1)^{mn} yx$ for $x \in E^m(X)$ and $y \in E^n(X)$). However, the converse fails dramatically: there are many examples of multiplicative cohomology theories which cannot be represented by $\mathbb{E}_\infty$-rings. Roughly speaking, one expects a cohomology theory E^* to be represented by an $\mathbb{E}_\infty$-ring if it is can be equipped with a multiplicative structure for which commutativity and associativity can be seen (at least up to coherent homotopy) at the level of cochains, rather than merely at the level of cohomology.

Let E be an $\mathbb{E}_\infty$-ring. Then the collection of homotopy groups $\pi_* E = E^{-*}(\{x\})$ has the structure of a graded-commutative ring. In particular, $\pi_0 E$ is a commutative ring and each $\pi_n E$ can be regarded as a module over $\pi_0 E$. We will say that E is *connective* if it is connective when regarded as a spectrum (that is, the homotopy groups $\pi_n E$ vanish for $n < 0$) and we will say that E is *discrete* if it is discrete when regarded as a spectrum (that is, the homotopy groups $\pi_n E$ vanish for $n \neq 0$). We let $\mathrm{CAlg}^{\mathrm{cn}}$ denote the full subcategory of CAlg spanned by the connective $\mathbb{E}_\infty$-rings, and we let $\mathrm{CAlg}^{\heartsuit}$ denote the full subcategory of CAlg spanned by the discrete $\mathbb{E}_\infty$-rings.

Remark 0.2.3.15. The construction $A \mapsto \Omega^\infty A$ induces an equivalence from the ∞-category of connective spectra to the ∞-category of grouplike $\mathbb{E}_\infty$-spaces. We can phrase this more informally as follows: giving a connective spectrum is equivalent to giving a space X which behaves like an abelian group *up to coherent homotopy*. This heuristic can be extended to $\mathbb{E}_\infty$-rings: a connective $\mathbb{E}_\infty$-ring A can be thought of as a space X which behaves like a commutative ring up to coherent homotopy.

Remark 0.2.3.16. The construction $A \mapsto \pi_0 A$ determines an equivalence from the ∞-category $\mathrm{CAlg}^{\heartsuit}$ to the ordinary category of commutative rings (regarded as an ∞-category via its nerve). We will generally abuse notation by using this equivalence to identify $\mathrm{CAlg}^{\heartsuit}$ with the category of commutative rings, so that every commutative ring R is regarded as an $\mathbb{E}_\infty$-ring. In terms of the heuristic of Remark 0.2.3.15, this corresponds to regarding R as a space equipped with the discrete topology.

By virtue of Remark 0.2.3.16, we can regard the theory of $\mathbb{E}_\infty$-rings as a generalization of classical commutative algebra. Moreover, it is a robust generalization: all of the basic results, constructions, and ideas that are needed to set up the foundations of classical algebraic geometry have analogues in the setting of $\mathbb{E}_\infty$-rings, which we will make use of throughout this book.

0.3 Overview

This book is divided into nine parts, each of which is devoted to exploring some facet of the relationship between algebraic geometry and structured ring spectra. Our first goal is to establish foundations for the subject. We begin in Part I by introducing "spectral" versions of various algebro-geometric objects (such as schemes, algebraic spaces, and Deligne-Mumford stacks) and studying how these objects are related to their classical counterparts. We also explain how to associate to every spectral scheme X (or, more generally, any spectral Deligne-Mumford stack) a stable ∞-category $\mathrm{QCoh}(X)$ of *quasi-coherent sheaves on* X, which is closely related to the abelian categories of quasi-coherent sheaves which appear in classical algebraic geometry.

Part II is concerned with proper morphisms in the setting of spectral algebraic geometry. In some sense, there is very little to say here: a morphism $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of spectral schemes is proper if and only if the underlying morphism of ordinary schemes $f_0 : (X, \pi_0 \mathscr{O}_X) \to (Y, \pi_0 \mathscr{O}_Y)$ is proper. However, some aspects of the theory work more smoothly in the spectral setting. For example, many important foundational results about proper morphisms between Noetherian schemes (for example, the direct image theorem, the theorem on formal functions, the Grothendieck existence theorem, and Grothendieck's formal GAGA principle) admit generalizations to the setting of spectral algebraic geometry which do not require any Noetherian assumptions (see Theorem 5.6.0.2, Lemma 8.5.1.1, Theorem 8.5.0.3, and Corollary 8.5.3.4).

The subject of Part III is the following general question: to what extent can an algebro-geometric object X can be recovered from the stable ∞-category $\mathrm{QCoh}(X)$? We address this question by proving several "Tannaka reconstruction" type results which assert that, in many circumstances, we can recover X as a kind of "spectrum" of $\mathrm{QCoh}(X)$ (much like an affine scheme $(Y, \mathscr{O}_Y)$ can be recovered as the spectrum of its coordinate ring $\Gamma(Y; \mathscr{O}_Y)$). We also show that there is a close relationship between stable ∞-categories $\mathcal{C}$ equipped with an action of $\mathrm{QCoh}(X)$ and *sheaves* of stable ∞-categories on X (categorifying the relationship between quasi-coherent sheaves on an affine scheme $(Y, \mathscr{O}_Y)$ and modules over the coordinate ring $\Gamma(Y; \mathscr{O}_Y)$.

A standard heuristic principle of deformation theory asserts that over a field κ of characteristic zero, one can describe a formal neighborhood of any algebro-geometric object X near a point $x \in X$ in terms of a differential graded Lie algebra. In Part IV, we will formulate this principle precisely by introducing an ∞-category Moduli_κ of *formal moduli problems* over κ and constructing an equivalence of Moduli_κ with an ∞-category of differential graded Lie algebras over κ. We also study variants of this principle in the setting of noncommutative geometry (which are valid in any characteristic). Part IV is mostly independent of the first three parts (they are relevant mainly because they provide examples of formal moduli problems which can be analyzed using the formalism of Part IV).

In Part **??** we study representability problems in the setting of spectral algebraic geometry. Suppose we are given a functor $h : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, where $\mathrm{CAlg}^{\mathrm{cn}}$ denotes the ∞-category of connective $\mathbb{E}_\infty$-rings and $\mathcal{S}$ denotes the ∞-category of spaces. We might then ask if there exists a spectral scheme X (or some other sort of algebro-geometric object) which *represents* the functor h, in the sense that there exist homotopy equivalences $h(A) \simeq \mathrm{Map}(\mathrm{Spec}\, A, X)$ depending functorially on A (such an X is uniquely determined up to equivalence, as we will see in Part I). In the setting of classical algebraic geometry, this sort of question can often be addressed using Artin's representability theorem, which gives necessary and sufficient conditions for a functor to be representable by an algebraic stack which is locally of finite presentation over a (sufficiently nice) commutative ring R. The main goal of Part **??** is to formulate and prove an analogous statement in the spectral setting.

The basic objects of study in classical and spectral algebraic geometry can be described in a very similar way: they are given by pairs $(X, \mathscr{O}_X)$, where X is a topological space (or some variant thereof: in the theory of Deligne-Mumford stacks, it is convenient to allow X to be a topos; when studying higher Deligne-Mumford stacks, it is convenient to allow X to be an ∞-topos) and $\mathscr{O}_X$ is a "structure sheaf" on X. The difference between classical and spectral algebraic geometry lies in what sort of sheaf $\mathscr{O}_X$ is: in the classical case, $\mathscr{O}_X$ is a sheaf of commutative rings; in the spectral case, it is a sheaf of $\mathbb{E}_\infty$-rings. In Part VI, we introduce general formalism of "∞-topoi with structure sheaves" which is intended to capture the spirit of these types of definitions in a broad degree of generality. Part VI does not depend on any of the earlier parts of this book (logically, it could precede Part I; however, most readers will probably find it easier to digest the theory of CAlg-valued sheaves than the general sheaf theory of Part VI).

In Part VII, we study several variants of spectral algebraic geometry:

- *derived differential topology*, whose basic objects (*derived manifolds*) are analogous to smooth manifolds in the same way that spectral schemes are analogous to smooth algebraic varieties.

- *derived complex analytic geometry*, whose basic objects (*derived complex-analytic spaces*) are analogous to complex-analytic manifolds in the same way that spectral schemes are analogous to smooth algebraic varieties.

- *derived algebraic geometry*, a variant of spectral algebraic geometry which uses simplicial commutative rings in place of $\mathbb{E}_\infty$-rings. The resulting theory is equivalent to spectral algebraic geometry in characteristic zero, but is quite different (and more closely connected to classical algebraic geometry) in positive and mixed characteristic.

Each of these variants can be regarded as an instance of the general paradigm of Part VI. However, we give an exposition in each instance which can be read independently, referring occasionally to the formalism of Part VI for the verification of some routine details.

The term "derived algebraic geometry" is meant to evoke an analogy with the theory of derived categories, which perhaps merits some explanation. To fix ideas, suppose that we are given a commutative ring A and an A-module M. Then:

- The construction $N \mapsto \mathrm{Hom}_A(M, N)$ determines a functor from the category of A-modules to itself which is left exact but generally not right exact. In order to account for the failure of right exactness, it is often useful to consider the right derived functors $\{N \mapsto \mathrm{Ext}^n_A(M, N)\}_{n \geqslant 0}$.

- The construction $N \mapsto M \otimes_A N$ determines a functor from the category of A-modules to itself which is right exact but generally not left exact. In order to account for the failure of left exactness, it is often useful to consider the left derived functors $\{N \mapsto \mathrm{Tor}^A_n(M, N)\}_{n \geqslant 0}$.

To study either of these derived functors, it is useful to consider the *derived category* $D(A)$ which obtained from the category of chain complexes of A-modules by formally inverting all quasi-isomorphisms. Let $D(A)_{\geqslant 0}$ denote the subcategory of $D(A)$ consisting of those chain complexes whose homology groups are concentrated in nonnegative degrees, and define $D(A)_{\leqslant 0}$ similarly. One can then consider *total derived functors*

$$\mathrm{RHom}_A(M, \bullet) : D(A)_{\leqslant 0} \to D(A)_{\leqslant 0} \qquad M \otimes^L_A \bullet : D(A)_{\geqslant 0} \to D(A)_{\geqslant 0}$$

which, when restricted to an ordinary A-module N (regarded as chain complexes concentrated in degree zero), yield chain complexes whose (co)homology groups recover the invariants $\mathrm{Ext}^n_A(M, N)$ and $\mathrm{Tor}^n_A(M, N)$, respectively.

The relationship between the theory of derived schemes and the classical theory of schemes is somewhat analogous to the relationship between the derived category $D(A)_{\leqslant 0}$

and the abelian category of A-modules. The ∞-category of derived schemes can be regarded as an enlargement of the category of schemes, and certain left exact constructions on schemes (such as the formation of fiber products) admit refinements in the setting of derived schemes in a way that retains additional (and often useful) information. The theory of algebraic stacks provides another enlargement of the category of schemes which is quite different, but in some sense formally dual. The 2-category of algebraic stacks is an enlargement of the category of schemes in which one has improved versions of certain right exact constructions, such as the formation of quotients by group actions (such quotients sometimes do not exist in the category of schemes, and when they do exist they are often badly behaved). In this respect, the relationship between the theory of algebraic stacks and the theory of schemes is somewhat analogous to the relationship between the derived category $D(A)_{\geqslant 0}$ and the abelian category of A-modules (to make the analogy stronger, one can further enlarge the category of schemes by considering algebraic n-stacks for $n > 1$).

In Part VIII we will discuss the theory of *derived stacks*, an extension of classical algebraic geometry which provides a common generalization of the theory of derived schemes and the theory of (higher) algebraic stacks (in much the same way that the full derived category $D(A)$ contains both $D(A)_{\leqslant 0}$ and $D(A)_{\geqslant 0}$ as full subcategories). Many important foundational results concerning ordinary algebraic stacks can be extended to the setting of derived stacks. In particular, we will prove a version of Artin's representability theorem, which establishes necessary and sufficient conditions for a functor to be representable by a derived stack (Theorem **??**) and which can be used to produce many examples of geometric objects which fit into the framework of this book.

In [170], Quillen showed that the homotopy theory of simply connected spaces whose homotopy groups are rational vector spaces is equivalent to the theory of *connected* differential graded Lie algebras over $\mathbf{Q}$. Quillen's work provided an early clue to the significance of differential graded Lie algebras, and partially inspired the study of their applications to deformation theory. In Part IX we will reverse this logic, explaining how Quillen's result is related to (and can be deduced from) the theory developed in Part IV. We also discuss Mandell's p-adic analogue of rational homotopy theory and describe a natural extension which makes use of algebro-geometric ideas. Part IX is primarily self-contained, and can be read independently of the rest of this book.

This book includes several appendices discussing background material needed in the body of the text. In Appendix A, we review of the theory of Grothendieck sites and sheaves in the setting of higher category theory, introduce the notion of a *coherent* ∞-topos (and prove an ∞-categorical analogue of Deligne's completeness theorem: for every coherent ∞-topos $\mathcal{X}$, the hypercompletion $\mathcal{X}^{\mathrm{hyp}}$ has enough points). In Appendix B we discuss several specific examples of Grothendieck topologies which arise in spectral algebraic geometry (such as the Nisnevich and étale topologies associated to a commutative ring) and their relationship to one another, reviewing some of the requisite commutative algebra along the way. Appendix C introduces the theory of *prestable ∞-categories*, a generalization of the notion of stable ∞-category which will play an important role in Part III. In Appendix D, we study the notion of an *R-linear* (pre)stable ∞-category, where R is a ring spectrum, and combine the results of Appendices B and C to prove several descent theorems (both for objects of R-linear ∞-categories and for R-linear ∞-categories themselves). Finally, Appendix E contains an exposition of profinite homotopy theory, which is needed for the discussion of p-adic homotopy theory in Part IX.

0.4 What is not in this book?

Even in a book as long as this, we could not hope to give a comprehensive account of the various ways in which homotopy theoretic ideas have influenced algebraic geometry. In this section, we present an (incomplete) list of ideas which are thematically related to the subject of spectral algebraic geometry, but which will not make an appearance in this book (at least in its present form).

Virtual fundamental classes: One of the primary motivations for developing the language of spectral algebraic geometry is to provide a natural setting for the theory of virtual fundamental classes described in the introduction. In this book, we will discuss some of the relevant formal ingredients (we discuss quasi-smoothness in §**??** and virtual dimension in §**??**). However, we will not discuss virtual fundamental classes (or the cohomological framework in which they naturally reside) here. The subject has been treated in the literature from a variety of perspectives: see, for example, [120], [18], [131], and [191].

Elliptic cohomology: The theory of spectral algebraic geometry plays a central role in understanding the moduli-theoretic interpretation of the theory of topological modular forms (see Example 0.1.1.15) and other related constructions in the setting of chromatic homotopy theory. We will discuss these applications in a sequel to this book. For an informal outline in the case of elliptic cohomology we refer the reader to [140]; see also [21] for a discussion of the more general theory of topological *automorphic* forms.

Higher stacks outside of derived algebraic geometry: Part VIII of this book is devoted to the theory of higher algebraic stacks in the setting of derived algebraic geometry. One can develop an analogous theory in any setting where one has a good notion of smooth morphism, including the theory of spectral algebraic geometry (using the notion of *differential smoothness* that we discuss in §11.2). However, we will not consider such objects in this book. A general framework which incorporates the theory of derived algebraic stacks and many other variants has been developed by Toën and Vezzosi; see [214].

Separated Deligne-Mumford stacks: Throughout this book, we will say an algebro-geometric object X is *separated* if the diagonal map $\delta : X \to X \times X$ is a closed immersion. This convention means that X is forbidden to exhibit any "stacky" behavior: an algebraic stack for which δ is a closed immersion is automatically an algebraic space. Many basic results about separated (spectral) algebraic spaces can be generalized to larger classes of (spectral) Deligne-Mumford stacks. We generally ignore such generalizations, except in cases where they require no additional effort to prove.

Global deformation theory: Let X be an algebro-geometric object (such as a scheme or a spectral scheme) defined over the field $\mathbf{C}$ of complex numbers. In Part IV of this book, we study the formal completion of X at a $\mathbf{C}$-valued point $x \in X(\mathbf{C})$ and show that it is determined by a differential graded Lie algebra. One could ask for something more ambitious: suppose we are given instead an arbitrary closed embedding $\iota : X_0 \hookrightarrow X$: how can one describe the formal completion of X along the image of ι in a manner that is completely intrinsic to X_0? We refer the reader to [77] for an approach this problem, using a theory of "derived" Lie algebroids.

Ind-Coherent Sheaves and Grothendieck Duality: Let $f : X \to Y$ a morphism between schemes of finite type over a field. To f, one can associate an *exceptional inverse image functor* $f^! : D^+(Y) \to D^+(X)$ which is a right adjoint to the (derived) pushforward f_* in the case where f is proper, and is left adjoint to f_* in the case where f is étale. Here $D^+(X)$ and $D^+(Y)$ denote the (cohomologically) "bounded below" derived categories of X and Y, respectively. The theory of Grothendieck duality can be generalized to the setting of spectral algebraic geometry. However, the assumption that objects be "cohomologically bounded below" objects is particularly annoying in the spectral setting (for example, it need not be satisfied by the structure sheaves of X and Y). To eliminate this assumption (and achieve a more robust theory), one needs to replace the stable ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves studied in this book by the closely related ∞-category $\mathrm{Ind}(\mathrm{Coh}(X))$ of Ind-*coherent sheaves on* X. For a discussion these issues (over fields of characteristic zero) we refer the reader to [76].

Shifted symplectic structures: One feature of spectral algebraic geometry that distinguishes it from classical algebraic geometry is that finiteness conditions on spectral schemes are more strongly reflected in their deformation theory. For example, if X is a spectral scheme of finite presentation over $\mathbf{C}$, then the cotangent complex $L_{X/\operatorname{Spec}\mathbf{C}}$ is perfect: that is, it is a dualizable object of $\mathrm{QCoh}(X)$ (the analogous statement for ordinary **C**-schemes holds only when X is a local complete intersection). There are some cases in which the cotangent complex $L_{X/\operatorname{Spec}\mathbf{C}}$ is not only dualizable, but *self-dual* (up to a shift). An antisymmetric identification of $L_{X/\operatorname{Spec}\mathbf{C}}$ with its shifted dual $\Sigma^n L^{\vee}_{X/\operatorname{Spec}\mathbf{C}}$ can be viewed as a kind of "shifted 2-form on X." When this 2-form is closed (in a suitable sense), we say that X has a *n-shifted symplectic structure*. The theory of shifted symplectic structures is still in its infancy (at the time of this writing), but seems to capture an essential feature which is common to many interesting moduli spaces (once they have been appropriately "derived"). For more details, see [164], [37], [166], [34].

0.5 Notation and Terminology

We will assume general familiarity with the terminology of [138] and [139]. For the reader's convenience, we now review some cases in which the conventions of this book differ from those [138], [139], or the established mathematical literature.

- We will generally not distinguish between a category $\mathcal{C}$ and its nerve $\mathrm{N}(\mathcal{C})$. In particular, we regard every category $\mathcal{C}$ as an ∞-category.

- We will generally abuse terminology by not distinguishing between an abelian group M and the associated Eilenberg-MacLane spectrum: that is, we view the ordinary category of abelian groups as a full subcategory of the ∞-category Sp of spectra. Similarly, we regard the ordinary category of commutative rings as a full subcategory of the ∞-category CAlg of $\mathbb{E}_\infty$-rings.

- Let A be an $\mathbb{E}_\infty$-ring. We will refer to A-module spectra simply as *A-modules*. The collection of A-modules can be organized into a stable ∞-category which we will denote by Mod_A and refer to as *the ∞-category of A-modules*. This convention has an unfortunate feature: when A is an ordinary commutative ring, it does not reduce to the usual notion of A-module. In this case, Mod_A is not the abelian category of A-modules but is closely related to it: the homotopy category hMod_A is equivalent to the derived category $D(A)$. *Unless otherwise specified, the term "A-module" will be used to refer to an object of* Mod_A*, even when A is an ordinary commutative ring.* When we wish to consider an A-module M in the usual sense, we will say that M is a *discrete* A-module or an *ordinary* A-module.

- Let A be a commutative ring and let $M \in \mathrm{Mod}_A$ be an A-module. Using the equivalence of categories $\mathrm{hMod}_A \simeq D(A)$, we can identify M with a chain complex of A-modules. This identification has the potential to lead to confusion: beware that the *homotopy groups* of M (if we regard M as a spectrum) correspond to the *homology groups* of

M (if we regard M as a chain complex). We will usually denote these groups by $\{\pi_n M\}_{n \in \mathbf{Z}}$ and refer to them as the homotopy groups of M: that is, we favor the perspective that M is a spectrum with an action of A, rather than a chain complex.

- Unless otherwise specified, all algebraic constructions we consider in this book should be understood in the "derived" sense. For example, if we are given discrete modules M and N over a commutative ring A, then the tensor product $M \otimes_A N$ denotes the *derived* tensor product $M \otimes_A^L N$. This may not be a discrete A-module: its homotopy groups are given by $\pi_n(M \otimes_A N) \simeq \mathrm{Tor}_n^A(M, N)$. When we wish to consider the usual tensor product of M with N over A, we will denote it by $\mathrm{Tor}_0^A(M, N)$ or by $\pi_0(M \otimes_A N)$.

- If M and N are spectra, we will denote the smash product of M with N by $M \otimes N$, rather than $M \wedge N$. More generally, if M and N are modules over an $\mathbb{E}_\infty$-ring A, then we will denote the smash product of M with N over A by $M \otimes_A N$, rather than $M \wedge_A N$. Note that when A is an ordinary commutative ring and the modules M and N are discrete, this agrees with the preceding convention.

- If $\mathcal{C}$ is a triangulated category (such as the homotopy category of a stable ∞-category), we will denote the shift functor on $\mathcal{C}$ by $X \mapsto \Sigma X$, rather than $X \mapsto X[1]$.

- If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories which admit finite limits, we let $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functor which are *left exact*: that is, those functors which preserve finite limits. If instead $\mathcal{C}$ and $\mathcal{D}$ admits finite colimits, we let $\mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})^{\mathrm{op}}$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the *right exact* functors: that is, those functors which preserve finite colimits.

- If $\mathcal{C}$ is an ∞-category, we let $\mathcal{C}^{\simeq}$ denote the largest Kan complex contained in $\mathcal{C}$: that is, the ∞-category obtained from $\mathcal{C}$ by discarding all non-invertible morphisms.

- We will say that a functor $f : \mathcal{C} \to \mathcal{D}$ between ∞-categories is *left cofinal* if, for every object $D \in \mathcal{D}$, the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible (this differs from the convention of [138], which refers to a functor with this property simply as *cofinal*; see Theorem HTT.4.1.3.1). We will say that f is *right cofinal* if the induced map $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is left cofinal, so that f is right cofinal if and only if the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ is weakly contractible for each $D \in \mathcal{D}$.

- We let $\mathcal{T}\mathrm{op}$ denote the category whose objects are topological spaces X, with maps given by continuous functions $f : X \to Y$. We let $\infty\mathcal{T}\mathrm{op}$ denote the ∞-category whose objects are ∞-topoi $\mathcal{X}$ (in the sense of [138]) and whose morphisms are geometric morphisms $f_* : \mathcal{X} \to \mathcal{Y}$ (that is, functors which admit a left adjoint f^* which preserves finite limits). This notation is intended to suggest that $\infty\mathcal{T}\mathrm{op}$ be viewed as an enlargement of $\mathcal{T}\mathrm{op}$ (which is roughly correct; see §1.5).

- If $\mathcal{X}$ is an ∞-topos, we will say that a collection of objects $\{U_\alpha \in \mathcal{X}\}$ is a *covering* of $\mathcal{X}$ if the coproduct $\amalg U_\alpha$ is 0-connective: that is, if the map $\amalg U_\alpha \to \mathbf{1}$ is an effective epimorphism, where $\mathbf{1}$ denotes a final object of $\mathcal{X}$.

- If $\mathcal{C}$ is an ∞-category, we will often write $\mathcal{C}^\heartsuit$ to denote some full subcategory of $\mathcal{C}$ which forms an ordinary category. We will mainly use this notation in the following three (closely related) cases:

  - If $\mathcal{X}$ is an ∞-topos, then we let $\mathcal{X}^\heartsuit$ denote the full subcategory of $\mathcal{X}$ spanned by the discrete objects of $\mathcal{X}$. This is an ordinary Grothendieck topos, which we will refer to as the *underlying topos of* $\mathcal{X}$.
  - If $\mathcal{C}$ is a stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$, then we let $\mathcal{C}^\heartsuit$ denote the intersection $\mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0}$. This is an abelian category, which we refer to as the *heart of* $\mathcal{C}$.
  - If R is an $\mathbb{E}_\infty$-ring, we let CAlg_R denote the ∞-category of $\mathbb{E}_\infty$-algebras over R. When R is connective, we let $\mathrm{CAlg}_R^\heartsuit$ denote the full subcategory of CAlg_R spanned by the discrete objects, so that $\mathrm{CAlg}_R^\heartsuit$ can be identified with the ordinary category of commutative rings A equipped with a ring homomorphism $\pi_0 R \to A$.

- If $\mathcal{C}$ is an essentially small ∞-category, we let $\mathrm{Ind}(\mathcal{C})$ denote the ∞-category of Ind-objects of $\mathcal{C}$ introduced in §HTT.5.3.5 . We will generally regard $\mathrm{Ind}(\mathcal{C})$ as a full subcategory of the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ by identifying each Ind-object of $\mathcal{C}$ with the functor that it represents on $\mathcal{C}$. Note that the Yoneda embedding $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ factors through the full subcategory $\mathrm{Ind}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. By slight abuse of terminology, we will refer to the induced map $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ also as the *Yoneda embedding*.

- If $\mathcal{C}$ is an essentially small ∞-category, we let $\mathrm{Pro}(\mathcal{C})$ denote the ∞-category $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. We will refer to $\mathrm{Pro}(\mathcal{C})$ as the *∞-category of* Pro-*objects of* $\mathcal{C}$. We will view $\mathrm{Pro}(\mathcal{C})$ as a full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$. As in the case of Ind-objects, there is a canonical map $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ which we will (by slight abuse of terminology) refer to as the Yoneda embedding.

- Let R be a commutative ring. In classical algebraic geometry, the notation $\mathrm{Spec}\, R$ is often used to refer to several different (but closely related) mathematical objects:

  (i) The topological space X whose points are prime ideals $\mathfrak{p} \subseteq R$, equipped with the *Zariski topology* having a basis consisting of open sets of the form $U_f = \{\mathfrak{p} \subseteq R : f \notin \mathfrak{p}\}$.

(ii) The affine scheme $(X, \mathscr{O}_X)$, where X is the topological space defined above and $\mathscr{O}_X$ is the sheaf of commutative rings given on basic open sets by the formula $\mathscr{O}_X(U_f) = R[f^{-1}]$.

(iii) The functor { commutative rings } → { sets } repesented by the affine scheme $(X, \mathscr{O}_X)$, which assigns to each commutative ring A the set $\mathrm{Hom}(R, A)$ of ring homomorphisms from R to A.

Note that (ii) and (iii) are equivalent data, and we will often abuse terminology by not distinguishing between them. To avoid confusing (i) and (ii), we denote the affine scheme $(X, \mathscr{O}_X)$ by $\operatorname{Spec} R$ and its underlying topological space X by $|\operatorname{Spec} R|$. More generally, we sometimes use the notation $|Y|$ to denote the "underlying topological space" of some geometric object Y (such as a scheme, algebraic space, or some variant thereof).

0.6 Acknowledgements

This book has been shaped by discussions with friends, colleagues, and collaborators over several years. I would like to thank David Ben-Zvi, Bhargav Bhatt, Roman Bezrukavnikov, David Carchedi, Brian Conrad, Vladimir Drinfeld, John Francis, Dennis Gaitsgory, Daniel Halpern-Leistner, David Nadler, Tony Pantev, Mauro Porta, and Carlos Simpson for many enlightening conversations on the subject matter of this book. I would also like to thank Benjamin Antieau, Anthony Blanc, Kestutis Cesnavicius, Adeel Khan, and David Roberts for offering suggestions and pointing out errors in earlier versions of this manuscript.

Special thanks are due to Bertrand Toën and Gabriele Vezzosi, who initiated the study of spectral algebraic geometry: much of this book is based on their writings and on ideas that they have generously shared over the years. Special thanks are due also to my advisor, Mike Hopkins: this book grew out of a doctoral dissertation written under his direction. Finally, I would also like to thank all of my students, past and present: Lukas Brantner, Omar Antolin Camarena, Dustin Clausen, Gijs Heuts, Akhil Mathew, Anatoly Preygel, and Nick Rozenblyum. With each I have had enlightening conversations about algebraic geometry, homotopy theory and their interaction, and many of the resulting insights have made their way into this book.

Part I

Fundamentals of Spectral Algebraic Geometry

Our goal in Part I is to set up foundations for the theory of spectral algebraic geometry that we will develop in this book. We begin in Chapter 1 by introducing the central definitions. After a brief review of Grothendieck's theory of schemes, we describe two of its extensions: the theory of *spectral schemes* (obtained from the classical theory by replacing ordinary commutative rings by $\mathbb{E}_\infty$-rings) and the theory of *Deligne-Mumford stacks* (obtained from the classical theory by replacing the Zariski topology with the étale topology). These extensions admit a common generalization: the theory of *spectral Deligne-Mumford stacks*, which are our principal objects of interest in this book. By definition, a spectral Deligne-Mumford stack X is a pair $(\mathcal{X}, \mathscr{O}_\mathcal{X})$, where $\mathcal{X}$ is an ∞-topos (in the sense of [138]) and $\mathscr{O}_\mathcal{X}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, which is required to satisfy a certain local condition (namely, we require that the pair $(\mathcal{X}, \mathscr{O}_\mathcal{X})$ can be described locally as the *étale spectrum* of a connective $\mathbb{E}_\infty$-ring: see Definition 1.4.2.5). The resulting theory has the following features:

- The collection of spectral Deligne-Mumford stacks can be organized into an ∞-category SpDM, which we will refer to as the *∞-category of spectral Deligne-Mumford stacks*.
- The ∞-category SpDM of spectral Deligne-Mumford stacks contains the 2-category DM of ordinary Deligne-Mumford stacks as a full subcategory (see Remark 1.4.8.3). In particular, it contains the ordinary category Sch of schemes as a full subcategory.
- To every spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathscr{O}_\mathcal{X})$, one can associate an ordinary Deligne-Mumford stack $(\mathcal{X}^\heartsuit, \pi_0 \mathscr{O}_\mathcal{X})$, which we all the *underlying Deligne-Mumford stack* of X (Remark 1.4.8.2). This construction determines a forgetful functor SpDM $\rightarrow$ DM, which is left homotopy inverse to the inclusion DM $\hookrightarrow$ SpDM).

In Chapter 2, we study quasi-coherent sheaves in spectral algebraic geometry. To each spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathscr{O}_\mathcal{X})$, we associate a stable ∞-category $\mathrm{QCoh}(\mathsf{X})$ whose objects we will refer to as *quasi-coherent sheaves* (Definition 2.2.2.1). The stable ∞-category $\mathrm{QCoh}(\mathsf{X})$ comes equipped with a t-structure whose heart $\mathrm{QCoh}(\mathsf{X})^\heartsuit$ can be identified with the abelian category of quasi-coherent sheaves on the ordinary Deligne-Mumford stack $(\mathcal{X}^\heartsuit, \pi_0 \mathscr{O})$. In some sense, the distinction between $\mathrm{QCoh}(\mathsf{X})$ and its heart $\mathrm{QCoh}(\mathsf{X})^\heartsuit$ measures the difference between classical and spectral algebraic geometry: when X arises from a "classical" geometric object (like a quasi-compact separated scheme), the stable ∞-category $\mathrm{QCoh}(\mathsf{X})$ can be obtained from the abelian category $\mathrm{QCoh}(\mathsf{X})^\heartsuit$ by passing to the derived ∞-category of §**??** (see Proposition **??**, or Corollary **??** for a closely related assertion).

Let X be a scheme. If R is a commutative ring, then we define an *R-valued point* of X to be a map of schemes $\mathrm{Spét}\, R \rightarrow X$. The collection of R-valued points of X forms a set $X(R) = \mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} R, X)$, and the construction $R \mapsto X(R)$ determines a functor from the category of commutative rings to the category of sets. The situation in spectral algebraic geometry is analogous. To every connective $\mathbb{E}_\infty$-ring R, one can associate a spectral Deligne-Mumford stack $\operatorname{Sp\acute{e}t} R$ which we call the *étale spectrum of* R (in the case where R is an ordinary commutative ring, this is simply the image of the affine scheme $\operatorname{Spec} R$ under the fully faithful embedding $\iota : \mathrm{Sch} \hookrightarrow \mathrm{SpDM}$). If X is a spectral Deligne-Mumford stack, then we define an *R-valued point* of X to be a morphism of spectral Deligne-Mumford stacks $\operatorname{Sp\acute{e}t} R \to \mathsf{X}$. However, there is a vital difference between classical and spectral algebraic geometry: the collection of spectral Deligne-Mumford stacks forms an ∞-category, rather than an ordinary category. Consequently, the collection of R-valued points of a spectral Deligne-Mumford stack X forms a space $\mathrm{Map}_{\mathrm{SpDM}}(\operatorname{Sp\acute{e}t} R, \mathsf{X})$, rather than a set. In Chapter 3, we will study the situation where the mapping space $\mathrm{Map}_{\mathrm{SpDM}}(\operatorname{Sp\acute{e}t} R, \mathsf{X})$ is discrete whenever R is an ordinary commutative ring; in this case, we say that X is a *spectral algebraic space*. Roughly speaking, this condition means that X is forbidden to exhibit any "stacky" behavior (beware, however, that the space $\mathrm{Map}_{\mathrm{SpDM}}(\operatorname{Sp\acute{e}t} R, \mathsf{X})$ will usually have a nontrivial homotopy type if R is not discrete, even if X arises from an ordinary scheme).

Remark 0.6.0.1. Our theory of spectral algebraic geometry is closely related to the theory of homotopical algebraic geometry introduced by Toën and Vezzosi, and the material presented here has substantial overlap with their work (see [202], [213], [214], and [215]). The primary difference in our exposition is that we stick closely to the classical view of scheme as a kind of ringed space, while Toën and Vezzosi put more emphasis on the "functor of points" philosophy described above.

Chapter 1

Schemes and Deligne-Mumford Stacks

In this section, we will introduce the basic objects of study in this book: spectral Deligne-Mumford stacks. The collection of spectral Deligne-Mumford stacks can be organized into an ∞-category SpDM, which contains the usual category of schemes as a full subcategory. Recall that a scheme is defined to be a topological space X together with a sheaf of commutative rings $\mathscr{O}$ on X for which the pair $(X, \mathscr{O})$ is isomorphic, locally on X, to the spectrum $\operatorname{Spec} R$ of a commutative ring R. Our definition of spectral Deligne-Mumford stack is similar in spirit, but differs in three significant ways:

(a) Rather than working with sheaves of ordinary commutative rings, we work with sheaves taking values in the larger ∞-category CAlg of $\mathbb{E}_\infty$-rings (see §HA.7 for an introduction to the theory of $\mathbb{E}_\infty$-rings, and Definition 1.3.1.4 for the definition of a CAlg-valued sheaf).

(b) In place of a topological space X, we consider an arbitrary ∞-topos $\mathcal{X}$. This affords us a great deal of flexibility in forming certain categorical constructions, like quotients by group actions, which can be useful for some applications (such as studying the moduli of objects which admit nontrivial symmetries).

(c) Rather than requiring $(\mathcal{X}, \mathscr{O})$ to be locally equivalent to an affine model of the form $\operatorname{Spec} R$, where R is a commutative ring, we consider instead models of the form $\operatorname{Spét} A$, where A is an $\mathbb{E}_\infty$-ring and $\operatorname{Spét} A$ is its *étale spectrum* (see Definitions 1.2.3.3 and 1.4.2.5). In other words, we will consistently work locally with respect to the étale topology, rather than the Zariski topology.

The ideas required to carry out modification (a) are logically independent from those required to carry out modifications (b) and (c), so we will first discuss them separately from one another. We will begin in §1.1 by considering (a) alone. This leads us to the notion of a *spectral scheme*: that is, a topological space X equipped with a sheaf of $\mathbb{E}_\infty$-rings $\mathscr{O}$ for which $(X, \mathscr{O})$ is locally equivalent to $\operatorname{Spec} A$, where A is a connective $\mathbb{E}_\infty$-ring (see Definition 1.1.2.8 and Corollary 1.1.6.2). We will show that the collection of spectral schemes can be organized into an ∞-category SpSch, which contains the usual category of schemes as a full subcategory (Proposition 1.1.8.4).

In §1.2, we will discuss a completely different enlargement of the category of schemes: the 2-category of Deligne-Mumford stacks. We will view Deligne-Mumford stacks as ringed topoi $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, which are locally equivalent to the étale spectrum of an affine scheme (Definition 1.2.4.1).

To adapt the notion of Deligne-Mumford stack to the setting of spectral algebraic geometry, we will need a theory of sheaves of $\mathbb{E}_\infty$-rings on topoi (or, more generally, on ∞-topoi). In §1.3, we will show how to associate to every ∞-topos $\mathcal{X}$ another ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$, whose objects are *sheaves of spectra* on $\mathcal{X}$ (or, equivalently, *spectrum objects* of $\mathcal{X}$). The ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is symmetric monoidal, and the commutative algebra objects of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ form an ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$. We will refer to the objects of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ as *sheaves of* $\mathbb{E}_\infty$*-rings on* $\mathcal{X}$. We then define a *spectrally ringed* ∞*-topos* to be a pair $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, where $\mathcal{X}$ is an ∞-topos and $\mathscr{O}_{\mathcal{X}}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

In §1.4, we will introduce the notion of a *spectral Deligne-Mumford stack*: that is, a spectrally ringed ∞-topos $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ which is locally equivalent to the étale spectrum of a (connective) $\mathbb{E}_\infty$-ring. The collection of spectral Deligne-Mumford stacks can be organized into an ∞-category SpDM, which we show to be an enlargement of the usual 2-category of Deligne-Mumford stacks (Remark 1.4.8.3).

In this book, we adopt the point of view that a Deligne-Mumford stack is a ringed topos $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ satisfying certain local assumptions. Another perspective (which is more common in the literature) is to view a Deligne-Mumford stack as a groupoid-valued functor on the category of commutative rings (or a certain type of fibered category). We will establish the equivalence of these perspectives in §1.2 (see Theorem 1.2.5.9). In §1.6, we will describe a similar approach to the theory of spectral Deligne-Mumford stack. To every spectral Deligne-Mumford stack (or spectral scheme) X, one can assign a $\mathcal{S}$-valued functor h_{X} on the ∞-category $\mathcal{S}$ of $\mathbb{E}_\infty$-rings. We will show that the construction $\mathsf{X} \mapsto h_{\mathsf{X}}$ is fully faithful (Proposition 1.6.4.2). Combining this fact with some elementary observations about the relationship between topological spaces and ∞-topoi (which we review in §1.5), we show that the ∞-category of spectral schemes can be identified with a full subcategory of the ∞-category of spectral Deligne-Mumford stacks. For this reason, we will primarily focus our attention on spectral Deligne-Mumford stacks throughout the rest of this book.

Contents

1.1 Spectral Schemes

1.1.1 Review of Scheme Theory

Our primary objective in this book is to develop the theory of *spectral algebraic geometry*: a variant of algebraic geometry which uses structured ring spectra in the place of ordinary commutative rings. Before we begin this undertaking, it will be helpful to review some of the foundational definitions of the classical theory.

Definition 1.1.1.1. A *ringed space* is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf of commutative rings on X. In this case, we will say that $\mathscr{O}_X$ is the *structure sheaf* of X. We will regard the collection of all ringed spaces as the objects of a category $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}$, where a morphism from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ in $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}$ consists of a pair (π, ϕ), where $\pi : X \to Y$ is a continuous map of topological spaces and $\phi : \mathscr{O}_Y \to \pi_* \mathscr{O}_X$ is a map between sheaves of commutative rings on Y.

Example 1.1.1.2. Let R be a commutative ring. We let $|\operatorname{Spec} R|$ denote the set of all prime ideals of R. For every ideal $I \subseteq R$, we set $V_I = \{\mathfrak{p} \in |\operatorname{Spec} R| : I \subseteq \mathfrak{p}\}$, and refer to V_I as the *vanishing locus* of I. We will regard $|\operatorname{Spec} R|$ as a topological space, where a subset of $|\operatorname{Spec} R|$ is closed if and only if it has the form V_I for some ideal $I \subseteq R$. We will refer to the resulting topology on $|\operatorname{Spec} R|$ as the *Zariski topology*.

For each element $x \in R$, let $U_x = \{\mathfrak{p} \in |\operatorname{Spec} R| : x \notin \mathfrak{p}\}$ denote the complement of the vanishing locus of the principal ideal (x). We will say that an open set $U \subseteq |\operatorname{Spec} R|$ is *elementary* if it has the form U_x, for some element $x \in R$. The collection of elementary open sets forms a basis for the topology of $|\operatorname{Spec} R|$.

The *structure sheaf* of $|\operatorname{Spec} R|$ is a sheaf of commutative rings $\mathscr{O}$ on $|\operatorname{Spec} R|$ with the following properties:

(a) There is a ring homomorphism $\phi : R \to \Gamma(|\operatorname{Spec} R|; \mathscr{O}) = \mathscr{O}(|\operatorname{Spec} R|)$.

(b) For each element $x \in R$, the composite map $R \xrightarrow{\phi} \mathscr{O}(|\operatorname{Spec} R|) \to \mathscr{O}(U_x)$ carries x to an invertible element of $\mathscr{O}(U_x)$, and induces an isomorphism of commutative rings $R[x^{-1}] \simeq \mathscr{O}(U_x)$. In particular, the map α itself is an isomorphism.

Since the open sets U_x form a basis for the topology of R, property (b) determines the structure sheaf $\mathscr{O}$ up to unique isomorphism, once the map $\alpha : R \to \mathscr{O}(|\operatorname{Spec} R|)$ has been fixed (for the existence of $\mathscr{O}$, see Example 1.1.4.7 below).

Remark 1.1.1.3. It is customary to abuse notation by identifying the spectrum $\operatorname{Spec} R$ of a commutative ring R with its underlying topological space $|\operatorname{Spec} R|$. However, we will avoid this abuse of notation for the time being.

Definition 1.1.1.4. Let $(X, \mathscr{O}_X)$ be a ringed space. For every open subset $U \subseteq X$, we let $\mathscr{O}_X|_U$ denote the restriction of $\mathscr{O}_X$ to open subsets of U (which we regard as a sheaf of commutative rings on U). Then $(X, \mathscr{O}_X|_U)$ is itself a locally ringed space.

We say that $(X, \mathscr{O}_X)$ is a *scheme* if, for every point $x \in X$, there exists an open subset $U \subseteq X$ containing x such that $(U, \mathscr{O}_X|_U)$ is isomorphic to $\operatorname{Spec} R$ for some commutative ring R (in the category of ringed spaces). We say that a scheme $(X, \mathscr{O}_X)$ is *affine* if it is isomorphic (in the category $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}$ of ringed spaces) to $\operatorname{Spec} R$, for some commutative ring R.

1.1.2 Spectrally Ringed Spaces

Our goal in this section is to introduce an ∞-categorical generalization of the theory of schemes, which we will refer to as the theory of *spectral schemes*. For this, we will need to work with topological spaces equipped with a sheaf of $\mathbb{E}_\infty$-rings, rather than a sheaf of ordinary commutative rings.

Definition 1.1.2.1. Let X be a topological space and let $\mathcal{U}(X)$ denote the partially ordered set of all open subsets of X (which we will regard as a category). For any ∞-category $\mathcal{C}$, a *$\mathcal{C}$-valued presheaf* on X is a functor $\mathscr{F} : \mathcal{U}(X)^{\mathrm{op}} \to \mathcal{C}$. We will say that a $\mathcal{C}$-valued presheaf $\mathscr{F}$ is a *sheaf* if it satisfies the following condition:

- Let $\{U_\alpha\}$ be a collection of open subsets of X having union U, and let $\mathcal{U}' = \{V \in \mathcal{U}(X) : (\exists \alpha)[V \subseteq U_\alpha]\}$. Then the functor $\mathscr{F}$ exhibits $\mathscr{F}(U)$ as a limit of the diagram $\mathscr{F}|_{\mathcal{U}'^{\mathrm{op}}}$ (in other words, $\mathscr{F}$ induces an equivalence $\mathscr{F}(U) \simeq \varprojlim_{V \in \mathcal{U}'} \mathscr{F}(V)$ in the ∞-category $\mathcal{C}$).

We let $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(X)$ denote the full subcategory of $\operatorname{Fun}(\mathcal{U}(X)^{\mathrm{op}}, \mathcal{C})$ spanned by those functors which are $\mathcal{C}$-valued sheaves on X..

Construction 1.1.2.2. Let $\mathcal{C}$ be an ∞-category, and let $\pi : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F} : \mathcal{U}(X)^{\mathrm{op}} \to \mathcal{C}$ be a $\mathcal{C}$-valued presheaf on X. Then we can define another $\mathcal{C}$-valued presheaf $(\pi_* \mathscr{F}) : \mathcal{U}(Y)^{\mathrm{op}} \to \mathcal{C}$, given on objects by the formula $(\pi_* \mathscr{F})(U) = \mathscr{F}(\pi^{-1}U)$. If $\mathscr{F}$ is a $\mathcal{C}$-valued sheaf on X, then $\pi_* \mathscr{F}$ is a $\mathcal{C}$-valued sheaf on Y. Moreover, the construction $\mathscr{F} \mapsto \pi_* \mathscr{F}$ determines a functor $\pi_* : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(X) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(Y)$, given by precomposition with the map of partially ordered sets $\pi^{-1} : \mathcal{U}(Y) \to \mathcal{U}(X)$. We will refer to π_* as the *pushforward* functor associated to π.

We can regard the construction $X \mapsto \mathcal{S}\mathrm{hv}_{\mathcal{C}}(X)^{\mathrm{op}}$ as a functor from the category $\mathcal{T}\mathrm{op}$ of topological spaces to the category of simplicial sets, which carries a continuous map map $\pi : X \to Y$ to the pushforward functor $\pi_* : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(X)^{\mathrm{op}} \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(Y)^{\mathrm{op}}$. We let $\mathcal{T}\mathrm{op}_{\mathcal{C}}$ denote the relative nerve of this functor, in the sense of Definition HTT.3.2.5.2 . Then $\mathcal{T}\mathrm{op}_{\mathcal{C}}$ is an ∞-category equipped with a coCartesian fibration $\mathcal{T}\mathrm{op}_{\mathcal{C}} \to \mathcal{T}\mathrm{op}$, having the property that each fiber $\mathcal{T}\mathrm{op}_{\mathcal{C}} \times_{\mathcal{T}\mathrm{op}} \{X\}$ is canonically isomorphic to the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(X)^{\mathrm{op}}$.

Remark 1.1.2.3. Let $\mathcal{C}$ be an ∞-category. Then the objects of $\mathcal{T}\mathrm{op}_{\mathcal{C}}$ are given by pairs $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a $\mathcal{C}$-valued sheaf on X. A morphism from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ in $\mathcal{T}\mathrm{op}_{\mathcal{C}}$ is given by a pair (π, α), where $\pi : X \to Y$ is a continuous map of topological spaces and $\alpha : \mathscr{O}_Y \to \pi_* \mathscr{O}_X$ is a morphism in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(Y)$.

Example 1.1.2.4. Let $\mathrm{CAlg}^{\heartsuit}$ denote the category of commutative rings (regarded as an ∞-category). Then $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^{\heartsuit}}$ is equivalent to the category of ringed spaces.

Definition 1.1.2.5. Let CAlg denote the ∞-category of $\mathbb{E}_\infty$-rings. A *spectrally ringed space* is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a CAlg-valued sheaf on X. In this case, we will refer to $\mathscr{O}_X$ as the *structure sheaf* of X. We will refer to $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ as the *∞-category of spectrally ringed spaces.*

Notation 1.1.2.6. Let X be a topological space, and let $\mathscr{F}$ be a sheaf on X with values in the ∞-category Sp of spectra. For each integer n, the construction $U \mapsto \pi_n(\mathscr{F}(U))$ determines a presheaf of abelian groups on X. We let $\pi_n \mathscr{F}$ denote the sheafification of this presheaf. If $\mathscr{O}_X$ is a sheaf of $\mathbb{E}_\infty$-rings on X, then $\pi_0 \mathscr{O}_X$ is a sheaf of commutative rings on X, and each $\pi_n \mathscr{O}_X$ can be regarded as a sheaf of $\pi_0 \mathscr{O}_X$-modules on X.

The construction $(X, \mathscr{O}_X) \mapsto (X, \pi_0 \mathscr{O}_X)$ determines a functor from the ∞-category $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ of spectrally ringed spaces to the category of ringed spaces. We will refer to $(X, \pi_0 \mathscr{O}_X)$ as the *underlying ringed space* of $(X, \mathscr{O}_X)$.

Warning 1.1.2.7. Let $(X, \mathscr{O}_X)$ be a spectrally ringed space. For every integer n and every open set $U \subseteq X$, there is a canonical map of abelian groups

$$\pi_n(\mathscr{O}_X(U)) \to (\pi_n \mathscr{O}_X)(U),$$

which is generally *not* an isomorphism.

We are now ready to introduce our main objects of interest.

Definition 1.1.2.8. A *spectral scheme* is a spectrally ringed space $(X, \mathscr{O}_X)$ which satisfies the following conditions:

(1) The underlying ringed space $(X, \pi_0 \mathscr{O}_X)$ is a scheme.

(2) Each of the sheaves $\pi_n \mathscr{O}_X$ is quasi-coherent (when viewed as a sheaf of $\pi_0 \mathscr{O}_X$-modules on X).

(3) Let U be an open subset of X for which the scheme $(U, (\pi_0 \mathscr{O}_X)|_U)$ is affine. Then, for each integer n, the canonical map $\pi_n(\mathscr{O}_X(U)) \to (\pi_n \mathscr{O}_X)(U)$ is an isomorphism.

(4) The sheaves $\pi_n \mathscr{O}_X$ vanish when $n < 0$.

Variant 1.1.2.9. We will say that a spectrally ringed space $(X, \mathscr{O}_X)$ is a *nonconnective spectral scheme* if it satisfies conditions (1), (2), and (3) of Definition 1.1.2.8.

Remark 1.1.2.10. If $(X, \mathscr{O}_X)$ is a nonconnective spectral scheme, then the ringed space $(X, \pi_0 \mathscr{O}_X)$ is a scheme. We will refer to $(X, \pi_0 \mathscr{O}_X)$ as the *underlying scheme* of $(X, \mathscr{O}_X)$.

1.1.3 Digression: Hypercompleteness

We next show that condition (3) of Definition 1.1.2.8 admits an alternate formulation, and is automatically satisfied in many cases of interest (see Corollary 1.1.3.6).

Definition 1.1.3.1. Let $\mathscr{F}$ be a spectrum-valued sheaf on a topological space X. We will say that $\mathscr{F}$ is *hypercomplete* if the functor $U \mapsto \Omega^\infty \mathscr{F}(U)$ determines a hypercomplete object of the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(X)$ (see §HTT.6.5.2).

Remark 1.1.3.2. Let X be a topological space. Then the collection of hypercomplete objects of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)$ is closed under small limits and under suspensions (see Proposition 1.3.3.3). Moreover, the condition that an object $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)$ be hypercomplete can be tested locally on X (Corollary 1.3.3.8).

Remark 1.1.3.3. Let $u : \mathscr{F} \to \mathscr{F}'$ be a morphism of hypercomplete spectrum-valued sheaves on a topological space X. Then u is an equivalence if and only if it induces an isomorphism $\pi_n \mathscr{F} \to \pi_n \mathscr{F}'$ (in the category of sheaves of abelian groups) for every integer n. The "only if" direction is obvious. To prove the converse, it suffices to show that the map $u_n : \Omega^\infty(\Sigma^n \mathscr{F}) \to \Omega^\infty(\Sigma^n \mathscr{F}')$ is an equivalence in the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(X)$ for every integer n. This is clear, since u_n is a morphism between hypercomplete objects of $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(X)$ which induces an isomorphism on homotopy sheaves.

Proposition 1.1.3.4. *Let $(X, \mathscr{O}_X)$ be a spectrally ringed space satisfying the following conditions:*

(1) *The underlying ringed space* $(X, \pi_0 \mathscr{O}_X)$ *is a scheme.*

(2) *Each of the sheaves* $\pi_n \mathscr{O}_X$ *is quasi-coherent (when viewed as a sheaf of* $\pi_0 \mathscr{O}_X$*-modules on* X*).*

(3′) *The structure sheaf* $\mathscr{O}_X$ *is hypercomplete.*

Then $(X, \mathscr{O}_X)$ *is a nonconnective spectral scheme.*

Proof. For each integer n, let $\tau_{\leqslant n} \mathscr{O}_X$ denote the n-truncation of $\mathscr{O}_X$ with respect to the natural t-structure on the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)$ (see Proposition 1.3.2.7). We then have fiber sequences

$$\Sigma^n(\pi_n \mathscr{O}_X) \to \tau_{\leqslant n} \mathscr{O}_X \to \tau_{\leqslant n-1} \mathscr{O}_X,$$

where we abuse notation by identifying the sheaf of abelian groups $\pi_n \mathscr{O}_X$ with the corresponding object in the heart of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)$. Passing to global sections and extracting homotopy groups, we obtain a long exact sequence

$$\mathrm{H}^{n-m}(U; (\pi_n \mathscr{O}_X)|_U) \to \pi_m(\tau_{\leqslant n} \mathscr{O}_X)(U) \to \pi_m(\tau_{\leqslant n-1} \mathscr{O}_X(U)) \to \mathrm{H}^{n-m+1}(U; (\pi_n \mathscr{O}_X)|_U)$$

for each open subset $U \subseteq X$. Assumption (2) implies that the cohomology groups $\mathrm{H}^i(U; (\pi_n \mathscr{O}_X)|_U)$ vanish whenever U is affine and $i > 0$. Moreover, for an open subset $U \subseteq X$, the spectra $(\tau_{\leqslant n} \mathscr{O}_X)(U)$ and $(\tau_{\leqslant n-1} \mathscr{O}_X)(U)$ are n-truncated and $(n-1)$-truncated, respectively. It follows that if U is affine, our long exact sequence degenerates to supply isomorphisms

$$\pi_m(\tau_{\leqslant n} \mathscr{O}_X)(U) \simeq \begin{cases} 0 & \text{if } m > n \\ (\pi_n \mathscr{O}_X)(U) & \text{if } m = n \\ (\tau_{\leqslant n-1} \mathscr{O}_X)(U) & \text{if } m < n. \end{cases}$$

Set $\mathscr{O}'_X = \varprojlim_n \tau_{\leqslant n} \mathscr{O}_X \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)$. We have an evident map $u : \mathscr{O}_X \to \mathscr{O}'_X$, and the above calculation shows that this map induces an equivalence $(\pi_n \mathscr{O}_X)(U) \to \pi_n(\mathscr{O}'_X(U))$ for every affine open subset $U \subseteq X$. In particular, u induces an isomorphism of sheaves $\pi_n \mathscr{O}_X \to \pi_n \mathscr{O}'_X$ for every integer n. The $\mathcal{S}$-valued sheaf $\Omega^\infty \mathscr{O}'_X$ is hypercomplete by construction, and the $\mathcal{S}$-valued sheaf $\Omega^\infty \mathscr{O}_X$ is hypercomplete by virtue of assumption (3′). It follows that the map u is an equivalence, so that $(\pi_n \mathscr{O}_X)(U) \simeq \pi_n(\mathscr{O}_X(U))$ for each affine open subset $U \subseteq X$. That is, $(X, \mathscr{O}_X)$ satisfies condition (3) of Definition 1.1.2.8, and is therefore a nonconnective spectral scheme. □

Remark 1.1.3.5. The converse of Proposition 1.1.3.4 is also true: if $(X, \mathscr{O}_X)$ is a nonconnective spectral scheme, then the structure sheaf $\mathscr{O}_X$ is hypercomplete (see Corollary 1.1.6.2).

Corollary 1.1.3.6. *Let $(X, \mathscr{O}_X)$ be a spectrally ringed space, and suppose that X is a Noetherian topological space of finite Krull dimension. Then $(X, \mathscr{O}_X)$ is a nonconnective spectral scheme if and only if $(X, \pi_0 \mathscr{O}_X)$ is a scheme and each homotopy group $\pi_n \mathscr{O}_X$ is a quasi-coherent sheaf on $(X, \pi_0 \mathscr{O}_X)$.*

Proof. If X is a Noetherian topological space of finite Krull dimension, then the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(X)$ has finite homotopy dimension, so that every object of $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(X)$ is hypercomplete (see §HTT.7.2.4 ; we will prove a generalization of this statement in §3.7). The desired result now follows from Proposition 1.1.3.4. □

Remark 1.1.3.7. Let $(X, \mathscr{O}_X)$ be a spectrally ringed ∞-topos which satisfies conditions (1) and (2) of Definition 1.1.2.8. Then we can obtain a nonconnective spectral scheme by replacing the sheaf $\mathscr{O}_X$ with its hypercompletion. This replacement does not change the underlying scheme $(X, \pi_0 \mathscr{O}_X)$ or any of the quasi-coherent sheaves $\pi_n \mathscr{O}_X$.

1.1.4 The Spectrum of an $\mathbb{E}_\infty$-Ring

Our next goal is to produce some examples of spectral schemes.

Definition 1.1.4.1. Let A be an $\mathbb{E}_\infty$-ring. We let $|\operatorname{Spec} A|$ denote the Zariski spectrum of the underlying commutative ring $R = \pi_0 A$. We will say that an open subset $U \subseteq |\operatorname{Spec} A|$ is *affine* if it is affine when regarded as an open subset of the affine scheme $\operatorname{Spec} R$.

Let A be an $\mathbb{E}_\infty$-ring. We wish to construct a CAlg-valued sheaf on the topological space $|\operatorname{Spec} A|$, analogous to the structure sheaf on the spectrum of an ordinary commutative ring. First, we review the theory of localizations in the setting of $\mathbb{E}_\infty$-rings.

Remark 1.1.4.2. Let $f : A \to B$ be a map of $\mathbb{E}_\infty$-rings, and let $a \in \pi_0 A$. We will say that *f exhibits B as a localization of A by $a \in \pi_0 A$* if the map f is étale and f induces an isomorphism of commutative rings $(\pi_0 A)[a^{-1}] \simeq \pi_0 B$. In this case, we will denote B by $A[a^{-1}]$. Theorem HA.7.5.0.6 guarantees that $A[a^{-1}]$ exists and is well-defined up to equivalence (in fact, up to a contractible space of choices). The localization map $A \to A[a^{-1}]$ can be characterized by either of the following conditions (see Corollary HA.7.5.4.6):

(1) The map $A \to A[a^{-1}]$ induces an isomorphism of graded rings $(\pi_* A)[a^{-1}] \to \pi_* A[a^{-1}]$.

(2) For every $\mathbb{E}_\infty$-ring B, the induced map $\operatorname{Map}_{\mathrm{CAlg}}(A[a^{-1}], B) \to \operatorname{Map}_{\mathrm{CAlg}}(A, B)$ restricts to a homotopy equivalence of $\operatorname{Map}_{\mathrm{CAlg}}(A[a^{-1}], B)$ with the summand of $\operatorname{Map}_{\mathrm{CAlg}}(A, B)$ spanned by those maps $A \to B$ which carry $a \in \pi_0 A$ to an invertible element of $\pi_0 B$.

Proposition 1.1.4.3. *Let A be an $\mathbb{E}_\infty$-ring and let $R = \pi_0 A$ be its underlying commutative ring. Then there exists a* CAlg*-valued sheaf $\mathscr{O}$ on the topological space $|\operatorname{Spec} A|$ and a morphism $\phi : A \to \mathscr{O}(|\operatorname{Spec} A|)$ of $\mathbb{E}_\infty$-rings which satisfies the following conditions:*

(a) *For every element $x \in R$ defining an elementary open subset $U_x = \{\mathfrak{p} \in |\operatorname{Spec} A| : x \notin \mathfrak{p}\}$, the composite map $A \xrightarrow{\phi} \mathscr{O}(|\operatorname{Spec} A|) \to \mathscr{O}(U_x)$ induces an equivalence of $\mathbb{E}_\infty$-rings $A[x^{-1}] \simeq \mathscr{O}(U_x)$.*

(b) *The canonical map*

$$R = \pi_0 A \xrightarrow{\alpha} \pi_0(\mathscr{O}(|\operatorname{Spec} A|)) \to (\pi_0 \mathscr{O})(|\operatorname{Spec} A|)$$

induces an isomorphism of $\pi_0 \mathscr{O}$ with the structure sheaf of the affine scheme $\operatorname{Spec} R$*.*

(c) *The pair $(|\operatorname{Spec} A|, \mathscr{O})$ is a nonconnective spectral scheme. If A is connective, then $(|\operatorname{Spec} A|, \mathscr{O})$ is a spectral scheme.*

We will deduce the existence of the sheaf $\mathscr{O}$ from the following general principle:

Proposition 1.1.4.4. *Let X be a topological space, let $\mathcal{C}$ be an ∞-category which admits small limits, and let $\mathcal{U}_e$ denote a collection of open subsets of X satisfying the following conditions:*

(i) *The sets belonging to $\mathcal{U}_e$ form a basis for the topology of X. That is, for every point $x \in X$ and every open set U containing x, there exists an open set $V \in \mathcal{U}_e$ such that $x \in V \subseteq U$.*

(ii) *For every pair of open sets $U, V \in \mathcal{U}_e$, the intersection $U \cap V$ belongs to $\mathcal{U}_e$.*

(iii) *Each of the open sets $U \in \mathcal{U}_e$ is quasi-compact.*

Then a functor $\mathscr{F} : \mathcal{U}(X)^{\mathrm{op}} \to \mathcal{C}$ is a $\mathcal{C}$-valued sheaf on X if and only if the following conditions are satisfied:

(1) *The functor $\mathscr{F}$ is a right Kan extension of $\mathscr{F}|_{\mathcal{U}_e^{\mathrm{op}}}$.*

(2) *Let $U_1, U_2, \ldots, U_n \in \mathcal{U}_e$ be a finite collection of open sets whose union $U = \bigcup U_i$ also belongs to $\mathcal{U}_e$. For each $S \subseteq \{1, \ldots, n\}$, let $U_S = \bigcap_{i \in S} U_i$. Then $\mathscr{F}$ induces an equivalence $\mathscr{F}(U) \to \varprojlim_{\emptyset \neq S} \mathscr{F}(U_S)$, where the limit is taken over the partially ordered set of all nonempty subsets $S \subseteq \{1, \ldots, n\}$.*

Corollary 1.1.4.5. *Under the hypotheses of Proposition 1.1.4.4, the restriction map*

$$\mathcal{S}\mathrm{hv}_{\mathcal{C}}(X) \to \operatorname{Fun}(\mathcal{U}_e^{\mathrm{op}}, \mathcal{C})$$

is a fully faithful embedding, whose essential image is the collection of those functors $\mathcal{U}_e^{\mathrm{op}} \to \mathcal{C}$ which satisfy condition (2) of Proposition 1.1.4.4.

Proof. Combine Propositions 1.1.4.4 and HTT.4.3.2.15. □

Before proving Proposition 1.1.4.4, let us describe some of its applications.

Example 1.1.4.6. Let X be a topological space and let $\mathcal{U}_e$ be a collection of open subsets of X satisfying conditions (i), (ii), and (iii) of Proposition 1.1.4.4. Applying Proposition 1.1.4.4 in the special case $\mathcal{C} = \mathcal{S}$, we obtain an equivalence of ∞-topoi $\mathcal{S}\mathrm{hv}(X) \simeq \mathcal{S}\mathrm{hv}(\mathcal{U}_e)$, where we regard $\mathcal{U}_e$ as equipped with the Grothendieck topology given by those sieves $\{U_\alpha \subseteq U\}$ for which $U = \bigcup U_\alpha$.

Example 1.1.4.7. Let R be a commutative ring, let $X = |\operatorname{Spec} R|$, let $\mathcal{U}_e$ denote the collection of all elementary open subsets of X, and let $\mathcal{C}$ be the category of commutative rings. Using Corollary 1.1.4.5, we see that the structure sheaf $\mathscr{O}$ of the affine scheme $\operatorname{Spec} R$ is essentially unique, and its existence reduces to the following basic assertion of commutative algebra: if $x_1, \ldots, x_n$ is a collection of elements of R which generate the unit ideal in R, then we have an equalizer diagram

$$R \longrightarrow \prod_{1 \leqslant i \leqslant n} R[x_i^{-1}] \rightrightarrows \prod_{1 \leqslant i < j \leqslant n} R[x_i^{-1}, x_j^{-1}]$$

in the category of commutative rings.

Proof of Proposition 1.1.4.3. Let $X = |\operatorname{Spec} A|$, let $\mathcal{U}_e$ denote the collection of open affine subsets of X, and let $\overline{\mathscr{O}}$ denote the structure sheaf of the affine scheme $\operatorname{Spec} R$. For every affine open subset $U \subseteq X$, the commutative ring $\overline{\mathscr{O}}(U)$ is étale as an R-algebra. Let $\overline{\mathscr{O}}_e$ denote the restriction of the sheaf $\overline{\mathscr{O}}$ to the partially ordered set $\mathcal{U}_e$, so that we can regard $\overline{\mathscr{O}}_e$ as a functor from $\mathcal{U}_e^{\mathrm{op}}$ to the category of étale R-algebras. Applying Theorem HA.7.5.0.6 , we see that there is an essentially unique functor $\mathscr{O}_e : \mathcal{U}_e^{\mathrm{op}} \to \mathrm{CAlg}_A$ such that $\overline{\mathscr{O}}_e(U) = \pi_0 \mathscr{O}_e(U)$, and each of the A-algebras $\mathscr{O}_e(U)$ is flat over A.

Let $\mathscr{O} : \mathcal{U}(X)^{\mathrm{op}} \to \mathrm{CAlg}$ be a right Kan extension of the functor $\mathscr{O}_e$. We claim that $\mathscr{O}$ is a CAlg-valued sheaf on X. To prove this, it will suffice to show that the functor $\mathscr{O}$ satisfies condition (2) of Proposition 1.1.4.4. Let $U_1, \ldots, U_n$ be a collection of open affine subsets of X whose union $U = \bigcup_{1 \leqslant i \leqslant n} U_i$ is also open. For each $S \subseteq \{1, \ldots, n\}$, let U_S denote the intersection $\bigcap_{i \in S} U_i$. We wish to show that the canonical map

$$\mu : \mathscr{O}(U) \to \varprojlim_{S \neq \varnothing} \mathscr{O}(U_S)$$

is an equivalence.

Since the open sets U_i form an affine open covering of U, the map $\overline{\mathscr{O}}(U) \to \prod_{1 \leqslant i \leqslant n} \overline{\mathscr{O}}(U_i)$ is faithfully flat. It follows that $\prod_{1 \leqslant i \leqslant n} \mathscr{O}(U_i)$ is faithfully flat over $\mathscr{O}(U)$. Consequently, to prove that μ is an equivalence, it will suffice to show that μ induces an equivalence

$$\mu_i : \mathscr{O}(U_i) \to \mathscr{O}(U_i) \otimes_{\mathscr{O}(U)} \varprojlim_{S \neq \varnothing} \mathscr{O}(U_S),$$

for $1 \leqslant i \leqslant n$. The operation $M \mapsto \mathscr{O}(U_i) \otimes_{\mathscr{O}(U)} M$ is an exact functor between stable ∞-categories, and therefore preserves finite limits. Consequently, we may identify μ_i with the canonical map

$$\mathscr{O}(U_i) \to \varprojlim_{S \neq \varnothing} (\mathscr{O}(U_i) \otimes_{\mathscr{O}(U)} \mathscr{O}(U_S)) \simeq \varprojlim_{S \neq \varnothing} \mathscr{O}(U_{S \cup \{i\}}).$$

Note that the functor $S \mapsto \mathscr{O}(U_{S \cup \{i\}})$ is a right Kan extension of its restriction to the partially ordered set $P = \{S \subseteq \{1, \ldots, n\} : i \in S\}$, so that μ_i is equivalent to the map $\mathscr{O}(U_i) \to \varprojlim_{S \in P} \mathscr{O}(U_S)$. This map is an equivalence, since the partially ordered set P contains $\{i\}$ as a least element. This completes the proof of (a).

By construction, the functor $U \mapsto \pi_0(\mathscr{O}(U))$ determines a presheaf of commutative rings on $|\operatorname{Spec} A|$ which agrees with $\overline{\mathscr{O}}$ on every affine open subset of $|\operatorname{Spec} A|$. Since the affine open subsets of $|\operatorname{Spec} A|$ form a basis for the topology of $|\operatorname{Spec} A|$, we obtain an isomorphism $\overline{\mathscr{O}} \simeq \pi_0 \mathscr{O}$ of sheaves of commutative rings on $|\operatorname{Spec} A|$. This proves (b).

We now prove (c). Let m be an integer, and let $M = \pi_m A$. Then we can regard M as a (discrete) R-module. Let $\mathscr{F}$ be the associated quasi-coherent sheaf on the affine scheme $\operatorname{Spec} R$, so that $\mathscr{F}(U) \simeq \overline{\mathscr{O}}(U) \otimes_R M$ for every affine open subset $U \subseteq |\operatorname{Spec} A|$. Since $\mathscr{O}(U)$ is flat over A, we obtain isomorphisms

$$\mathscr{F}(M) \simeq \overline{\mathscr{O}}(U) \otimes_R M \simeq \pi_0 \mathscr{O}(U) \otimes_{\pi_0 A} \pi_m A \simeq \pi_m \mathscr{O}(U),$$

depending functorially on $U \in \mathcal{U}_e$. Since $\mathcal{U}_e$ forms a basis for the topology of $|\operatorname{Spec} A|$, it follows that the sheaf $\pi_m \mathscr{O}$ is isomorphic to $\mathscr{F}$ (as a sheaf of $\overline{\mathscr{O}}$-modules), and is therefore quasi-coherent. Condition (3) of Definition 1.1.2.8 follows immediately from our construction, so that $(|\operatorname{Spec} A|, \mathscr{O})$ is a nonconnective spectral scheme. If A is connective, then the above calculation gives $\pi_m \mathscr{O} \simeq \mathscr{F} \simeq 0$ for $m < 0$, so that $(|\operatorname{Spec} A|, \mathscr{O})$ is a spectral scheme. □

Definition 1.1.4.8. Let A be an $\mathbb{E}_\infty$-ring, and let $\mathscr{O}$ be the sheaf of $\mathbb{E}_\infty$-rings on $|\operatorname{Spec} A|$ constructed in the proof of Proposition 1.1.4.3. We will refer to $\mathscr{O}$ as the *structure sheaf* of $|\operatorname{Spec} A|$. The pair $(|\operatorname{Spec} A|, \mathscr{O})$ is a spectrally ringed space, which we will denote by $\operatorname{Spec} A$ and refer to as the *spectrum* of A. We will say that a nonconnective spectral scheme $(X, \mathscr{O}_X)$ is *affine* if it is equivalent to $\operatorname{Spec} A$ for some $\mathbb{E}_\infty$-ring A.

Warning 1.1.4.9. Let R be a commutative ring. Then we can regard R as a discrete $\mathbb{E}_\infty$-ring. In this case, the notation introduced in Definition 1.1.4.8 is potentially ambiguous: we write $\operatorname{Spec} R$ to denote both the affine scheme $(|\operatorname{Spec} R|, \mathscr{O}_0)$ of Example 1.1.1.2 and the spectral scheme $(|\operatorname{Spec} R|, \mathscr{O})$ of Proposition 1.1.4.3. These two objects are not quite the same: $\mathscr{O}_0$ is a sheaf of commutative rings on $X = |\operatorname{Spec} R|$, while $\mathscr{O}$ is a sheaf of $\mathbb{E}_\infty$-rings on X. However, they are interchangeable data: the sheaf of commutative rings $\mathscr{O}_0$ isomorphic to $\pi_0 \mathscr{O}$, and the sheaf of $\mathbb{E}_\infty$-rings $\mathscr{O}$ can be recovered as the sheafification of $\mathscr{O}_0$ (regarded as a presheaf of $\mathbb{E}_\infty$-rings on X). Moreover, we have an equivalence $\mathscr{O}_0(U) \simeq \mathscr{O}(U)$ whenever U is an affine subset of X, which extends to an isomorphism $\mathscr{O}_0(U) \simeq \pi_0 \mathscr{O}(U)$ for every open subset $U \subseteq X$. However, the $\mathbb{E}_\infty$-rings $\mathscr{O}(U)$ are generally not discrete: we have canonical isomorphisms

$$\pi_{-n} \mathscr{O}(U) \simeq \mathrm{H}^n(U; \mathscr{O}_0|_U).$$

Proof of Proposition 1.1.4.4. Suppose first that $\mathscr{F}$ is a $\mathcal{C}$-valued sheaf on X. Let $\{U_\alpha\}_{\alpha \in A}$ be any collection of open subsets of X, and set $U = \bigcup_\alpha U_\alpha$, and let $\mathcal{U}'$ denote the subset of $\mathcal{U}(X)$ given by those open sets V which are contained in U_α for some $\alpha \in A$. Let $P(A)$ denote the partially ordered set of all nonempty finite subsets of A. For each $S \in P(A)$, we set $U_S = \bigcap_{\alpha \in S} U_\alpha$. The construction $S \mapsto U_S$ determines a right cofinal map $P(A) \to \mathcal{U}'^{\mathrm{op}}$. Invoking our hypothesis that $\mathscr{F}$ is a sheaf, we deduce that the canonical maps

$$\mathscr{F}(U) \to \varprojlim_{V \in \mathcal{U}'} \mathscr{F}(V) \to \varprojlim_{S \in P(A)} \mathscr{F}(U_S)$$

are equivalences in the ∞-category $\mathcal{C}$. This immediately implies condition (2).

To verify condition (1), we must show that for any open subset $U \subseteq X$, the canonical map

$$\mathscr{F}(U) \to \varprojlim_{V \in \mathcal{U}_e \cap \mathcal{U}(U)} \mathscr{F}(V)$$

is an equivalence. Since $\mathcal{U}_e$ forms a basis for the topology of X, we can write $U = \bigcup_{\gamma \in A} U_\gamma$ where each γ belongs to $\mathcal{U}_e$. Let $\mathcal{U}'$ be defined as above, so that we have a commutative diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \longrightarrow & \varprojlim_{V \in \mathcal{U}_e \cap \mathcal{U}(U)} \mathscr{F}(V) \\ \downarrow & & \downarrow \\ \varprojlim_{V \in \mathcal{U}'} \mathscr{F}(V) & \longrightarrow & \varprojlim_{V \in \mathcal{U}_e \cap \mathcal{U}'} \mathscr{F}(V). \end{array}$$

The left vertical map is an equivalence by virtue of our assumption that $\mathscr{F}$ is a $\mathcal{C}$-valued sheaf. Similarly, the assumption that $\mathscr{F}$ is a sheaf implies that the functor $\mathscr{F}|_{(\mathcal{U}_e \cap \mathcal{U}(U))^{\mathrm{op}}}$ is a right Kan extension of $\mathscr{F}|_{(\mathcal{U}_e \cap \mathcal{U}')^{\mathrm{op}}}$, which shows that the right vertical map is also an equivalence. We are therefore reduced to proving that the bottom horizontal map an equivalence. For this, it suffices to show that the inclusion of simplicial sets $\mathcal{U}_e \cap \mathcal{U}' \hookrightarrow \mathcal{U}'$ is left cofinal. This is equivalent to the assertion that for each $V \in \mathcal{U}'$, the partially ordered set $T = \{W \in \mathcal{U}_e \cap \mathcal{U}' : V \subseteq W\}$ has weakly contractible nerve. This is clear, since T is nonempty and closed under finite intersections.

Now suppose that $\mathscr{F}$ satisfies conditions (1) and (2); we will show that $\mathscr{F}$ is a $\mathcal{C}$-valued sheaf on X. Fix an open set $U \subseteq X$ and an open covering $\{U_\gamma\}_{\gamma \in A}$ of U. Let $\mathcal{U}'$ be as above; we wish to show that the canonical map $\mathscr{F}(U) \to \varprojlim_{V \in \mathcal{U}'} \mathscr{F}(V)$ is an equivalence in $\mathcal{C}$.

Since $\mathscr{F}$ satisfies condition (1), the functor $\mathscr{F}|_{\mathcal{U}'^{\mathrm{op}}}$ is a right Kan extension of $\mathscr{F}|_{(\mathcal{U}_e \cap \mathcal{U}')^{\mathrm{op}}}$. It will therefore suffice to show that the composite map

$$\mathscr{F}(U) \to \varprojlim_{V \in \mathcal{U}'} \mathscr{F}(V) \to \varprojlim_{V \in \mathcal{U}_e \cap \mathcal{U}'} \mathscr{F}(V)$$

is an equivalence. Since assumption (1) supplies an equivalence $\mathscr{F}(U) \simeq \varprojlim_{V \in \mathcal{U}_e \cap \mathcal{U}(U)} \mathscr{F}(V)$, it will suffice to show that $\mathscr{F}|_{(\mathcal{U}_e \cap \mathcal{U}(U))^{\mathrm{op}}}$ is a right Kan extension of $\mathscr{F}|_{(\mathcal{U}_e \cap \mathcal{U}')}{}^{\mathrm{op}}$. For this, let us fix an open set $V \in \mathcal{U}_e$ with $V \subseteq U$; we wish to show that the map

$$\theta : \mathscr{F}(V) \to \varprojlim_{W \in \mathcal{U}_e \cap \mathcal{U}', W \subseteq V} \mathscr{F}(W)$$

is an equivalence.

It follows from assumption (iii) that V is quasi-compact, so we can write V as a finite union $V_1 \cup \cdots \cup V_n$ where each V_i belongs to $\mathcal{U}' \cap \mathcal{U}_e$. Let $\mathcal{U}''$ denote the collection of all open subsets of X which belong to $\mathcal{U}_e$ and are contained in one of the open sets V_i. We will prove:

$(*)$ The functor $\mathscr{O}|_{(\mathcal{U}_e \cap \mathcal{U}(V))^{\mathrm{op}}}$ is a right Kan extension of $\mathscr{O}|_{\mathcal{U}''^{\mathrm{op}}}$.

Assuming $(*)$, the map θ fits into a commutative diagram

$$\begin{array}{ccc} \mathscr{O}(V) & \xrightarrow{\theta} & \varprojlim_{W \in \mathcal{U}_e \cap \mathcal{U}', W \subseteq V} \mathscr{O}(W) \\ & \searrow \quad \swarrow & \\ & \varprojlim_{V \in \mathcal{U}''} \mathscr{O}(V) & \end{array}$$

where the vertical maps are equivalences, so that θ is an equivalence. To prove $(*)$, we must show that for every open set $W \in \mathcal{U}_e$ with $W \subseteq V$, the canonical map

$$\phi : \mathscr{F}(W) \to \varprojlim_{W' \subseteq W, W' \in \mathcal{U}''} \mathscr{F}(W')$$

is an equivalence. Let Q denote the partially ordered set of all nonempty subsets of $\{1, \ldots, n\}$. For each subset $S \subseteq \{1, \ldots, n\}$, let W_S denote the intersection $W \cap \bigcap_{i \in S} V_i$. Since the construction $S \mapsto W_S$ induces a right cofinal functor $S \to \mathcal{U}(W) \cap \mathcal{U}''$, we can identify ϕ with the map

$$\mathscr{F}(W) \to \varprojlim_{\varnothing \neq S} \mathscr{F}(W_S),$$

which is an equivalence by virtue of assumption (2). $\square$

1.1.5 The Universal Property of Spec A

If R is a commutative ring, then the affine scheme $\operatorname{Spec} R$ can be characterized by a universal mapping property. To formulate it, we need to work in the setting of *locally* ringed spaces.

Definition 1.1.5.1. Let X be a topological space and let $\mathscr{O}_X$ be a sheaf of commutative rings on X. We say that $\mathscr{O}_X$ is *local* if, for every point $x \in X$, the stalk $\pi_0 \mathscr{O}_{X,x}$ is a local ring (we will give a quick review of the theory of local rings in §1.4). We say that a ringed space $(X, \mathscr{O}_X)$ is *locally ringed* if $\mathscr{O}_X$ is local.

The collection of locally ringed spaces can be organized into a category $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^\heartsuit}$, where a map of locally ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is given by a continuous map $\pi : X \to Y$ together with a map of sheaves $\mathscr{O}_Y \to \pi_* \mathscr{O}_X$ satisfying the following locality condition: for every point $x \in X$, the induced ring homomorphism $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is local (that is, it carries noninvertible elements of $\mathscr{O}_{Y,f(x)}$ to noninvertible elements of $\mathscr{O}_{X,x}$).

Example 1.1.5.2. Every scheme $(X, \mathscr{O}_X)$ is a locally ringed space. To prove this, we may work locally on X and thereby reduce to the case where $(X, \mathscr{O}_X) = \operatorname{Spec} R$ for some commutative ring R. In this case, the stalk of the structure sheaf $\mathscr{O}_X$ at a point $\mathfrak{p} \in |\operatorname{Spec} R|$ can be identified with the local ring $R_{\mathfrak{p}}$.

Definition 1.1.5.1 has an obvious generalization to the setting of CAlg-valued sheaves:

Definition 1.1.5.3. Let $(X, \mathscr{O}_X)$ be a spectrally ringed space. We will say that $(X, \mathscr{O}_X)$ is a *locally spectrally ringed space* if the underlying ringed space $(X, \pi_0 \mathscr{O}_X)$ is locally ringed. We let $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ denote the subcategory of $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ whose objects are locally spectrally ringed spaces $(X, \mathscr{O}_X)$, and whose morphisms are maps $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ for which the underlying map $(X, \pi_0 \mathscr{O}_X) \to (Y, \pi_0 \mathscr{O}_Y)$ is morphism in the category of locally ringed spaces.

Every spectral scheme $(X, \mathscr{O}_X)$ is a locally spectrally ringed space. This allows us to organize the collection of spectral schemes into an ∞-category:

Definition 1.1.5.4. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be nonconnective spectral schemes. A *morphism of nonconnective spectral schemes* from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ is a map $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ in the category $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ of locally spectrally ringed spaces. In other words, a morphism from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ consists of a pair (f, α), where $f : X \to Y$ is a continuous map of topological spaces and $\alpha : \mathscr{O}_Y \to f_* \mathscr{O}_X$ is a morphism in $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(Y)$ which induces a local homomorphism of commutative rings $(\pi_0 \mathscr{O}_Y)_{f(x)} \to (\pi_0 \mathscr{O}_X)_x$ for each $x \in X$.

Let $\mathrm{SpSch}^{\mathrm{nc}}$ denote the subcategory of $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ whose objects are nonconnective spectral schemes and whose morphisms are morphisms of nonconnective spectral schemes. Let

SpSch denote the full subcategory of $\mathrm{SpSch}^{\mathrm{nc}}$ spanned by the spectral schemes. We will refer to SpSch as the *∞-category of spectral schemes*, and to $\mathrm{SpSch}^{\mathrm{nc}}$ as the *∞-category of nonconnective spectral schemes*.

If $(X, \mathscr{O}_X)$ is a locally ringed space and R is a commutative ring, then there is a canonical bijection

$$\mathrm{Hom}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}}}((X, \mathscr{O}_X), \operatorname{Spec} R) \simeq \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}}(R, \mathscr{O}_X(X)).$$

In the setting of spectral algebraic geometry, we have an analogous statement:

Proposition 1.1.5.5. *Let $(X, \mathscr{O}_X)$ be a locally spectrally ringed space, let A be an $\mathbb{E}_\infty$-ring, and write $\operatorname{Spec}(A) = (|\operatorname{Spec} A|, \mathscr{O})$. Then composition with the canonical map $\phi : A \to \mathscr{O}(|\operatorname{Spec} A|)$ induces a homotopy equivalence*

$$\mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((X, \mathscr{O}_X), \operatorname{Spec} A) \to \mathrm{Map}_{\mathrm{CAlg}}(A, \mathscr{O}_X(X)).$$

Proof. Let $\phi : A \to \mathscr{O}_X(X)$ be a morphism of $\mathbb{E}_\infty$-rings; we wish to show that the homotopy fiber

$$Z = \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((X, \mathscr{O}_X), \operatorname{Spec} A) \times_{\mathrm{Map}_{\mathrm{CAlg}}(A, \mathscr{O}_X(X))} \{\phi\}$$

is contractible. Let R denote the commutative ring $\pi_0 R$. For each point $x \in X$, let $\kappa(x)$ denote the residue field of the local ring $(\pi_0 \mathscr{O}_X)_x$, so that ϕ determines a ring homomorphism $R \to \kappa(x)$ whose kernel is a prime ideal $\mathfrak{p}_x \subseteq R$. Let $f : X \to |\operatorname{Spec} A|$ be the map given by $f(x) = \mathfrak{p}_x$. We first claim that f is continuous. To prove this, it will suffice to show that for every element $r \in R$, the set $U = \{x \in X : r \notin \mathfrak{p}_x\}$ is an open subset of X. Suppose that $x \in U$, so that ϕ carries r to an invertible element of the local ring $(\pi_0 \mathscr{O}_X)_x$. Let $s \in (\pi_0 \mathscr{O}_X)_x$ denote the multiplicative inverse of this element. Then there exists an open set $V \subseteq X$ containing x and an element $\overline{s} \in \pi_0(\mathscr{O}_X(V))$ lifting s. Shrinking V if necessary, we may suppose that $\overline{s}$ is a multiplicative inverse of the image of r under the composite map $R \xrightarrow{\phi} \pi_0 \mathscr{O}_X(X) \to \pi_0 \mathscr{O}_X(V)$. This implies that $V \subseteq U$, so that U contains an open neighborhood of x.

Suppose we are given a map of spectrally ringed spaces from $(X, \mathscr{O}_X)$ to $\operatorname{Spec} A$, given by a map of topological spaces $g : X \to |\operatorname{Spec} A|$ and a morphism $\gamma : \mathscr{O} \to g_* \mathscr{O}_X$. For each $x \in X$, the composite map $R \to \pi_0 \mathscr{O}_X(X) \to (\pi_0 \mathscr{O}_X)_x \to \kappa(x)$ factors through $R_{g(x)}$, so that $f(x) \subseteq g(x)$ (as prime ideals of the commutative ring R). Moreover, (g, γ) is a morphism of locally spectrally ringed spaces if and only if equality holds for each $x \in X$. We may therefore identify Z with the homotopy fiber of the map $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(|\operatorname{Spec} A|)}(\mathscr{O}, g_* \mathscr{O}_X) \to \mathrm{Map}_{\mathrm{CAlg}}(A, \mathscr{O}_X(X))$ over the point ϕ.

Let $\mathcal{U}_e$ denote the collection of all elementary open subsets of $|\operatorname{Spec} A|$ and define $\mathcal{C} = \mathrm{Fun}(\mathcal{U}_e^{\mathrm{op}}, \mathrm{CAlg})$. It follows from Corollary 1.1.4.5 that the restriction functor $T : \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(|\operatorname{Spec} A|) \to \mathcal{C}$ is a fully faithful embedding. We may therefore identify Z with the homotopy fiber of the map $\mathrm{Map}_{\mathcal{C}}(T\mathscr{O}, Tg_* \mathscr{O}_X) \to \mathrm{Map}_{\mathrm{CAlg}}(A, \mathscr{O}_X(X))$ over the point ϕ. Let $\mathrm{CAlg}_R^{\heartsuit}$ denote the category of commutative R-algebras. Since $\mathscr{O}(U)$ is a localization of A for each $U \in \mathcal{U}_e$, we can use Theorem HA.7.5.4.2 to identify Z with the set of maps

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{U}_e^{\mathrm{op}}, \mathrm{CAlg}_R^{\heartsuit})}(\pi_0 T\mathscr{O}, \pi_0 Tg_* \mathscr{O}_X).$$

For each element $r \in R$, let U_r denote the elementary open subset $\{\mathfrak{p} \in |\operatorname{Spec} A| : r \notin \mathfrak{p}\} \subseteq |\operatorname{Spec} A|$, so that $\pi_0 \mathscr{O}(U_r) \simeq R[r^{-1}]$. Consequently, to prove that Z is contractible, it will suffice to show that for each $r \in R$, the image of r is invertible in the commutative ring $\pi_0(g_* \mathscr{O}_X)(U_r)$.

Let $\phi(r)$ denote the image of r under the map $R \to \pi_0 \mathscr{O}_X(X)$, so that multiplication by r induces a map m_r from $\mathscr{O}_X$ to itself (as a Sp-valued sheaf on X). Let $U = g^{-1} U_r$. For each point $x \in U$, the image of r in the local ring $(\pi_0 \mathscr{O}_X)_x$ is invertible. We may therefore choose an open set $V \subseteq X$ containing x such that the image of r in $\pi_0 \mathscr{O}_X(V)$ is invertible, so that m_r induces an equivalence from $\mathscr{O}_X|_V$ to itself. It follows that $\mathrm{fib}(m_r)$ vanishes on U, so that multiplication by r induces an isomorphism from $\pi_0 \mathscr{O}_X(U)$ to itself. It follows that the image of r in $\pi_0(g_* \mathscr{O}_X)(U_r)$ is invertible, as desired. □

Remark 1.1.5.6. The global sections functor $(X, \mathscr{O}_X) \mapsto \mathscr{O}_X(X)$ determines a forgetful functor $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{loc}} \to \mathrm{CAlg}^{\mathrm{op}}$. It follows from Proposition 1.1.5.5 that this functor admits a right adjoint, given on objects by $A \mapsto \operatorname{Spec} A$. In particular, the spectrally ringed space $\operatorname{Spec} A$ depends functorially on A.

Remark 1.1.5.7. Let A and B be $\mathbb{E}_\infty$-rings. Using the universal property of $\operatorname{Spec} B$, we obtain a homotopy equivalence $\mathrm{Map}_{\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{loc}}}(\operatorname{Spec} A, \operatorname{Spec} B) \simeq \mathrm{Map}_{\mathrm{CAlg}}(B, \mathscr{O}(|\operatorname{Spec} A|))$, where $\mathscr{O}$ denotes the structure sheaf of $\operatorname{Spec} A$. Under this homotopy equivalence, the canonical map $\mathrm{Map}_{\mathrm{CAlg}}(B, A) \to \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{loc}}}(\operatorname{Spec} A, \operatorname{Spec} B)$ is given by composition with the equivalence $\alpha : A \to \mathscr{O}(|\operatorname{Spec} A|)$ appearing in Proposition 1.1.4.3. It follows that the functor $\operatorname{Spec} : \mathrm{CAlg}^{\mathrm{op}} \to \mathrm{SpSch}^{\mathrm{nc}}$ is fully faithful.

1.1.6 Characterization of Affine Spectral Schemes

Let $(X, \mathscr{O}_X)$ be a locally spectrally ringed space. Proposition 1.1.5.5 asserts that every map of $\mathbb{E}_\infty$-rings $\alpha : A \to \mathscr{O}_X(X)$ induces a map $f : (X, \mathscr{O}_X) \to \operatorname{Spec} A$ in the ∞-category $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{loc}}$. Our next result gives a criterion for this map to be an equivalence:

Proposition 1.1.6.1. *Let $(X, \mathscr{O}_X)$ be a locally spectrally ringed space, let A be an $\mathbb{E}_\infty$-ring, and let $f : (X, \mathscr{O}_X) \to \operatorname{Spec} A$ be a morphism of locally spectrally ringed spaces. Assume that:*

(a) The map f induces an equivalence of $\mathbb{E}_\infty$-rings $\alpha : A \to \mathscr{O}_X(X)$.

(b) *The underlying ringed space* $(X, \pi_0 \mathscr{O}_X)$ *is an affine scheme.*

(c) *For each integer* n*, the sheaf* $\pi_n \mathscr{O}_X$ *is quasi-coherent.*

Then the following conditions are equivalent:

(1) *The map* f *is an equivalence.*

(2) *The sheaf* $\mathscr{O}_X$ *is hypercomplete.*

(3) *The spectrally ringed space* $(X, \mathscr{O}_X)$ *is a nonconnective spectral scheme.*

Corollary 1.1.6.2. *Let* $(X, \mathscr{O}_X)$ *be a spectrally ringed space. Suppose that the ringed space* $(X, \pi_0 \mathscr{O}_X)$ *is a scheme, and that* $\pi_n \mathscr{O}_X$ *is a quasi-coherent sheaf of* $(\pi_0 \mathscr{O}_X)$*-modules on* X *for every integer* n*. The following conditions are equivalent:*

(1) *The structure sheaf* $\mathscr{O}_X$ *is hypercomplete.*

(2) *The pair* $(X, \mathscr{O}_X)$ *is a nonconnective spectral scheme.*

(3) *For each point* $x \in X$*, there exists an open set* $U \subseteq X$ *containing* x *such that the spectrally ringed space* $(U, \mathscr{O}_X|_U)$ *is an affine nonconnective spectral scheme (that is, there exists an equivalence* $(U, \mathscr{O}_X|_U) \simeq \operatorname{Spec} A$ *for some* $\mathbb{E}_\infty$*-ring* A*).*

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 1.1.3.4. We next show that (2) $\Rightarrow$ (3). Assume that $(X, \mathscr{O}_X)$ is a nonconnective spectral scheme, and let $x \in X$. Since the ringed space $(X, \pi_0 \mathscr{O}_X)$ is a scheme, we can choose an open subset $U \subseteq X$ containing X such that $(U, (\pi_0 \mathscr{O}_X)|_U)$ is an affine scheme. Set $A = \mathscr{O}_X(U)$, so that Proposition 1.1.5.5 supplies a map of locally spectrally ringed spaces $f : (U, \mathscr{O}_X|_U) \to \operatorname{Spec} A$. Using condition (1) and Proposition 1.1.6.1, we conclude that f is an equivalence.

We now complete the proof by showing that (3) $\Rightarrow$ (1). The assertion that $\mathscr{O}_X$ is hypercomplete can be tested locally on X (see Corollary 1.3.3.8). Using (3), we are reduced to proving that $\mathscr{O}_X$ is hypercomplete in the special case where $(X, \mathscr{O}_X) \simeq \operatorname{Spec} A$ for some $\mathbb{E}_\infty$-ring A, which follows from Proposition 1.1.6.1. □

Corollary 1.1.6.3. *Let* $(X, \mathscr{O}_X)$ *be a nonconnective spectral scheme. Then* $(X, \mathscr{O}_X)$ *is affine if and only if the underlying scheme* $(X, \pi_0 \mathscr{O}_X)$ *is affine.*

Corollary 1.1.6.4. *Let* $(X, \mathscr{O}_X)$ *be a spectrally ringed space. Then the condition that* X *is a nonconnective spectral scheme can be tested locally on* X*. That is, if each point* $x \in X$ *has an open neighborhood* $U \subseteq X$ *such that* $(U, \mathscr{O}_X|_U)$ *is a nonconnective spectral scheme, then* $(X, \mathscr{O}_X)$ *is a nonconnective spectral scheme.*

Proof of Proposition 1.1.6.1. Let $\mathscr{O}$ denote the structure sheaf of the nonconnective spectral scheme $\operatorname{Spec} A$. To prove that (1) implies (2), it will suffice to show that $\mathscr{O}$ is hypercomplete. We will prove this by exhibiting $\mathscr{O}$ as an inverse limit $\varprojlim_n \mathscr{O}_n$, where each $\mathscr{O}_n$ is a sheaf of spectra on $|\operatorname{Spec} A|$ taking values in the full subcategory $\mathrm{Sp}_{\leqslant n} \subseteq \mathrm{Sp}$ of n-truncated spectra.

Let $\tau_{\geqslant 0} A$ denote the connective cover of A, and let $\mathscr{O}'$ denote the structure sheaf of $\operatorname{Spec}(\tau_{\geqslant 0} A)$ (which we also regard as a sheaf of $\mathbb{E}_\infty$-rings on the topological space $|\operatorname{Spec} A|$). Let $\mathcal{U}_e$ denote the collection of all affine open subsets of X, and define functors $\mathscr{F}^\circ_n : \mathcal{U}_e^{\mathrm{op}} \to \mathrm{Sp}$ by the formula $\mathscr{F}^\circ_n(U) = \tau_{\leqslant n} \mathscr{O}(U)$. We claim that each $\mathscr{F}^\circ_n$ satisfies hypothesis (2) of Proposition 1.1.4.4. To prove this, it suffices to observe that we have a canonical equivalence $\mathscr{F}^\circ_n(U) \simeq (\tau_{\leqslant n} A) \otimes_{\tau_{\geqslant 0} A} \mathscr{O}'(U)$, and that the $N \mapsto (\tau_{\leqslant n} A) \otimes_{\tau_{\geqslant 0} A} N$ commutes with finite limits. Let $\mathscr{F}_n : \mathcal{U}(|\operatorname{Spec} A|)^{\mathrm{op}} \to \mathrm{Sp}$ be a right Kan extension of $\mathscr{F}^\circ_n$, so that $\mathscr{F}_n$ is a Sp-valued sheaf on $|\operatorname{Spec} A|$ (by Proposition 1.1.4.4). By construction, we have a canonical map $\mathscr{O} \to \varprojlim_n \mathscr{F}_n$, which is an equivalence when evaluated on each affine open subset of X and therefore an equivalence (since the functors $\mathscr{O}$ and $\varprojlim_n \mathscr{F}_n$ are right Kan extensions of their restrictions to $\mathcal{U}_e^{\mathrm{op}}$, by Proposition 1.1.4.4). By construction, the spectrum $\mathscr{F}_n(U)$ is n-truncated for any affine open subset $U \subseteq X$. Since the collection of n-truncated spectra is closed under small limits, it follows that $\mathscr{F}_n(U)$ is n-truncated for all $U \subseteq X$, so that $\mathscr{F}_n$ is an n-truncated object of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(|\operatorname{Spec} A|)$. This completes the proof of the implication (1) $\Rightarrow$ (2).

The implication (2) $\Rightarrow$ (3) follows from Proposition 1.1.3.4. We will complete the proof by showing that (3) $\Rightarrow$ (1). Suppose that $(X, \mathscr{O}_X)$ is a nonconnective spectral scheme. Using (b), we can write $(X, \pi_0 \mathscr{O}_X) \simeq \operatorname{Spec} R$ for some commutative ring R. For each integer n, set $\mathcal{M}_n = \pi_n \mathscr{O}_X$. It follows from ($c$) that $\mathcal{M}_n$ is a quasi-coherent sheaf on the scheme $(X, \pi_0 \mathscr{O}_X)$. We may therefore choose a discrete R-module M_n such that $\mathcal{M}_n(U) = (\pi_0 \mathscr{O}_X)(U) \otimes_R M_n$ for each affine open subset $U \subseteq X$. Using (2), we deduce that the canonical map $\pi_n(\mathscr{O}_X(U)) \to \mathcal{M}_n(U)$ is an isomorphism whenever U is affine. Using (a), we obtain isomorphisms

$$\pi_n A \overset{\alpha}{\simeq} \pi_n \mathscr{O}_X(X) \simeq M_n.$$

Taking $n = 0$, we conclude that the canonical map $\pi_0 A \to R$ is an isomorphism, so that f induces a homeomorphism $X \to |\operatorname{Spec} R|$. Let $\mathscr{O}$ denote the structure sheaf of $\operatorname{Spec} A$; let us abuse notation by identifying $\mathscr{O}$ with a CAlg-valued sheaf on X. To complete the proof, it will suffice to show that f induces an equivalence $\mathscr{O} \to \mathscr{O}_X$ in $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)$. Using Corollary 1.1.4.5, we are reduced to proving that the map $\mathscr{O}(U) \to \mathscr{O}_X(U)$ is an equivalence for each open subset $U \subseteq X$. Equivalently, we must show that each of the maps $\theta_n : \pi_n \mathscr{O}(U) \to \pi_n \mathscr{O}_X(U)$ is an isomorphism. This is clear, since θ_n fits into a commutative

diagram

$$\begin{array}{ccc} & (\pi_0 \mathscr{O}_X)(U) \otimes_R M_n & \\ \swarrow & & \searrow \\ \pi_n \mathscr{O}(U) & \longrightarrow & \pi_n \mathscr{O}_X(U). \end{array}$$

where the vertical maps are isomorphisms. □

1.1.7 Truncations of Spectral Schemes

The difference between an $\mathbb{E}_\infty$-ring A and the ordinary commutative ring $\pi_0 A$ is controlled by the remaining homotopy groups $\{\pi_n A\}_{n \neq 0}$: if these vanish, then the data of A and $\pi_0 A$ are interchangeable. We now make some analogous remarks for *sheaves* of $\mathbb{E}_\infty$-rings.

Definition 1.1.7.1. Let X be a topological space and let $\mathscr{O}_X$ be a sheaf of $\mathbb{E}_\infty$-rings on X. We will say that $\mathscr{O}_X$ is *connective* if the sheaves $\pi_n \mathscr{O}_X$ vanish for $n < 0$. We let $\mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{CAlg}}(X)$ denote the full subcategory of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)$ spanned by the connective sheaves of $\mathbb{E}_\infty$-rings on X.

If $\mathscr{O}_X$ is a sheaf of $\mathbb{E}_\infty$-rings on X and n is an integer, then we will say that $\mathscr{O}_X$ is *n-truncated* if the $\mathbb{E}_\infty$-ring $\mathscr{O}_X(U)$ is n-truncated, for every open subset $U \subseteq X$. We let $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)^{\leqslant n}$ denote the full subcategory of $\mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{CAlg}}(X)$ spanned by the n-truncated sheaves of $\mathbb{E}_\infty$-rings on X, and $\mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{CAlg}}(X)^{\leqslant n}$ the intersection $\mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{CAlg}}(X) \cap \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)^{\leqslant n}$.

Warning 1.1.7.2. Let $\mathscr{O}_X$ be a sheaf of $\mathbb{E}_\infty$-rings on a topological space X. The condition that $\mathscr{O}_X$ is connective does *not* imply that the $\mathbb{E}_\infty$-rings $\mathscr{O}_X(U)$ are connective for $U \subseteq X$. Nevertheless, there is an equivalence of ∞-categories $\rho : \mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{CAlg}}(X) \to \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}}(X)$, given on objects by the formula $(\rho \mathscr{O})(U) = \tau_{\geqslant 0}(\mathscr{O}(U))$; see Proposition 1.3.5.7.

Remark 1.1.7.3. Let $(X, \mathscr{O}_X)$ be a nonconnective spectral scheme. Then $(X, \mathscr{O}_X)$ is a spectral scheme if and only if the structure sheaf $\mathscr{O}_X$ is connective.

Remark 1.1.7.4. Let $(X, \mathscr{O}_X)$ be a spectrally ringed space. If $\mathscr{O}_X$ is n-truncated for some integer n, then the sheaves $\pi_m \mathscr{O}_X$ are trivial for $m > n$. The converse holds provided that $\mathscr{O}_X$ is hypercomplete. In particular, if $(X, \mathscr{O}_X)$ is a nonconnective spectral scheme, then $\mathscr{O}_X$ is n-truncated if and only if $\pi_m \mathscr{O}_X \simeq 0$ for $m > n$.

Our next goal is to describe the relationship between the ∞-category SpSch of spectral schemes and the larger ∞-category $\mathrm{SpSch}^{\mathrm{nc}}$ of nonconnective spectral schemes. For this, we will need a few general remarks about CAlg-valued sheaves on topological space. We will defer the proofs until §1.3, where we will give a more systematic treatment of spectrum-valued sheaves.

Fix a topological space X. Then:

$(T1)$ The inclusion functor $\mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{CAlg}}(X) \hookrightarrow \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)$ admits a right adjoint, which we will denote by $\mathscr{O}_X \mapsto \tau_{\geqslant 0} \mathscr{O}_X$ (see Remark 1.3.5.4). Moreover, a map $\alpha : \mathscr{O}_X \to \mathscr{O}'_X$ induces an equivalence $\tau_{\geqslant 0} \mathscr{O}_X \to \tau_{\geqslant 0} \mathscr{O}'_X$ if and only if α induces an equivalence of $\mathrm{CAlg}^{\mathrm{cn}}$-valued sheaves on X (that is, if and only if the induced map $\tau_{\geqslant 0} \mathscr{O}_X(U) \to \tau_{\geqslant 0} \mathscr{O}'_X(U)$ is an equivalence, for each open set $U \subseteq X$; see Proposition 1.3.5.10). In particular, the canonical map $\tau_{\geqslant 0} \mathscr{O}_X \to \mathscr{O}_X$ induces isomorphisms $\pi_m(\tau_{\geqslant 0} \mathscr{O}_X) \to \pi_m \mathscr{O}_X$ for $m \geqslant 0$.

$(T2)$ For each $n \geqslant 0$, the inclusion functor $\mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{CAlg}}(X)^{\leqslant n} \hookrightarrow \mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{CAlg}}(X)$ admits a left adjoint, which we will denote by $\mathscr{O}_X \mapsto \tau_{\leqslant n} \mathscr{O}_X$. Moreover, for every connective CAlg-valued sheaf $\mathscr{O}_X$ on X, the canonical map $\pi_m \mathscr{O}_X \to \pi_m(\tau_{\leqslant n} \mathscr{O}_X)$ is an isomorphism for $m \leqslant n$ (see Remark 1.3.5.6).

$(T3)$ The construction $\mathscr{O}_X \mapsto \pi_0 \mathscr{O}_X$ induces an equivalence of the ∞-category $\mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{CAlg}}(X)^{\leqslant 0}$ with the ordinary category of sheaves of commutative rings on X (see Remark 1.3.5.6).

Proposition 1.1.7.5. *Let $(X, \mathscr{O}_X)$ be a nonconnective spectral scheme, and let $\tau_{\geqslant 0} \mathscr{O}_X$ denote a connective cover of $\mathscr{O}_X$ in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)$. Then $(X, \tau_{\geqslant 0} \mathscr{O}_X)$ is a spectral scheme. Moreover, it has the following universal property: for every spectrally ringed ∞-topos $(Y, \mathscr{O}_Y)$ where $\mathscr{O}_Y$ is connective, the canonical map*

$$\theta : \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((X, \tau_{\leqslant 0} \mathscr{O}_X), (Y, \mathscr{O}_Y)) \to \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((X, \mathscr{O}_X), (Y, \mathscr{O}_Y))$$

is a homotopy equivalence. If $(Y, \mathscr{O}_Y)$ is locally spectrally ringed, then the map θ restricts to a homotopy equivalence

$$\mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((X, \tau_{\leqslant 0} \mathscr{O}_X), (Y, \mathscr{O}_Y)) \to \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((X, \mathscr{O}_X), (Y, \mathscr{O}_Y)).$$

Corollary 1.1.7.6. *The inclusion functor* $\mathrm{SpSch} \hookrightarrow \mathrm{SpSch}^{\mathrm{nc}}$ *admits a left adjoint, given by* $(X, \mathscr{O}_X) \mapsto (X, \tau_{\geqslant 0} \mathscr{O}_X)$.

Proof of Proposition 1.1.7.5. Let $(X, \mathscr{O}_X)$ be an arbitrary spectrally ringed space, and let $(Y, \mathscr{O}_Y)$ be a spectrally ringed space where $\mathscr{O}_Y$ is connective. The canonical map $\tau_{\geqslant 0} \mathscr{O}_X \to \mathscr{O}_X$ is an equivalence after applying the functor $\Omega^\infty : \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X) \to \mathcal{S}\mathrm{hv}_{\mathcal{S}}(X)$. It follows that the canonical map $\alpha : f_*(\tau_{\geqslant 0} \mathscr{O}_X) \to f_* \mathscr{O}_X$ is an equivalence after applying the functor $\Omega^\infty : \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(Y) \to \mathcal{S}\mathrm{hv}_{\mathcal{S}}(Y)$. In particular, α induces an equivalence of connective covers, so that composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(Y)}(\mathscr{O}_Y, f_*(\tau_{\geqslant 0} \mathscr{O}_X)) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(Y)}(\mathscr{O}_Y, f_* \mathscr{O}_X).$$

Passing to the disjoint union over all continuous maps $f : X \to Y$, we conclude that the map

$$\theta : \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((X, \tau_{\geqslant 0} \mathscr{O}_X), (Y, \mathscr{O}_Y)) \to \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((X, \mathscr{O}_X), (Y, \mathscr{O}_Y)).$$

If $\mathscr{O}_X$ and $\mathscr{O}_Y$ are local, then $\tau_{\geqslant 0}\,\mathscr{O}_X$ is also local (since we have an isomorphism $\pi_0(\tau_{\geqslant 0}\,\mathscr{O}_X) \to \pi_0\,\mathscr{O}_X$), and it follows immediately from the definitions that θ restricts to a homotopy equivalence

$$\mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((X, \tau_{\geqslant 0}\,\mathscr{O}_X), (Y, \mathscr{O}_Y)) \to \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((X, \mathscr{O}_X), (Y, \mathscr{O}_Y)).$$

To complete the proof, it will suffice to show that if $(X, \mathscr{O}_X)$ is a nonconnective spectral scheme, then $(X, \tau_{\geqslant 0}\,\mathscr{O}_X)$ is a spectral scheme. This assertion can be tested locally on X: we may therefore assume without loss of generality that $(X, \mathscr{O}_X) = \operatorname{Spec} A$ for some $\mathbb{E}_\infty$-ring A. Let $B = \tau_{\geqslant 0}A$. Then $\operatorname{Spec} B$ is a spectral scheme, so the canonical map $(X, \mathscr{O}_X) \to \operatorname{Spec} B$ factors as a composition $(X, \mathscr{O}_X) \to (X, \tau_{\geqslant 0}\,\mathscr{O}_X) \xrightarrow{\phi} \operatorname{Spec} B$. The natural map $B \to A$ induces an isomorphism of commutative rings and therefore a homeomorphism $f : |\operatorname{Spec} A| \to |\operatorname{Spec} B|$. Let $\mathscr{O}'$ denote the structure sheaf of $\operatorname{Spec} B$; we wish to show that ϕ induces a map $\mathscr{O}' \to f_*\,\mathscr{O}_X$ which exhibits $\mathscr{O}'$ as a connective cover of $f_*\,\mathscr{O}_X$. Since B is connective, $\mathscr{O}'$ is connective (Proposition 1.1.4.3); by virtue of $(T1)$, it will suffice to show that the map $\mathscr{O}' \to f_*\,\mathscr{O}_X$ induces an equivalence of $\mathrm{CAlg}^{\mathrm{cn}}$-valued sheaves on X (after composing pointwise with the truncation functor $\tau_{\geqslant 0} : \mathrm{CAlg} \to \mathrm{CAlg}^{\mathrm{cn}}$). Using Proposition 1.1.4.4, we are reduced to showing that the map $\tau_{\geqslant 0}\,\mathscr{O}'(U) \to \tau_{\geqslant 0}\,\mathscr{O}_X(f^{-1}U)$ is an equivalence for every elementary open subset $U \subseteq |\operatorname{Spec} B|$. In other words, we must show that the map $\tau_{\geqslant 0}B[a^{-1}] \to \tau_{\geqslant 0}A[a^{-1}]$ is an equivalence for each $a \in \pi_0 A$, which is clear. □

Remark 1.1.7.7. The proof of Proposition 1.1.7.5 shows that if $(X, \mathscr{O}_X)$ is an affine nonconnective spectral scheme, then the associated spectral scheme $(X, \tau_{\geqslant 0}\,\mathscr{O}_X)$ is the spectrum of a connective $\mathbb{E}_\infty$-ring. In particular, if A is an $\mathbb{E}_\infty$-ring and $\operatorname{Spec} A$ is a spectral scheme, then A must be connective.

Definition 1.1.7.8. Let $(X, \mathscr{O}_X)$ be a spectral scheme, and let $n \geqslant 0$ be an integer. We will say that $(X, \mathscr{O}_X)$ is *n-truncated* if the structure sheaf $\mathscr{O}_X$ is n-truncated (when regarded as a connective CAlg-valued sheaf on X). We let $\mathrm{SpSch}^{\leqslant n}$ denote the full subcategory of SpSch spanned by the n-truncated spectral schemes.

Example 1.1.7.9. Let A be a connective $\mathbb{E}_\infty$-ring and write $\operatorname{Spec} A = (X, \mathscr{O}_X)$. For every elementary open subset $U \subseteq X$, the $\mathbb{E}_\infty$-ring $\mathscr{O}_X(U)$ has the form $A[a^{-1}]$ for some $a \in \pi_0 A$. It follows that if A is n-truncated, then $\mathscr{O}_X(U)$ is n-truncated for each elementary open subset $U \subseteq X$ (and therefore for every open subset $U \subseteq X$ by virtue of Proposition 1.1.4.4), so that $(X, \mathscr{O}_X)$ is an n-truncated spectral scheme. Conversely, if $(X, \mathscr{O}_X)$ is an n-truncated spectral scheme, then $A \simeq \mathscr{O}_X(X)$ is an n-truncated connective $\mathbb{E}_\infty$-ring.

Proposition 1.1.7.10. *Let $(X, \mathscr{O}_X)$ be a spectral scheme. For each $n \geqslant 0$, the truncation $(X, \tau_{\leqslant n}\,\mathscr{O}_X)$ is also a spectral scheme. Moreover, it has the following universal property: for* *every spectrally ringed space $(Y, \mathscr{O}_Y)$ for which $\mathscr{O}_Y$ is n-truncated, the canonical map*

$$\theta : \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((Y, \mathscr{O}_Y), (X, \tau_{\leqslant n}\, \mathscr{O}_X)) \to \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((Y, \mathscr{O}_Y), (X, \mathscr{O}_X))$$

is a homotopy equivalence. Moreover, if $(Y, \mathscr{O}_Y)$ is locally spectrally ringed, then θ restricts to a homotopy equivalence

$$\mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((Y, \mathscr{O}_Y), (X, \tau_{\leqslant n}\, \mathscr{O}_X)) \to \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((Y, \mathscr{O}_Y), (X, \mathscr{O}_X)).$$

Corollary 1.1.7.11. *For each integer $n \geqslant 0$, the inclusion* $\mathrm{SpSch}^{\leqslant n} \hookrightarrow \mathrm{SpSch}$ *admits a right adjoint, given on objects by $(X, \mathscr{O}_X) \mapsto (X, \tau_{\leqslant n}\, \mathscr{O}_X)$.*

Proof of Proposition 1.1.7.10. Suppose first that $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are arbitrary spectrally ringed space for which $\mathscr{O}_X$ is connective and $\mathscr{O}_Y$ is n-truncated. For every continuous map $f : Y \to X$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)}(\tau_{\leqslant n}\, \mathscr{O}_X, \tau_{\geqslant 0} f_*\, \mathscr{O}_Y) & \longrightarrow & \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)}(\mathscr{O}_X, \tau_{\geqslant 0} f_*\, \mathscr{O}_Y) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)}(\tau_{\leqslant n}\, \mathscr{O}_X, f_*\, \mathscr{O}_Y) & \longrightarrow & \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)}(\mathscr{O}_X, f_*\, \mathscr{O}_Y). \end{array}$$

Here the vertical maps are homotopy equivalences since $\mathscr{O}_X$ and $\tau_{\leqslant n}\, \mathscr{O}_X$ are connective, and the universal property of $\tau_{\leqslant n}\, \mathscr{O}_X$ guarantees that the upper horizontal map is a homotopy equivalence. It follows that the lower horizontal map is a homotopy equivalence. Passing to a disjoint union over all continuous maps $f : Y \to X$, we conclude that the map

$$\theta : \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((Y, \mathscr{O}_Y), (X, \tau_{\leqslant n}\, \mathscr{O}_X)) \to \mathrm{Map}_{\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((Y, \mathscr{O}_Y), (X, \mathscr{O}_X))$$

is a homotopy equivalence, from which we immediately deduce that θ restricts to a homotopy equivalence

$$\mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((Y, \mathscr{O}_Y), (X, \tau_{\leqslant n}\, \mathscr{O}_X)) \to \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((Y, \mathscr{O}_Y), (X, \mathscr{O}_X)).$$

We now complete the proof by showing that if $(X, \mathscr{O}_X)$ is a spectral scheme, then $(X, \tau_{\leqslant n}\, \mathscr{O}_X)$ is a spectral scheme. The assertion The assertion is local on X. We may therefore assume without loss of generality that $(X, \mathscr{O}_X)$ is affine, hence of the form $\mathrm{Spec}\, A$ for some $\mathbb{E}_\infty$-ring A. It follows from Remark 1.1.7.7 that A is connective. Let $B = \tau_{\leqslant n} A$. Then $\mathrm{Spec}\, B$ is an n-truncated spectral scheme (Example 1.1.7.9), so the canonical map $\mathrm{Spec}\, B \to \mathrm{Spec}\, A \simeq (X, \mathscr{O}_X)$ factors as a composition

$$\mathrm{Spec}\, B \xrightarrow{\phi} (X, \tau_{\leqslant n}\, \mathscr{O}_X) \to (X, \mathscr{O}_X).$$

To complete the proof, it will suffice to show that ϕ is an equivalence. We will prove this by showing that for every locally spectrally ringed space $(Y, \mathscr{O}_Y)$ where $\mathscr{O}_Y$ is connective and n-truncated, the horizontal map in the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((Y, \mathscr{O}_Y), \operatorname{Spec} B) & \longrightarrow & \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((Y, \mathscr{O}_Y), (X, \tau_{\leqslant n} \mathscr{O}_X)) \\ & \searrow \quad \swarrow & \\ & \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((Y, \mathscr{O}_Y), (X, \mathscr{O}_X)) & \end{array}$$

is a homotopy equivalence. Here the right vertical map is a homotopy equivalence by the first part of the proof, and we can use Proposition 1.1.5.5 to identify the left horizontal map with the map ψ appearing in the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}}(B, \tau_{\geqslant 0}(\mathscr{O}_Y(Y))) & \xrightarrow{\psi'} & \mathrm{Map}_{\mathrm{CAlg}}(A, \tau_{\geqslant 0}(\mathscr{O}_Y(Y))) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}}(B, \mathscr{O}_Y(Y)) & \xrightarrow{\psi} & \mathrm{Map}_{\mathrm{CAlg}}(A, \mathscr{O}_Y(Y)) \end{array}$$

Here the universal property of B ensures that ψ' is a homotopy equivalence (since $\tau_{\geqslant 0} \mathscr{O}_Y(Y)$ is n-truncated), and the vertical maps are homotopy equivalences because A and B are connective. □

1.1.8 Comparing Spectral Schemes with Schemes

We now consider the relationship between our theory of spectral schemes and the classical theory of schemes.

Proposition 1.1.8.1. *Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be spectrally ringed spaces. Suppose that $\mathscr{O}_X$ is 0-truncated and $\mathscr{O}_Y$ is connective. Then the canonical map*

$$\mathrm{Map}_{\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((X, \mathscr{O}_X), (Y, \mathscr{O}_Y)) \to \mathrm{Hom}((X, \pi_0 \mathscr{O}_X), (Y, \pi_0 \mathscr{O}_Y))$$

is a homotopy equivalence, where right hand side denotes the (discrete) set of morphisms from $(X, \pi_0 \mathscr{O}_X)$ to $(Y, \pi_0 \mathscr{O}_Y)$ in the category of ringed spaces.

Proof. Let $\mathcal{C}$ denote the category of sheaves of commutative rings on Y. Fix a continuous map $f : X \to Y$; we wish to show that the map

$$\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(Y)}(\mathscr{O}_Y, f_* \mathscr{O}_X) \to \mathrm{Hom}_{\mathcal{C}}(\pi_0 \mathscr{O}_Y, f_* \pi_0 \mathscr{O}_X).$$

Since $\mathscr{O}_X$ is 0-truncated, the presheaf of commutative rings $U \mapsto \pi_0(\mathscr{O}_X(U))$ is a sheaf, so that the canonical map $\pi_0(\mathscr{O}_X(U)) \to (\pi_0 \mathscr{O}_X)(U)$ is an isomorphism for every open subset $U \subseteq X$. For each open subset $V \subseteq Y$, we the canonical map

$$\pi_0((f_* \mathscr{O}_X)(V)) = \pi_0(\mathscr{O}_X(f^{-1}V)) \simeq (\pi_0 \mathscr{O}_X)(f^{-1}V) = (f_*(\pi_0 \mathscr{O}_X))(V)$$

is an isomorphism. It will therefore suffice to show that the canonical map

$$\theta_{\mathscr{A}} : \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(Y)}(\mathscr{O}_Y, \mathscr{A}) \to \mathrm{Hom}_{\mathcal{C}}(\pi_0 \mathscr{O}_Y, \pi_0 \mathscr{A})$$

is a homotopy equivalence, where $\mathscr{A} = f_* \mathscr{O}_X$. In fact, we will prove more generally that the map $\theta_{\mathscr{A}}$ is an equivalence whenever $\mathscr{A}$ is a 0-truncated object of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(Y)$. Since $\mathscr{O}_Y$ is connective, we can use $(T1)$ to replace $\mathscr{A}$ by its connective cover $\tau_{\geqslant 0} \mathscr{A}$ and thereby reduce to the case where $\mathscr{A}$ is connective. We can then use $(T2)$ to identify the mapping space $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(Y)}(\mathscr{O}_Y, \mathscr{A})$ with $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(Y)}(\tau_{\leqslant 0} \mathscr{O}_Y, \mathscr{A})$, which is equivalent to $\mathrm{Hom}_{\mathcal{C}}(\pi_0 \mathscr{O}_Y, \pi_0 \mathscr{A})$ by $(T3)$. □

Corollary 1.1.8.2. *Let $\mathcal{C}$ denote the full subcategory of $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ spanned by those spectrally ringed spaces $(X, \mathscr{O}_X)$ where $\mathscr{O}_X$ is connective and 0-truncated. Then the construction $(X, \mathscr{O}_X) \mapsto (X, \pi_0 \mathscr{O}_X)$ induces an equivalence from $\mathcal{C}$ to the category of ringed spaces.*

Proof. It follows from Proposition 1.1.8.1 that the construction $(X, \mathscr{O}_X) \mapsto (X, \pi_0 \mathscr{O}_X)$ is fully faithful when restricted to spectrally ringed spaces $(X, \mathscr{O}_X)$ for which $\mathscr{O}_X$ is connective and 0-truncated. To verify the essential surjectivity, it will suffice to show that for every topological space X and every sheaf of commutative rings $\mathscr{A}$ on X, there exists a connective 0-truncated object $\mathscr{O}_X \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)$ such that $\mathscr{A}$ is isomorphic to $\pi_0 \mathscr{O}_X$. This follows from assertion $(T3)$. □

Corollary 1.1.8.3. *Let $\mathcal{C}^{\mathrm{loc}}$ denote the full subcategory of $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ spanned by those locally spectrally ringed spaces $(X, \mathscr{O}_X)$ where $\mathscr{O}_X$ is connective and 0-truncated. Then the construction $(X, \mathscr{O}_X) \mapsto (X, \pi_0 \mathscr{O}_X)$ induces an equivalence from $\mathcal{C}^{\mathrm{loc}}$ to the ordinary category of locally ringed spaces.*

Let Sch denote the category of schemes.. Passage to the underlying scheme determines a functor from $\mathrm{SpSch}^{\mathrm{nc}} \to \mathrm{Sch}$, given on objects by $(X, \mathscr{O}_X) \mapsto (X, \pi_0 \mathscr{O}_X)$.

Proposition 1.1.8.4. *The construction $(X, \mathscr{O}_X) \mapsto (X, \pi_0 \mathscr{O}_X)$ induces an equivalence of ∞-categories $\mathrm{SpSch}^{\leqslant 0} \to \mathrm{Sch}$.*

Proof. Corollary 1.1.8.3 implies that the functor $(X, \mathscr{O}_X) \mapsto (X, \pi_0 \mathscr{O}_X)$ is fully faithful when restricted to $\mathrm{SpSch}^{\leqslant 0}$. Moreover, it implies that for every scheme $(X, \mathscr{O})$, we can write $\mathscr{O} = \pi_0 \mathscr{O}_X$ for some object $\mathscr{O}_X \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)$ which is connective and 0-truncated. To complete the proof, it will suffice to show that $(X, \mathscr{O}_X)$ is a spectral scheme. Since $\mathscr{O}_X$ is connective, it will suffice to show that $(X, \mathscr{O}_X$ is a nonconnective spectral scheme. We will prove this by verifying the hypotheses of Proposition 1.1.3.4. The ringed space $(X, \pi_0 \mathscr{O}_X) \simeq (X, \mathscr{O})$ is a scheme by construction. The sheaves $\pi_n \mathscr{O}_X$ either vanish (if $n \neq 0$) or are isomorphic to $\mathscr{O}$ (if $n = 0$), and are therefore quasi-coherent. Finally, the assumption that $\mathscr{O}_X$ is 0-truncated immediately implies that $\mathscr{O}_X$ is hypercomplete. □

Remark 1.1.8.5. We can regard a homotopy inverse to the equivalence $\mathrm{SpSch}^{\leqslant 0} \to \mathrm{Sch}$ as supplying a fully faithful functor $\mathrm{Sch} \to \mathrm{SpSch}$. In other words, we can identify the category of schemes with a full subcategory of the ∞-category of spectral schemes: namely, the full subcategory spanned by the 0-truncated spectral schemes.

1.2 Deligne-Mumford Stacks

Let E be an elliptic curve over the field $\mathbf{C}$ of complex numbers. Then E is classified up to isomorphism by its *j-invariant* $j(E) \in \mathbf{C}$. The theory of the j-invariant supplies a complete classification of elliptic curves over $\mathbf{C}$: two elliptic curves E and E' are isomorphic if and only if $j(E) = j(E')$, and every complex number arises as the j-invariant of some curve over $\mathbf{C}$.

If we wish to classify *families* of elliptic curves, the situation becomes more complicated. Suppose that X is an algebraic variety over $\mathbf{C}$ and that E is a family of elliptic curves over X (in other words, we have a proper smooth morphism $\pi : E \to X$ equipped with a section, whose fibers are elliptic curves). For each point $x \in X(\mathbf{C})$, let E_x denote the fiber product $\operatorname{Spec} \mathbf{C} \times_X E$, so that E_x is an elliptic curve over $\mathbf{C}$. The construction $x \mapsto j(E_x)$ is a regular function on X, which we can view as a map of algebraic varieties $j_E : X \to \mathbf{A}^1$. However, the function j_E *does not* determine E up to isomorphism in general.

Example 1.2.0.1. Let E be an elliptic curve over $\mathbf{C}$, let $\widetilde{X}$ be an algebraic variety over $\mathbf{C}$ equipped with a fixed-point free involution σ, and let X denote the quotient of $\widetilde{X}$ by the action of σ. Then σ determines involutions σ_+ and σ_- on the product $\widetilde{X} \times_{\operatorname{Spec} \mathbf{C}} E$, given on $\mathbf{C}$-points by the formulae

$$\sigma_+(\widetilde{x}, y) = (\sigma(\widetilde{x}), y) \qquad \sigma_-(\widetilde{x}, y) = (\sigma(\widetilde{x}), -y).$$

Let E_+ denote the quotient of $\widetilde{X} \times_{\operatorname{Spec} \mathbf{C}} E$ by the action of the involution σ_+, and define E_- similarly. We have obvious projection maps $\pi_\pm : E_\pm \to X$, each fiber of which is isomorphic to the original elliptic curve E. It follows that E_+ and E_- determine the same j-invariant

$$X \to \{j(E)\} \hookrightarrow \mathbf{A}^1 .$$

However, E_+ and E_- are *never* isomorphic as elliptic curves over X unless the variety $\widetilde{X}$ is disconnected.

It follows from Example 1.2.0.1 that there can be no *fine moduli space* of elliptic curves: that is, there cannot exist a scheme M such that elliptic curves over an arbitrary scheme X are classified up isomorphism by maps from X into M. In fact, this phenomenon is ubiquitous: a similar problem arises whenever we wish to classify objects which admit nontrivial symmetries. To address this issue, Deligne and Mumford introduced a new type of algebro-geometric object which now bears their names: the *Deligne-Mumford stack*. The collection of Deligne-Mumford stacks can be organized into a 2-category DM, which contains the usual category of schemes as a full subcategory. Moreover, it also contains more exotic objects such as the moduli stack of elliptic curves $\mathcal{M}_{1,1}$, which has the property that for any scheme X, the category $\mathrm{Hom}_{\mathrm{DM}}(X, \mathcal{M}_{1,1})$ can be identified with the category of elliptic curves $\pi : E \to X$ (with morphisms given by *isomorphisms* of elliptic curves).

Our goal in this section is to give a brief overview of the classical theory of Deligne-Mumford stacks. In §1.4, we will explain how the ideas presented here can be generalized to the setting of spectral algebraic geometry and thereby obtain a notion of *spectral Deligne-Mumford stack* (Definition 1.4.4.2), which will play a central role throughout this book.

Remark 1.2.0.2. For a detailed introduction to the theory of algebraic stacks, we refer the reader to [129].

Warning 1.2.0.3. Our presentation of the theory of Deligne-Mumford stacks differs from the presentation given in [129] (and most others in the literature) in two major respects:

(i) We will define a Deligne-Mumford stack to be a ringed topos $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ satisfying a "local affineness" condition which parallels the usual definition of a scheme (Definition 1.1.1.4). The equivalence of this perspective with the "functor-of-points" approach will be established at the end of this section (Theorem 1.2.5.9).

(ii) We do not include any separatedness hypotheses in our definition of Deligne-Mumford stack (so that we allow, for example, classifying stacks for infinite discrete groups).

Roughly speaking, the definition of a Deligne-Mumford stacks is obtained by modifying the definition of a scheme $(X, \mathscr{O}_X)$ in two ways:

(a) In place of the topological space X, we allow an arbitrary Grothendieck topos $\mathcal{X}$.

(b) In place of the requirement that $(X, \mathscr{O}_X)$ be locally isomorphic to the spectrum of a commutative ring, we require that $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be locally equivalent to $\mathrm{Spét}\, A$, where A is an $\mathbb{E}_\infty$-ring and $\mathrm{Spét}\, A$ denotes its spectrum with respect to the étale topology (see Construction 1.2.3.3).

1.2.1 Local Rings in a Topos

We begin by reviewing the theory of locally ringed topoi.

Definition 1.2.1.1. A *ringed topos* consists of a pair $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, where $\mathcal{X}$ is a Grothendieck topos and $\mathscr{O}_{\mathcal{X}}$ is a commutative ring object of $\mathcal{X}$. Given a pair of ringed topoi $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$, we define a category $\mathrm{Map}_{1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}))$ as follows:

- The objects of $\mathrm{Hom}_{1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}))$ are pairs (f_*, α), where $f_* : \mathcal{X} \to \mathcal{Y}$ is a geometric morphism of topoi (in other words, f_* is a functor which admits a left adjoint f^* which preserves finite limits) and $\alpha : \mathscr{O}_Y \to f_* \mathscr{O}_X$ is a morphism of commutative ring objects of $\mathcal{Y}$.

- A morphism from (f_*, α) to (f'_*, α') in $\mathrm{Hom}_{1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}))$ is a natural transformation of functors $\beta : f'_* \to f_*$ for which the diagram

$$\begin{array}{ccc} & \mathscr{O}_{\mathcal{Y}} & \\ {}^{\alpha'}\swarrow & & \searrow^{\alpha} \\ f'_* \mathscr{O}_{\mathcal{X}} & \xrightarrow{\beta(\mathscr{O}_{\mathcal{X}})} & f_* \mathscr{O}_{\mathcal{X}} \end{array}$$

  commutes.

We will regard the collection of all ringed topoi as a (strict) 2-category $1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}$, with the categories of morphisms defined above and the evident composition law.

Example 1.2.1.2. Let X be a topological space and let $\mathscr{O}_X$ be a sheaf of commutative rings on X. Then we can regard $\mathscr{O}_X$ as a commutative ring object of the topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X)$ of set-valued sheaves on X. The pair $(\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X), \mathscr{O}_X)$ is a ringed topos.

Notation 1.2.1.3. Let $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a ringed topos, and let $U \in \mathcal{X}$ be an object. We let $\mathscr{O}_{\mathcal{X}}|_U$ denote the product $U \times \mathscr{O}_{\mathcal{X}}$, which we view as a commutative ring object of the topos $\mathcal{X}_{/U}$. Then $(\mathcal{X}_{/U}, \mathscr{O}_{\mathcal{X}}|_U)$ is another ringed topos, equipped with an evident morphism $(\mathcal{X}_{/U}, \mathscr{O}_{\mathcal{X}}|_U) \to (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$.

We now review what it means for a commutative ring object of a topos $\mathcal{X}$ to be *local*. Let R be a commutative ring. For every element $r \in R$, we let (r) denote the principal ideal generated by r. If r is not a unit, then $(r) \neq R$, so (by Zorn's lemma) (r) is contained in a maximal ideal $\mathfrak{m} \subset R$. We say that R is *local* if it contains a unique maximal ideal $\mathfrak{m}_R$. In this case, the above reasoning shows that $\mathfrak{m}_R$ can be described as the collection of non-invertible elements of R. The ring R is local if and only if the collection of non-units $R - R^\times$ forms an ideal in R. Since $R - R^\times$ is clearly closed under multiplication by elements of R, this is equivalent to the requirement that $R - R^\times$ is an additive subgroup of R. That is, R is local if and only if the following pair of conditions is satisfied:

(a) The element 0 belongs to $R - R^\times$. In other words, 0 is not a unit in R: this is equivalent to the requirement that $0 \neq 1$ in R.

(b) If $r, r' \in R - R^\times$, then $r + r' \in R - R^\times$. Equivalently, if $r + r'$ is a unit, then either r or r' is a unit. This is equivalent to the following apparently weaker condition: if $s \in R$, then either s or $1 - s$ is a unit in R (to see this, take $s = \frac{r}{r+r'}$, so that s is invertible if and only if r is invertible and $1 - s \simeq \frac{r'}{r+r'}$ is invertible if and only if r' is invertible).

If R and R' are local commutative rings, then we say that a ring homomorphism $f : R \to R'$ is *local* if it carries $\mathfrak{m}_R$ into $\mathfrak{m}_{R'}$: that is, if an element $x \in R$ is invertible if and only if its image $f(x) \in R'$ is invertible.

All of these notions admit generalizations to the setting of commutative ring objects of an arbitrary Grothendieck topos:

Definition 1.2.1.4. Let $\mathcal{X}$ be a topos with final object $\mathbf{1}$, and let $\mathscr{O}$ be a commutative ring object of $\mathcal{X}$. Let $\mathscr{O}^{\times}$ denote the group object of $\mathcal{X}$ given by the units of $\mathscr{O}$, so that we have a pullback diagram

$$\begin{array}{ccc} \mathscr{O}^{\times} & \longrightarrow & \mathscr{O} \times \mathscr{O} \\ \downarrow & & \downarrow m \\ \mathbf{1} & \xrightarrow{1} & \mathscr{O} \end{array}$$

where m denotes the multiplication on $\mathscr{O}$,

We will say that $\mathscr{O}$ is *local* if the following conditions are satisfied:

(a) The sheaf $\mathscr{O}_{\mathcal{X}}$ is locally nontrivial. That is, if $0 : \mathbf{1} \to \mathscr{O}$ denotes the zero section of $\mathscr{O}$, then the fiber product $\mathbf{1} \times_{\mathscr{O}} \mathscr{O}^{\times}$ is an initial object of $\mathcal{X}$.

(b) Let $e : \mathscr{O}^{\times} \hookrightarrow \mathscr{O}$ denote the inclusion map. Then the maps e and $1 - e$ determine an effective epimorphism $\mathscr{O}^{\times} \amalg \mathscr{O}^{\times} \to \mathscr{O}$ in the topos $\mathcal{X}$.

If $\alpha : \mathscr{O} \to \mathscr{O}'$ is a map between commutative ring objects of $\mathcal{X}$, then we say that α is *local* if the diagram

$$\begin{array}{ccc} \mathscr{O}^{\times} & \longrightarrow & \mathscr{O}'^{\times} \\ \downarrow & & \downarrow \\ \mathscr{O} & \longrightarrow & \mathscr{O}' \end{array}$$

is a pullback square in $\mathcal{X}$.

We let $1\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}}$ denote the subcategory of $1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^{\heartsuit}}$ whose objects are ringed topoi $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ where $\mathscr{O}_{\mathcal{X}}$ is local, and whose morphisms maps $(f_*, \alpha) : (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ for which α classifies a local map $f^* \mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$. We will refer to $1\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}}$ as the *2-category of locally ringed topoi.*

Example 1.2.1.5. Let $(X, \mathscr{O}_X)$ be a ringed space. Then $\mathscr{O}_X$ is local (in the sense of Definition 1.2.1.4) if and only if $(X, \mathscr{O}_X)$ is a locally ringed space (in the sense of Definition 1.1.5.1): that is, if and only if each stalk $\mathscr{O}_{X,x}$ is a local ring. Moreover, if $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are locally ringed spaces, then a map of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a map of locally ringed spaces (in the sense of Definition 1.1.5.1) if and only if the induced map of ringed topoi $(\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X), \mathscr{O}_X) \to (\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(Y), \mathscr{O}_Y)$ is local (in the sense of Definition 1.2.1.4).

Remark 1.2.1.6. Let $\mathscr{O}$ be a commutative ring object of a topos $\mathcal{X}$, and let $U = \mathbf{1} \times_{\mathscr{O}} \mathscr{O}^{\times}$ be the fiber product appearing in condition (a) of Definition 1.2.1.4. Then U is a subobject of the final object $\mathbf{1} \in \mathcal{X}$ and is maximal among those subobjects $V \subseteq \mathbf{1}$ for which the restriction $\mathscr{O}|_V$ is trivial. If $f_* : \mathcal{Y} \to \mathcal{X}$ is a geometric morphism of topoi, then:

(i) The pullback $f^* \mathscr{O}$ is trivial (as a commutative ring object of $\mathcal{Y}$) if and only if the geometric morphism f_* factors through the open immersion of topoi $\mathcal{X}_{/U} \to \mathcal{X}$ determined by U.

(ii) The pullback $f^* \mathscr{O}$ satisfies condition (a) of Definition 1.2.1.4 if and only if the geometric morphism f_* factors through the closed subtopos of $\mathcal{X}$ complementary to U.

Remark 1.2.1.7. Let $\mathcal{X}$ be a topos and suppose we are given a commutative diagram

$$\begin{array}{ccccc} & & \mathscr{O}' & & \\ & \overset{\alpha}{\nearrow} & & \overset{\beta}{\searrow} & \\ \mathscr{O} & & \xrightarrow{\gamma} & & \mathscr{O}'' \end{array}$$

of commutative ring objects of $\mathcal{X}$. Then:

(a) If α and β are local, then γ is local.

(b) If β and γ are local, then α is local.

(c) If α and γ are local and α is an effective epimorphism, then β is local.

Proposition 1.2.1.8. *Let $\mathcal{X}$ be a topos and let $\alpha : \mathscr{O} \to \mathscr{O}'$ be a morphism between commutative ring objects of $\mathcal{X}$. Then:*

(1) *If $\mathscr{O}'$ is local and α is local, then $\mathscr{O}$ is local.*

(2) *If $\mathscr{O}$ is local and α is an effective epimorphism, then the following conditions are equivalent:*

   (a) *The commutative ring object $\mathscr{O}'$ is local.*

   (b) *The commutative ring object $\mathscr{O}'$ satisfies condition (a) of Definition 1.2.1.4.*

   (c) *The morphism α is local.*

Proof. Consider first the commutative diagram σ :

$$\begin{array}{ccc} \mathscr{O}^{\times} \amalg \mathscr{O}^{\times} & \longrightarrow & \mathscr{O}'^{\times} \amalg \mathscr{O}'^{\times} \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v'} \\ \mathscr{O} & \xrightarrow{\alpha} & \mathscr{O}' \end{array}$$

where the vertical maps are defined as condition (b) of Definition 1.2.1.4. If α is local, then σ is a pullback square. If, in addition, the commutative ring object $\mathscr{O}'$ is local, then v' is an effective epimorphism. It follows that v is also an effective epimorphism. Since $\mathbf{1} \times_{\mathscr{O}'} \mathscr{O}'^{\times}$ is an initial object of $\mathcal{X}$, the existence of a morphism

$$\mathbf{1} \times_{\mathscr{O}} \mathscr{O}^{\times} \to \mathbf{1} \times_{\mathscr{O}} \mathscr{O}^{\times}$$

shows that $\mathbf{1} \times_{\mathscr{O}} \mathscr{O}^{\times}$ is also initial in $\mathcal{X}$. This completest the proof of (1).

To prove (2), assume that $\mathscr{O}$ is local and that α is an effective epimorphism. The implication $(a) \Rightarrow (b)$ is trivial, and the implication $(b) \Rightarrow (a)$ follows by inspecting the commutative diagram σ (if v and α are effective epimorphisms, it follows that v' is an effective epmorphism as well). We next prove that $(c) \Rightarrow (b)$. Assume that α is local. Then the induced map

$$\beta : \mathbf{1} \times_{\mathscr{O}} \mathscr{O}^{\times} \to \mathbf{1} \times_{\mathscr{O}'} \mathscr{O}'^{\times}$$

is a pullback of α, and therefore an effective epimorphism. Our assumption that $\mathscr{O}$ is local guarantees that the domain of β is an initial object of $\mathcal{X}$. It follows that the codomain of β is also an initial object of $\mathcal{X}$, so that assertion (b) is satisfied.

We now complete the proof by showing that (b) implies (c). Fix an object $X \in \mathcal{X}$ and a morphism $f : X \to \mathscr{O}$, which we regard as an element of the commutative ring $\mathrm{Hom}_{\mathcal{X}}(X, \mathscr{O})$. Let $\alpha_X : \mathrm{Hom}_{\mathcal{X}}(X, \mathscr{O}) \to \mathrm{Hom}_{\mathcal{X}}(X, \mathscr{O}')$ be the homomorphism of commutative rings determined by α. We wish to show that if $\alpha_X(f)$ is invertible, then f is invertible. Let $\overline{g} : X \to \mathscr{O}'$ be a multiplicative inverse of $\alpha_X(f)$ in the commutative ring $\mathrm{Hom}_{\mathcal{X}}(X, \mathscr{O}')$. Since α is an effective epimorphism, we can (after passing to a covering of X) assume without loss of generality that $\overline{g} = \alpha_X(g)$ for some $g : X \to \mathscr{O}$. Since $\mathscr{O}$ is local, we can (after passing to a further covering of X) assume that either fg or $1 - fg$ is invertible in the commutative ring $\mathrm{Hom}_{\mathcal{X}}(X, \mathscr{O})$. In the first case, we conclude that f is invertible as desired. In the second case, it follows that $\alpha_X(1 - fg) = 0$ is invertible in the commutative ring $\mathrm{Hom}_{\mathcal{X}}(X, \mathscr{O}')$, so that condition (b) guarantees that X is an initial object of $\mathcal{X}$. In this case, $\mathrm{Hom}_{\mathcal{X}}(X, \mathscr{O})$ is the zero ring (so that f is tautologically invertible). □

1.2.2 Strictly Henselian Rings in a Topos

To develop the theory of Deligne-Mumford stacks, we will need to work with ringed topoi satisfying stronger locality requirements, related to the étale topology rather than the Zariski topology.

Notation 1.2.2.1. Let $\mathcal{X}$ be a topos, let $\mathscr{O}_{\mathcal{X}}$ be a commutative ring object of $\mathcal{X}$, and let $\mathrm{CAlg}^{\heartsuit}$ denote the category of commutative rings. For every commutative ring R, let $\mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}})$ denote an object of $\mathcal{X}$ having the following universal property: for every object

$U \in \mathcal{X}$, there is a canonical bijection

$$\mathrm{Hom}_{\mathcal{X}}(U, \mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}})) \simeq \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}}(R, \mathrm{Hom}_{\mathcal{X}}(U, \mathscr{O}_{\mathcal{X}})).$$

Remark 1.2.2.2. Let $f : (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be a map of ringed topoi and let R be a commutative ring. Then we have a canonical map $f^* \mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{Y}}) \to \mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}})$ in the topos $\mathcal{X}$, which is an isomorphism if $\mathscr{O}_{\mathcal{X}} \simeq f^* \mathscr{O}_{\mathcal{Y}}$ and R is finitely generated as a commutative ring.

Example 1.2.2.3. Let $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be as in Notation 1.2.2.1 and let R be the zero ring. Then $\mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}})$ can be identified with the fiber product $U = \mathbf{1} \times_{\mathscr{O}_{\mathcal{X}}} \mathscr{O}_{\mathcal{X}}^{\times}$ in Remark 1.2.1.6. In particular, if $\mathscr{O}_{\mathcal{X}}$ is local, then $\mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}})$ is an initial object of $\mathcal{X}$.

Example 1.2.2.4. Let $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be as in Notation 1.2.2.1 and let R be a commutative ring which factors as a Cartesian product $R_0 \times R_1$. The projection maps $R_0 \leftarrow R \to R_1$ induce morphisms

$$\mathcal{S}\mathrm{ol}_{R_0}(\mathscr{O}_{\mathcal{X}}) \to \mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}}) \leftarrow \mathcal{S}\mathrm{ol}_{R_1}(\mathscr{O}_{\mathcal{X}}).$$

Unwinding the definitions, we have

$$\begin{aligned} \mathcal{S}\mathrm{ol}_{R_i}(\mathscr{O}_{\mathcal{X}}) \times_{\mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}})} \mathcal{S}\mathrm{ol}_{R_j}(\mathscr{O}_{\mathcal{X}}) &\simeq \mathcal{S}\mathrm{ol}_{R_i \otimes_R R_j}(\mathscr{O}_{\mathcal{X}}) \\ &\simeq \begin{cases} \mathcal{S}\mathrm{ol}_{R_i}(\mathscr{O}_{\mathcal{X}}) & \text{if } i = j \\ \mathcal{S}\mathrm{ol}_0(\mathscr{O}_{\mathcal{X}}) & \text{if } i \neq j. \end{cases} \end{aligned}$$

In particular, if $\mathscr{O}_{\mathcal{X}}$ satisfies condition (a) of Definition 1.2.1.4, then the $\mathcal{S}\mathrm{ol}_{R_i}(\mathscr{O}_{\mathcal{X}})$ are disjoint subobjects of $\mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}})$, so we have a monomorphism

$$\mathcal{S}\mathrm{ol}_{R_0}(\mathscr{O}_{\mathcal{X}}) \amalg \mathcal{S}\mathrm{ol}_{R_1}(\mathscr{O}_{\mathcal{X}}) \to \mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}}).$$

If $\mathscr{O}_{\mathcal{X}}$ satisfies condition (b) of Definition 1.2.1.4, then this map is an epimorphism. Consequently, if $\mathscr{O}_{\mathcal{X}}$ is local, then the construction $R \mapsto \mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}})$ carries finite products in the category of commutative rings to finite coproducts in the topos $\mathcal{X}$.

Definition 1.2.2.5. Let $\mathcal{X}$ be a topos and let $\mathscr{O}_{\mathcal{X}}$ be a commutative ring object of $\mathcal{X}$. We will say that $\mathscr{O}_{\mathcal{X}}$ is *strictly Henselian* if the following condition is satisfied:

$(*)$ For every commutative ring R and every finite collection of étale maps $R \to R_\alpha$ which induce a faithfully flat map $R \to \prod_\alpha R_\alpha$, the induced map

$$\coprod_\alpha \mathcal{S}\mathrm{ol}_{R_\alpha}(\mathscr{O}_{\mathcal{X}}) \to \mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}})$$

is an effective epimorphism.

We let $1\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^{\heartsuit}}$ denote the full subcategory of $1\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}}$ spanned by those ringed topoi $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ where $\mathscr{O}_{\mathcal{X}}$ is strictly Henselian.

Remark 1.2.2.6. In the situation of Definition 1.2.2.5, it suffices to verify condition $(*)$ in the case where the commutative ring R is finitely generated (this follows immediately from the structure theory of étale ring homomorphisms; see Proposition B.1.1.1).

Remark 1.2.2.7. Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi, and let $\mathscr{O}_{\mathcal{Y}}$ be a commutative ring object of $\mathcal{Y}$. If $\mathscr{O}_{\mathcal{Y}}$ is strictly Henselian, then $f^* \mathscr{O}_{\mathcal{Y}}$ is a strictly Henselian commutative ring object of $\mathcal{X}$. This follows immediately from Remarks 1.2.2.2 and 1.2.2.6.

Example 1.2.2.8. Let $\mathcal{X} = \mathcal{S}\mathrm{et}$ be the category of sets, and let $\mathscr{O}_{\mathcal{X}}$ be a commutative ring object of $\mathcal{X}$, which we can identify with a commutative ring A. For every finitely generated commutative ring R, we can identify $\mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}})$ with the set $\mathrm{Hom}(R, A)$ of ring homomorphisms from R to A. Applying Corollary B.3.5.4, we deduce that $\mathscr{O}_{\mathcal{X}}$ is strictly Henselian (in the sense of Definition B.3.5.1) if and only if the commutative ring A is strictly Henselian (in the sense of Definition B.3.5.1).

Remark 1.2.2.9. Let $\mathcal{X}$ be a topos, and let $\mathscr{O}_{\mathcal{X}}$ be a commutative ring object of $\mathcal{X}$. If $\mathscr{O}_{\mathcal{X}}$ is strictly Henselian, then $\mathscr{O}_{\mathcal{X}}$ is local. Conversely, if $\mathscr{O}_{\mathcal{X}}$ is local, then (by virtue of Example 1.2.2.4) it is strictly Henselian if and only if it satisfies the following *a priori* weaker version of condition $(*)$:

$(*')$ For every commutative ring R and every faithfully flat étale map $R \to R'$, the induced map $\mathcal{S}\mathrm{ol}_{R'}(\mathscr{O}_{\mathcal{X}}) \to \mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{X}})$ is an effective epimorphism in the topos $\mathcal{X}$.

Remark 1.2.2.10. Let $\mathcal{X}$ be a topos, and suppose that $\mathcal{X}$ has enough points. If $\mathscr{O}_{\mathcal{X}}$ is a commutative ring object of $\mathcal{X}$, then $\mathscr{O}_{\mathcal{X}}$ is strictly Henselian if and only if, for each point $x^* : \mathcal{X} \to \mathcal{S}\mathrm{et}$ of $\mathcal{X}$, the pullback $x^* \mathscr{O}_{\mathcal{X}}$ is a strictly Henselian commutative ring. In particular, if X is a topological space and $\mathscr{O}_X$ is a sheaf of commutative rings on X, then $\mathscr{O}_X$ is strictly Henselian (as a commutative ring object of the topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X)$) if and only if each stalk $\mathscr{O}_{X,x}$ is a strictly Henselian ring.

Example 1.2.2.11. Let $(X, \mathscr{O}_X)$ be a complex analytic space. Then $\mathscr{O}_X$ is strictly Henselian (when viewed as a commutative ring object of the topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X)$).

Proposition 1.2.2.12. *Let $f : (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be a morphism of locally ringed topoi and let $v : A \to B$ be an étale morphism of commutative rings. If $\mathscr{O}_{\mathcal{Y}}$ is strictly Henselian and $\mathscr{O}_{\mathcal{X}}$ is local, then the induced diagram σ :*

$$\begin{array}{ccc} f^* \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}}) \\ \downarrow & & \downarrow \\ f^* \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}}) \end{array}$$

is a pullback square in the topos $\mathcal{X}$.

Proof. Using Proposition B.1.1.1, we can choose a pushout diagram

$$\begin{array}{ccc} A_0 & \longrightarrow & A \\ \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v} \\ B_0 & \longrightarrow & B \end{array}$$

in the category of commutative rings, where v_0 is étale and A_0 is finitely generated (so that B_0 is also finitely generated). In this case, σ fits into a commutative diagram

$$\begin{array}{ccccc} f^* \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_{B_0}(\mathscr{O}_{\mathcal{X}}) \\ \downarrow & & \downarrow & & \downarrow \\ f^* \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_{A_0}(\mathscr{O}_{\mathcal{X}}) \end{array}$$

where the square on the right is a pullback. It will therefore suffice to show that the outer rectangle in the diagram

$$\begin{array}{ccccc} f^* \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & f^* \mathcal{S}\mathrm{ol}_{B_0}(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_{B_0}(\mathscr{O}_{\mathcal{X}}) \\ \downarrow & & \downarrow & & \downarrow \\ f^* \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & f^* \mathcal{S}\mathrm{ol}_{A_0}(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_{A_0}(\mathscr{O}_{\mathcal{X}}) \end{array}$$

is a pullback square. Since the square on the left is a pullback, we are reduced to showing that the square on the right is a pullback. Using Remark 1.2.2.2, we can rewrite this diagram as

$$\begin{array}{ccc} \mathcal{S}\mathrm{ol}_{B_0}(f^* \mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_{B_0}(\mathscr{O}_{\mathcal{X}}) \\ \downarrow & & \downarrow \\ \mathcal{S}\mathrm{ol}_{A_0}(f^* \mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_{A_0}(\mathscr{O}_{\mathcal{X}}). \end{array}$$

Replacing v by v_0 and $\mathscr{O}_{\mathcal{Y}}$ by $f^* \mathscr{O}_{\mathcal{Y}}$, we are reduced to proving Proposition 1.2.2.12 in the special case where $\mathcal{X} = \mathcal{Y}$ (and f_* is the identity map).

For each point $x \in |\operatorname{Spec} A|$, let $\kappa(x)$ denote the residue field of A at x and let $d(x) = \dim_{\kappa(x)}(\kappa(x) \otimes_A B)$. Note that there exists an integer $n \geqslant 0$ such that $d(x) \leqslant n$ for each $x \in |\operatorname{Spec} A|$. Our proof will proceed by induction on n. Note that if $n = 0$, then $B \simeq A[0^{-1}]$ and the desired result follows from our assumption that f is local. We now treat the case $n = 0$. The ring homomorphism v induces a map of Zariski spectra $u : |\operatorname{Spec} B| \to |\operatorname{Spec} A|$. Let $U \subseteq |\operatorname{Spec} A|$ be the image of u, so that U is quasi-compact and open (Corollary B.2.2.5). Write $U = \bigcup_{1 \leqslant i \leqslant n} |\operatorname{Spec} A[\frac{1}{a_i}]|$ for some elements $a_1, \ldots, a_n \in A$. Since U contains the image of $|\operatorname{Spec} B|$, the elements $v(a_i)$ generate the unit ideal in B. Since $\mathscr{O}_{\mathcal{X}}$ is local, it follows that the map

$$\coprod_{1 \leqslant i \leqslant n} \mathcal{S}\mathrm{ol}_{B[v(a_i)^{-1}]}(\mathscr{O}_{\mathcal{X}}) \to \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}})$$

is an effective epimorphism in $\mathcal{X}$. It will therefore suffice to show that each of the induced maps

$$\theta_i : \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) \times_{\mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}}} \mathcal{S}\mathrm{ol}_{B[f(a_i)^{-1}]}(\mathscr{O}_{\mathcal{X}}) \to \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}}) \times_{\mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}})} \mathcal{S}\mathrm{ol}_{B[f(a_i)^{-1}]}(\mathscr{O}_{\mathcal{X}})$$

is an isomorphism. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{ol}_{B[f(a_i)^{-1}]}(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) \times_{\mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}})} \mathcal{S}\mathrm{ol}_{B[f(a_i)^{-1}]}(\mathscr{O}_{\mathcal{X}}) \\ \downarrow{\theta'_i} & & \downarrow{\theta_i} \\ \mathcal{S}\mathrm{ol}_{A[a_i^{-1}]}(\mathscr{O}_{\mathcal{Y}}) \times_{\mathcal{S}\mathrm{ol}_{A[a_i^{-1}]}(\mathscr{O}_{\mathcal{X}})} \mathcal{S}\mathrm{ol}_{B[f(a_i)^{-1}]}(\mathscr{O}_{\mathcal{X}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}}) \times_{\mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}})} \mathcal{S}\mathrm{ol}_{B[f(a_i)^{-1}]}(\mathscr{O}_{\mathcal{X}}), \end{array}$$

where the horizontal maps are isomorphisms by virtue of our assumption that the map $\mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$ is local. It will therefore suffice to show that each of the maps θ'_i is an isomorphism. Replacing A by $A[a_i^{-1}]$ and B by $B[f(a_i)^{-1}]$, we are reduced to the case where the map $u : |\operatorname{Spec} B| \to |\operatorname{Spec} A|$ is surjective. In this case, B is étale and faithfully flat over A. Since $\mathscr{O}_{\mathcal{Y}}$ is strictly Henselian, the map $\mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) \to \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}})$ is an effective epimorphism. We are therefore reduced to proving that the pullback map

$$\begin{aligned} \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) \times_{\mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}})} \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) &\xrightarrow{\psi} \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) \times_{\mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}})} (\mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}}) \times_{\mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}})} \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}})) \\ &\simeq \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) \times_{\mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}})} (\mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}}) \times_{\mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}})} \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}})) \end{aligned}$$

is an isomorphism. We may therefore replace v by the induced map $B \to B \otimes_A B$, and thereby reduce to proving Proposition 1.2.2.12 in the special case where v admits a left inverse $B \to A$. Set $C = B \otimes_A B$, so that ψ can be identified with the canonical map

$$\mathcal{S}\mathrm{ol}_C(\mathscr{O}_{\mathcal{Y}}) \to \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) \times_{\mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}})} \mathcal{S}\mathrm{ol}_C(\mathscr{O}_{\mathcal{X}}).$$

Our assumption that v is étale guarantees that C factors as a direct product $B \times C'$ in the category $\mathrm{CAlg}_B^{\heartsuit}$. Since both $\mathscr{O}_{\mathcal{Y}}$ and $\mathscr{O}_{\mathcal{X}}$ is local, we can factor ψ as a direct product of maps

$$\psi_0 : \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) \to \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) \times_{\mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}})} \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}})$$

$$\psi_1 : \mathcal{S}\mathrm{ol}_{C'}(\mathscr{O}_{\mathcal{Y}}) \to \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) \times_{\mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}})} \mathcal{S}\mathrm{ol}_{C'}(\mathscr{O}_{\mathcal{X}}).$$

Here the map ψ_0 is obviously an equivalence, and the map ψ_1 is an equivalence by virtue of our inductive hypothesis (note that $\dim_{\kappa(x)}(\kappa(x) \otimes_B C') = \dim_{\kappa(x)}(\kappa(x) \otimes_B C) - 1 < n$ for each $x \in |\operatorname{Spec} B|$). □

Remark 1.2.2.13. In the statement of Proposition 1.2.2.12, we do not need the full strength of our assumption that $\mathscr{O}_{\mathcal{X}}$ is local: for example, if $\mathcal{X}$ has enough points, it is enough to assume that each stalk $\mathscr{O}_{\mathcal{X},x}$ has the property that the Zariski spectrum $|\operatorname{Spec} \mathscr{O}_{\mathcal{X},x}|$ is connected.

Proposition 1.2.2.14. *Let $\mathcal{X}$ be a topos and let $\alpha : \mathscr{O} \to \mathscr{O}'$ be an effective epimorphism between local commutative ring objects of $\mathcal{X}$. If $\mathscr{O}$ is strictly Henselian, then $\mathscr{O}'$ is strictly Henselian.*

Proof. By virtue of Remark 1.2.2.9, it will suffice to show that for every faithfully flat étale morphism $\phi : A \to B$ of commutative rings, the induced map $\rho : \mathcal{S}\mathrm{ol}_B(\mathscr{O}') \to \mathcal{S}\mathrm{ol}_A(\mathscr{O}')$ is an effective epimorphism in $\mathcal{X}$. Using the structure theory of étale morphisms (see Proposition B.1.1.3) we can choose a pushout diagram of commutative rings

$$\begin{array}{ccc} A_0 & \longrightarrow & A \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi} \\ B_0 & \longrightarrow & B \end{array}$$

where ϕ_0 is étale and $A_0 \simeq \mathbf{Z}[x_1, \ldots, x_n]$ is a polynomial ring over $\mathbf{Z}$. Let $U \subseteq |\operatorname{Spec} A_0|$ denote the (open) image of the map $|\operatorname{Spec} B_0| \to |\operatorname{Spec} A_0|$ and choose elements $\{f_i \in A_0\}_{1 \leqslant i \leqslant m}$ such that $U = \bigcup_{1 \leqslant i \leqslant m} |\operatorname{Spec} A_0[f_i^{-1}]|$. Then the images of f_i in A generate the unit ideal. Since $\mathscr{O}'$ is local, the bottom horizontal map in the diagram

$$\begin{array}{ccc} \coprod_{1 \leqslant i \leqslant m} \mathcal{S}\mathrm{ol}_{B[f_i^{-1}]}(\mathscr{O}') & \longrightarrow & \mathcal{S}\mathrm{ol}_B(\mathscr{O}') \\ \downarrow & & \downarrow{\scriptstyle \rho} \\ \coprod_{1 \leqslant i \leqslant m} \mathcal{S}\mathrm{ol}_{A[f_i^{-1}]}(\mathscr{O}') & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\mathscr{O}') \end{array}$$

is an effective epimorphism. Consequently, to show that ρ is an effective epimorphism, it will suffice to show that each of the induced maps $\mathcal{S}\mathrm{ol}_{B[f_i^{-1}]}(\mathscr{O}') \to \mathcal{S}\mathrm{ol}_{A[f_i^{-1}]}(\mathscr{O}')$ is an effective epimorphism. Using the existence of a pullback square

$$\begin{array}{ccc} \mathcal{S}\mathrm{ol}_{B[f_i^{-1}]}(\mathscr{O}') & \longrightarrow & \mathcal{S}\mathrm{ol}_{A[f_i^{-1}]}(\mathscr{O}') \\ \downarrow & & \downarrow \\ \mathcal{S}\mathrm{ol}_{B_0[f_i^{-1}]}(\mathscr{O}') & \longrightarrow & \mathcal{S}\mathrm{ol}_{A_0[f_i^{-1}]}(\mathscr{O}'), \end{array}$$

we can replace A by $A_0[f_i^{-1}]$ (note that the map $A_0[f_i^{-1}] \to B_0[f_i^{-1}]$ is faithfully flat because $|\operatorname{Spec} A_0[f_i^{-1}]|$ is contained in U).

Consider the diagram

$$\begin{array}{ccccc} \mathcal{S}\mathrm{ol}_B(\mathscr{O}) & \xrightarrow{\overline{\rho}} & \mathcal{S}\mathrm{ol}_A(\mathscr{O}) & \longrightarrow & \mathcal{S}\mathrm{ol}_{A_0}(\mathscr{O}) \\ \downarrow & & \downarrow u & & \downarrow v \\ \mathcal{S}\mathrm{ol}_B(\mathscr{O}') & \xrightarrow{\rho} & \mathcal{S}\mathrm{ol}_A(\mathscr{O}') & \longrightarrow & \mathcal{S}\mathrm{ol}_{A_0}(\mathscr{O}'). \end{array}$$

Since A_0 is isomorphic to the polynomial ring $\mathbf{Z}[x_1, \ldots, x_n]$, we can identify v with the map $\alpha^n : \mathscr{O}^n \to \mathscr{O}'^n$. Our assumption that α is an effective epimorphism now guarantees that v is an effetive epimorphism. Since α is local, the right square in this diagram is a pullback, so that u is also an effective epimorphism. Our assumption that $\mathscr{O}$ is strictly Henselian guarantees that $\overline{\rho}$ is an effective epimorphism. It now follows by inspection of the above diagram that ρ is an effective epimorphism as desired. □

1.2.3 The Étale Spectrum of a Commutative Ring

Let R be a commutative ring and let $\operatorname{Spec} R = (X, \mathscr{O}_X)$ be the associated affine scheme. Then we can regard $\mathscr{O}_X$ as a commutative ring object of the topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X)$. This commutative ring object is local, but is usually not strictly Henselian. To remedy this, one can replace the Zariski spectrum $\operatorname{Spec} R$ by a slightly more sophisticated object, which we will refer to as the *étale spectrum* of R.

Definition 1.2.3.1 (The Étale Topos of a Commutative Ring)**.** Let R be a commutative ring. We let $\mathrm{CAlg}^{\heartsuit}_R$ denote the category of commutative R-algebras, and $\mathrm{CAlg}^{\text{ét}}_R$ the full subcategory of CAlg_R spanned by the étale R-algebras. The (opposite of) the ∞-category $\mathrm{CAlg}^{\text{ét}}_R$ is equipped with a Grothendieck topology, where a family of maps $\{A \to A_\alpha\}$ generates a covering sieve if and only if there exists some finite collection of indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that the map $A \to \prod_{1 \leqslant i \leqslant n} A_{\alpha_i}$ is faithfully flat (this is an immediate consequence of Proposition A.3.2.1). We will refer to this Grothendieck topology as the *étale topology.*

We let $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}^{\text{ét}}_R)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\text{ét}}_R, \mathcal{S}\mathrm{et})$ spanned by those functors which are sheaves with respect to the étale topology. We will refer to $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}^{\text{ét}}_R)$ as the *étale topos* of R.

Proposition 1.2.3.2. *Let R be a commutative ring, and let $\mathscr{O} : \mathrm{CAlg}^{\text{ét}}_R \to \mathcal{S}\mathrm{et}$ be the forgetful functor (which assigns to each étale R-algebra A its underlying set). Then $\mathscr{O}$ is a sheaf for the étale topology, and can therefore be identified with a commutative ring object of the topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}^{\text{ét}}_R)$. Moreover, $\mathscr{O}$ is strictly Henselian.*

Proof. The assertion that $\mathscr{O}$ is an étale sheaf is equivalent to the assertion that for every étale R-algebra R' and every faithfully flat étale map $R' \to R''$, the diagram

$$R' \to R'' \rightrightarrows R'' \otimes_{R'} R''$$

is an equalizer in the category of sets (see Proposition A.3.3.1). We now show that $\mathscr{O}$ is strictly Henselian. Suppose we are given a commutative ring A and a faithfully flat étale map $A \to \prod_{1 \leqslant i \leqslant n} A_i$. We wish to show that the induced map $\theta : \coprod_{1 \leqslant i \leqslant n} \mathcal{S}\mathrm{ol}_{A_i}(\mathscr{O}) \to \mathcal{S}\mathrm{ol}_A(\mathscr{O})$ is an epimorphism in the topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_R^{\text{ét}})$. To prove this, suppose we are given an étale R-algebra R' and a point $\eta \in \mathcal{S}\mathrm{ol}_A(\mathscr{O})(R')$, which we can identify with an algebra homomorphism $A \to R'$. For $1 \leqslant i \leqslant n$, let $R'_i = A_i \otimes_A R'$, and let η_i denote the image of η in $\mathcal{S}\mathrm{ol}_A(\mathscr{O})(R'_i)$. Then $\prod_{1 \leqslant i \leqslant n} R'_i$ is faithfully flat and étale over R, and is therefore generates a covering sieve on the object $R' \in (\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}$. Moreover, each of the points η_i can be lifted to a point $\overline{\eta}_i \in \mathcal{S}\mathrm{ol}_{A_i}(\mathscr{O})(R'_i)$, so that θ is an epimorphism as desired. $\square$

Definition 1.2.3.3. Let R be a commutative ring, and let $\mathscr{O}$ be as in Proposition 1.2.3.2. We will denote the ringed topos $(\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_R^{\text{ét}}), \mathscr{O})$ by $\mathrm{Spét}\, R$, and refer to it as the *étale spectrum* of the commutative ring R.

If $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a ringed topos, we let $\Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$ denote the commutative ring $\mathrm{Hom}_{\mathcal{X}}(\mathbf{1}, \mathscr{O}_{\mathcal{X}})$, where $\mathbf{1}$ denotes a final object of $\mathcal{X}$. We will refer to the construction $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \mapsto \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$ as the *global sections functor.*.

Proposition 1.2.3.4. *Let $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a ringed topos for which $\mathscr{O}_{\mathcal{X}}$ is strictly Henselian, and let R be a commutative ring. Then the global sections functor induces an equivalence of categories*

$$\mathrm{Map}_{1\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^\heartsuit}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), \mathrm{Spét}\, R) \to \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit}(R, \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}}))$$

(where the set on the right hand side is interpreted as a discrete category, having only identity morphisms).

Corollary 1.2.3.5. *The global sections functor*

$$1\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^\heartsuit} \to (\mathrm{CAlg}^\heartsuit)^{\mathrm{op}}$$

$$(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \mapsto \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$$

admits a right adjoint, given on objects by $R \mapsto \mathrm{Spét}\, R$.

Remark 1.2.3.6. If R is a commutative ring and $\mathrm{Spét}\, R = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, then the unit map $R \to \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$ is an equivalence. It follows that the construction $R \mapsto \mathrm{Spét}\, R$ determines a fully faithful functor from (the opposite of) the category of commutative rings to the 2-category $1\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^\heartsuit}$.

In particular, we can regard the construction $R \mapsto \mathrm{Spét}\, R$ as a contravariant functor from the category of commutative rings to the 2-category $1\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^\heartsuit} \subseteq 1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}$.

Remark 1.2.3.7. Let R be a commutative ring, and let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ denote the étale spectrum $\operatorname{Spét} R$. For every étale R-algebra A, let $h^A : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}\mathrm{et}$ denote the functor corepresented by A, which we regard as an object of the topos $\mathcal{X}$. Then there is a canonical equivalence of ringed topoi

$$(\mathcal{X}_{/h^A}, \mathcal{O}_\mathcal{X}|_{h^A}) \simeq \operatorname{Spét} A.$$

This can be deduced either from the universal properties of $\operatorname{Spét} R$ and $\operatorname{Spét} A$ (Proposition 1.2.3.4), or directly from the construction of the étale spectra.

Proof of Proposition 1.2.3.4. Fix a ring homomorphism $\phi : R \to \Gamma(\mathcal{X}; \mathcal{O}_\mathcal{X})$, and let $\mathcal{C}$ denote the fiber product

$$\mathrm{Map}_{1\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^\heartsuit}}((\mathcal{X}, \mathcal{O}_\mathcal{X}), \operatorname{Spét} R) \times_{\mathrm{Hom}_{\mathrm{CAlg}^\heartsuit}(R, \Gamma(\mathcal{X}; \mathcal{O}_\mathcal{X}))} \{\phi\}.$$

We will show that the category $\mathcal{C}$ is trivial (that is, it is equivalent to the category having a single object and a single morphism).

Write $\operatorname{Spét} R = (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$. Let $\mathbf{1}_\mathcal{X}$ and $\mathbf{1}_\mathcal{Y}$ denote final objects of the topoi $\mathcal{X}$ and $\mathcal{Y}$, respectively. For every étale R-algebra A, let $h^A \in \mathcal{Y} \subseteq \mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S}\mathrm{et})$ denote the functor corepresented by A, so that h^A fits into a pullback diagram σ_A:

$$\begin{array}{ccc} h^A & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\mathcal{O}_\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathbf{1}_\mathcal{Y} & \longrightarrow & \mathcal{S}\mathrm{ol}_R(\mathcal{O}_\mathcal{Y}). \end{array}$$

We also define an object $X_A \in \mathcal{X}$ so that we have a pullback diagram τ_A :

$$\begin{array}{ccc} X_A & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\mathcal{O}_\mathcal{X}) \\ \downarrow & & \downarrow \\ \mathbf{1}_\mathcal{X} & \longrightarrow & \mathcal{S}\mathrm{ol}_R(\mathcal{O}_\mathcal{X}), \end{array}$$

where the bottom horizontal map is determined by the ring homomorphism $\phi : R \to \mathrm{Hom}_\mathcal{X}(\mathbf{1}_\mathcal{X}, \mathcal{O}_\mathcal{X})$.

Let $f = (f_*, \alpha)$ be a map of ringed topoi from $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ to $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ which induces the ring homomorphism ϕ upon passage to global sections. For every étale R-algebra A, the map α determines a natural transformation of diagrams $f^*\sigma_A \to \tau_A$, which gives in particular a map $\nu_{f,A} : f^* h^A \to X_A$. If (f_*, α) belongs to $\mathcal{C}$, then the maps $\nu_{f,A}$ are isomorphisms (Proposition 1.2.2.12). In particular, if we are given two objects (f_*, α) and (f'_*, α') in $\mathcal{C}$, then for each $A \in \mathrm{CAlg}_R^{\text{ét}}$ there is a unique isomorphism $\theta_A : f^* h^A \simeq f'^* h^A$ for which the

diagram

$$\begin{array}{ccc} f^* h^A & \xrightarrow{\theta_A} & f'^* h^A \\ & {\scriptstyle \nu_{f,A}} \searrow \quad \swarrow {\scriptstyle \nu_{f',A}} & \\ & X_A & \end{array}$$

commutes. In particular, for each $X \in \mathcal{X}$, we have bijections

$$\begin{aligned} (f'_* X)(A) &\simeq \mathrm{Hom}_{\mathcal{Y}}(h^A, f'_* X) \\ &\simeq \mathrm{Hom}_{\mathcal{X}}(f'^* h^A, X) \\ &\simeq \mathrm{Hom}_{\mathcal{X}}(f^* h^A, X) \\ &\simeq \mathrm{Hom}_{\mathcal{Y}}(h^A, f_* X) \\ &\simeq (f_* X)(A). \end{aligned}$$

These bijections depend functorially on X and A, and therefore supply an isomorphism γ between the functors f'_* and f_*. We claim that γ is the unique morphism from (f_*, α) to (f'_*, α') in the category $\mathcal{C}$. Uniqueness is clear from the definition. To prove that γ is a morphism in $\mathcal{C}$, let us abuse notation by identifying γ with the adjoint natural transformation $f^* \to f'^*$; we must show that the diagram

$$\begin{array}{ccc} f^* \mathscr{O}_{\mathcal{Y}} & \xrightarrow{\gamma(\mathscr{O}_{\mathcal{Y}})} & f'^* \mathscr{O}_{\mathcal{Y}} \\ & \searrow \quad \swarrow & \\ & \mathscr{O}_{\mathcal{X}} & \end{array}$$

commutes. Writing $\mathscr{O}_{\mathcal{Y}}$ as a colimit of representable functors, we are reduced to proving that for each map $h^A \to \mathscr{O}_{\mathcal{Y}}$ in the topos $\mathcal{Y}$ (classified by an element $a \in A$), the diagram

$$\begin{array}{ccc} f^* h^A & \xrightarrow{\gamma(h^A)} & f'^* h^A \\ \downarrow & & \downarrow \\ f^* \mathscr{O}_{\mathcal{Y}} & & f'^* \mathscr{O}_{\mathcal{Y}} \\ & \searrow \quad \swarrow & \\ & \mathscr{O}_{\mathcal{X}} & \end{array}$$

commutes. This is because the vertical compositions can be identified (using the maps $\nu_{f,A}$ and $\nu_{f',A}$) with the composite map $X_A \hookrightarrow \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}}) \xrightarrow{a} \mathscr{O}_{\mathcal{X}}$.

It follows from the above argument that for every pair of objects $C, C' \in \mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(C, C')$ contains a unique element (which is also an isomorphism in $\mathcal{C}$). To complete the proof, it will suffice to show that the category $\mathcal{C}$ is nonempty. To this end, we define a functor $f_* : \mathcal{X} \to \mathrm{Fun}(\mathrm{CAlg}^{\text{ét}}_R, \mathcal{S}\mathrm{et})$ by the formula $f_*(X)(A) = \mathrm{Hom}_{\mathcal{X}}(X_A, X)$. The assumption that $\mathscr{O}_{\mathcal{X}}$ is strictly Henselian implies that f_* factors through the full subcategory $\mathcal{Y} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\text{ét}}_R, \mathcal{S}\mathrm{et})$. We claim that f_* is a geometric morphism of topoi: that is, it admits a left adjoint f^* which commutes with finite limits. The existence of a left adjoint f^* follows from the adjoint functor theorem (since f_* is an accessible functor which preserves small limits). By construction, we have a canonical isomorphism $f^* h^A \simeq X_A$ for each $A \in \mathrm{CAlg}^{\text{ét}}_R$. To prove that f^* preserves finite limits, it suffices to show that the construction $A \mapsto f^* h^A \simeq X_A$ carries finite colimits in $\mathrm{CAlg}^{\text{ét}}_A$ to finite limits in $\mathcal{X}$ (see Lemma HTT.6.4.5.6), which follows immediately from the construction.

Unwinding the definitions, we obtain for each $A \in \mathrm{CAlg}^{\text{ét}}_R$ a canonical ring homomorphism

$$A \to \mathrm{Hom}_{\mathcal{X}}(\mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}}), \mathscr{O}_{\mathcal{X}}) \to \mathrm{Hom}_{\mathcal{X}}(X_A, \mathscr{O}_{\mathcal{X}}) = (f_* \mathscr{O}_{\mathcal{X}})(A).$$

These ring homomorphisms depend functorially on R, and therefore give rise to a homomorphism $\alpha : \mathscr{O}_{\mathcal{Y}} \to f_* \mathscr{O}_{\mathcal{X}}$ of commutative ring objects of $\mathcal{Y}$. We may therefore regard (f_*, α) as a map of ringed topoi from $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ to $\mathrm{Spét}\, R$. It is clear from the construction that the induced ring homomorphism $R \simeq \Gamma(\mathcal{Y}; \mathscr{O}_{\mathcal{Y}}) \to \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$ coincides with ϕ. To complete the proof that (f_*, α) is an object of the category $\mathcal{C}$, it will suffice to show that the underlying map $f^* \mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$ is local. To prove this, fix an object $X \in \mathcal{X}$ and an element ζ of the commutative ring $\mathrm{Hom}_{\mathcal{X}}(X, f^* \mathscr{O}_{\mathcal{Y}})$; we wish to show that if the image of ζ in $\mathrm{Hom}_{\mathcal{X}}(X, \mathscr{O}_{\mathcal{X}})$ is invertible, then ζ is invertible. This assertion can be tested locally on X: we may therefore assume that the map ζ factors as a composition

$$X \xrightarrow{\zeta_0} X_A \simeq f^* h^A \xrightarrow{f^* \zeta_1} f^* \mathscr{O}_{\mathcal{Y}},$$

for some étale R-algebra A. In this case, we can identify ζ_1 with an element $a \in A$ and ζ_0 with a map of R-algebras $\phi_X : A \to \mathrm{Hom}_{\mathcal{X}}(X, \mathscr{O}_{\mathcal{X}})$, in which case the image of ζ in $\mathrm{Hom}_{\mathcal{X}}(X, \mathscr{O}_{\mathcal{X}})$ can be identified with $\phi_X(r)$. If this element of $\mathrm{Hom}_{\mathcal{X}}(X, \mathscr{O}_{\mathcal{X}})$ is invertible, then ϕ_X extends to an R-algebra homomorphism $\overline{\phi}_X : A[a^{-1}] \to \mathrm{Hom}_{\mathcal{X}}(X, \mathscr{O}_{\mathcal{X}})$. In this case, ζ_0 factors through the subobject $X_{A[a^{-1}]} \subseteq X_A$. We are therefore reduced to proving that the composite map $X_{A[a^{-1}]} \to X_A \simeq f^* h^A \xrightarrow{f^* \zeta_1} f^* \mathscr{O}_{\mathcal{Y}}$ determines an invertible element $\overline{r}$ of the commutative ring

$$\mathrm{Hom}_{\mathcal{X}}(X_{A[a^{-1}]}, f^* \mathscr{O}_{\mathcal{Y}}) \simeq \mathrm{Hom}_{\mathcal{Y}}(h^{A[a^{-1}]}, f_* f^* \mathscr{O}_{\mathcal{Y}}) \simeq (f_* f^* \mathscr{O}_{\mathcal{Y}})(A[a^{-1}]).$$

This is clear, since $\overline{r}$ is the image of r under the composite map

$$A \to A[a^{-1}] = \mathscr{O}_{\mathcal{Y}}(A[a^{-1}]) \to (f_* f^* \mathscr{O}_{\mathcal{Y}})(A[a^{-1}]).$$

□

1.2.4 Deligne-Mumford Stacks as Ringed Topoi

By definition, a scheme is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to the Zariski spectrum $\operatorname{Spec} R$ of a commutative ring R. We now consider an analogous definition, replacing the Zariski spectrum $\operatorname{Spec} R$ by the étale spectrum $\operatorname{Sp\acute{e}t} R$ of Definition 1.2.3.3.

Definition 1.2.4.1. Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a ringed topos. We will say that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a *Deligne-Mumford stack* if there exists a collection of objects $U_\alpha \in \mathcal{X}$ satisfying the following conditions:

(1) The objects U_α cover $\mathcal{X}$. That is, the natural map $\amalg_\alpha U_\alpha \to \mathbf{1}$ is an epimorphism in $\mathcal{X}$, where $\mathbf{1}$ denotes the final object of $\mathcal{X}$.

(2) For each index α, the ringed topos $(\mathcal{X}_{/U_\alpha}, \mathcal{O}_\mathcal{X}|_{U_\alpha})$ is equivalent (in the 2-category $1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}$) to a ringed topos of the form $\operatorname{Sp\acute{e}t} R_\alpha$, where R_α is a commutative ring.

Note that conditions (1) and (2) imply that $\mathcal{O}_\mathcal{X}$ is strictly Henselian. We let DM denote the full subcategory of $1\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^\heartsuit}$ spanned by those ringed topoi $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ which are Deligne-Mumford stacks.

Definition 1.2.4.2. Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a Deligne-Mumford stack. We will say that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is *affine* if it is equivalent to $\operatorname{Sp\acute{e}t} R$ for some commutative ring R. We will say that an object $U \in \mathcal{X}$ is *affine* if the ringed topos $(\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X}|_U)$ is affine.

Remark 1.2.4.3. Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a Deligne-Mumford stack. Then $\mathcal{X}$ is generated by affine objects. That is, for every object $X \in \mathcal{X}$, there exists an effective epimorphism $\amalg U_\alpha \to X$, where each U_α is affine. This assertion can be tested locally on $\mathcal{X}$: we may therefore assume without loss of generality that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is affine, in which case the desired result follows from Remark 1.2.3.7.

Remark 1.2.4.4. Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a ringed topos. Then $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a Deligne-Mumford stack if and only if, for each object $X \in \mathcal{X}$, there exists an effective epimorphism $\amalg U_\alpha \to X$ where each of the ringed topoi $(\mathcal{X}_{/U_\alpha}, \mathcal{O}_\mathcal{X}|_{U_\alpha})$ is an affine Deligne-Mumford stack. The "only if" direction follows immediately from the definition, and the "if" direction follows from Remark 1.2.4.3.

Remark 1.2.4.5. Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a ringed topos. The condition that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a Deligne-Mumford stack can be tested locally on $\mathcal{X}$. More precisely:

(*i*) If $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a Deligne-Mumford stack, then any ringed topos of the form $(\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X}|_U)$ is a Deligne-Mumford stack.

(*ii*) If the topos $\mathcal{X}$ admits a covering by objects U_α such that each of the ringed topoi $(\mathcal{X}_{/U_\alpha}, \mathcal{O}_\mathcal{X}|_{U_\alpha})$ is a Deligne-Mumford stack, then $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a Deligne-Mumford stack.

Both of these assertions follow immediately from the characterization of Deligne-Mumford stacks supplied by Remark 1.2.4.4.

Proposition 1.2.4.6. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a Deligne-Mumford stack. Then the collection of affine objects of $\mathcal{X}$ is closed under fiber products.*

Proof. For each object $U \in \mathcal{X}$, let X_U denote the Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathscr{O}_{\mathcal{X}}|_U)$. The functor $R \mapsto \operatorname{Spét} R$ determines a fully faithful embedding $(\mathrm{CAlg}^{\heartsuit})^{\mathrm{op}} \to \mathrm{DM}$ which preserves small limits (since it is a right adjoint), so that its essential image is closed under small limits. The desired result now follows from the observation that the construction $U \mapsto \mathsf{X}_U$ determines a functor from $\mathcal{X}$ to DM which preserves fiber products. □

Proposition 1.2.4.7. *Let $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be ringed topoi. If $\mathscr{O}_{\mathcal{X}}$ is strictly Henselian and $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ is a Deligne-Mumford stack, then the category $\mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^{\heartsuit}}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}))$ is a groupoid.*

Proof. Let (f_*, α) and (f'_*, α') be maps from $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ to $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ in $1\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^{\heartsuit}}$, and let γ be a morphism from (f_*, α) to (f'_*, α'). Let us identify γ with a natural transformation from f^* to f'^*. We wish to prove that γ is an isomorphism. Fix a covering of $\mathcal{Y}$ by objects $\{U_\beta\}_{\beta \in I}$ such that each $(\mathcal{Y}_{/U_\beta}, \mathscr{O}_{\mathcal{Y}}|_{U_\beta})$ is equivalent to $\operatorname{Spét} R_\beta$ for some commutative ring R_β. Then the objects $f'^* U_\beta$ cover the topos $\mathcal{X}$. The assertion that γ is an isomorphism can be tested locally on $\mathcal{X}$; we may therefore replace $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ by ringed topoi of the form $(\mathcal{X}_{/f^* U_\beta}, \mathscr{O}_{\mathcal{X}}|_{f^* U_\beta})$ and thereby reduce to the case where there exists a map $u : \mathbf{1}_{\mathcal{X}} \to f^* U_\beta$ in the topos $\mathcal{X}$, for some index β. In this case, the map u (and the induced map $u' : \mathbf{1}_{\mathcal{X}} \to f'^* U_\beta$) determine factorizations of f and f' through the ringed topos $(\mathcal{Y}_{/U_\beta}, \mathscr{O}_{\mathcal{Y}}|_{U_\beta})$. We may therefore reduce to the case where $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \simeq \operatorname{Spét} R_\beta$ is affine, in which case the desired result follows from Proposition 1.2.3.4. □

Corollary 1.2.4.8. *The 2-category* DM *of Deligne-Mumford stacks is a $(2,1)$-category. In other words, every 2-morphism in* DM *is invertible.*

Remark 1.2.4.9. Let $\mathcal{C}$ denote the simplicial category whose objects are Deligne-Mumford stacks $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, with morphism spaces given by

$$\mathrm{Map}_{\mathcal{C}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})) = \mathrm{N}(\mathrm{Map}_{\mathrm{DM}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}))).$$

It follows from Corollary 1.2.4.8 that $\mathcal{C}$ is a fibrant simplicial category, so that its homotopy coherent nerve $\mathrm{N}(\mathcal{C})$ is an ∞-category. We will denote this ∞-category by N(DM), and refer to it as the *∞-category of Deligne-Mumford stacks.*

1.2.5 Deligne-Mumford Stacks as Functors

We now compare Definition 1.2.4.1 with the more traditional "functor-of-points" approach to the theory of Deligne-Mumford stacks.

Notation 1.2.5.1. Let $\tau_{\leqslant 1}\mathcal{S}$ denote the ∞-category of 1-truncated spaces (in other words, the ∞-category of groupoids). Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a Deligne-Mumford stack. We define a functor $h_{\mathsf{X}} : \mathrm{CAlg}^{\heartsuit} \to \tau_{\leqslant 1}\mathcal{S}$ by the formula $h_{\mathsf{X}}(R) = \mathrm{Map}_{1\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^{\heartsuit}}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X})$. We will refer to h_{X} as the *functor of points* of X.

Proposition 1.2.5.2. *The construction* $\mathsf{X} \mapsto h_{\mathsf{X}}$ *determines a fully faithful embedding of* ∞*-categories* $\mathrm{N}(\mathrm{DM}) \to \mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \tau_{\leqslant 1}\mathcal{S})$.

Proof of Proposition 1.2.5.2. Let X and Y be Deligne-Mumford stacks. Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$. For each object $U \in \mathcal{X}$, set $\mathsf{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathcal{X}}|_U)$ and consider the canonical map

$$\theta_U : \mathrm{Map}_{\mathrm{N}(\mathrm{DM})}(\mathsf{X}_U, \mathsf{Y}) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \tau_{\leqslant 1}\mathcal{S})}(h_{\mathsf{X}_U}, h_{\mathsf{Y}}).$$

Let us say that an object $U \in \mathcal{X}$ is *good* if θ_U is a homotopy equivalence. We wish to show that the final object of $\mathcal{X}$ is good. In fact, we will show that every object of $\mathcal{X}$ is good. The proof proceeds in several steps:

(*i*) The construction $U \mapsto \theta_U$ carries coproducts in $\mathcal{X}$ to products. Consequently, the collection of good objects of $\mathcal{X}$ is closed under small coproducts.

(*ii*) Suppose that $f : U_0 \to X$ is an effective epimorphism of $\mathcal{X}$, and let $U_\bullet$ denote its Čech nerve. Then θ_X can be identified with the totalization of the cosimplicial object $\theta_{U_\bullet}$ in $\mathrm{Fun}(\Delta^1, \mathcal{S})$. Consequently, if each of the objects U_m is good, then X is good.

(*iii*) Every affine object of $\mathcal{X}$ is good (this follows from Yoneda's lemma).

(*iv*) Let $f : X \to Y$ be a monomorphism in $\mathcal{X}$. If Y is affine, then X is good. To prove this, choose an effective epimorphism $g : U_0 \to X$, where U_0 is a coproduct of affine objects U_α (see Remark 1.2.4.3). Let $U_\bullet$ be the Čech nerve of g. By virtue of (*ii*), it will suffice to show that each U_m is good. Using (*i*), we are reduced to showing that each fiber product $U_{\alpha_0} \times_X \cdots \times_X U_{\alpha_m}$ is good. Since f is a monomorphism, we have an equivalence

$$U_{\alpha_0} \times_X \cdots \times_X U_{\alpha_m} \simeq U_{\alpha_0} \times_Y \cdots \times_Y U_{\alpha_m}.$$

It follows from Proposition 1.2.4.6 that this object is affine, so that the desired result follows from (*iii*).

(*v*) Let $f : X \to Y$ be an arbitrary morphism in $\mathcal{X}$. If Y is affine, then X is good. To prove this, choose an effective epimorphism $g : U_0 \to X$, where U_0 is a coproduct of affine objects U_α (see Remark 1.2.4.3). Let $U_\bullet$ be the Čech nerve of g. By virtue of (*ii*), it will suffice to show that each U_m is good. Using (*i*), we are reduced to showing that each fiber product $U_{\alpha_0} \times_X \cdots \times_X U_{\alpha_m}$ is good. This follows from (*iv*), since there exists a monomorphism

$$U_{\alpha_0} \times_X \cdots \times_X U_{\alpha_m} \hookrightarrow U_{\alpha_0} \times_Y \cdots \times_Y U_{\alpha_m}$$

whose codomain is affine by virtue of Proposition 1.2.4.6.

(*vi*) Every object $X \in \mathcal{X}$ is good. To prove this, choose an effective epimorphism $g : U_0 \to X$, where U_0 is a coproduct of affine objects U_α (see Remark 1.2.4.3). Let $U_\bullet$ be the Čech nerve of g. By virtue of (*ii*), it will suffice to show that each U_m is good. Using (*i*), we are reduced to showing that each fiber product $U_{\alpha_0} \times_X \cdots \times_X U_{\alpha_m}$ is good. This follows from (*v*), since the projection map $U_{\alpha_0} \times_X \cdots \times_X U_{\alpha_m} \to U_{\alpha_0}$ has affine codomain.

□

For the remainder of this section, we will abuse notation by not distinguishing between a Deligne-Mumford stack and its image under the fully faithful embedding of Proposition 1.2.5.2 (in other words, we will identify a Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ with the functor h_X).

Definition 1.2.5.3. Let R be a commutative ring and write $\operatorname{Spét} R = (\mathcal{X}, \mathcal{O}_\mathcal{X})$. For each object $U \in \mathcal{X} = \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_R^{\text{ét}})$, we let $\operatorname{Spét}_U R$ denote the ringed topos $(\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X}|_U)$. Then $\operatorname{Spét}_U R$ is a Deligne-Mumford stack equipped with a canonical map $\operatorname{Spét}_U R \to \operatorname{Spét} R$.

If $f : X \to Y$ is an arbitrary morphism in $\mathrm{Fun}(\mathrm{CAlg}^\heartsuit, \tau_{\leqslant 1} \mathcal{S})$, we will say that f is *representable and étale* if, for every commutative ring R and every point $\eta \in Y(R)$, the fiber product $\operatorname{Spét} R \times_Y X$ is equivalent to $\operatorname{Spét}_U(R)$, for some object $U \in \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_R^{\text{ét}})$.

Remark 1.2.5.4. In the situation of Definition 1.2.5.3, a map $f : X \to Y$ is representable and étale if and only if, for each point $\eta \in Y(R)$, the fiber product $\operatorname{Spét} R \times_Y X$ is representable by an algebraic space which is étale over $\operatorname{Spét} R$. The veracity of this assertion requires that we adopt a slightly more general definition of algebraic space than the one given in [117].

Suppose that $X = \operatorname{Spét}_U(R)$ for some sheaf $U \in \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_R^{\text{ét}})$. Choosing a set of sections $\eta_\alpha \in U(R_\alpha)$ which generate U, we obtain an effective epimorphism $\amalg \operatorname{Spét} R_\alpha \to X$ in the category of étale sheaves on (the opposite of) the category $\mathrm{CAlg}_R^{\text{ét}}$. However, the maps $v_\alpha : \operatorname{Spét} R_\alpha \to X$ need not be relatively representable by schemes. Nevertheless, the maps v_α are relatively representable in the special case where there exists a monomorphism

$X \hookrightarrow \operatorname{Sp\acute{e}t} A$, for some étale R-algebra A. In this case, each fiber product $\operatorname{Sp\acute{e}t} B \times_X \operatorname{Sp\acute{e}t} R_\alpha$ can be identified with the functor corepresented by $B \otimes_A R_\alpha$.

In the general case, each fiber product $X_{\alpha,\beta} = \operatorname{Sp\acute{e}t} R_\alpha \times_X \operatorname{Sp\acute{e}t} R_\beta$ is again étale over $\operatorname{Sp\acute{e}t} R$ (in the sense of Definition 1.2.5.3), and admits a monomorphism

$$X_{\alpha,\beta} \hookrightarrow \operatorname{Sp\acute{e}t} R_\alpha \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Sp\acute{e}t} R_\beta \simeq \operatorname{Sp\acute{e}t}(R_\alpha \otimes_R R_\beta).$$

It follows that each $X_{\alpha,\beta}$ is a representable by an algebraic space in the sense of [117], so that the maps $v_\alpha : \operatorname{Sp\acute{e}t} R_\alpha \to X$ are relatively representable by algebraic spaces.

Remark 1.2.5.5. Let $Y = (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ be a Deligne-Mumford stack, and suppose we are given a morphism $f : X \to Y$ in $\operatorname{Fun}(\mathrm{CAlg}^\heartsuit, \tau_{\leqslant 1} \mathcal{S})$. Assume further that X is a sheaf for the étale topology. Then the following conditions are equivalent:

(1) The functor X is equivalent (as an object of the ∞-category $\operatorname{Fun}(\mathrm{CAlg}^\heartsuit, \tau_{\leqslant 1} \mathcal{S})_{/Y}$) to a Deligne-Mumford stack of the form $(\mathcal{Y}_{/U}, \mathcal{O}_\mathcal{Y}|_U)$ for some object $U \in \mathcal{Y}$.

(2) The map f is representable and étale (in the sense of Definition 1.2.5.3).

The implication (1) $\Rightarrow$ (2) follows immediately from the definitions. The converse follows by using (2) to construct the object $U \in \mathcal{Y}$ locally, and then invoking the assumption that $\mathcal{X}$ is an étale sheaf.

Remark 1.2.5.6. Let $f : X \to Y$ be a morphism in $\operatorname{Fun}(\mathrm{CAlg}^\heartsuit, \tau_{\leqslant 1} \mathcal{S})$, and suppose that both X and Y are sheaves for the étale topology. Then f is étale and representable if and only if, for every Deligne-Mumford stack $Z = (\mathcal{Z}, \mathcal{O}_\mathcal{Z})$ equipped with a map $Z \to Y$, the fiber product $Z \times_Y X$ is equivalent to to $(\mathcal{Z}_{/U}, \mathcal{O}_\mathcal{Z}|_U)$ for some object $U \in \mathcal{Z}$. The "if" direction is obvious, and the converse follows from Remark 1.2.5.5. It follows from this characterization that the collection of representable étale morphisms (between étale sheaves) is closed under composition.

Proposition 1.2.5.7. *Let $\phi : A \to B$ be a homomorphism of commutative rings. The following conditions are equivalent:*

(1) *The ring homomorphism ϕ is étale (see §B.1).*

(2) *The induced map of Deligne-Mumford stacks* $\operatorname{Sp\acute{e}t} B \to \operatorname{Sp\acute{e}t} A$ *is representable and étale (here we abuse terminology by identifying* $\operatorname{Sp\acute{e}t} A$ *and* $\operatorname{Sp\acute{e}t} B$ *with the functors they represent).*

In other words, if A is a commutative ring, then an object $U \in \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}^{\acute{e}t}_A)$ is affine if and only if U is corepresentable by an étale A-algebra B.

Proof. The implication (1) $\Rightarrow$ (2) follows immediately from the construction of $\operatorname{Sp\acute{e}t} B$. Conversely, suppose that (2) is satisfied. Write $\operatorname{Sp\acute{e}t} A = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, so that $\operatorname{Sp\acute{e}t} B \simeq (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ for some object $U \in \mathcal{X}$. For each étale A-algebra A', let $h^{A'} \in \mathcal{X}$ denote the functor corepresented by A'. The objects $h^{A'}$ generate the topos $\mathcal{X}$ under colimits. We may therefore choose a collection of étale A-algebras $\{A_\alpha\}_{\alpha \in I}$ and an effective epimorphism $\coprod_{\alpha \in I} h^{A_\alpha} \to U$ in $\mathcal{X}$. Under the identification of $\mathcal{X}_{/U}$ with $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_B^{\acute{e}t})$, we can identify each h^{A_α} with an object $V_\alpha \in \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_B^{\acute{e}t})$. These objects cover the topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_B^{\acute{e}t})$. We may therefore choose a finite collection of indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ and a faithfully flat étale map $B \to \prod_{1 \leqslant i \leqslant n} B_i$ such that each $V_{\alpha_i}(B_i)$ is nonempty. Set $A' = \prod_{1 \leqslant i \leqslant n} A_{\alpha_i}$ and $B' = \prod B_i$, so that we morphisms of Deligne-Mumford stacks

$$\operatorname{Sp\acute{e}t} B' \to \operatorname{Sp\acute{e}t} A' \to \operatorname{Sp\acute{e}t} B \to \operatorname{Sp\acute{e}t} A,$$

induced by a sequence of ring homomorphisms $A \xrightarrow{\phi} B \to A' \to B'$. It follows that B' is a retract of $A' \otimes_B B'$ in the category of A-algebras. Since B' is étale over B and A' is étale over A, $A' \otimes_B B'$ is étale over A, so that B' is étale over A. Since the map $B \to B'$ is étale and faithfully flat, we conclude that B is also étale over A (see Proposition B.1.4.1). $\square$

Definition 1.2.5.8. Let $f_\alpha : X_\alpha \to Y$ be a collection of representable étale morphisms in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \tau_{\leqslant 1} \mathcal{S})$. We will say that the set $\{f_\alpha\}$ is *jointly surjective* if, for every commutative ring R and every point $\eta \in Y(R)$, if we write $\operatorname{Sp\acute{e}t} R \times_Y X_\alpha$ as $\operatorname{Sp\acute{e}t}_{U_\alpha} R$ for some $U_\alpha \in \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_R^{\acute{e}t})$, then the objects U_α comprise a covering of the topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_R^{\acute{e}t})$.

We will say that a single representable étale morphism $f : X \to Y$ is *surjective* if the set $\{f\}$ is jointly surjective.

Remark 1.2.5.5 admits the following converse:

Theorem 1.2.5.9. *Let $X : \mathrm{CAlg}^{\heartsuit} \to \mathcal{S}_{\leqslant 1}$ be a functor. The following conditions are equivalent:*

(1) *The functor X is (representable by) a Deligne-Mumford stack.*

(2) *The functor X is a sheaf for the étale topology, and there exists a jointly surjective collection of representable étale morphisms $f_\alpha : U_\alpha \to X$, where each U_α is (representable by) an affine Deligne-Mumford stack* $\operatorname{Sp\acute{e}t} R_\alpha$.

(3) *The functor X is a sheaf for the étale topology, and there exists a jointly surjective collection of representable étale morphisms $f_\alpha : U_\alpha \to X$, where each U_α is (representable by) a Deligne-Mumford stack.*

(4) *The functor X is a sheaf for the étale topology, and there exists a representable étale surjection $f : U_0 \to X$, where U_0 is (representable by) a Deligne-Mumford stack.*

Example 1.2.5.10. Let $X : \mathrm{CAlg}^{\heartsuit} \to \mathcal{S}\mathrm{et}$ be a functor which is representable by a scheme. Then X satisfies condition (2) of Theorem 1.2.5.9 (in fact, we can take the maps $u_\alpha : \operatorname{Spec} R_\alpha \to X$ to be any open covering of X by affine schemes). Theorem 1.2.5.9 implies that the essential image of the functor $\mathrm{N}(\mathrm{DM}) \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \tau_{\leqslant 1}\mathcal{S})$ includes all functors which are representable by schemes. We therefore obtain a fully faithful embedding from the category Sch of schemes to the 2-category DM of Deligne-Mumford stacks, given on affine schemes by $\operatorname{Spec} R \mapsto \operatorname{Sp\acute{e}t} R$. Note that this embedding is *not* given by the formula $(X, \mathscr{O}_X) \mapsto (\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X), \mathscr{O}_X)$, because the structure sheaf of a scheme is usually not strictly Henselian. Instead, the Deligne-Mumford stack associated to a scheme $(X, \mathscr{O}_X)$ can be viewed as the classifying topos for strict Henselizations of $\mathscr{O}_X$: see §1.6 for more details).

Proof of Theorem 1.2.5.9. Let $X : \mathrm{CAlg}^{\heartsuit} \to \tau_{\leqslant 1}\mathcal{S}$ be the functor represented by a Deligne-Mumford stack $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$. Fix a commutative ring A and write $\operatorname{Sp\acute{e}t} A = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$. For every étale A-algebra A', let $h^{A'} \in \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_A^{\mathrm{\acute{e}t}})$ denote the functor corepresented by A'. Then the restriction of X to $\mathrm{CAlg}_A^{\mathrm{\acute{e}t}}$ is given by the formula

$$X(A') = \mathrm{Map}_{\mathrm{DM}}((\mathcal{Y}_{/h^{A'}}, \mathscr{O}_{\mathcal{Y}}|_{h^{A'}}), (\mathcal{X}, \mathscr{O}_{\mathcal{X}}),$$

from which it follows easily that the restriction of X to $\mathrm{CAlg}_A^{\mathrm{\acute{e}t}}$ is an étale sheaf. It follows that X is a sheaf for the étale topology, so that the implication (1) ⇒ (2) follows immediately from the definition of Deligne-Mumford stack.

The implication (2) ⇒ (3) is trivial, and the implication (3) ⇒ (4) follows by taking U_0 to be the coproduct of the U_α (in the 2-category of Deligne-Mumford stacks). We will complete the proof by showing that (4) ⇒ (1). Let U_0 be a functor which is representable by a Deligne-Mumford stack $(\mathcal{U}_0, \mathscr{O}_{\mathcal{U}_0})$, let $f : U_0 \to X$ be a representable étale surjection, and let $U_\bullet$ be the Čech nerve of f (formed in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \tau_{\leqslant 1}\mathcal{S})$). Since f is an effective epimorphism of étale sheaves, we can identify X with the geometric realization of $U_\bullet$ in the ∞-category of $\tau_{\leqslant 1}\mathcal{S}$-valued étale sheaves on $\mathrm{CAlg}^{\heartsuit}$. Each U_m admits a representable étale map to U_0, and is therefore representable by a Deligne-Mumford stack $(\mathcal{U}_m, \mathscr{O}_{\mathcal{U}_m})$. We can view $\mathcal{U}_\bullet$ as a simplicial object in the 2-category of Grothendieck topoi; let $\mathcal{X}$ denote its geometric realization (so that the objects of $\mathcal{X}$ can be identified with sequences $\{X_m \in \mathcal{U}_m\}_{m \geqslant 0}$ which are compatible with one another under pullback). Since each of the maps $U_m \to U_n$ is étale, the collection of structure sheaves $\{\mathscr{O}_{\mathcal{U}_m}\}_{m \geqslant 0}$ can be identified with a commutative ring object $\mathscr{O}_{\mathcal{X}} \in \mathcal{X}$.

For each integer m, the inclusion $[m] \hookrightarrow [m+1]$ determines a representable étale surjection map $U_{m+1} \to U_m$. We may therefore choose an object $V_m \in \mathcal{U}_m$ and an equivalence

$$(\mathcal{U}_{m+1}, \mathscr{O}_{\mathcal{U}_{m+1}}) \simeq (\mathcal{U}_{m/V_m}, \mathscr{O}_{\mathcal{U}_m}|_{V_m}).$$

The objects V_m are compatible under pullback, and therefore determine an object V of the topos $\mathcal{X}$. Since each V_m covers the final object of $\mathcal{U}_m$, the object V covers the final object of

$\mathcal{X}$. Moreover, we can identify $(\mathcal{X}_{/V}, \mathscr{O}_{\mathcal{X}}|_V)$ with the geometric realization of the simplicial ringed topos

$$(\mathcal{U}_{\bullet/V_\bullet}, \mathscr{O}_{\mathcal{U}_\bullet}|_{V_\bullet}) \simeq (\mathcal{U}_{\bullet+1}, \mathscr{O}_{\mathcal{U}_{\bullet+1}}),$$

which is equivalent to $(\mathcal{U}_0, \mathscr{O}_{\mathcal{U}_0})$. It follows from Remark 1.2.4.4 that $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a Deligne-Mumford stack. Let $X' : \mathrm{CAlg}^{\heartsuit} \to \tau_{\leqslant 1}\mathcal{S}$ denote the functor represented by $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$. Let $f' : U_0 \to X'$ be the canonical map. The natural isomorphism $f^*V \simeq V_0$ shows that $U_\bullet$ is the Čech nerve of the map f', so that f' factors as a composition

$$U_0 \xrightarrow{f} |U_\bullet| \simeq X \xrightarrow{g} X',$$

where g is a monomorphism. To complete the proof, it will suffice to show that f' is an epimorphism of étale sheaves. This follows from the observation that V covers the final object of the topos $\mathcal{X}$. □

1.2.6 Quasi-Coherent Sheaves on a Deligne-Mumford Stack

We close this section with a brief discussion of quasi-coherent sheaves on a Deligne-Mumford stack, which will play a role in §1.4:

Definition 1.2.6.1. Let $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a Deligne-Mumford stack, and let $\mathscr{F}$ be a $\mathscr{O}_{\mathcal{X}}$-module object of $\mathcal{X}$. For each object $U \in \mathcal{X}$, we let $\mathscr{O}_{\mathcal{X}}(U)$ denote the commutative ring $\mathrm{Hom}_{\mathcal{X}}(U, \mathscr{O}_{\mathcal{X}})$, and $\mathscr{F}(U)$ the module $\mathrm{Hom}_{\mathcal{X}}(U, \mathscr{F})$. We will say that $\mathscr{F}$ is *quasi-coherent* if the following condition is satisfied:

(∗) For every morphism $U \to V$ between affine object of $\mathcal{X}$, the induced map

$$\mathscr{O}_{\mathcal{X}}(U) \otimes_{\mathscr{O}_{\mathcal{X}}(V)} \mathscr{F}(V) \to \mathscr{F}(U)$$

is an isomorphism of modules over the commutative ring $\mathscr{O}_{\mathcal{X}}(U)$.

Example 1.2.6.2. Let A be a commutative ring and let write $\mathrm{Spét}\, A = (\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_A^{\text{ét}}), \mathscr{O})$. Proposition 1.2.5.7 implies that a $\mathscr{O}$-module $\mathscr{F}$ in $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_A^{\text{ét}})$ is quasi-coherent if and only if, for every morphism $B \to C$ of étale A-algebras, the induced map $C \otimes_B \mathscr{F}(B) \to \mathscr{F}(C)$ is an equivalence. In other words, $\mathscr{F}$ is quasi-coherent if and only if there exists a (discrete) A-module M for which $\mathscr{F}$ is given by the formula $\mathscr{F}(B) = B \otimes_A M$. Conversely, for any discrete A-module M, the theory of faithfully flat descent implies that the construction $B \mapsto B \otimes_A M$ determines a sheaf for the étale topology on $\mathrm{CAlg}_A^{\text{ét}}$. We summarize the situation as follows: the category of quasi-coherent sheaves on $\mathrm{Spét}\, A$ is equivalent to the category of (discrete) A-modules.

Proposition 1.2.6.3. *Let $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a Deligne-Mumford stack, and let $\mathscr{F}$ be a $\mathscr{O}_{\mathcal{X}}$-module object of $\mathcal{X}$. The condition that $\mathscr{F}$ be quasi-coherent can be tested locally on $\mathcal{X}$. More precisely, if there exists a covering of $\mathcal{X}$ by objects U_α such that each restriction $\mathscr{F}|_{U_\alpha}$ is a quasi-coherent sheaf on $(\mathcal{X}_{/U_\alpha}, \mathscr{O}_{\mathcal{X}}|_{U_\alpha})$, then $\mathscr{F}$ is quasi-coherent.*

Proof. Let $f : V \to W$ be a morphism between affine objects of $\mathcal{X}$; we wish to show that the induced map $\mathscr{O}_{\mathcal{X}}(V) \otimes_{\mathscr{O}_{\mathcal{X}}(W)} \mathscr{F}(W) \to \mathscr{F}(V)$ is an isomorphism. Replacing $\mathcal{X}$ by $(\mathcal{X}_{/W}, \mathscr{O}_{\mathcal{X}}|_W)$, we may assume that $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) = \operatorname{Sp\acute{e}t} A$ for some commutative ring A. Let us identify $\mathcal{X}$ with $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}_A^{\acute{e}t})$, so that we can view $\mathscr{F}$ as a functor from $\mathrm{CAlg}_A^{\acute{e}t}$ to the category of sets. To prove that $\mathscr{F}$ is quasi-coherent, it will suffice to show that for every étale A-algebra B, the natural map $B \otimes_A \mathscr{F}(A) \to \mathscr{F}(B)$ is an isomorphism (see Example 1.2.6.2).

We may assume without loss of generality that each U_α is affine and therefore associated to some étale A-algebra A_α. Since the objects U_α cover $\mathcal{X}$, we may choose a finite set of indices $\alpha_1, \ldots, \alpha_n$ for which the product $\prod_{1 \leqslant i \leqslant n} A_{\alpha_i}$ is faithfully flat over A. Our assumption on $\mathscr{F}$ guarantees that for any morphism of étale A-algebras $C \to C'$, if C admits the structure of an A_α-algebra for some index α, then the induced map $C' \otimes_C \mathscr{F}(C) \to \mathscr{F}(C')$ is an isomorphism.

For any étale A-algebra B, the hypothesis that $\mathscr{F}$ is an étale sheaf (and the flatness of B as an A-module) supplies a commutative diagram of short exact sequences

$$\begin{array}{ccccccc}
0 \longrightarrow & B \otimes_A \mathscr{F}(A) & \longrightarrow & \prod_{1 \leqslant i \leqslant n} B \otimes_A \mathscr{F}(A_{\alpha_i}) & \longrightarrow & \prod_{1 \leqslant i,j \leqslant n} B \otimes_A \mathscr{F}(A_{\alpha_i} \otimes_A A_{\alpha_j}) \\
& \downarrow \theta & & \downarrow \theta' & & \downarrow \theta'' \\
0 \longrightarrow & \mathscr{F}(B) & \longrightarrow & \prod_{1 \leqslant i \leqslant n} \mathscr{F}(B \otimes_A A_{\alpha_i}) & \longrightarrow & \prod_{1 \leqslant i,j \leqslant n} \mathscr{F}(B \otimes_A A_{\alpha_i} \otimes_A A_{\alpha_j}).
\end{array}$$

Since θ' and θ'' are isomorphisms, we conclude that θ is also an isomorphism. □

1.3 Sheaves of Spectra

In §1.1, we introduced the notion of a *spectrally ringed space*: that is, a pair $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a sheaf of $\mathbb{E}_\infty$-rings on X. The language of spectrally ringed spaces is adequate for describing many of the algebro-geometric objects which we are interested in studying (such as the *spectral schemes* of Definition 1.1.2.8). However, to accommodate more exotic objects (such as the *spectral Deligne-Mumford stacks* of §1.4), it will be useful to work with a more general notion of CAlg-valued sheaf. In this section, we will study pairs $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ where $\mathcal{X}$ is an ∞-topos and $\mathscr{O}_{\mathcal{X}}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

1.3.1 Sheaves on ∞-Topoi

Let $\mathcal{C}$ be an ∞-category. In §1.1, we introduced the notion of a $\mathcal{C}$-valued sheaf on a topological space X (Definition 1.1.2.1). This definition can be generalized to an arbitrary Grothendieck site:

Definition 1.3.1.1. Let $\mathcal{T}$ be an essentially small ∞-category. Recall that a *Grothendieck topology* on $\mathcal{T}$ is a Grothendieck topology on the homotopy category $\mathrm{h}\mathcal{T}$, in the sense of classical category theory (see §HTT.6.2.2 for a detailed discussion). Let $\mathcal{C}$ be an arbitrary ∞-category. We will say that a functor $\mathscr{O} : \mathcal{T}^{\mathrm{op}} \to \mathcal{C}$ is a *$\mathcal{C}$-valued sheaf on $\mathcal{T}$* if the following condition is satisfied: for every object $U \in \mathcal{T}$ and every covering sieve $\mathcal{T}^0_{/U} \subseteq \mathcal{T}_{/U}$, the composite map

$$(\mathcal{T}^0_{/U})^{\lhd} \subseteq (\mathcal{T}_{/U})^{\lhd} \to \mathcal{T} \xrightarrow{\mathscr{O}^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}}$$

is a colimit diagram in $\mathcal{C}^{\mathrm{op}}$. We let $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{T})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C})$ spanned by the $\mathcal{C}$-valued sheaves on $\mathcal{T}$.

More informally, a functor $\mathscr{O} : \mathcal{T}^{\mathrm{op}} \to \mathcal{C}$ is a $\mathcal{C}$-valued sheaf if, for every object $U \in \mathcal{T}$ and every covering sieve $\mathcal{T}^0_{/U}$ of U, the canonical map

$$\mathscr{O}(U) \to \varprojlim_{V \in \mathcal{T}^0_{/U}} \mathscr{O}(V)$$

is an equivalence in $\mathcal{C}$.

Example 1.3.1.2. Let X be a topological space, let $\mathcal{C}$ be an ∞-category, and let $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(X)$ be the ∞-category of $\mathcal{C}$-valued sheaves on X (Definition 1.1.2.1). Then $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(X) = \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{U}(X))$, where $\mathcal{U}(X)$ is the partially ordered set of all open subsets of X, which is endowed with the usual Grothendieck topology (so that a collection of inclusions $\{U_\alpha \subseteq U\}$ generates a covering sieve on U if and only if $U = \bigcup U_\alpha$).

Example 1.3.1.3. Let $\mathcal{T}$ be a small ∞-category equipped with a Grothendieck topology and let $\mathcal{S}$ denote the ∞-category of spaces. Then we will denote the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{T})$ simply by $\mathcal{S}\mathrm{hv}(\mathcal{T})$ and refer to it as the *∞-category of sheaves on $\mathcal{T}$*. The ∞-category $\mathcal{S}\mathrm{hv}(\mathcal{T})$ is an accessible left-exact localization of the presheaf ∞-category $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S})$, and is therefore an ∞-topos (see §HTT.6.2).

In the situation of Definition 1.3.1.1, the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{T})$ does not depend on the exact details of the Grothendieck site $\mathcal{T}$: it depends only on the associated ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{T})$. To see this, it will be convenient to introduce a site-independent version of Definition 1.3.1.1 (which also makes sense for ∞-topoi which do not arise as sheaves on a Grothendieck site).

Definition 1.3.1.4. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{C}$ be an arbitrary ∞-category. A *$\mathcal{C}$-valued sheaf on $\mathcal{X}$* is a functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{C}$ which preserves small limits. We let $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$ spanned by the $\mathcal{C}$-valued sheaves on $\mathcal{X}$.

Warning 1.3.1.5. Let $\mathcal{X}$ be an ∞-topos, and let $\mathcal{C}$ be an arbitrary ∞-category. Then the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ introduced in Definition 1.3.1.4 generally does *not* coincide with the ∞-category $\mathcal{C}$-valued sheaves with respect to a Grothendieck topology on $\mathcal{X}$ (for example, the canonical topology on $\mathcal{X}$). Consequently, the conventions of Definition 1.3.1.4 and 1.3.1.1 conflict with one another. However, there should be little danger of confusion: for example, an ∞-topos $\mathcal{X}$ is never essentially small as an ∞-category, unless $\mathcal{X}$ is a contractible Kan complex.

Remark 1.3.1.6. Let $\mathcal{C}$ be a presentable ∞-category and $\mathcal{X}$ an ∞-topos. Then the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ can be identified with the tensor product $\mathcal{C} \otimes \mathcal{X}$ introduced in §HA.4.8.1. In particular, $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ is a presentable ∞-category.

We now show that Definitions 1.3.1.1 and 1.3.1.4 are compatible with one another, at least when the ∞-category $\mathcal{C}$ admits small limits. For any ∞-category $\mathcal{T}$, we let $\mathcal{P}(\mathcal{T})$ denote the ∞-category $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S})$ of $\mathcal{S}$-valued presheaves on $\mathcal{T}$, and $j : \mathcal{T} \to \mathcal{P}(\mathcal{T})$ the Yoneda embedding.

Proposition 1.3.1.7. *Let $\mathcal{T}$ be a small ∞-category equipped with a Grothendieck topology. Let $j : \mathcal{T} \to \mathcal{P}(\mathcal{T})$ denote the Yoneda embedding and $L : \mathcal{P}(\mathcal{T}) \to \mathcal{S}\mathrm{hv}(\mathcal{T})$ a left adjoint to the inclusion. Let $\mathcal{C}$ be an arbitrary ∞-category which admits small limits. Then composition with $L \circ j$ induces an equivalence of ∞-categories $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{S}\mathrm{hv}(\mathcal{T})) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{T})$.*

Corollary 1.3.1.8. *Let X be a topological space and let $\mathcal{C}$ be an ∞-category which admits small limits. Then there is a canonical equivalence of ∞-categories $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(X) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{S}\mathrm{hv}(X))$, where the left hand side is given by Definition 1.1.2.1 and the right hand side by Definition 1.3.1.4.*

Proof of Proposition 1.3.1.7. According to Theorem HTT.5.1.5.6, composition with j induces an equivalence of ∞-categories $\mathrm{Fun}_0(\mathcal{P}(\mathcal{T})^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C})$, where $\mathrm{Fun}_0(\mathcal{P}(\mathcal{T})^{\mathrm{op}}, \mathcal{C})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{P}(\mathcal{T})^{\mathrm{op}}, \mathcal{C})$ spanned by those functors which preserve small limits. According to Proposition HTT.5.5.4.20, composition with L induces a fully faithful embedding $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{S}\mathrm{hv}(\mathcal{T})) \to \mathrm{Fun}_0(\mathcal{P}(\mathcal{T})^{\mathrm{op}}, \mathcal{C})$. The essential image of this embedding consists of those limit-preserving functors $F : \mathcal{P}(\mathcal{T})^{\mathrm{op}} \to \mathcal{C}$ such that, for every $X \in \mathcal{T}$ and every covering sieve $\mathcal{T}^0_{/X} \subseteq \mathcal{T}_{/X}$, the induced map $F(jX) \to F(Y)$ is an equivalence in $\mathcal{C}$, where Y is the subobject of jX corresponding to the sieve $\mathcal{T}^0_{/X}$. Unwinding the definitions, this translates into the condition that the composition

$$(\mathcal{T}^0_{/X})^{\triangleright} \subseteq (\mathcal{T}_{/X})^{\triangleright} \to \mathcal{T} \xrightarrow{j} \mathcal{P}(\mathcal{T}) \xrightarrow{F} \mathcal{C}^{\mathrm{op}}$$

is a colimit diagram. It follows that the composition

$$\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{S}\mathrm{hv}(\mathcal{T})) \to \mathrm{Fun}_0(\mathcal{P}(\mathcal{T})^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C})$$

is fully faithful, and its essential image is the full subcategory $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{T})$. □

1.3.2 Sheaves of Spectra

In this book, are primarily interested in $\mathcal{C}$-valued sheaves when $\mathcal{C} = \mathrm{Sp}$ is the ∞-category of spectra.

Definition 1.3.2.1. Let $\mathcal{X}$ be an ∞-topos. A *sheaf of spectra* on $\mathcal{X}$ is a sheaf on $\mathcal{X}$ with values in the ∞-category Sp of spectra. We let $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp})$ spanned by the sheaves of spectra on $\mathcal{X}$.

Remark 1.3.2.2. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{S})$ spanned by those functors which preserve small limits. Recall that the ∞-category Sp of spectra can be defined as the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ spanned by those functors $E : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ which are reduced and excisive; here $\mathcal{S}_*^{\mathrm{fin}}$ denotes the ∞-category of pointed finite spaces (Definition HA.1.4.3.1). We therefore obtain an isomorphism of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ with the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}))$ spanned by those functors which are reduced and excisive. Since the Yoneda embedding induces an equivalence of ∞-categories $\mathcal{X} \to \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})$ (Proposition HTT.5.5.2.2), we obtain an equivalence of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ with the ∞-category $\mathrm{Sp}(\mathcal{X})$ of spectrum objects of $\mathcal{X}$ (see Definition HA.1.4.2.8). In particular, $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is a presentable stable ∞-category. Moreover, we have a forgetful functor $\Omega^\infty : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{X}$, which is obtained by pointwise composition with the forgetful functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ (together with the identification $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})$).

Notation 1.3.2.3. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}^{\heartsuit} = \tau_{\leqslant 0}\mathcal{X}$ denote its underlying topos. Composing the forgetful functor functor $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}) \simeq \mathcal{X}$ with the truncation functor $\tau_{\leqslant 0} : \mathcal{X} \to \mathcal{X}^{\heartsuit}$, we obtain a functor $\pi_0 : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{X}^{\heartsuit}$. More generally, for any integer n, we let $\pi_n : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \tau_{\leqslant 0}\mathcal{X}$ denote the composition of the functor π_0 with the shift functor $\Omega^n : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Note that π_n can also be described as the composition

$$\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{X}) \xrightarrow{\Omega^{n-2}} \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{S}_*}(\mathcal{X}) \simeq \mathcal{X}_* \xrightarrow{\pi_2} \tau_{\leqslant 0}\mathcal{X}.$$

It follows that π_n can be regarded as a functor from $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ to the category of abelian groups objects of $\mathcal{X}^{\heartsuit}$.

Example 1.3.2.4. In the situation of Notation 1.3.2.3, suppose that $\mathcal{X} = \mathcal{S}\mathrm{hv}(X)$ for some topological space X, and let $\mathscr{F}$ be an object of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{S}\mathrm{hv}(X)) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)$. For each integer n, we can identify $\pi_n \mathscr{F}$ with the sheaf of abelian groups on X given by sheafifying the presheaf $U \mapsto \pi_n(\mathscr{F}(U))$.

Definition 1.3.2.5. For every integer n, the functor $\Omega^{\infty-n} : \mathrm{Sp} \to \mathcal{S}$ induces a functor $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}) \simeq \mathcal{X}$, which we will also denote by $\Omega^{\infty-n}$. We will say that an object $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is *n-truncated* if $\Omega^{\infty+n}\mathscr{F}$ is a discrete object of $\mathcal{X}$. We will say that a sheaf of spectra $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is *n-connective* if the homotopy groups $\pi_m \mathscr{F}$ vanish for $m < n$.

We will say that M is *connective* if it is 0-connective (equivalently, M is connective if the object $\Omega^{\infty-m}\mathscr{F} \in \mathcal{X}$ is m-connective for every integer m). We let $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant n}$ denote the the full subcategory of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ spanned by the n-connective objects, and $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant n}$ the full subcategory of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ spanned by the n-truncated objects. In the special case $n = 0$, we will denote $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant 0}$ by $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})^{\mathrm{cn}}$.

Remark 1.3.2.6. Let $\mathcal{X}$ be an ∞-topos and let $\mathscr{F}$ be a sheaf of spectra on $\mathcal{X}$. Then $\mathscr{F}$ is n-truncated if and only if, for every object $U \in \mathcal{X}$, the spectrum $\mathscr{F}(U)$ is n-truncated.

The classes of 0-truncated and 0-connective spectrum-valued sheaves determine a t-structure:

Proposition 1.3.2.7. *Let $\mathcal{X}$ be an ∞-topos.*

(1) *The full subcategories $(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant 0}, \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0})$ determine a t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$.*

(2) *The t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is compatible with filtered colimits (that is, the full subcategory $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0} \subseteq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is closed under filtered colimits).*

(3) *The t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is right complete.*

(4) *The functor π_0 of Notation 1.3.2.3 determines an equivalence of categories from the heart of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ to the category of abelian group objects of $\mathcal{X}^{\heartsuit}$.*

Proof. It follows from Proposition HA.1.4.3.4 that $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ admits a t-structure given by the pair $(\mathcal{C}, \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0})$, where $\mathcal{C}$ is the collection of objects $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ for which the mapping space $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})}(\mathscr{F}, \Omega(\mathscr{G}))$ is contractible for every coconnective object $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0}$. Fix $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$; using Remark 1.3.2.2 we can identify $\mathscr{F}$ with a sequence of pointed objects $\mathscr{F}(n) \in \mathcal{X}_*$ and equivalences $\gamma_n : \mathscr{F}(n) \simeq \Omega\mathscr{F}(n+1)$. Set $\mathscr{F}'(n) = \tau_{\leqslant n-1}\mathscr{F}(n)$; the equivalences γ_n induce equivalences $\gamma'_n : \mathscr{F}'(n) \simeq \Omega\mathscr{F}'(n+1)$, so we can regard $\{\mathscr{F}'(n)\}$ as an object $\mathscr{F}' \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. We have a canonical map $\mathscr{F} \to \mathscr{F}'$. If $\mathscr{G}$ is a coconnective object of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0}$, then we have

$$\begin{aligned} \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})}(\mathscr{F}, \Omega\mathscr{G}) &\simeq \varprojlim \mathrm{Map}_{\mathcal{X}_*}(\mathscr{F}(n), \Omega^{\infty+1-n}\mathscr{G}) \\ &\simeq \varprojlim \mathrm{Map}_{\mathcal{X}_*}(\mathscr{F}'(n), \Omega^{\infty+1-n}\mathscr{G}) \\ &\simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})}(\mathscr{F}', \mathscr{G}). \end{aligned}$$

On the other hand, $\Omega^{-1}M'$ is a 0-truncated object of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. It follows that $\mathscr{F} \in \mathcal{C}$ if and only if $\mathscr{F}' \simeq 0$. This is equivalent to the requirement that each $\mathscr{F}'(n) \simeq \tau_{\leqslant n-1}\mathscr{F}(n)$ is a final object of $\mathcal{X}_*$: that is, the requirement that each $\mathscr{F}'(n)$ is n-connective. This proves that $\mathcal{C} = \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant 0}$ so that assertion (1) holds.

We observe that the loop functor $\Omega : \mathcal{X}_* \to \mathcal{X}_*$ preserves filtered colimits (Example HTT.7.3.4.7), so that $\Omega^{\infty+1} : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{X}_*$ preserves filtered colimits for each n. It follows that the homotopy fiber of $\Omega^{\infty+1}$ (over the zero object $* \in \mathcal{X}_*$) is closed under filtered colimits, so that (2) is satisfied. It follows easily that $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0}$ is stable under countable coproducts. Any object $\mathscr{F} \in \bigcap_n \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant -n}$ has the property that $\Omega^{\infty-n}\mathscr{F} \in \mathcal{X}_*$ is final for each n, so that M is a zero object of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Assertion (3) now follows from Proposition HA.1.2.1.19 .

Let us identify $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ with the homotopy inverse limit of the tower of ∞-categories

$$\cdots \to \mathcal{X}_* \xrightarrow{\Omega_*} \mathcal{X}_* \xrightarrow{\Omega_*} \mathcal{X}_* .$$

Under this identification, the heart $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})^{\heartsuit} \subseteq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ corresponds to the homotopy inverse limit of the tower

$$\cdots \to \mathcal{EM}_2(\mathcal{X}) \xrightarrow{\Omega} \mathcal{EM}_1(\mathcal{X}) \xrightarrow{\Omega} \mathcal{EM}_0(\mathcal{X}) = \mathcal{X}^{\heartsuit},$$

where $\mathcal{EM}_n(\mathcal{X}) \subseteq \mathcal{X}_*$ denotes the full subcategory spanned by the Eilenberg-MacLane objects (that is, objects which are both n-truncated and n-connective; see Definition HTT.7.2.2.1). Assertion (4) follows from the observation that $\mathcal{EM}_n(\mathcal{X})$ is equivalent to the category of abelian group objects of the underlying topos of $\mathcal{X}$ for $n \geqslant 2$ (Proposition HTT.7.2.2.12). □

Remark 1.3.2.8. Let $g^* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi (that is, a functor which preserves small colimits and finite limits). Then g^* is left exact, and therefore induces a functor $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \simeq \mathrm{Sp}(\mathcal{X}) \to \mathrm{Sp}(\mathcal{Y}) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y})$. We will abuse notation by denoting this functor also by g^*. It is a left adjoint to the pushforward functor $g_* : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$, given by pointwise composition with $g^* : \mathcal{X} \to \mathcal{Y}$.

Since the functor $g^* : \mathcal{X} \to \mathcal{Y}$ preserves n-truncated objects and n-connective objects for every integer n, we conclude that the functor $g^* : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y})$ is t-exact: that is, it carries $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant n}$ into $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y})_{\geqslant n}$ and $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant n}$ into $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y})_{\leqslant n}$. It follows that g_* is left t-exact: that is, $g_* \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y})_{\leqslant n} \subseteq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant n}$. The functor g_* is usually not right t-exact.

1.3.3 ∞-Connective Sheaves of Spectra

The t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is not left complete in general. For example, there may exist nonzero objects $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ whose homotopy groups $\pi_n \mathscr{F}$ vanish for all integers n.

Definition 1.3.3.1. Let $\mathcal{X}$ be an ∞-topos and let $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ be a sheaf of spectra on $\mathcal{X}$. We will say that $\mathscr{F}$ is *∞-connective* if it is n-connective for every integer n. In other words, $\mathscr{F}$ is ∞-connective if $\pi_n \mathscr{F} \simeq 0$ for every integer n.

Remark 1.3.3.2. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}^{\mathrm{hyp}} \subseteq \mathcal{X}$ be the full subcategory spanned by the hypercomplete objects. Then the inclusion map $f_* : \mathcal{X}^{\mathrm{hyp}} \to \mathcal{X}$ is a geometric morphism of ∞-topoi, which admits a left exact left adjoint $f^* : \mathcal{X} \to \mathcal{X}^{\mathrm{hyp}}$. Applying Remark 1.3.2.8, we obtain a pair of adjoint functors

$$\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}^{\mathrm{hyp}}).$$

Note that an object $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is ∞-connective if and only if $f^* \mathscr{F} \simeq 0$. Since the inclusion $\mathcal{X}^{\mathrm{hyp}} \hookrightarrow \mathcal{X}$ is fully faithful, the functor $f_* : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}^{\mathrm{hyp}}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is also fully faithful.

Proposition 1.3.3.3. *Let $\mathcal{X}$ be an ∞-topos and let $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. The following conditions are equivalent:*

(1) *The object $\Omega^\infty \mathscr{F} \in \mathcal{X}$ is hypercomplete.*

(2) *The sheaf of spectra $\mathscr{F}$ belongs to the essential image of the fully faithful embedding $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}^{\mathrm{hyp}}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$.*

(3) *For every ∞-connective object $\mathscr{G} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$, the mapping space $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})}(\mathscr{G}, \mathscr{F})$ is contractible.*

(4) *For every ∞-connective object $\mathscr{G} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$, every map $u : \mathscr{G} \to \mathscr{F}$ is nullhomotopic.*

Proof. We first show that (1) and (2) are equivalent. The implication (2) ⇒ (1) is immediate. Conversely, suppose that $\mathscr{F}$ satisfies condition (1). To show that $\mathscr{F}$ belongs to the essential image of the fully faithful embedding, it will suffice to show that $\Omega^{\infty-n} \mathscr{F} \in \mathcal{X}$ is hypercomplete for each $n \geqslant 0$. Proceeding by induction on n, we are reduced to proving that $\Omega^{\infty-1} \mathscr{F}$ is hypercomplete. We have a fiber sequence

$$\Omega^{\infty-1} \mathscr{F} \to \pi_{-1} \mathscr{F} \to \Omega^\infty(\Sigma^2(\tau_{\geqslant 0} \mathscr{F})).$$

Since $\pi_{-1} \mathscr{F}$ is a discrete object of $\mathcal{X}$ (and therefore hypercomplete), we are reduced to proving that the object $U = \Omega^\infty(\Sigma^2(\tau_{\geqslant 0} \mathscr{F})) \in \mathcal{X}$ is hypercomplete. Let U' denote its hypercomplete; we wish to show that the map $U \to U'$ is an equivalence in $\mathcal{X}$. Since U and U' are 2-connective pointed objects of $\mathcal{X}$, it will suffice to show that the induced map $\Omega^2 U \to \Omega^2 U'$ is an equivalence. Since the formation of hypercompletions is left exact, we can identify $\Omega^2 U'$ with the hypercompletion of $\Omega^2 U$. We are therefore reduced to proving that $\Omega^2 U \simeq \Omega^\infty \mathscr{F}$ is hypercomplete, which follows from assumption (1).

The implication (2) ⇒ (3) follows from Remark 1.3.3.2, and the implication (3) ⇒ (4) is immediate. We will complete the proof by showing that (4) ⇒ (2). Suppose that $\mathscr{F}$ satisfies condition (4), and consider the adjoint functors

$$\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}^{\mathrm{hyp}}).$$

appearing in Remark 1.3.3.2. To prove (2), it will suffice to show that the unit map $u : \mathscr{F} \to f_* f^* \mathscr{F}$ is an equivalence. Since the functor f_* is fully faithful, the pullback $f^* u$ is an equivalence. Since f^* is an exact functor, we conclude that $f^* \operatorname{fib}(u) \simeq 0$: that is, the fiber $\operatorname{fib}(u)$ is ∞-connective. Using (4), we deduce that the canonical map $\operatorname{fib}(u) \to \mathscr{F}$ is nullhomotopic. It follows that $\operatorname{cofib}(u)$ is a retract of $f_* f^* \mathscr{F}$ in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. In particular, $\Omega^{\infty - n}(\operatorname{cofib}(u))$ is a retract of the hypercomplete object $\Omega^{\infty-n}(f_* f^* \mathscr{F})$ for every integer n, and is therefore hypercomplete. Since the homotopy groups of $\Omega^{\infty-n}(\operatorname{cofib}(u))$ vanish, we conclude that $\Omega^{\infty-n}(\operatorname{cofib}(u))$ is a final object of $\mathcal{X}$. It follows that $\operatorname{cofib}(u) \simeq 0$, so that u is an equivalence as desired. □

Definition 1.3.3.4. Let $\mathcal{X}$ be an ∞-topos. We will say that an object $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is *hypercomplete* if it satisfies the equivalent conditions of Proposition 1.3.3.3.

Remark 1.3.3.5. Let $\mathcal{X}$ be an ∞-topos. Then the full subcategories of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ spanned by the ∞-connective and hypercomplete objects determine a t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ (with trivial heart). In particular, every object $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ fits into an essentially unique fiber sequence $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ where $\mathscr{F}'$ is ∞-connective and $\mathscr{F}''$ is hypercomplete.

The condition of hypercompleteness can be tested locally:

Proposition 1.3.3.6. *Let $\mathcal{X}$ be an ∞-topos. The property that an object $X \in \mathcal{X}$ is hypercomplete can be tested locally on $\mathcal{X}$. In other words, if there exists a collection of objects $\{U_\alpha \in \mathcal{X}\}$ such that $\amalg_\alpha U_\alpha$ is 0-connective and each product $X \times U_\alpha$ is a hypercomplete object of the ∞-topos $\mathcal{X}_{/U_\alpha}$, then X is hypercomplete.*

Proof. Let $U = \amalg U_\alpha$, so that $X \times U$ is a hypercomplete object of the ∞-topos $\mathcal{X}_{/U} \simeq \prod_\alpha \mathcal{X}_{/U_\alpha}$. Let $U_\bullet$ be the simplicial object of $\mathcal{X}$ given by the Čech nerve of the effective epimorphism $U \to \mathbf{1}$, where $\mathbf{1}$ denotes a final object of $\mathcal{X}$.

Let $f : Y \to Z$ be an ∞-connected morphism in $\mathcal{X}$; we wish to prove that the induced map $\alpha : \mathrm{Map}_{\mathcal{X}}(Z, X) \to \mathrm{Map}_{\mathcal{X}}(Y, X)$ is a homotopy equivalence. We can obtain α as the totalization of a map of cosimplicial spaces

$$\alpha^\bullet : \mathrm{Map}_{\mathcal{X}}(Z \times U_\bullet, X) \to \mathrm{Map}_{\mathcal{X}}(Y \times U_\bullet, X).$$

It will therefore suffice to show that each α^n is an equivalence. Replacing Y by $Y \times U_n$ and Z by $Z \times U_n$, we can reduce to the case where $\alpha = 0$. In this case, α_0 can be identified with the map $\mathrm{Map}_{\mathcal{X}_{/U}}(Z \times U, X \times U) \to \mathrm{Map}_{\mathcal{X}_{/U}}(Y \times U, X \times U)$. This map is an equivalence because $X \times U$ is a hypercomplete object of $\mathcal{X}_{/U}$, and the map $Z \times U \to Y \times U$ is ∞-connected. □

Corollary 1.3.3.7. *Let $\mathcal{X}$ be an ∞-topos, and let $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. If there exists a 0-connective object $\amalg U_\alpha$ in $\mathcal{X}$ such that each pullback $u_\alpha^* \mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}_{/U_\alpha})$ is hypercomplete (where $u_\alpha : \mathcal{X}_{/U_\alpha} \to \mathcal{X}$ denotes the étale geometric morphism determined by U_α), then $\mathscr{F}$ is hypercomplete.*

Corollary 1.3.3.8. *Let X be a topological space and let $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)$. If there exists an open covering U_α of X such that each restriction $\mathscr{F}|_{U_\alpha}$ is hypercomplete, then $\mathscr{F}$ is hypercomplete.*

Under some mild hypotheses on $\mathcal{X}$, one can show that the t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is left complete.

Definition 1.3.3.9. Let $\mathcal{X}$ be an ∞-topos, and let $n \geqslant 0$ be an integer. We will say that $\mathcal{X}$ is *locally of cohomological dimension* $\leqslant n$ if there exists a collection of objects $\{U_\alpha\}$ of $\mathcal{X}$ which generate $\mathcal{X}$ under small colimits, such that each of the ∞-topoi $\mathcal{X}_{/U_\alpha}$ has cohomological dimension $\leqslant n$ (see Definition HTT.7.2.2.18).

Proposition 1.3.3.10. *Let $\mathcal{X}$ be an ∞-topos and let $n \geqslant 2$ be an integer. The following conditions are equivalent:*

(1) *The ∞-topos $\mathcal{X}$ is locally of homotopy dimension $\leqslant n$ (see Definition HTT.7.2.1.8).*

(2) *The ∞-topos $\mathcal{X}$ is hypercomplete and locally of cohomological dimension $\leqslant n$.*

If these conditions are satisfied and U is an object of $\mathcal{X}$, then $\mathcal{X}_{/U}$ is of homotopy dimension $\leqslant n$ if and only if it is of cohomological dimension $\leqslant n$.

Proof. Let U be an object of $\mathcal{X}$. If $\mathcal{X}_{/U}$ is of homotopy dimension $\leqslant n$, then $\mathcal{X}_{/U}$ is of cohomological dimension $\leqslant n$ (Corollary HTT.7.2.2.30). It follows that if $\mathcal{X}$ is locally of homotopy dimension $\leqslant n$, then $\mathcal{X}$ is locally of cohomological dimension $\leqslant n$. The implication (1) $\Rightarrow$ (2) now follows from Corollary HTT.7.2.1.12 . For the converse, suppose that (2) is satisfied. We will complete the proof by showing that for each object $U \in \mathcal{X}$, if $\mathcal{X}_{/U}$ has cohomological dimension $\leqslant n$, then $\mathcal{X}_{/U}$ is of homotopy dimension $\leqslant n$. Replacing $\mathcal{X}$ by $\mathcal{X}_{/U}$, we are reduced to the problem of showing that $\mathcal{X}$ has homotopy dimension $\leqslant n$.

Let $\mathbf{1}$ denote the final object of $\mathcal{X}$, and let $\mathscr{F} \in \mathcal{X}$ be n-connective. We wish to prove that the mapping space $\mathrm{Map}_{\mathcal{X}}(\mathbf{1}, \mathscr{F})$ is nonempty. We begin by constructing a compatible sequence of maps $\phi_m : \mathbf{1} \to \tau_{\leqslant m} \mathscr{F}$. The construction proceeds by induction on m, the case $m < n$ being trivial. If $m \geqslant n$ and ϕ_{m-1} has already been constructed then the fiber product $\mathbf{1} \times_{\tau_{\leqslant m-1} \mathscr{F}} \tau_{\leqslant m} \mathscr{F}$ is an m-gerbe in $\mathcal{X}$. Our assumption that $\mathcal{X}$ has cohomological dimension $\leqslant n \leqslant m$ guarantees that this gerbe is automatically trivial, so that ϕ_{m-1} lifts to a map ϕ_m. Together, the maps $\{\phi_m\}_{m \geqslant 0}$ determine a map $\phi : \mathbf{1} \to \widehat{\mathscr{F}}$, where $\widehat{\mathscr{F}}$ denotes the limit $\varprojlim_m \tau_{\leqslant m} \mathscr{F}$. To complete the proof, it will suffice to show that the canonical map $\theta : \mathscr{F} \to \widehat{\mathscr{F}}$ is an equivalence. Because $\mathcal{X}$ is hypercomplete, this is equivalent to the assertion that θ induces an equivalence $\theta_m : \tau_{\leqslant m} \mathscr{F} \to \tau_{\leqslant m} \widehat{\mathscr{F}}$ for each $m \geqslant 0$. Note that θ_m is a right homotopy inverse to the canonical map

$$\gamma : \tau_{\leqslant m} \widehat{\mathscr{F}} \to \tau_{\leqslant m}(\tau_{\leqslant m} \mathscr{F}) \simeq \tau_{\leqslant m} \mathscr{F} .$$

It will therefore suffice to prove that γ is an equivalence: that is, the projection $\overline{\gamma} : \widehat{\mathscr{F}} \to \tau_{\leqslant m} \mathscr{F}$ exhibits $\tau_{\leqslant m} \mathscr{F}$ as an m-truncation of $\widehat{\mathscr{F}}$. In fact, we claim that the map $\overline{\gamma}$ is $(m+1)$-connective.

The map γ factors as a composition $\widehat{\mathscr{F}} \xrightarrow{\overline{\gamma}'} \tau_{\leqslant m+n} \mathscr{F} \xrightarrow{\overline{\gamma}''} \tau_{\leqslant m} \mathscr{F}$. Because $\overline{\gamma}''$ is $(m+1)$-connective, we are reduced to showing that $\overline{\gamma}'$ is $(m+1)$-connective. To prove this, it will suffice to show that $\mathcal{X}$ is generated under small colimits by objects V for which the map

$$\mathrm{Map}_{\mathcal{X}}(V, \widehat{\mathscr{F}}) \to \mathrm{Map}_{\mathcal{X}}(V, \tau_{\leqslant m+n} \mathscr{F})$$

is $(m+1)$-connective. In fact, we claim that this condition holds whenever $\mathcal{X}_{/V}$ has cohomological dimension $\leqslant n$. To prove this, it suffices to show that each of the maps $\psi : \mathrm{Map}_{\mathcal{X}}(V, \tau_{\leqslant t+1+n} \mathscr{F}) \to \mathrm{Map}_{\mathcal{X}}(V, \tau_{\leqslant t+n} \mathscr{F})$ is $(t+1)$-connective. Choose a point $\eta \in \mathrm{Map}_{\mathcal{X}}(V, \tau_{\leqslant t+n} \mathscr{F})$ and let $\overline{V} = V \times_{\tau_{\leqslant t+n} \mathscr{F}} \tau_{\leqslant t+1+n} \mathscr{F}$, so that the homotopy fiber of ψ over the point η can be identified with $\mathrm{Map}_{\mathcal{X}_{/V}}(V, \overline{V})$. By construction, $\overline{V}$ is an $(t+1+n)$-gerbe in $\mathcal{X}_{/V}$ banded by some abelian group object $\mathscr{A}$ of $\mathcal{X}_{/V}$. Since $\mathcal{X}_{/V}$ has cohomological dimension $\leqslant n$, the cohomology group $\mathrm{H}^{t+2+n}(\mathcal{X}_{/V}; \mathscr{F})$ vanishes, so that $\overline{V}$ is a trivial gerbe. We therefore obtain isomorphisms

$$\pi_j \mathrm{Map}_{\mathcal{X}_{/V}}(V, \overline{V}) \simeq \mathrm{H}^{t+1+n-j}(\mathcal{X}_{/V}; \mathscr{A}).$$

Since $\mathcal{X}_{/V}$ has cohomological dimension $\leqslant n$, this group vanishes for $j \leqslant t$, so that $\mathrm{Map}_{\mathcal{X}_{/V}}(V, \overline{V})$ is $(t+1)$-connective as desired. □

Corollary 1.3.3.11. *Let $\mathcal{X}$ be an ∞-topos. Assume that $\mathcal{X}$ is hypercomplete and locally of cohomological dimension $\leqslant n$, for some integer n. Then the t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is left complete.*

Proof. Without loss of generality, we may assume that $n \geqslant 2$. According to Proposition 1.3.3.10, $\mathcal{X}$ is locally of homotopy dimension $\leqslant 2$. It follows from Proposition HTT.7.2.1.10 that $\mathcal{X}$ is Postnikov complete (in the sense of Definition A.7.2.1). From this we immediately deduce that $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant 0}$ is Postnikov complete, so that $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is left complete. □

1.3.4 Sheafification and Tensor Products

Our next objective is to describe a symmetric monoidal structure on the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Roughly speaking, this symmetric monoidal structure is given by levelwise smash product, followed by sheafification. We begin by discussing the latter procedure.

Remark 1.3.4.1. Let $\mathcal{D}$ and $\mathcal{C}$ be small ∞-categories, and assume that $\mathcal{D}$ admits finite colimits. Composition with the Yoneda embeddings $\mathcal{D}^{\mathrm{op}} \to \mathcal{P}(\mathcal{D}^{\mathrm{op}})$ and $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ yields functors

$$\mathrm{Fun}^*(\mathcal{P}(\mathcal{D}^{\mathrm{op}}), \mathcal{P}(\mathcal{C}^{\mathrm{op}})) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}^{\mathrm{op}}, \mathcal{P}(\mathcal{C}^{\mathrm{op}})) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Ind}(\mathcal{D})) \leftarrow \mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D})).$$

Here $\mathrm{Fun}^*(\mathcal{P}(\mathcal{D}^{\mathrm{op}}), \mathcal{P}(\mathcal{C}^{\mathrm{op}}))$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{P}(\mathcal{D}^{\mathrm{op}}), \mathcal{P}(\mathcal{C}^{\mathrm{op}}))$ spanned by those functors which preserve small colimits and finite limits, $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}^{\mathrm{op}}, \mathcal{P}(\mathcal{C}^{\mathrm{op}}))$ the full subcategory of $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{P}(\mathcal{C}^{\mathrm{op}}))$ spanned by those functors which preserve finite limits, and $\mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))$ the full subcategory of $\mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))$ spanned by those functors which preserve filtered colimits. Each of these functors is an equivalence of ∞-categories (see Propositions HTT.6.1.5.2 and HTT.5.3.5.10; the middle equivalence is an isomorphism of simplicial sets obtained by identifying both sides with a full subcategory of $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S})$).

Assume that both $\mathcal{C}$ and $\mathcal{D}$ admit finite colimits, so that $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Ind}(\mathcal{D})$ are compactly generated presentable ∞-categories. The presheaf ∞-categories $\mathcal{P}(\mathcal{C}^{\mathrm{op}})$ and $\mathcal{P}(\mathcal{D}^{\mathrm{op}})$ are classifying ∞-topoi for $\mathrm{Ind}(\mathcal{C})$-valued and $\mathrm{Ind}(\mathcal{D})$-valued sheaves, respectively. The above argument shows that every geometric morphism between classifying ∞-topoi arises from a functor $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ which preserves filtered colimits. Put more informally, every natural operation which takes $\mathrm{Ind}(\mathcal{C})$-valued sheaves and produces $\mathrm{Ind}(\mathcal{D})$-valued sheaves is determined by a functor $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ which preserves filtered colimits.

Suppose now that we are given ∞-categories $\mathcal{C}$ and $\mathcal{D}$ which admit finite colimits, and let $f : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ be a functor which preserves filtered colimits. Remark 1.3.4.1 guarantees the existence of an induced functor $\theta : \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{D})}(\mathcal{X})$ for an arbitrary ∞-topos $\mathcal{X}$, which depends functorially on $\mathcal{X}$. In the special case where $\mathcal{X} = \mathcal{P}(\mathcal{U})$ is an ∞-category of presheaves on some small ∞-category $\mathcal{U}$, we can write down the functor θ very explicitly: it fits into a homotopy commutative diagram

$$\begin{array}{ccc}
\mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{X}) & \longrightarrow & \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{X}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{C})) & \xrightarrow{\circ f} & \mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D})),
\end{array}$$

where the vertical maps are equivalences of ∞-categories given by composition with the Yoneda embedding $\mathcal{U} \to \mathcal{P}(\mathcal{U})$. More generally, if we assume only that we are given a geometric morphism $\mathcal{P}(\mathcal{U}) \to \mathcal{X}$, then we obtain a larger (homotopy commutative) diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{X}) & \longrightarrow & \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{D})}(\mathcal{X}) \\ \uparrow & & \uparrow \\ \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{P}(\mathcal{U})) & \longrightarrow & \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{D})}(\mathcal{P}(\mathcal{U})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{C})) & \xrightarrow{\circ f} & \mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{D})). \end{array}$$

The existence of this diagram immediately implies the following result:

Lemma 1.3.4.2. *Let $\mathcal{U}$ be a small ∞-category and suppose we are given a geometric morphism of ∞-topoi $g^* : \mathcal{P}(\mathcal{U}) \to \mathcal{X}$. Let $\mathcal{C}$ be a small ∞-category which admits finite colimits, and let $T_{\mathcal{C}}$ denote the functor* $\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{C})) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{P}(\mathcal{U})) \to \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{X})$ *induced by g^*. Let $\mathcal{D}$ be another small ∞-category which admits finite colimits, and define $T_{\mathcal{D}}$ similarly. Suppose that $f : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ is a functor which preserves small filtered colimits. Then if $\alpha : \mathscr{F} \to \mathscr{G}$ is a morphism in* $\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{C}))$ *such that $T_{\mathcal{C}}(\alpha)$ is an equivalence, then the induced map $\alpha' : (f \circ \mathscr{F}) \to (f \circ \mathscr{G})$ also has the property that $T_{\mathcal{D}}(\alpha')$ is an equivalence.*

Lemma 1.3.4.3. *Let $\mathcal{X}$ be an ∞-topos and $\mathcal{C}$ a presentable ∞-category. Then the inclusion $i : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$ admits a left adjoint L.*

Proof. The proof does not really require that $\mathcal{X}$ is an ∞-topos, only that $\mathcal{X}$ is a presentable ∞-category. Under this assumption, we may suppose without loss of generality that $\mathcal{X} = \mathrm{Ind}_{\kappa}(\mathcal{X}_0)$, where κ is a regular cardinal and $\mathcal{X}_0$ is a small ∞-category which admits κ-small colimits. Then i is equivalent to the composition

$$\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \xrightarrow{G_{\mathcal{C}}} \mathrm{Fun}'(\mathcal{X}_0^{\mathrm{op}}, \mathcal{C}) \overset{i'}{\subseteq} \mathrm{Fun}(\mathcal{X}_0^{\mathrm{op}}, \mathcal{C}) \xrightarrow{G'_{\mathcal{C}}} \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}),$$

where $\mathrm{Fun}'(\mathcal{X}_0^{\mathrm{op}}, \mathcal{C})$ is the full subcategory of $\mathrm{Fun}(\mathcal{X}_0^{\mathrm{op}}, \mathcal{C})$ spanned by those functors which preserve κ-small limits, $G_{\mathcal{C}}$ is the functor given by restriction along the Yoneda embedding $j : \mathcal{X}_0 \to \mathcal{X}$, and $G'_{\mathcal{C}}$ is given by right Kan extension along j. The functor $G_{\mathcal{C}}$ is an equivalence of ∞-categories (Proposition HTT.5.5.1.9), and the functor $G'_{\mathcal{C}}$ admits a left adjoint (given by composition with j). Consequently, it suffices to show that the inclusion i' admits a left adjoint. This follows immediately from Lemmas HTT.5.5.4.17 , HTT.5.5.4.18 , and HTT.5.5.4.19 . □

Lemma 1.3.4.4. *Let $\mathcal{X}$ be an ∞-topos, and let $f : \mathcal{C} \to \mathcal{D}$ be a functor between compactly generated presentable ∞-categories. Assume that f preserves small filtered colimits. Let $L_{\mathcal{C}}$:* $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ *and $L_{\mathcal{D}} : \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{D}) \to \mathcal{S}\mathrm{hv}_{\mathcal{D}}(\mathcal{X})$ be left adjoints to the inclusion* *functors. Then composition with f determines a functor $F : \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{D})$ which carries $L_{\mathcal{C}}$-equivalences to $L_{\mathcal{D}}$-equivalences.*

Remark 1.3.4.5. In the situation of Lemma 1.3.4.4, the functor F descends to a functor $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{D}}(\mathcal{X})$, given by the composition $L_{\mathcal{D}} \circ F$. This is simply another description of the construction arising from Remark 1.3.4.1.

Proof. We use notation as in the proof of Lemma 1.3.4.3. For κ sufficiently large, the full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ is stable under limits, so that (by Proposition HTT.6.1.5.2) we have a geometric morphism $g^* : \mathcal{P}(\mathcal{X}_0) \to \mathcal{X}$. Then the functor $L_{\mathcal{C}}$ can be realized as the composition of the restriction functor $r_{\mathcal{C}} : \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}_0^{\mathrm{op}}, \mathcal{C})$ with the functor $T_{\mathcal{C}} : \mathrm{Fun}(\mathcal{X}_0^{\mathrm{op}}, \mathcal{C}) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{P}(\mathcal{X}_0)) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ induced by g^*, and we can similarly write $L_{\mathcal{D}} = T_{\mathcal{D}} \circ r_{\mathcal{D}}$. If α is a morphism in the ∞-category $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$ such that $L_{\mathcal{C}}(\alpha) = T_{\mathcal{C}}(r_{\mathcal{C}}(\alpha))$ is an equivalence, then Lemma 1.3.4.2 shows that $L_{\mathcal{D}}(F(\alpha)) = T_{\mathcal{D}}(r_{\mathcal{D}}(F\alpha))$ is an equivalence, as required. □

We will regard the ∞-category Sp of spectra as endowed with the smash product monoidal structure defined in §HA.4.8.2. This symmetric monoidal structure induces a symmetric monoidal structure on the ∞-category $\mathrm{Fun}(K, \mathrm{Sp})$, for any simplicial set K (Remark HA.2.1.3.4); we will refer to this symmetric monoidal structure as the *pointwise smash product monoidal structure.*

Proposition 1.3.4.6. *Let $\mathcal{X}$ be an ∞-topos, and let $L : \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ be a left adjoint to the inclusion. Then L is compatible with the pointwise smash product monoidal structure, in the sense of Definition HA.2.2.1.6: that is, if $f : \mathscr{F} \to \mathscr{F}'$ is an L-equivalence in $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp})$ and $\mathscr{G} \in \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp})$, then the induced map $\mathscr{F} \otimes \mathscr{G} \to \mathscr{F}' \otimes \mathscr{G}$ is also an L-equivalence in $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp})$. Consequently, the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ inherits the structure of a symmetric monoidal ∞-category, with respect to which L is a symmetric monoidal functor (Proposition HA.2.2.1.9).*

Proof. Apply Lemma 1.3.4.4 to the tensor product functor $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$. □

We will henceforth regard the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ as endowed with the symmetric monoidal structure of Proposition 1.3.4.6, for any ∞-topos $\mathcal{X}$. We will abuse terminology by referring to this symmetric monoidal structure as the *smash product* symmetric monoidal structure.

Proposition 1.3.4.7. *Let $\mathcal{X}$ be an ∞-topos, and let $L : \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ be a left adjoint to the inclusion. Regard $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp})$ as endowed with the t-structure induced by the natural t-structure on Sp. Then:*

(1) *The functor L is t-exact: that is, L carries $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp}_{\geqslant 0})$ into $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant 0}$ and $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp}_{\leqslant 0})$ into $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0}$.*

(2) *The smash product symmetric monoidal structure on* $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ *is compatible with the t-structure on* $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. *In other words, the full subcategory* $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})^{\mathrm{cn}} \subseteq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ *contains the unit object and is stable under tensor products.*

Proof. The construction of Lemma 1.3.4.3 shows that (for sufficiently large κ) we can factor L as the composition of a restriction functor $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Fun}(\mathcal{X}_0^{\mathrm{op}}, \mathrm{Sp})$ with the functor $\mathrm{Fun}(\mathcal{X}_0^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{P}(\mathcal{X}_0)) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ induced by a geometric morphism $g^* : \mathcal{P}(\mathcal{X}_0) \to \mathcal{X}$. Assertion (1) now follows from Remark 1.3.2.8. To prove (2), we show that if we are given a finite collection of connective objects $\{\mathscr{F}_i\}_{1 \leqslant i \leqslant n}$ of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$, then the tensor product $\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n$ is connective. Choose fiber sequences $\mathscr{F}'_i \to \mathscr{F}_i \to \mathscr{F}''_i$ in $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp})$, where $\mathscr{F}'_i \in \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp}_{\geqslant 0})$ and $\mathscr{F}''_i \in \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp}_{\leqslant -1})$. It follows from (1) that $L\mathscr{F}'_i \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant 0}$ and $L\mathscr{F}''_i \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant -1}$. We have fiber sequences

$$L\mathscr{F}'_i \to L\mathscr{F}_i \to L\mathscr{F}''_i$$

in $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Since $L\mathscr{F}_i \simeq \mathscr{F}_i$ is connective, we deduce that the map $L\mathscr{F}'_i \to L\mathscr{F}_i \simeq \mathscr{F}_i$ is an equivalence for every index i. Using Proposition 1.3.4.6, we deduce that the tensor product $\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n$ in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ can be written as $L(\mathscr{F}'_1 \otimes \cdots \otimes \mathscr{F}'_n)$. By virtue of (1), it will suffice to show that $\mathscr{F}'_1 \otimes \cdots \otimes \mathscr{F}'_n$ is a connective object of $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp})$, which follows because the smash product monoidal structure on Sp is compatible with its t-structure (Lemma HA.7.1.1.7). □

1.3.5 Sheaves of $\mathbb{E}_\infty$-Rings

We now study sheaves with values in the ∞-category CAlg of $\mathbb{E}_\infty$-rings.

Remark 1.3.5.1. The forgetful functor $\mathrm{CAlg} = \mathrm{CAlg}(\mathrm{Sp}) \to \mathrm{Sp}$ is conservative and preserves small limits (see Lemma HA.3.2.2.6 and Corollary HA.3.2.2.5). It follows that for any ∞-topos $\mathcal{X}$, we have a canonical equivalence of ∞-categories (even an isomorphism of simplicial sets) $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X}) \simeq \mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$.

Remark 1.3.5.2. Let $\mathcal{X}$ be an ∞-topos and $\mathscr{O} : \mathcal{X}^{\mathrm{op}} \to \mathrm{CAlg}$ a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Composing with the forgetful functor $\mathrm{CAlg} \to \mathrm{Sp}$, we obtain a sheaf of spectra on $\mathcal{X}$; we will generally abuse notation by denoting this sheaf of spectra also by $\mathscr{O}$. In particular, we can define homotopy groups $\pi_n \mathscr{O}$ as in Notation 1.3.2.3. These homotopy groups have a bit more structure in this case: $\pi_0 \mathscr{O}$ is a commutative ring object in the underlying topos of $\mathcal{X}$, while each $\pi_n \mathscr{O}$ has the structure of a $\pi_0 \mathscr{O}$-module.

Definition 1.3.5.3. Let $\mathcal{X}$ be an ∞-topos. We will say that a sheaf $\mathscr{O}$ of $\mathbb{E}_\infty$-rings on $\mathcal{X}$ is *connective* if it is connective when regarded as a sheaf of spectra on $\mathcal{X}$: that is, if the homotopy groups $\pi_n \mathscr{O}$ vanish for $n < 0$. We let $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}$ denote the full subcategory of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ spanned by the connective sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

Remark 1.3.5.4. Let $\mathcal{X}$ be an ∞-topos. Combining Proposition 1.3.4.7 with Remark HA.2.2.1.5 , we deduce that the inclusion $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}} \hookrightarrow \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ admits a right adjoint. In other words, if $\mathscr{O}$ is an arbitrary sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, then we can find a connective sheaf of $\mathbb{E}_\infty$-rings $\mathscr{O}'$ equipped with a map $\alpha : \mathscr{O}' \to \mathscr{O}$ having the following universal property: for every object $\mathscr{A} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}$, composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})}(\mathscr{A}, \mathscr{O}') \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})}(\mathscr{A}, \mathscr{O}).$$

In this case, we will say that *$\mathscr{O}'$ is a connective cover of $\mathscr{O}$*, or that α *exhibits $\mathscr{O}'$ as a connective cover of $\mathscr{O}$*. Moreover, the map α exhibits $\mathscr{O}'$ as a connective cover of $\mathscr{O}$ in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$; in particular, it induces isomorphisms

$$\pi_m \mathscr{O}' \simeq \begin{cases} \pi_m \mathscr{O} & \text{if } m \geqslant 0 \\ 0 & \text{if } m < 0. \end{cases}$$

We will generally denote the connective cover of $\mathscr{O}$ by $\tau_{\geqslant 0} \mathscr{O}$.

Definition 1.3.5.5. Let $\mathcal{X}$ be an ∞-topos, let $\mathscr{O}$ be a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and let $n \geqslant 0$ be an integer. We will say that $\mathscr{O}$ is *n-truncated* if the underlying spectrum-valued sheaf of $\mathscr{O}$ is n-truncated. We will say that $\mathscr{O}$ is *discrete* if it is 0-truncated. We let $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}_{\leqslant n}$ denote the full subcategory of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}$ spanned by the n-truncated objects of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}$.

Remark 1.3.5.6. Let $\mathcal{X}$ be an ∞-topos and let $n \geqslant 0$ be an integer. Then we can identify $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}_{\leqslant n}$ with the ∞-category of commutative algebra objects of the symmetric monoidal ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})^{\mathrm{cn}}_{\leqslant n}$. In particular, when $n = 0$, we can identify $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}_{\leqslant n}$ with the ordinary category of commutative ring objects of the underlying topos of $\mathcal{X}$ (see Proposition 1.3.2.7).

Combining Proposition 1.3.4.7 with Proposition HA.2.2.1.9 , we deduce that the inclusion functor

$$\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}_{\leqslant n} \hookrightarrow \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}_{\leqslant n}$$

admits a left adjoint. In other words, if $\mathscr{O}$ is an arbitrary connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, then we can find an n-truncated connective sheaf of $\mathbb{E}_\infty$-rings $\mathscr{O}'$ equipped with a map $\alpha : \mathscr{O} \to \mathscr{O}'$ having the following universal property: for every object $\mathscr{A} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}_{\leqslant n}$, composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})}(\mathscr{O}', \mathscr{A}) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})}(\mathscr{O}, \mathscr{A}).$$

In this case, we will say that *$\mathscr{O}'$ is an n-truncation of $\mathscr{O}$*, or that α *exhibits $\mathscr{O}'$ as an n-truncation of $\mathscr{O}$*. Moreover, the map α exhibits $\mathscr{O}'$ as an n-truncation of $\mathscr{O}$ in the ∞-category

$\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$; in particular, it induces isomorphisms

$$\pi_m \mathscr{O}' \simeq \begin{cases} \pi_m \mathscr{O} & \text{if } m \leqslant n \\ 0 & \text{if } m > n. \end{cases}$$

We will generally denote the n-truncation of $\mathscr{O}$ by $\tau_{\leqslant n} \mathscr{O}$.

Let $\mathcal{X}$ be an ∞-topos and let $\mathscr{F}$ be a spectrum-valued sheaf on $\mathcal{X}$. The condition that $\mathscr{F}$ be connective *does not* guarantee that $\mathscr{F}(U)$ is connective for each object $U \in \mathcal{X}$. Nevertheless, there is a close relationship between connective Sp-valued sheaves on $\mathcal{X}$ and $\mathrm{Sp}^{\mathrm{cn}}$-valued sheaves on $\mathcal{X}$:

Proposition 1.3.5.7. *Let $\mathcal{X}$ be an ∞-topos. Then composition with the truncation functor $\tau_{\geqslant 0} : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{cn}}$ induces an equivalence of ∞-categories $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})^{\mathrm{cn}} \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{X})$.*

We will deduce Proposition 1.3.5.7 from the following more general principle:

Proposition 1.3.5.8. *Let $\mathcal{C}$ be a compactly generated presentable ∞-category. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory which is closed under the formation of colimits and which is generated under small colimits by compact objects of $\mathcal{C}$. Let $\mathcal{X}$ be an ∞-topos. Then:*

(1) *The ∞-category $\mathcal{C}_0$ is presentable and compactly generated.*

(2) *The inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$ admits a right adjoint g which commutes with filtered colimits.*

(3) *Composition with g determines a functor $G : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}_0}(\mathcal{X})$.*

(4) *The functor G admits a fully faithful left adjoint F.*

Remark 1.3.5.9. In the situation of Proposition 1.3.5.8, an object $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ belongs to the essential image of the full faithful embedding $\mathcal{S}\mathrm{hv}_{\mathcal{C}_0}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ if and only if the canonical map $G(\mathscr{F}) \to \mathscr{F}$ is an L-equivalence in $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$, where L denotes a left adjoint to the inclusion $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$.

Proof of Proposition 1.3.5.7. Let $\mathrm{Sp}^{\mathrm{cn}}$ denote the full subcategory of Sp spanned by the connective spectra. Then $\mathrm{Sp}^{\mathrm{cn}}$ is stable under small colimits in Sp, and is generated under small colimits by the sphere spectrum $S \in \mathrm{Sp}^{\mathrm{cn}}$ (which is a compact object of the ∞-category Sp). Consequently, Proposition 1.3.5.8 supplies a fully faithful embedding $F : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ for every ∞-topos $\mathcal{X}$, which is right homotopy inverse to the functor $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{X})$ given by composition with $\tau_{\geqslant 0} : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{cn}}$. To complete the proof, it will suffice to show that $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})^{\mathrm{cn}}$ is the essential image of the functor F.

Let $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$, so that we have a fiber sequence $\tau_{\geqslant 0} \mathscr{F} \xrightarrow{\phi} \mathscr{F} \to \tau_{\leqslant -1} \mathscr{F}$ in the ∞-category $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp})$. Let $L : \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ be a left adjoint to the inclusion.

According to Remark 1.3.5.9, the object $\mathscr{F}$ belongs to the essential image of F if and only if $L(\phi)$ is an equivalence. Since the functor L is t-exact, this is equivalent to the requirement that $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})^{\mathrm{cn}}$. □

Many variations on Proposition 1.3.5.7 are possible:

Proposition 1.3.5.10. *Let $\mathcal{X}$ be an ∞-topos. Then composition with the functor $\tau_{\geqslant 0}$: $\mathrm{CAlg} \to \mathrm{CAlg}^{\mathrm{cn}}$ induces an equivalence of ∞-categories $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}} \to \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathcal{X})$.*

It is possible to deduce Proposition 1.3.5.10 from the fact the the equivalence of Proposition 1.3.5.7 respects the symmetric monoidal structures on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})^{\mathrm{cn}}$ and $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{X})$. However, we will give an alternate argument which appeals to Proposition 1.3.5.8.

Proof of Proposition 1.3.5.10. The ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ is a colocalization of CAlg, which is generated under small colimits by the compact object $\mathrm{Sym}^*(S)$, where S denotes the sphere spectrum and $\mathrm{Sym}^* : \mathrm{Sp} \to \mathrm{CAlg}$ denotes a left adjoint to the forgetful functor. Proposition 1.3.5.8 gives a fully faithful embedding $F : \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ for every ∞-topos $\mathcal{X}$, which is right homotopy inverse to the functor $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathcal{X})$ given by composition with the functor $\tau_{\geqslant 0}\, \mathrm{CAlg} \to \mathrm{CAlg}$. To complete the proof, it will suffice to show that the essential image of F coincides with $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}$.

Let $\mathscr{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and let $\tau_{\geqslant 0}\,\mathscr{O} \in \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{CAlg})$ be the presheaf of $\mathbb{E}_\infty$-rings obtained by pointwise passage to the connective cover. Let $\mathscr{O}' \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ be a sheafification of the presheaf $\tau_{\geqslant 0}\,\mathscr{O}$, so that the evident map $\tau_{\geqslant 0}\,\mathscr{O} \to \mathscr{O}$ induces a map of sheaves $\alpha : \mathscr{O}' \to \mathscr{O}$. According to Remark 1.3.5.9, the sheaf $\mathscr{O}$ belongs to the essential image of F if and only if α is an equivalence. Let $u : \mathrm{CAlg} \to \mathrm{Sp}$ denote the forgetful functor. Since u preserves small limits, composition with u induces a forgetful functor $U : \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Since u is conservative, the functor U is also conservative, so that α is an equivalence if and only if $U(\alpha)$ is an equivalence. Since u preserves filtered colimits, Lemma 1.3.4.4 implies that $U(\mathscr{O}')$ can be identified with a sheafification of $u \circ \tau_{\geqslant 0}\,\mathscr{O} \simeq \tau_{\geqslant 0}(u \circ \mathscr{O})$. The proof of Proposition 1.3.5.7 guarantees that $U(\alpha)$ is an equivalence if and only if $U(\mathscr{O})$ is connective as a sheaf of spectra: that is, if and only if $\mathscr{O}$ belongs to $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}$. □

Proof of Proposition 1.3.5.8. Since $\mathcal{C}_0$ is stable under small colimits in $\mathcal{C}$, the inclusion $i : \mathcal{C}_0 \subseteq \mathcal{C}$ preserves small colimits so that i admits a right adjoint $g : \mathcal{C} \to \mathcal{C}_0$ by Corollary HTT.5.5.2.9 . Let $\mathcal{D} \subseteq \mathcal{C}_0$ be the full subcategory spanned by those objects of $\mathcal{C}_0$ which are compact in $\mathcal{C}$. Any such object is automatically compact in $\mathcal{C}_0$, so we have a fully faithful embedding $q : \mathrm{Ind}(\mathcal{D}) \to \mathcal{C}_0$ (Proposition HTT.5.3.5.11). Since $\mathcal{C}_0$ is generated under small colimits by objects of $\mathcal{D}$, we deduce that q is an equivalence of ∞-categories; this proves (1). Moreover, it shows that the collection of compact objects in $\mathcal{C}_0$ is an idempotent completion of $\mathcal{D}$; since $\mathcal{D}$ is already idempotent complete, we deduce that every compact object of $\mathcal{C}_0$ is also compact in $\mathcal{C}$. Assertion (2) now follows from Proposition HTT.5.5.7.2 . Assertion (3) is obvious (since g preserves small limits; see Proposition HTT.5.2.3.5).

Let $L : \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ be a left adjoint to the inclusion, and define L_0 similarly. We observe that G is equivalent to the composition

$$\mathcal{S}\mathrm{hv}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}) \xrightarrow{G'} \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}_0) \xrightarrow{L_0} \mathcal{S}\mathrm{hv}_{\mathcal{C}_0}(\mathcal{X}),$$

where G' is given by composition with g. It follows that G admits a left adjoint F, which can be described as the composition

$$\mathcal{S}\mathrm{hv}(\mathcal{C}) \xleftarrow{L} \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}) \supseteq \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}_0) \supseteq \mathcal{S}\mathrm{hv}_{\mathcal{C}_0}(\mathcal{X}).$$

To complete the proof, it suffices to show that F is fully faithful. In other words, we wish to show that for every object $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathcal{C}_0}(\mathcal{X})$, the unit map $\mathscr{F} \to (G \circ F)(\mathscr{F})$ is an equivalence. In other words, we wish to show that the map $\alpha : \mathscr{F} \to L\mathscr{F}$ becomes an equivalence after applying the functor G'. Since $G'(\mathscr{F}) \simeq \mathscr{F}$ and $G'(L\mathscr{F})$ belong to $\mathcal{S}\mathrm{hv}_{\mathcal{C}_0}(\mathcal{X})$, this is equivalent to the requirement that $G'(\alpha)$ is an L_0-equivalence in the ∞-category $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}_0)$. This follows from (3) and Lemma 1.3.4.4, since α is an L-equivalence in $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$. □

1.4 Spectral Deligne-Mumford Stacks

In §1.1 and §1.2 we introduced two different generalizations of the notion of scheme: the notion of spectral scheme (Definition 1.1.2.8) and the notion of Deligne-Mumford stack (Definition 1.2.4.1). These two generalizations serve rather different purposes:

- The ∞-category SpSch of spectral schemes can viewed roughly as a "left-derived" version of the category of schemes. More precisely, though the category Sch and the ∞-category SpSch both admit fiber products, the inclusion Sch $\hookrightarrow$ SpSch does *not* preserve fiber products. If $f : X \to Y$ and $f' : X' \to Y$ are morphisms of schemes, then we can form a fiber product $(Z, \mathscr{O}_Z)$ of X and X' over Y in the ∞-category SpSch, whose underlying ordinary scheme $(Z, \pi_0 \mathscr{O}_Z)$ is the fiber product $X \times_Y X'$ in the category of schemes. However, the structure sheaf $\mathscr{O}_Z$ need not be 0-truncated, so that $(Z, \mathscr{O}_Z)$ need not be an ordinary scheme. This is not a bug, but a feature: the sheaves $\pi_n \mathscr{O}_Z$ carry useful geometric information which can detect (and help correct for) the failure of the maps f and f' to be transversal with respect to one another.

- The collection of Deligne-Mumford stacks is organized into a 2-category DM which can be regarded as a "right-derived" enlargement of the category of schemes. More precisely, there is a fully faithful embedding Sch $\hookrightarrow \mathcal{C}$ which is not compatible with certain very basic colimit constructions, such as passage to quotients under the action of a finite group. If X is a scheme equipped with an action of a finite group G, then one can consider either the quotient X/G in the category of schemes (which exists under mild hypotheses on X), or the stack-theoretic quotient $X//G$ in the 2-category $\mathcal{C}$. The usual quotient X/G can be recovered as the "coarse moduli space" of the Deligne-Mumford stack $X//G$, but they are generally not the same unless G acts freely on X. Once again, this should be regarded as a feature rather than a bug: the stack-theoretic quotient $X//G$ carries useful geometric information about the subgroups of G which stabilize points of X; this information is forgotten when passing to the usual quotient X/G.

For some purposes, it is useful to enlarge the category of schemes simultaneously in both of these directions. To accomplish this, we will introduce the notion of a *spectral Deligne-Mumford stack* (Definition 1.4.4.2). Roughly speaking, the definition of a spectral Deligne-Mumford stacks is obtained by modifying the definition of a scheme $(X, \mathscr{O}_X)$ in three different ways:

(*i*) For every topological space X, the ∞-category $\mathcal{S}\mathrm{hv}(X)$ of sheaves of spaces on X is an ∞-topos. Moreover, if the topological space X is *sober* (that is, if every irreducible closed subset of X has a unique generic point), then we can recover X from $\mathcal{S}\mathrm{hv}(X)$: the points $x \in X$ can be identified with isomorphism classes of geometric morphisms $x^* : \mathcal{S}\mathrm{hv}(X) \to \mathcal{S}$, and open subsets of X can be identified with subobjects of the unit object $1 \in \mathcal{S}\mathrm{hv}(X)$. In other words, the space X and the ∞-topos $\mathcal{S}\mathrm{hv}(X)$ are interchangable: either one canonically determines the other. The situation described above can be summarized by saying that we can regard the theory of ∞-topoi as a *generalization* of the classical theory of topological spaces (more precisely, of the theory of sober topological spaces). For this reason, we opt to dispense with topological spaces altogether and work instead with a general ∞-topos $\mathcal{X}$.

(*ii*) In place of the sheaf $\mathscr{O}_X$ of commutative rings on X, we consider an arbitrary sheaf $\mathscr{O}_{\mathcal{X}}$ of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

(*iii*) In place of the requirement that $(X, \mathscr{O}_X)$ be locally isomorphic to the spectrum of a commutative ring, we require that $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be locally equivalent to $\mathrm{Sp\acute{e}t}\, A$, where A is an $\mathbb{E}_\infty$-ring and $\mathrm{Sp\acute{e}t}\, A$ denotes its spectrum with respect to the étale topology (see Proposition 1.4.2.4).

1.4.1 Spectrally Ringed ∞-Topoi

We begin with a discussion of CAlg-valued rings on ∞-topoi.

Definition 1.4.1.1. A *spectrally ringed ∞-topos* is a pair $(\mathcal{X}, \mathscr{O})$, where $\mathcal{X}$ is an ∞-topos and $\mathscr{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

Remark 1.4.1.2. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ be a spectrally ringed ∞-topos. We will often refer to $\mathcal{O}$ as the *structure sheaf* of X. We will often denote the structure sheaf $\mathcal{O}$ by $\mathcal{O}_{\mathcal{X}}$ or $\mathcal{O}_{\mathsf{X}}$ (the latter notation is convenient when we wish to distinguish between spectrally ringed ∞-topoi having the same underlying ∞-topos).

Construction 1.4.1.3. Precomposition with a geometric morphism of ∞-topoi $f^* : \mathcal{X} \to \mathcal{Y}$ induces a pushforward functor $f_* : \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})$. We may therefore view the construction $\mathcal{X} \mapsto \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{op}}$ as determining a functor $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}} : \infty\mathcal{T}\mathrm{op} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$, where $\infty\mathcal{T}\mathrm{op}$ denotes the ∞-category of ∞-topoi. This functor classifies a coCartesian fibration $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}} \to \infty\mathcal{T}\mathrm{op}$. More informally, the objects of $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ are spectrally ringed ∞-topoi $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and a morphism from $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ to $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ is given by a pair (f^*, ϕ), where $f_* : \mathcal{X} \to \mathcal{Y}$ is a geometric morphism of ∞-topoi and $\phi : \mathcal{O}_{\mathcal{Y}} \to f_* \mathcal{O}_{\mathcal{X}}$ is a morphism of sheaves of $\mathbb{E}_{\infty}$-rings on $\mathcal{Y}$. We will refer to $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ as the *∞-category of spectrally ringed ∞-topoi.*

Remark 1.4.1.4. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}^{\heartsuit}$ denote the underlying topos of $\mathcal{X}$. For any sheaf of $\mathbb{E}_{\infty}$-rings $\mathcal{O}_{\mathcal{X}}$ on $\mathcal{X}$, we can regard $\pi_0 \mathcal{O}_{\mathcal{X}}$ as a commutative ring object of $\mathcal{X}^{\heartsuit}$. We will refer to $(\mathcal{X}^{\heartsuit}, \pi_0 \mathcal{O}_{\mathcal{X}})$ as the *underlying ringed topos* of $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. The construction

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto (\mathcal{X}^{\heartsuit}, \pi_0 \mathcal{O}_{\mathcal{X}})$$

determines a functor from the homotopy 2-category of $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ to the 2-category $1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^{\heartsuit}}$ of ringed topoi.

Remark 1.4.1.5. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a ringed topos, and let $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})$ denote the 1-localic ∞-topos associated to $\mathcal{X}$ (see §HTT.6.4.5). Remark 1.3.5.6 supplies an equivalence from the category of commutative ring objects of $\mathcal{X}$ to the ∞-category of connective 0-truncated sheaves of $\mathbb{E}_{\infty}$-rings on $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})$. We let $\mathcal{O}$ denote the image of $\mathcal{O}_{\mathcal{X}}$ under this equivalence. Then $(\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}), \mathcal{O})$ is a spectrally ringed ∞-topos. Moreover, the construction $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto (\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}), \mathcal{O})$ determines a fully faithful embedding from the ∞-category of ringed topoi (obtained from the 2-category $1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^{\heartsuit}}$ by discarding noninvertible 2-morphisms) to the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ of spectrally ringed ∞-topoi. The essential image of this fully faithful embedding consists of those spectrally ringed ∞-topoi $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ where $\mathcal{Y}$ is 1-localic and the structure sheaf $\mathcal{O}_{\mathcal{Y}}$ is connective and 0-truncated.

Remark 1.4.1.6. For every topological space X, Example 1.3.1.2 supplies an equivalence of ∞-categories $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{S}\mathrm{hv}_{\mathcal{S}}(X)) \to \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)$, which depends functorially on X. It follows that there is a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{T}\mathrm{op}_{\mathrm{CAlg}} & \xrightarrow{\overline{\phi}} & \infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}} \\ \downarrow & & \downarrow \\ \mathcal{T}\mathrm{op} & \xrightarrow{\phi} & \infty\mathcal{T}\mathrm{op}, \end{array}$$

where $\phi : \mathcal{T}\mathrm{op} \to \infty\mathcal{T}\mathrm{op}$ is the functor which carries a topological space X to its associated ∞-topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(X)$.

1.4.2 The Étale Spectrum of an $\mathbb{E}_\infty$-Ring

In §1.2, we constructed the *étale spectrum* $\mathrm{Sp\acute{e}t}\, R$ of a commutative ring R (Definition 1.2.2.5). We now introduce an analogous construction in the setting of $\mathbb{E}_\infty$-rings.

Definition 1.4.2.1. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{O}_{\mathcal{X}}$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. We will say that $\mathcal{O}_{\mathcal{X}}$ is *local* if $\pi_0 \mathcal{O}_{\mathcal{X}}$ is local, when regarded as a commutative ring object of the topos $\mathcal{X}^{\heartsuit}$ (see Definition 1.2.1.4). We will say that $\mathcal{O}_{\mathcal{X}}$ is *strictly Henselian* if $\pi_0 \mathcal{O}_{\mathcal{X}}$ is strictly Henselian, in the sense of Definition 1.2.2.5.

If $f : \mathcal{O} \to \mathcal{O}'$ is a morphism of CAlg-valued sheaves on $\mathcal{X}$, we will say that f is *local* if it induces a local morphism $\pi_0 \mathcal{O} \to \pi_0 \mathcal{O}'$ of commutative ring objects of the topos $\mathcal{X}^{\heartsuit}$, in the sense of Definition 1.2.1.4.

We let $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ denote the subcategory of $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ whose objects are spectrally ringed ∞-topoi $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ for which $\mathcal{O}_{\mathcal{X}}$ is local, and whose morphisms are maps $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ for which the associated map $f^* \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ is a local morphism of CAlg-valued sheaves on $\mathcal{X}$. We let $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ denote the full subcategory of $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ spanned by those pairs $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where $\mathcal{O}_{\mathcal{X}}$ is strictly Henselian. We will say that a spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is *local* if it belongs to $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$, and *strictly Henselian* if it belongs to $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$.

Remark 1.4.2.2. Let $(X, \mathcal{O}_X)$ be a spectrally ringed space. If $\mathcal{O}_X$ is strictly Henselian (when regarded as a CAlg-valued sheaf on the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(X)$), then $\mathcal{O}_X$ is local (in the sense of Definition 1.1.5.3).

The equivalence of ∞-categories $\mathrm{CAlg}^{\mathrm{op}} \simeq \infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}} \times_{\infty\mathcal{T}\mathrm{op}} \{\mathcal{S}\}$ determines a fully faithful embedding $\mathrm{CAlg}^{\mathrm{op}} \hookrightarrow \infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$, which carries each $\mathbb{E}_\infty$-ring A to the spectrally ringed ∞-topos $(\mathcal{S}, \mathcal{O}_A)$ where $\mathcal{O}_A \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{S})$ is characterized by existence of an equivalence $\mathcal{O}_A(*) \simeq A$. This embedding admits a left adjoint, which carries a pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ to the $\mathbb{E}_\infty$-ring $\mathcal{O}_{\mathcal{X}}(\mathbf{1})$, where $\mathbf{1}$ is a final object of $\mathcal{X}$. We will denote this left adjoint by $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto \Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}})$, and refer to it as the *global sections functor*.

We will need the following analogue of Remark 1.1.5.6:

Proposition 1.4.2.3. *The global sections functor*

$$\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}} \to \mathrm{CAlg}^{\mathrm{op}}$$

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto \Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}})$$

admits a right adjoint $\mathrm{Sp\acute{e}t} : \mathrm{CAlg}^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$.

Proposition 1.4.2.3 asserts that for every $\mathbb{E}_\infty$-ring R, there exists a strictly Henselian spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ and a map $\theta : R \to \Gamma(\mathcal{X}; \mathcal{O}_\mathcal{X})$ with the following universal property: for every strictly Henselian spectrally ringed ∞-topos $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$, composition with θ induces a homotopy equivalence

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{Y}, \mathcal{O}_\mathcal{Y}), (\mathcal{X}, \mathcal{O}_\mathcal{X})) \to \mathrm{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{Y}; \mathcal{O}_\mathcal{Y})).$$

The spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is uniquely determined up to equivalence and depends functorially on R. A more explicit description of $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is given by the following:

Proposition 1.4.2.4. *Let R be an $\mathbb{E}_\infty$-ring, and let $\mathcal{O} : \mathrm{CAlg}^{\acute{e}t}_R \to \mathrm{CAlg}$ denote the forgetful functor. Then:*

(1) *The functor $\mathcal{O}$ is a sheaf with respect to the étale topology of Notation* **??**.

(2) *When regarded as a sheaf of $\mathbb{E}_\infty$-rings on the ∞-topos $\mathcal{S}\mathrm{hv}^{\acute{e}t}_R$ (see Proposition 1.3.1.7), the sheaf $\mathcal{O}$ is strictly Henselian.*

By construction, we have a canonical equivalence

$$\alpha : \Gamma(\mathcal{S}\mathrm{hv}^{\acute{e}t}_R; \mathcal{O}) \simeq \mathcal{O}(R) = R.$$

(3) *Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be an arbitrary object of $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$. Then composition with α induces a homotopy equivalence*

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{X}, \mathcal{O}_\mathcal{X}), (\mathcal{S}\mathrm{hv}^{\acute{e}t}_R, \mathcal{O})) \to \mathrm{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{X}; \mathcal{O}_\mathcal{X})).$$

We postpone the proof for the moment.

Definition 1.4.2.5. Let R be an $\mathbb{E}_\infty$-ring. We let $\mathrm{Sp\acute{e}t}\, R$ denote the spectrally ringed ∞-topos $(\mathcal{S}\mathrm{hv}^{\text{ét}}_R, \mathcal{O})$ appearing in Proposition 1.4.2.4. We will refer to $\mathrm{Sp\acute{e}t}\, R$ as the *étale spectrum of R*.

Warning 1.4.2.6. Let R be a commutative ring. We now have two definitions for the étale spectrum of R:

(a) We can regard R as an ordinary commutative ring, and consider the ringed topos $(\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}^{\text{ét}}_R), \mathcal{O}_0)$ introduced in Definition 1.2.3.3.

(b) We can regard R as a discrete $\mathbb{E}_\infty$-ring, and consider the spectrally ringed ∞-topos $(\mathcal{S}\mathrm{hv}^{\text{ét}}_R, \mathcal{O})$ of Definition 1.4.2.5.

However, there should be little risk of confusion: these two mathematical objects are essentially identical with one another. More precisely, the spectrally ringed ∞-topos $(\mathcal{S}\mathrm{hv}^{\text{ét}}_R, \mathcal{O})$ is the image of $(\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}^{\text{ét}}_R), \mathcal{O}_0)$ under the fully faithful embedding described in Remark **??**.

Remark 1.4.2.7. It follows from the third assertion of Proposition 1.4.2.4 that the construction $R \mapsto \operatorname{Spét} R$ can be regarded as a right adjoint to the global sections functor $\Gamma : \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}} \to \mathrm{CAlg}^{\mathrm{op}}$; in particular, the étale spectrum $\operatorname{Spét} R$ depends functorially on R.

1.4.3 Solution Sheaves

We now introduce some terminology which will be useful for the proof of Proposition 1.4.2.4.

Notation 1.4.3.1. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectrally ringed ∞-topos. For each $\mathbb{E}_\infty$-ring R, the construction $(U \in \mathcal{X}) \mapsto \mathrm{Map}_{\mathrm{CAlg}}(R, \mathcal{O}_{\mathcal{X}}(U))$ determines a $\mathcal{S}$-valued sheaf on $\mathcal{X}$. We let $\mathcal{S}\mathrm{ol}_R(\mathcal{O}_{\mathcal{X}})$ denote an object of $\mathcal{X}$ which represents this functor.

Example 1.4.3.2. Let $R = \mathrm{Sym}^*(S^n)$ denote the free $\mathbb{E}_\infty$-ring on a single generator in degree n. If $\mathcal{O}_{\mathcal{X}}$ is a sheaf of $\mathbb{E}_\infty$-rings on an ∞-topos $\mathcal{X}$, then there is a canonical equivalence $\mathcal{S}\mathrm{ol}_R(\mathcal{O}_{\mathcal{X}}) \simeq \Omega^{\infty+n}\,\mathcal{O}_{\mathcal{X}}$ in the ∞-topos $\mathcal{X}$.

Remark 1.4.3.3 (Functoriality in R)**.** Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{O}_{\mathcal{X}}$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then the construction $R \mapsto \mathcal{S}\mathrm{ol}_R(\mathcal{O}_{\mathcal{X}})$ is contravariantly functorial in R. Moreover, it carries colimits (in the ∞-category CAlg of $\mathbb{E}_\infty$-rings) to limits (in the ∞-topos $\mathcal{X}$).

Remark 1.4.3.4 (Functoriality in $\mathcal{O}_{\mathcal{X}}$)**.** Let $\mathcal{X}$ be an ∞-topos and let R be an $\mathbb{E}_\infty$-ring. Then the construction $\mathcal{O}_{\mathcal{X}} \mapsto \mathcal{S}\mathrm{ol}_R(\mathcal{O}_{\mathcal{X}})$ determines a functor $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}(\mathcal{X})$ which preserves small limits.

Remark 1.4.3.5 (Functoriality in $\mathcal{X}$)**.** Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi and let R be an $\mathbb{E}_\infty$-ring. Then the diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X}) & \xrightarrow{\mathcal{S}\mathrm{ol}_R} & \mathcal{X} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y}) & \xrightarrow{\mathcal{S}\mathrm{ol}_R} & \mathcal{Y} \end{array}$$

commutes (up to canonical equivalence).

Remark 1.4.3.6. In the situation of Notation 1.4.3.1, suppose that R is connective. Then for any object $\mathcal{O}_{\mathcal{X}} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$, the canonical map $\mathcal{S}\mathrm{ol}_R(\tau_{\geq 0}\,\mathcal{O}_{\mathcal{X}}) \to \mathcal{S}\mathrm{ol}_R(\mathcal{O}_{\mathcal{X}})$ is an equivalence in $\mathcal{X}$.

It follows from Remark 1.4.3.5 that for any geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{Y}$ and any $\mathbb{E}_\infty$-ring R, there is a canonical natural transformation of functors $f^*\,\mathcal{S}\mathrm{ol}_R \to \mathcal{S}\mathrm{ol}_R\,f^*$ from $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})$ to $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$.

Lemma 1.4.3.7. *Let R be a compact object of* CAlg*. Then, for any geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{Y}$ and any sheaf $\mathscr{O}_{\mathcal{Y}}$ of $\mathbb{E}_\infty$-rings on $\mathcal{Y}$, the canonical map $f^* \mathcal{S}\mathrm{ol}_R(\mathscr{O}_{\mathcal{Y}}) \to \mathcal{S}\mathrm{ol}_R(f^* \mathscr{O}_{\mathcal{Y}})$ is an equivalence in $\mathcal{X}$.*

Proof. Let $\mathcal{C} \subseteq \mathrm{CAlg}_R$ be the full subcategory spanned by those objects R for which the natural map $f^* \mathcal{S}\mathrm{ol}_R \to \mathcal{S}\mathrm{ol}_R f^*$ is an equivalence of functors from $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})$ to $\mathcal{X}$. Then $\mathcal{C}$ is closed under retracts, and it follows from Remark 1.4.3.3 (together with the left exactness of f^*) that $\mathcal{C}$ is closed under finite colimits in CAlg_R. To prove that $\mathcal{C}$ contains all compact objects of CAlg_R, it will suffice to show that it contains all free algebras of the form $\mathrm{Sym}^*(S^n)$, which follows from Example 1.4.3.2. □

Lemma 1.4.3.8. *Let $f : A \to B$ be an étale morphism between connective $\mathbb{E}_\infty$-rings, and let $\mathscr{O}_{\mathcal{X}}$ be a connective sheaf of $\mathbb{E}_\infty$-rings on an ∞-topos $\mathcal{X}$. Then the diagram*

$$\begin{array}{ccc} \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_B(\pi_0 \mathscr{O}_{\mathcal{X}}) \\ \downarrow & & \downarrow \\ \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\pi_0 \mathscr{O}_{\mathcal{X}}) \end{array}$$

is a pullback square in $\mathcal{X}$.

Proof. Using Proposition B.1.1.3, we can choose a pushout diagram

$$\begin{array}{ccc} A_0 & \longrightarrow & A \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ B_0 & \longrightarrow & B \end{array}$$

in CAlg, where f_0 is étale and the $\mathbb{E}_\infty$-rings A_0 and B_0 are compact and connective. Using Remark 1.4.3.3, we can replace f by f_0 and thereby reduce to the case where A and B are compact. Because $\mathcal{X}$ is an ∞-topos, we can choose a small ∞-category $\mathcal{C}$ and a geometric morphism $\phi_* : \mathcal{X} \to \mathcal{P}(\mathcal{C})$ which exhibits $\mathcal{X}$ as a left exact localization of $\mathcal{P}(\mathcal{C})$. We then have equivalences

$$\mathscr{O}_{\mathcal{X}} \simeq \phi^*(\tau_{\geqslant 0}\phi_* \mathscr{O}_{\mathcal{X}}) \qquad \pi_0 \mathscr{O}_{\mathcal{X}} \simeq \phi^*(\pi_0 \phi_* \mathscr{O}_{\mathcal{X}}).$$

Using Lemma 1.4.3.7, we can replace $\mathscr{O}_{\mathcal{X}}$ by $\tau_{\geqslant 0}(\phi_* \mathscr{O}_{\mathcal{X}})$ and thereby reduce to the case where $\mathcal{X} = \mathcal{P}(\mathcal{C})$ is an ∞-category of presheaves. In this case, we are reduced to proving that for every object $C \in \mathcal{C}$, the diagram of mapping spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}}(B, \mathscr{O}_{\mathcal{X}}(C)) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}}(B, \pi_0 \mathscr{O}_{\mathcal{X}}(C)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}}(A, \mathscr{O}_{\mathcal{X}}(C)) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}}(A, \pi_0 \mathscr{O}_{\mathcal{X}}(C)) \end{array}$$

is a pullback square, which follows from Theorem HA.7.5.4.2 . □

Lemma 1.4.3.9. *(a) Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a spectrally ringed ∞-topos. Then $\mathcal{O}_\mathcal{X}$ is strictly Henselian if and only if, for every $\mathbb{E}_\infty$-ring A and every faithfully flat étale morphism $A \to \prod_{1 \leqslant i \leqslant n} A_i$, the induced map*

$$\coprod_{1 \leqslant i \leqslant n} \mathcal{S}\mathrm{ol}_{A_i}(\mathcal{O}_\mathcal{X}) \to \mathcal{S}\mathrm{ol}_A(\mathcal{O}_\mathcal{X})$$

is an effective epimorphism in $\mathcal{X}$.

(b) Let $f : (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ be a morphism of spectrally ringed ∞-topoi, where $\mathcal{O}_\mathcal{X}$ and $\mathcal{O}_\mathcal{Y}$ are strictly Henselian. Then f is a morphism in $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ if and only if, for every étale morphism of $\mathbb{E}_\infty$-rings $A \to B$, the associated diagram

$$\begin{array}{ccc} f^* \mathcal{S}\mathrm{ol}_B(\mathcal{O}_\mathcal{Y}) & \longrightarrow & \mathcal{S}\mathrm{ol}_B(\mathcal{O}_\mathcal{X}) \\ \downarrow & & \downarrow \\ f^* \mathcal{S}\mathrm{ol}_A(\mathcal{O}_\mathcal{Y}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\mathcal{O}_\mathcal{X}). \end{array}$$

Proof. We first prove the "only if" direction of (a). Assume that $\mathcal{O}_\mathcal{X}$ is strictly Henselian and that we are given a collection of étale morphisms $\{A \to A_i\}_{1 \leqslant i \leqslant n}$ for which the induced map $A \to \prod_{1 \leqslant i \leqslant n} A_i$ is faithfully flat. We wish to prove that the induced map

$$\coprod_{1 \leqslant i \leqslant n} \mathcal{S}\mathrm{ol}_{A_i}(\mathcal{O}_\mathcal{X}) \to \mathcal{S}\mathrm{ol}_A(\mathcal{O}_\mathcal{X})$$

is an effective epimorphism. Note that each of the maps $A \to A_i$ is flat, and therefore fits into a pushout square of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} \tau_{\geqslant 0} A & \longrightarrow & \tau_{\geqslant 0} A_i \\ \downarrow & & \downarrow \\ A & \longrightarrow & A_i. \end{array}$$

Using Remark 1.4.3.3, we obtain a pullback diagram

$$\begin{array}{ccc} \coprod_{1 \leqslant i \leqslant n} \mathcal{S}\mathrm{ol}_{A_i}(\mathcal{O}_\mathcal{X}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\mathcal{O}_\mathcal{X}) \\ \downarrow & & \downarrow \\ \coprod_{1 \leqslant i \leqslant n} \mathcal{S}\mathrm{ol}_{\tau_{\geqslant 0} A_i}(\mathcal{O}_\mathcal{X}) & \longrightarrow & \mathcal{S}\mathrm{ol}_{\tau_{\geqslant 0} A}(\mathcal{O}_\mathcal{X}). \end{array}$$

Consequently, to prove the the upper horizontal map in this diagram is an effective epimorphism, it will suffice to show that the lower horizontal map is an effective epimorphism. We may therefore replace A by $\tau_{\geqslant 0} A$ (and each A_i with $\tau_{\geqslant 0} A_i$) and thereby reduce to the case where A is connective. In this case, there is no loss of generality in assuming that $\mathscr{O}_{\mathcal{X}}$ is connective (Remark 1.4.3.6). Lemma 1.4.3.8 now supplies a pullback diagram

$$\begin{array}{ccc} \coprod_{1 \leqslant i \leqslant n} \mathcal{S}\mathrm{ol}_{A_i}(\mathscr{O}_{\mathcal{X}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}}) \\ \downarrow & & \downarrow \\ \coprod_{1 \leqslant i \leqslant n} \mathcal{S}\mathrm{ol}_{A_i}(\pi_0 \mathscr{O}_{\mathcal{X}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\pi_0 \mathscr{O}_{\mathcal{X}}), \end{array}$$

where the bottom horizontal map is an effective epimorphism (between discrete objects of $\mathcal{X}$) by virtue of our assumption that $\pi_0 \mathscr{O}_{\mathcal{X}}$ is strictly Henselian.

We now prove the "if" direction of (a). Assume that $\mathscr{O}_{\mathcal{X}}$ has the property described by (a); we wish to show that for every commutative ring R and every finite collection of étale morphisms $\{R \to R_i\}_{1 \leqslant i \leqslant n}$ for which the induced map $R \to \prod R_i$ is faithfully flat, the induced map $\amalg \mathcal{S}\mathrm{ol}_{R_i}(\pi_0 \mathscr{O}_{\mathcal{X}}) \to \mathcal{S}\mathrm{ol}_R(\pi_0 \mathscr{O}_{\mathcal{X}})$ is an effective epimorphism (between discrete objects of $\mathcal{X}$). Writing R as a direct limit of its finitely generated subrings, we may assume without loss of generality that R is finitely generated: that is, there exists a surjection of commutative rings $P \to R$, where P is a polynomial ring over $\mathbf{Z}$. Using the structure theory of étale morphisms of commutative rings (see Proposition B.1.1.3), we can lift each R_i to an étale P-algebra P_i. Let $U \subseteq \operatorname{Spec} P$ be the union of the images of the maps $\operatorname{Spec} P_i \to \operatorname{Spec} P$. Since étale morphisms have open images, the set U is open with respect to the Zariski topology. It is clearly quasi-compact, so we can choose a collection of elements $t_1, \ldots, t_k \in P$ for which U is covered by the open subsets $\operatorname{Spec} P[t_j^{-1}] \subseteq \operatorname{Spec} P$. Because the map $R \to \prod R_i$ is faithfully flat, the map $\operatorname{Spec} R \to \operatorname{Spec} P$ factors through U. We may therefore choose coefficients $\overline{c}_j \in R$ for which the sum $\overline{c}_1 t_1 + \cdots + \overline{c}_k t_k = 1$ in R; here we abuse notation by identifying each t_j with its image in R. Using the surjectivity of the map $P \to R$, we can lift each $\overline{c}_j$ to an element $c_j \in P$. Let $u = c_1 t_1 + \cdots + c_k t_k$. Then the map $P[u^{-1}] \to \prod P_i[u^{-1}]$ is faithfully flat and the map $P \to R$ factors through $P[u^{-1}]$. We may therefore replace R by $P[u^{-1}]$ and thereby reduce to the case where the commutative ring R has the form $\mathbf{Z}[x_1, \ldots, x_m][u^{-1}]$ for some $u \in \mathbf{Z}[x_1, \ldots, x_m]$.

Let $\overline{R}$ denote the $\mathbb{E}_\infty$-ring given by $S\{x_1, \ldots, x_m\}[u^{-1}]$, so that $R = \pi_0 \overline{R}$. Since the composite map

$$\coprod_{1 \leqslant i \leqslant n} \mathcal{S}\mathrm{ol}_{\overline{R}_i}(\mathscr{O}_{\mathcal{X}}) \xrightarrow{\rho} \mathcal{S}\mathrm{ol}_{\overline{R}}(\mathscr{O}_{\mathcal{X}}) \xrightarrow{\rho'} \mathcal{S}\mathrm{ol}_{\overline{R}}(\pi_0 \mathscr{O}_{\mathcal{X}}) \simeq \mathcal{S}\mathrm{ol}_R(\pi_0 \mathscr{O}_{\mathcal{X}})$$

factors through $\amalg_{1 \leqslant i \leqslant n} \mathcal{S}\mathrm{ol}_{R_i}(\pi_0 \mathscr{O}_{\mathcal{X}})$, it will suffice to show that ρ and ρ' are effective epimorphisms. The map ρ is an effective epimorphism by virtue of our assumption that $\mathscr{O}_{\mathcal{X}}$ satisfies the condition described in (a). The map ρ' is an effective epimorphism because it is a pullback of the mth power of the effective epimorphism $\Omega^\infty \mathscr{O}_{\mathcal{X}} \to \pi_0 \mathscr{O}_{\mathcal{X}}$. This completes the proof of (a).

We now prove (*b*). Suppose first that f is a morphism in $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$; we wish to show that if $\phi : A \to B$ is an étale morphism of $\mathbb{E}_\infty$-rings, then the diagram

$$\begin{array}{ccc} f^* \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}}) \\ \downarrow & & \downarrow \\ f^* \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}}) \end{array}$$

is a pullback square in $\mathcal{X}$. As in the first part of the proof, it will suffice to prove this in the special case where A and B are connective. Remark 1.4.3.6 then allows us to replace $\mathscr{O}_{\mathcal{X}}$ and $\mathscr{O}_{\mathcal{Y}}$ by their connective covers, and thereby reduce to the case where they are connective as well. In this case, we have a commutative diagram

$$\begin{array}{ccccc} f^* \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_B(\pi_0 \mathscr{O}_{\mathcal{X}}) \\ \downarrow & & \downarrow & & \downarrow \\ f^* \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\pi_0 \mathscr{O}_{\mathcal{X}}) \end{array}$$

where the right square is a pullback by Lemma 1.4.3.8. It will therefore suffice to show that the outer rectangle is a pullback. This rectangle also appears in the commutative diagram

$$\begin{array}{ccccc} f^* \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & f^* \mathcal{S}\mathrm{ol}_B(\pi_0 \mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_B(\pi_0 \mathscr{O}_{\mathcal{X}}) \\ \downarrow & & \downarrow & & \downarrow \\ f^* \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & f^* \mathcal{S}\mathrm{ol}_A(\pi_0 \mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\pi_0 \mathscr{O}_{\mathcal{X}}) \end{array}$$

where the left square is a pullback (Lemma 1.4.3.8). We are then reduced to showing that the right square is a pullback diagram (of discrete objects of $\mathcal{X}$), which follows from Proposition 1.2.2.12.

For the converse, suppose that f satisfies the condition described in (*b*); we claim that induced map $f^* \mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$ is local. Fix an object $X \in \mathcal{X}$ and a map $e : X \to \pi_0 f^* \mathscr{O}_{\mathcal{Y}}$, and let $\overline{e}$ denote the composite map $X \xrightarrow{e} \pi_0 f^* \mathscr{O}_{\mathcal{Y}} \to \pi_0 \mathscr{O}_{\mathcal{X}}$. We must show that if $\overline{e}$ is invertible when regarded as an element of the commutative ring $\pi_0 \mathrm{Map}_{\mathcal{X}}(X, \pi_0 \mathscr{O}_{\mathcal{X}})$, then e is invertible when regarded as an element of the commutative ring $\pi_0 \mathrm{Map}_{\mathcal{X}}(X, \pi_0 f^* \mathscr{O}_{\mathcal{Y}})$. This assertion is local X: we may therefore assume without loss of generality that e factors through a map $X \to f^* \mathscr{O}_{\mathcal{Y}}$. In this case, the desired result follows by inspecting the diagram

$$\begin{array}{ccccc} f^* \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_B(\mathscr{O}_{\mathcal{X}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_B(\pi_0 \mathscr{O}_{\mathcal{X}}) \\ \downarrow & & \downarrow & & \downarrow \\ f^* \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}}) & \longrightarrow & \mathcal{S}\mathrm{ol}_A(\pi_0 \mathscr{O}_{\mathcal{X}}) \end{array}$$

in the special case $A = S\{x\}$ and $B = S\{x\}[x^{-1}]$; here the left square is a pullback by virtue of our hypothesis on f and the right square is a pullback by virtue of Lemma 1.4.3.8. □

Proof of Proposition 1.4.2.4. Let R be an $\mathbb{E}_\infty$-ring and let $\mathscr{O} : \mathrm{CAlg}_R^{\text{ét}} \to \mathrm{CAlg}$ be the forgetful functor. It follows from Theorem D.6.3.5 that $\mathscr{O}$ is a CAlg-valued sheaf (with respect to the étale topology on $\mathrm{CAlg}_R^{\text{ét}}$). Note that $\pi_0 \mathscr{O}$ can be identified with the sheafification of the presheaf of commutative rings given by the composite map $\mathrm{CAlg}_R^{\text{ét}} \xrightarrow{\pi_0} \mathrm{CAlg}^{\heartsuit}$, which is the structure sheaf of the Deligne-Mumford stack $\mathrm{Spét}\, R$. It follows from Proposition 1.2.3.2 that $\pi_0 \mathscr{O}$ is strictly Henselian, so that the sheaf $\mathscr{O}$ is strictly Henselian.

Let $\mathcal{X}$ be an arbitrary ∞-topos and let $\mathscr{O}_{\mathcal{X}}$ be a strictly Henselian CAlg-valued sheaf on $\mathcal{X}$. The construction $A \mapsto \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}})$ determines a functor $\mathcal{S}\mathrm{ol}_\bullet(\mathscr{O}_{\mathcal{X}}) : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{X}^{\mathrm{op}}$. Applying Proposition HTT.6.1.5.2 , we can identify the mapping space $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{S}\mathrm{hv}_R^{\text{ét}})$ with the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathcal{X})^{\simeq}$ spanned by those functors $\chi : (\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}} \to \mathcal{X}$ which satisfy the following conditions:

(i) The functor χ preserves finite limits.

(ii) For every faithfully flat étale morphism $A \to \prod_{1 \leqslant i \leqslant n} A_i$ in $\mathrm{CAlg}_R^{\text{ét}}$, the induced map $\amalg \chi(A_i) \to \chi(A)$ is an effective epimorphism in the ∞-topos $\mathcal{X}$.

If χ is a functor satisfying these conditions and $f_* : \mathcal{X} \to \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ is the associated geometric morphism, then we can identify the direct image $f_* \mathscr{O}_{\mathcal{X}}$ with the CAlg-valued sheaf on $\mathrm{CAlg}_R^{\text{ét}}$ given by $\mathrm{CAlg}_R^{\text{ét}} \xrightarrow{\chi} \mathcal{X}^{\mathrm{op}} \xrightarrow{\mathscr{O}_{\mathcal{X}}} \mathrm{CAlg}$. We may therefore identify the mapping space $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{S}\mathrm{hv}_R^{\text{ét}})}(\mathscr{O}, f_* \mathscr{O}_{\mathcal{X}})$ with the mapping space $\mathrm{Map}_{\mathrm{Fun}((\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}, \mathcal{X})}(\chi, \mathcal{S}\mathrm{ol}_\bullet(\mathscr{O}_{\mathcal{X}}))$.

Let $\mathbf{1}_{\mathcal{X}}$ denote a final object of $\mathcal{X}$. Fix a morphism of $\mathbb{E}_\infty$-rings $\phi : R \to \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$, which we can identify with a map $\mathbf{1}_{\mathcal{X}} \to \mathcal{S}\mathrm{ol}_{\mathbf{R}}(\mathscr{O}_{\mathcal{X}})$. For each object $A \in \mathrm{CAlg}_R^{\text{ét}}$, let $\mathcal{S}\mathrm{ol}_A^0(\mathscr{O}_{\mathcal{X}})$ denote the fiber product $\mathbf{1}_{\mathcal{X}} \times_{\mathcal{S}\mathrm{ol}_{\mathbf{R}}(\mathscr{O}_{\mathcal{X}})} \mathcal{S}\mathrm{ol}_{\mathbf{A}}(\mathscr{O}_{\mathcal{X}})$, and regard the construction $A \mapsto \mathcal{S}\mathrm{ol}_A^0(\mathscr{O}_{\mathcal{X}})$ as a functor $\mathcal{S}\mathrm{ol}_\bullet^0 : (\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}} \to \mathcal{X}$. The above analysis shows that the homotopy fiber of the canonical map

$$\theta : \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), \mathrm{Spét}\, R) \to \mathrm{Fun}((\mathrm{CAlg}_A^{\text{ét}})^{\mathrm{op}}, \mathcal{X})^{\simeq} \times \mathrm{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{X}, \mathscr{O}_{\mathcal{X}}))$$

over the pair (χ, ϕ) can be identified with the mapping space $\mathrm{Map}_{\mathrm{Fun}((\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}, \mathcal{X})}(\chi, \mathcal{S}\mathrm{ol}_\bullet^0(\mathscr{O}_{\mathcal{X}}))$. It follows from Lemma 1.4.3.9 that this identification carries the homotopy fiber of the restriction

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{sHen}}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), \mathrm{Spét}\, R) \to \mathrm{Fun}((\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}, \mathcal{X})^{\simeq} \times \mathrm{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{X}, \mathscr{O}_{\mathcal{X}}))$$

to the subspace of $\mathrm{Map}_{\mathrm{Fun}((\mathrm{CAlg}_A^{\text{ét}})^{\mathrm{op}}, \mathcal{X})}(\chi, \mathcal{S}\mathrm{ol}_\bullet^0(\mathscr{O}_{\mathcal{X}}))$ spanned by the equivalences. It follows that the fiber product

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{sHen}}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), \mathrm{Spét}\, R) \times_{\mathrm{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{X}, \mathscr{O}_{\mathcal{X}}))} \{\phi\}$$

is either empty or contractible. We will complete the proof by explicitly constructing a point of this fiber product, given by a morphism of ringed ∞-topoi $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to \mathrm{Spét}\, R$. Let

$\chi : (\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}} \to \mathcal{X}$ be the functor $A \mapsto \mathcal{S}\mathrm{ol}_A^0(\mathscr{O}_{\mathcal{X}})$. The construction $A \mapsto \mathcal{S}\mathrm{ol}_A(\mathscr{O}_{\mathcal{X}})$ carries colimits in CAlg to limits in $\mathcal{X}$. It follows that the functor $\mathcal{S}\mathrm{ol}_{\bullet}(\mathscr{O}_{\mathcal{X}})$ preserves pullbacks, so that $\mathcal{S}\mathrm{ol}_{\bullet}^0(\mathscr{O}_{\mathcal{X}})$ also preserves pullbacks. By construction, the functor $\mathcal{S}\mathrm{ol}_{\bullet}^0(\mathscr{O}_{\mathcal{X}})$ carries R to a final object of $\mathcal{X}$, so that $\mathcal{S}\mathrm{ol}_{\bullet}^0(\mathscr{O}_{\mathcal{X}})$ preserves finite limits. Since $\mathscr{O}_{\mathcal{X}}$ is strictly Henselian, Lemma 1.4.3.9 implies that the functor $\chi = \mathcal{S}\mathrm{ol}_{\bullet}^0(\mathscr{O}_{\mathcal{X}})$ satisfies condition (ii) above, and therefore determines a geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{S}\mathrm{hv}_R^{\text{ét}}$. The preceding analysis shows that the projection map $\mathcal{S}\mathrm{ol}_{\bullet}^0(\mathscr{O}_{\mathcal{X}}) \to \mathcal{S}\mathrm{ol}_{\bullet}(\mathscr{O}_{\mathcal{X}})$ determines a morphism of CAlg-valued sheaves $\alpha : \mathscr{O} \to f_* \mathscr{O}_{\mathcal{X}}$, so that we can regard $f = (f_*, \alpha)$ as a morphism of spectrally ringed topoi from $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ into $\mathrm{Sp\acute{e}t}\, A$. We will complete the proof by showing that f is a morphism in the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{sHen}}$. To this end, fix an object $U \in \mathcal{X}$ and an element $x \in \pi_0 \mathrm{Map}_{\mathcal{X}}(U, f^*(\pi_0 \mathscr{O}))$ whose image in $\pi_0 \mathrm{Map}_{\mathcal{X}}(U, \pi_0 \mathscr{O}_{\mathcal{X}})$ is invertible; we wish to show that x is invertible. For each object $A \in \mathrm{CAlg}_R^{\text{ét}}$, let $h^A \in \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ denote the sheaf corepresented by B. Since the objects h^A generate $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ under small colimits, we can therefore choose an effective epimorphism $\amalg h^{A_\alpha} \to \pi_0 \mathscr{O}$, which induces an effective epimorphism $\coprod f^* h^{A_\alpha} \to f^* \pi_0 \mathscr{O}$. Working locally on U, we may assume that the map $x : U \to f^* \pi_0 \mathscr{O}$ factors as a composition $U \xrightarrow{\psi} f^* h^A \xrightarrow{f^* y} f^* \pi_0 \mathscr{O}$, where $\psi : U \to f^* h^A \simeq \mathcal{S}\mathrm{ol}_A^0(\mathscr{O}_{\mathcal{X}})$ classifies an R-algebra morphism $\overline{\phi} : A \to \mathscr{O}_{\mathcal{X}}(U)$, and $y : h^A \to \pi_0 \mathscr{O}$ is a map of discrete objects of $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ which we can identify with an element of the commutative ring $\pi_0 \mathscr{O}(A) = \pi_0 A$. Applying $\overline{\phi}$ to y, we obtain an element $\overline{\phi}(y) \in \pi_0(\mathscr{O}_{\mathcal{X}}(U))$ whose image in $\mathrm{Map}_{\mathcal{X}}(U, \pi_0 \mathscr{O}_{\mathcal{X}})$ is an equivalence. It follows that multiplication by $\overline{\phi}(y)$ induces an equivalence from $\mathscr{O}_{\mathcal{X}}|_U$ to itself, so that $\overline{\phi}(y)$ is invertible in $\mathscr{O}_{\mathcal{X}}|_U$. Consequently, the map ψ admits an (essentially unique) lift to a map $\overline{\psi} : U \to \mathcal{S}\mathrm{ol}_{A[y^{-1}]}^0(\mathscr{O}_{\mathcal{X}}) \simeq f^* h^{A[y^{-1}]}$, so that x factors as a composition $U \xrightarrow{\overline{\psi}} f^* h^{A[y^{-1}]} \xrightarrow{f^* y} f^* \pi_0 \mathscr{O}$, and is therefore invertible, as desired. □

1.4.4 Spectral Deligne-Mumford Stacks

We are now ready to introduce the main objects of study in this book.

Notation 1.4.4.1. Let $\mathcal{X}$ be an ∞-topos, let $\mathcal{C}$ be an arbitrary ∞-category, and let $\mathscr{O}_{\mathcal{X}}$ be a $\mathcal{C}$-valued sheaf on $\mathcal{X}$. For each object $U \in \mathcal{X}$, we let $\mathscr{O}_{\mathcal{X}}|_U$ denote the composite functor

$$(\mathcal{X}_{/U})^{\mathrm{op}} \to \mathcal{X}^{\mathrm{op}} \xrightarrow{\mathscr{O}_{\mathcal{X}}} \mathcal{C},$$

so that $\mathscr{O}_{\mathcal{X}}|_U$ is a $\mathcal{C}$-valued sheaf on the ∞-topos $\mathcal{X}_{/U}$. We will refer to $\mathscr{O}_{\mathcal{X}}|_U$ as the *restriction of $\mathscr{O}_{\mathcal{X}}$ to U*.

Definition 1.4.4.2. A *nonconnective spectral Deligne-Mumford stack* is a spectrally ringed ∞-topos $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ for which there exists a collection of objects $U_\alpha \in \mathcal{X}$ satisfying the following conditions:

(*i*) The objects U_α cover $\mathcal{X}$. That is, the coproduct $\amalg_\alpha U_\alpha$ is 0-connective.

(*ii*) For each index α, there exists an $\mathbb{E}_\infty$-ring A_α and an equivalence of spectrally ringed ∞-topoi $(\mathcal{X}_{/U_\alpha}, \mathscr{O}_\mathcal{X}|_{U_\alpha}) \simeq \operatorname{Sp\acute{e}t} A_\alpha$.

We let $\mathrm{SpDM}^{\mathrm{nc}}$ denote the full subcategory of $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ spanned by the nonconnective spectral Deligne-Mumford stacks.

A *spectral Deligne-Mumford stack* is a nonconnective spectral Deligne-Mumford stack $(\mathcal{X}, \mathscr{O}_\mathcal{X})$ for which the structure sheaf $\mathscr{O}_\mathcal{X}$ is connective. We let SpDM denote the full subcategory of $\mathrm{SpDM}^{\mathrm{nc}}$ spanned by the spectral Deligne-Mumford stacks.

Remark 1.4.4.3. Let A be an $\mathbb{E}_\infty$-ring, and let $\mathscr{O}$ be the sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{S}\mathrm{hv}^{\acute{e}t}_A$ given in Proposition 1.4.2.4. If A is connective, then any étale A-algebra is also connective. It follows that $\mathscr{O}$ is a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{S}\mathrm{hv}^{\acute{e}t}_A$, so that $\operatorname{Sp\acute{e}t} A$ is a spectral Deligne-Mumford stack.

Remark 1.4.4.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be an étale morphism of spectrally ringed ∞-topoi. If Y is a (nonconnective) spectral Deligne-Mumford stack, then so is X. The converse holds if f is an étale surjection.

1.4.5 Connective Covers

Our next goal is to compare the theory of spectral Deligne-Mumford stacks with the more general theory of nonconnective spectral Deligne-Mumford stacks. To this end, we start by establishing an analogue of Proposition 1.1.7.5:

Proposition 1.4.5.1. *Let $(\mathcal{X}, \mathscr{O}_\mathcal{X})$ be a nonconnective spectral Deligne-Mumford stack. Then $(\mathcal{X}, \tau_{\geqslant 0}\mathscr{O}_\mathcal{X})$ is a spectral Deligne-Mumford stack. Moreover, it has the following universal property: for every object $(\mathcal{Y}, \mathscr{O}_\mathcal{Y}) \in \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$, if $\mathscr{O}_\mathcal{Y}$ is connective, then the canonical map*

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{X}, \tau_{\geqslant 0}\mathscr{O}_\mathcal{X}), (\mathcal{Y}, \mathscr{O}_\mathcal{Y})) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{X}, \mathscr{O}_\mathcal{X}), (\mathcal{Y}, \mathscr{O}_\mathcal{Y}))$$

is a homotopy equivalence.

Corollary 1.4.5.2. *The inclusion functor* $\mathrm{SpDM} \hookrightarrow \mathrm{SpDM}^{\mathrm{nc}}$ *admits a left adjoint, given on objects by $(\mathcal{X}, \mathscr{O}_\mathcal{X}) \mapsto (\mathcal{X}, \tau_{\geqslant 0}\mathscr{O}_\mathcal{X})$.*

Proof of Proposition 1.4.5.1. Let $(\mathcal{Y}, \mathscr{O}_\mathcal{Y})$ be an arbitrary spectrally ringed ∞-topos for which $\mathscr{O}_\mathcal{Y}$ is connective. For every geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$, the pullback $f^*\mathscr{O}_\mathcal{Y}$ is also connective so that the natural map

$$\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})}(f^*\mathscr{O}_\mathcal{Y}, \tau_{\geqslant 0}\mathscr{O}_\mathcal{X}) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})}(f^*\mathscr{O}_\mathcal{Y}, \mathscr{O}_\mathcal{X})$$

is a homotopy equivalence. It follows that the homotopy fibers of the vertical maps in the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((\mathcal{X}, \tau_{\geqslant 0}\,\mathscr{O}_{\mathcal{X}}), (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})) & \xrightarrow{\theta} & \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})) \\ & \searrow \quad \swarrow & \\ & \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{Y}) & \end{array}$$

are homotopy equivalent to one another, so that θ is a homotopy equivalence. If $\mathscr{O}_{\mathcal{Y}}$ is strictly Henselian, then θ restricts to a homotopy equivalence

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{X}, \tau_{\geqslant 0}\,\mathscr{O}_{\mathcal{X}}), (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}))$$

(since $\pi_0(\tau_{\geqslant 0}\,\mathscr{O}_{\mathcal{X}})$ is isomorphic to $\pi_0\,\mathscr{O}_{\mathcal{X}}$).

To complete the proof, it will suffice to show that if $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a nonconnective spectral Deligne-Mumford stack, then $(\mathcal{X}, \tau_{\geqslant 0}\,\mathscr{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford stack. The assertion is local on $\mathcal{X}$. We may therefore assume without loss of generality that $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ has the form $\mathrm{Spét}\,A$ for some $\mathbb{E}_\infty$-ring A. Let $B = \tau_{\geqslant 0}A$. Then $\mathrm{Spét}\,B$ is a spectral Deligne-Mumford stack (Remark 1.4.4.3), so the canonical map $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to \mathrm{Spét}\,B$ factors as a composition

$$(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \xrightarrow{\phi} (\mathcal{X}, \tau_{\geqslant 0}\,\mathscr{O}_{\mathcal{X}}) \xrightarrow{\psi} \mathrm{Spét}\,B.$$

To complete the proof, it will suffice to show that ψ is an equivalence. Using the explicit description of the functor Spét supplied by Proposition 1.4.2.4, we can identify both $\mathcal{X}$ and the underlying ∞-topos of $\mathrm{Spét}\,B$ with $\mathcal{S}\mathrm{hv}^{\mathrm{ét}}_A$. The structure sheaf $\mathscr{O}$ of $\mathrm{Spét}\,B$ is given by $\mathscr{O}(A') = \tau_{\geqslant 0}A'$, from which it immediately follows that $\mathscr{O}$ is a connective cover of $\mathscr{O}_{\mathcal{X}}$ (so that ψ is an equivalence). □

Corollary 1.4.5.3. *Let A be an $\mathbb{E}_\infty$-ring. If* Spét A *is a spectral Deligne-Mumford stack, then A is connective.*

Proof. Write $\mathrm{Spét}\,A = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$. The proof of Proposition 1.4.5.1 shows that $(\mathcal{X}, \tau_{\geqslant 0}\,\mathscr{O}_{\mathcal{X}}) \simeq \mathrm{Spét}(\tau_{\geqslant 0}A)$. Passing to global sections, we deduce that the canonical map $\tau_{\geqslant 0}A \simeq \Gamma(\mathcal{X}; \tau_{\geqslant 0}\,\mathscr{O}_{\mathcal{X}}) \to \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}}) \simeq A$ is an equivalence, so that A is connective. □

1.4.6 Truncated Spectral Deligne-Mumford Stacks

We now study the formation of Postnikov towers in the setting of spectral Deligne-Mumford stacks.

Definition 1.4.6.1. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack, and let $n \geqslant 0$ be an integer. We will say that X is *n-truncated* if $\mathscr{O}_{\mathcal{X}}$ is n-truncated when regarded as a sheaf of spectra on $\mathcal{X}$. We let $\mathrm{SpDM}^{\leqslant n}$ denote the full subcategory of SpDM spanned by the n-truncated spectral Deligne-Mumford stacks.

Example 1.4.6.2. Let A be an $\mathbb{E}_\infty$-ring. Then A can be recovered as the $\mathbb{E}_\infty$-ring of global sections of the structure sheaf of $\operatorname{Sp\acute{e}t} A$. Consequently, if $\operatorname{Sp\acute{e}t} A$ is an n-truncated spectral Deligne-Mumford stack, then A is connective (see Corollary 1.4.5.3) and n-truncated. Conversely, if A is connective and n-truncated, then the explicit description of $\operatorname{Sp\acute{e}t} A$ given by Proposition 1.4.2.4 shows that $\operatorname{Sp\acute{e}t} A$ is an n-truncated spectral Deligne-Mumford stack.

Proposition 1.4.6.3. *Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a spectral Deligne-Mumford stack. For each $n \geqslant 0$, the truncation $(\mathcal{X}, \tau_{\leqslant n} \mathcal{O}_\mathcal{X}$ is also a spectral Deligne-Mumford stack. Moreover, it has the following universal property: for every object $(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \in \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$, if $\mathcal{O}_\mathcal{Y}$ is connective and n-truncated, then the canonical map*

$$\operatorname{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{Y}, \mathcal{O}_\mathcal{Y}), (\mathcal{X}, \tau_{\leqslant n} \mathcal{O}_\mathcal{X})) \to \operatorname{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{Y}, \mathcal{O}_\mathcal{Y}), (\mathcal{X}, \mathcal{O}_\mathcal{X}))$$

is a homotopy equivalence.

Corollary 1.4.6.4. *For each integer $n \geqslant 0$, the inclusion $\mathrm{SpDM}^{\leqslant n} \hookrightarrow \mathrm{SpDM}$ admits a right adjoint, given on objects by $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \mapsto (\mathcal{X}, \tau_{\leqslant n} \mathcal{O}_\mathcal{X})$.*

Notation 1.4.6.5. If $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a spectral Deligne-Mumford stack and $n \geqslant 0$ is an integer, we let $\tau_{\leqslant n} \mathsf{X}$ denote the spectral Deligne-Mumford stack given by $(\mathcal{X}, \tau_{\leqslant n} \mathcal{O}_\mathcal{X})$. We will refer to $\tau_{\leqslant n} \mathsf{X}$ as the *n-truncation* of X

Proof of Proposition 1.4.6.3. Let $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ be an arbitrary spectrally ringed ∞-topos for which $\mathcal{O}_\mathcal{Y}$ is n-truncated. For every geometric morphism $f_* : \mathcal{Y} \to \mathcal{X}$, the pushforward $f_* \mathcal{O}_\mathcal{Y}$ is also n-truncated. We therefore have a commutative diagram

$$\begin{array}{ccc} \operatorname{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})}(\tau_{\leqslant n} \mathcal{O}_\mathcal{X}, \tau_{\geqslant 0} f_* \mathcal{O}_\mathcal{Y}) & \longrightarrow & \operatorname{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})}(\mathcal{O}_\mathcal{X}, \tau_{\geqslant 0} f_* \mathcal{O}_\mathcal{Y}) \\ \downarrow & & \downarrow \\ \operatorname{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})}(\tau_{\leqslant n} \mathcal{O}_\mathcal{X}, f_* \mathcal{O}_\mathcal{Y}) & \longrightarrow & \operatorname{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})}(\mathcal{O}_\mathcal{X}, f_* \mathcal{O}_\mathcal{Y}) \end{array}$$

where the vertical maps are homotopy equivalences (since $\mathcal{O}_\mathcal{X}$ is connective) and the upper horizontal map is a homotopy equivalence (since $\tau_{\geqslant 0} f_* \mathcal{O}_\mathcal{Y}$ is connective and n-truncated). It follows that the lower horizontal map is also an equivalence. Allowing f to vary, we deduce that the homotopy fibers of the vertical maps in the diagram

$$\begin{array}{ccc} \operatorname{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((\mathcal{Y}, \mathcal{O}_\mathcal{Y}), (\mathcal{X}, \tau_{\leqslant n} \mathcal{O}_\mathcal{X})) & \xrightarrow{\theta} & \operatorname{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}((\mathcal{Y}, \mathcal{O}_\mathcal{Y}), (\mathcal{X}, \mathcal{O}_\mathcal{X})) \\ & \searrow \quad \swarrow & \\ & \operatorname{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{Y}, \mathcal{X}) & \end{array}$$

are homotopy equivalent to one another, so that θ is a homotopy equivalence. If $\mathcal{O}_\mathcal{Y}$ is strictly Henselian, then θ restricts to a homotopy equivalence

$$\operatorname{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{Y}, \mathcal{O}_\mathcal{Y}), (\mathcal{X}, \tau_{\leqslant n} \mathcal{O}_\mathcal{X})) \to \operatorname{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{Y}, \mathcal{O}_\mathcal{Y}), (\mathcal{X}, \mathcal{O}_\mathcal{X}))$$

(since $\pi_0(\tau_{\leqslant n}\mathscr{O}_{\mathcal{X}})$ is isomorphic to $\pi_0\mathscr{O}_{\mathcal{X}}$).

To complete the proof, it will suffice to show that if $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford stack, then $(\mathcal{X}, \tau_{\leqslant n}\mathscr{O}_{\mathcal{X}})$ is also a spectral Deligne-Mumford stack. The assertion is local on $\mathcal{X}$. We may therefore assume without loss of generality that $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ has the form $\operatorname{Sp\acute{e}t} A$ for some $\mathbb{E}_\infty$-ring A. It follows from Corollary 1.4.5.3 that A is connective. Let $B = \tau_{\leqslant n}A$. Then $\operatorname{Sp\acute{e}t} B$ is an n-truncated spectral Deligne-Mumford stack (Example 1.4.6.2), so the canonical map $\operatorname{Sp\acute{e}t} B \to \operatorname{Sp\acute{e}t} A \simeq (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ factors as a composition

$$\operatorname{Sp\acute{e}t} B \xrightarrow{\phi} (\mathcal{X}, \tau_{\leqslant n}\mathscr{O}_{\mathcal{X}}) \to (\mathcal{X}, \mathscr{O}_{\mathcal{X}}).$$

To complete the proof, it will suffice to show that ϕ is an equivalence. Using the explicit description of the functor Spét supplied by Proposition 1.4.2.4 and Theorem HA.7.5.0.6 , we can identify both $\mathcal{X}$ and the underlying ∞-topos of $\operatorname{Sp\acute{e}t} B$ with $\mathcal{S}\mathrm{hv}_A^{\acute{e}t}$. Under this identification structure sheaf $\mathscr{O}$ of $\operatorname{Sp\acute{e}t} B$ corresponds to the functor $\mathrm{CAlg}_A^{\acute{e}t}$ given by $\mathscr{O}(A') = \tau_{\leqslant n}A'$, from which we immediately deduce that ϕ is an equivalence. □

1.4.7 Affine Spectral Deligne-Mumford Stacks

Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. We will say that X is *affine* it is equivalent to $\operatorname{Sp\acute{e}t} A$, for some $\mathbb{E}_\infty$-ring A.

Remark 1.4.7.1. Arguing as in Remark 1.1.5.7, we see that the construction $A \mapsto \operatorname{Sp\acute{e}t} A$ determines a fully faithful embedding $\operatorname{Sp\acute{e}t} : \mathrm{CAlg}^{\mathrm{op}} \to \mathrm{SpDM}^{\mathrm{nc}}$, whose essential image is the full subcategory of $\mathrm{SpDM}^{\mathrm{nc}}$ spanned by the affine nonconnective spectral Deligne-Mumford stacks.

Our main goal is to establish the following characterization of affine spectral Deligne-Mumford stacks:

Proposition 1.4.7.2. *Let $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectrally ringed ∞-topos which satisfies the following conditions:*

(a) *Let $\mathcal{X}^{\heartsuit}$ denote the underlying topos of $\mathcal{X}$. Then the ringed topos $(\mathcal{X}^{\heartsuit}, \pi_0\mathscr{O}_{\mathcal{X}})$ is equivalent to* $\operatorname{Sp\acute{e}t} R$ *for some commutative ring R.*

(b) *The ∞-topos $\mathcal{X}$ is* 1*-localic (that is, the natural geometric morphism $\mathcal{X} \to \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit}) \simeq \mathcal{S}\mathrm{hv}_R^{\acute{e}t}$ is an equivalence).*

(c) *For each integer n, the $(\pi_0\mathscr{O}_{\mathcal{X}})$-module $\pi_n\mathscr{O}_{\mathcal{X}}$ is quasi-coherent (in the sense of Definition 1.2.6.1).*

(d) *The sheaf $\mathscr{O}_{\mathcal{X}}$ is hypercomplete.*

Then $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *is equivalent to* $\operatorname{Sp\acute{e}t} A$*, for some* $\mathbb{E}_\infty$*-ring* A*.*

Corollary 1.4.7.3. *Let* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *be a nonconnective spectral Deligne-Mumford stack. Then* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *is affine if and only if the* 0*-truncated spectral Deligne-Mumford stack* $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}})$ *is affine.*

Corollary 1.4.7.4. *Suppose we are given a commutative diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{f} & \mathsf{Y} \\ & \searrow \quad \swarrow & \\ & \mathsf{Z}, & \end{array}$$

and suppose that the underlying map $\mathsf{X} \times_{\mathsf{Z}} \tau_{\leqslant 0} \mathsf{Z} \to \mathsf{Y} \times_{\mathsf{Z}} \tau_{\leqslant 0} \mathsf{Z}$ *is an equivalence. Then* f *is an equivalence.*

Proof. The assertion is local on Y; we may therefore assume without loss of generality that $\mathsf{Y} = \operatorname{Sp\acute{e}t} A$ and $\mathsf{Z} \simeq \operatorname{Sp\acute{e}t} R$ are affine. Our assumption guarantees that f induces an equivalence of 0-truncations, so that $\tau_{\leqslant 0} \mathsf{X}$ is affine. Applying Corollary 1.4.7.3, we deduce that $\mathsf{X} \simeq \operatorname{Sp\acute{e}t} B$ is affine. Let K denote the cofiber of the map $A \to B$ (formed in the ∞-category Mod_R). We wish to prove that $K \simeq 0$. Assume otherwise. Since K is connective, there exists a smallest integer n such that $\pi_n K$ is nontrivial. In this case, we have

$$\pi_n K \simeq \operatorname{Tor}_0^{\pi_0 R}(\pi_0 R, \pi_n K) \simeq \pi_n(\pi_0 R \otimes_R K) \simeq \pi_n \operatorname{cofib}(\pi_0 R \otimes_R A \to \pi_0 R \otimes_R B) \simeq 0,$$

and obtain a contradiction. □

The proof of Proposition 1.4.7.2 will require some preliminaries.

Lemma 1.4.7.5. *Let* $\mathcal{X}$ *be an* ∞*-topos and let* $n \geqslant -1$ *be an integer. Suppose we are given a collection of morphisms* $f_\alpha : U_\alpha \to X$ *in* $\mathcal{X}$ *with the following properties:*

(*i*) *Each of the morphisms* f_α *is* $(n-1)$*-truncated.*

(*ii*) *Each of the objects* U_α *is* n*-truncated.*

(*iii*) *The induced map* $f : \amalg_\alpha U_\alpha \to X$ *is an effective epimorphism in* $\mathcal{X}$*.*

Then X *is* n*-truncated.*

Proof. Without loss of generality, we may assume that $\mathcal{X}$ is given as a left exact localization of $\mathcal{P}(\mathcal{C}) = \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, for some small ∞-category $\mathcal{C}$. Let $L : \mathcal{P}(\mathcal{C}) \to \mathcal{X}$ be a left adjoint to the inclusion. For each object $C \in \mathcal{C}$, let $X'(C) \subseteq X(C)$ denote the union of those connected components which meet the image of one of the maps $U_\alpha(C) \to X(C)$, so that we have an effective epimorphism $f' : \amalg_\alpha U_\alpha \to X'$ in the ∞-topos $\mathcal{P}(\mathcal{C})$. It follows from (iii) that the functor L carries X' to X. Since L is left exact, it will suffice to prove that X' is n-truncated. We may therefore replace X by X', and thereby reduce to the case where $\mathcal{X} = \mathcal{P}(\mathcal{C})$ is an ∞-category of presheaves. Working objectwise, we may reduce to the case where $\mathcal{C} \simeq \Delta^0$, so that $\mathcal{X}$ is the ∞-category $\mathcal{S}$ of spaces.

If $n = -1$, then either X is empty or one of the maps h_α is an equivalence; in either case, we immediately conclude that X is (-1)-truncated. Suppose that $n \geqslant 0$. We wish to prove that $\pi_m(X, x) \simeq)$ for each integer $m > n$ and each base point $x \in X$. Using (iii), we may assume that $x = f_\alpha(\overline{x})$ for some point $\overline{x} \in U_\alpha$. In this case, condition (i) implies that the map $\pi_m(U_\alpha, \overline{x}) \to \pi_m(X, x)$ is an isomorphism, and condition (ii) implies that $\pi_m(U_\alpha, \overline{x}) \simeq 0$. □

Remark 1.4.7.6. In the situation of Lemma 1.4.7.5, we can replace (ii) by the following apparently weaker condition:

(ii') The map f_α factors as a composition $U_\alpha \to V_\alpha \to X$, where V_α is n-truncated.

Indeed, if this condition is satisfied, then U_α can be realized as a retract of $U_\alpha \times_X V_\alpha$. If f_α satisfies condition (i), then the projection map $U_\alpha \times_X V_\alpha \to V_\alpha$ is $(n-1)$-truncated. Since V_α is n-truncated, we conclude that $U_\alpha \times_X V_\alpha$ is n-truncated, and therefore U_α is n-truncated.

Lemma 1.4.7.7. *Let $\mathcal{X}$ be an ∞-topos containing an object X and let $n \geqslant 0$ be an integer. Then:*

(1) *If the ∞-topos $\mathcal{X}$ is $(n+1)$-localic and $\mathcal{X}_{/X}$ is n-localic, then the object X is n-truncated.*

(2) *If the ∞-topos $\mathcal{X}$ is n-localic and X is n-truncated, then the ∞-topos $\mathcal{X}_{/X}$ is n-localic.*

(3) *If the ∞-topos $\mathcal{X}_{/X}$ is $(n+1)$-localic and the object X is both n-truncated and 0-connective, then $\mathcal{X}$ is $(n+1)$-localic.*

Proof. We first prove (1). If $\mathcal{X}$ is $(n+1)$-localic, then we can choose an effective epimorphism $\amalg V_\alpha \to X$ where each V_α is an n-truncated object of $\mathcal{X}$. If $\mathcal{X}_{/X}$ is n-localic, then we can choose effective epimorphisms $\amalg U_{\alpha,\beta} \to V_\alpha$, where each $U_{\alpha,\beta}$ is an $(n-1)$-truncated object of $\mathcal{X}_{/X}$. Applying Remark 1.4.7.6, we conclude that X is n-truncated.

We now prove (2). If $\mathcal{X}$ is n-localic, then we can write $\mathcal{X}$ as a topological localization of $\mathcal{P}(\mathcal{C})$, for some small n-category $\mathcal{C}$ (see the proof of Proposition HTT.6.4.5.7). Let us identify $\mathcal{X}$ with the corresponding subcategory of $\mathcal{P}(\mathcal{C})$. Then $\mathcal{X}_{/X}$ is a topological localization of $\mathcal{P}(\mathcal{C})_{/X}$. According to Proposition HTT.6.4.5.9 , it will suffice to show that the ∞-topos $\mathcal{P}(\mathcal{C})_{/X}$ is n-localic. The presheaf $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ classifies a right fibration of ∞-categories $\theta : \widetilde{\mathcal{C}} \to \mathcal{C}$. Since X is n-truncated, the fibers of θ are n-truncated Kan complexes, so that $\widetilde{\mathcal{C}}$ is also an n-category. We complete the proof by observing that there is a canonical equivalence of ∞-categories $\mathcal{P}(\mathcal{C})_{/X} \simeq \mathcal{P}(\widetilde{\mathcal{C}})$.

We now prove (3). Suppose that X is n-truncated and 0-connective and that $\mathcal{X}_{/X}$ is $(n+1)$-localic. Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism which exhibits $\mathcal{Y}$ as an $(n+1)$-localic reflection of $\mathcal{X}$. Then the associated pullback functor f^* restricts to an equivalence $\tau_{\leqslant n} \mathcal{Y} \to \tau_{\leqslant n} \mathcal{X}$. In particular, we can assume without loss of generality that $X = f^* Y$ for some n-truncated object $Y \in \mathcal{Y}$. By construction, the pullback functor f^* induces an equivalence of ∞-categories $(\tau_{\leqslant n} \mathcal{Y})_{/Y} \to (\tau_{\leqslant n} \mathcal{X})_{/X}$. Restricting to n-truncated objects on both sides, we see that f^* induces an equivalence of ∞-categories $\tau_{\leqslant n}(\mathcal{Y}_{/Y}) \to \tau_{\leqslant n}(\mathcal{X}_{/X})$. It follows from (2) that $\mathcal{Y}_{/Y}$ is $(n+1)$-localic and the ∞-topos $\mathcal{X}_{/X}$ is $(n+1)$-localic by assumption, so that f^* induces an equivalence $\mathcal{Y}_{/Y} \to \mathcal{X}_{/X}$.

We will show that the counit map $v : f^* f_* \to \mathrm{id}_{\mathcal{X}}$ is an equivalence (that the unit map $u : \mathrm{id}_{\mathcal{Y}} \to f_* f^*$ is an equivalence can be proven by the same argument). Let X' be an object of $\mathcal{X}$; we wish to show that the natural map $v_{X'} : f^* f_* X' \to X'$ is an equivalence. Since X is 0-connective, it will suffice to show that

$$v_{X'} \times \mathrm{id}_X : (f^* f_* X') \times X \to X' \times X$$

is an equivalence. Unwinding the definitions, we see that this map factors as a composition

$$\begin{aligned} (f^* f_* X') \times X &\simeq f^*(f_* X' \times Y) \\ &\simeq f^* f_*(X' \times X) \\ &\to X' \times X, \end{aligned}$$

where the last map is an equivalence by virtue of our assumption that f^* and f_* induce mutually inverse equivalences between $\mathcal{Y}_{/Y}$ and $\mathcal{X}_{/X}$. □

Proof of Proposition 1.4.7.2. We proceed as in the proof of Proposition 1.1.3.4. For each integer n, we have a fiber sequence of spectrum-valued sheaves

$$\Sigma^n(\pi_n \mathscr{O}_{\mathcal{X}}) \to \tau_{\leqslant n} \mathscr{O}_{\mathcal{X}} \to \tau_{\leqslant n-1} \mathscr{O}_{\mathcal{X}},$$

where we abuse notation by identifying the sheaf of abelian groups $\pi_n \mathscr{O}_{\mathcal{X}}$ with the corresponding object in the heart of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Passing to global sections and extracting homotopy groups, we obtain a long exact sequence

$$\mathrm{H}^{n-m}(\mathcal{X}_{/U}; (\pi_n \mathscr{O}_{\mathcal{X}})|_U) \to \pi_m(\tau_{\leqslant n} \mathscr{O}_{\mathcal{X}})(U) \to \pi_m(\tau_{\leqslant n-1} \mathscr{O}_{\mathcal{X}}(U)) \to \mathrm{H}^{n-m+1}(\mathcal{X}_{/U}; (\pi_n \mathscr{O}_{\mathcal{X}})|_U)$$

for each object $U \in \mathcal{X}$. Using assumptions (a), (b), and (c), we see that the groups $\mathrm{H}^i(\mathcal{X}_{/U}; (\pi_n \mathscr{O}_{\mathcal{X}})|_U)$ vanish if $U \in \mathcal{X}^{\heartsuit}$ is affine and $i > 0$ (see §HTT.7.2.2 for a discussion of the cohomology of an ∞-topos, Remark HTT.7.2.2.17 for a comparison with the usual theory of sheaf cohomology, and §D.3 for a closely related discussion). Since $(\tau_{\leqslant n}\mathscr{O}_{\mathcal{X}})(U)$ and $(\tau_{\leqslant n-1}\mathscr{O}_{\mathcal{X}})(U)$ are n-truncated and $(n-1)$-truncated, respectively, we conclude that our long exact sequence degenerates to give isomorphisms

$$\pi_m(\tau_{\leqslant n}\mathscr{O}_{\mathcal{X}})(U) \simeq \begin{cases} 0 & \text{if } m > n \\ (\pi_n \mathscr{O}_{\mathcal{X}})(U) & \text{if } m = n \\ (\tau_{\leqslant n-1}\mathscr{O}_{\mathcal{X}})(U) & \text{if } m < n. \end{cases}$$

when $U \in \mathcal{X}^{\heartsuit}$ is affine.

Set $\mathscr{O}'_{\mathcal{X}} = \varprojlim_n \tau_{\leqslant n}\mathscr{O}_{\mathcal{X}} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. We have an evident map $u : \mathscr{O}_{\mathcal{X}} \to \mathscr{O}'_{\mathcal{X}}$, and the above calculation shows that this map induces an equivalence $(\pi_n \mathscr{O}_{\mathcal{X}})(U) \to \pi_n(\mathscr{O}'_{\mathcal{X}}(U))$ for every affine object $U \in \mathcal{X}^{\heartsuit}$. In particular, u induces an isomorphism of sheaves $\pi_n \mathscr{O}_{\mathcal{X}} \to \pi_n \mathscr{O}'_{\mathcal{X}}$ for every integer n. Since $\mathscr{O}'_{\mathcal{X}}$ is hypercomplete by construction, condition (d) implies that u is an equivalence. It follows that the canonical map $\pi_n(\mathscr{O}_{\mathcal{X}}(U)) \to (\pi_n \mathscr{O}_{\mathcal{X}})(U)$ is an isomorphism for each affine object $U \in \mathcal{X}^{\heartsuit}$.

Set $A = \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$. Then Proposition 1.4.2.4 supplies a map of spectrally ringed ∞-topoi $f : (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to \mathrm{Spét}\, A$. Condition (a) implies that the final object of $\mathcal{X}$ is affine, so that the canonical map $\pi_n A \to \Gamma(\mathcal{X}^{\heartsuit}; \pi_n \mathscr{O}_{\mathcal{X}})$ is an isomorphism for every integer n. In particular, $\pi_0 A$ can be identified with the ring of global sections of $\pi_0 \mathscr{O}_{\mathcal{X}}$, so that (a) supplies an equivalence $\mathcal{X}^{\heartsuit} \simeq \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}^{\text{ét}}_{\pi_0 A})$. Combining this observation with (b), we deduce that f induces an equivalence of the underlying ∞-topoi. Then f supplies a morphism of structure sheaves $\alpha : \mathscr{O} \to f_* \mathscr{O}_{\mathcal{X}}$; we wish to show that this map is an equivalence. Since $\mathcal{S}\mathrm{hv}^{\text{ét}}_A$ is generated under small colimits by corepresentable functors h^B, we are reduced to proving that α induces an equivalence

$$B \simeq \mathscr{O}(h^B) \to \mathscr{O}_{\mathcal{X}}(f^* h^B)$$

for each object $B \in \mathrm{CAlg}^{\text{ét}}_A$. We will prove that for each integer n, the map $\pi_n B \to \pi_n \mathscr{O}_{\mathcal{X}}(f^* h^B)$ is an isomorphism of abelian groups. Since $f^* h^B$ is an affine object of $\mathcal{X}^{\heartsuit}$, we can identify $\pi_n \mathscr{O}_{\mathcal{X}}(f^* h^B)$ with the abelian group $\mathrm{Hom}_{\mathcal{X}^{\heartsuit}}(f^* h^B, \pi_n \mathscr{O}_{\mathcal{X}})$. Assumption (b) implies that $\pi_n \mathscr{O}_{\mathcal{X}}$ is a quasi-coherent sheaf on the affine Deligne-Mumford stack $(\mathcal{X}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{X}})$, so that we can identify $\mathrm{Hom}_{\mathcal{X}^{\heartsuit}}(f^* h^B, \pi_n \mathscr{O}_{\mathcal{X}})$ with

$$\pi_0 B \otimes_{\pi_0 A} \Gamma(\mathcal{X}^{\heartsuit}, \pi_n \mathscr{O}_{\mathcal{X}}) \simeq \pi_0 B \otimes_{\pi_0 A} \pi_n A \simeq \pi_n B.$$

□

We close this section with a remark about affine "opens" in an arbitrary spectral Deligne-Mumford stack.

Definition 1.4.7.8. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. We will say that an object $U \in \mathcal{X}$ is *affine* if the nonconnective spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ is affine.

Proposition 1.4.7.9. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be an nonconnective spectral Deligne-Mumford stack, and let $\mathcal{X}_0$ be the full subcategory of $\mathcal{X}$ spanned by the affine objects. Then $\mathcal{X}$ is generated by $\mathcal{X}_0$ under small colimits (in other words, $\mathcal{X}$ is the smallest full subcategory of itself which contains $\mathcal{X}_0$ and is closed under small colimits).*

Proof. Let $\mathcal{X}' \subseteq \mathcal{X}$ be a full subcategory containing $\mathcal{X}_0$ and closed under small colimits. We wish to prove that $\mathcal{X}'$ contains every object $X \in \mathcal{X}$. We first prove this under the additional assumption that there exists a morphism $X \to Y$, where $Y \in \mathcal{X}$ is affine. In this case, we can replace $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ by $(\mathcal{X}_{/Y}, \mathcal{O}_{\mathcal{X}}|_Y)$, and thereby reduce to the case where $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq \operatorname{Spét} R$ is affine. In this case, we can identify $\mathcal{X}$ with $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$. It follows that $\mathcal{X}$ is generated under small colimits by corepresentable functors h^A (where A ranges over étale R-algebras), each of which is affine.

We now treat the general case. Let $\mathbf{1}$ denote the final object of $\mathcal{X}$. Choose an effective epimorphism $U = \amalg_\alpha U_\alpha \to \mathbf{1}$, where each U_α is affine. Let $U_\bullet$ be the Čech nerve of the map $U \to \mathbf{1}$, so that $|U_\bullet| \simeq \mathbf{1}$. Then X is the geometric realization of the simplicial object $|X \times U_\bullet|$. It will therefore suffice to show that each of the objects $X \times U_n$ belongs to $\mathcal{X}'$. Note that there exists a map $U_n \to U$, so that we can write $X \times U_n$ as a coproduct of objects of the form $X \times U_n \times_U U_\alpha$. We conclude by observing that each of these objects admits a morphism to the affine object $U_\alpha \in \mathcal{X}$, and therefore belongs to $\mathcal{X}'$ by the first part of the proof. □

1.4.8 A Recognition Criterion for Spectral Deligne-Mumford Stacks

We now give a concrete characterization of the class of spectral Deligne-Mumford stacks which is more in the spirit of Definition 1.1.2.8.

Theorem 1.4.8.1. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectrally ringed ∞-topos, and let $\mathcal{X}^\heartsuit$ denote underlying topos of $\mathcal{X}$. Then $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a nonconnective spectral Deligne-Mumford stack if and only if the following conditions are satisfied:*

(1) *The ringed topos $(\mathcal{X}^\heartsuit, \pi_0 \mathcal{O}_{\mathcal{X}})$ is a Deligne-Mumford stack, in the sense of Definition 1.2.4.1.*

(2) *The canonical geometric morphism $\phi_* : \mathcal{X} \to \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^\heartsuit)$ (which exhibits $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^\heartsuit)$ as a 1-localic reflection of $\mathcal{X}$) is étale.*

(3) *For each integer n, the homotopy group $\pi_n \mathcal{O}_{\mathcal{X}}$ is a quasi-coherent sheaf on $(\mathcal{X}^\heartsuit, \pi_0 \mathcal{O}_{\mathcal{X}})$, in the sense of Definition 1.2.6.1.*

(4) *The structure sheaf $\mathscr{O}_{\mathcal{X}}$ is hypercomplete (see Definition 1.3.3.4).*

Remark 1.4.8.2. Let $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. Then Theorem **??** implies that the ringed topos $(\mathcal{X}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{X}})$ is a Deligne-Mumford stack. We will refer to $(\mathcal{X}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{X}})$ as the *underlying Deligne-Mumford stack* of $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$.

Remark 1.4.8.3. Let $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a ringed topos, and let $(\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}), \mathscr{O})$ be the associated spectrally ringed ∞-topos (see Remark 1.4.1.5). Then the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})$ is 1-localic, the groups $\pi_n \mathscr{O}$ vanish for $n \neq 0$, and $\mathscr{O}$ is hypercomplete (since it is 0-truncated). Consequently, the spectrally ringed ∞-topos $(\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}), \mathscr{O})$ automatically satisfies conditions (2), (3), and (4) of Theorem 1.4.8.1. It follows that $(\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}), \mathscr{O})$ is a spectral Deligne-Mumford stack if and only if $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a Deligne-Mumford stack. In particular, the construction

$$(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \mapsto (\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}), \mathscr{O})$$

determines a fully faithful embedding $\mathrm{DM} \to \mathrm{SpDM}$, whose essential image consists of those spectral Deligne-Mumford stacks $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ for which the ∞-topos $\mathcal{X}$ is 1-localic and the structure sheaf $\mathscr{O}_{\mathcal{X}}$ is 0-truncated.

Remark 1.4.8.4. Let $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be an arbitrary spectral Deligne-Mumford stack, let $(\mathcal{X}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{X}})$ be its underlying Deligne-Mumford stack, and let $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be the spectral Deligne-Mumford stack associated to $(\mathcal{X}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{X}})$ (Remark 1.4.8.3). Then we have a canonical diagram of spectral Deligne-Mumford stacks

$$(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \xleftarrow{g} (\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}}) \xrightarrow{f} (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}).$$

Here the map f is étale (by Theorem 1.4.8.1), and the map g exhibits $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ as an "infinitesimal thickening" of $(\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}})$.

Remark 1.4.8.5. If $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a 0-truncated spectral Deligne-Mumford stack, then Remark 1.4.8.4 supplies an étale map $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$, where $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ is a 0-truncated, 1-localic spectral Deligne-Mumford stack.

Proof of Theorem 1.4.8.1. Suppose first that $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a nonconnective spectral Deligne-Mumford stack. Assertion (2) follows from Theorem **??**. Choose a covering of $\mathcal{X}$ by objects U_α for which each of the spectrally ringed ∞-topoi $(\mathcal{X}_{/U_\alpha}, \mathscr{O}_{\mathcal{X}}|_{U_\alpha})$ has the form $\mathrm{Sp\acute{e}t}\, R_\alpha$ for some $\mathbb{E}_\infty$-ring R_α. Using (2), we see that each of the geometric morphisms $\phi_{\alpha *} : \mathcal{X}_{/U_\alpha} \to \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})$ is étale, so that we can choose equivalences $\mathcal{X}_{/U_\alpha} \simeq \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})_{/V_\alpha}$ for some objects $V_\alpha \in \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})_{/V_\alpha}$. The underlying ∞-topos of $\mathrm{Sp\acute{e}t}\, R_\alpha$ is 1-localic by construction, so that each V_α is 1-truncated by virtue of Lemma 1.4.7.7. Since $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})$ is 1-localic, it is generated (under small colimits) by discrete objects. In particular, we can choose effective epimorphisms $u_\alpha : V'_\alpha \to V_\alpha$, where each V'_α is a discrete object of $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})$.

Since V_α is 1-truncated, we see that each of the maps $u_\alpha : V'_\alpha \to V_\alpha$ is 0-truncated. We therefore have an equivalence

$$\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^\heartsuit)_{/V'_\alpha} \simeq (\mathcal{S}\mathrm{hv}^{\text{ét}}_{R_\alpha})_{/W_\alpha}$$

for some discrete objects W_α in $\mathcal{S}\mathrm{hv}^{\text{ét}}_{R_\alpha}$. Let $\mathscr{O}$ denote the structure sheaf of $\operatorname{Spét} R_\alpha$, so that we obtain equivalences of ringed topoi

$$(\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}^{\text{ét}}_R)_{/W_\alpha}, (\pi_0 \mathscr{O})|_{W_\alpha}) \simeq (\mathcal{X}_{0/V'_\alpha}, (\pi_0 \mathscr{O}_{\mathcal{X}})_{/V'_\alpha}).$$

Since $(\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{CAlg}^{\text{ét}}_R), \pi_0 \mathscr{O}) \simeq \operatorname{Spét}(\pi_0 R)$ is a Deligne-Mumford stack, it follows from Remark 1.2.4.5 that $(\mathcal{X}^\heartsuit, \pi_0 \mathscr{O}_{\mathcal{X}})$ is also a Deligne-Mumford stack. This completes the proof of (1). Moreover, the above argument shows that we have a canonical isomorphism of sheaves $(\pi_n \mathscr{O}_{\mathcal{X}})|_{V'_\alpha} \simeq \mathscr{F}|_{W_\alpha}$, where $\mathscr{F}$ denotes the quasi-coherent sheaf on $\operatorname{Spét}(\pi_0 R)$ associated to the $\pi_0 R$-module $\pi_n R$ (see Example 1.2.6.2). Assertion (3) now follows from Proposition 1.2.6.3. To prove (4), we may work locally on $\mathcal{X}$ (by virtue of Proposition 1.3.3.6) and thereby reduce to the case where $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) = \operatorname{Spét} A$ for some $\mathbb{E}_\infty$-ring A. We may further assume that A is connective. In this case, $\mathscr{O}_{\mathcal{X}}$ can be written as the inverse limit of the structure sheaves of the spectral Deligne-Mumford stacks $\operatorname{Spét}(\tau_{\leqslant n} A)$, each of which is truncated and therefore hypercomplete.

Now suppose that conditions (1) through (4) are satisfied; we wish to prove that $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a nonconnective spectral Deligne-Mumford stack. Using (2), we can write $\mathcal{X} = \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^\heartsuit)_{/X}$ for some object $X \in \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^\heartsuit)$. Using (1), we can choose an effective epimorphism $\amalg U_\alpha \to X$ in $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^\heartsuit)$, where each U_α is an affine object of $\mathcal{X}^\heartsuit$. Then each U_α can be identified with an object $V_\alpha \in \mathcal{X}$. To complete the proof, it will suffice to show that each $(\mathcal{X}_{/V_\alpha}, \mathscr{O}_{\mathcal{X}}|_{V_\alpha})$ has the form $\operatorname{Spét} A$, for some $\mathbb{E}_\infty$-ring A. This follows from Proposition 1.4.7.2. □

1.4.9 Postnikov Towers of Spectral Deligne-Mumford Stacks

Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. In §1.4.6, we saw that each truncation $\tau_{\leqslant n} \mathsf{X} = (\mathcal{X}, \tau_{\leqslant n} \mathscr{O}_{\mathcal{X}})$ is also a spectral Deligne-Mumford stack. We now show that X is determined by the collection of truncations $\{\tau_{\leqslant n} \mathsf{X}\}_{n \geqslant 0}$.

Proposition 1.4.9.1. *The ∞-category* SpDM *is a homotopy limit of the tower*

$$\cdots \to \mathrm{SpDM}^{\leqslant 3} \xrightarrow{\tau_{\leqslant 2}} \mathrm{SpDM}^{\leqslant 2} \xrightarrow{\tau_{\leqslant 1}} \mathrm{SpDM}^{\leqslant 1} \xrightarrow{\tau_{\leqslant 0}} \mathrm{SpDM}^{\leqslant 0}.$$

Proof. Let G denote the evident functor $\mathrm{SpDM} \to \varprojlim \mathrm{SpDM}^{\leqslant n}$. We first claim that G is fully faithful. Unwinding the definitions, we must show that if $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ and $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ are spectral Deligne-Mumford stacks, then the canonical map

$$\theta : \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{X}, \mathsf{Y}) \to \varprojlim \mathrm{Map}_{\mathrm{SpDM}}((\mathcal{X}, \tau_{\leqslant n} \mathscr{O}_{\mathcal{X}}), \mathsf{Y})$$

is a homotopy equivalence. Let $K = \mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})^{\simeq}$ denote the space of geometric morphisms from the underlying ∞-topos of X to the underlying ∞-topos of Y. We will show that θ induces a homotopy equivalence after passing to the homotopy fiber over any point of K, corresponding to a geometric morphism $f^* : \mathcal{Y} \to \mathcal{X}$. In this case, we wish to show that the canonical map

$$\phi : \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})}(f^* \mathscr{O}_{\mathcal{Y}}, \mathscr{O}_{\mathcal{X}}) \to \varprojlim \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})}(f^* \mathscr{O}_{\mathcal{Y}}, \tau_{\leqslant n} \mathscr{O}_{\mathcal{X}})$$

induces a homotopy equivalence on the summands corresponding to *local* maps between strictly Henselian sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. This follows from the following pair of assertions:

(a) The map ϕ is a homotopy equivalence. In fact, the canonical map $\mathscr{O}_{\mathcal{X}} \to \varprojlim \tau_{\leqslant n} \mathscr{O}_{\mathcal{X}}$ is an equivalence of sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$: this follows from the proof of Theorem 1.4.8.1.

(b) A map $f^* \mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$ is local if and only if each of the induced maps $f^* \mathscr{O}_{\mathcal{Y}} \to \tau_{\leqslant n} \mathscr{O}_{\mathcal{X}}$ is local. Both conditions are equivalent to the assertion that the underlying map

$$\pi_0 f^* \mathscr{O}_{\mathcal{Y}} \to \pi_0 \mathscr{O}_{\mathcal{X}}$$

is local (Definition 1.4.2.1).

It remains to prove that G is essentially surjective. Suppose we are given an object of $\varprojlim_n \mathrm{SpDM}^{\leqslant n}$, given by a sequence of spectral Deligne-Mumford stacks

$$(\mathcal{X}_0, \mathscr{O}_0) \to (\mathcal{X}_1, \mathscr{O}_1) \to (\mathcal{X}_2, \mathscr{O}_2) \to \cdots$$

with the following property: each of the maps $(\mathcal{X}_i, \mathscr{O}_i) \to (\mathcal{X}_{i+1}, \mathscr{O}_{i+1})$ induces an equivalence $(\mathcal{X}_i, \mathscr{O}_i) \simeq \tau_{\leqslant i}(\mathcal{X}_{i+1}, \mathscr{O}_{i+1})$. It follows that the sequence of ∞-topoi $\mathcal{X}_0 \to \mathcal{X}_1 \to \cdots$ is equivalent to the constant sequence taking the value $\mathcal{X} = \mathcal{X}_0$. To complete the proof, it will suffice to verify the following:

(c) The spectrally ringed ∞-topos $\mathsf{X} = (\mathcal{X}, \varprojlim \mathscr{O}_n)$ is a spectral Deligne-Mumford stack.

(d) For every integer n, the canonical map $(\mathcal{X}, \mathscr{O}_n) \to \mathsf{X}$ induces an equivalence $(\mathcal{X}, \mathscr{O}_n) \to \tau_{\leqslant n} \mathsf{X}$.

Both of these assertions are local on X. We may therefore assume without loss of generality that $(\mathcal{X}, \mathscr{O}_0)$ is affine. It follows that each pair $(\mathsf{X}, \mathscr{O}_n)$ is affine (Corollary 1.4.7.3), so that the sequence of spectral Deligne-Mumford stacks above is determined by a tower of connective $\mathbb{E}_\infty$-rings

$$\cdots A_2 \to A_1 \to A_0$$

which induces equivalences $\tau_{\leqslant n} A_{n+1} \to A_n$ for each $n \geqslant 0$. Since the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ is Postnikov complete (see Definition A.7.2.1 and Proposition HA.7.1.3.19), we can the limit $A = \varprojlim_n A_n$ is a connective $\mathbb{E}_\infty$-ring with $A_n \simeq \tau_{\leqslant n} A$ for every integer n. A simple calculation now shows that $\mathsf{X} \simeq \mathrm{Sp\acute{e}t}\, A$, from which assertions (c) and (d) follow easily. □

1.4.10 Étale Morphisms of Spectral Deligne-Mumford Stacks

In the setting of spectral Deligne-Mumford stacks, there are two *a priori* different notions of étale morphism $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ that one could consider: one could require that f is étale as a morphism of spectrally ringed ∞-topoi (see Definition 1.4.10.1 below), or one could require that f is obtained locally by applying the spectrum functor Spét to an étale morphism of $\mathbb{E}_\infty$-rings. Our next goal is to show that these definitions are equivalent (Corollary 1.4.10.3).

Definition 1.4.10.1. Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of spectrally ringed ∞-topoi. We will say that f is *étale* if the following conditions are satisfied:

- (a) The underlying geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ is étale: that is, it induces an equivalence $\mathcal{X} \simeq \mathcal{Y}_{/U}$ for some object $U \in \mathcal{Y}$ (see §HTT.6.3.5).
- (b) The induced map $f^* \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ is an equivalence of sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

We will say that f is an *étale surjection* if, in addition, the object $U \in \mathcal{Y}$ appearing in (a) is 0-connective.

Theorem 1.4.10.2. *Let $\phi : A \to B$ be a map of $\mathbb{E}_\infty$-rings. Then ϕ is étale if and only if the induced map* $\operatorname{Spét} B \to \operatorname{Spét} A$ *is an étale morphism of nonconnective spectral Deligne-Mumford stacks.*

Corollary 1.4.10.3. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a map between nonconnective spectral Deligne-Mumford stacks. The following conditions are equivalent:*

- (i) *The map f is étale.*
- (ii) *For every commutative diagram*

$$\begin{array}{ccc} \operatorname{Spét} B & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow f \\ \operatorname{Spét} A & \longrightarrow & \mathsf{Y} \end{array}$$

  in which the horizontal maps are étale, the underlying map of $\mathbb{E}_\infty$-rings $A \to B$ is étale.

Proof of Theorem 1.4.10.2. We proceed as in the proof of Proposition 1.2.5.7. The implication (1) $\Rightarrow$ (2) follows immediately from the construction of $\operatorname{Spét} B$ (see Definition 1.2.3.3). Conversely, suppose that (2) is satisfied. Write $\operatorname{Spét} A = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, so that $\operatorname{Spét} B \simeq (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ for some object $U \in \mathcal{X}$. Choose an effective epimorphism $\amalg_{i \in I} V_i \to U$, where each $V_i \in \mathcal{X}$ is the functor corepresented by an étale A-algebra A_i. We can identify each V_i with an object of the ∞-topos $\mathcal{X}_{/U} \simeq \mathcal{S}\mathrm{hv}_B^{\text{ét}}$. We may therefore choose an effective epimorphism $\amalg_{j \in J} W_j \to U$, where each W_j is corepresented by an étale B-algebra B_j, and each of the maps $W_j \to U$ factors through $V_{\alpha(i)}$ for some map $\alpha : J \to I$. Without loss of generality, we may assume that the set J is finite. Replacing I by J and α by the identity map, we can assume that I is finite as well. Set $V = \amalg_{i \in I} V_i$ and $W = \amalg_{j \in J} W_j$, so that V is corepresented by an étale A-algebra A' (when regarded as an object of $\mathcal{S}\mathrm{hv}_A^{\text{ét}}$), and W is corepresented by an étale B-algebra B' (when regarded as an object of $\mathcal{S}\mathrm{hv}_B^{\text{ét}}$. We therefore have maps of spectral Deligne-Mumford stacks

$$\mathrm{Spét}\, B' \to \mathrm{Spét}\, A' \to \mathrm{Spét}\, B \to \mathrm{Spét}\, A.$$

Applying Remark 1.4.7.1, we obtain maps of $\mathbb{E}_\infty$-rings

$$A \to B \to A' \to B'.$$

It follows that B' is a retract of $A' \otimes_B B'$ in the ∞-category category of A-algebras. Since B' is étale over B and A' is étale over A, the algebra $A' \otimes_B B'$ is étale over A, so that B' is étale over A (Remark **??**). Since the map $B \to B'$ is étale and faithfully flat, we conclude that B is also étale over A (Proposition B.1.4.1). □

1.4.11 Limits of Spectral Deligne-Mumford Stacks

We close this section with a few remarks about the formation of limits in the ∞-category SpDM of spectral Deligne-Mumford stacks.

Proposition 1.4.11.1. (1) *The ∞-category* $\mathrm{SpDM}^{\mathrm{nc}}$ *of nonconnective spectral Deligne-Mumford stacks admits finite limits, and the inclusion functor* $\mathrm{SpDM}^{\mathrm{nc}} \hookrightarrow \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ *preserves finite limits.*

(2) *Suppose we are given a pullback diagram of nonconnective spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}. \end{array}$$

If ϕ is étale, so is ϕ'.

(3) *The functor* $\mathrm{Spét} : \mathrm{CAlg}^{\mathrm{op}} \to \mathrm{SpDM}^{\mathrm{nc}}$ *preserves finite products. That is, if R is an $\mathbb{E}_\infty$-ring and $A, B \in \mathrm{CAlg}_R$, then the canonical map*

$$\mathrm{Spét}(A \otimes_R B) \to (\mathrm{Spét}\, A) \times_{\mathrm{Spét}\, R} (\mathrm{Spét}\, B)$$

is an equivalence.

(4) *Suppose we are given a pullback diagram of nonconnective spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}. \end{array}$$

If X, Y, *and* Y' *are spectral Deligne-Mumford stacks, then* X' *is a spectral Deligne-Mumford stack.*

Proof. Assertion (1) is a special case of Proposition **??** (see Remark **??**), which we will prove in Part VI. Assertion (2) follows from Remark 21.4.6.7. Assertion (3) follows because the functor Spét is right adjoint to the global sections functor. To prove (4), we can use (2) to reduce to the case where X, Y, and Y' are affine. In this case, the desired result follows from (3). □

Corollary 1.4.11.2. *The* ∞*-category* $\mathrm{SpDM}^{\mathrm{nc}}$ *of nonconnective spectral Deligne-Mumford stacks is idempotent complete.*

Proof. The ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ admits filtered limits (this is a special case of Proposition 21.4.3.2, but is also easy to verify directly) and is therefore idempotent complete. Consequently, it will suffice to show that the ∞-category $\mathrm{SpDM}^{\mathrm{nc}}$ is closed under retracts in $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$. Suppose we are given a commutative diagram

$$\begin{array}{ccccc} & & (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) & & \\ & \overset{i}{\nearrow} & & \overset{r}{\searrow} & \\ (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) & & \xrightarrow{\mathrm{id}} & & (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \end{array}$$

in $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$, where $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ is a nonconnective spectral Deligne-Mumford stack; we wish to show that $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is also a nonconnective spectral Deligne-Mumford stack. We will show that $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ satisfies the criteria of Theorem 1.4.8.1:

(1) Let $\mathcal{X}^{\heartsuit}$ and $\mathcal{Y}^{\heartsuit}$ denote the topoi of discrete objects of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Then the ringed topos $(\mathcal{X}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{X}})$ is a retract of $(\mathcal{Y}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{Y}})$ in the 2-category $1\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^{\heartsuit}}$. Theorem 1.4.8.1 implies that $(\mathcal{Y}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{Y}})$ is a Deligne-Mumford stack. Since the 2-category of Deligne-Mumford stacks admits finite limits, it is idempotent complete; it follows that $(\mathcal{X}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{X}})$ is also a Deligne-Mumford stack.

(2) We have a commutative diagram of ∞-topoi

$$\begin{array}{ccccc} \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit}) & \longrightarrow & \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{Y}^{\heartsuit}) & \longrightarrow & \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit}). \end{array}$$

It follows from Theorem 1.4.8.1 that the middle vertical map is étale. Consequently, the fiber product $\mathcal{Z} = \mathcal{Y} \times_{\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{Y}^{\heartsuit})} \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})$ (formed in the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi) is étale over $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})$. Since $\mathcal{X}$ is a retract of $\mathcal{Z}$ in $\infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})}$, it follows that the geometric morphism $\mathcal{X} \to \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})$ is also étale.

(3) Let $j : (\mathcal{X}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{Y}})$ be the map of Deligne-Mumford stacks determined by i, and let j^* denote the associated pullback functor on quasi-coherent sheaves. Theorem 1.4.8.1 implies that $\pi_n \mathscr{O}_{\mathcal{Y}}$ is quasi-coherent (when regarded as an object of the abelian category of $\pi_0 \mathscr{O}_{\mathcal{Y}}$-modules in $\mathcal{Y}^{\heartsuit}$), so that $j^* \pi_n \mathscr{O}_{\mathcal{Y}}$ is a quasi-coherent sheaf on $(\mathcal{X}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{X}})$. The homotopy group $\pi_n \mathscr{O}_{\mathcal{X}}$ is a retract and therefore a direct summand of $j^* \pi_n \mathscr{O}_{\mathcal{Y}}$, and therefore also quasi-coherent.

(4) Theorem 1.4.8.1 implies that $\mathscr{O}_{\mathcal{Y}}$ is hypercomplete (when regarded as a sheaf of spectra on $\mathcal{Y}$), so that $r_* \mathscr{O}_{\mathcal{Y}}$ is hypercomplete (when regarded as a sheaf of spectra on $\mathcal{X}$). Since $\mathscr{O}_{\mathcal{X}}$ is a retract (and therefore a direct summand, when regarded as a sheaf of spectra) of the direct image $r_* \mathscr{O}_{\mathcal{Y}}$, it is also hypercomplete.

□

1.5 Digression: Topological Spaces and ∞-Topoi

To every topological space X, one can associate an ∞-topos $\mathcal{S}\mathrm{hv}(X) = \mathcal{S}\mathrm{hv}_{\mathcal{S}}(X)$ of $\mathcal{S}$-valued sheaves on X. In this section, we will review a closely related construction, which assigns to each ∞-topos $\mathcal{X}$ an underlying topological space $|\mathcal{X}|$. These ideas will play an important role when we discuss the relationship between spectral schemes and spectral Deligne-Mumford stacks in §1.6.

1.5.1 Locales

We begin with a brief review of "pointless" topology. For a more detailed discussion, we refer the reader to [105].

Definition 1.5.1.1. Let Λ be a partially ordered set. We say that Λ is a *locale* if it satisfies the following pair of conditions:

(1) Every subset $S \subseteq \Lambda$ has a least upper bound $\bigvee S \in \Lambda$.

Note that condition (1) implies that every subset $S \subseteq \Lambda$ also has a greatest lower bound $\bigwedge S \in \Lambda$ (namely, the least upper bound of the set $T = \{U \in \Lambda : (\forall V \in S) U \leqslant V\}$). In particular, every pair of elements $U, V \in \Lambda$ have a greatest lower bound $U \wedge V$.

(2) For every subset $\{U_\alpha\} \subseteq \Lambda$ and every element $V \in \Lambda$, we have $(\bigvee U_\alpha) \wedge V = \bigvee (U_\alpha \wedge V)$.

Remark 1.5.1.2. In the situation of Definition 1.5.1.1, we can replace (2) by the following apparently weaker condition:

(2′) For every subset $\{U_\alpha\} \subseteq \Lambda$ and every element $V \in \Lambda$, we have

$$(\bigvee U_\alpha) \wedge V \leqslant \bigvee (U_\alpha \wedge V).$$

The reverse inequality is automatic.

Remark 1.5.1.3. Let Λ be a locale. Then Λ is a distributive lattice (see Proposition A.1.4.4): that is, it satisfies a distributive law

$$(\bigwedge_{U \in S} U) \vee V = \bigwedge_{U \in S} (U \vee V)$$

whenever S is a *finite* subset of Λ.

Example 1.5.1.4. Let X be a topological space, and let $\mathcal{U}(X)$ denote the collection of all open subsets of X. Then $\mathcal{U}(X)$ is a locale (when regarded as partially ordered by inclusion).

Definition 1.5.1.5. Let Λ and Λ' be locales. A *morphism of locales* from Λ to Λ' is a functor $f^* : \Lambda' \to \Lambda$ satisfying the following conditions:

(1) The function f^* preserves joins. That is, for every subset $S \subseteq \Lambda'$, we have $f^*(\bigvee S) = \bigvee(f^* S)$.

(2) The function f^* preserves finite meets. That is, for every finite subset $S \subseteq \Lambda'$, we have $f^*(\bigwedge S) = \bigwedge(f^* S)$.

The collection of locale morphisms is closed under composition. We may therefore organize the collection of locales into a category, which we will denote by Loc.

Example 1.5.1.6. Let $f : X \to Y$ be a continuous map of topological spaces. Then f induces a morphism of locales $\mathcal{U}(X) \to \mathcal{U}(Y)$, which carries an open subset $U \subseteq Y$ to its inverse image $f^{-1}U \subseteq X$. We may therefore regard the construction $X \mapsto \mathcal{U}(X)$ as a functor from the category $\mathcal{T}\mathrm{op}$ of topological spaces to the category Loc of locales.

1.5.2 Points of a Locale

The functor described in Example 1.5.1.6 admits a left adjoint. To describe it, we need to introduce a bit more terminology.

Definition 1.5.2.1. Let Λ be a locale. We will say that an object $U \in \Lambda$ is *indecomposable* if, whenever $U = \bigwedge S$ for some finite subset $S \subseteq \Lambda$, we have $U \in S$.

We let $|\Lambda|$ denote the set of all indecomposable elements of Λ. For each element $U \in \Lambda$, we let $|\Lambda|_U$ denote the subset $\{V \in |\Lambda| : U \nleqslant V\}$.

Proposition 1.5.2.2. *Let Λ be a locale. Then the set $|\Lambda|$ of indecomposable elements of Λ has the structure of a topological space, whose open sets are those of the form $|\Lambda|_U$. Moreover, the construction $U \mapsto |\Lambda|_U$ determines a morphism of locales $v^* : \mathcal{U}(|\Lambda|) \to \Lambda$.*

Proof. We claim the following:

(1) For every subset $S \subseteq \Lambda$, we have

$$|\Lambda|_{\bigvee S} = \bigcup_{U \in S} |\Lambda|_U.$$

(2) For every finite subset $S \subseteq \Lambda$, we have

$$|\Lambda|_{\bigwedge S} = \bigcap_{U \in S} |\Lambda|_U.$$

It follows from (1) that the collection of subsets of $|\Lambda|$ which have the form $|\Lambda|_U$ is closed under unions, and from (2) that the same collection is closed under finite intersections. Moreover, (1) and (2) immediately imply that v^* is a morphism of locales.

Assertion (1) is an immediate consequence of the definitions: we have $\bigvee S \nleq V$ if and only if $U \nleq V$ for some $U \in S$. To prove (2), we must show that if $S \subseteq \Lambda$ is finite and V is indecomposable, then $\bigwedge S \nleq V$ if and only if $U \nleq V$ for all U in S. The "only if" direction is clear (and does not require the assumption that V is indecomposable). Conversely, suppose that $U \nleq V$ for all U in S. Then $V \neq U \vee V$ for all $U \in S$. Since V is indecomposable, we conclude that $\bigwedge_{U \in S}(U \vee V) \neq V$. Applying Remark 1.5.1.3, we can rewrite the left hand side as $(\bigwedge S) \vee V$, so that $\bigwedge S \nleq V$. □

Proposition 1.5.2.3. *Let Λ be a locale and let X be a topological space. Then composition with the locale morphism $v^* : \mathcal{U}(|\Lambda|) \to \Lambda$ of Proposition 1.5.2.2 induces a bijection*

$$\theta : \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, |\Lambda|) \to \mathrm{Hom}_{\mathrm{Loc}}(\mathcal{U}(X), \Lambda).$$

Proof. We first show that θ is injective. Suppose that $f, g : X \to |\Lambda|$ are continuous maps of topological spaces with $f \neq g$. Then there exists a point $x \in X$ such that $f(x) \neq g(x)$. Let us regard $f(x)$ and $g(x)$ as indecomposable elements of Λ. Without loss of generality we may assume that $f(x) \nleq g(x)$. We then ahve

$$f(x) \in |\Lambda|_{f(x)} \qquad g(x) \notin |\Lambda|_{f(x)}.$$

It follows that $f^{-1}|\Lambda|_{f(x)} \neq g^{-1}|\Lambda|_{f(x)}$, so that $\theta(f) \neq \theta(g)$.

We now prove that θ is surjective. Suppose we are given a morphism of locales from $\mathcal{U}(X)$ to Λ, given by a map $f^* : \Lambda \to \mathcal{U}(X)$. For each point $x \in X$, let $S_x = \{U \in \Lambda : x \notin f^*U\}$. Since f^* preserves infinite joins, S_x contains a largest element $U_x = \bigvee S_x$. We claim that

U_x is indecomposable. For suppose that $U_x = \bigwedge T$, where $T \subseteq \Lambda$ is finite. We then have $f^*(U_x) = \bigcap_{V \in T} f^*(V)$. Since $x \notin f^*(U_x)$, it follows that $x \notin f^*(V)$ for some $V \in T$. Then $V \in S_x$, so that $V = U_x$.

Let $f : X \to |\Lambda|$ be the map given by $f(x) = U_x$. For each point $x \in X$ and each element $V \in \Lambda$, we have

$$\begin{aligned} x \in f^{-1}|\Lambda|_V & \Leftrightarrow U_x \in |\Lambda|_V \\ & \Leftrightarrow V \nleq U_x \\ & \Leftrightarrow V \notin S_x \\ & \Leftrightarrow x \in f^*(V). \end{aligned}$$

It follows that for $V \in \Lambda$, the inverse image $f^{-1}|\Lambda|_V$ coincides with the open set $f^*(V) \subseteq X$, so that f is a continuous map. Moreover, our calculation immediately implies that $\theta(f) = f^*$. □

Corollary 1.5.2.4. *The functor $X \mapsto \mathcal{U}(X)$ admits a left adjoint, given on objects by the formula $\Lambda \mapsto |\Lambda|$. In particular, the topological space $|\Lambda|$ depends functorially on the locale Λ.*

Definition 1.5.2.5. Let Λ be a locale. We will say that Λ is *spatial* if the counit map $v^* : \mathcal{U}(|\Lambda|) \to \Lambda$ is an isomorphism of locales.

Remark 1.5.2.6. Let Λ be a locale. By definition, every open subset of $|\Lambda|$ has the form $|\Lambda|_U$, for some element $U \in \Lambda$. Consequently, the counit map $v^* : \mathcal{U}(|\Lambda|) \to \Lambda$ is automatically surjective (when regarded as a map of sets from Λ to $\mathcal{U}(|\Lambda|)$). The condition that Λ is spatial is equivalent to the condition that v^* is surjective: that is, that $|\Lambda|_U \neq |\Lambda|_V$ for $U \neq V$.

Corollary 1.5.2.7. *The construction $\Lambda \mapsto |\Lambda|$ determines a fully faithful embedding $\mathrm{Loc}^{\mathrm{spa}} \to \mathcal{T}\mathrm{op}$, where $\mathrm{Loc}^{\mathrm{spa}}$ denotes the full subcategory of Loc spanned by the spatial locales.*

Remark 1.5.2.8. Let X be a topological space. Then the locale $\mathcal{U}(X)$ is automatically spatial: an open subset $U \in \mathcal{U}(X)$ can be recovered as the inverse image of $|\mathcal{U}(X)|_U$ under the unit map $X \to |\mathcal{U}(X)|$. Consequently, if $|\mathcal{U}(X)|_U = |\mathcal{U}(X)|_V$, then $U = V$.

It follows that the construction $\Lambda \mapsto \mathcal{U}(|\Lambda|)$ determines a right adjoint to the inclusion functor $\mathrm{Loc}^{\mathrm{spa}} \hookrightarrow \mathrm{Loc}$. In particular, the category of spatial locales is a colocalization of the category of locales.

1.5.3 Sober Topological Spaces

Our next goal is to describe the essential image of the embedding $\mathrm{Loc}^{\mathrm{spa}} \hookrightarrow \mathcal{T}\mathrm{op}$.

Definition 1.5.3.1. Let X be a topological space. A closed subset $K \subseteq X$ is said to be *irreducible* if it is nonempty and cannot be written as a union $K_- \cup K_+$ of proper closed subsets $K_-, K_+ \subsetneq K$. A point $x \in K$ is said to be a *generic point* of K if K is the closure of $\{x\}$. The space X is said to be *sober* if every irreducible closed subset of X has a unique generic point.

Remark 1.5.3.2. Let X be a topological space. For every point $x \in X$, the closure of the set $\{x\}$ is irreducible.

Example 1.5.3.3. Every Hausdorff topological space X is sober (the only irreducible closed subsets of X are singletons).

Proposition 1.5.3.4. *Let Λ be a locale. Then the topological space $|\Lambda|$ is sober.*

Proof. Let U be an indecomposable element of Λ. For each $V \in \Lambda$, we have $U \notin |\Lambda|_V$ if and only if $V \leqslant U$. In particular, $|\Lambda|_U$ is the largest open subset of $|\Lambda|$ which does not contain U, and is therefore the complement of the closure $\{U\}$. If U' is another indecomposable element of Λ having the same closure in $|\Lambda|$, then for each $V \in \Lambda$ we have

$$U \notin |\Lambda|_V \Leftrightarrow U' \notin |\Lambda|_V,$$

so that $V \leqslant U \Leftrightarrow V \leqslant U'$ and therefore $U = U'$.

Let K be an irreducible closed subset of $|\Lambda|$. The above argument shows that K has at most one generic point. We will complete the proof by showing that there exists a generic point of K. Set $S = \{U \in \Lambda : K = |\Lambda| - |\Lambda|_U\}$. Since K is closed, the set S is nonempty. It follows that S contains a largest element U. We will complete the proof by showing that U is indecomposable, so that $K = |\Lambda| - |\Lambda|_U$ is the closure of U.

Suppose that T is a finite subset of Λ satisfying $U = \bigwedge_{V \in T} V$. We wish to show that $U \in T$. Suppose otherwise: then, by the maximality of U, each of the open sets $|\Lambda|_V$ has nontrivial intersection with K. Since K is irreducible, it follows that the intersection $\bigcap_{V \in T} |\Lambda|_V$ has nontrivial intersection with K, contradicting our assumption that $|\Lambda|_U \cap K = \varnothing$. □

Proposition 1.5.3.5. *Let X be a topological space. The following conditions are equivalent:*

(1) *The topological space X is sober.*

(2) *The unit map $u : X \to |\mathcal{U}(X)|$ is a homeomorphism.*

Proof of Proposition 1.5.3.5. The implication (2) ⇒ (1) follows immediately from Proposition 1.5.3.4. For the converse, suppose that (1) is satisfied. Note that an open set $U \in \mathcal{U}(X)$ is indecomposable if and only if the complement $X - U$ is irreducible. Consequently, we can identify $|\mathcal{U}(X)|$ with the collection of irreducible closed subsets of X. Under this identification, the map u carries a point $x \in X$ to the closure of $\{x\}$. The assumption that X is sober implies that the map u is bijective. To complete the proof that u is a homeomorphism, it will suffice to note that every open subset $U \subseteq X$ is the inverse image (under the map u) of the open subset $|\mathcal{U}(X)|_U \subseteq \mathcal{U}(X)$. □

Remark 1.5.3.6. Combining Propositions 1.5.3.4 and 1.5.3.5, we conclude that the category $\mathcal{T}\mathrm{op}^{\mathrm{sob}}$ of sober topological spaces is a localization of the category $\mathcal{T}\mathrm{op}$ of all topological spaces. The inclusion $\mathcal{T}\mathrm{op}^{\mathrm{sob}} \hookrightarrow \mathcal{T}\mathrm{op}$ admits a left adjoint, given by $X \mapsto |\mathcal{U}(X)|$. We refer to the topological space $|\mathcal{U}(X)|$ as the *soberification* of X. The points of $|\mathcal{U}(X)|$ can be identified with irreducible closed subsets of X.

The above arguments show that the adjoint functors

$$\mathcal{T}\mathrm{op} \underset{||}{\overset{\mathcal{U}}{\rightleftarrows}} \mathrm{Loc}$$

restrict to an equivalence between the category $\mathcal{T}\mathrm{op}^{\mathrm{sob}}$ of sober topological spaces and the category $\mathrm{Loc}^{\mathrm{spa}}$ of spatial locales. $\mathcal{T}\mathrm{op}^{\mathrm{sob}} \simeq \mathrm{Loc}^{\mathrm{spa}}$

Algebraic geometry furnishes plenty of examples of sober topological spaces.

Proposition 1.5.3.7. *Let X be a topological space. Then:*

(a) *If X is sober, then any open subset $U \subseteq X$ is sober.*

(b) *If X can be written as a union of sober open subsets U_α, then X is sober.*

Proof. We first prove (a). Let $K \subseteq U$ be an irreducible closed subset of U, and let $\overline{K}$ be its closure in X. We first claim that $\overline{K}$ is irreducible. Suppose otherwise: then we can write $\overline{K} = \overline{K}_- \cup \overline{K}_+$ for some proper closed subsets $\overline{K}_-, \overline{K}_+ \subseteq \overline{K}$. In this case, we also have $K = (U \cap \overline{K}_-) \cup (U \cap \overline{K}_+)$. The irreducibility of K then implies that $K = U \cap \overline{K}_-$ or $K = U \cap \overline{K}_+$. Since K is dense in $\overline{K}$, this contradicts our assumption that either $\overline{K}_-$ and $\overline{K}_+$ are proper subsets of $\overline{K}$.

If X is irreducible, we conclude that $\overline{K}$ has a unique generic point $x \in X$. Then x belongs to the nonempty open subset $U \cap \overline{K} = K$ of $\overline{K}$, and therefore belongs to U. It follows that x is a generic point of K in U. We claim that this generic point is unique. To see this, suppose that y is any other generic point of K in U. Then $y \in K \subseteq \overline{K}$, so that the closure of $\{y\}$ in X is contained in $\overline{K}$. However, the closure of $\{y\}$ in X contains K and therefore contains $\overline{K}$. It follows that y is a generic point of $\overline{K}$ in X, so that $y = x$ by virtue of our assumption that X is sober. This completes the proof of (a).

We now prove (b). Suppose that X admits a covering by sober open subsets $\{U_\alpha\}_{\alpha \in A}$. Let $K \subseteq X$ be an irreducible closed subset. Then K is nonempty, so that the intersection $K_\alpha = U_\alpha \cap K$ is nonempty for some $\alpha \in A$. Then we can write $K = \overline{K}_\alpha \cup (K \cap (X - U_\alpha))$. Using the irreducibility of K, we deduce that $K = \overline{K}_\alpha$: that is, K_α is dense in K. We now claim that K_α is an irreducible closed subset of U_α. To see this, suppose that $K_\alpha = K_- \cup K_+$ for some closed subsets $K_-, K_+ \subset K_\alpha$. Then $K = \overline{K}_- \cup \overline{K}_+$, so that the irreducibility of K implies that $K = \overline{K}_-$ or $\overline{K}_+$. Since K_- and K_+ are closed in U_α, we conclude that $K_\alpha = U_\alpha \cap K$ is equal to either K_- or K_+.

Since U_α is irreducible, the set K_α has a unique generic point $x \in U_\alpha$. Then the closure of $\{x\}$ in X is given by $\overline{K}_\alpha = K$, so that x is a generic point of K. Let y be any other generic point of K. Then y is contained in the nonempty open subset $K_\alpha \subseteq K$, so that $y \in U_\alpha$. It follows that y is a generic point of K_α in U_α, so that $y = x$ by virtue of our assumption that U_α is sober. This completes the proof of (b). □

Corollary 1.5.3.8. *Let $(X, \mathscr{O}_X)$ be a nonconnective spectral scheme. Then the topological space X is sober.*

Proof. Using Proposition 1.5.3.7 and Corollary 1.1.6.2, we can reduce to the case where $(X, \mathscr{O}_X) \simeq \operatorname{Spec} A$ for some connective $\mathbb{E}_\infty$-ring A, so that X is homeomorphic to $|\operatorname{Spec} R|$ for $R = \pi_0 A$. In this case, every closed set $K \subseteq X$ can be realized as the vanishing locus of a radical ideal $I \subseteq R$. Moreover, K is irreducible if and only if I is a prime ideal, in which case I is the unique generic point of K (when regarded as an element of $|\operatorname{Spec} R|$). □

1.5.4 Locales Associated to ∞-Topoi

We now discuss another class of examples of locales:

Definition 1.5.4.1. Let $\mathcal{X}$ be an ∞-topos. We let $\operatorname{Sub}(\mathcal{X})$ denote the collection of equivalence classes of (-1)-truncated objects of $\mathcal{X}$. For each (-1)-truncated object $U \in \mathcal{X}$, we let $[U] \in \operatorname{Sub}(\mathcal{X})$ denote the equivalence class of U. We regard $\operatorname{Sub}(\mathcal{X})$ as a partially ordered set, with $[U] \leqslant [V]$ if there exists a morphism from U to V in the ∞-topos $\mathcal{X}$. Then $\operatorname{Sub}(\mathcal{X})$ is a locale, which we refer to as the *underlying locale* of X (see §HTT.6.4.2).

Remark 1.5.4.2. The construction $\mathcal{X} \mapsto \operatorname{Sub}(\mathcal{X})$ determines a functor $\operatorname{Sub} : \infty\mathcal{T}\mathrm{op} \to \mathrm{Loc}$. This is a localization functor: that is, it admits a fully faithful right adjoint $\iota : \mathrm{Loc} \hookrightarrow \infty\mathcal{T}\mathrm{op}$, whose essential image is spanned by the 0-localic ∞-topoi (see §HTT.6.4.5 .

Definition 1.5.4.3. Let $\mathcal{X}$ be an ∞-topos. We let $|\mathcal{X}|$ denote the underlying topological space $|\operatorname{Sub}(\mathcal{X})|$ of the locale $\operatorname{Sub}(\mathcal{X})$. We will say that $\mathcal{X}$ is *spatial* if the locale $\operatorname{Sub}(\mathcal{X})$ is spatial.

Example 1.5.4.4. Let X be a topological space, and let $\mathcal{S}\mathrm{hv}(X)$ denote the ∞-topos of $\mathcal{S}$-valued sheaves on X. Then we can identify $\operatorname{Sub}(\mathcal{S}\mathrm{hv}(X))$ with the locale $\mathcal{U}(X)$ of open subsets of X. It follows that the topological space $|\mathcal{S}\mathrm{hv}(X)|$ can be identified with the soberification of X. In particular, there is a canonical map $X \to |\mathcal{S}\mathrm{hv}(X)|$, which is a homeomorphism if and only if X is sober.

Example 1.5.4.5. Let $\mathcal{X}$ be an ∞-topos. Every geometric morphism $f_* : \mathcal{S} \to \mathcal{X}$ induces a continuous map $|\,\mathcal{S}\,| \to |\,\mathcal{X}\,|$, which we can identify with a point η_f of the topological space $|\,\mathcal{X}\,|$. Let U be a (-1)-truncated object of $\mathcal{X}$. Then for each point $f_* : \mathcal{S} \to \mathcal{X}$, the space f^*U is either empty or contractible, depending on whether or not η_f belongs to $|\,\mathcal{X}\,|_U$. If V is another (-1)-truncated object of $\mathcal{X}$ satisfying $|\,\mathcal{X}\,|_V$, it follows that the canonical maps

$$f^*U \leftarrow f^*(U \times V) \to f^*(V)$$

are homotopy equivalences. If the ∞-topos $\mathcal{X}$ has enough points (see §A.4), we conclude that the projection maps $U \leftarrow U \times V \to V$ are equivalences, so that $[U] = [V] \in \mathrm{Sub}(\mathcal{X})$. It follows that every ∞-topos with enough points is spatial.

Proposition 1.5.4.6. *Let R be an $\mathbb{E}_\infty$-ring. Then the ∞-topos $\mathcal{S}\mathrm{hv}_R^{\acute{e}t}$ is spatial. Moreover, there is a canonical homeomorphism $|\,\mathcal{S}\mathrm{hv}_R^{\acute{e}t}\,| \simeq |\operatorname{Spec} R|$.*

Proof. Let $X = |\operatorname{Spec} R|$, and let $\mathcal{U}(X)$ be the collection of all open subsets of X. Let $\mathcal{C} = (\mathrm{CAlg}_R^{\acute{e}t})^{\mathrm{op}}$ denote the opposite of the ∞-category of étale R-algebras, so that $\mathcal{S}\mathrm{hv}_R^{\acute{e}t}$ is the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ spanned by the étale sheaves. It follows that $\mathrm{Sub}(\mathcal{S}\mathrm{hv}_R^{\acute{e}t})$ can be identified with the partially ordered set P of sieves $\mathcal{C}^{(0)} \subseteq \mathcal{C}$ which are *saturated* in the following sense: if A is an étale R-algebra, and there exists an étale covering $\{A \to A_i\}$ for which each A_i belongs to $\mathcal{C}^{(0)}$, then A belongs to $\mathcal{C}^{(0)}$.

For each open set $U \subseteq X$, let $\lambda(U)$ denote the full subcategory of $\mathcal{C}$ spanned by those objects A for which the map $|\operatorname{Spec} A| \to |\operatorname{Spec} R|$ factors through U. We will complete the proof by showing that the construction $U \mapsto \lambda(U)$ determines an isomorphism of partially ordered sets $\lambda : \mathcal{U}(X) \to P$.

For each element $a \in \pi_0 R$, let $U_a = \{\mathfrak{p} \in |\operatorname{Spec} R| : a \notin \mathfrak{p}\}$. We first claim that if U and V are open subsets of X such that $\lambda(U) \subseteq \lambda(V)$, then $U \subseteq V$. Since U is the union of basic open sets of the form U_a, we may assume that $U = U_a$ for some $a \in \pi_0 R$. Then $R[a^{-1}] \in \lambda(U) \subseteq \lambda(V)$, so that V contains the image of the map $|\operatorname{Spec} R[a^{-1}]| \to |\operatorname{Spec} R| = X$ (which coincides with U).

The above argument shows that λ is an isomorphism of $\mathcal{U}(X)$ onto a partially ordered subset of P. To complete the proof, it will suffice to show that λ is surjective. To this end, choose a saturated sieve $\mathcal{C}^{(0)} \subseteq \mathcal{C}$; we wish to show that $\mathcal{C}^{(0)}$ lies in the image of λ. For every étale R-algebra A, let U_A denote the image of the map $|\operatorname{Spec} A| \to |\operatorname{Spec} R|$; this is an open subset of X (Corollary B.2.2.5). Let U be the smallest open subset of X which contains U_A for each $A \in \mathcal{C}^{(0)}$. By construction, we have $\mathcal{C}^{(0)} \subseteq \lambda(U)$. To complete the proof, it suffices to show that this inclusion is an equality. That is, we must show that if A is an étale R-algebra such that the image of the map $\theta : |\operatorname{Spec} A| \to |\operatorname{Spec} R| = X$ is contained in U, then $A \in \mathcal{C}^{(0)}$. Since the image of θ is quasi-compact, it is contained in a finite union of $\bigcup_{1 \leqslant i \leqslant n} U_{B_i}$, where each $B_i \in \mathcal{C}^{(0)}$. It follows that the map $A \to \prod_{1 \leqslant i \leqslant n} (B_i \otimes_R A)$ is étale and faithfully flat. Since $\mathcal{C}^{(0)}$ is a saturated sieve containing each B_i, it must also contain A. □

Remark 1.5.4.7. Let $\phi : \mathcal{T}\mathrm{op} \to \infty\mathcal{T}\mathrm{op}$ be the functor of Remark 1.4.1.6, which carries a topological space X to the ∞-topos $\mathcal{S}\mathrm{hv}(X)$. Then ϕ factors as a composition

$$\mathcal{T}\mathrm{op} \xrightarrow{\mathcal{U}} \mathrm{Loc} \xrightarrow{\iota} \infty\mathcal{T}\mathrm{op},$$

where ι is the fully faithful embedding of Remark 1.5.4.2, and $\mathcal{U}$ is the functor of Example 1.5.1.6. It follows that ϕ is fully faithful when restricted to sober topological spaces (Remark 1.5.3.6). Consequently, the functor $\overline{\phi} : \mathcal{T}\mathrm{op}_{\mathrm{CAlg}} \to \infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ of Remark 1.4.1.6 is fully faithful when restricted to spectrally ringed spaces $(X, \mathscr{O}_X)$ for which X is sober. In particular, $\overline{\phi}$ is fully faithful when restricted to nonconnective spectral schemes (Corollary 1.5.3.8).

1.6 The Functor of Points

In classical algebraic geometry, we can often describe algebraic varieties (or schemes) as solutions to *moduli problems*. For example, the n-dimensional projective space $\mathbf{P}^n$ can be characterized as follows: it is universal among schemes over which there is a line bundle $\mathcal{L}$ generated by $(n+1)$ global sections. In particular, for any commutative ring A, the set $\mathrm{Hom}(\mathrm{Spec}\, A, \mathbf{P}^n)$ can be identified with the set of isomorphism classes of pairs $(L, \eta : A^{n+1} \to L)$ where L is an invertible A-module and η is a surjective A-module homomorphism (such a pair is determined up to unique isomorphism by the submodule $\ker(\eta) \subseteq A^{n+1}$).

More generally, *any* scheme X determines a covariant functor h_X from commutative rings to sets, given by the formula $h_X(A) = \mathrm{Hom}(\mathrm{Spec}\, A, X)$. We refer to $h_X(A)$ as the *set of A-valued points of* X, and to h_X as the *functor of points* of X. This functor determines X up to canonical isomorphism. More precisely, the construction $X \mapsto h_X$ determines a fully faithful embedding from the category of schemes to the presheaf category $\mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \mathcal{S}\mathrm{et})$. Consequently, it is possible to think of schemes as objects of $\mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \mathcal{S}\mathrm{et})$, rather than the category of locally ringed spaces. This point of view is often valuable: it is sometimes easier to describe the functor represented by a scheme X than it is to provide an explicit construction of X as a locally ringed space. Moreover, the "functor of points" perspective becomes essential when we wish to study more general algebro-geometric objects such as algebraic stacks.

1.6.1 The Case of a Spectrally Ringed Space

We begin by associating a functor to each spectrally ringed space.

Definition 1.6.1.1. Let $(X, \mathscr{O}_X)$ be a locally spectrally ringed space. We define a functor $h_X^{\mathrm{nc}} : \mathrm{CAlg} \to \mathcal{S}$ by the formula

$$h_X^{\mathrm{nc}}(R) = \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Spec}\, R, (X, \mathscr{O}_X)),$$

where $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ denotes the ∞-category of locally spectrally ringed spaces (see Definition 1.1.5.3). We let h_X denote the restriction of h_X^{nc} to the full subcategory $\mathrm{CAlg}^{\mathrm{cn}} \subseteq \mathrm{CAlg}$ spanned by the connective $\mathbb{E}_\infty$-rings. We will refer to both h_X and h_X^{nc} as the *functor of points* of $(X, \mathscr{O}_X)$.

Warning 1.6.1.2. The notation of Definition 1.6.1.1 is abusive: if $(X, \mathscr{O}_X)$ is a locally spectrally ringed ∞-topos, then the functors h_X^{nc} and h_X depend on the structure sheaf $\mathscr{O}_X$, and not only on the underlying topological space X.

1.6.2 Flat Descent

Our first main result in this section can be stated as follows:

Theorem 1.6.2.1. *Let $(X, \mathscr{O}_X)$ be a locally spectrally ringed space. Then the functor $h_X^{\mathrm{nc}} : \mathrm{CAlg} \to \mathcal{S}$ is a hypercomplete sheaf with respect to the fpqc topology of Proposition B.6.1.3.*

Since the construction $\mathrm{Spec} : \mathrm{CAlg}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ is fully faithful (Remark 1.1.5.7), the functor $A \mapsto h^{\mathrm{nc}}_{\mathrm{Spec}\, A}$ coincides with the Yoneda embedding $\mathrm{CAlg}^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$. We may therefore view Theorem D.6.3.5 (which asserts that the fpqc topology on $\mathrm{CAlg}^{\mathrm{op}}$ is subcanonical) as a special case of Theorem 1.6.2.1. This observation does not supply a new proof of Theorem D.6.3.5, because Theorem D.6.3.5 is one of the main ingredients in our proof of Theorem 1.6.2.1. The other main ingredient is the compatibility of the Zariski topology with flat descent, which can be formulated more precisely as follows:

Proposition 1.6.2.2. *For every $\mathbb{E}_\infty$-ring A, let $\mathcal{U}(A)$ be the collection set of open subsets of the topological space $|\mathrm{Spec}\, A|$. Then $A \mapsto \mathcal{U}(A)$ determines a functor $\mathcal{U} : \mathrm{CAlg} \to \mathcal{S}\mathrm{et}$. This functor is a sheaf (of sets) with respect to the fpqc topology on $\mathrm{CAlg}^{\mathrm{op}}$.*

Remark 1.6.2.3. The sheaf $\mathcal{U} : \mathrm{CAlg} \to \mathcal{S}\mathrm{et}$ of Proposition 1.6.2.2 can be regarded as a discrete object in the ∞-category of $\mathcal{S}$-valued sheaves on $\mathrm{CAlg}^{\mathrm{op}}$. Consequently, it is automatically hypercomplete.

Proof of Proposition 1.6.2.2. We will show that the functor $\mathcal{U} : \mathrm{CAlg} \to \mathcal{S}\mathrm{et}$ satisfies conditions (1) and (2) of Proposition A.3.3.1. To verify (1), we must show that for every finite collection of $\mathbb{E}_\infty$-rings $\{A_i\}_{1 \leqslant i \leqslant n}$, the map $\mathcal{U}(\prod A_i) \to \prod \mathcal{U}(A_i)$ is bijective. This follows from the observation that there is a canonical homeomorphism $|\mathrm{Spec}(\prod_i A_i)| \simeq \amalg_i |\mathrm{Spec}\, A_i|$.

We now prove (2). Let $f : A \to B$ be a faithfully flat morphism of $\mathbb{E}_\infty$-rings; we wish to prove that

$$\mathcal{U}(A) \longrightarrow \mathcal{U}(B) \rightrightarrows \mathcal{U}(B \otimes_A B)$$

is an equalizer diagram in the category of sets. We can divide this assertion into two parts:

(a) The map $\mathcal{U}(A) \to \mathcal{U}(B)$ is injective. To prove this, we must show that an open subset $U \subseteq |\operatorname{Spec} A|$ is determined by its inverse image in $|\operatorname{Spec} B|$. This is clear, since the assumption that $A \to B$ is faithfully flat guarantees the induced map $|\operatorname{Spec} B| \to |\operatorname{Spec} A|$ is surjective.

(b) Let $\phi_0, \phi_1 : |\operatorname{Spec} B \otimes_A B| \to |\operatorname{Spec} B|$ be the two projection maps. We claim that if $Z \subseteq |\operatorname{Spec} B|$ is a closed subset with $\phi_0^{-1} Z = \phi_1^{-1} Z$, then $Z = \phi^{-1} V$ for some closed subset $V \subseteq |\operatorname{Spec} B|$. Choose an ideal $I \subseteq \pi_0 B$ such that $Z = \{\mathfrak{p} \subseteq \pi_0 B : I \subseteq \mathfrak{p}\}$, and let $J = f^{-1} I \subseteq \pi_0 A$. Set $V = \{\mathfrak{q} \subseteq \pi_0 A : J \subseteq \mathfrak{q}\}$. Then $\phi^{-1} V = \{\mathfrak{p} \subseteq \pi_0 B : f(J)\pi_0 B \subseteq \mathfrak{p}\}$. To prove that $\phi^{-1} V = Z$, it suffices to show that $f(J)\pi_0 B$ and I have the same nilradical. Let R denote the commutative ring $\pi_0 A / J$ and R' the commutative ring $\pi_0 B / J \pi_0 B$, and let I' denote the image of I in R'. Then $R \to R'$ is faithfully flat and the composite map $R \to R' \to R'/I'$ is injective; we wish to prove that every element $x \in I'$ is nilpotent. Since $\phi_0^{-1} Z = \phi_1^{-1} Z$, we deduce that the ideals $I' \otimes_R R'$ and $R' \otimes_R I'$ have the same radical in $R' \otimes_R R'$. Consequently, since $x \otimes 1$ belongs to $I' \otimes_R R'$, some power $x^n \otimes 1$ belongs to $R' \otimes_R I'$. It follows that the image of x^n vanishes in $R' \otimes_R R'/I'$. Since R' is flat over R, the injection $R \to R'/I'$ induces an injection $R' \to R' \otimes_R R'/I'$. It follows that $x^n = 0$ in R', as desired.

□

Proposition 1.6.2.4. (1) *The functor* $\operatorname{Spec} : \mathrm{CAlg}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{loc}}$ *preserves finite coproducts.*

(2) *Let R be an $\mathbb{E}_\infty$-ring, and let $R^\bullet$ be a cosemisimplicial $\mathbb{E}_\infty$-ring which is a hypercovering of R with respect to the fpqc topology (see Definition A.5.7.1). Then* $\operatorname{Spec} R$ *is a colimit of the diagram* $\{\operatorname{Spec} R^\bullet\}$ *in* $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{loc}}$.

Proof. Let $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ denote the ∞-category of spectrally ringed spaces (Definition 1.1.2.5) and let $\mathcal{T}\mathrm{op}$ denote the ordinary category of topological spaces and continuous maps, so that we have forgetful functors

$$\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{loc}} \xrightarrow{j} \mathcal{T}\mathrm{op}_{\mathrm{CAlg}} \xrightarrow{q} \mathcal{T}\mathrm{op}.$$

We will deduce assertion (1) from the following three claims:

(1′) The functor $q \circ j \circ \operatorname{Spec} : \mathrm{CAlg}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}$ preserves finite coproducts.

(1″) The functor $j \circ \mathrm{Spec} : \mathrm{CAlg}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ carries finite coproducts to q-coproducts.

(1‴) The functor $\mathrm{Spec} : \mathrm{CAlg}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ carries finite coproducts to j-coproducts.

To prove these claims, let $\{R_i\}_{1 \leqslant i \leqslant n}$ be a finite collection of $\mathbb{E}_\infty$-rings having product R. Let $X_i = |\operatorname{Spec} R_i|$ and let $X = |\operatorname{Spec} R|$, so that we can write $\operatorname{Spec} R_i = (X_i, \mathscr{O}_{X_i})$ and $\operatorname{Spec} R = (X, \mathscr{O}_X)$. For each index i, let $\phi_i : X_i \to X$ denote the map induced by the projection $R \to R_i$. Assertion (1′) was established as part of Proposition 1.6.2.2. By virtue of Proposition HTT.4.3.1.9 , assertion (1″) is equivalent to the requirement that the canonical map $\mathscr{O}_X \to \prod_i (\phi_i)_* \mathscr{O}_{X_i}$ is an equivalence of CAlg-valued X. Note that X has a basis of open sets of the form $U_f = \{\mathfrak{p} \subset \pi_0 R : f \notin \mathfrak{p}\}$, where $f = (f_1, \ldots, f_n)$ ranges over the elements of $\pi_0 R \simeq \pi_0 R_1 \times \cdots \times \pi_0 R_n$. Since this basis is stable under finite intersections, it suffices to observe that the canonical map

$$R[f^{-1}] \simeq \mathscr{O}_X(U_f) \to (\prod (\phi_i)_* \mathscr{O}_{X_i})(U_f) \simeq \prod \mathscr{O}_{X_i}(U_f \times_X X_i) \simeq \prod R_i[f_i^{-1}]$$

is an equivalence of $\mathbb{E}_\infty$-rings.

Unwinding the definitions, we can formulate assertion (1‴) as follows: a morphism $g : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ in $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ belongs to $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ if and only if, for $1 \leqslant i \leqslant n$, the induced map $g_i : (X_i, \mathscr{O}_{X_i}) \to (Y, \mathscr{O}_Y)$ belongs to $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$. This follows immediately from the definitions, since $\mathscr{O}_{X_i}$ can be identified with the restriction $\mathscr{O}_X|_{X_i}$.

We now prove (2). Let $R^\bullet : \mathbf{\Delta}_{s,+} \to \mathrm{CAlg}_R$ be an fpqc hypercovering of $R = R^{-1}$ in the ∞-category $\mathrm{CAlg}^{\mathrm{op}}$. Reasoning as above, we are reduced to proving the following three assertions:

(2′) The composition $q \circ j \circ \operatorname{Spec} \circ R^\bullet$ is a colimit diagram in the ∞-category $\mathcal{T}\mathrm{op}$.

(2″) The composition $j \circ \operatorname{Spec} \circ R^\bullet$ is a q-colimit diagram in the ∞-category $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$.

(2‴) The composition $\operatorname{Spec} \circ R^\bullet$ is a j-colimit diagram in the ∞-category $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$.

By virtue of (1′) and Proposition A.5.7.2, assertion (2′) is equivalent to the requirement that the functor

$$q \circ j \circ \mathrm{Spec} : \mathrm{CAlg} \to \mathcal{T}\mathrm{op}^{\mathrm{op}}$$

is a hypercomplete sheaf with respect to the fpqc topology. Because $\mathcal{T}\mathrm{op}$ is an ordinary category, it will suffice to show that $q \circ j \circ \mathrm{Spec}$ is a sheaf with respect to the fpqc topology, which follows from Proposition 1.6.2.2. We now prove (2″). Let $X = |\operatorname{Spec} R|$, so that we can write $\operatorname{Spec} R = (X, \mathscr{O}_X)$. For every nonnegative integer n let $X_n = |\operatorname{Spec} R^n|$ and write $\operatorname{Spec} R^n = (X_n, \mathscr{O}_{X_n})$. Let $\mathscr{F}^n$ denote the pushforward of $\mathscr{O}_{X_n}$ along the canonical map $X_n \to X$. Then $\mathscr{F}^\bullet$ is a cosemisimplicial object in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)$. By virtue of Proposition HTT.4.3.1.9 , condition 23″) is equivalent to the requirement that the canonical map $\alpha : \mathscr{O}_X \to \varprojlim \mathscr{F}^\bullet$ is an equivalence. We note that X has a basis of open sets of the form $U_f = \{\mathfrak{p} \subset \pi_0 R^{-1} : f \notin \mathfrak{p}\}$. Since this collection is stable under finite intersection, to prove that α is an equivalence it suffices to show that α induces an equivalence of $\mathbb{E}_\infty$-rings $\mathscr{O}_X(U_f) \to \varprojlim \mathscr{F}^\bullet(U_f)$, for each $f \in \pi_0 R$. Replacing $R^\bullet$ by $R^\bullet[f^{-1}]$, we can reduce to the case where $U_f = X$. In this case, we need to show that the map

$$R \simeq \mathscr{O}(X) \to \varprojlim \mathscr{F}^\bullet(X) \simeq \varprojlim \mathscr{O}_{X_\bullet}(X_\bullet) \simeq \varprojlim R^\bullet$$

is an equivalence of $\mathbb{E}_\infty$-rings, which follows from Theorem D.6.3.5.

It remains to prove $(2''')$. Unwinding the definitions, we must show that if $(Y, \mathscr{O}_Y)$ is an object of $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$, then a map $\phi : \operatorname{Spec} R \to (Y, \mathscr{O}_Y)$ in $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ belongs to $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ if and only if the induced map $\phi^0 : (Y, \mathscr{O}_Y) \to \operatorname{Spec} R^0$ belongs to $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$. Let $f : X \to Y$ be the map of topological spaces underlying ϕ, and set $\mathscr{O}' = f^* \mathscr{O}_Y \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(X)$. Let $g : X_0 \to X$ denote the projection map. We are then reduced to proving the following: a morphism of sheaves of rings $\alpha : \pi_0 \mathscr{O}_X \to \pi_0 \mathscr{O}'$ is local if and only if the composite map $g^*\pi_0 \mathscr{O}' \xrightarrow{\alpha} g^*\pi_0 \mathscr{O}_X \to \pi_0 \mathscr{O}_{X_0}$ is a local map (between sheaves of local rings on X_0). This follows immediately from the observation that the map $g : X_0 \to X$ is surjective (since the underlying map of commutative rings $\pi_0 R \to \pi_0 R^0$ is assumed to be faithfully flat). □

Proof of Theorem 1.6.2.1. Combine Proposition 1.6.2.4 with Proposition A.5.7.2. □

1.6.3 The Functor of Points of a Spectral Scheme

We now use Theorem 1.6.2.1 to investigate the functor of points of a (possibly nonconnective) spectral scheme.

Proposition 1.6.3.1. *Let $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}} \subseteq \operatorname{Fun}(\mathrm{CAlg}, \mathcal{S})$ and $\mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{fpqc}} \subseteq \operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ denote the full subcategories spanned by those functors which are sheaves for the fpqc topology. Let $(X, \mathscr{O}_X)$ be a spectrally ringed space. For each open set $U \subseteq X$, let us regard $(U, \mathscr{O}_X|_U)$ as a spectrally ringed space which represents functors $h^{\mathrm{nc}}_U \in \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}$ and $h_U \in \mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{fpqc}}$. Then:*

(1) *The construction $U \mapsto h^{\mathrm{nc}}_U$ determines a $\mathcal{S}\mathrm{hv}^{\mathrm{op}}_{\mathrm{fpqc}}$-valued sheaf on X.*

(2) *The construction $U \mapsto h_U$ determines a $(\mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{fpqc}})^{\mathrm{op}}$-valued sheaf on X.*

Proof. We will prove (1); the proof of (2) is similar. Let $\{U_\alpha\}$ be a collection of open subsets of X, let $U = \bigcup U_\alpha$, and let $\mathcal{U}$ be the collection of open subsets of X which are contained in some U_α. We wish to prove that h^{nc}_U is a colimit of the diagram $\{h^{\mathrm{nc}}_V\}_{V \in \mathcal{U}}$ in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}$. Let $\mathcal{Y} \subseteq \operatorname{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})$ denote the full subcategory spanned by those functors which are sheaves with respect to the fpqc topology. Then $\mathcal{Y}$ is an ∞-topos (in a larger universe), and $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}$ is a full subcategory of $\mathcal{Y}$. It will therefore suffice to show that h^{nc}_U is a colimit of the diagram $\{h^{\mathrm{nc}}_V\}_{V \in \mathcal{U}}$ in the ∞-category $\mathcal{Y}$.

Let $j : \mathrm{CAlg}^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})$ denote the Yoneda embedding. Then j factors through $\mathcal{Y}$ (Theorem D.6.3.5), and sheaves of the form $j(R)$ generate $\mathcal{Y}$ under colimits. It will therefore suffice to show that for every map $\phi : j(R) \to h_U^{\mathrm{nc}}$, the canonical map

$$\varinjlim_{V \in \mathcal{U}} (j(R) \times_{h_U^{\mathrm{nc}}} h_V^{\mathrm{nc}}) \to j(R)$$

is an equivalence in $\mathcal{Y}$. Note that ϕ determines a continuous map of topological spaces $f : |\operatorname{Spec} R| \to U$, and that $j(R) \times_{h_U^{\mathrm{nc}}} h_V^{\mathrm{nc}}$ can be identified with the subfunctor $j_V(R) \subseteq j(R)$ which carries an $\mathbb{E}_\infty$-ring A to the summand of $\mathrm{Map}_{\mathrm{CAlg}}(R, A)$ spanned by those maps $R \to A$ for which the induced map of topological spaces $|\operatorname{Spec} A| \to |\operatorname{Spec} R|$ factors through $f^{-1}(V)$.

Let $\mathcal{C}^{(0)} \subseteq \mathcal{Y}_{/j(R)}$ denote the sieve generated by the collection of morphisms $\{j_V(R) \hookrightarrow j(R)\}_{V \in \mathcal{U}}$. Then the construction $V \mapsto j_V(R)$ determines a left cofinal map from $\mathcal{U}$ into $\mathcal{C}^{(0)}$. Consequently, to prove that $j(R) \simeq \varinjlim_{V \in \mathcal{U}} j_V(R)$, it will suffice to show that $\mathcal{C}^{(0)}$ is a covering sieve with respect to the canonical topology on $\mathcal{Y}$ (see §HTT.6.2.4). Because the open sets U_α cover U, we can choose elements $\{a_i \in \pi_0 R\}_{1 \leqslant i \leqslant n}$ which generate the unit ideal, such that each of the maps $|\operatorname{Spec} R[a_i^{-1}]| \to |\operatorname{Spec} R|$ factors through some $f^{-1} U_{\alpha_i}$. Then the sieve $\mathcal{C}^{(0)}$ contains the maps $j(R[a_i^{-1}]) \to j(R)$, and is therefore a covering sieve since the map $R \to \prod R[a_i^{-1}]$ is faithfully flat. □

Remark 1.6.3.2. In the statement and proof of Proposition 1.6.3.1, we can replace the fpqc topology with the Zariski topology.

Corollary 1.6.3.3. (1) *The construction* $(X, \mathscr{O}_X) \mapsto h_X^{\mathrm{nc}}$ *determines a fully faithful embedding* $\mathrm{SpSch}^{\mathrm{nc}} \to \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$.

(2) *The construction* $(X, \mathscr{O}_X) \mapsto h_X$ *determines a fully faithful embedding* $\mathrm{SpSch} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$.

Proof. We will prove (1); the proof of (2) is similar. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be nonconnective spectral schemes. For each open set $U \subseteq X$, let us regard $(U, \mathscr{O}_X|_U)$ as a nonconnective spectral scheme, which represents a functor $h_U^{\mathrm{nc}} : \mathrm{CAlg} \to \mathcal{S}$. Let us say that an open subset $U \subseteq X$ is *good* if the canonical map

$$\theta_U : \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((U, \mathscr{O}_X|_U), (Y, \mathscr{O}_Y)) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})}(h_U^{\mathrm{nc}}, h_Y^{\mathrm{nc}})$$

is an equivalence. We will complete the proof by showing that every open subset of X is good. The main step is the following:

(∗) Let U be an open subset of X which is given as a union $\bigcup_{\alpha \in A} U_\alpha$, and suppose that each finite intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$ is good. Then U is good.

Assertion $(*)$ follows immediately from Theorem 1.6.2.1 and Proposition 1.6.3.1, which implies that θ_U can be obtained as a limit of maps of the form θ_V where V has the form $U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$.

We now complete the argument as follows:

(a) Every affine open subset $U \subseteq X$ is good (this follows immediately from the definitions).

(b) Let $U \subseteq X$ be an open set which is contained in an affine open set V. Then we can write U as a union of affine open sets $\{U_\alpha\}$. Since the collection of affine open subsets of V is closed under finite intersections, it follows from (a) that every finite intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$ is good. Invoking $(*)$, we conclude that U is good.

(c) Let $U \subseteq X$ be an arbitrary open set, and write U as a union of affine open subsets U_α. Then each intersection $U_{\alpha_1} \cap \cdots U_{\alpha_n}$ is contained in U_{α_1}, and therefore good (by virtue of (b)). Applying $(*)$, we deduce that U is good.

□

1.6.4 The Functor of Points of a Spectrally Ringed ∞-Topos

We now introduce a variant of Definition 1.6.1.1, which is designed to accommodate spectral Deligne-Mumford stacks as well as spectral schemes.

Definition 1.6.4.1. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a locally spectrally ringed ∞-topos. For every $\mathbb{E}_\infty$-ring R, we let $h^{\mathrm{nc}}_{\mathsf{X}}(R)$ denote the mapping space $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X})$. Then the construction $R \mapsto h^{\mathrm{nc}}_{\mathsf{X}}(R)$ determines a functor $h^{\mathrm{nc}}_{\mathsf{X}} : \mathrm{CAlg} \to \widehat{\mathcal{S}}$. We let h_{X} denote the restriction of $h^{\mathrm{nc}}_{\mathsf{X}}$ to the full subcategory $\mathrm{CAlg}^{\mathrm{cn}} \subseteq \mathrm{CAlg}$ spanned by the connective $\mathbb{E}_\infty$-rings. We refer to both $h^{\mathrm{nc}}_{\mathsf{X}}$ and h_{X} as the *functor of points* of the spectrally ringed ∞-topos X.

We have the following analogue of Corollary 1.6.3.3:

Proposition 1.6.4.2. (1) *Let X be a nonconnective spectral Deligne-Mumford stack. Then, for every $\mathbb{E}_\infty$-ring R, the space $h^{\mathrm{nc}}_{\mathsf{X}}(R) = \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X})$ is essentially small.*

(2) *The construction $\mathsf{X} \mapsto h^{\mathrm{nc}}_{\mathsf{X}}$ determines a fully faithful embedding $\mathrm{SpDM}^{\mathrm{nc}} \to \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$.*

(3) *The construction $\mathsf{X} \mapsto h_{\mathsf{X}}$ determines a fully faithful embedding $\mathrm{SpDM} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$.*

Proof. This is a special case of Theorem **??**, which we will prove in Part VI. See Remarks **??** and **??**. (See also Theorem 8.1.5.1 for a generalization of (1) and (3)). □

1.6.5 The Spatial Case

We have now given introduced a "functor of points" in two different contexts: locally spectrally ringed spaces (Definition ??), and locally spectrally ringed ∞-topoi (Definition 1.6.4.1). These two definitions are not identical, but we will show that they are closely related to one another (Remark 1.6.5.13).

Construction 1.6.5.1. Let $\mathcal{X}$ be a spatial ∞-topos. Then the construction $U \mapsto |\mathcal{X}|_U$ determines an equivalence from the full subcategory $\tau_{\leqslant -1}\mathcal{X}$ of (-1)-truncated objects of $\mathcal{X}$ to the partially ordered set $\mathcal{U}(|\mathcal{X}|)$ of open subsets of $|\mathcal{X}|$.

Let $\mathcal{C}$ be an ∞-category, and let $\mathscr{O} \in \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ be a $\mathcal{C}$-valued sheaf on $\mathcal{X}$. We let $\mathscr{O}^{\mathrm{spa}}$ denote the composite functor

$$\mathcal{U}(|\mathcal{X}|)^{\mathrm{op}} \simeq (\tau_{\leqslant -1}\mathcal{X})^{\mathrm{op}} \hookrightarrow \mathcal{X}^{\mathrm{op}} \xrightarrow{\mathscr{O}} \mathcal{C}.$$

Proposition 1.6.5.2. *Let $\mathcal{X}$ be a spatial ∞-topos, let $\mathcal{C}$ be an ∞-category, and let $\mathscr{O} : \mathcal{X}^{\mathrm{op}} \to \mathcal{C}$ be a $\mathcal{C}$-valued sheaf on $\mathcal{X}$. Then $\mathscr{O}^{\mathrm{spa}} : \mathcal{U}(|\mathcal{X}|)^{\mathrm{op}} \to \mathcal{C}$ is a $\mathcal{C}$-valued sheaf on the topological space $|\mathcal{X}|$.*

Proof. Let $\{U_\alpha\}$ be a collection of open subsets of $|\mathcal{X}|$ with union U, and let $\mathcal{U}$ be the collection of open subsets of $|\mathcal{X}|$ which are contained in some U_α. We wish to show that the functor $\mathscr{O}^{\mathrm{spa}}$ exhibits $\mathscr{O}^{\mathrm{spa}}(U)$ as a limit of the diagram $\{\mathscr{O}^{\mathrm{spa}}(V)\}_{V \in \mathcal{U}}$.

Let us abuse notation by identifying open subsets of $|\mathcal{X}|$ with (-1)-truncated objects of $\mathcal{X}$, so that we are required to prove that $\mathscr{O}$ exhibits $\mathscr{O}(U)$ as a limit of the diagram $\{\mathscr{O}(V)\}_{V \in \mathcal{U}}$. Since $\mathscr{O}$ is a sheaf, it will suffice to show that the canonical map $\varinjlim_{V \in \mathcal{U}} V \to U$ is an equivalence in the ∞-topos $\mathcal{X}$. Since the U_α cover U, this is equivalent to the assertion that each of the induced maps $U_\alpha \times_U (\varinjlim_{V \in \mathcal{U}} V) \to U_\alpha$ is an equivalence in $\mathcal{X}$. Since colimits are universal in $\mathcal{X}$, we can rewrite the domain of this map as a colimit $\varinjlim_{V \in \mathcal{U}}(U_\alpha \times_U V)$. Let $\mathcal{U}_\alpha$ denote the subset of $\mathcal{U}$ consisting of those open sets which are contained in U_α. Since the functor $V \mapsto U_\alpha \times_U V$ is a left Kan extension of its restriction to $\mathcal{U}_\alpha$, we are reduced to proving that the canonical map $\theta : \varinjlim_{V \in \mathcal{U}_\alpha}(U_\alpha \times_U V) \to U_\alpha$ is an equivalence. Because $\mathcal{U}_\alpha$ contains U_α as a final object, we can identify θ with the projection map $U_\alpha \times_U U_\alpha \to U_\alpha$, which is an equivalence because the map $U_\alpha \to U$ is (-1)-truncated. □

Remark 1.6.5.3. Let $\mathcal{X}$ be an ∞-topos. If $\mathcal{X}$ is spatial, then we can identify $\mathcal{S}\mathrm{hv}(|\mathcal{X}|)$ with the 0-localic reflection of $\mathcal{X}$, so that we have an evident geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(|\mathcal{X}|)$. If $\mathcal{C}$ is an ∞-category which admits small limits and $\mathscr{O} \in \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$, then $\mathscr{O}^{\mathrm{spa}}$ can be identified with the pushforward $f_* \mathscr{O}$ under the equivalence of ∞-categories $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{S}\mathrm{hv}(|\mathcal{X}|)) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{X}}(|\mathcal{X}|)$ of Proposition 1.3.1.7.

Remark 1.6.5.4. Let $\mathcal{X}$ be an ∞-topos. There is an evident map from the locale of open subsets of $|\mathcal{X}|$ to the locale $\mathrm{Sub}(\mathcal{X})$, which is an equivalence if and only if $\mathcal{X}$ is spatial.

Passing to the associated ∞-topoi, we obtain a commutative diagram of geometric morphisms

$$\mathcal{X} \xrightarrow{f_*} \mathcal{X}_0 \xleftarrow{g_*} \mathcal{S}\mathrm{hv}(|\mathcal{X}|),$$

where f_* exhibits $\mathcal{X}_0$ as a 0-localic reflection of $\mathcal{X}$, and g_* is an equivalence if and only if $\mathcal{X}$ is spatial. In good cases, we can associate to each sheaf $\mathcal{O}$ on the ∞-topos $\mathcal{X}$ another sheaf $\mathcal{O}^{\mathrm{spa}}$ on $|\mathcal{X}|$, given by $g^* f_* \mathcal{O}$. However, unless $\mathcal{X}$ is spatial, there is no obvious description of the sections of $\mathcal{O}^{\mathrm{spa}}$ over an open subset $U \subseteq |\mathcal{X}|$.

Remark 1.6.5.5. Let X be a topological space and let $\mathcal{C}$ be an ∞-category. Then the canonical map $\epsilon : X \to |\mathcal{S}\mathrm{hv}(X)|$ induces an isomorphism from the lattice of open subsets of $|\mathcal{S}\mathrm{hv}(X)|$ to the lattice of open subsets of X, so that the pushforward functor $\epsilon_* : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(X) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(|\mathcal{S}\mathrm{hv}(X)|)$ is an isomorphism of simplicial sets. Suppose that $\mathcal{C}$ admits small limits. Then the pushforward functor ϵ_* can be identified with the composition of the equivalence $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(X) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{S}\mathrm{hv}(X))$ of Proposition 1.3.1.7 and the functor $\mathcal{O} \mapsto \mathcal{O}^{\mathrm{spa}}$ of Construction 1.6.5.1. In particular, we see that the construction $\mathcal{O} \mapsto \mathcal{O}^{\mathrm{spa}}$ induces an equivalence of ∞-categories $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{S}\mathrm{hv}(X)) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(|\mathcal{S}\mathrm{hv}(X)|)$.

Definition 1.6.5.6. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectrally ringed ∞-topos. We will say that X is *spatial* if the ∞-topos $\mathcal{X}$ is spatial (Definition 1.5.4.3). In this case, we let $\mathsf{X}^{\mathrm{spa}}$ denote the spectrally ringed space $(|\mathcal{X}|, \mathcal{O}_{\mathcal{X}}^{\mathrm{spa}})$.

Example 1.6.5.7. Let R be an $\mathbb{E}_\infty$-ring. Then $\mathrm{Sp\acute{e}t}\, R$ is spatial, and Proposition 1.5.4.6 supplies a canonical homeomorphism $|\mathcal{S}\mathrm{hv}_R^{\acute{e}t}| \simeq |\mathrm{Spec}\, R|$. Under this equivalence, an open set $U_a = \{\mathfrak{p} \in |\mathrm{Spec}\, R| : a \notin \mathfrak{p}\}$ corresponds to the sheaf $h^{R[a^{-1}]} \in \mathcal{S}\mathrm{hv}_R^{\acute{e}t}$ corepresented by $R[a^{-1}]$. Let $\mathcal{O}$ denote the structure sheaf of $\mathrm{Sp\acute{e}t}\, R$, so that $\mathcal{O}^{\mathrm{spa}}$ is given on basic open sets by the formula $\mathcal{O}^{\mathrm{spa}}(U_a) = R[a^{-1}]$, and therefore coincides with the structure sheaf of $\mathrm{Spec}\, R$ (see the proof of Proposition 1.1.4.3). Consequently, we obtain an equivalence of spectrally ringed spaces $(\mathrm{Sp\acute{e}t}\, R)^{\mathrm{spa}} \simeq \mathrm{Spec}\, R$, which depends functorially on R.

Proposition 1.6.5.8. *Let $\mathcal{X}$ be a spatial ∞-topos and let $X = |\mathcal{X}|$ denote its underlying topological space. If $\mathcal{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ is local (in the sense of Definition 1.4.2.1), then $(X, \mathcal{O}^{\mathrm{spa}})$ is a locally spectrally ringed space. The converse holds if $\mathcal{X}$ is* 0*-localic.*

Proof. Suppose that $\mathcal{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ is local; we wish to prove that $\mathcal{O}^{\mathrm{spa}}$ is local. To prove this, fix an open set $U \subseteq X$ and a collection of elements $\{f_i \in (\pi_0\, \mathcal{O}^{\mathrm{spa}})(U)\}_{1 \leqslant i \leqslant n}$ satisfying $\sum_{1 \leqslant i \leqslant n} f_i = 1$. We wish to prove that we can write U as a union of open sets $\{U_i\}_{1 \leqslant i \leqslant n}$, such that the image of f_i is invertible in $(\pi_0\, \mathcal{O}^{\mathrm{spa}})(U_i)$. The assertion is local on U; we may therefore assume without loss of generality that each f_i can be lifted to a section $\overline{f}_i \in \pi_0(\mathcal{O}^{\mathrm{spa}}(U))$, and that these sections satisfy $\sum_{1 \leqslant i \leqslant n} \overline{f}_i = 1 \in \pi_0(\mathcal{O}^{\mathrm{spa}}(U))$. Let us identify U with a (-1)-truncated object of $\mathcal{X}$, so that each $\overline{f}_i$ determines a morphism $U \to \pi_0 \mathscr{O}$ in the topos of discrete objects of $\mathcal{X}$. Form pullback diagrams

$$\begin{array}{ccc} U_i & \longrightarrow & U \\ \downarrow & & \downarrow \\ (\pi_0 \mathscr{O})^\times & \longrightarrow & \pi_0 \mathscr{O}. \end{array}$$

Then each of the maps $U_i \to U$ is (-1)-truncated, so that we can identify each U_i with an open subset of X. The assumption that $\pi_0 \mathscr{O}$ is local guarantees that the U_i form a covering of X. Replacing $\mathcal{X}$ by one of the $\mathcal{X}_{/U_i}$, we may assume that $X = U_i$ for some i. In this case, the image of $\overline{f}_i$ in $(\pi_0 \mathscr{O})(X)$ is invertible. Refining our covering if necessary, we may suppose that $\overline{f}_i$ is invertible in $\pi_0(\mathscr{O}(X)) \simeq \pi_0(\mathscr{O}^{\mathrm{spa}}(X))$, so that f_i is invertible in $(\pi_0 \mathscr{O}^{\mathrm{spa}})(X)$.

Now suppose that $\mathcal{X}$ is 0-localic and that $(X, \mathscr{O}^{\mathrm{spa}})$ is locally ringed; we will show that $\mathscr{O}$ is local. For this, it will suffice to show that for any discrete object $Y \in \mathcal{X}$ and any finite collection of sections $\{g_i \in (\pi_0 \mathscr{O})(Y)\}$ satisfying $\sum g_i = 1 \in (\pi_0 \mathscr{O})(Y)$, we can choose an effective epimorphism $\amalg Y_i \to Y$ such that each g_i is invertible when restricted to Y_i. The assertion is local on Y. Using the assumption that $\mathcal{X}$ is 0-localic, we can assume that Y is (-1)-truncated and that each g_i can be lifted to a section $\overline{g}_i \in \pi_0(\mathscr{O}(Y))$ satisfying $\sum \overline{g}_i = 1 \in \pi_0(\mathscr{O}(Y))$. In this case, we can identify Y with an open subset of X, and each $\overline{g}_i$ determines an element of $(\pi_0 \mathscr{O}^{\mathrm{spa}})(Y)$. Using the assumption that $\mathscr{O}^{\mathrm{spa}}$ is local, we conclude that Y can be written as a union of open subsets U_i, such that each $\overline{g}_i$ has invertible image in $(\pi_0 \mathscr{O}^{\mathrm{spa}})(U_i)$. Working locally on Y, we may assume that Y is equal to some U_i, and that $\overline{g}_i$ has invertible image in $\pi_0(\mathscr{O}^{\mathrm{spa}}(U_i)) = \pi_0(\mathscr{O}(U_i))$. It then follows that g_i is invertible. □

We will also need a variant of Proposition 1.6.5.8 for morphisms of spectrally ringed ∞-topoi:

Proposition 1.6.5.9. *Let $\phi : (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be a morphism of spectrally ringed ∞-topoi. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are spatial, and that $\mathscr{O}_{\mathcal{X}}$ and $\mathscr{O}_{\mathcal{Y}}$ are local. If f is local, then the induced map $(|\mathcal{X}|, \mathscr{O}^{\mathrm{spa}}_{\mathcal{X}}) \to (|\mathcal{Y}|, \mathscr{O}^{\mathrm{spa}}_{\mathcal{Y}})$ is a morphism of locally spectrally ringed spaces. The converse holds if $\mathcal{Y}$ is 0-localic.*

Proof. Let $X = |\mathcal{X}|$ and $Y = |\mathcal{Y}|$, so that ϕ induces a continuous map of topological spaces $f : X \to Y$. Suppose first that the underlying map $\alpha : \phi^* \mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$ is local. We wish to show that for each point $x \in X$, the induced map $(\pi_0 \mathscr{O}^{\mathrm{spa}}_{\mathcal{Y}})_{f(x)} \to (\pi_0 \mathscr{O}^{\mathrm{spa}}_{\mathcal{X}})_x$ is a local homomorphism of local commutative rings. Fix an element $u \in (\pi_0 \mathscr{O}^{\mathrm{spa}}_{\mathcal{Y}})_{f(x)}$ whose image in $(\pi_0 \mathscr{O}^{\mathrm{spa}}_{\mathcal{X}})_x$ is invertible. Choose an open subset $V \subseteq Y$ containing $f(x)$ and an element $\overline{u} \in \pi_0(\mathscr{O}^{\mathrm{spa}}_{\mathcal{Y}}(V)) \simeq \pi_0(\mathscr{O}_{\mathcal{Y}}(V))$ lifting u. Then there exists an open set $U \subseteq f^{-1}(V)$ containing x such that the image of $\overline{u}$ in $\pi_0(\mathscr{O}^{\mathrm{spa}}_{\mathcal{X}}(U)) \simeq \pi_0(\mathscr{O}_{\mathcal{X}}(U))$ is invertible. Since α is local, we conclude that there exists a subset $V' \subseteq V$ containing $f(x)$ such that the image of $\overline{u}$ in $\pi_0(\mathscr{O}^{\mathrm{spa}}_{\mathcal{Y}}(V')) \simeq \pi_0(\mathscr{O}_{\mathcal{Y}}(V'))$ is invertible, which immediately implies that u is invertible.

For the converse, assume that $\mathcal{X}$ is 0-localic and that that ϕ induces a morphism $\phi^{\mathrm{spa}} : (X, \mathscr{O}^{\mathrm{spa}}_{\mathcal{X}}) \to (Y, \mathscr{O}^{\mathrm{spa}}_{\mathcal{Y}})$ of locally spectrally ringed spaces; we wish to prove that α is local. Equivalently, we wish to show that the diagram

$$\begin{array}{ccc} \phi^*(\pi_0\,\mathscr{O}_{\mathcal{Y}})^\times & \longrightarrow & (\pi_0\,\mathscr{O}_{\mathcal{X}})^\times \\ \downarrow & & \downarrow \\ \phi^*\pi_0\,\mathscr{O}_{\mathcal{Y}} & \longrightarrow & \pi_0\,\mathscr{O}_{\mathcal{X}} \end{array}$$

is a pullback square of discrete objects of $\mathcal{X}$. We prove more generally that for every object $V \in \mathcal{Y}$ equipped with a map $u : V \to \pi_0\,\mathscr{O}_{\mathcal{Y}}$, if we let $V^\times$ denote the fiber product $V \times_{\pi_0\,\mathscr{O}_{\mathcal{Y}}} (\pi_0\,\mathscr{O}_{\mathcal{Y}})^\times$, then the diagram

$$\begin{array}{ccc} \phi^* V^\times & \longrightarrow & (\pi_0\,\mathscr{O}_{\mathcal{X}})^\times \\ \downarrow & & \downarrow \\ \phi^* V & \longrightarrow & \pi_0\,\mathscr{O}_{\mathcal{X}} \end{array}$$

is a pullback square in $\mathcal{X}$. This assertion can be tested locally on V. Using our assumption that $\mathcal{Y}$ is 0-localic, we may reduce to the case where V is (-1)-truncated (so that $V^\times$ is also (-1)-truncated) and where u lifts to an element $\overline{u} \in \pi_0(\mathscr{O}_{\mathcal{Y}}(V))$. Let U denote the fiber product $\phi^* V \times_{\pi_0\,\mathscr{O}_{\mathcal{X}}} (\pi_0\,\mathscr{O}_{\mathcal{X}})^\times$, so that U is a (-1)-truncated object of $\mathcal{X}$ (which we can identify with an open subset of X). We have an inclusion $f^{-1}(V^\times) \subseteq U$, and we wish to show that this inclusion is an equality. For this, it suffices to show that every point $x \in U \subseteq X$ is contained in $f^{-1}V^\times$. Note that if $x \in U$, then the image of $\overline{u}$ is invertible in the commutative ring $(\pi_0\,\mathscr{O}^{\mathrm{spa}}_{\mathcal{X}})_x$. Since ϕ^{spa} is local, we conclude that the image of $\overline{u}$ in $(\pi_0\,\mathscr{O}^{\mathrm{spa}}_{\mathcal{Y}})_{f(x)}$ is invertible, so that $f(x) \in V^\times$ as desired. □

Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spatial locally spectrally ringed ∞-topos, and write $\mathsf{X}^{\mathrm{spa}} = (|\mathcal{X}|, \mathscr{O}^{\mathrm{spa}}_{\mathcal{X}})$. By functoriality (and Example 1.6.5.7), we obtain a canonical map

$$\begin{aligned} h^{\mathrm{nc}}_{\mathsf{X}}(R) &= \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\,R, \mathsf{X}) \\ &\to \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((\mathrm{Sp\acute{e}t}\,R)^{\mathrm{spa}}, \mathsf{X}^{\mathrm{spa}}) \\ &\simeq \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Spec}\,R, \mathsf{X}^{\mathrm{spa}}) \\ &= h^{\mathrm{nc}}_{|\mathcal{X}|}(R), \end{aligned}$$

depending functorially on R. Here the domain is given by Definition 1.6.4.1, and the codomain by Definition **??**.

Theorem 1.6.5.10. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spatial locally spectrally ringed ∞-topos. If $\mathcal{X}$ is 0-localic, then the above construction induces an equivalence of functors $h^{\mathrm{nc}}_{\mathsf{X}} \to h^{\mathrm{nc}}_{|\mathcal{X}|}$ from* CAlg *to* $\widehat{\mathcal{S}}$.

Theorem 1.6.5.10 is an immediate consequence of the following more general assertion:

Proposition 1.6.5.11. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ and $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be locally spectrally ringed ∞-topoi. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are spatial and that $\mathcal{Y}$ is 0-localic. Then the canonical map*

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathsf{X}, \mathsf{Y}) \to \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathsf{X}^{\mathrm{spa}}, \mathsf{Y}^{\mathrm{spa}})$$

is a homotopy equivalence.

Corollary 1.6.5.12. *Let $\infty\mathcal{T}\mathrm{op}^{\circ}_{\mathrm{CAlg}}$ denote the full subcategory of $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ spanned by those spectrally ringed ∞-topoi $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ where $\mathcal{X}$ is spatial and 0-localic. Then the construction $\mathsf{X} \mapsto \mathsf{X}^{\mathrm{spa}}$ induces a fully faithful embedding $\infty\mathcal{T}\mathrm{op}^{\circ}_{\mathrm{CAlg}} \to \mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$, whose essential image consists of those spectrally ringed spaces $(X, \mathscr{O}_X)$ for which X is sober.*

Proof. The full faithfulness follows from Proposition 1.6.5.11. For the essential surjectivity, we note that if X is sober, then we have $X \simeq |\mathcal{S}\mathrm{hv}(X)|$ (Example 1.5.4.4), in which case the desired result follows from Remark 1.6.5.5 and Proposition 1.6.5.8. □

Remark 1.6.5.13. Let $(X, \mathscr{O}_X)$ be a locally spectrally ringed space. If X is sober, then Corollary 1.6.5.12 implies that we can write $(X, \mathscr{O}_X) = \mathsf{X}^{\mathrm{spa}}$, where X is a locally spectrally ringed ∞-topos which is spatial and 0-localic (moreover, X is unique up to a contractible space of choices). It follows from Theorem 1.6.5.10 that the functor of points h^{nc}_X of $(X, \mathscr{O}_X)$ is equivalent to the functor of points $h^{\mathrm{nc}}_{\mathsf{X}}$ of X. Restricting to the full subcategory $\mathrm{CAlg}^{\mathrm{cn}} \subseteq \mathrm{CAlg}$, we obtain also an equivalence $h_X \simeq h_{\mathsf{X}}$. It follows that we can regard Definition 1.6.1.1 as a special case of Definition 1.6.4.1 (at least if we restrict our attention to locally spectrally ringed spaces $(X, \mathscr{O}_X)$ for which X is sober).

Proof of Proposition 1.6.5.11. We have a commutative diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathsf{X}, \mathsf{Y}) & \xrightarrow{\theta} & \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathsf{X}^{\mathrm{spa}}, \mathsf{Y}^{\mathrm{spa}}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{Y}) & \xrightarrow{\theta_0} & \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(|\mathcal{X}|, |\mathcal{Y}|). \end{array}$$

To prove that θ is a homotopy equivalence, it will suffice to show that θ_0 is a homotopy equivalence, and that θ induces a homotopy equivalence after passing to vertical homotopy fibers over any chosen point $\phi_* \in \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{Y})$. To prove the latter claim, we first note that since $\mathcal{Y}$ is 0-localic, Remark 1.6.5.5 supplies a homotopy equivalence

$$\begin{aligned} \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})}(\mathscr{O}_{\mathcal{Y}}, \phi_* \mathscr{O}_{\mathcal{X}}) &\simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(|\mathcal{Y}|)}(\mathscr{O}_{\mathcal{Y}}^{\mathrm{spa}}, (\phi_* \mathscr{O}_{\mathcal{X}})^{\mathrm{spa}}) \\ &\simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(|\mathcal{Y}|)}(\mathscr{O}_{\mathcal{Y}}^{\mathrm{spa}}, f_* \mathscr{O}_{\mathcal{X}}^{\mathrm{spa}}), \end{aligned}$$

where $f : |\mathcal{X}| \to |\mathcal{Y}|$ denotes the continuous map of topological spaces determined by the geometric morphism ϕ_*. It now suffices to observe that a map of spectrally ringed ∞-topoi $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ is local if and only if the induced map of spectrally ringed spaces is local, by virtue of Proposition 1.6.5.9.

To complete the proof, it suffices to show that θ_0 is an equivalence. Note that θ_0 factors as a composition

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{Y}) \xrightarrow{\theta_0'} \mathrm{Hom}(\mathrm{Sub}(\mathcal{X}), \mathrm{Sub}(\mathcal{Y})) \xrightarrow{\theta_0''} \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(|\mathcal{X}|, |\mathcal{Y}|),$$

where the middle term is the set of maps from $\mathrm{Sub}(\mathcal{X})$ to $\mathrm{Sub}(\mathcal{Y})$ in the category of 0-topoi (see §HTT.6.4.2). Our assumption that $\mathcal{Y}$ is 0-localic guarantees that θ_0' is a homotopy equivalence, and our assumption that $\mathcal{X}$ and $\mathcal{Y}$ are spatial guarantees that θ_0'' is bijective (see Remark 1.5.3.6). □

1.6.6 Comparison of Zariski and Étale Topologies

Our next goal is to show that there is a fully faithful embedding from the ∞-category $\mathrm{SpSch}^{\mathrm{nc}}$ of nonconnective spectral schemes to the ∞-category $\mathrm{SpDM}^{\mathrm{nc}}$ of nonconnective spectral Deligne-Mumford stacks. In terms of the functor of points, this embedding is the identity: we will show that for any (nonconnective) spectral scheme X, the functor h_X represented by X is *also* representable by a spectral Deligne-Mumford stack (Corollary 1.6.6.3).

Remark 1.6.6.1. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. Then the ∞-topos $\mathcal{X}$ is spatial, so that the spectrally ringed space $\mathsf{X}^{\mathrm{spa}} = (|\mathcal{X}|, \mathscr{O}_{\mathcal{X}}^{\mathrm{spa}})$ is well-defined. To prove this, it suffices to show that the hypercompletion $\mathcal{X}^{\mathrm{hyp}}$ is spatial, which follows from Corollary **??** and Example 1.5.4.5.

Theorem 1.6.6.2. *Let $(X, \mathscr{O}_X)$ be a nonconnective spectral scheme. Then there exists a nonconnective spectral Deligne-Mumford stack X and an equivalence of spectrally ringed spaces $\alpha : \mathsf{X}^{\mathrm{spa}} \simeq (X, \mathscr{O}_X)$ with the following universal property: for every spectrally ringed ∞-topos $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ for which $\mathcal{Y}$ is spatial and $\mathscr{O}_{\mathcal{Y}}$ is strictly Henselian, composition with α induces a homotopy equivalence*

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}(\mathsf{Y}, \mathsf{X}) \to \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathsf{Y}^{\mathrm{spa}}, (X, \mathscr{O}_X)).$$

Corollary 1.6.6.3. *Let* $\mathrm{SpSch}'^{\mathrm{nc}} \subseteq \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$ *be the essential image of the fully faithful embedding* $h^{\mathrm{nc}} : \mathrm{SpSch}^{\mathrm{nc}} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$ *introduced in Definition 1.6.4.1, and let* $\mathrm{SpDM}'^{\mathrm{nc}} \subseteq \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$ *be the essential image of the fully faithful embedding* $h^{\mathrm{nc}} : \mathrm{SpSch}^{\mathrm{nc}} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$ *of Definition 1.6.4.1. Then* $\mathrm{SpSch}'^{\mathrm{nc}} \subseteq \mathrm{SpDM}'^{\mathrm{nc}}$*.*

Proof. Let $(X, \mathcal{O}_X)$ be a nonconnective spectral scheme representing a functor $h_X^{\mathrm{nc}} : \mathrm{CAlg} \to \mathcal{S}$, and let $\mathsf{X} \in \mathrm{SpDM}^{\mathrm{nc}}$ be as in Theorem 1.6.6.2, so that X represents a functor $h_{\mathsf{X}}^{\mathrm{nc}} : \mathrm{CAlg} \to \mathcal{S}$. Taking $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, R$ in the statement of Theorem 1.6.6.2, we conclude that the canonical map $h_{\mathsf{X}}^{\mathrm{nc}}(R) \to h_X^{\mathrm{nc}}(R)$ is a homotopy equivalence. It follows that the functor h_X^{nc} is representable by X, and therefore belongs to $\mathrm{SpDM}'^{\mathrm{nc}}$. $\square$

We now turn to the proof of Theorem 1.6.6.2. The main ingredient is the following general fact, which we will establish in Part VI (see Theorem **??**):

Proposition 1.6.6.4. *The inclusion functor* $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}} \hookrightarrow \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ *admits a right adjoint* $\mathrm{Spec}^{\acute{e}t}_{\mathrm{Zar}} : \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$*.*

Proof of Theorem 1.6.6.2. Let $(X, \mathcal{O}_X)$ be a nonconnective spectral scheme. Then X is sober (Corollary 1.5.3.8), so Corollary 1.6.5.12 allows us to write $(X, \mathcal{O}_X) \simeq \mathsf{Y}^{\mathrm{spa}}$ where $\mathsf{X}' = (\mathcal{S}\mathrm{hv}(X), \mathcal{O})$ is a locally spectrally ringed ∞-topos. Let $\mathsf{X} = \mathrm{Spec}^{\mathrm{\acute{e}t}}_{\mathrm{Zar}}\, \mathsf{X}'$, so that $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a spectrally ringed ∞-topos for which $\mathcal{O}_{\mathcal{X}}$ is locally ringed. We have evident map $\rho : \mathsf{X} \to \mathsf{X}'$ in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$. We will complete the proof by verifying the following:

(1) The spectrally ringed ∞-topos X is a nonconnective spectral Deligne-Mumford stack. In particular, X is spatial.

(2) The canonical map $\mathsf{X} \to \mathsf{Y}$ induces an equivalence of spectrally ringed spaces $\alpha : \mathsf{X}^{\mathrm{spa}} \simeq \mathsf{X}'^{\mathrm{spa}} \simeq (X, \mathcal{O}_X)$.

(3) For every spectrally ringed ∞-topos $\mathsf{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ for which $\mathcal{Y}$ is spatial and $\mathcal{O}_{\mathcal{Y}}$ is strictly Henselian, composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}(\mathsf{Y}, \mathsf{X}) \to \mathrm{Map}_{\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathsf{Y}^{\mathrm{spa}}, (X, \mathcal{O}_X)).$$

Note that, by virtue of Proposition **??**, assertion (3) is equivalent to the requirement that the canonical map $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}(\mathsf{Y}, \mathsf{X}) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathsf{Y}, \mathsf{X}')$ is a homotopy equivalence, which follows immediately from the construction of X.

Let $f_* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(X)$ be the geometric morphism underlying ρ. For each open subset $U \subseteq X$, we let f^*U denote the corresponding (-1)-truncated object of $\mathcal{X}$. It follows immediately from the definitions that we can identify $\mathsf{X}_{f^*U} = (\mathcal{X}_{/f^*U}, \mathcal{O}_{\mathcal{X}}|_{f^*U})$ with the relative spectrum $\mathrm{Spec}^{\mathrm{\acute{e}t}}_{\mathrm{Zar}}(\mathcal{S}\mathrm{hv}(U), \mathcal{O}|_U)$. Consequently, to prove assertions (1) and (2), we can work locally on X and thereby reduce to the case where $(X, \mathcal{O}_X) = \mathrm{Spec}\, R$ is affine.

In this case, the universal properties of $\operatorname{Spec} R$ and $\operatorname{Spét} R$ (see Propositions 1.1.5.5 and 1.4.2.4) supply an equivalence $\operatorname{Spec}^{\text{ét}}_{\text{Zar}} \mathsf{X}' \simeq \operatorname{Spét} R$. It follows that X is an affine spectral Deligne-Mumford stack, which proves (1). Assertion (2) follows from Example 1.6.5.7. □

Remark 1.6.6.5. The proof of Theorem 1.6.6.2 shows that the fully faithful embedding $\iota : \mathrm{SpSch}^{\mathrm{nc}} \to \mathrm{SpDM}^{\mathrm{nc}}$ carries SpSch into SpDM. Moreover, if $(X, \mathscr{O}_X)$ is an n-truncated spectral scheme, then $\iota(X, \mathscr{O}_X)$ is an n-truncated spectral Deligne-Mumford stack. In particular, we obtain a fully faithful embedding from the ∞-category of 0-truncated spectral schemes into the ∞-category of 0-truncated Deligne-Mumford stacks. This can be identified with the usual embedding of the category of schemes into the 2-category of Deligne-Mumford stacks, using the fully faithful embeddings of Proposition 1.1.8.4 and Remark 1.4.8.3.

1.6.7 Schematic Spectral Deligne-Mumford Stacks

It follows from Corollary 1.6.6.3 that there is a fully faithful embedding of ∞-categories $\iota : \mathrm{SpSch}^{\mathrm{nc}} \to \mathrm{SpDM}^{\mathrm{nc}}$, which carries each nonconnective spectral scheme $(X, \mathscr{O}_X)$ to a nonconnective spectral Deligne-Mumford stack X satisfying $h^{\mathrm{nc}}_{\mathsf{X}} \simeq h^{\mathrm{nc}}_X$. We will say that a nonconnective spectral Deligne-Mumford stack X is *schematic* if it belongs to the essential image of this fully faithful embedding.

Remark 1.6.7.1. It is often convenient to abuse notation and identify $\mathrm{SpSch}^{\mathrm{nc}}$ with its essential image under the functor ι: in other words, to not distinguish between a nonconnective spectral scheme and the associated schematic nonconnective spectral Deligne-Mumford stack.

We next give a characterization of the class of schematic spectral Deligne-Mumford stacks. For this, we need to introduce a bit of terminology.

Definition 1.6.7.2. Suppose that $j : \mathsf{U} \to \mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a map of nonconnective spectral Deligne-Mumford stacks. We will say that j is an *open immersion* if it factors as a composition

$$\mathsf{U} \xrightarrow{j'} (\mathcal{X}_{/U}, \mathscr{O}_{\mathcal{X}}|_U) \xrightarrow{j''} (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$$

where j' is an equivalence and j'' is the étale morphism associated to a (-1)-truncated object $U \in \mathcal{X}$. In this case, we will also say that U is an *open substack* of X.

Proposition 1.6.7.3. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. The following conditions are equivalent:*

(1) *There exists a collection of open immersions $\{j_\alpha : \mathsf{U}_\alpha \to \mathsf{X}\}$ which are mutually surjective (that is, each U_α has the form $(\mathcal{X}_{/U_\alpha}, \mathscr{O}_{\mathcal{X}}|_{U_\alpha})$, and the coproduct $\amalg U_\alpha$ is a 0-connective object of $\mathcal{X}$), where each U_α is affine.*

(2) *The spectrally ringed space* $(X, \mathscr{O}_X) \simeq \mathsf{X}^{\mathrm{spa}}$ *is a nonconnective spectral scheme, and the identity map* $\mathrm{id} : \mathsf{X}^{\mathrm{spa}} \simeq (X, \mathscr{O}_X)$ *satisfies the requirements of Theorem 1.6.6.2.*

(3) *The nonconnective spectral Deligne-Mumford stack* X *is schematic.*

Proof. The equivalence of (2) and (3) is a tautology. Let us abuse notation by identifying (-1)-truncated objects of $\mathcal{X}$ with open subsets of the topological space X. If (2) is satisfied, then $\mathcal{X}$ admits a covering by (-1)-truncated objects U_α such that each $(U_\alpha, \mathscr{O}_X|_{U_\alpha})$ is an affine nonconnective spectral scheme $\operatorname{Spec} R_\alpha$. Arguing as in the proof of Theorem 1.6.6.2, we conclude that $(\mathcal{X}_{/U_\alpha}, \mathscr{O}_{\mathcal{X}}|_{U_\alpha})$ is equivalent to the affine nonconnective spectral Deligne-Mumford stack $\operatorname{Spét} R$, so that (1) is satisfied.

We complete the proof by showing that (1) $\Rightarrow$ (2). Note that assertion (2) can be tested locally on the topological space X. We are therefore free to replace X by an open substack and thereby to reduce to the case where $\mathsf{X} = \operatorname{Spét} R$ is affine, in which case the desired result follows from Example 1.6.5.7. □

Corollary 1.6.7.4. *Let* $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ *be a nonconnective spectral Deligne-Mumford stack. Then* X *is schematic if and only if the* 0*-truncated spectral Deligne-Mumford stack* $(\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}})$ *is schematic.*

Proof. This follows from Proposition 1.6.7.3, since a nonconnective spectral Deligne-Mumford stack $(\mathcal{U}, \mathscr{O}_{\mathcal{U}})$ is affine if and only if $(\mathcal{U}, \pi_0 \mathscr{O}_{\mathcal{U}})$ is affine (see Corollary 1.4.7.3). □

According to Proposition 1.4.11.1, the ∞-category $\mathrm{SpDM}^{\mathrm{nc}}$ of nonconnective spectal Deligne-Mumford stacks admits finite limits. One can use the same reasoning to prove that the ∞-category $\mathrm{SpSch}^{\mathrm{nc}}$ of nonconnective spectral schemes admits finite limits. However, the details are somewhat tedious; instead, we can deduce the statement about schemes from Proposition 1.4.11.1, using the characterization of schematic spectral Deligne-Mumford stacks supplied by Proposition 1.6.7.3.

Corollary 1.6.7.5. *Let* $\mathcal{C}$ *denote the full subcategory of* $\mathrm{SpDM}^{\mathrm{nc}}$ *spanned by the schematic nonconnective spectral Deligne-Mumford stacks. Then* $\mathcal{C}$ *is closed under finite limits in* $\mathrm{SpDM}^{\mathrm{nc}}$ *(which exist by virtue of Proposition 1.4.11.1).*

Proof. The final object $\operatorname{Spét} S \in \mathrm{SpDM}^{\mathrm{nc}}$ is affine, and therefore schematic. To complete the proof, it will suffice to show that for every pullback diagram

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y} \end{array}$$

in $\mathrm{SpDM}^{\mathrm{nc}}$, if X, Y, and Y' are schematic, then X' is also schematic. Write $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$, write $\mathsf{X}' = (\mathcal{X}', \mathscr{O}_{\mathcal{X}'})$, and let $f^* : \mathcal{Y} \to \mathcal{X}'$ be the underlying geometric morphism of ∞-topoi.

Using Proposition 1.6.7.3, we see that $\mathcal{Y}$ admits a covering by affine (-1)-truncated objects U_α. Note that each pullback $f^* U_\alpha$ is a (-1)-truncated object of $\mathcal{X}'$. Using the criterion of Proposition 1.6.7.3, we see that to prove that X' is schematic, it will suffice to show that each $(\mathcal{X}'_{/f^* U_\alpha}, \mathscr{O}_{\mathcal{X}'}|_{f^* U_\alpha})$ is schematic. We may therefore replace Y by $(\mathcal{Y}_{/U_\alpha}, \mathscr{O}_{\mathcal{Y}}|_{U_\alpha})$, and thereby reduce to the case where $\mathsf{Y} \simeq \operatorname{Sp\acute{e}t} A$ is affine. Using a similar argument, we can reduce to the case where $\mathsf{X} \simeq \operatorname{Sp\acute{e}t} B$ and $\mathsf{Y}' \simeq \operatorname{Sp\acute{e}t} A'$ are affine. In this case, we conclude that $\mathsf{X}' \simeq (\operatorname{Sp\acute{e}t} A') \times_{\operatorname{Sp\acute{e}t} A} \operatorname{Sp\acute{e}t} B \simeq \operatorname{Sp\acute{e}t}(A' \otimes_A B)$ is affine, and therefore schematic. □

Corollary 1.6.7.6. *The ∞-category $\mathrm{SpSch}^{\mathrm{nc}}$ of nonconnective spectral schemes admits finite limits.*

1.6.8 Spectral Deligne-Mumford n-Stacks

Recall that a spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is said to be *n-localic* if the underlying ∞-topos $\mathcal{X}$ is n-localic (Definition **??**). We now show that for $n > 0$, this condition can be formulated directly in terms of the functor h_{X} represented by X.

Definition 1.6.8.1. Let $n \geqslant 0$. A *spectral Deligne-Mumford n-stack* is a spectral Deligne-Mumford stack X with the following property: for every commutative ring R, the mapping space $\mathrm{Map}_{\mathrm{SpDM}}(\operatorname{Sp\acute{e}t} R, \mathsf{X})$ is n-truncated. A *spectral algebraic space* is a spectral Deligne-Mumford 0-stack.

Example 1.6.8.2. For every connective $\mathbb{E}_\infty$-ring A, the étale spectrum $\operatorname{Sp\acute{e}t} A$ is a spectral algebraic space.

Remark 1.6.8.3. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. The condition that X be a spectral Deligne-Mumford n-stack depends only on the underlying 0-truncated spectral Deligne-Mumford stack $(\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}})$.

Remark 1.6.8.4. Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}. \end{array}$$

Assume that X and Y' are spectral Deligne-Mumford n-stacks for some $n \geqslant 0$, and that Y is a spectral Deligne-Mumford $(n+1)$-stack. Then X' is a spectral Deligne-Mumford n-stack. In particular, if Y' and Y are affine and X is a spectral Deligne-Mumford n-stack, then X' is a spectral Deligne-Mumford n-stack.

Proposition 1.6.8.5. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack, and let $n \geqslant 1$ be an integer. Then X is n-localic if and only if it is a spectral Deligne-Mumford n-stack.*

Proposition 1.6.8.5 does not hold when $n = 0$. For example, if $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ is affine, then X is a spectral algebraic space (Example 1.6.8.2), but is usually not 0-localic. However, we do have the following:

Corollary 1.6.8.6. *Let X be a spectral algebraic space. Then X is 1-localic.*

The proof of Proposition 1.6.8.5 depends on a few general observations about n-localic ∞-topoi.

Lemma 1.6.8.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi, and suppose that $\mathcal{X}$ is n-localic for some $n \geqslant 0$. Let $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{Y})$ spanned by the geometric morphisms $f^* : \mathcal{X} \to \mathcal{Y}$. Then $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ is equivalent to an n-category.*

Proof. Let $\tau_{\leqslant n-1}\,\mathcal{X}$ and $\tau_{\leqslant n-1}\,\mathcal{Y}$ denote the underlying n-topoi of $\mathcal{X}$ and $\mathcal{Y}$. Since $\mathcal{X}$ is n-localic, we can identify $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ with the full subcategory of $\mathrm{Fun}(\tau_{\leqslant n-1}\,\mathcal{X}, \tau_{\leqslant n-1}\,\mathcal{Y})$ spanned by those functors which preserve small colimits and finite limits. The desired result now follows from the observation that $\tau_{\leqslant n-1}\,\mathcal{Y}$ is equivalent to an n-category. □

Lemma 1.6.8.8. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ and $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be spectral Deligne-Mumford stacks. Assume that $\mathscr{O}_{\mathcal{Y}}$ is n-truncated, and that $\mathcal{X}$ is n-localic. Then the mapping space $\mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Y}, \mathsf{X})$ is n-truncated.*

Proof. There is an evident forgetful functor $\theta : \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Y}, \mathsf{X}) \to \mathrm{Fun}^*(\mathsf{X}, \mathsf{Y})^{\simeq}$, where the codomain of θ is n-truncated by Lemma 1.6.8.7. It will therefore suffice to show that the homotopy fiber of θ over every geometric morphism $f^* : \mathcal{X} \to \mathcal{Y}$ is n-truncated. Unwinding the definitions, we see that this fiber can be identified with a summand of the mapping space $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})}(f^*\,\mathscr{O}_{\mathcal{X}}, \mathscr{O}_{\mathcal{Y}})$, which is n-truncated by virtue of our assumption that $\mathscr{O}_{\mathcal{Y}}$ is n-truncated. □

Remark 1.6.8.9. Using exactly same argument, we can deduce an analogous result for spectral schemes. In particular, every spectral scheme represents a functor which carries discrete $\mathbb{E}_\infty$-rings to discrete spaces, so that every schematic spectral Deligne-Mumford stack is a spectral algebraic space.

Proof of Proposition 1.6.8.5. The implication (1) $\Rightarrow$ (2) follows from Lemma 1.6.8.8. Assume now that (2) is satisfied. Replacing $\mathscr{O}_{\mathcal{X}}$ by $\pi_0\,\mathscr{O}_{\mathcal{X}}$, we may assume that $\mathscr{O}_{\mathcal{X}}$ is discrete. It follows from Theorem 1.4.8.1 that there exists a 1-localic spectral Deligne-Mumford stack $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ and a 2-connective object $U \in \mathcal{Y}$ such that $\mathsf{X} \simeq (\mathcal{Y}_{/U}, \mathscr{O}_{\mathcal{Y}}|_U)$. To prove that X is n-localic, it will suffice to show that the object U is n-truncated (Lemma 1.4.7.7). Let $\mathcal{Y}_0$ be the full subcategory of $\mathcal{Y}$ spanned by those objects $Y \in \mathcal{Y}$ such that $\mathrm{Map}_{\mathcal{Y}}(Y, U)$ is n-truncated. We wish to show that $\mathcal{Y}_0 = \mathcal{Y}$. Since $\mathcal{Y}_0$ is closed under small colimits in $\mathcal{Y}$, it will suffice to show that $\mathcal{Y}_0$ contains every object Y for which

$(\mathcal{Y}_{/Y}, \mathscr{O}_{\mathcal{Y}}|_Y) \simeq \operatorname{Sp\acute{e}t} R$ is affine (note that in this case, R is automatically discrete). We now observe that $\operatorname{Map}_{\mathcal{Y}}(Y, U)$ can be identified with the homotopy fiber of the forgetful map $\operatorname{Map}_{\operatorname{SpDM}}(\operatorname{Sp\acute{e}t} R, \mathsf{X}) \to \operatorname{Map}_{\operatorname{SpDM}}(\operatorname{Sp\acute{e}t} R, \mathsf{Y})$. Here $\operatorname{Map}_{\operatorname{SpDM}}(\operatorname{Sp\acute{e}t} R, \mathsf{X})$ is n-truncated by assumption (2), and $\operatorname{Map}_{\operatorname{SpDM}}(\operatorname{Sp\acute{e}t} R, \mathsf{Y})$ is 1-truncated by Lemma 1.6.8.8, so that the homotopy fiber is also n-truncated. □

Chapter 2

Quasi-Coherent Sheaves

To every spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, one can associate a theory of *quasi-coherent sheaves* on X. The collection of quasi-coherent sheaves on X can be organized into a stable ∞-category which we will denote by $\mathrm{QCoh}(\mathsf{X})$. The ∞-category $\mathrm{QCoh}(\mathsf{X})$ is our principal object of study in this section.

We begin in a more general setting. Let $(\mathcal{X}, \mathscr{O})$ be an arbitrary spectrally ringed ∞-topos. In §2.1, we will introduce a stable ∞-category $\mathrm{Mod}_{\mathscr{O}}$, whose objects are $\mathscr{O}$-module objects in the ∞-category $\mathrm{Sp}(\mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ of sheaves of spectra on $\mathcal{X}$. Our main result (Corollary 2.1.2.4) implies that if $\mathscr{O}$ is discrete and $\mathcal{X}$ is 1-localic (for example, if $(\mathcal{X}, \mathscr{O})$ is an ordinary Deligne-Mumford stack: see Proposition **??**), then $\mathrm{Mod}_{\mathscr{O}}$ contains the (bounded below) derived ∞-category $\mathcal{D}(\mathcal{A})_{<\infty}$ as a full subcategory, where $\mathcal{A}$ is the abelian category of discrete sheaves of $\mathscr{O}$-modules on $\mathcal{X}$.

Suppose now that $\mathsf{X} = (\mathcal{X}, \mathscr{O})$ is a spectral Deligne-Mumford stack. In §2.2, we will introduce a full subcategory $\mathrm{QCoh}(\mathsf{X}) \subseteq \mathrm{Mod}_{\mathscr{O}}$, which we call the *$\infty$-category of quasi-coherent sheaves on* X. It is uniquely characterized by the following properties:

- (a) Let $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}}$ be a sheaf of $\mathscr{O}$-modules on $\mathcal{X}$. Then the condition that $\mathscr{F}$ be quasi-coherent is of a local nature on $\mathcal{X}$. In particular, $\mathscr{F}$ is quasi-coherent if and only if, for every affine $U \in \mathsf{X}$, the restriction $\mathscr{F}|_U$ is a quasi-coherent sheaf on the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathscr{O}|_U)$ (Remark 2.2.2.3).

- (b) Suppose that $\mathsf{X} = \mathrm{Spét}\, A$ is affine. Then the global sections functor $\Gamma : \mathrm{Mod}_{\mathscr{O}} \to \mathrm{Mod}_A$ admits a fully faithful left adjoint, whose essential image is the full subcategory $\mathrm{QCoh}(\mathsf{X}) \subseteq \mathrm{Mod}_{\mathscr{O}}$ (Proposition 2.2.3.3).

We will see that the collection of quasi-coherent sheaves admits several other characterizations (see Definition 2.2.2.1, Proposition 2.2.4.3, and Proposition 2.2.6.1).

In many situations, we will need to understand the global sections functor $(\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})) \mapsto \Gamma(\mathsf{X}; \mathscr{F})$ in the case where X is not affine. To ensure that this construction has reasonable behavior, one generally needs to make some assumptions about X. In §2.3, we discuss a hierarchy of "compactness" conditions on X (analogous to quasi-compactness and quasi-separatedness in the setting of classical algebraic geometry) which are relevant for this purpose.

If $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is an affine spectral Deligne-Mumford stack, then the global sections functor $\mathscr{F} \mapsto \Gamma(\mathsf{X}; \mathscr{F})$ induces an equivalence from the ∞-category $\mathrm{QCoh}(\mathsf{X})$ to the ∞-category $\mathrm{Mod}_{\Gamma(\mathsf{X}; \mathcal{O}_{\mathcal{X}})}$ of modules over the $\mathbb{E}_\infty$-ring $\Gamma(\mathsf{X}; \mathcal{O}_{\mathcal{X}})$. In §2.4, we will show that this is true more generally under the assumption that X is *quasi-affine*: that is, if X is quasi-compact and admits an open immersion into an affine spectral Deligne-Mumford stack (Proposition 2.4.1.4). In other words, if we work at the level of stable ∞-categories and $\mathbb{E}_\infty$-rings (rather than abelian categories and ordinary commutative rings), then quasi-affine spectral Deligne-Mumford stacks behave to a large extent as if they are affine. As another illustration of this philosophy, we show that if $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is quasi-affine, then its functor of points h_{X} is corepresented by the (possibly nonconnective) $\mathbb{E}_\infty$-ring $\Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}})$ (Corollary 2.4.2.2). In particular, the construction $\mathsf{X} \mapsto \Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}})$ determines a fully faithful embedding from (the opposite of) the ∞-category of quasi-affine spectral Deligne-Mumford stacks to the ∞-category of $\mathbb{E}_\infty$-rings. In §2.6, we give a description of the image of this fully faithful embedding, following work of Bhatt and Halpern-Leistner ([27]).

Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a map of spectrally ringed ∞-topoi. Then f determines a pair of adjoint functors

$$\mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}.$$

If $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathsf{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ are spectral Deligne-Mumford stacks, then the pullback functor f^* carries quasi-coherent sheaves on Y to quasi-coherent sheaves on X. In §2.5, we will study conditions on f which guarantee that the pushforward f_* also preserves quasi-coherence. In particular, we will show that f_* preserves coherence when the morphism f is quasi-affine (Corollary 2.5.4.6). We will deduce this from a more general statement (Proposition 2.5.4.3) which we will apply in Chapter 3 to prove analogous results in the setting of quasi-compact, quasi-separated spectral algebraic spaces.

The theory of quasi-coherent sheaves on spectral Deligne-Mumford stacks can be regarded as a global analogue of the theory of modules over $\mathbb{E}_\infty$-rings. In particular, if P is any property of modules which can be tested locally with respect to the étale topology (see Definition 2.8.4.1), then it makes sense to ask if a quasi-coherent sheaf has the property P. In §2.8, we will study a number of properties which can be defined in this way. Many of these properties involve finiteness conditions on modules over $\mathbb{E}_\infty$-rings, which we study in detail in §2.7. A particularly important example is the property of invertibility: a quasi-coherent sheaf $\mathscr{F}$ on a spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is said to be *invertible* if there exists another quasi-coherent sheaf $\mathscr{F}^{-1}$ such that $\mathscr{F} \otimes \mathscr{F}^{-1} \simeq \mathcal{O}_{\mathcal{X}}$. The collection of (equivalence classes of) invertible quasi-coherent sheaves can be organized into an abelian group $\mathrm{Pic}^{\dagger}(\mathsf{X})$, the *extended Picard group* of X, which we will study in §2.9.

Contents

2.1 Sheaves on a Spectrally Ringed ∞-Topos

Let X be a topological space and let $\mathscr{O}$ be a sheaf of commutative rings on X. A *sheaf of $\mathscr{O}$-modules* is a sheaf of abelian groups $\mathscr{F}$ on X such that $\mathscr{F}(U)$ is equipped with the structure of a module over the commutative ring $\mathscr{O}(U)$ for every open subset $U \subseteq X$, depending functorially on U. Our goal in this section is to introduce an ∞-categorical analogue of the theory of sheaves of modules. We will replace the topological space X with an arbitrary ∞-topos $\mathcal{X}$, and $\mathscr{O}$ by an arbitrary sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

Definition 2.1.0.1. Let $\mathcal{X}$ be an ∞-topos and let $\mathscr{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Recall that $\mathscr{O}$ can be identified with a commutative algebra object of the symmetric monoidal ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ of sheaves of spectra on $\mathcal{X}$ (see §**??**). We let $\mathrm{Mod}_{\mathscr{O}}$ denote the ∞-category $\mathrm{Mod}_{\mathscr{O}}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$ of $\mathscr{O}$-module objects of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Then $\mathrm{Mod}_{\mathscr{O}}$ can be regarded as a symmetric monoidal ∞-category with respect to the relative tensor product $\otimes_{\mathscr{O}}$ (see §HA.3.4.4). We will refer to the objects of $\mathrm{Mod}_{\mathscr{O}}$ as *sheaves of $\mathscr{O}$-modules on $\mathcal{X}$*, or sometimes just as *$\mathscr{O}$-modules.*

Warning 2.1.0.2. Let X be a topological space and let $\mathscr{O}$ be a sheaf of commutative rings on X. Then we can identify $\mathscr{O}$ with a sheaf of $\mathbb{E}_\infty$-rings on the ∞-topos $\mathcal{S}\mathrm{hv}(X)$. In this case, Definition 2.1.0.1 does not recover the classical theory of sheaves of $\mathscr{O}$-modules on X, because we allow ourselves to consider sheaves of spectra rather than sheaves of abelian groups. However, we will prove below that the ∞-category $\mathrm{Mod}_{\mathscr{O}}$ is stable and equipped with a natural t-structure (Proposition 2.1.1.1). The classical theory of sheaves of $\mathscr{O}$-modules can be recovered by taking the heart $\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}$ of the ∞-category $\mathrm{Mod}_{\mathscr{O}}$. Moreover, the ∞-category $\mathrm{Mod}_{\mathscr{O}}$ is closely related to the derived ∞-category of its heart (see Corollary 2.1.2.3).

The next proposition summarizes some of the basic formal properties of Definition **??**:

Proposition 2.1.0.3. *Let $\mathcal{X}$ be an ∞-topos and $\mathscr{O}$ a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then:*

(1) *The ∞-category $\mathrm{Mod}_{\mathscr{O}}$ is stable.*

(2) *The ∞-category $\mathrm{Mod}_{\mathscr{O}}$ is presentable and the tensor product $\otimes_{\mathscr{O}} : \mathrm{Mod}_{\mathscr{O}} \times \mathrm{Mod}_{\mathscr{O}} \to \mathrm{Mod}_{\mathscr{O}}$ preserves small colimits separately in each variable.*

(3) *The forgetful functor $\theta : \mathrm{Mod}_{\mathscr{O}} \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is conservative and preserves small limits and colimits.*

Proof. Assertion (1) follows from Proposition HA.7.1.1.4 , assertion (2) follows from Theorem HA.3.4.4.2 , and assertion (3) follows from Corollaries HA.3.4.3.2 and HA.3.4.4.6 . □

Notation 2.1.0.4. Let $(\mathcal{X}, \mathscr{O})$ be a spectrally ringed ∞-topos, and suppose we are given objects $\mathscr{F}, \mathscr{F}' \in \mathrm{Mod}_{\mathscr{O}}$. For every integer n, we let $\mathrm{Ext}^n_{\mathscr{O}}(\mathscr{F}, \mathscr{F}')$ denote the abelian group $\mathrm{Ext}^n_{\mathrm{Mod}_{\mathscr{O}}}(\mathscr{F}, \mathscr{F}')$ of homotopy classes of maps from $\mathscr{F}$ to $\Sigma^n \mathscr{F}'$ in $\mathrm{Mod}_{\mathscr{O}}$.

Remark 2.1.0.5. Let $(\mathcal{X}, \mathscr{O})$ be a spectrally ringed ∞-topos. Then the construction $(U \in \mathcal{X}) \mapsto \mathrm{Mod}_{\mathscr{O}|_U}$ determines a functor from $\mathcal{X}^{\mathrm{op}}$ into the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$ of (not necessarily small) ∞-categories. Moreover, this functor preserves small limits.

To see this, consider the coCartesian fibration $p : \mathrm{Fun}(\Delta^1, \mathcal{X}) \to \mathrm{Fun}(\{1\}, \mathcal{X}) \simeq \mathcal{X}$ given by evaluation at $\{1\} \subseteq \Delta^1$. This coCartesian fibration is classified by a functor $\chi : \mathcal{X} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$, which assigns to each object $U \in \mathcal{X}$ the ∞-topos $\mathcal{X}^{/U}$. We claim that this functor preserves small colimits. To prove this, it suffices to show that the opposite functor $\chi : \mathcal{X}^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}\,\mathrm{op}} \simeq \mathcal{P}\mathrm{r}^{\mathrm{R}}$ preserves small limits; this functor classifies p as a *Cartesian* fibration, and is a limit diagram by virtue of Theorems HTT.6.1.3.9 and HTT.5.5.3.18 together Proposition HTT.5.5.3.13 . For any presentable ∞-category $\mathcal{C}$, we obtain a new functor given by the composition

$$\mathcal{X} \xrightarrow{\chi} \mathcal{P}\mathrm{r}^{\mathrm{L}} \xrightarrow{\otimes \mathcal{C}} \mathcal{P}\mathrm{r}^{\mathrm{L}},$$

which assigns to each object $U \in \mathcal{X}$ the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}^{/U})$ (see Remark 1.3.1.6). The same reasoning yields a limit-preserving functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}\,\mathrm{op}} \simeq \mathcal{P}\mathrm{r}^{\mathrm{R}}$ which, by virtue of Theorem HTT.5.5.3.18, gives a limit-preserving functor $\chi[\mathcal{C}] : \mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$.

The evident forgetful functor Mod → CAlg determines a natural transformation of functors $\chi[\mathrm{Mod}] \to \chi[\mathrm{CAlg}]$ from $\mathcal{X}^{\mathrm{op}}$ to $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$. Every sheaf $\mathscr{O}$ of $\mathbb{E}_{\infty}$-rings on $\mathcal{X}$ determines a natural transformation $* \to \chi[\mathrm{CAlg}]$, where $*$ denotes the constant functor $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ taking the value Δ^0. Forming a pullback diagram

$$\begin{array}{ccc} \phi & \longrightarrow & \chi[\mathrm{Mod}] \\ \downarrow & & \downarrow \\ * & \longrightarrow & \chi[\mathrm{CAlg}], \end{array}$$

we obtain a new limit-preserving functor $\phi : \mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$. Unwinding the definitions, we see that ϕ assigns to each object $U \in \mathcal{X}$ the ∞-category $\mathrm{Mod}_{\mathscr{O}|_U}$, and to every morphism $f : U \to V$ in $\mathcal{X}$ the associated pullback functor $f^* : \mathrm{Mod}_{\mathscr{O}|_V} \to \mathrm{Mod}_{\mathscr{O}|_U}$. Since $\chi[\mathrm{CAlg}]$ and $\chi[\mathrm{Mod}]$ preserve small limits, so does ϕ.

2.1.1 The t-Structure on $\mathrm{Mod}_{\mathscr{O}}$

Let $\mathcal{X}$ be an ∞-topos and let $\mathscr{O}$ be a sheaf of $\mathbb{E}_{\infty}$-rings on $\mathcal{X}$. We will say that a $\mathscr{O}$-module $\mathscr{F}$ is *connective* if it is connective when viewed as a sheaf of spectra on $\mathcal{X}$: that is, if the homotopy sheaves $\pi_n \mathscr{F}$ vanish for $n < 0$. We let $\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{Mod}_{\mathscr{O}}$ spanned by the connective $\mathscr{O}$-modules. This notion is primarily useful in the case where the sheaf $\mathscr{O}$ is itself connective.

Proposition 2.1.1.1. *Let $\mathcal{X}$ be an ∞-topos and let $\mathscr{O}$ be a connective sheaf of $\mathbb{E}_{\infty}$-rings on $\mathcal{X}$. Then:*

(a) *The ∞-category $\mathrm{Mod}_{\mathscr{O}}$ admits a t-structure $(\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}}, (\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0})$, where $(\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0}$ is the inverse image of $\mathcal{S}\mathrm{hv}(\mathrm{Sp})_{\leqslant 0}$ under the forgetful functor $\theta : \mathrm{Mod}_{\mathscr{O}} \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$.*

(b) *The t-structure on $(\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}}, (\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0})$ is compatible with the symmetric monoidal structure on $\mathrm{Mod}_{\mathscr{O}}$. In other words, the full subcategory $(\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}} \subseteq \mathrm{Mod}_{\mathscr{O}}$ contains the unit object of $\mathrm{Mod}_{\mathscr{O}}$ and is closed under the relative tensor product $\otimes_{\mathscr{O}}$.*

(c) *The t-structure $(\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}}, (\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0})$ is right complete and compatible with filtered colimits (in other words, the full subcategory $(\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0}$ is stable under filtered colimits in $\mathrm{Mod}_{\mathscr{O}}$).*

Proof. We first prove (a). It follows immediately from the definitions that the full subcategory $\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}} \subseteq \mathrm{Mod}_{\mathscr{O}}$ is closed under small colimits and extensions. Using Proposition HA.1.4.4.11, we deduce the existence of an accessible t-structure $((\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}}, \mathrm{Mod}'_{\mathscr{O}})$ on $\mathrm{Mod}_{\mathscr{O}}$. To complete the proof, it will suffice to show that $\mathrm{Mod}'_{\mathscr{O}} = (\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0}$. Suppose first that $\mathscr{F} \in \mathrm{Mod}'_{\mathscr{O}}$. Then the mapping space $\mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}}}(\mathscr{G}, \mathscr{F})$ is discrete for every object $\mathscr{G} \in (\mathrm{Mod}_{\mathscr{O}})_{\geqslant 0}$. In particular, for every connective sheaf of spectra $\mathcal{M} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant 0}$, the mapping space $\mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}}}(\mathcal{M} \otimes \mathscr{O}, \mathscr{F}) \simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})}(\mathcal{M}, \theta(\mathscr{F}))$ is discrete, so that $\theta(\mathscr{F}) \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0}$ and therefore $\mathscr{F} \in (\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0}$.

Conversely, suppose that $\mathscr{F} \in (\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0}$. We wish to prove that $\mathscr{F} \in \mathrm{Mod}'_{\mathscr{O}}$. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Mod}_{\mathscr{O}}$ spanned by those objects $\mathscr{G} \in \mathrm{Mod}_{\mathscr{O}}$ for which the mapping space $\mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}}}(\mathscr{G}, \mathscr{F})$ is discrete. We wish to prove that $\mathcal{C}$ contains $\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}}$. Condition (3) shows that θ induces a functor $\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}} \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant 0}$ which is conservative and preserves small colimits; moreover, this functor has a left adjoint F, given informally by the formula $F(\mathcal{M}) \simeq \mathscr{O} \otimes \mathcal{M}$. Using Proposition HA.4.7.3.14 , we conclude that $\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}}$ is generated under geometric realizations by the essential image of F. Since $\mathcal{C}$ is stable under colimits, it will suffice to show that $\mathcal{C}$ contains the essential image of F. Unwinding the definitions, we are reduced to proving that the mapping space

$$\mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}}}(F(\mathcal{M}), \mathscr{F}) \simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})}(\mathcal{M}, \theta(\mathscr{F}))$$

is discrete for every connective sheaf of spectra $\mathcal{M}$ on $\mathcal{X}$, which is equivalent to our assumption that $\theta(\mathscr{F}) \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0}$. This completes the proof of (a).

We now prove (b). The unit object of $\mathrm{Mod}_{\mathscr{O}}$ is the sheaf $\mathscr{O}$ (regarded as a module over itself), which is connective by assumption. We claim that for every pair of objects $\mathscr{F}, \mathscr{G} \in \mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}}$, the relative tensor product $\mathscr{F} \otimes_{\mathscr{O}} \mathscr{G}$ is also connective. Note that, as a sheaf of spectra, the relative tensor prouct $\mathscr{F} \otimes_{\mathscr{O}} \mathscr{G}$ can be identified with the geometric realization of a simplicial object whose entires are iterated tensor products $\mathscr{F} \otimes \mathscr{O} \otimes \cdots \otimes \mathscr{O} \otimes \mathscr{G}$. Since $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{O}$ are connective, the above tensor product is connective (Proposition 1.3.4.7); because $\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant 0}$ is closed under colimits we conclude that $\mathscr{F} \otimes_{\mathscr{O}} \mathscr{G}$ is connective.

We now prove (c). Since the forgetful functor $\theta : \mathrm{Mod}_{\mathscr{O}} \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ preserves filtered colimits (Proposition 2.1.0.3) and the full subcategory $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0} \subseteq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is closed under filtered colimits (Proposition 1.3.2.7), it follows that the full subcategory $(\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0} \subseteq \mathrm{Mod}_{\mathscr{O}}$ is closed under filtered colimits. By virtue of Proposition HA.1.2.1.19 , to show that the t-structure $(\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}}, (\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0})$ is right-complete, it is sufficient to show that it is right-separated: that is, that the intersection

$$\bigcap (\mathrm{Mod}_{\mathscr{O}})_{\leqslant -n} \simeq \theta^{-1}(\bigcap \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant -n})$$

contains only zero objects of $\mathrm{Mod}_{\mathscr{O}}$. This follows from the conservativity of the functor θ, since the intersection $\bigcap \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant -n}$ contains only zero objects of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ (Proposition 1.3.2.7). □

Warning 2.1.1.2. The t-structure of Proposition 2.1.1.1 is generally not left complete or even left separated. However, it is left complete (left separated) whenever the t-structure $(\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant 0}, \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0})$ is left complete (left separated). For example, if $\mathcal{X}$ is hypercomplete, then $\mathrm{Mod}_{\mathscr{O}}$ is left separated; if Postnikov towers in $\mathcal{X}$ are convergent, then $\mathrm{Mod}_{\mathscr{O}}$ is left complete.

2.1.2 The Derived ∞-Category of $\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}$

Let $\mathcal{X}$ be an ∞-topos and let $\mathscr{O}$ be a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. We let $\mathrm{Mod}_{\mathscr{O}}^{\heartsuit} \subseteq \mathrm{Mod}_{\mathscr{O}}$ denote the heart of the t-structure described in Proposition 2.1.1.1 (that is, the intersection $\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}} \cap (\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0}$).

Remark 2.1.2.1. Unwinding the definitions, we can identify $\pi_0 \mathscr{O}$ as a commutative ring object in the underlying topos $\mathcal{X}^{\heartsuit}$, and $\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}$ with the abelian category of $(\pi_0 \mathscr{O})$-module objects of $\mathcal{X}^{\heartsuit}$.

According to Remark HA.1.3.5.23 , the inclusion $\iota : \mathrm{Mod}_{\mathscr{O}}^{\heartsuit} \hookrightarrow \mathrm{Mod}_{\mathscr{O}}$ admits an essentially unique extension to a t-exact functor $\mathcal{D}(\mathrm{Mod}_{\mathscr{O}}^{\heartsuit})_{<\infty} \to \mathrm{Mod}_{\mathscr{O}}$, where $\mathcal{D}(\mathrm{Mod}_{\mathscr{O}}^{\heartsuit})_{<\infty}$ denotes the derived ∞-category of $\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}$ (see §HA.1.3.2). If the ∞-topos $\mathcal{X}$ is hypercomplete, then the t-structure on $\mathrm{Mod}_{\mathscr{O}}$ is left separated (Warning 2.1.1.2), so Theorem C.5.4.9 implies that ι admits an essentially unique extension to a colimit-preserving t-exact functor $\rho : \mathcal{D}(\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}) \to \mathrm{Mod}_{\mathscr{O}}$.

Theorem 2.1.2.2. *Let $(\mathcal{X}, \mathscr{O})$ be a spectrally ringed ∞-topos satisfying the following conditions:*

(a) *The structure sheaf $\mathscr{O}$ is discrete.*

(b) *For each object $X \in \mathcal{X}$, there exists an effective epimorphism $U \to X$ where U is a discrete object of $\mathcal{X}$.*

(c) *The ∞-topos $\mathcal{X}$ is hypercomplete.*

Then the functor $\rho : \mathcal{D}(\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}) \to \mathrm{Mod}_{\mathscr{O}}$ supplied by Theorem C.5.4.9 is an equivalence of ∞-categories.

Before giving the proof of Theorem 2.1.2.2, let us note some of its consequences.

Corollary 2.1.2.3. *Let $(\mathcal{X}, \mathscr{O})$ be a spectrally ringed ∞-topos satisfying the following conditions:*

(a) *The structure sheaf $\mathscr{O}$ is discrete.*

(b) *For each object $X \in \mathcal{X}$, there exists an effective epimorphism $U \to X$ where U is a discrete object of $\mathcal{X}$.*

Then the inclusion $\mathrm{Mod}_{\mathscr{O}}^{\heartsuit} \hookrightarrow \mathrm{Mod}_{\mathscr{O}}$ *extends to a fully faithful embedding* $\iota : \mathcal{D}(\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}) \hookrightarrow \mathrm{Mod}_{\mathscr{O}}$*, whose essential image is the full subcategory of hypercomplete* $\mathscr{O}$*-module objects of* $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$*.*

Proof. Let $f^* : \mathcal{X} \to \mathcal{X}^{\mathrm{hyp}}$ be a left adjoint to the inclusion. Then the spectrally ringed ∞-topos $(\mathcal{X}^{\mathrm{hyp}}, f^* \mathscr{O})$ satisfies the hypotheses of Theorem 2.1.2.2, so that we have an equivalence of ∞-categories

$$\mathcal{D}(\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}) \simeq \mathcal{D}(\mathrm{Mod}_{f^* \mathscr{O}}^{\heartsuit}) \simeq \mathrm{Mod}_{f^* \mathscr{O}}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}^{\mathrm{hyp}})).$$

We now define ι to be the composition of this equivalence with the pushforward functor

$$f_* : \mathrm{Mod}_{f^* \mathscr{O}}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}^{\mathrm{hyp}})) \to \mathrm{Mod}_{\mathscr{O}}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})) = \mathrm{Mod}_{\mathscr{O}}$$

(which is a fully faithful embedding whose essential image is spanned by the hypercomplete $\mathscr{O}$-module objects of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$). □

Corollary 2.1.2.4. *Let* $(\mathcal{X}, \mathscr{O})$ *be a spectrally ringed* ∞*-topos satisfying the following conditions:*

- (a) *The structure sheaf* $\mathscr{O}$ *is discrete.*
- (b) *For each object* $X \in \mathcal{X}$*, there exists an effective epimorphism* $U \to X$ *where* U *is a discrete object of* $\mathcal{X}$*.*

Then the functor $\iota : \mathcal{D}(\mathrm{Mod}_{\mathscr{O}}^{\heartsuit})_{<\infty} \to \mathrm{Mod}_{\mathscr{O}}$ *supplied by Remark HA.1.3.5.23 is a fully faithful embedding, whose essential image is the union* $\bigcup_{n \geqslant 0} (\mathrm{Mod}_{\mathscr{O}})_{\leqslant n}$*.*

We begin by proving Theorem 2.1.2.2 in the special case where $\mathcal{X}$ is a presheaf ∞-topos.

Proposition 2.1.2.5. *Let* $\mathcal{C}$ *be a category, let* $\mathscr{O} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{CAlg}^{\heartsuit})$ *be a presheaf of commutative rings on* $\mathcal{C}$*, and let* $\mathrm{Mod}_{\mathscr{O}} = \mathrm{Mod}_{\mathscr{O}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}))$ *be the* ∞*-category of* $\mathscr{O}$*-modules on the hypercomplete* ∞*-topos* $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$*. Then the canonical map* $\rho : \mathcal{D}(\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}) \to \mathrm{Mod}_{\mathscr{O}}$ *(supplied by Corollary* **??***) is an equivalence of* ∞*-categories.*

Proof. For each object $C \in \mathcal{C}$, let $h_C : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et} \subset \mathcal{S}$ be the functor represented by C (given on objects by the formula $h_C(D) = \mathrm{Hom}_{\mathcal{C}}(D, C)$) and let $\mathscr{F}_C \in \mathrm{Mod}_{\mathscr{O}}$ be the tensor product $\mathscr{O} \otimes \Sigma^{\infty}_{+}(h_C)$, given on objects by the formula $\mathscr{F}_C(D) \simeq \bigoplus_{\eta: D \to C} \mathscr{O}(D)$. Note that $\mathscr{F}_C$ belongs to the heart of $\mathrm{Mod}_{\mathscr{O}}$.

For any object $\mathscr{G} \in \mathrm{Mod}_{\mathscr{O}}$, we have a canonical homotopy equivalence

$$\begin{aligned} \mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}}}(\mathscr{F}_C, \mathscr{G}) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})}(\Sigma^{\infty}_{+} h_C, \mathscr{G}) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(h_C, \Omega^{\infty} \mathscr{G}) \\ &\simeq \Omega^{\infty} \mathscr{G}(C). \end{aligned}$$

Note that if $\mathscr{G} \in (\mathrm{Mod}_{\mathscr{O}})_{\geqslant 1}$, then we have $\pi_0 \mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}}}(\mathscr{F}_C, \mathscr{G}) \simeq 0$: that is, $\mathscr{F}_C$ is a projective object of the ∞-category $\mathrm{Mod}_{\mathscr{O}}^{\mathrm{cn}}$, and in particular is a projective object of the abelian category $\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}$. For an arbitrary object $\mathscr{G} \in \mathrm{Mod}_{\mathscr{O}}$, the induced map $\bigoplus_{\eta \in \pi_0 \mathscr{G}(C)} \mathscr{F}_C \to \mathscr{G}$ is an epimorphism on π_0. It follows that the abelian category $\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}$ has enough projective objects, so that the derived ∞-category $\mathcal{D}(\mathrm{Mod}_{\mathscr{O}}^{\heartsuit})$ is left complete (see Proposition HA.1.3.5.24). The presheaf ∞-topos $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is Postnikov complete (in the sense of Definition A.7.2.1), the t-structure on $\mathrm{Mod}_{\mathscr{O}}$ is also left complete (Warning 2.1.1.2). Consequently, to show that the functor ρ is an equivalence of ∞-categories, it will suffice to show that the underlying map

$$\rho_{<\infty} : \mathcal{D}(\mathrm{Mod}_{\mathscr{O}}^{\heartsuit})_{<\infty} \to (\mathrm{Mod}_{\mathscr{O}})_{<\infty}$$

is an equivalence of ∞-categories. Using the dual of Proposition HA.1.3.3.7 , we are reduced to proving the following:

(∗) For every pair of objects $\mathscr{G}, \mathscr{G}' \in \mathrm{Mod}_{\mathscr{O}}^{\heartsuit}$, there exists an epimorphism $\theta : \mathscr{F} \to \mathscr{G}$ in $\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}$ for which the groups $\mathrm{Ext}^n_{\mathscr{O}}(\mathscr{F}, \mathscr{G}')$ vanish for $n > 0$.

This is clear: we can take θ to be the epimorphism $\bigoplus_{\eta \in \pi_0 \mathscr{G}(C)} \mathscr{F}(C) \to \mathscr{G}$ described above. □

The proof of Theorem 2.1.2.2 will require a brief digression. Let $\mathcal{X}$ be an ∞-topos and $\mathscr{O}$ a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then for every object $X \in \mathcal{X}$, we let $\mathscr{O}|_X$ denote the composition of $\mathscr{O}$ with the forgetful functor $\pi : \mathcal{X}_{/X} \to \mathcal{X}$, so that $\mathscr{O}|_X$ is a sheaf of $\mathbb{E}_\infty$-rings on the ∞-topos $\mathcal{X}_{/X}$. Composition with π determines a pullback functor $\mathrm{Mod}_{\mathscr{O}} \to \mathrm{Mod}_{\mathscr{O}|_X}$, which we will denote by π^*. The functor π^* preserves small limits and colimits, and therefore admits a left adjoint $\pi_! : \mathrm{Mod}_{\mathscr{O}|_X} \to \mathrm{Mod}_{\mathscr{O}}$ (Corollary HTT.5.5.2.9).

Lemma 2.1.2.6. *Let $\mathcal{X}$ be an ∞-topos, $\mathscr{O}$ a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and X a discrete object of $\mathcal{X}$. Then the functor $\pi_! : \mathrm{Mod}_{\mathscr{O}|_X} \to \mathrm{Mod}_{\mathscr{O}}$ is t-exact (with respect to the t-structures introduced in Proposition 2.1.1.1).*

Proof. The functor $\pi_!$ is obviously right t-exact (since it is the left adjoint of the t-exact pullback functor $\pi^* : \mathrm{Mod}_{\mathscr{O}} \to \mathrm{Mod}_{\mathscr{O}|_X}$). It will therefore suffice to show that $\pi_!$ is left t-exact: that is, that $\pi_!$ carries $(\mathrm{Mod}_{\mathscr{O}|_X})_{\leqslant 0}$ to $(\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0}$.

Without loss of generality, we may assume that $\mathcal{X}$ is an accessible left-exact localization of a presheaf ∞-category $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ for some small ∞-category $\mathcal{C}$; we will identify X with the corresponding discrete object of $\mathcal{P}(\mathcal{C})$. Then $\mathscr{O}$ can be obtained as the pullback of a connective sheaf of $\mathbb{E}_\infty$-rings $\mathscr{O}' \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{P}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{CAlg})$. We have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Mod}_{\mathscr{O}'|_X} & \xrightarrow{\pi'_!} & \mathrm{Mod}_{\mathscr{O}'} \\ \downarrow{g^*} & & \downarrow{f^*} \\ \mathrm{Mod}_{\mathscr{O}|_X} & \xrightarrow{\pi_!} & \mathrm{Mod}_{\mathscr{O}}, \end{array}$$

where the vertical maps are given by pullback along the geometric morphisms

$$f_* : \mathcal{X} \hookrightarrow \mathcal{P}(\mathcal{C}) \qquad g_* : \mathcal{X}_{/X} \hookrightarrow \mathcal{P}(\mathcal{C})_{/X}$$

(and are therefore t-exact). For any object $\mathscr{F} \in (\mathrm{Mod}_{\mathscr{O}|_X})_{\leqslant 0}$, there exists an object $\mathscr{F}' \in (\mathrm{Mod}_{\mathscr{O}'|_X})_{\leqslant 0}$ such that $\mathscr{F} \simeq g^* \mathscr{F}'$: for example, we can take $\mathscr{F}'$ to be the pushforward $g_* \mathscr{F}$. Since the functor f^* is t-exact, to prove that $\pi_! \mathscr{F} \in (\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0}$, it will suffice to show that $\pi'_! \mathscr{F}' \in (\mathrm{Mod}_{\mathscr{O}'})_{\leqslant 0}$. In other words, we wish to show that for every object $C \in \mathcal{C}$, the $\mathscr{O}'(C)$-module spectrum $(\pi'_! \mathscr{F}')(C)$ belongs to $\mathrm{Sp}_{\leqslant 0}$. Since X is discrete, we may assume without loss of generality that X is a $\mathcal{S}$et-valued functor on $\mathcal{C}^{\mathrm{op}}$. Note that $(\pi'_! \mathscr{F}')(C)$ can be written as a coproduct $\amalg_{\eta \in X(C)} \mathscr{F}'(C_\eta)$ where $C_\eta \in \mathcal{P}(\mathcal{C})_{/X}$ denotes map $j(C) \to X$ representing $\eta \in X(C)$, where $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is the Yoneda embedding. Since $\mathscr{F}' \in (\mathrm{Mod}_{\mathscr{O}'|_X})_{\leqslant 0}$, each of the spectra $\mathscr{F}'(C_\eta) \in (\mathrm{Sp})_{\leqslant 0}$, so that $(\pi_! \mathscr{F}')(C) \in \mathrm{Sp}_{\leqslant 0}$ as desired. □

Proposition 2.1.2.7. *Let $(\mathcal{X}, \mathscr{O})$ be a spectrally ringed ∞-topos and let $n \geqslant 0$ be an integer. Suppose that the following conditions are satisfied:*

- (a) *For every object $X \in \mathcal{X}$, there exists an effective epimorphism $U \to X$ where U is a discrete object of $\mathcal{X}$.*
- (b) *The structure sheaf $\mathscr{O}$ is connective and n-truncated.*

Then the ∞-category $\mathrm{Mod}^{\mathrm{cn}}_{\mathscr{O}}$ is n-complicial (see Definition C.5.3.1).

Proof. Using (a) and the presentability of $\mathcal{X}$, we can choose a collection of discrete objects $\{U_\alpha\}$ of $\mathcal{X}$ with the following property: for every object $X \in \mathcal{X}$, there exists an effective epimorphism $\overline{X} \to X$, where $\overline{X}$ can be written as a coproduct of objects belonging to $\{U_\alpha\}$. For each index α, let $\pi_{\alpha!} : \mathrm{Mod}_{\mathscr{O}|_{U_\alpha}} \to \mathrm{Mod}_{\mathscr{O}}$ denote the functor of Lemma 2.1.2.6 and set $\mathscr{O}_\alpha = \pi_{\alpha!}(\mathscr{O}|_{U_\alpha})$. Since each U_α is discrete, Lemma 2.1.2.6 (and assumption (b)) imply that each $\mathscr{O}_\alpha$ is an n-truncated object of $\mathrm{Mod}^{\mathrm{cn}}_{\mathscr{O}}$.

Let $\mathscr{F}$ be an object of $\mathrm{Mod}_{\mathscr{O}}$. Unwinding the definitions, we see that each element $\eta \in \pi_0 \mathscr{F}(U_\alpha)$ determines a homotopy class of maps $f_\eta : \mathscr{O}_\alpha \to \mathscr{F}$ in the ∞-category $\mathrm{Mod}_{\mathscr{O}}$. Amalgamating these maps as α and η vary, we obtain a map $f : \bigoplus_{\eta \in \pi_0 \mathscr{F}(U_\alpha)} \mathscr{O}_\alpha \to \mathscr{F}$. By construction, the morphism f induces an epimorphism on π_0 and the domain of f is n-truncated. Allowing $\mathscr{F}$ to vary, we conclude that $\mathrm{Mod}^{\mathrm{cn}}_{\mathscr{O}}$ is n-complicial. □

Proof of Theorem 2.1.2.2. Let $(\mathcal{X}, \mathcal{O})$ be a spectrally ringed ∞-topos. It follows from Theorem C.5.4.9 that the inclusion $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit} \hookrightarrow \mathrm{Mod}_{\mathcal{O}}^{\mathrm{cn}}$ admits an essentially unique extension to a functor $\lambda : \mathcal{D}(\mathrm{Mod}_{\mathcal{O}}^{\heartsuit})_{\geqslant 0} \to \mathrm{Mod}_{\mathcal{O}}^{\mathrm{cn}}$ which preserves small colimits and finite limits. Suppose that $\mathcal{O}$ is discrete and that for every object $X \in \mathcal{X}$, there exists an effective epimorphism $U \to X$ where U is discrete. Applying Proposition 2.1.2.7, we deduce that the ∞-category $\mathrm{Mod}_{\mathcal{O}}^{\mathrm{cn}}$ is 0-complicial (in the sense of Definition C.5.3.1). If the ∞-topos $\mathcal{X}$ is hypercomplete, then $\mathrm{Mod}_{\mathcal{O}}^{\mathrm{cn}}$ is also separated (in the sense of Definition C.1.2.12). Since $\mathcal{D}(\mathrm{Mod}_{\mathcal{O}}^{\heartsuit})_{\geqslant 0}$ is also 0-complicial (Proposition C.5.3.2) and separated, Proposition C.5.4.5 implies that λ is an equivalence of ∞-categories. Passing to stabilizations, we obtain a t-exact equivalence

$$\mathcal{D}(\mathcal{A}) \simeq \mathrm{Sp}(\mathcal{D}(\mathcal{A})_{\geqslant 0}) \xrightarrow{\sim} \mathrm{Sp}(\mathrm{Mod}_{\mathcal{O}}^{\mathrm{cn}}) \simeq \mathrm{Mod}_{\mathcal{O}}.$$

$\square$

We close this section by proving a variant of Theorem 2.1.2.2 for morphisms between spectrally ringed ∞-topoi.

Theorem 2.1.2.8. *Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of spectrally ringed ∞-topoi which satisfy conditions (a) and (b) of Corollary 2.1.2.4 and let $f_*^{\heartsuit} : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}^{\heartsuit} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}^{\heartsuit}$ be the functor of abelian categories given at the level of objects by the formula $f_*^{\heartsuit}\mathscr{F} = \pi_0(f_*\mathscr{F})$. Then the pushforward functor*

$$\mathcal{D}(\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}^{\heartsuit})_{<\infty} \simeq \bigcup_n (\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}})_{\leqslant n} \xrightarrow{f_*} \bigcup_n (\mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}})_{\leqslant n} \simeq \mathcal{D}(\mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}^{\heartsuit})_{<\infty}$$

is a right derived functor of $f_*^{\heartsuit}$ (see Example HA.1.3.3.4).

Proof. Let $\mathscr{F}$ be an injective object of the abelian category $\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}^{\heartsuit}$. We wish to show that the pushforward $f_*\mathscr{F}$ belongs to the heart of $\mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}$. Fix an object $Y \in \mathcal{Y}$ and an element $x \in \pi_n((f_*\mathscr{F})(Y))$; we wish to prove that if $n \neq 0$, then we can choose an effective epimorphism $Y' \to Y$ such that the image of x vanishes in $\pi_n((f_*\mathscr{F})(Y'))$. In fact, we will prove something stronger: the group $\pi_n((f_*\mathscr{F})(Y'))$ vanishes whenever Y' is a discrete object of $\mathcal{Y}$ (note that there exists an effective epimorphism $Y' \to Y$ with Y' discrete by virtue of our assumption that $\mathcal{Y}$ satisfies condition (b) of Corollary 2.1.2.4). Set $X = f^*Y'$; we wish to prove that $\pi_n\mathscr{F}(X) \in \mathcal{X}$ vanishes. Let $\pi_! : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}|_X} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}$ be as Lemma 2.1.2.6. Since $X = f^*Y'$ is discrete, the functor $\pi_!$ is t-exact. It follows that $\pi_!(\mathcal{O}_{\mathcal{X}}|_X)$ belongs to $\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}^{\heartsuit}$. Using Corollary 2.1.2.4 and the injectivity of $\mathscr{F}$, we obtain

$$\pi_n\mathscr{F}(X) \simeq \mathrm{Ext}^{-n}_{\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}|_X}}(\mathcal{O}_{\mathcal{X}}|_X, \mathscr{F}|_X) \simeq \mathrm{Ext}^{-n}_{\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}}(\pi_!\mathcal{O}_{\mathcal{X}}|_X, \mathscr{F}) \simeq 0.$$

$\square$

2.2 Quasi-Coherent Sheaves on Spectral Deligne-Mumford Stacks

Let $(X, \mathscr{O}_X)$ be a scheme. Recall that a (discrete) sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules on X is said to be *quasi-coherent* if it satisfies the following condition:

$(*)$ For every pair of affine open subsets $U \subseteq V \subseteq X$, the canonical map

$$\mathscr{O}_X(U) \otimes_{\mathscr{O}_X(V)} \mathscr{F}(V) \to \mathscr{F}(U)$$

is an isomorphism.

The theory of quasi-coherent sheaves plays an essential role in classical algebraic geometry. Our goal in this section is to introduce an analogous theory in the setting of spectral algebraic geometry.

Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_\mathcal{X})$ be a spectrally ringed ∞-topos. In §2.1, we defined a stable ∞-category $\mathrm{Mod}_{\mathscr{O}_\mathcal{X}}$ whose objects are sheaves of $\mathscr{O}_\mathcal{X}$-module spectra on $\mathcal{X}$. In the special case where X is a nonconnective spectral Deligne-Mumford stack, we will define a full subcategory $\mathrm{QCoh}(\mathsf{X}) \subseteq \mathrm{Mod}_{\mathscr{O}_\mathcal{X}}$, which we will refer to as the ∞-category of *quasi-coherent* sheaves on X. The condition that a sheaf $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_\mathcal{X}}$ is quasi-coherent can be expressed in several different ways:

(a) The triple $(\mathcal{X}, \mathscr{O}_\mathcal{X}, \mathscr{F})$ is locally equivalent to the spectrum (in the sense of Corollary 2.2.1.5 below) of a pair (A, M), where A is an $\mathbb{E}_\infty$-ring and M is an A-module spectrum.

(b) The sheaf $\mathscr{F}$ satisfies the analogue of condition $(*)$ above: for any morphism $U \to V$ between affine objects of $\mathcal{X}$, the induced map $\mathscr{O}_X(U) \otimes_{\mathscr{O}_X(V)} \mathscr{F}(V) \to \mathscr{F}(U)$ is an equivalence.

(c) Each homotopy group $\pi_n \mathscr{F}$ is a quasi-coherent sheaf over the underlying ordinary Deligne-Mumford stack $(\mathcal{X}^\heartsuit, \pi_0 \mathscr{O}_\mathcal{X})$, and the sheaf $\mathscr{F}$ is hypercomplete.

We will adopt characterization (a) as our definition of quasi-coherent sheaf, and prove its equivalence with (b) and (c) later in this section (Propositions 2.2.4.3 and 2.2.6.1, respectively).

2.2.1 The Étale Spectrum of a Module

Let $(\mathcal{X}, \mathscr{O})$ be a spectrally ringed ∞-topos and let $\mathscr{F} \in \mathrm{Mod}_\mathscr{O}$ be a $\mathscr{O}$-module object of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Then we can view the pair $(\mathscr{O}, \mathscr{F})$ as a single sheaf on $\mathcal{X}$:

Notation 2.2.1.1. Let $\mathrm{Mod} = \mathrm{Mod}(\mathrm{Sp})$ denote the ∞-category of pairs (A, M), where A is an $\mathbb{E}_\infty$-ring and M is an A-module spectrum. We let $\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}$ denote the ∞-category whose objects are ∞-topoi $\mathcal{X}$ together with a Mod-valued sheaf on $\mathcal{X}$. More precisely, we let $\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}$ denote an ∞-category equipped with a coCartesian fibration $\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}} \to \infty\mathcal{T}\mathrm{op}$, which is classified by the functor

$$\infty\mathcal{T}\mathrm{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$$

$$\mathcal{X} \mapsto \mathcal{S}\mathrm{hv}_{\mathrm{Mod}}(\mathcal{X})^{\mathrm{op}}.$$

We can describe the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}$ more informally as follows:

- The objects of $\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}$ are triples $(\mathcal{X}, \mathcal{O}, \mathcal{F})$, where $\mathcal{X}$ is an ∞-topos, $\mathcal{O}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and $\mathcal{F}$ is a sheaf of $\mathcal{O}$-module spectra on $\mathcal{X}$.

- A morphism from $(\mathcal{X}, \mathcal{O}, \mathcal{F})$ to $(\mathcal{X}', \mathcal{O}', \mathcal{F}')$ in $\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}$ consists of a triple (ϕ_*, α, β), where $\phi_* : \mathcal{X} \to \mathcal{X}'$ is a geometric morphism of ∞-topoi, $\alpha : \mathcal{O}' \to \phi_* \mathcal{O}$ is a morphism of CAlg-valued sheaves, and $\beta : \mathcal{F}' \to \phi_* \mathcal{F}$ is a morphism of $\mathcal{O}'$-modules.

Notation 2.2.1.2. We can identify the fiber product $\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}} \times_{\mathcal{T}\mathrm{op}_\infty} \{\mathcal{S}\}$ with the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Mod}}(\mathcal{S})^{\mathrm{op}} \simeq \mathrm{Mod}^{\mathrm{op}}$. The induced functor $\mathrm{Mod}^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}$ admits a left adjoint, given on objects by $(\mathcal{X}, \mathcal{O}, \mathcal{F}) \mapsto (\Gamma(\mathcal{X}; \mathcal{O}), \Gamma(\mathcal{X}; \mathcal{F}))$. We will refer to this left adjoint as the *global sections functor*, and denote it by $\Gamma : \infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}} \to \mathrm{Mod}^{\mathrm{op}}$.

Notation 2.2.1.3. There is an evident forgetful functor $\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}} \to \infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$, given on objects by $(\mathcal{X}, \mathcal{O}, \mathcal{F}) \mapsto (\mathcal{X}, \mathcal{O})$. We let $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{Mod}}$ denote the fiber product

$$\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}} \times_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}} \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}},$$

so that $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{Mod}}$ is a subcategory of $\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}$ whose objects are triples $(\mathcal{X}, \mathcal{O}, \mathcal{F})$ for which the sheaf $\mathcal{O}$ is strictly Henselian.

Proposition 2.2.1.4. *Let A be an $\mathbb{E}_\infty$-ring and let M be an A-module. Let*

$$\rho : \mathrm{CAlg}^{\acute{e}t}_A \to \mathrm{Mod}$$

denote the functor given on objects by $B \mapsto (B, B \otimes_A M)$. Then:

(1) *The functor ρ is a sheaf with respect to the étale topology on $\mathrm{CAlg}^{\acute{e}t}_A$, and can therefore be identified with a* Mod*-valued sheaf $(\mathcal{O}, \mathcal{F})$ on the ∞-topos $\mathcal{S}\mathrm{hv}^{\acute{e}t}_A$ (see Proposition 1.3.1.7).*

(2) *The sheaf of $\mathbb{E}_\infty$-rings $\mathcal{O}$ is strictly Henselian, so that we can view $(\mathcal{S}\mathrm{hv}^{\acute{e}t}_A, \mathcal{O}, \mathcal{F})$ as an object of $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{Mod}}$.*

(3) *Let* $\Gamma : \infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}} \to \mathrm{Mod}^{\mathrm{op}}$ *be the global sections functor, so that* $\Gamma(\mathcal{S}\mathrm{hv}_A^{\acute{e}t}, \mathscr{O}, \mathscr{F}) \simeq (A, M)$*. For every object* $\mathsf{X} \in \infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}^{\mathrm{sHen}}$*, the canonical map*

$$\theta : \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}^{\mathrm{sHen}}}(\mathsf{X}, (\mathcal{S}\mathrm{hv}_A^{\acute{e}t}, \mathscr{O}, \mathscr{F})) \to \mathrm{Map}_{\mathrm{Mod}}((A, M), \Gamma(\mathsf{X}))$$

is a homotopy equivalence.

Proof of Proposition 2.2.1.4. Assertion (1) follows from Theorems D.6.3.5 and **??**. Assertion (2) from Proposition **??**. To prove (3), write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}}, \mathscr{F}_{\mathcal{X}})$ and observe that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}^{\mathrm{sHen}}}(\mathsf{X}, (\mathcal{S}\mathrm{hv}_A^{\acute{e}t}, \mathscr{O}, \mathscr{F})) & \xrightarrow{\theta} & \mathrm{Map}_{\mathrm{Mod}}((A, M), \Gamma(\mathsf{X})) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{sHen}}}((\mathcal{X}, \mathscr{O}_{\mathcal{X}}), (\mathcal{S}\mathrm{hv}_A^{\acute{e}t}, \mathscr{O})) & \xrightarrow{\theta_0} & \mathrm{Map}_{\mathrm{CAlg}}(A, \Gamma(\mathcal{X}; \mathscr{O})) \end{array}$$

where the map θ_0 is a homotopy equivalence by virtue of Proposition 1.4.2.4. Consequently, in order to prove that θ is a homotopy equivalence, it will suffice to show that it induces a homotopy equivalence after taking the homotopy fibers of the vertical maps over a point corresponding to a map of spectrally ringed ∞-topoi $\eta : (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{S}\mathrm{hv}_A^{\acute{e}t}, \mathscr{O})$. Unwinding the definitions, we are reduced to proving that the formation of global sections induces a homotopy equivalence

$$\phi_M : \mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}}}(\mathscr{F}, \eta_* \mathscr{F}_{\mathcal{X}}) \to \mathrm{Map}_{\mathrm{Mod}_A}(M, \Gamma(\mathcal{X}; \mathscr{F}_{\mathcal{X}})).$$

Let us say that an A-module M is *good* if the map ϕ_M is a homotopy equivalence for every choice of sheaf $\mathscr{F}_{\mathcal{X}} \in \mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}}$. We wish to show that every A-module M is good. The collection of good A-modules span a stable subcategory of Mod_A which is closed under colimits. Consequently, we can reduce to the case $M = A$, in which case $\mathscr{F}$ is equivalent to the structure sheaf of Spét A and the desired result is obvious. □

Corollary 2.2.1.5. *Let* $\Gamma^{\mathrm{sHen}} : \infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}^{\mathrm{sHen}} \to \mathrm{Mod}^{\mathrm{op}}$ *denote the restriction of the global sections functor* $\Gamma : \infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}} \to \mathrm{Mod}^{\mathrm{op}}$ *to the subcategory* $\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}^{\mathrm{sHen}} \subseteq \infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}$*. Then* Γ^{sHen} *admits a right adjoint*

$$\mathrm{Sp\acute{e}t}_{\mathrm{Mod}} : \mathrm{Mod}^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}^{\mathrm{sHen}}.$$

Remark 2.2.1.6. Given an $\mathbb{E}_\infty$-ring A and an A-module M, the object $\mathrm{Sp\acute{e}t}_{\mathrm{Mod}}(A, M)$ can be described explicitly using the construction of Proposition 2.2.1.4. In particular, we see that the underlying spectrally ringed ∞-topos of $\mathrm{Sp\acute{e}t}_{\mathrm{Mod}}(A, M)$ can be identified with the étale spectrum Spét A. It follows that the diagram of ∞-categories

$$\begin{array}{ccc} \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{Mod}} & \longrightarrow & \mathrm{Mod}^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}} & \longrightarrow & \mathrm{CAlg}^{\mathrm{op}} \end{array}$$

is right adjointable, so that the diagram

$$\begin{array}{ccc} \mathrm{Mod}^{\mathrm{op}} & \xrightarrow{\mathrm{Sp\acute{e}t}_{\mathrm{Mod}}} & \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{Mod}} \\ \downarrow & & \downarrow \\ \mathrm{CAlg}^{\mathrm{op}} & \xrightarrow{\mathrm{Sp\acute{e}t}} & \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}} \end{array}$$

commutes up to canonical homotopy.

2.2.2 Quasi-Coherence

We are now ready to introduce our main objects of interest.

Definition 2.2.2.1. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O})$ be a nonconnective spectral Deligne-Mumford stack and let $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules on $\mathcal{X}$. We will say that $\mathscr{F}$ is *quasi-coherent* if there exists a collection of objects $U_\alpha \in \mathcal{X}$ which cover $\mathcal{X}$ (that is, the map $\amalg_\alpha U_\alpha \to \mathbf{1}$ is an effective epimorphism) satisfying the following condition:

($*$) For each α, there exists an $\mathbb{E}_\infty$-ring A_α, an A_α-module M_α, and an equivalence

$$(\mathcal{X}_{/U_\alpha}, \mathscr{O}|_{U_\alpha}, \mathscr{F}|_{U_\alpha}) \simeq \mathrm{Sp\acute{e}t}_{\mathrm{Mod}}(A_\alpha, M_\alpha)$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{Mod}}$.

We let $\mathrm{QCoh}(\mathsf{X})$ denote the full subcategory of $\mathrm{Mod}_{\mathscr{O}}$ spanned by the quasi-coherent sheaves of $\mathscr{O}$-modules on $\mathcal{X}$.

Remark 2.2.2.2. The existence of a covering $\{U_\alpha\}$ satisfying condition ($*$) guarantees that $(\mathcal{X}, \mathscr{O})$ is a nonconnective spectral Deligne-Mumford stack.

Remark 2.2.2.3. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O})$ be a nonconnective spectral Deligne-Mumford stack, and let $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules on $\mathcal{X}$. The condition that $\mathscr{F}$ be quasi-coherent is local on $\mathcal{X}$. In other words:

- For every morphism $U \to V$ in $\mathcal{X}$, if $\mathscr{F}|_V$ is a quasi-coherent sheaf on $(\mathcal{X}_{/V}, \mathscr{O}|_V)$, then $\mathscr{F}|_U$ is a quasi-coherent sheaf on $(\mathcal{X}_{/U}, \mathscr{O}|_U)$.
- Conversely, if we are given an effective epimorphism $\amalg_\alpha U_\alpha \to V$ and each restriction $\mathscr{F}|_{U_\alpha}$ is a quasi-coherent sheaf on $(\mathcal{X}_{/U_\alpha}, \mathscr{O}|_{U_\alpha})$, then $\mathscr{F}|_V$ is a quasi-coherent sheaf on $(\mathcal{X}_{/V}, \mathscr{O}|_V)$.

2.2.3 The Affine Case

Our next goal is to describe quasi-coherent sheaves over affine spectral Deligne-Mumford stacks. We begin with a few general remarks.

Lemma 2.2.3.1. *Let A be an $\mathbb{E}_\infty$-ring, let M be an A-module, and set $(\mathcal{X}, \mathcal{O}, \mathcal{F}) = \mathrm{Sp\acute{e}t}_{\mathrm{Mod}}(A, M)$. Let $U \in \mathcal{X}$ be affine. Then the canonical map $\mathcal{O}(U) \otimes_A M \to \mathcal{F}(U)$ is an equivalence.*

Proof. Since U is affine, Theorem 1.4.10.2 implies that $U \in \mathcal{X} \simeq \mathcal{S}\mathrm{hv}_A^{\acute{e}t}$ can be identified with the functor corepresented by an étale A-algebra B. In this case, the desired result follows immediately from the construction of $(\mathcal{X}, \mathcal{O}, \mathcal{F})$ supplied by Proposition 2.2.1.4. □

Lemma 2.2.3.2. *Let $(\mathcal{X}, \mathcal{O}) \simeq \mathrm{Sp\acute{e}t}\, A$ be an affine nonconnective spectral Deligne-Mumford stack. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}$-module and let $M = \Gamma(\mathcal{X}; \mathcal{F})$ be the global sections of $\mathcal{F}$, regarded as an $A \simeq \Gamma(\mathcal{X}; \mathcal{O})$-module. Then the canonical map $(\mathcal{X}, \mathcal{O}, \mathcal{F}) \to \mathrm{Sp\acute{e}t}_{\mathrm{Mod}}(A, M)$ is an equivalence (in the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{Mod}}^{\mathrm{sHen}}$).*

Proof. Using Proposition 1.4.2.4, we can identify $\mathcal{X}$ with the ∞-category $\mathcal{S}\mathrm{hv}_A^{\acute{e}t}$. Since $\mathcal{F}$ is quasi-coherent, there exists a collection of objects $U_\alpha \in \mathcal{X}$ covering $\mathcal{X}$ for which each $(\mathcal{X}_{/U_\alpha}, \mathcal{O}|_{U_\alpha}, \mathcal{F}|_{U_\alpha})$ has the form $\mathrm{Sp\acute{e}t}_{\mathrm{Mod}}(A_\alpha, M_\alpha)$ for some object $(A_\alpha, M_\alpha) \in \mathrm{Mod}$. In particular, $(\mathcal{X}_{/U_\alpha}, \mathcal{O}|_{U_\alpha}) \simeq \mathrm{Sp\acute{e}t}(A_\alpha)$ is an affine nonconnective spectral Deligne-Mumford stack, so that A_α is an étale A-algebra by Theorem 1.4.10.2. Without loss of generality, we may assume that the set of indices α is finite. Let $B = \prod B_\alpha$ so that $\mathrm{Sp\acute{e}t}\, B \simeq (\mathcal{X}_{/U}, \mathcal{O}|_U)$ for $U = \amalg_\alpha U_\alpha$. We now observe that $(\mathcal{X}_{/U}, \mathcal{O}|_U, \mathcal{F}|_U) \simeq \mathrm{Sp\acute{e}t}_{\mathrm{Mod}}(B, N)$ where $B = \prod A_\alpha$ and $N = \prod M_\alpha$.

Let us abuse notation by identifying the pair $(\mathcal{O}, \mathcal{F})$ with the underlying functor $\mathrm{CAlg}_A^{\acute{e}t} \to \mathrm{Mod}$. Using Lemma 2.2.3.1, we deduce that the canonical map $\mathcal{F}(R) \otimes_R R' \to \mathcal{F}(R')$ is an equivalence whenever $R \to R'$ is a morphism in $\mathrm{CAlg}_A^{\acute{e}t}$ for which the étale map $A \to R$ factors through B. Let $B^\bullet$ be the Čech nerve of the faithfully flat morphism $A \to B$. Since $\mathcal{F}$ is a sheaf, we have $M = \mathcal{F}(A) = \varprojlim \mathcal{F}(B^\bullet)$. The proof of Theorem D.6.3.1 shows that the canonical map $M \otimes_A B \to \mathcal{F}(B)$ is an equivalence, so that $M \otimes_A R \to \mathcal{F}(R)$ is an equivalence for any étale map $A \to R$ which factors through B. Let $\mathrm{Sp\acute{e}t}_{\mathrm{Mod}}(A, M) \simeq (\mathcal{X}, \mathcal{O}, \mathcal{F}')$, so that the map $M \to \mathcal{F}(A)$ induces a morphism of sheaves of $\mathcal{O}$-modules $\mathcal{F}' \to \mathcal{F}$. Using Lemma 2.2.3.1, we deduce that α induces an equivalence $\mathcal{F}'(R) \to \mathcal{F}(R)$ whenever $A \to R$ is an étale map which factors through B. Since $\mathcal{F}'$ and $\mathcal{F}$ are sheaves, they are determined by their restriction to the sieve generated by B, so that α is an equivalence as desired. □

Proposition 2.2.3.3. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}) \simeq \mathrm{Sp\acute{e}t}\, A$ be an affine nonconnective spectral Deligne-Mumford stack. Then the global sections functor $\Gamma(\mathcal{X}; \bullet) : \mathrm{Mod}_{\mathcal{O}} \to \mathrm{Mod}_A$ admits a fully faithful left adjoint, whose essential image is the full subcategory $\mathrm{QCoh}(\mathsf{X}) \subseteq \mathrm{Mod}_{\mathcal{O}}$.*

Proof. Let $F : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}}$ denote the functor given on objects by by the formula $\mathrm{Sp\acute{e}t}_{\mathrm{Mod}}(A, M) \simeq (\mathcal{X}, \mathcal{O}, F(M))$. Unwinding the definitions, we deduce immediately that F is a left adjoint to $\Gamma(\mathcal{X}; \bullet)$. It is clear that F carries A-modules to quasi-coherent $\mathcal{O}$-modules. Conversely, Lemma 2.2.3.2 implies that every quasi-coherent $\mathcal{O}$-module belongs to the essential image of F. To prove that F is fully faithful, it suffices to show that for every A-module M, the unit map $M \to \Gamma(\mathcal{X}; F(M))$ is an equivalence, which is a special case of Lemma 2.2.3.1. □

2.2.4 The General Case

We now use Proposition 2.2.3.3 together with the fact that quasi-coherence can be tested locally to establish some pleasant features of quasi-coherent sheaves on *arbitrary* nonconnective spectral Deligne-Mumford stacks.

Proposition 2.2.4.1. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ be a nonconnective spectral Deligne-Mumford stack. Then:*

(1) *The ∞-category $\mathrm{QCoh}(\mathsf{X})$ is closed under small colimits in $\mathrm{Mod}_{\mathcal{O}}$.*

(2) *The ∞-category $\mathrm{QCoh}(\mathsf{X})$ is stable.*

(3) *The ∞-category $\mathrm{QCoh}(\mathsf{X})$ is presentable.*

Proof. We first prove (1). Suppose we are given a small diagram $\{\mathscr{F}_\alpha\}$ of quasi-coherent $\mathcal{O}$-modules, having a colimit $\mathscr{F} \in \mathrm{Mod}_{\mathcal{O}}$. We wish to prove that $\mathscr{F}$ is quasi-coherent. The assertion is local on $\mathcal{X}$: it therefore suffices to show that $\mathscr{F}|_U \simeq \varinjlim \mathscr{F}_\alpha|_U$ is a quasi-coherent sheaf on $\mathcal{X}_{/U}$ whenever $(\mathcal{X}_{/U}, \mathcal{O}|_U)$ is affine. Replacing $\mathcal{X}$ by $\mathcal{X}_{/U}$, we may assume that $(\mathcal{X}, \mathcal{O})$ is affine. In this case, the desired result follows from Proposition 2.2.3.3. Using exactly the same argument, we deduce that $\mathrm{QCoh}(\mathsf{X})$ is closed under shifts in the stable ∞-category $\mathrm{Mod}_{\mathcal{O}}$. Assertion (2) now follows from Lemma HA.1.1.3.3 .

To prove (3), we let $\mathcal{X}_0 \subseteq \mathcal{X}$ denote the full subcategory spanned by those objects U for which the ∞-category $\mathrm{QCoh}(\mathsf{X}_U)$ is presentable. We wish to prove that $\mathcal{X}_0 = \mathcal{X}$. According to Remark 2.1.0.5, the construction $U \mapsto \mathrm{Mod}_{\mathcal{O}|_U}$ defines a limit-preserving functor $\chi_{\mathcal{O}} : \mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. This functor is classified by a Cartesian fibration $p : \widetilde{\mathcal{X}} \to \mathcal{X}$. Let $\widetilde{\mathcal{X}}'$ denote the full subcategory of $\widetilde{\mathcal{X}}$ spanned by those objects X which correspond to quasi-coherent sheaves on $\mathcal{X}_{/p(X)}$. Remark 2.2.2.3 guarantees that $p|\widetilde{\mathcal{X}}'$ is also a Cartesian fibration, which is classified by another functor $\chi'_{\mathcal{O}} : \mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ (given informally by $U \mapsto \mathrm{QCoh}(\mathsf{X}_U)$). Since the condition of quasi-coherence is local (Remark 2.2.2.3), Proposition HTT.3.3.3.1 shows that $\chi'_{\mathcal{O}}$ is again a limit diagram. The functor $\chi'_{\mathcal{O}}$ evidently factors through the subcategory $\widehat{\mathcal{C}\mathrm{at}}'_\infty \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$ spanned by those ∞-categories which admit small colimits and those functors which preserve small colimits. The ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories can be identified with a full subcategory of $\widehat{\mathcal{C}\mathrm{at}}'_\infty$, so that $\mathcal{X}_0^{\mathrm{op}} = (\chi'_{\mathcal{O}})^{-1}\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Since $\chi'_{\mathcal{O}}$ preserves small limits, it follows from Proposition HTT.5.5.3.13 that $\mathcal{X}_0$ is stable under small colimits in $\mathcal{X}$. It will therefore suffice to show that $\mathcal{X}_0$ contains every object $U \in \mathcal{X}$ such that $(\mathcal{X}_{/U}, \mathcal{O}|_U)$ is affine, which follows immediately from Proposition 2.2.3.3. □

Proposition 2.2.4.2. *Let* $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ *be a nonconnective spectral Deligne-Mumford stack. Then the full subcategory* $\mathrm{QCoh}(\mathsf{X}) \subseteq \mathrm{Mod}_{\mathcal{O}}$ *contains the unit object* $\mathcal{O}$ *and is stable under tensor products, and therefore inherits a symmetric monoidal structure from the symmetric monoidal structure on* $\mathrm{Mod}_{\mathcal{O}}$ *(see Proposition HA.2.2.1.1).*

Proof. The assertion is local, so we may assume that $\mathsf{X} = \mathrm{Spét}\, A$ is an affine nonconnective spectral Deligne-Mumford stack. Let $F : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}}$ be the functor described in Proposition 2.2.3.3, so that the essential image of F is the full subcategory $\mathrm{QCoh}(\mathsf{X}) \subseteq \mathrm{Mod}_{\mathcal{O}}$. Then $F(A) \simeq \mathcal{O}$, so that $\mathcal{O}$ is quasi-coherent. To show that $\mathrm{QCoh}(\mathsf{X})$ is stable under tensor products, it suffices to show that $F(M) \otimes_{\mathcal{O}} F(N)$ is quasi-coherent, for every pair of A-modules $M, N \in \mathrm{Mod}_A$.

Let us identify the ∞-topos $\mathcal{X}$ with $\mathcal{S}\mathrm{hv}_A^{\text{ét}} \subseteq \mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathcal{S})$. For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$, we can identify the pair $(\mathcal{O}, \mathcal{F})$ with a Mod-valued sheaf $\mathrm{CAlg}_A^{\text{ét}} \to \mathrm{Mod}$. Using the construction supplied by Proposition 2.2.1.4, we see that $F(M)$ and $F(N)$ are given by the formulas

$$F(M)(B) = M \otimes_A B \qquad F(N)(B) = N \otimes_A B.$$

It follows that $F(M) \otimes_{\mathcal{O}} F(N)$ is the sheafification of the presheaf

$$B \mapsto F(M)(B) \otimes_{\mathcal{O}(B)} F(N)(B) \simeq (M \otimes_A B) \otimes_B (N \otimes_A B) \simeq (M \otimes_A N) \otimes_A B.$$

According to Proposition 2.2.1.4, this presheaf is already a sheaf which we will denote by $\mathcal{F}$. We have $\mathcal{F}(A) \simeq M \otimes_A N$ so the above formula shows that the canonical map $\mathcal{F}(A) \otimes_A B \to \mathcal{F}(B)$ is an equivalence for every étale A-algebra B; in other words, the counit map $F(\Gamma(\mathcal{X}; \mathcal{F})) \to \mathcal{F}$ is an equivalence, so that $\mathcal{F}$ belongs to the essential image $\mathrm{QCoh}(\mathsf{X}) \subseteq \mathrm{Mod}_{\mathcal{O}}$ of the functor F. □

Our next result can be regarded as a non-affine analogue of Proposition 2.2.3.3:

Proposition 2.2.4.3. *Let* $(\mathcal{X}, \mathcal{O})$ *be a nonconnective spectral Deligne-Mumford stack and let* $\mathcal{F}$ *be a sheaf of* $\mathcal{O}$*-modules on* $\mathcal{X}$*. The following conditions are equivalent:*

(1) *The sheaf* $\mathcal{F}$ *is quasi-coherent.*

(2) *Let* $f : U \to V$ *be a morphism in* $\mathcal{X}$ *such that* $(\mathcal{X}_{/U}, \mathcal{O}|_U)$ *and* $(\mathcal{X}_{/V}, \mathcal{O}|_V)$ *are affine. Then the canonical map* $\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \to \mathcal{F}(U)$ *is an equivalence.*

Proof. Assume first that (1) is satisfied. To prove (2), we are free to replace $\mathcal{X}$ by $\mathcal{X}_{/V}$ and thereby reduce to the case where $(\mathcal{X}, \mathscr{O})$ is affine. It follows that $(\mathcal{X}, \mathscr{O}, \mathscr{F}) \simeq \mathrm{Sp\acute{e}t}_{\mathrm{Mod}}(A, M)$ for some $(A, M) \in \mathrm{Mod}$ (Lemma 2.2.3.2), so that assertion (2) follows from Lemma 2.2.3.1.

Now suppose that (2) is satisfied; we wish to prove that $\mathscr{F}$ is quasi-coherent. The assertion is local on $\mathcal{X}$: we may therefore assume without loss of generality that $(\mathcal{X}, \mathscr{O})$ is an affine nonconnective spectral Deligne-Mumford stack $\mathrm{Sp\acute{e}t}\, A$. Let $M = \Gamma(\mathcal{X}; \mathscr{F})$, regarded as an $A \simeq \Gamma(\mathcal{X}; \mathscr{O})$-module. Then the identity map $M \to \Gamma(\mathcal{X}; \mathscr{F})$ induces a morphism $(\mathcal{X}, \mathscr{O}, \mathscr{F}) \to \mathrm{Sp\acute{e}t}_{\mathrm{Mod}}(A, M) \simeq (\mathcal{X}, \mathscr{O}, \mathscr{F}')$ in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{Mod}}$. To complete the proof, it will suffice to show that this map induces an equivalence of spectrum-valued sheaves $\mathscr{F}' \to \mathscr{F}$. Since $\mathcal{X}$ is generated under small colimits by the full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ spanned by those objects $U \in \mathcal{X}$ for which $(\mathcal{X}_{/U}, \mathscr{O}|_U)$ is affine (Lemma **??**), it will suffice to show that $\mathscr{F}'(U) \to \mathscr{F}(U)$ is an equivalence when U is affine. This follows from the observation that we have a commutative diagram

$$\begin{array}{ccc} & M \otimes_A \mathscr{O}(U) & \\ {}^{\psi}\swarrow & & \searrow^{\phi} \\ \mathscr{F}'(U) & \longrightarrow & \mathscr{F}(U) \end{array}$$

where ϕ is an equivalence by Lemma 2.2.3.1 and ψ is an equivalence by assumption (2). □

2.2.5 Truncations of Quasi-Coherent Sheaves

We now restrict our attention to the case of spectral Deligne-Mumford stacks $\mathsf{X} = (\mathcal{X}, \mathscr{O})$: that is, we assume that the structure sheaf $\mathscr{O}$ is connective. In this case, the ∞-category $\mathrm{QCoh}(\mathsf{X})$ inherits a t-structure.

Lemma 2.2.5.1. *Let A be a connective $\mathbb{E}_\infty$-ring, let $\mathrm{Sp\acute{e}t}\, A = (\mathcal{X}, \mathscr{O})$, and let $F : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathscr{O}}$ be the fully faithful embedding of Proposition 2.2.3.3. Then F is t-exact.*

Proof. The functor F is left adjoint to the global sections functor $\mathscr{F} \mapsto \Gamma(\mathcal{X}; \mathscr{F})$, which is obviously left t-exact. It follows formally that F is right t-exact. To complete the proof, we will show that F is left t-exact: that is, if $M \in (\mathrm{Mod}_A)_{\leqslant 0}$, then $F(M) \in (\mathrm{Mod}_{\mathscr{O}})_{\leqslant 0}$. Let $\mathcal{X}_0$ be the full subcategory of $\mathcal{X}$ spanned by those objects $U \in \mathcal{X}$ such that $F(M)(U) \in \mathrm{Sp}_{\leqslant 0}$. We wish to prove that $\mathcal{X}_0 = \mathcal{X}$. Since $\mathscr{F}$ is a sheaf and the full subcategory $\mathrm{Sp}_{\leqslant 0} \subseteq \mathrm{Sp}$ is stable under limits, we deduce that $\mathcal{X}_0$ is stable under colimits in $\mathcal{X}$. It will therefore suffice to show that $\mathcal{X}_0$ contains all objects $U \in \mathcal{X}$ such that $(\mathcal{X}_{/U}, \mathscr{O}|U)$ is an affine spectral Deligne-Mumford stack $\mathrm{Sp\acute{e}t}\, B$ (Lemma **??**). We have a canonical equivalence $F(M)(U) \simeq M \otimes_A B$. The desired result now follows from Theorem HA.7.2.2.15 , since Theorem 1.4.10.2 guarantees that B is étale (and in particular flat) over A. □

Proposition 2.2.5.2. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack. Then the full subcategory $\mathrm{QCoh}(\mathsf{X}) \subseteq \mathrm{Mod}_{\mathcal{O}}$ is compatible with the t-structure of Proposition 2.1.1.1. More precisely, if $\mathscr{F} \in \mathrm{Mod}_{\mathcal{O}}$ is quasi-coherent, then the truncations $\tau_{\geqslant n} \mathscr{F}$ and $\tau_{\leqslant n} \mathscr{F}$ are quasi-coherent, for every integer n. Consequently, the full subcategories $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0} = \mathrm{QCoh}(\mathsf{X}) \cap (\mathrm{Mod}_{\mathcal{O}})_{\geqslant 0}$ and $\mathrm{QCoh}(\mathsf{X})_{\leqslant 0} = \mathrm{QCoh}(\mathsf{X}) \cap (\mathrm{Mod}_{\mathcal{O}})_{\leqslant 0}$ determine a t-structure on the ∞-category $\mathrm{QCoh}(\mathsf{X})$.*

Proof. Replacing $\mathscr{F}$ by its translates if necessary, it will suffice to show that if $\mathscr{F}$ is quasi-coherent, then $\tau_{\geqslant 0} \mathscr{F}$ and $\tau_{\leqslant -1} \mathscr{F}$ are quasi-coherent. This assertion is local on $\mathcal{X}$; we may therefore assume that $(\mathcal{X}, \mathcal{O}) \simeq \mathrm{Sp\acute{e}t}\, A$ is an affine spectral Deligne-Mumford stack (where A is a connective $\mathbb{E}_\infty$-ring). Let $F : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}}$ be the functor described in Proposition 2.2.3.3. Since $\mathscr{F}$ is quasi-coherent, we may assume without loss of generality that $\mathscr{F} = F(M)$ for some A-module M. Since A is connective, there is a fiber sequence

$$M' \to M \to M''$$

where M' is a connective A-module and $M'' \in (\mathrm{Mod}_A)_{\leqslant -1}$. Applying the exact functor F, we obtain a fiber sequence

$$F(M') \to \mathscr{F} \to F(M'')$$

in $\mathrm{Mod}_{\mathcal{O}}$. Lemma 2.2.5.1 guarantees that $F(M') \in (\mathrm{Mod}_{\mathcal{O}})_{\geqslant 0}$ and $F(M'') \in (\mathrm{Mod}_{\mathcal{O}})_{\leqslant -1}$. We therefore obtain identifications $F(M') \simeq \tau_{\geqslant 0} \mathscr{F}$ and $F(M'') \simeq \tau_{\leqslant -1} \mathscr{F}$ which proves that $\tau_{\geqslant 0} \mathscr{F}$ and $\tau_{\leqslant -1} \mathscr{F}$ are quasi-coherent. □

Notation 2.2.5.3. If $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ is a spectral Deligne-Mumford stack, we let $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ denote the full subcategory $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0} \subseteq \mathrm{QCoh}(\mathsf{X})$ defined in Proposition 2.2.5.2. We will say that a quasi-coherent sheaf $\mathscr{F}$ is *connective* if it belongs to $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$.

The basic properties of the t-structure on $\mathrm{QCoh}(\mathsf{X})$ can be summarized as follows:

Proposition 2.2.5.4. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack. Then:*

(1) *The t-structure on $\mathrm{QCoh}(\mathsf{X})$ is accessible (see Definition HA.1.4.4.12).*

(2) *The t-structure on $\mathrm{QCoh}(\mathsf{X})$ is compatible with filtered colimits: that is, the full subcategory $\mathrm{QCoh}(\mathsf{X})_{\leqslant 0}$ is closed under filtered colimits.*

(3) *The t-structure on $\mathrm{QCoh}(\mathsf{X})$ is both right and left complete.*

Proof. Assertion (1) is equivalent to the statement that $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is presentable (Proposition HA.1.4.4.13). This follows from Proposition HTT.5.5.3.12 , since $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ can be identified with the fiber product $\mathrm{QCoh}(\mathsf{X}) \times_{\mathrm{Mod}_{\mathcal{O}}} \mathrm{Mod}_{\mathcal{O}}^{\mathrm{cn}}$. Assertion (2) follows from Proposition 2.2.4.1 together with the corresponding result for $\mathrm{Mod}_{\mathcal{O}}$ (Proposition 2.1.1.1).

We now prove (3). Since $\mathrm{Mod}_{\mathcal{O}}$ is right-complete (Proposition 2.1.1.1), we deduce that $\bigcap_n \mathrm{QCoh}(\mathsf{X})_{\leqslant -n} \subseteq \bigcap_n (\mathrm{Mod}_{\mathcal{O}})_{\leqslant -n}$ contains only zero objects. Combining this observation with (2), we deduce that $\mathrm{QCoh}(\mathsf{X})$ is right-complete (see Proposition HA.1.2.1.19).

The proof that $\mathrm{QCoh}(\mathsf{X})$ is left-complete requires a bit more effort (note that $\mathrm{Mod}_{\mathcal{O}}$ need not be left complete). Consider the full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ spanned by those objects $U \in \mathcal{X}$ for which the t-structure on $\mathrm{QCoh}(\mathcal{X}_{/U})$ is left-complete. To complete the proof, it will suffice to show that $\mathcal{X}_0 = \mathcal{X}$. Using Proposition 2.2.3.3, Lemma 2.2.5.1, and Proposition HA.7.1.1.13 , we deduce that $\mathcal{X}_0$ contains every affine object $U \in \mathcal{X}$. It will therefore suffice to show that $\mathcal{X}_0$ is closed under small colimits in $\mathcal{X}$ (Lemma **??**). Since the conditions of being quasi-coherent and n-truncated are local, the proof of Proposition 2.2.4.1 shows that the constructions

$$U \mapsto \mathrm{QCoh}(\mathsf{X}_U) \qquad U \mapsto \mathrm{QCoh}(\mathsf{X}_U)_{\leqslant n}$$

determine limit-preserving functors $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. If $\{U_\alpha\}$ is a diagram in $\mathcal{X}$ having a colimit $U \in \mathcal{X}$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X}_U) & \longrightarrow & \varprojlim_\alpha \mathrm{QCoh}(\mathsf{X}_{U_\alpha}) \\ \downarrow & & \downarrow \theta \\ \varprojlim_n \mathrm{QCoh}(\mathsf{X}_U)_{\leqslant n} & \longrightarrow & \varprojlim_{n,\alpha} \mathrm{QCoh}(\mathsf{X}_{U_\alpha})_{\leqslant n}. \end{array}$$

where the vertical maps are equivalences. If each U_α belongs to $\mathcal{X}_0$, then the right vertical map is also an equivalence, so the left vertical map is an equivalence as well and $U \in \mathcal{X}_0$ as desired. □

2.2.6 Discrete Quasi-Coherent Sheaves

Let $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack. The truncation map $\mathcal{O} \to \pi_0 \mathcal{O}$ induces an equivalence of abelian categories $\mathrm{Mod}^{\heartsuit}_{\pi_0 \mathcal{O}} \to \mathrm{Mod}^{\heartsuit}_{\mathcal{O}}$. Using the criterion for quasi-coherence supplied by Proposition 2.2.4.3, we see that this equivalence respects the property of quasi-coherence: that is, we have an equivalence of abelian categories

$$\mathrm{QCoh}(\mathcal{X}, \pi_0 \mathcal{O})^{\heartsuit} \simeq \mathrm{QCoh}(\mathsf{X})^{\heartsuit}.$$

(for a more general assertion of this nature, see Corollary 2.5.9.2).

Let $\mathcal{X}^{\heartsuit}$ be the underlying topos of $\mathcal{X}$, so that we can identify $\pi_0 \mathcal{O}$ with a commutative ring object of $\mathcal{X}^{\heartsuit}$. Then we can identify $\mathrm{Mod}^{\heartsuit}_{\mathcal{O}}$ with the abelian category of $\pi_0 \mathcal{O}$-module objects of $\mathcal{X}^{\heartsuit}$. Under this identification, the full subcategory $\mathrm{QCoh}(\mathsf{X})^{\heartsuit} \simeq \mathrm{QCoh}(\mathcal{X}, \pi_0 \mathcal{O})^{\heartsuit} \subseteq \mathrm{Mod}^{\heartsuit}_{\pi_0 \mathcal{O}}$ corresponds to the abelian category of quasi-coherent sheaves on the ordinary Deligne-Mumford stack $(\mathcal{X}^{\heartsuit}, \pi_0 \mathcal{O})$, in the sense of Definition 1.2.6.1 (this follows immediately from the characterization of Proposition 2.2.4.3).

In the non-discrete case, there is a similar characterization of quasi-coherence:

Proposition 2.2.6.1. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O})$ be a nonconnective spectral Deligne-Mumford stack and let $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules on $\mathcal{X}$. Then $\mathscr{F}$ is quasi-coherent if and only if it satisfies the following conditions:*

(1) *For every integer n, the homotopy sheaf $\pi_n \mathscr{F}$ is quasi-coherent (in the sense of Definition 1.2.6.1) when viewed as a $\pi_0 \mathscr{O}$-module object of the underlying topos $\mathcal{X}^{\heartsuit}$.*

(2) *The object $\Omega^\infty \mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}) \simeq \mathcal{X}$ is hypercomplete.*

Corollary 2.2.6.2. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O})$ be a spectral Deligne-Mumford stack. Assume that $\mathcal{X}$ is 1-localic and that the structure sheaf $\mathscr{O}$ is discrete. Then there is a canonical equivalence of ∞-categories $\mathrm{QCoh}(\mathsf{X}) \simeq \mathcal{D}(\mathrm{Mod}^{\heartsuit}_{\mathscr{O}})_{\mathrm{qc}}$, where $\mathcal{D}(\mathrm{Mod}^{\heartsuit}_{\mathscr{O}})_{\mathrm{qc}}$ denotes the full subcategory of the derived ∞-category $\mathcal{D}(\mathrm{Mod}^{\heartsuit}_{\mathscr{O}})$ spanned by those chain complexes of $\mathscr{O}$-module objects of $\mathcal{X}^{\heartsuit}$ whose homologies are quasi-coherent (in the sense of Definition 1.2.6.1).*

Proof. Combine Proposition 2.2.6.1 with Corollary 2.1.2.3. □

Remark 2.2.6.3. Under some slightly stronger hypotheses, one can show that the ∞-category $\mathrm{QCoh}(\mathsf{X})$ is equivalent to the derived ∞-category of its heart: see Corollary 10.3.4.13.

Proof of Proposition 2.2.6.1. Replacing $\mathscr{O}$ by its connective cover if necessary, we may assume that $\mathscr{O}$ is connective. If $\mathscr{F}$ is quasi-coherent, then Proposition 2.2.5.2 implies that each homotopy group $\pi_n \mathscr{F}$ is quasi-coherent as a $\mathscr{O}$-module. To prove that (2) is satisfied, it suffices to work locally on $\mathcal{X}$; we may therefore assume that $(\mathcal{X}, \mathscr{O}) \simeq \mathrm{Sp\acute{e}t}\, A$ for some connective $\mathbb{E}_\infty$-ring A. Let F be the functor of Proposition 2.2.3.3, so that $\mathscr{F} \simeq F(M)$ for some A-module M. Let us identify the pair $(\mathscr{O}, \mathscr{F})$ with the sheaf $\mathrm{CAlg}^{\text{ét}}_A \to \mathrm{Mod}$ given by $B \mapsto (B, B \otimes_A M)$. We note that for each $B \in \mathrm{CAlg}^{\text{ét}}_A$, we have

$$F(M)(B) \simeq B \otimes_A M \simeq \varprojlim \tau_{\leqslant n}(B \otimes_A M) \simeq \varprojlim B \otimes_A (\tau_{\leqslant n} M) \simeq \varprojlim F(\tau_{\leqslant n} M)(B).$$

It follows that $\mathscr{F} \simeq F(M) \simeq \varprojlim F(\tau_{\leqslant n} M)$ is a limit of truncated objects of $\mathrm{Mod}_{\mathscr{O}}$ (Lemma 2.2.5.1), so that $\Omega^\infty \mathscr{F}$ is a limit of truncated objects of $\mathcal{X}$ and therefore hypercomplete.

Now suppose that $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}}$ satisfies conditions (1) and (2). We wish to prove that $\mathscr{F}$ is quasi-coherent. Note that $\mathscr{F} \simeq \varinjlim \tau_{\geqslant -n} \mathscr{F}$ by Proposition 2.2.5.4. Since the collection of quasi-coherent sheaves is closed under colimits in $\mathrm{Mod}_{\mathscr{O}}$, it suffices to prove that each $\tau_{\geqslant -n} \mathscr{F}$ is quasi-coherent. Replacing $\mathscr{F}$ by $\Sigma^n(\tau_{\geqslant -n} \mathscr{F})$, we may assume that $\mathscr{F}$ is connective. Since the condition of being quasi-coherent is local on $\mathcal{X}$, we may suppose that $(\mathcal{X}, \mathscr{O}) \simeq \mathrm{Sp\acute{e}t}\, A$ is an affine spectral Deligne-Mumford stack, where A is a connective $\mathbb{E}_\infty$-ring; let $\mathcal{C}$ and $F : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathscr{O}}$ be defined as above.

We now argue by induction on m that each truncation $\tau_{\leqslant m} \mathscr{F}$ is quasi-coherent. For $m < 0$, this is obvious. If $m \geqslant 0$, it follows from the existence of a fiber sequence

$$\tau_{\leqslant m-1} \mathscr{F} \to \tau_{\leqslant m} \mathscr{F} \to \Sigma^m(\pi_m \mathscr{F}).$$

Using Proposition 2.2.3.3, we may suppose that the tower $\{\tau_{\leqslant m} \mathscr{F}\}$ is obtained from a tower of A-modules $\{M_m\}_{m \geqslant 0}$. Using Lemma 2.2.5.1, we deduce that for $m \leqslant m'$, the map $M_{m'} \to M_m$ exhibits M_m as an m-truncation $\tau_{\leqslant m} M_{m'}$. Since Mod_A is left complete (Proposition HA.7.1.1.13), the A-module $M \simeq \varprojlim M_m$ has the property that $\tau_{\leqslant m} M \simeq M_m$ for every integer m. For every flat A-algebra B, we also obtain an equivalence $\tau_{\leqslant m}(M \otimes_A B) \simeq M_m \otimes_A B$, so that $M \otimes_A B \simeq \varprojlim (M_m \otimes_A B)$. It follows that $F(M) \simeq \varprojlim_m F(M_m) \simeq \varprojlim \tau_{\leqslant m} \mathscr{F}$ in the ∞-category $\mathrm{Mod}_{\mathscr{O}}$. In particular, we obtain a map $\alpha : \mathscr{F} \to F(M)$. To prove that $\mathscr{F}$ is quasi-coherent, it will suffice to show that α is an equivalence. Since $\mathscr{F}$ and $F(M)$ are both connective, this is equivalent to the requirement that α induces an equivalence $\Omega^\infty \mathscr{F} \to \Omega^\infty F(M)$ in $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}) \simeq \mathcal{X}$. Since $\Omega^\infty \mathscr{F}$ is hypercomplete (by (2)) and $\Omega^\infty F(M) \simeq \varprojlim \Omega^\infty F(M_m)$ is hypercomplete (since it is an inverse limit of truncated objects of $\mathcal{X}$), it will suffice to show that the map $\Omega^\infty(\alpha) : \Omega^\infty \mathscr{F} \to \Omega^\infty F(M)$ is ∞-connective. This is clear, since for every integer $m \geqslant 0$, the truncation $\tau_{\leqslant m} \Omega^\infty(\alpha)$ is homotopic to the composition of equivalences

$$\tau_{\leqslant m} \Omega^\infty \mathscr{F} \simeq \Omega^\infty(\tau_{\leqslant m} \mathscr{F}) \simeq \Omega^\infty F(M_m) \simeq \Omega^\infty F(\tau_{\leqslant m} M) \simeq \Omega^\infty \tau_{\leqslant m} F(M) \simeq \tau_{\leqslant m} \Omega^\infty F(M).$$

□

2.3 Compactness Hypotheses on Spectral Deligne-Mumford Stacks

Let $(X, \mathscr{O}_X)$ be a scheme. Recall that X is said to be *quasi-compact* if every open covering of X has a finite subcovering, and *quasi-separated* if the collection of quasi-compact open subsets of X is closed under pairwise intersections. In this section, we will study analogous conditions in the setting of spectral Deligne-Mumford stacks.

2.3.1 Quasi-Compactness (Absolute Case)

Throughout this section, we will assume that the reader is familiar with the theory of coherent ∞-topoi developed in §A.2 (see Definition A.2.0.12).

Definition 2.3.1.1. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack and let $n \geqslant 0$ be an integer. We will say that X is *n-quasi-compact* if the ∞-topos $\mathcal{X}$ is n-coherent (see §A.2). We will say that X is *quasi-compact* if it is 0-quasi-compact. We will say that X is *∞-quasi-compact* if it is n-quasi-compact for every integer n.

Proposition 2.3.1.2. *Let A be an $\mathbb{E}_\infty$-ring. Then the nonconnective spectral Deligne-Mumford stack* Spét A *is ∞-quasi-compact.*

Proof. Combine Propositions A.3.1.3 and 1.4.2.4. □

Corollary 2.3.1.3. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. Then the ∞-topos $\mathcal{X}$ is locally coherent (see Definition A.2.1.6).*

Corollary 2.3.1.4. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. Then the hypercompletion $\mathcal{X}^{\mathrm{hyp}}$ has enough points.*

Proof. Combine Corollary 2.3.1.3 with Theorem A.4.0.5. □

2.3.2 Quasi-Compactness (Relative Case)

Recall that if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a nonconnective spectral Deligne-Mumford stack, then we say an object $U \in \mathcal{X}$ is *affine* if the nonconnective spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ is affine.

Proposition 2.3.2.1. *Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of nonconnective spectral Deligne-Mumford stacks. Let $n \geqslant 0$ be an integer. The following conditions are equivalent:*

(1) *For every n-coherent object $U \in \mathcal{Y}$, the pullback f^*U is n-coherent.*

(2) *For every affine object $U \in \mathcal{Y}$, the pullback f^*U is an n-coherent object of $\mathcal{X}$.*

(3) *There exists a full subcategory $\mathcal{Y}_0 \subseteq \mathcal{Y}$ with the following properties:*

   (a) *Each object $U \in \mathcal{Y}_0$ is n-coherent.*

   (b) *For each $U \in \mathcal{Y}_0$, the pullback $f^*(U)$ is n-coherent.*

   (c) *For each object $Y \in \mathcal{Y}$, there exists an effective epimorphism $\amalg Y_i \to Y$, where each $Y_i \in \mathcal{Y}_0$.*

Moreover, if $n > 0$, then these conditions imply:

(4) *For every relatively $(n-1)$-coherent morphism $u : U \to Y$ in $\mathcal{Y}$, the pullback $f^*(u)$ is a relatively $(n-1)$-coherent morphism in $\mathcal{X}$.*

Definition 2.3.2.2. Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of nonconnective spectral Deligne-Mumford stacks. We will say that f is *n-quasi-compact* if it satisfies the equivalent conditions of Proposition 2.3.2.1. We will say that f is *quasi-compact* if it is 0-quasi-compact, and *∞-quasi-compact* if it is n-quasi-compact for every integer $n \geqslant 0$.

Proof of Proposition 2.3.2.1. We proceed by induction on n. The implication $(1) \Rightarrow (2)$ is immediately from Proposition 2.3.1.2. To see that $(2) \Rightarrow (3)$, we take $\mathcal{Y}_0$ to be the collection of all objects $U \in \mathcal{Y}$ such that $(\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}}|_U)$ is affine. We next show that $(3) \Rightarrow (4)$ if $n > 0$. Let $u : U \to Y$ be an $(n-1)$-coherent morphism in $\mathcal{Y}$; we wish to show that $f^*(u)$ is a relatively $(n-1)$-coherent morphism in $\mathcal{X}$. Choose an effective epimorphism $\amalg_{i \in I} Y_i \to Y$, where each $Y_i \in \mathcal{Y}$. Using Corollary A.2.1.5, we are reduced to proving that the induced map $f^*(U \times_Y \amalg_{i \in I} Y_i) \to f^*(\amalg_{i \in I} Y_i)$ is relatively $(n-1)$-coherent. We may therefore replace Y by some Y_i and thereby reduce to the case where Y is n-coherent. Then U is $(n-1)$-coherent. Using (2) together with the inductive hypothesis, we deduce that f^*Y is n-coherent and that f^*U is $(n-1)$-coherent, so that $f^*(u)$ is relatively $(n-1)$-coherent as desired.

We now prove that (3) implies (1). Fix an n-coherent object $U \in \mathcal{Y}$; we wish to prove that $f^*(U)$ is an n-coherent object of $\mathcal{X}$. Choose an effective epimorphism $\amalg_{i \in I} U_i \to U$ where each $U_i \in \mathcal{Y}_0$. Since U is quasi-compact, we may assume without loss of generality that I is finite. Using (2) and Remark A.2.0.16, we conclude that $\amalg f^*(U_i)$ is an n-coherent object of $\mathcal{X}$. Moreover, the map $\amalg f^*(U_i) \to f^*(U)$ is an effective epimorphism which is $(n-1)$-coherent if $n > 0$ (by virtue of (4)). Using Proposition A.2.1.3 we conclude that f^*U is n-coherent as desired. □

Remark 2.3.2.3. Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of nonconnective spectral Deligne-Mumford stacks and let $n \geqslant 0$ be an integer. Suppose that f is n-quasi-compact. Then:

- (a) The pullback functor f^* carries compact objects of $\tau_{\leqslant n-1}\,\mathcal{Y}$ to compact objects of $\tau_{\leqslant n-1}\,\mathcal{X}$.
- (b) The pushforward functor $f_* : \tau_{\leqslant n-1}\,\mathcal{X} \to \tau_{\leqslant n-1}\,\mathcal{Y}$ commutes with filtered colimits.

Assertion (a) follows from Proposition 2.3.2.1 and Remark A.2.3.3, and assertion (b) follows from (a) and Proposition HTT.5.5.7.2 .

Example 2.3.2.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map between affine nonconnective spectral Deligne-Mumford stacks. Then f is ∞-quasi-compact; this follows immediately from Proposition 2.3.1.2.

Remark 2.3.2.5. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of nonconnective spectral Deligne-Mumford stacks and let $0 \leqslant n \leqslant \infty$. The following conditions are equivalent:

1. The map f is n-coherent.
2. For every étale morphism $\operatorname{Spét} A \to \mathsf{Y}$, the fiber product $\operatorname{Spét} A \times_{\mathsf{Y}} \mathsf{X}$ is n-coherent.

Proposition 2.3.2.6. *Let* X *be a quasi-compact nonconnective spectral Deligne-Mumford stack, and let* $n > 0$. *The following conditions on* X *are equivalent:*

1. *For every pair of maps* $\operatorname{Spét} A \to \mathsf{X} \leftarrow \operatorname{Spét} B$, *the fiber product* $\operatorname{Spec} A \times_{\mathsf{X}} \operatorname{Spét} B$ *is* $(n-1)$*-coherent.*
2. *Every map* $f : \operatorname{Spét} A \to \mathsf{X}$ *is* $(n-1)$*-coherent.*

(3) *For every pair of maps* $\operatorname{Spét} A \to \mathsf{X} \xleftarrow{u} \operatorname{Spét} B$ *where* u *is étale, the fiber product* $\operatorname{Spét} A \times_{\mathsf{X}} \operatorname{Spét} B$ *is* $(n-1)$*-coherent.*

(4) *Every étale map* $f : \operatorname{Spét} A \to \mathsf{X}$ *is* $(n-1)$*-coherent.*

(5) *For every pair of étale maps* $\operatorname{Spét} A \to \mathsf{X} \leftarrow \operatorname{Spét} B$*, the fiber product* $\operatorname{Spét} A \times_{\mathsf{X}} \operatorname{Spét} B$ *is* $(n-1)$*-coherent.*

(6) *The nonconnective spectral Deligne-Mumford stack* X *is* n*-coherent.*

Proof. The equivalences (1) ⇔ (2) and (3) ⇔ (4) are obvious. The equivalences (2) ⇔ (3) and (4) ⇔ (5) follow from Remark 2.3.2.5, and the equivalence (5) ⇔ (6) follows from Corollary A.2.1.4. □

Proposition 2.3.2.7. *Let* X *be a spectral Deligne-Mumford* n*-stack. If* X *is* $(n+1)$*-quasi-compact, then* X *is* ∞*-quasi-compact.*

Proof. It will suffice to show that if X is $(n+1)$-quasi-compact, then it is also $(n+2)$-quasi-compact. We proceed by induction on n. The case $n = 0$ will be established in §3.4 (Proposition 3.4.1.2). Assume that $n > 0$. To prove that X is $(n+2)$-quasi-compact, it will suffice to show that for every pullback diagram

$$\begin{array}{ccc} \mathsf{Y} & \longrightarrow & \operatorname{Spét} A \\ \downarrow & & \downarrow \\ \operatorname{Spét} B & \longrightarrow & \mathsf{X}, \end{array}$$

the spectral Deligne-Mumford stack Y is $(n+1)$-quasi-compact (Proposition 2.3.2.6). This follows from the inductive hypothesis, since Y is a spectral Deligne-Mumford $(n-1)$-stack (Remark 1.6.8.4). □

2.3.3 Pullbacks of Quasi-Compact Morphisms

Our next result describes the behavior of n-quasi-compact morphisms with respect to base change:

Proposition 2.3.3.1. *Suppose we are given a pullback diagram of nonconnective spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{g'} & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y}, \end{array}$$

and let $0 \leqslant n \leqslant \infty$*. If* f *is* n*-quasi-compact, then* f' *is* n*-quasi-compact. The converse holds if* g *is quasi-compact and surjective (see Definition 3.5.5.5).*

We begin by establishing the first half of Proposition 2.3.3.1.

Lemma 2.3.3.2. *Suppose we are given a pullback diagram σ :*

$$\begin{array}{ccc} (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) & \xrightarrow{f'} & (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}) \\ \downarrow{g'} & & \downarrow{g} \\ (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) & \xrightarrow{f} & (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \end{array}$$

of nonconnective spectral Deligne-Mumford stacks. If f is n-quasi-compact, then f' is n-quasi-compact.

Proof. Let $\mathcal{Y}'_0$ be the full subcategory of $\mathcal{Y}'$ spanned by those objects $Y' \in \mathcal{Y}'$ with the following properties:

(*i*) The pair $(\mathcal{Y}'_{/Y'}, \mathcal{O}_{\mathcal{Y}'}|_{Y'})$ is affine.

(*ii*) There exists an object $Y \in \mathcal{Y}$ and a map $Y' \to g^*Y$, where $(\mathcal{Y}_{/Y}, \mathcal{O}_{\mathcal{Y}}|_Y)$ is affine.

This subcategory satisfies requirements (a), (b), and (c) of Proposition 2.3.2.1; it will therefore suffice to show that f'^*Y is an n-coherent object of $\mathcal{X}'$.

Replacing σ by the diagram

$$\begin{array}{ccc} (\mathcal{X}'_{/f'^*Y'}, \mathcal{O}_{\mathcal{X}'}|_{f'^*Y'}) & \xrightarrow{f'} & (\mathcal{Y}'_{/Y'}, \mathcal{O}_{\mathcal{Y}'}|_{Y'}) \\ \downarrow & & \downarrow{g} \\ (\mathcal{X}_{/f^*Y}, \mathcal{O}_{\mathcal{X}}|_{f^*Y}) & \xrightarrow{f} & (\mathcal{Y}_{/Y}, \mathcal{O}_{\mathcal{Y}}|_Y), \end{array}$$

we can reduce to the case where $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ and $(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'})$ are affine. Since f is n-quasi-compact, the ∞-topos $\mathcal{X}$ is n-coherent; we wish to prove that $\mathcal{X}'$ is n-coherent. To prove this, it suffices to show that the map g' is n-quasi-compact. This assertion is local on $\mathcal{X}$; we may therefore assume that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is affine. Since σ is a pullback diagram, we conclude that $(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is affine and the desired result follows from Example 2.3.2.4. □

Corollary 2.3.3.3. *Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a map of nonconnective spectral Deligne-Mumford stacks. Assume that $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is affine. Then f is n-quasi-compact if and only if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is n-quasi-compact.*

Proof. The "only if" direction is obvious (and requires only that $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be n-quasi-compact). Conversely, suppose that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is affine. Let $U \in \mathcal{Y}$ be such that $(\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}}|_U)$ is affine. We wish to prove that f^*U is an n-coherent object of $\mathcal{X}$. We have a pullback diagram

$$\begin{array}{ccc} (\mathcal{X}_{/f^*U}, \mathcal{O}_{\mathcal{X}}|_{f^*U}) & \longrightarrow & (\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}}|_U) \\ \downarrow{g} & & \downarrow{g'} \\ (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) & \longrightarrow & (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}). \end{array}$$

The map g' is n-quasi-compact by Example 2.3.2.4, so that Lemma 2.3.3.2 guarantees that g is n-quasi-compact. Since the final object $\mathbf{1} \in \mathcal{X}$ is n-coherent, we conclude that $g^* \mathbf{1} \in \mathcal{X}_{/f^* U}$ is n-coherent: that is, $f^* U$ is an n-coherent object of $\mathcal{X}$. □

Proof of Proposition 2.3.3.1. The first assertion follows from Proposition 2.3.3.2. To prove the second, we may assume without loss of generality that $\mathsf{Y} = \operatorname{Sp\acute{e}t} A$ is affine. If the map $g : \mathsf{Y}' \to \mathsf{Y}$ is quasi-compact, then it follows that Y' is also quasi-compact. We may therefore choose an étale surjection $\operatorname{Sp\acute{e}t} B \to \mathsf{Y}'$. Replacing Y' by $\operatorname{Sp\acute{e}t} B$, we may assume that Y' is also affine.

Write $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathsf{X}' = (\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. Our assumption that f' is n-quasi-compact is equivalent to the assumption that $\mathcal{X}'$ is n-coherent, and we wish to prove that $\mathcal{X}$ is also n-coherent (Corollary 2.3.3.3). We claim more generally that if $X \in \mathcal{X}$ is an object such that $g'^* X \in \mathcal{X}'$ is m-coherent, then X is m-coherent. The proof proceeds by induction on m. We begin with the case $m = 0$. Suppose that we are given an effective epimorphism $\coprod_{i \in I} U_i \to X$ in the ∞-topos $\mathcal{X}$. Then the induced map $\coprod_{i \in I} g'^* U_i \to g'^* X$ is an effective epimorphism in $\mathcal{X}'$. If $g'^* X$ is quasi-compact, then we can choose a finite subset $I_0 \subseteq I$ such that the induced map $\coprod_{i \in I_0} g'^* U_i \to g'^* X$ is an effective epimorphism. Since the map g' is surjective, it follows that the map $\coprod_{i \in I_0} U_i \to X$ is also an effective epimorphism.

We now treat the case $m > 0$. According to Corollary A.2.1.4, it will suffice to show that if we are given affine objects $U, V \in \mathcal{X}_{/X}$, then the fiber product $U \times_X V$ is $(m-1)$-coherent. By the inductive hypothesis, it suffices to show that $g'^*(U \times_X V) \simeq g'^* U \times_{g'^* X} g'^* V$ is $(m-1)$-coherent. This follows from the m-coherence of $g'^* X$, since $f^* U$ and $f^* V$ are affine. □

2.3.4 The Schematic Case

We now discuss the relationship of Definition 2.3.1.1 with classical scheme theory.

Lemma 2.3.4.1. *Let X be a topological space. The following conditions are equivalent:*

(1) *The ∞-topos $\mathcal{S}\mathrm{hv}(X)$ is coherent.*

(2) *The ∞-topos $\mathcal{S}\mathrm{hv}(X)$ is 1-coherent.*

(3) *The collection of quasi-compact open subsets of X is closed under finite intersections and forms a basis for the topology of X.*

Proof. The implication (1) ⇒ (2) is obvious. We prove that (2) ⇒ (3). For each $U \subseteq X$, let $\chi_U \in \mathcal{S}\mathrm{hv}(X)$ be the sheaf given by the formula

$$\chi_U(V) = \begin{cases} \Delta^0 & \text{if } V \subseteq U \\ \emptyset & \text{otherwise.} \end{cases}$$

We note that χ_U is a quasi-compact object of $\mathcal{X}$ if and only if U is quasi-compact as a topological space. If $\mathcal{S}\mathrm{hv}(X)$ is 1-coherent, then the collection of quasi-compact objects of $\mathcal{S}\mathrm{hv}(X)$ are closed under products. Since the construction $U \mapsto \chi_U$ carries finite intersections to finite products, we conclude that the collection of quasi-compact open subsets of X is closed under finite intersections. We claim that the quasi-compact open subsets form a basis for the topology of X. To prove this, choose an arbitrary open subset $U \subseteq X$. Since $\mathcal{S}\mathrm{hv}(X)$ is 1-coherent, there exists an effective epimorphism $\theta : \amalg \mathscr{F}_i \to \chi_U$, where each $\mathscr{F}_i$ is a quasi-compact object of $\mathcal{S}\mathrm{hv}(X)$. For each index i, we have $\tau_{\leqslant -1} \mathscr{F}_i \simeq \chi_{U_i}$ for some open set $U_i \subseteq X$. It follows that θ induces an effective epimorphism $\amalg_i \chi_{U_i} \to \chi_U$, so that $U = \bigcup U_i$. We claim that each U_i is quasi-compact: equivalently, each of the sheaves χ_{U_i} is a quasi-compact object of $\mathcal{S}\mathrm{hv}(X)$. This follows from the observation that we have effective epimorphisms $\mathscr{F}_i \to \chi_{U_i}$.

We now complete the proof by showing that (3) implies (1). Assume that X is a coherent topological space. Let $\mathcal{C} \subseteq \mathcal{S}\mathrm{hv}(X)$ be the full subcategory spanned by objects of the form χ_U, where U is a quasi-compact open subset of X. Since the quasi-compact open subsets of X form a basis for the topology of X, the ∞-category $\mathcal{C}$ generates $\mathcal{S}\mathrm{hv}(X)$ under small colimits. It will therefore suffice to show that $\mathcal{C}$ consists of coherent objects of $\mathcal{S}\mathrm{hv}(X)$. We prove by induction on n that the objects of $\mathcal{C}$ are n-coherent. The case $n = 0$ is clear. Assume that the objects of $\mathcal{C}$ are known to be n-coherent for $n \geqslant 0$. We wish to prove that if $U \subseteq X$ is a quasi-compact open subset, then χ_U is $(n+1)$-coherent. According to Corollary A.2.1.4, it will suffice to show that for every pair of objects $\chi_V, \chi_{V'} \in \mathcal{C}$, every fiber product $\chi_V \times_{\chi_U} \chi_{V'}$ is n-coherent. Unwinding the definitions, this is equivalent to the statement that $V \cap V'$ is quasi-compact, which follows from our assumption that the quasi-compact open subsets of X are closed under finite intersections. □

Proposition 2.3.4.2. *Let $(X, \mathscr{O}_X)$ be a spectral scheme and let $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) = \mathrm{Spec}^{\acute{e}t}_{\mathrm{Zar}}(X, \mathscr{O}_X)$ be the associated schematic spectral Deligne-Mumford stack. Then:*

(1) *The spectral Deligne-Mumford stack $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is quasi-compact if and only if the topological space X is quasi-compact.*

(2) *For $1 \leqslant n \leqslant \infty$, the spectral Deligne-Mumford stack $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is n-quasi-compact if and only if X is quasi-compact and quasi-separated.*

Proof. Assertion (1) and the "only if" direction of (2) follow immediately from Lemma **??**. To prove the converse, we prove by induction on $n > 0$ that if X is quasi-compact and quasi-separated, then $\mathcal{X}$ is n-coherent. To carry out the inductive step, it will suffice to show that for every pullback diagram

$$\begin{array}{ccc} (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) & \longrightarrow & \operatorname{Sp\acute{e}t} A \\ \downarrow & & \downarrow u \\ \operatorname{Sp\acute{e}t} B & \xrightarrow{v} & (\mathcal{X}, \mathscr{O}_{\mathcal{Y}}), \end{array}$$

the spectral Deligne-Mumford stack $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ is $(n-1)$-quasi-compact (Proposition 2.3.2.6). By virtue of Corollary **??**, the maps u and v are induced by maps of spectral schemes $u_0 : \operatorname{Spec} A \to (X, \mathscr{O}_X)$ and $v_0 : \operatorname{Spec} B \to (X, \mathscr{O}_X)$. It follows that we can write $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \simeq \operatorname{Spec}^{\text{ét}}_{\text{Zar}}(Y, \mathscr{O}_Y)$, where $(Y, \mathscr{O}_Y)$ denotes the fiber product $\operatorname{Spec} A \times_{(X, \mathscr{O}_X)} \operatorname{Spec} B$ in the ∞-category of spectral schemes. We then have a pullback diagram of ordinary schemes

$$\begin{array}{ccc} (Y, \pi_0 \mathscr{O}_Y) & \longrightarrow & \operatorname{Spec} \pi_0 A \\ \downarrow & & \downarrow \\ \operatorname{Spec} \pi_0 B & \longrightarrow & (X, \pi_0 \mathscr{O}_X). \end{array}$$

Since X is quasi-compact and quasi-separated, Y is also quasi-compact and quasi-separated. Applying the inductive hypothesis (or part (1), if $n = 1$), we conclude that $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ is $(n-1)$-quasi-compact as desired. □

2.3.5 Transitivity Properties of Quasi-Compactness

We now discuss the closure of n-quasi-compact morphisms under composition.

Proposition 2.3.5.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of nonconnective spectral Deligne-Mumford stacks and let $0 \leqslant n \leqslant \infty$. Then:*

(1) *If Y is n-quasi-compact and f is n-quasi-compact, then X is n-quasi-compact.*

(2) *If X is n-quasi-compact and Y is $(n+1)$-quasi-compact, then f is n-quasi-compact.*

Proof. We proceed by induction on n. We begin with assertion (1). Assume that f and Y are n-quasi-compact; we wish to prove that X is n-quasi-compact. Choose an étale surjection $\operatorname{Sp\acute{e}t} R \to \mathsf{Y}$. Then the fiber product $\mathsf{X}' = \operatorname{Sp\acute{e}t} R \times_{\mathsf{Y}} \mathsf{X}$ is n-coherent. We have an étale surjection $\mathsf{X}' \to \mathsf{X}$, so that X is quasi-compact. This completes the proof when $n = 0$. Assume now that $n > 0$. By virtue of Proposition 2.3.2.6, it will suffice to show that every map $\operatorname{Sp\acute{e}t} A \to \mathsf{X}$ is $(n-1)$-quasi-compact. Using Proposition 2.3.3.1, we are reduced to showing that the induced map

$$u : \operatorname{Sp\acute{e}t} R \times_{\mathsf{Y}} \operatorname{Sp\acute{e}t} A \to \mathsf{X}'$$

is $(n-1)$-quasi-compact. Since X' is n-quasi-compact and $\mathrm{Sp\acute{e}t}\, R \times_{\mathsf{Y}} \mathrm{Sp\acute{e}t}\, A$ is $(n-1)$-quasi-compact (using Proposition 2.3.2.6 and the n-quasi-compact of Y), the $(n-1)$-quasi-compactness of u follows from the inductive hypothesis.

We now prove (2). Assume that X is n-quasi-compact and that Y is $(n+1)$-quasi-compact. We wish to show that for every map $\mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$, the fiber product $\mathsf{X}' = \mathrm{Sp\acute{e}t}\, R \times_{\mathsf{Y}} \mathsf{X}$ is n-quasi-compact. By (1), it will suffice to show that the projection map $\mathsf{X}' \to \mathsf{X}$ is n-quasi-compact. This follows from Proposition 2.3.3.1, since the map $\mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$ is n-quasi-compact by Corollary 2.3.2.4. □

Corollary 2.3.5.2. *Suppose we are given maps $f : \mathsf{X} \to \mathsf{Y}$ and $g : \mathsf{Y} \to \mathsf{Z}$ of nonconnective spectral Deligne-Mumford stacks. Then:*

(i) If f and g are n-quasi-compact, then $g \circ f$ is n-quasi-compact.

(ii) If g is $(n+1)$-quasi-compact and $g \circ f$ is n-quasi-compact, then f is n-quasi-compact.

Corollary 2.3.5.3. *Let Z be a quasi-compact nonconnective spectral Deligne-Mumford stack and let $n \geqslant 0$. Then Z is $(n+1)$-quasi-compact if and only if the following condition is satisfied: for every pair of maps $\mathsf{X} \to \mathsf{Z} \leftarrow \mathsf{Y}$ where X and Y are n-quasi-compact, the fiber product $\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y}$ is n-quasi-compact.*

Proof. The "if" direction follows immediately from Proposition 2.3.2.6 (take X and Y to be affine). Conversely, suppose that Z is $(n+1)$-quasi-compact. If X is n-quasi-compact, then the map $\mathsf{X} \to \mathsf{Z}$ is n-quasi-compact. It follows from Proposition 2.3.3.1 that the projection map $\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y} \to \mathsf{Y}$ is n-quasi-compact. Since Y is also n-quasi-compact, Proposition 2.3.5.1 implies that $\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y}$ is n-quasi-compact. □

Corollary 2.3.5.4. *The collection of ∞-quasi-compact nonconnective spectral Deligne-Mumford stacks is closed under the formation of fiber products.*

2.4 Quasi-Affine Spectral Deligne-Mumford Stacks

Recall that a scheme X is said to be *quasi-affine* if it is quasi-compact and there exists an open immersion $j : X \hookrightarrow Y$, where Y is an affine scheme. In this section, we will study the analogous condition in the setting of spectral Deligne-Mumford stacks. Our main results are that quasi-affine spectral Deligne-Mumford stacks behave, in many respects, as if they were affine:

(a) If $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a quasi-affine spectral Deligne-Mumford stack, then X can be functorially recovered from the $\mathbb{E}_\infty$-ring $\Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}})$ of global sections of its structure sheaf (Corollary 2.4.2.2).

(b) If $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a quasi-affine spectral Deligne-Mumford stack and $\mathscr{F}$ is a quasi-coherent sheaf on $\mathcal{X}$, then $\mathscr{F}$ can be functorially recovered from its spectrum of global sections $\Gamma(\mathcal{X}; \mathscr{F})$, regarded as a module over the $\mathbb{E}_\infty$-ring $\Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}})$ (Proposition 2.4.1.4).

2.4.1 The Nonconnective Case

We with a study of quasi-affine objects in the setting of nonconnective spectral Deligne-Mumford stacks.

Definition 2.4.1.1. Let X be a nonconnective spectral Deligne-Mumford stack. We say that X is *quasi-affine* if X is quasi-compact and there exists an open immersion $j : \mathsf{X} \hookrightarrow \operatorname{Sp\acute{e}t} R$ for some $\mathbb{E}_\infty$-ring R (see Definition 1.6.7.2).

Remark 2.4.1.2. Let X be a nonconnective spectral Deligne-Mumford stack. If X is quasi-affine, then it is schematic.

Suppose that X is a quasi-affine nonconnective spectral Deligne-Mumford stack. Then there exists an open immersion $j : \mathsf{X} \to \mathsf{X}'$, where X' is affine. The following pair of results asserts that there is a canonical choice of X', for which the ∞-categories $\operatorname{QCoh}(\mathsf{X})$ and $\operatorname{QCoh}(\mathsf{X}')$ are equivalent.

Proposition 2.4.1.3. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-compact nonconnective spectral Deligne-Mumford stack. The following conditions are equivalent:*

(1) *The nonconnective spectral Deligne-Mumford stack X is quasi-affine.*

(2) *The canonical map $\mathsf{X} \to \operatorname{Sp\acute{e}t} \Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}})$ is an open immersion.*

Proposition 2.4.1.4. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-affine nonconnective spectral Deligne-Mumford stack. The global sections functor $\mathscr{F} \mapsto \Gamma(\mathcal{X}; \mathscr{F})$ induces an equivalence of ∞-categories $e : \operatorname{QCoh}(\mathsf{X}) \to \operatorname{Mod}_{\Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}})}$.*

The proofs of Propositions 2.4.1.3 and 2.4.1.4 depend on the following technical result:

Proposition 2.4.1.5. *Let X be a quasi-compact nonconnective spectral Deligne-Mumford stack and let $j : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ be an open immersion. Then:*

(1) *The global sections functor $\Gamma : \operatorname{QCoh}(\mathsf{X}) \to \operatorname{Mod}_R$ commutes with small colimits.*

(2) *Suppose that R is connective. Then there exists an integer n such that $\Gamma(\operatorname{QCoh}(\mathsf{X})_{\geqslant 0}) \subseteq (\operatorname{Mod}_R)_{\geqslant -n}$.*

(3) *Suppose we are given a pullback diagram of nonconnective spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{j'} & \operatorname{Sp\acute{e}t} R' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{X} & \xrightarrow{j} & \operatorname{Sp\acute{e}t} R. \end{array}$$

Then the associated diagram of ∞-categories

$$\begin{array}{ccc} \operatorname{Mod}_R & \xrightarrow{j^*} & \operatorname{QCoh}(\mathsf{X}) \\ \downarrow & & \downarrow \\ \operatorname{Mod}_{R'} & \xrightarrow{j'^*} & \operatorname{QCoh}(\mathsf{X}') \end{array}$$

is right adjointable.

Corollary 2.4.1.6. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a quasi-compact nonconnective spectral Deligne-Mumford stack and let $j : \mathsf{X} \to \operatorname{Spét} R$ be an open immersion. Then the global sections functor $\Gamma : \operatorname{QCoh}(\mathsf{X}) \to \operatorname{Mod}_R$ is fully faithful.*

Proof. Let $j^* : \operatorname{Mod}_R \to \operatorname{QCoh}(\mathsf{X})$ denote a left adjoint to Γ and let $\mathscr{F} \in \operatorname{QCoh}(\mathsf{X})$; we will show that the counit map $j^*\Gamma(\mathcal{X}; \mathscr{F}) \to \mathscr{F}$ is an equivalence. The open immersion j is determined by an open subset $U \subseteq |\operatorname{Spec} R|$. Write U as a union $\bigcup_{1 \leqslant i \leqslant n} U_i$, where each U_i is the open subset given by $|\operatorname{Spec} R[x_i^{-1}]|$ for some $x_i \in \pi_0 R$. For $1 \leqslant i \leqslant n$, let $g_i : \mathsf{U}_i \to \mathsf{X}$ be the open immersion determined by the inclusion $U_i \subseteq U$. It will therefore suffice to show that each of the induced maps $\theta_i : (j \circ g_i)^* \Gamma(\mathcal{X}; \mathscr{F}) \to g_i^* \Gamma(\mathcal{X}; \mathscr{F})$ is an equivalence. This follows immediately from Proposition 2.4.1.5, since the projection map $\mathsf{U}_i \times_{\operatorname{Spét} R} \mathsf{X} \to \mathsf{U}_i$ is an equivalence. □

Proof of Proposition 2.4.1.3. The implication (2) ⇒ (1) is obvious. We will show that (1) ⇒ (2). Assume therefore that there exists an open immersion $j : \mathsf{X} \to \operatorname{Spét} R$. Set $A = \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$, so that j determines a map of $\mathbb{E}_\infty$-rings $\phi : R \to A$. Then $\mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét} A$ is an open substack of $\operatorname{Spét} A$. We will complete the proof by showing that the projection map $p : \mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét} A \to \mathsf{X}$ is an equivalence. The map j determines an open subset U of the Zariski spectrum $|\operatorname{Spec} R|$. Since X is quasi-compact, this open subset can be written as a union $\bigcup_{1 \leqslant i \leqslant n} |\operatorname{Spec} R[x_i^{-1}]|$ for some elements $x_i \in \pi_0 R$. To show that p is an equivalence, it will suffice to show that each of the induced projection maps

$$p_i : \operatorname{Spét} R[x_i^{-1}] \times_{\operatorname{Spét} R} \operatorname{Spét} A \to \operatorname{Spét} R[x_i^{-1}]$$

is an equivalence.

Let $x = x_i$. We wish to prove that the map $\theta : R[x^{-1}] \to R[x^{-1}] \otimes_R A$ is an equivalence of $\mathbb{E}_\infty$-rings. Let $\mathcal{O}'$ denote the structure sheaf of $\operatorname{Spét} R$. For every open subset $V \subseteq |\operatorname{Spec} R|$, let V_0 denote the intersection of V with the open set $|\operatorname{Spec} R[x^{-1}]|$, and let f_V denote the canonical map $R[x^{-1}] \otimes_R \mathcal{O}'(V) \to \mathcal{O}'(V_0)$. We note that f_U is left inverse to θ. It will therefore suffice to show that f_U is an equivalence, which is a special case of Proposition 2.4.1.5. $\square$

Proof of Proposition 2.4.1.4. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a quasi-affine nonconnective spectral Deligne-Mumford stack. Proposition 2.4.1.3 implies that the map $j : \mathsf{X} \to \operatorname{Spét} \Gamma(\mathcal{X}; \mathcal{O}_\mathcal{X})$ is an open immersion, so that the global sections functor $\Gamma : \operatorname{QCoh}(\mathsf{X}) \to \operatorname{Mod}_{\Gamma(\mathcal{X};\mathcal{O}_\mathcal{X})}$ is fully faithful by Corollary 2.4.1.6. Consequently, to prove that Γ is an equivalence of ∞-categories, it will suffice to show that the unit map $u_M : M \to \Gamma(\mathcal{X}; j^* M)$ is an equivalence for every A-module M. Since Γ commutes with small colimits (Proposition 2.4.1.5), the collection of those A-modules M for which u_M is an equivalence is closed under small colimits. It will therefore suffice to show that u_M is an equivalence in the case where M has the form $\Sigma^n \Gamma(\mathcal{X}; \mathcal{O}_\mathcal{X})$, for some integer n. We may easily reduce to the case $n = 0$, in which case the desired result is a tautology. $\square$

Proof of Proposition 2.4.1.5. The open immersion j is determined by an open subset $U \subseteq |\operatorname{Spec} R|$. For every open subset $V \subseteq U$, let $\Gamma_V : \operatorname{QCoh}(\mathsf{X}) \to \operatorname{Mod}_R$ be the functor given by evaluation at V (which we can identify with a (-1)-truncated object of the underlying ∞-topos $\mathcal{X}$ of X). Given a pair of open sets $V', V'' \subseteq U$, we obtain a pullback diagram of functors σ:

$$\begin{array}{ccc} \Gamma_{V' \cup V''} & \longrightarrow & \Gamma_{V''} \\ \downarrow & & \downarrow \\ \Gamma_{V'} & \longrightarrow & \Gamma_{V' \cap V''}. \end{array}$$

To prove (1), it will suffice to show that for every quasi-compact open subset $V \subseteq U$, the functor Γ_V commutes with filtered colimits. Since V is quasi-compact, we can write V as a union $\bigcup_{1 \leqslant i \leqslant n} V_i$ where each $V_i \subseteq |\operatorname{Spec} R|$ is given by $|\operatorname{Spec} R[x_i^{-1}]|$ for some $x_i \in \pi_0 R$. We proceed by induction on n. If $n = 0$, then V is empty and the result is obvious. If $n > 0$, we let $V' = V_1$ and $V'' = \bigcup_{1 < i \leqslant n} V_i$ so that $V = V' \cup V''$. The inductive hypothesis implies that $\Gamma_{V''}$ and $\Gamma_{V' \cap V''}$ commute with filtered colimits. Using the pullback diagram σ, we are reduced to proving that $\Gamma_{V'}$ commutes with filtered colimits. This is clear, since $\Gamma_{V'}$ is given by the composition

$$\operatorname{QCoh}(\mathsf{X}) \to \operatorname{QCoh}(\operatorname{Spét} R[x_1^{-1}]) \simeq \operatorname{Mod}_{R[x_1^{-1}]} \to \operatorname{Mod}_R .$$

We now prove (2). Assume that R is connective. We will show that if $V \subseteq U$ is an open subset which can be written as a union $\bigcup_{1 \leqslant i \leqslant n} V_i$, where each V_i is of the form $|\operatorname{Spec} R[x_i^{-1}]|$, then Γ_V carries $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$ to $(\mathrm{Mod}_R)_{\geqslant 1-n}$. We proceed by induction on n. In the case $n = 0$, $V = \varnothing$ and there is nothing to prove. Assume therefore that $n > 0$ and define subsets $V', V'' \subseteq V$ as above. If $M \in \mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$, then the pullback diagram σ gives a fiber sequence

$$\Gamma_V(M) \to \Gamma_{V'}(M) \oplus \Gamma_{V''}(M) \to \Gamma_{V' \cap V''}(M)$$

and therefore an exact sequence of abelian groups

$$\pi_{m+1}\Gamma_{V' \cap V''}(M) \to \pi_m \Gamma_V(M) \to \pi_m \Gamma_{V'}(M) \oplus \pi_m \Gamma_{V''}(M).$$

The functor $\Gamma_{V'}$ is given by the composition

$$\mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R[x_1^{-1}]) \simeq \mathrm{Mod}_{R[x_1^{-1}]} \to \mathrm{Mod}_R\,.$$

and is therefore t-exact. Using the inductive hypothesis, we deduce that if $m \leqslant -n$, then

$$\pi_{m+1}\Gamma_{V' \cap V''}(M) \simeq \pi_m \Gamma_{V''}(M) \simeq 0,$$

from which it follows that $\pi_m \Gamma_V(M) \simeq 0$.

We now prove (3). Let $\pi : |\operatorname{Spec} R'| \to |\operatorname{Spec} R|$ be the continuous map of topological spaces induced by the map of $\mathbb{E}_\infty$-rings $R \to R'$. For every open set $V \subseteq U$, let $\Gamma_{\pi^{-1}V} : \mathrm{QCoh}(\mathsf{X}') \to \mathrm{Mod}_{R'}$ be defined as above. Let us say that an open subset $V \subseteq U$ is *good* if the canonical map $R' \otimes_R \Gamma_V \to \Gamma_{\pi^{-1}V}$ is an equivalence of functors from $\mathrm{QCoh}(\mathsf{X})$ to $\mathrm{Mod}_{R'}$. Note that if $V', V'' \subseteq U$, then the canonical map

$$\Gamma_{\pi^{-1}(V' \cup V'')} \to \Gamma_{\pi^{-1}(V')} \times_{\Gamma_{\pi^{-1}(V' \cap V'')}} \Gamma_{\pi^{-1}(V'')}$$

is an equivalence. It follows that if V', V'', and $V' \cap V''$ are good, then $V' \cup V''$ is good. We will prove that every quasi-compact open subset $V \subseteq U$ is good. Write $V = \bigcup_{1 \leqslant i \leqslant n} V_i$ as above; we proceed by induction on n. When $n = 0$, $V = \varnothing$ and there is nothing to prove. If $n > 0$, we define $V', V'' \subseteq V$ as above, so that V'' and $V' \cap V''$ are good by the inductive hypothesis. We may therefore replace V by V' and thereby reduce to the case $\mathsf{X}_V = \mathrm{Sp\acute{e}t}\, R[x_1^{-1}]$, in which case the desired result follows from Lemma D.3.5.6. □

2.4.2 The Connective Case

Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a quasi-affine spectral Deligne-Mumford stack and let $A = \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$. Then the canonical map $j : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, A$ is an open immersion (Proposition 2.4.1.3). However, A is usually not connective:

Proposition 2.4.2.1. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a quasi-affine spectral Deligne-Mumford stack. If $A = \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$ is connective, then X is affine.*

Proof. The open immersion $j : \mathsf{X} \hookrightarrow \operatorname{Spét} A$ of Proposition 2.4.1.3 determines a quasi-compact open subset $U \subseteq |\operatorname{Spec} A|$, consisting of those prime ideals which fail to contain some finitely generated ideal $I = (x_1, \ldots, x_n) \subseteq \pi_0 A$. Let $M = (\pi_0 A)/I$, which we regard as a discrete A-module. Then $M[x_i^{-1}] \simeq 0$ for $1 \leqslant i \leqslant n$, so that M is annihilated by the pullback functor $\operatorname{Mod}_A \simeq \operatorname{QCoh}(\operatorname{Spét} A) \xrightarrow{j^*} \operatorname{QCoh}(\mathsf{X})$. Proposition 2.4.1.4 implies that the pullback functor j^* is an equivalence of ∞-categories, so that $M \simeq 0$. It follows that I is the unit ideal in $\pi_0 A$, so that j is an equivalence. □

Corollary 2.4.2.2. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a quasi-affine nonconnective spectral Deligne-Mumford stack, let $A = \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$, and let $j : \mathsf{X} \to \operatorname{Spét} A$ be the open immersion of Proposition 2.4.1.3. For every spectral Deligne-Mumford stack Y, composition with j induces a homotopy equivalence*

$$\theta : \operatorname{Map}_{\operatorname{SpDM}^{\mathrm{nc}}}(\mathsf{Y}, \mathsf{X}) \to \operatorname{Map}_{\operatorname{SpDM}^{\mathrm{nc}}}(\mathsf{Y}, \operatorname{Spét} A).$$

Proof. The assertion is local on Y; we may therefore assume that Y is affine, so that $\mathsf{Y} \simeq \operatorname{Spét} R$ for some connective $\mathbb{E}_\infty$-ring R. Since j is an open immersion, the map θ exhibits $\operatorname{Map}_{\operatorname{SpDM}}(\mathsf{Y}, \mathsf{X})$ as a summand of $\operatorname{Map}_{\operatorname{SpDM}}(\mathsf{Y}, \operatorname{Spét} A)$. It will therefore suffice to show that every map $f : \operatorname{Spét} R \to \operatorname{Spét} A$ factors through j. Form a pullback diagram

$$\begin{array}{ccc} (\mathcal{X}', \mathscr{O}_{\mathcal{X}'}) & \xrightarrow{j'} & \operatorname{Spét} R \\ \downarrow & & \downarrow \\ \mathsf{X} & \longrightarrow & \operatorname{Spét} A \end{array}$$

so that j' is an open immersion. Since $A \simeq \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$, Proposition 2.4.1.5 implies that the induced map $R \to \Gamma(\mathcal{X}'; \mathscr{O}_{\mathcal{X}'})$ is an equivalence. Since R is connective, it follows from Proposition 2.4.2.1 that j' is an equivalence. □

In spite of Proposition 2.4.2.1, every quasi-affine spectral Deligne-Mumford stack admits an open immersion into the spectrum of a *connective* $\mathbb{E}_\infty$-ring:

Proposition 2.4.2.3. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. The following conditions are equivalent:*

(1) *There exists a connective $\mathbb{E}_\infty$-ring R and an open immersion $j : \mathsf{X} \to \operatorname{Spét} R$.*

(2) *The spectral Deligne-Mumford stack X is quasi-affine.*

(3) *The discrete spectral Deligne-Mumford stack $(\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}})$ is quasi-affine.*

Proof. The implication (1) ⇒ (2) is obvious. To prove (2) ⇒ (3), we note that if $j : \mathsf{X} \to \operatorname{Spét} R$ is an open immersion, then j induces an open immersion $(\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}}) \to \operatorname{Spét}(\pi_0 R)$.

It remains to prove that (3) ⇒ (1). For each $i \geqslant 0$, we let R_i denote the $\mathbb{E}_\infty$-ring $\Gamma(\mathcal{X}; \tau_{\leqslant i} \mathscr{O}_{\mathcal{X}})$, and let $R = \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}}) \simeq \varprojlim_i R_i$. Applying Proposition 2.4.1.5 to the quasi-affine spectral Deligne-Mumford stack $(\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}})$, we deduce that there exists an integer n such that $\Gamma(\mathcal{X}; \mathscr{F}) \in \mathrm{Sp}_{\geqslant -n}$ whenever $\mathscr{F}$ belongs to the heart of $\operatorname{QCoh}(\mathsf{X})$. The fiber sequence

$$\Sigma^m(\pi_m \mathscr{O}_{\mathcal{X}}) \to \tau_{\leqslant m} \mathscr{O}_{\mathcal{X}} \to \tau_{\leqslant m-1} \mathscr{O}_{\mathcal{X}}$$

yields a fiber sequence of spectra

$$\Sigma^m \Gamma(\mathcal{X}; \pi_m \mathscr{O}_{\mathcal{X}}) \to R_m \to R_{m-1}$$

so that the map $\pi_i R_m \to \pi_i R_{m-1}$ is an isomorphism for $m > n + i$. It follows that each of the maps $\pi_i R \to \pi_i R_{n+i}$ is an isomorphism.

Since condition (3) is satisfied, Proposition 2.4.1.3 implies that the canonical map $j_0 : (\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}}) \to \operatorname{Spét} R_0$ is an open immersion, corresponding to some quasi-compact open subset $U \subseteq |\operatorname{Spec} R_0|$. For each $x \in \pi_0 R_0$, let $U_x = \{\mathfrak{p} \in |\operatorname{Spec} R_0| : x \notin \mathfrak{p}\}$. We next prove the following:

(∗) Let x be an element of $\pi_0 R_0$ such that $U_x \subseteq U$. Then there exists an integer $m > 0$ such that x^m can be lifted to an element of $\pi_0 R$.

For every pair of integers $i \leqslant i'$, let $\phi_{i',i} : \pi_0 R_{i'} \to \pi_0 R_i$ be the canonical map. To prove (∗), we show that for each $i \geqslant 0$, some positive power x^m of x lies in the image of the map $\phi_{i,0} : \pi_0 R_i \to \pi_0 R_0$. Since $\pi_0 R \simeq \pi_0 R_n$, (∗) will follow if we prove this in the case $i = n$. We proceed by induction on i, the case $i = 0$ being trivial. Assume therefore that there exists an integer $m > 0$ such that $x^m = \phi_{i,0}(y)$ for some $y \in \pi_0 R_i$. We will prove that some positive power of y lies in the image of the map $\phi_{i+1,i}$. Using Theorem HA.7.4.1.26 , we deduce that $\tau_{\leqslant i+1} \mathscr{O}_{\mathcal{X}}$ is a square-zero extension of $\tau_{\leqslant i} \mathscr{O}_{\mathcal{X}}$ by the module $\Sigma^{i+1}(\pi_{i+1} \mathscr{O}_{\mathcal{X}})$. It follows that R_{i+1} is a square-zero extension of R_i by the module $\Gamma(\mathcal{X}; \Sigma^{i+1} \pi_{i+1} \mathscr{O}_{\mathcal{X}})$. In particular, the image of the map $\phi_{i+1,i}$ is the kernel of a derivation $d : \pi_0 R_i \to \pi_{-i-2} \Gamma(\mathcal{X}; \pi_{i+1} \mathscr{O}_{\mathcal{X}})$. We wish to prove that $d(y^{m'}) = 0$ for some $m' > 0$. Since d is a derivation, we have $d(y^{m'}) = m' y^{m'-1} dy$. It will therefore suffice to show that $dy \in \pi_{-i-2} \Gamma(\mathcal{X}; \pi_{i+1} \mathscr{O}_{\mathcal{X}})$ is annihilated by some power of y. Note that $\Gamma(\mathcal{X}; \pi_{i+1} \mathscr{O}_{\mathcal{X}})$ has the structure of a module over R_0. Moreover, Corollary 2.4.1.6 implies that $j_0^* \Gamma(\mathcal{X}; \pi_{i+1} \mathscr{O}_{\mathcal{X}})$ is equivalent to $\pi_{i+1} \mathscr{O}_{\mathcal{X}}$, which a discrete sheaf of spectra on $\mathcal{X}$. Since $U_x \subseteq U$, we deduce that $\Gamma(\mathcal{X}; \pi_{i+1} \mathscr{O}_{\mathcal{X}})[x^{-1}]$ is discrete. Since $i + 2 \neq 0$, it follows that every element of $\pi_{-i-2} \Gamma(\mathcal{X}; \pi_{i+1} \mathscr{O}_{\mathcal{X}})$ is annihilated by a power of x, and therefore by a power of y. This completes the proof of (∗).

Write U as a union of open sets $\bigcup_{1 \leqslant i \leqslant n} U_{x_i}$ for some elements $x_i \in \pi_0 R_0$. Using (∗), we may assume without loss of generality that each x_i is the image of some element $y_i \in \pi_0 R$.

For $1 \leqslant i \leqslant n$, let V_i denote the open subset $\{\mathfrak{p} \in |\operatorname{Spec} R| : y_i \notin \mathfrak{p}\}$, and let $V = \bigcup_{1 \leqslant i \leqslant n} V_i$. Let V denote the open substack of $\operatorname{Spét} R$ corresponding to V, and for $1 \leqslant i \leqslant n$ let V_i denote the open substack of V corresponding to V_i. Since V is the inverse image of U in $|\operatorname{Spec} R|$, the canonical map $j : \mathsf{X} \to \operatorname{Spét} R$ factors through V. We claim that j induces an equivalence $\mathsf{X} \to \mathsf{V}$. To prove this, it suffices to show that each of the induced maps $\mathsf{X} \times_{\mathsf{V}} \mathsf{V}_i \to \mathsf{V}_i$ is an equivalence. By virtue of Proposition 2.4.1.5, this is equivalent to the assertion that $\mathsf{X} \times$ is affine. This follows from Remark ??, since the 0-truncation of $\mathsf{X} \times_{\mathsf{V}} \mathsf{V}_i$ is given by $\operatorname{Spét} R_0[x_i^{-1}]$. □

2.4.3 Descent

Let X be a quasi-affine spectral Deligne-Mumford stack. Combining Corollary 2.4.2.2 with Theorem D.6.3.5, we deduce that the functor $R \mapsto \operatorname{Map}_{\operatorname{SpDM}}(\operatorname{Spét} R, \mathsf{X})$ is a hypercomplete sheaf with respect to the flat topology on $\operatorname{CAlg}^{\operatorname{cn}}$. In fact, we have the following stronger assertion:

Proposition 2.4.3.1. *Let X be quasi-affine nonconnective spectral Deligne-Mumford stack, and let $h_{\mathsf{X}} = h_{\mathsf{X}}^{\mathrm{nc},\acute{e}t}$ denote the functor of points of X (see Definition ??). Then h_{X} is a hypercomplete sheaf with respect to the flat topology on* CAlg*.*

Proof. Choose an open immersion $j : \mathsf{X} \hookrightarrow \operatorname{Spét} A$, for some $\mathbb{E}_\infty$-ring A. It follows from Theorem D.6.3.5 that the functor

$$R \mapsto \operatorname{Map}_{\operatorname{SpDM}^{\mathrm{nc}}}(\operatorname{Spét} R, \operatorname{Spét} A) \simeq \operatorname{Map}_{\operatorname{CAlg}}(A, R)$$

is a hypercomplete sheaf with respect to the flat topology on CAlg. According to Lemma D.4.3.2, it will suffice to show that for every map $\eta : \operatorname{Spét} R \to \operatorname{Spét} A$, the fiber product $\mathsf{X}' = \mathsf{X} \times_{\operatorname{Spét} A} \operatorname{Spét} R$ represents a hypercomplete sheaf with respect to the flat topology on CAlg_R. We can identify X' with an open substack of $\operatorname{Spét} R$, classified by an open set $U \subseteq |\operatorname{Spec} R|$. Unwinding the definitions, we are reduced to showing that if $\phi : B \to B'$ is a faithfully flat morphism in CAlg_R such that the map $|\operatorname{Spec} B'| \to |\operatorname{Spec} R|$ factors through U, then $|\operatorname{Spec} B| \to |\operatorname{Spec} R|$ also factors through U. This is clear, since the map $|\operatorname{Spec} B'| \to |\operatorname{Spec} B|$ is a surjection. □

In fact, we can prove an even stronger version of Proposition 2.4.3.1. For every $\mathbb{E}_\infty$-ring R, let $\operatorname{SpDM}^{\mathrm{nc}}_R$ denote the ∞-category $\operatorname{SpDM}^{\mathrm{nc}}_{/\operatorname{Spét} R}$ of nonconnective spectral Deligne-Mumford stacks X equipped with a map $f : \mathsf{X} \to \operatorname{Spét} R$. Let $\operatorname{QAff}^{\mathrm{nc}}_R$ denote the full subcategory of $\operatorname{SpDM}^{\mathrm{nc}}_R$ spanned by those maps $f : \mathsf{X} \to \operatorname{Spét} R$ where X is quasi-affine. If R is connective, we let QAff_R denote the full subcategory of $\operatorname{QAff}^{\mathrm{nc}}_R$ spanned by those morphisms where X is a spectral Deligne-Mumford stack.

Proposition 2.4.3.2 (Effective Descent for Quasi-Affine Morphisms)**.** *The functor $R \mapsto \mathrm{QAff}^{\mathrm{nc}}_R$, is a hypercomplete sheaf (with values in $\widehat{\mathcal{C}\mathrm{at}}_\infty$) with respect to the flat topology on* CAlg*. The functor $R \mapsto \mathrm{QAff}_R$ is a hypercomplete sheaf with respect to the flat topology on* $\mathrm{CAlg}^{\mathrm{cn}}$.

Proof. For every $\mathbb{E}_\infty$-ring R, let $\mathrm{Aff}^{\mathrm{nc}}_R$ denote the full subcategory of $\mathrm{SpDM}^{\mathrm{nc}}_R$ spanned by those morphisms $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ where X is an affine nonconnective spectral Deligne-Mumford stack. We have an equivalence of ∞-categories $(\mathrm{Aff}^{\mathrm{nc}}_R)^{\mathrm{op}} \simeq \mathrm{CAlg}^{\mathrm{op}}_R$. Using Corollary D.6.3.3, we deduce that the functor $R \mapsto \mathrm{Aff}^{\mathrm{nc}}_R$ is a hypercomplete sheaf with respect to the flat topology.

For every $\mathbb{E}_\infty$-ring R, let $Y(R)$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{SpDM}^{\mathrm{nc}}_R)$ spanned by those morphisms $f : \mathsf{U} \to \mathsf{X}$, where U is affine and f is an open immersion. Let us regard Y as a functor $\mathrm{CAlg} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. We claim that Y is a hypercomplete sheaf with respect to the flat topology. Evaluation at $\{1\} \subseteq \Delta^1$ determines a map $Y(R) \to \mathrm{Aff}^{\mathrm{nc}}_R$, depending functorially on R. Using Lemma D.4.3.2, we are reduced to verifying the following assertion:

- $(*)$ Let R be an $\mathbb{E}_\infty$-ring, let $f : \mathrm{Sp\acute{e}t}\, A \to \mathrm{Sp\acute{e}t}\, R$ be a map of affine spectral Deligne-Mumford stacks, and let $F : \mathrm{CAlg}_R \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the functor which assigns to each R-algebra R' the ∞-category of open substacks of $\mathrm{Sp\acute{e}t}\, R' \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, A$. Then F is a hypercomplete sheaf with respect to the flat topology.

This follows easily from Proposition 1.6.2.2.

For every $\mathbb{E}_\infty$-ring R, let $Y'(R)$ denote the full subcategory of $Y(R)$ spanned by those morphisms $f : \mathsf{U} \to \mathsf{X}$ where U is quasi-compact. Let us regard Y' as a functor $\mathrm{CAlg} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. We claim that Y' is a hypercomplete sheaf with respect to the flat topology. Since Y is a sheaf with respect to the flat topology, we may use Lemma D.4.3.2 to reduce to proving the following concrete assertion:

- $(*')$ Let $f : A \to A'$ be a faithfully flat map of $\mathbb{E}_\infty$-rings, and let $U \subseteq |\mathrm{Spec}\, A|$ be an open subset. If the inverse image of U in $|\mathrm{Spec}\, A'|$ is quasi-compact, then U is quasi-compact.

This is clear, since the map $|\mathrm{Spec}\, A'| \to |\mathrm{Spec}\, A|$ is surjective.

For every $\mathbb{E}_\infty$-ring R, let $Y''(R)$ denote the full subcategory of $Y'(R)$ spanned by those morphisms $f : (\mathcal{U}, \mathcal{O}_{\mathcal{U}}) \to \mathrm{Sp\acute{e}t}\, A$ which induce an equivalence of $\mathbb{E}_\infty$-rings $A \to \Gamma(\mathcal{U}; \mathcal{O}_{\mathcal{U}})$. Let us regard Y'' as a functor $\mathrm{CAlg} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. We claim that Y'' is a hypercomplete sheaf with respect to the flat topology. This follows easily from Lemma D.4.3.2 and Corollary 2.5.4.6.

Evaluation at $\{0\} \subseteq \Delta^1$ induces a functor $\phi_R : Y''(R) \to \mathrm{QAff}^{\mathrm{nc}}_R$, depending functorially on R. Proposition 2.4.1.3 implies that each of these functors is an equivalence of ∞-categories. It follows that $R \mapsto \mathrm{QAff}^{\mathrm{nc}}_R$ is a hypercomplete sheaf with respect to the flat topology on

CAlg. To prove that the functor $R \mapsto \mathrm{QAff}_R$ is a hypercomplete sheaf with respect to the flat topology on $\mathrm{CAlg}^{\mathrm{cn}}$, we invoke Lemma D.4.3.2 to reduce to the following assertion:

$(*'')$ Suppose we are given a map of nonconnective spectral Deligne-Mumford stacks $\mathsf{U} \to \mathrm{Sp\acute{e}t}\, R$. Assume that R is connective and that there exists a faithfully flat morphism $R \to R'$ such that the fiber product $\mathsf{U} \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, R'$ is a spectral Deligne-Mumford stack. Then U is a spectral Deligne-Mumford stack (that is, its structure sheaf is connective).

This follows immediately from Example 2.8.3.8. □

2.4.4 Affine and Quasi-Affine Morphisms

We conclude this section by introducing a relative version of the notion of a quasi-affine spectral Deligne-Mumford stack.

Definition 2.4.4.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of nonconnective spectral Deligne-Mumford stacks. We will say that f is *affine* if, for every map $\mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$, the fiber product $\mathsf{X} \times_{\mathsf{Y}} \mathrm{Sp\acute{e}t}\, R$ is affine. We will say that f is *quasi-affine* if, for every map $\mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$, the fiber product $\mathsf{X} \times_{\mathsf{Y}} \mathrm{Sp\acute{e}t}\, R$ is quasi-affine.

Remark 2.4.4.2. Let $f : (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be a morphism of spectral Deligne-Mumford stacks. Then f is affine (quasi-affine) if and only if the underlying map of 0-truncated spectral Deligne-Mumford stacks $(\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathscr{O}_{\mathcal{Y}})$ is affine (quasi-affine); see Corollary 1.4.7.3 (Proposition 2.4.2.3).

The following assertions regarding affine and quasi-affine morphisms follow immediately from the definition:

Proposition 2.4.4.3. (1) *Any equivalence of nonconnective spectral Deligne-Mumford stacks is affine. Any affine morphism is quasi-affine.*

(2) *Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of nonconnective spectral Deligne-Mumford stacks, and suppose that Y is affine. Then f is affine (quasi-affine) if and only if X is affine (quasi-affine).*

(3) *Suppose we are given a pullback diagram of nonconnective spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}. \end{array}$$

If f is affine (quasi-affine), then f' is affine (quasi-affine).

2.5 Pullbacks and Pushforwards of Quasi-Coherent Sheaves

Let $f : (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a map of spectrally ringed ∞-topoi. Combining the pushforward functor $f_* : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ with restriction of scalars along the underlying map $\mathscr{O}_{\mathcal{X}} \to f_* \mathscr{O}_{\mathcal{Y}}$ of CAlg-valued sheaves on $\mathcal{X}$, we obtain a pushforward functor $f_* : \mathrm{Mod}_{\mathscr{O}_{\mathcal{Y}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}}$. This functor admits a left adjoint $\mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathcal{Y}}}$ which we will generally denote by f^*.

Warning 2.5.0.1. Any geometric morphism of ∞-topoi $f_* : \mathcal{Y} \to \mathcal{X}$ induces a pullback functor $f^*_{\mathrm{Sp}} : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ on spectrum-valued sheaves. If f_* is promoted to a morphism of spectrally ringed ∞-topoi $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, then the pullback functor $f^* : \mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathcal{Y}}}$ is usually not compatible with f^*_{Sp}. That is, the diagram of ∞-categories σ :

$$\begin{array}{ccc} \mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}} & \xrightarrow{f^*} & \mathrm{Mod}_{\mathscr{O}_{\mathcal{Y}}} \\ \downarrow & & \downarrow \\ \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) & \xrightarrow{f^*_{\mathrm{Sp}}} & \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y}) \end{array}$$

generally does not commute. Instead we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mod}_{\mathscr{O}_{\mathcal{Y}}} & \xrightarrow{f_*} & \mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}} \\ \downarrow & & \downarrow \\ \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y}) & \xrightarrow{f_*} & \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}), \end{array}$$

which is left adjointable if and only if the morphism f exhibits $\mathscr{O}_{\mathcal{Y}}$ as the pullback of $\mathscr{O}_{\mathcal{X}}$.

Proposition 2.5.0.2. *Let $f : (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a morphism of nonconnective spectral Deligne-Mumford stacks. Then the pullback functor $f^* : \mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathcal{Y}}}$ carries quasi-coherent sheaves on $\mathcal{X}$ to quasi-coherent sheaves on $\mathcal{Y}$.*

Proof. The assertion is local on $\mathcal{X}$ and $\mathcal{Y}$. We may therefore assume that both $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ are affine, in which case the desired result follows immediately from the characterization of quasi-coherent sheaves given by Proposition 2.2.3.3. □

Our goal in this section is to study conditions which guarantee that a pushforward functor f_* preserves quasi-coherence.

2.5.1 The Affine Case

We begin by proving Proposition 2.5.0.2 in the case where the morphism $f : \mathsf{Y} \to \mathsf{X}$ is affine, in the sense of Definition 2.4.4.1.

Proposition 2.5.1.1. *Let $f : \mathsf{Y} \to \mathsf{X}$ be an affine morphism of nonconnective spectral Deligne-Mumford stacks. Then:*

(1) *The direct image functor $f_* : \mathrm{Mod}_{\mathscr{O}_\mathsf{Y}} \to \mathrm{Mod}_{\mathscr{O}_\mathsf{X}}$ carries quasi-coherent sheaves on Y to quasi-coherent sheaves on X.*

(2) *The functor $f_* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ preserves small colimits.*

(3) *If Y and X are spectral Deligne-Mumford stacks, then the functor f_* is t-exact.*

Proof. The assertion is local on X, so we may assume without loss of generality that $\mathsf{X} = \mathrm{Spét}\, A$ is affine. In this case, the affineness of f guarantees that $\mathsf{Y} = \mathrm{Spét}\, B$ is also affine. Let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ be the quasi-coherent sheaf associated to some B-module M, and let $\mathscr{F}' \in \mathrm{QCoh}(\mathsf{X})$ denote the quasi-coherent sheaf associated to the image of M under the forgetful functor $\mathrm{Mod}_B \to \mathrm{Mod}_A$. The counit map $B \otimes_A M \to M$ determines a map $f^* \mathscr{F}' \to \mathscr{F}$, which is adjoint to a map of $\mathscr{O}_\mathsf{Y}$-modules $\mathscr{F}' \to f_* \mathscr{F}$. We claim that this map is an equivalence. For this, we must show that $u : \mathscr{F}'(U) \to (f_* \mathscr{F})(U) \simeq \mathscr{F}(f^* U)$ is an equivalence of spectra for each object $U \in \mathcal{X}$; here $\mathcal{X}$ denotes the underlying ∞-topos of X. The collection of those objects U which satisfy this condition is stable under colimits. We may therefore assume that U is representable by an étale A-algebra A'. In this case, u can be identified with the canonical equivalence $M \otimes_A A' \simeq M \otimes_B (B \otimes_A A')$. This proves (1), and shows that the induced map $f_* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ can be identified with the forgetful functor $\mathrm{Mod}_B \to \mathrm{Mod}_A$. Assertions (2) and (3) follow immediately from this identification. □

We can use Proposition 2.5.1.1 to classify affine morphisms of spectral Deligne-Mumford stacks. Let X be a spectral Deligne-Mumford stack and let Aff_X denote the full subcategory of $\mathrm{SpDM}_{/\mathsf{X}}$ spanned by the affine morphisms $f : \mathsf{Y} \to \mathsf{X}$. It follows from Proposition **??** that the construction

$$(f : \mathsf{Y} \to \mathsf{X}) \mapsto (f_* \mathscr{O}_\mathsf{Y} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}))$$

determines a functor $\lambda : \mathrm{Aff}_\mathsf{X}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}})$.

Proposition 2.5.1.2. *Let X be a spectral Deligne-Mumford stack. Then the functor $\lambda : \mathrm{Aff}_\mathsf{X}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}})$ described above is an equivalence of ∞-categories.*

Proof. The assertion is local on X. We may therefore reduce to the case where $\mathsf{X} = \mathrm{Spét}\, A$ is affine, in which case λ is a homotopy inverse to the evident equivalence

$$\begin{aligned} \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}) &\simeq \mathrm{CAlg}(\mathrm{Mod}_A^{\mathrm{cn}}) \\ &\simeq \mathrm{CAlg}_A^{\mathrm{cn}} \\ &\xrightarrow{\mathrm{Spét}} \mathrm{Aff}_{\mathrm{Spét}\, A}\,. \end{aligned}$$

□

Construction 2.5.1.3. [The Relative Spectrum] Let X be a spectral Deligne-Mumford stack. We let

$$\mathrm{Sp\acute{e}t}_{\mathsf{X}} : \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}) \to \mathrm{Aff}_{\mathsf{X}}^{\mathrm{op}} \subseteq \mathrm{SpDM}_{/\mathsf{X}}^{\mathrm{op}}$$

denote a homotopy inverse to the equivalence λ of Proposition 2.5.1.2. Given an object $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}})$, we will refer to $\mathrm{Sp\acute{e}t}_{\mathsf{X}}(\mathscr{A})$ as the *spectrum of* $\mathscr{A}$ *relative to* X.

Example 2.5.1.4. For any spectral Deligne-Mumford stack X, we have a canonical equivalence $\mathrm{Sp\acute{e}t}_{\mathsf{X}}\, \mathscr{O}_{\mathsf{X}} \simeq \mathsf{X}$.

Example 2.5.1.5. Let $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ be an affine spectral Deligne-Mumford stack. Then the relative spectrum functor $\mathrm{Sp\acute{e}t}_{\mathsf{X}}$ can be identified with the functor

$$\mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}) \simeq \mathrm{CAlg}_A^{\mathrm{cn}} \xrightarrow{\mathrm{Sp\acute{e}t}} \mathrm{SpDM}_{/\mathrm{Sp\acute{e}t}\, A}^{\mathrm{op}}$$

appearing in the proof of Proposition 2.5.1.2.

2.5.2 Excision Squares

Let $\mathcal{X}$ be an ∞-topos, and let $U \in \mathcal{X}$ be a (-1)-truncated object. We let $\mathcal{X}/U$ denote the full subcategory of $\mathcal{X}$ spanned by those objects $X \in \mathcal{X}$ for which the projection map $X \times U \to U$ is an equivalence. Then $\mathcal{X}/U$ is itself an ∞-topos, and the inclusion functor $i_* : \mathcal{X}/U \to \mathcal{X}$ is a geometric morphism of ∞-topoi. Recall that a geometric morphism $f_* : \mathcal{Y} \to \mathcal{X}$ is a *closed immersion* if it factors as a composition

$$\mathcal{Y} \xrightarrow{g_*} \mathcal{X}/U \xrightarrow{i_*} \mathcal{X},$$

where U is a (-1)-truncated object of $\mathcal{X}$ and g_* is an equivalence. For a more thorough discussion, we refer the reader to §HTT.7.3.2 .

Proposition 2.5.2.1. *Let $\mathcal{X}$ be an ∞-topos and suppose we are given a diagram σ :*

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ \downarrow & & \downarrow \\ V & \xrightarrow{f'} & V', \end{array}$$

in $\mathcal{X}$. The following conditions are equivalent:

(1) *The diagram σ is both a pushout square and a pullback square, and the map f' is (-1)-truncated.*

(2) *The diagram σ is a pushout square and the map f is (-1)-truncated.*

(3) *The diagram σ is a pullback square, f' is (-1)-truncated, and if we let $i^* : \mathcal{X}_{/V'} \to \mathcal{X}_{/V'}/V$ denote the corresponding closed immersion, then $i^* U'$ is a final object of $\mathcal{X}_{/V'}/V$.*

Proof. The equivalence of (1) and (3) is a matter of unwinding definitions, and the implication (1) $\Rightarrow$ (2) is obvious. We will show that (2) $\Rightarrow$ (1). Since $\mathcal{X}$ is an ∞-topos, there exists a fully faithful geometric morphism $i_* : \mathcal{X} \to \mathcal{P}(\mathcal{C})$, for some small ∞-category $\mathcal{C}$. Form a pushout diagram τ :

$$\begin{array}{ccc} i_* U & \longrightarrow & i_* U' \\ \downarrow & & \downarrow \\ i_* V & \xrightarrow{g'} & W \end{array}$$

in $\mathcal{P}(\mathcal{C})$. Then $\sigma \simeq i^*(\tau)$. It will therefore suffice to show that τ is a pullback diagram and that g' is (-1)-truncated. In other words, we may replace $\mathcal{X}$ by $\mathcal{P}(\mathcal{C})$ and thereby reduce to the case where $\mathcal{X}$ is an ∞-category of presheaves. Working pointwise, we can reduce to the case $\mathcal{X} = \mathcal{S}$. In this case, the condition that f is (-1)-truncated guarantees that $U' \simeq U \amalg X$ for some space X, in which case $V' \simeq V \amalg X$ and the result is obvious. $\square$

Definition 2.5.2.2. We will say that a diagram

$$\begin{array}{ccc} U & \longrightarrow & U' \\ \downarrow & & \downarrow \\ V & \longrightarrow & V' \end{array}$$

in an ∞-topos $\mathcal{X}$ is an *excision square* if it satisfies the equivalent conditions of Proposition 2.5.2.1.

Variant 2.5.2.3. Suppose we are given a commutative diagram σ :

$$\begin{array}{ccc} \mathsf{U} & \longrightarrow & \mathsf{U}' \\ \downarrow & & \downarrow \\ \mathsf{V} & \longrightarrow & \mathsf{V}' \end{array}$$

of spectrally ringed ∞-topoi. We will say that σ is an *excision square* if it is equivalent to a diagram of the form

$$\begin{array}{ccc} (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U) & \longrightarrow & (\mathcal{X}_{/U'}, \mathcal{O}_{\mathcal{X}}|_{U'}) \\ \downarrow & & \downarrow \\ (\mathcal{X}_{/V}, \mathcal{O}_{\mathcal{X}}|_V) & \longrightarrow & (\mathcal{X}_{/V'}, \mathcal{O}_{\mathcal{X}}|_{V'}) \end{array}$$

for some spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and some excision square

$$\begin{array}{ccc} U & \longrightarrow & U' \\ \downarrow & & \downarrow \\ V & \longrightarrow & V' \end{array}$$

in $\mathcal{X}$. In this case, we may assume without loss of generality that V' is a final object of $\mathcal{X}$ (otherwise, we can replace $\mathcal{X}$ by $\mathcal{X}_{/V'}$).

2.5.3 Scallop Decompositions

Let X be a quasi-compact quasi-separated scheme. Then we can choose a finite collection of affine open subsets $U_1, \ldots, U_n \subseteq X$ which cover X. Many basic results in the theory of schemes can be proven by considering the filtration of X by open subschemes

$$\emptyset \subseteq U_1 \subseteq U_1 \cup U_2 \subseteq U_1 \cup U_2 \cup U_3 \subseteq \cdots \subseteq U_1 \cup \cdots \cup U_n = X.$$

We now introduce an analogous device for analyzing (spectral) algebraic spaces which are not schematic.

Definition 2.5.3.1. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. A *scallop decomposition* of $\mathcal{X}$ consists of a sequence of (-1)-truncated morphisms

$$U_0 \to U_1 \to \cdots \to U_n$$

in $\mathcal{X}$ satisfying the following conditions:

(a) The object $U_0 \in \mathcal{X}$ is initial and the object $U_n \in \mathcal{X}$ is final.

(b) For $1 \leqslant i \leqslant n$, there exists an excision square

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow \\ U_{i-1} & \longrightarrow & U_i \end{array}$$

where X is affine and V is quasi-compact.

In this case, we will refer to n as the *length* of the scallop decomposition.

Remark 2.5.3.2. In the situation of Definition 2.5.3.1, each of the objects U_i in $\mathcal{X}$ determines an open substack $\mathsf{U}_i = (\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}}|_{U_i})$ of X. In this case, we will also refer to the sequence of open immersions

$$\emptyset \simeq \mathsf{U}_0 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n \simeq \mathsf{X}$$

as a *scallop decomposition* of X.

Example 2.5.3.3. Let X be a quasi-affine nonconnective spectral Deligne-Mumford stack. Then X admits a scallop decomposition.

Remark 2.5.3.4. We will show later that a spectral Deligne-Mumford stack admits a scallop decomposition if and only if it is a quasi-compact, quasi-separated spectral algebraic space (Theorem 3.4.2.1).

Before stating our next result, we need to introduce a bit of terminology. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. We say that an object $U \in \mathcal{X}$ is *semiaffine* if it is quasi-compact and there exists a (-1)-truncated map $U \to X$ in $\mathcal{X}$, where X is affine. We will say that a morphism $f : U \to V$ in $\mathcal{X}$ is *semiaffine* if the fiber product $U \times_V X$ is semiaffine, whenever $X \in \mathcal{X}$ is affine.

Proposition 2.5.3.5. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack which admits a scallop decomposition. Suppose that $\mathcal{C} \subseteq \mathcal{X}$ is a full subcategory satisfying the following conditions:*

(0) *The ∞-category $\mathcal{C}$ is closed under equivalence in $\mathcal{X}$.*

(1) *Initial objects of $\mathcal{X}$ belong to $\mathcal{C}$.*

(2) *If we are given an excision square*

$$\begin{array}{ccc} U & \longrightarrow & U' \\ \downarrow & & \downarrow \\ V & \longrightarrow & V' \end{array}$$

of semiaffine morphisms in $\mathcal{X}$ where U' is affine and $U, V \in \mathcal{C}$, then $V' \in \mathcal{C}$.

Then $\mathcal{C}$ contains the final objects of $\mathcal{X}$.

Corollary 2.5.3.6. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack which admits a scallop decomposition. Suppose that $\mathcal{C} \subseteq \mathcal{X}$ is a full subcategory which contains all affine objects of $\mathcal{X}$ and is closed under pushouts. Then $\mathcal{C}$ contains the final object of $\mathcal{X}$.*

Proof of Proposition 2.5.3.5. It follows immediately from (1) and (2) that every affine object of $\mathcal{X}$ belongs to $\mathcal{C}$. We next show that if $U \in \mathcal{X}$ is semiaffine, then $U \in \mathcal{C}$. Choose a (-1)-truncated map $j : U \to X$ where X is affine, so that $(\mathcal{X}_{/X}, \mathcal{O}_{\mathcal{X}} | X) \simeq \operatorname{Spét} R$. Then we can identify U with an open subset of the topological space $|\operatorname{Spec} R|$. Since U is quasi-compact, we can write U as a finite union $\bigcup_{1 \leqslant i \leqslant n} |\operatorname{Spec} R[x_i^{-1}]|$ for some elements $x_i \in \pi_0 R$. Choose n as small as possible. We proceed by induction on n. If $n = 0$, then U is an initial object of $\mathcal{X}$ and therefore $U \in \mathcal{C}$ by virtue of (1). Assume therefore that $n > 0$. Let

$U_0 = \bigcup_{1 \leqslant i < n} |\operatorname{Spec} R[x_i^{-1}]|$, let $U_1 = |\operatorname{Spec} R[x_n^{-1}]|$, and let $U_{01} = U_0 \cap U_1$. We identify U_0, U_1, and U_{01} with (-1)-truncated objects of $\mathcal{X}$, so that we have an excision square

$$\begin{array}{ccc} U_{01} & \longrightarrow & U_1 \\ \downarrow & & \downarrow \\ U_0 & \longrightarrow & U. \end{array}$$

Since $U_{01}, U_0 \in \mathcal{C}$ be the inductive hypothesis and U_1 is affine, we deduce that $U \in \mathcal{C}$ by (2).

Choose a scallop decomposition

$$U_0 \to U_1 \to \cdots \to U_n$$

for $\mathcal{X}$. We prove by induction on i that each U_i belongs to $\mathcal{C}$. When $i = 0$, this follows from (1). Taking $i = n$ we will obtain the result. To carry out the inductive step, suppose that $U_i \in \mathcal{C}$. Choose an excision square

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & U_{i+1} \end{array}$$

where X is affine and V is quasi-compact. The map $V \to X$ is (-1)-truncated, so that V is semiaffine and therefore $V \in \mathcal{C}$. It follows from (2) that $U_{i+1} \in \mathcal{C}$, as desired. □

2.5.4 Pushforwards of Quasi-Coherent Sheaves

We next introduce a hypothesis on morphisms $f : \mathsf{X} \to \mathsf{Y}$ which will guarantee that the pushforward f_* preserves quasi-coherence (Proposition 2.5.4.3).

Definition 2.5.4.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of nonconnective spectral Deligne-Mumford stacks. We will say that f is *relatively scalloped* if, for every map $\operatorname{Spét} R \to \mathsf{Y}$, the fiber product $\mathsf{X} \times_{\mathsf{Y}} \operatorname{Spét} R$ admits a scallop decomposition.

Example 2.5.4.2. Every quasi-affine morphism is relatively scalloped (see Example 2.5.3.3).

Proposition 2.5.4.3. *Let $f : \mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \mathsf{Y}$ be a relatively scalloped map of nonconnective spectral Deligne-Mumford stacks. Then the pushforward functor $f_* : \operatorname{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \operatorname{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ carries quasi-coherent sheaves to quasi-coherent sheaves. Moreover, the induced functor $f_* : \operatorname{QCoh}(\mathsf{X}) \to \operatorname{QCoh}(\mathsf{Y})$ commutes with small colimits.*

Proof. The assertion is local on $\mathcal{Y}$; we may therefore assume without loss of generality that $\mathsf{Y} = \operatorname{Spét} R$ is affine. For each object $U \in \mathcal{X}$, let $\Gamma_U : \operatorname{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \operatorname{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ denote the composite functor

$$\operatorname{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \operatorname{Mod}_{\mathcal{O}_{\mathcal{X}}|U}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}_{/U})) \to \operatorname{Mod}_{\mathcal{O}_{\mathcal{Y}}}.$$

Let us say that U is *good* if Γ_U restricts to a colimit-preserving functor from $\mathrm{QCoh}(\mathsf{X})$ into $\mathrm{QCoh}(\mathsf{Y})$. The construction $U \mapsto \Gamma_U$ carries pushout square to pullback squares. It follows that the collection of good objects of $\mathcal{X}$ is stable under finite colimits. Since every affine object of $\mathcal{X}$ is good (Proposition 2.5.1.1) and X admits a scallop decomposition, Corollary 2.5.3.6 implies that the final object of $\mathcal{X}$ is good. □

In the situation of Proposition 2.5.4.3, we can also bound the cohomological amplitude of the pushforward functor f_*:

Proposition 2.5.4.4. *Let $f : \mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \mathsf{Y}$ be a relatively scalloped map of spectral Deligne-Mumford stacks. Assume that Y is quasi-compact. Then there exists an integer n such that the pushforward functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ carries $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$ into $\mathrm{QCoh}(\mathsf{Y})_{\geqslant -n}$.*

Proof. Since Y is quasi-compact, we can choose an étale surjection $\mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$ for some connective $\mathbb{E}_\infty$-ring R. Replacing Y by $\mathrm{Sp\acute{e}t}\, R$, we may assume that Y is affine so that X admits a scallop decomposition. We define the class of *good* objects $U \in \mathcal{X}$ as in the proof of Proposition 2.5.4.3. For every good object $U \in \mathcal{X}$, let $\Gamma_U : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ be defined as in the proof of Proposition 2.5.4.3. Let us say that U is *n-good* if $\Gamma_U(\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}) \subseteq \mathrm{QCoh}(\mathsf{Y})_{\geqslant -n}$. Note that if we are given a pushout diagram

$$\begin{array}{ccc} U & \longrightarrow & U' \\ \downarrow & & \downarrow \\ V & \longrightarrow & V' \end{array}$$

in $\mathcal{X}$, then we have a fiber sequence of functors

$$\Gamma_{U'} \oplus \Gamma_V \to \Gamma_{V'} \to \Sigma \Gamma_U.$$

It follows that if U' and V are n-good and U is $(n-1)$-good, then V' is also n-good. Let us say that a good object $U \in \mathcal{X}$ is *very good* if it n-good for some integer $n \geqslant 0$. It follows that the collection of very good objects of $\mathcal{X}$ is closed under pushouts. Any affine object of $\mathcal{X}$ is 0-good, and therefore very good. Using Corollary 2.5.3.6, we deduce that the final object of $\mathcal{X}$ is very good, which implies the desired result. □

The formation of pushforwards along a relatively scalloped morphism is compatible with base change:

Proposition 2.5.4.5. *Suppose we are given a pullback diagram of nonconnective spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{g'} & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y} \end{array}$$

where f (and therefore f') is relatively scalloped. Then the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{Y}) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathsf{X}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{Y}') & \xrightarrow{f'^*} & \mathrm{QCoh}(\mathsf{X}') \end{array}$$

is right adjointable. In other words, for every object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, the canonical map $\lambda : g^* f_* \mathscr{F} \to f'_* g'^* \mathscr{F}$ is an equivalence in $\mathrm{QCoh}(\mathsf{Y}')$.

Proof. The assertion is local on $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ and $\mathsf{Y}' = (\mathcal{Y}', \mathscr{O}_{\mathcal{Y}'})$; we may therefore assume that $\mathsf{Y} = \mathrm{Spét}\, R$ and $\mathsf{Y}' = \mathrm{Spét}\, R'$ are affine. Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ and $\mathsf{X}' = (\mathcal{X}', \mathscr{O}_{\mathcal{X}'})$. Let $U \in \mathcal{X}$ be an object and $U' = g'^* U$ its pullback to X'. Define functors $\Gamma_U : \mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathcal{Y}}}$ and $\Gamma_{U'} : \mathrm{Mod}_{\mathscr{O}_{\mathcal{X}'}} \to \mathrm{Mod}_{\mathscr{O}_{\mathcal{Y}'}}$ as in the proof of Proposition 2.5.4.3. Let us say that $U \in \mathcal{X}$ is *good* if the canonical map $\lambda_U : g^* \Gamma_U(\mathscr{F}) \to \Gamma_{U'}(f'^* \mathscr{F})$ is an equivalence of $\mathscr{O}_{\mathcal{Y}'}$-modules. Since the construction $U \mapsto \lambda_U$ carries finite colimits to finite limits, the collection of good objects of $\mathcal{X}$ is closed under finite colimits. We wish to prove that the final object of $\mathcal{X}$ is good. Since $\mathcal{X}$ admits a scallop decomposition, it will suffice to show that every affine object of $\mathcal{X}$ is good (Corollary 2.5.3.6). We may therefore reduce to the case where $\mathcal{X}$ (and therefore also $\mathcal{X}'$) are affine, in which case the desired assertion is a special case of Lemma D.3.5.6. □

Corollary 2.5.4.6. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a quasi-affine map of nonconnective spectral Deligne-Mumford stacks. Then the pushforward functor f_* restricts to a colimit-preserving functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$. Moreover, for every pullback diagram*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}, \end{array}$$

the induced diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{Y}) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathsf{X}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{Y}') & \xrightarrow{f'^*} & \mathrm{QCoh}(\mathsf{X}') \end{array}$$

is right adjointable.

Proof. Combine Proposition 2.5.4.3, Proposition 2.5.4.5, and Example 2.5.4.2. □

2.5.5 Categorical Digression

If $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is an arbitrary morphism of spectrally ringed ∞-topoi, then the direct image functor $f_* : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ is lax symmetric monoidal. This is a consequence of the following general categorical observation:

Proposition 2.5.5.1. *Let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a symmetric monoidal functor between symmetric monoidal ∞-categories, and suppose that the underlying functor $f : \mathcal{C} \to \mathcal{D}$ admits a right adjoint g. Then g is lax symmetric monoidal: that is, it extends naturally to a map of ∞-operads $G : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$.*

Proof. Consider the diagram

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \xrightarrow{F} & \mathcal{D}^{\otimes} \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & \mathcal{F}\mathrm{in}_* . & \end{array}$$

For every object $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$, the induced map $\mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{D}^{\otimes}_{\langle n \rangle}$ can be identified with $f^n : \mathcal{C}^n \to \mathcal{D}^n$, and therefore admits a right adjoint $g^n : \mathcal{D}^n \to \mathcal{C}^n$. Since F carries p-coCartesian morphisms to q-coCartesian morphisms, Proposition HA.7.3.2.6 guarantees the existence of a functor $G : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ which is a right adjoint of F relative to $\mathcal{F}\mathrm{in}_*$. In particular, $G|_{\mathcal{D}}$ is a right adjoint to f and we may therefore assume that $G|_{\mathcal{D}} = g$. To see that G is a map of ∞-operads, it suffices to observe that for every injection $\langle m \rangle^{\circ} \hookrightarrow \langle n \rangle^{\circ}$, the diagram

$$\begin{array}{ccc} \mathcal{C}^n & \longrightarrow & \mathcal{D}^n \\ \downarrow & & \downarrow \\ \mathcal{C}^m & \longrightarrow & \mathcal{D}^m \end{array}$$

is right adjointable. □

Remark 2.5.5.2. In the situation of Proposition 2.5.5.1, F and G determine adjoint functors

$$\mathrm{CAlg}(\mathcal{C}) \rightleftarrows \mathrm{CAlg}(\mathcal{D}).$$

Corollary 2.5.5.3. *Let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a symmetric monoidal functor between symmetric monoidal ∞-categories, and suppose that the underlying functor $f : \mathcal{C} \to \mathcal{D}$ admits a right adjoint g; let $G : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ be the resulting map of ∞-operads. Then:*

(1) *If $\mathbf{1}$ denotes the unit object of $\mathcal{D}$, then $A = g(\mathbf{1})$ has the structure of a commutative algebra object of $\mathcal{C}$.*

(2) *The functor G factors as a composition*

$$\mathcal{D}^{\otimes} \simeq \mathrm{Mod}_{\mathbf{1}}(\mathcal{D})^{\otimes} \to \mathrm{Mod}_A(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes} .$$

Example 2.5.5.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectrally ringed ∞-topoi. Then the direct image functor $f_* : \mathrm{Mod}_{\mathscr{O}_\mathsf{X}} \to \mathrm{Mod}_{\mathscr{O}_\mathsf{Y}}$ is lax symmetric monoidal, and therefore factors as a composition

$$\mathrm{Mod}_{\mathscr{O}_\mathsf{X}} \to \mathrm{Mod}_{f_* \mathscr{O}_\mathsf{X}} \to \mathrm{Mod}_{\mathscr{O}_\mathsf{Y}}.$$

If X and Y are spectral Deligne-Mumford stacks and f is relatively scalloped, then the induced map $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ is lax symmetric monoidal, and canonically factors as a composition $\mathrm{QCoh}(\mathsf{X}) \to \mathrm{Mod}_{f_* \mathscr{O}_\mathsf{X}}(\mathrm{QCoh}(\mathsf{Y})) \to \mathrm{QCoh}(\mathsf{Y})$.

2.5.6 The Quasi-Affine Case

We now describe some consequences of Proposition 2.5.4.5 in the case where $f : \mathsf{X} \to \mathsf{Y}$ is a quasi-affine morphism of spectral Deligne-Mumford stacks.

Proposition 2.5.6.1. *Let $f : \mathsf{X} = (\mathcal{X}, \mathscr{O}_\mathcal{X}) \to \mathsf{Y}$ be a quasi-affine map of spectral Deligne-Mumford stacks. Then the induced functor $\mathrm{QCoh}(\mathsf{X}) \to \mathrm{Mod}_{f_* \mathscr{O}_\mathcal{X}}(\mathrm{QCoh}(\mathsf{Y}))$ is an equivalence of ∞-categories.*

Proof. The assertion is local on Y. We may therefore assume that $\mathsf{Y} = \mathrm{Spét}\, R$ is affine, so that X is quasi-affine and the desired result follows from Proposition 2.4.1.4. □

Proposition 2.5.6.2. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a quasi-affine morphism between spectral Deligne-Mumford stacks, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then $\mathscr{F}$ belongs to $\mathrm{QCoh}(\mathsf{X})_{\leqslant 0}$ if and only if the direct image $f_* \mathscr{F}$ belongs to $\mathrm{QCoh}(\mathsf{Y})_{\leqslant 0}$.*

Proof. Since the pullback functor f^* is right t-exact, the right adjoint f_* is left t-exact. Consequently, the "only if" direction is tautological (and does not require the assumption that f is quasi-affine). Conversely, suppose that $f_* \mathscr{F}$ belongs to $\mathrm{QCoh}(\mathsf{Y})_{\leqslant 0}$; we wish to show that $\mathscr{F}$ belongs to $\mathrm{QCoh}(\mathsf{X})_{\leqslant 0}$. By virtue of Proposition 2.5.4.5, we can work locally on Y and thereby reduce to the case where $\mathsf{Y} = \mathrm{Spét}\, A$ for some connective $\mathbb{E}_\infty$-ring A. Since f is quasi-affine, we can factor f as a composition

$$\mathsf{X} \xrightarrow{f'} \mathrm{Spét}\, B \xrightarrow{f''} \mathrm{Spét}\, A$$

for some connective $\mathbb{E}_\infty$-ring B, where f' is a quasi-compact open immersion. Since the functor f''_* is t-exact and conservative, it follows that $f'_* \mathscr{F}$ belongs to $\mathrm{QCoh}(\mathrm{Spét}\, B)_{\leqslant 0}$. Because f' is an open immersion, the diagram

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{\mathrm{id}} & \mathsf{X} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f'} \\ \mathsf{X} & \xrightarrow{f'} & \mathrm{Spét}\, B \end{array}$$

is a pullback square. Applying Proposition 2.5.4.5, we deduce that $\mathscr{F}$ can be identified with the pullback $f'^* f'_* \mathscr{F}$, and therefore belongs to $f'^* \operatorname{QCoh}(\operatorname{Spét} B)_{\leqslant 0} \subseteq \operatorname{QCoh}(\mathsf{X})_{\leqslant 0}$ as desired. □

Corollary 2.5.6.3. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a quasi-affine morphism between spectral Deligne-Mumford stacks. Then $\operatorname{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is the smallest full subcategory of $\operatorname{QCoh}(\mathsf{X})$ which is closed under colimits and extensions and contains the pullback $f^* \mathscr{F}$ for each $\mathscr{F} \in \operatorname{QCoh}(\mathsf{Y})^{\mathrm{cn}}$.*

Proof. By virtue of Proposition HA.1.4.4.11 , the ∞-category $\operatorname{QCoh}(\mathsf{X})$ admits a t-structure $(\operatorname{QCoh}'(\mathsf{X}), \operatorname{QCoh}''(\mathsf{X}))$, where $\operatorname{QCoh}'(\mathsf{X})$ is the smallest full subcategory of $\operatorname{QCoh}(\mathsf{X})$ which is closed under colimits and extensions and contains $f^* \operatorname{QCoh}(\mathsf{Y})^{\mathrm{cn}}$, and $\operatorname{QCoh}''(\mathsf{X})$ is the full subcategory of $\operatorname{QCoh}(\mathsf{X})$ spanned by those sheaves $\mathscr{G}$ having the property that the groups $\operatorname{Ext}^n_{\operatorname{QCoh}(\mathsf{X})}(f^* \mathscr{F}, \mathscr{G})$ vanish for $n < 0$ and every connective object $\mathscr{F} \in \operatorname{QCoh}(\mathsf{Y})$. Using the identification $\operatorname{Ext}^n_{\operatorname{QCoh}(\mathsf{X})}(f^* \mathscr{F}, \mathscr{G}) \simeq \operatorname{Ext}^n_{\operatorname{QCoh}(\mathsf{Y})}(\mathscr{F}, f_* \mathscr{G})$, we deduce that $\mathscr{G}$ belongs to $\operatorname{QCoh}''(\mathsf{X})$ if and only if $f_* \mathscr{G} \in \operatorname{QCoh}(\mathsf{Y})_{\leqslant 0}$. Applying Proposition 2.5.6.2, we deduce that $\operatorname{QCoh}''(\mathsf{X}) = \operatorname{QCoh}(\mathsf{X})_{\leqslant 0}$, so that the t-structure $(\operatorname{QCoh}'(\mathsf{X}), \operatorname{QCoh}''(\mathsf{X}))$ coincides with the t-structure of Proposition 2.2.5.2 and therefore $\operatorname{QCoh}'(\mathsf{X}) = \operatorname{QCoh}(\mathsf{X})^{\mathrm{cn}}$. □

Corollary 2.5.6.4. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a quasi-affine spectral Deligne-Mumford stack. Then $\operatorname{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is the smallest full subcategory of $\operatorname{QCoh}(\mathsf{X})$ which contains the unit object $\mathscr{O}_{\mathcal{X}}$ and is closed under colimits and extensions.*

Proof. Apply Corollary 2.5.6.3 in the special case where $Y = \operatorname{Spét} S$, where S denotes the sphere spectrum. □

2.5.7 Compositions of Quasi-Affine Morphisms

We now show that the collection of quasi-affine morphisms is closed under composition.

Lemma 2.5.7.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a quasi-affine morphism of spectral Deligne-Mumford stacks, and set $\mathscr{A} = \tau_{\geqslant 0} f_* \mathscr{O}_{\mathsf{X}}$. Then the canonical map $\mathsf{X} \to \operatorname{Spec}^{\acute{e}t}_{\mathsf{Y}} \mathscr{A}$ (see Construction 2.5.1.3) is a quasi-compact open immersion.*

Proof. The assertion is local on Y. We may therefore assume that Y is affine, in which case the desired result follows from Proposition 2.4.1.3 (and the proof of Proposition 2.4.2.3). □

Lemma 2.5.7.2. *Let $f : \mathsf{X} \to \mathsf{Y}$ be an affine morphism of spectral Deligne-Mumford stacks. If Y is quasi-affine, then X is quasi-affine.*

Proof. Set $A = \Gamma(\mathsf{Y}; \mathscr{O}_{\mathsf{Y}})$ and $B = \Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}})$, so that we have a commutative diagram of nonconnective spectral Deligne-Mumford stacks σ :

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{g} & \operatorname{Sp\acute{e}t} B \\ \downarrow^{f} & & \downarrow \\ \mathsf{Y} & \xrightarrow{g_0} & \operatorname{Sp\acute{e}t} A. \end{array}$$

Note that B can be identified with the image of $f_* \mathscr{O}_{\mathsf{X}}$ under the equivalence $\mathrm{QCoh}(\mathsf{Y}) \simeq \mathrm{Mod}_A$ of Proposition 2.5.6.1. Write $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}})$. For every affine object $U \in \mathcal{Y}$, we obtain an equivalence

$$\mathscr{O}_{\mathsf{X}}(f^* U) \simeq (f_* \mathscr{O}_{\mathsf{X}})(U) \simeq \mathscr{O}_{\mathsf{Y}}(U) \otimes_A B,$$

so that the outer rectangle in the associated diagram

$$\begin{array}{ccccc} \mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}_U & \longrightarrow & \mathsf{X} & \xrightarrow{g} & \operatorname{Sp\acute{e}t} B \\ \downarrow & & \downarrow^{f} & & \downarrow \\ \mathsf{Y}_U & \longrightarrow & \mathsf{Y} & \xrightarrow{g_0} & \operatorname{Sp\acute{e}t} A \end{array}$$

is a pullback square, where $\mathsf{Y}_U = (\mathcal{Y}_{/U}, \mathscr{O}_{\mathsf{Y}} |_U)$. Allowing U to vary, we deduce that σ is a pullback square. Since the map g_0 is a quasi-compact open immersion (Proposition 2.4.1.3), it follows that g is a quasi-compact open immersio, so that X is quasi-affine as desired. □

Proposition 2.5.7.3. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Suppose that Y is quasi-affine and that f is quasi-affine. Then X is quasi-affine.*

Proof. Set $\mathscr{A} = \tau_{\geqslant 0} f_* \mathscr{O}_{\mathsf{X}}$. Regard $\mathscr{A}$ as a a commutative algebra object of $\mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}$, and set $\mathsf{X}' = \operatorname{Sp\acute{e}t}_{\mathsf{Y}} \mathscr{A}$ (see Construction 2.5.1.3). Lemma 2.5.7.1 implies that the canonical map $\mathsf{X} \to \mathsf{X}'$ is a quasi-compact open immersion. It will therefore suffice to show that X' is quasi-affine, which follows from Lemma 2.5.7.2. □

Proposition 2.5.7.4. *Suppose we are given a commutative diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccccc} & & \mathsf{Y} & & \\ & \nearrow^{f} & & \searrow^{g} & \\ \mathsf{X} & & \xrightarrow{h} & & \mathsf{Z}, \end{array}$$

where g is quasi-affine. Then f is quasi-affine if and only if h is quasi-affine.

Proof. Suppose first that f is quasi-affine; we wish to show that h is quasi-affine. Equivalently, we wish to show that for every map $\operatorname{Sp\acute{e}t} R \to \mathsf{Z}$, the fiber product $\operatorname{Sp\acute{e}t} R \times_{\mathsf{Z}} \mathsf{X}$ is quasi-affine. Our hypothesis on g guarantees that $\operatorname{Sp\acute{e}t} R \times_{\mathsf{Z}} \mathsf{Y}$ is quasi-affine. The projection map $\operatorname{Sp\acute{e}t} R \times_{\mathsf{Z}} \mathsf{X} \to \operatorname{Sp\acute{e}t} R \times_{\mathsf{Z}} \mathsf{Y}$ is a pullback of f and is therefore quasi-affine, so that $\operatorname{Sp\acute{e}t} R \times_{\mathsf{Z}} \mathsf{X}$ is quasi-affine by virtue of Proposition 2.5.7.3.

Now suppose that h is quasi-affine; we wish to show that f has the same property. Choose a map $\eta : \operatorname{Sp\acute{e}t} R \to \mathsf{Y}$; we wish to show that the fiber product $\operatorname{Sp\acute{e}t} R \times_{\mathsf{Y}} \mathsf{X}$ is quasi-affine. Our hypotheses on g and h imply that the fiber products $\mathsf{X}_R = \operatorname{Sp\acute{e}t} R \times_{\mathsf{Z}} \mathsf{X}$ and $\mathsf{Y}_R = \operatorname{Sp\acute{e}t} R \times_{\mathsf{Z}} \mathsf{Y}$ are quasi-affine, and we have a pullback diagram

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} R \times_{\mathsf{Y}} \mathsf{X} & \longrightarrow & \operatorname{Sp\acute{e}t} R \\ \downarrow & & \downarrow \\ \mathsf{X}_R & \longrightarrow & \mathsf{Y}_R . \end{array}$$

Since Y_R is quasi-affine, the right vertical map is affine. It follows that the left vertical map is also affine, so that $\operatorname{Sp\acute{e}t} R \times_{\mathsf{Y}} \mathsf{X}$ is quasi-affine by virtue of Lemma 2.5.7.2. □

2.5.8 Pushforwards of Truncated Quasi-Coherent Sheaves

If we are willing to restrict our attention to truncated quasi-coherent sheaves on spectral Deligne-Mumford stacks, then we can verify the quasi-coherence of direct images under conditions much weaker than those of Proposition 2.5.4.3:

Theorem 2.5.8.1. *Let $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a map of spectral Deligne-Mumford stacks which is ∞-quasi-compact. Then the induced functor $f_* : \operatorname{Mod}_{\mathcal{O}_{\mathcal{Y}}} \to \operatorname{Mod}_{\mathcal{O}_{\mathcal{X}}}$ carries $\operatorname{QCoh}(\mathcal{Y})_{\leqslant 0}$ into $\operatorname{QCoh}(\mathcal{X})_{\leqslant 0}$.*

The proof of Theorem 2.5.8.1 will require the following somewhat technical definition:

Definition 2.5.8.2. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. We let $\operatorname{Mod}^{\heartsuit}_{\mathcal{O}_{\mathcal{X}}}$ denote the heart of the ∞-category $\operatorname{Mod}_{\mathcal{O}_{\mathcal{X}}}$: it can be identified with (the nerve of) the abelian category of sheaves of discrete modules over $\pi_0 \mathcal{O}_{\mathcal{X}}$. We will say that an object of $\operatorname{Mod}^{\heartsuit}_{\mathcal{O}_{\mathcal{X}}}$ is *quasi-coherent* if it belongs to $\operatorname{QCoh}(\mathsf{X})^{\heartsuit} = \operatorname{QCoh}(\mathsf{X}) \cap \operatorname{Mod}^{\heartsuit}_{\mathcal{O}_{\mathcal{X}}}$. We will say that an object $\mathscr{F} \in \operatorname{Mod}^{\heartsuit}_{\mathcal{O}_{\mathcal{X}}}$ is *semicoherent* if, for every affine object $U \in \mathcal{X}$, there exists a composition series

$$0 = \mathscr{F}_0 \hookrightarrow \mathscr{F}_1 \hookrightarrow \cdots \hookrightarrow \mathscr{F}_n = \mathscr{F}|_U$$

such that each quotient $\mathscr{F}_i / \mathscr{F}_{i-1}$ is a subobject of some quasi-coherent object $\mathscr{G}_i \in \operatorname{QCoh}(\mathsf{U})^{\heartsuit}$, where $\mathsf{U} = (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$.

Theorem 2.5.8.1 admits the following refinement:

Theorem 2.5.8.3. *Let $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be an n-quasi-compact morphism between spectral Deligne-Mumford stacks. Let $\mathscr{F} \in (\operatorname{Mod}_{\mathcal{O}_{\mathcal{Y}}})_{\leqslant 0}$ be sheaf of $\mathcal{O}_{\mathcal{Y}}$-modules satisfying the following conditions:*

(*a*) *For* $0 \leqslant i < n$, $\pi_{-i}\mathscr{F}$ *is quasi-coherent.*

(*b*) *The sheaf* $\pi_{-n}\mathscr{F}$ *is semicoherent.*

Then the direct image $f_*\mathscr{F}$ *also satisfies conditions* (*a*) *and* (*b*).

Proof of Theorem 2.5.8.1. Combine Theorem 2.5.8.3 with the quasi-coherence criterion of Proposition 2.2.6.1. □

The proof of Theorem 2.5.8.3 will require a few preliminaries. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. Since the t-structure on $\mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}}$ restricts to a t-structure on the full subcategory $\mathrm{QCoh}(\mathsf{X})$, we can identify $\mathrm{QCoh}(\mathsf{X})^{\heartsuit}$ with a full subcategory of the abelian category $\mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}}^{\heartsuit}$ which is closed under the formation of kernels, cokernels, and extensions. Our first goal is to extend these observations to semicoherent sheaves.

Lemma 2.5.8.4. *Let* $\mathsf{X} = (\mathcal{X}, \mathscr{O})$ *be a spectral Deligne-Mumford stack, and suppose we are given a morphism* $\alpha : \mathscr{F} \to \mathscr{G}$ *in the abelian category* $\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}$. *If* $\mathscr{F}$ *is quasi-coherent and* $\mathscr{G}$ *is semicoherent, then the kernel* $\ker(\alpha)$ *and the image* $\mathrm{im}(\alpha)$ *(formed in the abelian category* $\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}$*) are quasi-coherent.*

Proof. The assertion is local on $\mathcal{X}$; we may therefore assume that X is affine so that there exists a finite filtration

$$0 = \mathscr{G}_0 \hookrightarrow \mathscr{G}_1 \hookrightarrow \cdots \hookrightarrow \mathscr{G}_n = \mathscr{G}$$

such that each quotient $\mathscr{G}_i / \mathscr{G}_{i-1}$ is a subobject of a quasi-coherent object $\mathscr{H}_i \in \mathrm{QCoh}(\mathsf{X})^{\heartsuit}$. Let $\mathcal{K}_i$ denote the kernel of the composite map

$$\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \to \mathscr{G} / \mathscr{G}_i .$$

For each index i, α induces a monomorphism

$$\mathcal{K}_i / \mathcal{K}_{i-1} \hookrightarrow \mathscr{G}_i / \mathscr{G}_{i-1} \to \mathscr{H}_i .$$

Thus $\mathcal{K}_{i-1}$ can be identified with the kernel of a map $\mathcal{K}_i \to \mathscr{H}_i$. Note that $\mathcal{K}_n \simeq \mathscr{F}$ is quasi-coherent. It follows by descending induction on i that each $\mathcal{K}_i$ is quasi-coherent. In particular, $\mathcal{K}_0 = \ker(\alpha)$ is quasi-coherent. Using the exact sequence

$$0 \to \ker(\alpha) \to \mathscr{F} \to \mathrm{im}(\alpha) \to 0,$$

we see that $\mathrm{im}(\alpha)$ is quasi-coherent as well. □

Lemma 2.5.8.5. *Let* $\mathsf{X} = (\mathcal{X}, \mathscr{O})$ *be a spectral Deligne-Mumford stack, and suppose we are given an exact sequence*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

in the abelian category $\mathrm{Mod}_{\mathscr{O}}^{\heartsuit}$.

(a) *If* $\mathscr{F}'$ *and* $\mathscr{F}''$ *are psuedo-coherent, then* $\mathscr{F}$ *is semicoherent.*

(b) *If* $\mathscr{F}$ *is semicoherent, then* $\mathscr{F}'$ *is semicoherent.*

(c) *If* $\mathscr{F}'$ *is quasi-coherent and* $\mathscr{F}$ *is semicoherent, then* $\mathscr{F}''$ *is semicoherent.*

Proof. Assertion (a) follows immediately from the definitions. We next prove (b). Without loss of generality, we may assume that X is affine. Then $\mathscr{F}$ admits a finite filtration

$$0 = \mathscr{F}_0 \hookrightarrow \cdots \hookrightarrow \mathscr{F}_n = \mathscr{F}$$

and a collection of monomorphisms $\mathscr{F}_i / \mathscr{F}_{i-1} \hookrightarrow \mathscr{G}_i$, where $\mathscr{G}_i \in \mathrm{QCoh}(\mathsf{X})^{\heartsuit}$. Let us regard $\mathscr{F}_i$ and $\mathscr{F}'$ as subobjects of $\mathscr{F}$, and set $\mathscr{F}'_i = \mathscr{F}_i \cap \mathscr{F}'$. Then we have a filtration

$$0 = \mathscr{F}'_0 \hookrightarrow \cdots \hookrightarrow \mathscr{F}'_n = \mathscr{F}'$$

where each quotient $\mathscr{F}'_i / \mathscr{F}'_{i-1}$ is equivalent to a subobject of $\mathscr{F}_i / \mathscr{F}_{i-1}$, and therefore to a subobject of $\mathscr{G}_i$. This proves that $\mathscr{F}'$ is semicoherent.

It remains to prove (c). Again we may assume without loss of generality that X is affine, so that $\mathscr{F}$ and $\mathscr{F}'$ admit composition series as indicated above. We first prove by descending induction on i that each $\mathscr{F}'_i$ is quasi-coherent. The result is obvious for $i = n$, since $\mathscr{F}'_n \simeq \mathscr{F}'$. For the inductive step, we note that $\mathscr{F}'_i$ can be described as the kernel of a map $\mathscr{F}'_{i+1} \to \mathscr{F}_{i+1} / \mathscr{F}_i \hookrightarrow \mathscr{G}_i$, and is therefore quasi-coherent. It follows that each of the quotients $\mathscr{F}'_i / \mathscr{F}'_{i-1}$ is quasi-coherent. Form a short exact sequence

$$0 \to \mathscr{F}'_i / \mathscr{F}'_{i-1} \to \mathscr{G}_i \to \mathscr{H}_i \to 0,$$

so that each $\mathscr{H}_i$ is quasi-coherent. Let $\mathscr{F}''_i$ denote the image of $\mathscr{F}_i$ in $\mathscr{F}''$. Then we have a finite filtration

$$0 = \mathscr{F}''_0 \hookrightarrow \cdots \hookrightarrow \mathscr{F}''_n = \mathscr{F}''.$$

For each index i, the monomorphism $\mathscr{F}_i / \mathscr{F}_{i-1} \hookrightarrow \mathscr{G}_i$ induces a monomorphism $\mathscr{F}''_i / \mathscr{F}''_{i-1} \to \mathscr{H}_i$. It follows that $\mathscr{F}''$ is semicoherent, as desired. □

Lemma 2.5.8.6. *Let* $f : (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ *be an affine morphism between spectral Deligne-Mumford stacks. Let* $\mathscr{F} \in (\mathrm{Mod}_{\mathscr{O}_{\mathcal{Y}}})_{\leqslant 0}$ *be such that* $\pi_0 \mathscr{F}$ *is semicoherent. Then* $(f_* \mathscr{F}) \in (\mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}})_{\leqslant 0}$, *and* $\pi_0(f_* \mathscr{F})$ *is semicoherent.*

Proof. We first note that the pushforward functor f_* is left t-exact. Let $\mathscr{F} \in (\mathrm{Mod}_{\mathscr{O}_{\mathcal{Y}}})_{\leqslant 0}$ be such that $\pi_0 \mathscr{F}$ is semicoherent; we wish to prove that $\pi_0(f_* \mathscr{F})$ is semicoherent. Since f_* is left t-exact, the map $f_*(\tau_{\geqslant 0} \mathscr{F}) \to f_* \mathscr{F}$ induces an equivalence $\pi_0 f_*(\tau_{\geqslant 0} \mathscr{F}) \to \pi_0(f_* \mathscr{F})$. We may therefore replace $\mathscr{F}$ by $\tau_{\geqslant 0} \mathscr{F}$ and thereby reduce to the case $\mathscr{F} \in \mathrm{Mod}^{\heartsuit}_{\mathscr{O}_{\mathcal{Y}}}$. We may assume without loss of generality that $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is affine. Since f is affine, we deduce that

$(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ is affine, so that $\mathscr{F} \simeq \pi_0 \mathscr{F}$ admits a composition series $\mathscr{F} \simeq \pi_0 \mathscr{F}$ is semicoherent, we can choose a composition series

$$0 = \mathscr{F}_0 \hookrightarrow \cdots \hookrightarrow \mathscr{F}_n \simeq \mathscr{F}$$

where each quotient admits a monomorphism $\mathscr{F}_i / \mathscr{F}_{i-1} \hookrightarrow \mathscr{G}_i$ foe some quasi-coherent object $\mathscr{G}_i \in \mathrm{Mod}^{\heartsuit}_{\mathscr{O}_{\mathcal{Y}}}$. Since f_* is left t-exact, we get an induced filtration

$$0 = \pi_0 f_* \mathscr{F}_0 \hookrightarrow \cdots \hookrightarrow \pi_0 f_* \mathscr{F}_n = \pi_0 f_* \mathscr{F}$$

where each successive quotient $(\pi_0 f_* \mathscr{F}_i)/(\pi_0 f_* \mathscr{F}_{i-1})$ admits a monomorphism

$$(\pi_0 f_* \mathscr{F}_i)/(\pi_0 f_* \mathscr{F}_{i-1}) \hookrightarrow \pi_0 f_*(\mathscr{F}_i / \mathscr{F}_{i-1}) \hookrightarrow \pi_0 f_* \mathscr{G}_i .$$

It now suffices to observe that $\pi_0 f_* \mathscr{G}_i$ is a quasi-coherent $\mathscr{O}_{\mathcal{X}}$-module (Corollary 2.5.4.6). □

Proof of Theorem 2.5.8.3. Without loss of generality, we may assume that $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is affine. We proceed by induction on n. Then the ∞-topos $\mathcal{Y}$ is n-coherent, and in particular quasi-compact. We may therefore choose an effective epimorphism $u : U_0 \to \mathbf{1}$ in $\mathcal{Y}$, where $\mathbf{1}$ denotes the final object and $(\mathcal{Y}_{/U_0}, \mathscr{O}_{\mathcal{Y}} |U_0)$ is affine. Let $U_\bullet$ denote the Čech nerve of u. For each $k \geqslant 0$, let $f^k : (\mathcal{Y}_{/U_k}, \mathscr{O}_{\mathcal{Y}} |U_k) \to (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be the map induced by f, and let $\mathscr{G}^k = f^k_*(\mathscr{F} |U_k) \in (\mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}})_{\leqslant 0}$. We obtain a cosimplicial object $\mathscr{G}^\bullet$ of $(\mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}})_{\leqslant 0}$ whose totalization is equivalent to $f_* \mathscr{F}$. Applying Proposition HA.1.2.4.5 and Variant HA.1.2.4.9 , we deduce the existence of a spectral sequence $\{E_r^{p,q}, d_r\}_{r \geqslant 1}$ in the abelian category $\mathrm{Mod}^{\heartsuit}_{\mathscr{O}_{\mathcal{X}}}$ with the following properties:

- (i) We have $E_1^{p,q} \simeq \pi_{-q} \mathscr{G}^p$ for $p, q \geqslant 0$, and $E_1^{p,q} \simeq 0$ otherwise.
- (ii) The differentials d_r have bidegree $(r, 1-r)$: that is, they carry $E_r^{p,q}$ into $E_r^{p+r,q-r+1}$.
- (iii) The spectral sequence $\{E_r^{p,q}, d_r\}_{r \geqslant 1}$ converges to $\pi_{-p-q} f_* \mathscr{F}$ in the following sense: for every integer $k \geqslant 0$, there exists a finite filtration

$$0 = F^{-1}\pi_{-k}(f_* \mathscr{F}) \hookrightarrow F^0 \pi_{-k}(f_* \mathscr{F}) \hookrightarrow \cdots \hookrightarrow F^k \pi_{-k}(f_* \mathscr{F}) = \pi_{-k}(f_* \mathscr{F})$$

  in the abelian category $\mathrm{Mod}^{\heartsuit}_{\mathscr{O}_{\mathcal{X}}}$ such that each successive quotient $F^q \pi_0(f_* \mathscr{F})/F^{q-1}\pi_0(f_* \mathscr{F})$ is isomorphic to $E_r^{k-q,q}$ for $r \gg 0$.

Since $\mathcal{Y}$ is n-coherent, each of the objects $U_q \in \mathcal{Y}$ is $(n-1)$-coherent. Using the inductive hypothesis and (i), we deduce:

- (iv) The objects $E_1^{p,q}$ are quasi-coherent for $q < n-1$ and semicoherent for $q = n-1$.

When $p = 0$ we can do a bit better: since $(\mathcal{Y}_{/U_0}, \mathscr{O}_{\mathcal{Y}} |U_0)$ is affine, Lemma 2.5.8.6 gives:

(*v*) The objects $E_1^{0,q}$ are quasi-coherent for $q < n$ and semicoherent for $q = n$.

We now prove the following statement by induction on r:

(∗) The object $E_r^{p,q}$ is psuedo-coherent if $p+q = n$, and quasi-coherent if $p+q < n$.

In the case $r = 1$, assertion (∗) follows from (*iv*), (*v*), and (*i*). In the general case, we can describe $E_r^{p,q}$ as the cohomology of a cochain complex

$$E_{r-1}^{p-r,q+r-1} \xrightarrow{\alpha} E_{r-1}^{p,q} \xrightarrow{\beta} E_{r-1}^{p+r,q-r+1}.$$

so that we have an exact sequence

$$0 \to \mathrm{im}(\alpha) \to \ker(\beta) \to E_r^{p,q} \to 0.$$

If $p+q < n$, then $E_{r-1}^{p,q}$ and $E_{r-1}^{p-r,q+r-1}$ are quasi-coherent and $E_{r-1}^{p+r,q-r+1}$ is semicoherent (by the inductive hypothesis). It follows that $\mathrm{im}(\alpha)$ and $\ker(\beta)$ are quasi-coherent (Lemma 2.5.8.5), so that $E_r^{p,q}$ is quasi-coherent. If $p+q = n$, then the inductive hypothesis guarantees instead that $E_{r-1}^{p-r,q+r-1}$ is quasi-coherent and $E_r^{p,q}$ is semicoherent. Lemma 2.5.8.4 then guarantees that $\mathrm{im}(\alpha)$ is quasi-coherent and Lemma 2.5.8.5 guarantees that $\ker(\beta)$ is semicoherent, so that $E_r^{p,q}$ is semicoherent by Lemma 2.5.8.5.

Using (∗) and (3), we deduce that $\pi_{-k} f_* \mathscr{F}$ admits a finite filtration by objects of $\mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}}^{\heartsuit}$ which are quasi-coherent if $k < n$ and semicoherent if $k = n$. Since the classes of quasi-coherent and semicoherent objects of $\mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}}^{\heartsuit}$ are stable under extensions (Lemma 2.5.8.5), we conclude that $\pi_{-k} f_* \mathscr{F}$ is quasi-coherent for $k < n$ and semicoherent for $k = n$, as desired. □

2.5.9 Connectivity Hypotheses

We conclude this section with a few remarks about the behavior of the pushforward functor f_* in the case where $f : \mathsf{X} \to \mathsf{Y}$ is a "highly connected" affine morphism.

Proposition 2.5.9.1. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ and $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be spectral Deligne-Mumford stacks. Let $f : \mathsf{X} \to \mathsf{Y}$ be an affine morphism, let $n \geqslant 0$ be an integer, and suppose that the fiber of the map $\mathscr{O}_{\mathcal{Y}} \to f_* \mathscr{O}_{\mathcal{X}}$ is n-connective. Then:*

(1) *The pushforward functor $f_* : \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}_{\leqslant n} \to \mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}_{\leqslant n}$ is fully faithful.*

(2) *The pushforward functor $f_* : \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}_{\leqslant n-1} \to \mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}_{\leqslant n-1}$ is an equivalence of ∞-categories.*

Proof. The assertion is local on Y, so we may assume without loss of generality that Y is affine. Write $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, A$ for some connective $\mathbb{E}_\infty$-ring A. Since f is affine, we can assume

$\mathsf{X} = \operatorname{Sp\acute{e}t} B$ for some connective A-algebra B. Let $u : A \to B$ denote the underlying map of $\mathbb{E}_\infty$-rings, so that $\mathrm{fib}(u)$ is an n-connective spectrum. To prove (1), we must show that the forgetful functor $\phi_n : (\mathrm{Mod}_B^{\mathrm{cn}})_{\leqslant n} \to (\mathrm{Mod}_A^{\mathrm{cn}})_{\leqslant n}$ is fully faithful. We observe that ϕ_n has a left adjoint ψ_n, given by $M \mapsto \tau_{\leqslant n}(B \otimes_A M)$. We wish to show that the counit map $\psi_n \circ \phi_n \to \mathrm{id}$ is an equivalence. Unwinding the definitions, we must show that if $M \in (\mathrm{Mod}_B^{\mathrm{cn}})_{\leqslant n}$, then the canonical map $\theta : B \otimes_A M \to M$ exhibits M as an n-truncation of $B \otimes_A M$. Since M is n-truncated, this is equivalent to the requirement that $\mathrm{fib}(\theta)$ is $(n+1)$-connective. Let θ_0 be the multiplication map $B \otimes_A B \to B$, so that $\mathrm{fib}(\theta) \simeq \mathrm{fib}(\theta_0) \otimes_B M$. Since M is connective, it will suffice to show that $\mathrm{fib}(\theta_0)$ is $(n+1)$-connective. Note that θ_0 admits a section s, so we can identify $\mathrm{fib}(\theta_0)$ with the $\mathrm{cofib}(s) = B \otimes_A \mathrm{cofib}(u)$. We complete the proof of (1) by observing that $\mathrm{cofib}(u) = \Sigma \, \mathrm{fib}(u)$ is $(n+1)$-connective.

We now prove (2). Let ϕ_{n-1} and ψ_{n-1} be defined as above; we wish to show that the unit map $\mathrm{id} \to \phi_{n-1} \circ \psi_{n-1}$ is an equivalence. In other words, we wish to show that if $N \in (\mathrm{Mod}_A^{\mathrm{cn}})_{\leqslant n-1}$, then the canonical map $N \to B \otimes_A N$ induces an isomorphism $\pi_i N \to \pi_i(B \otimes_A N)$ for $i < n$. We have a long exact sequence

$$\pi_i(\mathrm{fib}(u) \otimes_A N) \to \pi_i N \to \pi_i(B \otimes_A N) \to \pi_{i-1}(\mathrm{fib}(u) \otimes_A N).$$

It therefore suffices to show that the homotopy groups $\pi_i(\mathrm{fib}(u) \otimes_A N)$ vanish for $i < n$. This is clear, since $\mathrm{fib}(u)$ is n-connective and N is connective. □

Corollary 2.5.9.2. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ and $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be spectral Deligne-Mumford stacks. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism which induces an equivalence of ∞-topoi $\mathsf{X} \simeq \mathsf{Y}$ and an equivalence of n-truncations $\tau_{\leqslant n} \mathscr{O}_{\mathcal{Y}} \simeq \tau_{\leqslant n} f_* \mathscr{O}_{\mathcal{X}}$. Then the pushforward functor f_* induces an equivalence of ∞-categories $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}_{\leqslant n} \simeq \mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}_{\leqslant n}$.*

Proof. Let $\tau_{\leqslant n} \mathsf{X} = (\mathcal{X}, \tau_{\leqslant n} \mathscr{O}_{\mathcal{X}})$ and define $\tau_{\leqslant n} \mathsf{Y}$ similarly, so that we have a commutative diagram

$$\begin{array}{ccc} \tau_{\leqslant n} \mathsf{X} & \longrightarrow & \tau_{\leqslant n} \mathsf{Y} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ \mathsf{X} & \longrightarrow & \mathsf{Y} \end{array}$$

where the upper horizontal map is an equivalence. It will therefore suffice to show that the vertical maps induce equivalences of ∞-categories

$$\phi_* : \mathrm{QCoh}(\tau_{\leqslant n} \mathsf{X})^{\mathrm{cn}}_{\leqslant n} \to \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}_{\leqslant n} \qquad \psi_* : \mathrm{QCoh}(\tau_{\leqslant n} \mathsf{Y})^{\mathrm{cn}}_{\leqslant n} \to \mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}_{\leqslant n}.$$

Both of these assertions follow immediately from Proposition 2.5.9.1. □

Corollary 2.5.9.3. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack. For each $n \geqslant 0$, set $\mathsf{X}_n = (\mathcal{X}, \tau_{\leqslant n} \mathscr{O}_{\mathsf{X}})$. Then the canonical maps $i_n : \mathsf{X}_n \to \mathsf{X}$ induce an equivalence $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \to \varprojlim \mathrm{QCoh}(\mathsf{X}_n)^{\mathrm{cn}}$.*

Proof. Combine Corollary 2.5.9.2 with Proposition 2.2.5.4. □

2.6 Classification of Quasi-Affine Spectral Deligne-Mumford Stacks

It follows from Corollary 2.4.2.2 that a quasi-affine spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ can be recovered from the $\mathbb{E}_\infty$-ring $\Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}})$ of global sections of its structure sheaf. Our goal in this section is to prove a more precise result of Bhatt and Halpern-Leistner (see [27]). First, we recall a bit of terminology.

Definition 2.6.0.1. Let R be an $\mathbb{E}_\infty$-ring. We will say that an object $A \in \mathrm{CAlg}_R$ is *idempotent* if the multiplication map $m : A \otimes_R A \to A$ is an equivalence.

Theorem 2.6.0.2. *Let* QAff *denote the full subcategory of* SpDM *spanned by the quasi-affine spectral Deligne-Mumford stacks. Then the construction* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto \Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}})$ *determines fully faithful functor* $\rho : \mathrm{QAff}^{\mathrm{op}} \to \mathrm{CAlg}$*. Moreover, the following conditions on an object* $A \in \mathrm{CAlg}$ *are equivalent:*

(1) *The* $\mathbb{E}_\infty$*-ring* A *belongs to the essential image of the functor* ρ.

(2) *The* $\mathbb{E}_\infty$*-ring* A *is* $(-n)$*-connective for* $n \gg 0$ *and* A *is a compact idempotent object of* $\mathrm{CAlg}_{\tau_{\geqslant 0} A}$.

(3) *The* $\mathbb{E}_\infty$*-ring* A *is* $(-n)$*-connective for* $n \gg 0$ *and there exists a connective* $\mathbb{E}_\infty$*-ring* R *and a morphism* $R \to A$ *which exhibits* A *as a compact idempotent object of* CAlg_R.

We will deduce Theorem 2.6.0.2 from the following more precise result, which we will prove at the end of this section.

Proposition 2.6.0.3. *Let* R *be a connective* $\mathbb{E}_\infty$*-ring and let* $\operatorname{Spec} R = (|\operatorname{Spec} R|, \mathcal{O})$ *be its Zariski spectrum (Example 1.1.1.2). Then the construction* $U \mapsto \mathcal{O}(U)$ *determines a fully faithful embedding from the set of quasi-compact open subsets of* $|\operatorname{Spec} A|$ *(ordered by reverse inclusion) to the full subcategory of* CAlg_R *spanned by those object which are compact, idempotent, and* $(-n)$*-connective for some* $n \gg 0$.

Proof of Theorem 2.6.0.2 from Proposition 2.6.0.3. If $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathsf{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ are spectral Deligne-Mumford stacks for which $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is quasi-affine, then the canonical map

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Y}, \mathsf{X}) \to \mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(\mathsf{Y}, \operatorname{Sp\acute{e}t} \Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}})) \simeq \mathrm{Map}_{\mathrm{CAlg}}(\Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}}), \Gamma(\mathcal{Y}; \mathcal{O}_{\mathcal{Y}})$$

is a homotopy equivalence by virtue of Corollary 2.4.2.2. This proves that the global sections functor $\rho : \mathrm{QAff}^{\mathrm{op}} \to \mathrm{CAlg}$ is fully faithful.

According to Proposition 2.4.2.3, a spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is quasi-affine if and only if there exists a quasi-compact open immersion $\mathsf{X} \hookrightarrow \operatorname{Sp\acute{e}t} R$, for some connective $\mathbb{E}_\infty$-ring R. Consequently, the equivalence of conditions (1) and (3) follows from Proposition 2.6.0.3. The implication (2) ⇒ (3) is obvious, and the implication (1) ⇒ (2) follows from Proposition 2.6.0.3 together with the observation that if X is quasi-affine, then the canonical map the canonical map $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, \tau_{\geqslant 0}\Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$ is an open immersion (see Proposition 2.4.1.3). □

2.6.1 The Case of a Noetherian Commutative Ring

We begin by discussing the classification of idempotent R-algebras in the case where R is a Noetherian commutative ring, following Hopkins and Neeman (see [98], [160], and [197]).

Notation 2.6.1.1. Let R be a connective $\mathbb{E}_\infty$-ring. For each point $x \in |\operatorname{Spec} R|$, we let $\kappa(x)$ denote the residue field of the commutative ring $\pi_0 R$ at the point x. If $A \in \mathrm{CAlg}_R$ is idempotent, we let $\mathrm{Supp}(A)$ denote the set $\{x \in |\operatorname{Spec} R| : A \otimes_R \kappa(x) \neq 0\}$. We will refer to $\mathrm{Supp}(A)$ as the *support* of A.

Theorem 2.6.1.2. *Let R be a commutative Noetherian ring. Then the construction $A \mapsto \mathrm{Supp}(A)$ induces an equivalence from the full subcategory $\mathrm{CAlg}_R^{\mathrm{Idem}} \subseteq \mathrm{CAlg}_R$ spanned by the idempotent objects to the collection of $U \subseteq |\operatorname{Spec} R|$ which are closed under generalization (meaning that $x \in \overline{\{y\}} \cap U$ implies $y \in U$), ordered by reverse inclusion.*

The proof of Theorem 2.6.1.2 will require some preliminaries.

Lemma 2.6.1.3. *Let R be a Noetherian commutative ring and let $M \in \mathrm{Mod}_R$. Assume that, for every residue field κ of R, the tensor product $\kappa \otimes_R M$ vanishes. Then $M \simeq 0$.*

Proof. We will prove that for every ideal $I \subseteq R$, the tensor product $R/I \otimes_R M$ vanishes. Since R is Noetherian, we may proceed by induction: to prove that $R/I \otimes_R M \simeq 0$, we may assume that $R/J \otimes_R M \simeq 0$ for every ideal $I' \subseteq R$ containing I. Replacing R by R/I and M by $R/I \otimes_R M$, we may assume that $R/I' \otimes_R M \simeq 0$ for every nonzero ideal $I' \subseteq R$, and we wish to prove that $M \simeq 0$.

Let $J \subseteq R$ be maximal among those ideals for which the tensor product $J \otimes_R M$ vanishes. We wish to prove that $J = R$. Assume otherwise. Then R/J has an associated prime $\mathfrak{p}$ of R, so that $\mathfrak{p}$ occurs as the annihilator of an element element $x \in R/J$. Let $J' \subseteq R$ denote the inverse image of the cyclic module $Rx \subseteq R/I$, so that J' is an ideal of R which properly contains J. We have a short exact sequence of discrete R-modules

$$0 \to J \to J' \to R/\mathfrak{p} \to 0,$$

hence a fiber sequence

$$J \otimes_R M \to J' \otimes_R M \to R/\mathfrak{p} \otimes_R M.$$

To obtain a contradiction, it will suffice to show that $J' \otimes_R M$ vanishes. Since $J \otimes_R M \simeq 0$, we are reduced to proving that $R/\mathfrak{p} \otimes_R M \simeq 0$. We may therefore replace R by $R/\mathfrak{p}$ and thereby reduce to the case where R is an integral domain.

Since R is an integral domain, we have a short exact sequence of discrete R-modules

$$0 \to R \xrightarrow{y} R \to R/Ry \to 0.$$

It follows that $R/Ry \otimes_R M$ is the cofiber of the map $\phi_y : M \to M$ given by multiplication by y. Since Ry is a nonzero ideal of R, the tensor product $R/Ry \otimes_R M$ vanishes. It follows that ϕ_y is invertible.

Let K denote the fraction field of R. Then K is flat as an R-module, so the canonical map $K \otimes_R \pi_* M \to \pi_*(K \otimes_R M)$ is an isomorphism. Since every nonzero element of R acts invertibly on $\pi_* R$, it follows that the map $\pi_* M \to \pi_*(K \otimes_R M)$ is an isomorphism. Then $M \simeq K \otimes_R M \simeq 0$, since K is a residue field of R. □

Corollary 2.6.1.4. *Let R be a Noetherian $\mathbb{E}_\infty$-ring and let M be an almost connective R-module. Suppose that for every residue field κ of $\pi_0 R$, the tensor product $\kappa \otimes_R M$ vanishes. Then $M \simeq 0$.*

Proof. Assume that $M \neq 0$. There exists some smallest integer n such that $\pi_n M$ is nonzero. Then $\pi_n(\pi_0 R \otimes_R M) \simeq \pi_n M$ is also nonzero. Invoking Lemma 2.6.1.3, we deduce that there exists a residue field κ of $\pi_0 R$ such that

$$\kappa \otimes_{\pi_0 R} (\pi_0 R \otimes_R M) \simeq \kappa \otimes_R M$$

is nonzero. □

Warning 2.6.1.5. In the situation of Corollary 2.6.1.4, the hypothesis that M is almost connective cannot be removed. For example, let K denote the complex K-theory spectrum and let K/p denote the cofiber of the map $K \to K$ given by multiplication by a prime number p. Then K/p can be regarded as a module over the sphere spectrum S, and this module has the property that $\kappa \otimes_S K/p = \kappa \otimes K/p \simeq 0$ for every field κ (in other words, spectrum K/p has vanishing homology with coefficients in any field).

Lemma 2.6.1.6. *Let κ be a field and let $A \in \mathrm{CAlg}_\kappa$ be an idempotent object. Then either $A \simeq 0$ or the unit map $e : \kappa \to A$ is an equivalence.*

Proof. Since A is idempotent, the canonical map $A \simeq A \otimes_\kappa \kappa \to A \otimes_\kappa A$ is an equivalence. Passing to cofibers, we deduce that $A \otimes_\kappa \mathrm{cofib}(e) \simeq 0$. If A is nonzero, then it contains κ as a direct summand (in the ∞-category Mod_κ) so that $\kappa \otimes_\kappa \mathrm{cofib}(e) \simeq \mathrm{cofib}(e)$ vanishes and therefore e is an equivalence. □

Lemma 2.6.1.7. *Let R be a Noetherian commutative ring and let $\phi : A \to B$ be a morphism in CAlg_R. If A and B are idempotent and $\mathrm{Supp}(A) = \mathrm{Supp}(B)$, then ϕ is an equivalence.*

Proof. By virtue of Lemma 2.6.1.3, it will suffice to show that ϕ induces an equivalence $A \otimes_R \kappa \to B \otimes_R \kappa$ for every residue field κ of R. This follows from Lemma 2.6.1.6 and our assumption that $\mathrm{Supp}(A) = \mathrm{Supp}(B)$. □

Proof of Theorem 2.6.1.2. We first note that if $A, B \in \mathrm{CAlg}_R$ and A is idempotent, then the mapping space $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B)$ is either empty or contractible: this follows from the observation that the diagonal map

$$\mathrm{Map}_{\mathrm{CAlg}_R}(A, B) \to \mathrm{Map}_{\mathrm{CAlg}_R}(A, B) \times \mathrm{Map}_{\mathrm{CAlg}_R}(A, B)) \simeq \mathrm{Map}_{\mathrm{CAlg}_R}(A \otimes_R A, B)$$

is a homotopy equivalence. It will therefore suffice to verify the following:

(a) For $A, B \in \mathrm{CAlg}_R^{\mathrm{Idem}}$, the mapping space $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B)$ is nonempty if and only if $\mathrm{Supp}(A) \supseteq \mathrm{Supp}(B)$.

(b) A subset $U \subseteq |\operatorname{Spec} R|$ has the form $\mathrm{Supp}(A)$ for some $A \in \mathrm{CAlg}_R^{\mathrm{Idem}}$ if and only if U is closed under generalization.

We first prove (a). Let A and B be idempotent $\mathbb{E}_\infty$-algebras over R. If there exists a morphism $\phi : A \to B$, then for every residue field κ of $\pi_0 A$ there is an induced map $A \otimes_R \kappa \to B \otimes_R \kappa$. Consequently, if $A \otimes_R \kappa \simeq 0$, then we also have $B \otimes_R \kappa$. This proves that $\mathrm{Supp}(A) \supseteq \mathrm{Supp}(B)$.

For the converse, suppose that $\mathrm{Supp}(A) \supseteq \mathrm{Supp}(B)$. Then $\mathrm{Supp}(A \otimes_R B) = \mathrm{Supp}(A) \cap \mathrm{Supp}(B) = \mathrm{Supp}(B)$. It follows from Lemma 2.6.1.7 that the natural map $B \to A \otimes_R B$ is an equivalence, so that there exists a morphism $A \to A \otimes_R B \simeq B$. This completes the proof of (a).

We now prove (b). Suppose first that $A \in \mathrm{CAlg}_R^{\mathrm{Idem}}$: we will show that the support $\mathrm{Supp}(A)$ is closed under generalization. Choose points $x, y \in |\operatorname{Spec} R|$ such that $x \in \overline{\{y\}} \cap \mathrm{Supp}(A)$; we wish to show that $y \in \mathrm{Supp}(A)$. Using Lemma **??**, we can choose a ring homomorphism $R \to R'$, where R' is a discrete valuation ring and the induced map $|\operatorname{Spec} R'| \to |\operatorname{Spec} R|$ carries the closed point of $|\operatorname{Spec} R'|$ to x and the generic point of $|\operatorname{Spec} R'|$ to y. Replacing A by $A \otimes_R R' \in \mathrm{CAlg}_{R'}$, we can reduce to the case where $R = R'$ is a discrete valuation ring. Let π be a uniformizer of R. If $\mathrm{Supp}(A)$ does not contain the generic point of $|\operatorname{Spec} R|$, then $A[\pi^{-1}] \simeq 0$ and therefore multiplication by π is locally nilpotent on $\pi_0 A$. Since $A \neq 0$, it follows that $\pi_0 A$ contains a nonzero element which is annihilated by π. Using the exactness of the sequence

$$\pi_1(A \otimes_R (R/\pi)) \to \pi_0 A \xrightarrow{\pi} \pi_0 A,$$

we conclude that $\pi_1(A \otimes_R (R/\pi)) \neq 0$. This contradicts Lemma 2.6.1.6, since $A \otimes_R (R/\pi)$ is an idempotent $\mathbb{E}_\infty$-algebra over the residue field R/π.

Now suppose that $U \subseteq |\operatorname{Spec} R|$ is stable under generalization; we wish to prove that $U = \operatorname{Supp}(A_U)$ for some $A_U \in \mathrm{CAlg}_R^{\mathrm{Idem}}$. We first consider the case where U is an open subset of $|\operatorname{Spec} R|$. Since R is Noetherian, the open set U is automatically quasi-compact and therefore determines a quasi-affine open substack $\mathsf{U} \subseteq \operatorname{Sp\acute{e}t} R$. Let A_U denote the $\mathbb{E}_\infty$-algebra of global sections of the structure sheaf of U. Note that the diagonal map $\mathsf{U} \to \mathsf{U} \times_{\operatorname{Sp\acute{e}t} R} \mathsf{U}$ is an equivalence. Using Proposition 2.5.4.5 (and Example 2.5.3.3), we deduce that A_U is an idempotent object of CAlg_R satisfying $\operatorname{Supp}(A_U) = U$.

We now treat the general case. Let U be a subset of $|\operatorname{Spec} R|$ which is stable under generalization. For every open set $V \subseteq |\operatorname{Spec} R|$ containing U, let $A_V \in \mathrm{CAlg}_R^{\mathrm{Idem}}$ be defined as above. Then the construction $V \mapsto A_V$ is functorial (this follows from assertion (a); alternatively, we can observe that A_V is the value of the structure sheaf of $\operatorname{Sp\acute{e}t} R$ on V). Then $A = \varinjlim_{U \subseteq V} A_V$ is a filtered colimit of idempotent objects of CAlg_R, hence idempotent. Unwinding the definitions, we deduce that $\operatorname{Supp}(A) = \bigcap_{U \subseteq V} \operatorname{Supp}(A_V) = \bigcap_{U \subseteq V} V$, where the intersection is taken over all open neighborhoods of U in $|\operatorname{Spec} R|$. Since U is closed under generalization, this intersection is equal to U. □

Warning 2.6.1.8. If R is a non-Noetherian commutative ring, then the support of an idempotent object $A \in \mathrm{CAlg}_R$ need not be stable under generalization. For example, if R is a valuation ring with value group $\mathbf{Q}$, then the residue field of R is an idempotent R-algebra which is supported at the closed point of $|\operatorname{Spec} R|$.

2.6.2 The Case of a Commutative Ring

If R is a non-Noetherian commutative ring, then it is not so easy to describe the idempotent objects of CAlg_R. However, there is still a simple classification of *compact* idempotent objects:

Proposition 2.6.2.1. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathscr{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(|\operatorname{Spec} R|)$ denote the structure sheaf of* $\operatorname{Spec} R$ *(regarded as a spectral scheme). Then the construction $U \mapsto \mathscr{O}(U)$ induces a fully faithful embedding from the category of quasi-compact open subsets of $|\operatorname{Spec} R|$ (ordered by reverse inclusion) to the full subcategory of* CAlg_R *spanned by the compact idempotent objects. If R is discrete, then this functor is an equivalence.*

Proof. Unwinding the definitions, we must verify the following:

- (a) For every quasi-compact open subset $U \subseteq |\operatorname{Spec} R|$, the algebra $\mathscr{O}(U) \in \mathrm{CAlg}_R$ is compact and idempotent.
- (b) For every pair of quasi-compact open subsets $U, V \subseteq |\operatorname{Spec} R|$, the mapping space $\mathrm{Map}_{\mathrm{CAlg}_R}(\mathscr{O}(U), \mathscr{O}(V))$ is contractible if $V \subseteq U$ and empty otherwise.

(c) If R is discrete, then every compact idempotent object $A \in \mathrm{CAlg}_R$ has the form $\mathscr{O}(U)$ for some quasi-compact open subset $U \subseteq |\operatorname{Spec} R|$.

We begin by verifying (a). Note that each open subset $U \subseteq |\operatorname{Spec} R|$ determines an open substack U of $\operatorname{Spét} R$, and we can identify $\mathscr{O}(U)$ with the $\mathbb{E}_\infty$-ring of global sections of the structure sheaf of U. If U is quasi-compact, then U is quasi-affine. Applying Proposition 2.5.4.5 and Example 2.5.3.3, we can identify $\mathscr{O}(U) \otimes_R \mathscr{O}(U)$ with the $\mathbb{E}_\infty$-ring of global sections of the structure sheaf of $\mathsf{U} \times_{\operatorname{Spét} R} \mathsf{U}$. Since the diagonal map $\mathsf{U} \to \mathsf{U} \times_{\operatorname{Spét} R} \mathsf{U}$ is an equivalence, it follows that $\mathscr{O}(U) \in \mathrm{CAlg}_R$ is idempotent.

We next show that $\mathscr{O}(U)$ is a compact object of CAlg_R. Since U is quasi-compact, it can be identified with the complement of the vanishing locus of a finitely generated ideal $I = (x_1, \ldots, x_n) \subseteq \pi_0 R$. For $1 \leqslant i \leqslant n$, let $\operatorname{cofib}(x_i) \in \mathrm{Mod}_R$ denote the cofiber of the map $x_i : R \to R$, and let K denote the tensor product $\operatorname{cofib}(x_1) \otimes_R \cdots \otimes_R \operatorname{cofib}(x_n)$. We will prove the following:

($*$) For any object $A \in \mathrm{CAlg}_R$, we have

$$\operatorname{Map}_{\mathrm{CAlg}_R}(\mathscr{O}(U), A) \simeq \begin{cases} \emptyset & \text{if } A \otimes_R K \neq 0 \\ * & \text{if } A \otimes_R K \simeq 0. \end{cases}$$

It follows from this description (and the fact that $K \in \mathrm{Mod}_R$ is perfect) that the functor $A \mapsto \operatorname{Map}_{\mathrm{CAlg}_R}(\mathscr{O}(U), A)$ commutes with filtered colimits.

We now prove ($*$). By construction, the image of K under the equivalence $\mathrm{Mod}_R \simeq \operatorname{QCoh}(\operatorname{Spét} R)$ is a quasi-coherent sheaf which vanishes on U, so the tensor product $\mathscr{O}(U) \otimes_R K$ vanishes. It follows that if $A \otimes_R K \neq 0$, then there do not exist any R-algebra maps $\mathscr{O}(U) \to A$. We wish prove the converse: if $A \otimes_R K = 0$, then the mapping space $\operatorname{Map}_{\mathrm{CAlg}_R}(\mathscr{O}(U), A)$ is nonempty (it is then automatically contractible, since $\mathscr{O}(U)$ is an idempotent object of CAlg_R). Note that we have canonical maps $\mathscr{O}(U) \to A \otimes_R \mathscr{O}(U) \xleftarrow{v} A$. Consequently, to show that there exists an R-algebra morphism from $\mathscr{O}(U)$ to A, it will suffice to show that v is an equivalence, or equivalently that $A \otimes_R \operatorname{cofib}(R \to \mathscr{O}(U)) \simeq 0$. This is a consequence of the following more general assertion:

($*'_m$) Let M be an R-module satisfying $M[x_i^{-1}] \simeq 0$ for $1 \leqslant i \leqslant m$. Then $A \otimes_R \operatorname{cofib}(x_{m+1}) \otimes_R \cdots \otimes_R \operatorname{cofib}(x_n) \otimes_R M \simeq 0$.

We prove ($*'_m$) using induction on m. The case $m = 0$ follows from our assumption that $A \otimes_R K \simeq 0$. To carry out the inductive step, suppose that $m > 0$ and that M satisfies $M[x_i^{-1}]$ for $1 \leqslant i \leqslant m$, and set $N = A \otimes_R \operatorname{cofib}(x_{m+1}) \otimes_R \cdots \otimes_R \operatorname{cofib}(x_n) \otimes_R M \simeq 0$. Our inductive hypothesis implies that $N \otimes_R \operatorname{cofib}(x_m) \simeq 0$: that is, the multiplication map $x_m : N \to N$ is an equivalence. It follows that the unit map $N \to N[x_m^{-1}]$ is an equivalence.

Since $M[x_m^{-1}]$ vanishes, we conclude that $N[x_m^{-1}] \simeq 0$, so that $N \simeq 0$ as desired. This completes the proof of (a).

We now prove (b). Because $\mathscr{O}$ is a functor, it is immediate that the mapping space $\mathrm{Map}_{\mathrm{CAlg}_R}(\mathscr{O}(U), \mathscr{O}(V))$ is nonempty whenever $V \subseteq U$ (in which case it is automatically contractible, since $\mathscr{O}(U)$ is idempotent). It will therefore suffice to show that if V is not contained in U, then the mapping space $\mathrm{Map}_{\mathrm{CAlg}_R}(\mathscr{O}(U), \mathscr{O}(V))$ is empty. Choose a point $x \in V$ which does not belong to U, and let $\kappa(x)$ denote the residue field of the commutative ring $\pi_0 R$ at x. Since the mapping space $\mathrm{Map}_{\mathrm{CAlg}_R}(\mathscr{O}(V), \kappa(x))$ is nonempty, it will suffice to show that $\mathrm{Map}_{\mathrm{CAlg}_R}(\mathscr{O}(U), \kappa(x))$ is empty. This follows immediately from $(*)$, since

$$\pi_0(\kappa(x) \otimes_R K) \simeq \kappa(x)/(x_1, \ldots, x_n) \simeq \kappa(x) \neq 0$$

by virtue of our assumption that $x \notin U$.

We now prove (c). Assume that R is discrete and let $A \in \mathrm{CAlg}_R$ be a compact object. Writing R as a direct limit of its finitely generated subrings, we can apply Theorem **??** to write $A = R \otimes_{R_0} A_0$, where $R_0 \subseteq R$ is a finitely generated subalgebra and A_0 is a compact object of CAlg_{R_0}. Let $m_0 : A_0 \otimes_{R_0} A_0 \to A_0$ be the multiplication map. If A is idempotent, then the morphism m_0 becomes an equivalence after applying the extension-of-scalars functor $\mathrm{CAlg}_{R_0} \to \mathrm{CAlg}_R$. Applying Theorem **??** again, we deduce that there is a finitely generated subring $R_1 \subseteq R$ containing R_0 such that the multiplication $A_1 \otimes_{R_1} A_1 \to A_1$ is an equivalence, where $A_1 = R_1 \otimes_{R_0} A_0$. Then A_1 is a compact idempotent object of CAlg_{R_1}. We may therefore replace the object $A \in \mathrm{CAlg}_R$ by $A_1 \in \mathrm{CAlg}_{R_1}$ and thereby reduce to the case where the commutative ring R is Noetherian. Let $U = \mathrm{Supp}(A)$. The proof of Theorem 2.6.1.2 shows that we can write $A = \varinjlim \mathscr{O}(V)$ where the colimit is taken over all open subsets $V \subseteq |\operatorname{Spec} R|$ which contain U. Since $A \in \mathrm{CAlg}_R$ is compact, it follows that A is a retract of some $\mathscr{O}(V)$ and is therefore equivalent to $\mathscr{O}(V)$ (since every morphism from $\mathscr{O}_V$ to itself is homotopic to the identity, every retract of $\mathscr{O}(V)$ is equivalent to $\mathscr{O}(V)$ itself). □

2.6.3 The General Case

It follows from Proposition 2.6.2.1 that if R is a commutative ring, then every compact idempotent object $A \in \mathrm{CAlg}_R$ arises from a quasi-compact open subset $U \subseteq |\operatorname{Spec} R|$. For more general (connective) $\mathbb{E}_\infty$-rings, this need not be true:

Example 2.6.3.1. Let $R = \mathrm{Sym}^*_{\mathbf{Q}}(\Sigma^2 \mathbf{Q})$ be the free $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$ on a single generator of degree 2, so that we have a canonical isomorphism $\pi_* R \simeq \mathbf{Q}[t]$ where $t \in \pi_2 R$. Then $|\operatorname{Spec} R|$ consists of a single point. However, the localization $R[t^{-1}]$ is a compact idempotent object of CAlg_R which is not equivalent to 0 or R.

Example 2.6.3.2. Let $R = S_{(p)}$ be the p-local sphere spectrum. Then $|\operatorname{Spec} R| \simeq |\operatorname{Spec} \mathbf{Z}_{(p)}|$ has exactly three open subsets, which determine compact idempotent objects

$0, \mathbf{Q}, S_{(p)} \in \mathrm{CAlg}_R$. However, there are many other compact idempotent objects of CAlg_R, given by *telescopic localizations* of the sphere spectrum (see, for example, [171]).

We can rule out the sort of "wild" idempotents appearing in Examples 2.6.3.1 and 2.6.3.2 by imposing connectivity hypotheses.

Proof of Proposition 2.6.0.3. Let R be a connective $\mathbb{E}_\infty$-ring and let A be a compact idempotent object of CAlg_R. By virtue of Proposition 2.6.2.1, it will suffice to prove that the following conditions on A are equivalent:

- (i) The $\mathbb{E}_\infty$-ring A is $(-n)$-connective for $n \gg 0$.
- (ii) There exists a quasi-compact open subset $U \subseteq |\operatorname{Spec} R|$ and an equivalence $A \simeq \mathscr{O}(U)$, where $\mathscr{O}$ denotes the structure sheaf of the spectral scheme $\operatorname{Spec} R$.

The implication $(ii) \Rightarrow (i)$ follows immediately from Proposition 2.5.4.4. We now prove the converse. Let A be a compact idempotent object of CAlg_R and set $A_0 = (\pi_0 R) \otimes_R A$, so that A_0 is a compact idempotent object of $\mathrm{CAlg}_{\pi_0 R}$. Let $\mathscr{O}_0$ denote the structure sheaf of the spectral scheme $\operatorname{Spec} \pi_0 R$. Applying Proposition 2.6.2.1, we deduce that $A_0 = \mathscr{O}_0(U)$ for some quasi-compact open subset $U \subseteq |\operatorname{Spec} \pi_0 R| \simeq |\operatorname{Spec} R|$. Set $A' = \mathscr{O}(U)$. We will complete the proof by showing that if A satisfies (i), then A and A' are equivalent objects of CAlg_R. More precisely, we will show that the canonical maps

$$A \xrightarrow{u} A \otimes_R A' \xleftarrow{v} A'$$

are both equivalences in CAlg_R. Using the implication $(ii) \Rightarrow (i)$, we see that A' also satisfies condition (i), so that the tensor product $A \otimes_R A'$ satisfies (i) as well. It follows that $\mathrm{cofib}(u)$ and $\mathrm{cofib}(v)$ are $(-n)$-connective for $n \gg 0$. Consequently, to show that $\mathrm{cofib}(u)$ and $\mathrm{cofib}(v)$ vanish, it will suffice to show that the tensor products $\mathrm{cofib}(u) \otimes_R (\pi_0 R)$ and $\mathrm{cofib}(v) \otimes_R (\pi_0 R)$ vanish. In other words, it suffices to show that u and v are equivalences after extension of scalars along the map $R \to \pi_0 R$, which is immediate from our construction. □

2.7 Finiteness Properties of Modules

Let A be an $\mathbb{E}_\infty$-ring and let M be an A-module. Recall that M is said to be *perfect* if it is a compact object of Mod_A (see §HA.7.2.4). If A is discrete, then we can identify Mod_A with the derived ∞-category of the abelian category $\mathrm{Mod}_A^\heartsuit$ of discrete A-modules (Remark HA.7.1.1.16). In this case, an object $K \in \mathrm{Mod}_A \simeq \mathcal{D}(\mathrm{Mod}_A^\heartsuit)$ is perfect if and only if it can be represented by a bounded chain complex of finitely generated projective A-modules. In this section, we will study some weaker finiteness conditions on A-modules, which correspond (in the case where A is discrete) to chain complexes which are required to be finitely generated and projective only in a certain range of degrees.

Definition 2.7.0.1. Let A be a connective $\mathbb{E}_1$-ring and let M be a left A-module. We will say that M is *perfect to order n* if, for every filtered diagram $\{N_\alpha\}$ in $(\mathrm{LMod}_A)_{\leqslant 0}$, the canonical map $\varinjlim_\alpha \mathrm{Ext}^i_A(M, N_\alpha) \to \mathrm{Ext}^i_A(M, \varinjlim N_\alpha)$ injective for $i = n$ and bijective for $i < n$.

Remark 2.7.0.2. If A is a connective $\mathbb{E}_1$-ring, then a left A-module M is almost perfect (see Definition HA.7.2.4.10) if and only if it is perfect to order n for every integer n.

Example 2.7.0.3. Let A be a connective $\mathbb{E}_1$-ring. If $M \in \mathrm{LMod}_A$ is $(n+1)$-connective, then it is perfect to order n.

The following reformulation of Definition 2.7.0.1 is often convenient:

Proposition 2.7.0.4. *Let A be a connective $\mathbb{E}_1$-ring, let M be a left A-module, and let n be an integer. The following conditions are equivalent:*

(1) *The left A-module M is perfect to order n, in the sense of Definition 2.7.0.1.*

(2) *For every filtered diagram $\{N_\alpha\}$ in LMod_A where each N_α is 0-truncated and each transition map $\pi_0 N_\alpha \to \pi_0 N_\beta$ is a monomorphism, the canonical map $\varinjlim_\alpha \mathrm{Map}_{\mathrm{LMod}_A}(M, \Sigma^n N_\alpha) \to \mathrm{Map}_{\mathrm{LMod}_A}(M, \varinjlim \Sigma^n N_\alpha)$ is a homotopy equivalence.*

(3) *The module M is almost connective. Moreover, if $\{N_\alpha\}$ is a filtered diagram in $\mathrm{LMod}_A^\heartsuit$ having the property that each transition map $N_\alpha \to N_\beta$ is a monomorphism, then the canonical map $\varinjlim_\alpha \mathrm{Map}_{\mathrm{LMod}_A}(M, \Sigma^n N_\alpha) \to \mathrm{Map}_{\mathrm{LMod}_A}(M, \varinjlim \Sigma^n N_\alpha)$ is a homotopy equivalence.*

Proof. Without loss of generality we may assume $n = 0$. We first show that (1) implies (2). Suppose first that M is perfect to order n, and let $\{N_\alpha\}$ be as in (2). Since each of the transition maps $\pi_0 N_\alpha \to \pi_0 N_\beta$ is an monomorphism, it follows that each of the maps $\pi_0 N_\alpha \to \pi_0 N$ is a monomorphism. For each index α, form a cofiber sequence $N_\alpha \to N \to K_\alpha$, so that K_α is 0-truncated. Applying our hypothesis that M is perfect to order 0 to the filtered diagram $\{K_\alpha\}$, we conclude that the canonical map

$$\varinjlim \mathrm{Ext}^i_A(M, K_\alpha) \to \mathrm{Ext}^i_A(M, \varinjlim K_\alpha) \simeq \mathrm{Ext}^i_A(M, 0) \simeq 0$$

is injective for $i = 0$ and bijective for $i < 0$: that is, the abelian groups $\varinjlim \mathrm{Ext}^i_A(M, K_\alpha)$ vanish for $i \leqslant 0$. It follows that the space $\varinjlim_\alpha \mathrm{Map}_{\mathrm{LMod}_A}(M, K_\alpha)$ is contractible. Using the evident fiber sequence

$$\varinjlim_\alpha \mathrm{Map}_{\mathrm{LMod}_A}(M, N_\alpha) \xrightarrow{\rho} \mathrm{Map}_{\mathrm{LMod}_A}(M, N) \to \varinjlim_\alpha \mathrm{Map}_{\mathrm{LMod}_A}(M, K_\alpha),$$

we deduce that ρ is a homotopy equivalence.

Note that if M satisfies condition (2), then the canonical map $\varinjlim_k \mathrm{Map}_{\mathrm{LMod}_A}(M, \tau_{\geqslant -k}(\tau_{\leqslant 0} M)) \to \mathrm{Map}_{\mathrm{LMod}_A}(M, \tau_{\leqslant 0} M)$ is a homotopy equivalence. In particular, the canonical map $M \to \tau_{\leqslant 0} M$ factors through some truncation $\tau_{\geqslant -k} \tau_{\leqslant 0} M$, which implies that M is $(-k)$-connective. This shows that (2) ⇒ (3).

We now complete the proof by showing that (3) ⇒ (1). Assume that M satisfies condition (3), and let $\{N_\alpha\}$ be a filtered diagram of 0-truncated A-modules having some colimit N. We wish to show that the canonical map $\varinjlim \mathrm{Ext}^i_A(M, N_\alpha) \to \mathrm{Ext}^i_A(M, N)$ is bijective for $i < 0$ and injective for $i = 0$. Equivalently, we wish to show that the canonical map $\theta : \varinjlim_\alpha \mathrm{Map}_{\mathrm{LMod}_A}(M, N_\alpha) \to \mathrm{Map}_{\mathrm{LMod}_A}(M, N)$ has (-1)-truncated homotopy fibers. Choose an integer $k \gg 0$ for which M is $(-k)$-connective. We can then replace $\{N_\alpha\}$ by $\{\tau_{\geqslant -k} N_\alpha\}$, and thereby reduce to the case where each N_α is $(-m)$-connective for some fixed $m \geqslant 0$. We now proceed by induction on m. Assume that $m > 0$. Set $N'_\alpha = \tau_{\geqslant 1-m} N_\alpha$ and $N''_\alpha = \tau_{\leqslant -m} N_\alpha$, so that we have a commutative diagram of fiber sequences

$$\begin{array}{ccc}
\varinjlim_\alpha \mathrm{Map}_{\mathrm{LMod}_A}(M, N'_\alpha) & \xrightarrow{\theta'} & \mathrm{Map}_{\mathrm{LMod}_A}(M, N') \\
\downarrow & & \downarrow \\
\varinjlim_\alpha \mathrm{Map}_{\mathrm{LMod}_A}(M, N_\alpha) & \xrightarrow{\theta} & \mathrm{Map}_{\mathrm{LMod}_A}(M, N) \\
\downarrow & & \downarrow \\
\varinjlim_\alpha \mathrm{Map}_{\mathrm{LMod}_A}(M, N''_\alpha) & \xrightarrow{\theta''} & \mathrm{Map}_{\mathrm{LMod}_A}(M, N'')
\end{array}$$

where $N' = \varinjlim N'_\alpha$ and $N'' = \varinjlim N''_\alpha$. Since the map θ' has (-1)-truncated homotopy fibers by our inductive hypothesis, we are reduced to proving that the map θ'' has (-1)-truncated homotopy fibers. The map θ'' is obtained by applying the functor Ω^m to the canonical map $\varinjlim_\alpha \mathrm{Map}_{\mathrm{LMod}_A}(M, \pi_{-m} N_\alpha) \to \mathrm{Map}_{\mathrm{LMod}_A}(M, \pi_{-m} N)$. Replacing $\{N_\alpha\}$ by $\{\pi_{-m} N_\alpha\}$, we may reduce to the case $m = 0$.

Let us regard each N_α as a discrete left module over the associative ring $\pi_0 A$. Without loss of generality, we may assume that the diagram $\{N_\alpha\}$ is indexed by a filtered partially ordered set I. For $\alpha \leqslant \beta$ in I, let $K_{\alpha,\beta}$ denote the kernel of the transition map $N_\alpha \to N_\beta$. For fixed α, we can regard $\{K_{\alpha,\beta}\}_{\beta \geqslant \alpha}$ as a filtered diagram of discrete A-modules whose transition maps are monomorphisms, whose colimit $K_\alpha = \varinjlim_{\beta \geqslant \alpha} K_{\alpha,\beta}$ can be identified with the kernel of the map $N_\alpha \to N$. Applying assumption (3), we deduce that the canonical map $\varinjlim_{\beta \geqslant \alpha} \mathrm{Map}_{\mathrm{LMod}_A}(M, K_{\alpha,\beta}) \to \mathrm{Map}_{\mathrm{LMod}_A}(M, K_\alpha)$ is a homotopy equivalence. Set $N'_\alpha = N_\alpha / K_\alpha \simeq \mathrm{im}(N_\alpha \to N)$, so that we have a fiber sequence of spaces

$$\varinjlim_{\beta \geqslant \alpha} \mathrm{Map}_{\mathrm{LMod}_A}(M, K_{\alpha,\beta}) \to \mathrm{Map}_{\mathrm{LMod}_A}(M, N_\alpha) \to \mathrm{Map}_{\mathrm{LMod}_A}(M, N'_\alpha).$$

Applying assumption (3) to the diagram $\{N'_\alpha\}_{\alpha \in I}$, we deduce that the canonical map $\varinjlim_\alpha \mathrm{Map}_{\mathrm{LMod}_A}(M, N'_\alpha) \to \mathrm{Map}_{\mathrm{LMod}_A}(M, N)$ is a homotopy equivalence. It follows that the map θ fits into a fiber sequence

$$\varinjlim_{\alpha \in I} \varinjlim_{\beta \geqslant \alpha} \mathrm{Map}_{\mathrm{LMod}_A}(M, K_{\alpha,\beta}) \to \varinjlim_{\alpha} \mathrm{Map}_{\mathrm{LMod}_A}(M, N_\alpha) \xrightarrow{\theta} \mathrm{Map}_{\mathrm{LMod}_A}(M, N).$$

The first term is contractible, since the transition map $K_{\alpha,\beta} \to K_{\alpha',\beta'}$ vanishes for $\alpha' \geqslant \beta$. It follows that θ is (-1)-truncated, as desired. □

Remark 2.7.0.5. Let A be a connective $\mathbb{E}_1$-ring, let M be a left A-module, and let n be an integer. Then:

(a) If $\tau_{\leqslant n} M$ is a compact object of $\tau_{\leqslant n} \mathrm{LMod}_A$, then M is perfect to order n.

(b) If M is perfect to order n, then $\tau_{\leqslant n-1} M$ is a compact object of $\tau_{\leqslant n-1} \mathrm{LMod}_A$.

Remark 2.7.0.6. Let A be a connective $\mathbb{E}_1$-ring, and let M be a left A-module which is perfect to order n for some integer n. Then $M \in (\mathrm{LMod}_A)_{\geqslant -m}$ for some m. This follows immediately from Remark 2.7.0.5.

Remark 2.7.0.7. Let A be a connective $\mathbb{E}_1$-ring and suppose we are given a fiber sequence of left A-modules $M' \to M \to M''$. If M' is perfect to order n, then M is perfect to order n if and only if M'' is perfect to order n. This follows immediately from an inspection of the associated long exact sequence of Ext-groups.

Remark 2.7.0.8. Suppose we are given a finite collection of connective $\mathbb{E}_1$-rings $\{A_i\}_{1 \leqslant i \leqslant n}$ having product A. Let M be a left A-module, so that $M \simeq \prod M_i$ where each M_i is a left module over A_i. Then M is perfect to order n if and only if each M_i is perfect to order n.

2.7.1 Finitely n-Presented Modules

We now consider a slight variant of Definition 2.7.0.1.

Definition 2.7.1.1. Let A be a connective $\mathbb{E}_1$-ring, let M be a left A-module, and let n be an integer. We will say that M is *finitely n-presented* if M is n-truncated and perfect to order $(n+1)$.

Remark 2.7.1.2. Let A be a connective $\mathbb{E}_1$-ring, and suppose we are given a map of left A-modules $f : M \to M'$ such that the induced map $\pi_i M \to \pi_i M'$ is surjective when $i = n$ and bijective for $i < n$. Let $N \in (\mathrm{LMod}_A)_{\leqslant 0}$. Then the induced map $\mathrm{Ext}^i_A(M', N) \to \mathrm{Ext}^i_A(M, N)$ is injective for $i = n$ and bijective for $i < n$. It follows that if M is perfect to order n, so is M'.

Remark 2.7.1.3. Let A be a connective $\mathbb{E}_1$-ring. If M is a left A-module which is perfect to order $n+1$, then $\tau_{\leqslant n} M$ is also perfect to order $n+1$ (this is a special case of Remark 2.7.1.2), and is therefore finitely n-presented.

Remark 2.7.1.4. Let A be a connective $\mathbb{E}_1$-ring and let M be a compact object of $(\mathrm{LMod}_A)_{\leqslant n}$. Since LMod_A is compactly generated, we deduce that M is a retract of $\tau_{\leqslant n} M'$ for some compact object $M' \in \mathrm{LMod}_A$. It follows from Remark 2.7.1.3 that M is finitely n-presented. Conversely, if M is finitely n-presented, then it is a compact object of $(\mathrm{LMod}_A)_{\leqslant n}$ by virtue of Remark 2.7.0.5.

Proposition 2.7.1.5. *Let A be discrete $\mathbb{E}_1$-ring and let M be a left A-module. The following conditions are equivalent:*

(1) *The left A-module M is finitely 0-presented and, for every discrete right A-module N, the abelian group $\pi_1(N \otimes_A M)$ vanishes.*

(2) *The left A-module M is perfect and of* Tor*-amplitude $\leqslant 0$.*

Proof. We will show that (1) implies (2); the reverse implication is obvious. It follows from Remark **??** that the module M is $(-n)$-connective for $n \gg 0$. We proceed by induction on n. If $n = 0$, then M is a discrete A-module. Condition (1) then implies that the group $\mathrm{Tor}_1^A(N, M) \simeq 0$ for every discrete A-module N, so that M is flat over A. Since it is also finitely presented as a left A-module, it is a projective module of finite rank, hence perfect and of Tor-amplitude $\leqslant 0$.

We now carry out the inductive step. Suppose that M is $(-n)$-connective for $n > 0$. It follows from Remark 2.7.0.5 that $\pi_{-n} M$ is a finitely generated module over $\pi_0 A$. We can therefore choose a map $\alpha : \Sigma^{-n} A^k \to M$ which is surjective on π_{-n}, so that $\mathrm{cofib}(\alpha)$ is $(1-n)$-connective. Remark 2.7.0.7 implies that $\mathrm{cofib}(\alpha)$ is perfect to order 1, and for $m > 0$ the exact sequence $\pi_m M \to \pi_m \mathrm{cofib}(\alpha) \to \pi_{m-1} \Sigma^{-n} A^k$ shows that $\mathrm{cofib}(\alpha)$ is 0-truncated. If N is a discrete right A-module, the existence of a short exact sequence

$$\pi_1(N \otimes_A M) \to \pi_1(N \otimes_A \mathrm{cofib}(\alpha)) \to \pi_0(N \otimes_A \Sigma^{-n} A)$$

implies that $\pi_1(N \otimes_A \mathrm{cofib}(\alpha)) \simeq 0$. Applying our inductive hypothesis, we deduce that $\mathrm{cofib}(\alpha)$ is perfect of Tor-amplitude $\leqslant 0$. Using the fiber sequence $\Sigma^{-n} A^k \to M \to \mathrm{cofib}(\alpha)$, we deduce that M is also perfect of Tor-amplitude $\leqslant 0$. □

2.7.2 Alternate Characterizations

Our next result gives a formulation of Definition 2.7.0.1 which is well-adapted to making inductive arguments:

Proposition 2.7.2.1. *Let A be a connective $\mathbb{E}_1$-ring and let M be a connective left A-module. Then:*

(1) *The module M is perfect to order 0 if and only if $\pi_0 M$ is finitely generated as a module over $\pi_0 A$.*

(2) *Let $n > 0$ and suppose we are given a map of A-modules $\phi : A^k \to M$ which induces a surjection $\pi_0 A^k \to \pi_0 M$. Then M is perfect to order n if and only if* $\mathrm{fib}(\phi)$ *is perfect to order $(n-1)$.*

Proof. We first prove (1). For each $N \in (\mathrm{LMod}_A)_{\leqslant 0}$, we have $\mathrm{Ext}^i_A(M, N) \simeq 0$ for $i < 0$, and $\mathrm{Ext}^0_A(M, N)$ is the abelian group of $\pi_0 A$-module homomorphisms from $\pi_0 M$ into $\pi_0 N$. Consequently, M is perfect to order 0 if and only if, for every filtered diagram of discrete $\pi_0 A$-modules N_α having colimit N, the canonical map

$$\varinjlim_{\alpha} \mathrm{Ext}^0_{\pi_0 A}(\pi_0 M, N_\alpha) \to \mathrm{Ext}^0_{\pi_0 A}(\pi_0 M, N)$$

is injective. If $\pi_0 M$ is finitely generated as a $\pi_0 A$-module, then we can choose a surjection $(\pi_0 A)^k \to \pi_0 M$, in which case the domain and codomain of θ can be identified with subgroups of the abelian group N^k; this proves the "if" direction of (1). For the converse, suppose that M is perfect to order 0. Let $\{N_\alpha\}$ be the (filtered) diagram of all quotients of the form $(\pi_0 M)/S$, where S is a finitely generated submodule of $\pi_0 M$. Then $\varinjlim N_\alpha \simeq 0$. It follows that $\varinjlim_\alpha \mathrm{Ext}^0_{\pi_0 A}(\pi_0 M, N_\alpha) \simeq 0$, so that the canonical epimorphism $\pi_0 M \to N_\alpha$ is the zero map for some index α. This implies that $\pi_0 M$ is finitely generated.

We now prove (2). Choose a fiber sequence of connective A-modules

$$M' \to A^k \to M,$$

and suppose we are given a filtered diagram $\{N_\alpha\}$ in $(\mathrm{LMod}_A)_{\leqslant 0}$ having a colimit N. For every pair of object $X, Y \in \mathrm{LMod}_A$, let $\mathrm{Mor}(X, Y)$ denote the spectrum of maps from X to Y in LMod_A, so that $\mathrm{Ext}^i_A(X, Y) = \pi_{-i} \mathrm{Mor}(X, Y)$. Let $F(X)$ denote the fiber of the canonical map $\varinjlim_\alpha \mathrm{Mor}(X, N_\alpha) \to \mathrm{Mor}(X, N)$. Note that $F(A) \simeq 0$. We have a fiber sequence of spectra

$$F(M') \to F(A^k) \to F(M),$$

so that $F(M)$ can be identified with the suspension of $F(M')$. In particular, $\pi_i F(M) \simeq 0$ for $i \geqslant n$ if and only if $\pi_i F(M') \simeq 0$ for $i \geqslant n-1$, from which (2) follows. □

Corollary 2.7.2.2. *Let A be a connective $\mathbb{E}_1$-ring and let M be a left A-module. The following conditions are equivalent:*

(1) *The left module M is perfect to order n.*

(2) *There exists a perfect left A-module P of* Tor*-amplitude $\leqslant n$ and a morphism $P \to M$ whose fiber is n-connective.*

(3) *There exists a perfect left A-module P and a morphism $P \to M$ whose fiber is n-connective.*

Proof. The implication (2) ⇒ (3) is obvious. We next show that (3) ⇒ (1). Let $\alpha : P \to M$ be a morphism of left A-modules, where P is perfect and $\mathrm{fib}(\alpha)$ is n-connective. Then $\mathrm{cofib}(\alpha) \simeq \Sigma\,\mathrm{fib}(\alpha)$ is perfect to order n (Example 2.7.0.3). Applying Remark 2.7.0.7 to the fiber sequence $P \to M \to \mathrm{cofib}(\alpha)$, we deduce that M is perfect to order n.

Note that if (1) is satisfied Remark D.7.7.4 implies that M is k-connective for some $k \ll 0$. To prove that (1) ⇒ (2), we may (after replacing M by $\Sigma^{-k}M$ and n by $n-k$) reduce to the case where M is connective. We now proceed by induction on n. If $n < 0$, there is nothing to prove (by Example 2.7.0.3). Let us therefore suppose that $n \geqslant 0$. Using Proposition 2.7.2.1, we can choose a finitely generated free A-module F and a fiber sequence

$$M' \xrightarrow{\alpha} F \to M$$

where M' is connective. Applying Remark 2.7.0.7 to the shifted sequence

$$F \to M \to \Sigma M',$$

we deduce that $\Sigma M'$ is perfect to order n, so that M' is perfect to order $(n-1)$. By virtue of the inductive hypothesis, there exists a perfect A-module P of Tor-amplitude $\leqslant n-1$ and a map $\beta : P \to M$ with $(n-1)$-connective fiber. Set $Q = \mathrm{cofib}(\beta \circ \alpha)$. Then Q is a perfect left A-module of Tor-amplitude $\leqslant n$ equipped with a map $\gamma : Q \to M$. We have a commutative diagram of fiber sequences

$$\begin{array}{ccccc} \mathrm{fib}(\beta) & \longrightarrow & P & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{fib}(\gamma) & \longrightarrow & Q & \longrightarrow & M \end{array}$$

which supplies an equivalence $\mathrm{fib}(\gamma) \simeq \Sigma\,\mathrm{fib}(\beta)$, so that $\mathrm{fib}(\gamma)$ is n-connective. □

Corollary 2.7.2.3. *Let A be a connective $\mathbb{E}_1$-ring and let M be a left A-module. Assume that A is left Noetherian (that is, each homotopy group $\pi_n A$ is Noetherian when viewed as a left module over $\pi_0 A$). Then M is perfect to order n if and only if it satisfies the following conditions:*

(1) *The homotopy groups $\pi_i M$ vanish for $i \ll 0$.*

(2) *For each $m \leqslant n$, the homotopy group $\pi_m M$ is finitely generated as a module over $\pi_0 A$.*

Proof. The necessity of condition (1) follows from Remark **??**. We may therefore assume without loss of generality that (1) is satisfied. Replacing M by $\Sigma^p M$ and n by $n+p$ and thereby reduce to the case where M is connective. We now proceed by induction on n. If $n < 0$ there is nothing to prove and if $n = 0$ the desired result follows from Proposition 2.7.2.1. Let us therefore assume that $n > 0$. Note that if M is perfect to order n, then it is perfect to order 0 and therefore $\pi_0 M$ is finitely generated as a module over $\pi_0 A$. We may therefore assume without loss of generality that there exists a fiber sequence

$$M' \to A^m \to M$$

of connective left A-modules. For each $i \geqslant 0$, we have a short exact sequence

$$(\pi_{i+1} A)^m \to \pi_{i+1} M \to \pi_i M' \to (\pi_i A)^m.$$

Since $\pi_0 A$ is a left Noetherian ring and the modules $\pi_i A$ and $\pi_{i+1} A$ are finitely generated, it follows that $\pi_{i+1} A$ is finitely generated over $\pi_0 A$ if and only if $\pi_i M'$ is finitely generated over $\pi_0 A$. Allowing i to vary, we deduce that M satisfies condition (2) if and only if the homotopy groups $\pi_i M'$ are finitely generated for $i < n$. By virtue of the inductive hypothesis, this is equivalent to the requirement that M' is perfect to order $(n-1)$. The desired result now follows from Proposition 2.7.2.1. □

Corollary 2.7.2.4. *Let A be a connective $\mathbb{E}_1$-ring, let M be a connective left module over A, and let $n \geqslant 0$ be an integer. The following conditions are equivalent:*

- (a) *The module M is perfect to order n.*
- (b) *There exists a simplicial object $M_\bullet$ of the ∞-category $\mathrm{LMod}_A^{\mathrm{cn}}$ such that $|M_\bullet| \simeq M$ and each of the modules M_k is finitely generated and free for $k \leqslant n$.*
- (c) *There exists a simplicial object $M_\bullet$ of the ∞-category $\mathrm{LMod}_A^{\mathrm{cn}}$ such that $|M_\bullet| \simeq M$ and, for each $0 \leqslant k \leqslant n$, the module M_k is perfect to order $n-k$.*

Proof. Suppose first that (a) is satisfied. Using Proposition 2.7.2.1, we can choose a free A-module of finite rank P_0 and a map $\phi_0 : P_0 \to M$ whose fiber is 0-connective. We will extend this to a diagram

$$P_0 \to P_1 \to P_2 \to \cdots$$

in $(\mathrm{Mod}_A)_{/M}$, where each P_m is perfect and each of the structure morphisms $\phi_m : P_m \to M$ has m-connective fiber. The construction proceeds by induction. Assume that $\phi_m : P_m \to M$ has been constructed for some $m \geqslant 0$. If $m < n$, then the cofiber $\mathrm{cofib}(\phi_m)$ is $(m+1)$-connective and perfect to order $(m+1)$, so we can choose a free left A-module F of finite rank and a morphism $\Sigma^{m+1} F \to \mathrm{cofib}(\phi_m)$ which is surjective on π_{m+1}. We then define P_{m+1} to be the fiber product $\Sigma^{m+1} F \times_{\mathrm{cofib}(\phi_m)} M$, so that we have a canonical fiber sequence

$$P_m \to P_{m+1} \to \Sigma^{m+1} F$$

which immediately shows that P_{m+1} is perfect. If $m \geqslant n$, we simply set $P_{m+1} = M$.

Applying Theorem HA.1.2.4.1 , we see that there exists a simplicial object $M_\bullet$ of LMod_A for which the diagram

$$P_0 \to P_1 \to P_2 \to \cdots \to P_n \to M \to M \to M \to \cdots$$

agrees with the skeletal filtration of $|M_\bullet|$. In particular, we have $|M_\bullet| \simeq M$, and each M_k can be identified with a direct sum of finitely many copies of modules of the form P_0 and $\Sigma^{-m}\,\mathrm{cofib}(P_{m-1} \to P_m)$ for $m \leqslant k$. It follows from the above construction that M_k is a free left A-module of finite rank for $k \leqslant n$, so that condition (b) is satisfied.

The implication $(b) \Rightarrow (c)$ is obvious. We will complete the proof by showing that (c) implies (a). Let $M_\bullet$ be a simplicial object of $\mathrm{LMod}_A^{\mathrm{cn}}$ satisfying the requirements of (c), and let

$$P_0 \to P_1 \to P_2 \to P_3 \to \cdots$$

be the associated filtered object of LMod_A (see Theorem HA.1.2.4.1). We then have a fiber sequence

$$P_n \to M \to K$$

where K is $(n+1)$-connective. It follows from Example 2.7.0.3 that K is perfect to order n. Consequently, to prove that M is perfect to order n, it will suffice to show that P_n is perfect to order n (Remark 2.7.0.7). We will prove more generally that P_m is perfect to order n for each $m \leqslant n$. The proof proceeds by induction on m. In the case $m = 0$, we simply note that $P_0 \simeq M_0$ is perfect to order n by assumption. To carry out the inductive step, we note that if $m > 0$ then we have a fiber sequence

$$P_{m-1} \to P_m \to \Sigma^m N$$

where N denotes the mth term in the normalized chain complex associated to the simplicial object $M_\bullet$ of the homotopy category hLMod_A. If $m \leqslant n$, then N is perfect to order $(n-m)$ (since it is a direct summand of M_n) and therefore the suspension $\Sigma^m N$ is perfect to order n. Since P_{m-1} is also perfect to order n by our inductive hypothesis, it follows from Remark 2.7.0.7 that P_m is also perfect to order n. □

2.7.3 Extension of Scalars

We now study the behavior of finiteness conditions on modules with respect to base change.

Proposition 2.7.3.1. *Let $f : A \to B$ be a map of connective $\mathbb{E}_1$-rings and let M be a left A-module. If M is perfect to order n as an A-module, then $B \otimes_A M$ is perfect to order n as a B-module. The converse holds if B is faithfully flat (as a right module) over A.*

Proof. Assume first that M is perfect to order n as an A-module. We wish to show that $B \otimes_A M$ is perfect to order n as a B-module. Remark **??** implies that $M \in (\mathrm{LMod}_A)_{\geqslant -m}$ for some $m \gg 0$. Replacing M by $\Sigma^m M$ (and n by $n+m$) we may assume that M is connective and that $n \geqslant 0$. We proceed by induction on n. If $n = 0$, then we are reduced to proving that $\pi_0(B \otimes_A M) \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M)$ is finitely generated as a module over $\pi_0 B$, which follows because $\pi_0 M$ is finitely generated over $\pi_0 A$. If $n > 0$, then we can choose a fiber sequence of connective left A-modules

$$M' \to A^k \to M.$$

Tensoring with B, we obtain a fiber sequence of connective left B-modules

$$B \otimes_A M' \to B^k \to B \otimes_A M$$

Using Proposition 2.7.2.1, we deduce that M' is perfect to order $(n-1)$ as an A-module. The inductive hypothesis implies that $B \otimes_A M'$ is perfect to order $(n-1)$ as a B-module. Using Proposition 2.7.2.1 again, we deduce that $B \otimes_A M$ is perfect to order n as a B-module.

We now prove the converse. Assume that f is faithfully flat and that $B \otimes_A M$ is perfect to order n as a B-module. Then there exists an integer m such that $\pi_i(B \otimes_A M) \simeq \mathrm{Tor}_i^{\pi_0 A}(\pi_0 B, \pi_i M)$ vanishes for $i < -m$. Since $\pi_0 B$ is faithfully flat over $\pi_0 A$, we deduce that $\pi_i M \simeq 0$ for $i < -m$. Replacing M by $\Sigma^m(M)$ and n by $n+m$, we may assume that M is connective and that $n \geqslant 0$. We prove that M is perfect to order n using induction on n. We first treat the case $n = 0$. We must show that $\pi_0 M$ is finitely generated as a module over $\pi_0 A$. Our assumption that $B \otimes_A M$ is perfect to order 0 guarantees that $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M)$ is finitely generated as a module over $\pi_0 B$. We may therefore choose a finitely generated submodule $M_0 \subseteq \pi_0 M$ such that the induced map $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, M_0) \to \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M)$ is surjective, so that $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, (\pi_0 M)/M_0)) \simeq 0$. Since $\pi_0 B$ is faithfully flat over $\pi_0 A$, we deduce that $(\pi_0 M)/M_0 \simeq 0$. It follows that $\pi_0 M \simeq M_0$ is finitely generated.

Now suppose that $n > 0$. The argument above shows that $\pi_0 M$ is finitely generated, so we can choose a fiber sequence of connective left A-modules

$$M' \to A^k \to M.$$

Tensoring with B, we obtain a fiber sequence of connective left B-modules

$$B \otimes_A M' \to B^k \to B \otimes_A M.$$

Since $B \otimes_A M$ is perfect to order n, Proposition 2.7.2.1 implies that $B \otimes_A M'$ is perfect to order $n-1$. It follows from the inductive hypothesis that M' is perfect to order $n-1$, so that M is perfect to order n by Proposition 2.7.2.1. □

Proposition 2.7.3.1 admits the following converse:

Proposition 2.7.3.2. *Let $f : A \to A'$ be a map of connective $\mathbb{E}_1$-rings. Assume that the underlying map of associative rings $\pi_0 A \to \pi_0 A'$ is a surjection whose kernel is a nilpotent ideal $I \subseteq \pi_0 A$. Let M be a connective left A-module, and set $M' = A' \otimes_A M$. Then:*

- (a) *The A-module M is perfect to order n over A if and only if the A'-module M' is perfect to order n.*
- (b) *The A-module M is almost perfect if and only if the A'-module M' is almost perfect.*
- (c) *The A-module M has* Tor*-amplitude $\leqslant k$ if and only if the A'-module M' has* Tor*-amplitude $\leqslant k$.*
- (d) *The A-module M is perfect if and only if the A'-module M' is perfect.*
- (e) *The A-module M is n-connective if and only if the A'-module M' is n-connective.*

Proof. The "only if" directions of assertions (a) through $(de$ are clear. To complete the proof of (a), suppose that M' is perfect to order n as a left A'-module. We will prove that M is perfect to order n as an A-module. The proof proceeds by induction on n. If $n < 0$, there is nothing to prove (since M is assumed to be connective). If $n = 0$, we must show that $\pi_0 M$ is finitely generated as a module over $\pi_0 A$. Our assumption that M' is perfect to order 0 guarantees that $\pi_0 M' \simeq (\pi_0 M)/I(\pi_0 M)$ is finitely generated as a module over $\pi_0 A' \simeq (\pi_0 A)/I$. The desired result now follows from our assumption that I is nilpotent.

Assume now that $n > 0$ and that M' is perfect to order n over A. The argument above shows that $\pi_0 M$ is finitely generated as a module over $\pi_0 A$, so we can choose a fiber sequence

$$N \to A^k \to M$$

where N is connective. Applying Proposition 2.7.2.1, we deduce that $A' \otimes_A N$ is perfect to order $(n-1)$ over A'. The inductive hypothesis now shows that N is perfect to order $(n-1)$ over A, so that M is perfect to order n over A (by Proposition 2.7.2.1 again). This completes the proof of (a).

Assertion (b) follows immediately from (a). We now prove (c). Suppose that M' has Tor-amplitude $\leqslant k$ over A', and let N be a discrete right A-module. Then N can be written as a finite extension of (discrete) modules over the form $I^k N / I^{k+1} N$, each of which can be regarded as a module over A'. Our assumption that M' has Tor-amplitude $\leqslant k$ guarantees that each tensor product $(I^k N/I^{k+1} N) \otimes_A M \simeq (I^k N/I^{+1} N) \otimes_{A'} M'$ is k-truncated. Since the collection of k-truncated spectra is closed under extensions, we conclude that $N \otimes_A M$ is k-truncated. Allowing N to vary, we conclude that M has Tor-amplitude $\leqslant k$ over A.

Assertion (d) follows from (b) and (c), by virtue of Proposition HA.7.2.4.23 . To prove (e), we proceed by induction on n. The case $n \leqslant 0$ is trivial (since M is assumed to be connective). To carry out the inductive step, assume that M' is $(n+1)$-connective for some

$n \geqslant 0$; we will prove that M is also $(n+1)$-connective. Our inductive hypothesis guarantees that M is n-connective. It follows that we have an isomorphism $\pi_n M' \simeq (\pi_n M)/I(\pi_n M)$. The $(n+1)$-connectivity of M' then guarantees that $\pi_n M = I(\pi_n M)$. Since I is a nilpotent ideal, it follows that $\pi_n M \simeq 0$, so that M is $(n+1)$-connective as desired. □

Proposition 2.7.3.3. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_1$-rings and let $n \geqslant 0$ be an integer. The following conditions are equivalent:*

(1) *The morphism f exhibits B as a left A-module which is perfect to order n.*

(2) *Let M be a connective left B-module. If M is perfect to order n as a left B-module, then it is also perfect to order n as a left A-module.*

Proof. The implication (2) ⇒ (1) is immediate (take $M = B$). For the converse, we proceed by induction on n. Assume that (1) is satisfied and let M be a connective left B-module which is perfect to order n over B. If $n = 0$, then $\pi_0 B$ is finitely generated as a left module over $\pi_0 B$, and therefore also as a left module over $\pi_0 A$. Consequently, M is perfect to order 0 over A (Proposition 2.7.2.1). Let us therefore assume that $n > 0$. Choose a map $\alpha : B^r \to M$ which is surjective on π_0. Then $\operatorname{fib}(\alpha)$ is connective and perfect to order $(n-1)$ over B, so our inductive hypothesis guarantees that $\operatorname{fib}(\alpha)$ is perfect to order $(n-1)$ over A. We then have a fiber sequence $B^r \to M \to \Sigma \operatorname{fib}(\alpha)$ where both B^r and $\Sigma \operatorname{fib}(\alpha)$ are perfect to order n over A, so that M is also perfect to order n over A. □

2.7.4 Fiberwise Connectivity Criterion

Let A be a commutative ring and let M be a finitely generated (discrete) A-module. It follows from Nakayama's lemma that if the fiber of M vanishes at some point of $x \in |\operatorname{Spec} A|$, then the module M itself vanishes in some neighborhood of x. We now formulate a "derived" analogue of this observation:

Proposition 2.7.4.1. *Let A be a connective $\mathbb{E}_\infty$-ring, let M be an A-module which is perfect to order n for some integer $n \geqslant 0$, and let κ be the residue field of A at some prime ideal $\mathfrak{p} \subseteq A$. The following conditions are equivalent:*

(a) *There exists an element $a \in \pi_0 A$ which is not contained in $\mathfrak{p}$ for which the localization $M[a^{-1}]$ is $(n+1)$-connective.*

(b) *The tensor product $\kappa \otimes_A M$ is $(n+1)$-connective.*

Proof. The implication (a) ⇒ (b) is trivial. Suppose that (b) is satisfied. For each $k \leqslant n$, we will prove that there exists an element $a \in (\pi_0 A) - \mathfrak{p}$ such that the localization $M[a^{-1}]$ is $(k+1)$-connective. Note that this condition is automatically satisfied for $k \ll 0$ by virtue of Remark **??**. To carry out the inductive step, let us assume that $k \leqslant n$ and that there exists $b \in (\pi_0 A) - \mathfrak{p}$ such that $M[b^{-1}]$ is k-connective. Replacing A by $A[b^{-1}]$ and M by $M[b^{-1}]$, we may assume that M itself is k-connective. It follows from Proposition 2.7.2.1 that $\pi_k M$ is a finitely generated module over $\pi_0 A$. Assumption (b) guarantees that

$$\mathrm{Tor}_0^{\pi_0 A}(\kappa, \pi_k M) \simeq \pi_k(\kappa \otimes_A M) \simeq 0.$$

Applying Nakayama's lemma, we deduce that $\pi_k M$ is annihilated by some element $a \in (\pi_0 A) - \mathfrak{p}$, so that $M[a^{-1}]$ is $(k+1)$-connective. □

Corollary 2.7.4.2. *Let A be a connective $\mathbb{E}_\infty$-ring and let M be an A-module which is perfect to order n for some integer $n \geqslant 0$. The following conditions are equivalent:*

- (a) *The module M is $(n+1)$-connective.*
- (b) *For every residue field κ of A, the tensor product $\kappa \otimes_A M$ is $(n+1)$-connective.*

Corollary 2.7.4.3. *Let A be a connective $\mathbb{E}_\infty$-ring, let M be an A-module which is almost perfect, and let n be an integer. Then M is n-connective if and only if, for every residue field κ of A, the tensor product $\kappa \otimes_A M$ is n-connective.*

Corollary 2.7.4.4. *Let A be a connective $\mathbb{E}_\infty$-ring and let M be an A-module which is almost perfect. Then $M \simeq 0$ if and only if, for every residue field κ of A, the tensor product $\kappa \otimes_A M$ vanishes.*

2.8 Local Properties of Quasi-Coherent Sheaves

Let X be a spectral Deligne-Mumford stack. In §2.2, we introduced the ∞-category $\mathrm{QCoh}(\mathsf{X})$ of *quasi-coherent* sheaves on X. In this section, we will study some examples properties of quasi-coherent sheaves on X which can be tested *locally* on X.

2.8.1 Étale-Local Properties of Spectral Deligne-Mumford Stacks

We begin with some general remarks about local conditions on spectral Deligne-Mumford stacks.

Definition 2.8.1.1. Let Y be a nonconnective spectral Deligne-Mumford stack. We will say that a collection of morphisms $\{f_\alpha : \mathsf{X}_\alpha \to \mathsf{Y}\}$ is *jointly surjective* if the induced map $\amalg_\alpha \mathsf{X}_\alpha \to \mathsf{Y}$ is surjective (see Definition 3.5.5.5).

Definition 2.8.1.2. Let P be a property of nonconnective spectral Deligne-Mumford stacks. We will say that P is *local for the étale topology* if the following conditions hold:

- (i) For every étale morphism of nonconnective spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$, if Y has the property P, then X also has the property P.

(*ii*) Given a jointly surjective collection of étale morphisms $\{\mathsf{X}_\alpha \to \mathsf{Y}\}$, if each X_α has the property P, then Y has the property P.

Remark 2.8.1.3. Let P be a property of nonconnective spectral Deligne-Mumford stacks which is local for the étale topology. A nonconnective spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ has the property P if and only if, for every affine $U \in \mathcal{X}$, the affine nonconnective spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X}|_U)$ has the property P. Consequently, P is determined by the full subcategory $\mathrm{CAlg}(P) \subseteq \mathrm{CAlg}$ spanned by those $\mathbb{E}_\infty$-rings A such that $\mathrm{Sp\acute{e}t}\, A$ has the property P. Moreover, the full subcategory $\mathrm{CAlg}(P)$ has the following properties:

(*i*) If $f : A \to A'$ is an étale morphism of $\mathbb{E}_\infty$-rings and $A \in \mathrm{CAlg}(P)$, then $A' \in \mathrm{CAlg}(P)$.

(*ii*) Given a finite collection of étale maps $\{A \to A_\alpha\}$ such that $A \to \prod_\alpha A_\alpha$ is faithfully flat, if each $A_\alpha \in \mathrm{CAlg}(P)$, then $A \in \mathrm{CAlg}(P)$.

Conversely, given a full subcategory $\mathrm{CAlg}(P) \subseteq \mathrm{CAlg}$, we obtain a property P of nonconnective spectral Deligne-Mumford stacks as follows: a pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ has the property P if and only if, whenever we have an equivalence $(\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X}|_U) \simeq \mathrm{Sp\acute{e}t}\, A$, the $\mathbb{E}_\infty$-ring A belongs to $\mathrm{CAlg}(P)$. If $\mathrm{CAlg}(P)$ satisfies conditions (*i*) and (*ii*), then the property P is local for the étale topology.

Recall that an $\mathbb{E}_\infty$-ring A is said to be *Noetherian* if A is connective, $\pi_0 A$ is a Noetherian commutative ring, and $\pi_n A$ is a finitely generated module over $\pi_0 A$ for every integer n.

Definition 2.8.1.4. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a spectral Deligne-Mumford stack. We will say that X is *locally Noetherian* if, whenever $U \in \mathcal{X}$ is affine so that $(\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X}|_U) \simeq \mathrm{Sp\acute{e}t}\, A$, the $\mathbb{E}_\infty$-ring A is Noetherian.

Proposition 2.8.1.5. *The property of being a locally Noetherian spectral Deligne-Mumford stack is local for the étale topology.*

Lemma 2.8.1.6. *Let $f : A \to B$ be a faithfully flat map of $\mathbb{E}_\infty$-rings. If B is Noetherian, then A is Noetherian.*

Proof. We first show that $\pi_0 A$ is a Noetherian commutative ring. We claim that the collection of ideals in $\pi_0 A$ satisfies the ascending chain condition. To prove this, it will suffice to show that the construction $I \mapsto I(\pi_0 B)$ determines an injection from the partially ordered set of ideals of $\pi_0 A$ to the partially ordered set of ideals of $\pi_0 B$. Since $\pi_0 B$ is flat over $\pi_0 A$, the map $I \otimes_{\pi_0 A} \pi_0 B$ is an injection with image $I(\pi_0 B)$. Given a pair of ideals $I, J \subseteq \pi_0 A$ we have an exact sequence of $\pi_0 A$-modules

$$0 \to I \cap J \to I \oplus J \to I + J \to 0.$$

This sequence remains exact after tensoring with $\pi_0 B$, so that $(I \cap J)(\pi_0 B) = I(\pi_0 B) \cap J(\pi_0 B)$. It follows that if $I(\pi_0 B) = J(\pi_0 B)$, then the inclusion $(I \cap J)(\pi_0 B) \hookrightarrow I(\pi_0 B)$ is bijective, so that $I/(I \cap J) \otimes_{\pi_0 A} \pi_0 B \simeq 0$. Since $\pi_0 B$ is faithfully flat over $\pi_0 A$, this implies that $I/(I \cap J) = 0$, so that $I \subseteq J$. A similar argument shows that $J \subseteq I$, so that $I = J$. This completes the proof that $\pi_0 A$ is a Noetherian commutative ring.

Since f is faithfully flat, we have $\pi_n B \simeq \pi_n A \otimes_{\pi_0 A} \pi_0 B$. Since $\pi_n B \simeq 0$ for $n < 0$, the faithful flatness of $\pi_0 B$ over $\pi_0 A$ implies that $\pi_n A \simeq 0$. This proves that A is connective. To complete the proof, we must show that each $\pi_n A$ is finitely generated as a module over $\pi_0 A$. Since $\pi_n B$ is finitely generated as a $\pi_0 B$-module, we can choose a finitely generated submodule $M \subseteq \pi_n A$ such that the map

$$M \otimes_{\pi_0 A} \pi_0 B \to \pi_n A \otimes_{\pi_0 A} \pi_0 B \simeq \pi_n B$$

is surjective. The cokernel of this map is given by $(\pi_n A)/M \otimes_{\pi_0 A} \pi_n B$. Since $\pi_0 B$ is faithfully flat over $\pi_0 A$, we deduce that $(\pi_n A)/M \simeq 0$, so that $\pi_n A \simeq M$ is finitely generated as a module over $\pi_0 A$. □

Proof of Proposition 2.8.1.5. By virtue of Remark 2.8.1.3, it will suffice to prove the following assertions:

- (i) If $f : A \to A'$ is an étale morphism of $\mathbb{E}_\infty$-rings and A is Noetherian, then A' is also Noetherian.
- (ii) Given a finite collection of étale maps $\{A \to A_\alpha\}$ such that $A \to \prod_\alpha A_\alpha$ is faithfully flat, if each A_α is Noetherian, then A is Noetherian.

Assertion (i) is obvious, and assertion (ii) follows immediately from Lemma 2.8.1.6. □

We now turn our attention to properties of morphisms of spectral Deligne-Mumford stacks.

Definition 2.8.1.7. Let P be a property of morphisms between nonconnective spectral Deligne-Mumford stacks. We will say that P is *local on the source with respect to the étale topology.* if the following conditions hold:

- (i) For every composable pair of morphisms $\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z}$. if f is étale and g has the property P, then $g \circ f$ has the property P.
- (ii) Given a jointly surjective collection of étale maps $\{f_\alpha : \mathsf{X}_\alpha \to \mathsf{Y}\}$ and a morphism $g : \mathsf{Y} \to \mathsf{Z}$, if each of the composite maps $g \circ f_\alpha$ has the property P, then g has the property P.

Example 2.8.1.8. Let P be the property of being an étale morphism between nonconnective spectral Deligne-Mumford stacks. Then P is local on the source with respect to the étale topology.

2.8.2 Flat Morphisms

Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of schemes. Recall that f is said to be *flat* if the underlying map of local commutative rings $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is flat for each $x \in X$. We now study an analogous condition in the setting of spectral algebraic geometry.

Definition 2.8.2.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of nonconnective spectral Deligne-Mumford stacks. We will say that f is *flat* if the following condition is satisfied:

(∗) For every commutative diagram

$$\begin{array}{ccc} \operatorname{Spét} B & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow f \\ \operatorname{Spét} A & \longrightarrow & \mathsf{Y} \end{array}$$

in which the horizontal maps are étale, the underlying map of $\mathbb{E}_\infty$-rings $A \to B$ is flat.

Lemma 2.8.2.2. *Let $\phi : A \to B$ be an étale morphism of $\mathbb{E}_\infty$-rings, and let M be a B-module spectrum. If M is flat over A, then it is flat over B.*

Proof. If M is flat over A, then the tensor product $B \otimes_A M \simeq (B \otimes_A B) \otimes_B M$ is flat over B. Since ϕ is étale, B is a retract of $B \otimes_A B$, so that $M \simeq B \otimes_B M$ is a retract of $(B \otimes_A B) \otimes_B M$ and therefore also flat over B. □

Proposition 2.8.2.3. *Let $\phi : A \to B$ be a map of $\mathbb{E}_\infty$-rings. The following conditions are equivalent:*

(1) *The map ϕ is flat.*

(2) *The map ϕ induces a flat morphism $\operatorname{Spét} B \to \operatorname{Spét} A$ of nonconnective spectral Deligne-Mumford stacks.*

Proof. The implication (2) ⇒ (1) is obvious. Conversely, suppose that (1) is satisfied. Suppose we are given a commutative diagram

$$\begin{array}{ccc} \operatorname{Spét} B' & \longrightarrow & \operatorname{Spét} B \\ \downarrow & & \downarrow \\ \operatorname{Spét} A' & \longrightarrow & \operatorname{Spét} A \end{array}$$

where the horizontal maps are étale; we wish to prove that B' is flat over A'. Using Theorem 1.4.10.2, we deduce that the map of $\mathbb{E}_\infty$-rings $B \to B'$ is étale. It follows that B' is flat over A. Since A' is étale over A (Theorem 1.4.10.2), the desired result follows from Lemma 2.8.2.2. □

Proposition 2.8.2.4. *The property of being a flat morphism (between nonconnective spectral Deligne-Mumford stacks) is local on the source with respect to the étale topology.*

Proof of Proposition 2.8.2.4. Condition (i) of Definition 2.8.1.7 follows immediately from the definition. To prove (ii), suppose we are given a morphism $g : \mathsf{Y} \to \mathsf{Z}$ and a jointly surjective collection of étale maps $f_\alpha : \mathsf{X}_\alpha \to \mathsf{Z}$ such that each composition $g \circ f_\alpha$ is flat. We wish to show that g is flat. Choose a commutative diagram

$$\begin{array}{ccc} \operatorname{Spét} B & \longrightarrow & \mathsf{Y} \\ \downarrow & & \downarrow g \\ \operatorname{Spét} A & \longrightarrow & \mathsf{Z} \end{array}$$

where the horizontal maps are étale. We wish to show that B is flat over A. Since the maps f_α are jointly surjective, we can choose a finite collection of étale maps $\{B \to B_\beta\}$ such that $B \to \prod_\beta B_\beta$ is faithfully flat, and each of the induced maps $\operatorname{Spét} B_\beta \to \operatorname{Spét} B \to \mathsf{Y}$ factors through some X_α. Since $g \circ f_\alpha$ is assumed to be flat, we deduce that B_β is flat as an A-module. It follows that $\prod_\beta B_\beta$ is flat as an A-module. Using Lemma B.1.4.2, we deduce that B is flat over A. $\square$

Remark 2.8.2.5. Let k be a commutative ring, regarded as a discrete $\mathbb{E}_\infty$-ring. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a nonconnective spectral Deligne-Mumford stack equipped with a morphism $\phi : \mathsf{X} \to \operatorname{Spét} k$. We will say that X is *flat over* k if the morphism ϕ is flat, in the sense of Definition 2.8.2.1. In this case, the structure sheaf $\mathcal{O}_\mathcal{X}$ is automatically discrete. Suppose that $\mathcal{X}$ is 1-localic and $\mathcal{O}_\mathcal{X}$ is discrete, so we may identify $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ with an ordinary Deligne-Mumford stack X. Then X is flat over k if and only if X is flat over k (in the sense of classical algebraic geometry). Consequently, Proposition **??** furnishes an equivalence between the ∞-category of spectral Deligne-Mumford 1-stacks which are flat over k with the ∞-category of ordinary Deligne-Mumford stacks which are flat over k.

Remark 2.8.2.6. Suppose we are given a pullback diagram

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow f' & & \downarrow f \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y} \end{array}$$

of nonconnective spectral Deligne-Mumford stacks. If f is flat, then f' is flat.

Remark 2.8.2.7. Suppose we are given morphisms of nonconnective spectral Deligne-Mumford stacks $\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z}$. If f and g are flat, then the composition $g \circ f$ is flat. To prove this, suppose we are given a commutative diagram

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} C & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow \\ \operatorname{Sp\acute{e}t} A & \longrightarrow & \mathsf{Z} \end{array}$$

where the horizontal maps are étale; we wish to show that C is flat over A. This assertion is local on $\operatorname{Sp\acute{e}t} C$ with respect to the étale topology (Proposition 2.8.2.4), so we may assume that the map $\operatorname{Sp\acute{e}t} C \to \operatorname{Sp\acute{e}t} A \times_{\mathsf{Z}} \mathsf{Y}$ factors as a composition $\operatorname{Sp\acute{e}t} C \to \operatorname{Sp\acute{e}t} B \xrightarrow{u} \operatorname{Sp\acute{e}t} A \times_{\mathsf{Z}} \mathsf{Y}$ where u is étale. Since f is flat, C is flat over B. Because g is flat, B is flat over A. It follows from Lemma B.1.4.2 that C is flat over A.

Remark 2.8.2.8. Let k' be an $\mathbb{E}_\infty$-ring and let $k = \tau_{\geqslant 0} k'$ be its connective cover. The proof of Proposition **??** shows that if A is an $\mathbb{E}_\infty$-algebra over k' with $\operatorname{Sp\acute{e}t} A = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, then $(\mathcal{X}, \tau_{\geqslant 0} \mathscr{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford stack which can be identified with the spectrum of $\tau_{\geqslant 0 A}$. In particular, if $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is flat over k', then $(\mathcal{X}, \tau_{\geqslant 0} \mathscr{O}_{\mathcal{X}})$ is flat over k. By reduction to the affine case, we obtain the more general global assertion: if $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a flat over k', then $(\mathcal{X}, \tau_{\geqslant 0} \mathscr{O}_{\mathcal{X}})$ is flat over k.

Definition 2.8.2.9. Let A be an $\mathbb{E}_\infty$-ring. A *(nonconnective) spectral Deligne-Mumford stack over* A is a (nonconnective) spectral Deligne-Mumford stack X equipped with a map $\phi : \mathsf{X} \to \operatorname{Sp\acute{e}t} A$. We let $\mathrm{SpDM}^{\mathrm{nc}}_A$ denote the ∞-category $\mathrm{SpDM}^{\mathrm{nc}}_{/\operatorname{Sp\acute{e}t} A}$ of nonconnective spectral Deligne-Mumford stacks over A. Let $\mathrm{SpDM}^{\flat}_A$ denote the full subcategory of $\mathrm{SpDM}^{\mathrm{nc}}_A$ spanned by those objects for which the map ϕ is flat.

Proposition 2.8.2.10. *Let $f : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings, and suppose that f induces an isomorphism $\pi_n(A) \to \pi_n(B)$ for $n \geqslant 0$. Then the pullback functor*

$$\mathsf{X} \mapsto \mathsf{X} \times_{\operatorname{Sp\acute{e}t} A} \operatorname{Sp\acute{e}t} B$$

induces an equivalence of ∞-categories $f^* : \mathrm{SpDM}^{\flat}_B \to \mathrm{SpDM}^{\flat}_A$.

Proof. It follows from Remark 2.8.2.6 that if X is flat over A, then $f^* \mathsf{X}$ is flat over B. Let $\overline{A}$ denote a connective cover of A (which is also a connective cover of B, since $\pi_n A \simeq \pi_n B$ for $n \geqslant 0$). We have a commutative diagram of pullback functors

$$\begin{array}{ccc} \mathrm{SpDM}^{\flat}_B & \longrightarrow & \mathrm{SpDM}^{\flat}_A \\ & \searrow \quad \swarrow & \\ & \mathrm{SpDM}^{\flat}_{\overline{A}}. & \end{array}$$

It will therefore suffice to prove that the vertical functors are equivalences of ∞-categories. We may therefore reduce to the case where f exhibits A as a connective cover of B. In this case, the functor f^* has a right adjoint G, given informally by the formula $G(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) = (\mathcal{X}, \tau_{\geqslant 0}\, \mathscr{O}_{\mathcal{X}})$ (this functor preserves flatness by Remark 2.8.2.8). Consequently, it suffices to show that the unit and counit transformations

$$FG(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \qquad (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \to GF(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$$

are equivalences whenever $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a nonconnective spectral Deligne-Mumford stack which is flat over B or $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ is a spectral Deligne-Mumford stack which is flat over A. These assertions are local on $\mathcal{X}$ and $\mathcal{Y}$; we may therefore reduce to the affine case, where the desired result follows from Proposition HA.7.2.2.24 . □

2.8.3 Fpqc-Local Properties of Spectral Deligne-Mumford Stacks

We now study an analogue of Definition 2.8.1.2 for the flat topology.

Definition 2.8.3.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks. We will say that f is a *flat covering* if the following conditions are satisfied:

(1) The map f is flat.

(2) For every quasi-compact open substack $\mathsf{V} \hookrightarrow \mathsf{Y}$, there exists a quasi-compact open substack $\mathsf{U} \hookrightarrow \mathsf{X}$ such that f induces a surjection $\mathsf{U} \to \mathsf{V}$.

Example 2.8.3.2. Let $f : \mathsf{X} \to \mathsf{Y}$ be an étale surjection. Then f is a flat covering. Condition (1) of Definition 2.8.3.1 is obvious. To prove (2), we first replace Y by V and thereby reduce to the case where Y is quasi-compact. Choose an étale surjection $\operatorname{Spét} A \to \mathsf{Y}$. Write $\operatorname{Spét} A \times_{\mathsf{Y}} \mathsf{X}$ as $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$. For every affine object $X \in \mathcal{X}$, we can write $(\mathcal{X}_{/X}, \mathscr{O}_{\mathcal{X}}|_X) \simeq \operatorname{Spét} B$ for some étale A-algebra B, so that the map $|\operatorname{Spec} B| \to |\operatorname{Spec} A|$ has image given by some open subset $U_X \subseteq |\operatorname{Spec} A|$ (Proposition **??**). Since f is surjective, the open sets U_X cover $|\operatorname{Spec} A|$. Since $|\operatorname{Spec} A|$ is quasi-compact, this open cover has a finite subcover. Taking the disjoint union of the corresponding objects of $\mathcal{X}$, we obtain an affine object $X \in \mathcal{X}$ such that the induced map $(\mathcal{X}_{/X}, \mathscr{O}_{\mathcal{X}}|_X) \to \operatorname{Spét} A$ is surjective. Since X is quasi-compact, the image of $(\mathcal{X}_{/X}, \mathscr{O}_{\mathcal{X}}|_X)$ in X is a quasi-compact open substack $\mathsf{U} \subseteq \mathsf{X}$ having the desired properties.

Proposition 2.8.3.3. *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{Y}' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \mathsf{X} & \xrightarrow{f} & \mathsf{Y}, \end{array}$$

where g' is étale and f is a flat covering. Then g is étale.

Proof. Without loss of generality we may assume that $\mathsf{Y} = \operatorname{Sp\acute{e}t} A$ and $\mathsf{Y}' = \operatorname{Sp\acute{e}t} A'$ are affine. Replacing X by a quasi-compact open substack, we can assume that there exists an étale surjection $\operatorname{Sp\acute{e}t} B \to \mathsf{X}$, so that B is faithfully flat over A. Our hypothesis that g' is étale guarantees that $B' = B \otimes_A A'$ is étale over B. In particular, B' is locally of finite presentation over B, so that A' is locally of finite presentation over A (see Proposition 4.2.1.5). We have $B \otimes_A L_{A'/A} \simeq L_{B'/B} \simeq 0$, so that $L_{A'/A} \simeq 0$ by virtue of our assumption that B is faithfully flat over A. It follows from Lemma B.1.3.3 that A' is étale over A, as desired. □

Corollary 2.8.3.4. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{Y}' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \mathsf{X} & \xrightarrow{f} & \mathsf{Y}, \end{array}$$

where g' is an equivalence and f is a flat covering. Then the following conditions are equivalent:

- (i) *The morphism g is an equivalence.*
- (ii) *There exists an integer $n \gg 0$ such that g is a relative spectral Deligne-Mumford n-stack (that is, $\mathsf{Y}' \times_{\mathsf{Y}} \operatorname{Sp\acute{e}t} R$ is a spectral Deligne-Mumford n-stack for every map $\operatorname{Sp\acute{e}t} R \to \mathsf{Y}$).*

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. Assume that (ii) is satisfied. It follows from Proposition 2.8.3.3 that g is étale. Set $\mathsf{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, so that we can write Y' as $(\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}}|_U)$ for some object $U \in \mathcal{Y}$. Assumption (ii) implies that U is n-truncated for some integer $n \gg 0$. We proceed by induction on n. Applying our inductive hypothesis to the diagonal map $\mathsf{Y}' \to \mathsf{Y}' \times_{\mathsf{Y}} \mathsf{Y}'$, we deduce that the diagonal map $U \to U \times U$ is an equivalence: that is, U is a (-1)-truncated object of Y, so that g is an open immersion. The map f is surjective and factors up to homotopy through g, so that g is a surjective open immersion and therefore an equivalence. □

Warning 2.8.3.5. In the situation of Corollary 2.8.3.4, condition (ii) is not automatic. For example, let κ be a field and write $\operatorname{Sp\acute{e}t} \kappa = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. If $U \in \mathcal{X}$ is an ∞-connective object, then the projection map $g : (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \operatorname{Sp\acute{e}t} \kappa$ becomes an equivalence after pulling back along the faithfully flat $\operatorname{Sp\acute{e}t} \overline{\kappa} \to \operatorname{Sp\acute{e}t} \kappa$, where $\operatorname{Sp\acute{e}t} \overline{\kappa}$ is an algebraic closure of κ. However, g is not an equivalence unless U is a final object of $\mathcal{X}$ (which need not be the case: see Warning HTT.7.2.2.31).

Definition 2.8.3.6. Let P be a property of nonconnective spectral Deligne-Mumford stacks. We will say that P is *local for the flat topology* if the following conditions hold:

(*i*) For every flat morphism of nonconnective spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$, if Y has the property P, then X also has the property P.

(*ii*) Given a collection of flat morphisms $\{\mathsf{X}_\alpha \to \mathsf{Y}\}$, if each X_α has the property P and the induced map $\amalg \mathsf{X}_\alpha \to \mathsf{Y}$ is a flat covering, then Y has the property P.

Proposition 2.8.3.7. *Let P be a property of nonconnective spectral Deligne-Mumford stacks. Then P is local for the flat topology if and only if the following conditions are satisfied:*

(1) *The property P is local for the étale topology (Definition 2.8.3.6).*

(2) *If $f : A \to B$ is a flat morphism of $\mathbb{E}_\infty$-rings such that* $\operatorname{Sp\acute{e}t} A$ *has the property P, then* $\operatorname{Sp\acute{e}t} B$ *has the property P. The converse holds provided that f is faithfully flat.*

Proof. If P is local for the flat topology, then condition (2) is obvious and condition (1) follows from Example 2.8.3.2. Conversely, suppose that conditions (1) and (2) are satisfied. We first verify condition (*i*) of Definition 2.8.3.6. Let $f : \mathsf{X} \to \mathsf{Y}$ be a flat morphism of spectral Deligne-Mumford stacks, and assume that Y has the property P. We wish to show that X has the property P. By virtue of assumption (1), this condition is local with respect to the étale topology on X. We may therefore assume that $\mathsf{X} \simeq \operatorname{Sp\acute{e}t} B$ and that the map f factors as a composition $\mathsf{X} \to \operatorname{Sp\acute{e}t} A \xrightarrow{f''} \mathsf{Y}$ where f'' is étale. Then $\operatorname{Sp\acute{e}t} A$ has the property P. Since f is flat, B is flat over A. It then follows from (2) that $\mathsf{X} \simeq \operatorname{Sp\acute{e}t} B$ has the property P, as desired.

We now verify condition (*ii*). Let $f : \mathsf{X} \to \mathsf{Y}$ be a flat covering and suppose that X has the property P; we wish to show that Y has the property P. The assertion is local with respect to the étale topology on Y, so we may suppose that $\mathsf{Y} \simeq \operatorname{Sp\acute{e}t} A$ is affine. In particular, Y is quasi-compact. Replacing X by an open substack if necessary (and using (1)), we can reduce to the case where X is quasi-compact. We can then choose an étale surjection $\operatorname{Sp\acute{e}t} B \to \mathsf{X}$. Then $\operatorname{Sp\acute{e}t} B$ has the property P (by (1)) and B is faithfully flat over A, so that $\operatorname{Sp\acute{e}t} A$ has the property P by (2). □

Example 2.8.3.8. Let P be the property of being a spectral Deligne-Mumford stack, so that a nonconnective spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ has the property P if and only if the structure sheaf $\mathcal{O}_\mathcal{X}$ is connective. Then P is local with respect to the flat topology.

Example 2.8.3.9. Let P be the property of being an n-truncated spectral Deligne-Mumford stack, so that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ has the property P if and only if the structure sheaf $\mathcal{O}_\mathcal{X}$ is connective and n-truncated. Then P is local with respect to the flat topology.

We now consider a relative version of Definition 2.8.3.6.

Definition 2.8.3.10. Let P be a property of morphisms between nonconnective spectral Deligne-Mumford stacks. We will say that P is *local on the source with respect to the flat topology* if the following conditions hold:

(*i*) For every composable pair of morphisms $\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z}$, if f is flat and g has the property P, then $g \circ f$ has the property P.

(*ii*) Given a collection of flat morphisms $\{f_\alpha : \mathsf{X}_\alpha \to \mathsf{Y}\}$ which determine a flat covering $\amalg_\alpha \mathsf{X}_\alpha \to \mathsf{Y}$ and a morphism $g : \mathsf{Y} \to \mathsf{Z}$, if each of the composite maps $g \circ f_\alpha$ has the property P, then g has the property P.

Proposition 2.8.3.7 has the following analogue for properties of morphisms between nonconnective spectral Deligne-Mumford stacks, which is proven in the same way:

Proposition 2.8.3.11. *Let P be a property of morphisms between nonconnective spectral Deligne-Mumford stacks. Then P is local on the source for the flat topology if and only if the following conditions are satisfied:*

(1) *The property P is local on the source for the étale topology (Definition 2.8.3.6).*

(2) *Suppose we are given a pair of maps $\mathrm{Spét}\, B \xrightarrow{f} \mathrm{Spét}\, A \xrightarrow{g} \mathsf{Z}$ such that B is flat over A. If g has the property P, then $g \circ f$ has the property P. The converse holds if B is faithfully flat over A.*

Example 2.8.3.12. The property of being a flat morphism is local on the source with respect to the flat topology. By virtue of Proposition 2.8.2.4, it will suffice to show that the property of flatness satisfies condition (2) of Proposition 2.8.3.11. The first assertion is obvious (since the collection of flat morphisms is closed under composition, by Remark 2.8.2.7). Conversely, suppose that B is faithfully flat over A and that we are given a map $g : \mathrm{Spét}\, A \to \mathsf{Z}$ such that the composite map $\mathrm{Spét}\, B \to \mathrm{Spét}\, A \to \mathsf{Z}$ is flat. We wish to show that g is flat. This follows immediately from the definitions, together with Lemma B.1.4.2.

2.8.4 Fpqc-Local Properties of Modules

We now consider properties of quasi-coherent sheaves which can be tested locally with respect to the flat topology.

Definition 2.8.4.1. Let P be a property of pairs (A, M), where A is a connective $\mathbb{E}_\infty$-ring and M is an A-module. We will say that P is *local for the flat topology* if the following conditions are satisfied:

(1) Let $f : A \to B$ be a flat morphism of connective $\mathbb{E}_\infty$-rings, let M be an A-module. If (A, M) has the property P, then $(B, B \otimes_A M)$ has the property P. The converse holds if f is faithfully flat.

(2) Suppose we are given a finite collection of pairs $(A_i, M_i) \in \mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$, each of which has the property P. Then the product $(\prod A_i, \prod M_i) \in \mathcal{C}$ has the property P.

We now illustrate Definition 2.8.4.1 with some examples:

Proposition 2.8.4.2. *The following conditions on a pair* $(A, M) \in \mathrm{Mod} \times_{\mathrm{CAlg}} \mathrm{CAlg}^{\mathrm{cn}}$ *are local for the flat topology:*

(1) *The condition that* M *is* n*-connective (when regarded as a spectrum), where* n *is a fixed integer.*

(2) *The condition that* M *is almost connective: that is, that* M *is* $(-n)$*-connective for* $n \gg 0$.

(3) *The condition that* M *is* n*-truncated (that is, that* $\pi_i M \simeq 0$ *for* $i > n$*), where* n *is a fixed integer.*

(4) *The condition that* M *is truncated (that is, that* $\pi_i M \simeq 0$ *for* $i \gg 0$*).*

(5) *The condition that* M *is has* Tor*-amplitude* $\leqslant n$*, where* n *is a fixed integer.*

(6) *The condition that* M *is flat.*

(7) *The condition that* M *is perfect to order* n *over* A*, where* n *is a fixed integer.*

(8) *The condition that* M *is finitely* n*-presented over* A*, where* $n \geqslant 0$ *is a fixed integer.*

(9) *The condition that* M *is almost perfect over* A*.*

(10) *The condition that* M *is perfect over* A*.*

Lemma 2.8.4.3. *Let* $f : A \to B$ *be a faithfully flat morphism of connective* $\mathbb{E}_\infty$*-rings and let* M *be a left* A*-module. If* $B \otimes_A M$ *has* Tor*-amplitude* $\leqslant n$ *over* B*, then* M *has* Tor*-amplitude* $\leqslant n$ *over* A *(see Definition HA.7.2.4.21).*

Proof. Let N be a discrete A-module; we wish to show that $M \otimes_A N$ is n-truncated. Since B is faithfully flat over A, it suffices to show that

$$B \otimes_A (M \otimes_A N) \simeq (B \otimes_A M) \otimes_B (B \otimes_A N)$$

is n-truncated. This follows from our assumption that $B \otimes_A M$ has Tor-amplitude $\leqslant n$, since $B \otimes_A N$ is a discrete B-module. □

Proof of Proposition 2.8.4.2. Assertions (1) and (3) follow from Proposition HA.7.2.2.13, and assertions (2) and (4) follow immediately from (1) and (3). Assertion (5) follows from Lemma 2.8.4.3. Assertion (6) follows from (5) and (1). Assertion (7) follows from Proposition 2.7.3.1 and Remark 2.7.0.8. Assertion (8) follows from (1), (3), and (7). Assertion (9) follows from (7). Assertion (10) follows (5), (9), and Proposition HA.7.2.4.23. □

Definition 2.8.4.4. Let P be a property of pairs (A, M), where A is a connective $\mathbb{E}_\infty$-ring and M is an A-module, and assume that P is local for the flat topology. Let X be a spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$. We will say that $\mathscr{F}$ *has the property* P if, for every étale morphism $f : \mathrm{Spét}\, A \to \mathscr{F}$, the pullback $f^* \mathscr{F} \in \mathrm{QCoh}(\mathrm{Spét}\, A) \simeq \mathrm{Mod}_A$ corresponds to an A-module M for which the pair (A, M) has the property P.

Let us spell out the contents of Definition 2.8.4.4 in the examples provided by Proposition 2.8.4.2. Let $\mathscr{F}$ be a quasi-coherent sheaf on a spectral Deligne-Mumford stack X. We say that $\mathscr{F}$ is:

(1) *n-connective* if, for every étale morphism $f : \mathrm{Spét}\, A \to \mathsf{X}$, the pullback $f^* \mathscr{F} \in \mathrm{Mod}_A$ is n-connective. This is equivalent to the requirement that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})_{\geqslant n}$.

(2) *almost connective* if, for every étale morphism $f : \mathrm{Spét}\, A \to \mathsf{X}$, the pullback $f^* \mathscr{F} \in \mathrm{Mod}_A$ is almost connective. If X is quasi-compact, then this condition is equivalent to the requirement that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})_{\geqslant -n}$ for some integer n.

(3) *n-truncated* if, for every étale morphism $f : \mathrm{Spét}\, A \to \mathsf{X}$, the pullback $f^* \mathscr{F} \in \mathrm{Mod}_A$ is n-truncated. This is equivalent to the requirement that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})_{\leqslant n}$.

(4) *locally truncated* if, for every étale morphism $f : \mathrm{Spét}\, A \to \mathsf{X}$, the pullback $f^* \mathscr{F} \in \mathrm{Mod}_A$ is n-truncated for some integer n. If X is quasi-compact, this is equivalent to the requirement that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})_{\leqslant n}$ for some integer n.

(5) *of* Tor-*amplitude* $\leqslant n$ if, for every étale morphism $f : \mathrm{Spét}\, A \to \mathsf{X}$, the pullback $f^* \mathscr{F} \in \mathrm{Mod}_A$ is of Tor-amplitude $\leqslant n$.

(6) *flat* if, for every étale morphism $f : \mathrm{Spét}\, A \to \mathsf{X}$, the pullback $f^* \mathscr{F} \in \mathrm{Mod}_A$ is flat.

(7) *perfect to order n* if, for every étale morphism $f : \mathrm{Spét}\, A \to \mathsf{X}$, the pullback $f^* \mathscr{F} \in \mathrm{Mod}_A$ is perfect to order n.

(8) *finitely n-presented* if, for every étale morphism $f : \mathrm{Spét}\, A \to \mathsf{X}$, the pullback $f^* \mathscr{F} \in \mathrm{Mod}_A$ is finitely n-presented.

(9) *almost perfect* if, for every étale morphism $f : \mathrm{Spét}\, A \to \mathsf{X}$, the pullback $f^* \mathscr{F} \in \mathrm{Mod}_A$ is almost perfect.

(10) *perfect* if, for every étale morphism $f : \mathrm{Spét}\, A \to \mathsf{X}$, the pullback $f^* \in \mathrm{Mod}_A$ is perfect.

Remark 2.8.4.5. In case (4), our terminology does not quite conform to the general convention of Definition 2.8.4.4. We use the term "locally truncated" rather than "truncated" to emphasize the possibility that if $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is a quasi-coherent sheaf whose pullback $f^* \mathscr{F}$ is truncated for every étale map $f : \mathrm{Spét}\, A \to \mathsf{X}$, then $\mathscr{F}$ need not belong to $\bigcup \mathrm{QCoh}(\mathsf{X})_{\leqslant n}$ unless we assume that X is quasi-compact.

Remark 2.8.4.6. In the setting of classical scheme theory, the notions of *perfect to order* n and *almost perfect* were introduced by Illusie in [101], who refers to an object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ as *n-pseudo-coherent* if it is perfect to order n (in our terminology), and as *pseudo-coherent* it is almost perfect (in our terminology).

We have the following analogue of Proposition 2.8.1.7, which can be proven by exactly the same argument:

Proposition 2.8.4.7. *Let P be a property of pairs (A, M), where A is a connective $\mathbb{E}_\infty$-ring and M is an A-module, and assume that P is local for the flat topology. Let X be a spectral Deligne-Mumford stack and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then:*

(1) *If $f : \mathsf{Y} \to \mathsf{X}$ is a flat morphism and $\mathscr{F}$ has the property P, then $f^* \mathscr{F}$ has the property P. In particular, if $\mathscr{F}$ is n-connective (almost connective, n-truncated, locally truncated, of* Tor*-amplitude $\leqslant n$, flat, perfect to order n, finitely n-presented, almost perfect, perfect), then $f^* \mathscr{F}$ has the same property.*

(2) *If we are given a collection of flat maps $\{f_\alpha : \mathsf{Y}_\alpha \to \mathsf{X}\}$ which induces a flat covering $\amalg \mathsf{Y}_\alpha \to \mathsf{X}$, and each pullback $f_\alpha^* \mathscr{F}$ has the property P, then $\mathscr{F}$ has the property P. In particular, if each $f_\alpha^* \mathscr{F}$ is n-connective (almost connective, n-truncated, locally truncated, of* Tor*-amplitude $\leqslant n$, flat, perfect to order n, finitely n-presented, almost perfect, perfect), then $\mathscr{F}$ has the same property.*

2.9 Vector Bundles and Invertible Sheaves

Let X be a scheme. One of the most important invariants of X is its *Picard group* $\mathrm{Pic}(X)$: the group of isomorphism classes of invertible sheaves on X. Our goal in this section is to extend the theory of the Picard group to the setting of spectral algebraic geometry.

2.9.1 Locally Free Modules

We begin with a more general discussion of locally free sheaves.

Definition 2.9.1.1. Let A be a connective $\mathbb{E}_\infty$-ring and let M be an A-module. We will say that M is *locally free of finite rank* if there exists an integer n such that M is a direct summand of A^n.

Remark 2.9.1.2. Let A be a connective $\mathbb{E}_\infty$-ring. Using Proposition HA.7.2.4.20 , we conclude that an A-module M is locally free of finite rank if and only if M is flat and almost perfect.

Corollary 2.9.1.3. *Let A be a connective $\mathbb{E}_1$-ring and let M be a left A-module of* Tor*-amplitude $\leqslant 0$. If M is perfect to order 1, then it is perfect. If, in addition, M is connective, then it is locally free of finite rank.*

Proof. Using Proposition 2.7.3.2, we can reduce to the case where A is discrete. In this case, the first assertion follows from Proposition 2.7.1.5, and the second from Remark 2.9.1.2. □

Combining Proposition 2.8.4.2, Proposition 6.2.5.2, and Remark 2.9.1.2, we obtain the following:

Proposition 2.9.1.4. *The property of being a locally free module of finite rank is stable under base change and local with respect to the flat topology.*

Our next result gives another characterization of the class of locally free modules:

Proposition 2.9.1.5. *Let R be a connective $\mathbb{E}_\infty$-ring and let M be a connective R-module. The following conditions are equivalent:*

(1) *The module M is locally free of finite rank.*

(2) *The module M is a dualizable object of the symmetric monoidal ∞-category $\mathrm{Mod}_R^{\mathrm{cn}}$.*

Proof. The collection of dualizable objects of $\mathrm{Mod}_R^{\mathrm{cn}}$ is evidently closed under the formation of retracts and direct sums. Since the unit object $R \in \mathrm{Mod}_R^{\mathrm{cn}}$ is dualizable, we conclude that (1) $\Rightarrow$ (2). Conversely, suppose that M is a dualizable object of $\mathrm{Mod}_R^{\mathrm{cn}}$. Then M is a dualizable object of Mod_R, and therefore a perfect R-module (Proposition 6.2.6.2). Let $M^\vee$ denote the dual of M. For any discrete R-module N, we have isomorphisms $\pi_i(M \otimes_R N) \simeq \pi_i \mathrm{Map}_{\mathrm{Mod}_R}(M^\vee, N)$ for $i \geqslant 0$. Since $M^\vee$ is connective, we deduce that $\pi_i(M \otimes_R N)$ vanishes for $i > 0$. It follows that M is flat, so that M is locally free of finite rank by Proposition HA.7.2.4.20. □

2.9.2 The Rank of a Locally Free Module

We now restrict our attention to modules which are locally free of a particular rank $n \in \mathbf{Z}_{\geqslant 0}$.

Definition 2.9.2.1. Let R be a connective $\mathbb{E}_\infty$-ring, let M be an R-module, and let $n \geqslant 0$ be an integer. We will say that M is *locally free of rank n* if the following conditions are satisfied:

(a) The module M is locally free of finite rank (equivalently, M is flat and almost perfect as an R-module: see Proposition HA.7.2.4.20).

(b) For every field κ and every map of $\mathbb{E}_\infty$-rings $R \to \kappa$, the vector space $\pi_0(\kappa \otimes_R M)$ has dimension n over κ.

Remark 2.9.2.2. To verify condition (b) of Definition 2.9.2.1, we are free to pass to any field extension of κ. We may therefore assume without loss of generality that κ is algebraically closed.

The terminology of Definitions 2.9.1.1 and 2.9.2.1 is motivated by the following observation:

Proposition 2.9.2.3. *Let R be a connective $\mathbb{E}_\infty$-ring and let M be an R-module which is locally free of finite rank. Then there exists a sequence of elements $x_1, \ldots, x_m \in \pi_0 R$ which generate the unit ideal, such that each of the modules $M[x_i^{-1}] = R[x_i^{-1}] \otimes_R M$ is free of rank n_i over $R[x_i^{-1}]$. If M is locally free of rank n, then we can assume that $n_i = n$ for every integer i.*

Proof. Let us say that an element $x \in \pi_0 R$ is *good* if $M[x^{-1}]$ is a free module of finite rank over $R[x^{-1}]$ (which is of rank n in the case where M is locally free of rank n). To complete the proof, it will suffice to show that the collection of good elements of $\pi_0 R$ generate the unit ideal in $\pi_0 R$. Assume otherwise; then there exists a maximal ideal $\mathfrak{m}$ of $\pi_0 R$ which contains every good element of R. Let $\kappa = (\pi_0 R)/\mathfrak{m}$ denote the residue field of $\pi_0 R$ at $\mathfrak{m}$. Then $\pi_0(\kappa \otimes_R M)$ is a finite dimensional vector space over k (which is of dimension n in the case M is locally free of rank n). Let n' be the dimension of this vector space, and choose elements $y_1, \ldots, y_{n'} \in \pi_0 M$ whose images form a basis for $\pi_0(\kappa \otimes_R M)$. Since $\pi_0 M$ is finitely generated as a module over $\pi_0 R$, Nakayama's lemma implies that the images of the elements y_i generate the localization $(\pi_0 M)_{\mathfrak{m}}$. We may therefore choose an element $x \in (\pi_0 R) - \mathfrak{m}$ such that the elements y_i generate the module $(\pi_0 M)[x^{-1}]$. It follows that there is a map a map $\phi : R[x^{-1}]^{n'} \to M$ which induces a surjection $(\pi_0 R[x^{-1}])^{n'} \to \pi_0 M[x^{-1}]$. Since M is projective, the map ϕ admits a right homotopy inverse $\psi : M[x^{-1}] \to R[x^{-1}]^{n'}$. The composite map $\psi \circ \phi : R[x^{-1}]^{n'} \to R[x^{-1}]^{n'}$ determines an n'-by-n' matrix A_{ij} with values in the commutative ring $\pi_0 R[x^{-1}]$. Let D denote the determinant of this matrix, and choose an element $x' \in \pi_0 R$ with $x^a D = x'$. Since ϕ induces an isomorphism of vector spaces $k^{n'} \simeq \pi_0(\kappa \otimes_R M)$, the element x' does not belong to $\mathfrak{m}$. We note that the image of D in $\pi_0 R[x^{-1}, x'^{-1}]$ is invertible, so that ϕ induces an equivalence $R[x^{-1}, x'^{-1}]^{n'} \to M[x^{-1}, x'^{-1}]$. It follows that $xx' \in \pi_0 R$ is good. Since the product xx' does not belong to $\mathfrak{m}$, we obtain a contradiction. □

Proposition 2.9.2.4. *The condition that an R-module M be locally free of rank n is stable under base change and local with respect to the flat topology (see Definitions 2.8.4.1 and 6.2.5.1).*

Proof. According to Proposition 2.9.1.4, the condition of being locally free of finite rank is stable under base change and local with respect to the flat topology. It will therefore suffice to prove the following:

(∗) Let R be a connective $\mathbb{E}_\infty$-ring, M a locally free R-module of finite rank, and $R \to \prod_{1 \leqslant i \leqslant m} R_i$ a faithfully flat map of $\mathbb{E}_\infty$-rings. If each tensor product $R_i \otimes_R M$ satisfies condition (b) of Definition 2.9.2.1, then so does M.

To prove (∗), let us suppose we are given a field κ and a map $R \to \kappa$; we wish to prove that $\pi_0(\kappa \otimes_R M)$ is a κ-vector space of dimension n. Since $R \to \prod_{1 \leqslant i \leqslant m} R_i$ is faithfully flat, there exists an index i such that $\kappa \otimes_R R_i \neq 0$. Let κ' be a residue field of the commutative ring $\kappa \otimes_R R_i$. Then κ' is a field extension of κ, so it will suffice to show that $\pi_0(\kappa' \otimes_R M)$ has dimension n over κ' (Remark 2.9.2.2). This follows from the existence of an isomorphism

$$\pi_0(\kappa' \otimes_R M) \simeq \pi_0(\kappa' \otimes_{R_i} (R_i \otimes_R M)),$$

since $R_i \otimes_R M$ is locally free of rank n over R_i. □

2.9.3 Locally Free Sheaves

Using Proposition 2.9.2.4, we can introduce a global version of Definition 2.9.2.1.

Notation 2.9.3.1. Let X be a spectral Deligne-Mumford stack. We say that a quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is *locally free of rank n* if, for every étale morphism $f : \mathrm{Spét}\, A \to \mathsf{X}$, the pullback $f^* \mathscr{F} \in \mathrm{QCoh}(\mathrm{Spét}\, A) \simeq \mathrm{Mod}_A$ is locally free of rank n when regarded as an A-module. Note that if this condition is satsified, then $g^* \mathscr{F} \in \mathrm{Mod}_B$ is locally free of rank n for every map $g : \mathrm{Spét}\, B \to \mathsf{X}$.

For perfect quasi-coherent sheaves, local freeness is an open condition:

Proposition 2.9.3.2. *Let X be a spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ be perfect. Then:*

(1) *For every integer n, there exists a largest open substack $i_n : \mathsf{X}_n \hookrightarrow \mathsf{X}$ such that $i_n^* \mathscr{F}$ is locally free of rank n.*

(2) *The sheaf $\mathscr{F}$ is locally free of finite rank if and only if the canonical map $\theta : \amalg_n \mathsf{X}_n \to \mathsf{X}$ is an equivalence of spectral Deligne-Mumford stacks.*

(3) *Let $f : \mathsf{Y} \to \mathsf{X}$ be a map of spectral Deligne-Mumford stacks. Then f factors through X_n if and only if $f^* \mathscr{F}$ is locally free of rank n.*

In particular, if $\mathscr{F}$ is locally free of finite rank, then each of the maps i_n is a clopen immersion (see Definition 3.1.7.2).

Lemma 2.9.3.3. *Let X be a spectral Deligne-Mumford stack, and let $\mathscr{F}$ be an almost perfect quasi-coherent sheaf on X. For every integer n, there exists quasi-compact open immersion $i : \mathsf{U} \to \mathsf{X}$ with the following property: a morphism of spectral Deligne-Mumford stacks $f : \mathsf{X}' \to \mathsf{X}$ factors through U if and only if $f^* \mathscr{F}$ is n-connective.*

Proof. The assertion is local on X. We may therefore assume without loss of generality that X is quasi-compact, so that $\mathscr{F}$ is m-connective for some integer m. We proceed by induction on the difference $n - m$. If $n - m \leqslant 0$, then we can take $\mathsf{U} = \mathsf{X}$. Assume that $m < n$. Using the inductive hypothesis, we can choose a quasi-compact open immersion $j : \mathsf{V} \to \mathsf{X}$ such that a map $\mathsf{X}' \to \mathsf{X}$ factors through j if and only if $f^* \mathscr{F}$ is $(n-1)$-connective. Replacing X by V, we may assume that $\mathscr{F}$ is $(n-1)$-connective. Since the assertion is local on X, we may assume without loss of generality that $\mathsf{X} = \operatorname{Spét} R$ is affine, so that $\mathscr{F}$ corresponds to an $(n-1)$-connective R-module M. Since $\mathscr{F}$ is almost perfect, $\pi_{n-1} M$ is finitely presented over $\pi_0 R$. We may therefore choose a presentation

$$(\pi_0 R)^{m'} \xrightarrow{T} (\pi_0 R)^m \to \pi_{n-1} M \simeq 0.$$

Let $I \subseteq \pi_0 R$ be the ideal generated by all m-by-m minors of the matrix representing the map T. Let $U = \{\mathfrak{p} \in |\operatorname{Spec} R| : I \nsubseteq \mathfrak{p}\} \subseteq |\operatorname{Spec} R|$ and let U be the corresponding open substack of X. We claim that U has the desired properties. To prove this, it suffices to observe that a map $\operatorname{Spét} R' \to \mathsf{X}$ factors through U if and only if the abelian group $\operatorname{Tor}_0^{\pi_0 R}(\pi_0 R', \pi_{n-1} M) \simeq \pi_{n-1}(R' \otimes_R M)$ vanishes. □

Lemma 2.9.3.4. *Let X be a spectral Deligne-Mumford stack, and let $\mathscr{F}$ be a perfect quasi-coherent sheaf on $\mathscr{F}$. Then there exists a quasi-compact open immersion $i : \mathsf{U} \to \mathsf{X}$ with the following property: a morphism of spectral Deligne-Mumford stacks $f : \mathsf{X}' \to \mathsf{X}$ factors through U if and only if $f^* \mathscr{F}$ is locally free of finite rank.*

Proof. Since $\mathscr{F}$ is perfect, it is a dualizable object of $\operatorname{QCoh}(\mathsf{X})$ (Proposition 6.2.6.2); let us denote its dual by $\mathscr{F}^\vee$. Note that $\mathscr{F}$ is locally free of finite rank if and only if both $\mathscr{F}$ and $\mathscr{F}^\vee$ are connective (Proposition HA.7.2.4.20). The desired result now follows from Lemma 2.9.3.3. □

Proof of Proposition 2.9.3.2. Assertion (1) follows immediately from Proposition 2.9.2.4, and the "if" direction of (2) is immediate. Conversely, suppose that $\mathscr{F}$ is locally free of finite rank; we wish to show that the map θ is an equivalence. The map θ is evidently étale, and is surjective by virtue of Proposition 2.9.2.3. To prove that θ is an equivalence, it will suffice to show that the diagonal map

$$\amalg_n \mathsf{X}_n \to (\amalg_n \mathsf{X}_n) \times_{\mathsf{X}} (\amalg_n \mathsf{X}_n) \simeq \amalg_{m,n} (\mathsf{X}_m \times_{\mathsf{X}} \mathsf{X}_n)$$

is an equivalence. Since each i_n is an open immersion, each of the maps $\mathsf{X}_n \to \mathsf{X}_n \times_{\mathsf{X}} \mathsf{X}_n$ is an equivalence. It will therefore suffice to show that $\mathsf{X}_m \times_{\mathsf{X}} \mathsf{X}_n$ is trivial for $m \neq n$. Equivalently, we must show that if R is a connective $\mathbb{E}_\infty$-ring and M is a locally free R-module of rank m which is also of rank $n \neq m$, then $R \simeq 0$. Assume otherwise, and let κ be a residue field of $\pi_0 R$. We obtain an immediate contradiction, since $\pi_0(\kappa \otimes_R M)$ is a vector space over k which is dimension m and also of dimension $n \neq m$.

It remains to prove (3). Using Lemma 2.9.3.4, we may reduce to the case where $\mathscr{F}$ is locally free of finite rank, so that the above argument shows that θ is an equivalence. Let $f : \mathsf{Y} \to \mathsf{X}$ be such that $f^* \mathscr{F}$ is locally free of rank n. Then we can write f as a coproduct of maps $f_m : \mathsf{Y}_m \to \mathsf{X}_m$. Then $(i_m \circ f_m)^* \mathscr{F}$ is locally free of rank m and locally free of rank n. It follows that Y_m is empty for $m \neq n$, so that $\mathsf{Y}_n \simeq \mathsf{Y}$ and f factors through i_n as desired. □

2.9.4 Line Bundles

We now consider quasi-coherent sheaves which are locally free of rank 1.

Definition 2.9.4.1. Let X be a spectral Deligne-Mumford stack. We will say that a quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is a *line bundle* if it is locally free of rank 1. We let $\mathscr{P}\mathrm{ic}(\mathsf{X})$ denote the full subcategory of $\mathrm{QCoh}(\mathsf{X})^{\simeq}$ spanned by the line bundles on X, and we let $\mathrm{Pic}(\mathsf{X}) = \pi_0 \mathscr{P}\mathrm{ic}(\mathsf{X})$ collection of homotopy equivalences classes of line bundles on X. We will refer to $\mathrm{Pic}(\mathsf{X})$ as the *Picard group* of X.

Proposition 2.9.4.2. *Let X be a spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$. The following conditions are equivalent:*

(1) *The quasi-coherent sheaf $\mathscr{F}$ is a line bundle on X.*

(2) *The quasi-coherent sheaf $\mathscr{F}$ is an invertible object of the symmetric monoidal ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$.*

Proof. Suppose first that (1) is satisfied. Then $\mathscr{F}$ is locally free of finite rank, and therefore a dualizable object of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ (see Proposition 2.9.1.5). Let us denote its dual by $\mathscr{F}^{\vee}$, and let $e : \mathscr{F} \otimes \mathscr{F}^{\vee} \to \mathscr{O}$ be the evaluation map (where $\mathscr{O}$ denotes the structure sheaf of X). We claim that e is an equivalence. To prove this, it suffices to show that e induces an equivalence of R-modules $\eta^*(\mathscr{F} \otimes \mathscr{F}^{\vee}) \to R$ for every map $\eta : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$. The assertion is local on $|\operatorname{Spec} R|$, so we can apply Proposition 2.9.2.3 to reduce further to the case where $\eta^* \mathscr{F} \simeq R$, in which case the result is obvious.

We now prove (2). Assume that $\mathscr{F}$ is an invertible object in $\mathrm{QCoh}(X)^{\mathrm{cn}}$. We wish to show that for every point $\eta \in X(R)$, the pullback $M = \eta^* \mathscr{F} \in \mathrm{QCoh}(\operatorname{Spec} R) \simeq \mathrm{Mod}_R$ is locally free of rank 1. Note that M is a dualizable object of $\mathrm{Mod}_R^{\mathrm{cn}}$ and therefore locally free of finite rank (Proposition 2.9.1.5). In particular, for every map from R to a field κ, the tensor product $\kappa \otimes_R M$ can be identified with a finite dimensional vector space over κ, which is invertible as an object of $\mathrm{Mod}_\kappa^{\mathrm{cn}}$. It follows easily that $\kappa \otimes_R M$ has dimension 1 over κ, so that M is locally free of rank 1. □

2.9.5 Invertible Sheaves

For some purposes, it is convenient to consider a slight enlargement of the Picard group $\mathrm{Pic}(\mathsf{X})$.

Definition 2.9.5.1. Let X be a spectral Deligne-Mumford stack, and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$. We will say that $\mathscr{F}$ is *invertible* if it is invertible as an object of the symmetric monoidal ∞-category $\mathrm{QCoh}(\mathsf{X})$: that is, if there exists another object $\mathscr{F}' \in \mathrm{QCoh}(\mathsf{X})$ such that $\mathscr{F} \otimes \mathscr{F}'$ is equivalent to the unit object of $\mathrm{QCoh}(\mathsf{X})$. We let $\mathscr{P}\mathrm{ic}^{\dagger}(\mathsf{X})$ denote the full subcategory of $\mathrm{QCoh}(\mathsf{X})^{\simeq}$ spanned by the invertible objects, and we let $\mathrm{Pic}^{\dagger}(\mathsf{X})$ denote the set of connected components $\pi_0 \mathscr{P}\mathrm{ic}^{\dagger}(\mathsf{X})$. We will refer to $\mathrm{Pic}^{\dagger}(\mathsf{X})$ as the *extended Picard group* of X.

Remark 2.9.5.2. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. Then $\mathscr{P}\mathrm{ic}^{\dagger}(\mathsf{X}) \subseteq \mathrm{QCoh}(\mathsf{X})$ is essentially small. To prove this, choose a regular cardinal κ for which the global sections functor $\mathscr{F} \mapsto \Gamma(\mathsf{X}; \mathscr{F})$ commutes with κ-filtered colimits. If $\mathcal{L}$ is an invertible object of $\mathrm{QCoh}(\mathsf{X})$, then the canonical equivalence $\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathcal{L}, \mathscr{F}) \simeq \Gamma(\mathsf{X}; \mathcal{L}^{-1} \otimes \mathscr{F})$ shows that $\mathcal{L}$ is a κ-compact object of $\mathrm{QCoh}(\mathsf{X})$. Since the full subcategory of $\mathrm{QCoh}(\mathsf{X})$ spanned by the κ-compact objects is essentially small, it follows that $\mathscr{P}\mathrm{ic}^{\dagger}(\mathsf{X})$ is essentially small.

Remark 2.9.5.3. Let X be a spectral Deligne-Mumford stack. It follows from Proposition 2.9.4.2 that every line bundle on X is an invertible object of $\mathrm{QCoh}(\mathsf{X})$, so that we have inclusions

$$\mathscr{P}\mathrm{ic}(\mathsf{X}) \subseteq \mathscr{P}\mathrm{ic}^{\dagger}(\mathsf{X}) \qquad \mathrm{Pic}(\mathsf{X}) \subseteq \mathrm{Pic}^{\dagger}(\mathsf{X}).$$

It follows from Remark 2.9.5.2 that $\mathscr{P}\mathrm{ic}(\mathsf{X})$ is essentially small, and that the sets $\mathrm{Pic}(\mathsf{X})$ and $\mathrm{Pic}^{\dagger}(\mathsf{X})$ are small.

Remark 2.9.5.4. Let $\mathcal{C}$ be a symmetric monoidal ∞-category, and let $\mathcal{E}$ denote the full subcategory of $\mathcal{C}^{\simeq}$ spanned by the invertible objects of $\mathcal{C}$. Then $\mathcal{E}$ inherits the structure of a symmetric monoidal ∞-category. It can therefore be viewed as grouplike commutative monoid object of the ∞-category $\mathcal{S}$ of spaces. It follows from Remark HA.5.2.6.26 that $\mathcal{E}$ is an infinite loop space; in particular, the symmetric monoidal structure on $\mathcal{C}$ determines an abelian group structure on $\pi_0 \mathcal{E}$. Applying this observation to the symmetric monoidal ∞-categories $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ and $\mathrm{QCoh}(\mathsf{X})$ (where X is a spectral Deligne-Mumford stack), we conclude that $\mathscr{P}\mathrm{ic}(\mathsf{X})$ and $\mathscr{P}\mathrm{ic}^{\dagger}(\mathsf{X})$ are infinite loop spaces, and that the sets $\mathrm{Pic}(\mathsf{X})$ and $\mathrm{Pic}^{\dagger}(\mathsf{X})$ can be regarded as abelian groups.

Remark 2.9.5.5. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack, and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$. Then $\mathscr{F}$ is invertible if and only if $\mathscr{F}$ is dualizable, and the evaluation map $\mathscr{F} \otimes \mathscr{F}^{\vee} \to \mathscr{O}_{\mathcal{X}}$ is an equivalence. It follows that the condition that $\mathscr{F}$ be invertible is local with respect to the flat topology on X.

We next describe the relationship between line bundles and invertible sheaves.

Proposition 2.9.5.6. *Let X be a spectral Deligne-Mumford stack, and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ be perfect. For each integer $n \in \mathbf{Z}$, let X_n denote the largest open substack of X for which the restriction of $\Sigma^n \mathscr{F}$ to X_n is a line bundle (see Proposition 2.9.3.2). Then:*

(1) *The canonical map $j : \amalg_{n \in \mathbf{Z}} \mathsf{X}_n \to \mathsf{X}$ is an open immersion.*

(2) *Let $g : \mathsf{Y} \to \mathsf{X}$ be a map of spectral Deligne-Mumford stacks. Then g factors through j if and only if $g^* \mathscr{F}$ is invertible.*

Corollary 2.9.5.7. *Let X be a spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$. The following conditions are equivalent:*

(1) *There exists an equivalence $\mathsf{X} \simeq \amalg_{n \in \mathbf{Z}} \mathsf{X}_n$ such that, for each $n \in \mathbf{Z}$, the restriction of $\Sigma^n \mathscr{F}$ to X_n is a line bundle.*

(2) *There exists a mutually surjective collection of étale maps $f_\alpha : \mathsf{U}_\alpha \to \mathsf{X}$, integers n_α, and line bundles $\mathcal{L}_\alpha \in \mathscr{P}\mathrm{ic}(\mathsf{U}_\alpha)$ such that $f_\alpha^* \mathscr{F} \simeq \Sigma^{n_\alpha} \mathcal{L}_\alpha$.*

(3) *There exists a mutually surjective collection of étale maps $f_\alpha : \mathsf{U}_\alpha \to \mathsf{X}$, such that each $f_\alpha^* \mathscr{F}$ is invertible.*

(4) *The object $\mathscr{F}$ is invertible.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are immediate, and (3) ⇒ (4) follows from Remark 2.9.5.5. Suppose that (4) is satisfied. Then $\mathscr{F}$ is perfect, so that condition (1) follows from Proposition 2.9.5.6. □

Remark 2.9.5.8. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. Using Corollary 2.9.5.7, we obtain an isomorphism of abelian groups

$$\mathrm{Pic}^{\dagger}(\mathsf{X}) \simeq \mathrm{Pic}(\mathsf{X}) \times \Gamma(\mathcal{X}; \underline{\mathbf{Z}}),$$

where $\underline{\mathbf{Z}}$ denotes the constant sheaf on $\mathcal{X}$ taking the value $\mathbf{Z}$. In other words, $\mathrm{Pic}^{\dagger}(\mathsf{X})$ is the product of $\mathrm{Pic}(\mathsf{X})$ with the abelian group of locally constant $\mathbf{Z}$-valued functions on X. In particular, if X is connected, we obtain an isomorphism $\mathrm{Pic}^{\dagger}(\mathsf{X}) \simeq \mathrm{Pic}(\mathsf{X}) \times \mathbf{Z}$: that is, every invertible object of $\mathrm{QCoh}(\mathsf{X})$ can be written uniquely in the form $\Sigma^n \mathcal{L}$, where n is an integer and $\mathcal{L}$ is a line bundle on X.

Proof of Proposition 2.9.5.6. We first claim that j is an open immersion. Arguing as in the proof of Proposition 2.9.3.2, we are reduced to proving that the fiber product $\mathsf{X}_m \times_{\mathsf{X}} \mathsf{X}_n$ is empty for $m \neq n$. Suppose otherwise; then there exists a field κ and a map $\eta : \mathrm{Sp\acute{e}t}\, \kappa \to \mathsf{X}$ which factors through both X_m and X_n. Then $\pi_0 \eta^* \Sigma^m \mathscr{F}$ is a 1-dimensional vector space over κ (since η factors through X_m) and also a 0-dimensional vector space over κ (since η factors through X_n), so we obtain a contradiction. This proves (1).

The "only if" direction of (2) is obvious. For the converse, suppose we are given a map $g : \mathsf{Y} \to \mathsf{X}$ such that $g^* \mathscr{F}$ is invertible; we wish to show that g factors through j. Without loss of generality, we may suppose that $\mathsf{Y} = \operatorname{Spét} R$ is affine. Let us identify $g^* \mathscr{F}$ with an invertible R-module M. To prove that g factors through j, it will suffice to show that for every maximal ideal $\mathfrak{m} \subseteq \pi_0 R$, there exists an element $a \in \pi_0 R$ which does not belong to $\mathfrak{m}$ and an integer n such that $\Sigma^{-n} M[a^{-1}]$ is locally free of rank 1.

Let $\kappa = (\pi_0 R)/\mathfrak{m}$. Then $\kappa \otimes_R M$ is an invertible κ-module, so that $\pi_*(\kappa \otimes_R M)$ is an invertible object in the symmetric monoidal category of graded vector spaces over κ. It follows that there exists an integer n such that

$$\pi_i(\kappa \otimes_R M) \simeq \begin{cases} \kappa & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Since M is an invertible R-module, it is nonzero and perfect. It follows that there exists a smallest integer m such that $\pi_m M \neq 0$. If $m < n$, then $(\pi_m M)/\mathfrak{m}(\pi_m M) \simeq \pi_m(\kappa \otimes_R M) \simeq 0$. Since $\pi_m M$ is finitely generated as a module over $\pi_0 R$, it follows from Nakayama's lemma that there exists an element $a \in (\pi_0 R) - \mathfrak{m}$ such that $\pi_m M[a^{-1}] = 0$. Replacing R by the localization $R[a^{-1}]$, we may reduce to the case where $\pi_m M = 0$. Iterating this procedure, we may reduce to the case where $\pi_m M \simeq 0$ for $m < n$. Applying a similar argument to the dual $M^\vee$ of M, we may suppose that $\pi_m M^\vee \simeq 0$ for $m < -n$: that is, M has Tor-amplitude $\geqslant n$. It then follows from Remark 2.9.1.2 that $\Sigma^{-n} M$ is locally free of finite rank. Using Proposition 2.9.3.2 we may suppose (after further localization) that $\Sigma^{-n} M$ is locally free of rank 1, as desired. □

2.9.6 The Affine Case

Let X be a spectral Deligne-Mumford stack and let X_0 be its underlying ordinary Deligne-Mumford stack. Every line bundle on X determines a line bundle on X_0, so there is a natural map from the Picard group of X (in the sense of Definition 2.9.4.1) to the Picard group of X_0 (in the sense of classical algebraic geometry). In general, this map need not be an isomorphism. However, it is an isomorphism whenever X is affine (Proposition 2.9.6.2).

Notation 2.9.6.1. Let R be a connective $\mathbb{E}_\infty$-ring. We define

$$\mathscr{P}\mathrm{ic}(R) = \mathscr{P}\mathrm{ic}(\operatorname{Spét} R) \qquad \mathscr{P}\mathrm{ic}^\dagger(R) = \mathscr{P}\mathrm{ic}^\dagger(\operatorname{Spét} R)$$

$$\operatorname{Pic}(R) = \operatorname{Pic}(\operatorname{Spét} R) \qquad \operatorname{Pic}^\dagger(R) = \operatorname{Pic}^\dagger(\operatorname{Spét} R).$$

We will refer to $\operatorname{Pic}(R)$ and $\operatorname{Pic}^\dagger(R)$ as the *Picard group of* R and the *extended Picard group of* R, respectively.

Proposition 2.9.6.2. *Let R be a connective $\mathbb{E}_\infty$-ring. Then the canonical maps*

$$\alpha : \mathrm{Pic}(R) \to \mathrm{Pic}(\pi_0 R) \qquad \beta : \mathrm{Pic}^\dagger(R) \to \mathrm{Pic}^\dagger(\pi_0 R)$$

are isomorphisms.

Proof. By virtue of Remark 2.9.5.8, it will suffice to prove that α is an isomorphism, which follows from Corollary HA.7.2.2.19 . □

Corollary 2.9.6.3. *Let R be a connective $\mathbb{E}_\infty$-ring. Then the canonical maps*

$$\mathscr{P}\mathrm{ic}(R) \to \varprojlim \mathscr{P}\mathrm{ic}(\tau_{\leqslant n} R) \qquad \mathscr{P}\mathrm{ic}^\dagger(R) \to \varprojlim \mathscr{P}\mathrm{ic}(\tau_{\leqslant n} R)$$

are homotopy equivalences.

Proof. Using Proposition 2.9.6.2, we see that the maps

$$\pi_m \mathscr{P}\mathrm{ic}(R) \to \pi_m \mathscr{P}\mathrm{ic}(\tau_{\leqslant n} R) \qquad \pi_m \mathscr{P}\mathrm{ic}^\dagger(R) \to \pi_m \mathscr{P}\mathrm{ic}^\dagger(\tau_{\leqslant n} R)$$

are isomorphisms as soon as $n \geqslant m - 1$. □

Chapter 3

Spectral Algebraic Spaces

Recall that a spectral Deligne-Mumford stack X is a *spectral algebraic space* if the mapping space $\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X})$ is discrete for every commutative ring R (Definition 1.6.8.1). In this section, we will develop some foundations for the theory of spectral algebraic spaces. Our main technical result (Theorem 3.4.2.1) asserts that every quasi-compact, quasi-separated spectral algebraic space admits a scallop decomposition, in the sense of Definition 2.5.3.1. This has a number of pleasant consequences: for example, it implies that if $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a morphism of quasi-compact, quasi-separated spectral algebraic spaces, then the pushforward functor $f_* : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ preserves quasi-coherence (Corollary 3.4.2.2).

Our proof of Theorem 3.4.2.1 will require a long detour through some of the fundamentals of spectral algebraic geometry. We begin in §3.1 by introducing the notion of a *closed immersion* between spectral Deligne-Mumford stacks (Definition 3.1.0.1). We will say that a spectral algebraic space X is *separated* if the diagonal map $\mathsf{X} \to \mathsf{X} \times \mathsf{X}$ is a closed immersion. In §3.2, we prove a slightly weaker version of Theorem 3.4.2.1: every quasi-compact, separated spectral algebraic space admits a scallop decomposition (Proposition 3.2.3.1). As a consequence, we will see that for any morphism between quasi-compact, separated spectral algebraic spaces $f : \mathsf{X} \to \mathsf{Y}$, there is a well-behaved pushforward operation f_* on quasi-coherent sheaves (Corollary 3.2.3.3). We will apply this result in §3.3 to develop a theory of quasi-finite morphisms between spectral Deligne-Mumford stacks. In particular, we will prove the following version of Zariski's Main Theorem: if $f : \mathsf{X} \to \mathsf{Y}$ is quasi-compact, separated, and locally quasi-finite, then f is quasi-affine (Theorem 3.3.0.2). We will apply this result in §3.4 to prove the existence of scallop decompositions for arbitrary quasi-compact, quasi-separated spectral algebraic spaces, and describe some applicatoins.

In §3.6, we study the underlying topological space $|\mathsf{X}|$ of a spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. We define $|\mathsf{X}|$ to be the space of points of the locale underlying the ∞-topos $\mathcal{X}$. More or less by definition, we can identify open subsets of $|\mathsf{X}|$ with open substacks of X. However, it is not immediately clear how to describe the *points* of X. In the special case where X is a quasi-separated spectral algebraic space, we will use Theorem 3.4.2.1 to show that there is a bijection between points of $|\mathsf{X}|$ and (equivalence classes of) *geometric points* of X: that is, maps $\iota : \text{Spét}\, \kappa \to \mathsf{X}$, where κ is a separably closed field. The latter are closely related to points of the underlying ∞-topos $\mathcal{X}$, which we study in §3.5.

The topology on the space $|\mathsf{X}|$ can be regarded as an analogue of the Zariski topology in the setting of spectral algebraic spaces. There is a version of the Nisnevich topology as well, which we will discuss in §3.7. In particular, we prove that the class of Nisnevich sheaves can be characterized by an excision property, just as with ordinary schemes (Theorem 3.7.5.1).

Contents

3.1 Closed Immersions

Let $f : X \to Y$ be a morphism of schemes. Recall that f is said to be a *closed immersion* if it induces a homeomorphism from the underlying topological space of X to a closed subset of the underlying topological space of Y, and an epimorphism of structure sheaves $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. In this section, we will develop an analogous theory of closed immersions in the setting of spectral algebraic geometry.

Definition 3.1.0.1. Let $f : (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be a morphism of spectrally ringed ∞-topoi. We will say that f is a *closed immersion* if the following conditions are satisfied:

(*a*) The underlying geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ is a closed immersion of ∞-topoi. That is, f_* induces an equivalence $\mathcal{X} \to \mathcal{Y}/U$, for some (-1)-truncated object $U \in \mathcal{Y}$ (see Notation **??**).

(*b*) The structure sheaves $\mathscr{O}_{\mathcal{X}}$ and $\mathscr{O}_{\mathcal{Y}}$ are connective.

(*c*) Let $f^{-1} : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ denote the left adjoint of the pushforward functor f_*. Then the underlying map $f^{-1}\mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$ has a connective fiber. In other words, the induced map $f^{-1}\pi_0\mathscr{O}_{\mathcal{Y}} \to \pi_0\mathscr{O}_{\mathcal{X}}$ is an epimorphism (in the abelian category of abelian group objects of $\mathcal{X}^{\heartsuit}$).

3.1.1 Classification of Closed Immersions

In the setting of *locally* spectrally ringed ∞-topoi, closed immersions having a fixed codomain admit a reasonably simple description.

Proposition 3.1.1.1. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a locally spectrally ringed ∞-topos. Assume that the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is connective and let $\alpha : \mathcal{O}_{\mathcal{X}} \to \mathcal{O}'$ be a morphism of connective sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$ which induces an epimorphism $\pi_0 \mathcal{O}_{\mathcal{X}} \to \pi_0 \mathcal{O}'$. Then there exists a closed immersion $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ and an equivalence $\beta : \mathcal{O}' \simeq f_* \mathcal{O}_{\mathcal{Y}}$ in $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\mathcal{O}_{\mathcal{X}}/}$ with the following universal property: for every morphism of locally spectrally ringed ∞-topos $h : (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, composition with β induces a homotopy equivalence*

$$\begin{aligned}\mathrm{Map}_{(\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}})_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}}((\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})) &\to \mathrm{Map}_{(\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}})_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}}((\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}), (\mathcal{X}, \mathcal{O}'))\\ &\simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\mathcal{O}_{\mathcal{X}}/}}(\mathcal{O}', h_* \mathcal{O}_{\mathcal{Z}}).\end{aligned}$$

Proof. Let U denote the (-1)-truncated object of $\mathcal{X}$ given by $\mathbf{1} \times_{\pi_0 \mathcal{O}'} (\pi_0 \mathcal{O}')^\times$ (see Remark 1.2.1.6), let $f_* : \mathcal{Y} \to \mathcal{X}$ be the closed immersion of ∞-topoi complementary to U (so that $\mathcal{Y}$ can be identified with the full subcategory $\mathcal{X}/U \subseteq \mathcal{X}$ spanned by those objects X for which $U \times X$ is initial in $\mathcal{X}$), and let $\mathcal{O}_{\mathcal{Y}} = f^* \mathcal{O}'$. Since the restriction $\mathcal{O}'|_U$ vanishes, the unit map $\beta : \mathcal{O}' \to f_* \mathcal{O}_{\mathcal{Y}}$ is an equivalence. It follows from Proposition 1.2.1.8 that the spectrally ringed ∞-topos $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is local.

Note that if $h : (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \to (\mathcal{X}, \mathcal{O}')$ is any morphism of spectrally ringed ∞-topoi, the pullback h^*U admits a morphism to the fiber product $V = \mathbf{1} \times_{\pi_0 \mathcal{O}_{\mathcal{Z}}} (\pi_0 \mathcal{O}_{\mathcal{Z}})^\times$ in the ∞-topos $\mathcal{Z}$. If $\mathcal{O}_{\mathcal{Z}}$ is local, then V is initial in $\mathcal{Z}$, so that U is also initial in $\mathcal{Z}$ so that the underlying map of ∞-topoi $h_* : \mathcal{Z} \to \mathcal{X}$ must factor as a composition $\mathcal{Z} \xrightarrow{g_*} \mathcal{Y} \xrightarrow{f_*} \mathcal{X}$. In this case, we can promote g_* to a morphism of spectrally ringed ∞-topoi $g : (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. To complete the proof, it will suffice to show that g is local if and only if h is local, which is an immediate consequence of Remark 1.2.1.7. □

Remark 3.1.1.2. Let $h : (\mathcal{Y}, \mathcal{O}_{\mathcal{Z}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be an arbitrary closed immersion in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$. Then the induced map $\mathcal{O}_{\mathcal{X}} \to h_* \mathcal{O}_{\mathcal{Z}}$ induces an epimorphism $\alpha : \pi_0 \mathcal{O}_{\mathcal{X}} \to \pi_0 h_* \mathcal{O}_{\mathcal{Z}}$. Let $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be the closed immersion obtained by applying Proposition 3.1.1.1 to α. The universal property of f guarantees that h factors as a composition $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \xrightarrow{g} (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \xrightarrow{f} (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. A simple calculation shows that g is an equivalence. It follows that every closed immersion $h : (\mathcal{Y}, \mathcal{O}_{\mathcal{Z}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ can be obtained using the construction of Proposition 3.1.1.1.

Corollary 3.1.1.3. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a locally spectrally ringed ∞-topos where $\mathcal{O}_{\mathcal{X}}$ is connective. Let $\mathcal{C} \subseteq (\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}})_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}$ denote the subcategory spanned by those maps f :*

$(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ which are closed immersions. Then the construction

$$(f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})) \mapsto f_* \mathcal{O}_{\mathcal{Y}}$$

determines a fully faithful embedding $\theta : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\mathcal{O}_{\mathcal{X}}/}$. Moreover, the essential image of θ is the full subcategory $\mathcal{C}' \subseteq \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\mathcal{O}_{\mathcal{X}}/}$ spanned by those maps $\alpha : \mathcal{O}_{\mathcal{X}} \to \mathcal{O}'$ where $\mathcal{O}'$ is connective and $\mathrm{fib}(\alpha)$ *is connective.*

Proof. It follows from Proposition 3.1.1.1 that the functor $\theta : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}'$ is essentially surjective. To prove that it is fully faithful, suppose we are given closed immersions

$$f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \qquad g : (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$; we wish to show that θ induces a homotopy equivalence

$$\rho : \mathrm{Map}_{(\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}})_{/(\mathcal{X}, \mathcal{O}_{\mathcal{X}})}}((\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\mathcal{O}_{\mathcal{X}}/}}(f_* \mathcal{O}_{\mathcal{Y}}, g_* \mathcal{O}_{\mathcal{Z}}).$$

To prove this, it suffices to show that the closed immersion f enjoys the universal property described in Proposition 3.1.1.1, which follows from Remark 3.1.1.2. □

Remark 3.1.1.4. Let $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a closed immersion in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$. If $\mathcal{O}_{\mathcal{X}}$ is strictly Henselian, then $\mathcal{O}_{\mathcal{Y}}$ is strictly Henselian (this follows immediately from Proposition 1.2.2.14).

3.1.2 Closed Immersions of Spectral Deligne-Mumford Stacks

Let R be a commutative ring. Then the construction $I \mapsto \mathrm{Spec}\, R/I$ induces a bijective correspondence between ideals $I \subseteq R$ and closed subschemes of the affine scheme $\mathrm{Spec}\, R$. Our main goal in this section is to establish an analogous correspondence in the setting of spectral algebraic geometry:

Theorem 3.1.2.1. *Let A be a connective $\mathbb{E}_\infty$-ring and let $f : (\mathcal{X}, \mathcal{O}) \to \mathrm{Sp\acute{e}t}\, A$ be a map of spectral Deligne-Mumford stacks. The following conditions are equivalent:*

(1) *The map f is a closed immersion.*

(2) *The spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O})$ is affine and the canonical map $A \to \Gamma(\mathcal{X}; \mathcal{O})$ induces a surjection of commutative rings $\pi_0 A \to \pi_0 \Gamma(\mathcal{X}; \mathcal{O})$.*

Remark 3.1.2.2. The condition that a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks be a closed immersion is local on the target with respect to the étale topology (see Definition 6.3.1.1). Together with Theorem 3.1.2.1, this observation completely determines the class of closed immersions between spectral Deligne-Mumford stacks.

Before giving the proof of Theorem 3.1.2.1, let us describe some of its consequences.

Corollary 3.1.2.3. *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}. \end{array}$$

If f is a closed immersion, so is f'.

Proof. The assertion is local on Y and Y'. We may therefore assume without loss of generality that $\mathsf{Y} \simeq \operatorname{Spét} A$ is affine, so that Theorem 3.1.2.1 implies that X has the form $\operatorname{Spét} B$ for some connective $\mathbb{E}_\infty$-ring A' for which the map $\phi : A \to B$ is 0-connective. Similarly, we may assume that $\mathsf{Y}' \simeq \operatorname{Spét} A'$ for some connective $\mathbb{E}_\infty$-ring A'. According to Theorem 3.1.2.1, it will suffice to show that the map $\phi' : A' \to B \otimes_A A'$ is connective. This is clear, since $\operatorname{fib}(\phi') \simeq \operatorname{fib}(\phi) \otimes_A A'$. □

Corollary 3.1.2.4. *Suppose we are given a commutative diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccccc} \mathsf{X} & & \xrightarrow{f} & & \mathsf{Y} \\ & {\scriptstyle h}\searrow & & \swarrow{\scriptstyle g} & \\ & & \mathsf{Z}. & & \end{array}$$

If g is a closed immersion, then f is a closed immersion if and only if h is a closed immersion.

Proof. The assertion is local on Z. We may therefore assume without loss of generality that $\mathsf{Z} \simeq \operatorname{Spét} C$ for some connective $\mathbb{E}_\infty$-ring C. It follows from Theorem 3.1.2.1 that $\mathsf{Y} \simeq \operatorname{Spét} B$ is affine, and that the map $\pi_0 C \to \pi_0 B$ is surjective. Theorem 3.1.2.1 also implies that if either f or h is a closed immersion, then X has the form $\operatorname{Spét} A$, for some connective $\mathbb{E}_\infty$-ring A. The desired result now follows from the observation that a map $\pi_0 B \to \pi_0 A$ is surjective if and only if the composite map $\pi_0 C \to \pi_0 B \to \pi_0 A$ is surjective. □

Remark 3.1.2.5. Corollaries 3.1.2.3 and 3.1.2.4 are true in the more general setting of spectrally ringed ∞-topoi. See Corollary **??** and Remark **??**.

Warning 3.1.2.6. In the setting of classical algebraic geometry, every closed immersion is a monomorphism. The analogous statement is not true in spectral algebraic geometry. If $f : \mathsf{Y} \to \mathsf{X}$ is a closed immersion of spectral Deligne-Mumford stacks and R is a connective $\mathbb{E}_\infty$-ring, then it is generally not true that the induced map

$$\theta : \operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} R, \mathsf{Y}) \to \operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} R, \mathsf{X})$$

is the inclusion of a summand. However, this *is* true in the special case where R is discrete. To prove this, fix a map $\eta : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$ and apply Theorem 3.1.2.1 to write the fiber product $\mathrm{Sp\acute{e}t}\, R \times_{\mathsf{X}} \mathsf{Y}$ can be written as $\mathrm{Sp\acute{e}t}\, A$, where A is an $\mathbb{E}_\infty$-algebra over R for which the map $\phi : R \to \pi_0 A$ is surjective. It follows that the fiber product

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{Y}) \times_{\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, X, \mathsf{X})} \{\eta\}$$

is either empty (if $\ker(\phi) \neq 0$) or contractible (if $\ker(\phi) \simeq 0$).

Remark 3.1.2.7. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack, and suppose we are given a closed immersion $f : (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ of locally spectrally ringed ∞-topoi. The following conditions are equivalent:

(1) The spectrally ringed ∞-topos $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ is a spectral Deligne-Mumford stack.

(2) The direct image $f_* \mathscr{O}_{\mathcal{Y}}$ is quasi-coherent as an $\mathscr{O}_{\mathcal{X}}$-module.

To prove this, we may assume that $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ is affine. If condition (1) is satisfied, then Theorem 3.1.2.1 guarantees that $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \simeq \mathrm{Sp\acute{e}t}\, B$ for some connective $\mathbb{E}_\infty$-algebra A over B, in which case the direct image $f_* \mathscr{O}_{\mathcal{Y}}$ can be identified with the quasi-coherent sheaf associated to the A-module B. Conversely, suppose that (2) is satisfied. Then $f_* \mathscr{O}_{\mathcal{Y}}$ can be regarded as a commutative algebra object of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \simeq \mathrm{Mod}_A^{\mathrm{cn}}$ which we can identify with a connective A-algebra B for which the map $\pi_0 A \to \pi_0 B$ is surjective. It then follows from Corollary 3.1.1.3 that $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ is equivalent to the affine spectral Deligne-Mumford stack $\mathrm{Sp\acute{e}t}\, B$.

3.1.3 Closed Immersion of ∞-Topoi

The proof of Theorem 3.1.2.1 will require a number of preliminaries. We begin by with general remarks about the topological properties of closed immersions.

Lemma 3.1.3.1. *Let $\mathcal{X}$ be an ∞-topos and let $f_* : \mathcal{Y} \to \mathcal{X}$ be a closed immersion of ∞-topoi. If $\mathcal{X}$ is n-coherent (for some $n \geqslant 0$), then $\mathcal{Y}$ is n-coherent.*

Proof. The proof proceeds by induction on n. In the case $n = 0$, we must show that if $\mathcal{X}$ is quasi-compact then $\mathcal{Y}$ is also quasi-compact. Assume that $\mathcal{X}$ is quasi-compact and choose an effective epimorphism $\theta : \coprod_{i \in I} Y_i \to \mathbf{1}$ in $\mathcal{Y}$, where $\mathbf{1}$ denotes the final object of $\mathcal{Y}$. We wish to show that there exists a finite subset $I_0 \subseteq I$ such that the induced map $\coprod_{i \in I_0} Y_i \to \mathbf{1}$ is an effective epimorphism. If I is empty, we can take $I_0 = I$. Otherwise, we can use Lemma 3.1.3.2 (since f_* is a closed immersion) to see that the induced map $\coprod_{i \in I} f_*(Y_i) \to f_* \mathbf{1}$ is effective epimorphism in $\mathcal{X}$. Since $\mathcal{X}$ is quasi-compact, there exists a finite subset $I_0 \subseteq I$ such that the map $\coprod_{i \in I_0} f_*(Y_i) \to f_* \mathbf{1}$ is an effective epimorphism, so that $\coprod_{i \in I_0} Y_i \to \mathbf{1}$ is an effective epimorphism in $\mathcal{Y}$.

Now suppose that $n > 0$. The inductive hypothesis guarantees that the pullback functor f^* carries $(n-1)$-coherent objects of $\mathcal{X}$ to $(n-1)$-coherent objects of $\mathcal{Y}$. Assume that $\mathcal{X}$ is $(n-1)$-coherent. Let $\mathcal{Y}_0 \subseteq \mathcal{Y}$ be the collection of all objects of the form f^*X, where $X \in \mathcal{X}$ is $(n-1)$-coherent. Then $\mathcal{Y}_0$ is stable under products in $\mathcal{Y}$ and every object $Y \in \mathcal{Y}$ admits an effective epimorphism $\amalg Y_i \to Y$, where each $Y_i \in \mathcal{Y}_0$. Applying Corollary A.2.1.4, we deduce that $\mathcal{Y}$ is n-coherent. □

Lemma 3.1.3.2. *Let $\mathcal{X}$ be an ∞-topos containing a (-1)-truncated object U, and let $i^* : \mathcal{X} \to \mathcal{X}/U$ and $j^* : \mathcal{X} \to \mathcal{X}_{/U}$ be the associated geometric morphisms. Suppose we are given a family of morphisms $\{\alpha_i : X_i \to X\}$ in $\mathcal{X}$. Then the composite map $\amalg X_i \to X$ is an effective epimorphism if and only if the corresponding maps*

$$\amalg i^* X_i \to f^* X \qquad \amalg\, j^* X_i \to g^* X$$

are effective epimorphisms in $\mathcal{X}/U$ and $\mathcal{X}_{/U}$, respectively.

Proof. Since i^* and j^* commute with coproducts, we can replace the family $\{X_i \to X\}$ by the single map $\alpha : X_0 \to X$, where $X_0 = \amalg_i X_i$. Let $X_\bullet$ be the Čech nerve of α and let X' be its geometric realization. Then α is an effective epimorphism if and only if it induces an equivalence $\beta : X' \to X$. Similarly, $f^*(\alpha)$ and $g^*(\alpha)$ are effective epimorphisms if and only if $i^*(\beta)$ and $j^*(\beta)$ are equivalences. The desired result now follows from Lemma HA.A.5.11 . □

Lemma 3.1.3.3. *Let $f_* : \mathcal{Y} \to \mathcal{X}$ be a closed immersion of ∞-topoi. Then:*

(1) *The functor f_* carries n-connective objects of $\mathcal{Y}$ to n-connective objects of $\mathcal{X}$.*

(2) *The functor f_* carries n-connective morphisms of $\mathcal{Y}$ to n-connective morphisms of $\mathcal{X}$.*

(3) *The functor f_* induces a left t-exact pushforward functor $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$.*

Proof. Since f_* carries final objects of $\mathcal{Y}$ to final objects of $\mathcal{X}$, assertion (1) follows from (2). Assertion (3) is an immediate consequence of (1). It will therefore suffice to prove (2). We proceed by induction on n: when $n = 0$, the desired result follows from Lemma 3.1.3.2. Assume that $n > 0$, and let $u : X \to Y$ be an n-connective morphism in $\mathcal{Y}$. We wish to show that $f_*(u)$ is n-connective. We have already seen that $f_*(u)$ is an effective epimorphism; it will therefore suffice to show that the diagonal map $f_*(X) \to f_*(X) \times_{f_*(Y)} f_*(X)$ is $(n-1)$-connective. This follows from the inductive hypothesis, since the diagonal map $X \to X \times_Y X$ is $(n-1)$-connective. □

Lemma 3.1.3.4. *Suppose we are given a commutative diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f_*} & \mathcal{Y} \\ & {\scriptstyle h_*}\searrow \quad \swarrow{\scriptstyle g_*} & \\ & \mathcal{Z} & \end{array}$$

in $\mathcal{T}\mathrm{op}_\infty$*, where* g_* *and* h_* *are closed immersions of* ∞*-topoi. Then* f_* *is a closed immersion of* ∞*-topoi.*

Proof. Without loss of generality, we may assume that $\mathcal{X} = \mathcal{Z}_{/U}$ and $\mathcal{Y} = \mathcal{Z}_{/V}$ for some (-1)-truncated objects $U, V \in \mathcal{Z}$. The commutativity of the diagram implies $\mathcal{Z}_{/U} \subseteq \mathcal{Z}_{/V}$. An object $Z \in \mathcal{Z}$ belongs to $\mathcal{Z}_{/U}$ if and only if the projection map $p : Z \times U \to U$ is an equivalence. Using Lemma HA.A.5.11 , we see that this is equivalent to the assertion that $Z \in \mathcal{Y}_{/V}$ and that $g^*(p)$ is an equivalence. Thus f_* induces an equivalence from $\mathcal{X}$ to $\mathcal{Y}_{/g^*U}$ and is therefore a closed immersion. □

Lemma 3.1.3.5. *Let* $\mathcal{X}$ *be an* n*-localic* ∞*-topos. Then any closed subtopos of* $\mathcal{X}$ *is also* n*-localic.*

Proof. Any closed subtopos of $\mathcal{X}$ is a topological localization of $\mathcal{X}$ (Proposition HTT.7.3.2.3) and therefore n-localic by virtue of Proposition HTT.6.4.5.9 . □

3.1.4 The Proof of Theorem 3.1.2.1

We will deduce Theorem 3.1.2.1 from the following slightly weaker assertion:

Proposition 3.1.4.1. *Let* $\phi : A \to B$ *be a map of connective* $\mathbb{E}_\infty$*-rings which induces a surjection* $\pi_0 A \to \pi_0 B$*. Then the corresponding map* $f : \mathrm{Sp\acute{e}t}\, B \to \mathrm{Sp\acute{e}t}\, A$ *induces a closed immersion of the underlying* ∞*-topoi.*

Remark 3.1.4.2. Proposition 3.1.4.1 is essentially a theorem of commutative algebra: it follows from (and implies) the statement that a surjection of commutative rings induces a closed immersion of their étale topoi.

Proof of Proposition 3.1.4.1. Using Proposition 1.4.2.4, we can identify the underlying ∞-topoi of $\mathrm{Sp\acute{e}t}\, A$ and $\mathrm{Spec}^{\acute{e}t B}$ with $\mathcal{S}\mathrm{hv}^{\acute{e}t}_A \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\acute{e}t}_A, \mathcal{S})$ and $\mathcal{S}\mathrm{hv}^{\acute{e}t}_B \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\acute{e}t}_B, \mathcal{S})$, respectively. Under these identifications, the functor f_* is given by composition with the extension-of-scalars functor

$$\mathrm{CAlg}^{\acute{e}t}_A \to \mathrm{CAlg}^{\acute{e}t}_B \qquad A' \mapsto A' \otimes_A B.$$

Let $U \in \mathrm{CAlg}_A^{\text{ét}} \to \mathcal{S}$ be the functor described by the formula

$$U(A') = \begin{cases} \Delta^0 & \text{if } A' \otimes_A B \simeq 0 \\ \varnothing & \text{otherwise.} \end{cases}$$

Then U is a (-1)-truncated object of $\mathcal{S}\mathrm{hv}_A^{\text{ét}}$, and the pushforward functor f_* carries the ∞-category $\mathcal{S}\mathrm{hv}_B^{\text{ét}}$ into the full subcategory $\mathcal{S}\mathrm{hv}_A^{\text{ét}}/U \subseteq \mathcal{S}\mathrm{hv}_A^{\text{ét}}$. We will complete the proof by showing that the adjoint functors

$$\mathcal{S}\mathrm{hv}_A^{\text{ét}}/U \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{S}\mathrm{hv}_B^{\text{ét}}$$

are mutually inverse equivalences of ∞-categories. Note that $\mathrm{CAlg}_A^{\text{ét}}$ and $\mathrm{CAlg}_B^{\text{ét}}$ are equivalent to the nerves of ordinary categories, by Theorem HA.7.5.0.6 . It follows that $\mathcal{S}\mathrm{hv}_A^{\text{ét}}$ and $\mathcal{S}\mathrm{hv}_B^{\text{ét}}$ are 1-localic ∞-topoi. Using Lemma 3.1.3.5, we conclude that $\mathcal{S}\mathrm{hv}_A^{\text{ét}}/U$ is also a 1-localic ∞-topos. Consequently, it will suffice to show that the adjoint functors (f^*, f_*) induce mutually inverse equivalences when restricted to discrete objects of $\mathcal{S}\mathrm{hv}_A^{\text{ét}}/U$ and $\mathcal{S}\mathrm{hv}_B^{\text{ét}}$.

We begin by showing that the functor f_* is conservative. Let $\alpha : \mathscr{F} \to \mathscr{F}'$ be a morphism in $\mathcal{S}\mathrm{hv}_B^{\text{ét}}$ such that $f_*(\alpha)$ is an equivalence. We will show that α induces an equivalence $\mathscr{F}(B') \to \mathscr{F}'(B')$ for every étale B-algebra B'. Using Proposition B.1.1.3, we deduce the existence of a pushout diagram

$$\begin{array}{ccc} A\{x_1, \ldots, x_m\} & \xrightarrow{g} & B \\ \downarrow & & \downarrow \\ A\{y_1, \ldots, y_m\}[\Delta^{-1}] & \longrightarrow & B', \end{array}$$

where the left vertical map carries each x_i to some polynomial $f_i(y_1, \ldots, y_m) \in \pi_0 A\{y_1, \ldots, y_m\}$ and $\Delta \in \pi_0 A\{y_1, \ldots, y_m\}$ denotes the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1 \leqslant i,j \leqslant m}$. Since the map $A \to B$ is surjective on π_0, the map g factors through A. Set $A' = A \otimes_{A\{x_1,\ldots,x_m\}} A\{y_1, \ldots, y_m\}[\Delta^{-1}]$. Then A' is an étale A-algebra. Since $f_*(\alpha)$ is an equivalence, we deduce that the natural map

$$\mathscr{F}(B') \simeq (f_* \mathscr{F})(A') \to (f_* \mathscr{F}')(A') \simeq \mathscr{F}'(B')$$

is an equivalence, as desired.

To complete the proof, it will suffice to show that if $\mathscr{F}$ is a discrete object of $\mathcal{S}\mathrm{hv}_A^{\text{ét}})/U$, then the unit map $u : \mathscr{F} \to f_* f^* \mathscr{F}$ is an equivalence. Since both $\mathscr{F}$ and $f_* f^* \mathscr{F}$ are discrete objects of $\mathcal{S}\mathrm{hv}_A^{\text{ét}}$, they are hypercomplete: it will therefore suffice to show that the map u is ∞-connective. According to Theorem A.4.0.5, the ∞-topos $(\mathcal{S}\mathrm{hv}_A^{\text{ét}})^{\mathrm{hyp}}$ has enough points. It will therefore suffice to show that for every geometric morphism $\eta^* : \mathcal{S}\mathrm{hv}_A^{\text{ét}} \to \mathcal{S}$, the map $\eta^*(u)$ is a homotopy equivalence.

According to Corollary 3.5.2.2, the geometric morphism η^* corresponds to a strictly Henselian A-algebra A' which can be written as a filtered colimit $\varinjlim A'_\alpha$, where each A'_α is étale over A. More precisely, the functor η^* is given by the formula $\eta^* \mathscr{G} \simeq \varinjlim \mathscr{G}(A'_\alpha)$. There are two cases to consider:

(1) Suppose that $A' \otimes_A B \simeq 0$. Then $1 = 0$ in $\pi_0(A' \otimes_A B) \simeq \varinjlim \pi_0(A'_\alpha \otimes_A B)$, so that $A'_\alpha \otimes_A B \simeq 0$ for some α. Reindexing our diagram, we may suppose that $A'_\alpha \otimes_A B \simeq 0$ for all α. Thus $\eta^* \mathscr{G} \simeq \varinjlim \mathscr{G}(A'_\alpha)$ is contractible whenever $\mathscr{G} \in \mathcal{S}\mathrm{hv}^{\text{ét}}_A /U$. In particular, $\eta^*(u)$ is a map between contractible spaces and therefore a homotopy equivalence.

(2) The tensor product $A' \otimes_A B \neq 0$. Note that $A' \otimes_A B \simeq \varinjlim A'_\alpha \otimes_A B$ is a filtered colimit of étale B-algebras. The map $A' \to A' \otimes_A B$ induces a surjection $\pi_0 A' \to \pi_0(A' \otimes_A B)$. It follows from Corollary B.3.3.2 that $A' \otimes_A B$ is strictly Henselian, and therefore determines a map $\eta'^* : \mathcal{S}\mathrm{hv}^{\text{ét}}_B \to \mathcal{S}$. Moreover, the composite map $(\mathcal{S}\mathrm{hv}(*), A' \otimes_A B) \to (\mathcal{S}\mathrm{hv}(*), A') \to \mathrm{Spét}\, A$ determines the same point of the ∞-topos $\mathcal{S}\mathrm{hv}^{\text{ét}}_A$; it follows that $\eta^* \simeq \eta'^* \circ f^*$. We therefore have a chain of equivalences

$$\begin{aligned} \eta^*(f_* f^* \mathscr{F}) &\simeq \varinjlim (f_* f^* \mathscr{F})(A'_\alpha) \\ &\simeq \varinjlim (f^* \mathscr{F})(A'_\alpha \otimes_A B) \\ &\simeq \eta'^* f^* \mathscr{F} \\ &\simeq \eta^* \mathscr{F} \end{aligned}$$

whose composition is a homotopy inverse to $\eta^*(u)$.

□

Lemma 3.1.4.3. *Let $f : (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ be a map of spectral Deligne-Mumford stacks, and assume that the underlying geometric morphism $\mathcal{X} \to \mathcal{Y}$ is a closed immersion in $\mathcal{T}\mathrm{op}_\infty$. Then the pushforward functor $f_* : \mathrm{Mod}_{\mathcal{O}_\mathcal{X}} \to \mathrm{Mod}_{\mathcal{O}_\mathcal{Y}}$ carries quasi-coherent sheaves to quasi-coherent sheaves.*

Proof. Note that f_* is left t-exact and commutes with small limits. It follows that if $\mathscr{F} \in \mathrm{Mod}_{\mathcal{O}_\mathcal{X}}$ is quasi-coherent, then

$$f_* \mathscr{F} \simeq f_* \varprojlim (\tau_{\leqslant n} \mathscr{F}) \simeq \varprojlim f_* \tau_{\leqslant n} \mathscr{F}$$

is the limit of a tower of truncated objects of $\mathrm{Mod}_{\mathcal{O}_\mathcal{X}}$, and therefore hypercomplete. According to Proposition 2.2.6.1, it will suffice to show that each homotopy group $\pi_n f_* \mathscr{F}$ is quasi-coherent. Since f_* is right t-exact (Lemma 3.1.3.3), we can replace $\mathscr{F}$ by $\tau_{\leqslant n} \mathscr{F}$. The desired result now follows from Corollary **??**, since Lemma 3.1.3.1 implies that f is k-quasi-separated for every integer $k \geqslant 0$. □

Proof of Theorem 3.1.2.1. We must prove two things:

(1) If $\phi : A \to B$ is a morphism of connective $\mathbb{E}_\infty$-rings which induces a surjective ring homomorphism $\pi_0 A \to \pi_0 B$, then the induced map $\operatorname{Sp\acute{e}t} B \to \operatorname{Sp\acute{e}t} A$ is a closed immersion of spectral Deligne-Mumford stacks.

(2) Every closed immersion $(\mathcal{X}, \mathcal{O}) \to \operatorname{Sp\acute{e}t} A$ arises in this way.

We first prove (1). For every connective $\mathbb{E}_\infty$-ring R, let $\operatorname{Sp\acute{e}t} R = (\mathcal{X}_R, \mathcal{O}_R)$. It follows from Proposition 3.1.4.1 that if $\phi : A \to B$ induces a surjection $\pi_0 A \to \pi_0 B$, then the corresponding map $f : (\mathcal{X}_B, \mathcal{O}_B) \to (\mathcal{X}_A, \mathcal{O}_A)$ induces a closed immersion of ∞-topoi $\mathcal{X}_B \to \mathcal{X}_A$. It remains to show that the map $f^{-1} \mathcal{O}_A \to \mathcal{O}_B$ has a connective fiber. Since f_* is a closed immersion, it suffices to show that the adjoint map $\mathcal{O}_A \simeq f_* f^{-1} \mathcal{O}_A \to f_* \mathcal{O}_B$ has a connective fiber. It follows from Lemma 3.1.4.3 that $f_* \mathcal{O}_B$ is a quasi-coherent sheaf on $(\mathcal{X}_A, \mathcal{O}_A)$. Since the equivalence $\operatorname{QCoh}(\mathcal{X}_A) \simeq \operatorname{Mod}_A$ is t-exact, it suffices to show that the map

$$A \simeq \Gamma(\mathcal{X}_A; \mathcal{O}_A) \to \Gamma(\mathcal{X}_A; f_* \mathcal{O}_B) \simeq \Gamma(\mathcal{X}_B; \mathcal{O}_B) \simeq B$$

has a connective fiber, which follows from our assumption on ϕ.

We now prove (2). Suppose we are given a closed immersion of spectral Deligne-Mumford stacks $f : (\mathcal{X}, \mathcal{O}) \to (\mathcal{X}_A, \mathcal{O}_A)$. Let $B = \Gamma(\mathcal{X}; \mathcal{O}) \simeq \Gamma(\mathcal{X}_A; f_* \mathcal{O})$. Using Lemmas 3.1.3.3 and 3.1.4.3, we deduce that $f_* \mathcal{O}$ is a connective quasi-coherent sheaf on $(\mathcal{X}_A, \mathcal{O}_A)$, so that B is a connective A-algebra. Moreover, since f is a closed immersion, the map $f^* \mathcal{O}_A \to \mathcal{O}$ is an epimorphism on π_0. Since f is a closed immersion, the adjoint map $\mathcal{O}_A \simeq f_* f^* \mathcal{O}_A \to f_* \mathcal{O}$ is a map of quasi-coherent sheaves with connective fiber. Since the equivalence $\operatorname{Mod}_A \simeq \operatorname{QCoh}(\mathcal{X}_A)$ is t-exact, we conclude that the map $A \to B$ is surjective on π_0. The universal property of $\operatorname{Sp\acute{e}t} B$ gives a commutative diagram

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{O}) & \xrightarrow{f'} & \operatorname{Sp\acute{e}t} B \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle f''} & \\ & \operatorname{Sp\acute{e}t} A. & \end{array}$$

Part (1) shows that f'' is a closed immersion of spectral Deligne-Mumford stacks. Using Lemma 3.1.3.4, we see that f' induces a closed immersion of ∞-topoi $\mathcal{X} \to \mathcal{X}_B$. Moreover, f' induces a map of quasi-coherent sheaves $\mathcal{O}_B \to f'_* \mathcal{O}$ which induces an equivalence on global sections; it follows that $\mathcal{O}_B \simeq f'_* \mathcal{O}$. We will complete the proof by showing that f' induces an equivalence of ∞-topoi $\mathcal{X} \simeq \mathcal{X}_B$. We have an equivalence $\mathcal{X} \simeq \mathcal{X}_B / U$ for some (-1)-truncated object $U \in \mathcal{X}_B$; we wish to show that U is an initial object of $\mathcal{X}_B$. The proof of Proposition 1.4.2.4 shows that we can identify $\mathcal{X}_B$ with the ∞-topos $\mathcal{S}\mathrm{hv}_B^{\acute{e}t}$. If U is not an initial object, then $U(B')$ is nonempty for some nonzero étale B-algebra B'. The equivalences $B' \simeq \mathcal{O}_B(B') \simeq (f'_* \mathcal{O})(B') \simeq 0$ now yield a contradiction. $\square$

3.1.5 Example: Schematic Images

Let $f : X \to Y$ be a quasi-compact morphism of schemes. Then there is a smallest closed subscheme $Y_0 \subseteq Y$ such that f factors through Y_0. The closed subscheme $Y_0 \subseteq Y$ is called the *schematic image* (or *scheme-theoretic image*) of the morphism f. We now describe a slight generalization of this construction.

Construction 3.1.5.1 (Schematic Images)**.** Let $f : \mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) = \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Assume that the map f is ∞-quasi-compact and that the structure sheaf $\mathscr{O}_{\mathcal{X}}$ is discrete. Then the direct image $f_* \mathscr{O}_{\mathcal{X}} \in \mathrm{Mod}_{\mathscr{O}_{\mathcal{Y}}}$ is quasi-coherent (Theorem 2.5.8.1) and 0-truncated. It follows that the unit map $\mathscr{O}_{\mathcal{Y}} \to f_* \mathscr{O}_{\mathcal{X}}$ factors as a composition

$$\mathscr{O}_{\mathcal{Y}} \to \pi_0 \mathscr{O}_{\mathcal{Y}} \xrightarrow{\alpha} \pi_0 \mathscr{O}_{\mathcal{X}} \to \mathscr{O}_{\mathcal{X}} .$$

Let $\mathscr{A}$ denote the image of α: then $\mathscr{A}$ is a connective commutative algebra object of $\mathrm{QCoh}(\mathsf{Y})$ for which the unit map $\mathscr{O}_{\mathcal{Y}} \to \mathscr{A}$ is an epimorphism. Using Corollary 3.1.1.3 and Remark 3.1.2.7, we see that there exists an essentially unique closed immersion $\iota : \mathsf{Y}_0 \hookrightarrow (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ for which $\mathscr{A}$ is the direct image of the structure sheaf of Y_0. Note that Y_0 is 0-truncated, and Proposition 3.1.1.1 shows that f admits an essentially unique factorization $\mathsf{X} \xrightarrow{f'} \mathsf{Y}_0 \xrightarrow{\iota} \mathsf{Y}$. We will refer to Y_0 as the *schematic image of* f.

Warning 3.1.5.2. The term "schematic image" is potentially misleading: in general, the spectral Deligne-Mumford stack Y_0 which appears in Construction 3.1.5.1 will not be schematic. However, it is schematic if Y is schematic (since it is affine over Y).

Remark 3.1.5.3. In the situation of Construction 3.1.5.1, it is not necessary to assume that f is ∞-quasi-compact: we really only need the quasi-coherence of the ideal sheaf $\ker(\alpha)$, which requires only that f is quasi-compact.

Remark 3.1.5.4. In the situation of Construction 3.1.5.1, we do not need to require that Y is 0-truncated. However, there is no harm in doing so: if X is 0-truncated, then any map $f : \mathsf{X} \to \mathsf{Y}$ automatically factors through the 0-truncation $\tau_{\leqslant 0} \mathsf{Y}$, and the schematic image of f is equivalent to the schematic image of the induced map $\mathsf{X} \to \tau_{\leqslant 0} \mathsf{Y}$. In other words, Construction 3.1.5.1 really belongs to the setting of classical algebraic geometry.

Remark 3.1.5.5. Let X be a 0-truncated spectral Deligne-Mumford stack. Then the schematic image of an ∞-quasi-compact morphism $f : \mathsf{X} \to \mathsf{Y}$ is initial among 0-truncated closed substacks Y_0 of Y such that f factors through Y_0.

Remark 3.1.5.6. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Assume that X is 0-truncated and that f is ∞-quasi-compact, and let Y_0 denote the schematic image of f. Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{f'} & \mathsf{Y}' \\ \downarrow & & \downarrow g \\ \mathsf{X} & \xrightarrow{f} & \mathsf{Y}. \end{array}$$

If g is flat, then X' is also 0-truncated, and we can identify the fiber product $\mathsf{Y}'_0 = \mathsf{Y}' \times_{\mathsf{Y}} \mathsf{Y}_0$ with the schematic image of f'. Beware that this is generally not true if g is not flat.

3.1.6 Reduced Closed Substacks

If X is a topological space, then there is a bijection from the set of closed subsets of X to the set of open subsets of X, given by $Y \mapsto X - Y$. In algebraic geometry, the situation is more subtle. Every closed subscheme Y of a scheme X has an open complement $U = X - Y$. However, this construction is not bijective: a closed subset of X generally admits many different scheme structures. Nevertheless, we can recover a bijective correspondence by restricting our attention to *reduced* closed subschemes of X. There is an entirely analogous picture in the setting of spectral algebraic geometry.

Definition 3.1.6.1. Let X be a spectral Deligne-Mumford stack. We will say that X is *reduced* if, for every étale map $\operatorname{Spét} R \to \mathsf{X}$, the $\mathbb{E}_\infty$-ring R is discrete and the underlying commutative ring.

Remark 3.1.6.2. The condition that a spectral Deligne-Mumford stack X be reduced is local with respect to the étale topology.

Proposition 3.1.6.3. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{SpDM}_{/\mathsf{X}}$ spanned by the closed immersions $i : \mathsf{X}_0 \to \mathsf{X}$, where X_0 is reduced. For every object $(i : \mathsf{X}_0 \to \mathsf{X})$ of $\mathrm{SpDM}_{/\mathsf{X}}$, let X_0^c denote the pushforward $i_*(\emptyset)$, where $\emptyset$ denotes an initial object in the underlying ∞-topos of X_0. Then the construction $(i : \mathsf{X}_0 \to \mathsf{X}) \mapsto \mathsf{X}_0^c$ determines an equivalence of ∞-categories $\mathcal{C} \simeq (\tau_{\leqslant -1} \mathcal{X})^{\mathrm{op}}$.*

Remark 3.1.6.4. We can state Proposition 3.1.6.3 more informally as follows: there is an order-reversing bijection between equivalence classes of reduced closed substacks of X and open substacks of X.

3.1.7 Clopen Immersions

In some cases, closed immersions are also open immersions:

Proposition 3.1.7.1. *Let $\mathcal{X}$ be an ∞-topos containing a (-1)-truncated object U, and let $i_* : \mathcal{X}_{/U} \to \mathcal{X}$ be the corresponding closed immersion of ∞-topoi. The following conditions are equivalent:*

(1) *The geometric morphism i_* is étale.*

(2) *The (-1)-truncated object U is complemented: that is, there exists an object $U' \in \mathcal{X}$ such that the coproduct $U \amalg U'$ is a final object of $\mathcal{X}$.*

Proof. The implication (2) ⇒ (1) is clear: if $U \amalg U'$ is a final object of $\mathcal{X}$, then the construction $X \mapsto U \amalg X$ determines an equivalence of ∞-categories $\mathcal{X}_{/U'} \to \mathcal{X}/U$. Conversely, suppose that (2) is satisfied. Then the pullback functor $i^* : \mathcal{X} \to \mathcal{X}/U$ admits a left adjoint $i_!$. Let $\mathbf{1}$ denote a final object of $\mathcal{X}$ and let $V = i_! i^* \mathbf{1}$. Then

$$U \times V \simeq U \times i_! i^* \mathbf{1} \simeq i_!(i^* U \times i^* \mathbf{1}) \simeq i_! i^*(U).$$

Since $i^*(U)$ is an initial object of $\mathcal{X}/U$, the object $U \times V \simeq i_! i^* U$ is an initial object of $\mathcal{X}$. Let $U' = \tau_{\leqslant -1} V$, so that $U \times U'$ is an initial object of $\mathcal{X}$. It follows that $U \amalg U'$ is also a (-1)-truncated object of $\mathcal{X}$. The identity map $\mathrm{id} : i_! i^* \mathbf{1} \to V$ induces a map $i^* \mathbf{1} \to i^* V$ in $\mathcal{X}/U$, which determines a map

$$\mathbf{1} \simeq i_* i^* \mathbf{1} \to i_* i^* V \simeq U \amalg_{U \times V} V \simeq U \amalg V.$$

Composing with the projection map $V \to \tau_{\leqslant -1} V$, we obtain a map $\mathbf{1} \to U \amalg U'$, so that $U \amalg U'$ is a (-1)-truncated, 0-connective object of $\mathcal{X}$ and therefore a final object of $\mathcal{X}$. □

Definition 3.1.7.2. Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. We will say that f_* is a *clopen immersion* if it satisfies the equivalent conditions of Proposition 3.1.7.1: that is, if it is both étale and a closed immersion. We will say that a map of nonconnective spectral Deligne-Mumford stacks $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a *clopen immersion* if it is étale and the underlying geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ is a clopen immersion of ∞-topoi.

Remark 3.1.7.3. Every clopen immersion of spectral Deligne-Mumford stacks is also a closed immersion; in particular, it is an affine map. Let R be a connective $\mathbb{E}_\infty$-ring. A map of spectral Deligne-Mumford stacks $\mathsf{X} \to \mathrm{Spét}\, R$ is a clopen immersion if and only if X has the form $\mathrm{Spét}\, R[e^{-1}]$, where e is an idempotent element in the commutative ring $\pi_0 R$.

Proof. The assertion is local on X. We may therefore assume without loss of generality that $\mathsf{X} = \mathrm{Spét}\, R$ is affine. Using Theorem 3.1.2.1, we see that every closed immersion $i : \mathsf{X}_0 \to \mathsf{X}$ is induced by a map of connective $\mathbb{E}_\infty$-rings $R \to R'$ which induces a surjection $\pi_0 R \to \pi_0 R'$. Moreover, X_0 is reduced if and only if R' is a discrete commutative ring of the form $(\pi_0 R)/I$ for some radical ideal $I \subseteq \pi_0 R$. It follows that $\mathrm{Sub}^{\mathrm{red}}(\mathsf{X})$ is equivalent to the the nerve of the partially ordered set of closed subsets of the Zariski spectrum $|\mathrm{Spec}\, R|$. The desired equivalence now follows from Lemma **??**. □

3.2 Separated Morphisms

Recall that a scheme X is said to be *separated* if the diagonal map $X \to X \times X$ is a closed immersion. In this section, we will study the analogous condition in the setting of spectral algebraic geometry.

Definition 3.2.0.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. We will say that f is *separated* if the diagonal morphism $\mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ is a closed immersion. We will say that a spectral Deligne-Mumford stack X is *separated* if it the morphism $\mathsf{X} \to \operatorname{Sp\acute{e}t} S$ is separated; here S denotes the sphere spectrum. In other words, X is separated if the absolute diagonal $\mathsf{X} \to \mathsf{X} \times \mathsf{X}$ is a closed immersion.

Warning 3.2.0.2. Our Definition 3.2.0.1 is somewhat nonstandard. In the theory of algebraic stacks, it is traditional to say that a map $\mathsf{X} \to \mathsf{Y}$ is separated if the diagonal map $\mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ is proper, rather than a closed immersion. Our Definition 3.2.0.1 corresponds to the usual notion of a *separated representable* morphism of stacks: that is, a separated morphism whose fibers are algebraic spaces.

3.2.1 Properties of Separated Morphisms

We begin with some general observations about Definition 3.2.0.1.

Remark 3.2.1.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a separated morphism of spectral Deligne-Mumford stacks. It follows from Warning 3.1.2.6 that for every discrete commutative ring R, the map $\operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Sp\acute{e}t} R, \mathsf{X}) \to \operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Sp\acute{e}t} R, \mathsf{Y})$ is 0-truncated. In particular, if Y is a spectral algebraic space, then X is also a spectral algebraic space. In particular, every separated spectral Deligne-Mumford stack is a spectral algebraic space.

Remark 3.2.1.2. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks, and let $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ be the diagonal map. Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ and $\mathsf{X} \times_{\mathsf{Y}} \mathsf{X} = (\mathcal{Z}, \mathscr{O}_{\mathcal{Z}})$. The map $\delta^{-1} \mathscr{O}_{\mathcal{Z}} \to \mathscr{O}_{\mathcal{X}}$ admits a right inverse, and therefore induces an epimorphism $\pi_0 \delta^{-1} \mathscr{O}_{\mathcal{Z}} \to \pi_0 \mathscr{O}_{\mathcal{X}}$. It follows f is separated if and only if the underlying geometric morphism of ∞-topoi $\delta_* : \mathcal{X} \to \mathcal{Z}$ is a closed immersion.

Remark 3.2.1.3. If $j : \mathsf{U} \to \mathsf{X}$ is an open immersion of spectral Deligne-Mumford stacks, then the diagonal map $\mathsf{U} \to \mathsf{U} \times_{\mathsf{X}} \mathsf{U}$ is an equivalence. It follows that every open immersion between spectral Deligne-Mumford stacks is separated. In particular, if X is a separated spectral algebraic space, then U is also a separated spectral algebraic space.

Remark 3.2.1.4. Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{f'} & \mathsf{X} \\ \downarrow & & \downarrow f \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y}. \end{array}$$

If f is separated, then so is f'; this follows immediately from Corollary 3.1.2.3. The converse holds if g is an étale surjection (see Remark 3.1.2.2).

Remark 3.2.1.5. Suppose we are given a commutative diagram

$$\begin{array}{ccc} & \mathsf{X} & \\ {}^{f}\swarrow & & \searrow^{h} \\ \mathsf{Y} & \xrightarrow{g} & \mathsf{Z} \end{array}$$

of spectral Deligne-Mumford stacks. If g is separated, then f is separated if and only if h is separated. In particular, the collection of separated morphisms is closed under composition. To see this, consider the diagram

$$\begin{array}{ccccc} \mathsf{X} & \xrightarrow{\delta} & \mathsf{X} \times_{\mathsf{Y}} \mathsf{X} & \xrightarrow{\delta'} & \mathsf{X} \times_{\mathsf{Z}} \mathsf{X} \\ & & \downarrow & & \downarrow \\ & & \mathsf{Y} & \xrightarrow{\delta''} & \mathsf{Y} \times_{\mathsf{Z}} \mathsf{Y}. \end{array}$$

Since g is separated, δ'' is a closed immersion. It follows from Corollary 3.1.2.3 that δ' is a closed immersion, so that δ is a closed immersion if and only if $\delta' \circ \delta$ is a closed immersion (Corollary 3.1.2.4).

Remark 3.2.1.6. Suppose we are given morphisms of spectral Deligne-Mumford stacks $\mathsf{X} \xrightarrow{f} \mathsf{Y} \xleftarrow{g} \mathsf{Z}$. If f and g are separated, then the induced map $\mathsf{X} \times_{\mathsf{Y}} \mathsf{Z} \to \mathsf{Y}$ is also separated. This follows immediately from Remarks 3.2.1.5 and 3.2.1.4.

Remark 3.2.1.7. Let $f : (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be a morphism of spectral Deligne-Mumford stacks. The condition that f be separated depends only the underlying morphism of 0-truncated spectral Deligne-Mumford stacks $(\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathscr{O}_{\mathcal{Y}})$.

Example 3.2.1.8. Let R be a connective $\mathbb{E}_\infty$-ring. Then the multiplication map $R \otimes R \to R$ induces a surjection $\pi_0(R \otimes R) \to \pi_0 R$. It follows from Theorem 3.1.2.1 that the diagonal map $\operatorname{Spét} R \to \operatorname{Spét} R \times \operatorname{Spét} R$ is a closed immersion, so that $\operatorname{Spét} R$ is separated.

Example 3.2.1.9. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks. Suppose that, for every commutative ring R, the induced map $\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X}) \to \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{Y})$ is (-1)-truncated. It follows that the diagonal map

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X}) \to \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X} \times_{\mathsf{Y}} \mathsf{X})$$

is a homotopy equivalence for every discrete commutative ring R. The map $\mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ induces an equivalence between the underlying 0-truncated spectral Deligne-Mumford stacks, and is therefore a closed immersion (Remark 3.2.1.7). It follows that f is separated.

Remark 3.2.1.10. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks. Assume that Y is separated. Then X is separated if and only if f is separated: this follows immediately from Remark 3.2.1.5. In particular, if we are given a map $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ for some connective $\mathbb{E}_\infty$-ring R, then X is separated if and only if f is separated: that is, if and only if the diagonal map $\mathsf{X} \to \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ is a closed immersion.

Example 3.2.1.11. Let $f : \mathsf{X} \to \mathsf{Y}$ be a quasi-affine morphism between spectral Deligne-Mumford stacks. Then f is separated. To prove this, we may work locally on Y to reduce to the case where Y is affine, in which case the result follows from Remark 3.2.1.10 and Examples 3.2.1.8 and 3.2.1.9.

3.2.2 Configuration Spaces

Let X be a separated spectral Deligne-Mumford stack. Then the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times \mathsf{X}$ is a closed immersion, which is complementary to an open substack $\mathrm{Conf}^2(\mathsf{X}) \subseteq \mathsf{X} \times \mathsf{X}$. Informally, we can regard $\mathrm{Conf}^2(\mathsf{X})$ as a "parameter space" for pairs of distinct points of X. We now consider a variant of this construction where we work relative to a base Y, and contemplate n-tuples of points in place of pairs of points:

Construction 3.2.2.1. Suppose we are given a separated morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathsf{X} \times_{\mathsf{Y}} \mathsf{X} = (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$, so that the diagonal map induces a closed immersion of ∞-topoi $\delta_* : \mathcal{X} \to \mathcal{Z}$. Let $\emptyset$ denote an initial object of $\mathcal{X}$ and let $U = \delta_*(\emptyset)$, so that U is a (-1)-truncated object of $\mathcal{Z}$ and δ_* induces an equivalence of ∞-topoi $\mathcal{X} \to \mathcal{Z}_{/U}$.

For every finite set I, let $\overline{\mathrm{Conf}}^I_{\mathsf{Y}}(\mathsf{X})$ denote the I-fold product of X with itself in the ∞-category $\mathrm{SpDM}_{/\mathsf{Y}}$. For every pair of distinct elements $i, j \in I$, we obtain an evaluation map

$$p_{i,j} : \overline{\mathrm{Conf}}^I_{\mathsf{Y}}(\mathsf{X}) \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}.$$

Let V denote the product $\prod_{i \neq j} p^*_{i,j}(U)$ in the underlying ∞-topos of $\overline{\mathrm{Conf}}^I_{\mathsf{Y}}(\mathsf{X})$. We let $\mathrm{Conf}^I_{\mathsf{Y}}(\mathsf{X})$ denote the open substack of $\overline{\mathrm{Conf}}^I_{\mathsf{Y}}(\mathsf{X})$ corresponding to the (-1)-truncated object

V. We will refer to $\mathrm{Conf}^I_{\mathsf{Y}}(\mathsf{X})$ as the *spectral Deligne-Mumford stack of I-configurations in* X (relative to Y).

Note that $\mathrm{Conf}^I_{\mathsf{Y}}(\mathsf{X})$ depends functorially on I. In particular, it is acted on by the group of all permutations of I, and (up to equivalence) depends only on the cardinality of the set I. When $I = \{1, 2, \ldots, n\}$, we will denote $\mathrm{Conf}^I_{\mathsf{Y}}(\mathsf{X})$ by $\mathrm{Conf}^n_{\mathsf{Y}}(\mathsf{X})$, so that $\mathrm{Conf}^n_{\mathsf{Y}}(\mathsf{X})$ carries an action of the symmetric group Σ_n.

Remark 3.2.2.2. In the situation of Construction 3.2.2.1, the projection map $\overline{\mathrm{Conf}}^I_{\mathsf{Y}}(\mathsf{X}) \to \mathsf{Y}$ is separated by Remark 3.2.1.6. The open immersion $\mathrm{Conf}^I_{\mathsf{Y}}(\mathsf{X}) \to \overline{\mathrm{Conf}}^I_{\mathsf{Y}}(\mathsf{X})$ is separated (Remark 3.2.1.3), so the projection $\mathrm{Conf}^I_{\mathsf{Y}}(\mathsf{X}) \to \mathsf{Y}$ is also separated (Remark 3.2.1.5).

Notation 3.2.2.3. Let X be a spectral Deligne-Mumford stack. If G is a discrete group, an *action* of G on X is a diagram $\chi : BG \to \mathrm{SpDM}$ carrying the base point of BG to X. Since every morphism in BG is an equivalence, χ is automatically a diagram consisting of étale morphisms in SpDM, so there exists a colimit $\varinjlim(\chi)$ of the diagram χ (Proposition **??**). We will denote this colimit by X/G, and refer to it as the *quotient of* X *by the action of* G. There is an evident étale surjection $\mathsf{X} \to \mathsf{X}/G$. Moreover, there is a canonical equivalence

$$\mathsf{X} \times_{\mathsf{X}/G} \mathsf{X} \simeq \amalg_{g \in G} \mathsf{X}.$$

The main property of Construction 3.2.2.1 we will need is the following:

Proposition 3.2.2.4. *Let Y be a separated spectral algebraic space and let $f : \mathsf{X} \to \mathsf{Y}$ be an étale map, where X is affine. For every $n > 0$, the quotient $\mathrm{Conf}^n_{\mathsf{Y}}(\mathsf{X})/\Sigma_n$ is affine.*

We will deduce Proposition 3.2.2.4 from the following general observation:

Proposition 3.2.2.5. *Let R be an $\mathbb{E}_\infty$-ring equipped with an action of a finite group G, and let R^G denote the $\mathbb{E}_\infty$-ring of invariants. Suppose that the action of G on the commutative ring $\pi_0 R$ is free (see Definition B.7.1.2). Then the canonical map $(\mathrm{Sp\acute{e}t}\, R)/G \to \mathrm{Sp\acute{e}t}\, R^G$ is an equivalence of spectral Deligne-Mumford stacks. In particular, the quotient $(\mathrm{Sp\acute{e}t}\, R)/G$ is affine.*

Proof. Let $\mathsf{X}_\bullet$ be the Čech nerve of the map $\mathrm{Sp\acute{e}t}\, R \to (\mathrm{Sp\acute{e}t}\, R)/G$, and let $\mathsf{Y}_\bullet$ be the Čech nerve of the map $\mathrm{Sp\acute{e}t}\, R \to \mathrm{Sp\acute{e}t}\, R^G$. It follows from Corollary **??** that the map $R^G \to R$ is faithfully flat and étale, so that the vertical maps in the diagram

$$\begin{array}{ccc} |\mathsf{X}_\bullet| & \longrightarrow & |\mathsf{Y}_\bullet| \\ \downarrow & & \downarrow \\ (\mathrm{Sp\acute{e}t}\, R)/G & \longrightarrow & \mathrm{Sp\acute{e}t}\, R^G \end{array}$$

are equivalences. It will therefore suffice to show that the canonical map $\mathsf{X}_n \to \mathsf{Y}_n$ is an equivalence for every integer n. Since $\mathsf{X}_\bullet$ and $\mathsf{Y}_\bullet$ are groupoid objects of SpDM, we only need to consider the cases $n = 0$ and $n = 1$. When $n = 0$, the result is obvious. When $n = 1$, we must show that the canonical map

$$\amalg_{g \in G} \mathsf{X} \rightarrow \operatorname{Spec}(R \otimes_{R^G} R)$$

is an equivalence. Equivalently, we must show that the canonical map

$$R \otimes_{R^G} R \rightarrow \prod_{g \in G} R$$

is an equivalence of $\mathbb{E}_\infty$-rings, which follows from Corollary **??**. □

Lemma 3.2.2.6. *Let $f : \mathsf{X} \rightarrow \mathsf{Y}$ be a map of spectral algebraic spaces. If f is separated, then the mapping space $\operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} R, \operatorname{Conf}^n_{\mathsf{Y}}(\mathsf{X}))$ is discrete for every commutative ring R and every integer $n \geqslant 0$. If R is nonzero, then the symmetric group Σ_n acts freely on the set $\pi_0 \operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} R, \operatorname{Conf}^n_{\mathsf{Y}}(\mathsf{X}))$.*

Proof. For any commutative ring R, the map $\theta : \operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} R, \mathsf{X}) \rightarrow \operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} R, \mathsf{Y})$ has discrete homotopy fibers (Remark 3.2.1.1). Since the codomain of θ is discrete, we conclude that the domain of θ is also discrete. Let $S = \pi_0 \operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} R, \mathsf{X})$ and $T = \pi_0 \operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} R, \mathsf{Y})$. There is an evident injection from $\pi_0 \operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} A, \operatorname{Conf}^n_{\mathsf{Y}}(\mathsf{X}))$ to the set $K = S \times_T \cdots \times_T S$ given by the n-fold fiber power of S over T. If $\sigma \in \Sigma_n$ is a nontrivial permutation which fixes an element $(s_1, \ldots, s_n)$ of K, then we must have $s_i = s_j$ for some $i \neq j$, in which case the corresponding map $\operatorname{Spét} R \rightarrow \overline{\operatorname{Conf}}^n_{\mathsf{Y}}(\mathsf{X}) \rightarrow \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ factors through the diagonal. By construction, the fiber product $\mathsf{X} \times_{\mathsf{X} \times_{\mathsf{Y}} \mathsf{X}} \operatorname{Conf}^n_{\mathsf{Y}}(\mathsf{X})$ is empty, which is impossible unless $R \simeq 0$. □

Remark 3.2.2.7. Let $f : \mathsf{X} \rightarrow \mathsf{Y}$ be a separated étale morphism of spectral Deligne-Mumford stacks. Then the diagonal map $\mathsf{X} \rightarrow \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ is a clopen immersion (see Definition 3.1.7.2). If we write $\mathsf{X} \times_{\mathsf{Y}} \mathsf{X} = (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ and define $U \in \mathcal{Z}$ as in Construction 3.2.2.1, then it follows that U has a complement (in the underlying locale of $\mathcal{Z}$). It follows that for any finite set I, the object $V = \prod_{i \neq j} p^*_{i,j}(U)$ appearing in Construction 3.2.2.1 has a complement in the underlying locale of $\overline{\operatorname{Conf}}^I_{\mathsf{Y}}(\mathsf{X})$, so that the open immersion $\operatorname{Conf}^I_{\mathsf{Y}}(\mathsf{X}) \rightarrow \overline{\operatorname{Conf}}^I_{\mathsf{Y}}(\mathsf{X})$ is also a clopen immersion.

Proposition 3.2.2.8. *Let Y be a separated spectral algebraic space. Suppose we are given an étale map $\mathsf{X} \rightarrow \mathsf{Y}$. If X is affine, then $\operatorname{Conf}^n_{\mathsf{Y}}(\mathsf{X})$ is affine for every $n > 0$.*

Proof. Since the diagonal of Y is affine, the fiber product

$$\overline{\operatorname{Conf}}^n_{\mathsf{Y}}(\mathsf{X}) \simeq \mathsf{X} \times_{\mathsf{Y}} \cdots \times_{\mathsf{Y}} \mathsf{X}$$

is affine. The desired result now follows from Remarks 3.2.2.7 and 3.1.7.3. □

Proof of Proposition 3.2.2.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be an étale morphism of separated spectral algebraic spaces, where X is affine. Proposition 3.2.2.8 implies that $\mathrm{Conf}^n_{\mathsf{Y}}(\mathsf{X})$ is affine, hence of the form $\mathrm{Spét}\, R$ for some connective $\mathbb{E}_\infty$-ring R. According to Proposition 3.2.2.5, it will suffice to show that the action of the symmetric group Σ_n on R is free, which follows from Lemma 3.2.2.6. □

3.2.3 Existence of Scallop Decompositions (Separated Case)

We can now state the main result of this section:

Proposition 3.2.3.1. *Let Y be a quasi-compact separated spectral algebraic space. Then Y admits a scallop decomposition (see Definition 2.5.3.1).*

Remark 3.2.3.2. In §3.4, we will prove that the hypothesis of separatedness can be replaced by quasi-separatedness; see Theorem 3.4.2.1.

Corollary 3.2.3.3. *Let $f : \mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathsf{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a quasi-compact separated morphism of spectral Deligne-Mumford stacks. Then:*

(1) *The pushforward functor $f_* : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ carries quasi-coherent sheaves to quasi-coherent sheaves.*

(2) *The induced functor $\mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ commutes with small colimits.*

(3) *For every pullback diagram*

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{g'} & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y}, \end{array}$$

the associated diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{Y}) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathsf{X}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{Y}') & \xrightarrow{f'^*} & \mathrm{QCoh}(\mathsf{X}') \end{array}$$

is right adjointable. In other words, for every object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, the canonical map $\lambda : g^* f_* \mathscr{F} \to f'_* g'^* \mathscr{F}$ is an equivalence in $\mathrm{QCoh}(\mathsf{Y}')$.

Proof. Combine Propositions 2.5.4.3, 2.5.4.5, and 3.2.3.1. □

We now turn to the proof of Proposition 3.2.3.1. We will need a bit of commutative algebra.

Lemma 3.2.3.4. *Let $f : R \to R'$ be an étale morphism of commutative rings. For every prime ideal $\mathfrak{p} \subseteq R$, let $\kappa(\mathfrak{p})$ denote the residue field of R at the prime ideal $\mathfrak{p}$, and let $r(\mathfrak{p})$ denote the dimension of the $\kappa(\mathfrak{p})$-vector space $R' \otimes_R \kappa(\mathfrak{p})$. Then:*

(1) *For every integer n, the set $\{\mathfrak{p} \in |\operatorname{Spec} R| : r(\mathfrak{p}) > n\}$ is quasi-compact and open in $|\operatorname{Spec} R|$.*

(2) *The function r is constant with value $n \in \mathbf{Z}$ if and only if f exhibits R' as a locally free R-module of degree n.*

(3) *The function r is bounded above.*

Proof. We will prove (1) and (2) using induction on n. We begin with the case $n = 0$. In this case, assertion (2) is obvious, and assertion (1) follows from the fact that the map $|\operatorname{Spec} R'| \to |\operatorname{Spec} R|$ has quasi-compact open image.

Now suppose $n > 0$. We first prove (1). Let $U = \{\mathfrak{p} \in |\operatorname{Spec} R| : r(\mathfrak{p}) > 0\}$. and let $V = \{\mathfrak{p} \in |\operatorname{Spec} R| : r(\mathfrak{p}) > n\}$. The inductive hypothesis implies that U is open, so it will suffice to show that V is a quasi-compact open subset of U. Using Proposition 1.6.2.2, we deduce that the map $\phi : |\operatorname{Spec} R'| \to U$ is a quotient map; it will therefore suffice to show that $\phi^{-1}U$ is a quasi-compact open subset of $|\operatorname{Spec} R'|$. Since f is étale, the tensor product $R' \otimes_R R'$ factors as a product $R' \times R''$. Then the set

$$\phi^{-1}U = \{\mathfrak{q} \in |\operatorname{Spec} R'| : \dim_{\kappa(\mathfrak{q})}(R'' \otimes_{R'} \kappa(\mathfrak{q})) > n-1\}$$

is open by the inductive hypothesis.

It remains to prove (2). The "only if" direction is obvious. For the converse, assume that r is a constant function with value $n > 0$. Then $U = |\operatorname{Spec} R|$, so f is faithfully flat. It therefore suffices to show that $R' \otimes_R R'$ is a locally free R'-module of rank n. This is equivalent to the requirement that R'' be a locally free R'-module of rank $(n-1)$, which follows from the inductive hypothesis.

We now prove (3). Using (1), we see that each of the sets $\{\mathfrak{p} \in |\operatorname{Spec} R| : r(\mathfrak{p}) > n\}$ is closed with respect to the constructible topology on $|\operatorname{Spec} R|$ (see §4.3). Since

$$\bigcap_n \{\mathfrak{p} \in |\operatorname{Spec} R| : r(\mathfrak{p}) > n\} = \varnothing$$

and $|\operatorname{Spec} R|$ is compact with respect to the constructible topology, we conclude that there exists an integer n such that $\{\mathfrak{p} \in |\operatorname{Spec} R| : r(\mathfrak{p}) > n\} = \varnothing$. □

Proof of Proposition 3.2.3.1. Since Y is quasi-compact, we can choose an étale surjection $u : \mathsf{X} \to \mathsf{Y}$, where X is affine. For every map $\eta : \operatorname{Spét} A \to \mathsf{Y}$, the pullback $\mathsf{X} \times_{\mathsf{Y}} \operatorname{Spét} A$ has the form $\operatorname{Spét} A'$, for some étale A-algebra A'. Let $r_\eta : |\operatorname{Spec} A| \to \mathbf{Z}_{\geqslant 0}$ be defined by the formula

$$r_\eta(\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})}(A' \otimes_A \kappa(\mathfrak{p})).$$

Using Lemma 3.2.3.4, we can define open immersions $\mathsf{V}_i \hookrightarrow \mathsf{Y}$ so that the following universal property is satisfied: a map $\eta : \operatorname{Spét} A \to \mathsf{Y}$ factors through V_i if and only if $r_\eta(\mathfrak{p}) \geqslant i$ for every prime ideal $\mathfrak{p} \subseteq \pi_0 A$. Lemma 3.2.3.4 implies that the fiber product $\operatorname{Spét} A \times_{\mathsf{Y}} \mathsf{V}_i$ is empty for $i \gg 0$. Using the quasi-compactness of Y, we conclude that there exists an integer n such that V_{n+1} is empty. The surjectivity of u guarantees that $\mathsf{V}_1 \simeq \mathsf{Y}$. For $0 \leqslant i \leqslant n$, let $U_i \in \mathcal{Y}$ be the (-1)-truncated object corresponding to the open substack V_{n+1-i}. We claim that the sequence of morphisms

$$U_0 \to U_1 \to \cdots \to U_n$$

gives a scallop decomposition of Y.

Note that each $0 \leqslant i < n$, the étale map $\operatorname{Conf}^{n-i}_{\mathsf{Y}}(\mathsf{X})/\Sigma_{n-i} \to \mathsf{Y}$ determines an object $X_i \in \mathcal{Y}$. It follows from Proposition 3.2.2.4 that X_i is affine. Choose an equivalence $\operatorname{Conf}^{n-i}_{\mathsf{Y}}(\mathsf{X})/\Sigma_{n-i} \simeq \operatorname{Spét} R_i$, so that we have an étale map $v_i : \operatorname{Spét} R_i \to \mathsf{Y}$. For every map $\eta : \operatorname{Spét} A \to \mathsf{Y}$, choose an equivalence $\operatorname{Spét} A \times_{\mathsf{Y}} (\operatorname{Conf}^{n-i}_{\mathsf{Y}}(\mathsf{X})/\Sigma_i) \simeq \operatorname{Spét} A^{(i)}$, and define a function $r^{(i)}_\eta : |\operatorname{Spec} A| \to \mathbf{Z}_{\geqslant 0}$ by the formula

$$r^{(i)}_\eta(\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})}(A^{(i)} \otimes_A \kappa(\mathfrak{p})).$$

An easy calculation shows that $r^{(i)}_\eta(\mathfrak{p})$ is equal to the binomial coefficient $\binom{r_\eta(\mathfrak{p})}{n-i}$. In particular, $r^{(i)}_\eta(\mathfrak{p})$ takes positive values if and only if $r^{(1)}_\eta(\mathfrak{p}) \geqslant n-i$ for every $\mathfrak{p} \in |\operatorname{Spec} A|$. It follows that the map v_i factors through V_i. Form a pullback diagram σ_i:

$$\begin{array}{ccc} X_i \times_{U_{i+1}} U_i & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & U_{i+1}. \end{array}$$

Note that there is an effective epimorphism

$$\amalg_{0 \leqslant j < i} X_i \times X_j \to X_i \times_{U_{i+1}} U_i.$$

Since Y is separated, each product $X_i \times X_j$ is affine and therefore quasi-compact, so that $X_i \times_{U_{i+1}} U_i$ is quasi-compact.

To complete the proof, it will suffice to show that each σ_i is an excision square. For this, we may replace Y by the reduced closed substack K_i of U_{i+1} which is complementary to U_i, and thereby reduce to the case where the function r_η takes the constant value i, for every $\eta : \operatorname{Spét} A \to \mathsf{Y}$. In this case, the function $r^{(i)}_\eta$ is constant with value 1, so that the map $A \to A^{(i)}$ is finite étale of degree 1 and therefore an equivalence (Lemma 3.2.3.4). It follows that the map $\operatorname{Spét} R_i \to \mathsf{Y}$ is also an equivalence, as desired. $\square$

3.3 Quasi-Finite Morphisms

Recall that a morphism $\phi : A \to B$ of commutative rings is said to be *quasi-finite* if the following conditions are satisfied:

(i) The commutative ring B is finitely generated as an A-algebra.

(ii) For each residue field κ of A, the fiber $\mathrm{Tor}_0^A(B, \kappa)$ is a finite-dimensional vector space over κ.

Remark 3.3.0.1. Assuming (i), condition (ii) is equivalent to the requirement that the induced map of topological spaces $|\operatorname{Spec} B| \to |\operatorname{Spec} A|$ has finite fibers.

A morphism of schemes $f : X \to Y$ is said to be *locally quasi-finite* if, for every point $x \in X$, there exist affine open neighborhoods $\operatorname{Spec} B \simeq U \subseteq X$ of x and $\operatorname{Spec} A \simeq V \subseteq Y$ such that $f(U) \subseteq V$ and the induced ring homomorphism $A \to B$ is quasi-finite. Our goal in this section is generalize the notion of locally quasi-finite morphism to the setting of spectral algebraic geometry. Our main result is the following version of Zariski's Main Theorem:

Theorem 3.3.0.2. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Assume that f is quasi-compact, separated, and locally quasi-finite. Then f is quasi-affine.*

3.3.1 Relative Dimension

We will assume that the reader is familiar with the theory of relative dimension for homomorphisms of commutative rings (see §B.2 for an overview). We now describe a "globalized" version of the relative dimension:

Definition 3.3.1.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks and let $d \geqslant 0$ be an integer. We will say that f is *of relative dimension* $\leqslant d$ if the following condition is satisfied: for every commutative diagram

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} B & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow f \\ \operatorname{Sp\acute{e}t} A & \longrightarrow & \mathsf{Y} \end{array}$$

in which the horizontal maps are étale, the induced map of commutative rings $\pi_0 A \to \pi_0 B$ is of relative dimension $\leqslant d$ (Definition B.2.3.1). We will say that f is *locally quasi-finite* if it is locally of relative dimension $\leqslant 0$.

Example 3.3.1.2. Every étale morphism of spectral Deligne-Mumford stacks is locally quasi-finite.

Example 3.3.1.3. Every closed immersion of spectral Deligne-Mumford stacks is locally quasi-finite.

Proposition 3.3.1.4. *For every nonnegative integer $d \geqslant 0$, the condition that a map of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ be of relative dimension $\leqslant d$ is local on the source with respect to the étale topology (see Definition 2.8.1.7). In particular, the condition that f is locally quasi-finite is local on the source with respect to the étale topology.*

Proof. It is clear that if $f : \mathsf{X} \to \mathsf{Y}$ is of relative dimension $\leqslant d$ and $g : \mathsf{U} \to \mathsf{X}$ is étale, then the composite map $f \circ g$ is of relative dimension $\leqslant d$. To complete the proof, let us suppose that $f : \mathsf{X} \to \mathsf{Y}$ is arbitrary and that we are given a jointly surjective collection of étale morphisms $\{g_\alpha : \mathsf{U}_\alpha \to \mathsf{X}\}$ such that each composition $f \circ g_\alpha$ is locally of relative dimension $\leqslant d$. We wish to show that f has the same property. Choose a commutative diagram

$$\begin{array}{ccc} \operatorname{Spét} B & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow f \\ \operatorname{Spét} A & \longrightarrow & \mathsf{Y} \end{array}$$

where the horizontal maps are étale. We wish to show that $\pi_0 B$ is of relative dimension d over $\pi_0 A$. It follows from Proposition 4.2.1.1 that $\pi_0 B$ is finitely generated as an algebra over $\pi_0 A$. Fix a prime ideal $\mathfrak{p} \subseteq \pi_0 B$ and let $\mathfrak{q}$ be its inverse image in $\pi_0 A$. We wish to show that the local ring $R = (\pi_0 B)_{\mathfrak{p}}/\mathfrak{q}(\pi_0 B)_{\mathfrak{p}}$ has dimension $\leqslant d$ (see Corollary B.2.3.10). Since the maps g_α are jointly surjective, we can choose an étale map $B \to B'$ and a prime ideal $\mathfrak{p}' \subseteq \pi_0 B'$ lying over $\mathfrak{p}$ for which the composite map $\operatorname{Spét} B' \to \operatorname{Spét} B \to \mathsf{X}$ factors through some U_α. Invoking our assumption that $f \circ g_\alpha$ is of relative dimension $\leqslant d$, we conclude that the local ring $R' = (\pi_0 B')_{\mathfrak{p}'}/\mathfrak{q}(\pi_0 A)_{\mathfrak{p}'}$ has dimension $\leqslant d$. Since R' is étale and faithfully flat over R, it follows that R also has dimension $\leqslant d$ (Variant B.2.2.4). □

Corollary 3.3.1.5. *Suppose we are given a morphisms of spectral Deligne-Mumford stacks*

$$\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z}.$$

If f is of relative dimension $\leqslant d$ and g is of relative dimension $\leqslant d'$, then the composition $g \circ f$ is of relative dimension $\leqslant (d + d')$. In particular, if f and g are locally quasi-finite, then so is $g \circ f$.

Proof. Suppose we are given a commutative diagram

$$\begin{array}{ccc} \operatorname{Spét} C & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow \\ \operatorname{Spét} A & \longrightarrow & \mathsf{Z} \end{array}$$

where the horizontal maps are étale. We wish to show that $\pi_0 C$ is of relative dimension $\leqslant (d+d')$ over $\pi_0 A$. The proof of Proposition 3.3.1.4 shows that this condition is étale local on C; we may therefore assume that the map $\mathrm{Sp\acute{e}t}\, C \to \mathrm{Sp\acute{e}t}\, A \times_{\mathsf{Z}} \mathsf{Y}$ factors through some étale map $\mathrm{Sp\acute{e}t}\, B \to \mathrm{Sp\acute{e}t}\, A \times_{\mathsf{Z}} \mathsf{Y}$. Since f and g are of relative dimension $\leqslant d$ and $\leqslant d'$, the ring homomorphisms $\pi_0 B \to \pi_0 C$ and $\pi_0 A \to \pi_0 B$ have relative dimensions $\leqslant d$ and $\leqslant d'$, respectively. The desired result now follows from Proposition B.2.3.7. □

Proposition 3.3.1.6. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks and let $d \geqslant 0$ be an integer. Then:*

(1) *The map f is of relative dimension $\leqslant d$ if and only if, for every étale map $\mathrm{Sp\acute{e}t}\, A \to \mathsf{Y}$, the induced map $\mathrm{Sp\acute{e}t}\, A \times_{\mathsf{Y}} \mathsf{X} \to \mathrm{Sp\acute{e}t}\, A$ is of relative dimension $\leqslant d$.*

(2) *Assume that $\mathsf{Y} \simeq \mathrm{Sp\acute{e}t}\, A$ is affine. Then f is of relative dimension $\leqslant d$ if and only if, for every étale map $\mathrm{Sp\acute{e}t}\, B \to \mathsf{X}$, the induced map of commutative rings $\pi_0 A \to \pi_0 B$ is of relative dimension $\leqslant d$.*

(3) *Assume that $\mathsf{Y} \simeq \mathrm{Sp\acute{e}t}\, A$ and $\mathsf{X} \simeq \mathrm{Sp\acute{e}t}\, B$ are both affine. Then f is of relative dimension $\leqslant d$ if and only if the underlying map of commutative rings $\pi_0 A \to \pi_0 B$ is of relative dimension $\leqslant d$.*

Proof. Assertion (1) follows immediately from the definition, and the "only if" directions of (2) and (3) are obvious. To complete the proof of (2), assume that $\mathsf{Y} \simeq \mathrm{Sp\acute{e}t}\, A$ and consider an arbitrary commutative diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}\, B & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Sp\acute{e}t}\, A' & \longrightarrow & \mathrm{Sp\acute{e}t}\, A \end{array}$$

where the horizontal maps are étale. If $\pi_0 B$ is of relative dimension $\leqslant d$ over $\pi_0 A$, then it is also of relative dimension $\leqslant d$ over $\pi_0 A'$ (Remark B.2.3.4). The proof of (3) is similar. □

Proposition 3.3.1.7. *The condition that a map of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ be of relative dimension $\leqslant d$ (locally quasi-finite) is local on the target with respect to the étale topology. That is, if we are given a jointly surjective collection of étale maps $\mathsf{U}_\alpha \to \mathsf{Y}$ for which each of the projections $\mathsf{X} \times_{\mathsf{Y}} \mathsf{U}_\alpha \to \mathsf{U}_\alpha$ is of relative dimension $\leqslant d$ (locally quasi-finite), then f is of relative dimension $\leqslant d$ (locally quasi-finite).*

Proof. Using Example 3.3.1.2 and Corollary 3.3.1.5, we see that each of the induced maps $\mathsf{U}_\alpha \times_{\mathsf{Y}} \mathsf{X} \to \mathsf{Y}$ is of relative dimension $\leqslant d$. Applying Proposition 3.3.1.4, we deduce that f is of relative dimension $\leqslant d$. □

Proposition 3.3.1.8. *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y}. \end{array}$$

If f is of relative dimension $\leqslant d$ (locally quasi-finite), then so is f'. The converse holds if g is faithfully flat and quasi-compact.

Proof. Assume first that f is of relative dimension $\leqslant d$; we wish to show that f' has the same property. Using Proposition 3.3.1.7 we can reduce to the case where $\mathsf{Y} = \operatorname{Sp\acute{e}t} A$ is affine, and the map g factors as a composition

$$\mathsf{Y}' \to \operatorname{Sp\acute{e}t} A_0 \xrightarrow{g'} \mathsf{Y},$$

where g' is étale. Replacing Y by $\operatorname{Sp\acute{e}t} A_0$, we may assume that Y is affine. Using Proposition 3.3.1.4, we may further suppose that $\mathsf{X} = \operatorname{Sp\acute{e}t} B_0$ is affine, so that $\mathsf{X}' \simeq \operatorname{Sp\acute{e}t}(A \otimes_{A_0} B_0)$ is also affine. In this case, the desired result follows from Proposition B.2.3.5.

Now suppose that g is faithfully flat and quasi-compact and that f' is locally quasi-finite; we wish to show that f is locally quasi-finite. Using Propositions 3.3.1.4 and 3.3.1.7, we may assume that $\mathsf{Y} = \operatorname{Sp\acute{e}t} A_0$ and $\mathsf{X} = \operatorname{Sp\acute{e}t} B_0$ are affine. Replacing Y' by an étale cover if necessary, we may suppose that $\mathsf{Y}' = \operatorname{Sp\acute{e}t} A$ for some flat A_0-algebra A. In this case, the desired result again follows from Proposition B.2.3.5. □

We have the following converse to Corollary 3.3.1.5:

Proposition 3.3.1.9. *Suppose we are given morphisms of spectral Deligne-Mumford stacks*

$$\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z}.$$

If $g \circ f$ is of relative dimension $\leqslant d$ (locally quasi-finite), then so is f.

Proof. Using Propositions 3.3.1.4 and 3.3.1.7, we can reduce to the case where $\mathsf{X} = \operatorname{Sp\acute{e}t} C$, $\mathsf{Y} = \operatorname{Sp\acute{e}t} B$, and $\mathsf{Z} = \operatorname{Sp\acute{e}t} A$ are affine, in which case the desired result follows from Remark B.2.3.4. □

3.3.2 Zariski's Main Theorem

The essential step in the proof of Theorem 3.3.0.2 is the following:

Proposition 3.3.2.1. *Let $f : \mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) = \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Suppose that:*

(a) *The map f is quasi-compact, separated and locally quasi-finite.*

(b) *The unit map $\mathscr{O}_{\mathcal{Y}} \to f_* \mathscr{O}_{\mathcal{X}}$ exhibits $\mathscr{O}_{\mathcal{Y}}$ as a connective cover of $f_* \mathscr{O}_{\mathcal{X}}$.*

Then f is an open immersion.

Remark 3.3.2.2. It follows from Corollary 3.2.3.3 that condition (b) of Proposition 3.3.2.1 is stable under flat base change.

The proof will require a few preliminaries.

Definition 3.3.2.3. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks. We will say that f is *finite étale of degree n* if, for every morphism $\operatorname{Spét} R \to \mathsf{Y}$, the fiber product $\mathsf{X} \times_{\mathsf{Y}} \operatorname{Spét} R$ has the form $\operatorname{Spét} R'$, where R' is a étale R-algebra which is locally free of degree n.

Example 3.3.2.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be a finite étale map of spectral Deligne-Mumford stacks of degree n. Then f is separated, so the configuration stack $\operatorname{Conf}^n_{\mathsf{Y}}(\mathsf{X})$ is defined and carries an action of the symmetric group Σ_n. We claim that the canonical map $\operatorname{Conf}^n_{\mathsf{Y}}(\mathsf{X})/\Sigma_n \to \mathsf{Y}$ is an equivalence. To prove this, we may work locally on Y and thereby reduce to the case where $\mathsf{Y} = \operatorname{Spét} R$ and $\mathsf{X} = \operatorname{Spét} R^n$. In this case, the result follows from a simple calculation (note that $\operatorname{Conf}^n_{\mathsf{Y}}(\mathsf{X}) \simeq \amalg_{\sigma \in \Sigma_n} \operatorname{Spét} R$).

Proposition 3.3.2.5. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a finite étale morphism of degree $n > 0$ between spectral Deligne-Mumford stacks. Assume that Y is a spectral algebraic space and that X is affine. Then Y is affine.*

Proof. Example 3.3.2.4 implies that Y can be described as the quotient $\operatorname{Conf}^n_{\mathsf{Y}}(\mathsf{X})$ by the action of the symmetric group Σ_n. Note that $\operatorname{Conf}^n_{\mathsf{Y}}(\mathsf{X})$ admits a clopen immersion (Definition 3.1.7.2) into the iterated fiber product $\mathsf{X} \times_{\mathsf{Y}} \mathsf{X} \times \cdots \times_{\mathsf{Y}} \mathsf{X}$. Since $n > 0$, we have a finite étale projection map $\mathsf{X} \times_{\mathsf{Y}} \cdots \times_{\mathsf{Y}} \mathsf{X} \to \mathsf{X}$. Since X is affine, it follows that $\mathsf{X} \times_{\mathsf{Y}} \cdots \times_{\mathsf{Y}} \mathsf{X}$ is affine and therefore $\operatorname{Conf}^n_{\mathsf{Y}}(\mathsf{X}) \simeq \operatorname{Spét} A$ is affine. To complete the proof, it will suffice to show that the action of Σ_n on $\operatorname{Spét} A$ is free, which follows from Lemma 3.2.2.6. □

Remark 3.3.2.6. Let $f : \mathsf{X} \to \mathsf{Y}$ be a finite étale map of spectral Deligne-Mumford stacks. Then f determines a decomposition $\mathsf{Y} \simeq \amalg_{n \geqslant 0} \mathsf{Y}_n$, where each of the induced maps $\mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}_n \to \mathsf{Y}_n$ is finite étale of degree n.

Proof of Proposition 3.3.2.1. The assertion is local on Y; we may therefore reduce to the case where $\mathsf{Y} = \operatorname{Spét} R$ is affine (so that X is a separated spectral algebraic space). Then X is quasi-compact, so we can choose an étale surjection $u : \operatorname{Spét} A \to \mathsf{X}$. Let $\mathfrak{p} \in |\operatorname{Spec} A|$ and let $\mathfrak{q}$ be its image in $|\operatorname{Spec} R|$. We will show that there exists an open set $U_{\mathfrak{q}} \subseteq |\operatorname{Spec} R|$ such that, if $\mathsf{U}_{\mathfrak{q}}$ denotes the corresponding open substack of Y, then the projection map

$\mathsf{X} \times_{\mathsf{Y}} \mathsf{U}_{\mathfrak{q}} \to \mathsf{U}_{\mathfrak{q}}$ is an equivalence. Let $U = \bigcup_{\mathfrak{p} \in |\operatorname{Spec} A|} U_{\mathfrak{q}}$ and let U be the corresponding open substack of Y. Then the projection $\mathsf{X} \times_{\mathsf{Y}} \mathsf{U} \to \mathsf{U}$ is an equivalence. Moreover, the open substack $\mathsf{U} \times_{\mathsf{Y}} \operatorname{Spét} A$ is equivalent to $\operatorname{Spét} A$. Since u is surjective, it follows that $\mathsf{X} \times_{\mathsf{Y}} \mathsf{U} \simeq \mathsf{X}$, so that we can identify f with the open immersion $\mathsf{U} \hookrightarrow \mathsf{Y}$.

It remains to construct the open set $U_{\mathfrak{q}}$. Let κ denote the residue field of $\pi_0 R$ at the prime ideal $\mathfrak{q}$. Since f is locally quasi-finite and u is étale, the map of commutative rings $\pi_0 R \to \pi_0 A$ is quasi-finite. It follows from Corollary B.3.4.5 that the map $\pi_0 R \to \kappa$ factors as a composition $\pi_0 R \to R'_0 \to \kappa$, where R'_0 is an étale $(\pi_0 R)$-algebra and $(\pi_0 A) \otimes_{\pi_0 R} R'_0$ decomposes as a product $B'_0 \times B''_0$, where B' is a finite R'_0-module and $\operatorname{Tor}_0^{R'_0}(B'', \kappa) \simeq 0$. Using Theorem HA.7.5.0.6 , we can choose an étale R-algebra R' with $\pi_0 R' \simeq R'_0$, so that $A \otimes_R R'$ decomposes as a product $B' \times B''$ with $\pi_0 B' \simeq B'_0$ and $\pi_0 B'' \simeq B''_0$. Since the map $|\operatorname{Spec} R'| \to |\operatorname{Spec} R|$ is open and its image contains $\mathfrak{q}$, we can replace R by R' and thereby assume that $A \simeq B' \times B''$, where $\pi_0 B'$ is a finitely generated module over $\pi_0 R$ and $B'' \otimes_R \kappa \simeq 0$. Since $A \otimes_R \kappa \neq 0$, it follows that $B' \otimes_R \kappa \neq 0$. The composite map

$$u' : \operatorname{Spét} B' \to \operatorname{Spét} A \to \mathsf{X}$$

is étale. Since X is separated, the map u' is affine. Since $\pi_0 B'$ is finitely generated as a $\pi_0 R$-module, we deduce that u' is finite étale. Using Remark 3.3.2.6, we deduce that X admits a decomposition $\mathsf{X} \simeq \amalg_{n \geqslant 0} \mathsf{X}_n$, where each of the induced maps $\operatorname{Spét} B' \times_{\mathsf{X}} \mathsf{X}_n \to \mathsf{X}_n$ is finite étale of degree n. Each fiber product $\operatorname{Spét} B' \times_{\mathsf{X}} \mathsf{X}_n$ is a summand of $\operatorname{Spét} B'$, and therefore affine. It follows from Proposition 3.3.2.5 that X_n is also affine for $n > 0$. Since X is quasi-compact, the stacks X_n are empty for $n \gg 0$. It follows that $\mathsf{X}' = \amalg_{n > 0} \mathsf{X}_n$ is an affine open substack of X. Note that since $\operatorname{Spét} \kappa \times_{\operatorname{Spét} R} \operatorname{Spét} B'$ is nonempty, the fiber product $\operatorname{Spét} \kappa \times_{\operatorname{Spét} R} \mathsf{X}'$ is also nonempty.

Using (b), we can choose an idempotent element $e \in \pi_0 R$ which vanishes on X_0 but not on X'. Since $\operatorname{Spét} \kappa \times_{\operatorname{Spét} R} \mathsf{X}' \neq \emptyset$, we must have $e \notin \mathfrak{q}$. We may therefore replace R by $R[e^{-1}]$ and thereby reduce to the case where X_0 is empty. In this case, $\mathsf{X} \simeq \mathsf{X}'$ is affine. Using (b) again, we deduce that f is an equivalence. □

Proof of Theorem 3.3.0.2. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is separated, quasi-compact, and locally quasi-finite; we wish to show that f is quasi-affine. The assertion is local on Y; we may therefore assume that $\mathsf{Y} \simeq \operatorname{Spét} R$ is affine. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and let A be the connective cover $f_* \mathcal{O}_{\mathcal{X}} \in \operatorname{CAlg}_R$. Then f factors as a composition $\mathsf{X} \xrightarrow{f'} \operatorname{Spét} A \xrightarrow{f''} \operatorname{Spét} R$. Since f is locally quasi-finite, the morphism f' is also locally quasi-finite (Proposition 3.3.1.9). Using Proposition 3.3.2.1, we deduce that f' is an open immersion, so that X can be identified with a quasi-compact open substack of $\operatorname{Spét} A$ and is therefore quasi-affine. □

3.4 Quasi-Separated Morphisms

Recall that a scheme X is said to be *quasi-separated* if the diagonal map $X \to X \times X$ is quasi-compact. In this section, we will investigate the analogous condition in the setting of spectral Deligne-Mumford stacks (Definition 3.4.0.1). Our main result (Theorem 3.4.2.1) asserts that a spectral Deligne-Mumford stack X admits a scallop decomposition (see Definition 2.5.3.1) if and only it is is a quasi-compact, quasi-separated spectral algebraic space. From this we will deduce a number of consequences concerning the global sections functor $\Gamma(\mathsf{X}; \bullet)$ on the ∞-category $\mathrm{QCoh}(\mathsf{X})$ of quasi-coherent sheaves on X.

Definition 3.4.0.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of nonconnective spectral Deligne-Mumford stacks. We will say that f is *quasi-separated* if the diagonal map $\mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ is quasi-compact. We say that a nonconnective spectral Deligne-Mumford stack X is *quasi-separated* if the map $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, S$ is quasi-separated, where S denotes the sphere spectrum. In other words, X is quasi-separated if the absolute diagonal $\mathsf{X} \to \mathsf{X} \times \mathsf{X}$ is quasi-compact.

Example 3.4.0.2. If $f : \mathsf{X} \to \mathsf{Y}$ is a separated morphism of spectral Deligne-Mumford stacks. Then the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ is a closed immersion, hence affine, and in particular quasi-compact. It follows that f is quasi-separated.

3.4.1 Quasi-Separatedness

We begin by summarizing some easy formal properties of Definition 3.4.0.1.

Proposition 3.4.1.1. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. The following conditions are equivalent:*

(1) *The nonconnective spectral Deligne-Mumford stack X is quasi-separated.*

(2) *For every $\mathbb{E}_\infty$-ring R and every pair of maps $f, g : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$, the fiber product $\mathrm{Sp\acute{e}t}\, R \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, R$ is quasi-compact.*

(3) *For every pair of maps $f : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$, $g : \mathrm{Sp\acute{e}t}\, R' \to \mathsf{X}$, the fiber product $\mathrm{Sp\acute{e}t}\, R \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, R'$ is quasi-compact.*

(4) *For every pair of étale maps maps $f : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$, $g : \mathrm{Sp\acute{e}t}\, R' \to \mathsf{X}$, the fiber product $\mathrm{Sp\acute{e}t}\, R \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, R'$ is quasi-compact.*

(5) *For every pair of affine objects $U, V \in \mathcal{X}$, the product $U \times V \in \mathcal{X}$ is quasi-compact.*

(6) *For every pair of quasi-compact objects $U, V \in \mathcal{X}$, the product $U \times V \in \mathcal{X}$ is quasi-compact.*

Proof. The implications (1) ⇔ (2) ⇐ (3) ⇒ (4) ⇔ (5) ⇐ (6) are obvious. We next prove that (2) ⇒ (3). Suppose we are given a pair of maps $f : \mathrm{Spét}\, R \to \mathsf{X}$, $g : \mathrm{Spét}\, R' \to \mathsf{X}$. Let $A = R \otimes R'$, so that f and g define maps $f', g' : \mathrm{Spét}\, A \to \mathsf{X}$. Note that

$$\mathrm{Spét}\, R \times_{\mathsf{X}} \mathrm{Spét}\, R' \simeq (\mathrm{Spét}\, A \times_{\mathsf{X}} \mathrm{Spét}\, A) \times_{\mathrm{Spét}(A \otimes A)} \mathrm{Spét}\, A.$$

If (2) is satisfied, then there exists an étale surjectiion $\mathrm{Spét}\, B \to \mathrm{Spét}\, A \times_{\mathsf{X}} \mathrm{Spét}\, A$. It follows that there is an étale surjection

$$\mathrm{Spét}(B \otimes_{A \otimes A} A) \to \mathrm{Spét}\, R \times_{\mathsf{X}} \mathrm{Spét}\, R',$$

so that $\mathrm{Spét}\, R \times_{\mathsf{X}} \mathrm{Spét}\, R'$ is quasi-compact.

We next show that (4) ⇒ (3). Assume we are given arbitrary maps $f : \mathrm{Spét}\, R \to \mathsf{X}$ and $g : \mathrm{Spét}\, R' \to \mathsf{X}$. Choose a faithfully flat étale map $R \to A$ such that the composite map $\mathrm{Spét}\, A \to \mathrm{Spét}\, R \xrightarrow{f} \mathsf{X}$ factors through some étale map $\mathrm{Spét}\, B \to \mathsf{X}$, and a faithfully flat étale map $R' \to A'$ such that the composite map $\mathrm{Spét}\, A' \to \mathrm{Spét}\, R' \xrightarrow{g} \mathsf{X}$ factors through an étale map $\mathrm{Spét}\, B' \to \mathsf{X}$. Condition (4) implies that $\mathrm{Spét}\, B \times_{\mathsf{X}} \mathrm{Spét}\, B'$ is quasi-compact, so there is an étale surjection $\mathrm{Spét}\, T \to \mathrm{Spét}\, B \times_{\mathsf{X}} \mathrm{Spét}\, B'$. It follows that the composite map

$$\mathrm{Spét}(T \otimes_{B \otimes B'} (A \otimes A'')) \to \mathrm{Spét}\, A \times_{\mathsf{X}} \mathrm{Spét}\, A' \to \mathrm{Spét}\, R \times_{\mathsf{X}} \mathrm{Spét}\, R'$$

is an étale surjection, so that $\mathrm{Spét}\, R \times_{\mathsf{X}} \mathrm{Spét}\, R'$ is also quasi-compact.

We complete the proof by showing that (5) ⇒ (6). Assume $U, V \in \mathcal{X}$ are quasi-compact. Then there exist effective epimorphisms $U' \to U$ and $V' \to V$, where U' and V' are affine. Condition (5) implies that $U' \times V'$ is quasi-compact. Since we have an effective epimorphism $U' \times V' \to U \times V$, it follows that $U \times V$ is quasi-compact. □

Proposition 3.4.1.2. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a quasi-compact, quasi-separated spectral algebraic space. Then the ∞-topos $\mathcal{X}$ is coherent.*

Proof. We first suppose that X is separated. Using Corollary A.2.1.4, it suffices to show that if we are given affine objects $U, V \in \mathcal{X}$, then the product $U \times V \in \mathcal{X}$ is coherent. Let U and V be the spectral Deligne-Mumford stacks determined by U and V. We claim that $U \times V$ is affine. This follows from Theorem 3.1.2.1, since $\mathsf{Y} \simeq \mathsf{U} \times_{\mathsf{X}} \mathsf{V}$ admits a closed immersion into the affine spectral Deligne-Mumford stack $\mathsf{U} \times \mathsf{V}$.

We now treat the general case. Once again, it suffices to show that if $U, V \in \mathcal{X}$ are affine, then $U \times V$ is coherent. By the first part of the proof, we are reduced to proving that $\mathsf{U} \times_{\mathsf{X}} \mathsf{V}$ is separated. For this, it suffices to show that the map $\mathsf{U} \times_{\mathsf{X}} \mathsf{V} \to \mathsf{U} \times \mathsf{V}$ is separated, which follows from Example 3.2.1.9 (since X is a spectral algebraic space). □

Proposition 3.4.1.3. *Let X be a quasi-separated spectral algebraic space. Suppose we are given étale maps $\mathrm{Spét}\, R \to \mathsf{X} \leftarrow \mathrm{Spét}\, R'$. Then the fiber product $\mathsf{Y} \simeq \mathrm{Spét}\, R \times_{\mathsf{X}} \mathrm{Spét}\, R'$ is quasi-affine.*

Proof. Since X is quasi-separated, the spectral Deligne-Mumford stack Y is quasi-compact. Because X is a spectral algebraic space, the canonical map

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, A, \mathsf{Y}) \to \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, A, \mathrm{Sp\acute{e}t}(R \otimes R'))$$

is (-1)-truncated for any commutative ring A. In particular, $\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, A, \mathsf{Y})$ is discrete, so that Y is a spectral algebraic space. It follows from Example 3.2.1.9 that the map Y is separated. The projection map $\mathsf{Y} \to \mathrm{Sp\acute{e}t}\, R$ is étale and therefore locally quasi-finite. It follows from Theorem 3.3.0.2 that Y is quasi-affine. □

Remark 3.4.1.4. In the situation of Proposition 3.4.1.3, Remark 2.4.1.2 implies that the fiber product $\mathrm{Sp\acute{e}t}\, R \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, R'$ is schematic. It follows that the full subcategory of SpDM spanned by the quasi-separated, 0-truncated spectral algebraic spaces is equivalent to the category of algebraic spaces introduced in [117].

3.4.2 Existence of Scallop Decompositions (Quasi-Separated Case)

We can now state the main result of this section.

Theorem 3.4.2.1. *Let $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be a spectral Deligne-Mumford stack. Then Y admits a scallop decomposition if and only if it is a quasi-compact, quasi-separated spectral algebraic space.*

We defer the proof of Theorem 3.4.2.1 until later in this section. First, let us summarize some of its consequences.

Corollary 3.4.2.2. *Let $f : \mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to \mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be a morphism of spectral Deligne-Mumford stacks. Assume that f is a relative spectral algebraic space which is quasi-compact and quasi-separated. Then:*

(1) *The pushforward functor $f_* : \mathrm{Mod}_{\mathscr{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathcal{Y}}}$ carries quasi-coherent sheaves to quasi-coherent sheaves.*

(2) *The induced functor $\mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ commutes with small colimits.*

(3) *For every pullback diagram*

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{g'} & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y}, \end{array}$$

the associated diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{Y}) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathsf{X}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{Y}') & \xrightarrow{f'^*} & \mathrm{QCoh}(\mathsf{X}') \end{array}$$

is right adjointable. In other words, for every object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, the canonical map $\lambda : g^* f_* \mathscr{F} \to f'_* g'^* \mathscr{F}$ is an equivalence in $\mathrm{QCoh}(\mathsf{Y}')$.

Proof. Combine Theorem 3.2.3.1 with Propositions 2.5.4.3 and 2.5.4.5. □

Corollary 3.4.2.3. *Let X be a quasi-compact, quasi-separated spectral algebraic space. Then there exists an integer n such that the global sections functor $\Gamma : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Sp}$ carries $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$ into $\mathrm{Sp}_{\geqslant -n}$.*

Proof. Combine Proposition 2.5.4.4 with Theorem 3.4.2.1. □

Corollary 3.4.2.4. *Let X be a quasi-separated spectral algebraic space. If X is nonempty, then there exists an open immersion $j : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$ for some nonzero connective $\mathbb{E}_\infty$-ring R.*

Proof. Replacing X by an open substack if necessary, we may suppose that X is quasi-compact. Choose a scallop decomposition $\mathsf{U}_0 \to \mathsf{U}_1 \to \cdots \to \mathsf{U}_n$ of X. Let i be the smallest integer such that U_i is nonempty. Then U_i is an affine open substack of X. □

Another consequence of Theorem 3.4.2.1 is that it is possible to choose a "Nisnevich neighborhood" around any point of quasi-separated spectral algebraic space.

Corollary 3.4.2.5. *Let Y be a quasi-separated spectral algebraic space. Let κ be a field, and suppose we are given a map $\eta : \mathrm{Sp\acute{e}t}\, \kappa \to \mathsf{Y}$. Then η admits a factorization*

$$\mathrm{Sp\acute{e}t}\, \kappa \xrightarrow{\eta'} \mathrm{Sp\acute{e}t}\, R \xrightarrow{\eta''} \mathsf{Y},$$

where η' is étale.

Remark 3.4.2.6 (Projection Formula)**.** Let $f : \mathsf{X} \to \mathsf{Y}$ be a quasi-compact, quasi-separated morphism of spectral Deligne-Mumford stacks, and assume that $\mathrm{Sp\acute{e}t}\, R \times_{\mathsf{Y}} \mathsf{X}$ is a spectral algebraic space for every map $\mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$. Suppose we are given quasi-coherent sheaves $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ and $\mathscr{G} \in \mathrm{QCoh}(\mathsf{Y})$. The counit map $f^* f_* \mathscr{F} \to \mathscr{F}$ induces a morphism $f^*(f_* \mathscr{F} \otimes \mathscr{G}) \simeq f^* f_* \mathscr{F} \otimes f^* \mathscr{G} \to \mathscr{F} \otimes f^* \mathscr{G}$, which is adjoint to a map $\theta : f_* \mathscr{F} \otimes \mathscr{G} \to f_*(\mathscr{F} \otimes f^* \mathscr{G})$. We claim that θ is an equivalence. To prove this, we may work locally on Y and thereby reduce to the case where $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, R$ is affine. The collection of those objects $\mathscr{G} \in \mathrm{QCoh}(\mathsf{Y}) \simeq \mathrm{Mod}_R$ for which θ is an equivalence is stable under shifts and colimits in $\mathrm{QCoh}(\mathsf{Y})$. It will therefore suffice to show that θ is an equivalence in the special case where $\mathscr{G}$ corresponds to the unit object $R \in \mathrm{Mod}_R \simeq \mathrm{QCoh}(\mathsf{Y})$, which is obvious.

3.4.3 The Proof of Theorem 3.4.2.1

Our proof of Theorem 3.4.2.1 will require a number of preliminaries.

Lemma 3.4.3.1. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-affine spectral Deligne-Mumford stack acted on by a finite group G. Assume that the action of G is free in the following sense: for every nonzero commutative ring R, G acts freely on the set $\pi_0 \operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Sp\acute{e}t} R, \mathsf{X})$. Then there exist a finite collection of G-equivariant (-1)-truncated objects $\{U_i \in \mathcal{X}\}_{1 \leqslant i \leqslant n}$ with the following properties:*

(1) *For $1 \leqslant i \leqslant n$, let U_i denote the open substack $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}} |U_i)$ of X. Then each of the quotients U_i / G is affine.*

(2) *The objects U_i cover $\mathcal{X}$. That is, if $\mathbf{1}$ denotes a final object of $\mathcal{X}$, then the canonical map $\amalg_{1 \leqslant i \leqslant n} U_i \to \mathbf{1}$ is an effective epimorphism.*

Proof. Let R denote the connective cover of the $\mathbb{E}_\infty$-ring $\Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}}) \simeq \mathcal{O}_{\mathcal{X}}(\mathbf{1})$. Since X is quasi-affine, the canonical map $j : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ is an open immersion (Proposition 2.4.1.3), classified by some quasi-compact open subset $U \subseteq |\operatorname{Spec} R|$. Then $U = \{\mathfrak{p} \in |\operatorname{Spec} R| : I \nsubseteq \mathfrak{p}\}$ for some radical ideal $I \subseteq \pi_0 R$. Note that the finite group G acts on the commutative ring $\pi_0 R$ and the ideal I is G-invariant. For every point $\mathfrak{p} \in U$, none of the prime ideals $\{\sigma(\mathfrak{p})\}_{\sigma \in G}$ contains I. Consequently, there exists an element $x \in I$ such that $\sigma(x) \notin \mathfrak{p}$ for each $\sigma \in G$. Replacing x by $\prod_{\sigma \in G} \sigma(x)$ if necessary, we may suppose that x is G-invariant. Let $U_{\mathfrak{p}} = \{\mathfrak{q} \in |\operatorname{Spec} R| : x \notin \mathfrak{q}\}$. Then $U_{\mathfrak{p}}$ is an open subset of U containing the point $\mathfrak{p}$. The collection of open sets $\{U_{\mathfrak{p}}\}_{\mathfrak{p} \in U}$ is an open covering of U. Since U is quasi-compact, there exists a finite subcovering by open sets $U_{\mathfrak{p}_1}, \ldots, U_{\mathfrak{p}_n}$, which we can identify with (-1)-truncated objects $U_1, \ldots, U_n \in \mathcal{X}$. It is clear that these objects satisfy condition (2). To verify (1), we note that each of the open substacks $\mathsf{U}_i = (\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}} |U_i)$ of X has the form $|\operatorname{Spec} R[x^{-1}]|$ for some G-invariant element $x \in \pi_0 R$. Since G acts freely on X, it also acts freely on the open substack U_i, so that U_i / G is affine by virtue of Lemma 3.4.3.1. □

Lemma 3.4.3.2. *Let $u : \mathsf{X} \to \mathsf{Y}$ be a map of spectral algebraic spaces. If X is separated, then u is separated.*

Proof. The map u factors as a composition $\mathsf{X} \xrightarrow{u'} \mathsf{X} \times \mathsf{Y} \xrightarrow{u''} \mathsf{Y}$. Since X is separated, the morphism u'' is separated. The morphism u' is a pullback of the diagonal map $\delta : \mathsf{Y} \to \mathsf{Y} \times \mathsf{Y}$. Since Y is a spectral algebraic space, Example 3.2.1.9 implies that δ is separated. It follows that u' is separated, so that $u = u'' \circ u'$ is also separated. □

Lemma 3.4.3.3. *Let $j : \mathsf{U} \to \mathsf{X}$ be a map of spectral Deligne-Mumford stacks. Assume that j is separated, quasi-compact, and that for every map $\operatorname{Sp\acute{e}t} \kappa \to \mathsf{X}$ where κ is a field, the fiber product $\mathsf{U} \times_{\mathsf{X}} \operatorname{Sp\acute{e}t} \kappa$ is either empty or equivalent to $\operatorname{Sp\acute{e}t} \kappa$. Then j is an open immersion.*

Proof. The assertion is local on X, so we may assume that X is affine. In this case, Theorem 3.3.0.2 implies that U is quasi-affine. Choose a covering of U by affine open substacks U_i, and for each index i let V_i be the open substack of X given by the image of U_i. Then each V_i is quasi-affine and therefore a separated spectral algebraic space. Since U_i is affine, the maps $u_i : \mathsf{U}_i \to \mathsf{V}_i$ are affine and étale. Our condition on the fibers of j guarantee that each u_i is finite étale of degree 1 and therefore an equivalence. It follows that j induces an equivalence from U to the open substack of X given by the union of the open substacks V_i. □

We will say that a diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \widetilde{\mathsf{U}} & \xrightarrow{j} & \widetilde{\mathsf{Y}} \\ \downarrow g & & \downarrow \\ \mathsf{U} & \longrightarrow & \mathsf{Y} \end{array}$$

is an *excision square* if it is a pushout square, where j is an open immersion and g is étale (see Definition 2.5.2.2).

Lemma 3.4.3.4. *Let Y be a spectral Deligne-Mumford stack. Suppose that there exists an excision square of spectral Deligne-Mumford stacks σ :*

$$\begin{array}{ccc} \widetilde{\mathsf{U}} & \longrightarrow & \widetilde{\mathsf{Y}} \\ \downarrow & & \downarrow \\ \mathsf{U} & \longrightarrow & \mathsf{Y} \end{array}$$

where U is a quasi-compact quasi-separated spectral algebraic space, $\widetilde{\mathsf{Y}}$ is affine, and $\widetilde{\mathsf{U}}$ is quasi-compact. Then Y is a quasi-compact quasi-separated spectral algebraic space.

Proof. The map $\mathsf{U} \amalg \widetilde{\mathsf{Y}} \to \mathsf{Y}$ is an étale surjection. Since $\widetilde{\mathsf{Y}}$ and U are quasi-compact, it follows immediately that Y is quasi-compact. We next prove that Y is quasi-separated. Choose maps $\mathsf{V}_0, \mathsf{V}_1 \to \mathsf{Y}$, where V_0 and V_1 are affine. We wish to prove that the fiber product $\mathsf{V}_0 \times_{\mathsf{Y}} \mathsf{V}_1$ is quasi-compact. Passing to an étale covering of V_0 and V_1 if necessary we may suppose that the maps $\mathsf{V}_i \to \mathsf{Y}$ factor through either U or $\widetilde{\mathsf{Y}}$. There are three cases to consider:

(*a*) Suppose that both of the maps $\mathsf{V}_i \to \mathsf{Y}$ factor through U. Then $\mathsf{V}_0 \times_{\mathsf{Y}} \mathsf{V}_1 \simeq \mathsf{V}_0 \times_{\mathsf{U}} \mathsf{V}_1$ is quasi-compact by virtue of our assumption that U is quasi-separated.

(*b*) Suppose that the map $\mathsf{V}_0 \to \mathsf{Y}$ factors through U and the map $\mathsf{V}_1 \to \mathsf{Y}$ factors through $\widetilde{\mathsf{Y}}$.

$$\mathsf{V}_0 \times_{\mathsf{Y}} \mathsf{V}_1 \simeq \mathsf{V}_0 \times_{\mathsf{U}} (\mathsf{U} \times_{\mathsf{Y}} \mathsf{V}_1) \simeq \mathsf{V}_0 \times_{\mathsf{U}} (\widetilde{\mathsf{U}} \times_{\widetilde{\mathsf{Y}}} \mathsf{V}_1).$$

Since $\widetilde{\mathsf{U}}$ is quasi-compact and $\widetilde{\mathsf{Y}}$ is quasi-separated, the fiber product $\widetilde{\mathsf{U}} \times_{\widetilde{\mathsf{Y}}} \mathsf{V}_1$ is quasi-compact. Using the quasi-separateness of U we deduce that $\mathsf{V}_0 \times_{\mathsf{Y}} \mathsf{V}_1$ is quasi-compact.

(c) Suppose that both of the maps $\mathsf{V}_i \to \mathsf{Y}$ factor through $\widetilde{\mathsf{Y}}$. Since σ is an excision square, the map

$$\widetilde{\mathsf{Y}} \amalg (\widetilde{\mathsf{U}} \times_{\mathsf{Y}} \widetilde{\mathsf{U}}) \to \widetilde{\mathsf{Y}} \times_{\mathsf{Y}} \widetilde{\mathsf{Y}}$$

is an étale surjection. We therefore obtain an étale surjection

$$(\mathsf{V}_0 \times_{\widetilde{\mathsf{Y}}} \mathsf{V}_1) \amalg ((\mathsf{V}_0 \times_{\widetilde{\mathsf{Y}}} \widetilde{\mathsf{U}}) \times_{\mathsf{U}} (\mathsf{V}_1 \times_{\widetilde{\mathsf{Y}}} \widetilde{\mathsf{U}})) \to \mathsf{V}_0 \times_{\mathsf{Y}} \mathsf{V}_1 .$$

The fiber product $\mathsf{V}_0 \times_{\widetilde{\mathsf{Y}}} \mathsf{V}_1$ is affine and therefore quasi-compact. Since $\widetilde{\mathsf{U}}$ is quasi-compact, the fiber products $\mathsf{V}_i \times_{\widetilde{\mathsf{Y}}} \widetilde{\mathsf{U}}$ are quasi-compact. Using the quasi-separateness of U, we deduce that the fiber product $(\mathsf{V}_0 \times_{\widetilde{\mathsf{Y}}} \widetilde{\mathsf{U}}) \times_{\mathsf{U}} (\mathsf{V}_1 \times_{\widetilde{\mathsf{Y}}} \widetilde{\mathsf{U}})$ is quasi-compact, so that $\mathsf{V}_0 \times_{\mathsf{Y}} \mathsf{V}_1$ is also quasi-compact.

It remains to prove that Y is a spectral algebraic space. We wish to show that the mapping space $\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{Y})$ is discrete for every commutative ring R. For every map $f : \mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$, the fiber product $\mathsf{U} \times_{\mathsf{Y}} \mathrm{Sp\acute{e}t}\, R$ is an open substack of $\mathrm{Sp\acute{e}t}\, R$ corresponding to an open subset $V_f \subseteq |\operatorname{Spec} R|$. Fix an open set $V \subseteq |\operatorname{Spec} R|$, and let $\mathrm{Map}^V_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{Y})$ denote the summand of $\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{Y})$ spanned by those maps f with $V_f = V$. Then $\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{Y}) \simeq \amalg_V \mathrm{Map}^V_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{Y})$, so it will suffice to show that each $\mathrm{Map}^V_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{Y})$ is discrete.

Let V denote the open substack of $\mathrm{Sp\acute{e}t}\, R$ corresponding to V. Write $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ and $\mathrm{Sp\acute{e}t}\, R = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, so we can identify V with a (-1)-truncated object of $\mathcal{X}$. The étale map $\widetilde{\mathsf{Y}} \to \mathsf{Y}$ determines an object $\widetilde{Y} \in \mathcal{Y}$. Every map $f : \mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$ determines an object $f^* \widetilde{Y} \in \mathcal{X}$. This construction determines a functor θ fitting into a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}^V_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{Y}) & \longrightarrow & \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{V}, \mathsf{U}) \\ \downarrow \theta & & \downarrow \theta_0 \\ \mathcal{X} & \longrightarrow & \mathcal{X}_{/V} . \end{array}$$

Since $\widetilde{\mathsf{U}} \to \mathsf{U}$ is a map between spectral algebraic spaces, the homotopy fibers of the induced map

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R', \widetilde{\mathsf{U}}) \to \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R', \mathsf{U})$$

are discrete for every étale R-algebra R'. It follows that θ_0 factors through the full subcategory $\tau_{\leqslant 0}\, \mathcal{X}_{/V} \subseteq \mathcal{X}_{/V}$ spanned by the discrete objects. Let $\mathcal{X}_0$ denote the full subcategory of $\mathcal{X}$ spanned by those objects X such that the image of X in $\mathcal{X}/V$ is a final object, so that θ factors through $\mathcal{X}_0$. Using Proposition HA.A.8.15 , we deduce that the homotopy fiber of the forgetful functor $\mathcal{X}_0 \to \mathcal{X}_{/V}$ over an object $\widetilde{V} \in \mathcal{X}_{/V}$ can be identified with the space $\mathrm{Map}_{\mathcal{X}_{/V}}(V, \widetilde{V})$; in particular, it is discrete if $\widetilde{V}$ is discrete. It follows that the map

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathsf{V}, \mathsf{U}) \times_{\mathcal{X}_{/V}} \mathcal{X} \to \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{V}, \mathsf{U})$$

has discrete homotopy fibers. Since the structure sheaf of V is discrete and U is a spectral algebraic space, the mapping space $\mathrm{Map}_{\mathrm{SpDM}}(\mathsf{V},\mathsf{U})$ is discrete. We conclude that the Kan complex $\mathrm{Map}_{\mathrm{SpDM}}(\mathsf{V},\mathsf{U}) \times_{\mathcal{X}_{/V}} \mathcal{X}$ is discrete. To complete the proof, it will suffice to show that the canonical map

$$\phi : \mathrm{Map}^V_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{Y}) \to \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{V},\mathsf{U}) \times_{\mathcal{X}_{/V}} \mathcal{X}$$

has discrete homotopy fibers. To this end, we fix an object $\widetilde{X} \in \mathcal{X}$ having image $\widetilde{V} \in \mathcal{X}_{/V}$; we will show that the map

$$\phi_{\widetilde{X}} : \mathrm{Map}^V_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{Y}) \times_{\mathcal{X}} \{\widetilde{X}\} \to \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{V},\mathsf{U}) \times_{\mathcal{X}_{/V}} \{\widetilde{V}\}$$

has discrete homotopy fibers. To prove this, we observe that $\phi_{\widetilde{X}}$ is a pullback of the map

$$\mathrm{Map}^V_{\mathrm{SpDM}}(\widetilde{\mathsf{X}},\widetilde{\mathsf{Y}}) \to \mathrm{Map}_{\mathrm{SpDM}}(\widetilde{\mathsf{V}},\widetilde{\mathsf{U}}),$$

where $\mathrm{Map}^V_{\mathrm{SpDM}}(\widetilde{\mathsf{X}},\widetilde{\mathsf{Y}})$ is the summand of $\mathrm{Map}_{\mathrm{SpDM}}(\widetilde{\mathsf{X}},\widetilde{\mathsf{Y}})$ corresponding to those maps satisfying $\widetilde{\mathsf{V}} \simeq \widetilde{\mathsf{U}} \times_{\widetilde{\mathsf{Y}}} \widetilde{\mathsf{X}}$. It now suffices to observe that $\mathrm{Map}^V_{\mathrm{SpDM}}(\widetilde{\mathsf{X}},\widetilde{\mathsf{Y}})$ and $\mathrm{Map}_{\mathrm{SpDM}}(\widetilde{\mathsf{V}},\widetilde{\mathsf{U}})$ are both discrete, since both $\widetilde{\mathsf{X}}$ and $\widetilde{\mathsf{V}}$ have discrete structure sheaves and both $\widetilde{\mathsf{Y}}$ and $\widetilde{\mathsf{U}}$ are spectral algebraic spaces. □

Proof of Theorem 3.4.2.1. If Y is a spectral Deligne-Mumford stack which admits a scallop decomposition, then Lemma 3.4.3.2 immediately implies that Y is a quasi-compact, quasi-separated spectral algebraic space (using induction on the length of the scallop decomposition). We will prove the converse using a slightly more complicated version of the proof of Proposition 3.2.3.1. Assume that Y is a quasi-compact, quasi-separated spectral algebraic space. Since Y is quasi-compact, we can choose an étale surjection $u : \mathsf{X} \to \mathsf{Y}$ where X is affine. Lemma 3.4.3.2 implies that u is separated. For $i \geqslant 1$, each of the evaluation maps $\mathrm{Conf}^i_{\mathsf{Y}}(\mathsf{X}) \to \mathsf{X}$ is étale, separated, and quasi-compact (since Y is assumed to be quasi-separated). Since X is affine, we conclude that $\mathrm{Conf}^i_{\mathsf{Y}}(\mathsf{X})$ is quasi-affine (Theorem 3.3.0.2). Using the quasi-compactness of Y, we deduce the existence of an integer n such that $\mathrm{Conf}^{n+1}_{\mathsf{Y}}(\mathsf{X})$ is empty. For $0 \leqslant i < n$, we can use Lemma 3.4.3.1 to obtain a finite covering of $\mathrm{Conf}^{n-i}_{\mathsf{Y}}(\mathsf{X})$ by Σ_i-invariant open substacks $\{\mathsf{U}_{i,j}\}_{1 \leqslant j \leqslant m_i}$ such that each quotient $\mathsf{U}_{i,j}/\Sigma_{n-i}$ is affine. Let $m = \sum_{0 \leqslant i < n} m_i$. If $1 \leqslant k \leqslant m$, then we can write $k = m_0 + \cdots + m_{i-1} + j$ where $1 \leqslant j \leqslant m_i$, and we let U_k denote the spectral Deligne-Mumford stack $\mathsf{U}_{i,j}$. For $0 \leqslant k \leqslant m$, we let V_k denote the open substack of Y given by the image of the étale map $\amalg_{1 \leqslant k' \leqslant k} \mathsf{U}_{k'} \to \mathsf{Y}$. We claim that the sequence of open immersions

$$\mathsf{V}_0 \to \mathsf{V}_1 \to \cdots \to \mathsf{V}_m$$

is a scallop decomposition of Y. Since u is surjective, it is clear that $\mathsf{V}_m \simeq \mathsf{Y}$, and V_0 is empty by construction. Let $0 < k \leqslant m$, and write $k = m_0 + \cdots + m_{i-1} + j$ for $1 \leqslant j \leqslant m_i$.

Form a pullback square

$$\begin{array}{ccc} \mathsf{W} & \longrightarrow & \mathsf{U}_{i,j}/\Sigma_{n-i} \\ \downarrow & & \downarrow q \\ \mathsf{V}_{k-1} & \longrightarrow & \mathsf{V}_k . \end{array}$$

We claim that this diagram is an excision square. To prove this, can replace Y by the reduced closed substack complementary to V_{k-1}, and thereby reduce to the case where $\mathrm{Conf}^{n-i+1}_{\mathsf{Y}}(\mathsf{X})$ is empty. In this case, we wish to show that q is an equivalence. Since q is an étale surjection by construction, it suffices to show that the map $\mathsf{U}_{i,j}/\Sigma_{n-i} \to \mathsf{Y}$ is an open immersion. In fact, we claim that the map $j : \mathrm{Conf}^{n-i}_{\mathsf{Y}}(\mathsf{X})/\Sigma_{n-i} \to \mathsf{Y}$ is an open immersion: this follows from Lemma 3.4.3.3 (since $\mathrm{Conf}^{n-i+1}_{\mathsf{Y}}(\mathsf{X})$ is empty). □

3.5 Geometric Points

Let $\mathcal{X}$ be an ∞-topos. Recall that a *point* of $\mathcal{X}$ is a geometric morphism $x^* : \mathcal{X} \to \mathcal{S}$, where $\mathcal{S}$ denotes the ∞-category of spaces. Our goal in this section is to give an explicit description of the points of $\mathcal{X}$ in the special case where $\mathcal{X}$ is the underlying ∞-topos of a spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

3.5.1 Strictly Henselian $\mathbb{E}_\infty$-Rings

We begin by introducing some terminology.

Definition 3.5.1.1. Let A be an $\mathbb{E}_\infty$-ring. We will say that A is *strictly Henselian* if the commutative ring $\pi_0 A$ is strictly Henselian, in the sense of Definition B.3.5.1.

Remark 3.5.1.2. An $\mathbb{E}_\infty$-ring A is strictly Henselian if and only if it is strictly Henselian when regarded as a sheaf of $\mathbb{E}_\infty$-rings on the ∞-topos $\mathcal{S}$, in the sense of Definition 1.4.2.1.

Remark 3.5.1.3. Let A be an $\mathbb{E}_\infty$-ring. Then A is strictly Henselian if and only if it satisfies the following condition: for every collection of étale morphisms $\{\phi_\alpha : A \to A_\alpha\}$ which generate a covering sieve on A with respect to the étale topology, one of the maps ϕ_α admits a left homotopy inverse. To prove this, we can use Theorem HA.7.5.0.6 to reduce to the case where A is discrete, in which case it follows from Proposition B.3.5.3.

Proposition 3.5.1.4. *Let $f : A \to A'$ be a map of $\mathbb{E}_\infty$-rings which induces a surjective ring homomorphism $\pi_0 A \to \pi_0 A'$. If A is strictly Henselian and $A' \neq 0$, then A' is strictly Henselian.*

Proof. This follows from Corollary B.3.3.2, since the local rings $\pi_0 A$ and $\pi_0 A'$ have the same residue field. □

3.5.2 Points of Affine Spectral Deligne-Mumford Stacks

Fix an $\mathbb{E}_\infty$-ring A. Let $\mathrm{CAlg}_A^{\text{ét}}$ denote the full subcategory of CAlg_A spanned by the $\mathbb{E}_\infty$-algebras which are étale over A, and $\mathcal{S}\mathrm{hv}_A^{\text{ét}}$ the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathcal{S})$ spanned by those functors which are sheaves with respect to the étale topology. By definition, a *point* of the ∞-topos $\mathcal{S}\mathrm{hv}_A^{\text{ét}}$ is a geometric morphism $f^* : \mathcal{S}\mathrm{hv}_A^{\text{ét}} \to \mathcal{S}$. Composition with the Yoneda embedding $(\mathrm{CAlg}_A^{\text{ét}})^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathcal{S})$ induces an equivalence between the ∞-category of points of the presheaf ∞-topos $\mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathcal{S})$ with the full subcategory $\mathrm{Ind}(\mathrm{CAlg}_A^{\text{ét}}) \subseteq \mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}})^{\mathrm{op}}, \mathcal{S})$ (Proposition HTT.6.1.5.2). We will say that an A-algebra B is Ind-*étale* if it is a filtered colimit of étale A-algebras. Since every étale A-algebra is a compact object of CAlg_A (Corollary HA.7.5.4.4), we can identify $\mathrm{Ind}(\mathrm{CAlg}_A^{\text{ét}})$ with a full subcategory $\mathrm{CAlg}_A^{\mathrm{Ind}-\text{ét}} \subseteq \mathrm{CAlg}_A$ spanned by the Ind-étale A-algebras.

Proposition 3.5.2.1. *Let A be an $\mathbb{E}_\infty$-ring, let B be an* Ind-*étale A-algebra, and let $\eta_* : \mathcal{S} \to \mathrm{Fun}(\mathrm{CAlg}_A^{ét}, \mathcal{S})$ be the geometric morphism determined by B. The following conditions are equivalent:*

(1) *The geometric morphism η_* factors through the full subcategory $\mathcal{S}\mathrm{hv}_A^{\text{ét}} \subseteq \mathrm{Fun}(\mathrm{CAlg}_A^{ét}, \mathcal{S})$.*

(2) *The $\mathbb{E}_\infty$-ring B is strictly Henselian.*

Proof. Using Proposition HTT.6.2.3.20 , we see that (1) is equivalent to the following condition:

$(*)$ Let A' be an étale$A$-algebra, and suppose we are given a finite collection of étale maps $\{A' \to A'_\alpha\}$ such that $A \to \prod_\alpha A_\alpha$ is faithfully flat (see Definition B.6.1.1). Then any A-algebra map $A' \to B$ factors (up to homotopy) through A'_α for some index α.

The implication (2) $\Rightarrow$ $(*)$ follows immediately by applying Remark 3.5.1.3 to the family of morphisms $\{B \to B \otimes_{A'} A'_\alpha\}$ (which generate a covering sieve on B with respect to the étale topology). Conversely, suppose that $(*)$ is satisfied. We will prove that B is strictly Henselian by verifying the criterion of Remark 3.5.1.3. Suppose we are given a finite collection of étale morphisms $\{B \to B_\alpha\}$ which induce a faithfully flat map $\theta : B \to \prod_\alpha B_\alpha$; we wish to show that there is an index α and a map of B-algebras $B_\alpha \to B$.

Write B as a filtered colimit of étale A-algebras $B(\beta)$. Using the structure theorem for étale morphisms (Proposition B.1.1.3), we can choose an index β and étale morphisms $\{B(\beta) \to B(\beta)_\alpha\}$ such that $B_\alpha \simeq B \otimes_{B(\beta)} B(\beta)_\alpha$. The image of the induced map

$$\amalg_\alpha |\operatorname{Spec} B(\beta)_\alpha| \to |\operatorname{Spec} B(\beta)|$$

is a quasi-compact open subset $U \subseteq |\operatorname{Spec} B(\beta)|$ (Proposition **??**) corresponding to a radical ideal $I \subseteq \pi_0 B(\beta)$. Since θ is faithfully flat, the image of I generates the unit ideal in

$\pi_0 B$. Changing our index β, we may suppose that I is the unit ideal, so that the map $B(\beta) \to \prod_\alpha B(\beta)_\alpha$ is faithfully flat. It follows from $(*)$ that there exists an index α and a map of $B(\beta)$-algebras $B(\beta)_\alpha \to B$, which determines a map of B-algebras $B_\alpha \to B$. $\square$

Proposition 3.5.2.1 yields the following description for points of an *affine* spectral Deligne-Mumford stacks:

Corollary 3.5.2.2. *Let A be an $\mathbb{E}_\infty$-ring, and let $\mathcal{C}$ be the full subcategory of $\mathrm{Fun}(\mathcal{S}\mathrm{hv}_A^{\acute{e}t}, \mathcal{S})$ spanned by those functors which are left exact and preserve small colimits. Then composition with the Yoneda embedding $(\mathrm{CAlg}_A^{\acute{e}t})^{\mathrm{op}} \hookrightarrow \mathcal{S}\mathrm{hv}_A^{\acute{e}t}$ induces an equivalence of $\mathcal{C}$ with the full subcategory of $\mathrm{Ind}(\mathrm{CAlg}_A^{\acute{e}t}) \simeq \mathrm{CAlg}_A^{\mathrm{Ind}-\acute{e}t} \subseteq \mathrm{CAlg}_A$ spanned by those A-algebras which are strictly Henselian and* Ind-*étale over A.*

Remark 3.5.2.3. Let A be an $\mathbb{E}_\infty$-ring and let $(\mathcal{X}, \mathcal{O}) = \mathrm{Sp\acute{e}t}(A)$ be the corresponding spectral Deligne-Mumford stack. Let $f^* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(*) = \mathcal{S}$ be a point of $\mathcal{X}$, which corresponds under the equivalence of Corollary 3.5.2.2 to a strictly Henselian A-algebra $A' \simeq \varinjlim A'_\alpha$, where each A'_α is an étale A-algebra. Let $\mathcal{C}$ be an arbitrary compactly generated ∞-category, and let $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}}((\mathrm{CAlg}_A^{\acute{e}t})^{\mathrm{op}})$. Unwinding the definitions, we obtain a canonical equivalence $f^* \mathscr{F} \simeq \varinjlim \mathscr{F}(A'_\alpha)$ in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(*) \simeq \mathcal{C}$. In particular, A' can be identified with the stalk of the structure sheaf $f^* \mathcal{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{S}) \simeq \mathrm{CAlg}$.

Remark 3.5.2.4. Let A be an $\mathbb{E}_\infty$-ring and let $(\mathcal{X}, \mathcal{O}) = \mathrm{Sp\acute{e}t}(A)$. Let B be a strictly Henselian $\mathbb{E}_\infty$-ring; let us identify B with the corresponding object of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{S})$, so that $(\mathcal{S}, B)$ is an object of $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\acute{e}t}$. We then have a canonical homotopy equivalence

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\acute{e}t}}((\mathcal{S}, B), (\mathcal{X}, \mathcal{O})) \to \mathrm{Map}_{\mathrm{CAlg}}(A, B).$$

Under the equivalence of Corollary 3.5.2.2, this assertion translates as follows: every map of $\mathbb{E}_\infty$-rings $f : A \to B$ factors uniquely as a composition

$$A \xrightarrow{f'} A' \xrightarrow{f''} B,$$

where A' is strictly Henselian, f' is Ind-étale, and f'' is local. We will refer to A' as the *strict Henselization of A along f*.

3.5.3 Minimal Geometric Points

It is often convenient to describe points of spectral Deligne-Mumford stacks by Henselizing the spectra of separably closed fields.

Definition 3.5.3.1. Let X be a spectral Deligne-Mumford stack. A *geometric point* of X is a morphism of spectral Deligne-Mumford stacks $\eta : \mathrm{Sp\acute{e}t}\, \kappa \to \mathsf{X}$, where κ is a separably closed field. We will say that a geometric point $\eta : \operatorname{Sp\acute{e}t} \kappa \to \mathsf{X}$ is *minimal* if it factors as a composition

$$\operatorname{Sp\acute{e}t} \kappa \xrightarrow{\eta'} \operatorname{Sp\acute{e}t} A \xrightarrow{\eta''} \mathsf{X}$$

where η'' is étale and η' induces a map of commutative rings $\phi : \pi_0 A \to \kappa$ which exhibits κ as a separable algebraic extension of some residue field of $\pi_0 A$.

For each object $\mathsf{X} \in \mathrm{SpDM}$, we let $\mathrm{Pt}_{\mathrm{g}}(\mathsf{X})$ denote the full subcategory of $\mathrm{SpDM}_{/\mathsf{X}}$ spanned by the minimal geometric points $\eta : \mathsf{X}_0 \to \mathsf{X}$.

Remark 3.5.3.2. Let X be a spectral Deligne-Mumford stack and let $\eta : \operatorname{Sp\acute{e}t} \kappa \to \mathsf{X}$ be a minimal geometric point of X. For *every* factorization

$$\operatorname{Sp\acute{e}t} \kappa \xrightarrow{\eta'} \operatorname{Sp\acute{e}t} A \xrightarrow{\eta''} \mathsf{X}$$

of η where η'' is étale, the map η' exhibits κ as a separable closure of the residue field $\kappa(\mathfrak{p})$ for prime ideal $\mathfrak{p} \subseteq \pi_0 A$.

Remark 3.5.3.3. Let $\phi : \mathsf{U} \to \mathsf{X}$ be an étale map of spectral Deligne-Mumford stacks, and suppose we are given a geometric point $\eta : \operatorname{Sp\acute{e}t} \kappa \to \mathsf{U}$. Then η is a minimal geometric point of U if and only if $\phi \circ \eta$ is a minimal geometric point of X.

Remark 3.5.3.4. Suppose we are given a commutative diagram of fields

$$\begin{array}{ccccc} & & \kappa & & \\ & \swarrow^{\phi} & & \searrow^{\psi} & \\ \kappa & & \xrightarrow{\theta} & & \kappa''. \end{array}$$

If ϕ and ψ exhibit κ' and κ'' as separable closures of κ, then θ is an isomorphism. It follows that if X is a spectral Deligne-Mumford stack, then every morphism between minimal geometric points of X is an equivalence: that is, the ∞-category $\mathrm{Pt}_{\mathrm{g}}(\mathsf{X})$ is a Kan complex.

3.5.4 Comparison of Points with Geometric Points

Let $\eta : \mathsf{X}_0 \to \mathsf{X}$ be a geometric point of a connective spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$. Then X_0 is the spectrum of a separably closed field, so that underlying ∞-topos of X_0 is canonically equivalent to $\mathcal{S}$. Consequently, the pullback functor η^* can be viewed as a geometric morphism $\mathcal{X} \to \mathcal{S}$.

Remark 3.5.4.1. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne stack. Let $U \in \mathcal{X}$ be an object of let $\mathsf{U} = (\mathcal{X}_{/U}, \mathscr{O}_{\mathcal{X}}|_U)$. The étale map of spectral Deligne-Mumford stacks $\phi : \mathsf{U} \to \mathsf{X}$ induces a map of Kan complexes $\theta : \mathrm{Pt}_{\mathrm{g}}(\mathsf{U}) \to \mathrm{Pt}_{\mathrm{g}}(\mathsf{X})$. Using Remark 3.5.3.3, we deduce that the homotopy fiber of θ over a point $\eta \in \mathrm{Pt}_{\mathrm{g}}(\mathsf{X})$ can be identified with the space $\eta^*(U) \in \mathcal{S}$.

Proposition 3.5.4.2. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. The construction*

$$(\eta : \mathsf{X}_0 \to \mathsf{X}) \mapsto (\eta^* \in \mathrm{Fun}(\mathcal{X}, \mathcal{S}))$$

determines an equivalence from the ∞-category $\mathrm{Pt}_{\mathrm{g}}(\mathsf{X})$ of minimal geometric points of X to the subcategory $\mathrm{Fun}^*(\mathcal{X}, \mathcal{S})^{\simeq} \subseteq \mathrm{Fun}(\mathcal{X}, \mathcal{S})$ whose objects are functors which preserve small colimits and finite limits and whose morphisms are equivalences.

Proof. For each object $U \in \mathcal{X}$, let X_U denote the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$. We let $\mathrm{Fun}^*(\mathcal{X}_{/U}, \mathcal{S})^{\simeq}$ denote the subcategory of $\mathrm{Fun}(\mathcal{X}_{/U}, \mathcal{S})$ whose objects are functors which preserve small colimits and finite limits and whose morphisms are equivalences. For each $U \in \mathcal{X}$, we have a canonical map of Kan complexes $\mathrm{Pt}_{\mathrm{g}}(\mathsf{X}_U) \to \mathrm{Fun}(\mathcal{X}_{/U}, \mathcal{S})^{\simeq}$. Let us say that U is *good* if θ_U is a homotopy equivalence. For any morphism $U \to V$ in $\mathcal{X}$, Remark 3.5.4.1 implies that the diagram

$$\begin{array}{ccc} \mathrm{Pt}_{\mathrm{g}}(\mathsf{X}_U) & \xrightarrow{\theta_U} & \mathrm{Fun}(\mathcal{X}_{/U}, \mathcal{S})^{\simeq} \\ \downarrow & & \downarrow \\ \mathrm{Pt}_{\mathrm{g}}(\mathsf{X}_V) & \xrightarrow{\theta_V} & \mathrm{Fun}(\mathcal{X}_{/V}, \mathcal{S})^{\simeq} \end{array}$$

is a pullback square. Since colimits in $\mathcal{S}$ are universal, we conclude that for every diagram $\{U_\alpha\}$ in $\mathcal{X}$ having colimit U, the induced diagram

$$\begin{array}{ccc} \varinjlim \mathrm{Pt}_{\mathrm{g}}(\mathsf{X}_{U_\alpha}) & \longrightarrow & \varinjlim \mathrm{Fun}(\mathcal{X}_{/U_\alpha}, \mathcal{S})^{\simeq} \\ \downarrow & & \downarrow \\ \mathrm{Pt}_{\mathrm{g}}(U) & \xrightarrow{\theta_U} & \mathrm{Fun}(\mathcal{X}_{/U}, \mathcal{S})^{\simeq} \end{array}$$

is a pullback square. It follows from Proposition HTT.6.3.5.5 that the right vertical map in this diagram is a homotopy equivalence, so that the left vertical map is a homotopy equivalence as well. Consequently, if each U_α is good, then U is good. We wish to prove that every object $U \in \mathcal{X}$ is good. By virtue of Proposition 1.4.7.9, it will suffice to treat the case where U is affine. Replacing X by X_U, we may assume that $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ for some connective $\mathbb{E}_\infty$-ring A.

Let $\mathsf{X}_0 = \mathrm{Sp\acute{e}t}(\pi_0 A)$. The underlying ∞-topoi of X and X_0 are the same, and the canonical map

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, \kappa, \mathsf{X}_0) \to \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, \kappa, \mathsf{X})$$

is a homotopy equivalence for every discrete $\mathbb{E}_\infty$-ring κ. We may therefore replace A by $\pi_0 A$ and thereby reduce to the case where A is a discrete $\mathbb{E}_\infty$-ring.

Let $\mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{S})$ spanned by those functors which preserve small colimits and finite limits. Using Corollary 3.5.2.2, we can identify

$\mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$ with the full subcategory of $\mathcal{C} \subseteq \mathrm{CAlg}_A$ whose objects are A-algebras B which are strictly Henselian and can be written as a filtered colimit of étale A-algebras. We can identify $\mathrm{Pt_g}(\mathsf{X})$ with the groupoid consisting of those A-algebras κ which are separable closures of some residue field of A. We will denote the functor $\theta : \mathrm{Pt_g}(\mathsf{X}) \to \mathcal{C}$ by $\kappa \mapsto A_\kappa$. This construction can be characterized by the following universal property: for every object $\kappa \in \mathrm{Pt_g}(\mathsf{X})$ and every étale A-algebra B, we have a canonical bijection

$$\mathrm{Hom}_A(B, A_\kappa) \simeq \mathrm{Hom}_k(B \otimes_A \kappa, \kappa) \simeq \mathrm{Hom}_A(B, \kappa).$$

Here $\mathrm{Hom}_R(R', R'')$ denotes the set of R-algebra maps from R' to R''. Note that this bijection extends naturally to the case where B is a filtered colimit of étale A-algebras.

For every object $B \in \mathcal{C}$, let $\mathfrak{p}_B$ denote the inverse image in A of the maximal ideal of B. Any morphism $B \to B'$ in $\mathcal{C}$ determines an inclusion of prime ideals $\mathfrak{p}_{B'} \subseteq \mathfrak{p}_B$. We let $\mathcal{C}_0$ denote the subcategory of $\mathcal{C}$ consisting of those morphisms $B \to B'$ for which $\mathfrak{p}_B = \mathfrak{p}_{B'}$. Since $\mathcal{C}_0$ contains all equivalences in $\mathcal{C}$, the map θ factors through $\mathcal{C}_0$. We will show that θ induces an equivalence $\mathrm{Pt_g}(\mathsf{X}) \to \mathcal{C}_0$. From this, it will follow that $\mathcal{C}_0$ is a Kan complex, hence that $\mathcal{C}_0$ is the largest Kan complex contained in $\mathcal{C}$ and therefore that θ exhibits $\mathrm{Pt_g}(\mathsf{X})$ as equivalent to the largest Kan complex contained in $\mathcal{C}$.

Let $\kappa \in \mathrm{Pt_g}(\mathsf{X})$, and let $\mathfrak{p} = \mathfrak{p}_{A_k}$. Note that $\mathrm{Hom}_A(A[u^{-1}], A_\kappa)$ is empty if and only if $u \in \mathfrak{p}$. It follows that $\mathfrak{p}$ is the kernel of the map $A \to \kappa$, so that κ is a separable closure of the residue field $\kappa(\mathfrak{p})$ of A. Since A_κ is a filtered colimit of étale A-algebras, the quotient $(A_\kappa/\mathfrak{p}A_\kappa) \simeq A_\kappa \otimes_A \kappa(\mathfrak{p})$ is a filtered colimit of finite étale algebras over $\kappa(\mathfrak{p})$. Since A_κ is strictly Henselian, the quotient $A_\kappa/\mathfrak{p}A_\kappa$ is also strictly Henselian and therefore a separable closure of the residue field $\kappa(\mathfrak{p})$. Let κ' be another separable closure of $\kappa(\mathfrak{p})$. The canonical map

$$v : \mathrm{Hom}_A(\kappa, \kappa') \simeq \mathrm{Map}_{\mathrm{Pt_g}(\mathsf{X})}(\kappa, \kappa') \to \mathrm{Map}_{\mathcal{C}}(A_\kappa, A_{\kappa'}) \simeq \mathrm{Hom}_A(A_\kappa, A_{\kappa'}) \simeq \mathrm{Hom}_A(A_\kappa, \kappa')$$

is given by composition with a map $v_0 : A_\kappa/\mathfrak{p}A_\kappa \to \kappa$. Here v_0 is a $\kappa(\mathfrak{p})$-algebra map between separable closures of $\kappa(\mathfrak{p})$, and therefore an isomorphism. It follows that v is bijective, which proves that $\theta : \mathrm{Pt_g}(\mathsf{X}) \to \mathcal{C}_0$ is fully faithful.

It remains to prove that θ is essentially surjective. Let $B \in \mathcal{C}$ and let $\mathfrak{p} = \mathfrak{p}_B$. Then $\kappa = B/\mathfrak{p}B \simeq B \otimes_A \kappa(\mathfrak{p})$ is a filtered colimit of étale $\kappa(\mathfrak{p})$-algebras. Since κ is strictly Henselian, we deduce that κ is a separable closure of $\kappa(\mathfrak{p})$. In particular, $\mathfrak{p}B$ is the maximal ideal of B, and we can identify κ with an object of $\mathrm{Pt_g}(\mathsf{X})$. For every étale A-algebra B', we have a canonical map

$$\mathrm{Hom}_A(B', B) \to \mathrm{Hom}_A(B', B/\mathfrak{p}B) = \mathrm{Hom}_A(B', \kappa) \simeq \mathrm{Hom}_A(B', A_\kappa).$$

Since B is Henselian, this map is bijective. Since B and A_κ can both be obtained as a filtered colimit of étale A-algebras, we conclude that $B \simeq A_\kappa$. □

Remark 3.5.4.3. Let X be a spectral Deligne-Mumford stack, and suppose we are given a morphism $\eta : \text{Spét}\, \kappa \to \mathsf{X}$, where κ is a separably closed field. Then η factors as a composition $\text{Spét}\, \kappa \xrightarrow{\eta'} \mathsf{U} \xrightarrow{\eta''} \mathsf{X}$, where U is affine and η'' is étale. Write $\mathsf{U} = \text{Spét}\, A$, so that η' determines a map of $\mathbb{E}_\infty$-rings $A \to \kappa$. The image of the map of commutative rings $\pi_0 A \to \kappa$ generates a subfield of $\kappa' \subseteq \kappa$. Let $\kappa_0 \subseteq \kappa$ denote the separable closure of κ' in κ. Then η factors as a composition $\text{Spét}\, \kappa \to \text{Spét}\, \kappa_0 \xrightarrow{\eta_0} \mathsf{X}$, where η_0 is a minimal geometric point of X and the inclusion $\kappa_0 \subseteq \kappa$ is an extension of separably closed fields.

3.5.5 Existence of Geometric Points

Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. We will say that X is *empty* if $\mathcal{X}$ is a contractible Kan complex (that is, if $\mathcal{X}$ is equivalent to the ∞-category of sheaves on the empty topological space). Otherwise, we will say that X is *nonempty*.

Remark 3.5.5.1. A spectral Deligne-Mumford stack X is empty if and only if it is an initial object of the ∞-category SpDM of spectral Deligne-Mumford stacks.

Lemma 3.5.5.2. *Let X be a spectral Deligne-Mumford stack. The following conditions are equivalent:*

(1) *The spectral Deligne-Mumford stack X is not empty.*

(2) *There exists a nonzero connective $\mathbb{E}_\infty$-ring A and an étale map $\text{Spét}\, A \to \mathsf{X}$.*

(3) *There exists a minimal geometric point $\text{Spét}\, \kappa \to \mathsf{X}$.*

Proof. The implications (2) ⇒ (3) and (3) ⇒ (1) are obvious. We prove that (1) ⇒ (2). Assume that (1) is satisfied. Write $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, so that there exists an object of $\mathcal{X}$ which is not initial. It follows that there exists an affine object $U \in \mathcal{X}$ which is not initial. Then $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ is equivalent to $\text{Spét}\, A$ for some connective $\mathbb{E}_\infty$-ring A. We therefore have an étale map $\text{Spét}\, A \to \mathsf{X}$. Since U is not an initial object of $\mathcal{X}$, A is nonzero; this proves (2). □

Remark 3.5.5.3. The implication (1) ⇒ (3) of Lemma 3.5.5.2 can also be deduced from Proposition 3.5.4.2 and Theorem A.4.0.5.

Proposition 3.5.5.4. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. The following conditions are equivalent:*

(1) *Let $\mathcal{X}$ and $\mathcal{Y}$ denote the underlying ∞-topoi of X and Y, respectively. Then every geometric morphism of ∞-topoi $\eta_* : \mathcal{S} \to \mathcal{Y}$ factors through the geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$.*

(2) *For every field* κ *and every map* $\eta : \mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Y}$*, there exists a field extension* κ' *of* κ *such that the composite map* $\mathrm{Sp\acute{e}t}\,\kappa' \to \mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Y}$ *factors through* f*.*

(3) *For every field* κ *and every map* $\eta : \mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Y}$*, the fiber product* $\mathrm{Sp\acute{e}t}\,\kappa \times_{\mathsf{Y}} \mathsf{X}$ *is nonempty.*

Definition 3.5.5.5. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. We will say that f is *surjective* if it satisfies the equivalent conditions of Proposition 3.5.5.4.

Remark 3.5.5.6. Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}. \end{array}$$

If f is surjective, then f' is surjective.

Remark 3.5.5.7. Suppose we are given maps of spectral Deligne-Mumford stacks

$$\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z},$$

where f is surjective. Then g is surjective if and only if $g \circ f$ is surjective.

Example 3.5.5.8. Let $f : \mathsf{X} \to \mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be an étale morphism of spectral Deligne-Mumford stacks, so that $\mathsf{X} \simeq (\mathcal{Y}_{/U}, \mathscr{O}_{\mathcal{Y}}|_U)$ for some object $U \in \mathsf{Y}$. The following conditions are equivalent:

(i) The object $U \in \mathcal{Y}$ is 0-connective.

(ii) The morphism f is surjective.

Proof of Proposition 3.5.5.4. The implication (2) ⇒ (3) is obvious, and the implication (3) ⇒ (1) follows from Lemma 3.5.5.2. We will show that (1) ⇒ (2). Let $\eta : \mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. We wish to show that, after enlarging κ if necessary, the map η factors through X. Without loss of generality, we may assume that κ is separably closed. Note that η determines a geometric morphism of ∞-topoi $\eta_* : \mathcal{S} \to \mathcal{Y}$. Using condition (1), we deduce that η_* factors as a composition $\mathcal{S} \xrightarrow{\eta'_*} \mathcal{X} \xrightarrow{f_*} \mathcal{Y}$. According to Proposition 3.5.4.2, the geometric morphism η'_* is determined by a minimal geometric point $\eta' : \mathrm{Sp\acute{e}t}\,k' \to \mathsf{X}$. Using Remark 3.5.4.3, we see that η and $f \circ \eta'$ admit factorizations

$$\mathrm{Sp\acute{e}t}\,\kappa \to \mathrm{Sp\acute{e}t}\,\kappa_0 \xrightarrow{\eta_0} \mathsf{Y} \qquad \mathrm{Sp\acute{e}t}\,\kappa' \to \mathrm{Sp\acute{e}t}\,\kappa'_0 \xrightarrow{\eta'_0} \mathsf{Y},$$

where κ_0 and κ'_0 are separably closed subfields of κ and κ', respectively, and η_0 and η'_0 are minimal geometric points of Y. By construction, the pushforward functors $(\eta_0)_*, (\eta'_0)_* : \mathcal{S} \to$

$\mathcal{Y}$ are homotopic. It follows from Proposition 3.5.4.2 that there is an isomorphism of fields $\kappa_0 \simeq \kappa_0'$ such that the diagram

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} \kappa_0 & \longrightarrow & \operatorname{Sp\acute{e}t} \kappa_0' \\ & {\scriptstyle \eta_0} \searrow \quad \swarrow {\scriptstyle \eta_0'} & \\ & \mathsf{Y} & \end{array}$$

commutes up to homotopy. Let κ'' be any residue field of the tensor product $\kappa \otimes_{\kappa_0} \kappa'$. Then the composite map $\operatorname{Sp\acute{e}t} \kappa'' \to \operatorname{Sp\acute{e}t} \kappa \xrightarrow{\eta} \mathsf{Y}$ factors through $f \circ \eta$, and therefore lifts to a map $\operatorname{Sp\acute{e}t} \kappa'' \to \mathsf{X}$. □

3.6 Points of Spectral Algebraic Spaces

Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. In §3.5, we studied the Kan complex $\operatorname{Fun}^*(\mathcal{X}, \mathcal{S})^{\simeq}$ of *points* of the ∞-topos X, and proved that it was equivalent to the ∞-category $\operatorname{Pt}_{\mathrm{g}}(\mathsf{X})$ of minimal geometric points of X (Proposition 3.5.4.2). In this section, we will study an *a priori* different notion of point, given by points of the locale $\tau_{\leqslant -1} \mathcal{X}$ of (-1)-truncated points of X. In good cases, one can show that $|\mathsf{X}|$ can be identified with the set of isomorphism classes of minimal geometric points of X (Corollary 3.6.4.4).

3.6.1 The Underlying Topological Space

In §1.5, we associated to each ∞-topos $\mathcal{X}$ a topological space $|\mathcal{X}|$ (see Definition 1.5.4.3). In this section, we will be interested in the case where $\mathcal{X}$ is the underlying ∞-topos of a spectral Deligne-Mumford stack.

Definition 3.6.1.1. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. We $|\mathsf{X}|$ denote the underlying topological $|\mathcal{X}|$ of the ∞-topos $\mathcal{X}$. We will refer to $|\mathsf{X}|$ as the *underlying topological space* of X.

More concretely, an element $x \in |\mathsf{X}|$ can be identified with an equivalence class of open substacks $\mathsf{U}_x \hookrightarrow \mathsf{X}$ having the following property:

$(*)$ Given a finite collection $\{\mathsf{V}_i \subseteq \mathsf{X}\}_{1 \leqslant i \leqslant n}$ of open substacks of X having intersection U_x, we have $\mathsf{U}_x = \mathsf{V}_i$ for some i.

Here we should think of U_x as the open substack of X given by the complement of the closure of the point x. We will regard $|\mathsf{X}|$ as a topological space, having open sets of the form $\{x \in |\mathsf{X}| : \mathsf{U} \nsubseteq \mathsf{U}_x\}$ where U ranges over open substacks of X.

Remark 3.6.1.2. In this section, we are primarily interested in studying the topological space $|\mathsf{X}|$ in the case where X is a quasi-separated spectral algebraic space. All of our results can be deduced immediately from their counterparts in the classical theory of algebraic spaces (see, for example, [117]). We include proofs here for the sake of completeness.

Remark 3.6.1.3. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. The ∞-topos $\mathcal{X}$ is locally coherent, so that the hypercompletion $\mathcal{X}^{\mathrm{hyp}}$ has enough points (Theorem A.4.0.5). It follows immediately that the locale of open substacks of X has enough points: that is, there is a one-to-one correspondence between open subsets of the topological space $|\mathsf{X}|$ and equivalence classes of open substacks of X.

Example 3.6.1.4. Let R be an $\mathbb{E}_\infty$-ring. Then there is a bijective correspondence between equivalence classes of open substacks of $\operatorname{Sp\acute{e}t} R$ and open subsets of the topological space $|\operatorname{Spec} R|$ (see Lemma **??**). Since every irreducible closed subset of $|\operatorname{Spec} R|$ has a unique generic point, we obtain a canonical homeomorphism $|\operatorname{Sp\acute{e}t} R| \simeq |\operatorname{Spec} R|$. In particular, if R is a field, then the topological space $|\operatorname{Sp\acute{e}t} R|$ consists of a single point.

Example 3.6.1.5. Let X be a schematic spectral algebraic space, and let $(X, \mathcal{O}_X)$ be the associated spectral scheme (see Definition **??**). Then $|\mathsf{X}|$ is homeomorphic to X (see Remark 1.1.2.10).

Proposition 3.6.1.6. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of quasi-separated spectral algebraic spaces. Assume that X is 0-truncated and quasi-compact, and let Y_0 be the schematic image of f (see Construction 3.1.5.1). Then the image of the closed immersion $|\mathsf{Y}_0| \hookrightarrow |\mathsf{Y}|$ is the closure of the image of the map $|f| : |\mathsf{X}| \to |\mathsf{Y}|$.*

Proof. Replacing Y by Y_0, we can reduce to the case where Y is 0-truncated and the unit map $u : \mathcal{O}_{\mathcal{Y}} \to \pi_0 f_* \mathcal{O}_{\mathcal{X}}$ is a monomorphism (here $\mathcal{O}_{\mathcal{X}}$ and $\mathcal{O}_{\mathcal{Y}}$ denote the structure sheaves of X and Y, respectively). In this case, we wish to show that the map $|\mathsf{X}| \to |\mathsf{Y}|$ has dense image. Assume otherwise: then there exists a nonempty open subset $U \subseteq |\mathsf{Y}|$ which does not intersect the image of f. Let U denote the open substack of Y corresponding to U. Using Remark 3.1.5.6, we can replace Y by U and thereby reduce to the case where X is empty. In this case, the assertion that u is a monomorphism guarantees that $\mathcal{O}_{\mathcal{Y}} \simeq 0$, contradicting our assumption that $U \neq \emptyset$. □

3.6.2 Points

Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. Every point of the ∞-topos $\mathcal{X}$ determines a point of the topological space $|\mathsf{X}|$. This observation determines a map of sets $\theta : \pi_0 \operatorname{Pt_g}(\mathsf{X}) \to |\mathsf{X}|$, where $\operatorname{Pt_g}(\mathsf{X})$ denotes the space of geometric points of X (Proposition 3.5.4.2). In good cases, one can show that the map θ is bijective. One of our goals in this section is prove this in the case where X is a quasi-separated spectral algebraic space. To do so, it will be convenient to describe the elements of $|\mathsf{X}|$ in a different way.

Definition 3.6.2.1. Let X be a spectral algebraic space. A *point* of X is a morphism $\eta : \mathsf{X}_0 \to \mathsf{X}$ with the following properties:

(1) The object $\mathsf{X}_0 \in \mathrm{SpDM}$ is equivalent to the spectrum of a field κ.

(2) For every commutative ring R, the map

$$\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}}(\kappa, R) \simeq \pi_0 \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X}_0) \to \pi_0 \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, X)$$

is injective.

Example 3.6.2.2. Let R be a connective $\mathbb{E}_\infty$-ring. A map $\eta : \mathrm{Sp\acute{e}t}\, \kappa \to \mathrm{Sp\acute{e}t}\, R$ is a point of R if and only if κ is a field and the underlying map $R \to \kappa$ exhibits κ as the residue field of the commutative ring $\pi_0 R$ at some prime ideal $\mathfrak{p} \subseteq \pi_0 R$.

Remark 3.6.2.3. Let X be a quasi-separated spectral algebraic space, and suppose we are given a point $\eta : \mathrm{Sp\acute{e}t}\, \kappa \to \mathsf{X}$. Choose any étale map $u : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$, and let Y denote the fiber product $\mathrm{Sp\acute{e}t}\, \kappa \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, R$. Since the diagonal of X is quasi-affine, Y has the form $\mathrm{Sp\acute{e}t}\, \kappa'$ for some étale κ-algebra κ'. We may therefore write κ' as a finite product $\prod_\alpha \kappa'_\alpha$, where each κ'_α is a finite separable extension of the field κ. Each of the induced maps $\mathrm{Sp\acute{e}t}\, \kappa'_\alpha \to \mathrm{Sp\acute{e}t}\, \kappa' \to \mathrm{Sp\acute{e}t}\, R$ is a point of $\mathrm{Sp\acute{e}t}\, R$, so that each κ'_α can be identified with a residue field of the commutative ring $\pi_0 R$ (Example 3.6.2.2) at some prime ideal $\mathfrak{p}_\alpha$. Moreover, the prime ideals $\mathfrak{p}_\alpha$ are distinct from one another.

Remark 3.6.2.4. Let X be a quasi-separated spectral algebraic space, and suppose we are given a commutative diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}\, \kappa & \xrightarrow{\theta} & \mathrm{Sp\acute{e}t}\, \kappa' \\ & {\scriptstyle \eta} \searrow \quad \swarrow {\scriptstyle \eta'} & \\ & \mathsf{X} & \end{array}$$

where η is a point of X and κ' is a field. Choose an étale map $u : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$ such that $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, \kappa' \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, R$ is nonempty, so that Y has the form $\mathrm{Sp\acute{e}t}\, \kappa''$ for some nonzero étale κ'-algebra κ''. Then $\mathrm{Sp\acute{e}t}\, \kappa \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, R$ is the spectrum of the commutative ring $\kappa \otimes_{\kappa'} \kappa''$. It follows from Remark 3.6.2.3 that the composite map $\pi_0 R \to \kappa'' \to \kappa \otimes_{\kappa'} \kappa''$ is surjective. In particular, the map $\kappa'' \to \kappa \otimes_{\kappa'} \kappa''$ is surjective, so we must have $\kappa' \simeq \kappa$: that is, the map θ is an equivalence.

Notation 3.6.2.5. Let X be a quasi-separated spectral algebraic space, and let $\mathrm{Pt}(\mathsf{X})$ be the full subcategory of $\mathrm{SpDM}_{/\mathsf{X}}$ spanned by the points of X. Remark 3.6.2.4 implies that $\mathrm{Pt}(\mathsf{X})$ is a Kan complex, and it follows immediately from the definition that all mapping spaces in $\mathrm{Pt}(\mathsf{X})$ are either empty or contractible. It follows that $\mathrm{Pt}(\mathsf{X})$ is homotopy equivalent to the discrete space $\pi_0 \mathrm{Pt}(\mathsf{X})$. We will generally abuse notation by identifying $\mathrm{Pt}(\mathsf{X})$ with $\pi_0 \mathrm{Pt}(\mathsf{X})$. If $\eta \in \mathrm{Pt}(\mathsf{X})$ corresponds to a morphism $\mathrm{Sp\acute{e}t}\, \kappa \to \mathsf{X}$, we will refer to κ as the *residue field* of X at the point η, and denote it by $\kappa(\eta)$.

Remark 3.6.2.6. Let $i : \mathsf{X}_0 \to \mathsf{X}$ be a closed immersion of quasi-separated spectral algebraic spaces, and let $j : \mathsf{U} \to \mathsf{X}$ be the complementary open immersion. Then the induced maps $\pi_0 \mathrm{Pt}(\mathsf{X}_0) \to \pi_0 \mathrm{Pt}(\mathsf{X}) \leftarrow \pi_0 \mathrm{Pt}(\mathsf{U})$ determine a bijection $\pi_0 \mathrm{Pt}(\mathsf{X}_0) \amalg \pi_0 \mathrm{Pt}(\mathsf{U}) \to \pi_0 \mathrm{Pt}(\mathsf{X})$.

3.6.3 Comparison of $\mathrm{Pt}(\mathsf{X})$ with $|\mathsf{X}|$

Let X be a spectral algebraic space. For every point $\eta : \mathrm{Sp\acute{e}t}\, \kappa \to \mathsf{X}$, the induced map of topological spaces $|\mathrm{Sp\acute{e}t}\, \kappa| \to |\mathsf{X}|$ determines an element of $|\mathsf{X}|$ (see Example 3.6.1.4). This construction determines a map of sets $\pi_0 \mathrm{Pt}(\mathsf{X}) \to |\mathsf{X}|$. Under mild hypotheses, this map is bijective:

Proposition 3.6.3.1. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-separated spectral algebraic space. Then construction above determines a bijection of sets $\theta : \pi_0 \mathrm{Pt}(\mathsf{X}) \to |\mathsf{X}|$.*

Proof. The topological space $|\mathsf{X}|$ is sober: that is, every irreducible closed subset of $|\mathsf{X}|$ has a unique generic point. It will therefore suffice to show that for every irreducible closed subset $K \subseteq |\mathsf{X}|$, there is a unique equivalence class of points $\eta : \mathrm{Sp\acute{e}t}\, \kappa \to \mathsf{X}$ which determine a generic point of K. Using Proposition 3.1.6.3, we can assume that K is the image of $|\mathsf{X}_0|$ for some closed immersion $\mathsf{X}_0 \to \mathsf{X}$. Replacing X by X_0, we may suppose X is reduced and that $|\mathsf{X}|$ is itself irreducible. In particular, X is nonempty; we may therefore choose an open immersion $j : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$ for some nonzero $\mathbb{E}_\infty$-ring R (Corollary 3.4.2.4). Since X is reduced, R is an ordinary commutative ring. Then the Zariski spectrum $|\mathrm{Spec}\, R|$ is homeomorphic to nonempty open subset of $|\mathsf{X}|$ and is therefore irreducible. It follows that R is an integral domain. Let $\eta \in |\mathsf{X}|$ be the image of the zero ideal $(0) \in |\mathrm{Spec}\, R|$, so that η corresponds to the point given by the composition

$$\mathrm{Sp\acute{e}t}\, \kappa \to \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$$

where κ is the fraction field of R. We claim that η is a generic point of $|\mathsf{X}|$: that is, that η belongs to every nonempty open subset $V \subseteq |\mathsf{X}|$. To see this, we note that because $|\mathsf{X}|$ is irreducible, the inverse image of V in $|\mathrm{Spec}\, R|$ is nonempty and therefore contains the ideal (0). This proves the surjectivity of θ. To prove injectivity, let us suppose we are given any other point $\eta' : \mathrm{Sp\acute{e}t}\, \kappa' \to \mathsf{X}$ which determines a generic point of $|\mathsf{X}'|$. Since η' determines a generic point of $|\mathsf{X}|$, it must factor through the nonempty open substack $\operatorname{Sp\acute{e}t} R$ of X. We may therefore identify κ' with the residue field of R at some prime ideal $\mathfrak{p} \subseteq R$. which belongs to every nonempty open subset of $|\operatorname{Spec} R|$. It follows that for every nonzero element $x \in R$, $\mathfrak{p} \in |\operatorname{Spec} R[x^{-1}]|$ and therefore $x \notin \mathfrak{p}$. This proves that $\mathfrak{p}$ coincides with the zero ideal (0), so that $\eta' \simeq \eta$. □

We now describe some consequences of Proposition 3.6.3.1.

Corollary 3.6.3.2. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y} \end{array}$$

of quasi-separated spectral algebraic spaces. Then the induced map $|\mathsf{X}'| \to |\mathsf{X}| \times_{|\mathsf{Y}|} |\mathsf{Y}'|$ is a surjection of topological spaces.

Proof. Every point $\eta : |\mathsf{X}| \times_{|\mathsf{Y}|} |\mathsf{Y}'|$ can be lifted to a commutative diagram

$$\begin{array}{ccccc} \operatorname{Sp\acute{e}t} \kappa & \longrightarrow & \operatorname{Sp\acute{e}t} \kappa' & \longleftarrow & \operatorname{Sp\acute{e}t} \kappa'' \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{X} & \longrightarrow & \mathsf{Y} & \longleftarrow & \mathsf{Y}' \end{array}$$

where κ, κ', and κ'' are fields. To prove that η can be lifted to a point of $|\mathsf{X}|$, it suffices to observe that $|\operatorname{Sp\acute{e}t} \kappa \times_{\operatorname{Sp\acute{e}t} \kappa'} \operatorname{Sp\acute{e}t} \kappa''|$ is nonempty: that is, that commutative ring $\kappa \otimes_{\kappa'} \kappa''$ is nonzero. □

Proposition 3.6.3.3. *Let X be a quasi-separated spectral algebraic space. Then:*

(1) *The topological space $|\mathsf{X}|$ is sober, and is quasi-compact if X is quasi-compact.*

(2) *The topological space $|\mathsf{X}|$ has a basis consisting of quasi-compact open sets.*

(3) *The topological space $|\mathsf{X}|$ is quasi-separated (that is, if U and V are quasi-compact open subsets of $|\mathsf{X}|$, then the intersection $U \cap V$ is also quasi-compact).*

Moreover, if $f : \mathsf{X} \to \mathsf{Y}$ is a quasi-compact morphism of quasi-separated spectral algebraic spaces, and $U \subseteq |\mathsf{Y}|$ is quasi-compact, then $f^{-1}U \subseteq |\mathsf{X}|$ is quasi-compact.

Corollary 3.6.3.4. *Let X be a quasi-compact, quasi-separated spectral algebraic space. Then the topological space $|\mathsf{X}|$ is coherent. Consequently, $|\mathsf{X}|$ is homeomorphic to the spectrum $\operatorname{Sp}(\mathcal{U}(\mathsf{X}))$, where $\mathcal{U}(\mathsf{X})$ denotes the collection of all quasi-compact open subsets of $|\mathsf{X}|$ (see Proposition A.1.5.10).*

Remark 3.6.3.5. Let X be a spectral Deligne-Mumford stack. Assume that X is locally Noetherian and quasi-compact. Then the topological space $|\mathsf{X}|$ is Noetherian: that is, every open subset of $|\mathsf{X}|$ is quasi-compact. To prove this, choose an $\mathbb{E}_\infty$-ring R and an étale surjection $\operatorname{Spét} R \to \mathsf{X}$. Since the lattice of open subsets of $|\mathsf{X}|$ injects into the lattice of open subsets of $|\operatorname{Spét} R| \simeq |\operatorname{Spec} R|$, we are reduced to proving that the topological space $|\operatorname{Spec} R|$ is locally Noetherian. This follows immediately from our assumption that $\pi_0 R$ is Noetherian.

Proof of Proposition 3.6.3.3. Assertion (1) follows immediately from the definitions. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$, and identify the collection of open sets in $|\mathsf{X}|$ with the collection of (equivalence classes of) (-1)-truncated objects of $\mathcal{X}$. For every object $U \in \mathcal{X}$, if we write $\mathsf{X}_U = (\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X}|_U)$, then the open subset of $|\mathsf{X}|$ corresponding to $\tau_{\leqslant -1} U$ can be described as the image of the map $|\mathsf{X}_U| \to |\mathsf{X}|$. Since $\mathcal{X}$ is generated under small colimits by affine objects, we see that $|\mathsf{X}|$ has a basis of open sets given by the images of maps $|\mathsf{U}| \to |\mathsf{X}|$, where U is an affine spectral algebraic space which is étale over X. In this case, $|\mathsf{U}|$ is quasi-compact by (1), so that $|\mathsf{X}|$ has a basis of quasi-compact open sets.

We now prove that $|\mathsf{X}|$ is quasi-separated. Suppose we are given quasi-compact open sets $U, V \subseteq |\mathsf{X}|$; we wish to show that $U \cap V$ is quasi-compact. Without loss of generality, we may assume that U and V are the images of maps $|\mathsf{U}| \to |\mathsf{X}|$ and $|\mathsf{V}| \to |\mathsf{X}|$, where U and V are affine spectral algebraic spaces which are étale over X. Using Corollary 3.6.3.2, we see that $U \cap V$ is the image of the map $\theta : |\mathsf{U} \times_\mathsf{X} \mathsf{V}| \to |\mathsf{X}|$. Since X is quasi-separated, the fiber product $\mathsf{U} \times_\mathsf{X} \mathsf{V}$ is quasi-compact, so that the underlying topological space $|\mathsf{U} \times_\mathsf{X} \mathsf{V}|$ is also quasi-compact by (1). It follows that the image of θ is quasi-compact, as desired.

Now suppose that $f : \mathsf{X} \to \mathsf{Y}$ is a quasi-compact morphism between quasi-separated spectral algebraic spaces and let $U \subseteq |\mathsf{Y}|$ be a quasi-compact open set; we wish to show that its inverse image is a quasi-compact open subset of $|\mathsf{X}|$. Without loss of generality, we may suppose that U is the image of a map $|\mathsf{U}| \to |\mathsf{Y}|$, where U is affine and étale over Y. Using Corollary 3.6.3.2 we see that the inverse image of U is the image of the map $\theta : |\mathsf{U} \times_\mathsf{Y} \mathsf{X}| \to |\mathsf{X}|$. Since f is quasi-compact, $\mathsf{U} \times_\mathsf{Y} \mathsf{X}$ is quasi-compact. It follows from (1) that the topological space $|\mathsf{U} \times_\mathsf{Y} \mathsf{X}|$ is quasi-compact, from which it follows that the image of θ is quasi-compact. □

Proposition 3.6.3.6. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a faithfully flat, quasi-compact morphism between quasi-separated spectral algebraic spaces. Then the induced map $|\mathsf{X}| \to |\mathsf{Y}|$ is a quotient map of topological spaces.*

Proof. Writing Y as a union of its quasi-compact open substacks, we can reduce to the case where Y (and therefore also X) is quasi-compact. Choose an étale surjection $\operatorname{Spét} R \to \mathsf{Y}$ and an étale surjection $\operatorname{Spét} R' \to \operatorname{Spét} R \times_\mathsf{Y} \mathsf{X}$. Then R' is faithfully flat over R, so that $|\operatorname{Spec} R'| \to |\operatorname{Spec} R|$ is a quotient map (Proposition 1.6.2.2). It will therefore suffice to show that the vertical maps appearing in the diagram

$$\begin{array}{ccc} |\operatorname{Spec} R'| & \longrightarrow & |\operatorname{Spec} R| \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ |\mathsf{X}| & \longrightarrow & |\mathsf{Y}| \end{array}$$

are quotient maps. We will prove that ϕ is a quotient map; the proof for ϕ' is similar. Fix a subset $U \subseteq \mathsf{Y}$, and suppose that $\phi^{-1}U$ is an open subset of $|\operatorname{Spec} R|$. Then the inverse images of $\phi^{-1}U$ under the two projection maps $|\operatorname{Sp\acute{e}t} R \times_{\mathsf{Y}} \operatorname{Sp\acute{e}t} R| \to |\operatorname{Sp\acute{e}t} R|$ coincide, so that $\phi^{-1}U = \phi^{-1}V$ for some open set $V \subseteq |\mathsf{Y}|$. Since ϕ is surjective, we obtain

$$U = \phi(\phi^{-1}U) = \phi(\phi^{-1}V) = V,$$

so that U is open. □

3.6.4 Comparison of $\mathrm{Pt}(\mathsf{X})$ with $\mathrm{Pt}_{\mathrm{g}}(\mathsf{X})$

We close this section with a discussion of the relationship between points of a spectral algebraic space X and geometric points of X.

Notation 3.6.4.1. Let X be a quasi-separated spectral algebraic space. Recall that $\mathrm{Pt}_{\mathrm{g}}(\mathsf{X})$ denotes the full subcategory of $\mathrm{SpDM}_{/\mathsf{X}}$ spanned by the minimal geometric points of X (see Definition 3.5.3.1). We let $\mathrm{Pt}_{\mathrm{g}}{}'(\mathsf{X})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{SpDM}_{/\mathsf{X}})$ whose objects are equivalent to commutative diagrams

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} \overline{\kappa} & \longrightarrow & \operatorname{Sp\acute{e}t} \kappa \\ & {\scriptstyle \eta'} \searrow \quad \swarrow {\scriptstyle \eta} & \\ & \mathsf{X} & \end{array}$$

where η is a point of X and η' is a minimal geometric point of X.

Proposition 3.6.4.2. *Let X be a quasi-separated spectral algebraic space. Then the forgetful functor $\mathrm{Pt}_{\mathrm{g}}{}'(\mathsf{X}) \to \mathrm{Pt}_{\mathrm{g}}(\mathsf{X})$ is an equivalence of ∞-categories.*

More informally, Proposition 3.6.4.2 asserts that every geometric point of a quasi-separated spectral algebraic space X determines a point of X.

Proof. It is clear that the forgetful functor $\theta : \mathrm{Pt}_{\mathrm{g}}{}'(\mathsf{X}) \to \mathrm{Pt}_{\mathrm{g}}(\mathsf{X})$ is fully faithful. We must prove that θ is essentially surjective. Fix a geometric point $\eta : \operatorname{Sp\acute{e}t} \overline{\kappa} \to \mathsf{X}$. Replacing X by an open substack if necessary, we may suppose that X is quasi-compact. Using Theorem 3.4.2.1, we can choose a scallop decomposition

$$\emptyset = \mathsf{U}_0 \to \mathsf{U}_1 \to \cdots \to \mathsf{U}_n \simeq \mathsf{X}.$$

Let i be the smallest integer such that η factors through U_i. Let K be the reduced closed substack of U_i complementary to U_{i-1}. Since $\overline{\kappa}$ is a field and η does not factor through U_{i-1}, it must factor through K. Note that $\mathsf{K} \simeq \operatorname{Spét} R$ is affine. It follows that η factors as a composition

$$\operatorname{Spét} \overline{\kappa} \to \operatorname{Spét} \kappa \to \operatorname{Spét} R \to \mathsf{U}_i \to \mathsf{X}$$

where κ is the residue field of $\pi_0 R$ at some prime ideal $\mathfrak{p} \subseteq \pi_0 R$. We now observe that the map $\operatorname{Spét} \kappa \to \mathsf{X}$ is a point of X. □

Proposition 3.6.4.3. *Let X be a quasi-separated spectral algebraic space, let $\eta : \operatorname{Spét} \kappa \to \mathsf{X}$ be a point of X, and let $\overline{\kappa}$ be a field extension of κ. The following conditions are equivalent:*

(1) *The field $\overline{\kappa}$ is a separable closure of κ.*

(2) *The composite map $\eta' : \operatorname{Spét} \overline{\kappa} \to \operatorname{Spét} \kappa \to \mathsf{X}$ is a minimal geometric point of X.*

Proof. We may assume without loss of generality that $\overline{\kappa}$ is separably closed (since this follows from both (1) and (2)). Choose an étale map $u : \operatorname{Spét} R \to \mathsf{X}$ such that the fiber product $\operatorname{Spét} R \times_{\mathsf{X}} \operatorname{Spét} \kappa$ is nonempty. Using Remark 3.6.2.3, we deduce that there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Spét} \kappa' & \xrightarrow{u'} & \operatorname{Spét} R \\ \downarrow & & \downarrow \\ \operatorname{Spét} \kappa & \longrightarrow & \mathsf{X} \end{array}$$

where κ' is a finite separable extension of κ, and u' exhibits κ' as a residue field of the commutative ring $\pi_0 R$. Since $\overline{\kappa}$ is separably closed, we can choose a map of κ-algebras $\kappa' \to \overline{\kappa}$, so that η' factors as a composition

$$\operatorname{Spét} \overline{\kappa} \xrightarrow{v} \operatorname{Spét} R \xrightarrow{v'} \mathsf{X}.$$

Then η' is a geometric point of X if and only if v exhibits $\overline{\kappa}$ as a separable closure of the residue field of $\pi_0 R$: that is, if and only if $\overline{\kappa}$ is a separable closure of κ'. Since κ' is a separably algebraic extension of κ, this is equivalent to the requirement that $\overline{\kappa}$ be a separable closure of κ. □

Combining Proposition 3.6.4.3, Proposition 3.6.4.2, and the discussion of Notation 3.6.2.5, we deduce:

Corollary 3.6.4.4. *Let X be a quasi-separated spectral algebraic space. Then the ∞-category $\operatorname{Pt}_{\mathrm{g}}(\mathsf{X})$ is canonically equivalent to the groupoid whose objects are pairs $(\eta, \overline{\kappa})$, where $\eta \in \pi_0 \operatorname{Pt}(\mathsf{X})$ is a point of X and $\overline{\kappa}$ is a separable closure of the residue field $\kappa(\eta)$.*

Corollary 3.6.4.5. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a surjective map between quasi-separated spectral algebraic spaces. Then the induced map $|\,\mathsf{X}\,| \to |\,\mathsf{Y}\,|$ is surjective.*

3.7 The Nisnevich Topology of a Spectral Algebraic Space

Let X be a quasi-projective algebraic variety of dimension d over a field κ. Then, for any (discrete) quasi-coherent sheaf $\mathscr{F}$ on X, the cohomology groups $\mathrm{H}^n(X; \mathscr{F})$ vanish for $n > d$. This is a special case of a much more general assertion:

Theorem 3.7.0.1 (Grothendieck)**.** *Let X be a Noetherian topological space of Krull dimension $\leqslant d$. Then the cohomology groups $\mathrm{H}^n(X; \mathscr{F})$ vanish for every integer $n > d$ and every sheaf $\mathscr{F}$ of abelian groups on X.*

Unfortunately, Theorem 3.7.0.1 does not apply directly in the case where X is an algebraic space, rather than a scheme. Nevertheless, we do have the following result:

Theorem 3.7.0.2. *Let X be a quasi-compact locally Noetherian spectral algebraic space and let $\mathscr{F}$ be a connective quasi-coherent sheaf on X. If the underlying topological space $|\mathsf{X}|$ has Krull dimension $\leqslant d$, then the homotopy groups $\pi_{-n}\Gamma(\mathsf{X}; \mathscr{F})$ vanish for $n > d$.*

Our goal in this section is to give a proof of Theorem 3.7.0.2 by deducing it from a variant of Theorem 3.7.0.1. Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, so that we have a geometric morphism of ∞-topoi $\pi_* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(|\mathsf{X}|)$ which induces a pushforward functor on spectrum-valued sheaves $\pi_* : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(|\mathsf{X}|)$. Note that if $U \subseteq |\mathsf{X}|$ is an open set which determines an *affine* substack of X, then our assumption that $\mathscr{F}$ is connective and quasi-coherent guarantees that the spectrum $(\pi_* \mathscr{F})(U)$ is connective. Consequently, when X is schematic, the pushforward $\pi_* \mathscr{F}$ is connective when viewed as a sheaf of spectra on $|\mathsf{X}|$, so the desired result follows (modulo issues of convergence) from Theorem 3.7.0.1. If X is not schematic, then the pushforward $\pi_* \mathscr{F}$ is not necessarily connective and this argument does not apply. We will prove Theorem 3.7.0.2 in the general case by instead comparing $\mathcal{X}$ with the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})$ of *Nisnevich sheaves* on X. This ∞-topos shares the good features of the Zariski site in the schematic case:

- Every quasi-compact, quasi-separated spectral algebraic space X admits a finite Nisnevich covering by affine spectral algebraic spaces (Example 3.7.1.5), so that connective quasi-coherent sheaves on X remain quasi-coherent when viewed as a sheaves for the Nisnevich topology (Example **??**).

- If X is quasi-compact, locally Noetherian, and $|\mathsf{X}|$ has Krull dimension $\leqslant d$, then the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})$ has homotopy dimension $\leqslant d$ (Theorem 3.7.7.1).

3.7.1 Nisnevich Coverings

We begin by defining the Nisnevich site of a spectral algebraic space $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$. Note that there is no loss of generality in restricting to the case of an ordinary algebraic space: replacing X by its 0-truncation $(\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}})$ has no effect on the theory of Nisnevich sheaves.

Definition 3.7.1.1. Let X be a quasi-compact, quasi-separated spectral algebraic space, and suppose that we are given a collection of étale morphisms $\{u_\alpha : \mathsf{X}_\alpha \to \mathsf{X}\}$. We will say that the morphisms u_α generate a *Nisnevich covering* of X if there exists a sequence of open immersions

$$\varnothing \simeq \mathsf{U}_0 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n \simeq \mathsf{X}$$

with the following property:

(∗) For $1 \leqslant i \leqslant n$, let K_i denote the reduced closed substack of U_i complementary to U_{i-1}. Then the composite map $\mathsf{K}_i \to \mathsf{U}_i \to \mathsf{X}$ factors through some u_α.

Notation 3.7.1.2. Let X be a quasi-compact, quasi-separated spectral algebraic space. We let $\mathcal{T}_{\mathsf{X}}$ denote the full subcategory of $\mathrm{SpDM}_{/\mathsf{X}}$ spanned by those maps $u : \mathsf{U} \to \mathsf{X}$ where u is étale and U is a quasi-compact, quasi-separated spectral algebraic space.

Remark 3.7.1.3. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-compact, quasi-separated spectral algebraic space. Then the construction $U \mapsto (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ determines a fully faithful embedding $\mathcal{X} \to \mathrm{SpDM}_{/\mathsf{X}}$, whose essential image is spanned by the étale maps $\mathsf{U} \to \mathsf{X}$. Under this equivalence of ∞-categories, $\mathcal{T}_{\mathsf{X}}$ corresponds to the full subcategory of $\mathcal{X}$ spanned by those objects which are coherent and discrete. In particular, $\mathcal{T}_{\mathsf{X}}$ is essentially small.

Remark 3.7.1.4 (Functoriality)**.** Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism between quasi-compact, quasi-separated spectral algebraic spaces. If $\{\mathsf{Y}_\alpha \in \mathcal{T}_{\mathsf{Y}}\}$ is a Nisnevich covering of Y, then the collection of fiber products $\{\mathsf{Y}_\alpha \times_{\mathsf{Y}} \mathsf{X} \in \mathcal{T}_{\mathsf{X}}\}$ is a Nisnevich covering of X.

Example 3.7.1.5. Let X be a quasi-compact, quasi-separated spectral algebraic space. Choose a scallop decomposition $\varnothing \simeq \mathsf{U}_0 \hookrightarrow \mathsf{U}_1 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n \simeq \mathsf{X}$, so that each of the open immersions $\mathsf{U}_{i-1} \hookrightarrow \mathsf{U}_i$ fits into an excision square

$$\begin{array}{ccc} \mathsf{V}_i & \longrightarrow & \operatorname{Sp\acute{e}t} R_i \\ \downarrow & & \downarrow \\ \mathsf{U}_{i-1} & \longrightarrow & \mathsf{U}_i \end{array}$$

(such a decomposition of X can always be found, by virtue of Theorem 3.4.2.1). Then the collection of maps $\operatorname{Sp\acute{e}t} R_i \to \mathsf{X}$ generate a Nisnevich covering of X. In particular, X admits a Nisnevich covering by finitely many affine spectral algebraic spaces.

3.7.2 The Affine Case

Definition 3.7.1.1 is closely related to the theory of Nisnevich coverings of $\mathbb{E}_\infty$-rings which was introduced in §B.4:

Proposition 3.7.2.1. *Suppose we are given a collection of étale morphisms $\{u_\alpha : \mathsf{X}_\alpha \to \mathsf{X}\}$ between quasi-compact, quasi-separated spectral algebraic spaces. The following conditions are equivalent:*

(1) *The morphisms u_α generate a Nisnevich covering of X, in the sense of Definition 3.7.1.1.*

(2) *For every map $\mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, where A is a connective $\mathbb{E}_\infty$-ring, there exists a collection of étale maps $\{A \to A_\beta\}$ such that each of the composite maps $\mathrm{Sp\acute{e}t}\, A_\beta \to \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$ factors through some u_α, and the underlying ring homomorphisms $\{\pi_0 A \to \pi_0 A_\beta\}$ are a Nisnevich covering in the sense of Definition B.4.1.1.*

(3) *For every étale map $\mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, where A is a connective $\mathbb{E}_\infty$-ring, there exists a collection of étale maps $\{A \to A_\beta\}$ such that each of the composite maps $\mathrm{Sp\acute{e}t}\, A_\beta \to \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$ factors through some u_α, and the underlying ring homomorphisms $\{\pi_0 A \to \pi_0 A_\beta\}$ are a Nisnevich covering in the sense of Definition B.4.1.1.*

Proof. We first prove that (1) $\Rightarrow$ (2). Assume that the morphisms u_α generate a Nisnevich covering of X. Then we can choose a sequence of open immersions

$$\emptyset \simeq \mathsf{U}_0 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n \simeq \mathsf{X}$$

having the property that for $1 \leqslant i \leqslant n$, if we let $\mathcal{K}_i$ denote the reduced closed substack of U_i complementary to U_{i-1}, then each of the maps $\mathcal{K}_i \to \mathsf{X}$ factors through some u_{α_i}.

Fix a map $f : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, where A is connective $\mathbb{E}_\infty$-ring. For $1 \leqslant i \leqslant n$, the fiber product $\mathsf{U}_i \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, A$ can be identified with an open substack of $\mathrm{Sp\acute{e}t}\, A$, corresponding to a quasi-compact open set $U_i \subseteq |\operatorname{Spec} A|$. Choose a scallop decomposition

$$\emptyset \simeq \mathsf{V}_{i,0} \hookrightarrow \cdots \hookrightarrow \mathsf{V}_{i,m_i} \simeq \mathsf{X}_{\alpha_i} \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, A,$$

so that we have excision squares

$$\begin{array}{ccc} \mathsf{W}_{i,j} & \longrightarrow & \mathrm{Sp\acute{e}t}\, A_{i,j} \\ \downarrow & & \downarrow \\ \mathsf{V}_{i,j-1} & \longrightarrow & \mathsf{V}_{i,j}\,. \end{array}$$

We can identify each $\mathsf{V}_{i,j}$ with an open substack of $\mathrm{Sp\acute{e}t}\, A$, which corresponds to a quasi-compact open subset $V_{i,j} \subseteq |\operatorname{Spec} A|$. Write each union $V'_{i,j} = U_{i-1} \cup V_{i,j}$ as the complement of the vanishing locus of a finitely generated ideal $I_{i,j} \subseteq \pi_0 A$. Without loss of generality, we may assume that there are inclusions

$$(0) = I_{1,0} \subseteq I_{1,1} \subseteq \cdots \subseteq I_{1,m_1} = I_{2,0} \subseteq \cdots \subseteq I_{n,m_n} = \pi_0 A.$$

We may therefore choose a finite set of elements $a_1, \ldots, a_\ell \in \pi_0 A$, such that each of the ideals $I_{i,j}$ is generated by the elements $\{a_k\}_{1 \leqslant k \leqslant k_0}$ for some integer k_0 (which depends on i and j). For $1 \leqslant k \leqslant \ell$, set $B_k = (\pi_0 A)/(a_1, \ldots, a_{k-1})[a_k^{-1}]$. By construction, each of the maps $\operatorname{Spét} B_k \rightarrow \operatorname{Spét} A \xrightarrow{f} \mathsf{X}$ factors through $\operatorname{Spét} R_{i,j}$ for some pair of integers i and j. It follows that the the maps $\{\pi_0 A \rightarrow \pi_0 A_{i,j}\}$ form a Nisnevich covering in the sense of Definition B.4.1.1, so that (2) is satisfied.

The implication (2) $\Rightarrow$ (3) is obvious. We will complete the proof by showing that (3) $\Rightarrow$ (1). Using Theorem 3.4.2.1, we can choose a scallop decomposition

$$\varnothing \simeq \mathsf{U}_0 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n \simeq \mathsf{X},$$

so that each of the maps $\mathsf{U}_{i-1} \hookrightarrow \mathsf{U}_i$ fits into an excision square

$$\begin{array}{ccc} \mathsf{U}'_{i-1} & \longrightarrow & \operatorname{Spét} B_i \\ \downarrow & & \downarrow \\ \mathsf{U}_{i-1} & \longrightarrow & \mathsf{U}_i . \end{array}$$

If condition (3) is satisfied, then for each of the $\mathbb{E}_\infty$-rings B_i we can choose a finite collection of étale maps $B_i \rightarrow B_{i,j}$ for which the underlying ring homomorphisms $\{\pi_0 B_i \rightarrow \pi_0 B_{i,j}\}$ are a Nisnevich covering (in the sense of Definition B.4.1.1) and each of the composite maps

$$\operatorname{Spét} B_{i,j} \rightarrow \operatorname{Spét} B_i \rightarrow \mathsf{U}_i \rightarrow \mathsf{X}$$

factors through some u_α. We can therefore assume that for each $1 \leqslant i \leqslant n$, we can choose a sequence of elements $b_{i,1}, \ldots, b_{i,m_i} \in \pi_0 B_i$ which generate the unit ideal, such that each of the induced maps

$$\operatorname{Spét}(\pi_0 B_i)/(b_{i,1}, \ldots, b_{i,j-1})[b_{i,j}^{-1}] \rightarrow \operatorname{Spét} B_i \rightarrow \mathsf{U}_i \rightarrow \mathsf{X}$$

factors through some U_α. Passing to refinement of our original scallop decomposition, we may assume that for $1 \leqslant i \leqslant n$, the composite map $\mathsf{K}_i \rightarrow \mathsf{U}_i \rightarrow \mathsf{X}$ factors through some u_α, where K_i denotes the reduced closed substack of U_i complementary to U_{i-1}. This completes the proof of (1). □

3.7.3 The Noetherian Case

We will primarily be interested in studying Nisnevich coverings in the setting where X is locally Noetherian. In this case, Definition 3.7.1.1 can be simplified:

Proposition 3.7.3.1. *Let $\{u_\alpha : \mathsf{X}_\alpha \rightarrow \mathsf{X}\}$ be a collection of étale morphisms between quasi-compact, quasi-separated spectral algebraic spaces. If X is locally Noetherian, then the following conditions are equivalent:*

(1) *For each point* $x \in |\mathsf{X}|$*, there exists an index* α *and a point* $x_\alpha \in |\mathsf{X}_\alpha|$ *such that* $u_\alpha(x_\alpha) = x$*, and* u_α *induces an isomorphism of residue fields* $\kappa(x) \to \kappa(x_\alpha)$*.*

(2) *The collection of morphisms* u_α *determine a Nisnevich covering of* X*.*

Proof. Suppose that (2) is satisfied. Then the collection of morphisms $\operatorname{Sp\acute{e}t} \kappa(x) \times_{\mathsf{X}} \mathsf{X}_\alpha$ is a Nisnevich covering of $\operatorname{Sp\acute{e}t} \kappa(x)$ (Remark 3.7.1.4). It follows immediately from the definitions that one of the projection maps $\operatorname{Sp\acute{e}t} \kappa(x) \times_{\mathsf{X}} \mathsf{X}_\alpha \to \operatorname{Sp\acute{e}t} \kappa(x)$ admits a section, which determines a point $x_\alpha \in |\mathsf{X}_\alpha|$ having the desired properties. This proves that (2) $\Rightarrow$ (1) (note that this argument does not require the assumption that X is locally Noetherian).

We now prove that (1) $\Rightarrow$ (2). Assume that (1) is satisfied. We will prove that the collection of morphisms u_α form a Nisnevich covering of X by verifying the third condition of Proposition 3.7.2.1. Let $f : \operatorname{Sp\acute{e}t} A \to \mathsf{X}$ be an étale map. Since X is locally Noetherian, the $\mathbb{E}_\infty$-ring A is Noetherian. It will therefore suffice to show that for every point $y \in |\operatorname{Spec} A|$, there exists an étale A-algebra A' and a point $y' \in |\operatorname{Spec} A'|$ lying over y such that $\kappa(y') \simeq \kappa(y)$, and the induced map $\operatorname{Sp\acute{e}t} A' \to \operatorname{Sp\acute{e}t} A \to \mathsf{X}$ factors through some X_α (Proposition B.4.3.1). Let $x \in |\mathsf{X}|$ denote the image of y. Using assumption (1), we can factor the composite map $\operatorname{Sp\acute{e}t} \kappa(y) \to \operatorname{Sp\acute{e}t} \kappa(x) \to \mathsf{X}$ through u_α for some α, so that we obtain a map $\operatorname{Sp\acute{e}t} \kappa(y) \to \operatorname{Sp\acute{e}t} A \times_{\mathsf{X}} \mathsf{X}_\alpha$. The desired result now follows from Corollary 3.4.2.5. □

3.7.4 Nisnevich Sheaves

Using Definition 3.7.1.1, we can associate a Grothendieck site to every quasi-compact, quasi-separated spectral algebraic space.

Proposition 3.7.4.1. *Let* X *be a quasi-compact, quasi-separated spectral algebraic space. Then there is a Grothendieck topology on the* ∞*-category* $\mathcal{T}_{\mathsf{X}}$ *which can be characterized as follows: for each object* $\mathsf{X}' \in \mathcal{T}_{\mathsf{X}}$*, a sieve* $\mathcal{C}^{(0)} \subseteq \mathcal{T}_{\mathsf{X}'} \simeq (\mathcal{T}_{\mathsf{X}})_{/\mathsf{X}'}$ *is covering if and only if it is is a Nisnevich covering of* X'*, in the sense of Definition 3.7.1.1.*

We will refer to the Grothendieck topology of Proposition 3.7.4.1 as the *Nisnevich topology* on $\mathcal{T}_{\mathsf{X}}$.

Remark 3.7.4.2. Let X be a quasi-compact, quasi-separated spectral algebraic space. Then the Nisnevich topology on $\mathcal{T}_{\mathsf{X}}$ is finitary, in the sense of Definition A.3.1.1.

Proof of Proposition 3.7.4.1. It follows immediately from the definitions that each $\mathcal{T}_{\mathsf{X}'}$ is a covering sieve of itself, and Remark 3.7.1.4 implies that the collection of covering sieves is closed under pullbacks. To verify transitivity, it will suffice to establish the following:

(∗) Suppose we are given a Nisnevich covering $\{\mathsf{Y}_\alpha \to \mathsf{Y}\}_{\alpha \in S}$ and, for each index $\alpha \in S$, another Nisnevich covering $\{\mathsf{Y}_{\alpha,\beta} \to \mathsf{Y}_\alpha\}$. Then the collection of composite maps $\mathsf{Y}_{\alpha,\beta} \to \mathsf{Y}_\alpha \to \mathsf{Y}$ is a Nisnevich covering of Y.

We will prove (∗) by verifying that the collection of maps $\{\mathsf{Y}_{\alpha,\beta} \to \mathsf{Y}\}$ satisfies condition (3) of Proposition 3.7.2.1. Fix a map $f : \operatorname{Spét} A \to \mathsf{Y}$. Applying Proposition 3.7.2.1, we conclude that A admits a Nisnevich covering $\{A \to A_i\}$ such that each of the composite maps $\operatorname{Spét} A_i \to \operatorname{Spét} A \to \mathsf{Y}$ factors through some Y_{α_i}. Applying Proposition 3.7.2.1 again, we conclude that each A_i admits a Nisnevich covering $\{A_i \to A_{i,j}\}$ for which each of the composite maps $\operatorname{Spét} A_{i,j} \to \operatorname{Spét} A_i \to \mathsf{Y}_{\alpha_i}$ factors through some $\mathsf{Y}_{\alpha_i,\beta}$. We now conclude by observing that the collection of maps $\{A \to A_{i,j}\}$ is a Nisnevich covering (Proposition B.4.2.3). □

Definition 3.7.4.3. Let X be a quasi-compact, quasi-separated spectral algebraic space and let $\mathscr{F} : \mathcal{T}_{\mathsf{X}}^{\mathrm{op}} \to \mathcal{C}$ be a functor from $\mathcal{T}_{\mathsf{X}}^{\mathrm{op}}$ to another ∞-category $\mathcal{C}$. We will say that $\mathscr{F}$ is a *Nisnevich sheaf* if it is a sheaf with respect to the Nisnevich topology of Proposition 3.7.4.1. We let $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})$ denote the full subcategory of $\operatorname{Fun}(\mathcal{T}_{\mathsf{X}}^{\mathrm{op}}, \mathcal{S})$ spanned by the Nisnevich sheaves; we will refer to $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})$ as the *Nisnevich ∞-topos of* X.

Example 3.7.4.4. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a quasi-compact, quasi-separated spectral algebraic space and let us identify $\mathcal{T}_{\mathsf{X}}$ with the full subcategory of $\mathcal{X}$ spanned by the coherent 0-truncated objects (Remark 3.7.1.3). Then the ∞-topos $\mathcal{X}$ is 1-localic (Corollary 1.6.8.6) and the underlying topos $\mathcal{X}^{\heartsuit}$ is coherent. It follows that the restricted Yoneda embedding

$$(X \in \mathcal{X}) \mapsto (h_X \in \operatorname{Fun}(\mathcal{T}_{\mathsf{X}}^{\mathrm{op}}, \mathcal{S})) \qquad h_X(U) = \operatorname{Map}_{\mathcal{X}}(U, X)$$

is fully faithful. The essential image of the construction $X \mapsto h_X$ consists of those functors $\mathcal{T}_{\mathsf{X}}^{\mathrm{op}} \to \mathcal{S}$ which are sheaves with respect to the étale topology. Since every covering for the Nisnevich topology is also a covering for the étale topology, we obtain a fully faithful geometric morphism $\rho_* : \mathcal{X} \to \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})$.

In the special case where X is affine, Definition 3.7.4.3 recovers the theory studied in §B.4:

Proposition 3.7.4.5. *Let A be a connective $\mathbb{E}_\infty$-ring, let $\mathsf{X} = \operatorname{Spét} A$, and let $\mathscr{F} : \mathcal{T}_{\mathsf{X}}^{\mathrm{op}} \to \mathcal{S}$ be a functor. Then $\mathscr{F}$ is a sheaf with respect to the Nisnevich topology of Proposition 3.7.4.1 if and only if it satisfies the following conditions:*

(1) *Let $\theta : \mathrm{CAlg}_A^{\text{ét}} \to \mathcal{T}_{\mathsf{X}}^{\mathrm{op}}$ denote the functor given by $B \mapsto \operatorname{Spét} B$. Then $\mathscr{F} \circ \theta \in \operatorname{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathcal{S})$ is a sheaf with respect to the Nisnevich topology of Definition B.4.2.2.*

(2) *The functor $\mathscr{F}$ is a right Kan extension of its restriction to the essential image of θ.*

Corollary 3.7.4.6. *Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathsf{X} = \operatorname{Sp\acute{e}t} A$. Then we have a canonical equivalence of ∞-categories $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X}) \simeq \mathcal{S}\mathrm{hv}_A^{\mathrm{Nis}}$.*

Proof. Combine Propositions 3.7.4.5 and HTT.4.3.2.15 . □

Proof of Proposition 3.7.4.5. Suppose first that $\mathscr{F}$ satisfies conditions (1) and (2); we will show that $\mathscr{F}$ is a sheaf with respect to the Nisnevich topology. Let $\mathsf{Y} \in \mathcal{T}_{\mathsf{X}}$ and let $\mathcal{C} \subseteq \mathcal{T}_{\mathsf{Y}}$ be a covering sieve for the Nisnevich topology; we must show that the canonical map $\mathscr{F}(\mathsf{Y}) \to \varprojlim_{\mathsf{Z} \in \mathcal{C}} \mathscr{F}(\mathsf{Z})$ is an equivalence. Let $\mathcal{D}$ be the the full subcategory of $\mathcal{T}_{\mathsf{Y}}$ spanned by those maps $\mathsf{Z} \to \mathsf{Y}$ where Z is affine. Using condition (2), we deduce that $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a right Kan extension of $\mathscr{F}|_{(\mathcal{C} \cap \mathcal{D})^{\mathrm{op}}}$. It will therefore suffice to show that $\mathscr{F}(\mathsf{Y})$ is a limit of the diagram $\mathscr{F}|_{(\mathcal{C} \cap \mathcal{D})^{\mathrm{op}}}$ (Lemma HTT.4.3.2.7). Using condition (1) and Proposition 3.7.2.1, we see that $\mathscr{F}|_{\mathcal{D}^{\mathrm{op}}}$ is a right Kan extension of $\mathscr{F}|_{(\mathcal{C} \cap \mathcal{D})^{\mathrm{op}}}$. Using Lemma HTT.4.3.2.7 again, we are reduced to proving that the map $\mathscr{F}(\mathsf{Y}) \to \varprojlim \mathscr{F}|_{\mathcal{D}^{\mathrm{op}}}$ is an equivalence, which follows from (2).

Now suppose that $\mathscr{F}$ is a sheaf with respect to the Nisnevich topology. We first claim that $\mathscr{F}$ satisfies (1). Let B be an étale A-algebra, and let $\mathcal{C} \subseteq (\mathrm{CAlg}_B^{\acute{e}t})^{\mathrm{op}}$ be a covering sieve with respect to the Nisnevich topology; we wish to show that the canonical map

$$\mathscr{F}(\operatorname{Sp\acute{e}t} B) \to \varprojlim_{C \in \mathcal{C}} \mathscr{F}(\operatorname{Sp\acute{e}t} C)$$

is a homotopy equivalence. To this end, we let $\mathcal{D} \subseteq \mathcal{T}_{\operatorname{Sp\acute{e}t} B}$ denote the sieve generated by the objects $\{\operatorname{Sp\acute{e}t} C\}_{C \in \mathcal{C}}$. Then the construction $C \mapsto \operatorname{Sp\acute{e}t} C$ determines a left cofinal functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, so it will suffice to show that $\mathscr{F}(\operatorname{Sp\acute{e}t} B)$ is a limit of the diagram $\mathscr{F}|_{\mathcal{D}^{\mathrm{op}}}$. This follows from our assumption that $\mathscr{F}$ is a Nisnevich sheaf, since $\mathcal{D}$ is a covering sieve for the Nisnevich topology (by virtue of Proposition 3.7.2.1).

We now prove that $\mathscr{F}$ satisfies (2). Let $\mathscr{F}' : \mathcal{T}_{\mathsf{X}}^{\mathrm{op}} \to \mathcal{S}$ be a right Kan extension of $\mathscr{F} \circ \theta$ along the functor θ, so that we have a canonical map $\alpha : \mathscr{F} \to \mathscr{F}'$ which is an equivalence on affine objects of $\mathcal{T}_{\mathsf{X}}$. Since $\mathscr{F}$ satisfies condition (1), the functor $\mathscr{F}'$ satisfies conditions (1) and (2), and is therefore a Nisnevich sheaf by the first part of the proof. Invoking Propositions 3.7.5.3 and 3.7.5.2, we conclude that α is an equivalence, so that $\mathscr{F}$ is a right Kan extension of $\mathscr{F} \circ \theta$ along θ. □

3.7.5 Nisnevich Excision

We next prove a global analogue of Theorem B.5.0.3.

Theorem 3.7.5.1. *[Morel-Voevodsky] Let X be a quasi-compact, quasi-separated spectral algebraic space and let $\mathscr{F} : \mathcal{T}_{\mathsf{X}}^{\mathrm{op}} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

(1) *The functor $\mathscr{F}$ is a sheaf with respect to the Nisnevich topology of Proposition 3.7.4.1.*

(2) *The space* $\mathscr{F}(\emptyset)$ *is contractible and for every excision square*

$$\begin{array}{ccc} \mathsf{U}' & \longrightarrow & \mathsf{Y}' \\ \downarrow & & \downarrow \\ \mathsf{U} & \longrightarrow & \mathsf{Y} \end{array}$$

in $\mathcal{T}_{\mathsf{X}}$, *the diagram of spaces*

$$\begin{array}{ccc} \mathscr{F}(\mathsf{U}') & \longleftarrow & \mathscr{F}(\mathsf{Y}') \\ \uparrow & & \uparrow \\ \mathscr{F}(\mathsf{U}) & \longleftarrow & \mathscr{F}(\mathsf{Y}) \end{array}$$

is a pullback square.

The proof of Theorem 3.7.5.1 will require some preliminaries. We first show that (1) ⇒ (2):

Proposition 3.7.5.2. *Let* X *be a quasi-compact, quasi-separated spectral algebraic space, and let* $\mathscr{F} : \mathcal{T}_{\mathsf{X}}^{\mathrm{op}} \to \mathcal{S}$ *be a sheaf for the Nisnevich topology. Then for every excision square*

$$\begin{array}{ccc} \mathsf{U}' & \longrightarrow & \mathsf{Y}' \\ \downarrow & & \downarrow \\ \mathsf{U} & \longrightarrow & \mathsf{Y} \end{array}$$

in $\mathcal{T}_{\mathsf{X}}$, *the diagram of spaces*

$$\begin{array}{ccc} \mathscr{F}(\mathsf{U}') & \longleftarrow & \mathscr{F}(\mathsf{Y}') \\ \uparrow & & \uparrow \\ \mathscr{F}(\mathsf{U}) & \longleftarrow & \mathscr{F}(\mathsf{Y}) \end{array}$$

is a pullback square.

Proof. For each object $\mathsf{Y} \in \mathcal{T}_{\mathsf{X}}$, let $h_{\mathsf{Y}} : \mathcal{T}_{\mathsf{X}}^{\mathrm{op}} \to \mathcal{S}$ denote the functor represented by Y. To prove Proposition 3.7.5.2, it will suffice to show that the construction $\mathsf{Y} \mapsto h_{\mathsf{Y}}$ carries excision square in $\mathcal{T}_{\mathsf{X}}$ to pushout diagrams in $\mathcal{S}\mathrm{hv}(\mathcal{T}_{\mathsf{X}})$. This follows immediately from Proposition 2.5.2.1. □

Proposition 3.7.5.3. *Let* X *be a quasi-compact, quasi-separated spectral algebraic space, and let* $\alpha : \mathscr{F} \to \mathscr{F}'$ *be a natural transformation between functors* $\mathscr{F}, \mathscr{F}' : \mathcal{T}_{\mathsf{X}}^{\mathrm{op}} \to \mathcal{S}$. *Assume that* $\mathscr{F}$ *and* $\mathscr{F}'$ *satisfy condition* (2) *of Theorem 3.7.5.1. If* α *induces a homotopy equivalence* $\mathscr{F}(\mathsf{Y}) \to \mathscr{F}'(\mathsf{Y})$ *whenever* $\mathsf{Y} \in \mathcal{T}_{\mathsf{X}}$ *is affine, then* α *is an equivalence.*

Proof. Let Y be an arbitrary object of $\mathcal{T}_{\mathsf{X}}$; we wish to show that α induces a homotopy equivalence $\alpha_{\mathsf{Y}} : \mathscr{F}(\mathsf{Y}) \to \mathscr{F}'(\mathsf{Y})$. We first treat the case where Y is quasi-affine. In this case, Y admits an open covering by finitely many affine open substacks $\{\mathsf{U}_i\}_{1 \leqslant i \leqslant n}$. We proceed by induction on n, the case $n \leqslant 1$ being trivial. To carry out the inductive step, we let Y' denote the open substack of Y given by the union of the open substacks $\{\mathsf{U}_i\}_{1 \leqslant i < n}$, so that we have an excision square

$$\begin{array}{ccc} \mathsf{U}_n \times_{\mathsf{Y}} \mathsf{Y}' & \longrightarrow & \mathsf{U}_n \\ \downarrow & & \downarrow \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}. \end{array}$$

It follows from the inductive hypothesis that $\alpha_{\mathsf{Y}'}$, α_{U_n}, and $\alpha_{\mathsf{U}_n \times_{\mathsf{Y}} \mathsf{Y}'}$ are homotopy equivalences. Since $\mathscr{F}$ and $\mathscr{F}'$ satisfy condition (2) of Theorem 3.7.5.1, we conclude that α_{Y} is a homotopy equivalence.

We now treat the general case. Using Theorem 3.4.2.1, we can choose a scallop decomposition $\varnothing = \mathsf{V}_0 \hookrightarrow \cdots \hookrightarrow \mathsf{V}_m \simeq \mathsf{Y}$ where each V_i fits into an excision square

$$\begin{array}{ccc} \mathsf{W}_i & \longrightarrow & \operatorname{Sp\acute{e}t} R_i \\ \downarrow & & \downarrow \\ \mathsf{V}_{i-1} & \longrightarrow & \mathsf{V}_i. \end{array}$$

The argument given above shows that the maps α_{W_i} and $\alpha_{\operatorname{Sp\acute{e}t} R_i}$ are homotopy equivalences. Using our assumption on $\mathscr{F}$ and $\mathscr{F}'$ together with induction on i, we conclude that each of the maps α_{V_i} is a homotopy equivalence. Taking $i = m$, we deduce that α_{Y} is a homotopy equivalence, as desired. □

Proof of Theorem 3.7.5.1. The implication (1) ⇒ (2) follows immediately from Proposition 3.7.5.2. Conversely, suppose that $\mathscr{F}$ satisfies (2). We begin by treating the case where $\mathsf{X} = \operatorname{Sp\acute{e}t} A$ is affine. Let $\mathscr{F}_0 : \mathrm{CAlg}_A^{\acute{e}t} \to \mathcal{S}$ be the functor given by $\mathscr{F}_0(B) = \mathscr{F}(\operatorname{Sp\acute{e}t} B)$. If $\mathscr{F}$ satisfies condition (2), then $\mathscr{F}_0$ satisfies Nisnevich excision (Definition B.5.0.1) and is therefore a sheaf with respect to the Nisnevich topology on $\mathrm{CAlg}_A^{\acute{e}t}$ (Theorem B.5.0.3). Let $\mathscr{F}' : \mathcal{T}_{\mathsf{X}}^{op} \to \mathcal{S}$ be a right Kan extension of $\mathscr{F}_0$ along the fully faithful embedding $\operatorname{Sp\acute{e}t} : \mathrm{CAlg}_A^{\acute{e}t} \hookrightarrow \mathcal{T}_{\mathsf{X}}^{op}$. Then $\mathscr{F}'$ is a Nisnevich sheaf (Proposition 3.7.4.5) and the canonical map $\alpha : \mathscr{F} \to \mathscr{F}'$ is an equivalence on all affine objects $\mathsf{U} \in \mathcal{T}_{\mathsf{X}}$. It follows from Proposition 3.7.5.3 that α is an equivalence, so that $\mathscr{F}$ is also a Nisnevich sheaf.

We now treat the general case. Let $\alpha : \mathscr{F} \to \mathscr{F}'$ be a natural transformation of functors which exhibits $\mathscr{F}'$ as a sheafification of $\mathscr{F}$ with respect to the Nisnevich topology. The first part of the proof shows that for every affine object $\mathsf{Y} \in \mathcal{T}_{\mathsf{X}}$, the restriction of $\mathscr{F}$ to $\mathcal{T}_{\mathsf{Y}}^{op}$ is already a Nisnevich sheaf. It follows that α induces an equivalence $\mathscr{F}(\mathsf{Y}) \to \mathscr{F}'(\mathsf{Y})$. Invoking Proposition 3.7.5.3, we conclude that α is an equivalence, so that $\mathscr{F}$ is a Nisnevich sheaf. □

3.7.6 Height and Krull Dimension

We now study the dimension theory of spectral algebraic spaces.

Definition 3.7.6.1. Let X be a quasi-separated spectral algebraic space. We will say that a point $x \in |\mathsf{X}|$ has *height* $\leqslant n$ if the following condition is satisfied:

($*$) For every sequence of points $x = x_0, x_1, \ldots, x_m \in |\mathsf{X}|$ having the property that $x_i \in \overline{\{x_{i+1}\}} - \{x_{i+1}\}$ for $0 \leqslant i < m$, we have $m \leqslant n$.

We say that x is of *finite height* if it has height $\leqslant n$ for some integer $n \geqslant 0$. In this case, we define the *height* of x to be the smallest nonnegative integer $n \geqslant 0$ such that x has height $\leqslant n$. We let $\mathrm{ht}(x)$ denote the height of x.

Proposition 3.7.6.2. *Let X be a quasi-separated spectral algebraic space, let A be a Noetherian $\mathbb{E}_\infty$-ring, and let $f : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$ be an étale map. Let $\mathfrak{p} \subseteq \pi_0 A$ be a prime ideal, and let x denote the image of $\mathfrak{p}$ under the continuous map $|\operatorname{Spec} A| \to |\mathsf{X}|$ determined by f. Then the height of x is equal to the Krull dimension of the local commutative ring $\pi_0 A_{\mathfrak{p}}$.*

Corollary 3.7.6.3. *Let X be a locally Noetherian spectral algebraic space. Then every point $x \in |\mathsf{X}|$ has finite height.*

Corollary 3.7.6.4. *Let $f : \mathsf{X} \to \mathsf{Y}$ be an étale morphism between locally Noetherian spectral algebraic spaces. Then for every point $x \in |\mathsf{X}|$, we have $\mathrm{ht}(x) = \mathrm{ht}(f(x))$.*

Proof. Combine Proposition 3.7.6.2 with Variant B.2.2.4. □

Proof of Proposition 3.7.6.2. Replacing X by an open substack if necessary, we may assume without loss of generality that X is quasi-compact. Let n denote the Krull dimension of the local ring $\pi_0 A_{\mathfrak{p}}$. Suppose we are given a chain of distinct points $x = x_0, x_1, \ldots, x_m \in |\mathsf{X}|$, where each x_i is contained in the closure of x_{i+1}. Applying Corollary 4.3.4.4 repeatedly, we can choose a descending chain of ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_m$$

in $\pi_0 A$, so that $m \leqslant n$. It follows that x has height $\leqslant n$. To prove that equality holds, we note that $\pi_0 A$ contains a descending chain of ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n,$$

whose image in $|\mathsf{X}|$ is a chain of points $x = x_0, x_1, \ldots, x_n$ for which each x_i is contained in the closure of x_{i+1}. To complete the proof, it suffices to show that the points x_i are distinct from one another. Suppose that $x_i = x_j$. Then the pair $(\mathfrak{p}_i, \mathfrak{p}_j)$ can be lifted to a point of $|\mathrm{Sp\acute{e}t}\, A \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, A|$ (Corollary 3.6.3.2). We can therefore choose an étale map

$u : \mathrm{Spét}\, B \to \mathrm{Spét}\, A \times_{\mathsf{X}} \mathrm{Spét}\, A$ and a prime ideal $\mathfrak{q} \subseteq \pi_0 B$ lying over $(\mathfrak{p}_i, \mathfrak{p}_j)$. Since the map $f : \mathrm{Spét}\, A \to \mathsf{X}$ is étale, the map u determines a pair of étale morphisms $A \to B$. Using Variant B.2.2.4 we obtain

$$n - i = \dim \pi_0 A_{\mathfrak{p}_i} = \dim \pi_0 B_{\mathfrak{q}} = \dim \pi_0 A_{\mathfrak{p}_j} = n - j,$$

so that $i = j$ as desired. □

Definition 3.7.6.5. Let X be a locally Noetherian spectral algebraic space. We will say that X has *Krull dimension* $\leqslant n$ if every point $x \in |\mathsf{X}|$ has height $\leqslant n$.

Remark 3.7.6.6. Let X be a spectral algebraic space which is quasi-compact and locally Noetherian. Then $|\mathsf{X}|$ is a Noetherian topological space (Remark 3.6.3.5). The spectral algebraic space X has Krull dimension $\leqslant n$ if and only if every chain of irreducible closed subsets

$$K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_m \subseteq |\mathsf{X}|$$

has length $m \leqslant n$. In other words, the Krull dimension of X is the same as the Krull dimension of the topological space $|\mathsf{X}|$; see §HTT.7.2.4 .

Remark 3.7.6.7. Let X be a locally Noetherian spectral algebraic space. The condition that X be of Krull dimension $\leqslant n$ is local with respect to the étale topology (Corollary 3.7.6.4).

3.7.7 A Vanishing Theorem for Nisnevich Sheaves

Let X be a spectral algebraic space which is quasi-compact, locally Noetherian, and of Krull dimension $\leqslant n$. Combining Remark 3.7.6.6 with Corollary HTT.7.2.4.17 , we see that the ∞-topos $\mathcal{S}\mathrm{hv}(|\mathsf{X}|)$ has homotopy dimensional $\leqslant n$. At the end of this section, we will prove the following analogue for Nisnevich sheaves:

Theorem 3.7.7.1. *Let X be a spectral algebraic space which is quasi-compact, locally Noetherian, and of Krull dimension $\leqslant n$. Then the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})$ has homotopy dimension $\leqslant n$.*

In the situation of Theorem 3.7.7.1, the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})$ is generated under colimits by objects U for which $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})_{/U} \simeq \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{U})$, where U is étale over X and therefore also of Krull dimension $\leqslant n$. We therefore obtain the following variant:

Corollary 3.7.7.2. *Let X be a spectral algebraic space which is quasi-compact, locally Noetherian, and of Krull dimension $\leqslant n$. Then the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})$ is locally of homotopy dimension $\leqslant n$.*

Corollary 3.7.7.3. *Let* X *be a spectral algebraic space which is quasi-compact, locally Noetherian, and of finite Krull dimension. Then the* ∞*-topos* $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})$ *is Postnikov complete (Definition A.7.2.1). In particular, the* ∞*-topos* $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})$ *is hypercomplete.*

Proof. Combine Theorem 3.7.7.1, Proposition HTT.7.2.1.10 , and Proposition **??**. □

Proof of Theorem 3.7.0.2. Let X be a quasi-compact locally Noetherian spectral algebraic space of Krull dimension $\leqslant d$ and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$. We wish to show that the homotopy groups $\pi_{-n}(\mathsf{X}; \mathscr{F})$ vanish for $n > d$. The quasi-coherent sheaf $\mathscr{F}$ determines a Nisnevich sheaf

$$\mathscr{G} : \mathcal{T}_{\mathsf{X}}^{\mathrm{op}} \to \mathcal{S} \qquad \mathscr{G}(\mathsf{Y}) = \Omega^{\infty-n}\Gamma(\mathsf{Y}; \mathscr{F}|_{\mathsf{Y}}).$$

Since $\mathscr{F}$ is connective and quasi-coherent, the space $\mathscr{G}(\mathsf{Y})$ is n-connective whenever Y is affine. Since every object of $\mathcal{T}_{\mathsf{X}}$ admits a Nisnevich covering by affine objects of $\mathcal{T}_{\mathsf{X}}$ (Example 3.7.1.5), the sheaf $\mathscr{G}$ is n-connective when viewed as an object of the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})$. Theorem 3.7.7.1 asserts that $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathsf{X})$ has homotopy dimension $\leqslant d$, so that the space $\mathscr{G}(\mathsf{X})$ is $(n-d)$-connective. In particular, we have $\pi_{-n}\Gamma(\mathsf{X}; \mathscr{F}) \simeq \pi_0 \mathscr{G}(\mathsf{X}) \simeq 0$. □

3.7.8 Proof of the Vanishing Theorem

We now give the proof of Theorem 3.7.7.1. It involves some variants of the ideas used in §B.5 which might be of independent interest (from the proof, one can extract stronger versions of Theorem 3.7.0.2).

Definition 3.7.8.1. Let X be a spectral algebraic space which is quasi-compact and locally Noetherian. Given a morphism $\alpha : \mathscr{F} \to \mathscr{F}'$ in $\mathcal{S}\mathrm{hv}(\mathcal{T}_{\mathsf{X}})$ and a point $u \in \mathscr{F}'(\mathsf{U})$, we let $\mathscr{F}_u \in \mathcal{S}\mathrm{hv}(\mathcal{T}_{\mathsf{U}})$ denote the sheaf given by $\mathscr{F}_u(\mathsf{V}) = \mathscr{F}(\mathsf{V}) \times_{\mathscr{F}'(\mathsf{V})} \{u\}$. We will say that α is *weakly* n*-connective* if the following condition is satisfied:

- (∗) For each point $u \in \mathscr{F}'(\mathsf{U})$, each $x \in |\,\mathsf{U}\,|$, and each map of spaces $S^k \to \mathscr{F}_u(\mathsf{U})$ where $-1 \leqslant k < n - \mathrm{ht}(x)$, there exists a map $g : \mathsf{U}' \to \mathsf{U}$, a point $x' \in |\,\mathsf{U}'\,|$ with $g(x') = x$ and $\kappa(x) \simeq \kappa(x')$ for which the composite map $S^k \to \mathscr{F}_u(\mathsf{U}) \to \mathscr{F}_u(\mathsf{U}')$ is nullhomotopic (when $k = -1$, this means that $\mathscr{F}_u(\mathsf{U}')$ is nonempty).

Remark 3.7.8.2. In the situation of Definition 3.7.8.1, the collection of weakly n-connective morphisms in $\mathcal{S}\mathrm{hv}(\mathcal{T}_{\mathsf{X}})$ is closed under pullbacks.

Lemma 3.7.8.3. *Let* X *be a spectral algebraic space which is quasi-compact and locally Noetherian, and let* $n \geqslant 0$*. Then a morphism* $\theta : \mathscr{F} \to \mathscr{F}'$ *in* $\mathcal{S}\mathrm{hv}(\mathcal{T}_{\mathsf{X}})$ *is weakly* n*-connective if and only if the following conditions are satisfied:*

(1) *For every point $u \in \mathscr{F}'(\mathsf{U})$, and every point $x \in |\,\mathsf{U}\,|$ of height $\leqslant n$, there exists a map there exists a map $g : \mathsf{U}' \to \mathsf{U}$, a point $x' \in |\,\mathsf{U}'\,|$ with $g(x') = x$ and $\kappa(x) \simeq \kappa(x')$ for whcih $\mathscr{F}_u(\mathsf{U}')$ is nonempty.*

(2) *If $n > 0$, then the diagonal map $\mathscr{F} \to \mathscr{F} \times_{\mathscr{F}'} \mathscr{F}$ is weakly $(n-1)$-connective.*

The proof is a simple matter of unravelling definitions (see the proof of Lemma B.5.2.5).

Remark 3.7.8.4. Let X be a quasi-compact, locally Noetherian spectral algebraic space. Then every n-connective morphism in $\mathcal{S}\mathrm{hv}(\mathcal{T}_{\mathsf{X}})$ is weakly n-connective. This follows immediately by induction on n, using the criteria of Lemma 3.7.8.3.

Lemma 3.7.8.5. *Let X be a spectral algebraic space which is quasi-compact and locally Noetherian. For each object $\mathsf{U} \in \mathcal{T}_{\mathsf{X}}$, let $h_{\mathsf{U}} \in \mathcal{S}\mathrm{hv}(\mathcal{T}_{\mathsf{X}}) \subseteq \mathrm{Fun}(\mathcal{T}_{\mathsf{X}}^{\mathrm{op}}, \mathcal{S})$ denote the functor represented by U. If $\theta : \mathscr{F} \to h_{\mathsf{X}}$ is a weakly n-connective morphism in $\mathcal{S}\mathrm{hv}(\mathcal{T}_{\mathsf{X}})$, then there exists a finite collection of points $\{x_1, \ldots, x_m\} \subseteq |\,\mathsf{X}\,|$ of height $> n$ and a commutative diagram σ :*

$$\begin{array}{ccc} & \mathscr{F} & \\ \nearrow & & \searrow^{\theta} \\ h_{\mathsf{U}} & \longrightarrow & h_{\mathsf{X}} \end{array}$$

where U denote the open substack of X corresponding to the open subset $U = |\,\mathsf{X}\,| - \bigcup_{1 \leqslant i \leqslant m} \overline{\{x_i\}}$.

Proof. We proceed as in the proof of Lemma 3.7.8.5, using induction on n. When $n = -1$, we take $x_1, \ldots, x_m$ to be the set of generic points of X, so that $U = \emptyset$ and the existence of the diagram σ is automatic. Assume now that $n \geqslant 0$ and that the result is known for the integer $n-1$, so that we can choose a commutative diagram

$$\begin{array}{ccc} & \mathscr{F} & \\ {}^{\phi}\nearrow & & \searrow^{\theta} \\ h_{\mathsf{V}} & \longrightarrow & h_{\mathsf{X}}. \end{array}$$

Here V denotes the open substack of X corresponding to an open set $V = |\,\mathsf{X}\,| - \bigcup_{1 \leqslant i \leqslant m} \overline{\{x_i\}}$ where the points x_i have height $\geqslant n$. Reordering the points x_i if necessary, we may assume that $x_1, x_2, \ldots, x_k$ have height n while $x_{k+1}, \ldots, x_m$ have height $> n$. We assume that this data has been chosen so that k is as small as possible. We will complete the induction by showing that $k = 0$. Otherwise, the point x_1 has height n. Since θ is weakly n-connective, there exists a map $f : \mathsf{X}' \to \mathsf{X}$ and a point $x' \in |\,\mathsf{X}'\,|$ such that $f(x') = x_1$, $\kappa(x) \simeq \kappa(x')$, and

$\mathscr{F}(|\mathsf{X}'|)$ is nonempty. We may therefore choose a commutative diagram

$$\begin{array}{ccc} & \mathscr{F} & \\ {\scriptstyle\psi}\nearrow & & \searrow{\scriptstyle\theta} \\ h_{\mathsf{X}'} & \xrightarrow{f} & h_{\mathsf{X}} \end{array}$$

Let V' denote the open substack of X corresponding to the open set $|\mathsf{X}| - \bigcup_{2 \leqslant i \leqslant m} \overline{\{x_i\}}$. Replacing X' by an open substack if necessary, we may suppose that there is an excision square

$$\begin{array}{ccc} \mathsf{V} \times_{\mathsf{X}} \mathsf{X}' & \longrightarrow & \mathsf{X}' \\ \downarrow & & \downarrow \\ \mathsf{V} & \longrightarrow & \mathsf{V}'. \end{array}$$

It follows from Lemma 3.7.8.3 that the canonical map

$$h_{\mathsf{V}} \times_{\mathscr{F}} h_{\mathsf{X}'} \to h_{\mathsf{V} \times_{\mathsf{X}} \mathsf{X}'}$$

is weakly $(n-1)$-connective. Applying the inductive hypothesis, we deduce that there is a finite collection of points $y_1, \ldots, y_{m'} \in |\mathsf{V} \times_{\mathsf{X}} \mathsf{X}'|$ of height $\geqslant n$ and a commutative diagram

$$\begin{array}{ccc} & h_{\mathsf{V}} \times_{\mathscr{F}} h_{\mathsf{X}'} & \\ \nearrow & & \searrow \\ h_{\mathsf{W}} & \longrightarrow & h_{\mathsf{V} \times_{\mathsf{X}} \mathsf{X}'}, \end{array}$$

where W is the open substack of $\mathsf{V} \times_{\mathsf{X}} \mathsf{X}'$ corresponding to the open subset $|\mathsf{V} \times_{\mathsf{X}} \mathsf{X}'| - \bigcup \overline{\{y_j\}}$. Replacing X' by the open substack complementary to the closures of the points y_j (which contains x', since x' is a point of height n and therefore cannot lie in the closure of any other point of height n), we may assume that $\mathsf{W} = \mathsf{V} \times_{\mathsf{X}} \mathsf{X}'$, so that the maps ϕ and ψ are homotopic when restricted to $\mathsf{V} \times_{\mathsf{X}} \mathsf{X}'$. It follows that ϕ and ψ can be amalgamated to a map $h_{\mathsf{V}'} \to \mathscr{F}$, contradicting the minimality of k. □

Proof of Theorem 3.7.7.1. Let $\mathscr{F}$ be an n-connective object of $\mathcal{S}\mathrm{hv}(\mathcal{T}_{\mathsf{X}})$; we wish to prove that the canonical map $\alpha : \mathscr{F} \to h_{\mathsf{X}}$ admits a section. Remark 3.7.8.4 shows that α is weakly n-connective. The desired result now follows from Lemma 3.7.8.5, combined with the observation that $|\mathsf{X}|$ does not contain any points of height $> n$ (by virtue of our assumption that X has Krull dimension $\leqslant n$). □

Part II

Proper Morphisms

Recall that a continuous map $f : X \to Y$ of Hausdorff topological spaces is said to be *proper* if it is closed (that is, for every closed subset $K \subseteq X$, the image $f(K) \subseteq Y$ is also closed) and, for every point $y \in Y$, the fiber $X_y = f^{-1}\{y\}$ is compact. In the setting of schemes, Grothendieck introduced the following analogous notion:

Definition 3.7.0.1. Let $f : X \to Y$ be a morphism of schemes. We say that f is *proper* if it satisfies the following conditions:

(1) The morphism f is *quasi-compact*: that is, for every quasi-compact open subset $V \subseteq Y$, the inverse image $f^{-1}V \subseteq X$ is also quasi-compact.

(2) The morphism f is *separated*: that is, the diagonal map $\delta : X \to X \times_Y X$ is a closed immersion.

(3) The morphism f is *locally of finite type*: that is, for every affine open subscheme $\operatorname{Spec} B \simeq U \subseteq X$ for which $f|_U$ factors through an affine open subscheme $\operatorname{Spec} A \simeq V \subseteq Y$, the commutative ring B is finitely generated as an A-algebra.

(4) The morphism f is *universally closed*: that is, for every pullback diagram of schemes

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y, \end{array}$$

the underlying map of topological spaces $X' \to Y'$ is closed.

Definition 3.7.0.1 plays a central role in the theory of schemes, thanks in part to the following foundational results of Grothendieck:

Theorem 3.7.0.2 (Direct Image Theorem)**.** *Let $f : X \to Y$ be a proper morphism between Noetherian schemes, and let $\mathscr{F}$ be a coherent sheaf on X. Then, for every integer $n \geqslant 0$, the higher direct image $R^n f_* \mathscr{F}$ is a coherent sheaf on Y.*

Theorem 3.7.0.3 (Grothendieck Existence Theorem)**.** *Let R be a Noetherian ring which is complete with respect to an ideal I, let $f : X \to \operatorname{Spec} R$ be a proper morphism of schemes, and for $m \geqslant 0$ set $X_m = \operatorname{Spec}(R/I^m) \times_{\operatorname{Spec} R} X$. Let $\operatorname{Coh}(X)$ and $\operatorname{Coh}(X_m)$ denote the abelian categories of coherent sheaves on X and X_m, respectively. Then the canonical map $\operatorname{Coh}(X) \to \varprojlim \{\operatorname{Coh}(X_m)\}$ is an equivalence of categories.*

Our primary goal in Part II is to adapt Definition 3.7.0.1 to the setting of spectral algebraic geometry and to prove analogues of Theorems 3.7.0.2 and 3.7.0.3 in the spectral setting. Some of the groundwork has already been laid in Part I, where we discussed the notions of *quasi-compact* and *separated* morphisms of spectral algebraic spaces (see Definitions 2.3.2.2 and 3.2.0.1). The next thing we need is an analogue of condition (3) of Definition 3.7.0.1. Here, we take our cue from classical algebraic geometry: we say that a morphism of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ is *locally of finite type* if the underlying map of ordinary algebraic spaces locally of finite type. However, this definition comes with an important caveat. In classical algebraic geometry, if $f : X \to Y$ is locally of finite type and Y is locally Noetherian, then X is also locally Noetherian. The analogous statement in spectral algebraic geometry is false: if $f : \mathsf{X} \to \mathsf{Y}$ is a morphism of spectral algebraic spaces, then the hypothesis that f is locally of finite type imposes a finiteness condition only on $\pi_0 \mathscr{O}_{\mathsf{X}}$, and not on the higher homotopy sheaves on $\mathscr{O}_{\mathsf{X}}$. To ensure that finiteness properties of the structure sheaf $\mathscr{O}_{\mathsf{Y}}$ are inherited by $\mathscr{O}_{\mathsf{X}}$, one needs to impose the stronger condition that f is *locally almost of finite presentation* (see Definition 4.2.0.1). There is a significant difference between demanding that f be locally of finite type and demanding that it be locally almost of finite presentation: in fact, there is an entire hierarchy of finiteness conditions intermediate between the two, which we will study in detail in Chapter 4.

In Chapter 5, we define the notion of proper morphism of spectral algebraic spaces by direct analogy with Definition 3.7.0.1: a morphism $f : \mathsf{X} \to \mathsf{Y}$ is said to be *proper* if it is quasi-compact, separated, locally of finite type, and universally closed (equivalently, f is proper if the underlying map of ordinary algebraic spaces is proper; see Definition 5.1.2.1). To provide examples, we give a construction of projective spaces over an arbitrary connective $\mathbb{E}_\infty$-ring R (Construction 5.4.1.3). We then prove an analogue of Chow's lemma, which shows that arbitrary proper morphism of spectral algebraic spaces can be approximated by projective morphisms (Theorem 5.5.0.1). Using this result, we prove a spectral analogue of Theorem 3.7.0.2: if $f : \mathsf{X} \to \mathsf{Y}$ is proper and locally almost of finite presentation, then the direct image functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ carries almost perfect objects of $\mathrm{QCoh}(\mathsf{X})$ to almost perfect objects of $\mathrm{QCoh}(\mathsf{Y})$ (Theorem 5.6.0.2).

In practice, the assertion that the direct image functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ preserves almost perfect objects is often not good enough: for many applications, one wants to know that f_* carries perfect objects of $\mathrm{QCoh}(\mathsf{X})$ to perfect objects of $\mathrm{QCoh}(\mathsf{Y})$. This is generally not true if we assume only that f is proper (even if X and Y are Noetherian). In Chapter 6, we introduce the notion of a *morphism of finite* Tor-*amplitude* (Definition 6.1.1.1) and show that the direct image functor f_* preserves perfect objects whenever f is proper, locally almost of finite presentation, and of finite Tor-amplitude (Theorem 6.1.3.2; see also Theorem 11.1.4.1 for a converse). Under the same assumptions, we develop a version of Grothendieck duality: there is a quasi-coherent sheaf $\omega_{\mathsf{X}/\mathsf{Y}} \in \mathrm{QCoh}(\mathsf{X})$, which we call the *dualizing sheaf of* f, which satisfies $(f_* \mathscr{F})^\vee \simeq f_*(\mathscr{F}^\vee \otimes \omega_{\mathsf{X}/\mathsf{Y}})$ for each $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}$.

The remainder Part **??** is devoted to establishing a spectral analogue of Theorem 3.7.0.3. Let X be a spectral algebraic space (or, more generally, a spectral Deligne-Mumford stack). In Chapter 8, we will associate to each (cocompact) closed subset $K \subseteq |\mathsf{X}|$ another algebro-geometric object $\mathsf{X}^\wedge_K$, which we call the *formal completion of* X *along* K (Definition **??**). To this object, we can associate an ∞-category of quasi-coherent sheaves $\mathrm{QCoh}(\mathsf{X}^\wedge_K)$ and a restriction functor $\rho : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X}^\wedge_K)$. Our main result asserts that if $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ is proper and locally almost of finite presentation, the closed set $K \subseteq |\mathsf{X}|$ is defined by the vanishing locus of a finitely generated ideal $I \subseteq \pi_0 R$, and the $\mathbb{E}_\infty$-ring R is I-complete (see Definition 7.3.0.5), then ρ restricts to an equivalence of ∞-categories $\mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}} \to \mathrm{QCoh}(\mathsf{X}^\wedge_K)^{\mathrm{aperf}}$, where $\mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$ denote the full subcategory of $\mathrm{QCoh}(\mathsf{X})$ spanned by the almost perfect objects and $\mathrm{QCoh}(\mathsf{X}^\wedge_K)^{\mathrm{aperf}}$ is defined similarly (Theorem 8.5.0.3). The proof will require a detailed study of completions in the setting of $\mathbb{E}_\infty$-rings and their modules, which we carry out in Chapter 7 (in somewhat greater generality).

Remark 3.7.0.4. If $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ is a proper morphism between locally Noetherian spectral algebraic spaces, then f is automatically locally almost of finite presentation (Remark 4.2.0.4). In this case, it is not difficult to deduce Theorems 5.6.0.2 and 8.5.0.3 from Theorems 3.7.0.2 and 3.7.0.3 (or, more precisely, from the analogous statements for algebraic spaces), applied to the underlying map of ordinary algebraic spaces $\tau_{\leqslant 0} \mathsf{X} \to \mathrm{Spec}(\pi_0 R)$. However, one pleasant feature of the spectral setting is that Theorems 5.6.0.2 and 8.5.0.3 do not require any Noetherian hypotheses, provided that F is locally almost of finite presentation. In the setting of classical algebraic geometry, the resulting generalization of Theorem 3.7.0.2 is due to Illusie (in the projective case) and Kiehl (in general); see [101] and [116].

Chapter 4

Morphisms of Finite Presentation

Let A be a commutative ring and let X be an A-scheme. Recall that X is said to be *locally of finite type* over A if, for every affine open subscheme $\operatorname{Spec} B \subseteq X$, the commutative ring B is finitely generated as an A-algebra. The scheme X is said to be *locally of finite presentation* over A if, for every open affine subscheme $\operatorname{Spec} B \subseteq X$, the commutative ring B is finitely presented as an A-algebra. Our goal in this section is to study some analogous finiteness conditions in the setting of spectral algebraic geometry.

We begin by considering finiteness conditions which can be imposed on a morphism $\phi : A \to B$ of connective $\mathbb{E}_\infty$-rings. Here the situation is more subtle than in classical commutative algebra. If A is a commutative ring, then one can specify a commutative A-algebra B by writing down generators and relations: that is, by writing B as the cofiber of a map of polynomial algebras $A[\vec{y}] \to A[\vec{x}]$. In the setting of $\mathbb{E}_\infty$-rings, the naive analogue of this statement is false: not every $\mathbb{E}_\infty$-algebra B over A can be obtained as the cofiber of a map of free A-algebras $A\{\vec{y}\} \to A\{\vec{x}\}$. In general, one needs to specify not only generators and relations, but relations among the relations, relations among those, and so forth. As a consequence, there is an entire hierarchy of natural finiteness conditions which can be thought of as intermediate between "finite type" and "finite presentation." We will study these conditions (and their interrelatinoships) in §4.2.

In §4.2, we apply the ideas of §4.1 to introduce several conditions on a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks (see Definition 4.2.0.1):

(a) The condition that f is locally of finite presentation.

(b) The condition that f is locally almost of finite presentation.

(c) The condition that f is locally of finite generation to order n, for some integer $n \geqslant 0$.

Condition (b) will be the most useful to us in this book: the role of "almost finite presentation" in spectral algebraic geometry can be regarded as analogous to the role of "finite presentation" in classical scheme theory (condition (a) is a bit too strong to play this role: for example, if $f : X \to Y$ is a morphism of finite presentation of Noetherian schemes, then f need not satisfy condition (a) when regarded as a morphism of spectral Deligne-Mumford stacks, but will always satisfy condition (b)). The role of condition (c) is more technical: it is equivalent to (b) if both X and Y are locally Noetherian, but otherwise provides a sequence of weaker finiteness conditions which "converge" to (b) (a morphism f is locally almost of finite presentation if and only if it is locally of finite generation to order n for *every* $n \geqslant 0$).

One of the useful features of finite presentation in classical algebraic geometry is that it enables arguments by *Noetherian approximation*. If A is a commutative ring and X is an A-scheme of finite presentation, then one can always find a finitely generated subring $A_0 \subseteq A$ and an isomorphism $X \simeq \operatorname{Spec} A \times_{\operatorname{Spec} A_0} X_0$, where X_0 is an A_0-scheme of finite presentation. This observation can be used to reduce many questions about arbitrary schemes to questions about Noetherian schemes, where they can be addressed using tools that are specific to the Noetherian case (such as dimension theory). In §4.4, we will adapt the technique of Noetherian approximation to the setting of spectral algebraic geometry. Here we encounter a new complication: the structure sheaf of a spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ has infinitely many homotopy sheaves $\{\pi_m \mathcal{O}_{\mathcal{X}}\}_{m \geqslant 0}$, and it is hard to approximate all of them simultaneously. We will therefore restrict our attention to Noetherian approximation in the setting of n-truncated spectral Deligne-Mumford stacks, where $n \geqslant 0$ is a fixed integer. For this purpose, the refined finiteness conditions studied in §4.2 will play an essential role.

Let $A_0 \subseteq A$ and $X \simeq \operatorname{Spec} A \times_{\operatorname{Spec} A_0} X_0$ be as above. To employ Noetherian approximation effectively, we often want to arrange that X_0 reflects whatever features of X we are interested in. This leads to questions of the following sort:

Question 4.0.0.1. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Under what conditions can we arrange that $\mathscr{F}$ is the pullback of a quasi-coherent sheaf $\mathscr{F}_0$ on X_0?

Question 4.0.0.2. Suppose that the scheme X has some property P. Under what conditions can we arrange that the scheme X_0 also has the property P?

We will discuss the analogues of Questions 4.0.0.1 and 4.0.0.2 in spectral algebraic geometry in §4.5 and §4.6, respectively. In the latter case, we will need some elementary facts about constructible subsets of (spectral) algebraic spaces, which we review in §4.3.

Contents

4.1 Finiteness Conditions on $\mathbb{E}_\infty$-Algebras

In the setting of classical commutative algebra, there are two natural finiteness conditions one can impose on a ring homomorphism $\phi : A \to B$:

(i) One can demand that ϕ is *of finite type*: that is, that it exhibits B as a finitely generated A-algebra (so that B is isomorphic, as an A-algebra, to a quotient $A[x_1, \ldots, x_n]/I$ for some $n \gg 0$ and some ideal $I \subseteq A[x_1, \ldots, x_n]$).

(ii) One can demand that ϕ is *of finite presentation*: that is, it exhibits B as a finitely presented A-algebra (so that B is isomorphic, as an A-algebra, to a quotient $A[x_1, \ldots, x_n]/I$ for some $n \gg 0$ and some finitely generated ideal $I \subseteq A[x_1, \ldots, x_n]$).

There are analogous demands that one can make on a morphism $\phi : A \to B$ of connective $\mathbb{E}_\infty$-rings:

(i') We will say that ϕ is *of finite type* if the underlying ring homomorphism $\pi_0 A \to \pi_0 B$ is of finite type. Equivalently, ϕ is of finite type if there exists a morphism of $A\{x_1, \ldots, x_n\} \to B$ of $\mathbb{E}_\infty$-algebras over A which is surjective on π_0.

(ii') We say that ϕ is *of finite presentation* if B can be built (as an object of the ∞-category CAlg_A) from the free algebra $A\{x\}$ using finite colimits (see Definition HA.7.2.4.26).

However, the passage from ordinary commutative algebra to the theory of (connective) $\mathbb{E}_\infty$-rings introduces a number of subtleties which will require us to study finiteness conditions that are intermediate between conditions (i') and (ii'):

- Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. The condition that ϕ is of finite presentation is not local (either on A or on B) with respect to the Zariski topology. It is often more convenient to require that $\phi : A \to B$ is *locally of finite presentation*: that is, that the functor $C \mapsto \mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$ commutes with filtered colimits (Definition HA.7.2.4.26). This condition is local with respect to the étale topology on A and B (see Remark 4.1.4.2), and is closely related to the condition that ϕ is of finite presentation (the morphism ϕ is locally of finite presentation if and only if it becomes of finite presentation after passing to some Zariski covering of B).

- If $\phi : A \to B$ is a homomorphism of commutative Noetherian rings, then conditions (i) and (ii) are equivalent (the Hilbert basis theorem guarantees that every ideal $I \subseteq A[x_1, \ldots, x_n]$). However, these conditions do not guarantee that ϕ is of finite presentation (or even locally of finite presentation) when regarded as a morphism of $\mathbb{E}_\infty$-rings. Nevertheless, they *do* guarantee that ϕ is *almost of finite presentation* (see Definition HA.7.2.4.26): that is, that the construction $C \mapsto \mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$ preserves filtered colimits when restricted to n-truncated connective A-algebras, for each $n \gg 0$.

- If $\phi : A \to B$ is a homomorphism of commutative rings, then we can think of B as built from A by generators and relations: condition (i) asserts that we need only finitely many generators, and condition (ii) asserts that we need finitely many generators *and* finitely many relations. In the setting of connective $\mathbb{E}_\infty$-rings, one should instead imagine that B is built from A by successively attaching n-cells for different values of $n \geqslant 0$: these can be thought as generators in the case $n = 0$ and relations in the case $n = 1$, but we generally need higher-dimensional cells as well. We can use the language of cell attachments to give informal descriptions of the finiteness conditions introduced above:

– The morphism ϕ is of finite type if B can be built using only finitely many 0-cells (but possibly infinitely many cells of each positive dimension).

– The morphism ϕ is almost of finite presentation if B can be built using only finitely many cells of each dimension.

– The morphism ϕ is of finite presentation if B can be built using only finitely many cells in total.

For many applications (for example, in our study of Noetherian approximation in §4.4) it will be useful to consider a hierarchy of finiteness conditions which interpolate between (i') and (ii'). For each $n \geqslant 0$, we will define a class of morphisms which we call *morphisms of finite generation to order* n (Definition 4.1.1.1). Roughly speaking, a morphism $\phi : A \to B$ is of finite generation to order n if the $\mathbb{E}_\infty$-ring B can be built from A using only finitely many cells of dimension $\leqslant n$ (but possibly infinitely many cells of higher dimension). When $n = 0$, this reduces to the condition that ϕ is of finite type. On the other hand, the morphism ϕ is almost of finite presentation if and only if it is of finite generation to order n for all $n \geqslant 0$ (Remark 4.1.1.5).

The relationship between the various finiteness conditions on a morphism $\phi : A \to B$ are summarized in the following diagram:

$$\begin{array}{ccc}
\phi \text{ of finite generation to order } 0 & \Longleftrightarrow & \pi_0\phi \text{ of finite type} \\
\Uparrow & & \Uparrow \\
\phi \text{ of finite generation to order } 1 & \Longrightarrow & \pi_0\phi \text{ of finite presentation} \\
\Uparrow & \overset{(*)}{\swarrow} & \\
\cdots & & \\
\Uparrow & & \\
\phi \text{ almost of finite presentation} & & \\
\Uparrow & & \\
\phi \text{ locally of finite presentation} & & \\
\Uparrow & & \\
\phi \text{ of finite presentation} & &
\end{array}$$

where the conditions on the right hand side are formulated in classical commutative algebra, and the implication $(*)$ holds under the assumption that both A and B are Noetherian.

4.1.1 Morphisms of Finite Generation

We now give a precise formulation of the finiteness conditions to be studied in this section.

Definition 4.1.1.1. Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings and let $n \geqslant 0$. We will say that ϕ is *of finite generation to order* n if the following condition is satisfied:

$(*)$ Let $\{C_\alpha\}$ be a filtered diagram of connective $\mathbb{E}_\infty$-rings over A having colimit C. Assume that each C_α is n-truncated and that each of the transition maps $\pi_n C_\alpha \to \pi_n C_\beta$ is a monomorphism. Then the canonical map

$$\varinjlim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$$

is a homotopy equivalence.

In this case, we will also say that B is *of finite generation to order n over A*.

We will say that ϕ is *of finite type* if it is of finite generation to order 0. In this case, we will also say that B *is of finite type over A*.

Remark 4.1.1.2. Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. It is possible formulate the condition that ϕ is of finite generation to order n using the language of cell attachments. However, it will be more convenient to adopt Definition 4.1.1.1, which does not require us to make auxiliary choices (such as presentation of B as an object of $\mathrm{CAlg}_A^{\mathrm{cn}}$).

In the case $n = 0$, Definition 4.1.1.1 can be made very concrete:

Proposition 4.1.1.3. *Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. The following conditions are equivalent:*

(a) *The morphism ϕ is of finite type (in the sense of Definition* **??***).*

(b) *The commutative ring $\pi_0 B$ is finitely generated as an algebra over $\pi_0 A$.*

Proof. Write $\pi_0 B$ as a (filtered) direct limit of subalgebras $\{C_\alpha\}$ which are finitely generated as algebras over $\pi_0 A$. If condition (a) is satisfied, then the canonical map $\varinjlim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, \pi_0 B)$ is a homotopy equivalence. In particular, the canonical map $B \to \pi_0 B$ factors through some C_α. It follows that $C_\alpha = \pi_0 B$, so that $\pi_0 B$ is finitely generated as an algebra over $\pi_0 A$.

Conversely, suppose that (b) is satisfied. Let $\{C_\alpha\}$ be *any* filtered diagram in CAlg_A, where each C_α is discrete and each of the transition maps $C_\alpha \to C_\beta$ is a monomorphism. Set $C = \varinjlim C_\alpha$. By abuse of notation, we can identify each C_α with its image in C. Since C is discrete, we have a commutative diagram

$$\begin{array}{ccc} \varinjlim \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{\pi_0 A}}(\pi_0 B, C_\alpha) & \xrightarrow{\overline{\rho}} & \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{\pi_0 A}}(\pi_0 B, C) \\ \downarrow & & \downarrow \\ \varinjlim \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) & \xrightarrow{\rho} & \mathrm{Map}_{\mathrm{CAlg}_A}(B, C) \end{array}$$

where the vertical maps are homotopy equivalences. Consequently, to show that ρ is a homotopy equivalence, it will suffice to show that $\overline{\rho}$ is bijective. In other words, we must show that every morphism $\phi : \pi_0 B \to C$ of $(\pi_0 A)$-algebras factors through some C_α. This is clear: condition (b) implies that there exists a finite collection of elements $b_1, \ldots, b_k \in \pi_0 B$ which generate $\pi_0 B$ as an algebra over $\pi_0 A$, so it suffices to consider any C_α which contains the elements $\{\phi(b_i)\}_{1 \leqslant i \leqslant k}$. □

Remark 4.1.1.4. Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings and let $n \geqslant 0$. Then:

(1) If $\tau_{\leqslant n} B$ is a compact object of $\tau_{\leqslant n} \mathrm{CAlg}_A$, then ϕ is of finite generation to order n.

(2) If ϕ is of finite generation to order $(n+1)$, then $\tau_{\leqslant n} B$ is a compact object of $\tau_{\leqslant n} \mathrm{CAlg}_A$.

In particular, if ϕ is of finite generation to order n for any $n > 0$, then $\pi_0 B$ is finitely presented as a commutative algebra over $\pi_0 A$ in the sense of classical commutative algebra.

Remark 4.1.1.5. Using Remark 4.1.1.4, we see that a morphism $\phi : A \to B$ of connective $\mathbb{E}_\infty$-rings is almost of finite presentation (see Definition HA.7.2.4.26) if and only if it is of finite generation to order n, for each $n \geqslant 0$.

Remark 4.1.1.6. Suppose we are given a finite collection $\{f_\alpha : A_\alpha \to B_\alpha\}$ of morphisms between connective $\mathbb{E}_\infty$-rings. Let $f : \prod_\alpha A_\alpha \to \prod_\alpha B_\alpha$ be the induced map. Then f is of finite generation to order n (almost of finite presentation, locally of finite presentation) if and only if each of the morphisms f_α is of finite generation to order n (almost of finite presentation, locally of finite presentation).

Remark 4.1.1.7. Suppose we are given a commutative diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccccc} & & B & & \\ & \overset{\phi}{\nearrow} & & \overset{\psi}{\searrow} & \\ A & & \xrightarrow{\phi'} & & B'. \end{array}$$

Let $n \geqslant 0$, and assume that the map ψ induces a surjection $\pi_n B \to \pi_n B'$ and a bijection $\pi_i B \to \pi_i B'$ for $i < n$. If ϕ is of finite generation to order n, then so is ϕ'. To prove this, consider any filtered diagram $\{C_\alpha\}$ of n-truncated objects of CAlg_A having colimit C, and suppose that each of the transition maps $\pi_n C_\alpha \to \pi_n C_\beta$ is a monomorphism. Then each of the maps $\pi_n C_\alpha \to \pi_n C$ is a monomorphism. It follows from our assumption on ψ that the diagram

$$\begin{array}{ccc} \varinjlim \mathrm{Map}_{\mathrm{CAlg}_A}(B', C_\alpha) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}_A}(B', C) \\ \downarrow & & \downarrow \\ \varinjlim \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}_A}(B, C) \end{array}$$

is a pullback square. If ϕ is of finite generation to order n, then the bottom horizontal map is a homotopy equivalence, so the upper horizontal map is a homotopy equivalence as well.

Remark 4.1.1.8. Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings which is of finite generation to order $n > 0$. Then the induced map $A \to \tau_{\leqslant n-1} B$ is also of finite generation to order n (this is a special case of Remark 4.1.1.7).

Remark 4.1.1.9. Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings which exhibits B as a compact object of $\tau_{\leqslant n-1}\operatorname{CAlg}_A^{\mathrm{cn}}$ (for some $n > 0$). Then B is a retract of $\tau_{\leqslant n-1} B'$ for some compact object $B' \in \operatorname{CAlg}_A^{\mathrm{cn}}$. Since the map $A \to B'$ is of finite generation to order n, so is the map $A \to B$ (by Remark 4.1.1.8). It follows that an object $B \in \tau_{\leqslant n-1}\operatorname{CAlg}_A^{\mathrm{cn}}$ is compact if and only if the map $A \to B$ is of finite generation to order n.

4.1.2 Differential Characterization

Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. According to Theorem HA.7.4.3.18 , the morphism ϕ is almost of finite presentation if and only if it exhibits $\pi_0 B$ as a finitely presented algebra over $\pi_0 A$ and the relative cotangent complex $L_{B/A}$ is almost perfect. We will need the following more refined statement:

Proposition 4.1.2.1. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings and let $n > 0$. Then f is of finite generation to order n if and only if the following conditions are satisfied:*

(1) *The commutative ring $\pi_0 B$ is finitely presented over $\pi_0 A$.*

(2) *The relative cotangent complex $L_{B/A}$ is perfect to order n as an B-module (see Definition 2.7.0.1).*

Proof. First suppose that f is of finite generation to order n. Since $n > 0$, condition (1) follows from Remark 4.1.1.4. To prove (2), we will show that the relative cotangent complex $L_{B/A}$ satisfies the third criterion of Proposition 2.7.0.4. Choose a filtered diagram $\{N_\alpha\}$ in $\operatorname{Mod}_B^\heartsuit$, where each of the transition maps $N_\alpha \to N_\beta$ is a monomorphism. We wish to show that the canonical map

$$\rho : \varinjlim_{\alpha} \operatorname{Map}_{\operatorname{Mod}_B}(L_{B/A}, \Sigma^n N_\alpha) \to \operatorname{Map}_{\operatorname{Mod}_B}(L_{B/A}, \Sigma^n \varinjlim N_\alpha)$$

is a homotopy equivalence. This follows from the observation that ρ fits into a pullback square

$$\begin{array}{ccc} \varinjlim \operatorname{Map}_{\operatorname{Mod}_B}(L_{B/A}, \Sigma^n N_\alpha) & \xrightarrow{\rho} & \operatorname{Map}_{\operatorname{Mod}_B}(L_{B/A}, \Sigma^n \varinjlim N_\alpha) \\ \downarrow & & \\ \varinjlim \operatorname{Map}_{\operatorname{CAlg}_A}(B, (\tau_{\leqslant n} B) \oplus \Sigma^n N_\alpha) & \longrightarrow & \operatorname{Map}_{\operatorname{CAlg}_A}(B, (\tau_{\leqslant n} B) \oplus \Sigma^n \varinjlim N_\alpha), \end{array}$$

where the lower horizontal map is a homotopy equivalence by virtue of our assumption that f is of finite generation to order n.

Now suppose that conditions (1) and (2) are satisfied and choose a filtered diagram $\{C_\alpha\}$ in $\mathrm{CAlg}_A^{\mathrm{cn}}$ having colimit C. Assume that each C_α is n-truncated and that the transition maps $\pi_n C_\alpha \to \pi_n C_\beta$ are monomorphisms. For $0 \leqslant i \leqslant n$, let θ_i denote the canonical map

$$\varinjlim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\leqslant i} C_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\leqslant i} C).$$

We will prove that θ_i is a homotopy equivalence for $0 \leqslant i \leqslant n$ using induction on i. When $i = 0$, the desired result follows from (1). Assume that $0 < i \leqslant n$ and that θ_{i-1} is a homotopy equivalence. Using the results of §HA.7.4.1 , we obtain a map of fiber sequences

$$\begin{array}{ccc} \varinjlim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\leqslant i} C_\alpha) & \xrightarrow{\theta_i} & \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\leqslant i} C) \\ \downarrow & & \downarrow \\ \varinjlim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\leqslant i-1} C_\alpha) & \xrightarrow{\theta_{i-1}} & \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\leqslant i-1} C) \\ \downarrow & & \downarrow \\ \varinjlim_\alpha \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, \Sigma^{i+1} \pi_i C_\alpha)) & \xrightarrow{\phi} & \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, \Sigma^{i+1} \pi_i C)). \end{array}$$

Set $D_\alpha = \mathrm{fib}(\Sigma^{i+1} \pi_i C_\alpha \to \Sigma^{i+1} \pi_i C)$, so that each D_α is n-truncated and $\varinjlim D_\alpha \simeq 0$. Assumption (2) implies that the canonical map $\varinjlim \mathrm{Ext}^i_B(L_{B/A}, D_\alpha) \to \mathrm{Ext}^i_B(L_{B/A} \varinjlim D_\alpha)$ is bijective for $i < 0$ and injective when $i = 0$: that is, the groups $\varinjlim \mathrm{Ext}^i_B(L_{B/A}, D_\alpha)$ vanish, so that the direct limit $Z = \varinjlim \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, D_\alpha)$ is contractible. Each nonempty homotopy fiber of ϕ is homotopy equivalent to Z, so the morphism ϕ is (-1)-truncated. Our hypothesis that θ_{i-1} is a homotopy equivalence now implies that θ_i is also a homotopy equivalence, as desired. □

4.1.3 Persistence of Finite Generation

We now use Proposition 4.1.2.1 to establish some elementary formal properties of Definition 4.1.1.1.

Proposition 4.1.3.1. *Suppose we are given maps $f : A \to B$ and $g : B \to C$ of connective $\mathbb{E}_\infty$-rings. Assume that f is of finite generation to order n. Then g is of finite generation to order n if and only if $g \circ f$ is of finite generation to order n.*

Proof. If $n = 0$, then the desired result follows immediately from Proposition 4.1.1.3. Let us therefore assume that $n > 0$. Then $\pi_0 B$ is finitely presented as an algebra over $\pi_0 A$. It follows that $\pi_0 C$ is finitely presented over $\pi_0 B$ if and only if it is finitely presented over $\pi_0 A$. Using Proposition 4.1.2.1, we are reduced to proving that $L_{C/A}$ is perfect to order n if and only if $L_{C/B}$ is perfect to order n. This follows by applying Remark 2.7.0.7 to the fiber sequence $C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B}$, since $C \otimes_B L_{B/A}$ is perfect to order n by Proposition 4.1.2.1 and Proposition 2.7.3.1. □

Proposition 4.1.3.2. *Suppose we are given a pushout diagram of connective $\mathbb{E}_\infty$-rings σ :*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B'. \end{array}$$

If f is of finite generation to order n, then f' is of finite generation to order n.

Proof. Using Proposition HA.7.2.2.13 , we obtain an isomorphism of commutative rings $\pi_0 B' \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 A')$. It follows that if $\pi_0 B$ is finitely generated (finitely presented) as a commutative ring over $\pi_0 A$, then $\pi_0 B'$ is finitely generated (finitely presented) as a commutative ring over $\pi_0 A'$. This completes the proof when $n = 0$ (Proposition 4.1.1.3). If $n > 0$, we must also show that $L_{B'/A'} \simeq B' \otimes_B L_{B/A}$ is perfect to order n as a B'-module (Proposition 4.1.2.1), which follows from Proposition 2.7.3.1 (since $L_{B/A}$ is perfect to order n as a B-module, by Proposition 4.1.2.1). □

Corollary 4.1.3.3. *Let R be a connective $\mathbb{E}_\infty$-ring, and suppose we are given a pushout diagram of connective $\mathbb{E}_\infty$-algebras over R:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B'. \end{array}$$

If A, B, and A' are of finite generation to order n over R, then B' is of finite generation to order n over R.

Proof. Since A' is of finite generation to order n over R, it will suffice to show that f' is of finite generation to order n (Proposition 4.1.3.1). Using Proposition 4.1.3.2, we are reduced to proving that f is of finite generation to order n. This follows from Proposition 4.1.3.1, since A and B are both of finite generation to order n over R. □

Proposition 4.1.3.4. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings, let $B_0 = \pi_0 A \otimes_A B$, and let $n \geqslant 0$. Then f is of finite generation to order n if and only if the induced map $f_0 : \pi_0 A \to B_0$ is of finite generation to order n.*

Proof. We have $\pi_0 B_0 = \pi_0(\pi_0 A \otimes_A B) \simeq \pi_0 B$, so that $\pi_0 B_0$ is finitely generated (finitely presented) over $\pi_0 A$ if and only if $\pi_0 B$ is finite generated (finitely presented) over $\pi_0 A$. This completes the proof when $n = 0$ (see Proposition 4.1.1.3). If $n > 0$, then the assertion that either f or f_0 is of finite generation to order n guarantees that $\pi_0 B$ is finitely presented over $\pi_0 A$. Using Proposition 4.1.2.1, we are reduced to proving that $L_{B/A}$ is perfect to order n over B if and only if $L_{B_0/\pi_0 A} \simeq B_0 \otimes_B L_{B/A}$ is perfect to order n is perfect to order n

over B_0. According to Proposition 2.9.4.2, both of these conditions are equivalent to the requirement that $\pi_0 B \otimes_B L_{B/A}$ is perfect to order n as a module over $\pi_0 B$. □

Corollary 4.1.3.5. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings, let $B_0 = \pi_0 A \otimes_A B$, and let $n \geqslant 0$. Then f is almost of finite presentation if and only if the induced map $f_0 : \pi_0 A \to B_0$ is almost of finite presentation.*

4.1.4 Local Nature of Finite Generation

For applications in spectral algebraic geometry, we need to understand how the finiteness conditions introduced in Definition 4.1.1.1 behave with respect to the étale topology. Note that if $f : A \to B$ is a morphism of connective $\mathbb{E}_\infty$-rings which exhibits B as a compact object of CAlg_A, then f is of finite generation to order n. In particular, any étale morphism of of finite generation to order n (Corollary HA.7.5.4.4).

Proposition 4.1.4.1. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. Then:*

(1) *For every étale morphism $B \to B'$, if f is of finite generation to order n, then so is the composite map $A \to B \to B'$.*

(2) *Suppose we are given a finite collection of étale maps $B \to B_\alpha$ such that the composition $B \to \prod_\alpha B_\alpha$ is faithfully flat. If each of the composite maps $A \to B \to B_\alpha$ is of finite generation to order n, then f is of finite generation to order n.*

Proof. Assertion (1) is immediate from Proposition 4.1.3.1, since étale morphisms are of finite generation to order n. Let us prove (2). We first treat the case $n = 0$. Let $B' = \prod_\alpha B_\alpha$. Since each $\pi_0 B_\alpha$ is finitely generated as an algebra over $\pi_0 A$, we deduce that $\pi_0 B'$ is finitely generated over $\pi_0 A$. The map of commutative rings $\pi_0 B \to \pi_0 B'$ is étale. Using the structure theory of étale morphisms (see Proposition B.1.1.3), we can choose a subalgebra $R \subseteq \pi_0 B$ which is finitely generated over $\pi_0 A$ and a faithfully flat étale R-algebra R' fitting into a pushout diagram

$$\begin{array}{ccc} \pi_0 B & \longrightarrow & \pi_0 B' \\ \downarrow & & \downarrow \\ R & \longrightarrow & R'. \end{array}$$

Since $\pi_0 B'$ is finitely generated, we may assume (after enlarging R if necessary) that the map $R' \to \pi_0 B'$ is surjective. Then $R' \otimes_R (\pi_0 B/R) \simeq 0$, so (by faithful flatness) we deduce that $\pi_0 B = R$ is finitely generated as a $\pi_0 A$-algebra.

Assume now that $n > 0$. We will show that the map $A \to B$ satisfies conditions (1) and (2) of Proposition 4.1.2.1. We first show that $\pi_0 B$ is finitely presented as a $\pi_0 A$-algebra. We have already seen that $\pi_0 B$ is finitely generated as a $\pi_0 A$-algebra, so we can choose an isomorphism $\pi_0 B \simeq R/I$ where R is a polynomial algebra over $\pi_0 A$. Write I as a filtered colimit of finitely generated ideals $I_\alpha \subseteq I$. Using the structure theory of étale morphisms, we can choose a finitely generated ideal $J \subseteq I$ and a faithfully flat étale map $R/J \to S$ such that $\pi_0 B' \simeq S \otimes_{R/J} R/I$. Then $\pi_0 B'$ is the quotient of S by the ideal IS. Since $\pi_0 B'$ is finitely presented as an $\pi_0 A$-algebra, we deduce that $\pi_0 B' \simeq S/I_0 S$ for some finitely generated ideal $I_0 \subseteq I$ containing J. Since S is faithfully flat over R/J, the map $I/I_0 \to IS/I_0 S \simeq 0$ is injective. Thus $I = I_0$ is a finitely generated ideal and $\pi_0 B$ is finitely presented over $\pi_0 A$ as desired.

It remains to verify condition (2) of Proposition 4.1.2.1. We have a fiber sequence of B'-modules $B' \otimes_B L_{B/A} \to L_{B'/A} \to L_{B'/B}$. Since B' is étale over B, the relative cotangent complex $L_{B'/B}$ vanishes. It follows that $L_{B'/A} \simeq B' \otimes_B L_{B/A}$. Proposition 4.1.2.1 implies that $L_{B'/A}$ is perfect to order n as a B'-module, so that $L_{B/A}$ is perfect to order n as a B-module by Proposition 2.7.3.1. □

Remark 4.1.4.2. In the situation of part (2) of Proposition 4.1.4.1, suppose that each B_α is locally of finite presentation over A (see Definition HA.7.2.4.26). Then B is locally of finite presentation over A. To prove this, we note that it follows from Proposition 4.1.4.1 that B is almost of finite presentation over A. It will therefore suffice to show that $L_{B/A}$ is perfect as a module over B. Using Proposition 2.8.4.2, we are reduced to proving that each $B_\alpha \otimes_B L_{B/A}$ is perfect as a module over B_α. This is clear, since the vanishing of $L_{B_\alpha/B}$ implies that the canonical map $B_\alpha \otimes_B L_{B/A} \to L_{B_\alpha/A}$ is an equivalence.

Proposition 4.1.4.3. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. Suppose that there exists a finite collection of flat morphisms $A \to A_\alpha$ with the following properties:*

(1) *The map $A \to \prod_\alpha A_\alpha$ is faithfully flat.*

(2) *Each of the maps $f_\alpha : A_\alpha \to A_\alpha \otimes_A B$ is of finite generation to order n (almost of finite presentation, locally of finite presentation).*

Then f is of finite generation to order n (almost of finite presentation, locally of finite presentation).

Proof. Let $A' = \prod_\alpha A_\alpha$, let $B' = A' \otimes_A B$, and let $f' : A' \to B'$ be the induced map. Let us first suppose that each f_α is of finite type. Then f' is of finite type (Remark 4.1.1.6), so that $\pi_0 B' = \pi_0 A' \otimes_{\pi_0 A} \pi_0 B$ is finitely generated as a commutative algebra over $\pi_0 A'$. We may therefore choose a finite collection of elements $x_1, \ldots, x_n \in \pi_0 B$ which generate $\pi_0 B'$ as an algebra over $\pi_0 A'$. Let R denote the polynomial ring $(\pi_0 A)[x_1, \ldots, x_n]$, so we have a map of commutative rings $\phi : R \to \pi_0 B$ which induces a surjection $\pi_0 A' \otimes_{\pi_0 A} R \to \pi_0 A' \otimes_{\pi_0 A} \pi_0 B$. Since $\pi_0 A'$ is faithfully flat over $\pi_0 A$, we deduce that ϕ is surjective, so that $\pi_0 B$ is finitely generated as a commutative ring over $\pi_0 A$ and therefore f is of finite type.

Now suppose that each f_α is of finite generation to order 1. Using Remark 4.1.1.6 and Proposition 4.1.2.1, we deduce that $\pi_0 B'$ is finitely presented as an algebra over $\pi_0 A'$. Define $\phi : R \to \pi_0 B$ be the surjection defined above. Let $I = \ker(\phi)$. Since A' is flat over A, we can identify $I \otimes_{\pi_0 A} \pi_0 A'$ with the kernel of the surjection $\pi_0 A' \otimes_{\pi_0 A} R \to \pi_0 B'$. The assumption that $\pi_0 B'$ is of finite presentation over $\pi_0 A'$ implies that this kernel is finitely generated. We may therefore choose a finitely generated submodule $J \subseteq I$ such that the induced map $J \otimes_{\pi_0 A} \pi_0 A' \to I \otimes_{\pi_0 A} \pi_0 A'$ is surjective. Since A' is faithfully flat over A, we deduce that $I = J$ is finitely generated, so that $\pi_0 B \simeq R/I$ is finitely presented as a commutative ring over $\pi_0 A$. Suppose that we wish to prove that f is of finite generation to order $n \geqslant 1$ (almost of finite presentation, locally of finite presentation). Using Proposition 4.1.2.1 and Theorem HA.7.4.3.18 , we are reduced to proving that the relative cotangent complex $L_{B/A}$ is perfect to order n (almost perfect, perfect) as a module over B. Using Remark 4.1.1.6 we deduce that $B' \otimes_B L_{B/A} \simeq L_{B'/A'}$ is perfect to order n (almost perfect, perfect) as a module over B'. Since B' is faithfully flat over B, the desired result follows from Proposition 2.8.4.2. □

4.2 Finiteness Conditions on Spectral Deligne-Mumford Stacks

Our goal in this section is to translate the ideas of §4.1 into the language of spectral Deligne-Mumford stacks.

Definition 4.2.0.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. We will say that f is *locally of finite generation to order n* (*locally almost of finite presentation*, *locally of finite presentation*) if the following condition is satisfied: for every commutative diagram

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} B & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow f \\ \operatorname{Sp\acute{e}t} A & \longrightarrow & \mathsf{Y} \end{array}$$

where the horizontal maps are étale, the $\mathbb{E}_\infty$-ring B is of finite generation to order n (almost of finite presentation, locally of finite presentation) over A. We will say that f is *locally of finite type* if it is locally of finite generation to order 0.

Example 4.2.0.2. A map of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ is locally of finite type if and only if the underlying map of ordinary Deligne-Mumford stacks is locally of finite type (in the sense of classical algebraic geometry).

Example 4.2.0.3. If $f : \mathsf{X} \to \mathsf{Y}$ is a morphism of spectral Deligne-Mumford stacks which is of finite generation to order 1, then the underlying map of ordinary Deligne-Mumford stacks is locally of finite presentation (in the sense of classical algebraic geometry). The converse holds if the structure sheaf of X is 0-truncated.

Remark 4.2.0.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks, and suppose that Y is locally Noetherian. Then the following conditions are equivalent:

(a) The morphism f is locally of finite type and X is locally Noetherian.

(b) The morphism f is locally almost of finite presentation.

To prove this, we can reduce to the case where X and Y are affine, in which case the desired result follows from Theorem HA.7.2.4.31 .

Warning 4.2.0.5. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of Deligne-Mumford stacks which is locally of finite presentation in the sense of classical algebraic geometry. If Y is not locally Noetherian, then f need not be locally almost of finite presentation when regarded as a morphism of 0-truncated spectral Deligne-Mumford stacks. The latter condition corresponds, in the terminology of [101], to the condition that f is *pseudo-coherent*: see Corollary 5.2.2.3 and Remark 5.2.2.4.

4.2.1 Morphisms Locally of Finite Generation

We begin by establishing some formal properties of Definition 4.2.0.1.

Proposition 4.2.1.1. *The condition that a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks be locally of finite generation to order n (locally almost of finite presentation, locally of finite presentation) is local on the source and target with respect to the étale topology (see Definition 2.8.1.7).*

Proof. It is clear that if f is locally of finite generation to order n (locally almost of finite presentation, locally of finite presentatiion) and $g : \mathsf{U} \to \mathsf{X}$ is étale, $f \circ g$ is locally of finite generation to order n (locally almost of finite presentation, locally of finite presentation). To complete the proof, let us suppose that $g : \mathsf{Y} \to \mathsf{Z}$ is arbitrary and that we are given a jointly surjective collection of étale maps $\{f_\alpha : \mathsf{X}_\alpha \to \mathsf{Y}\}$ such that each composition $g \circ f_\alpha$ is locally of finite generation to order n (locally almost of finite presentation, locally of finite presentation). We wish to show that g has the same property. Choose a commutative diagram

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} B & \longrightarrow & \mathsf{Y} \\ \downarrow & & \downarrow \\ \operatorname{Sp\acute{e}t} A & \longrightarrow & \mathsf{Z} \end{array}$$

where the horizontal maps are étale. We wish to show that B is of finite generation to order n (almost of finite presentation, locally of finite presentation) over A. Since the maps f_α are jointly surjective, we can choose an étale covering $\{B \to B_i\}$ such that each of the composite maps $\mathrm{Sp\acute{e}t}\, B_i \to \mathrm{Sp\acute{e}t}\, B \to \mathsf{Y}$ factors through some X_α. Using our assumption on $g \circ f_\alpha$, we deduce that each B_i is of finite generation to order n (almost of finite presentation, locally of finite presentation) over A. The desired result now follows from Proposition 4.1.4.1 and Remark 4.1.4.2. □

Remark 4.2.1.2. Let $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ be a map of spectral Deligne-Mumford stacks. Then f is locally of finite generation to order n (locally almost of finite presentation, locally of finite presentation) if and only if the following condition is satisfied:

(∗) For every étale map $\mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, the $\mathbb{E}_\infty$-ring A is of finite generation to order n (almost of finite presentation, locally of finite presentation) over R.

The "only if" direction is obvious. Conversely, suppose that (∗) is satisfied and we are given a commutative diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}\, A & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}\, R' & \longrightarrow & \mathrm{Sp\acute{e}t}\, R \end{array}$$

where the horizontal maps are étale. Using condition (∗), we deduce that A is of finite generation to order n (almost of finite presentation, locally of finite presentation) over R. It follows from Theorem 1.4.10.2 that R' is étale over R, and therefore locally of finite presentation over R. The desired result now follows from Proposition 4.1.3.1 and Remark HA.7.2.4.29 .

Remark 4.2.1.3. Let $f : \mathrm{Sp\acute{e}t}\, A \to \mathrm{Sp\acute{e}t}\, R$ be a map of affine spectral Deligne-Mumford stacks. Then f is locally of finite generation to order n (locally almost of finite presentation, locally of finite presentation) if and only if A is of finite generation to order n (almost of finite presentation, locally of finite presentation) over R. The "only if" direction follows immediately from the definitions. To prove the converse, it suffices to observe that if A is of finite generation to order n (almost of finite presentation, locally of finite presentation) over R and A' is an étale A-algebra, then A' is of finite generation to order n (almost of finite presentation, locally of finite presentation) over R.

Remark 4.2.1.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks. Then f is locally of finite generation to order n (locally almost of finite presentation, locally of finite presentation) if and only if, for every étale map $\mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$, the induced map $\mathsf{X} \times_{\mathsf{Y}} \mathrm{Sp\acute{e}t}\, R \to \mathrm{Sp\acute{e}t}\, R$ is locally of finite generation to order n (locally almost of finite presentation, locally of finite presentation).

Proposition 4.2.1.5. *The condition that a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks be locally of finite generation to order n (locally almost of finite presentation,* *locally of finite presentation) is local on the target with respect to the flat topology. That is, if there exists a flat covering* $\{\mathsf{Y}_\alpha \to \mathsf{Y}\}$ *such that each of the induced maps* $f_\alpha : \mathsf{Y}_\alpha \times_{\mathsf{Y}} \mathsf{X} \to \mathsf{Y}_\alpha$ *is locally of finite generation to order* n *(locally almost of finite presentation, locally of finite presentation), then* f *has the same property.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} \operatorname{Spét} B & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow f \\ \operatorname{Spét} A & \longrightarrow & \mathsf{Y} \end{array}$$

where the horizontal maps are étale; we wish to show that A is of finite generation to order n (almost of finite presentation, locally of finite presentation) over A. Since the map $\amalg \mathsf{Y}_\alpha \to \mathsf{Y}$ is a flat covering, there exists a faithfully flat map $A \to \prod_{1 \leqslant i \leqslant n} A_i$ such that each of the maps $\operatorname{Spét} A_i \to \operatorname{Spét} A \to \mathsf{Y}$ factors through an étale map $\operatorname{Spét} A_i \to \mathsf{Y}_\alpha$. Using our hypothesis on Y_α, we deduce that $A_i \otimes_R B$ is of finite generation to order n (almost of finite presentation, locally of finite presentation) over A_i. We now conclude by invoking Proposition 4.1.4.3. □

Proposition 4.2.1.6. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow f' & & \downarrow f \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y} \end{array}$$

of spectral Deligne-Mumford stacks. If f *is locally of finite generation to order* n *(locally almost of finite presentation, locally of finite presentation), then so is* f'*.*

Proof. Proposition 4.2.1.5 implies that the assertion is local on Y'; we may therefore assume without loss of generality that $\mathsf{Y}' = \operatorname{Spét} R'$ is affine and that the map $\mathsf{Y}' \to \mathsf{Y}$ factors as a composition $\operatorname{Spét} R' \to \operatorname{Spét} R \xrightarrow{u} \mathsf{Y}$, where u is étale. Replacing Y by $\operatorname{Spét} R$, we may assume that Y is also affine. Proposition 4.2.1.1 implies that the assertion is local on X' and therefore local on X; we may therefore suppose that $\mathsf{X} = \operatorname{Spét} A$ is affine. Then $\mathsf{X}' = \operatorname{Spét} A'$, where $A' = R' \otimes_R A$. Using Remark 4.2.1.3, we are reduced to proving that A' is of finite generation to order n (almost of finite presentation, locally of finite presentation) over R'. This follows from Proposition 4.1.3.2, since A is of finite generation to order n (almost of finite presentation, locally of finite presentation) over R. □

Proposition 4.2.1.7. *Let* $f : \mathsf{X} \to \mathsf{Y}$ *be a morphism of spectral Deligne-Mumford stacks which satisfies the following conditions:*

(1) *The spectral Deligne-Mumford stack* Y *is locally Noetherian.*

(2) *The map* f *is quasi-compact, separated, and locally of finite type.*

(3) *For every field* κ *and every morphism* $\mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Y}$*, the projection map* $\mathsf{X} \times_{\mathsf{Y}} \mathrm{Sp\acute{e}t}\,\kappa \to \mathrm{Sp\acute{e}t}\,\kappa$ *is an equivalence.*

Then f *is an equivalence.*

Proof. The assertion is local on Y. We may therefore assume without loss of generality that $\mathsf{Y} = \mathrm{Sp\acute{e}t}\,R$ for some Noetherian $\mathbb{E}_\infty$-ring R. For every étale map $\mathrm{Sp\acute{e}t}\,A \to \mathsf{X}$ and every residue field κ of $\pi_0 R$, condition (3) implies that $A \otimes_R \kappa$ is étale over κ. Using this together with (2), we deduce that f is locally quasi-finite. Invoking Theorem 3.3.0.2, we deduce that X is quasi-affine. Let R' be the $\mathbb{E}_\infty$-ring of global sections of the structure sheaf of X. Corollary 3.4.2.3 now guarantees that R' is almost connective. Using condition (3) and Proposition 2.5.4.5, we deduce that the canonical map $\theta : R \to R'$ induces an equivalence $\kappa \to R' \otimes_R \kappa$ for every residue field κ of $\pi_0 R$. Corollary 2.6.1.4 implies that θ is an equivalence, so that f is an equivalence by Proposition **??**. □

4.2.2 Digression: Bounds on the Relative Dimension

Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and let $d \geqslant 0$ be a nonnegative integer. The condition that f has relative dimension $\leqslant d$ is local with respect to the étale topology on X (see Proposition 3.3.1.4). Consequently, there exists a largest open substack $\mathsf{U} \subseteq \mathsf{X}$ such that $f|_{\mathsf{U}} : \mathsf{U} \to \mathsf{X}$ is of relative dimension $\leqslant d$. We will refer to U as the *relative dimension* $\leqslant d$ *locus of the morphism* f. Note that if we are given any pullback diagram

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}, \end{array}$$

then we can identify $\mathsf{U} \times_{\mathsf{X}} \mathsf{X}'$ with an open substack of X', and the restriction of f' to this open substack is also of relative dimension d (Corollary 3.3.1.5). It follows that $\mathsf{U} \times_{\mathsf{X}} \mathsf{X}'$ is contained in the relative dimension $\leqslant d$ locus of f'. Under mild hypotheses, we have equality:

Proposition 4.2.2.1 (Universality of the Relative Dimension $\leqslant d$ Locus)**.** *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{g} & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y} \end{array}$$

where f and f' are locally of finite type. If $\mathsf{U} \subseteq \mathsf{X}$ is the relative dimension $\leqslant d$ locus of f, then $\mathsf{U} \times_{\mathsf{X}} \mathsf{X}'$ is the relative dimension $\leqslant d$ locus of f'.

Proof. Without loss of generality, we may assume that $\mathsf{Y} \simeq \operatorname{Spét} A$ and $\mathsf{Y}' \simeq \operatorname{Spét} A'$ are affine. Similarly, we may assume that $\mathsf{X} \simeq \operatorname{Spét} B$ is affine, so that $\mathsf{X}' \simeq \operatorname{Spét} B'$ for $B' = A' \otimes_A B$. Let $x' \in |\operatorname{Spec} B'|$ be a point which belongs to the relative dimension $\leqslant d$ locus of f'; we wish to show that $x = g(x') \in |\operatorname{Spec} B|$ belongs to the relative dimension $\leqslant d$ locus of f. Let $\kappa(x')$ denote the residue field of B' at x' and let $\kappa(x)$ denote the residue field of B at x. Set $R = \pi_0(\kappa(x) \otimes_A B)$. Our assumption that f is locally of finite type guarantees that R is finitely generated as an algebra over $\kappa(x)$. Moreover, the canonical map $B \to \kappa(x)$ determines an augmentation $\epsilon : R \to \kappa(x)$ whose kernel is a maximal ideal $\mathfrak{m} \subseteq R$. Set $R' = \kappa(x') \otimes_{\kappa(x)} R$, so that ϵ determines an augmentation $\epsilon' : R' \to \kappa(x')$ whose kernel is a maximal ideal $\mathfrak{m}' \subseteq R'$. Let h and h' denote the Krull dimensions of the local rings $R_{\mathfrak{m}}$ and $R'_{\mathfrak{m}'}$. It follows from Theorem B.2.1.2 that $h = h'$ (note that $R_{\mathfrak{m}}$ and $R'_{\mathfrak{m}'}$ have the same Hilbert-Samuel polynomials). Our assumption that x' belongs to the relative dimension $\leqslant d$ locus of f' guarantees that $h' \leqslant d$. It follows that $h \leqslant d$. Applying Corollary B.2.4.8, we deduce that there exists an element $b \in \pi_0 B$ which does not vanish at x such that $B[b^{-1}]$ has relative dimension $\leqslant d$ over A, so that x belongs to the relative dimension $\leqslant d$ locus of f as desired. □

4.2.3 Morphisms Locally of Finite Presentation

Roughly speaking, the condition that a morphism of spectral Deligne-Mumford stacks $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be locally of finite generation to order $(n+1)$ can be viewed as a finiteness condition on the homotopy groups $\pi_i \mathcal{O}_{\mathcal{X}}$ for $i \leqslant n$. In practice, it is often useful to use this notion in conjunction with another hypothesis which controls the homotopy groups $\pi_i \mathcal{O}_{\mathcal{X}}$ for $i > n$.

Definition 4.2.3.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks. We will say that $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is *locally finitely n-presented* over Y if the following conditions are satisfied:

(i) The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is n-truncated.

(ii) The map f is locally of finite generation to order $n+1$.

In this case, we will also say that the morphism f is locally finitely n-presented.

Example 4.2.3.2. Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of spectral Deligne-Mumford stacks. Then f is locally finitely 0-presented if and only if the following conditions are satisfied:

(i) The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is discrete.

(ii) The induced map of ordinary Deligne-Mumford stacks $(\mathcal{X}^{\heartsuit}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}^{\heartsuit}, \pi_0 \mathscr{O}_{\mathcal{Y}})$, when is locally of finite presentation, in the sense of classical algebraic geometry.

We now summarize some of the formal properties of Definition 4.2.3.1.

Proposition 4.2.3.3. *Fix an integer $n \geqslant 0$.*

(1) *The condition that a map $f : \mathsf{X} \to \mathsf{Y}$ be locally finitely n-presented is local on the source with respect to the étale topology.*

(2) *The condition that a map $f : \mathsf{X} \to \mathsf{Y}$ be locally finitely n-presented is local on the target with respect to the flat topology.*

(3) *Suppose given a pair of maps $f : \mathsf{X} \to \mathsf{Y}$ and $g : \mathsf{Y} \to \mathsf{Z}$. Assume that g is locally finitely n-presented. Then f is locally finitely n-presented if and only if $g \circ f$ is locally finitely n-presented.*

Proof. Assertion (1) follows from Proposition 4.2.1.1 and Example 2.8.3.9, assertion (2) follows from Proposition 4.2.1.5 and Example 2.8.3.9, and assertion (3) follows from Proposition 4.1.3.1. □

We conclude with the following observation:

Proposition 4.2.3.4. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Then f is étale if and only if the following three conditions are satisfied:*

(1) *The morphism f is flat.*

(2) *The diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ is flat.*

(3) *The morphism f is locally almost of finite presentation.*

Proof. The necessity of conditions (1), (2), and (3) is clear. Conversely, suppose that (1), (2) and (3) are satisfied; we wish to show that f is étale. Choose étale maps $\operatorname{Spét} A \to \mathsf{Y}$ and $\operatorname{Spét} B \to \operatorname{Spét} A \times_{\mathsf{Y}} \mathsf{X}$; we wish to show that B is étale over A. Then the induced map $\operatorname{Spét} B \to \operatorname{Spét}(B \otimes_A B) \times_{\mathsf{X} \times_{\mathsf{Y}} \mathsf{X}} \mathsf{X}$ is étale, so condition (2) guarantees that B is flat over $B \otimes_A B$. Set $R = \pi_0(B \otimes_A B)$ and let I deote the kernel of the underlying ring homomorphism $R \to \pi_0 B$. Condition (3) guarantees that $\pi_0 B$ is finitely generated as an algebra over $\pi_0 A$, so the ideal I is finitely generated. It follows that $\pi_0 B$ is finitely presented and flat as a module R. Using Lazard's theorem, we deduce that $\pi_0 B$ is a projective module over R. It follows that the canonical map $R \to \pi_0 B$ splits, so we can write $I = (e)$ for some idempotent element $e \in R$. It now follows from (1) and (3) that B is étale over A, as desired (see Definition HA.7.5.0.4). □

4.2.4 The Locally Noetherian Case

In the setting of locally Noetherian spectral Deligne-Mumford stacks, Definition 4.2.3.1 can be dramatically simplified:

Proposition 4.2.4.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks. Assume that Y is locally Noetherian. Then f is locally finitely n-presented if and only if the following conditions are satisfied:*

(1) *The morphism f is locally of finite type.*

(2) *The spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is locally Noetherian.*

(3) *The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is n-truncated.*

Warning 4.2.4.2. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks which is locally finitely n-presented. If Y is not locally Noetherian, then f need not be locally finitely m-presented for $m > n$.

We will deduce Proposition 4.2.4.1 from the following general categorical fact:

Lemma 4.2.4.3. *Let $\mathcal{C}$ be a projectively generated ∞-category containing an object X and let $n \geqslant 0$ be an integer. The following conditions are equivalent:*

(*i*) *The truncation $\tau_{\leqslant n} X$ is compact when viewed as an object of $\tau_{\leqslant n} \mathcal{C}$.*

(*ii*) *There exists a compact object $Y \in \mathcal{C}$ and a morphism $Y \to X$ which induces an equivalence $\tau_{\leqslant n} Y \to \tau_{\leqslant n} X$.*

In particular, every compact object of $\tau_{\leqslant n} \mathcal{C}$ has the form $\tau_{\leqslant n} Y$ for some compact object $Y \in \mathcal{C}$.

Proof of Proposition 4.2.4.1. We may assume without loss of generality that $\mathsf{Y} \simeq \operatorname{Spét} A$ and $\mathsf{X} \simeq \operatorname{Spét} B$ are affine. Assume first that f is locally finitely n-presented. Conditions (1) and (3) are obvious. To prove (2), we note that B is a compact object of $\tau_{\leqslant n} \mathrm{CAlg}_A$ (Remark 4.1.1.9), so that $B \simeq \tau_{\leqslant n} B'$ for some A-algebra B' which is of finite presentation over A (Lemma 4.2.4.3). It follows from Proposition HA.7.2.4.31 that B' is Noetherian, so that B is also Noetherian.

Now suppose that (1), (2), and (3) are satisfied. Using (1), (2), and Proposition HA.7.2.4.31 , we deduce that the map $A \to B$ is locally almost of finite presentation, and in particular of finite generation to order $(n+1)$ over A. Combining this with (3), we deduce that f is finitely n-presented as desired. □

Remark 4.2.4.4. Let $\mathcal{C}$ be a projectively generated ∞-category. Recall that a morphism $X \to Y$ in $\mathcal{C}$ exhibits Y as an n-truncation of X if and only if, for every compact projective object $P \in \mathcal{C}$, the induced map $\mathrm{Map}_{\mathcal{C}}(P, X) \to \mathrm{Map}_{\mathcal{C}}(P, Y)$ exhibits the space $\mathrm{Map}_{\mathcal{C}}(P, Y)$ as an n-truncation of $\mathrm{Map}_{\mathcal{C}}(P, X)$. Consequently, if we are given a pullback diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ Y_0 & \longrightarrow & Y_1 \end{array}$$

in $\mathcal{C}$ where Y_0 is n-truncated and β exhibits Y_1 as an n-truncation of X_1, then α exhibits Y_0 as an n-truncation of X_0 (to prove this, it suffices to verify the analogous assertion in the ∞-category of spaces, which follows from the observation that the homotopy fibers of α are $(n+1)$-connective provided that the homotopy fibers of β are $(n+1)$-connective).

Proof of Lemma 4.2.4.3. The implication $(ii) \Rightarrow (i)$ is clear. To prove the converse, let $\mathcal{E} \subseteq \tau_{\leqslant n}\mathcal{C}$ denote the full subcategory spanned by those objects $\overline{X}$ with the following property:

- $(*)$ For every object $X \in \mathcal{C}$ equipped with an equivalence $\tau_{\leqslant n} X \simeq \overline{X}$, there exists a compact object $Y \in \mathcal{C}$ and a morphism $Y \to X$ which induces an equivalence $\tau_{\leqslant n} Y \to \tau_{\leqslant n} X \simeq \overline{X}$.

It is clear that every object of $\mathcal{E}$ is compact in $\tau_{\leqslant n}\mathcal{C}$. To complete the proof, it will suffice to show that every compact object of $\tau_{\leqslant n}\mathcal{C}$ belongs to $\mathcal{E}$. The main step will be to show that $\mathcal{E}$ is closed under finite colimits in $\tau_{\leqslant n}\mathcal{C}$. Assuming this for the moment, let $\overline{\mathcal{E}}$ denote the full subcategory of $\mathcal{C}$ spanned by those compact objects X for which $\tau \leqslant nX$ belongs to $\mathcal{E}$. Note that $\mathcal{E}$ contains the truncation $\tau_{\leqslant n} P$ for every compact projective object $P \in \mathcal{C}$ (for any object X equipped with an equivalence $\alpha : \tau_{\leqslant n} X \simeq \tau_{\leqslant n} P$, the projectivity of P guarantees that the map $P \to \tau_{\leqslant n} P \xrightarrow{\alpha^{-1}} \tau_{\leqslant n} X$ can be lifted to a map $P \to X$). Since $\mathcal{C}$ is generated under small colimits by its compact projective objects, the ∞-category $\tau_{\leqslant n}\mathcal{C}$ is generated by $\mathcal{E}$ under filtered colimits so that every compact object of $\tau_{\leqslant n}\mathcal{C}$ is a retract of an object of $\mathcal{E}$. But $\mathcal{E}$ is equivalent to an $(n+1)$-category, so the existence of finite limits in $\mathcal{E}$ guarantees that it is idempotent complete.

It remains to prove that $\mathcal{E}$ is closed under finite colimits. We have already observed that it contains the initial object of $\tau_{\leqslant n}\mathcal{C}$ (since the initial object can be computed as the n-truncation of the initial object of $\mathcal{C}$, which is compact and projective). It will therefore suffice to show that $\mathcal{E}$ is closed under pushouts. Suppose we are given a pushout diagram

$$\begin{array}{ccc} \overline{X}_{01} & \longrightarrow & \overline{X}_0 \\ \downarrow & & \downarrow \\ \overline{X}_1 & \longrightarrow & \overline{X} \end{array}$$

where $\overline{X}_{01}$, $\overline{X}_0$, and $\overline{X}_1$ belong to $\mathcal{E}$. We wish to show that $\overline{X}$ belongs to $\mathcal{E}$. Choose an object X equipped with a map $X \to \overline{X}$ which exhibits $\overline{X}$ as an n-truncation of X. Set $X_0 = X \times_{\overline{X}} \overline{X}_0$ and $X_1 = X \times_{\overline{X}} \overline{X}_1$. It follows from Remark 4.2.4.4 that the maps $X_0 \to \overline{X}_0$ and $X_1 \to \overline{X}_1$ exhibit $\overline{X}_0$ and $\overline{X}_1$ as n-truncations of X_0 and X_1, respectively. Our assumption that $\overline{X}_0$ and $\overline{X}_1$ satisfy condition $(*)$ guarantees that we can choose compact objects $Y_0, Y_1 \in \mathcal{C}$ equipped with maps $Y_0 \to X_0$ and $Y_1 \to X_1$ which induce equivalences $\tau_{\leqslant n} Y_0 \to \tau_{\leqslant n} X_0$ and $\tau_{\leqslant n} Y_1 \to \tau_{\leqslant n} X_1$. Set

$$X_{01} = Y_0 \times_{\overline{X}_0} \overline{X}_{01} \times_{\overline{X}_1} Y_1.$$

It follows from two applications of Remark 4.2.4.4 that the projection map $X_{01} \to \overline{X}_{01}$ exhibits $\overline{X}_{01}$ as an n-truncation of X_{01}. Because $\overline{X}_{01}$ satisfies condition $(*)$, we can choose a compact object $Y_{01} \in \mathcal{C}$ and a map $Y_{01} \to X_{01}$ which induces an equivalence $\tau_{\leqslant n} Y_{01} \simeq \tau_{\leqslant n} X_{01}$. Let Y denote the pushout $Y_0 \amalg_{Y_{01}} Y_1$. Then Y is a compact object of $\mathcal{C}$ equipped with a map $Y \to X$ which induces an equivalence $\tau_{\leqslant n} Y \simeq \tau_{\leqslant n} X$. □

4.2.5 Morphisms of Finite Presentation

We now combine the local finiteness condition of Definition 4.2.3.1 with some global considerations.

Definition 4.2.5.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks and let $n \geqslant 0$ be an integer. We will say that f is *finitely n-presented* if the following conditions are satisfied:

(1) The map f is locally finitely n-presented (see Definition 4.2.3.1).

(2) For every map Spét $A \to \mathsf{Y}$, the fiber product Spét $A \times_{\mathsf{Y}} \mathsf{X}$ is a spectral Deligne-Mumford m-stack for some integer $m \geqslant 0$.

(3) The morphism f is ∞-quasi-compact (see Definition 2.3.2.2).

Remark 4.2.5.2. In the situation of Definition 4.2.5.1, assume that Y is quasi-compact. Then condition (2) is equivalent to the following:

(2′) The map $f : \mathsf{X} \to \mathsf{Y}$ is a relative spectral Deligne-Mumford m-stack for some $m \geqslant 0$.

In this case, Proposition 2.3.2.7 implies that condition (3) is equivalent to the following *a priori* weaker condition:

(3′) The morphism f is $(m + 1)$-quasi-compact.

If $m = 0$ (in other words, if $f : \mathsf{X} \to \mathsf{Y}$ is a relative spectral algebraic space), then (3′) is equivalent to the following:

(3″) The morphism f is quasi-compact and quasi-separated.

Remark 4.2.5.3. Suppose we are given maps of spectral Deligne-Mumford stacks

$$\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z}.$$

Assume that g is finitely n-presented. Then f is finitely n-presented if and only if g is finitely n-presented (combine Proposition 4.2.3.3 with Corollary 2.3.5.2).

The property of being finitely n-presented is not stable under arbitrary base change. Given a pullback diagram

$$\begin{array}{ccc} \mathsf{Y}' & \longrightarrow & \mathsf{Y} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{X}' & \xrightarrow{g} & \mathsf{X} \end{array}$$

where f is finitely n-presented, the morphism f' need not be finitely n-presented without some flatness assumption on the morphism g. We can correct this difficulty by truncating the structure sheaf of the spectral Deligne-Mumford stack Y'.

Proposition 4.2.5.4. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y} \end{array}$$

of spectral Deligne-Mumford stacks. If $\tau_{\leqslant n}\mathsf{X}$ *is finitely* n*-presented over* Y*, then* $\tau_{\leqslant n}\mathsf{X}'$ *is finitely* n*-presented over* Y'*.*

Proof. We may assume without loss of generality that $\mathsf{Y} \simeq \operatorname{Sp\acute{e}t} A$ and $\mathsf{Y}' \simeq \operatorname{Sp\acute{e}t} A'$ are affine. Then X is an ∞-quasi-compact spectral Deligne-Mumford m-stack for some $m \geqslant 0$. It follows that X' is also an ∞-quasi-compact spectral Deligne-Mumford m-stack (see Remark 1.6.8.4 and Corollary 2.3.5.4). To complete the proof, it will suffice to show that f' is locally finitely n-presented. Replacing X by $\tau_{\leqslant n}\mathsf{X}$, we may assume that $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is locally of finite generation to order $(n+1)$ over A. It follows that $\mathsf{X}' = (\mathcal{X}', \mathscr{O}_{\mathcal{X}'})$ is locally of finite generation to order $(n+1)$ over $\operatorname{Sp\acute{e}t} A'$. Using Remark 4.1.1.8, we deduce that $(\mathcal{X}', \tau_{\leqslant n}\mathscr{O}_{\mathcal{X}'})$ is also locally of finite generation to order $(n+1)$ over $\operatorname{Sp\acute{e}t} A'$, hence locally finitely n-presented over $\operatorname{Sp\acute{e}t} A'$. □

Corollary 4.2.5.5. *Suppose we are given a commutative diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccccc} \mathsf{X}_0 & \longrightarrow & \mathsf{X} & \longleftarrow & \mathsf{X}_1 \\ & \searrow & \downarrow & \swarrow & \\ & & \mathsf{Y} & & \end{array}$$

where X_0*,* X_1*, and* X *are finitely* n*-presented over* Y*. Then* $\tau_{\leqslant n}(\mathsf{X}_0 \times_{\mathsf{X}} \mathsf{X}_1)$ *is finitely* n*-presented over* Y*.*

Proof. Using Remark 4.2.5.3, we see that X_0 is finitely n-presented over X. It follows from Proposition 4.2.5.4 that $\tau_{\leqslant n}(\mathsf{X}_0 \times_{\mathsf{X}} \mathsf{X}_1)$ is finitely n-presnented over X_1, and hence also finitely n-presented over X (Remark 4.2.5.3). □

4.3 Constructible Sets

Let $(X, \mathscr{O}_X)$ be a quasi-compact, quasi-separated scheme. A subset $K \subseteq X$ is said to be *constructible* if it can be written as a finite union of sets of the form $U - V$, where U and $V \subseteq U$ are quasi-compact open subsets of X. Constructible sets are ubiquitous in algebraic geometry: for many important properties P of points of X, one can show that the locus of points $x \in X$ satisfying P is constructible. For example, if $f : Y \to X$ is a morphism of schemes which is of finite presentation, then Chevalley's *constructibility theorem* asserts that the image of f is a constructible subset of X (see Theorem 4.3.3.1 and Corollary 4.3.4.2).

Our goal in this section is to review the theory of constructible sets. We will work in the setting of spectral algebraic spaces, though this represents no real gain in generality (the underlying topological space $|\mathsf{X}|$ of a spectral algebraic space X depends only on the ordinary algebraic space $\tau_{\leqslant 0}\mathsf{X}$).

4.3.1 The Constructible Topology

We begin with some general remarks. We will assume that the reader is familiar with the theory of distributive lattices and Boolean algebras; for a brief review, see §A.1.

Proposition 4.3.1.1. *Let* Lat *denote the category of distributive lattices (see Definition A.1.5.3) and let* BAlg $\subseteq$ Lat *denote the full subcategory spanned by the Boolean algebras (Definition A.1.6.4). Then* BAlg *is a localization of* Lat*: that is, the inclusion functor* BAlg $\hookrightarrow$ Lat *admits a left adjoint.*

Proof. Note that the categories Lat and BAlg are presentable. By virtue of the adjoint functor theorem (Corollary HTT.5.5.2.9), it will suffice to show that the inclusion functor BAlg $\hookrightarrow$ Lat preserves small limits and filtered colimits. The first of these assertions follows immediately from the definitions. To prove the second, it will suffice to show that if $\{\Lambda_\alpha\}$ is a filtered diagram of Boolean algebras having colimit $\Lambda = \varinjlim \Lambda_\alpha$ in the category Lat, then Λ is also a Boolean algebra. Note that every element $x \in \Lambda$ is the image of an element x_α of some Λ_α. Since x_α complemented, x is complemented (Remark A.1.6.3). □

Combining Theorem A.1.6.11 and Proposition A.1.5.10, we obtain the following consequence of Proposition 4.3.1.1:

Corollary 4.3.1.2. *Let $\mathcal{T}\mathrm{op}_{\mathrm{coh}}$ denote the category whose objects are coherent topological spaces and whose morphisms are quasi-compact continuous maps (Definition A.1.5.9) and let $\mathcal{T}\mathrm{op}_{\mathrm{St}} \subseteq \mathcal{T}\mathrm{op}_{\mathrm{coh}}$ denote the full subcategory spanned by the Stone spaces (Definition A.1.6.8). Then the inclusion functor $\mathcal{T}\mathrm{op}_{\mathrm{St}} \hookrightarrow \mathcal{T}\mathrm{op}_{\mathrm{coh}}$ admits a right adjoint.*

Notation 4.3.1.3. We will denote the right adjoint to the inclusion functor $\mathcal{T}\mathrm{op}_{\mathrm{St}} \hookrightarrow \mathcal{T}\mathrm{op}_{\mathrm{coh}}$ by $X \mapsto X_{\mathrm{c}}$. If X is a coherent topological space, then X_{c} is another topological space equipped with a continuous map $\phi : X_{\mathrm{c}} \to X$, which is characterized up to homeomorphism by the following pair of conditions:

(a) The topological space X_{c} is a Stone space and the map ϕ is quasi-compact.

(b) For every Stone space Y, composition with ϕ induces a bijection from the set of continuous maps $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(Y, X_{\mathrm{c}})$ to the set of quasi-compact continuous maps $\mathrm{Hom}_{\mathcal{T}\mathrm{op}_{\mathrm{coh}}}(Y, X)$.

Proposition 4.3.1.4. *Let X be a coherent topological space. Then the canonical map $\phi : X_{\mathrm{c}} \to X$ is a bijection.*

Proof. Let $*$ denote the topological space consisting of a single point, so that $* \in \mathcal{T}\mathrm{op}_{\mathrm{St}}$. As a map of sets, ϕ is given by the composition of bijections

$$X_{\mathrm{c}} \simeq \mathrm{Hom}_{\mathcal{T}\mathrm{op}_{\mathrm{St}}}(*, X_{\mathrm{c}}) \simeq \mathrm{Hom}_{\mathcal{T}\mathrm{op}_{\mathrm{coh}}}(*, X) \simeq X.$$

□

Remark 4.3.1.5. Let X be a coherent topological space. We will use Proposition 4.3.1.4 to identify the underlying sets of the topological spaces X and X_{c}. We may therefore view X_{c} as the space X endowed with a new topology, which we refer to as the *constructible topology*. We say that a subset $K \subseteq X$ is *constructible* if it is compact and open when regarded as a subset of X_{c}.

4.3.2 Constructible Open Sets

Let X be a coherent topological space. Then we can describe the class of constructible sets as the smallest Boolean algebra of subsets of X which contains every quasi-compact open subset of X. In particular, every quasi-compact open subset of X is constructible. The following converse is often useful:

Proposition 4.3.2.1. *Let X be a coherent topological space and let U be a subset of X. Then U is quasi-compact and open if and only if it satisfies the following conditions:*

(a) *The set U is constructible.*

(b) *The set U is stable under generalization. That is, if $x \in U$ and x belongs to the closure of a point $y \in X$, then $y \in U$.*

Corollary 4.3.2.2. *Let X be a coherent topological space, and let U be an open subset of X. Then U is constructible if and only if it is quasi-compact.*

Proof of Proposition 4.3.2.1. It is clear that conditions (a) and (b) necessary. We will show that they are sufficient using the following:

(∗) For every constructible subset $V \subseteq X$, we have $\overline{V} = \bigcup_{x \in V} \overline{\{x\}}$.

Assertion (∗) implies that for any constructible subset $U \subseteq X$, the interior of U is given by $X - \bigcup_{x \notin U} \overline{\{x\}}$, so that U is open if U satisfies (b).

The collection of constructible sets K satisfying condition (∗) is closed under finite unions. Consequently, to prove (∗) in general, it will suffice to prove (∗) in the special case where $K = V' \cap (X - W)$, where V' and W are quasi-compact open subsets of X. Replacing X by $X - W$, we may assume that V is a quasi-compact open subset of X.

Fix a point $y \in \overline{V}$. Let $\{V_\alpha\}$ denote the collection of all open subsets of V of the form $U \cap V$, where U is an open subset of X which contains y. Then each V_α is quasi-compact and nonempty (by virtue of our assumption that $y \in \overline{V}$). Moreover, the collection of open sets $\{V_\alpha\}$ is closed under finite intersections. Applying Remark A.1.5.11, we conclude that there exists a point $x \in \bigcap V_\alpha$. Then x is a point of V which belongs to every quasi-compact open subset of X which contains y, so that $y \in \overline{\{x\}}$. □

4.3.3 Chevalley's Constructibility Theorem

We will need the following classical theorem of Chevalley:

Theorem 4.3.3.1. *Let $f : A \to B$ be a homomorphism of commutative rings such that B is finitely presented as an A-algebra. Then the induced map $\phi : |\operatorname{Spec} B| \to |\operatorname{Spec} A|$ carries constructible subsets of $|\operatorname{Spec} B|$ to constructible subsets of $|\operatorname{Spec} A|$.*

Proof. Let Y be a constructible subset of $|\operatorname{Spec} B|$; we wish to show that $\phi(Y)$ is a constructible subset of $|\operatorname{Spec} A|$. Since B is finitely presented over A, we can choose a finitely generated subring $A_0 \subseteq A$ and a pushout diagram of commutative rings

$$\begin{array}{ccc} A_0 & \longrightarrow & A \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ B_0 & \longrightarrow & B, \end{array}$$

where B_0 is a finitely presented A_0-algebra. Enlarging A_0 if necessary, we may assume that Y is the inverse image of a constructible subset $Y_0 \subseteq |\operatorname{Spec} B_0|$ (Corollary 4.3.5.2). Let

$\phi_0 : |\operatorname{Spec} B_0| \to |\operatorname{Spec} A_0|$ be the map determined by f_0, so that $\phi(Y)$ is the inverse image of $\phi_0(Y_0)$. It will then suffice to show that $\phi_0(Y_0)$ is a constructible subset of $|\operatorname{Spec} A_0|$. Replacing f by f_0, we are reduced to proving Theorem 4.3.3.1 in the special case where the ring A is finitely generated. In particular, we may assume that A is Noetherian.

Write f as a composition of maps

$$A = A_0 \to A_1 \to \cdots \to A_n = B,$$

where each A_i is generated as an A_{i-1}-algebra by a single element. It will then suffice to show that each of the maps $|\operatorname{Spec} A_i| \to |\operatorname{Spec} A_{i-1}|$ carries constructible sets to constructible sets. We may therefore assume without loss of generality that $B = A[x]/I$ for some ideal $I \subseteq A[x]$. Since A is Noetherian, the ideal I is finitely generated. It follows that the closed immersion $|\operatorname{Spec} B| \to |\operatorname{Spec} A[x]|$ carries constructible sets to constructible sets. We may therefore reduce to the case where $B = A[x]$.

Let $Y \subseteq |\operatorname{Spec} A[x]|$ be a constructible set; we wish to show that $\phi(Y) \subseteq |\operatorname{Spec} A|$ is constructible. Writing Y as a finite union of locally closed subsets of $|\operatorname{Spec} A[x]|$, we can reduce to the case where Y is locally closed. We now proceed by Noetherian induction on A. Let us say that an ideal $I \subseteq A$ is *good* if the inverse image of $\phi(Y)$ in $|\operatorname{Spec} A/I|$ is constructible. We claim that every ideal $I \subseteq |\operatorname{Spec} A|$ is good. Assume otherwise. Since A is Noetherian, we can choose an ideal $I \subseteq A$ which is maximal among those ideals which are not good. Replacing A by A/I, we may reduce to the case where every nonzero ideal in A is good. It follows immediately that the ring A is reduced. If $A \simeq 0$, there is nothing to prove. Otherwise, we can choose a minimal prime ideal $\mathfrak{p} \subseteq A$, so that the localization $A_{\mathfrak{p}}$ is a field K.

Since Y is locally closed, we can write $Y = Z - Z'$, where Z is the vanishing locus of a radical ideal $J \subseteq A[x]$ and Z' is the vanishing locus of a radical ideal $J' \subseteq A[x]$ which contains J. Then the localizations $J_{\mathfrak{p}}$ and $J'_{\mathfrak{p}}$ are ideals in the polynomial ring $K[x]$. We distinguish three cases:

(*a*) The ideals $J_{\mathfrak{p}}$ and $J'_{\mathfrak{p}}$ are equal to one another. In this case, we can choose an element $a \in A - \mathfrak{p}$ such that $J[a^{-1}] = J'[a^{-1}]$. Let $U \subseteq |\operatorname{Spec} A|$ denote the open subset given by $|\operatorname{Spec} A[a^{-1}]|$, so that $U \times_{|\operatorname{Spec} A|} Z = U \times_{|\operatorname{Spec} A|} Z'$ and therefore $\phi(Y) \cap U = \varnothing$. The inductive hypothesis implies that $\phi(Y)$ is a constructible when regarded as a subset of $|\operatorname{Spec} A/(a)|$, and therefore constructible when regarded as a subset of $|\operatorname{Spec} A|$.

(*b*) The ideal $J_{\mathfrak{p}}$ is zero, and $J'_{\mathfrak{p}}$ is not zero. Since $K[x]$ is a principal ideal domain, the ideal $J'_{\mathfrak{p}}$ is generated by some (square-free) monic polynomial $f(x) \in K[x]$. Let us lift $f(x)$ to a monic polynomial $\overline{f}(x)$ with coefficients in the ring $A[a^{-1}]$, for some $a \in A - \mathfrak{p}$. Modifying the choice of a if necessary, we may assume that $J[a^{-1}] = 0$ and $J'[a^{-1}]$ is the ideal generated by $\overline{f}(x)$. In this case, $\phi(Y)$ contains the open set $U = |\operatorname{Spec} A[a^{-1}]|$. Since $\phi(Y) \cap |\operatorname{Spec} A/(a)|$ is constructible by our inductive hypothesis, it follows that $\phi(Y) = U \cup (\phi(Y) \cap |\operatorname{Spec} A/(a)|)$ is also constructible.

(c) The ideal $J'_{\mathfrak{p}}$ is generated by some (square-free) monic polynomial $f(x)$, the ideal $J_{\mathfrak{p}}$ is generated by a product $f(x)g(x)$, where $g(x)$ is a square-free monic polynomial which is relatively prime to $f(x)$ having degree > 0. In this case, the polynomials $f(x)$ and $g(x)$ generate the unit ideal in $K[x]$. Let us lift $f(x)$ and $g(x)$ to monic polynomial $\overline{f}(x)$ and $\overline{g}(x)$ with coefficients in the ring $A[a^{-1}]$, for some $a \in A - \mathfrak{p}$. Modifying a if necessary, we may assume that $J[a^{-1}]$ and $J'[a^{-1}]$ are generated by $\overline{f}(x)\overline{g}(x)$ and $\overline{f}(x)$, respectively, and that the polynomials $\overline{f}(x)$ and $\overline{g}(x)$ generate the unit ideal in $A[a^{-1}][x]$. Let $U = |\operatorname{Spec} A[a^{-1}]|$. Then we can identify $Y \times_{|\operatorname{Spec} A|} U$ with the vanishing locus of the polynomial $\overline{g}(x)$. Since $\overline{g}$ is a monic polynomial of positive degree, it follows that the projection map $Y \times_{|\operatorname{Spec} A|} U \to U$ is surjective. Since $\phi(Y) \cap |\operatorname{Spec} A/(a)|$ is constructible by our inductive hypothesis, it follows that $\phi(Y) = U \cup (\phi(Y) \cap |\operatorname{Spec} A/(a)|)$ is also constructible.

□

4.3.4 Constructibility in Algebraic Geometry

Let X be a quasi-compact, quasi-separated spectral algebraic space. Then the topological space $|\mathsf{X}|$ is coherent (Proposition 3.6.3.3). It therefore makes sense to consider constructible subsets of $|\mathsf{X}|$. It follows from Chevalley's constructibility theorem that this condition can be tested locally in a very strong sense:

Proposition 4.3.4.1. *Let X be a spectral algebraic space which is quasi-compact and quasi-separated, and let K be a subset of $|\mathsf{X}|$. The following conditions are equivalent:*

(1) *The set K is constructible.*

(2) *For every quasi-compact, quasi-separated spectral algebraic space Y and every map $f : \mathsf{Y} \to \mathsf{X}$, the inverse image $f^{-1}K \subseteq |\mathsf{Y}|$ is constructible.*

(3) *There exists an étale surjection $f : \operatorname{Spét} R \to \mathsf{X}$ for which the inverse image $f^{-1}K \subseteq |\operatorname{Spét} R| \simeq |\operatorname{Spec} R|$ is constructible.*

(4) *There exists a quasi-compact, quasi-separated algebraic space Y and a surjection $f : \mathsf{Y} \to \mathsf{X}$ such that $\tau_{\leqslant 0}\mathsf{Y}$ is locally finitely 0-presented over X (see Definition 4.2.3.1), and $f^{-1}K$ is a constructible subset of $|\mathsf{Y}|$.*

Proof. The implication (1) ⇒ (2) follows from Proposition 3.6.3.3, and the implications (2) ⇒ (3) and (3) ⇒ (4) are obvious. We will complete the proof by showing that (4) ⇒ (1).

Suppose that $f : \mathsf{Y} \to \mathsf{X}$ is a surjective morphism of quasi-compact, quasi-separated spectral algebraic spaces for which $\tau_{\leqslant 0}\,\mathsf{Y}$ is finitely 0-presented over X, and that $K \subseteq |\,\mathsf{X}\,|$ is a subset for which $f^{-1}K \subseteq |\,\mathsf{Y}\,|$ is constructible. We wish to prove that K is constructible. Applying Theorem 3.4.2.1, we can choose a scallop decomposition

$$\varnothing = \mathsf{U}_0 \hookrightarrow \mathsf{U}_1 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n = \mathsf{X}$$

of the spectral algebraic space $|\,\mathsf{X}\,|$. Then each $|\,\mathsf{U}_i\,|$ determines a quasi-compact open subset $U_i \subseteq |\,\mathsf{X}\,|$. We will prove that the intersection $K \cap U_i$ is a constructible subset of $|\,\mathsf{X}\,|$, using induction on i. The case $i = 0$ is trivial. To carry out the inductive step, assume that $i > 0$ and that $K \cap U_{i-1}$ is a constructible subset of $|\,\mathsf{X}\,|$. Choose an excision square

$$\begin{array}{ccc} \mathsf{V} & \xrightarrow{j} & \operatorname{Sp\acute{e}t} A \\ \downarrow & & \downarrow{\scriptstyle v} \\ \mathsf{U}_{i-1} & \longrightarrow & \mathsf{U}_i \end{array}$$

for some connective $\mathbb{E}_\infty$-ring A. Let $V = |\,\mathsf{V}\,|$, so that the open immersion j allows us to identify V with a quasi-compact open subset of $|\operatorname{Spec} A|$. Then v induces a continuous map $\phi : |\operatorname{Spec} A| \to U_i$, which restricts to a homeomorphism from $|\operatorname{Spec} A| - V$ to $U_i - U_{i-1}$. It follows that

$$K \cap U_i = (K \cap U_{i-1}) \cup \phi(\phi^{-1}(K \cap U_i) - V)$$

is constructible if and only if $\phi^{-1}(K \cap U_i) - V$ is a constructible subset of $|\operatorname{Spec} A|$. To prove this, it suffices to show that $\phi^{-1}K \subseteq |\operatorname{Spec} A|$ is constructible. Replacing X by $\operatorname{Sp\acute{e}t} A$, Y by $\operatorname{Sp\acute{e}t} A \times_{\mathsf{X}} \mathsf{Y}$, and K by $\phi^{-1}K$, we may reduce to the case where $\mathsf{X} = \operatorname{Sp\acute{e}t} A$ is affine.

Since Y is quasi-compact, we can choose an étale surjection $g : \operatorname{Sp\acute{e}t} B \to \mathsf{Y}$. Since $f^{-1}K$ is a constructible subset of $|\,\mathsf{Y}\,|$, Proposition 3.6.3.3 implies that $L = g^{-1}(f^{-1}K)$ is a constructible subset of $|\operatorname{Spec} B|$. Since f and g are surjective, we can identify K with the image of L under the continuous map $|\operatorname{Spec} B| \to |\operatorname{Spec} A|$. Our assumption that $\tau_{\leqslant 0}\,\mathsf{Y}$ is locally finitely 0-presented over X guarantees that $\pi_0 B$ is finitely presented as a commutative algebra over $\pi_0 A$, so that the constructibility of K follows from Theorem 4.3.3.1. □

Using Proposition 4.3.4.1, we immediately deduce the following "global" version of Theorem 4.3.3.1:

Corollary 4.3.4.2. *Let $f : \mathsf{Y} \to \mathsf{X}$ be a map of quasi-compact, quasi-separated algebraic spaces which is locally of finite generation to order* 1*. Then the induced map $\phi : |\,\mathsf{Y}\,| \to |\,\mathsf{X}\,|$ carries constructible subsets of $|\,\mathsf{Y}\,|$ to constructible subsets of $|\,\mathsf{X}\,|$.*

Proof. Let $K \subseteq |\,\mathsf{Y}\,|$ be constructible; we wish to show that $\phi(K)$ is a constructible subset of $|\,\mathsf{X}\,|$. Since X is quasi-compact and quasi-separated, we can choose étale surjections

$$\operatorname{Sp\acute{e}t} A \to \mathsf{X} \qquad \operatorname{Sp\acute{e}t} B \to \operatorname{Sp\acute{e}t} A \times_{\mathsf{X}} \mathsf{Y},$$

which determine a commutative diagram of topological spaces

$$\begin{array}{ccc} |\operatorname{Spec} B| & \xrightarrow{\phi'} & |\operatorname{Spec} A| \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} \\ |\mathsf{Y}| & \xrightarrow{\phi} & |\mathsf{X}|. \end{array}$$

Our assumption on f implies that $\pi_0 B$ is finitely presented as a commutative algebra over $\pi_0 A$, so that Theorem 4.3.3.1 guarantees that $\psi'(\psi'^{-1}K)$ is a constructible subset of $|\operatorname{Spec} A|$. Corollary 3.6.3.2 supplies an equality $\psi^{-1}\phi(K) = \phi'(\psi'^{-1}K)$, so that $\psi^{-1}\phi(K)$ is constructible. It follows from Proposition 4.3.4.1 that $\phi(K) \subseteq |\mathsf{X}|$ is constructible. □

Corollary 4.3.4.3. *Let $f : \mathsf{Y} \to \mathsf{X}$ be a morphism of quasi-compact, quasi-separated spectral algebraic spaces. Suppose that f is flat and exhibits $\tau_{\leqslant 0}\mathsf{Y}$ as locally finitely 0-presented over X. Then the induced map of topological spaces $\phi : |\mathsf{Y}| \to |\mathsf{X}|$ is open.*

Proof. Let $U \subseteq |\mathsf{Y}|$ be an open subset of $|\mathsf{Y}|$; we wish to prove that $\phi(U) \subseteq |\mathsf{X}|$ is open. Writing U as a union of quasi-compact open subsets of $|\mathsf{Y}|$, we may assume that U is quasi-compact. Then U determines a quasi-compact open substack $\mathsf{U} \hookrightarrow \mathsf{Y}$. Replacing Y by U, we are reduced to proving that the map ϕ has open image.

Choose an étale surjection $\operatorname{Spét} A \to \mathsf{X}$, which induces a continuous map $\psi : |\operatorname{Spec} A| \to |\mathsf{X}|$. It follows from Proposition 3.6.3.6 that ψ is a quotient map. It will therefore suffice to show that $\psi^{-1}\phi(|\mathsf{Y}|)$ is an open subset of $|\operatorname{Spec} A|$. Using Corollary 3.6.3.2, we can replace X by $\operatorname{Spét} A$ and Y by $\operatorname{Spét} A \times_{\mathsf{X}} \mathsf{Y}$, and thereby reduce to the case where X is affine. Choosing an étale surjection $\operatorname{Spét} B \to \mathsf{Y}$, we are reduced to proving that the map $\nu : |\operatorname{Spec} B| \to |\operatorname{Spec} A|$ has open image. Since $\pi_0 B$ is finitely presented as an algebra over $\pi_0 A$, the image of ν is a constructible subset of $|\operatorname{Spec} A|$ (Theorem 4.3.3.1). It will therefore suffice to show that the image of ν is stable under generalization (Proposition 4.3.2.1). This follows from Theorem B.2.2.1, since $\pi_0 B$ is flat over $\pi_0 A$. □

Corollary 4.3.4.4. *Let $f : \mathsf{Y} \to \mathsf{X}$ be an étale morphism between quasi-compact, quasi-separated spectral algebraic spaces. Then f has the going down property: that is, if $y \in |\mathsf{Y}|$ is a point for which $f(y) \in |\mathsf{X}|$ lies in the closure of of some point $x \in |\mathsf{X}|$, then we can write $x = f(\overline{x})$ for some point $\overline{x} \in |\mathsf{Y}|$ for which x lies in the closure of $\{\overline{x}\}$.*

Proof. Let K be the closure of $\{x\}$ in $|\mathsf{X}|$. Then $f^{-1}K \subseteq |\mathsf{Y}|$ is a closed set containing $f^{-1}\{x\}$. Since f is an open map (Corollary 4.3.4.3), any open subset $U \subseteq |\mathsf{Y}|$ which intersects $f^{-1}K$ has the property that K intersects $f(U)$, so that $x \in f(U)$ and therefore $f^{-1}\{x\}$ intersects U. It follows that $f^{-1}K$ is the closure of the set $f^{-1}\{x\}$. Since f is étale and quasi-compact, the set $f^{-1}\{x\}$ is finite. Consequently, $y \in K$ belongs to the closure of some point $\overline{x} \in f^{-1}\{x\}$, as desired. □

4.3.5 Constructible Subsets of Inverse Limits

We conclude this section with a few observations about the behavior of the constructible topology with respect to direct limits.

Proposition 4.3.5.1. *For every commutative ring R, let $\mathcal{U}(R)$ denote the distributive lattice of quasi-compact open subsets of the Zariski spectrum $|\operatorname{Spec} R|$. Then the functor $R \mapsto \mathcal{U}(R)$ commutes with filtered colimits.*

Proof. The partially ordered set of *all* open subsets of $|\operatorname{Spec} R|$ is isomorphic to the partially ordered set of radical ideals $I \subseteq R$. Under this isomorphism, $\mathcal{U}(R)$ corresponds to the collection of radical ideals I such that $I = \sqrt{J}$ for some finitely generated ideal $J \subseteq R$.

Let $\{R_\alpha\}_{\alpha \in A}$ be a diagram of commutative rings indexed by a filtered partially ordered set A, and let R be a colimit of this diagram. We wish to show that the canonical map $\phi : \varinjlim \mathcal{U}(R_\alpha) \to \mathcal{U}(R)$ is surjective. The surjectivity of ϕ follows from the observation that every finitely generated ideal $J \subseteq R$ has the form $J_\alpha R$, where J_α is a finitely generated ideal in R_α for some $\alpha \in A$. To prove the injectivity, we must show that if $J, J' \subseteq R_\alpha$ are two finitely generated ideals such that JR and $J'R$ have the same radical, then JR_β and $J'R_\beta$ have the same radical for some $\beta \geqslant \alpha$. Choose generators $x_1, \ldots, x_n \in R_\alpha$ for the ideal J, and generators $y_1, \ldots, y_m \in R_\alpha$ for the ideal J'. Let $\psi : R_\alpha \to R$ be the canonical map. The equality $\sqrt{JR} = \sqrt{J'R}$ implies that there are equations of the form

$$\psi(x_i)^{c_i} = \sum_j \lambda_{i,j} \psi(y_j) \qquad \psi(y_j)^{d_j} = \sum_i \mu_{i,j} \psi(x_i)$$

in the commutative ring R, where c_i and d_j are positive integers. Choose $\beta \geqslant \alpha$ such that the coefficients $\lambda_{i,j}$ and $\mu_{i,j}$ can be lifted to elements $\overline{\lambda}_{i,j}, \overline{\mu}_{i,j} \in R_\beta$. Let $\psi_\beta : R_\alpha \to R_\beta$ be the canonical map. Enlarging β if necessary, we may assume that the equations

$$\psi_\beta(x_i)^{c_i} = \sum_j \overline{\lambda}_{i,j} \psi_\beta(y_j) \qquad \psi_\beta(y_j)^{d_j} = \sum_i \overline{\mu}_{i,j} \psi_\beta(x_i)$$

hold in the commutative ring R_β, so that $\sqrt{JR_\beta} = \sqrt{J'R_\beta}$ as desired. □

Corollary 4.3.5.2. *For every commutative ring R, let $\mathcal{B}(R)$ denote the Boolean algebra consisting of constructible subsets of $|\operatorname{Spec} R|$. Then the functor $R \mapsto \mathcal{B}(R)$ commutes with filtered colimits.*

Proof. Let $R \mapsto \mathcal{U}(R)$ be the functor of Proposition 4.3.5.1. Using Proposition 4.3.1.4, we see that $\mathcal{B}$ is given by the composition $\mathrm{CAlg}^{\heartsuit} \xrightarrow{\mathcal{U}} \mathrm{Lat} \xrightarrow{U} \mathrm{BAlg}$, where U is as in Proposition 4.3.1.1. The functor U commutes with all colimits (since it is a left adjoint), and the functor $\mathcal{U}$ commutes with filtered colimits by Proposition 4.3.5.1. □

Corollary 4.3.5.3. *Let $\{R_\alpha\}_{\alpha \in A}$ be a diagram of commutative rings having colimit R, indexed by a filtered partially ordered set A Let $\alpha \in A$ and let $K \subseteq |\operatorname{Spec} R_\alpha|$ be a constructible subset. Suppose that the inverse image of K in $|\operatorname{Spec} R|$ is empty. Then there exists $\beta \geqslant \alpha$ in A such that the inverse image of K in $|\operatorname{Spec} R_\beta|$ is empty.*

Corollary 4.3.5.4. *Let A_0 be a commutative ring, and let $\{A_\alpha\}$ be a filtered diagram of commutative A_0-algebras having colimit A. Suppose that B_0 is an A_0-algebra of finite presentation, set $B = \pi_0(B_0 \otimes_{A_0} A)$, and assume that the map $|\operatorname{Spec} B| \to |\operatorname{Spec} A|$ is surjective. Then there exists an index α such that the map $|\operatorname{Spec} B_\alpha| \to |\operatorname{Spec} A_\alpha|$ is surjective, where $B_\alpha = \pi_0(B_0 \otimes_{A_0} A_\alpha)$.*

Proof. Combine Theorem 4.3.3.1 with Corollary 4.3.5.3. □

Proposition 4.3.5.5. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be a quasi-compact quasi-separated spectral algebraic space over R. For every map of connective $\mathbb{E}_\infty$-rings $R \to R'$, let $\mathsf{X}_{R'} = \operatorname{Spét} R' \times_{\operatorname{Spét} R} \mathsf{X}$, and let $\mathcal{U}_{\mathsf{X}}(R')$ denote the distributive lattice of quasi-compact open subsets of $|\mathsf{X}_{R'}|$. Then:*

(1) *The functor $R' \mapsto \mathcal{U}_{\mathsf{X}}(R')$ commutes with filtered colimits.*

(2) *The functor $R' \mapsto |\mathsf{X}_{R'}|$ carries filtered colimits of R-algebras to filtered limits of topological spaces.*

Proof. By virtue of Remark A.1.5.12 and Corollary 3.6.3.4, assertion (2) follows from (1). We now prove (1). Since X is quasi-compact, we can choose an étale surjection $\operatorname{Spét} A^0 \to \mathsf{X}$. Since X is quasi-separated, we can choose an étale surjection $\operatorname{Spét} A^1 \to \operatorname{Spét} A^0 \times_{\mathsf{X}} \operatorname{Spét} A^0$. For every commutative ring B, let $\mathcal{U}(B)$ be defined as in Proposition 4.3.5.1. Then for $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$, we have an equalizer diagram of sets

$$\mathcal{U}_{\mathsf{X}}(R') \longrightarrow \mathcal{U}(\pi_0(R' \otimes_R A^0)) \rightrightarrows \mathcal{U}(\pi_0(R' \otimes_R A^1)).$$

Since $\mathcal{U}$ commutes with filtered colimits, we conclude that $\mathcal{U}_{\mathsf{X}}$ commutes with filtered colimits. □

4.4 Noetherian Approximation

Let X be a scheme of finite presentation over a commutative ring R. Then there exists a finitely generated subring $R_0 \subseteq R$, an R_0-scheme X_0 of finite presentation, and an isomorphism of schemes $X \simeq \operatorname{Spec} R \times_{\operatorname{Spec} R_0} X_0$. This observation is the basis of a technique called *Noetherian approximation*: one can often reduce questions about the scheme X to questions about the scheme X_0, which may be easier to answer because X_0 is Noetherian.

Our goal in this section is to adapt the technique of Noetherian approximation to the setting of spectral algebraic geometry. More specifically, we would like to address questions like the following:

Question 4.4.0.1. Let X be a spectral Deligne-Mumford stack over a connective $\mathbb{E}_\infty$-ring R, and suppose that R is given as a filtered colimit $\varinjlim R_\alpha$ of connective $\mathbb{E}_\infty$-rings. Can we find an index α, a spectral Deligne-Mumford stack X_α over R_α, and an equivalence of spectral Deligne-Mumford stacks $\mathsf{X} \simeq \operatorname{Spét} R \times_{\operatorname{Spét} R_\alpha} \mathsf{X}_\alpha$?

To have a chance at an affirmative answer, we need to make some finiteness assumptions on X. However, even with finiteness assumptions in place, Question 4.4.0.1 is more subtle than its classical analogue. The main issue is that the data of a spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is infinitary in nature, because the structure sheaf $\mathcal{O}_{\mathcal{X}}$ may have an infinite number of nonzero homotopy sheaves $\pi_m \mathcal{O}_{\mathcal{X}}$. When looking for "approximations" to X, we can generally only control finitely many of these homotopy groups at one time. We can attempt to avoid the issue by studying *truncations* of X. Recall that for each $n \geqslant 0$, $\tau_{\leqslant n} \mathsf{X}$ denotes the spectral Deligne-Mumford stack $(\mathcal{X}, \tau_{\leqslant n} \mathcal{O}_{\mathcal{X}})$. A more reasonable version of Question 4.4.0.1 is the following:

Question 4.4.0.2. Let X be a spectral Deligne-Mumford stack over a connective $\mathbb{E}_\infty$-ring R. Suppose that R is given as a filtered colimit $\varinjlim R_\alpha$, and let $n \geqslant 0$ be an integer. Can we find an index α, a spectral Deligne-Mumford stack X_α over R_α, and an equivalence of spectral Deligne-Mumford stacks $\tau_{\leqslant n} \mathsf{X} \simeq \tau_{\leqslant n}(\operatorname{Spét} R \times_{\operatorname{Spét} R_\alpha} \mathsf{X}_\alpha)$?

Our main result (Theorem 4.4.2.2) supplies an affirmative answer in the case where X is finitely n-presented over R, in the sense of Definition 4.2.5.1. Moreover, in this case, we can assume that X_α is finitely n-presented over R_α, in which case X_α is essentially unique (more precisely, any two choices for X_α become equivalent after extending scalars along some transition map $R_\alpha \to R_\beta$ of our filtered diagram).

4.4.1 Approximation in the Affine Case

We begin by discussing the special case of Questions 4.4.0.1 and 4.4.0.2 in which X is assumed to be affine. In this case, we can formulate the following slight variant of Question 4.4.0.1:

Question 4.4.1.1. Let $\phi : R \to A$ be a morphism of connective $\mathbb{E}_\infty$-rings, and suppose that R is given as the colimit of a filtered diagram of connective $\mathbb{E}_\infty$-rings $\{R_\alpha\}$. Can we find an index α, an object $A_\alpha \in \operatorname{CAlg}^{\mathrm{cn}}_{R_\alpha}$, and an equivalence $A \simeq R \otimes_{R_\alpha} A_\alpha$?

We will address Question 4.4.1.1 using the following general categorical principle:

Proposition 4.4.1.2. *Let $\mathcal{C}$ be a compactly generated ∞-category. Then:*

(1) *For each object $C \in \mathcal{C}$, the ∞-category $\mathcal{C}_{C/}$ is compactly generated.*

(2) *Let $f : C \to D$ be a morphism in $\mathcal{C}$. Then f is compact when viewed as an object of $\mathcal{C}_{C/}$ if and only if there exists a pushout diagram*

$$\begin{array}{ccc} C_0 & \longrightarrow & D_0 \\ \downarrow & & \downarrow \\ C & \longrightarrow & D, \end{array}$$

where C_0 and D_0 are compact objects of $\mathcal{C}$.

(3) *Let $\mathcal{D}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms $f : C \to D$ which are compact when viewed as objects of $\mathcal{C}_{C/}$. Then evaluation at $\{0\} \subseteq \Delta^1$ induces a coCartesian fibration $\mathcal{D} \to \mathcal{C}$. Moreover, this coCartesian fibration is classified by a functor $\chi : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ which commutes with filtered colimits.*

Before giving the proof of Proposition 4.4.1.2, let us describe some consequences.

Corollary 4.4.1.3. *For every connective $\mathbb{E}_\infty$-ring R, let $\mathrm{CAlg}^{\mathrm{fp}}_R$ denote the full subcategory of $\mathrm{CAlg}^{\mathrm{cn}}_R$ spanned by those $\mathbb{E}_\infty$-algebras which are locally of finite presentation over R. Then the construction $R \mapsto \mathrm{CAlg}^{\mathrm{fp}}_R$ commutes with filtered colimits.*

Corollary 4.4.1.4. *For every connective $\mathbb{E}_\infty$-ring R, let CAlg^{n-fp}_R denote the full subcategory of CAlg_R spanned by those connective $\mathbb{E}_\infty$-algebras over R which are n-truncated and of finite generation to order $(n+1)$. Then the construction $R \mapsto \mathrm{CAlg}^{n-fp}_R$ determines a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$ which commutes with filtered colimits.*

Proof. Using Remark 4.1.1.9, we can identify CAlg^{n-fp}_R with the full subcategory of $\tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}_R \simeq \tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\leqslant n} R}$ spanned by the compact objects. The desired result now follows by applying Proposition 4.4.1.2 to the ∞-category $\tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}$ of n-truncated connective $\mathbb{E}_\infty$-rings (together with the observation that the truncation functor $\tau_{\leqslant n} : \mathrm{CAlg}^{\mathrm{cn}} \to \tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}$ commutes with filtered colimits). □

We will deduce Proposition 4.4.1.2 from the following:

Proposition 4.4.1.5. *Let $\mathcal{C}$ be a presentable ∞-category, and let $\chi : \mathcal{C} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ classify the coCartesian fibration $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}$ (so that $\chi(C) \simeq \mathcal{C}_{C/}$). Then χ preserves K-indexed colimits, for every small weakly contractible simplicial set K.*

Proof. The forgetful functor $q : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}$ as a Cartesian fibration which is classified by a map $\chi' : \mathcal{C}^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{R}}$. Let K be a small simplicial set which is weakly contractible. To show that χ preserves K-indexed colimits, it will suffice to show that χ' preserves K^{op}-indexed limits (Corollary HTT.5.5.3.4). For this purpose, it does not matter whether we regard χ' as a functor taking values in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ or in the larger ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$ (Theorem HTT.5.5.3.18). Without loss of generality, we may suppose that K is an ∞-category. Choose a colimit diagram $p : K^{\triangleright} \to \mathcal{C}$. We will show that the induced map

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}} K^{\triangleright} \to K^{\triangleright}$$

satisfies the dual version of the hypotheses of Proposition HA.5.2.2.36:

(*a*) Let v denote the cone point of $K^{\triangleright}$. We must show that the collection of forgetful functors $\mathcal{C}_{q(v)/} \to \mathcal{C}_{q(k)/}$ is jointly conservative (here k varies over the collection of all vertices of k). This is clear, since K is nonempty and each of these functors is individually conservative.

(*b*) Let $\overline{p}_0 : K \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ be a functor which carries each edge of K to a q-Cartesian morphism in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ and satisfies $q \circ \overline{p}_0 = p|_K$. Since q is a presentable fibration, we can extend $\overline{p}_0$ to a q-colimit diagram $\overline{p}$ with $q \circ \overline{p} = p$. We claim that $\overline{p}$ carries every morphism in $K^{\triangleleft}$ to a q-Cartesian morphism in $\mathrm{Fun}(\Delta^1, \mathcal{C})$. To prove this, let us identify $\overline{p}$ with a map $P : K^{\triangleright} \times \Delta^1 \to \mathcal{C}$, so that $p = P|_{K^{\triangleright} \times \{0\}}$. Let L denote the full subcategory of $K^{\triangleright} \times \Delta^1$ obtained by removing the final vertex. Since p is a colimit diagram, $P|_L$ is a left Kan extension of $P|_{K \times \Delta^1}$. Because $\overline{p}$ is a q-colimit diagram, P is a colimit diagram. Using Lemma HTT.4.3.2.7, we conclude that P exhibits $P(v, 1)$ as a colimit of $P|_{K \times \Delta^1}$. Because the inclusion $K \times \{1\} \hookrightarrow K \times \Delta^1$ is left cofinal, this implies that $P|_{K^{\triangleright} \times \{1\}}$ is a colimit diagram. Since $\overline{p}$ carries each morphism in K to a q-Cartesian morphism in $\mathrm{Fun}(\Delta^1, \mathcal{C})$, P carries each morphism in $K \times \{1\}$ to an equivalence in $\mathcal{C}$. Applying Corollary HTT.4.4.4.10 (and the weak contractibility of K), we conclude that P carries each morphism in $K^{\triangleright} \times \{1\}$ to an equivalence in $\mathcal{C}$, so that $\overline{p}$ carries each morphism in $K^{\triangleleft}$ to a q-Cartesian morphism in $\mathcal{C}$.

□

Proof of Proposition 4.4.1.2. First note that if $f_0 : C_0 \to D_0$ is a morphism between compact objects of $\mathcal{C}$, then f_0 is compact when viewed as an object of $\mathcal{C}_{C_0/}$, since the functor corepresented by f is given by

$$E \mapsto \mathrm{fib}(\mathrm{Map}_{\mathcal{C}}(D_0, E) \to \mathrm{Map}_{\mathcal{C}}(C_0, E)).$$

If $g : C_0 \to C$ is any map between objects of $\mathcal{C}$, then composition with g determines a functor $\mathcal{C}_{C/} \to \mathcal{C}_{C_0/}$ which preserves filtered colimits (Proposition HTT.4.4.2.9), so that the construction $D_0 \mapsto C \amalg_{C_0} D_0$ preserves compact objects (Proposition HTT.5.5.7.2). This proves the "if" direction of (2).

We now prove (1). Fix an object $C \in \mathcal{C}$, and let $\mathcal{E} \subseteq \mathcal{C}_{C/}$ denote the full subcategory spanned by the compact objects. Then the inclusion $\mathcal{E} \hookrightarrow \mathcal{C}_{C/}$ extends to a fully faithful embedding $F : \mathrm{Ind}(\mathcal{E}) \hookrightarrow \mathcal{C}_{C/}$ which preserves small colimits (Proposition HTT.5.5.1.9). To prove (1), it will suffice to show that F is an equivalence of ∞-categories. By virtue of Corollary HTT.5.5.2.9 , the functor F admits a right adjoint G. We are therefore reduced to proving that G is conservative. Let $u : D \to E$ be a morphism in $\mathcal{C}_{C/}$ such that $G(u)$ is an equivalence. Then u inducs a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}_{C/}}(X, D) \to \mathrm{Map}_{\mathcal{C}_{C/}}(X, E)$$

for every compact object $X \in \mathcal{C}_{C/}$. In particular, if X_0 is a compact object of $\mathcal{C}$, then we can take X to be the coproduct $C \amalg X_0$ to deduce that u induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(X_0, D) \to \mathrm{Map}_{\mathcal{C}}(X_0, E)$. Since $\mathcal{C}$ is compactly generated, this proves that u is an equivalence.

Assertion (3) follows from (1), Proposition 4.4.1.5, and Lemma HA.7.3.5.11 . We complete the proof by verifying the "only if" direction of (2). Let $f : C \to D$ be a morphism in $\mathcal{C}$ which is compact as an object of $\mathcal{C}_{C/}$; we wish to prove the existence of a pushout diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{f_0} & D_0 \\ \downarrow & & \downarrow \\ C & \xrightarrow{f} & D \end{array}$$

where C_0 and D_0 are compact. Since $\mathcal{C}$ is compactly generated, we can write C as the colimit of a filtered diagram $\{C_\alpha\}$ of compact objects of $\mathcal{C}$. Using (3), we see that there exists an index α and a pushout diagram

$$\begin{array}{ccc} C_\alpha & \xrightarrow{f_\alpha} & D_\alpha \\ \downarrow & & \downarrow \\ C & \xrightarrow{f} & D \end{array}$$

where f_α is compact when viewed as an object of $\mathcal{C}_{C_\alpha/}$. We will complete the proof by showing that D_α is a compact object of $\mathcal{C}$. To prove this, suppose we are given a filtered diagram $\{E_\beta\}$ in $\mathcal{C}$, having colimit E. We wish to show that the upper horizontal map in the diagram

$$\begin{array}{ccc} \varinjlim \mathrm{Map}_{\mathcal{C}}(D_\alpha, E_\beta) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(D_\alpha, E) \\ \downarrow & & \downarrow \\ \varinjlim_{\mathcal{C}}(C_\alpha, E_\beta) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(C_\alpha, E) \end{array}$$

is a homotopy equivalence. Since C_α is a compact object of $\mathcal{C}$, the lower horizontal map is a homotopy equivalence. We are therefore reduced to proving that the upper horizontal map induces a homotopy equivalence after passing to the homotopy fiber over any point of the direct limit $\varinjlim_{\mathcal{C}}(C_\alpha, E_\beta)$, which follows from our assumption that f_α is compact as an object of $\mathcal{C}_{C_\alpha/}$. $\square$

4.4.2 Approximation in the General Case

We now formulate an answer to Question 4.4.0.2 in the case where X is not assumed to be affine.

Construction 4.4.2.1. Let R be a connective $\mathbb{E}_\infty$-ring. We let $\mathrm{DM}_n^{\mathrm{fp}}(R)$ denote the full subcategory of $\mathrm{SpDM}_{/\mathrm{Sp\acute{e}t}\, R}$ spanned by those maps $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ which exhibit X as finitely n-presented over R. We will regard the construction $R \mapsto \mathrm{DM}_n^{\mathrm{fp}}(R)$ as a functor from $\mathrm{CAlg}^{\mathrm{cn}}$ to the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$. More precisely, we let $\mathrm{DM}_n^{\mathrm{fp}}$ be a functor which classifies the Cartesian fibration $p : \mathcal{C} \to (\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}}$, where $\mathcal{C}$ denotes the full subcategory of

$$\mathrm{Fun}(\Delta^1, \mathrm{SpDM}) \times_{\mathrm{Fun}(\{1\}, \mathrm{SpDM})} (\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}}$$

spanned by those morphisms $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ where R is a connective $\mathbb{E}_\infty$-ring and X is a spectral Deligne-Mumford stack which is finitely n-presented over $\mathrm{Sp\acute{e}t}\, R$ (it follows from Proposition 4.2.5.4 that p is indeed a Cartesian fibration). To every morphism $f : A \to B$ of connective $\mathbb{E}_\infty$-rings, the functor $\mathrm{DM}_n^{\mathrm{fp}}$ associates the map $\mathrm{DM}_n^{\mathrm{fp}}(A) \to \mathrm{DM}_n^{\mathrm{fp}}(B)$ given on objects by the formula $\mathsf{X} \mapsto \tau_{\leqslant n}(\mathrm{Sp\acute{e}t}\, B \times_{\mathrm{Sp\acute{e}t}\, A} \mathsf{X})$.

Our main result can be stated as follows:

Theorem 4.4.2.2. *Let $n \geqslant 0$ be an integer, and let $\mathrm{DM}_n^{\mathrm{fp}} : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Construction 4.4.2.1. Then:*

(1) *For every connective $\mathbb{E}_\infty$-ring R, the ∞-category $\mathrm{DM}_n^{\mathrm{fp}}(R)$ is essentially small.*

(2) *The functor $\mathrm{DM}_n^{\mathrm{fp}}$ commutes with small filtered colimits.*

Before embarking on the proof of Theorem 4.4.2.2, let us describe one of its consequences. Fix a connective $\mathbb{E}_\infty$-ring R and finitely n-presented morphism $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$. For any n-truncated R-algebra A, we have a canonical homotopy equivalence

$$\begin{aligned} \mathrm{Map}_{\mathrm{DM}_n^{\mathrm{fp}}(A)}(\mathrm{Sp\acute{e}t}\, A, \tau_{\leqslant n}(\mathrm{Sp\acute{e}t}\, A \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X})) &\simeq \mathrm{Map}_{\mathrm{SpDM}_{/\mathrm{Sp\acute{e}t}\, A}}(\mathrm{Sp\acute{e}t}\, A, \mathrm{Sp\acute{e}t}\, A \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}) \\ &\simeq \mathrm{Map}_{\mathrm{SpDM}_{/\mathrm{Sp\acute{e}t}\, R}}(\mathrm{Sp\acute{e}t}\, A, \mathsf{X}). \end{aligned}$$

The domain and codomain of this equivalence depend functorially on A, and Theorem 4.4.2.2 implies that the domain commutes with filtered colimits (as a functor of A). We therefore deduce the following consequence:

Proposition 4.4.2.3. *Let R be a connective $\mathbb{E}_\infty$-ring, let $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ be a finitely n-presented morphism of spectral Deligne-Mumford stacks, and let $h_{\mathsf{X}} : \tau_{\leqslant n} \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by X, given by the formula $h_{\mathsf{X}}(A) = \mathrm{Map}_{\mathrm{SpDM}_{/\mathrm{Sp\acute{e}t}\, R}}(\mathrm{Sp\acute{e}t}\, A, \mathsf{X})$. Then the functor h_{X} commutes with filtered colimits.*

In what follows, we will show that the converse is true: that is, Theorem 4.4.2.2 can be deduced from Proposition 4.4.2.3. We will give an independent proof of Proposition 4.4.2.3 in §17.4. In fact, we will prove a stronger statement: the conclusion of Proposition 4.4.2.3 is valid provided that f is locally of finite generation to order $(n+1)$ and X is a Deligne-Mumford m-stack for $m \gg 0$ (in other words, we need not assume that X is n-truncated or ∞-quasi-compact); see Proposition 17.4.3.1.

Remark 4.4.2.4. In the special case where X is affine, Proposition 4.4.2.3 follows immediately from Remarks 4.1.1.9 and 4.2.1.3.

4.4.3 Deduction of Theorem 4.4.2.2 from Proposition 4.4.2.3

Let us say that a map $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford is a *quasi-monomorphism* if, for every commutative ring A, the induced map

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, A, \mathsf{X}) \to \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, A, \mathsf{Y})$$

is (-1)-truncated. For $m \geqslant 0$ and any connective $\mathbb{E}_\infty$-ring R, let $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ denote the full subcategory of $\mathrm{DM}^{\mathrm{fp}}_n(R)$ spanned by those maps $f : \mathsf{X} \to \mathrm{Spec}\, R$ which are finitely n-presented, where X is a spectral Deligne-Mumford m-stack. Let $\mathrm{DM}^{\mathrm{fp}}_{n,-1}(R) \subseteq \mathrm{DM}^{\mathrm{fp}}_n(R)$ be the full subcategory spanned by those maps $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ which fit into a commutative diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{u} & \mathrm{Sp\acute{e}t}\, A \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & \mathrm{Sp\acute{e}t}\, R & \end{array}$$

where u is a quasi-monomorphism and A is finitely n-presented over R. Let $\mathrm{DM}^{\mathrm{fp}}_{n,-2}(R) \subseteq \mathrm{DM}^{\mathrm{fp}}_n(R)$ denote the full subcategory spanned by those maps $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ where X is affine. Note that if $R \to R'$ is a map of connective $\mathbb{E}_\infty$-rings, then the associated base-change functor $\mathrm{DM}^{\mathrm{fp}}_n(R) \to \mathrm{DM}^{\mathrm{fp}}_n(R')$ carries $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ into $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R')$, for each $m \geqslant -2$. Consequently, we have functors $\mathrm{DM}^{\mathrm{fp}}_{n,m} : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ for each $m \geqslant -2$, and $\mathrm{DM}^{\mathrm{fp}}_n \simeq \varinjlim_m \mathrm{DM}^{\mathrm{fp}}_{n,m}$. Theorem 4.4.2.2 is therefore an immediate consequence of the following:

Proposition 4.4.3.1. *Let $n \geqslant 0$ and $m \geqslant -2$ be integers. Then:*

(1) *For every connective $\mathbb{E}_\infty$-ring R, the ∞-category $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ is essentially small.*

(2) *The functor $R \mapsto \mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ commutes with filtered colimits.*

Proof. We proceed by induction on m. In the case $m = -2$, the desired result follows from Corollary 4.4.1.4. Suppose therefore that $m > -2$, and let $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ be an object of

$\mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ for some connective $\mathbb{E}_\infty$-ring R. Then X is quasi-compact, so we can choose an étale surjection $u : \mathsf{X}_0 \to \mathsf{X}$ where X_0 is affine. Let $\mathsf{X}_\bullet$ denote the Čech nerve of u in the ∞-category of n-truncated spectral Deligne-Mumford stacks. We claim that each X_i belongs to $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R)$. For $m > 0$ this is clear. When $m = 0$, we let Y denote n-truncation of the i-fold fiber power of X_0 over $\mathrm{Spét}\, R$. Then Y is affine, and the canonical map $\mathsf{X}_i \to \mathsf{Y}$ is a quasi-monomorphism. If $m = -1$, we can choose a commutative diagram

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{v} & \mathrm{Spét}\, A \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & \mathrm{Spét}\, R & \end{array}$$

where v is a quasi-monomorphism and A is finitely n-presented over R. Let $\mathsf{X}'_\bullet$ be the Čech nerve of the composite map $(v \circ u) : \mathsf{X}_0 \to \mathrm{Spét}\, A$. Since v is a quasi-monomorphism, the induced map $\mathsf{X}_\bullet \to \mathsf{X}'_\bullet$ induces an equivalence of 0-truncations. Since each X'_i is affine, each X_i is affine by Theorem 1.4.8.1.

We now prove (1). Fix a connective $\mathbb{E}_\infty$-ring R. Since $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R)$ is essentially small by the inductive hypothesis, the ∞-category of simplicial objects of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R)$ is also essentially small. The above argument shows that every object of $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ can be obtained as the geometric realization (in $\mathrm{SpDM}_{/\,\mathrm{Spét}\, R}$) of a simplicial object $\mathsf{X}_\bullet$ of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R)$, so that $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ is essentially small.

We now prove (2). Choose a filtered diagram of connective $\mathbb{E}_\infty$-rings $\{R_\alpha\}$ having colimit R, and consider the functor $\theta : \varinjlim_\alpha \mathrm{DM}^{\mathrm{fp}}_{n,m}(R_\alpha) \to \mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$. We first show that θ is fully faithful. We may assume without loss of generality that the diagram $\{R_\alpha\}$ is indexed by the nerve of a filtered partially ordered set P (Proposition HTT.5.3.1.18). Fix objects $\mathsf{X}_\alpha, \mathsf{Y}_\alpha \in \mathrm{DM}^{\mathrm{fp}}_{n,m}(R_\alpha)$. For $\beta \geqslant \alpha$, let X_β and Y_β denote the images of X_α and Y_α in $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R_\beta)$, and let X and Y denote the images of X_α and Y_α in $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$. We wish to show that the canonical map

$$\rho : \varinjlim_{\beta \geqslant \alpha} \mathrm{Map}_{\mathrm{SpDM}_{/\,\mathrm{Spét}\, R_\beta}}(\mathsf{X}_\beta, \mathsf{Y}_\beta) \to \mathrm{Map}_{\mathrm{SpDM}_{/\,\mathrm{Spét}\, R}}(\mathsf{X}, \mathsf{Y})$$

is a homotopy equivalence. Unwinding the definitions, we see that ρ can be identified with the canonical map

$$\varinjlim_{\beta \geqslant \alpha} \mathrm{Map}_{\mathrm{SpDM}_{/\,\mathrm{Spét}\, R_\alpha}}(\mathsf{X}_\beta, \mathsf{Y}_\alpha) \to \mathrm{Map}_{\mathrm{SpDM}_{/\,\mathrm{Spét}\, R_\alpha}}(\mathsf{X}, \mathsf{Y}_\alpha)$$

In what follows, let us regard the object Y_α as fixed; we wish to prove the following:

$(*)$ For every object $\mathsf{X}_\alpha \in \mathrm{DM}^{\mathrm{fp}}_{n,m'}(R_\alpha)$, the canonical map

$$\varinjlim_{\beta \geqslant \alpha} \mathrm{Map}_{\mathrm{SpDM}_{/\,\mathrm{Spét}\, R_\alpha}}(\mathsf{X}_\beta, \mathsf{Y}_\alpha) \to \mathrm{Map}_{\mathrm{SpDM}_{/\,\mathrm{Spét}\, R_\alpha}}(\mathsf{X}, \mathsf{Y}_\alpha)$$

is a homotopy equivalence.

The proof of $(*)$ proceeds by induction on m'. Suppose first that $m' = -2$, so that $\mathsf{X}_\alpha \simeq \operatorname{Sp\acute{e}t} A_\alpha$ for some connective $\mathbb{E}_\infty$-algebra A_α over R_α. Unwinding the definitions, we see that $\mathsf{X}_\beta \simeq \tau_{\leqslant n}(A_\alpha \otimes_{R_\alpha} R_\beta)$ and that $\mathsf{X} \simeq \tau_{\leqslant n}(A_\alpha \otimes_{R_\alpha} R)$. Since $R \simeq \varinjlim R_\beta$, we conclude that the canonical map

$$\varinjlim_{\beta \geqslant \alpha} \tau_{\leqslant n}(A_\alpha \otimes_{R_\alpha} R_\beta) \to \tau_{\leqslant n}(A_\alpha \otimes_{R_\alpha} R)$$

is an equivalence. Assertion $(*)$ now follows from Proposition 4.4.2.3.

Now suppose that $m' > -2$. Choose an étale surjection $u : \mathsf{X}_{\alpha,0} \to \mathsf{X}_\alpha$ where $\mathsf{X}_{\alpha,0}$ is affine, and let $\mathsf{X}_{\alpha,\bullet}$ be the Čech nerve of u. Define $\mathsf{X}_{\beta,\bullet}$ and $\mathsf{X}_\bullet$ as above. We wish to show that the canonical map

$$\phi : \varinjlim_{\beta \geqslant \alpha} \varprojlim_{[p] \in \mathbf{\Delta}} \operatorname{Map}_{\mathrm{SpDM}_{/ \operatorname{Sp\acute{e}t} R_\alpha}}(\mathsf{X}_{\beta,p}, \mathsf{Y}_\alpha) \to \varprojlim_{[p] \in \mathbf{\Delta}} \operatorname{Map}_{\mathrm{SpDM}_{/ \operatorname{Sp\acute{e}t} R}}(\mathsf{X}_p, \mathsf{Y}_\alpha)$$

is a homotopy equivalence. Choose an integer $k \geqslant m, n$, so all of the mapping spaces above are k-truncated (Lemma 1.6.8.8). Arguing as in the proof of Lemma HA.1.3.3.10 , we can identify ϕ with the map

$$\varinjlim_{\beta \geqslant \alpha} \varprojlim_{[p] \in \mathbf{\Delta}_{s, \leqslant k+1}} \operatorname{Map}_{\mathrm{SpDM}_{/ \operatorname{Sp\acute{e}t} R}}(\mathsf{X}_{\beta,p}, \mathsf{Y}) \to \varprojlim_{[p] \in \mathbf{\Delta}_{s, \leqslant k+1}} \operatorname{Map}_{\mathrm{SpDM}_{/ \operatorname{Sp\acute{e}t} R}}(\mathsf{X}_p, \mathsf{Y}).$$

Since filtered colimits in $\mathcal{S}$ commute with finite limits, ϕ is a finite limit of morphisms

$$\phi_p : \varinjlim_{\beta \geqslant \alpha} \operatorname{Map}_{\mathrm{SpDM}_{/ \operatorname{Sp\acute{e}t} R}}(\mathsf{X}_{\beta,p}, \mathsf{Y}) \to \operatorname{Map}_{\mathrm{SpDM}_{/ \operatorname{Sp\acute{e}t} R}}(\mathsf{X}_p, \mathsf{Y}).$$

Since $\mathsf{X}_{\alpha,p} \in \mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R_\alpha)$, the map ϕ_p is an equivalence by the inductive hypothesis. This completes the proof that θ is fully faithful.

It remains to prove that θ is essentially surjective. Fix an object $\mathsf{X} \in \mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ and choose an étale surjection $u : \mathsf{X}_0 \to \mathsf{X}$, where X_0 is affine. Let $\mathsf{X}_\bullet$ be the Čech nerve of u. Choose $k \geqslant m, n$, so that the ∞-category $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R)$ is equivalent to a $(k+1)$-category (Lemma 1.6.8.8). It follows that $\mathsf{X}_\bullet$ is a right Kan extension of $\mathsf{X}^t_\bullet = \mathsf{X}_\bullet |_{\mathbf{\Delta}^{\mathrm{op}}_{\leqslant k+3}}$ (Proposition A.8.2.6), which is an $(k+2)$-skeletal category object of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R)$ (see Definition A.8.2.2). Since $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R) \simeq \varinjlim_\alpha \mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R_\alpha)$ by the inductive hypothesis and the simplicial set $\mathrm{N}(\mathbf{\Delta}^{\mathrm{op}}_{\leqslant k+3})$ has a finite $(k+2)$-skeleton, $\mathsf{X}^t_\bullet$ is the image of a $(k+3)$-skeletal simplicial object $\overline{\mathsf{X}}^t_\bullet$ of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R_\alpha)$ for some index α. Enlarging α if necessary, we may assume that $\overline{\mathsf{X}}^t_\bullet$ is a $(k+3)$-skeletal category object of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R_\alpha)$. Let $\overline{\mathsf{X}}_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{DM}^{\mathrm{fp}}_{n,m-1}$ be a right Kan extension of $\overline{\mathsf{X}}^t_\bullet$. Using Propositions 4.4.2.3 and A.8.2.7, we deduce that $\overline{\mathsf{X}}_\bullet$ is a category object of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}$. Enlarging α if necessary, we may assume that $\overline{\mathsf{X}}_\bullet$ is a groupoid object of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R_\alpha)$, and that the projection maps $\overline{\mathsf{X}}_1 \to \overline{\mathsf{X}}_0$ are étale. Then $\overline{\mathsf{X}}_\bullet$ has a geometric realization $\overline{\mathsf{X}}$ in $\mathrm{SpDM}_{/\operatorname{Sp\acute{e}t} R_\alpha}$. We will prove that, after enlarging α if necessary, we have $\overline{\mathsf{X}} \in \mathrm{DM}^{\mathrm{fp}}_{n,m}(R_\alpha)$. It will then follow that $\overline{\mathsf{X}}$ is a preimage of $\mathsf{X} \in \mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ and the proof will be complete.

Since we have an étale surjection $\overline{\mathsf{X}}_0 \to \overline{\mathsf{X}}$, it is clear that $\overline{\mathsf{X}}$ is locally finitely n-presented over R_α. We next prove that the underlying ∞-topos $\mathcal{X}$ of $\overline{\mathsf{X}}$ is coherent. The étale map $\overline{\mathsf{X}}_0 \to \overline{\mathsf{X}}$ corresponds to an object $U \in \mathcal{X}$. Since $\overline{\mathsf{X}}_0$ and $\overline{\mathsf{X}}_1$ belong to $\mathrm{DM}^{\mathrm{fp}}_n(R_\alpha)$, U and $U \times U$ are coherent objects of $\mathcal{X}$. Example A.2.1.2 shows that the projection map $U \times U \to U$ is relatively i-coherent for every integer i. Let $\mathbf{1}$ denote the final object of $\mathcal{X}$. Since $p : U \to \mathbf{1}$ is an effective epimorphism, we deduce that p is relatively i-coherent for every integer i (Corollary A.2.1.5). Using the i-coherence of U, we deduce that $\mathcal{X}$ is i-coherent. Allowing i to vary, we deduce that $\mathcal{X}$ is coherent.

Let us now treat the case $m = -1$. Choose a commutative diagram

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{u} & \operatorname{Sp\acute{e}t} A \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & \operatorname{Sp\acute{e}t} R & \end{array}$$

where u is a quasi-monomorphism and A is finitely n-presented over R. Applying Corollary 4.4.1.4, we may assume that $A = \tau_{\leqslant n}(\overline{A} \otimes_{R_\alpha} R)$ for some n-truncated $\overline{A}$ which is finitely n-presented over R_α. It follows from $(*)$ that we may assume (after enlarging α if necessary) that u is the image of a map $\overline{u} : \overline{\mathsf{X}} \to \operatorname{Sp\acute{e}t} \overline{A}$ in $\mathrm{SpDM}_{/\operatorname{Sp\acute{e}t} R_\alpha}$. To complete the proof, it will suffice to show that $\overline{u}$ is a quasi-monomorphism. That is, we must show that the diagonal map $\overline{\mathsf{X}} \to \overline{\mathsf{X}} \times_{\operatorname{Sp\acute{e}t} \overline{A}} \overline{\mathsf{X}}$ induces an equivalence of 0-truncations. This assertion is local on $\overline{\mathsf{X}}$: it therefore suffices to show that the map $\phi : \overline{\mathsf{X}}_1 \simeq \overline{\mathsf{X}}_0 \times_{\overline{\mathsf{X}}} \overline{\mathsf{X}}_0 \to \overline{\mathsf{X}}_0 \times_{\operatorname{Spec} \overline{A}} \overline{\mathsf{X}}_0$. Note that the domain and codomain of ϕ are affine and that the image of ϕ under the natural map $\mathrm{DM}^{\mathrm{fp}}_{0,-2}(R_\alpha) \to \mathrm{DM}^{\mathrm{fp}}_{0,-2}(R)$ is an equivalence (by virtue of our assumption that u is a quasi-monomorphism). Applying Corollary 4.4.1.4 again, we deduce that there exists $\beta \geqslant \alpha$ such that the image of ϕ in $\mathrm{DM}^{\mathrm{fp}}_{0,-2}(R_\beta)$ is an equivalence. Replacing α by β, we deduce that $\overline{u}$ is a quasi-monomorphism as desired.

Assume now that $m \geqslant 0$. To complete the proof that $\overline{\mathsf{X}} \in \mathrm{DM}^{\mathrm{fp}}_{n,m}$, we must show that for every discrete commutative ring A, the mapping space $\mathrm{Map}_{\mathrm{SpDM}}(\operatorname{Sp\acute{e}t} A, \overline{\mathsf{X}})$ is m-truncated. For every étale map $A \to A'$, set

$$\mathscr{F}(A') = \mathrm{Map}_{\mathrm{SpDM}}(\operatorname{Sp\acute{e}t} A', \overline{\mathsf{X}}) \qquad \mathscr{F}_\bullet(A') = \mathrm{Map}_{\mathrm{SpDM}}(\operatorname{Sp\acute{e}t} A', \overline{\mathsf{X}}_\bullet).$$

Then $\mathscr{F}$ is a sheaf for the étale topology; we will prove that it is m-truncated. The projection $\overline{\mathsf{X}}_0 \to \overline{\mathsf{X}}$ induces an effective epimorphism of étale sheaves $\mathscr{F}_0 \to \mathscr{F}$. Since X_0 is affine, we may assume (enlarging α if necessary) that $\overline{\mathsf{X}}_0$ is affine, so that $\mathscr{F}_0$ is 0-truncated. If $m > 0$, it suffices to prove that $\mathscr{F}_1 = \mathscr{F}_0 \times_{\mathscr{F}} \mathscr{F}_0$ is $(m-1)$-truncated, which follows because $\overline{\mathsf{X}}_1 \in \mathrm{DM}^{\mathrm{fp}}_{n,m-1}$. If $m = 0$, we must work a bit harder: to show that $\mathscr{F}$ is discrete, we must show that $\mathscr{F}_1$ is an equivalence relation on $\mathscr{F}_0$ (note that each $\mathscr{F}_i$ is a *discrete* étale sheaf on A when $m \leqslant 1$). For this, it suffices to show that the diagonal map $v : \mathscr{F}_1 \to \mathscr{F}_1 \times_{\mathscr{F}_0 \times \mathscr{F}_0} \mathscr{F}_1$ is an equivalence. Consider the diagonal map $\overline{\delta} : \overline{\mathsf{X}}_1 \to \overline{\mathsf{X}}_1 \times_{\overline{\mathsf{X}}_0 \times \overline{\mathsf{X}}_0} \overline{\mathsf{X}}_1$. Since X is a spectral algebraic space, the map $\delta : \mathsf{X}_1 \to \mathsf{X}_1 \times_{\mathsf{X}_0 \times \mathsf{X}_0} \mathsf{X}_1$ induces an equivalence of 0-truncations. Our earlier argument shows that $\mathrm{DM}^{\mathrm{fp}}_{0,-1}(R) \simeq \varinjlim_{\beta \geqslant \alpha} \mathrm{DM}^{\mathrm{fp}}_{0,-1}(R_\beta)$, so we may assume (after enlarging α if necessary) that $\overline{\delta}$ also induces an equivalence of 0-truncations. It follows that v is an equivalence, as desired. □

4.4.4 Approximation without Truncation

We conclude this section by describing a useful variant of Theorem 4.4.2.2.

Proposition 4.4.4.1. *Let R be a connective $\mathbb{E}_\infty$-ring, let $f : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ be a morphism of spectral algebraic spaces, and let $n \geqslant 0$ be an integer. The following conditions are equivalent:*

(a) *The truncation $\tau_{\leqslant n} \mathsf{X}$ is finitely n-presented over R.*

(b) *There exists a commutative diagram of spectral algebraic spaces σ :*

$$\begin{array}{ccc} \mathsf{X} & \longrightarrow & \mathsf{X}_0 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} \\ \operatorname{Sp\acute{e}t} R & \longrightarrow & \operatorname{Sp\acute{e}t} R_0 \end{array}$$

where R_0 is finitely presented over the sphere spectrum S, the morphism f_0 is almost of finite presentation, and σ induces an equivalence $\tau_{\leqslant n} \mathsf{X} \to \tau_{\leqslant n}(\mathsf{X}_0 \times_{\operatorname{Sp\acute{e}t} R_0} \operatorname{Sp\acute{e}t} R)$.

Proof. The implication $(b) \Rightarrow (a)$ is obvious. For the converse, assume that (a) is satisfied. Write R as a filtered colimit $\varinjlim R_\alpha$, where each R_α is a connective $\mathbb{E}_\infty$-ring which is of finite presentation over the sphere spectrum S. Using Theorem 4.4.2.2, we deduce that there exists an index α and a diagram σ':

$$\begin{array}{ccc} \tau_{\leqslant n} \mathsf{X} & \xrightarrow{g} & \mathsf{X}_\alpha \\ \downarrow & & \downarrow{\scriptstyle f_\alpha} \\ \operatorname{Sp\acute{e}t} R & \longrightarrow & \operatorname{Sp\acute{e}t} R_\alpha \end{array}$$

with the following properties:

(a) The morphism f_α exhibits X_α as finitely n-presented over $\operatorname{Sp\acute{e}t} R_\alpha$.

(b) The diagram σ' induces an equivalence $\tau_{\leqslant n} \mathsf{X} \to \tau_{\leqslant n}(\mathsf{X}_\alpha \times_{\operatorname{Sp\acute{e}t} R_0} \operatorname{Sp\acute{e}t} R)$.

It follows from (b) and Remark 2.4.4.2 that the morphism g is affine. Set $R_0 = R_\alpha$. Since R_0 is finitely presented over the sphere spectrum S, it is Noetherian. Consequently, the morphism f_α is almost of finite presentation and X_α is locally Noetherian (see Proposition 4.2.4.1). Applying Proposition 17.1.6.1, we can extend σ' to a commutative diagram $\overline{\sigma}'$:

$$\begin{array}{ccc} \tau_{\leqslant n} \mathsf{X} & \longrightarrow & \mathsf{X}_\alpha \\ \downarrow & & \downarrow f'_\alpha \\ \mathsf{X} & \longrightarrow & \mathsf{X}_0 \\ \downarrow f & & \downarrow f''_\alpha \\ \operatorname{Sp\acute{e}t} R & \longrightarrow & \operatorname{Sp\acute{e}t} R_0 \end{array}$$

where X_0 is locally Noetherian and the morphism f'_α induces an equivalence $\tau_{\leqslant n} \mathsf{X}_\alpha \simeq \tau_{\leqslant n} \mathsf{X}_0$. Since f_α is of finite type, it follows that f''_α is of finite type. Using Remark 4.2.0.4, we deduce that f''_α is almost of finite presentation, so that the lower half of the diagram $\overline{\sigma}'$ witnesses assertion (b). □

4.5 Approximation of Quasi-Coherent Sheaves

Let X be spectral Deligne-Mumford stack which is finitely n-presented over a connective $\mathbb{E}_\infty$-ring R, and suppose that R is given as a filtered colimit of connective $\mathbb{E}_\infty$-rings $\{R_\alpha\}$. Theorem 4.4.2.2 implies that we can write X as a truncated fiber product $\tau_{\leqslant n}(\operatorname{Sp\acute{e}t} R \times_{\operatorname{Sp\acute{e}t} R_\alpha} \mathsf{X}_\alpha)$ for some index α and some spectral Deligne-Mumford stack X_α which is finitely n-presented over R_α. In this section, we will prove a linear analogue of Theorem 4.4.2.2, which relates quasi-coherent sheaves on X (satisfying suitable finiteness hypotheses) to quasi-coherent sheaves on X_α.

4.5.1 Approximation of Modules

We begin by studying the quasi-coherent sheaves on affine spectral Deligne-Mumford stacks. In this case, we wish to answer the following:

Question 4.5.1.1. Let $\{R_\alpha\}$ be a filtered diagram of connective $\mathbb{E}_\infty$-rings having colimit $R = \varinjlim R_\alpha$. What is the relationship between the ∞-category Mod_R and the diagram of ∞-categories $\{\operatorname{Mod}_{R_\alpha}\}$?

To answer Question 4.5.1.1, it is convenient to begin by studying the analogous question for $\mathbb{E}_1$-rings. Note that if M is a spectrum and R is an $\mathbb{E}_1$-ring, then giving a left action of R on M is equivalent to giving a morphism of $\mathbb{E}_1$-rings $\phi : R \to \operatorname{End}(M)$. If R is given as the colimit of a diagram $\{R_\alpha\}$ in the ∞-category Alg of $\mathbb{E}_1$-rings, then we can think of a morphism $\phi : R \to \mathrm{End}(M)$ as equivalent to supplying a compatible family of maps $\phi_\alpha : R_\alpha \to \mathrm{End}(M)$: that is, to equipping M with a compatible family of left R_α-module structures. We can formulate this idea more precisely as follows:

Proposition 4.5.1.2. *Let $\{R_\alpha\}$ be a diagram of $\mathbb{E}_1$-rings indexed by a contractible simplicial set K, having colimit R. Then the forgetful functor $\mathrm{LMod}_R \to \varprojlim\{\mathrm{LMod}_{R_\alpha}\}$ is an equivalence of ∞-categories.*

Proof. Combine Lemma HA.7.3.5.12 , Corollary HTT.5.5.3.4 , and Proposition HTT.5.5.3.18 . □

In the situation of Proposition 4.5.1.2, suppose that each R_α is connective, so that R is likewise connective. It follows that the forgetful functor $\mathrm{LMod}_R^{\mathrm{cn}} \to \varprojlim\{\mathrm{LMod}_{R_\alpha}^{\mathrm{cn}}\}$ is an equivalence of ∞-categories, and that for each $n \geqslant 0$ the forgetful functor $(\mathrm{LMod}_R)_{\leqslant n} \to \varprojlim\{(\mathrm{LMod}_{R_\alpha})_{\leqslant n}\}$ is an equivalence of ∞-categories. This immediately implies the following:

Corollary 4.5.1.3. *The construction $R \mapsto \mathrm{LMod}_R^{\mathrm{cn}}$ determines a functor $\mathrm{Alg}^{\mathrm{cn}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ which preserves colimits indexed by small weakly contractible simplicial sets.*

Corollary 4.5.1.4. *For each integer n, the construction $R \mapsto (\mathrm{LMod}_R^{\mathrm{cn}})_{\leqslant n}$ determines a functor $\mathrm{Alg}^{\mathrm{cn}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ which preserves colimits indexed by small weakly contractible simplicial sets.*

Since the forgetful functor CAlg → Alg commutes with filtered colimits, we immediately obtain the following variants:

Corollary 4.5.1.5. *The construction $R \mapsto \mathrm{Mod}_R$ determines a functor $\mathrm{CAlg} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ which commutes with filtered colimits.*

Corollary 4.5.1.6. *The construction $R \mapsto \mathrm{Mod}_R^{\mathrm{cn}}$ determines a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ which preserves small filtered colimits.*

Corollary 4.5.1.7. *For each integer n, the construction $R \mapsto (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant n}$ determines a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ which preserves small filtered colimits.*

Combining these results with Lemma HA.7.3.5.11 , we obtain the following:

Corollary 4.5.1.8. *The construction $R \mapsto \mathrm{Mod}_R^{\mathrm{perf}}$ determines a functor $\mathrm{CAlg} \to \mathcal{C}\mathrm{at}_\infty$ which commutes with filtered colimits.*

Corollary 4.5.1.9. *The construction $R \mapsto \mathrm{Mod}_R^{\mathrm{cn}} \cap \mathrm{Mod}_R^{\mathrm{perf}}$ determines a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$ which commutes with filtered colimits.*

Corollary 4.5.1.10. *Let $n \geqslant 0$ be an integer. For every connective $\mathbb{E}_\infty$-ring R, let Mod_R^{n-fp} denote the full subcategory of Mod_R spanned by those R-modules which are connective, n-truncated, and perfect to order $(n+1)$ (that is, the full subcategory spanned by the compact objects of $\tau_{\leqslant n} \mathrm{Mod}_R^{\mathrm{cn}}$). Then the construction $R \mapsto \mathrm{Mod}_R^{n-fp}$ determines a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$ which commutes with filtered colimits.*

4.5.2 The Non-Affine Case

Let R be a connective $\mathbb{E}_\infty$-ring and $n \geqslant 0$ an integer. Recall that a connective R-module M is said to be *finitely n-presented* if it is a compact object of the ∞-category $(\mathrm{Mod}_R)_{\leqslant n}$: equivalently, M is finitely n-presented if it is n-truncated and perfect to order $(n+1)$ (Definition 2.7.1.1). If X is a spectral Deligne-Mumford stack and $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, we see that $\mathscr{F}$ is *finitely n-presented* if, for every étale map $\eta : \mathrm{Spét}\, R \to \mathsf{X}$, the pullback $\eta^* \mathscr{F}$ is finitely n-presented when regarded as an object of $\mathrm{Mod}_R \simeq \mathrm{QCoh}(\mathrm{Spét}\, R)$ (see Definition 2.8.4.4).

Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks, and let $\mathscr{F}$ be a quasi-coherent sheaf on Y which is finitely n-presented. If f is flat, then $f^* \mathscr{F}$ is a finitely n-presented quasi-coherent sheaf on X. In the non-flat case, this need not be true. However, we do have the following analogue of Proposition 4.2.5.4:

Proposition 4.5.2.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks, and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ be finitely n-presented. Then $\tau_{\leqslant n} f^* \mathscr{F}$ is finitely n-presented.*

Proof. Since $\mathscr{F}$ is perfect to order $(n+1)$, we deduce that $f^* \mathscr{F}$ is perfect to order $(n+1)$. It follows that $\tau_{\leqslant n} f^* \mathscr{F}$ is also perfect to order $(n+1)$ (see Remark 2.7.1.3). Since $\tau_{\leqslant n} f^* \mathscr{F}$ is obviously n-truncated, it is finitely n-presented. □

Construction 4.5.2.2. The functor $\mathsf{X} \mapsto \mathrm{QCoh}(\mathsf{X})$ classifies a Cartesian fibration $\theta : \mathcal{C} \to \mathrm{SpDM}$. We can identify objects of $\mathcal{C}$ with pairs $(\mathsf{X}, \mathscr{F})$, where X is a spectral Deligne-Mumford stack and $\mathscr{F}$ is a quasi-coherent sheaf on X. Let $n \geqslant 0$ be an integer, and let $\mathcal{C}^{n-fp}$ denote the full subcategory of $\mathcal{C}$ spanned by those pairs $(\mathsf{X}, \mathscr{F})$, where $\mathscr{F}$ is connective and finitely n-presented. It follows from Proposition 4.5.2.1 that θ restricts to a Cartesian fibration $\mathcal{C}^{n-fp} \to \mathrm{SpDM}$. This Cartesian fibration is classified by a functor $\mathrm{QCoh}^{n-fp} : \mathrm{SpDM}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. We can describe this functor concretely as follows:

(a) To every spectral Deligne-Mumford stack X, $\mathrm{QCoh}^{n-fp}(\mathsf{X})$ can be identified with the full subcategory of $\mathrm{QCoh}(\mathsf{X})$ spanned by those quasi-coherent sheaves $\mathscr{F}$ which are connective and finitely n-presented.

(b) To every map $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks, the functor

$$\mathrm{QCoh}^{n-fp}(f) : \mathrm{QCoh}^{n-fp}(\mathsf{Y}) \to \mathrm{QCoh}^{n-fp}(\mathsf{X})$$

is given on objects by the construction $\mathscr{F} \mapsto \tau_{\leqslant n} f^* \mathscr{F}$.

The remainder of this section is devoted to the proof of the following result:

Theorem 4.5.2.3. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be a spectral Deligne-Mumford m-stack over R, for some integer $m < \infty$. Assume that X is ∞-quasi-compact and let $n \geqslant 0$ be an integer. Then:*

(1) *The ∞-category $\mathrm{QCoh}^{n-fp}(\mathsf{X})$ is essentially small.*

(2) *Suppose we are given a filtered diagram $\{R_\alpha\}$ of connective $\mathbb{E}_\infty$-algebras over R having colimit R'. Let $\mathsf{X}_\alpha = \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, R_\alpha$, and let $\mathsf{X}' = \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, R'$. Then the canonical map*
$$\theta : \varinjlim_{\alpha} \mathrm{QCoh}^{n-fp}(\mathsf{X}_\alpha) \to \mathrm{QCoh}^{n-fp}(\mathsf{X}')$$
is an equivalence of ∞-categories.

4.5.3 The Proof of Theorem 4.5.2.3

The proof of Theorem 4.5.2.3 will require the following general observation.

Lemma 4.5.3.1. *Filtered colimits are left exact in the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of small ∞-categories.*

Proof. Let $G : \mathcal{C}\mathrm{at}_\infty \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$ be the fully faithful embedding of Proposition HA.A.7.10 , given by $G(\mathcal{C})([n]) = \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\Delta^n, \mathcal{C})$. Since filtered colimits in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$ are left exact (Example HTT.7.3.4.7), it will suffice to show that G preserves finite limits and filtered colimits. The first assertion is obvious, and the second follows from the observation that each Δ^n is a compact object of $\mathcal{C}\mathrm{at}_\infty$. □

Proof of Theorem 4.5.2.3. Let X be a spectral Deligne-Mumford stack over a connective $\mathbb{E}_\infty$-ring R. Consider the following hypothesis for $m \geqslant -2$:

($*_m$) If $m \geqslant 0$, then X is a spectral Deligne-Mumford m-stack. If $m = -1$, then there exists a quasi-monomorphism $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, A$ for some connective $\mathbb{E}_\infty$-algebra A over R (see the proof of Theorem 4.4.2.2). If $m = -2$, then X is affine.

The hypothesis of Theorem 4.5.2.3 is that X satisfies ($*_m$) for m sufficiently large. We proceed by induction on m, beginning with the case $m = -2$. In this case, we can write $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ for some connective $\mathbb{E}_\infty$-ring A. Using Remark 2.7.1.4, we deduce that $\mathrm{QCoh}^{n-fp}(\mathsf{X})$ is equivalent to the ∞-category of compact objects of the presentable ∞-category $(\mathrm{Mod}_A^{\mathrm{cn}})_{\leqslant n}$, which proves (1). Assertion (2) follows from Corollary 4.5.1.10.

Now suppose that $m \geqslant -1$. Since X is quasi-compact, we can choose an étale surjection $u : \mathsf{X}_0 \to \mathsf{X}$, where X_0 is affine. Let $\mathsf{X}_\bullet$ be the Čech nerve of u. Then each X_p satisfies ($*_{m-1}$). Using the equivalence of ∞-categories $\mathrm{QCoh}^{n-fp}(\mathsf{X}) \simeq \varprojlim \mathrm{QCoh}^{n-fp}(\mathsf{X}_\bullet)$ and the inductive hypothesis, we deduce that $\mathrm{QCoh}^{n-fp}(\mathsf{X})$ is essentially small; this proves (1). To prove (2), we let $\mathsf{X}_{\bullet,\alpha}$ denote the simplicial spectral Deligne-Mumford stack given by $\mathsf{X}_\bullet \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, R_\alpha$, and $\mathsf{X}'_\bullet$ the simplicial spectral Deligne-Mumford stack given by $\mathsf{X}_\bullet \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, R'$. We have a commutative diagram

$$\begin{array}{ccc}
\varinjlim_\alpha \mathrm{QCoh}^{n-fp}(\mathsf{X}_\alpha) & \xrightarrow{\theta} & \mathrm{QCoh}^{n-fp}(\mathsf{X}') \\
\downarrow & & \downarrow \\
\varinjlim_\alpha \varprojlim_{[k]\in\mathbf{\Delta}} \mathrm{QCoh}^{n-fp}(\mathsf{X}_{k,\alpha}) & \longrightarrow & \varprojlim_{[k]\in\mathbf{\Delta}} \mathrm{QCoh}^{n-fp}(\mathsf{X}'_k) \\
\downarrow & & \downarrow \\
\varinjlim_\alpha \varprojlim_{[k]\in\mathbf{\Delta}_{s,\leqslant n+2}} \mathrm{QCoh}^{n-fp}(\mathsf{X}_{k,\alpha}) & \xrightarrow{\phi} & \varprojlim_{[k]\in\mathbf{\Delta}_{s,\leqslant n+2}} \mathrm{QCoh}^{n-fp}(\mathsf{X}'_k).
\end{array}$$

Here the vertical maps are equivalences by virtue of the observation that the functor QCoh^{n-fp} takes values in the full subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ spanned by those ∞-categories which are equivalent to $(n+1)$-categories (since $\mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}_{\leqslant n}$ is equivalent to an $(n+1)$-category, for every spectral Deligne-Mumford stack Y), and that this subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ is itself equivalent to an $(n+2)$-category. Consequently, to prove that θ is an equivalence of ∞-categories, it will suffice to show that ϕ is an equivalence of ∞-categories. The functor ϕ fits into a commutative diagram

$$\begin{array}{ccc}
\varinjlim_\alpha \varprojlim_{[k]\in\mathbf{\Delta}_{s,\leqslant n+2}} \mathrm{QCoh}^{n-fp}(\mathsf{X}_{k,\alpha}) & & \\
 & \searrow^{\phi'} & \\
\downarrow \phi & & \varprojlim_{[k]\in\mathbf{\Delta}_{s,\leqslant n+2}} \varinjlim_\alpha \mathrm{QCoh}^{n-fp}(\mathsf{X}_{k,\alpha}). \\
 & \swarrow_{\phi''} & \\
\varprojlim_{[k]\in\mathbf{\Delta}_{s,\leqslant n+2}} \mathrm{QCoh}^{n-fp}(\mathsf{X}'_k) & &
\end{array}$$

Here ϕ' is an equivalence of ∞-categories by Lemma 4.5.3.1, and ϕ'' is an equivalence of ∞-categories by the inductive hypothesis. □

4.6 Descent of Properties along Filtered Colimits

Let $f : \mathsf{X} \to \mathsf{Y}$ be a a map of spectral Deligne-Mumford stack which are finitely n-presented over a connective $\mathbb{E}_\infty$-ring A, and suppose that A is given as the colimit of a filtered diagram of connective $\mathbb{E}_\infty$-rings $\{A_\alpha\}$. Using Theorem 4.4.2.2, we deduce the existence of an index α and a map $f_\alpha : \mathsf{X}_\alpha \to \mathsf{Y}_\alpha$ of spectral Deligne-Mumford stacks which are finitely n-presented over A_α, such that f is equivalent to the induced map

$$\tau_{\leqslant n}(\operatorname{Sp\acute{e}t} A \times_{\operatorname{Sp\acute{e}t} A_\alpha} \mathsf{X}_\alpha) \to \tau_{\leqslant n}(\operatorname{Sp\acute{e}t} A \times_{\operatorname{Sp\acute{e}t} A_\alpha} \mathsf{Y}_\alpha).$$

Our goal in this section is to prove a variety of results which assert that if f has some property P, then we can arrange that f_α also has the property P. Our results can be summarized as follows:

(a) If f is affine, then f_α can be chosen to be affine (Proposition 4.6.1.1).

(b) If f is surjective, then f_α can be chosen to be surjective (Proposition 4.6.1.2).

(c) If f is a closed immersion, then f_α can be chosen to be a closed immersion (Proposition 4.6.1.3).

(d) If f is separated, then f_α can be chosen to be separated (Corollary 4.6.1.4).

(e) If f is étale, then f_α can be chosen to be étale (Proposition 4.6.2.1).

(f) If f is an open immersion, then f_α can be chosen to be an open immersion (Corollary 4.6.2.2).

4.6.1 Examples of Descent

We begin by handling the easy cases.

Proposition 4.6.1.1. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geqslant 0$ be an integer, let $f_0 : \mathsf{X}_0 \to \mathsf{Y}_0$ be a morphism in $\mathrm{DM}_n^{\mathrm{fp}}(A_0)$. Let $f : \mathsf{X} \to \mathsf{Y}$ be the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Suppose that f is affine. Then there exists an index α such that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is affine.*

Proof. Since Y_0 is quasi-compact, we can choose an étale surjection $\mathsf{Y}'_0 \to \mathsf{Y}_0$, where $\mathsf{Y}'_0 \simeq \operatorname{Sp\acute{e}t} B_0$ is affine. Replacing Y_0 by Y'_0, we can reduce to the case where Y_0 is affine, so that X is also affine. Corollary 4.4.1.4 shows that there exists an index α and an affine spectral Deligne-Mumford stack $\mathsf{Z} \in \mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ having image X in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Let X_α denote the image of X_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$. Then X_α and Z have equivalent images in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Altering our choice of α, we may assume with X_α is equivalent to Z (Theorem 4.4.2.2), so that X_α is affine and therefore the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is affine. □

Proposition 4.6.1.2. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geqslant 0$ be an integer, let $f_0 : \mathsf{X}_0 \to \mathsf{Y}_0$ be a morphism in $\mathrm{DM}_n^{\mathrm{fp}}(A_0)$. Let $f : \mathsf{X} \to \mathsf{Y}$ be the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Suppose that f is surjective. Then there exists an index α such that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is surjective.*

Proof. We may assume without loss of generality that the diagram $\{A_\alpha\}$ is indexed by a filtered partially ordered set. Since Y_0 is quasi-compact, we can choose an étale surjection $\mathsf{Y}'_0 \to \mathsf{Y}_0$, where Y'_0 is quasi-compact. Replacing Y_0 by Y'_0, we can reduce to the case where $\mathsf{Y}_0 \simeq \operatorname{Spét} B_0$ is affine. Similarly, we can assume that $\mathsf{X}_0 \simeq \operatorname{Spét} C_0$ is affine. Our assumption implies that the map of topological spaces

$$|\operatorname{Spec} \operatorname{Tor}_0^{\pi_0 A_0}(\pi_0 C_0, \pi_0 A)| \to |\operatorname{Spec} \operatorname{Tor}_0^{\pi_0 A_0}(\pi_0 B_0, \pi_0 A)|$$

is surjective. Using Corollary 4.3.5.4, we deduce the existence of an index α such that the map

$$|\operatorname{Spec} \operatorname{Tor}_0^{\pi_0 A_0}(\pi_0 C_0, \pi_0 A_\alpha)| \to |\operatorname{Spec} \operatorname{Tor}_0^{\pi_0 A_0}(\pi_0 B_0, \pi_0 A_\alpha)|$$

is surjective. It follows that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is a surjective map of spectral Deligne-Mumford stacks. □

Proposition 4.6.1.3. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geqslant 0$ be an integer, let $f_0 : \mathsf{X}_0 \to \mathsf{Y}_0$ be a morphism in $\mathrm{DM}_n^{\mathrm{fp}}(A_0)$. Let $f : \mathsf{X} \to \mathsf{Y}$ be the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Suppose that f is a closed immersion. Then there exists an index α such that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is a closed immersion.*

Proof. Since Y_0 is quasi-compact, we can choose an étale surjection $\mathsf{Y}'_0 \to \mathsf{Y}_0$, where $\mathsf{Y}'_0 \simeq \operatorname{Spét} B_0$ is affine. Replacing Y_0 by Y'_0, we can reduce to the case where Y_0 is affine. Using Proposition 4.6.1.1, we may assume that $\mathsf{X}_0 \simeq \operatorname{Spét} C_0$ is also affine. The condition that f is a closed immersion guarantees that the map

$$\varinjlim \operatorname{Tor}_0^{\pi_0 A_0}(\pi_0 A_\alpha, \pi_0 B_0) \to \varinjlim \operatorname{Tor}_0^{\pi_0 A_0}(\pi_0 A_\alpha, \pi_0 C_0)$$

is surjective. Since $\pi_0 C_0$ is a finitely presented algebra over $\pi_0 B_0$, we deduce that there is an index α such that the image of the map

$$\operatorname{Tor}_0^{\pi_0 A_0}(\pi_0 A_\alpha, \pi_0 B_0) \to \operatorname{Tor}_0^{\pi_0 A_0}(\pi_0 A_\alpha, \pi_0 C_0)$$

contains the image of $\pi_0 C_0$, and is therefore surjective. It follows that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is a closed immersion. □

Corollary 4.6.1.4. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geqslant 0$ be an integer, let $f_0 : \mathsf{X}_0 \to \mathsf{Y}_0$ be a morphism in $\mathrm{DM}_n^{\mathrm{fp}}(A_0)$. Let $f : \mathsf{X} \to \mathsf{Y}$ be the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Suppose that f is separated. Then there exists an index α such that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is separated.*

Proof. Apply Proposition 4.6.1.3 to the diagonal map $\mathsf{X}_0 \to \mathsf{X}_0 \times_{\mathsf{Y}_0} \mathsf{X}_0$. □

4.6.2 Descending Étale Morphisms

The final result of this section is somewhat more difficult.

Proposition 4.6.2.1. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geqslant 0$ be an integer, let $f_0 : \mathsf{X}_0 \to \mathsf{Y}_0$ be a morphism in $\mathrm{DM}_n^{\mathrm{fp}}(A_0)$. Let $f : \mathsf{X} \to \mathsf{Y}$ be the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Suppose that f is étale. Then there exists an index α such that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is étale.*

Corollary 4.6.2.2. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geqslant 0$ be an integer, let $f_0 : \mathsf{X}_0 \to \mathsf{Y}_0$ be a morphism in $\mathrm{DM}_n^{\mathrm{fp}}(A_0)$. Let $f : \mathsf{X} \to \mathsf{Y}$ be the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Suppose that f is an open immersion. Then there exists an index α such that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is an open immersion.*

Proof. Using Proposition 4.6.2.1, we can reduce to the case where f_0 is étale. Let δ_0 denote the diagonal map $\mathsf{X}_0 \to \mathsf{X}_0 \times_{\mathsf{Y}_0} \mathsf{X}_0$. Then the image of δ_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$ is an equivalence. Theorem 4.4.2.2 implies that there exists an index α such that the image of δ_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is an equivalence. It follows that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is an open immersion. □

The proof of Proposition 4.6.2.1 will require some preliminaries.

Remark 4.6.2.3. Let B be a connective $\mathbb{E}_\infty$-ring and let $n \geqslant 0$ be an integer. The truncation map $B \to \tau_{\leqslant n} B$ is $(n+1)$-connective. Consequently, Theorem HA.7.4.3.1 supplies an $(2n+3)$-connective map $(\tau_{\leqslant n} B \otimes_B \tau_{\geqslant n+1} B) \to \Sigma^{-1} L_{\tau_{\leqslant n} B/B}$. The map $\tau_{\geqslant n+1} B \to (\tau_{\leqslant n} B \otimes_B \tau_{\geqslant n+1} B)$ is $(2n+2)$-connective, so that the induced map $\tau_{\geqslant n+1} B \to \Sigma^{-1} L_{\tau_{\leqslant n} B/B}$ determines isomorphisms $\theta_m : \pi_m B \to \pi_{m+1} L_{\tau_{\leqslant n} B/B}$ for $n < m < 2n+2$ and a surjection when $m = 2n+2$.

Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings, and let $\partial : \Sigma^{-1} L_{\tau_{\leqslant n} B/B} \to L_{B/A}$ be the associated boundary map. Unwinding the definitions, we see that for $m > n$, the composition

$$\pi_m B \xrightarrow{\theta_m} \pi_{m+1} L_{\tau_{\leqslant n} B/B} \xrightarrow{\partial} \pi_m L_{B/A}$$

is induced by the universal A-linear derivation $d : B \to L_{B/A}$. In particular (taking $n = 0$), we conclude that the induced maps $\pi_m B \to \pi_m L_{B/A}$ is $\pi_0 B$-linear for $m > 0$.

Lemma 4.6.2.4. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings, and let $n \geqslant 0$. The induced map $\tau_{\leqslant n} A \to \tau_{\leqslant n} B$ is étale if and only if the following conditions are satisfied:*

(1) *The commutative ring $\pi_0 B$ is finitely presented over $\pi_0 A$.*

(2) *The relative cotangent complex $L_{B/A}$ is $(n+1)$-connective.*

(3) *Let $d : B \to L_{B/A}$ be the universal derivation (so that d is a map of A-module spectra). Then d induces a surjection $\pi_{n+1}B \to \pi_{n+1}L_{B/A}$.*

Proof. Suppose first that $\tau_{\leqslant n}B$ is étale over $\tau_{\leqslant n}A$. Then $\pi_0 B$ is étale over $\pi_0 A$, which immediately implies (1). Using Theorem HA.7.5.0.6 , we can choose an étale A-algebra A' and an isomorphism $\alpha : \pi_0 A' \simeq \pi_0 B$. Theorem HA.7.5.4.2 implies that α can be lifted to a map of A-algebras $\overline{\alpha} : A' \to B$. Since $L_{A'/A} \simeq 0$, we conclude that the canonical map $L_{B/A} \to L_{B/A'}$ is an equivalence. Note that $\overline{\alpha}$ induces an equivalence $\tau_{\leqslant n}A' \to \tau_{\leqslant n}B$, and is therefore n-connective. Using Corollary HA.7.4.3.2 , we deduce that $L_{B/A} \simeq L_{B/A'}$ is $(n+1)$-connective, thereby proving (2). To prove (3), we note that the composite map $A' \to B \to \tau_{\leqslant n}B$ is $(n+1)$-connective, so that $L_{\tau_{\leqslant n}B/A} \simeq L_{\tau_{\leqslant n}B/A'}$ is $(n+2)$-connective. We have a fiber sequence

$$(\tau_{\leqslant n}B) \otimes_B L_{B/A} \to L_{\tau_{\leqslant n}B/A} \to L_{\tau_{\leqslant n}B/B}.$$

The vanishing of $\pi_{n+1}L_{\tau_{\leqslant n}B/A}$ implies that the boundary map

$$\theta : \pi_{n+2}L_{\tau_{\leqslant n}B/B} \to \pi_{n+1}(\tau_{\leqslant n}B \otimes_B L_{B/A}) \simeq \pi_{n+1}L_{B/A}$$

is surjective. Using Remark 4.6.2.3, we deduce that the universal derivation $d : B \to L_{B/A}$ induces a surjection $\pi_{n+1}B \to \pi_{n+1}L_{B/A}$, so that (3) is satisfied.

Now suppose that conditions (1), (2), and (3) hold. We wish to prove that $\tau_{\leqslant n}B$ is étale over $\tau_{\leqslant n}A$. Consider the fiber sequence

$$(\tau_{\leqslant n}B) \otimes_B L_{B/A} \to L_{\tau_{\leqslant n}B/A} \to L_{\tau_{\leqslant n}B/B}.$$

It follows from condition (2) that $(\tau_{\leqslant n}B) \otimes_B L_{B/A}$ is $(n+1)$-connective, and we have a canonical isomorphism $\pi_{n+1}(\tau_{\leqslant n}B \otimes_B L_{B/A}) \simeq \pi_{n+1}L_{B/A}$. Using Remark 4.6.2.3, we deduce that $L_{\tau_{\leqslant n}B/B}$ is $(n+2)$-connective and obtain a canonical isomorphism $\pi_{n+2}L_{\tau_{\leqslant n}B/B} \simeq \pi_{n+1}B$. It follows that $L_{\tau_{\leqslant n}B/A}$ is $(n+1)$-connective, and we have a short exact sequence of abelian groups

$$\pi_{n+1}B \to \pi_{n+1}L_{B/A} \to \pi_{n+1}L_{\tau_{\leqslant n}B/A} \to 0.$$

Using condition (3), we conclude that $L_{\tau_{\leqslant n}B/A}$ is $(n+2)$-connective. Invoking Lemma B.1.3.2, we see that f factors as a composition $A \xrightarrow{f'} A' \xrightarrow{f''} B$ where f' is étale and f'' is $(n+1)$-connective. It follows that $\tau_{\leqslant n}B \simeq \tau_{\leqslant n}A'$ is étale over $\tau_{\leqslant n}A$, as desired. □

We now prove Proposition 4.6.2.1 in the affine case.

Lemma 4.6.2.5. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $f : A_0 \to B_0$ be a map of connective $\mathbb{E}_\infty$-rings which is of finite generation to order $n+1$ for some $n \geqslant 0$. Let $B_\alpha = A_\alpha \otimes_{A_0} B_0$ and let $B = \varinjlim B_\alpha \simeq A \otimes_{A_0} B_0$. Suppose that $\tau_{\leqslant n}B$ is étale over $\tau_{\leqslant n}A$. Then there exists an index α such that $\tau_{\leqslant n}B_\alpha$ is étale over $\tau_{\leqslant n}A_\alpha$.*

Proof. Using Lemma 4.6.2.4, we see that $L_{B/A} \simeq B \otimes_{B_0} L_{B_0/A_0}$ is $(n+1)$-connective. Using Theorem 4.5.2.3, we deduce that there exists an index α such that $B_\alpha \otimes_{B_0} L_{B_0/A_0}$ is $(n+1)$-connective. Since B_0 is of finite generation to order $(n+1)$ over A_0, the relative cotangent complex L_{B_0/A_0} is perfect to order $(n+1)$ so that $\pi_{n+1} L_{B_0/A_0}$ is finitely generated as a module over $\pi_0 B_0$. Choose a finite collection of generators $x_1, \ldots, x_k$ for $\pi_{n+1} L_{B_0/A_0}$, and let $x'_1, \ldots, x'_k$ denote their images in $\pi_{n+1} L_{B/A}$. Lemma 4.6.2.4 implies that the universal derivation $B \to L_{B/A}$ induces a surjection

$$\phi : \varinjlim \pi_{n+1} B_\alpha \simeq \pi_{n+1} B \to \pi_{n+1} L_{B/A}.$$

It follows that there exists an index α and elements $y_1, \ldots, y_k \in \pi_{n+1} B_\alpha$ such that $\phi(y_i) = x'_i$ for $1 \leqslant i \leqslant k$. Let x''_i denote the image of x_i in $\pi_{n+1} L_{B_\alpha/A_\alpha}$. Enlarging α if necessary, we may assume that the universal derivation $d_\alpha : B_\alpha \to L_{B_\alpha/A_\alpha}$ carries each y_i to x''_i. Note that $\pi_{n+1} L_{B_\alpha/A_\alpha} \simeq \mathrm{Tor}_0^{\pi_0 B_0}(\pi_0 B, \pi_{n+1} L_{B_0/A_0})$, so that the elements x''_i generate $\pi_{n+1} L_{B_\alpha/A_\alpha}$ as a module over $\pi_0 B_\alpha$. Since the universal derivation induces a $\pi_0 B_\alpha$-linear map $\pi_{n+1} B_\alpha \to \pi_{n+1} L_{B_\alpha/A_\alpha}$ (see Remark 4.6.2.3), we deduce that this map is surjective. Applying Lemma 4.6.2.4, we conclude that $\tau_{\leqslant n} B_\alpha$ is étale over $\tau_{\leqslant n} A_\alpha$, as desired. □

Proof of Proposition 4.6.2.1. Since Y_0 is quasi-compact, we can choose an étale surjection $\mathsf{Y}'_0 \to \mathsf{Y}_0$, where Y'_0 is affine. Replacing Y_0 by Y'_0 and X_0 by the fiber product $\mathsf{X}_0 \times_{\mathsf{Y}_0} \mathsf{Y}'_0$, we can assume that Y_0 is affine. Since X_0 is quasi-compact, we can choose an étale surjection $\mathsf{X}'_0 \to \mathsf{X}_0$, where X'_0 is affine. We may therefore replace X_0 by X'_0 and thereby reduce to the case where X_0 is also affine. The desired result now follows immediately from Theorem 1.4.10.2 and Lemma 4.6.2.5. □

Chapter 5

Proper Morphisms in Spectral Algebraic Geometry

Let $f : X \to Y$ be a morphism of schemes. Recall that f is said to be *proper* if it is separated, of finite type, and universally closed (Definition 3.7.0.1). Our goal in this section is to study the same hypothesis in the setting of spectral algebraic geometry. We begin in §5.1 by introducing the notion of a *proper morphism* between spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ (Definition 5.1.2.1).

In the setting of classical algebraic geometry, there are two important sources of examples of proper morphisms:

(*a*) Every closed immersion of schemes is proper. More generally, any finite morphism of schemes $f : X \to Y$ is proper (recall that $f : X \to Y$ is said to be *finite* if, for every affine open subscheme $\operatorname{Spét} A \simeq U \subseteq Y$, the inverse image $f^{-1}(U) \subseteq X$ is an affine scheme of the form $\operatorname{Spec} B$, where B is finitely generated as an A-module).

(*b*) For any commutative ring R, the projection map $\mathbf{P}^n_R \to \operatorname{Spec} R$ is proper, where $\mathbf{P}^n_R$ denotes projective space of dimension n over R.

In §5.2, we will see that the theory of finite morphisms can be generalized to the setting of spectral algebraic geometry in a straightforward way. The case of projective space is more subtle. In §5.4, we will associate to each connective $\mathbb{E}_\infty$-ring R a spectral algebraic space $\mathbf{P}^n_R$, which we call *projective space of dimension n over R* (Construction 5.4.1.3). These projective spaces share many of the pleasant features of their classical counterparts: for example, Serre's calculation of the cohomology of line bundles on projective space extends to the spectral setting without essential change (Theorem 5.4.2.6). However, there are a few surprises: for example, the projective spaces of Construction 5.4.1.3 are somewhat rigid objects (unlike projective spaces in classical algebraic geometry, which are homogeneous spaces).

Most (but not all) examples of proper morphisms of schemes can be obtained by combining (a) and (b). Recall that a complex algebraic variety X is said to be *proper* if the projection map $X \to \operatorname{Spec} \mathbf{C}$ is proper, and is said to be *projective* if there exists a closed immersion $X \hookrightarrow \mathbf{P}^n_{\mathbf{C}}$ for some $n \geqslant 0$ (or, equivalently, if there exists a finite morphism $X \to \mathbf{P}^n_{\mathbf{C}}$ for some $n \geqslant 0$). Every projective algebraic variety is proper, but not every proper algebraic variety is projective. However, a general proper algebraic variety X is not far from being projective: according to Chow's lemma, one can always choose a projective birational map $\pi : \overline{X} \to X$, where the variety $\overline{X}$ is projective. In §5.5, we establish a version of Chow's lemma in the setting of spectral algebraic geometry (Theorem 5.5.0.1).

Recall that scheme X is determined (up to canonical isomorphism) by its functor of points $h_X : \mathrm{CAlg}^{\heartsuit} \to \mathcal{S}\mathrm{et}$. In particular, the condition that a morphism of schemes $f : X \to Y$ is proper depends only the induced natural transformation $h_X \to h_Y$. If we assume that f is of finite type, then we can be more explicit: according to Grothendieck's *valuative criterion of properness*, f is proper if and only if, for every valuation ring V with fraction field K, the induced map $h_X(V) \to h_X(K) \times_{h_Y(K)} h_Y(V)$ is a bijection. In §5.3, we discuss an analogous valuative criterion in the setting of spectral algebraic spaces (see Theorem 5.3.0.1 and Corollary 5.3.1.2).

Remark 5.0.0.1. The material described in §5.1 through §5.3 really has nothing to do with spectral algebraic geometry. If $f : \mathsf{X} \to \mathsf{Y}$ is a morphism of spectral algebraic spaces and $f_0 : \mathsf{X}_0 \to \mathsf{Y}_0$ is the underlying map of classical algebraic spaces, then:

- The morphism f is proper if and only if f_0 is proper (Remark 5.1.2.2).
- The morphism f is finite if and only if f_0 is finite (Remark 5.2.0.2).
- The morphism f satisfies the valuative criterion of properness (or separatedness) if and only if the morphism f_0 satisfies the valuative criterion of properness (or separatedness).

Let X be a topological space and let $\mathcal{O}$ be a sheaf of commutative rings on X. Recall that a sheaf $\mathscr{F}$ of (discrete) $\mathcal{O}$-modules is said to be *coherent* if $\mathscr{F}$ is locally finitely generated and, for any open subset $U \subseteq X$ and any map $\alpha : \mathcal{O}^n |_U \to \mathscr{F} |_U$, the kernel of α is also locally finitely generated. In §**??**, we extend the definition of coherence to sheaves of module spectra on a spectrally ringed ∞-topos $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ (Definition **??**). In the case where X is a locally Noetherian spectral Deligne-Mumford stack, we show that an object $\mathscr{F} \in \mathrm{Mod}_{\mathcal{O}}$ is coherent and hypercomplete if and only if it is quasi-coherent and almost perfect (Proposition **??**). Consequently, we can view the theory of almost perfect sheaves as providing an extension of the theory of coherent sheaves to non-Noetherian settings. Alternatively, we can view the theory of coherent sheaves as an extension of the theory of almost perfect sheaves to settings outside of algebraic geometry (such as the setting of derived complex analytic geometry, which we study in §**??**).

If $f : X \to Y$ is a proper morphism between Noetherian schemes, then Grothendieck's direct image theorem asserts that the derived direct image functors $R^i f_*$ carry coherent sheaves on X to coherent sheaves on Y. This theorem was generalized to the setting of algebraic spaces by Knutson ([117]). In the language of spectral algebraic geometry, Knutson's result asserts that if $f : \mathsf{X} \to \mathsf{Y}$ is a proper morphism between 0-truncated spectral algebraic spaces, then the direct image functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ carries coherent objects to coherent objects, or equivalently (by virtue of Proposition **??**) that f_* carries almost perfect objects of $\mathrm{QCoh}(\mathsf{X})$ to almost perfect objects of $\mathrm{QCoh}(\mathsf{Y})$. In §5.6, we will prove a more general assertion: for any morphism of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ which is proper and locally almost of finite presentation, the direct image functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ carries almost perfect objects to almost perfect objects (Theorem 5.6.0.2). When X and Y are locally Noetherian, this is an elementary consequence of the direct image theorem of classical algebraic geometry (in the setting of algebraic spaces). We will prove Theorem 5.6.0.2 in general by reduction to the Noetherian case, using the approximation techniques of Chapter 4.

Remark 5.0.0.2. In the setting of classical algebraic geometry, Theorem 5.6.0.2 was proven by Illusie in the case where $f : \mathsf{X} \to \mathsf{Y}$ is projective, and by Kiehl in general. See [101] and [116].

Warning 5.0.0.3. The notion of proper morphism that we consider in this section is more restrictive than the standard notion in the stack-theoretic literature: for a morphism $f : \mathsf{X} \to \mathsf{Y}$ to be proper, we require that the relative diagonal $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ be a closed immersion. This excludes many examples of geometric interest (such as moduli stacks of stable curves).

Contents

5.1 Properness

5.1.1 Universally Closed Morphisms

Let $f : \mathsf{X} \to \mathsf{Y}$ be a separated morphism of spectral algebraic spaces. Then, for every pullback diagram

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y} \end{array}$$

the map f' is also separated. It follows that if Y' is a quasi-separated spectral algebraic space, then X' is also a quasi-separated spectral algebraic space. In this case, f' induces a map of topological spaces $|\mathsf{X}'| \to |\mathsf{Y}'|$.

Proposition 5.1.1.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a separated morphism of spectral algebraic spaces. The following conditions are equivalent:*

(*a*) *For every pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Sp\acute{e}t} R & \longrightarrow & \mathsf{Y}, \end{array}$$

the induced map of topological spaces $|\,\mathsf{X}'\,| \to |\operatorname{Sp\acute{e}t} R| \simeq |\operatorname{Spec} R|$ *is closed.*

(*b*) *For every pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y} \end{array}$$

where Y' *is a quasi-separated spectral algebraic space, the induced map of topological spaces* $|\,\mathsf{X}'\,| \to |\,\mathsf{Y}'\,|$ *is closed.*

Definition 5.1.1.2. Let $f : \mathsf{X} \to \mathsf{Y}$ be a separated morphism of spectral Deligne-Mumford stacks. We will say that f is *universally closed* if it satisfies the equivalent conditions of Proposition 5.1.1.1.

Proof of Proposition 5.1.1.1. The implication $(b) \Rightarrow (a)$ is obvious. Conversely, suppose that (a) is satisfied and that we are given a pullback square

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}, \end{array}$$

where Y' is a quasi-separated spectral algebraic space. We wish to show that the natural map $|\,\mathsf{X}'\,| \to |\,\mathsf{Y}'\,|$ is closed. Writing Y' as a union of its quasi-compact open substacks, we can reduce to the case where Y' is quasi-compact. Choose an étale surjection $\operatorname{Sp\acute{e}t} R \to \mathsf{Y}'$, and form a pullback diagram

$$\begin{array}{ccc} \mathsf{X}'' & \longrightarrow & \mathsf{X}' \\ \downarrow & & \downarrow \\ \operatorname{Sp\acute{e}t} R & \longrightarrow & \mathsf{Y}'. \end{array}$$

We then obtain a diagram of topological spaces

$$\begin{array}{ccc} |\,\mathsf{X}''\,| & \xrightarrow{\psi} & |\,\mathsf{X}'\,| \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ |\operatorname{Spec} R| & \xrightarrow{\psi'} & |\,\mathsf{Y}'\,|. \end{array}$$

Let $K \subseteq |\mathsf{X}'|$ be closed; we wish to show that $\phi(K)$ is closed. Since $\psi^{-1}K$ is a closed subset of $|\mathsf{X}'|$, condition (a) implies that $\phi'(\psi^{-1}K)$ is a closed subset of $|\operatorname{Spec} R|$. Corollary 3.6.3.2 gives $\psi'^{-1}(\phi(K)) = \phi'(\psi^{-1}K)$, so that $\psi'^{-1}(\phi(K))$ is a closed subset of $|\operatorname{Spec} R|$. Since ψ' is a quotient map (Proposition 3.6.3.6), we conclude that $\phi(K)$ is a closed subset of $|\mathsf{Y}'|$. □

Remark 5.1.1.3. Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}, \end{array}$$

where the vertical maps are separated. If f is universally closed, then so is f'.

Remark 5.1.1.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of quasi-compact, quasi-separated algebraic spaces, and let $\phi : |\mathsf{X}| \to |\mathsf{Y}|$ be the underlying map of topological spaces. Then the fibers of ϕ are quasi-compact. It follows that for any filtered collection of closed subsets $\{K_\alpha \subseteq |\mathsf{X}|\}$, we have $\phi(\bigcap K_\alpha) = \bigcap \phi(K_\alpha)$. Since $|\mathsf{X}|$ has a basis of quasi-compact open subsets, every closed set $K \subseteq |\mathsf{X}|$ can be obtained as a (filtered) intersection of closed subsets with quasi-compact complements. Consequently, to prove that ϕ is closed, it will suffice to show that $\phi(K) \subseteq |\mathsf{Y}|$ is closed whenever $K \subseteq |\mathsf{X}|$ is the complement of a quasi-compact open subset of $|\mathsf{X}|$.

Remark 5.1.1.5. Let $f : \mathsf{X} \to \operatorname{Spét} R$ be a morphism of quasi-compact, quasi-separated spectral algebraic spaces. Suppose we wish to verify that f is universally closed. Let $R \to R'$ be an arbitrary map of connective $\mathbb{E}_\infty$-rings and set $\mathsf{X}' = \mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét} R'$; we wish to prove that the map $\phi : |\mathsf{X}'| \to |\operatorname{Spec} R'|$ is closed. According to Remark 5.1.1.4, it suffices to show that $\phi(K) \subseteq |\operatorname{Spec} R'|$ is closed whenever $K \subseteq |\mathsf{X}'|$ is the complement of a quasi-compact open subset of $|\mathsf{X}'|$. Write $R' = \varinjlim R_\alpha$ in $\mathrm{CAlg}^{\mathrm{cn}}_R$, where each R_α is of finite presentation over R, and set $\mathsf{X}_\alpha = \mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét} R_\alpha$. According to Proposition 4.3.5.5, every quasi-compact open subset of $|\mathsf{X}'|$ is the inverse image of a quasi-compact open subset of some $|\mathsf{X}_\alpha|$. It will therefore suffice to show that the map $\phi_\alpha : |\mathsf{X}_\alpha| \to |\operatorname{Spec} R_\alpha|$ is closed. In other words, to verify condition (a) of Proposition 5.1.1.1, it suffices to treat the case where R' is finitely presented over R. In particular, if R is Noetherian, we may assume that R' is also Noetherian (Proposition HA.7.2.4.31).

5.1.2 Proper Morphisms

We can now introduce the main definition of interest to us in this section:

Definition 5.1.2.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. We will say that f is *proper* if it is quasi-compact (Definition 2.3.2.2, separated (Definition 3.2.0.1), locally of finite type (Definition 4.2.0.1), and universally closed (Definition 5.1.1.2).

Remark 5.1.2.2. The condition that a morphism of spectral Deligne-Mumford stacks $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be proper depends only on the induced map of 0-truncated spectral Deligne-Mumford stacks $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathcal{O}_{\mathcal{Y}})$. If $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ are spectral algebraic spaces, then f is proper if and only if the induced map $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathcal{O}_{\mathcal{Y}})$ is proper when regarded as a map of algebraic spaces, in the sense of [117].

Warning 5.1.2.3. When restricted to 0-truncated, 1-localic spectral Deligne-Mumford stacks, Definition 5.1.2.1 does not recover the usual notion of proper morphism between ordinary Deligne-Mumford stacks (because Definition 5.1.2.1 requires that the diagonal of a proper morphism $\mathsf{X} \to \mathsf{Y}$ is a closed immersion). It is possible to introduce a less restrictive version of Definition 5.1.2.1, which agrees with the usual notion of proper morphism in the classical case. However, this more general concept of properness will not be needed in this book.

Example 5.1.2.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be a closed immersion of spectral Deligne-Mumford stacks. Then f is proper.

5.1.3 Pullbacks of Proper Morphisms

We now study the behavior of proper morphisms under base change.

Proposition 5.1.3.1. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y} \end{array}$$

of spectral Deligne-Mumford stacks. Then:

(*a*) *If f is proper, then f' is also proper.*

(*b*) *If f' is proper, f is separated, and g is a flat covering (Definition 2.8.3.1), then f is proper.*

Proof. Assertion (*a*) follows from the fact that the collections of separated, locally finite type, quasi-compact, and universally closed morphisms are stable under pullbacks (Remark 3.2.1.4, Proposition 4.2.1.6, Proposition 2.3.3.1, and Remark 5.1.1.3). Let us prove (*b*). Assume that f' is proper, f is separated, and that g is a flat covering. We wish to prove that f is proper. It follows from Proposition 4.2.1.5 that f is locally of finite type. We wish to show that f is quasi-compact and universally closed. In other words, we wish to show that for any map $\mathsf{Y}_0 \to \mathsf{Y}$ where Y_0 is affine, the fiber product $\mathsf{X}_0 = \mathsf{Y}_0 \times_{\mathsf{Y}} \mathsf{X}$ is quasi-compact and the projection map $|\mathsf{X}_0| \to |\mathsf{Y}_0|$ is closed. Since g is a flat covering, we can choose an étale map $\mathsf{Y}'_0 \to \mathsf{Y}' \times_{\mathsf{Y}} \mathsf{Y}_0$ where Y'_0 is affine and the map $\mathsf{Y}'_0 \to \mathsf{Y}_0$ is flat and surjective. Set $\mathsf{X}'_0 = \mathsf{Y}'_0 \times_{\mathsf{Y}} \mathsf{X}$. Since f' is quasi-compact, we conclude that X'_0 is quasi-compact, so that X_0 is quasi-compact by virtue of Proposition 2.3.3.1. We have a commutative diagram of topological spaces

$$\begin{array}{ccc} |\mathsf{X}'_0| & \xrightarrow{\psi} & |\mathsf{X}_0| \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ |\mathsf{Y}'_0| & \xrightarrow{\psi'} & |\mathsf{Y}_0|. \end{array}$$

Let $K \subseteq |\mathsf{X}_0|$ be closed; we wish to show that $\phi(K) \subseteq |\mathsf{Y}_0|$ is closed. Since $\psi^{-1}K$ is a closed subset of $|\mathsf{X}'_0|$, our hypothesis that f' is proper guarantees that $\phi'(\psi^{-1}K)$ is a closed subset of $|\operatorname{Spec} R|$. Corollary 3.6.3.2 gives $\psi'^{-1}(\phi(K)) = \phi'(\psi^{-1}K)$, so that $\psi'^{-1}(\phi(K))$ is a closed subset of $|\mathsf{Y}'_0|$. Since ψ' is a quotient map (Proposition 3.6.3.6), we conclude that $\phi(K)$ is a closed subset of $|\mathsf{Y}_0|$ as desired. □

Corollary 5.1.3.2. *The condition that a morphism $f : \mathsf{X} \to \mathsf{Y}$ be proper is local on the target with respect to the étale topology. That is, if we are given an étale surjection for which the projection map $\mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}' \to \mathsf{Y}'$ is proper, then f is proper.*

Proof. Combine Remark 3.2.1.4 with Proposition 5.1.3.1. □

Remark 5.1.3.3. Suppose we are given a collection of proper morphisms $\{f_\alpha : \mathsf{X}_\alpha \to \mathsf{Y}_\alpha\}$. Then the induced map $\amalg \mathsf{X}_\alpha \to \amalg \mathsf{Y}_\alpha$ is proper.

5.1.4 Composition of Proper Morphisms

We conclude this section by studying the behavior of proper morphisms under composition.

Proposition 5.1.4.1. *Suppose we are given a commutative diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{f} & \mathsf{Y} \\ & \searrow \quad \swarrow{\scriptstyle g} & \\ & \mathsf{Z}. & \end{array}$$

Then:

(1) *If f and g are proper, then $g \circ f$ is proper.*

(2) *If $g \circ f$ is proper and g is separated, then f is proper.*

(3) *If $g \circ f$ is proper, g is separated and locally of finite type, and f is surjective, then g is proper.*

Proof. We first prove (1). Assume that f and g are proper. Using Remark 3.2.1.6, Proposition 2.3.5.1, and Proposition 4.1.3.1, we see that $g \circ f$ is separated, quasi-compact, and locally of finite type. It will therefore suffice to show that $g \circ f$ is universally closed. Choose a morphism $\mathsf{Z}_0 \to \mathsf{Z}$, where Z_0 is quasi-separated spectral algebraic space; we wish to show that the composite map

$$|\,\mathsf{Z}_0 \times_{\mathsf{Z}} \mathsf{X}\,| \xrightarrow{\phi} |\,\mathsf{Z}_0 \times_{\mathsf{Z}} \mathsf{Y}\,| \xrightarrow{\psi} |\,\mathsf{Z}_0\,|$$

is closed. This is clear: our assumption that f is universally closed guarantees that ϕ is closed, and our assumption that g is universally closed guarantees that ψ is closed.

We now prove (2). Assume that g is separated and that $g \circ f$ is proper. The map f factors as a composition

$$\mathsf{X} \xrightarrow{f'} \mathsf{X} \times_{\mathsf{Z}} \mathsf{Y} \xrightarrow{f''} \mathsf{Y}.$$

The map f'' is a pullback of $g \circ f$ and therefore proper (Proposition 5.1.3.1). The map f' is a pullback of the diagonal map $\mathsf{Y} \to \mathsf{Y} \times_{\mathsf{Z}} \mathsf{Y}$, and is therefore a closed immersion by virtue of our assumption that g is separated. It follows that f' is proper (Example 5.1.2.4), so that $f = f'' \circ f'$ is proper by assertion (1).

We now prove (3). The assertion is local on Z, so we may assume without loss of generality that $\mathsf{Z} = \operatorname{Spét} R$ is affine. Then X is quasi-compact. Since f is surjective, we deduce that Y is quasi-compact (Proposition 2.3.3.1), so that the morphism g is quasi-compact. We will complete the proof by showing that g is universally closed. Let R' be a connective R-algebra; we wish to show that the map $|\operatorname{Spét} R' \times_{\operatorname{Spét} R} \mathsf{Y}\,| \to |\operatorname{Spec} R'|$ is closed. Replacing Z by $\operatorname{Spét} R'$, we are reduced to proving that the map $|\,\mathsf{Y}\,| \to |\,\mathsf{Z}\,|$ is closed. Let $K \subseteq |\,\mathsf{Y}\,|$ be a closed subset. Since f is surjective, and $g \circ f$ is proper, we deduce that $g(K) = (g \circ f)(f^{-1}K)$ is a closed subset of $|\,\mathsf{Z}\,|$, as desired. □

5.2 Finite Morphisms

Let $f : X \to Y$ be a morphism of schemes. Recall that f is said to be *finite* if, for every map $\operatorname{Spec} A \to Y$, the fiber product $X \times_Y \operatorname{Spec} A$ is isomorphic to an affine scheme $\operatorname{Spec} B$, where B is finitely generated as an A-module. This notion admits a straightforward extension to the setting of spectral algebraic geometry:

Definition 5.2.0.1. Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of spectral Deligne-Mumford stacks. We will say that f is *finite* if it satisfies the following pair of conditions:

(1) The morphism f is affine.

(2) The pushforward $f_* \mathcal{O}_{\mathcal{X}}$ is perfect to order 0 (as a quasi-coherent sheaf on $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$).

Remark 5.2.0.2. Let $f : (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be a morphism of spectral Deligne-Mumford stacks. Then f is finite if and only if the induced map $f_0 : (\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathscr{O}_{\mathcal{Y}})$ is finite (see Remark 2.4.4.2). In particular, a morphism of spectral algebraic spaces is finite if and only if it induces a finite morphism between the underlying ordinary algebraic spaces, in the sense of classical algebraic geometry.

5.2.1 Finite Morphisms and Proper Morphisms

Definitions 5.2.0.1 and 5.1.2.1 are closely related:

Proposition 5.2.1.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. The following conditions are equivalent:*

(1) *The morphism f is finite.*

(2) *The morphism f is proper and locally quasi-finite.*

(3) *The morphism f is proper and quasi-affine.*

(4) *The morphism f is proper and affine.*

The proof of Proposition 5.2.1.1 depends on the following bit of commutative algebra:

Proposition 5.2.1.2 ("Lying Over")**.** *Let $f : R \to R'$ be a homomorphism of commutative rings. Suppose that f is injective and integral (that is, every element of R' is integral over R). Then the induced map $|\operatorname{Spec} R'| \to |\operatorname{Spec} R|$ is surjective.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R; we wish to show that $\mathfrak{p}$ lies in the image of the map $|\operatorname{Spec} R'| \to |\operatorname{Spec} R|$. Replacing R by the localization $R_{\mathfrak{p}}$ (and R' by the ring $R_{\mathfrak{p}} \otimes_R R'$), we may assume that R is a local ring and that $\mathfrak{p}$ is its maximal ideal. We wish to show that there is a prime ideal of R' which lies over $\mathfrak{p}$: in other words, that the quotient $R'/\mathfrak{p}R'$ is not zero. Assume otherwise: then $\mathfrak{p}R' = R'$. We can therefore choose a finitely generated R'-subalgebra $R'' \subseteq R'$ such that $\mathfrak{p}R'' = R''$. Since every element of R' is integral over R, the ring R'' is finitely generated as an R-module. Applying Nakayama's lemma, we deduce that $R'' \simeq 0$. Since the composite map $R \to R'' \to R'$ is injective, we conclude that $R \simeq 0$, contradicting our assumption that R is local. □

Corollary 5.2.1.3. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. The following conditions are equivalent:*

(*a*) *Every element of the commutative ring $\pi_0 B$ is integral over $\pi_0 A$.*

(*b*) *The induced map $\operatorname{Spét} B \to \operatorname{Spét} A$ is universally closed. In other words, for every connective A-algebra A', the induced map of topological spaces $|\operatorname{Spec} A' \otimes_A B| \to |\operatorname{Spec} A'|$ is closed.*

Proof. We first show that $(a) \Rightarrow (b)$. Replacing A and B by A' and B', it will suffice to show that condition (a) implies that the map $\phi : |\operatorname{Spec} B| \rightarrow |\operatorname{Spec} A|$ is closed. Let K be a closed subset of $|\operatorname{Spec} B|$; we wish to show that $\phi(K) \subseteq |\operatorname{Spec} A|$ is closed. Write K as the image of $|\operatorname{Spec} B/I|$ for some ideal $I \subseteq B$. Set $J = f^{-1}(I) \subseteq A$. We wish to show that the composite map $K \simeq |\operatorname{Spec} B/I| \xrightarrow{\phi_0} |\operatorname{Spec} A/J| \xrightarrow{\psi} |\operatorname{Spec} A|$ has closed image. This is clear, since ψ has closed image and the map ϕ_0 is surjective by virtue of Proposition 5.2.1.2.

We now show that (b) implies (a). Assume that f satisfies condition (b) and let x be an element of $\pi_0 B$; we wish to show that x is integral over $\pi_0 A$. Since f is universally closed, the map

$$|\operatorname{Spec}(\pi_0 B)[y]| \simeq |\operatorname{Spec}(\pi_0 A)[y] \otimes_A B| \rightarrow |\operatorname{Spec}(\pi_0 A)[y]|$$

is closed. Set $R = (\pi_0 B)[x^{-1}]$, so that we can identify R with the quotient of $(\pi_0 B)[y]$ by the ideal generated by the element $1 - xy$. It follows that the map $|\operatorname{Spec} R| \rightarrow |\operatorname{Spec}(\pi_0 A)[y]|$ has closed image, which can be described as the vanishing locus of some ideal $J \subseteq (\pi_0 A)[y]$. Note that K does not intersect the vanishing locus of the principal ideal $(y) \subseteq (\pi_0 A)(y)$ (since the image of y in R is invertible), so that J and (y) generate the unit ideal in $(\pi_0 A)[y]$. It follows that J contains some polynomial of the form $1 + a_1 y + \cdots + a_n y^n$, where the coefficients a_i belong to $\pi_0 A$. Raising this polynomial to a suitable power, we may assume that its image in R vanishes: that is, we have $1 + a_1 x^{-1} + \cdots + a_n x^{-n} = 0$ in $R = (\pi_0 B)[x^{-1}]$. It follows that It follows that $x^{m+n} + a_1 x^{m+n-1} + \cdots + a_n x^m$ vanishes in $\pi_0 B$ for $m \gg 0$, so that x is integral over $\pi_0 A$ as desired. □

Proof of Proposition 5.2.1.1. Let $f : \mathsf{X} \rightarrow \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Note that each of the conditions appearing in the statement of Proposition 5.2.1.1 can be tested locally on Y; we may therefore assume without loss of generality that $\mathsf{Y} \simeq \operatorname{Spét} A$ is affine.

We first show that $(1) \Rightarrow (2)$. Assume that f is finite, so that $\mathsf{X} \simeq \operatorname{Spét} B$ for some connective $\mathbb{E}_\infty$-ring B for which $\pi_0 B$ is finitely generated as a module over $\pi_0 A$. We claim that f is proper. It is clear that f is quasi-compact, separated, and locally of finite type. We claim that it is universally closed: that is, for every connective A-algebra A', the induced map of topological spaces $|\operatorname{Spec}(A' \otimes_A B)| \rightarrow |\operatorname{Spec} A'|$ is closed. This is an immediate consequence of Corollary 5.2.1.3 (since every element of $\pi_0(A' \otimes_A B) \simeq \operatorname{Tor}_0^{\pi_0 A}(\pi_0 A', \pi_0 B)$ is integral over $\pi_0 A'$). To complete the proof of the implication $(1) \Rightarrow (2)$, it suffices to observe that every finite morphism of commutative rings is quasi-finite (Example B.2.4.4).

The implication $(2) \Rightarrow (3)$ follows from Theorem 3.3.0.2. We next show that $(3) \Rightarrow (4)$. Assume that f is proper and quasi-affine, so there exists a quasi-compact open immersion $j : \mathsf{X} \rightarrow \operatorname{Spét} B$ for some connective $\mathbb{E}_\infty$-algebra B over A. The projection map $\operatorname{Spét} A \rightarrow \operatorname{Spét} R$ is separated, so that j induces a closed immersion $\gamma : \mathsf{X} \rightarrow \mathsf{X} \times_{\operatorname{Spét} A} \operatorname{Spét} B$. Since f is proper, the canonical map $|\mathsf{X} \times_{\operatorname{Spét} A} \operatorname{Spét} B| \rightarrow |\operatorname{Spét} B|$ is closed. It follows that j has closed image in the topological space $|\operatorname{Sp\acute{e}t} B| \simeq |\operatorname{Spec} B|$, so that j is a clopen immersion and therefore X is affine.

We now complete the proof by showing that (4) ⇒ (1). Assume that f is proper and affine, so that we can write $\mathsf{X} = \operatorname{Sp\acute{e}t} B$ for some connective $\mathbb{E}_\infty$-ring B. Since f is locally of finite type, the commutative ring $\pi_0 B$ is finitely generated as an algebra over $\pi_0 A$. Corollary 5.2.1.3 shows that every element of $\pi_0 B$ is integral over $\pi_0 A$, so that $\pi_0 B$ is also finitely generated as a $\pi_0 A$-module. □

5.2.2 Stronger Finiteness Conditions

Let $f : A \to B$ be a homomorphism of commutative rings which exhibits B as a finitely generated A-module. Then B is finitely presented as an A-module if and only if it is finitely presented as an A-algebra. This observation generalizes to the setting of spectral algebraic geometry:

Proposition 5.2.2.1. *Let $f : \mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to \mathsf{Y}$ be a finite morphism of spectral Deligne-Mumford stacks and let $n \geqslant 0$. The following conditions are equivalent:*

(1) *The morphism f is locally finitely n-presented (Definition 4.2.3.1)*

(2) *The direct image $f_* \mathcal{O}_\mathcal{X}$ is finitely n-presented as an object of* $\operatorname{QCoh}(\mathsf{Y})$ *(Definition 2.8.4.4).*

Corollary 5.2.2.2. *Let $f : \mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to \mathsf{Y}$ be a finite morphism of spectral Deligne-Mumford stacks. The following conditions are equivalent:*

(1) *The morphism f is locally almost of finite presentation.*

(2) *The direct image $f_* \mathcal{O}_\mathcal{X}$ is almost perfect as an object of* $\operatorname{QCoh}(\mathsf{Y})$.

Corollary 5.2.2.3. *Let $\phi : A \to B$ be a morphism of commutative rings. Then the following conditions are equivalent:*

(a) *The morphism ϕ is almost of finite presentation when regarded as a morphism of $\mathbb{E}_\infty$-rings.*

(b) *The morphism ϕ factors as a composition $A \to A[x_1, \ldots, x_n] \xrightarrow{\phi'} B$, where ϕ exhibits B as an almost perfect module over $A[x_1, \ldots, x_n]$.*

Proof. Suppose first that (b) is satisfied. Corollary 5.2.2.2 implies that the map $\phi' : A[x_1, \ldots, x_n] \to B$ is almost of finite presentation, so we are reduced to showing that the map $A \to A[x_1, \ldots, x_n]$ is almost of finite presentation. Since the condition of being almost of finite presentation is stable under base change, we may assume without loss of generality that $A = \mathbf{Z}$, in which case the desired result follows from Theorem HA.7.2.4.31 .

Now suppose that (a) is satisfied. Then B is of finite type over A, so there exists a surjection of A-algebras $\phi' : A[x_1, \ldots, x_n] \to B$. Since B and $A[x_1, \ldots, x_n]$ are both almost of finite presentation over A, the morphism ϕ' is almost of finite presentation (Corollary HA.7.4.3.19). It follows from Corollary 5.2.2.2 that ϕ' exhibits B as an almost perfect $A[x_1, \ldots, x_n]$-module, so that condition (b) is satisfied. □

Remark 5.2.2.4. In the language of [101], Corollary 5.2.2.3 asserts that ϕ is almost of finite presentation if and only if the induced map of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is *pseudo-coherent*.

Corollary 5.2.2.5. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings which exhibits A as a square-zero extension of B by a connective B-module M. If M is almost perfect as a B-module, then f is almost of finite presentation.*

Proof. By virtue of Corollary 5.2.2.2, it will suffice to show that B is almost perfect as an A-module. We argue that, for every integer $n \geqslant 0$, B is perfect to order n as an A-module. The proof proceeds by induction on n. The case $n = 0$ is trivial (since the map $\pi_0 A \to \pi_0 B$ is surjective). To carry out the inductive step, let us assume $n > 0$ and that B is perfect to order $(n-1)$ as an A-module. It follows from Proposition 2.7.3.3 that M is also perfect to order $(n-1)$ as an A-module. The cofiber sequence of A-modules $A \to B \to \Sigma(M)$ then shows that B is perfect to order n as an A-module, as desired. □

The proof of Proposition 5.2.2.1 will require the following category-theoretic fact:

Lemma 5.2.2.6. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category which admits filtered colimits, and suppose that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves filtered colimits. Let $A \in \mathrm{CAlg}(\mathcal{C})$. Assume that $\mathcal{C}$ is equivalent to an n-category for some integer $n \geqslant 1$, and that each tensor power $A^{\otimes m}$ is a compact object of $\mathcal{C}$. Then A is a compact object of $\mathrm{CAlg}(\mathcal{C})$.*

Proof. Let P be a filtered partially ordered set, let $P^{\rhd}$ be the partially ordered set obtained from P by adjoining a new largest element which we will denote by ω, and suppose we are given a diagram $B : \mathrm{N}(P^{\rhd}) \to \mathrm{CAlg}(\mathcal{C})$. We wish to prove that the canonical map

$$\varinjlim_{\alpha \in P} \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(A, B(\alpha)) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(A, B(\omega))$$

is a homotopy equivalence.

For each element $\alpha \in P^{\rhd}$, we let $P^{\rhd}_{\geqslant \alpha}$ denote the subset of $P^{\rhd}$ consisting of elements which are $\geqslant \alpha$, and let $B^{\geqslant \alpha}$ be the restriction of B to $\mathrm{N}(P^{\rhd}_{\geqslant \alpha})$. Let $\mathrm{Comm}^{\otimes}$ denote the commutative ∞-operad, and let $q : \mathcal{C}^{\otimes} \to \mathrm{Comm}^{\otimes}$ exhibit $\mathcal{C}^{\otimes}$ as a symmetric monoidal ∞-category. Since $\mathcal{C}$ is equivalent to an n-category, we may assume without loss of generality that the simplicial set $\mathcal{C}^{\otimes}$ is $(n+2)$-coskeletal.

For each index $\alpha \in P^{\rhd}$, let $\mathcal{C}(\alpha)^{\otimes}$ denote the ∞-operad $\mathcal{C}^{\otimes}_{/B^{\geqslant\alpha}_{\mathrm{Comm}^{\otimes}}}$ described in Theorem HA.2.2.2.4 , so that we have canonical equivalences

$$\mathrm{CAlg}(\mathcal{C}(\alpha)) \simeq \mathrm{CAlg}(\mathcal{C})_{/B^{\geqslant\alpha}} \simeq \mathrm{CAlg}(\mathcal{C})_{/B(\alpha)}$$

for each $\alpha \in P^{\rhd}$. We are therefore reduced to proving that the canonical map

$$\varinjlim_{\alpha\in P} \mathrm{CAlg}(\mathcal{C}(\alpha)) \times_{\mathrm{CAlg}(\mathcal{C})} \{A\} \to \mathrm{CAlg}(\mathcal{C}(\omega)) \times_{\mathrm{CAlg}(\mathcal{C})} \{A\}$$

is a homotopy equivalence.

For each α in $P^{\rhd}$, let $\mathcal{B}(\alpha)^{\otimes}$ denote the fiber product $\mathrm{Comm}^{\otimes} \times_{\mathcal{C}^{\otimes}} \mathcal{C}(\alpha)^{\otimes}$. We wish to prove that the canonical map $\varinjlim_{\alpha\in P} \mathrm{CAlg}(\mathcal{B}(\alpha)) \to \mathrm{CAlg}(\mathcal{B}(\omega))$ is a homotopy equivalence. Since the $(n+2)$-skeleton of $\mathrm{Comm}^{\otimes}$ is a finite simplicial set, the construction $\mathcal{D}^{\otimes} \mapsto \mathrm{CAlg}(\mathcal{D})$ commutes with filtered colimits when restricted to ∞-operads which are $(n+2)$-coskeletal. We are therefore reduced to proving that the map $\theta : \varinjlim_{\alpha\in P} \mathcal{B}(\alpha)^{\otimes} \to \mathcal{B}(\omega)^{\otimes}$ is an equivalence of ∞-operads. This follows easily from our assumption that each tensor power $A^{\otimes m}$ is compact when regarded as an object of $\mathcal{C}$. □

Proof of Proposition 5.2.2.1. Let $f : \mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to \mathsf{Y}$ be a finite morphism of spectral Deligne-Mumford stacks; we wish to show that f is locally finitely n-presented if and only if the direct image $f_* \mathscr{O}_{\mathcal{X}} \in \mathrm{QCoh}(\mathsf{Y})$ is finitely n-presented. Both assertions can be tested locally on Y, so we may assume without loss of generality that $\mathsf{Y} \simeq \mathrm{Sp\acute{e}t}\, A$ is affine. Since f is finite, we can write $\mathsf{X} \simeq \mathrm{Sp\acute{e}t}\, B$ for some connective $\mathbb{E}_{\infty}$-algebra B over A. Suppose first that f is locally almost of finite presentation; we wish to show that B is finitely n-presented as an A-module. Write A as a filtered colimit $\varinjlim A_{\alpha}$, where each A_{α} is a compact object of $\mathrm{CAlg}^{\mathrm{cn}}$ and therefore Noetherian. Using Corollary **??**, we can choose an index α and an equivalence $B \simeq \tau_{\leqslant n}(A \otimes_{A_{\alpha}} B_{\alpha})$, where B_{α} is finitely n-presented over A_{α}. Let $x_1, \ldots, x_n \in \pi_0 B_{\alpha}$ be a collection of elements which generate $\pi_0 B_{\alpha}$ as an algebra over $\pi_0 A_{\alpha}$. Let $\overline{x}_i$ denote the image of x_i in $\pi_0 B$. Since $\pi_0 B$ is a finitely generated module over $\pi_0 A$, each $\overline{x}_i$ is integral over $\pi_0 A$ and therefore is therefore the solution to some polynomial equation $\overline{f}_i(\overline{x}_i) = 0$ where the coefficients of $\overline{f}_i$ lie in $\pi_0 A$. Enlarging α if necessary, we can assume that each $\overline{f}_i$ can be lifted to a polynomial f_i having coefficients in $\pi_0 A_{\alpha}$ and that $f_i(x_i) = 0$. It follows that each x_i is integral over $\pi_0 A_{\alpha}$, so that $\pi_0 B_{\alpha}$ is finitely generated as a module over $\pi_0 A_{\alpha}$. We may therefore replace A by A_{α} and B by B_{α}, thereby reducing to the case where A is Noetherian. In this case, B is also Noetherian. It follows that each homotopy group $\pi_i B$ is finitely generated as a module over $\pi_0 B$, and therefore also as a module over $\pi_0 A$ (since $\pi_0 B$ is a finitely generated $\pi_0 A$-module). It follows from Corollary 2.7.2.3 that B is finitely n-presented over A, as desired.

Now suppose that B is finitely n-presented as an A-module; we wish to show that f is locally finitely n-presented. Let $\mathcal{C}$ denote the full subcategory of Mod_A spanned by those

A-modules which are connective and n-truncated. Note that $\mathcal{C}$ inherits a symmetric monoidal structure, whose tensor product is given by the construction $(M, N) \mapsto \tau_{\leqslant n}(M \otimes_A N)$. Then B is compact when viewed as an object of $\mathcal{C}$, and we wish to show that B is compact when viewed as an object of $\mathrm{CAlg}(\mathcal{C})$. This follows from Lemma 5.2.2.6, since $\mathcal{C}$ is equivalent to an $(n+1)$-category and the collection of compact objects of $\mathcal{C}$ is stable under the tensor product on $\mathcal{C}$. □

5.2.3 Finite Flat Morphisms

We close this section by introducing the class of finite flat morphisms between spectral Deligne-Mumford stacks.

Definition 5.2.3.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. We will say that f is *finite flat* if f is finite and the direct image $f_* \mathscr{O}_{\mathsf{X}} \in \mathrm{QCoh}(\mathsf{Y})$ is locally free of finite rank (see Notation 2.9.3.1). We will say that f is *finite flat of degree d* if it is finite flat and the direct image $f_* \mathscr{O}_{\mathsf{X}}$ is locally free of rank d.

Example 5.2.3.2. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. If f is finite étale of degree d (Definition 3.3.2.3), then it is finite flat of degree d.

Proposition 5.2.3.3. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Then f is finite flat (in the sense of Definition 5.2.3.1) if and only if f is proper, flat, locally quasi-finite, and locally almost of finite presentation over Y.*

Proof. Suppose first that f is finite flat. Then f is finite, hence proper and locally quasi-finite (Proposition 5.2.1.1). Since f is affine and the direct image $f_* \mathscr{O}_{\mathsf{X}}$ is flat, we conclude that f is flat. Finally, f is locally almost of finite presentation because $f_* \mathscr{O}_{\mathsf{X}} \in \mathrm{QCoh}(\mathsf{Y})$ is almost perfect (Corollary 5.2.2.2).

For the converse, suppose that f is proper, flat, locally quasi-finite, and locally almost of finite presentation over Y. Then f is finite (Proposition 5.2.1.1), and in particular affine. The flatness of f then guarantees that $\mathscr{A} = f_* \mathscr{O}_{\mathsf{X}}$ is flat. Moreover, our assumption that f is locally almost of finite presentation guarantees that $f_* \mathscr{O}_{\mathsf{X}}$ is almost perfect. Applying Proposition HA.7.2.4.20 , we deduce that $\mathscr{A}$ is locally free of finite rank.

□

Warning 5.2.3.4. The hypothesis that f be locally almost of finite presentation is necessary in Proposition 5.2.3.3. A morphism $f : \mathsf{X} \to \mathsf{Y}$ which is both finite and flat need not be finite flat. For example, if X is a totally disconnected compact Hausdorff space, R is the commutative ring of locally constant $\mathbf{C}$-valued functions on X, and $\epsilon : R \to \mathbf{C}$ is the map given by evaluation at some point $x \in X$, then the induced map $\mathrm{Sp\acute{e}t}\, \mathbf{C} \to \mathrm{Sp\acute{e}t}\, R$ is both finite and flat, but exhibits $\mathbf{C}$ as a projective R-module only when x is an isolated point of X (otherwise, the field $\mathbf{C}$ is not finitely presented over R).

Remark 5.2.3.5. Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}. \end{array}$$

If f is finite flat (of degree d), then f' is finite flat (of degree d). The converse holds if g is a flat covering.

Remark 5.2.3.6. Let $f : \mathsf{X} \to \mathsf{Y}$ and $g : \mathsf{Y} \to \mathsf{Z}$ be morphisms of spectral Deligne-Mumford stacks. If f is finite flat (of degree d) and g is finite flat (of degree d'), then the composition $g \circ f$ is also finite flat (of degree dd'). To prove this, we can assume without loss of generality that $\mathsf{Z} \simeq \operatorname{Spét} A$ is affine, so that $\mathsf{Y} \simeq \operatorname{Spét} B$ for some $\mathbb{E}_\infty$-algebra which is locally free of rank d' as an A-module, and $\mathsf{X} \simeq \operatorname{Spét} C$ for some $\mathbb{E}_\infty$-algebra C which is locally free rank d as a B-module; it then follows that C is locally free of rank dd' as an A-module.

5.3 Valuative Criteria

Let $f : X \to Y$ be a morphism of schemes which is separated and of finite type. According to the *valuative criterion of properness*, the morphism f is proper if and only if it satisfies the following condition:

$(*)$ For every valuation ring V with fraction field K and every commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \operatorname{Spec} V & \longrightarrow & Y, \end{array}$$

there exists a dotted arrow as indicated, rendering the diagram commutative (since f is separated, the dotted arrow is essentially unique).

Our goal in this section is to prove the following analogue in the setting of spectral Deligne-Mumford stacks:

Theorem 5.3.0.1 (Valuative Criterion for Properness)**.** *Let $f : \mathsf{X} \to \mathsf{Y}$ be a quasi-compact, separated map of spectral Deligne-Mumford stacks which is locally of finite type. Then f is proper if and only if the following condition is satisfied:*

$(*)$ *For every valuation ring V with fraction field K and every commutative diagram*

$$\begin{array}{ccc} \operatorname{Spét} K & \longrightarrow & \mathsf{X} \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spét} V & \longrightarrow & \mathsf{Y}, \end{array}$$

there exists a dotted arrow as indicated, rendering the diagram commutative.

Moreover, if Y *is locally Noetherian, then it suffices to verify condition* $(*)$ *in the special case where* V *is a discrete valuation ring.*

Remark 5.3.0.2. In the statement of Theorem 5.3.0.1, there is no loss of generality in assuming that Y is a spectral algebraic space (or even that Y is affine), in which case X is also a spectral algebraic space. In this case, the condition that $f : \mathsf{X} \to \mathsf{Y}$ is proper depends only on the underlying map $f_0 : \tau_{\leqslant 0} \mathsf{X} \to \tau_{\leqslant 0} \mathsf{Y}$ of ordinary algebraic spaces. Consequently, Theorem 5.3.0.1 follows immediately from the analogous statement for ordinary algebraic spaces (see [129] for a more general statement, which does not require that the diagonal of f is a closed immersion).

5.3.1 The Valuative Criterion for Separatedness

Before giving the proof of Theorem 5.3.0.1, let us describe some easy consequences:

Proposition 5.3.1.1 (Valuative Criterion for Separatedness)**.** *Let* $f : \mathsf{X} \to \mathsf{Y}$ *be a quasi-separated map of spectral Deligne-Mumford stacks which represent functors* $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$*. Assume that* f *is a relative spectral algebraic space (that is, that the induced map* $X(R) \to Y(R)$ *has discrete homotopy fibers, for every commutative ring* R*). The following conditions are equivalent:*

(1) *The map* f *is separated.*

(2) *The diagonal map* $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ *is proper.*

(3) *For every valuation ring* V *with fraction field* K*, the canonical map* $X(V) \to X(K) \times_{Y(K)} Y(V)$ *is* (-1)*-truncated (that is, it is the inclusion of a summand).*

Moreover, if f *is locally of finite type and* Y *is locally Noetherian, then it suffices to verify condition* (3) *in the special case where* V *is a discrete valuation ring.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate (since any closed immersion is proper). Let $\mathsf{Z} = \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$, and let $\delta' : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Z}} \mathsf{X}$ be the diagonal of the map δ. Since f is a relative spectral algebraic space, the map δ' induces an equivalence between the underlying 0-truncated spectral Deligne-Mumford stacks, and is therefore a closed immersion. It follows that δ is separated. Since X is quasi-separated, the map δ is quasi-compact. Since δ admits a left homotopy inverse, it is locally of finite type. Using Theorem 5.3.0.1, we see that δ is proper if and only if the following condition is satisfied:

($2'$) Let $Z : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by Z. Then, for every valuation ring V with fraction field K, the canonical map $X(V) \to X(K) \times_{Z(K)} Z(V)$ is surjective on connected components.

The equivalence $(2') \Leftrightarrow (3)$ now follows by inspection.

Write $\mathsf{Z} = (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ and $\mathsf{Z}_0 = (\mathcal{Z}, \pi_0 \mathcal{O}_{\mathcal{Z}})$. Note that δ is proper if and only if the induced map $\mathsf{X} \times_{\mathsf{Z}} \mathsf{Z}_0 \to \mathsf{Z}_0$ is proper. If f is locally of finite type and Y is locally Noetherian, then Z_0 is locally Noetherian. Using Theorem 5.3.0.1, we deduce that δ is proper if and only if condition $(2')$ is satisfied whenever V is a discrete valuation ring (which is equivalent to the requirement that (3) is satisfied whenever V is a discrete valuation ring).

To complete the proof, it will suffice to show that $(2) \Rightarrow (1)$. Assume that δ is proper. Since δ is locally quasi-finite, we conclude that δ is finite (Proposition 5.2.1.1). Choose a map $\operatorname{Sp\acute{e}t} R \to \mathsf{Z}$, so that $\mathsf{X} \times_{\mathsf{Z}} \operatorname{Sp\acute{e}t} R \simeq \operatorname{Sp\acute{e}t} R'$ for some R-algebra R'. We wish to prove that the underlying map of commutative rings $\pi_0 R \to \pi_0 R'$ is surjective. Replacing R by $\pi_0 R$, we may assume that R is a commutative ring. Since a map of discrete R-modules $M \to N$ is surjective if and only if it surjective after localization at any prime ideal $\mathfrak{p}$ of R, we may replace R by $R_{\mathfrak{p}}$ and thereby reduce to the case where R is local. Since $\pi_0 R'$ is finitely generated as a module over R, we may use Nakayama's lemma to replace R by its residue field and thereby reduce to the case where R is a field κ. Then $\pi_0 R'$ is a finite dimensional algebra over κ. We will complete the proof by showing that the dimension of $\pi_0 R'$ is $\leqslant 1$. For this, it suffices to show that the inclusion of the first factor induces an isomorphism

$$\pi_0 R' \to (\pi_0 R') \otimes_{\kappa} (\pi_0 R') \simeq \pi_0 (R' \otimes_{\kappa} R').$$

This follows immediately from our observation that δ' induces an equivalence on the underlying 0-truncated spectral Deligne-Mumford stacks. □

Corollary 5.3.1.2. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of quasi-compact, quasi-separated morphism of spectral Deligne-Mumford stacks which is locally of finite type. Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functors represented by X and Y, and suppose that f is a relative spectral algebraic space. Then f is proper if and only if, for every valuation ring V with fraction field K, the induced map $X(V) \to X(K) \times_{Y(K)} Y(V)$ is a homotopy equivalence. Moreover, if Y is locally Noetherian, then it suffices to verify this condition in the special case where V is a discrete valuation ring.*

Proof. Combine Theorem 5.3.0.1 with Proposition 5.3.1.1. □

5.3.2 Interlude

We now review some commutative algebra which will be needed in the proof of Theorem 5.3.0.1.

Theorem 5.3.2.1 (Krull-Akizuki)**.** *Let R be a Noetherian integral domain of Krull dimension $\leqslant 1$, let K denote the fraction field of R, and let R' be the integral closure of R in K. Then R' is a Dedekind ring.*

Proof. We first show that R' is Noetherian. Let $I \subseteq R'$ be an ideal; we wish to show that I is finitely generated. If $I = (0)$, there is nothing to prove. Otherwise, we can choose some nonzero element $x \in I \subseteq R'$. The element x is integral over R, so we have $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ for some elements $\{a_i\}_{1 \leqslant i \leqslant n}$ in R. Choose n as small as possible, so that $a_n \neq 0$. Then $t = r_n$ is a nonzero element of $I \cap R$.

Since R has Krull dimension $\leqslant 1$, the quotient R/tR is an R-module of finite length. For every R-module M of finite length, let $\ell(M)$ denote the length of M. Note that if $A \subseteq R'$ is a finitely generated R-subalgebra, then A is finitely generated as an R-module (since every element of R' is integral over R), so that $A/t^m A$ is a finitely generated module over $R/t^m R$ and is therefore an R-module of finite length for each $m \geqslant 0$. We have an exact sequence of R-modules $R/t^m R \to A/t^m A \to A/(R + t^m A) \to 0$, which gives inequalities

$$m\ell(A/tA) = \ell(A/t^m A) \leqslant \ell(R/t^m R) + \ell(A/(R + t^m A)) \leqslant m\ell(R/tR) + \ell(A/R).$$

Since R-module A/R has finite length (it is finitely generated and vanishes after extending scalars to K), it follows that $\ell(A/tA) \leqslant \ell(R/tR)$. In particular, the image of the map $A \to R'/tR'$ is a module of finite length which is bounded above by an integer $\ell(R/tR)$ which does not depend on A. Writing R' as a union of its finitely generated R-subalgebras, we deduce that R'/tR' is an R-module of finite length (bounded above by $\ell(R/tR)$. It follows that the quotient I/tR' is finitely generated as an R-module (and therefore also as an R'-module), so that the ideal I is finitely generated as desired.

To complete the proof, we must show that the Noetherian ring R' is a Dedekind ring. Since R' is integrally closed by construction, it will suffice to show that R' has Krull dimension $\leqslant 1$. To prove this, it will suffice to show that if $I \subseteq R'$ is a *prime* ideal and $t \in R$ is chosen as above, then I is minimal among prime ideals of R' containing (t). In fact, the quotient R'/tR' is an Artinian ring, since it of finite length as an R-module (and therefore also as an (R'/tR')-module). □

Proposition 5.3.2.2. *Let R be a commutative ring, let K be a field, and let $\phi : R \to K$ be a ring homomorphism. Let $\mathfrak{p} \subseteq R$ be a prime ideal containing* $\ker(\phi)$. *Then there exists a valuation subring $V \subseteq K$ (with fraction field K) such that $\phi(R) \subseteq V$ and $\mathfrak{p} = \phi^{-1}\mathfrak{m}$, where $\mathfrak{m}$ denotes the maximal ideal of V. Moreover, if R is Noetherian, $\mathfrak{p} \neq \ker(\phi)$, and K is a localization of a finitely generated R-algebra, then we can arrange that V is a discrete valuation ring.*

Proof. We first treat the general case (where R is not assumed to be Noetherian). Replacing R by the localization $R_{\mathfrak{p}}$, we may assume that R is a local ring with maximal ideal $\mathfrak{p}$. Let P denote the partially ordered consisting of subrings $A \subseteq K$ which contain $\phi(R)$ and satisfy $\mathfrak{p}A \neq A$. Using Zorn's lemma, we deduce that P has a maximal element, which we will denote by V. We will show that V has the desired properties.

We first claim that V is a local ring. Choose elements $x, y \in V$ with $x + y = 1$; we must show that either x or y is invertible in V. Since $V/\mathfrak{p}V \neq 0$, one of the localizations $(V/\mathfrak{p}V)[x^{-1}]$ and $(V/\mathfrak{q}V)[y^{-1}]$ must be nonzero. Without loss of generality, we may assume that $(V/\mathfrak{q}V)[x^{-1}] \neq 0$, so that $V[x^{-1}] \neq \mathfrak{q}V[x^{-1}]$. The maximality of V then implies that $V = V[x^{-1}]$, so that x is invertible in V.

Let $\mathfrak{m}$ denote the maximal ideal of V. Since $\mathfrak{p}V$ is a proper ideal of V, we have $\mathfrak{p}V \subseteq \mathfrak{m}$ and therefore $\mathfrak{p} \subseteq \phi^{-1}\mathfrak{m}$. Since $\mathfrak{p}$ is a maximal ideal of R, we conclude that $\mathfrak{p} = \phi^{-1}\mathfrak{m}$.

We now complete the proof by showing that V is a valuation ring with fraction field K. Let x be a nonzero element of K; we wish to show that either x or x^{-1} belongs to V. If x^{-1} does not belong to V, then the subring $V' \subseteq K$ generated by V and x^{-1} is strictly larger than V and therefore satisfies $V' = \mathfrak{p}V'$. In particular, we can write $1 = \sum_{0 \leqslant i \leqslant n} c_i x^{-i}$ for some coefficients $c_i \in \mathfrak{p}V \subseteq \mathfrak{m}$. Then $x^n = \sum_{1 \leqslant i \leqslant n} \frac{c_i}{1-c_0} x^{n-i}$ so that x is integral over V. If x does not belong to V, then the subring $V'' \subseteq K$ generated by V and x properly contains V and is finitely generated as an V-module. The maximality of V implies that $V'' = \mathfrak{p}V''$. Using Nakayama's lemma, we deduce that $V'' = 0$ and obtain a contradiction. This completes the proof of the first assertion.

Now suppose that R is Noetherian, $\mathfrak{p} \neq \ker(\phi)$, and that K is finitely generated over R. Replacing R by its image in K, we may suppose that R is a subring of K. Let $x_1, \ldots, x_n \in K$ be a transcendence basis for K over the fraction field of R. Replacing R by $R[x_1, \ldots, x_n]$ and $\mathfrak{p}$ by $\mathfrak{p}[x_1, \ldots, x_n]$, we may reduce to the case where K is a finite algebraic extension of the fraction field of R. Replacing R by the localization $R_{\mathfrak{p}}$, we may assume that R is a local ring with maximal ideal $\mathfrak{p}$. Since R is Noetherian, we can choose a finite set of generators $y_1, \ldots, y_m \in \mathfrak{p}$ for the ideal $\mathfrak{p}$. For $1 \leqslant i \leqslant m$, let R_i denote the subring of K generated by R together with the elements $\frac{y_j}{y_i}$. We now claim:

$(*)$ There exists $1 \leqslant i \leqslant m$ such that y_i is not invertible in R_i.

Suppose that $(*)$ is not satisfied: that is, $y_i^{-1} \in R_i$ for every index i. Then each y_i^{-1} can be written as a polynomial (with coefficients in R) in the variables $\frac{y_j}{y_i}$. Clearing denominators, we deduce that there exists an integer a such that $y_i^a \in \mathfrak{p}^{a+1}$ for every index i. It follows that $\mathfrak{p}^b \subseteq \mathfrak{p}^{b+1}$ for $b > a(m-1)$. Since R is Noetherian with maximal ideal $\mathfrak{p}$, the Krull intersection theorem (Corollary 7.3.6.10) implies that $\bigcap_{b \geqslant 0} \mathfrak{p}^b = 0$, so that $\mathfrak{p}^b = 0$ for $b > a(m-1)$. In particular, $\mathfrak{p}$ consists of nilpotent elements of R. Since $R \subseteq K$ is an integral domain, we deduce that $\mathfrak{p} = 0$, contradicting our assumption that $\mathfrak{p} \neq \ker(\phi)$. This completes the proof of $(*)$.

Using $(*)$, let us choose an index i such that y_i is not invertible in R_i. Let $\mathfrak{q}$ be minimal among prime ideals of R_i which contain y_i. Then $\mathfrak{q}$ contains each y_j, so that $\mathfrak{q} \cap R$ contains $\mathfrak{p}$. Since $\mathfrak{p}$ is a maximal ideal of R, we deduce that $\mathfrak{q} \cap R = \mathfrak{p}$. We may therefore replace R by $(R_i)_{\mathfrak{q}}$ (which is Noetherian, since it is finitely generated over R) and thereby reduce to the case where the prime ideal $\mathfrak{p}$ of R is minimal among prime ideals containing some element $x \in R$. It follows that R is a local Noetherian ring of dimension $\leqslant 1$ (Theorem B.2.1.2). Let R' denote the integral closure of R in K, so that R' is a Dedekind domain (Theorem 5.3.2.1). Since R' is integral over R, the maximal ideal $\mathfrak{p}$ of R can be lifted to a maximal ideal $\mathfrak{p}'$ of R' (Proposition 5.2.1.2). Then $V = R'_{\mathfrak{p}'}$ is a discrete valuation ring having the desired properties. □

Lemma 5.3.2.3. *Let V and $\{W_i\}_{1 \leqslant i \leqslant n}$ be valuation rings having the same fraction field K. If $W_i \nsubseteq V$ for $1 \leqslant i \leqslant n$, then $\bigcap W_i \nsubseteq V$.*

Proof. Set $R = V \cap \bigcap_{1 \leqslant i \leqslant n} W_i$. Let $\mathfrak{p} \subseteq R$ denote the inverse image of the maximal ideal of V, and for $1 \leqslant i \leqslant n$ let $\mathfrak{q}_i \subseteq R$ denote the inverse image of the maximal ideal of W_i. We will show that $\mathfrak{p} \nsubseteq \mathfrak{q}_i$ for $1 \leqslant i \leqslant n$. It then follows from prime avoidance that there exists an element $x \in \mathfrak{p}$ which does not belong to any $\mathfrak{q}_i$. Then $\frac{1}{x} \in K$ belongs to each W_i but does not belong to V.

It remains to show that $\mathfrak{p} \nsubseteq \mathfrak{q}_i$. To prove this, choose an element $y \in W_i$ which does not belong to V. We claim that there exists an integer d such that $\frac{y-1}{y^d-1}$ belongs to $\mathfrak{q}_i$ but does not belong to $\mathfrak{p}$ (note that $y \notin V$ guarantees that y is not a root of unity in K, so that $\frac{y-1}{y^d-1}$ is well-defined). To guarantee this, it will suffice to ensure the following:

(*a*) The element $\frac{y-1}{y^d-1}$ belongs to V and to each W_i (and is therefore an element of A).

(*b*) The product $y\frac{y-1}{y^d-1}$ belongs to V.

Note that $\frac{y-1}{y^d-1}$ has a multiplicative inverse $y^{d-1} + \cdots + 1 \in W_i$, and therefore cannot belong to the prime ideal $\mathfrak{q}_i$. On the other hand, if (*a*) and (*b*) are satisfied, then $\frac{y-1}{y^d-1}$ does belong to $\mathfrak{p}$: otherwise, it would be an invertible element of the valuation ring V, so that (*b*) would imply $y \in V$. We now complete the proof by observing that conditions (*a*) and (*b*) are satisfied for any $d \geqslant 2$ for which y does not represent a dth root of unity in the residue field of any of the valuation rings V, $\{W_i\}_{1 \leqslant i \leqslant n}$ to which it belongs. □

Proposition 5.3.2.4. *Let V be a valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K, let K' be a finite product of finite algebraic extension fields of K, and let R be a subring of K' containing V. Then there are at most finitely many prime ideals $\mathfrak{q} \subseteq R$ such that $\mathfrak{q} \cap V = \mathfrak{m}$.*

Proof. Write $K' = \prod_{1 \leqslant i \leqslant m} K_i$, where each K_i is finite algebraic extension field of K. For $1 \leqslant i \leqslant n$, let $\mathfrak{p}_i \subseteq R$ denote the kernel of the composite map $R \hookrightarrow K' \to K_i$. Note that every prime ideal $\mathfrak{q} \subseteq R$ contains one of the prime ideals $\mathfrak{p}_i$ (otherwise, we could choose elements $x_i \in \mathfrak{p}_i - \mathfrak{q}$ for $1 \leqslant i \leqslant m$, in which case we have $0 = \prod_{1 \leqslant i \leqslant m} x_i \notin \mathfrak{q}$). It will therefore suffice to show that for $1 \leqslant i \leqslant m$, there are only finitely many prime ideals $\mathfrak{q} \subseteq R$

such that $\mathfrak{p}_i \subseteq \mathfrak{q}$ and $\mathfrak{q} \cap V = \mathfrak{m}$. To prove this, we can replace K' by K_i and R by its image in K_i, and thereby reduce to the case where K' is a finite algebraic extension of K.

If K is a field of characteristic p, let K^{perf} denote the perfect closure of K, given as the direct limit of the sequence $K \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K \to \cdots$, and define ${K'}^{\mathrm{perf}}$, V^{perf} and R^{perf} similarly. Then we have a commutative diagram

$$\begin{array}{ccc} |\operatorname{Spec} R^{\mathrm{perf}}| & \longrightarrow & |\operatorname{Spec} R| \\ \downarrow & & \downarrow \\ |\operatorname{Spec} V^{\mathrm{perf}}| & \longrightarrow & |\operatorname{Spec} V| \end{array}$$

where the horizontal maps are bijective. We may therefore replace K by K^{perf} in the statement of Corollary 5.3.2.4, and thereby reduce to the case where K is perfect. Then K' is a separable extension of K. Enlarging K' if necessary, we may assume that K' is a Galois extension of K with Galois group G.

For every prime ideal $\mathfrak{q} \subseteq R$, there exists a valuation ring $V' \subseteq K'$ with fraction field K' and maximal ideal $\mathfrak{m}'$ such that $R \subseteq V'$ and $\mathfrak{q} = R \cap \mathfrak{m}'$ (Proposition 5.3.2.2). In particular, it follows that $\mathfrak{m} = \mathfrak{m}' \cap V$, so we can recover V as $V' \cap K$. We will complete the proof by showing that there are only finitely many such valuation rings $V' \subseteq K'$. More precisely, we claim that if W_0 and W_1 are valuation rings in K' satisfying $W_0 \cap K = W_1 \cap K$, then W_0 and W_1 are conjugate by the action of the Galois group G. To prove this, we first note that if $g(W_0) \nsubseteq W_1$ for all $g \in G$, then Lemma 5.3.2.3 supplies an element $x \in \bigcap_{g \in G} g(W_0)$ which does not belong to W_1. This is a contradiction, since the product $\prod_{g \in G} g(x)$ belongs to $W_0 \cap K$ but not to $W_1 \cap K$. We therefore have $g(W_0) \subseteq W_1$ for some $g \in G$, and similarly $h(W_1) \subseteq W_0$ for some $h \in G$. This yields a chain of inclusions

$$W_0 \supseteq h(W_1) \supseteq hg(W_0) \supseteq hgh(W_1) \supseteq \cdots \supseteq (hg)^n(W_0) = W_0,$$

where n denotes the order of the element $hg \in G$. It follows that equality holds throughout, so that W_1 and W_0 are conjugate by the action of G. □

5.3.3 The Proof of Theorem 5.3.0.1

Fix a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks which is separated, quasi-compact, and locally of finite type. We wish to show that f is proper if and only if it satisfies the valuative criterion:

$(*)$ For every valuation ring V with fraction field K and every commutative diagram

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} K & \longrightarrow & \mathsf{X} \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Sp\acute{e}t} V & \longrightarrow & \mathsf{Y}, \end{array}$$

there exists a dotted arrow as indicated, rendering the diagram commutative.

We first prove that condition $(*)$ is satisfied when f is proper. Without loss of generality, we may replace Y by $\operatorname{Spét} V$. Let K denote the fraction field of V; we wish to show that if $f : \mathsf{X} \to \operatorname{Spét} V$ is proper, then every map $\phi : \operatorname{Spét} K \to \mathsf{X}$ (of spectral algebraic spaces over V) can be extended to a map $\operatorname{Spét} V \to \mathsf{X}$. The map ϕ determines a point $\eta \in |\mathsf{X}|$. Let Z denote the smallest closed subset of $|\mathsf{X}|$ containing the point η. Let Z denote the reduced closed substack of X corresponding to the subset Z.

Lemma 5.3.3.1. *In the situation above, the composite map* $\mathsf{Z} \hookrightarrow \mathsf{X} \xrightarrow{f} \operatorname{Spét} V$ *is locally quasi-finite.*

Proof. This assertion is local on X. Let us therefore choose an étale map $g : \operatorname{Spét} A \to \mathsf{X}$, and set

$$\operatorname{Spét} K \times_{\mathsf{X}} \operatorname{Spét} A \simeq \operatorname{Spét} K' \qquad \mathsf{Z} \times_{\mathsf{X}} \operatorname{Spét} A \simeq \operatorname{Spét} B.$$

Since g is étale, K' is a product $\prod_{1 \leqslant i \leqslant n} K_i$ of separable algebraic extension fields of K. Similarly, $\operatorname{Spét} B \to \mathsf{Z}$ is étale, so that B is a reduced commutative ring. By construction, the map $\operatorname{Spét} K \to \mathsf{Z}$ induces an injective map of commutative rings $B \hookrightarrow K'$. Since f is locally of finite type, B is finitely generated as a commutative ring over V. To show that $\mathsf{Z} \to \operatorname{Spét} V$ is locally quasi-finite, we wish to that for every prime ideal $\mathfrak{p} \subseteq V$, there are only finitely many prime ideals of B lying over $\mathfrak{p}$. Replacing V by $V_{\mathfrak{p}}$, we may reduce to the case where $\mathfrak{p}$ is the maximal ideal of V. In this case, the desired result follows from Proposition 5.3.2.4. □

Returning the proof of Theorem 5.3.0.1, we note that the map $f|_{\mathsf{Z}} : \mathsf{Z} \to \operatorname{Spét} V$ is both proper and locally quasi-finite. It follows that it is *finite*: that is, we can write $\mathsf{Y} \simeq \operatorname{Spét} R$ for some commutative ring R which is finitely generated as a V-module (see Proposition 5.2.1.1). Moreover, the map $\operatorname{Spét} K \to \mathsf{Y}$ induces an injection $R \to K$. We may therefore identify R with a subalgebra of K which is finitely generated as a module over V. Since V is a valuation ring of K, it is integrally closed in K. It follows that $R \simeq V$, so that the inclusion $\mathsf{Z} \hookrightarrow \mathsf{X}$ gives the desired extension of ϕ.

Suppose now that $(*)$ is satisfied and that we are given a pullback diagram

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow \\ \operatorname{Spét} R & \longrightarrow & \mathsf{Y}; \end{array}$$

we wish to prove that the induced map of topological spaces $|\mathsf{X}'| \to |\operatorname{Spec} R|$ is closed. Let Z be a closed subset of $|\mathsf{X}'|$ and let Z be the corresponding reduced closed substack of X'. Choose an étale surjection $\operatorname{Spét} B \to \mathsf{Z}$ (so that B is a reduced commutative ring) and let I

denote the kernel of the induced map of commutative rings $\pi_0 R \to B$. We will prove that $\psi(Z) \subseteq |\operatorname{Spec} R|$ agrees with the image of the closed embedding $|\operatorname{Spec}(\pi_0 R)/I| \hookrightarrow |\operatorname{Spec} R|$. To this end, let $\mathfrak{q}$ be a prime ideal of $\pi_0 R$ containing the ideal I; we wish to show that $\mathfrak{q}$ belongs to $\psi(Z)$. Using Zorn's lemma, we see that there is a prime ideal $\mathfrak{p} \subseteq \mathfrak{q}$ of $\pi_0 R$ which is minimal among prime ideals which contain I. The injection of commutative rings $(\pi_0 R)/I \hookrightarrow B$ induces an injection $((\pi_0 R)/I)_{\mathfrak{p}} \to B_{\mathfrak{p}}$, so that the localization $B_{\mathfrak{p}}$ is nonzero. It follows that $B_{\mathfrak{p}}$ contains a prime ideal, which is the localization of a prime ideal $\mathfrak{p}' \subseteq B$. Note that the image of $\mathfrak{p}'$ in $|\operatorname{Spec} R|$ belongs to the image of the inclusion $|\operatorname{Spec}((\pi_0 R)/I)_{\mathfrak{p}}| \hookrightarrow |\operatorname{Spec} R|$. By construction, the ring $((\pi_0 R)/I)_{\mathfrak{p}}$ contains a unique prime ideal, whose image in $|\operatorname{Spec} R|$ coincides with $\mathfrak{p}$. It follows that the map $|\operatorname{Spec} B| \to |\operatorname{Spec} R|$ carries $\mathfrak{p}'$ to $\mathfrak{p}$.

Let K denote the fraction field of $B/\mathfrak{p}'$ and let $\psi : \pi_0 R \to K$ be the induced map. Using Proposition 5.3.2.2, we can choose a valuation ring $V \subseteq K$ with fraction field K and maximal ideal $\mathfrak{m}$, such that $\psi^{-1}\mathfrak{m} = \mathfrak{q}$. This determines a commutative diagram

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} K & \longrightarrow & \mathsf{X}' \\ \downarrow & \overset{i}{\nearrow} & \downarrow \\ \operatorname{Sp\acute{e}t} V & \longrightarrow & \operatorname{Sp\acute{e}t} R. \end{array}$$

Applying condition $(*)$, we deduce the existence of a dotted arrow as indicated in the diagram. Since the map $\operatorname{Sp\acute{e}t} K \to \mathsf{X}'$ factors through the closed immersion $\mathsf{Z} \hookrightarrow \mathsf{X}'$, the map i also factors through Z. It follows that $\psi(Y)$ contains the image of the map $|\operatorname{Spec} V| \to |\operatorname{Spec} R|$, which includes the point $\mathfrak{q} \in |\operatorname{Spec} R|$. This completes the proof that f is proper.

Now let us assume that Y is locally Noetherian, and that condition $(*)$ is satisfied whenever V is a discrete valuation ring. We wish to show that f is proper. The assertion is local on Y; we may therefore assume that $\mathsf{Y} = \operatorname{Sp\acute{e}t} R$ for some Noetherian $\mathbb{E}_\infty$-ring R. We wish to show that for every pullback diagram

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow \\ \operatorname{Sp\acute{e}t} R' & \longrightarrow & \operatorname{Sp\acute{e}t} R, \end{array}$$

the induced map of topological spaces $|\mathsf{X}'| \to |\operatorname{Spec} R'|$ is closed. Using Remark 5.1.1.5, we assume without loss of generality that R' is Noetherian. Replacing R by R', we are reduced to proving that the map $|\mathsf{X}| \to |\operatorname{Spec} R|$ is closed. The proof now proceeds as in the previous case, using the second part of Proposition 5.3.2.2 to arrange that the valuation ring V is actually discrete.

5.4 Projective Spaces

For each $n \geqslant 0$, the complex projective space $\mathbf{CP}^n$ can be defined as the set of 1-dimensional subspaces of the vector space $\mathbf{C}^{n+1}$. The set $\mathbf{CP}^n$ can be regarded as an algebraic variety: more precisely, it can be identified with the set of $\mathbf{C}$-valued points of a smooth $\mathbf{C}$-scheme $\mathbf{P}^n_{\mathbf{C}}$, which (by slight abuse of terminology) we also refer to as complex projective space. In fact, this scheme can be defined over $\mathbf{Z}$: that is, there exists a smooth $\mathbf{Z}$-scheme $\mathbf{P}^n_{\mathbf{Z}}$ and an isomorphism $\mathbf{P}^n_{\mathbf{C}} \simeq \operatorname{Spec} \mathbf{C} \times_{\operatorname{Spec} \mathbf{Z}} \mathbf{P}^n_{\mathbf{Z}}$. In the setting of spectral algebraic geometry, one can ask for more:

Question 5.4.0.1. Can projective spaces be defined over the sphere spectrum? In other words, can we write $\mathbf{P}^n_{\mathbf{Z}} = \operatorname{Spec} \mathbf{Z} \times_{\operatorname{Spec} S} \mathbf{P}^n_S$ for some spectral scheme $\mathbf{P}^n_S$ defined over S?

Our first goal in this section is to supply an affirmative answer to Question 5.4.0.1 by means of an explicit construction (Construction 5.4.1.3). However, this answer comes with some caveats:

(a) The projective spaces $\mathbf{P}^n_S$ of Construction 5.4.1.3 are not as nicely behaved as their classical counterparts. For example, projective spaces in classical algebraic geometry are homogeneous: if k is a field, then for any two k-valued points x and y of $\mathbf{P}^n_{\mathbf{Z}}$, there exists an automorphism of $\mathbf{P}^n_k = \operatorname{Spec} k \times_{\operatorname{Spec} \mathbf{Z}} \mathbf{P}^n_{\mathbf{Z}}$ which carries x to y. However, the spectral schemes $\mathbf{P}^n_S$ are much more rigid: for example, one can define S-valued points $\{0, 1, \infty\}$ of $\mathbf{P}^1_S$, but there is no automorphism of $\mathbf{P}^1_S$ which exchanges 0 and 1.

(b) The projective space $\mathbf{P}^n_{\mathbf{Z}}$ can be characterized by a universal property: it represents a functor $F : \mathrm{CAlg}^{\heartsuit} \rightarrow \mathcal{S}\mathrm{et}$ which assigns to each commutative ring R the collection of all direct summands of R^{n+1} which are locally free of rank 1. The functor F admits a canonical extension $\overline{F} : \mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathcal{S}$. However, the functor $\overline{F}$ is *not* represented by the spectral scheme $\mathbf{P}^n_S$ of Construction 5.4.1.3 (the functor $\overline{F}$ *is* representable by a spectral scheme $\mathbf{P}^n_{\mathrm{sm}}$, but $\mathbf{P}^n_{\mathrm{sm}}$ is not flat over S: it follows that we cannot recover the classical projective space $\mathbf{P}^n_{\mathbf{Z}}$ as the fiber product $\operatorname{Spec} \mathbf{Z} \times_{\operatorname{Spec} S} \mathbf{P}^n_{\mathrm{sm}}$).

Remark 5.4.0.2. In §19.2.6, we will describe a different notion of projective space in the setting of spectral algebraic geometry which does not share problems (a) and (b) (but suffers from its own defects; see Remark 19.2.6.7.

5.4.1 Projective Spaces in Spectral Algebraic Geometry

We begin by introducing some definitions.

Notation 5.4.1.1. Let R be an $\mathbb{E}_\infty$-ring. For every space X, we let $R[X]$ denote the R-module given by $R \otimes \Sigma^\infty_+ X$. The construction $X \mapsto R[X]$ determines a symmetric monoidal functor $\mathcal{S} \to \mathrm{Mod}_R$. Consequently, it induces a functor $\mathrm{CAlg}(\mathcal{S}) \to \mathrm{CAlg}(\mathrm{Mod}_R) \simeq \mathrm{CAlg}_R$, which we will also denote by $X \mapsto R[X]$. In particular, if X is a commutative monoid (regarded as a commutative algebra object of $\mathcal{S}$ by endowing X with the discrete topology), then we can regard $R[X]$ as an $\mathbb{E}_\infty$-algebra over R. We will refer to $R[X]$ as the *monoid algebra of X over R*.

Remark 5.4.1.2. Let R be an $\mathbb{E}_\infty$-ring and let X be a set (regarded a space equipped with the discrete topology). Then we can identify $R[X]$ with the coproduct $\bigoplus_{x \in X} R$. If X has the structure of a commutative monoid, then $R[X]$ is a flat R-algebra, whose underlying commutative ring $\pi_0 R[X]$ can be identified with the monoid algebra $(\pi_0 R)[X]$ (in the sense of classical commutative algebra).

Construction 5.4.1.3 (Projective Space)**.** Let $[n]$ denote the set $\{0 < 1 < \ldots < n\}$, let $P([n])$ denote the collection of all subsets of $[n]$, and let $P^\circ([n]) \subseteq P([n])$ denote the collection of all nonempty subsets of $[n]$. For every subset $I \subseteq [n]$, set M_I denote the subset of $\mathbf{Z}^{n+1}$ consisting of those tuples $(k_0, \ldots, k_n)$ satisfying $k_0 + \cdots + k_n = 0$ and $k_i \geqslant 0$ for $i \notin I$. Then M_I is a commutative monoid which depends functorially on I. If R is a connective $\mathbb{E}_\infty$-ring, we let $R[M_I]$ denote the associated monoid algebra (Notation 5.4.1.1). Note that the construction $I \mapsto R[M_I]$ determines a functor from $P([n])$ to the ∞-category $\mathrm{CAlg}_R^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-algebras over R, so the construction $I \mapsto \mathrm{Sp\acute{e}t}\, R[M_I]$ determines a functor $P([n])^{\mathrm{op}} \to \mathrm{SpDM}$. Note that if $\varnothing \neq I \subseteq J \subseteq [n]$, then the map $R[M_I] \to R[M_J]$ exhibits $R[M_J]$ as a localization of $R[M_I]$, so the induced map $\mathrm{Sp\acute{e}t}\, R[M_J] \to \mathrm{Sp\acute{e}t}\, R[M_I]$ is étale (in fact, an open immersion). We let $\mathbf{P}^n_R$ denote the colimit $\varinjlim_{I \in P^\circ([n])^{\mathrm{op}}} \mathrm{Sp\acute{e}t}\, R[M_I]$, formed in the ∞-category SpDM of spectral Deligne-Mumford stacks. We will refer to $\mathbf{P}^n_R$ as *projective space of dimension n over R*.

Remark 5.4.1.4. Let R be a connective $\mathbb{E}_\infty$-ring. Then the commutative ring $\pi_0 R[\mathbf{Z}^{n+1}]$ can be identified with the ring of Laurent polynomials $(\pi_0 R)[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. For every subset $I \subseteq [n]$, the inclusion $M_I \hookrightarrow \mathbf{Z}^{n+1}$ induces a monomorphism $\pi_0 R[M_I] \to (\pi_0 R)[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$ whose image is the $(\pi_0 R)$-subalgebra generated by $\frac{x_j}{x_i}$ where $i \in I$. In what follows, we will generally abuse notation by identifying $\frac{x_j}{x_i}$ with an element of $\pi_0 R[M_I]$.

If we are given subsets $I \subseteq J \subseteq [n]$ and an element $i \in I$, then we can identify $R[M_J]$ with the localization of $R[M_I]$ obtained by inverting the elements $\frac{x_j}{x_i}$, where $j \in J$. If follows that for any pair of subsets $I, J \subseteq [n]$ having nonempty intersection, the diagram

$$\begin{array}{ccc} R[M_{I \cap J}] & \longrightarrow & R[M_I] \\ \downarrow & & \downarrow \\ R[M_J] & \longrightarrow & R[M_{I \cup J}] \end{array}$$

is a pushout diagram of $\mathbb{E}_\infty$-rings.

Remark 5.4.1.5. When the set $I \subseteq [n]$ is empty, the monoid M_I is trivial and we have $R[M_I] \simeq R$. Consequently, the diagram

$$P([n])^{\mathrm{op}} \to \mathrm{SpDM} \qquad I \mapsto \mathrm{Sp\acute{e}t}\, R[M_I]$$

determines a tautological map $q : \mathbf{P}^n_R = \varinjlim_{I \neq \varnothing} \mathrm{Sp\acute{e}t}\, R[M_i] \to \mathrm{Sp\acute{e}t}\, R[M_\varnothing] \simeq \mathrm{Sp\acute{e}t}\, R$. It follows from Remark 5.4.1.4 that for each $I \subseteq [n]$, the $\mathbb{E}_\infty$-algebra $R[M_I]$ is flat over R; consequently, the spectral Deligne-Mumford stack $\mathbf{P}^n_R$ is flat over R.

Remark 5.4.1.6. Construction 5.4.1.3 is compatible with base change in R: for any morphism $R \to R'$ of connective $\mathbb{E}_\infty$-rings, we have a canonical pullback diagram

$$\begin{array}{ccc} \mathbf{P}^n_{R'} & \longrightarrow & \mathbf{P}^n_R \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ \mathrm{Sp\acute{e}t}\, R' & \longrightarrow & \mathrm{Sp\acute{e}t}\, R, \end{array}$$

where q is defined as in Remark 5.4.1.5 and q' is defined similarly.

Proposition 5.4.1.7. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathbf{P}^n_R$ be projective space of dimension n over R. For each nonempty subset $I \subseteq [n]$, let $R[M_I]$ be as in Construction 5.4.1.3, so that we have a tautological map $\phi_I : \mathrm{Sp\acute{e}t}\, R[M_I] \to \mathbf{P}^n_R$.*

- (a) *For $0 \leqslant i \leqslant n$, the map $\phi_{\{i\}} : \mathrm{Sp\acute{e}t}\, R[M_{\{i\}}] \to \mathbf{P}^n_R$ is an open immersion.*
- (b) *For each nonempty subset $I \subseteq [n]$, the map $\phi_I : \mathrm{Sp\acute{e}t}\, R[M_I] \to \mathbf{P}^n_R$ is an open immersion, whose image is the intersection of the images of the maps $\{\phi_{\{i\}}\}_{i \in I}$.*
- (c) *The spectral Deligne-Mumford stack $\mathbf{P}^n_R$ is schematic. In particular, it is a spectral algebraic space.*

Proof. We first prove (a). Fix an element $i \in [n]$, let $P^\circ([n] - \{i\})$ be the collection of all nonempty subsets of $[n] - \{i\}$, and let X denote the colimit $\varinjlim_{I \in P^\circ([n]-\{i\})^{\mathrm{op}}} \mathrm{Sp\acute{e}t}\, R[M_I]$. For each $I \in P^\circ([n] - \{i\})$, Remark 5.4.1.4 implies that $\mathrm{Sp\acute{e}t}\, R[M_{I \cup \{i\}}]$ can be identified with an open substack of $\mathrm{Sp\acute{e}t}\, R[M_I]$, and that for $I \subseteq J$ the induced diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}\, R[M_{J \cup \{i\}}] & \longrightarrow & \mathrm{Sp\acute{e}t}\, R[M_{I \cup \{i\}}] \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}\, R[M_J] & \longrightarrow & \mathrm{Sp\acute{e}t}\, R[M_I] \end{array}$$

is a pullback square. It follows that the colimit $\varinjlim_{I \in P^\circ([n]-\{i\})^{\mathrm{op}}} \mathrm{Sp\acute{e}t}\, R[M_{I \cup \{i\}}]$. can be identified with an open substack $\mathsf{U} \subseteq \mathsf{X}$, and that for each $I \in P^\circ([n] - \{i\})$ the diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}\, R[M_{I \cup \{i\}}] & \longrightarrow & \mathrm{Sp\acute{e}t}\, R[M_I] \\ \downarrow & & \downarrow \\ \mathsf{U} & \xrightarrow{\ j\ } & \mathsf{X} \end{array}$$

is a pullback square. Unwinding the definitions, we see the projective space $\mathbf{P}^n_R$ fits into a pushout diagram of spectral Deligne-Mumford stacks σ :

$$\begin{array}{ccc} \mathsf{U} & \xrightarrow{j} & \mathsf{X} \\ \downarrow & & \downarrow \\ \operatorname{Sp\acute{e}t} R[M_{\{i\}}] & \xrightarrow{\phi_{\{i\}}} & \mathbf{P}^n_R \end{array}$$

in which all morphisms are étale. Since j is an open immersion, it follows that σ is an excision square: in particular, it is a pullback square and the map $\phi_{\{i\}}$ is an open immersion (see Proposition 2.5.2.1). This proves (a), and shows that for every nonempty subset $I \subseteq [n]-\{i\}$, the diagram

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} R[M_{I\cup\{i\}}] & \longrightarrow & \operatorname{Sp\acute{e}t} R[M_I] \\ \downarrow & & \downarrow \\ \operatorname{Sp\acute{e}t} R[M_{\{i\}}] & \longrightarrow & \mathbf{P}^n_R \end{array}$$

is a pullback square. Assertion (b) now follows by induction on the size of I, and assertion (c) follows from (a) together with Proposition 1.6.7.3 (since the open immersions $\phi_{\{i\}}$ are mutually surjective). $\square$

Example 5.4.1.8. Let R be a commutative ring. Then $\mathbf{P}^n_R$ is schematic (Proposition 5.4.1.7) and 0-truncated (since it is flat over R by virtue of Remark 5.4.1.5), and can therefore be identified with an ordinary scheme. Proposition 5.4.1.7 shows that this scheme can be covered by affine open subschemes $U_0, \ldots, U_n \subseteq \mathbf{P}^n_R$, where $U_i \simeq \operatorname{Spec} R[M_{\{i\}}] \simeq R[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}]$ is an n-dimensional affine space over R, which are "glued together" along the intersections $U_i \cap U_j \simeq \operatorname{Spec} R[M_{\{i,j\}}]$. From this description, we see that Construction 5.4.1.3 recovers the usual definition of projective space of dimension n over R.

Remark 5.4.1.9. Using a more elaborate version of Construction 5.4.1.3, one can develop a theory of toric varieties over any connective $\mathbb{E}_\infty$-ring R, generalizing the classical theory of toric varieties over commutative rings (as described, for example, in [72]).

5.4.2 Line Bundles on Projective Space

Let $n \geqslant 0$ be an integer. By definition, there is a bijective correspondence between points $x \in \mathbf{CP}^n$ and 1-dimensional subspaces $L_x \subseteq \mathbf{C}^{n+1}$. The construction $x \mapsto L_x$ determines a complex analytic vector bundle of rank 1 over $\mathbf{CP}^n$. This vector bundle is algebraic: that is, it arises from line bundle $\mathscr{O}(-1)$ on the scheme $\mathbf{P}^n_{\mathbf{C}}$. For every integer m, we let $\mathscr{O}(m)$ denote the tensor power $\mathscr{O}(-1)^{\otimes -m}$. Our next goal is to show that the line bundles

$\mathscr{O}(m)$ are "defined over the sphere": that is, they are the pullbacks of line bundles on the spectral algebraic space $\mathbf{P}^n_S$. To produce these line bundles, we will need a slight variant of Construction 5.4.1.3.

Construction 5.4.2.1. Let $n \geqslant 0$ be an integer and let R be a connective $\mathbb{E}_\infty$-ring. For each subset $I \subseteq [n]$ and each integer m, we let $M_I(m)$ denote the subset of $\mathbf{Z}^{n+1}$ consisting of those tuples $(k_0, \ldots, k_n)$ satisfying $k_0 + \cdots + k_n = m$ and $k_i \geqslant 0$ for $i \notin I$. We regard $M_I(m)$ as a set equipped with an action of the monoid $M_I = M_I(0)$ appearing in Construction 5.4.1.3.

Note that if $I \neq \varnothing$, then $M_I(m)$ is free module over M_I on a single generator. More precisely, it is freely generated by any tuple $(k_0, \ldots, k_n) \in M_I(m)$ which satisfies $k_i = 0$ for $i \notin I$. From this description, we see that if $(k_0, \ldots, k_n)$ freely generates $M_I(m)$ as a module over M_I, then it freely generates $M_J(m)$ as a module over M_J for any $J \supseteq I$. It follows that the inclusion $M_I(m) \hookrightarrow M_J(m)$ induces an equivalence of $R[M_J] \otimes_{R[M_I]} R[M_I(m)] \to R[M_J(m)]$ in the ∞-category of $R[M_J]$-modules. We may therefore regard the diagram $I \mapsto R[M_I(m)]$ as an object of the ∞-category

$$\varprojlim_{I \in P^\circ([n])} \mathrm{Mod}_{R[M_i]} \simeq \varprojlim_{I \in P^\circ([n])} \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R[M_i]) \simeq \mathrm{QCoh}(\mathbf{P}^n_R).$$

We will denote this object by $\mathscr{O}(m)$.

Remark 5.4.2.2. In the situation of Construction 5.4.2.1, the module $R[M_I(m)]$ is free of rank 1 over $R[M_I]$ for every nonempty subset $I \subseteq [n]$. It follows that $\mathscr{O}(m) \in \mathrm{QCoh}(\mathbf{P}^n_R)$ is a line bundle on $\mathbf{P}^n_R$.

Remark 5.4.2.3. Construction 5.4.2.1 does not really depend on the choice of $\mathbb{E}_\infty$-ring R: if $f : R \to R'$ is a morphism of $\mathbb{E}_\infty$-rings and we let

$$\mathscr{O}(m)_R \in \mathrm{QCoh}(\mathbf{P}^n_R) \qquad \mathscr{O}(m)_{R'} \in \mathrm{QCoh}(\mathbf{P}^n_{R'})$$

denote the line bundles obtained by applying Construction 5.4.2.1 to R and R', respectively, then we have a canonical equivalence $\mathscr{O}(m)_{R'} \simeq F^* \mathscr{O}(m)_R$, where $F : \mathbf{P}^n_{R'} \to \mathbf{P}^n_R$ denotes the map induced by f.

Example 5.4.2.4. In Construction 5.4.2.1, we have $M_\varnothing(m) = \{(k_0, k_1, \ldots, k_n) \in \mathbf{Z}^{n+1}_{\geqslant 0} : k_0 + \cdots + k_n = m\}$. It follows that $M_\varnothing(m)$ is a finite set of cardinality $\binom{m+n}{n}$ if $m \geqslant 0$, and is empty for $m < 0$.

In the situation of Construction 5.4.2.1, we let $\Gamma(\mathbf{P}^n_R; \mathscr{O}(m))$ denote the spectrum of global section of the line bundle $\mathscr{O}(m)$ on $\mathbf{P}^n_R$. In the case where R is an ordinary commutative ring, we can regard $\mathscr{O}(m)$ as a sheaf of abelian groups on $\mathbf{P}^n_R$, and the homotopy groups of $\Gamma(\mathbf{P}^n_R; \mathscr{O}(m))$ are given by

$$\pi_* \Gamma(\mathbf{P}^n_R; \mathscr{O}(m)) = \mathrm{H}^{-*}(\mathbf{P}^n_R; \mathscr{O}(m)).$$

The cohomology groups on the right hand side were computed by Serre in [187] (with two slight caveats: [187] considers cohomology with respect to the Zariski topology rather than the étale topology, and restricts attention to the case where R is an algebraically closed field).

Our next goal is to reproduce Serre's calculation over an arbitrary connective $\mathbb{E}_\infty$-ring R. We begin by constructing some global sections of the line bundles $\mathscr{O}(m)$.

Construction 5.4.2.5. Let $n \geqslant 0$ be an integer and let R be a connective $\mathbb{E}_\infty$-ring. Let $\vec{k} = (k_0, \ldots, k_n) \in \mathbf{Z}^{n+1}_{\geqslant 0}$, and set $m = k_0 + \cdots + k_n$. Then $\vec{k}$ can be regarded as an element of $M_I(m)$ for every subset $I \subseteq [n]$, and therefore determines a map

$$R \simeq \varprojlim_{I \in P^\circ([n])} R[\{\vec{k}\}] \rightarrow \varprojlim_{I \in P^\circ([n])} R[M_I(m)] \simeq \Gamma(\mathbf{P}^n_R; \mathscr{O}(m)).$$

We will denote this map by $x^{\vec{k}} : R \rightarrow \Gamma(\mathbf{P}^n_R; \mathscr{O}(m))$.

We can now formulate Serre's result as follows:

Theorem 5.4.2.6 (Serre)**.** *Let R be a connective $\mathbb{E}_\infty$-ring, let $n \geqslant 0$ be an integer, and let $\mathscr{O}(m)$ be the line bundle on $\mathbf{P}^n_R$ given by Construction 5.4.2.1. Then:*

1. *If $m \geqslant 0$, then the maps $x^{\vec{k}} : R \rightarrow \Gamma(\mathbf{P}^n_R; \mathscr{O}(m))$ induce an equivalence $\bigoplus_{\vec{k} \in M_\varnothing(m)} R \simeq \Gamma(\mathbf{P}^n_R; \mathscr{O}(m))$. In particular, $\Gamma(\mathbf{P}^n_R; \mathscr{O}(m))$ is a free R-module of rank $\binom{m+n}{n}$ (see Example 5.4.2.4).*

2. *If $m < 0$, then $\Gamma(\mathbf{P}^n_R; \mathscr{O}(m))$ is equivalent to a direct sum of $\binom{-m-1}{n}$ copies of $\Sigma^{-n} R$. In particular, $\Gamma(\mathbf{P}^n_R; \mathscr{O}(m))$ vanishes for $-n \leqslant m \leqslant -1$.*

Proof. For each element $\vec{k} = (k_0, \ldots, k_n) \in \mathbf{Z}^{n+1}$ and each $I \subseteq [n]$, set $M_I(\vec{k}) = M_I \cap \{\vec{k}\}$. Then $M_I(\vec{k})$ is either empty (if $k_i < 0$ for some $i \notin I$) or consists of the single element $\{\vec{k}\}$. Let $Q \subseteq P^\circ([n])$ be the partially ordered subset consisting of those nonempty sets $I \subseteq [n]$ such that $M_I(\vec{k})$ is empty.

Let us regard the construction $I \mapsto R[M_I(\vec{k})]$ as a functor $\lambda_{\vec{k}} : P^\circ([n]) \rightarrow \mathrm{Mod}_R$. Let $\underline{R} : P^\circ([n]) \rightarrow \mathrm{Mod}_R$ denote the constant functor with the value R. We have a canonical map $u : \lambda_{\vec{k}} \rightarrow \underline{R}$. Note that the canonical map $\underline{R} \rightarrow \mathrm{cofib}(u)$ is an equivalence when restricted to Q, and that $\mathrm{cofib}(u)$ is a right Kan extension of its restriction to Q. Consequently, we have $\varprojlim_{I \in P^\circ([n])} \mathrm{cofib}(u)(I) \simeq \varprojlim_{I \in Q} R \simeq R^{\mathrm{N}(Q)}$, where $R^{\mathrm{N}(Q)}$ denotes the function spectrum of maps $\mathrm{N}(Q) \rightarrow R$. It follows that $\varprojlim \lambda_{\vec{k}}$ can be identified with the fiber of the diagonal map $R \rightarrow R^{\mathrm{N}(Q)}$. We now distinguish three cases:

(a) All of the integers k_i are nonnegative. In this case, $Q = \varnothing$, and we have $\varprojlim \lambda_{\vec{k}} \simeq R$.

(b) Some of the integers k_i are negative and some are not. In this case, we claim that the simplicial set $\mathrm{N}(Q)$ is weakly contractible, so that $\varprojlim \lambda_{\vec{k}} \simeq 0$. To prove this, fix $i \in [n]$ such that $k_i \geqslant 0$, and let $Q' \subseteq Q$ consist of those elements $I \in Q$ which contain i. Then the inclusion $Q' \hookrightarrow Q$ admits a left adjoint (given by $I \hookrightarrow I \cup \{i\}$) and therefore induces a weak homotopy equivalence $\mathrm{N}(Q') \hookrightarrow \mathrm{N}(Q)$. We are therefore reduced to proving that $\mathrm{N}(Q')$ is weakly contractible. This is clear, since the singleton $\{i\}$ is a least element of Q'.

(c) All of the integers k_i are negative. In this case, Q is the collection of all nonempty, proper subsets of $[n]$. It follows that the simplicial set $\mathrm{N}(Q)$ can be identified with the subdivision of $\partial \Delta^n$, so we obtain an equivalence $\varprojlim \lambda_{\vec{k}} \simeq \mathrm{fib}(R \to R^{\mathrm{N}(Q)} \simeq R^{\partial \Delta^n}) \simeq \Sigma^{-n} R$.

Unwinding the definitions, we have

$$\begin{aligned}
\Gamma(\mathbf{P}^n_R; \mathscr{O}(m)) &\simeq \varprojlim_{I \in P^{\circ}([n])} R[M_I(m)] \\
&\simeq \varprojlim_{I \in P^{\circ}([n])} \bigoplus_{\vec{k} \in M_{[n]}(m)} R[M_I(\vec{k})] \\
&\simeq \bigoplus_{\vec{k} \in M_{[n]}(m)} \varprojlim_{I \in P^{\circ}([n])} R[M_I(\vec{k})] \\
&\simeq \bigoplus_{\vec{k} \in M_{[n]}(m)} \varprojlim \lambda_{\vec{k}}.
\end{aligned}$$

We now distinguish two cases:

(1) If $m \geqslant 0$, then no element $\vec{k} \in M_{[n]}(m)$ can satisfy (c), and for those which satisfy (b) the corresponding summand $\varprojlim \lambda_{\vec{k}}$ vanishes. The remaining summands are indexed by the finite set $M_{\varnothing}(m)$, and each summand is equivalent to R. We therefore obtain a decomposition $\Gamma(\mathbf{P}^n_R; \mathscr{O}(m)) \simeq \bigoplus_{\vec{k} \in M_{\varnothing}(m)}$, given by the maps $x^{\vec{k}} : R \to \Gamma(\mathbf{P}^n_R; \mathscr{O}(m))$ of Construction 5.4.2.5.

(2) If $m < 0$, then no element $\vec{k} \in M_{[n]}(m)$ can satisfy (a), and for thiose which satisfy (b) the corresponding summand $\varprojlim \lambda_{\vec{k}}$ vanishes. The remaining summands are indexed by the finite set $T = \{(k_0, \ldots, k_n) \in \mathbf{Z}^{n+1}_{<0} : k_0 + \cdots + k_n = m\}$ of size $\binom{-m-1}{n}$, and each summand is equivalent to $\Sigma^{-n} R$. We therefore obtain an equivalence $\Gamma(\mathbf{P}^n_R; \mathscr{O}(m)) \simeq \bigoplus_{\vec{k} \in T} \Sigma^{-n} R$.

□

5.4.3 The Universal Property of Projective Space

Let $\mathbf{P}^n_R$ be projective space of dimension n over a connective $\mathbb{E}_\infty$-ring R. We will refer to the line bundle $\mathscr{O}(-1) \in \mathrm{QCoh}(\mathbf{P}^n_R)$ as the *tautological line bundle* over $\mathbf{P}^n_R$. Construction 5.4.2.5 produces a finite collection of elements

$$x_0, \ldots, x_n \in \pi_0 \Gamma(\mathbf{P}^n_R; \mathscr{O}(1)) = \pi_0 \operatorname{Map}_{\mathrm{QCoh}(\mathbf{P}^n_R)}(\mathscr{O}(-1), \mathscr{O}),$$

which we can identify with a single map $e : \mathscr{O}(-1) \to \mathscr{O}^{n+1}$ (here $\mathscr{O} = \mathscr{O}(0)$ denote the structure sheaf of $\mathbf{P}^n_R$). It is not hard to see that the cofiber $\mathrm{cofib}(e)$ is a vector bundle of rank n on $\mathbf{P}^n_R$. In the setting of classical algebraic geometry, the projective space $\mathbf{P}^n_R$ is universal with respect to these features:

Theorem 5.4.3.1. *Let A be a commutative ring and let $X(A)$ denote the set of submodules $L \subseteq A^{n+1}$ for which the quotient A^{n+1}/L is a locally free A-module of rank n (from which it follows that L is locally free of rank 1). Then the construction*

$$(f : \operatorname{Spét} A \to \mathbf{P}^n_S) \mapsto (\operatorname{im}(f^*(e) \subseteq A^{n+1})$$

induces a homotopy equivalence (of discrete spaces) $\operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} A, \mathbf{P}^n_S) \simeq X(A)$; here S denotes the sphere spectrum and $e : \mathscr{O}(-1) \to \mathscr{O}^{n+1}$ is defined as above.

Remark 5.4.3.2. In the statement of Theorem 5.4.3.1, we can replace the sphere spectrum S with the ring $\mathbf{Z}$ of integers (or any connective $\mathbb{E}_\infty$-ring R satisfying $\pi_0 R \simeq \mathbf{Z}$).

Warning 5.4.3.3. Theorem 5.4.3.1 does not extend (at least in a naive way) to the case where A is a connective $\mathbb{E}_\infty$-ring. A slight refinement of Construction 5.4.2.1 shows that we can regard the construction $m \mapsto \mathscr{O}(m)$ as a symmetric monoidal functor from the set $\mathbf{Z}$ of integers (regarded as a category with no non-identity morphisms) to the ∞-category $\mathscr{P}\mathrm{ic}(\mathbf{P}^n_S)$ of line bundles on the projective space $\mathbf{P}^n_S$ (see Definition 2.9.4.1). Consequently, for any map $f : \operatorname{Spét} A \to \mathbf{P}^n_S$, the construction $m \mapsto f^* \mathscr{O}(m)$ determines a symmetric monoidal functor $\mathbf{Z} \to \mathscr{P}\mathrm{ic}(A)$. This symmetric monoidal functor equips the invertible A-module $L = f^* \mathscr{O}(-1)$ with some additional structures: for example, it supplies a nullhomotopy of the automorphism of $L \otimes_A L$ given by "swapping" the factors of L. In the setting of spectral algebraic geometry, this structure is not automatic.

In §19.2, we will show that there is a variant of the projective space $\mathbf{P}^n_S$ (which is not given by Construction 5.4.1.3) for which Theorem 5.4.3.1 does extend to the setting of $\mathbb{E}_\infty$-rings (see Definition 19.2.6.3).

Proof of Theorem 5.4.3.1. For every commutative ring A, set $Y(A) = \operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} A, \mathbf{P}^n_S)$, which we will regard as a set (since A is discrete and $\mathbf{P}^n_S$ is a spectral algebraic space), so that the construction $f \mapsto \operatorname{im}(f^*(e))$ determines a map $\gamma(A) : Y(A) \to X(A)$; we wish to show that $\gamma(A)$ is a bijection.

For every nonempty $I \subseteq [n]$, let $X_I(A)$ denote the subset of $X(A)$ consisting of those submodules $L \subseteq A^{n+1}$ for which the composite map $L \hookrightarrow A^{n+1} \xrightarrow{p_i} A$ is an isomorphism for each $i \in I$; here $p_i : A^{n+1} \to A$ denotes the projection onto the ith factor. Let $Y_I(A)$ denote the subset of $Y(A)$ consisting of those maps $\operatorname{Spét} A \to \mathbf{P}^n_S$ which factor through the open immersion $\phi_I : \operatorname{Spét} S[M_I] \hookrightarrow \mathbf{P}^n_S$ of Proposition 5.4.1.7. Note that the map $\gamma(A) : Y(A) \to X(A)$ restricts to give maps $\gamma_I(A) : Y_I(A) \to X_I(A)$ for every nonempty $I \subseteq [n]$. Given an element $L \in X_I(A)$, the construction

Given an element $(L, \alpha_0, \ldots, \alpha_n)$ in $X_I(A)$, the construction $(k_0, \ldots, k_n) \mapsto (\bigotimes_{0 \leqslant i \leqslant n} (p_i|_L)^{\otimes k_i})$ determines a monoid homomorphism

$$M_I \to \operatorname{Hom}_A(L^{\otimes k_0 + \cdots + k_n}, A^{\otimes k_0 + \cdots + k_n}) \simeq \operatorname{Hom}_A(A, A) \simeq A$$

(where we regard A as monoid with respect to multiplication multiplication). This construction determines a map

$$X_I(A) \to \operatorname{Map}_{\operatorname{CAlg}^\heartsuit}(\mathbf{Z}[M_I], A) \simeq \operatorname{Map}_{\operatorname{CAlg}}(S[M_I], A) \simeq Y_I(A)$$

which is easily seen to be inverse to $\gamma_I(A)$. It follows that each of the maps $\gamma_I(A)$ is a bijection.

We now prove that the map $\gamma(A)$ is injective. Suppose we are given elements $f, g \in Y(A)$ satisfying $\gamma(A)(f) = \gamma(A)(g)$; we wish to show that $f = g$. To prove this, we can work locally with respect to the Zariski topology on $\operatorname{Spec} A$ and thereby reduce to the case where $f \in Y_{\{i\}}(A)$ and $g \in Y_{\{j\}}(A)$ for some $i, j \in [n]$. Then $\gamma(A)(f) = \gamma(A)(g)$ belongs to $X_{\{i\}}(A) \cap X_{\{j\}}(A) = X_{\{i,j\}}(A)$. It follows from the above arguments that we can write $\gamma(A)(f) = \gamma(A)(h) = \gamma(A)(g)$ for some $h \in Y_{\{i,j\}}(A)$. The injectivity of $\gamma_{\{i\}}(A)$ then shows that $f = h$, and the injectivity of $\gamma_{\{j\}}(A)$ shows that $h = g$. It follows by transitivity that $f = g$, as desired.

Note that the constructions $A \mapsto X(A)$ and $A \mapsto Y(A)$ are both sheaves with respect to the Zariski topology, and the construction $A \mapsto \gamma(A)$ determines a morphism of sheaves. Consequently, to complete the proof, it will suffice to show that the natural transformation $\gamma : X \to Y$ is an effective epimorphism of Zariski sheaves. For each $i \in [n]$, the constructions $A \mapsto X_{\{i\}}(A)$ and $A \mapsto Y_{\{i\}}(A)$ determine Zariski sheaves $X_{\{i\}}$ and $Y_{\{i\}}$, and the maps $\gamma_{\{i\}}(A)$ induce an isomorphism $X_{\{i\}} \simeq Y_{\{i\}}$. It follows that the natural map $\rho : \amalg_{0 \leqslant i \leqslant n} Y_{\{i\}} \to Y$ factors through γ. Since the map ρ is an effective epimorphism of Zariski sheaves, it follows that γ is also an effective epimorphism of Zariski sheaves. $\square$

Corollary 5.4.3.4. *Let R be a connective $\mathbb{E}_\infty$-ring. Then the projection map $\mathbf{P}^n_R \to \operatorname{Spét} R$ is proper and locally almost of finite presentation.*

Warning 5.4.3.5. If the $\mathbb{E}_\infty$-ring R is not a $\mathbf{Q}$-algebra, then the morphism $\mathbf{P}^n_R \to \operatorname{Spét} R$ is not locally of finite presentation.

Proof of Corollary 5.4.3.4. Using Remark 5.4.1.6, we can reduce to the case where $R = S$ is there sphere spectrum. For $0 \leqslant i \leqslant n$, the monoid $M_{\{i\}}$ of Construction 5.4.1.3 is isomorphic to $\mathbf{Z}^n_{\geqslant 0}$, so that $\pi_* S[M_{\{i\}}]$ is isomorphic to a polynomial ring $(\pi_* S)[y_1 \ldots, y_n]$. It follows that $S[M_{\{i\}}]$ is Noetherian and of finite type over S. Since $\mathbf{P}^n_S$ can be covered by open substacks of the form $\operatorname{Spét} S[M_{\{i\}}]$, it is locally Noetherian and locally of finite type over S. Applying Remark 4.2.0.4, we deduce that $\mathbf{P}^n_S$ is locally almost of finite presentation over S.

It follows from Proposition 5.4.1.7 that $\mathbf{P}^n_S$ is a spectral algebraic space. Moreover, it is covered by finitely many open immersions $\phi_{\{i\}} : \operatorname{Spét} S[M_{\{i\}}] \to \mathbf{P}^n_S$ for which the fiber products

$$\operatorname{Spét} S[M_{\{i\}}] \times_{\mathbf{P}^n_S} \operatorname{Spét} S[M_{\{j\}}] \simeq \operatorname{Spét} S[M_{\{i,j\}}]$$

are affine. It follows that $\mathbf{P}^n_S$ is quasi-compact and that the diagonal of $\mathbf{P}^n_S$ is affine. In particular, $\mathbf{P}^n_S$ is quasi-separated.

To complete the proof of Corollary 5.4.3.4, it will suffice to show that the projection map $\mathbf{P}^n_S \to S$ satisfies the valuative criterion of properness (see Corollary 5.3.1.2). That is, we must show that if V is a valuation ring with fraction field K, then the induced map $\operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} V, \mathbf{P}^n_S) \to \operatorname{Map}_{\mathrm{SpDM}}(\operatorname{Spét} K, \mathbf{P}^n_S)$ is a homotopy equivalence. Equivalently, we must show that the restriction map $X(V) \to X(K)$ is bijective, where X is defined as in the statement of Theorem 5.4.3.1. We must show that if L is a 1-dimensional subspace of K^{n+1}, then there is a unique V-submodule $L_0 \subseteq L \cap V^{n+1}$ such that V^{n+1}/L_0 is locally free of rank n. Choose a nonzero vector $x \in L \subseteq K^{n+1}$, and write $x = (x_0, \ldots, x_n)$. Since V is a valuation ring, the fractional ideal generated by the elements $\{x_i\}_{0 \leqslant i \leqslant n}$ is principal: that is, we can choose $i \in [n]$ such that $x_j = \lambda_j x_i$ for some coefficients $\lambda_j \in V$. A simple calculation shows that the unique candidate for L_0 is the free V-submodule of V^{n+1} generated by the vector $(\lambda_0, \lambda_1, \ldots, \lambda_n)$. □

5.5 Chow's Lemma

In the setting of classical algebraic geometry (even over the field $\mathbf{C}$ of complex numbers), there are many examples of algebraic varieties which are not quasi-projective. However, *Chow's lemma* asserts that for every complex algebraic variety X, there exists a projective birational map $\pi : \widetilde{X} \to X$, where $\widetilde{X}$ is quasi-projective. Our goal in this section is to establish the following variant of Chow's lemma in the setting of spectral algebraic geometry:

Theorem 5.5.0.1 (Chow's Lemma for Spectral Algebraic Spaces)**.** *Let R be a connective $\mathbb{E}_\infty$-ring, let X be a quasi-compact separated algebraic space over R, and suppose that the underlying ordinary algebraic space $\tau_{\leqslant 0} \mathsf{X}$ is finitely* 0*-presented over R. Then there exists a finite sequence of closed immersions*

$$\varnothing = \mathsf{Y}_0 \to \mathsf{Y}_1 \to \mathsf{Y}_2 \to \cdots \to \mathsf{Y}_n \to \mathsf{X}$$

with the following properties:

(a) *Each of the closed immersions* $\mathsf{Y}_i \to \mathsf{X}$ *is locally almost of finite presentation.*

(b) *The closed immersion* $\mathsf{Y}_n \to \mathsf{X}$ *induces an equivalence of the underlying ordinary algebraic spaces* $\tau_{\leqslant 0}\mathsf{Y}_n \to \tau_{\leqslant 0}\mathsf{X}$.

(c) *For* $1 \leqslant i \leqslant n$, *there exists an integer* $d_i \geqslant 0$ *and a closed immersion* $h_i : \widetilde{\mathsf{Y}}_i \to \mathsf{Y}_i \times_{\operatorname{Sp\acute{e}t} R} \mathbf{P}^{d_i}_R$ *such that* h_i *is locally almost of finite presentation, the composite map* $\widetilde{\mathsf{Y}}_i \xrightarrow{h_i} \mathsf{Y}_i \times_{\operatorname{Sp\acute{e}t} R} \mathbf{P}^{d_i}_R \to \mathbf{P}^{d_i}_R$ *is locally quasi-finite, and the projection map* $\widetilde{\mathsf{Y}}_i \times_{\mathsf{Y}_i} \mathsf{U}_i \to \mathsf{U}_i$ *is an equivalence, where* U_i *denotes the open substack of* Y_i *complementary to the closed immersion* $\mathsf{Y}_{i-1} \to \mathsf{Y}_i$.

Remark 5.5.0.2. Many refinements of Theorem 5.5.0.1 are possible. For example, we can arrange that each of the maps $\widetilde{\mathsf{Y}}_i \xrightarrow{h_i} \mathsf{Y}_i \times_{\operatorname{Sp\acute{e}t} R} \mathbf{P}^{d_i}_R \to \mathbf{P}^{d_i}_R$ is an immersion (meaning that it factors as as a closed immersion followed by an open immersion), and the hypothesis that the base $\operatorname{Sp\acute{e}t} R$ be affine can be considerably weakened. However, Theorem 5.5.0.1 will be sufficient for our needs in this book.

Remark 5.5.0.3. In the situation of Theorem 5.5.0.1, each of the natural maps $\widetilde{\mathsf{Y}}_i \to \mathsf{X}$ factors as a composition

$$\widetilde{\mathsf{Y}_i} \xrightarrow{h_i} \mathsf{Y}_i \times_{\operatorname{Sp\acute{e}t} R} \mathbf{P}^{d_i}_R \to \mathsf{X} \times_{\operatorname{Sp\acute{e}t} R} \mathbf{P}^{d_i}_R \xrightarrow{q} \mathsf{X},$$

where the first two maps are closed immersions which are locally almost of finite presentation and q is a pullback of the projection map $\mathbf{P}^{d_i}_R \to \operatorname{Sp\acute{e}t} R$, which is proper and locally almost of finite presentation by virtue of Corollary 5.4.3.4. It follows that each map $\widetilde{\mathsf{Y}}_i \to \mathsf{X}$ is proper and locally almost of finite presentation.

Remark 5.5.0.4. In the situation of Theorem 5.5.0.1, each of the natural maps $\rho_i : \widetilde{\mathsf{Y}}_i \to \mathbf{P}^{d_i}_R$ factors as a composition

$$\widetilde{\mathsf{Y}_i} \xrightarrow{h_i} \mathsf{Y}_i \times_{\operatorname{Sp\acute{e}t} R} \mathbf{P}^{d_i}_R \to \mathsf{X} \times_{\operatorname{Sp\acute{e}t} R} \mathbf{P}^{d_i}_R \xrightarrow{q'} \mathbf{P}^{d_i}_R,$$

where the first two maps are closed immersions which are locally almost of finite presentation and q' is given by projection onto the second factor. Consequently:

(i) If X is proper over R, then the map $\rho_i : \widetilde{\mathsf{Y}}_i \to \mathbf{P}^{d_i}_R$ is proper. Since ρ_i is also locally quasi-finite, it follows that ρ_i is finite.

(ii) If X is locally almost of finite presentation over R, then the map ρ_i is locally almost of finite presentation.

The remainder of this section is devoted to the proof of Theorem 5.5.0.1. Our strategy is to use the Noetherian approximation techniques of Chapter 4 to reduce to the case where R is Noetherian and X is locally Noetherian. In this case, the canonical map $\tau_{\leqslant 0}\mathsf{X} \to \mathsf{X}$ is locally almost of finite presentation. We can therefore replace X by $\tau_{\leqslant 0}\mathsf{X}$ and thereby reduce to proving Theorem 5.5.0.1 for ordinary algebraic spaces. Here we follow the strategy of Knutson in [117] (where a variant of Theorem 5.5.0.1 appears as Theorem IV.3.1).

5.5.1 Digression on Projective Space

We begin by reviewing some auxiliary constructions which will be useful for proving the analogue of Theorem 5.5.0.1 in the setting of classical algebraic geometry.

Notation 5.5.1.1. Let $n \geqslant 0$ be an integer. For every commutative ring R, let $R[x]_{\leqslant n}$ denote the subset of $R[x]$ consisting of polynomials of degree $\leqslant n$. We let $\mathbf{P}^n(R)$ denote the collection of R-submodules $L \subseteq R[x]_{\leqslant n}$ which have the following properties:

- (a) As an R-module, L is projective of rank 1.
- (b) The R-module L is a direct summand of $R[x]_{\leqslant n}$ (in other words, the quotient $R[x]_{\leqslant n}/L$ is also projective).

We will regard the construction $R \mapsto \mathbf{P}^n(R)$ as a functor from the category of commutative rings to the category of sets.

Remark 5.5.1.2. Since $R[x]_{\leqslant n}$ is isomorphic (as an R-module) to R^{n+1}, Theorem 5.4.3.1 shows that the functor $\mathbf{P}^n : \mathrm{CAlg}^{\heartsuit} \to \mathcal{S}\mathrm{et}$ of Notation 5.5.1.1 is represented by the algebraic space $\mathbf{P}^n_{\mathbf{Z}}$.

Remark 5.5.1.3. Let R be a commutative ring and let $f(x) \in R[x]$ be a polynomial of degree $\leqslant n$. Then the cyclic submodule $Rf(x) \subseteq R[x]_{\leqslant n}$ is an element of $\mathbf{P}^n(R)$ if and only if the coefficients of $f(x)$ generate the unit ideal in R (in other words, if and only if $f(x)$ has nonzero image in $\kappa[x]$, for each residue field κ of R.

Construction 5.5.1.4. Let R be a commutative ring. For every pair of integers $m, n \geqslant 0$, multiplication of polynomials determines a map $R[x]_{\leqslant m} \otimes_R R[x]_{\leqslant n} \to R[x]_{\leqslant m+n}$. If $L \subseteq R[x]_{\leqslant m}$ and $L' \subseteq R[x]_{\leqslant n}$ are submodules satisfying conditions (a) and (b) of Notation 5.5.1.1, then the image of the composite map

$$L \otimes L' \to R[x]_{\leqslant m} \otimes_R R[x]_{\leqslant n} \to R[x]_{\leqslant m+n}$$

is a submodule which also satisfies conditions (a) and (b) of Notation 5.5.1.1. We will denote this image by LL'. The construction $(L, L') \mapsto LL'$ determines a map $\mathbf{P}^m(R) \times \mathbf{P}^n(R) \to \mathbf{P}^{m+n}(R)$. Iterating this construction, we obtain maps

$$\mathbf{P}^{k_1}(R) \times \mathbf{P}^{k_2}(R) \times \cdots \times \mathbf{P}^{k_n}(R) \to \mathbf{P}^{k_1+\cdots+k_n}(R).$$

We will be particularly interested in the case $k_1 = k_2 = \cdots = k_n = k$ for some integer k, in which case we obtain a map $\gamma(R) : \mathbf{P}^k(R)^n \to \mathbf{P}^{kn}(R)$. The maps $\gamma(R)$ determine a natural transformation of functors $\gamma : (\mathbf{P}^1)^n \to \mathbf{P}^n$, which (by virtue of Remark 5.5.1.2) we can view as a morphism of algebraic spaces $\mathbf{P}^k_{\mathbf{Z}} \times_{\mathrm{Spét}\,\mathbf{Z}} \cdots \times_{\mathrm{Spét}\,\mathbf{Z}} \mathbf{P}^k_{\mathbf{Z}} \to \mathbf{P}^{kn}_{\mathbf{Z}}$.

Proposition 5.5.1.5. *For each $k, n \geqslant 0$, the morphism of algebraic spaces $\gamma : \mathbf{P}^k_{\mathbf{Z}} \times_{\mathrm{Spec}\,\mathbf{Z}} \cdots \times_{\mathrm{Spec}\,\mathbf{Z}} \mathbf{P}^k_{\mathbf{Z}} \to \mathbf{P}^{kn}_{\mathbf{Z}}$ of Construction 5.5.1.4 is finite.*

Proof. Since the domain and codomain of γ are proper over $\mathbf{Z}$ (Corollary 5.4.3.4), the morphism γ is automatically proper. It will therefore suffice to show that the fibers of γ are finite. This can be checked at the level of κ-valued points, where κ is an algebraically closed field. The desired result now follows from the observation that, if $f(x) \in \kappa[x]$ is a nonzero polynomial of degree $\leqslant kn$, then up to rescaling there are only finitely many factorizations $f(x) = f_1(x) \cdots f_n(x)$ into polynomials of degree $\leqslant k$ (since $\kappa[x]$ is a unique factorization domain). □

Notation 5.5.1.6. Let $\phi : A \to B$ be a homomorphism of commutative rings which is finite flat of rank n. Then the induced map of polynomial rings $A[x] \to B[x]$ is also finite flat of rank n. We therefore have a well-defined norm map $Nm_{B[x]/A[x]} : B[x] \to A[x]$. Unwinding the definitions, we see that the norm map $Nm_{B[x]/A[x]}$ carries $B[x]_{\leqslant k}$ into $A[x]_{\leqslant kn}$.

Lemma 5.5.1.7. *Let $\phi : A \to B$ be as in Remark 5.5.1.6, and let $L \subseteq B[x]_{\leqslant k}$ be an element of $\mathbf{P}^k(B)$, and let $Nm_{B/A}(L)$ denote the A-submodule of $A[x]_{\leqslant nk}$ generated by elements of the form $Nm_{B[x]/A[x]}(f)$ where $f \in L$. Then $Nm_{B/A}(L) \in \mathbf{P}^{nk}(A)$.*

Proof. The assertion is local with respect to the Zariski topology on A. We may therefore assume without loss of generality that L is free of rank 1, generated by a polynomial $f(x) \in B[x]$ of degree $\leqslant k$. Since $Nm_{B[x]/A[x]}(\lambda f(x)) = Nm_{B/A}(\lambda) Nm_{B[x]/A[x]}(f(x))$ for $\lambda \in B$, we can identify $Nm_{B/A}(L)$ with the A-submodule of $A[x]_{\leqslant kn}$ generated by $Nm_{B[x]/A[x]}(f(x))$. We wish to show that this submodule is an element of $\mathbf{P}^{nk}(A)$. Using Remark 5.5.1.3, we can reduce to the case where $A = \kappa$ is a field, in which case we wish to show that $Nm_{B[x]/A[x]}(f(x))$ has invertible image in the fraction field $\kappa(x)$ of the polynomial ring $\kappa[x]$. To prove this, it will suffice to show that $f(x)$ has invertible image in the tensor product $B[x] \otimes_{\kappa[x]} \kappa(x)$, which follows from the criterion of Remark 5.5.1.3. □

Construction 5.5.1.8. Let $u : \mathsf{U} \to \mathsf{X}$ be a morphism of algebraic spaces which is finite flat of degree n, and suppose we are given a morphism $f : \mathsf{U} \to \mathbf{P}^k_{\mathbf{Z}}$. For every A-valued point η of X, our assumption that q is finite flat of degree n implies that we can write $\mathsf{U} \times_{\mathsf{X}} \mathrm{Spét}\,A = \mathrm{Spét}\,B$ for some B which is finite flat of degree n over A. The composite map $\mathrm{Spét}\,B \to \mathsf{U} \xrightarrow{f} \mathbf{P}^k_{\mathbf{Z}}$ then determines an element $L_\eta \in \mathbf{P}^k(B)$, so that Lemma 5.5.1.7 produces an element $Nm_{B/A}(L_\eta) \in \mathbf{P}^{kn}(A)$. The construction $\eta \mapsto Nm_{B/A}(L_\eta)$ depends functorially on A and is therefore classified by a map of algebraic spaces $Nm_q(f) : \mathsf{X} \to \mathbf{P}^{kn}_{\mathbf{Z}}$.

We will need the following compatibility between Constructions 5.5.1.4 and 5.5.1.8:

Proposition 5.5.1.9. *Let $q : \mathsf{U} \to \mathsf{X}$ be a morphism of algebraic spaces which is finite étale of rank n, let $f : \mathsf{U} \to \mathbf{P}^k_{\mathbf{Z}}$ be a morphism of algebraic spaces, and let $\mathrm{Conf}^n_{\mathsf{X}}(\mathsf{U})$ be as in Construction 3.2.2.1. Then the diagram*

$$\begin{array}{ccccc} \mathrm{Conf}^n_{\mathsf{X}}(\mathsf{U}) & \longrightarrow & \mathsf{U} \times_{\mathsf{X}} \cdots \times_{\mathsf{X}} \mathsf{U} & \xrightarrow{f} & \mathbf{P}^k_{\mathbf{Z}} \times_{\mathrm{Spét}\,\mathbf{Z}} \cdots \times_{\mathrm{Spét}\,\mathbf{Z}} \mathbf{P}^k\,\mathbf{Z} \\ \downarrow & & & & \downarrow \gamma \\ \mathsf{X} & & \xrightarrow{Nm_q(f)} & & \mathbf{P}^{kn}_{\mathbf{Z}} \end{array}$$

commutes, where $Nm_q(f)$ is the map defined in Construction 5.5.1.8 and γ is the map defined in Construction 5.5.1.4.

Proof. The assertion is local with respect to the étale topology on X. We may therefore assume without loss of generality that $\mathsf{X} = \mathrm{Spét}\,A$ is affine, that $\mathsf{U} = \mathrm{Spét}\,A^n$, and that the map $f : \mathsf{U} \to \mathbf{P}^k_{\mathbf{Z}}$ is given by an element $L \in \mathbf{P}^k(A^n)$ which is free as an A^n-module: that is, it is given by an n-tuple of elements $f_1(x), \ldots, f_n(x) \in A[x]_{\leqslant k}$. The result now follows from the observation that for every permutation σ of the set $\{1, \ldots, n\}$, the norm $Nm_{A^n[x]/A[x]}(f_1(x), \ldots, f_n(x))$ coincides with the product $\prod_{1 \leqslant i \leqslant n} f_{\sigma(i)}(x)$. □

Warning 5.5.1.10. The constructions of this section (specifically, Constructions 5.5.1.4 and 5.5.1.8) have no obvious analogue in the setting of spectral algebraic geometry: they use special features of commutative rings that do not hold for $\mathbb{E}_\infty$-rings.

5.5.2 Chow's Lemma in Classical Algebraic Geometry

Our next goal is to prove the following version of Chow's lemma in classical algebraic geometry:

Proposition 5.5.2.1. *Let R be a commutative ring and let X be an algebraic space which is nonempty, quasi-compact, separated, and locally of finite type over R. Then there exists a commutative diagram of algebraic spaces σ :*

$$\begin{array}{ccc} \widetilde{\mathsf{X}} & \xrightarrow{\phi} & \mathbf{P}^n_R \\ \downarrow \psi & & \downarrow \\ \mathsf{X} & \longrightarrow & \mathrm{Spét}\,R \end{array}$$

with the following properties:

(*a*) *The diagram σ induces a closed immersion $\widetilde{\mathsf{X}} \hookrightarrow \mathsf{X} \times_{\mathrm{Spét}\,R} \mathbf{P}^n_R$.*

(*b*) *There exists a nonempty open substack* $\mathsf{U} \subseteq \mathsf{X}$ *for which the projection map* $\psi_{\mathsf{U}} : \widetilde{\mathsf{X}} \times_{\mathsf{X}} \mathsf{U} \to \mathsf{U}$ *is an equivalence.*

(*c*) *The map* ϕ *is locally quasi-finite.*

Proof. Let us regard the commutative ring R as fixed. For each $k \geqslant 0$, we let $\mathbf{P}^k$ denote the projective space $\mathbf{P}^k_R$ of dimension k over R, and $\mathbf{A}^k$ the affine space $\operatorname{Spét} R[t_1, \ldots, t_k]$ of dimension k over R. All products are formed in the (ordinary) category of algebraic spaces over R (so that we write $\mathsf{X} \times \mathsf{Y}$ in place of $\tau_{\leqslant 0}(\mathsf{X} \times_{\operatorname{Spét} R} \mathsf{Y})$, if X and Y are 0-truncated spectral algebraic spaces over R).

We now follow the proof of Chow's lemma given in [117]. Since X is quasi-compact, we can choose a surjective étale map $u : \mathsf{U} \to \mathsf{X}$, where U is affine. Since X is separated, the morphism u is affine. For each point $x \in |\mathsf{X}|$, let $\kappa(x)$ denote the residue field of X at x (see Notation 3.6.2.5). Then the fiber $u^{-1}(x) = \operatorname{Spét} \kappa(x) \times_{\mathsf{X}} \mathsf{U}$ has the form $\operatorname{Spét} T_x$, where T_x is an étale $\kappa(x)$-algebra (that is, a finite product of finite separable field extensions of $\kappa(x)$). Let $r(x)$ denote the dimension of T_x as a $\kappa(x)$-algebra. It follows from Lemma 3.2.3.4 that the function $r : |\mathsf{X}| \to \mathbf{Z}_{\geqslant 0}$ is lower semicontinuous and bounded above. Set $n = \sup\{r(x) : x \in |\mathsf{X}|\}$, and note that $n > 0$ (since U is nonempty). Let $\mathsf{X}_0 \subseteq \mathsf{X}$ be the open substack corresponding to the open subset Choose a nonempty open subset $\{x \in |\mathsf{X}| : r(x) = n\} \subseteq |\mathsf{X}|$. Note that X_0 is quasi-compact (Lemma 3.2.3.4). Let $\mathsf{U}_0 \subseteq \mathsf{U}$ denote the inverse image of X_0 in U. Then u restricts to a map $u_0 : \mathsf{U}_0 \to \mathsf{X}_0$, and Lemma 3.2.3.4 implies that u_0 is finite étale of rank n (in particular, it is finite flat of rank n).

Since U is affine and locally of finite type over R, we can choose a closed immersion $e : \mathsf{U} \to \mathbf{A}^k$ for some $k \gg 0$. Let $\overline{e}$ denote the composite map $\mathsf{X} \xrightarrow{e} \mathbf{A}^k \hookrightarrow \mathbf{P}^k$, and let $\rho : \mathsf{X}_0 \to \mathbf{P}^{kn}$ denote the norm $Nm_{u_0}(\overline{e}|_{\mathsf{U}_0})$ (see Construction 5.5.1.8). Let $v : \mathsf{X}_0 \hookrightarrow \mathsf{X}$ denote the inclusion map. We define $\widetilde{\mathsf{X}}$ to be the schematic image (see Construction 3.1.5.1) of the map $(v, \rho) : \mathsf{X}_0 \to \mathsf{X} \times \mathbf{P}^{kn}$. By construction, we have an evident commutative diagram

$$\begin{array}{ccc} \widetilde{\mathsf{X}} & \xrightarrow{\phi} & \mathbf{P}^{kn} \\ \downarrow{\scriptstyle \psi} & & \downarrow \\ \mathsf{X} & \longrightarrow & \operatorname{Spét} R. \end{array}$$

We claim that this diagram satisfies the requirements of Theorem 5.5.2.1. Condition (*a*) is obvious (since $\widetilde{\mathsf{X}}$ is defined as a closed substack of $\mathsf{X} \times \mathbf{P}^{kn}$). To verify (*b*), it will suffice to show that the projection map $\psi_0 : \mathsf{X}_0 \times_{\mathsf{X}} \widetilde{\mathsf{X}} \to \mathsf{X}_0$ is an isomorphism. Using Remark 3.1.5.6, we see that the fiber product $\mathsf{X}_0 \times_{\mathsf{X}} \widetilde{\mathsf{X}}$ can be identified with the schematic image of the map $(\mathrm{id}, \rho) : \mathsf{X}_0 \to \mathsf{X}_0 \times \mathbf{P}^{kn}$. This map is already a closed immersion (since $\mathbf{P}^{kn}$ is separated), so its schematic image is equivalent to X_0.

It remains to prove that the map $\phi : \mathsf{X} \to \mathbf{P}^{kn}$ is locally quasi-finite. Let $\mathsf{Y}_0 = \mathrm{Conf}^n_{\mathsf{X}_0}(\mathsf{U}_0)$ (see Construction 3.2.2.1). Proposition 5.5.1.9 supplies a commutative diagram

$$\begin{array}{ccc} \mathsf{Y}_0 & \xrightarrow{\overline{\rho}} & (\mathbf{P}^k)^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \gamma} \\ \mathsf{X}_0 & \xrightarrow{\rho} & \mathbf{P}^n \end{array}$$

where $\gamma : (\mathbf{P}^k)^n \to \mathbf{P}^{kn}$ is defined in Construction 5.5.1.4. Let Y denote the schematic image of the map $(v \circ \pi, \overline{\rho}) : \mathsf{Y}_0 \to \mathsf{X} \times (\mathbf{P}^k)^n$. We then have a commutative diagram

$$\begin{array}{ccccccc} \mathsf{Y}_0 & \longrightarrow & \mathsf{Y} & \xrightarrow{\overline{i}} & \mathsf{X} \times (\mathbf{P}^k)^n & \longrightarrow & (\mathbf{P}^k)^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} & & \downarrow & & \downarrow{\scriptstyle \gamma} \\ \mathsf{X}_0 & \longrightarrow & \widetilde{\mathsf{X}} & \xrightarrow{i} & \mathsf{X} \times \mathbf{P}^{kn} & \longrightarrow & \mathbf{P}^{kn}. \end{array}$$

The map γ is finite (Proposition 5.5.1.5) and therefore induces a closed map of topological spaces $|\mathsf{X} \times (\mathbf{P}^k)^n| \to |\mathsf{X} \times \mathbf{P}^{kn}|$ (Proposition 5.2.1.1). It follows that $|\mathsf{Y}|$ has closed image in $|\mathsf{X} \times \mathbf{P}^{kn}|$. Since π is surjective, it follows from Proposition 3.6.1.6 that the image of $|\mathsf{Y}|$ contains $|\mathsf{X}|$: that is, the map $\overline{\pi} : \mathsf{Y} \to \mathsf{X}$ is surjective. Consequently, to show that to show that ϕ is locally quasi-finite, it will suffice to show that $\phi \circ \overline{\pi}$ is quasi-finite. Let $\overline{\phi}$ denote the composite map $\mathsf{Y} \xrightarrow{\overline{i}} \mathsf{X} \times (\mathbf{P}^k)^n \to (\mathbf{P}^k)^n$. Then $\phi \circ \overline{\pi} = \gamma \circ \overline{\phi}$. Since γ is finite, it will suffice to show that $\overline{\phi}$ is locally quasi-finite.

For $1 \leqslant j \leqslant n$, let $\mathsf{V}_j \subseteq (\mathbf{P}^k)^n$ denote the inverse image of $\mathbf{A}^k$ under the jth projection $(\mathbf{P}^k)^n = \mathbf{P}^k$, and let Y_j denote the open substack of Y given by $\mathsf{Y}_j \times_{(\mathbf{P}^k)^n} \mathsf{V}_j$. To complete the proof of Theorem 5.5.2.1, it will suffice to verify the following:

(a) For $1 \leqslant j \leqslant n$, the restriction $\overline{\phi}|_{\mathsf{Y}_j} : \mathsf{Y}_j \to \mathsf{V}_j$ is locally quasi-finite.

(b) The open substacks $\{\mathsf{Y}_j\}_{1 \leqslant j \leqslant n}$ cover Y.

We begin with the proof of (a). Using Remark 3.1.5.6, we can identify Y_j with the schematic image of the map $(v \circ \pi, \overline{\rho}) : \mathsf{Y}_0 \to \mathsf{X} \times \mathsf{V}_j$. Unwinding the definitions (and identifying V_j with the product $\mathbf{A}^k \times (\mathbf{P}^k)^{n-1}$, we see that this map factors as a composition

$$\mathsf{Y}_0 \to \mathsf{U} \times (\mathbf{P}^k)^{n-1} \xrightarrow{h_j} \mathsf{X} \times \mathbf{A}^k \times (\mathbf{P}^k)^{n-1},$$

where h_j is the product of the map $(u, e) : \mathsf{U} \to \mathsf{X} \times \mathbf{A}^k$ with the identity map on $(\mathbf{P}^k)^{n-1}$. The map (u, e) is a closed immersion (since e is a closed immersion and X is separated), so h_j is a closed immersion. We can therefore identify Y_j with a closed substack of the product

$\mathsf{U} \times (\mathbf{P}^k)^{n-1}$. Consequently, to show that $\overline{\phi}|_{\mathsf{Y}_j}$ is locally quasi-finite, it will suffice to show that the composition

$$\mathsf{U} \times (\mathbf{P}^k)^{n-1} \xrightarrow{h_j} \mathsf{X} \times \mathbf{A}^k \times (\mathbf{P}^k)^{n-1} \to \mathbf{A}^k \times (\mathbf{P}^k)^{n-1}$$

is locally quasi-finite. In fact, this composition is a closed immersion (since it is a pullback of the closed immersion $e : \mathsf{U} \hookrightarrow \mathbf{A}^k$).

We now prove (b). Since the map $u : \mathsf{U} \to \mathsf{X}$ is surjective, it will suffice to show that $\mathsf{U} \times_{\mathsf{X}} \mathsf{Y}$ is covered by the open substacks $\{\mathsf{U} \times_{\mathsf{X}} \mathsf{Y}_j\}_{1 \leqslant j \leqslant n}$. Using Remark 3.1.5.6, we see that $\mathsf{U} \times_{\mathsf{X}} \mathsf{Y}_0 \simeq \mathsf{U}_0 \times_{\mathsf{X}_0} \mathsf{Y}_0$ is dense in $\mathsf{U} \times_{\mathsf{X}} \mathsf{Y}$. Since the map $u_0 : \mathsf{U}_0 \to \mathsf{X}_0$ is finite étale of rank n, it is split after base change along the map $\mathsf{Y}_0 = \mathrm{Conf}^n_{\mathsf{X}_0}(\mathsf{U}_0)$: that is, we can identify the fiber product $\mathsf{U}_0 \times_{\mathsf{X}_0} \mathsf{Y}_0$ with a disjoint union $\amalg_{1 \leqslant j \leqslant n} \mathsf{Y}_0$. For $1 \leqslant j \leqslant n$, let $Z_j \subseteq |\, \mathsf{U}_0 \times_{\mathsf{X}_0} \mathsf{Y}_0 \,|$ denote the jth summand, and let $\overline{Z}_j$ denote the closure of Z_j in $|\, \mathsf{U} \times_{\mathsf{X}} \mathsf{Y} \,|$. Using Proposition 3.6.1.6, we see that the closed sets $\overline{Z}_j$ cover $|\, \mathsf{U} \times_{\mathsf{X}} \mathsf{Y} \,|$. Consequently, it will suffice to show that each $\overline{Z}_j$ is contained in $|\, \mathsf{U} \times_{\mathsf{X}} \mathsf{Y}_j \,|$. To prove this, we observe that $|\, \mathsf{U} \times_{\mathsf{X} \times \mathbf{P}^k} \mathsf{Y} \,|$ is a closed subspace of $|\, \mathsf{U} \times_{\mathsf{X}} \mathsf{Y} \,|$ which is contained in $|\, \mathsf{U} \times_{\mathsf{X}} \mathsf{Y}_j \,|$ and contains Z_j. □

5.5.3 Chow's Lemma in Spectral Algebraic Geometry

We now turn to the proof of Theorem 5.5.0.1. We begin by considering the case of ordinary algebraic spaces over a Noetherian commutative ring. In this case, we will use Proposition 5.5.2.1 to establish the following special case of Theorem 5.5.0.1.

Proposition 5.5.3.1. *Let R be a Noetherian commutative ring and let X be an algebraic space which is quasi-compact, separated, and locally of finite type over R. In the category of algebraic spaces over R, there exist closed immersions*

$$\varnothing = \mathsf{Y}_0 \hookrightarrow \mathsf{Y}_1 \hookrightarrow \mathsf{Y}_2 \hookrightarrow \cdots \hookrightarrow \mathsf{Y}_n = \mathsf{X}$$

and $h_i : \widetilde{\mathsf{Y}}_i \hookrightarrow \mathsf{Y}_i \times_{\mathrm{Sp\acute{e}t}\, R} \mathbf{P}^{d_i}_R$ such that each composite map $\widetilde{\mathsf{Y}}_i \xrightarrow{h_i} \mathsf{Y}_i \times_{\mathrm{Sp\acute{e}t}\, R} \mathbf{P}^{d_i}_R \to \mathbf{P}^{d_i}_R$ is locally quasi-finite and each projection map $\widetilde{\mathsf{Y}}_i \times_{\mathsf{Y}_i} \mathsf{U}_i \to \mathsf{U}_i$ is an equivalence, where U_i denotes the open substack of Y_i complementary to the closed immersion $\mathsf{Y}_{i-1} \hookrightarrow \mathsf{Y}_i$.

Proof. Proceeding by Noetherian induction, we may suppose that the conclusion of Proposition 5.5.3.1 is satisfied for every closed subspace $\mathsf{X}' \subsetneq \mathsf{X}$. If X is empty, then there is nothing to prove. Otherwise, let σ :

$$\begin{array}{ccc} \widetilde{\mathsf{X}} & \xrightarrow{\phi} & \mathbf{P}^k_R \\ \downarrow{\scriptstyle \psi} & & \downarrow \\ \mathsf{X} & \longrightarrow & \mathrm{Sp\acute{e}t}\, R \end{array}$$

and $\mathsf{U} \subseteq \mathsf{X}$ satisfy the requirements of Proposition 5.5.2.1 and let X' be the reduced closed substack of X complementary to U. Applying our inductive hypothesis to X', we deduce the existence of closed immersions

$$\varnothing = \mathsf{Y}_0 \hookrightarrow \mathsf{Y}_1 \hookrightarrow \mathsf{Y}_2 \hookrightarrow \cdots \hookrightarrow \mathsf{Y}_n = \mathsf{X}'$$

and $h_i : \widetilde{\mathsf{Y}}_i \hookrightarrow \mathsf{Y}_i \times_{\operatorname{Sp\acute{e}t} R} \mathbf{P}^{d_i}_R$ satisfying the requirements of Proposition 5.5.3.1 for X'. We then obtain a proof of Proposition 5.5.3.1 by setting $\mathsf{Y}_{n+1} = \mathsf{X}$ and taking $h_{n+1} : \widetilde{\mathsf{X}} \to \mathsf{Y}_{n+1} \times_{\operatorname{Sp\acute{e}t} R} \mathbf{P}^k_R$ to be the map classifying σ. □

Proof of Theorem 5.5.0.1. Let $f : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ be a quasi-compact, separated morphism of spectral algebraic spaces which exhibits $\tau_{\leqslant 0} \mathsf{X}$ as finitely 0-presented over R. Applying Proposition 4.4.4.1, we can choose a diagram σ :

$$\begin{array}{ccc} \mathsf{X} & \longrightarrow & \mathsf{X}_0 \\ \downarrow f & & \downarrow f_0 \\ \operatorname{Sp\acute{e}t} R & \longrightarrow & \operatorname{Sp\acute{e}t} R_0 \end{array}$$

where R_0 is finitely presented over the sphere spectrum S, the morphism f_0 is almost of finite presentation, and σ induces an equivalence $\tau_{\leqslant 0} \mathsf{X} \to \tau_{\leqslant 0}(\mathsf{X}_0 \times_{\operatorname{Sp\acute{e}t} R_0} \operatorname{Sp\acute{e}t} R)$. Using Corollary 4.6.1.4, we may further assume (after replacing R_0 by an extension if necessary) that X_0 is a separated spectral algebraic space. Applying Proposition 5.5.3.1 to the map of ordinary algebraic spaces $\tau_{\leqslant 0} \mathsf{X}_0 \to \operatorname{Sp\acute{e}t} \pi_0 R$, we deduce the existence of a sequence of closed immersions

$$\varnothing = \mathsf{Y}'_0 \hookrightarrow \mathsf{Y}'_1 \hookrightarrow \mathsf{Y}'_2 \hookrightarrow \cdots \hookrightarrow \mathsf{Y}'_n = \tau_{\leqslant 0} \mathsf{X}_0 .$$

and

$$h'_i : \widetilde{\mathsf{Y}}'_i \hookrightarrow \mathsf{Y}'_i \times_{\operatorname{Sp\acute{e}t} \pi_0 R_0} \mathbf{P}^{d_i}_{\pi_0 R_0} \simeq \mathsf{Y}' \times_{\operatorname{Sp\acute{e}t} R_0} \mathbf{P}^{d_i}_{R_0}$$

of 0-truncated algebraic spaces over R_0 for which the induced maps $e_i : \widetilde{\mathsf{Y}}'_i \to \mathbf{P}^{d_i}_{\pi_0 R_0}$ are locally quasi-finite and the projection maps $\widetilde{\mathsf{Y}}'_i \times_{\mathsf{Y}'_i} \mathsf{U}'_i \to \mathsf{U}'_i$ are equivalences, where U'_i denotes the open substack of Y'_i complementary to Y'_{i-1}. We now set $\mathsf{Y}_i = \mathsf{X} \times_{\mathsf{X}_0} \mathsf{Y}'_i$ and $\widetilde{\mathsf{Y}}'_i = \mathsf{X} \times_{\mathsf{X}_0} \widetilde{\mathsf{Y}}'_i$. By construction, we have closed immersions of spectral algebraic spaces

$$\varnothing = \mathsf{Y}_0 \to \mathsf{Y}_1 \to \cdots \to \mathsf{Y}_n = \mathsf{X} \times_{\mathsf{X}_0} \tau_{\leqslant 0} \mathsf{X}_0 \xrightarrow{g} \mathsf{X},$$

where g induces an equivalence of the underlying ordinary algebraic spaces. Each of the closed immersions $\mathsf{Y}_i \to \mathsf{X}$ is a pullback of the closed immersion $\mathsf{Y}'_i \to \mathsf{X}_0$, which is automatically almost of finite presentation because Y'_i and X_0 are locally Noetherian (Remark 4.2.0.4).

Finally, each of the maps h'_i fits into a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathsf{Y}}_i & \xrightarrow{h_i} & \mathsf{Y}_i \times_{\mathrm{Sp\acute{e}t}\, R} \mathbf{P}^{d_i}_R \\ \downarrow & & \downarrow \\ \widetilde{\mathsf{Y}}'_i & \xrightarrow{h'_i} & \mathsf{Y}'_i \times_{\mathrm{Sp\acute{e}t}\, R_0} \mathbf{P}^{d_i}_{R_0}. \end{array}$$

Since h'_i is a closed immersion of locally Noetherian algebraic spaces, it is automatically locally almost of finite presentation when regarded as a morphism of spectral algebraic spaces (Remark refontinet), so that h_i is also a closed immersion which is locally almost of finite presentation. Since the projection maps $\widetilde{\mathsf{Y}}'_i \times_{\mathsf{Y}'_i} \mathsf{U}'_i \to \mathsf{U}'_i$ are equivalences, it follows that the maps h_i induce equivalences $\widetilde{\mathsf{Y}}_i \times_{\mathsf{Y}_i} \mathsf{U}_i \simeq \mathsf{U}_i$, where $\mathsf{U}_i \simeq \mathsf{U}'_i \times_{\mathsf{Y}'_i} \mathsf{Y}_i$ is the open substack of Y_i complementary to the image of Y_{i-1}. We conclude by observing that each of the composite maps $\widetilde{\mathsf{Y}}_i \xrightarrow{h_i} \mathsf{Y}_i \times_{\mathrm{Sp\acute{e}t}\, R} \mathbf{P}^{d_i}_R$ is locally quasi-finite, since it is a pullback of the composition $\widetilde{\mathsf{Y}}'_i \xrightarrow{e_i} \mathbf{P}^{d_i}_{\pi_0 R_0} \xrightarrow{u} \mathbf{P}^{d_i}_R$ where e_i is locally quasi-finite by construction and the map u is locally quasi-finite since it induces an equivalence of 0-truncations. □

5.5.4 Application: Noetherian Approximation for Properness

We close this section by describing typical application of Chow's lemma.

Proposition 5.5.4.1. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geqslant 0$ be an integer, let $\mathsf{X}_0 \in \mathrm{DM}^{\mathrm{fp}}_n(A_0)$. For each index α, let X_α denote the image of X_0 in $\mathrm{DM}^{\mathrm{fp}}_n(A_\alpha)$, and let X denote the image of X_0 in $\mathrm{DM}^{\mathrm{fp}}_n(A)$. If X is proper over $\mathrm{Sp\acute{e}t}\, A$, then there exists an index α such that X_α is proper over $\mathrm{Sp\acute{e}t}\, A_\alpha$.*

We begin with an easier version of Proposition 5.5.4.1.

Lemma 5.5.4.2. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geqslant 0$ be an integer, let $\mathsf{X}_0 \in \mathrm{DM}^{\mathrm{fp}}_n(A_0)$. For each index α, let X_α denote the image of X_0 in $\mathrm{DM}^{\mathrm{fp}}_n(A_\alpha)$, and let X denote the image of X_0 in $\mathrm{DM}^{\mathrm{fp}}_n(A)$. If X is finite over $\mathrm{Sp\acute{e}t}\, A$, then there exists an index α such that X_α is finite over $\mathrm{Sp\acute{e}t}\, A_\alpha$.*

Remark 5.5.4.3. By virtue of Proposition 5.2.1.1, Lemma 5.5.4.2 is an immediate consequence of Propositions 5.5.4.1 and 4.6.1.1. However, we will use Lemma 5.5.4.2 in our proof of Proposition 5.5.4.1.

Proof of Lemma 5.5.4.2. Using Proposition 4.6.1.1, we see that there exists an index α such that X_α is affine: that is, we can write $\mathsf{X}_\alpha = \mathrm{Sp\acute{e}t}\, B_\alpha$ for some B_α which is finitely n-presented over A. Choose a finite collection of elements $x_1, \ldots, x_n \in \pi_0 B_\alpha$ which generate $\pi_0 B_\alpha$ as an algebra over $\pi_0 A_\alpha$. Since X is finite over A, the commutative ring $\pi_0(A \otimes_{A_\alpha} B_\alpha)$ is finitely generated as a module over $\pi_0 A$. It follows that the image of each x_i in $\pi_0(A \otimes_{A_\alpha} B_\alpha)$ is integral over $\pi_0 A$. Modifying α if necessary, we may assume that each x_i is integral over $\pi_0 A_\alpha$. It then follows that $\pi_0 B_\alpha$ is finitely generated as a module over $\pi_0 A_\alpha$, so that X_α is finite over A_α. □

Proof of Proposition 5.5.4.1. Using Remark 5.1.2.2, we may reduce to the case where $n = 0$. Using Corollary 4.6.1.4, we may assume without loss of generality that X_0 is a separated spectral algebraic space of finite type over A_0. Choose closed immersions

$$\emptyset = \mathsf{Y}'_0 \to \cdots \to \mathsf{Y}'_n \to \mathsf{X}_0$$

and $h'_i : \widetilde{\mathsf{Y}}'_i \to \mathsf{Y}'_i \times_{\mathrm{Sp\acute{e}t}\, A_0} \mathbf{P}^{d_i}_{A_0}$. satisfying the requirements of Theorem 5.5.0.1. Set $\mathsf{Y}_i = \mathsf{Y}'_i \times_{\mathsf{X}_0} \mathsf{X}$ and $\widetilde{\mathsf{Y}}_i = \widetilde{\mathsf{Y}}'_i \times_{\mathsf{X}_0} \mathsf{X}$, so that the maps h'_i induce closed immersions $h_i : \widetilde{\mathsf{Y}}_i \to \mathsf{Y}_i \times_{\mathrm{Sp\acute{e}t}\, A} \mathbf{P}^{d_i}_A$. Since X is proper over A, it follows from Remark 5.5.0.4 that each of the induced maps $\widetilde{\mathsf{Y}}_i \to \mathbf{P}^{d_i}_A$ is finite. Using Lemma 5.5.4.2, we can assume (after replacing A_0 by some A_α) that each of the maps $\widetilde{\mathsf{Y}}'_i \to \mathbf{P}^{d_i}_{A_0}$ is finite. Combining Proposition 5.2.1.1 with Corollary 5.4.3.4, we deduce that each $\widetilde{\mathsf{Y}}'_i$ is proper over $\mathrm{Sp\acute{e}t}\, A_0$. Since the map $\amalg \widetilde{\mathsf{Y}}'_i \to \mathsf{X}_0$ is surjective, it follows from Proposition 5.1.4.1 that X_0 is also proper over $\mathrm{Sp\acute{e}t}\, A_0$. □

5.6 The Direct Image Theorem

Recall the direct image theorem for proper morphisms of Noetherian algebraic spaces (see [117]):

Theorem 5.6.0.1. *Let $f : X \to Y$ be a proper morphism between Noetherian algebraic spaces and let $\mathscr{F}$ be an object in the abelian category of coherent sheaves on X. Then, for each $i \geqslant 0$, the higher direct image $R^i f_* \mathscr{F}$ is a coherent sheaf on Y.*

If we regard X and Y as spectral algebraic spaces, then the higher direct images $R^i f_* \mathscr{F}$ are just the homotopy sheaves of the (derived) direct image $f_* \mathscr{F}$. By virtue of Corollary 2.7.2.3, the requirement that these homotopy sheaves are coherent (in the sense of classical algebraic geometry) is equivalent to the requirement that $f_* \mathscr{F}$ is almost perfect (in the sense of Definition 2.8.4.4). Consequently, Theorem 5.6.0.1 can be regarded as a special case of the following more general assertion:

Theorem 5.6.0.2. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation. Then the pushforward functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ carries almost perfect objects of $\mathrm{QCoh}(\mathsf{X})$ to almost perfect objects of $\mathrm{QCoh}(\mathsf{Y})$.*

Our goal in this section is to give a proof of Theorem 5.6.0.2. We will employ the same basic strategy used by Knutson to prove Theorem 5.6.0.1 in [117] (and earlier by Grothendieck to establish the special case of Theorem 5.6.0.1 where X and Y are schemes): namely, we use Chow's lemma to reduce to the case where X is a projective space over Y, in which case the desired result can be deduced from Serre's calculation of the cohomology of line bundles on projective space (Theorem 5.4.2.6). However, the proof of Theorem 5.6.0.2 is a bit more subtle, because we do not make any Noetherian assumptions on X and Y.

Remark 5.6.0.3. In the setting of classical algebraic geometry, Theorem 5.6.0.2 was proven by Illusie in the special case where f is projective (see [101]), and by Kiehl for a general proper morphism (see [116]).

5.6.1 The Case of a Finite Morphism

We begin with an easy special case of Theorem 5.6.0.2.

Proposition 5.6.1.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is finite and locally almost of finite presentation. Then an object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is almost perfect if and only if the direct image $f_* \mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ is almost perfect.*

We will deduce Proposition 5.6.1.1 from the following:

Lemma 5.6.1.2. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_1$-rings, and let M be a left B-module. Suppose that B is almost perfect when regarded as a left A-module. Then, for every integer n, the module M is perfect to order n over B if and only if it is perfect to order n over A. In particular, M is almost perfect when regarded as a left B-module if and only if it is almost perfect when regarded as a left A-module.*

Proof. Note that if M is perfect to order n as a left module over either A or B, then M is $(-m)$-connective for $m \gg 0$. Replacing M by $\Sigma^m M$ (and n by $n+m$), we may assume that M is connective. We now proceed by induction on n. If $n < 0$, there is nothing to prove. Let us therefore assume that $n \geqslant 0$. Note that since B is almost perfect as a left A-module, the homotopy group $\pi_0 B$ is finitely generated as a left A-module. It follows that $\pi_0 M$ is finitely generated as a left module over $\pi_0 A$ if and only if it is finitely generated as a module over $\pi_0 B$. If neither of these conditions is satisfied, then M is not perfect to order n as either a left A-module or as a left B-module, and the proof is complete. Otherwise, we can choose a fiber sequence of left B-modules $M' \to B^n \to M$, where M' is connective. Note that B^n is almost perfect both as a left A-module and as a left B-module. It follows that M is perfect to order n (as a left module over either A or B) if and only if M' is perfect to order $(n-1)$ (as a left module over either A or B); see Remark 2.7.0.7. The desired result now follows by applying our inductive hypothesis to M'. □

Proof of Proposition 5.6.1.1. Let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$; we wish to show that $\mathscr{F}$ is almost perfect if and only if $f_* \mathscr{F}$ is almost perfect. Both conditions are local on Y, so we may assume without loss of generality that $\mathsf{Y} = \operatorname{Sp\acute{e}t} A$ is affine. In this case, our assumption that f is finite guarantees that $\mathsf{X} \simeq \operatorname{Sp\acute{e}t} B$ is also affine. Since f is locally almost of finite presentation, the $\mathbb{E}_\infty$-ring B is almost perfect when regarded as an A-module (Proposition 5.2.2.2). The desired result now follows from Lemma **??**. □

5.6.2 The Case of Projective Space

We begin by establishing the following special case of Theorem 5.6.0.2:

Proposition 5.6.2.1. *Let R be a connective $\mathbb{E}_\infty$-ring, let $n \geqslant 0$ be an integer, and let $q : \mathbf{P}^n_R \to \operatorname{Sp\acute{e}t} R$ denote the projection map. Then the direct image functor carries almost perfect objects of $\mathrm{QCoh}(\mathbf{P}^n_R)$ to almost perfect objects of $\mathrm{QCoh}(\operatorname{Sp\acute{e}t} R) \simeq \mathrm{Mod}_R$.*

We will deduce Proposition 5.6.2.1 from Theorem 5.4.2.6 and the following:

Lemma 5.6.2.2. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathscr{F} \in \mathrm{QCoh}(\mathbf{P}^n_R)^{\mathrm{cn}}$. Then there exists a fiber sequence $\mathscr{F}' \to \mathscr{G} \to \mathscr{F}$ in $\mathrm{QCoh}(\mathbf{P}^n_R)$, where $\mathscr{G} \simeq \bigoplus_\alpha \mathscr{O}(d_\alpha)$ is a coproduct of copies of the line bundles $\mathscr{O}(d)$ described in Construction 5.4.2.1, and $\mathscr{F}'$ is connective.*

Proof. Let us denote the monoid algebra $R[\mathbf{Z}^{n+1}_{\geqslant 0}]$ by $R[x_0, \ldots, x_n]$. Set $\mathsf{X} = \operatorname{Sp\acute{e}t} R[x_0, \ldots, x_n]$. Let $\mathsf{U} \subseteq \mathsf{X}$ denote the open substack complementary to the vanishing locus of the ideal $(x_0, \ldots, x_n) \subseteq \pi_0 R[x_0, \ldots, x_n]$. For $0 \leqslant i \leqslant n$, let $\mathsf{U}_i \subseteq \mathsf{U} \subseteq \mathsf{X}$ denote the open substack complementary to the vanishing locus of (x_i), and for each $I \subseteq [n]$ let U_I denote the intersection $\bigcap_{i \in I} \mathsf{U}_i$. Since the open stack U is covered by $\{\mathsf{U}_i\}_{0 \leqslant i \leqslant n}$, we can write U as the colimit $\varinjlim_{\varnothing \neq I \subseteq [n]} \mathsf{U}_I$ (formed in the ∞-category of spectral Deligne-Mumford stacks).

For each subset $I \subseteq [n]$, set $M_I^+ = \{(k_0, \ldots, k_n) \in \mathbf{Z}^{n+1} : (k_i < 0) \Rightarrow (i \in I)\}$, so we have a canonical equivalence $\mathsf{U}_I \simeq \operatorname{Sp\acute{e}t} R[M_I^+]$. Let $M_I \subseteq M_I^+$ denote the subset consisting of those tuples $(k_0, \ldots, k_n)$ satisfying $k_0 + \cdots + k_n = 0$. The inclusion $M_I \hookrightarrow M_I^+$ induces a morphism of $\mathbb{E}_\infty$-algebras $R[M_I] \to R[M_I^+]$. These morphisms depend functorially on I and therefore induce a map

$$q : \mathsf{U} \simeq \varinjlim_{\varnothing \neq I \subseteq [n]} \mathsf{U}_I = \varinjlim_{\varnothing \neq I \subseteq [n]} \operatorname{Sp\acute{e}t} R[M_I^+] \to \varinjlim_{\varnothing \neq I \subseteq [n]} \operatorname{Sp\acute{e}t} R[M_I] \simeq \mathbf{P}^n_R.$$

Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ and $\mathsf{U} = (\mathcal{U}, \mathscr{O}_{\mathcal{U}})$, and let $j : \mathsf{U} \hookrightarrow \mathsf{X}$ denote the inclusion map. Since X is affine, there exists a set A and a morphism $\rho_{\mathsf{X}} : \bigoplus_{\alpha \in A} \mathscr{O}_{\mathcal{X}} \to j_* q^* \mathscr{F}$ which induces an epimorphism $\pi_0(\bigoplus \mathscr{O}_{\mathcal{X}}) \to \pi_0(j_* q^* \mathscr{F})$ in the abelian category $\mathrm{QCoh}(\mathsf{X})^{\heartsuit}$. Restricting to U, we obtain a map $\rho_{\mathsf{U}} : \bigoplus_{\alpha \in A} \mathscr{O}_{\mathcal{U}} \to q^* \mathscr{F}$ in $\mathrm{QCoh}(\mathsf{U})$ which induces an epimorphism on π_0. Let us identify ρ_{U} with a map

$$\bigoplus_{\alpha \in A} \mathscr{O}(0) \to q_* q^* \mathscr{F} \simeq \mathscr{F} \otimes q_* \mathscr{O}_{\mathcal{U}},$$

which classifies a family of maps $\{\rho_\alpha : \mathscr{O}(0) \to \mathscr{F} \otimes q_* \mathscr{O}_{\mathcal{U}}\}_{\alpha \in A}$. Unwinding the definitions, we see that $q_* \mathscr{O}_{\mathcal{U}}$ can be identified with the direct sum $\bigoplus_{m \in \mathbf{Z}} \mathscr{O}(m)$. Since $\mathbf{P}^n_R$ is a quasi-compact and quasi-separated spectral algebraic space, the global section functor on $\mathbf{P}^n_R$ commutes with filtered colimits. It follows that for each $\alpha \in A$, there exists a constant $c_\alpha \geqslant 0$ such that ρ_α factors as a composition

$$\mathscr{O}(0) \xrightarrow{\rho'_\alpha} \bigoplus_{|m| \leqslant c_\alpha} \mathscr{F} \otimes \mathscr{O}(m) \to \bigoplus_{m \in \mathbf{Z}} \mathscr{F} \otimes \mathscr{O}(m) \simeq \mathscr{F} \otimes q_* \mathscr{O}_{\mathcal{U}}.$$

Each of the maps ρ'_α can in turn be identified with a finite collection of maps $\{\rho'_{\alpha,m} : \mathscr{O}(-m) \to \mathscr{F}\}_{|m| \leqslant c_\alpha}$ in the ∞-category $\mathrm{QCoh}(\mathbf{P}^n_R)$. Set $\mathscr{G} = \bigoplus_{\alpha \in A, |m| \leqslant c_\alpha} \mathscr{O}(-m)$, and let $\mu : \mathscr{G} \to \mathscr{F}$ be the amalgam of the maps $\rho'_{\alpha,m}$. We claim that the fiber of μ is connective. Since the map $q : \mathsf{U} \to \mathbf{P}^n_R$ is faithfully flat, it will suffice to show that the pullback $q^* \mu$ has connective fiber: that is, that $q^* \mu$ induces an epimorphism $\pi_0 q^* \mathscr{G} \to \pi_0 q^* \mathscr{F}$. This is clear, since the map ρ_{U} factors through $q^* \mu$. □

Remark 5.6.2.3. In the situation of Lemma 5.6.2.2, suppose that the quasi-coherent sheaf $\mathscr{F}$ is perfect to order 0. Using the quasi-compactness of $\mathbf{P}^n_R$, we can choose a finite set of indices $\{\alpha_1, \ldots, \alpha_k\}$ such that the composite map $\bigoplus_{1 \leqslant i \leqslant k} \mathscr{O}(d_{\alpha_i}) \to \mathscr{G} \to \mathscr{F}$ has connective fiber. In other words, we can assume that $\mathscr{G}$ is a coproduct of *finitely many* line bundles of the form $\mathscr{O}(n)$. In this case, the existence of a fiber sequence $\mathscr{F}' \to \mathscr{G} \to \mathscr{F}$ implies that if $\mathscr{F}$ is perfect to order $m+1$ for $m \geqslant 0$, then $\mathscr{F}'$ is perfect to order m (see Proposition 2.7.2.1). In particular, if $\mathscr{F}$ is almost perfect, then $\mathscr{F}'$ is almost perfect.

Proof of Proposition 5.6.2.1. Let $\mathscr{F} \in \mathrm{QCoh}(\mathbf{P}^n_R)$ be almost perfect and let $q : \mathbf{P}^n_R \to \mathrm{Sp\acute{e}t}\, R$ be the projection map. We wish to show that the direct image $q_* \mathscr{F} \in \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R) \simeq \mathrm{Mod}_R$ is almost perfect. Replacing $\mathscr{F}$ by a suspension if necessary, we may assume without loss of generality that $\mathscr{F}$ is connective. In this case, the desired result follows from the following family of assertions.

$(*_m)$ Let $\mathscr{F} \in \mathrm{QCoh}(\mathbf{P}^n_R)$ be connective and almost perfect. Then $q_* \mathscr{F} \in \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R) \simeq \mathrm{Mod}_R$ is perfect to order m.

Corollary 3.4.2.3 implies that there exists an integer $k \ll 0$ such that for every connective object $\mathscr{F} \in \mathrm{QCoh}(\mathbf{P}^n_R)$, the direct image $q_* \mathscr{F}$ is k-connective (in fact, we can take $k = -n$, since $\mathbf{P}^n_R$ is a separated spectral algebraic space which admits an open cover by $(n+1)$ affine open substacks). Consequently, assertion $(*_m)$ is automatic for $m < k$. We prove $(*_m)$ in general using induction on m. Suppose that $\mathscr{F} \in \mathrm{QCoh}(\mathbf{P}^n_R)$ is connective and almost perfect. Using Remark 5.6.2.3, we can choose a fiber sequence $\mathscr{F}' \to \mathscr{G} \to \mathscr{F}$, where $\mathscr{G}$ is a direct sum of finitely many line bundles of the form $\mathscr{O}(a)$ and $\mathscr{F}'$ is also connective and almost perfect. We then have a fiber sequence of R-modules $q_* \mathscr{F}' \to q_* \mathscr{G} \to q_* \mathscr{F}$.

Theorem 5.4.2.6 shows that $q_* \mathscr{G}$ is a perfect R-module and our inductive hypothesis implies that $q_* \mathscr{F}'$ is perfect to order $(m-1)$, so that $q_* \mathscr{F}$ is perfect to order m as desired. $\square$

Remark 5.6.2.4. The proof of Proposition 5.6.2.1 yields the following slightly stronger assertion:

$(*)$ Let $\mathscr{F} \in \mathrm{QCoh}(\mathbf{P}^n_R)$ be connective and perfect to order m for $m \geqslant 0$, and let $q : \mathbf{P}^n_R \to \mathrm{Sp\acute{e}t}\, R$ denote the projection map. Then $q_* \mathscr{F} \in \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R) \simeq \mathrm{Mod}_R$ is perfect to order $m-n$.

5.6.3 Sheaves Supported on a Closed Subset

Let $f : X \to Y$ be a morphism of complex algebraic varieties, and let $\mathscr{F}$ be a coherent sheaf on X. To guarantee the coherence of the higher direct images $R^i f_* \mathscr{F}$, it is not necessary to assume that f is proper: it is enough to know that the restriction of $f|_K$ is proper, where $K \subseteq X$ is the support of $\mathscr{F}$. To prove an analogous generalization of Theorem 5.6.0.2, we will need the following:

Proposition 5.6.3.1. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a quasi-compact, quasi-separated spectral algebraic space, let $i : (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \to \mathsf{X}$ be a closed immersion which is locally almost of finite presentation, let $j : \mathsf{U} \hookrightarrow \mathsf{X}$ be the complementary open immersion, and let $\mathcal{I}$ denote the fiber of the unit map $\mathscr{O}_{\mathcal{X}} \to i_* \mathscr{O}_{\mathcal{Y}}$. Suppose we are given quasi-coherent sheaves $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$ such that $\mathscr{F}$ is almost perfect, $\mathscr{G}$ is truncated, and $j^* \mathscr{G} \simeq 0$. Then the direct limit $\varinjlim \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathcal{I}^{\otimes n} \otimes \mathscr{F}, \mathscr{G})$ vanishes.*

Proof. For each object $U \in \mathcal{X}$, set $\mathsf{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathcal{X}}|_U)$ and let $T_{\mathscr{G}}(U)$ denote the direct limit $\varinjlim \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X}|_U)}(\mathcal{I}^{\otimes n}|_U \otimes \mathscr{F}|_U, \mathscr{G}|_U)$. Since filtered colimits in $\mathcal{S}$ are left exact, the construction $U \mapsto T_{\mathscr{G}}(U)$ carries finite colimits in $\mathcal{X}$ to finite limits in $\mathcal{S}$. It follows that the collection of those objects $U \in \mathcal{X}$ for which $T_{\mathscr{G}}(U)$ is contractible is closed under finite colimits. We wish to show that $T_{\mathscr{G}}(\mathbf{1})$ is contractible, where $\mathbf{1}$ denotes a final object of $\mathcal{X}$. By virtue of Proposition 2.5.3.5 (and Theorem 3.4.2.1), it will suffice to show that $T_{\mathscr{G}}(U)$ is contractible when $U \in \mathcal{X}$ is affine. Replacing X by X_U, we may assume that X is affine. Let us now denote $T_{\mathscr{G}}(\mathbf{1})$ simply by $T_{\mathscr{G}}$.

Our assumption that i is locally almost of finite presentation guarantees that the sheaf $\mathcal{I}$ is almost perfect (Corollary 5.2.2.2). Consequently, each of the sheaves $\mathcal{I}^{\otimes n} \otimes \mathscr{F}$ is almost perfect. It follows that the the construction $\mathscr{G} \mapsto T_{\mathscr{G}}$ commutes with filtered colimits when restricted to n-truncated objects of $\mathrm{QCoh}(\mathsf{X})$, for any integer n. It will therefore suffice to show that the space $T_{\tau_{\geqslant m} \mathscr{G}}$ is contractible for every integer m. This is automatic for $m \gg 0$ (since $\mathscr{G}$ is truncated). We handle the general case by descending induction on m. To carry out the inductive step, we note that the fiber sequence $\tau_{\geqslant m+1} \mathscr{G} \to \tau_{\geqslant m} \mathscr{G} \to \Sigma^m \pi_m \mathscr{G}$

induces a fiber sequence of spaces

$$T_{\tau_{\geqslant m+1}\mathscr{G}} \to T_{\tau_{\geqslant m}\mathscr{G}} \to T_{\Sigma^m \pi_m \mathscr{G}}.$$

We are therefore reduced to showing that $T_{\Sigma^m \pi_m \mathscr{G}}$ is contractible. Replacing $\mathscr{F}$ by $\Sigma^{-m}\mathscr{F}$ and $\mathscr{G}$ by $\pi_m \mathscr{G}$, we are reduced to proving that $T_{\mathscr{G}}$ is contractible in the special case where $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X})^{\heartsuit}$.

Write $\mathsf{X} = \mathrm{Spét}\, A$, so that $\mathscr{F}$ corresponds to an almost perfect A-module M and $\mathscr{G}$ to a discrete A-module N. Writing N as a colimit of its finitely generated submodules, we can reduce to the case where N is finitely generated. Let $I \subseteq \pi_0 A$ be the ideal given by the kernel of the map $\pi_0 A \to \pi_0 \Gamma(\mathcal{Y}; \mathscr{O}_{\mathcal{Y}})$. Since $\mathcal{I}$ is almost perfect, the ideal I is finitely generated. Our assumption that $j^* \mathscr{G} \simeq 0$ guarantees that every element $x \in I$ has a locally nilpotent action on N. Since I and N are finitely generated, it follows that $I^k N = 0$ for some $k \gg 0$. Then the multiplication map $\mathcal{I}^{\otimes k} \otimes \mathscr{G} \to \mathscr{G}$ is nullhomotopic, so the transition map $\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathcal{I}^{\otimes n} \otimes \mathscr{F}, \mathscr{G}) \to \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathcal{I}^{\otimes n+k} \otimes \mathscr{F}, \mathscr{G})$ is nullhomotopic. Passing to the direct limit, we deduce that $T_{\mathscr{G}} \simeq \varinjlim \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathcal{I}^{\otimes n} \otimes \mathscr{F}, \mathscr{G})$ is contractible, as desired. □

Example 5.6.3.2. In the situation of Proposition 5.6.3.1, suppose that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is almost perfect and $j^* \mathscr{F} \simeq 0$. Then, for every integer n, there exists $k \gg 0$ such that the composite map $\mathcal{I}^{\otimes k} \otimes \mathscr{F} \to \mathscr{F} \to \tau_{\leqslant n} \mathscr{F}$ is nullhomotopic.

Proposition 5.6.3.3. *Let $i : \mathsf{Y} \to \mathsf{X}$ and $f : \mathsf{X} \to \mathsf{Z}$ be morphisms of quasi-compact, quasi-separated spectral algebraic spaces. Suppose that i is a closed immersion which is locally almost of finite presentation, let $j : \mathsf{U} \hookrightarrow \mathsf{X}$ be the complementary open immersion, and let $K \subseteq |\mathsf{X}|$ denote the image of $|\mathsf{Y}|$. Assume that the direct image functor $(f \circ i)_* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{Z})$ carries almost perfect objects of $\mathrm{QCoh}(\mathsf{Y})$ to almost perfect objects of $\mathrm{QCoh}(\mathsf{Z})$. Suppose that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is almost perfect and that $j^* \mathscr{F} \simeq 0$. Then $f_* \mathscr{F} \in \mathrm{QCoh}(\mathsf{Z})$ is almost perfect.*

Proof. Fix an integer n; we will show that $\tau_{\leqslant n} f_* \mathscr{F} \in \mathrm{QCoh}(\mathsf{Z})$ is finitely n-presented. Enlarging n if necessary, we may assume that $n \geqslant 0$ and that the functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Z})$ carries $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$ into $\mathrm{QCoh}(\mathsf{Z})_{\geqslant -n}$. Let $\mathcal{I}$ be as in the statement of Proposition 5.6.3.1. Using Example 5.6.3.2, we deduce that there exists an integer $k \gg 0$ such that the multiplication map $\mathcal{I}^{\otimes k} \otimes \mathscr{F} \xrightarrow{v} \mathscr{F} \to \tau_{\leqslant 2n} \mathscr{F}$ is nullhomotopic. It follows that the truncation map $u : \mathscr{F} \to \tau_{\leqslant 2n} \mathscr{F}$ factors through $\mathrm{cofib}(v)$. We have a fiber sequence $f_*(\tau_{\geqslant 2n+1} \mathscr{F}) \to f_* \mathscr{F} \xrightarrow{f_* u} f_*(\tau_{\leqslant 2n} \mathscr{F})$ whose first term belongs to $\mathrm{QCoh}(\mathsf{Z})_{\geqslant n+1}$, so that $f_*(u)$ induces an equivalence $\tau_{\leqslant n} f_* \mathscr{F} \simeq \tau_{\leqslant n} f_*(\tau_{\leqslant 2n} \mathscr{F})$. It follows that $\tau_{\leqslant n} f_* \mathscr{F}$ is a retract of $\tau_{\leqslant n} f_* \, \mathrm{cofib}(v)$. It will therefore suffice to show that $\tau_{\leqslant n} f_*(\mathrm{cofib}(v))$ is finitely n-presented. In fact, we claim that $f_*(\mathrm{cofib}(v))$ is almost perfect. This a special case of the following assertion:

(∗) For every $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$ almost perfect and each $k \geqslant 0$, the functor f_* carries $\mathrm{cofib}(\mathcal{I}^{\otimes k} \otimes \mathscr{G} \to \mathscr{G})$ to an almost perfect object of $\mathrm{QCoh}(\mathsf{Z})$.

The proof of (∗) proceeds by induction on k, the case $k = 0$ being trivial. To carry out the inductive step, we observe that the multiplication map $v : \mathcal{I}^{\otimes k} \otimes \mathscr{G} \to \mathscr{G}$ factors as a composition

$$\mathcal{I}^{\otimes k} \otimes \mathscr{G} \xrightarrow{v'} \mathcal{I} \otimes \mathscr{G} \xrightarrow{v''} \mathscr{G},$$

so we have a fiber sequence $\mathrm{cofib}(v') \to \mathrm{cofib}(v) \to \mathrm{cofib}(v'')$. Since the direct image $f_* \mathrm{cofib}(v')$ is almost perfect by our inductive hypothesis, we are reduced to showing that the direct image $f_* \mathrm{cofib}(v'')$. This is clear, since we have an equivalence $f_* \mathrm{cofib}(v) \simeq (f \circ i)_* i^* \mathscr{G}$, and the functors $(f \circ i)_*$ and i^* both preserve almost perfect objects. □

5.6.4 The Proof of Theorem 5.6.0.2

We are now ready to give the proof of Theorem 5.6.0.2. Suppose that $f : \mathsf{X} \to \mathsf{Z}$ is a morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation. We wish to show that the direct image functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Z})$ carries almost perfect objects of $\mathrm{QCoh}(\mathsf{X})$ to almost perfect objects of $\mathrm{QCoh}(\mathsf{Z})$. To prove this, we can assume without loss of generality that $\mathsf{Z} = \mathrm{Sp\acute{e}t}\, R$ is affine. Applying Theorem 5.5.0.1, we deduce the existence of a sequence of closed immersions

$$\varnothing = \mathsf{Y}_0 \to \mathsf{Y}_1 \to \mathsf{Y}_2 \to \cdots \to \mathsf{Y}_n \to \mathsf{X}$$

and $h_i : \widetilde{\mathsf{Y}}_i \to \mathsf{Y}_i \times_{\mathrm{Sp\acute{e}t}\, R} \mathbf{P}^{d_i}_R$ which are locally almost of finite presentation, where the map $\mathsf{Y}_n \to \mathsf{X}$ induces a homeomorphism $|\mathsf{Y}_n| \simeq |\mathsf{X}|$, where each of the induced maps $\rho_i : \widetilde{\mathsf{Y}}_i \to \mathbf{P}^{d_i}_R$ is finite and locally almost of finite presentation (Remark 5.5.0.4), and each of the projection maps $\widetilde{\mathsf{Y}}_i \times_{\mathsf{Y}_i} \mathsf{U}_i \to \mathsf{U}_i$ is an equivalence, where U_i denotes the open substack of Y_i complementary to Y_{i-1}. For $0 \leqslant i \leqslant n$, let $v_i : \mathsf{Y}_i \to \mathsf{X}$ be the corresponding closed immersion. By virtue of Proposition 5.6.3.3, to prove that the functor f_* carries almost perfect objects to almost perfect objects, it will suffice to show that the functor $(v_n \circ f)_* : \mathrm{QCoh}(\mathsf{Y}_n) \to \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R)$ carries almost perfect objects to almost perfect objects. This is a special case of the following:

($*_i$) The functor $(f \circ v_i)_* : \mathrm{QCoh}(\mathsf{Y}_i) \to \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R)$ carries almost perfect objects to almost perfect objects.

The proof of ($*_i$) proceeds by induction on i, the case $i = 0$ being trivial. To carry out the inductive step, let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ be almost perfect. Let $u : \widetilde{\mathsf{Y}}_i \to \mathsf{Y}_i$ denote the composition

$$\widetilde{\mathsf{Y}}_i \xrightarrow{h_i} \mathsf{Y}_i \times_{\mathrm{Sp\acute{e}t}\, R} \mathbf{P}^{d_i}_R \to \mathsf{Y}_i\,.$$

It follows from Propositions 5.6.1.1 and 5.6.2.1 that the functor $u_* : \mathrm{QCoh}(\widetilde{\mathsf{Y}}_i) \to \mathrm{QCoh}(\mathsf{Y}_i)$ carries almost perfect objects to almost perfect objects. Form a fiber sequence $\mathscr{F}' \to \mathscr{F} \to u_* u^* \mathscr{F}$. Note that $(f \circ v_i)_* u_* u^* \mathscr{F}$ can be identified with the direct image of $u^* \mathscr{F}$ under the composite map $\widetilde{\mathsf{Y}}_i \xrightarrow{\rho_i} \mathbf{P}^{d_i}_R \to \mathrm{Sp\acute{e}t}\, R$, and is therefore almost perfect by virtue of Propositions 5.6.1.1 and 5.6.2.1. Consequently, to show that $(f \circ v_i)_* \mathscr{F}$ is almost perfect, it will suffice to show that $(f \circ v_i)_* \mathscr{F}'$ is almost perfect. This follows from $(*_{i-1})$ and Proposition 5.6.3.3, since the restriction $\mathscr{F}'|_{\mathsf{U}_i}$ vanishes (by virtue our assumption that the projection map $\widetilde{\mathsf{Y}}_i \times_{\mathsf{Y}_i} \mathsf{U}_i \to \mathsf{U}_i$ is an equivalence). This completes the proof of Theorem 5.6.0.2.

5.6.5 The Direct Image Theorem for Sheaves with Proper Support

We close this section with a slight generalization of Theorem 5.6.0.2.

Definition 5.6.5.1. Let X be a spectral Deligne-Mumford stack, let $K \subseteq |\mathsf{X}|$ be a closed subset, and let $j : \mathsf{U} \to \mathsf{X}$ be the complementary open immersion. We will say that K is *cocompact* if the map j is quasi-compact. Note that this is equivalent to the assertion that, for every map $\phi : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, the inverse image of K in $|\mathrm{Spec}\, A|$ can be written as the vanishing locus of a finitely generated ideal $I \subseteq \pi_0 A$.

Proposition 5.6.5.2. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is locally almost of finite presentation, let $K \subseteq |\mathsf{X}|$ be a cocompact closed subset, let K be the reduced closed substack of X corresponding to K, and suppose that the composite map $\mathsf{K} \to \mathsf{X} \xrightarrow{f} \mathsf{Y}$ is proper. If $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is almost perfect and supported on K (meaning that $j^* \mathscr{F} \simeq 0$, where $j : \mathsf{U} \hookrightarrow \mathsf{X}$ is an open immersion complementary to K), then the direct image $f_* \mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ is almost perfect.*

Proof. The assertion is local on Y, so we may assume without loss of generality that $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, R$ is affine. Using Proposition 4.4.4.1, we can choose a commutative diagram

$$\begin{array}{ccc} \mathsf{X} & \longrightarrow & \mathsf{X}_0 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} \\ \mathrm{Sp\acute{e}t}\, R & \longrightarrow & \mathrm{Sp\acute{e}t}\, R_0 \end{array}$$

where R_0 is Noetherian, f_0 is locally almost of finite presentation, and the map $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, R \times_{\mathrm{Sp\acute{e}t}\, R_0} \mathsf{X}_0$ induces an equivalence of 0-truncations. Using Proposition 4.3.5.5 we can assume (after modifying R_0 if necessary) that K is the inverse image of a closed subset $K_0 \subseteq |\mathsf{X}_0|$ (automatically cocompact, since X_0 is locally Noetherian). Let K_0 denote the reduced closed substack of X_0 corresponding to K_0. Since X_0 is locally Noetherian, the closed immersion $\mathsf{K}_0 \to \mathsf{X}_0$ is locally almost of finite presentation. Set $\mathsf{K}' = \mathsf{K}_0 \times_{\mathsf{X}_0} \mathsf{X}$, so we have a closed immersion $i : \mathsf{K}' \to \mathsf{X}$ which locally almost of finite presentation. Note that we can identify K with the underlying reduced substack of K', so our assumption that

K is proper over R guarantees that K' is also proper over R. Applying Theorem 5.6.0.2, we deduce that the functor $(f \circ i)_* : \mathrm{QCoh}(\mathsf{K}') \to \mathrm{QCoh}(\mathsf{Y})$ carries almost perfect objects to almost perfect objects. It now follows from Proposition 5.6.3.3 that if $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is almost perfect and supported on K, then $f_* \mathscr{F}$ is almost perfect. □

5.6.6 Application: Proper Descent for Quasi-Coherent Sheaves

Let $f : X \to Y$ be a morphism of schemes which is faithfully flat and quasi-compact. Using Grothendieck's theory of faithfully flat descent, one can identify the abelian category $\mathrm{QCoh}(Y)^{\heartsuit}$ of quasi-coherent sheaves on Y with the abelian category of quasi-coherent sheaves on X which are equipped with descent data: in other words, $\mathrm{QCoh}(Y)^{\heartsuit}$ is equivalent to the totalization of the cosimplicial abelian category

$$\mathrm{QCoh}(X)^{\heartsuit} \rightrightarrows \mathrm{QCoh}(X \times_Y X)^{\heartsuit} \rightrightarrows \mathrm{QCoh}(X \times_Y X \times_Y X) \rightrightarrows \cdots$$

It follows from Corollary D.6.3.3 that a similar statement holds at the "derived" level: that is, we can identify the stable ∞-category $\mathrm{QCoh}(\mathsf{Y})$ with the totalization of the cosimplicial ∞-category $\mathrm{QCoh}(\mathsf{X}_\bullet)$; here X and Y denote the spectral algebraic spaces associated to X and Y, and $\mathsf{X}_\bullet$ denotes the Čech nerve of the map $f : \mathsf{X} \to \mathsf{Y}$.

Let us now consider the analogous descent questions in the case where $f : X \to Y$ is a proper surjection of schemes. Here, the failure of f to be flat creates two (related) difficulties:

(a) The pullback functor $f^* : \mathrm{QCoh}(Y)^{\heartsuit} \to \mathrm{QCoh}(X)^{\heartsuit}$ need not be an exact functor, and therefore need not coincide with the "derived" pullback functor $f^* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$.

(b) The fiber product $X \times_Y X$, formed in the category of schemes, need not agree with the fiber product $\mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ formed in the ∞-category of spectral algebraic spaces.

Because of (a) and (b), it is difficult to formulate a theory of proper descent purely in the language of classical algebraic geometry. However, in the setting of spectral algebraic geometry, we have the following:

Theorem 5.6.6.1 (Proper Descent For Quasi-Coherent Sheaves)**.** *Let $f : \mathsf{X} \to \mathsf{Y}$ be a surjective proper morphism of spectral Deligne-Mumford stacks, and let $\mathsf{X}_\bullet$ denote the Čech nerve of f. Assume that Y is locally Noetherian and that the structure sheaf $\mathscr{O}_{\mathsf{Y}}$ is truncated. Then the pullback functor $\mathrm{QCoh}(\mathsf{Y}) \to \mathrm{Tot}\,\mathrm{QCoh}(\mathsf{X}_\bullet)$ is an equivalence of ∞-categories.*

Warning 5.6.6.2. In the statement of Theorem 5.6.6.1, the structure sheaves of the iterated fiber products $\mathsf{X} \times_{\mathsf{Y}} \cdots \times_{\mathsf{Y}} \mathsf{X}$ need not be truncated, even if the structure sheaves $\mathscr{O}_{\mathsf{X}}$ and $\mathscr{O}_{\mathsf{Y}}$ are truncated.

The proof of Theorem 5.6.6.1 is based on the following:

Proposition 5.6.6.3. *Let R be a truncated Noetherian $\mathbb{E}_\infty$-ring, let $f : \mathsf{X} \to \operatorname{Spét} R$ be a proper surjection of spectral algebraic spaces, and let $\mathcal{C}$ denote the smallest stable subcategory of Mod_R which contains $\Gamma(\mathsf{X}; \mathscr{F})$ for each $\mathscr{F} \in \operatorname{QCoh}(\mathsf{X})$ and is closed under retracts. Then $\mathcal{C} = \operatorname{Mod}_R$.*

Proof. Since R is truncated, every R-module M admits a finite filtration whose successive quotients are of the form $(\pi_n R) \otimes_R M$. Because $\mathcal{C}$ is closed under extensions, it will suffice to show that $\mathcal{C}$ contains the essential image of the forgetful functor $\operatorname{Mod}_{\pi_0 R} \to \operatorname{Mod}_R$. We may therefore replace R by $\pi_0 R$ and thereby reduce to the case where R is discrete.

Let us say that an ideal $I \subseteq R$ is *good* if $\mathcal{C}$ contains the essential image of the forgetful functor $\operatorname{Mod}_{R/I} \to \operatorname{Mod}_R$. To complete the proof, it will suffice to show that the zero ideal $(0) \subseteq R$ is good. Assume otherwise: then, since R is Noetherian, we can choose an ideal $I \subseteq R$ which is maximal among those ideals which are not good. Replacing R by R/I, we can reduce to the case where every nonzero ideal of R is good.

Since R is Noetherian, every finitely generated discrete R-module N admits a finite filtration whose successive quotients have the form $R/\mathfrak{p}$, for some prime ideal $\mathfrak{p} \subseteq R$. Applying this observation in the case $R = N$, we deduce that each object $M \in \operatorname{Mod}_R$ admits a finite filtration whose successive quotients have the form $(R/\mathfrak{p}) \otimes_R M$. Consequently, to show that $M \in \mathcal{C}$, it will suffice to show that each tensor product $(R/\mathfrak{p}) \otimes_R M$ belongs to $\mathcal{C}$. By virtue of our assumption on R, this is automatic unless $\mathfrak{p} = (0)$. We may therefore assume without loss of generality that R is an integral domain.

Let K be the fraction field of R. Our assumption that f is surjective guarantees that we can choose a closed point x of the generic fiber $|\mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét} K|$. Let X' be the schematic closure of x in X, and let us abuse notation by identifying the structure sheaf $\mathscr{O}_{\mathsf{X}'}$ with a (discrete) quasi-coherent sheaf on X. Set $N = \Gamma(\mathsf{X}'; \mathscr{O}_{\mathsf{X}'})$. It follows from Theorem 5.6.0.2 (or from Theorem 5.6.0.1) that N is an almost perfect R-module. Moreover, $K \otimes_R N \simeq \kappa(x)$ is a finite extension field of K; in particular, it is a free K-module of finite rank. We can therefore choose elements $x_1, \ldots, x_d \in \pi_0 N$ whose images form a basis for $K \otimes_R N$ as a vector space over K. These elements determine a map $\rho : R^d \to N$ whose fiber $\operatorname{fib}(\rho)$ satisfies $K \otimes_R \operatorname{fib}(\rho) \simeq 0$. Note that the R-module $\operatorname{fib}(\rho)$ is 0-truncated and almost perfect, so that the homotopy groups $\pi_i \operatorname{fib}(\rho)$ are finitely generated R-modules which vanish for all but finitely many integers i. It follows that $\operatorname{fib}(\rho)$ admits a finite filtration whose successive quotients have the form $\Sigma^i (R/\mathfrak{p})$, where $\mathfrak{p}$ is a nonzero prime ideal in R. Our hypothesis on R then guarantees that for any R-module M, the tensor product $\operatorname{fib}(\rho) \otimes_R M$ belongs to $\mathcal{C}$. Note that the projection formula for f gives an equivalence $N \otimes_R M \simeq f_*(\mathscr{O}_{\mathsf{X}'} \otimes f^* M)$, so that $N \otimes_R M$ also belongs to $\mathcal{C}$. Using the evident fiber sequence $\operatorname{fib}(\rho) \otimes_R M \to R^d \otimes_R M \to N \otimes_R M$, we conclude that $R^d \otimes_R M \simeq M^d$ belongs to $\mathcal{C}$. Since $\mathcal{C}$ is closed under retracts, we conclude that $M \in \mathcal{C}$. $\square$

Proposition 5.6.6.4. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a proper surjection of spectral Deligne-Mumford stacks. Assume that Y is locally Noetherian and that the structure sheaf $\mathscr{O}_{\mathsf{Y}}$ is truncated. Then the adjunction* $\mathrm{QCoh}(\mathsf{Y}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{QCoh}(\mathsf{X})$ *is comonadic: that is, it induces an equivalence of* $\mathrm{QCoh}(\mathsf{Y})$ *with the ∞-category of comodules for the comonad $f^* f_*$ on* $\mathrm{QCoh}(\mathsf{X})$ *(see Definition HA.4.7.3.4).*

Proof. By virtue of the Barr-Beck theorem (Theorem HA.4.7.3.5), it will suffice to verify the following:

(i) The pullback functor $f^* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ is conservative.

(ii) The pullback functor $f^* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ preserves totalizations of f^*-split cosimplicial objects.

Both of these assertions are local on Y, so we may assume without loss of generality that $\mathsf{Y} \simeq \mathrm{Sp\acute{e}t}\, R$ is affine.

We first prove (i). Suppose we are given an R-module $N \in \mathrm{Mod}_R \simeq \mathrm{QCoh}(\mathsf{Y})$ such that $f^*N \simeq 0$. Let $\mathcal{C} \subseteq \mathrm{Mod}_R$ be the full subcategory spanned by those R-modules M such that $M \otimes_R N \simeq 0$. For each $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, we have $\Gamma(\mathsf{X}; \mathscr{F}) \otimes_R N \simeq \Gamma(\mathsf{X}; \mathscr{F} \otimes f^*N) \simeq 0$, so that $\Gamma(\mathsf{X}; \mathscr{F})$ belongs to $\mathcal{C}$. Since $\mathcal{C}$ is evidently a stable subcategory of Mod_R which is closed under retracts, Proposition 5.6.6.3 guarantees that $R \in \mathcal{C}$, so that $N \simeq R \otimes_R N \simeq 0$.

The proof of (ii) is similar. Suppose that $N^\bullet$ is an f^*-split cosimplicial object of $\mathrm{Mod}_R \simeq \mathrm{QCoh}(\mathsf{Y})$. Let $\mathcal{D} \subseteq \mathrm{Mod}_R$ be the full subcategory spanned by those R-modules M for which the tautological map $f^*(\mathrm{Tot}(M \otimes_R N^\bullet)) \to \mathrm{Tot}(f^*(M \otimes_R N^\bullet))$ is an equivalence. Note that for each $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, the cosimplicial R-module $\Gamma(\mathsf{X}; \mathscr{F}) \otimes_R N^\bullet \simeq \Gamma(\mathsf{X}; \mathscr{F} \otimes f^* N^\bullet)$ is split, so that $\Gamma(\mathsf{X}; \mathscr{F})$ belongs to $\mathcal{D}$. Because $\mathcal{D}$ is a stable subcategory of Mod_R which is closed under retracts, Proposition 5.6.6.3 guarantees that $R \in \mathcal{D}$, so that the canonical map $f^*(\mathrm{Tot}(N^\bullet)) \to \mathrm{Tot}(f^* N^\bullet)$ is an equivalence. □

Proof of Theorem 5.6.6.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a proper morphism of spectral Deligne-Mumford stacks, where Y is locally Noetherian and $\mathscr{O}_{\mathsf{Y}}$ is truncated. Let $\mathsf{X}_\bullet$ denote the Čech nerve of f, which we regard as an augmented simplicial object of SpDM (so that $\mathsf{X}_0 \simeq \mathsf{X}$ and $\mathsf{X}_{-1} \simeq \mathsf{Y}$). We wish to show that the augmented cosimplicial ∞-category $\mathrm{QCoh}(\mathsf{X}_\bullet)$ determines a limit diagram $\mathbf{\Delta}_+ \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. Using (the dual of) Corollary HA.4.7.5.3 , it will suffice to verify the following:

(a) The adjunction $\mathrm{QCoh}(\mathsf{Y}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{QCoh}(\mathsf{X})$ is comonadic.

(b) The augmented cosimplicial ∞-category $\mathrm{QCoh}(\mathsf{X}_\bullet)$ satisfies the Beck-Chevalley condition. More precisely, for every morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}_+$, the diagram of pullback functors

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X}_m) & \longrightarrow & \mathrm{QCoh}(\mathsf{X}_{m+1}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{X}_n) & \longrightarrow & \mathrm{QCoh}(\mathsf{X}_{n+1}) \end{array}$$

is right adjointable.

Assertion (a) follows from Proposition 5.6.6.4, and assertion (b) is a special case of Corollary 3.4.2.2. □

In the statement of Theorem 5.6.6.1, the truncatedness hypothesis on $\mathscr{O}_{\mathsf{Y}}$ is necessary. For example, let $R = \mathrm{Sym}^*_{\mathbf{Q}}(\Sigma^2 \mathbf{Q})$ be the free $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$ on a single generator $t \in \pi_2 R$. Then the truncation map $R \to \pi_0 R \simeq \mathbf{Q}$ induces a proper morphism $f : \mathrm{Sp\acute{e}t}\, \mathbf{Q} \to \mathrm{Sp\acute{e}t}\, R$. However, the pullback functor $f^* : \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R) \to \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, \mathbf{Q})$ is not conservative: for example, it annihilates the localization $R[t^{-1}]$. If we restrict our attention to *connective* quasi-coherent sheaves, this difficulty does not arise:

Corollary 5.6.6.5. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a proper morphism of spectral Deligne-Mumford stacks and let $\mathsf{X}_\bullet$ denote the Čech nerve of f. If Y is locally Noetherian, then the induced map $\mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}} \to \mathrm{Tot}(\mathrm{QCoh}(\mathsf{X}_\bullet)^{\mathrm{cn}})$ is fully faithful.*

Warning 5.6.6.6. In the setting of Corollary 5.6.6.5, the pulback functor $\mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}} \to \mathrm{Tot}\, \mathrm{QCoh}(\mathsf{X}_\bullet)^{\mathrm{cn}}$ need not be an equivalence of ∞-categories, even when the structure sheaves of $\mathscr{O}_{\mathsf{X}}$ and $\mathscr{O}_{\mathsf{Y}}$ are both discrete. In other words, given an object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ whose pullback $f^* \mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is connective, we cannot conclude that $\mathscr{F}$ is connective (for a counterexample, take $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, \mathbf{C}[x, y]$ to be an affine space of dimension 2, X to be the $\mathbf{C}$-scheme obtained from Y by a blow-up at the origin, and $\mathscr{F}$ to be the (derived) pushforward $j_* \mathscr{O}_{\mathsf{U}}$, where $j : \mathsf{U} \hookrightarrow \mathsf{Y}$ is the open immersion complementary to the origin).

Proof of Corollary 5.6.6.5. Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ and $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}})$. For each $n \geqslant 0$, set $\mathsf{X}(n) = (\mathcal{X}, \tau_{\leqslant n} \mathscr{O}_{\mathsf{X}})$ and $\mathsf{Y}(n) = (\mathcal{Y}, \tau_{\leqslant n} \mathscr{O}_{\mathsf{Y}})$. Then f induces a proper morphism $f(n) : \mathsf{X}(n) \to \mathsf{Y}(n)$ having a Čech nerve $\mathsf{X}(n)_\bullet$. We then have a commutative diagram of pullback functors

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}} & \longrightarrow & \mathrm{Tot}\, \mathrm{QCoh}(\mathsf{X}_\bullet)^{\mathrm{cn}} \\ \downarrow & & \downarrow \\ \varprojlim_n \mathrm{QCoh}(\mathsf{Y}(n))^{\mathrm{cn}} & \longrightarrow & \varprojlim_n \mathrm{Tot}\, \mathrm{QCoh}(\mathsf{X}(n)_\bullet). \end{array}$$

It follows from Corollary 2.5.9.3 that the vertical maps are equivalences of ∞-categories. Since each $\mathsf{Y}(n)$ has a truncated structure sheaf, the bottom horizontal map is fully faithful by virtue of Theorem 5.6.6.1. □

Remark 5.6.6.7. In the statements of Theorem 5.6.6.1 and Corollary 5.6.6.5, we do not need the full strength of the assumption that Y is locally Noetherian: it is sufficient to assume that the 0-truncation of Y is locally Noetherian.

Chapter 6

Grothendieck Duality

Let X be a smooth projective variety of dimension n over a field κ, and let Ω^n_X denote the *canonical bundle* of X: that is, the top exterior power of the cotangent bundle of X. The celebrated *Serre duality theorem* asserts the following:

(a) There is a canonical trace map $\mathrm{tr} : \mathrm{H}^n(X; \Omega^n_X) \to \kappa$, which is an isomorphism if X is connected.

(b) If $\mathscr{E}$ is any vector bundle on X, then the bilinear map

$$\mathrm{H}^i(X; \mathscr{E}) \times \mathrm{H}^{n-i}(X; \Omega^n_X \otimes \mathscr{E}^\vee) \to \mathrm{H}^n(X; \Omega^n_X) \xrightarrow{\mathrm{tr}} \kappa$$

is a perfect pairing for every integer i: that is, we have canonical isomorphisms $\mathrm{H}^i(X; \mathscr{E})^\vee \simeq \mathrm{H}^{n-i}(X; \Omega^n_X \otimes \mathscr{E}^\vee)$.

Serre duality was generalized by Grothendieck to the setting where X is not assumed to be smooth. In general, one needs to replace the line bundle Ω^n_X by the *dualizing complex* ω_X, which is an object of the derived category $D(X)$ (when X is smooth of dimension n, we take $\omega_X = \Omega^n_X[n]$). In this case, assertion (b) generalizes as follows:

(b') For every quasi-coherent sheaf $\mathscr{F}$ on X, there is a canonical isomorphism $\mathrm{H}^i(X; \mathscr{F})^\vee \simeq \mathrm{Hom}_{D(X)}(\mathscr{F}[i], \omega_X)$.

Moreover, Grothendieck's work placed the duality theory of coherent sheaves in a more general context: rather than restricting attention to projective algebraic varieties over a field, he developed a *relative* theory of duality for any proper morphism $f : X \to S$ of separated Noetherian schemes. In this case, one should replace the dualizing complex ω_X by the *relative dualizing complex* $\omega_{X/Y}$ of the morphism f, in which case (b') admits the following further generalization:

(b'') For every object $\mathscr{F}$ of the derived category $D(X)$, there is a canonical isomorphism $\mathrm{Hom}_{D(X)}(\mathscr{F}, \omega_{X/Y}) \simeq \mathrm{Hom}_{D(Y)} Rf_* \mathscr{F}, \mathscr{O}_Y)$.

The relative dualizing sheaf $\omega_{X/Y} \in D(X)$ is characterized up to isomorphism by the property described in (b'): more precisely, it can be described as the image of $\mathscr{O}_Y$ under a functor $f^! : D(Y) \to D(X)$ which is right adjoint to the (derived) direct image functor $Rf_* : D(X) \to D(Y)$. We refer the reader to [**?**] for a detailed exposition.

Our goal in this chapter is to describe an generalization of Grothendieck duality to the setting of spectral algebraic geometry. Let $f : \mathsf{X} \to \mathsf{Y}$ be any morphism of quasi-compact, quasi-separated spectral algebraic spaces. Then the direct image functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ preserves small colimits (Corollary 3.4.2.2). It follows from the adjoint functor theorem (Corollary HTT.5.5.2.9) that f_* admits a right adjoint. In general, this right adjoint is a somewhat pathological construction (for example, it cannot be computed locally, even on Y). Roughly speaking, one would expect the functor f_* to have a *well-behaved* right adjoint only in cases where we can guarantee that the construction $\mathscr{F} \mapsto f_* \mathscr{F}$ preserves finiteness conditions (see, for example, Proposition HTT.5.5.7.2). One such case was studied in Chapter 5: if the morphism $f : \mathsf{X} \to \mathsf{Y}$ is proper and locally almost of finite presentation, then the direct image functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ carries almost perfect objects of $\mathrm{QCoh}(\mathsf{X})$ to almost perfect objects of $\mathrm{QCoh}(\mathsf{Y})$ (Theorem 5.6.0.2). In this case, we will denote the right adjoint to f_* by $f^!$ and refer to it as the *exceptional inverse image functor associated to* f. In §6.4 we will use Theorem 5.6.0.2 to show that the functor $f^!$ is well-behaved, at least when restricted to truncated objects of $\mathrm{QCoh}(\mathsf{Y})$ (see Proposition 6.4.1.4 and Corollary 6.4.1.9).

To obtain a robust duality theory for *unbounded* quasi-coherent sheaves, it is necessary to impose stronger finiteness assumptions on the direct image functor f_*. If we assume that $f : \mathsf{X} \to \mathsf{Y}$ is proper and locally almost of finite presentation and that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is perfect, then Theorem 5.6.0.2 guarantees that the direct image $f_* \mathscr{F}$ is *almost* perfect. However, it need not be perfect (this fails, for example, when f is the closed immersion $\mathrm{Sp\acute{e}t}\, \mathbf{Z}/2\mathbf{Z} \hookrightarrow \mathrm{Sp\acute{e}t}\, \mathbf{Z}/4\mathbf{Z}$). To address this issue, we introduce in §6.1 the notion of a *morphism of finite* Tor-*amplitude*. Our main goal is to show that if f is proper, locally almost of finite presentation, and of finite Tor-amplitude, then the direct image functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ carries perfect objects to perfect objects (Theorem 6.1.3.2; for a converse, see Theorem 11.1.4.1). In §6.4.1, we will show that under the same assumptions, the exceptional inverse image functor $f^!$ is given by the formula $f^! \mathscr{F} \simeq \omega_{\mathsf{X}/\mathsf{Y}} \otimes f^* \mathscr{F}$, (Corollary 6.4.2.7), where $\omega_{\mathsf{X}/\mathsf{Y}} \in \mathrm{QCoh}(\mathsf{X})$ is the *relative dualizing sheaf of* f (Definition 6.4.2.4).

The second half of this chapter is devoted to studying the notion of an *absolute* dualizing sheaf in the setting of Noetherian spectral algebraic spaces. Roughly speaking, an object $\omega_{\mathsf{X}} \in \mathrm{QCoh}(\mathsf{X})$ is a *dualizing sheaf for* X if the construction $\mathscr{F} \mapsto \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}, \omega_{\mathsf{X}})$ is involutive when restricted to *coherent* sheaves (see Definition 6.6.1.1 and Theorem 6.6.1.8). In §6.6, we show that such objects often exist (see Theorems 6.6.4.1 and 6.6.4.3) and are unique up to invertible twists (Proposition 6.6.2.1). Our exposition will rely on a few facts about injective dimension of quasi-coherent sheaves in the setting of spectral algebraic geometry, which we discuss in §6.5.

Warning 6.0.0.1. In this book, we define the exceptional inverse image functor $f^!$: $\mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ only in the case where $f : \mathsf{X} \to \mathsf{Y}$ is proper (and locally almost of finite presentation). It is possible to develop a more elaborate theory of Grothendieck duality for non-proper morphisms, (which is essential if one wishes to carry out a local study of relative dualizing sheaves $\omega_{\mathsf{X}/\mathsf{Y}}$), but the construction requires some rather elaborate categorical constructions which we do not wish to undertake here. For more details (in a different but closely related context), we refer the reader to the work of Gaitsgory and Rozenblyum ([77]).

Contents

6.1 Morphisms of Finite Tor-Amplitude

Let $f : \mathsf{X} \to \mathsf{Y}$ be a proper map of spectral Deligne-Mumford stacks which is locally almost of finite presentation. According to Theorem 5.6.0.2, the pushforward functor f_* carries almost perfect objects of $\mathrm{QCoh}(\mathsf{X})$ to almost perfect objects of $\mathrm{QCoh}(\mathsf{Y})$. It is natural for additional conditions which guarantee that the pushforward functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ also preserves perfect objects. In this section, we will show that it is sufficient to assume that the morphism f has finite Tor-amplitude (Theorem 6.1.3.2). In §11.1, we will see that this condition is also necessary, at least if Y is affine (see Theorem 11.1.4.1).

6.1.1 Tor-Amplitude

Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. Recall that ϕ is said to be *flat* if it exhibits B as a flat A-module. More generally, if $n \geqslant 0$ is an integer, we will say that ϕ has Tor-*amplitude* $\leqslant n$ if B has Tor-amplitude $\leqslant n$ as an A-module: in other words, if the extension of scalars functor

$$\mathrm{Mod}_A \to \mathrm{Mod}_B \qquad M \mapsto B \otimes_A M$$

carries $\mathrm{Mod}_A^\heartsuit$ into $(\mathrm{Mod}_B)_{\leqslant n}$. This condition can be globalized as follows:

Definition 6.1.1.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and let $n \geqslant 0$ be an integer. We will say that f has Tor-*amplitude* $\leqslant n$ if, for every commutative diagram

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} B & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow f \\ \operatorname{Sp\acute{e}t} A & \longrightarrow & \mathsf{Y} \end{array}$$

where the horizontal maps are étale, the $\mathbb{E}_\infty$-ring B has Tor-amplitude $\leqslant n$ as an A-module (see Definition HA.7.2.4.21). We say that f has *finite* Tor-*amplitude* if it has Tor-amplitude $\leqslant n$ for some integer n.

Example 6.1.1.2. Let κ be a field, and let $f : \mathsf{X} \to \operatorname{Sp\acute{e}t} \kappa$ be a morphism of spectral Deligne-Mumford stacks. Then f has Tor-amplitude $\leqslant n$ if and only if X is n-truncated.

Example 6.1.1.3. A morphism of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ has Tor-amplitude $\leqslant 0$ if and only if f is flat.

When restricted to affine spectral Deligne-Mumford stacks, Definition 6.1.1.1 recovers the algebraic notion of Tor-amplitude:

Proposition 6.1.1.4. *Let $f : \operatorname{Sp\acute{e}t} B \to \operatorname{Sp\acute{e}t} A$ be a morphism between affine spectral Deligne-Mumford stacks and let $n \geqslant 0$. Then f has* Tor-*amplitude $\leqslant n$ if and only if B has* Tor-*amplitude $\leqslant n$ as an A-module.*

Lemma 6.1.1.5. *Let $f : \mathsf{X} \to \operatorname{Sp\acute{e}t} A$ be a morphism of spectral Deligne-Mumford stacks, and let $n \geqslant 0$. The following conditions are equivalent:*

(1) *The morphism f has* Tor-*amplitude $\leqslant n$.*

(2) *For every étale morphism* $\operatorname{Sp\acute{e}t} B \to \mathsf{X}$*, B has* Tor-*amplitude $\leqslant n$ as an A-module.*

Proof. It is clear that (1) $\Rightarrow$ (2). Conversely, suppose that (2) is satisfied, and suppose we are given an étale map $g : \operatorname{Sp\acute{e}t} B \to \mathsf{X}$ such that $f \circ g$ factors as a composition $\operatorname{Sp\acute{e}t} B \to \operatorname{Sp\acute{e}t} A' \to \operatorname{Sp\acute{e}t} A$, for some A' which is étale over A. We wish to show that B has Tor-amplitude $\leqslant n$ as an A'-module. Since A' is étale over A, B is a retract (as an A'-module) of $A' \otimes_A B$, which is of Tor-amplitude $\leqslant n$ over A' by virtue of (2). □

Lemma 6.1.1.6. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_1$-rings, and let M be a left B-module. Suppose that B has* Tor-*amplitude $\leqslant m$ as a left A-module, and the M has* Tor-*amplitude $\leqslant n$ as a left B-module. Then M has* Tor-*amplitude $\leqslant m + n$ as a left A-module.*

Proof. Let $N \in (\mathrm{LMod}_A)_{\leqslant p}$; we wish to show that $N \otimes_R M$ is $(p+m+n)$-truncated. We have $N \otimes_A M \simeq (N \otimes_A B) \otimes_B M$. The desired result now follows from the observation that $N \otimes_A B$ is $(p+m)$-truncated. □

Proof of Proposition 6.1.1.4. It follows immediately from the definitions that if $f : \operatorname{Spét} B \to \operatorname{Spét} A$ is a morphism of Tor-amplitude $\leqslant n$, then B has Tor-amplitude $\leqslant n$ as an A-module. Conversely, suppose that B has Tor-amplitude $\leqslant n$ as an A-module; we wish to show that f has Tor-amplitude $\leqslant n$. By virtue of Lemma 6.1.1.5, it will suffice to show that every étale B-algebra B' has Tor-amplitude $\leqslant n$ as an A-module. This is an immediate consequence of Lemma 6.1.1.6, since B' is flat (and therefore of Tor-amplitude $\leqslant 0$) over B. □

6.1.2 Pullback and Composition of Morphisms of Finite Tor-Amplitude

We now summarize some of the formal properties enjoyed by the class of morphisms of finite Tor-amplitude.

Proposition 6.1.2.1. *For each $n \geqslant 0$, the condition that a morphism of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ be of* Tor*-amplitude $\leqslant n$ is local on the source with respect to the flat topology (see Definition 2.8.3.10).*

Proof. First suppose that f has Tor-amplitude $\leqslant n$, and that we are given a flat map $g : \mathsf{X}' \to \mathsf{X}$. We wish to show that $g \circ f$ has Tor-amplitude $\leqslant n$. Consider a commutative diagram

$$\begin{array}{ccc} \operatorname{Spét} C & \longrightarrow & \mathsf{X}' \\ \downarrow & & \downarrow \\ \operatorname{Spét} A & \longrightarrow & \mathsf{Y}; \end{array}$$

we wish to show that C has Tor-amplitude $\leqslant n$ as an A-module. In other words, we wish to show that if M is a discrete A-module, then $C \otimes_A M$ is n-truncated. This assertion is local on C with respect to the étale topology. We may therefore suppose that the map $\operatorname{Spét} C \to \operatorname{Spét} A \times_{\mathsf{Y}} \mathsf{X}$ factors as a composition

$$\operatorname{Spét} C \to \operatorname{Spét} B \xrightarrow{u} \operatorname{Spét} A \times_{\mathsf{Y}} \mathsf{X},$$

where u is étale. Since f has Tor-amplitude $\leqslant n$, we see that $B \otimes_A M$ is n-truncated. Then $C \otimes_A M \simeq C \otimes_B (B \otimes_A M)$ is n-truncated because C is flat over B.

Now suppose that we are given a flat covering $\{g_\alpha : \mathsf{X}_\alpha \to \mathsf{X}\}$ such that each $g_\alpha \circ f$ has Tor-amplitude $\leqslant n$; we wish to show that f has Tor-amplitude $\leqslant n$. We may assume without loss of generality that $\mathsf{Y} = \operatorname{Spét} A$ is affine. Choose an étale map $\operatorname{Spét} B \to \mathsf{X}$; we wish to show that B has Tor-amplitude $\leqslant n$ over A (see Lemma 6.1.1.5). Since the g_α form a flat covering, we can find finitely many étale maps $\operatorname{Spét} C_\alpha \to \mathsf{X}_\alpha \times_{\mathsf{X}} \operatorname{Spét} B$ such that $C = \prod C_\alpha$

is faithfully flat over B. If M is a discrete A-module, then $C \otimes_A M \simeq C \otimes_B (B \otimes_A M)$ is n-truncated; it follows that $B \otimes_A M$ is n-truncated so that B has Tor-amplitude $\leqslant n$ over A. □

Proposition 6.1.2.2. *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y}. \end{array}$$

If f has Tor*-amplitude $\leqslant n$, so does f'. The converse holds if g is a flat covering.*

Proof. Suppose first that f has Tor-amplitude $\leqslant n$. To prove that f' has Tor-amplitude $\leqslant n$, we may assume without loss of generality that $\mathsf{Y}' = \operatorname{Sp\acute{e}t} A'$ is affine. Choose a faithfully flat étale morphism $A' \to A''$ such that the composite map

$$\operatorname{Sp\acute{e}t} A'' \to \operatorname{Sp\acute{e}t} A' \to \mathsf{Y}$$

factors through an étale map $\operatorname{Sp\acute{e}t} A \to \mathsf{Y}$. Using Proposition 6.1.2.1, we are reduced to proving that for every étale map $\operatorname{Sp\acute{e}t} B' \to \mathsf{X}_{\operatorname{Sp\acute{e}t} A'} \operatorname{Sp\acute{e}t} A''$, B' has Tor-amplitude $\leqslant n$ over A'. Using Proposition 6.1.2.1 we may further reduce to the case where the map

$$\operatorname{Sp\acute{e}t} B' \to \operatorname{Sp\acute{e}t} A \times_{\mathsf{Y}} \mathsf{X}$$

factors through some étale map $\operatorname{Sp\acute{e}t} B \to \operatorname{Sp\acute{e}t} A \times_{\mathsf{Y}} \mathsf{X}$. The map $A' \to B'$ factors as a composition $A' \to A'' \to A'' \otimes_A B \to B'$, where the first and third map are étale, and the middle map has Tor-amplitude $\leqslant n$. It follows that B' has Tor-amplitude $\leqslant n$ over A', as desired.

Now suppose that g is a flat covering and that f' has Tor-amplitude $\leqslant n$; we wish to show that f has the same property. We may assume without loss of generality that $\mathsf{Y} = \operatorname{Sp\acute{e}t} A$ is affine. Using Proposition 6.1.2.1 we can further reduce to the case where $\mathsf{X} = \operatorname{Sp\acute{e}t} B$ is affine. Since g is a flat covering, we can choose an étale map $\operatorname{Sp\acute{e}t} A' \to \mathsf{Y}'$ such that A' is faithfully flat over B. Because f' has Tor-amplitude $\leqslant n$, we deduce that $A' \otimes_A B$ has Tor-amplitude $\leqslant n$ over A'. It then follows from Lemma 2.8.4.3 that B has Tor-amplitude $\leqslant n$ over A. □

In the locally Noetherian case, Proposition 6.1.2.2 admits a converse:

Proposition 6.1.2.3. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism spectral Deligne-Mumford stacks and let $n \geqslant 0$ be an integer. Assume that Y is locally Noetherian. The following conditions are equivalent:*

(*a*) *The morphism* f *has* Tor-*amplitude* $\leqslant n$.

(*b*) *For every field* κ *and every pullback diagram*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \operatorname{Sp\acute{e}t}(\kappa) & \longrightarrow & \mathsf{Y}, \end{array}$$

the morphism f' *has* Tor-*amplitude* $\leqslant n$.

(*c*) *For every field* κ *and every morphism* $\operatorname{Sp\acute{e}t}(\kappa) \to \mathsf{Y}$, *the fiber product* $\operatorname{Sp\acute{e}t}(\kappa) \times_{\mathsf{Y}} \mathsf{X}$ *is* n*-truncated.*

The equivalence $(b) \Leftrightarrow (c)$ follows from Proposition 6.1.1.2, and the implication $(a) \Rightarrow (b)$ from Proposition 6.1.2.2. The nontrivial implication $(b) \Rightarrow (a)$ reduces immediately to the affine case, which follows from the following more general assertion:

Lemma 6.1.2.4. *Let* A *be a Noetherian* $\mathbb{E}_\infty$*-ring and let* M *be an* A*-module. The following conditions are equivalent:*

(*i*) *The module* M *has* Tor-*amplitude* $\leqslant n$.

(*ii*) *For each residue field* κ *of* A, *the tensor product* $\kappa \otimes_A M$ *is* n*-truncated.*

Proof. The implication $(i) \Rightarrow (ii)$ is immediate. For the converse, assume that (ii) is satisfied. We wish to show that, for every discrete A-module N, the tensor product $N \otimes_A M$ is n-truncated. Writing N as a filtered colimit of finite generated submodules, we may assume that N is finitely generated as a module over $\pi_0(A)$. Let $Y \subseteq |\operatorname{Spec}(A)|$ be the support of N. Proceeding by Noetherian induction, we may assume that $N' \otimes_A M$ is n-truncated for every discrete A-module N' whose support is strictly contained in Y. Writing N as a finite extension of modules of the form $\pi_0(A)/\mathfrak{p}_i$, where $\mathfrak{p}_i \subseteq \pi_0(A)$ is prime, we can reduce to the case where $N = \pi_0(A)/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq \pi_0(A)$. Replacing A by $\pi_0(A)/\mathfrak{p}$ (and M by $(\pi_0(A)/\mathfrak{p}) \otimes_A M$), we can reduce to the case where A is an integral domain (regarded as a discrete $\mathbb{E}_\infty$-ring) and $N = A$; in this case, we wish to show that M itself is n-truncated. Let K denote the fraction field of A. Then K/A can be written as a filtered colimit of finitely generated submodules having support smaller than Y. It follows from our inductive hypothesis that the tensor product $(K/A) \otimes_A M$ is n-truncated. Assumption (2) guarantees that $K \otimes_A M$ is also n-truncated, so that $M \simeq \operatorname{fib}(K \otimes_A M \to (K/A) \otimes_A M)$ is n-truncated as desired. □

Proposition 6.1.2.5. *Let* $f : \mathsf{X} \to \mathsf{Y}$ *and* $g : \mathsf{Y} \to \mathsf{Z}$ *be maps of spectral Deligne-Mumford stacks. If* f *has* Tor-*amplitude* $\leqslant m$ *and* g *has* Tor-*amplitude* $\leqslant n$, *then* $g \circ f$ *has* Tor-*amplitude* $\leqslant m + n$.

Proof. Using Propositions 6.1.2.1 and 6.1.2.2, we can reduce to the case where X, Y, and Z are affine. In this case, the desired result follows from Lemma 6.1.1.6 and Proposition 6.1.1.4. □

Remark 6.1.2.6. Let $f : \mathsf{X} \to \operatorname{Spét} R$ be a map of spectral Deligne-Mumford stacks, and form a pullback square

$$\begin{array}{ccc} \mathsf{X}_0 & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ \operatorname{Spét} \pi_0 R & \longrightarrow & \operatorname{Spét} R. \end{array}$$

Then f has Tor-amplitude $\leqslant n$ if and only if f_0 has Tor-amplitude $\leqslant n$.

6.1.3 Direct Images of Perfect Quasi-Coherent Sheaves

Our main result is a simple consequence of Theorem 5.6.0.2 together with the following:

Proposition 6.1.3.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks which is of* Tor*-amplitude $\leqslant n$. Assume that f is quasi-compact, quasi-separated, and exhibits X are a relative spectral algebraic space over Y. Let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ be a quasi-coherent sheaf which is locally of* Tor*-amplitude $\leqslant k$. Then the pushforward $f_* \mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ has* Tor*-amplitude $\leqslant n + k$.*

Proof. The assertion is local on Y; we may therefore suppose that $\mathsf{Y} \simeq \operatorname{Spét} R$ is affine. Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$. Let us say that an object $U \in \mathcal{X}$ is *good* if $\mathscr{F}(U)$ is of Tor-amplitude $\leqslant n + k$ over R. It follows from Lemma 6.1.1.6 that every affine object of $\mathcal{X}$ is good, and Proposition HA.7.2.4.23 implies that the collection of good objects of $\mathcal{X}$ is closed under pushouts. Using Theorem 3.4.2.1 and Corollary 2.5.3.6, we conclude that the final object of $\mathcal{X}$ is good, so that $f_* \mathscr{F}$ has Tor-amplitude $\leqslant n + k$. □

Theorem 6.1.3.2. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks which is proper, locally almost of finite presentation, and of finite* Tor*-amplitude. Then the pushforward functor f_* carries perfect objects of $\mathrm{QCoh}(\mathsf{X})$ to perfect objects of $\mathrm{QCoh}(\mathsf{Y})$.*

Proof. Let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ be perfect; we wish to show that $f_* \mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ is perfect. This assertion is local on Y, so we may assume without loss of generality that Y is affine. Since the morphism f is proper, X is quasi-compact. It follows that $\mathscr{F}$ has Tor-amplitude $\leqslant n$ for some n, so that $f_* \mathscr{F}$ has finite Tor-amplitude by virtue of Proposition 6.1.3.1. Since $f_* \mathscr{F}$ is almost perfect (by virtue of Theorem 5.6.0.2), it follows from Proposition HA.7.2.4.23 that $f_* \mathscr{F}$ is perfect. □

Remark 6.1.3.3. In the setting of classical algebraic geometry, Theorem 6.1.3.2 appears in [134].

6.1.4 Tor Amplitude at a Point

Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of schemes, let $x \in X$ be a point, and let $y = f(x) \in Y$. Recall that f is said to be *flat at x* if the induced map of local rings $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is flat. We then have the following:

(i) The morphism f is flat if and only if it is flat at x, for each point $x \in X$.

(ii) If f is locally of finite presentation, then the collection of points $x \in X$ at which f is flat is an open subset of X.

We now describe generalizations of (i) and (ii), where flatness is replaced by a bound on Tor-amplitude. For simplicity, we will restrict our attention to the affine case.

Notation 6.1.4.1. If A is an $\mathbb{E}_\infty$-ring and $\mathfrak{p} \subseteq \pi_0 A$ is a prime ideal, we let $A_\mathfrak{p}$ denote the localization $A[S^{-1}]$, where $S = \pi_0 A - \mathfrak{p}$. For any A-module M, we let $A_\mathfrak{p}$ denote the tensor product $A_\mathfrak{p} \otimes_A M$.

Definition 6.1.4.2. Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings and let M be a B-module. For each integer n and each prime ideal $\mathfrak{p} \subseteq \pi_0 B$, we say that M *has* Tor-*amplitude* $\leqslant n$ *over A at* $\mathfrak{p}$ if the localization $M_\mathfrak{p}$ has Tor-amplitude $\leqslant n$ as an A-module. If M is connective, then we M *is flat over A at* $\mathfrak{p}$ if the localization $M_\mathfrak{p}$ is a flat A-module.

Remark 6.1.4.3. In the situation of Definition 6.1.4.2, we can regard $M_\mathfrak{p}$ as a module over the localization $A_\mathfrak{q}$, where $\mathfrak{q} \subseteq \pi_0 A$ is the inverse image of $\mathfrak{p}$. Then M has Tor-amplitude $\leqslant n$ over A at $\mathfrak{p}$ if and only if the localization $M_\mathfrak{p}$ has Tor-amplitude $\leqslant n$ as a module over $A_\mathfrak{q}$.

Proposition 6.1.4.4. *Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings, let M be an B-module, and let n be an integer. Then the following conditions are equivalent:*

(a) *The A-module M has* Tor-*amplitude* $\leqslant n$.

(b) *The module M has* Tor-*amplitude* $\leqslant n$ *over A at $\mathfrak{p}$ for each prime ideal $\mathfrak{p} \subseteq \pi_0 B$.*

Proof. Suppose that (a) is satisfied, and let $\mathfrak{p} \subseteq \pi_0 B$ be a prime ideal. Since $M_\mathfrak{p}$ can be obtained as a filtered colimit of copies of M, it follows that $M_\mathfrak{p}$ also has Tor-amplitude $\leqslant n$ as an A-module. Conversely, suppose that (b) is satisfied. Let N be a discrete A-module; we wish to prove that $\pi_m(M \otimes_A N) \simeq 0$ for $m > n$. Let us regard $M \otimes_A N$ as a module over B. It will therefore suffice to show that $\pi_m(M \otimes_R N)_\mathfrak{p}$ for each prime ideal $\mathfrak{p} \subseteq \pi_0 B$. This follows from ($b$), since we have a canonical isomorphism $\pi_m(M \otimes_A N)_\mathfrak{p} \simeq \pi_m(M_\mathfrak{p} \otimes_A N)$. □

We have the following generalization of (ii):

Proposition 6.1.4.5. *Let $m \geqslant n \geqslant 0$ be integers, let A be a commutative ring, let B be an $\mathbb{E}_\infty$-algebra over A which is of finite generation to order $n+1$, and let M be a connective B-module which is finitely m-presented. Let $\mathfrak{p} \subseteq \pi_0 B$ be a prime ideal, and let $\mathfrak{q} \subseteq A$ denote its inverse image, and let κ denote the residue field of A at $\mathfrak{q}$. Then the following conditions are equivalent:*

(a) *The module M has* Tor*-amplitude $\leqslant n$ over A at $\mathfrak{p}$.*

(b) *The homotopy groups $\pi_i(M \otimes_A \kappa)_{\mathfrak{p}}$ vanish for $n < i \leqslant m+1$.*

Moreover, the collection of prime ideals $\mathfrak{p} \subseteq \pi_0 B$ which satisfy these conditions is an open subset of $|\operatorname{Spec} B|$.

Before giving the proof of Proposition 6.1.4.5, let us describe some of its consequences. Taking $B = A$ in Proposition 6.1.4.5, we obtain the following:

Corollary 6.1.4.6. *Let $m \geqslant n \geqslant 0$ be integers, let A be a commutative ring, and let M be a connective A-module which is finitely m-presented. For each prime ideal $\mathfrak{p} \subseteq A$, the following conditions are equivalent:*

(a) *The localization $M_{\mathfrak{p}}$ has* Tor*-amplitude $\leqslant n$ over A.*

(b) *The homotopy groups $\pi_i(M \otimes_A \kappa)_{\mathfrak{p}}$ vanish for $n < i \leqslant m+1$, where κ denotes the residue field of A at $\mathfrak{p}$.*

Moreover, the collection of those prime ideals $\mathfrak{p} \subseteq A$ which satisfy these conditions is an open subset of $|\operatorname{Spec} A|$.

Corollary 6.1.4.7. *Let $n \geqslant 0$ be an integer, let A be a connective $\mathbb{E}_\infty$-ring, and let M be an almost perfect A-module. Then M has* Tor*-amplitude $\leqslant n$ if and only if, for every maximal ideal $\mathfrak{m} \subseteq \pi_0 A$ with residue field $\kappa = (\pi_0 A)/\mathfrak{m}$, the homotopy groups $\pi_i(M \otimes_A \kappa)$ vanish for $i > n$.*

Proof. The "only if" direction is obvious. For the converse, we can use Remark 6.1.2.6 to reduce to the case where A is discrete. Fix $m \geqslant n$. For every maximal ideal $\mathfrak{m} \subseteq \pi_0 A$ with residue field $\kappa = (\pi_0 A)/\mathfrak{m}$, the natural map $\pi_i(M \otimes_A \kappa) \to \pi_i(\tau_{\leqslant m} M \otimes_A \kappa)$ is surjective for $i = m+1$ and bijective for $i \leqslant m$. It follows that $\pi_i(\tau_{\leqslant m} M \otimes_A \kappa) \simeq 0$ for $n < i \leqslant m+1$. Applying Corollary 6.1.4.6 and Proposition 6.1.4.4, we deduce that $\tau_{\leqslant m} M$ has Tor-amplitude $\leqslant n$ over A. Since A is discrete, this implies that $\tau_{\leqslant m} M$ is n-truncated. Allowing m to vary, we deduce that M is n-truncated, so that $M \simeq \tau_{\leqslant n} M$ has Tor-amplitude $\leqslant n$ over A. □

Corollary 6.1.4.8. *Let $n \geqslant 0$ and let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism between spectral Deligne-Mumford stacks which is locally almost of finite presentation. The following conditions are equivalent:*

(1) *The morphism f is of* Tor*-amplitude* $\leqslant n$.

(2) *For every field κ and every map* $\operatorname{Sp\acute{e}t} \kappa \to \mathsf{Y}$*, the fiber* $\operatorname{Sp\acute{e}t} \kappa \times_{\mathsf{Y}} \mathsf{X}$ *is n-truncated.*

Proof. The implication (1) ⇒ (2) follows from Proposition 6.1.2.2 and Example 6.1.1.2. To prove the converse, we can work locally and thereby reduce to the case where $\mathsf{Y} = \operatorname{Sp\acute{e}t} A$ and $\mathsf{X} = \operatorname{Sp\acute{e}t} B$ are affine. Assume that (2) is satisfied; we wish to prove that B has Tor-amplitude $\leqslant n$ over A. Using Remark 6.1.2.6, we can replace A by $\pi_0 A$ and B by $\pi_0 A \otimes_A B$, and thereby reduce to the case where A is discrete. It follows from Propositions 6.1.4.5 and 6.1.4.4 that for each $m \geqslant n$, the truncation $\tau_{\leqslant m} B$ has Tor-amplitude $\leqslant n$ over A. In particular, we deduce that $\pi_i B \simeq 0$ for $n < i \leqslant m$. Since m is arbitrary, we conclude that A is n-truncated. It follows that $A \simeq \tau_{\leqslant n} A$ has Tor-amplitude $\leqslant n$ over R. □

Corollary 6.1.4.8 immediately implies the following classical result:

Corollary 6.1.4.9 (Fiberwise Flatness Criterion)**.** *Suppose given a commutative diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{f} & \mathsf{Y} \\ & \searrow \quad \swarrow & \\ & \mathsf{Z}, & \end{array}$$

where f is locally almost of finite presentation. The following conditions are equivalent:

(1) *The morphism f is flat.*

(2) *For every field κ and every map* $\operatorname{Sp\acute{e}t} \kappa \to \mathsf{Z}$*, the induced map* $f_\kappa : \operatorname{Sp\acute{e}t} \kappa \times_{\mathsf{Z}} \mathsf{X} \to \operatorname{Sp\acute{e}t} \kappa \times_{\mathsf{Z}} \mathsf{Y}$ *is flat.*

Corollary 6.1.4.10. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation. The following conditions are equivalent:*

(1) *The morphism f is finite flat.*

(2) *For every field κ and every morphism* $\operatorname{Sp\acute{e}t} \kappa \to \mathsf{Y}$*, the projection map* $\operatorname{Sp\acute{e}t} \kappa \times_{\mathsf{Y}} \mathsf{X} \to \operatorname{Sp\acute{e}t} \kappa$ *is finite flat.*

Proof. Combine Corollary 6.1.4.9 with Proposition 5.2.3.3. □

Corollary 6.1.4.11. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism between spectral Deligne-Mumford stacks. Then f is étale if and only if it satisfies the following conditions:*

(1) *The morphism f is locally almost of finite presentation.*

(2) *For every field κ and every map* $\mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Y}$*, the projection map* $\mathrm{Sp\acute{e}t}\,\kappa \times_{\mathsf{Y}} \mathsf{X} \to \mathrm{Sp\acute{e}t}\,\kappa$ *is étale.*

Proof. Combine Corollary 6.1.4.8 with Proposition 4.2.3.4. □

Corollary 6.1.4.12. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism between spectral Deligne-Mumford stacks. Then f is an equivalence if and only if it satisfies the following conditions:*

(1) *The morphism f is locally almost of finite presentation.*

(2) *For every field κ and every map* $\mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Y}$*, the projection map* $\mathrm{Sp\acute{e}t}\,\kappa \times_{\mathsf{Y}} \mathsf{X} \to \mathrm{Sp\acute{e}t}\,\kappa$ *is an equivalence.*

(3) *The morphism f is a relative n-stack for some $n \gg 0$ (see Definition 6.3.1.11).*

Proof. The necessity of conditions (1) through (3) is obvious. To prove the converse, suppose that conditions (1) through (3) are satisfied; we wish to show that f is an equivalence. Using (1) and (2), we deduce that f is étale. Write $\mathsf{Y} = (\mathcal{Y}, \mathcal{O}_{\mathsf{Y}})$, so that we have $\mathsf{X} \simeq (\mathcal{Y}_{/U}, \mathcal{O}_{\mathsf{Y}}|_U)$ for some object $U \in \mathcal{Y}$. Using (2) and Proposition 3.5.4.2, we deduce that $\eta^* U$ is contractible for every point $\eta^* \in \mathrm{Fun}^*(\mathcal{Y}, \mathcal{S})$. Since the hypercompletion of $\mathcal{Y}$ has enough points (Theorem A.4.0.5), it folows that U is ∞-connective (see Proposition A.4.2.1). Condition (3) guarantees that U is n-truncated for $n \gg 0$, so that U is a final object of $\mathcal{Y}$ and f is an equivalence as desired. □

6.1.5 The Proof of Proposition 6.1.4.5

The proof of Proposition 6.1.4.5 is fairly technical. We begin by discussing the case where A is Noetherian, in which case the desired result is a minor modification of Theorem 11.1.1 of [90].

Lemma 6.1.5.1. *Let R be a commutative ring, let M be a connective R-module spectrum, and suppose that M is $(n+1)$-truncated for some integer $n \geqslant 0$. The following conditions are equivalent:*

(a) *The R-module spectrum M has* Tor*-amplitude $\leqslant n$ over R.*

(b) *For every discrete R-module N, the homotopy group $\pi_{n+1}(M \otimes_R N)$ vanishes.*

Proof. The implication $(a) \Rightarrow (b)$ is immediate. We will prove the converse using induction on n. Suppose first that $n = 0$. Applying (b) in the case $N = R$, we deduce that M is a discrete R-module. Condition (b) then asserts that that $\mathrm{Tor}_1^R(M, N) \simeq 0$ for every discrete R-module N (see Corollary HA.7.2.1.22), so that the functor $N \mapsto \mathrm{Tor}_0^R(M, N)$ is exact and therefore M is flat over R. Assume now that $n > 0$. Choose a free R-module P and a map $\phi : P \to M$ which induces a surjection $\pi_0 P \to \pi_0 M$. Then ϕ fits into a fiber sequence $M' \to P \to M$. Consequently, to prove that M has Tor-amplitude $\leqslant n$, it will suffice to show that M' has Tor-amplitude $\leqslant n-1$. Using the exact sequence $\pi_{m+1} M \to \pi_m M' \to \pi_m P$, we deduce that M' is $(n-1)$-truncated. By virtue of the inductive hypothesis, we are reduced to proving that $\pi_n(M' \otimes_R N) \simeq 0$ for any discrete R-module N. We have a short exact sequence $\pi_{n+1}(M \otimes_R N) \to \pi_n(M' \otimes_R N) \to \pi_n(P \otimes_R N)$. Since P is a free R-module and N is discrete, $\pi_n(P \otimes_R N)$. Consequently, if condition (b) is satisfied, then $\pi_n(M' \otimes_R N) \simeq 0$ as desired. □

Lemma 6.1.5.2. *Let $m > n \geqslant 0$ be integers, let A be a Noetherian commutative ring, let B be a connective $\mathbb{E}_\infty$-algebra which is almost of finite presentation over A, and let M be a B-module which is of finite presentation to order m. Let $\mathfrak{p} \subseteq \pi_0 B$ be a prime ideal, let $\mathfrak{q}$ denote its inverse image in A, and let $\kappa = \kappa(\mathfrak{q})$ denote the residue field of A at $\mathfrak{q}$. Then the following conditions are equivalent:*

- (a) *The module M has* Tor*-amplitude $\leqslant n$ over A at $\mathfrak{p}$.*
- (b) *The homotopy groups $\pi_i(M \otimes_A \kappa)_{\mathfrak{p}}$ vanish for $n < i \leqslant m$.*

Proof. The implication $(a) \Rightarrow (b)$ is immediate. Conversely, suppose that (b) is satisfied; we will show that $M_{\mathfrak{p}}$ has Tor-amplitude $\leqslant n$ over A. Proceeding by descending induction on n, we can reduced to the case $m = n + 1$. According to Lemma 6.1.5.1 (and Remark 6.1.4.3), it will suffice to show that $\pi_{n+1}(M_{\mathfrak{p}} \otimes_{A_{\mathfrak{q}}} N) \simeq 0$ for every discrete $A_{\mathfrak{q}}$-module N. Writing N as a filtered colimit of finitely generated submodules, we may reduce to the case where N is finitely generated. The collection of those $A_{\mathfrak{q}}$-modules N for which $\pi_{n+1}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{q}}} N) \simeq 0$ is closed under extensions. We may therefore reduce to the case $N = A_{\mathfrak{q}}/I$, where I is a prime ideal in $A_{\mathfrak{q}}$. Proceeding by Noetherian induction, we may assume that $\pi_{n+1}(M_{\mathfrak{p}} \otimes_{A_{\mathfrak{q}}} A_{\mathfrak{q}}/J) \simeq 0$ whenever J is a prime ideal of $R_{\mathfrak{q}}$ properly containing I.

Let $A' = A_{\mathfrak{q}}/I$. If A' is a field, then I is the maximal ideal of $A_{\mathfrak{q}}$ and the desired result follows from (b). Otherwise, we can choose a nonzero element x belonging to the maximal ideal of A'. Note that $A'/(x)$ can be written as a finite extension of $A_{\mathfrak{q}}$-modules of the form $A_{\mathfrak{q}}/J$, where J is a prime ideal of $A_{\mathfrak{q}}$ properly containing I. It follows that $\pi_{n+1}(M_{\mathfrak{p}} \otimes_{A_{\mathfrak{q}}} A'/(x)) \simeq 0$, so that multiplication by x induces a surjection from $\pi_{n+1}(M_{\mathfrak{p}} \otimes_{A_{\mathfrak{q}}} A')$ to itself. Note that $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{q}}} A'$ is almost perfect as a module over $B_{\mathfrak{p}}$. Proposition HA.7.2.4.31 implies that B is Noetherian, so that $B_{\mathfrak{p}}$ is Noetherian and therefore $\pi_{n+1}(M_{\mathfrak{p}} \otimes_{A_{\mathfrak{q}}} A')$ is finitely generated as a discrete module over the local ring $\pi_0 B_{\mathfrak{p}}$. The image of x in $\pi_0 B_{\mathfrak{p}}$ belongs to the maximal ideal. Applying Nakayama's lemma, we deduce that $\pi_{n+1}(M_{\mathfrak{p}} \otimes_{A_{\mathfrak{q}}} A') \simeq 0$, as desired. □

We now wish to study the loci of prime ideals $\mathfrak{p} \subseteq \pi_0 B$ which satisfy the conditions appearing in Lemma 6.1.5.2.

Lemma 6.1.5.3. *Let X be a sober Noetherian topological space and let $U \subseteq X$ be a subset. Then U is open if and only if it satisfies the following conditions:*

(a) *The set U is stable under generalization: that is, if points $x, y \in X$ satisfy $x \in \overline{\{y\}}$ and x belongs to U, then y also belongs to U.*

(b) *For every point $x \in U$, the intersection $U \cap \overline{\{x\}}$ contains an nonempty open subset of $\overline{\{x\}}$.*

Proof. The necessity of conditions (a) and (b) are obvious. Conversely, suppose that (a) and (b) are satisfied; we wish to show that U is open. Let U° denote the interior of U. Replacing X by $X - U^{\circ}$ and U by $U - U^{\circ}$, we can reduce to the case where $U^{\circ} = \varnothing$. Writing X as a union of its irreducible components, we may assume that X is irreducible. If U is empty, there is nothing to prove; otherwise, there exists a point $x \in U$. Using (a), we deduce that the generic point of X belongs to U. It then follows from (b) that the interior of U is nonempty, contrary to our assumptions. □

Lemma 6.1.5.4 (Generic Freeness)**.** *Let A be a Noetherian integral domain, let B be a finitely generated (discrete) A-algebra, and let M be a finitely generated B-module. Then there exists a nonzero element $a \in A$ such that $M[a^{-1}]$ is a free $A[a^{-1}]$-module.*

Proof. Choose a finite collection of elements $b_1, \ldots, b_n \in B$ which generate B as an A-algebra. We proceed by induction on n. In the case $n = 0$, M is a finitely generated A-module. Let K denote the fraction field of A, so that $K \otimes_A M$ is a finitely generated vector space over K. We can then choose finitely many elements $x_1, \ldots, x_k \in M$ whose images form a basis for $K \otimes_A M$. The elements $\{x_i\}_{1 \leqslant i \leqslant k}$ determine a map $\phi : A^k \to M$ such that $K \otimes_A \ker(\phi) \simeq 0 \simeq K \otimes_A \operatorname{coker}(\phi)$. Since A is Noetherian, the kernel and cokernel of ϕ are finitely generated. We can therefore choose an element $a \in A$ which annihilates $\ker(\phi)$ and $\operatorname{coker}(\phi)$, so that ϕ induces an isomorphism $A[a^{-1}]^k \simeq M[a^{-1}]$.

We now treat the case $n > 0$. Let B_0 denote the A-subalgebra of B generated by $b_1, \ldots, b_{n-1}$. Let $M_0 \subseteq M$ be a finitely generated B_0-submodule which generates M as a B-module. For $i \geqslant 0$, set $M_i = M_0 + b_n M_0 + \cdots + b_m^i M_0 \subseteq M$, so that we have $M = \bigcup M_i$. For each $i \geqslant 0$, multiplication by b_n^{i+1} induces a surjection $M_0 \to M_{i+1}/M_i$ whose kernel is a submodule $N_i \subseteq M_0$. The ring B_0 is Noetherian (by virtue of the Hilbert basis theorem) and M_0 is a finitely generated module over B_0. It follows that the ascending chain of submodules $N_0 \subseteq N_1 \subseteq \cdots$ eventually stabilizes. In other words, there exists an integer $i \geqslant 0$ for which the maps $b_n^j : M_{i+1}/M_i \to M_{i+j+1}/M_{i+j}$ are isomorphisms for each $j \geqslant 0$. Using our inductive hypothesis repeatedly, we can choose a nonzero element $a \in A$ for which each of the localizations $M_0[a^{-1}], (M_1/M_0)[a^{-1}], (M_2/M_1)[a^{-1}], \ldots, (M_{i+1}/M_i)[a^{-1}]$ is a free module over $A[a^{-1}]$, so that $(M_{j+1}/M_j)[a^{-1}]$ is a free $A[a^{-1}]$-module for all i. It follows that each of the exact sequences $0 \to M_j[a^{-1}] \to M_{j+1}[a^{-1} \to (M_{j+1}/M_j)[a^{-1}] \to 0$ splits, so that $M[a^{-1}] \simeq M_0[a^{-1}] \oplus \bigoplus_{j \geqslant 0}(M_{j+1}/M_j)[a^{-1}]$ is also a free module over $A[a^{-1}]$, as desired. □

Lemma 6.1.5.5. *Let $m > n \geqslant 0$ be integers, let A be a Noetherian commutative ring, let B be a connective $\mathbb{E}_\infty$-algebra which is almost of finite presentation over A, and let M be a B-module which is of finite presentation to order m. Let $U \subseteq |\operatorname{Spec} B|$ denote the set consisting of those prime ideals $\mathfrak{p} \subseteq \pi_0 B$ such that M has* Tor*-amplitude $\leqslant n$ over A at $\mathfrak{p}$. Then U is open.*

Proof. Since A is Noetherian, the $\mathbb{E}_\infty$-ring B is also Noetherian (Theorem HA.7.2.4.31) so that the topological space $|\operatorname{Spec} B|$ is Noetherian. It will therefore suffice to show that the set U satisfies conditions (a) and (b) of Lemma 6.1.5.3. Condition (a) is obvious: if we are given prime ideals $\mathfrak{p} \subseteq \mathfrak{p}' \subseteq \pi_0 B$, then the localization $M_{\mathfrak{p}}$ can be obtained as a filtered colimit of copies of $M_{\mathfrak{p}'}$, so that if $M_{\mathfrak{p}'}$ has Tor-amplitude $\leqslant n$ as an A-module then $M_{\mathfrak{p}}$ also has Tor-amplitude $\leqslant n$ as an A-module.

We now verify condition (b). Choose $\mathfrak{p} \in U$, and let $\mathfrak{q} \subseteq A$ be the inverse image of $\mathfrak{p}$. Set $A' = A/\mathfrak{q}$, $B' = A' \otimes_A B$, and $M' = B' \otimes_B M$. Our assumption that $\mathfrak{p}$ belongs to U guarantees that the homotopy groups $\pi_i M'_{\mathfrak{q}}$ vanish for $n < i \leqslant m$. Since each of these homotopy groups is a finitely generated module over $\pi_0 B$, we can choose an element $b \in (\pi_0 B) - \mathfrak{q}$ such that $\pi_i M'[b^{-1}]$ vanishes for $n < i \leqslant m$. Using Lemma 6.1.5.4, we can choose a nonzero element $\overline{a} \in A'$ such that $\pi_i M'[b^{-1}][\overline{a}^{-1}]$ is a free $A'[\overline{a}^{-1}]$-module for $0 \leqslant i \leqslant n$. Choose an element $a \in A$ lifting $\overline{a} \in A'$, and let us abuse notation by identifying a with its image in $\pi_0 B$. We will complete the proof by showing that U contains every prime ideal $\mathfrak{p}' \subseteq (\pi_0 B)$ such that $\mathfrak{p} \subseteq \mathfrak{p}'$ and $a, b \notin \mathfrak{p}'$ (note that the collection of such prime ideals forms an open subset of the closure of $\mathfrak{p}$ in $|\operatorname{Spec} B|$). To prove this, let $\mathfrak{q}' \subseteq A$ be the inverse image of $\mathfrak{p}'$, and let κ be the residue field of A at $\mathfrak{q}'$. Using the criterion of Lemma 6.1.5.2, we are reduced to showing that the homotopy groups $\pi_i(M \otimes_A \kappa)_{\mathfrak{q}'} \simeq \pi_i(M' \otimes_{A'} \kappa)_{\mathfrak{q}'}$ vanishes for $n < i \leqslant m$. Note that these homotopy groups are localizations of $\pi_i(M'[b^{-1}\overline{a}^{-1}] \otimes_{A'[\overline{a}^{-1}]} \kappa)$. Proposition HA.7.2.1.19 supplies a spectral sequence $\{E_r^{s,t}, d_r\}_{r \geqslant 2}$ with second page $E_2^{s,t} = \operatorname{Tor}_s^{A'[\overline{a}^{-1}]}(\pi_t M'[b^{-1}\overline{a}^{-1}], \kappa)$ converging to $\pi_{s+t}(M'[b^{-1}\overline{a}^{-1}] \otimes_{A'[\overline{a}^{-1}]} \kappa$ It now suffices to observe that $E_2^{s,t}$ vanishes for $n < s+t \leqslant m$ (since the modules $\pi_t M'[b^{-1}\overline{a}^{-1}]$ are free for $0 \leqslant t \leqslant n$ and vanish for $n < t \leqslant m$). □

Lemma 6.1.5.6. *Let $m \geqslant n \geqslant 0$ be integers, let A_0 be a Noetherian commutative ring, let B_0 be an $\mathbb{E}_\infty$-algebra over A_0 which is finitely m-presented, and let M_0 be a connective B_0-module which is finitely m-presented. Suppose we are given a diagram $\{A_\alpha\}_{\alpha \in I}$ of discrete Noetherian A_0-algebras indexed by a filtered partially ordered set I, and set $A = \varinjlim A_\alpha$ (so that A need not be Noetherian). For each $\alpha \in I$, set $B_\alpha = \tau_{\leqslant m}(A_\alpha \otimes_{A_0} B_0)$ and $M_\alpha = \tau_{\leqslant m}(B_\alpha \otimes_{B_0} M_0)$, and set $B = \tau_{\leqslant m}(A \otimes_{A_0} B_0)) \simeq \varinjlim B_\alpha$ and $M = \tau_{\leqslant m}(B \otimes_{B_0} M_0) \simeq \varinjlim M_\alpha$. Let $\mathfrak{p} \subseteq \pi_0 BA$ be a prime ideal having inverse images $\mathfrak{q} \subseteq A$, $\mathfrak{p}_\alpha \subseteq \pi_0 B_\alpha$, and $\mathfrak{q}_\alpha \subseteq A_\alpha$. Let κ denote the* *residue field of R at $\mathfrak{q}$, and for each $\alpha \in I$ let κ_α denote the residue field of A_α at $\mathfrak{q}_\alpha$. Then the following conditions are equivalent:*

- (a) *The module M has* Tor*-amplitude $\leqslant n$ over A at $\mathfrak{p}$.*
- (a') *There exists an index $\alpha \in I$ such that M_α has* Tor*-amplitude $\leqslant n$ over A_α at $\mathfrak{p}_\alpha$.*
- (b) *The homotopy groups $\pi_i(M \otimes_A \kappa)_{\mathfrak{p}}$ vanish for $n < i \leqslant m+1$.*
- (b') *There exists an index $\alpha \in I$ such that the homotopy groups $\pi_i(M_\alpha \otimes_{A_\alpha} \kappa_\alpha)_{\mathfrak{p}_\alpha}$ vanish for $n < i \leqslant m+1$.*

Proof. We will show that $(a) \Rightarrow (b) \Rightarrow (b') \Rightarrow (a') \Rightarrow (a)$. The implication $(a) \Rightarrow (b)$ is immediate from the definitions.

We now show that $(b') \Leftrightarrow (a')$. For each $\alpha \in I$, the commutative ring A_α is Noetherian and B_α is finitely m-presented over A_α; it follows from Proposition 4.2.4.1 that B_α is almost of finite presentation over A_α. In particular, B_α is Noetherian. Consequently, the assumption that M_α is finitely m-presented over B_α guarantees that it is also finitely $(m+1)$-presented over B_α. The desired result now follows from Lemma 6.1.5.2 (applied to the pair of integers $m+1 > n$).

Note that for each $\alpha \in I$, we have an equivalence $M \simeq \tau_{\leqslant m}(A \otimes_{A_\alpha} M_\alpha)$, and therefore an equivalence $M_{\mathfrak{p}} \simeq \tau_{\leqslant m}(A \otimes_{A_\alpha} M_\alpha \mathfrak{p}_\alpha)_{\mathfrak{p}}$. If M_α has Tor-amplitude $\leqslant n$ over A_α at $\mathfrak{p}_\alpha$, then the tensor product $A \otimes_{A_\alpha} M_{\alpha \mathfrak{p}_\alpha}$ is n-truncated, and therefore also m-truncated. It then follows that $M_{\mathfrak{p}}$ is a localization of $A \otimes_{A_\alpha} M_{\alpha \mathfrak{p}_\alpha}$, and is therefore of Tor-amplitude $\leqslant n$ over A. This proves that $(a') \Rightarrow (a)$.

We now complete the proof by showing that (b) implies (b'). Let $\mathfrak{p}_0$ and $\mathfrak{q}_0$ denote the inverse images of $\mathfrak{p}$ in the commutative rings $\pi_0 B_0$ and A_0, respectively. Let κ_0 denote the residue field of A_0 at $\mathfrak{p}$ and define

$$K_0 = \kappa_0 \otimes_{A_0} M_0 \qquad K_\alpha = \kappa_\alpha \otimes_{A_\alpha} M_\alpha \qquad K = \kappa \otimes_A M.$$

By construction, we have $(m+1)$-connective maps $A_\alpha \otimes_{A_0} M_0 \to M_\alpha$ and $A \otimes_{A_0} M_0 \to M$, so that the induced maps

$$\rho_i : \kappa \otimes_{\kappa_0} (\pi_i K_0) \to \pi_i K \qquad \rho_{i,\alpha} : \kappa \otimes_{\kappa_0} (\pi_i K_0) \to \kappa \otimes_{\kappa_\alpha} (\pi_i K_\alpha)$$

are isomorphisms for $i \leqslant m$ and surjections for $i = m+1$. For $n < i \leqslant m+1$, condition (b) implies that $(\pi_i K)_{\mathfrak{p}} \simeq 0$. It follows that the map ρ_i vanishes after localization at $\mathfrak{p}$. Since $\pi_i K_0$ is a finitely generated module over $\pi_0(\kappa_0 \otimes_{A_0} B_0)$, we conclude that the image of ρ_i is annihilated by some element $b \in (\pi_0 B) - \mathfrak{p}$. Without loss of generality, we may assume that there exists an element $\alpha_i \in I$ such that b is the image of some element $\overline{b} \in \pi_0 B_{\alpha_i}$. Enlarging α_i if necessary, we may assume that multiplication by $\overline{b}$ annihilates the image of the map ρ_{i,α_i}. Consequently, condition (b') is satisfied for any $\alpha \in I$ which is an upper bound for $\{\alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{m+1}\}$. □

Proof of Proposition 6.1.4.5. Let $m \geqslant n \geqslant 0$ be integers, let A be a commutative ring, let B be a connective $\mathbb{E}_\infty$-algebra over A which is of finite generation to order $(m+1)$, and let M be a connective B-module which is finitely m-presented. Let $\mathfrak{p} \subseteq \pi_0 B$ be a prime ideal, let $\mathfrak{q} \subseteq A$ be its inverse image, and let κ denote the residue field of A at $\mathfrak{q}$. We wish to show that M has Tor-amplitude $\leqslant n$ over A at $\mathfrak{p}$ if and only if the homotopy groups $\pi_i(\kappa \otimes_A M_{\mathfrak{p}})$ vanish for $n < i \leqslant m+1$. To prove this, we are free to replace B by $\tau_{\leqslant m} B$ and thereby reduce to the case where B is finitely m-presented over A.

Write A as a union of its finitely generated subalgebras $\{A_\alpha\}_{\alpha \in I}$. Using Corollary 4.4.1.4, we see that there exists an index $\alpha \in I$, an $\mathbb{E}_\infty$-algebra B_α which is finitely m-presented over A_α, and an equivalence $B \simeq \tau_{\leqslant m}(A \otimes_{A_\alpha} B_\alpha)$. Using Corollary 4.5.1.10, we can (after enlarging α if necessary) assume that there exists a B_α-module M_α which is connective and finitely m-presented and an equivalence $M \simeq \tau_{\leqslant m}(B \otimes_{B_\alpha} M_\alpha)$. The desired equivalence now follows from Lemma 6.1.5.6

For each $\beta \geqslant \alpha$, set $B_\beta = \tau_{\leqslant m}(A_\beta \otimes_{A_\alpha} B_\alpha)$ and $M_\beta = \tau_{\leqslant m}(B_\beta \otimes_{B_\alpha} M_\alpha)$. Let U denote the subset of $|\operatorname{Spec} B|$ consisting of those prime ideals $\mathfrak{p}$ such that M has Tor-amplitude $\leqslant n$ over A at $\mathfrak{p}$, and let U_β denote the subset of $|\operatorname{Spec} B_\beta|$ consisting of those prime ideals $\mathfrak{p}$ such that M_β has Tor-amplitude $\leqslant n$ over A_β at $\mathfrak{p}$. It follows from Lemma 6.1.5.5 that each U_β is an open subset of $|\operatorname{Spec} B_\beta|$. Lemma 6.1.5.6 implies that U is the union of the inverse images of the sets U_β, so that U is an open subset of $|\operatorname{Spec} B|$. □

6.1.6 Tor-Amplitude and Filtered Colimits

We close this section by mentioning a few consequences of Proposition 6.1.4.5.

Proposition 6.1.6.1. *Let $m \geqslant n \geqslant 0$ be integers, let A_0 be a commutative ring and let B_0 be an $\mathbb{E}_\infty$-algebra over A_0 which is finitely m-presented. Suppose we are given a diagram $\{A_\alpha\}_{\alpha \in I}$ of discrete A_0-algebras indexed by a filtered partially ordered set I, and set $A = \varinjlim A_\alpha$. Set $B_\alpha = \tau_{\leqslant m}(A_\alpha \otimes_{A_0} B_0)$ and $B = \tau_{\leqslant m}(A \otimes_{A_0} B_0) \simeq \varinjlim B_\alpha$. If B has* Tor-*amplitude $\leqslant n$ over A, then there exists an index α such that B_α has* Tor-*amplitude $\leqslant n$ over A.*

Warning 6.1.6.2. In the statement of Proposition 6.1.6.1, the hypothesis that the diagram $\{A_\alpha\}$ consists of commutative rings (rather than connective $\mathbb{E}_\infty$-rings) cannot be eliminated.

Proof of Proposition 6.1.6.1. Write A as a union of finitely generated subalgebras $\{R_\beta\}_{\beta \in J}$. Using Corollary 4.4.1.4, we conclude that there exists an index $\beta \in J$, an $\mathbb{E}_\infty$-algebra S_β which is is finitely m-presented over R_β, and an equivalence $B \simeq \tau_{\leqslant m}(A \otimes_{R_\beta} S_\beta)$. For each $\beta' \geqslant \beta$, set $S_{\beta'} = \tau_{\leqslant m}(R_{\beta'} \otimes_{R_\beta} S_\beta)$, and let $U_{\beta'} \subseteq |\operatorname{Spec} S_{\beta'}|$ be the set consisting of those prime ideals $\mathfrak{p}$ such that $S_{\beta'}$ has Tor-amplitude $\leqslant n$ over $R_{\beta'}$ at $\mathfrak{p}$. Then each $U_{\beta'}$ is an open subset of $|\operatorname{Spec} S_{\beta'}|$ (Lemma 6.1.5.5). It follows from Lemma 6.1.5.6 (and our assumption that B has Tor-amplitude $\leqslant n$ over A) that $|\operatorname{Spec} B|$ is covered by the the inverse images of the open sets $U_{\beta'}$. Since $|\operatorname{Spec} B|$ is quasi-compact, it follows that there exists $\beta' \geqslant \beta$ for which the map $|\operatorname{Spec} B| \to |\operatorname{Spec} S_{\beta'}|$ factors through $U_{\beta'}$. Using Proposition 4.3.5.1, we deduce that there exists $\beta'' \geqslant \beta'$ such that the map $|\operatorname{Spec} S_{\beta''}| \to |\operatorname{Spec} S_{\beta'}|$ factors through $U_{\beta'}$. It then follows that $U_{\beta''} = |\operatorname{Spec} S_{\beta''}|$, so that $S_{\beta''}$ has Tor-amplitude $\leqslant n$ over $R_{\beta''}$ (Proposition 6.1.4.4).

Since $R_{\beta''}$ is finitely presented as a commutative ring, the inclusion $R_{\beta'} \to A$ must factor through A_α for some $\alpha \in I$. Set $B'_\alpha = A_\alpha \otimes_{R_{\beta''}} S_{\beta''}$, so that B'_α has Tor-amplitude $\leqslant n$ over A_α. Note that B_α and B'_α are $\mathbb{E}_\infty$-algebras which are finitely m-presented over A_α and we have a canonical equivalence $\tau_{\leqslant m}(A \otimes_{A_\alpha} B_\alpha) \simeq B \simeq \tau_{\leqslant m}(A \otimes_{A_\alpha} B'_\alpha)$. Using Corollary 4.4.1.4, we can assume (after enlarging α if necessary) that B_α and B'_α are equivalent as $\mathbb{E}_\infty$-algebras over A_α, so that B_α has Tor-amplitude $\leqslant n$ over A_α. □

Corollary 6.1.6.3. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Suppose that there exists an integer n such that f is locally of finite generation to order $(n+1)$ and of* Tor*-amplitude $\leqslant n$. Then f is locally almost of finite presentation.*

Proof. The assertion is local on X and Y, so we may assume without loss of generality that $\mathsf{Y} = \operatorname{Spét} A$ and $\mathsf{X} = \operatorname{Spét} B$ are affine. We wish to show that B is almost of finite presentation over A. By virtue of Proposition 4.1.3.4, we can replace A by $\pi_0 A$ and B by the tensor product $B \otimes_A \pi_0 A$, and thereby reduce to the case where A is discrete. Since B is of Tor-amplitude $\leqslant n$ over A, it is n-truncated. It is therefore finitely n-presented over A.

Write A as a filtered colimit its finitely generated subrings and applying Corollary 4.4.1.4, we deduce that there exists a finitely generated subring $A_0 \subseteq A$ and an equivalence $B \simeq \tau_{\leqslant n}(A \otimes_{A_0} B_0)$, where B_0 is finitely n-presented over A_0. By virtue of Proposition 6.1.6.1, we may assume (after enlarging A_0 if necessary) that B_0 has Tor-amplitude $\leqslant n$ over A_0. It follows that the tensor product $A \otimes_{A_0} B_0$ is n-truncated, and is therefore equivalent to B. Replacing A by A_0 and B by B_0, we can reduce to the case where A is Noetherian. In this case, Proposition 4.2.4.1 implies that B is Noetherian and of finite type over A, so that B is almost of finite presentation over A by virtue of Remark 4.2.0.4. □

Corollary 6.1.6.4. *Let $f : \mathsf{X} \to \mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}})$ be a morphism of spectral Deligne-Mumford stacks. The following conditions are equivalent:*

(1) *The map f is flat and locally almost of finite presentation.*

(2) *Let $\mathsf{Y}_0 = (\mathcal{Y}, \pi_0 \mathscr{O}_{\mathsf{Y}})$, and let $\mathsf{X}_0 = \mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}_0$. Then the projection map $\mathsf{X}_0 \to \mathsf{Y}_0$ is flat and locally finitely* 0*-presented.*

Proof. The implication (1) ⇒ (2) is clear. For the converse, we may assume without loss of generality that $\mathsf{X} \simeq \operatorname{Spét} B$ and $\mathsf{Y} \simeq \operatorname{Spét} A$ are affine. Condition (2) then guarantees that $B_0 = B \otimes_A (\pi_0 A)$ is flat and finitely 0-presented over $\pi_0 A$. It follows immediately that B is flat over A. We will complete the proof by showing that B is almost of finite presentation over A. Using assumption (2) and Proposition 4.1.2.1, we deduce that the cotangent complex $L_{B_0/\pi_0 A} \simeq B_0 \otimes_B L_{B/A}$ is perfect to order 1 as a B_0-module (Proposition 4.1.2.1). Using Proposition 2.7.3.2, we see that $L_{B/A}$ is perfect to order 1 as a module over B. Since $\pi_0 B$ is finitely presented as an algebra over $\pi_0 A$, Proposition 4.1.2.1 implies that B is of finite generation to order 1 over A. Using the flatness of B over A and Corollary 6.1.6.3, we deduce that B is almost of finite presentation over A, as desired. □

Corollary 6.1.6.5. *Let $f : R \to A$ be a morphism of connective $\mathbb{E}_\infty$-rings. Suppose that there exists a faithful flat morphism $g : A \to B$ such that g and $g \circ f$ are almost of finite presentation. Then f is almost of finite presentation.*

Proof. We first show that $\pi_0 A$ is finitely generated as an algebra over $\pi_0 R$. Write $\pi_0 A$ as a filtered colimit of commutative rings $\{A_\alpha\}$ which are finitely presented over $\pi_0 R$. Since $\pi_0 B \simeq \pi_0 A \otimes_A B$ is almost of finite presentation over $\pi_0 A$, we can write $\pi_0 B \simeq \tau_{\leqslant 0}(\pi_0 A \otimes_{A_\alpha} B_\alpha)$ for some B_α which is finitely 0-presented over A_α. Using Proposition 6.1.6.1 and Proposition 4.6.1.2, we may assume without loss of generality that B_α is faithfully flat over A_α; in particular, B_α is discrete. Choose a finite collection of elements $x_1, \ldots, x_n$ which generate $\pi_0 B$ as an algebra over $\pi_0 R$. Changing α if necessary, we may suppose that each x_i can be lifted to an element $\overline{x}_i$ in B_α. Then the map $\psi : B_\alpha \to \pi_0 B$ is surjective. Since B_α is faithfully flat over A_α, we conclude that the map $A_\alpha \to \pi_0 A$ is surjective, so that $\pi_0 A$ is finitely generated over $\pi_0 R$.

Let I denote the kernel of the map $A_\alpha \to \pi_0 A$. Then $I B_\alpha$ is the kernel of ψ. Since ψ is a surjection between finitely presented $\pi_0 R$-algebras, $\ker(\psi)$ is finitely generated. Since B_α is faithfully flat over A_α, we conclude that I is finitely generated, so that $\pi_0 A \simeq A_\alpha / I$ is finitely presented as a commutative ring over $\pi_0 R$.

To complete the proof that A is almost of finite presention over R, it will suffice to show that $L_{A/R}$ is almost perfect as an A-module (Proposition 4.1.2.1). Since B is faithfully flat over A, it will suffice to prove that $B \otimes_A L_{A/R}$ is almost perfect as a B-module (Proposition 2.8.4.2). This follows from the existence of a fiber sequence $B \otimes_A L_{A/R} \to L_{B/R} \to L_{B/A}$, since $L_{B/A}$ and $L_{B/R}$ are almost perfect by virtue of Proposition 4.1.2.1. □

6.2 Digression: Quasi-Coherent Sheaves on a Functor

In §2.2, we introduced the notion of a *quasi-coherent sheaf* on a spectral Deligne-Mumford stack X. Every spectral Deligne-Mumford stack X determines a "functor of points" $h_{\mathsf{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, given on objects by the formula $h_{\mathsf{X}}(R) = \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Spét}\, R, \mathsf{X})$. According to Proposition **??**, the spectral Deligne-Mumford stack X is determined (up to canonical equivalence) from the functor h_{X}. In particular, the ∞-category $\mathrm{QCoh}(\mathsf{X})$ is depends only on the functor h_{X}. Our goal in this section is to make the passage from h_{X} to $\mathrm{QCoh}(\mathsf{X})$ explicit. To accomplish this, we will associate to any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ an ∞-category $\mathrm{QCoh}(X)$, which we call the *∞-category of quasi-coherent sheaves on the functor X*. Roughly speaking, an object $\mathscr{F} \in \mathrm{QCoh}(X)$ can be viewed as a rule which assigns to each point $\eta \in X(R)$ an R-module $\mathscr{F}(\eta)$, which depends functorially on R in the following sense: if $\phi : R \to R'$ is a map of connective $\mathbb{E}_\infty$-rings and η' denotes the image of η in $X(R')$, then we have a canonical equivalence $R' \otimes_R \mathscr{F}(\eta) \simeq \mathscr{F}(\eta')$. Moreover, we require that this equivalence is compatible with composition of morphisms in $\mathrm{CAlg}^{\mathrm{cn}}$, up to coherent homotopy. The first part of this section is devoted to converting the above discussion into a precise definition (Definition 6.2.2.1). We then show that, in the special case where X is the functor represented by a spectral Deligne-Mumford stack X, there is a canonical equivalence of ∞-categories $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(\mathsf{X})$ (Proposition 6.2.4.1).

6.2.1 Categorical Digression

Our first goal is to make turn the heuristic description of $\mathrm{QCoh}(X)$ sketched above into a precise definition. For this, we will need to develop a bit of categorical machinery. In what follows, we will assume that the reader is familiar with the theory of Cartesian fibrations developed in §HTT.2.4 .

Definition 6.2.1.1. Let $q : \mathcal{D} \to \mathcal{E}$ be a Cartesian fibration of simplicial sets. Given another map of simplicial sets $e : \mathcal{C} \to \mathcal{E}$, we let $\mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ denote the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C},\mathcal{E})} \{e\}$ whose objects are maps $F : \mathcal{C} \to \mathcal{D}$ which fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & {}_{e}\searrow \quad \swarrow_{q} & \\ & \mathcal{E}. & \end{array}$$

We will say that such an object F is *q-Cartesian* if it carries every edge of $\mathcal{C}$ to a q-Cartesian edge of $\mathcal{D}$. $\mathrm{Fun}^{\mathrm{cart}}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ spanned by the Cartesian maps.

Remark 6.2.1.2. As our notation suggests, we will be primarily interested in the special case of Definition 6.2.1.1 where $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are ∞-categories. However, Definition 6.2.1.1 makes sense in greater generality: the assumption that q is a Cartesian fibration guarantees that $\mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}^{\mathrm{cart}}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ are ∞-categories, regardless of whether or not $\mathcal{C}$, $\mathcal{D}$, or $\mathcal{E}$ are ∞-categories.

Remark 6.2.1.3. In the situation of Definition 6.2.1.1, suppose that $q : \mathcal{D} \to \mathcal{E}$ is a right fibration. Then every edge of $\mathcal{D}$ is q-Cartesian, so we have $\mathrm{Fun}^{\mathrm{cart}}_{\mathcal{E}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$.

In the situation of Definition 6.2.1.1, the assumption that $q : \mathcal{D} \to \mathcal{E}$ is a Cartesian fibration means that we can regard the construction $(E \in \mathcal{E}) \mapsto \mathcal{D}_E = \mathcal{D} \times_{\mathcal{E}} \{E\}$ as a contravariant functor from $\mathcal{D}$ to the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of small ∞-categories. Using Proposition HTT.3.3.3.1 , we can identify the ∞-category $\mathrm{Fun}^{\mathrm{cart}}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ with a limit of the composite functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$. We now study the naturality of this identification, focusing on the special case where the map $e : \mathcal{C} \to \mathcal{E}$ is a right fibration.

Notation 6.2.1.4. Let $\mathcal{S}\mathrm{et}_\Delta$ denote the category of simplicial sets. Given a simplicial set $\mathcal{E}$, we let $(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$ denote the full subcategory of $(\mathcal{S}\mathrm{et}_\Delta)_{/\mathcal{E}}$ whose objects are right fibrations $e : \mathcal{C} \to \mathcal{E}$. We will regard $(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$ as a simplicial category, with mapping spaces given by

$$\mathrm{Map}_{(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}^{\mathrm{cart}}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$$

(see Remark 6.2.1.3). Note that the simplicial category $(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$ is fibrant: that is, for every pair of objects $\mathcal{C}, \mathcal{D} \in (\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$, the mapping space $\mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ is a Kan complex. Let $\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$ denote the homotopy coherent nerve of the simplicial category $(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$. It follows from Proposition HTT.1.1.5.10 that $\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$ is an ∞-category. We will refer to $\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$ as *the ∞-category of right fibrations over $\mathcal{E}$*.

Remark 6.2.1.5. Let $\mathcal{E}$ be a simplicial set. Then we can regard $(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$ as the full subcategory of $(\mathcal{S}\mathrm{et}_\Delta)_{/\mathcal{E}}$ spanned by those objects which are fibrant with respect to the *contravariant model structure* described in §HTT.2.1.4 (note that such objects are also cofibrant: every object $(\mathcal{S}\mathrm{et}_\Delta)_{/\mathcal{E}}$ is cofibrant with respect to the contravariant model structure). Consequently, we can regard $\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$ is the underlying ∞-category of the simplicial model category $(\mathcal{S}\mathrm{et}_\Delta)_{/\mathcal{E}}$.

Remark 6.2.1.6. Let $\mathcal{E}$ be a simplicial set. Then Proposition HTT.5.1.1.1 supplies canonical equivalences of ∞-categories $\mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S}) \xleftarrow{\sim} \mathcal{P}''(\mathcal{E}) \xrightarrow{\sim} \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$, where $\mathcal{P}''(\mathcal{E})$ denotes the ∞-category underlying the model category of simplicial presheaves on the simplicial category $\mathfrak{C}[\mathcal{E}]$ (see §HTT.1.1.5)

Construction 6.2.1.7. Recall that the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of small ∞-categories is defined as the homotopy coherent nerve of the simplicial category $\mathcal{C}\mathrm{at}^\Delta_\infty$, where the objects of $\mathcal{C}\mathrm{at}^\Delta_\infty$ are (small) ∞-categories and the mapping spaces are given by $\mathrm{Map}_{\mathcal{C}\mathrm{at}^\Delta_\infty}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}(\mathcal{C}, \mathcal{D})^\simeq$

Let $q : \mathcal{D} \to \mathcal{E}$ be a Cartesian fibration of simplicial sets. Then the construction $\mathcal{C} \mapsto \mathrm{Fun}^{\mathrm{cart}}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ determines a contravariant simplicial functor $(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}} \to (\mathcal{C}\mathrm{at}^\Delta_\infty)$. Passing to homotopy coherent nerves, we obtain a functor of ∞-categories $\Phi_0 : (\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}})^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$. Composing this functor with the equivalence $\mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$ of Remark 6.2.1.6, we obtain a functor $\Phi(q) : \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$, which is well-defined up to (canonical) homotopy.

Remark 6.2.1.8. Let $q : \mathcal{D} \to \mathcal{E}$ be a Cartesian fibration of simplicial sets and let $X : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}$ be a functor. Then the functor $\Phi(q)$ of Construction 6.2.1.7 carries X to the ∞-category $\mathrm{Fun}^{\mathrm{cart}}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$, where $e : \mathcal{C} \to \mathcal{E}$ is a right fibration classified by X. More informally, the ∞-category $\Phi(q)(X)$ can be described as the limit $\varprojlim_{E \in \mathcal{E}, \eta \in X(E)} \mathcal{D}_E$: in other words, an object of $\Phi(q)(X)$ is a rule $\mathscr{F}$ which assigns to each object $E \in \mathcal{E}$ and each point $\eta \in X(E)$ an object $\mathscr{F}(\eta) \in \mathcal{D}_E$, which depends functorially on E and η.

The functor $\Phi(q)$ of Construction 6.2.1.7 can be characterized by a universal mapping property:

Proposition 6.2.1.9. *Let $q : \mathcal{D} \to \mathcal{E}$ be a Cartesian fibration of simplicial sets which is classified by a functor $\chi : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$, let $j : \mathcal{E}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}}$ be the Yoneda embedding, and let let $\Phi(q) : \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ be as in Construction 6.2.1.7. Then $\Phi(q)$ is a right Kan extension of χ along j.*

Remark 6.2.1.10. There are evident dual versions of the constructions described above. If $q : \mathcal{D} \to \mathcal{E}$ is a coCartesian fibration of simplicial sets and we are given a map $e : \mathcal{C} \to \mathcal{E}$, then we will say that a functor $\mathscr{F} \in \mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ is *q-coCartesian* if it carries every edge of $\mathcal{C}$ to a q-coCartesian edge of $\mathcal{D}$. We let $\mathrm{Fun}^{\mathrm{ccart}}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ spanned by the q-coCartesian maps. Using a dual version of Construction 6.2.1.7, we see that the construction $\mathcal{C} \mapsto \mathrm{Fun}^{\mathrm{ccart}}_{\mathcal{E}}(\mathcal{C}, \mathcal{E})$ determines a functor $\Phi'(q) : \mathrm{Fun}(\mathcal{E}, \mathcal{S})^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$. If $\chi : \mathcal{E} \to \mathcal{C}\mathrm{at}_\infty$ is a functor classifying the coCartesian fibration q, then Proposition 6.2.1.9 implies that $\Phi'(q)$ is a right Kan extension of χ along the Yoneda embedding $j : \mathcal{E} \to \mathrm{Fun}(\mathcal{E}, \mathcal{S})^{\mathrm{op}}$.

Remark 6.2.1.11. Remark 6.2.1.10 has an evident analogue in the setting of simplicial sets and ∞-categories that are not necessarily small. If $q : \mathcal{D} \to \mathcal{E}$ is a coCartesian fibration between simplicial sets which are not necessarily small, then q is classified by a functor $\chi : S \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. Performing Construction 6.2.1.7 in a larger universe, we obtain a functor $\Phi'(q) : \mathrm{Fun}(\mathcal{E}, \widehat{\mathcal{S}})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$, which we can identify with a right Kan extension of χ along the Yoneda embedding $\mathcal{E} \to \mathrm{Fun}(\mathcal{E}, \widehat{\mathcal{S}})^{\mathrm{op}}$.

We now turn to the proof of Proposition 6.2.1.9. By virtue of Lemma HTT.5.1.5.5 , it will suffice to verify the following pair of assertions:

Lemma 6.2.1.12. *Let $q : \mathcal{D} \to \mathcal{E}$ be a Cartesian fibration of simplicial sets. Then the functor $\Phi(q) : \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ of Construction 6.2.1.7 commutes with small limits.*

Lemma 6.2.1.13. *Let $q : \mathcal{D} \to \mathcal{E}$ be a Cartesian fibration of simplicial sets, let $\Phi(q) : \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ be as in Construction 6.2.1.7, and let $j : \mathcal{E} \to \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S})$ denote the Yoneda embedding. Then the composition $(\Phi(q) \circ j^{\mathrm{op}}) : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ classifies the Cartesian fibration q.*

Note that the statement of Lemma 6.2.1.13 involves the functor $\Phi(q)$ of Construction 6.2.1.7, which is in turn defined using the equivalence $\mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$ of Remark 6.2.1.6. Consequently, any proof of Lemma 6.2.1.13 will require us to analyze this equivalence, which was established using the straightening and unstraightening construction of §HTT.3.2 . In the proof that follows, we will assume that the reader is familiar with these constructions (a reader who does not wish to be burdened with these technical matters can regard Proposition 6.2.1.9 as providing alternative definition of the functor $\Phi(q)$ and proceed directly to Definition 6.2.2.1, where we will return to the study of algebraic geometry).

Proof of Lemma 6.2.1.13. We begin by recalling the definition of the Yoneda embedding j. Choose a weak equivalence of simplicial categories $\phi : \mathfrak{C}[\mathcal{E}] \to \mathcal{C}^{\mathrm{op}}$, where $\mathcal{C}$ is fibrant (that is, the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex for every pair of objects $X, Y \in \mathcal{C}$). The construction $X \mapsto \mathrm{Map}_{\mathcal{C}}(X, \bullet)$ determines a simplicial functor $F : \mathcal{C}^{\mathrm{op}} \to (\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}})^\circ$, which (after composing with ϕ) yields a map of simplicial sets $f : \mathcal{E} \to \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}})^\circ$; here we regard $\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}$ as endowed with the projective model structure and we let $(\mathcal{S}\mathrm{et})_\Delta^{\mathcal{C}})^\circ$ denote the full of $\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}$ spanned by the fibrant-cofibrant objects. The Yoneda embedding j is obtained by composing f with the equivalences $\theta : \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}})^\circ \to \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathfrak{C}[\mathcal{E}^{\mathrm{op}}]})^\circ$ and $\theta' : \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathfrak{C}[\mathcal{E}^{\mathrm{op}}]})^\circ \to \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S})$ The functor $\Phi(q)$ is obtained by composing a homotopy inverse to θ', the equivalence $U : \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathfrak{C}[\mathcal{E}^{\mathrm{op}}]})^\circ \to \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}}$ induced by the unstraightening functor $\mathrm{Un}_{\mathcal{E}}$ of §HTT.3.2.1 , and the functor Φ_0 of Construction 6.2.1.7. It therefore suffices to show that the composition $\Phi(q) \circ j \simeq \Phi_0 \circ U \circ \theta \circ f = \Phi_0 \circ \mathrm{Un}_\phi \circ f$ classifies the Cartesian fibration q.

Without loss of generality, we may suppose that $\mathcal{D} \simeq \mathrm{Un}_\phi^+ \chi$, where χ is a fibrant-cofibrant object of $(\mathcal{S}\mathrm{et}_\Delta^+)^{\mathcal{C}}$ (Theorem HTT.3.2.0.1). Then the composition $\Phi_0 \circ \mathrm{Un}_\phi : \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}})^\circ \to \mathcal{C}\mathrm{at}_\infty$ is given by the homotopy coherent nerve of the simplicial functor $F \mapsto \mathrm{Map}_{(\mathcal{S}\mathrm{et}_\Delta^+)_{/S}}(\mathrm{Un}_\phi^+ F^\sharp, \mathrm{Un}_\phi^+ \chi)$, which is equivalent to the functor $F \mapsto \mathrm{Map}_{(\mathcal{S}\mathrm{et}_\Delta^+)^{\mathcal{C}}}(F^\sharp, \chi)$; here $F^\sharp : \mathcal{C} \to \mathcal{S}\mathrm{et}_\Delta^+$ denotes the functor given by $F^\sharp(C) = F(C)^\sharp$. In particular, if F is representable by an object $C \in \mathcal{C}$, the classical (simplicially enriched) version of Yoneda's lemma gives a canonical isomorphism $\mathrm{Map}_{(\mathcal{S}\mathrm{et}_\Delta^+)^{\mathcal{C}}}(F^\sharp, \chi) \simeq \chi(C)$. We conclude that $\Phi(q) \circ j \simeq \Phi_0 \circ \mathrm{Un}_\phi \circ f$ is adjoint to the simplicial functor $\mathfrak{C}[\mathcal{E}^{\mathrm{op}}] \to \mathcal{C} \xrightarrow{\chi} \mathcal{C}\mathrm{at}_\infty^\Delta$, so that $\Phi(q) \circ j$ classifies the Cartesian fibration q as desired. □

We now turn to Lemma 6.2.1.12. In comparison with Lemma 6.2.1.13, the proof is relatively formal.

Lemma 6.2.1.14. *Let S be a simplicial set. Let $(\mathcal{S}\mathrm{et}_\Delta^+)_{/S}$ denote the category of marked simplicial sets equipped with a map to S, which we regard as endowed with the Cartesian model structure (see §HTT.3.1). Let $F : (\mathcal{S}\mathrm{et}_\Delta)_{/S} \to (\mathcal{S}\mathrm{et}_\Delta^+)_{/S}$ be the functor given by*

$X \mapsto X^\sharp$, and regard $(\mathcal{S}\mathrm{et}_\Delta)_{/S}$ as endowed with the contravariant model structure (see §HTT.2.1.4). Then:

(1) *The functor F carries fibrant objects of $(\mathcal{S}\mathrm{et}_\Delta)_{/S}$ (with respect to the contravariant model structure) to fibrant objects of $(\mathcal{S}\mathrm{et}^+_\Delta)_{/S}$, and therefore induces a functor of ∞-categories $f : \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/S} \to \mathrm{N}(\mathcal{S}\mathrm{et}^+_\Delta)^\circ_{/S}$. Here $(\mathcal{S}\mathrm{et}^+_\Delta)^\circ_{/S}$ denotes the simplicial category of fibrant objects of $(\mathcal{S}\mathrm{et}^+_\Delta)_{/S}$, and $\mathrm{N}(\mathcal{S}\mathrm{et}^+_\Delta)^\circ_{/S}$ denotes its homotopy coherent nerve.*

(2) *The functor f preserves small limits and colimits.*

Proof. Assertion (1) follows immediately from Proposition HTT.2.4.2.4 . To prove (2), we observe that f fits into a homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta)^{\mathfrak{C}[S]^{\mathrm{op}}})^\circ & \xrightarrow{f'} & \mathrm{N}((\mathcal{S}\mathrm{et}^+_\Delta)^{\mathfrak{C}[S]^{\mathrm{op}}})^\circ \\ \downarrow & & \downarrow \\ \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/S} & \xrightarrow{f} & \mathrm{N}(\mathcal{S}\mathrm{et}^+_\Delta)^\circ_{/S} \end{array}$$

where the vertical maps are given by the unstraightening functors of §HTT.2.2.1 and §HTT.3.2.1 , and therefore equivalences of ∞-categories. It therefore suffices to prove that the map f' preserves small limits. Using Proposition HTT.4.2.4.4 , we can identify f'' with the map $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(S^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty)$ induced by the inclusion $i : \mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$. It therefore suffices to show that i preserves small and colimits, which follows from the observation that i admits left and right adjoints. □

Proof of Lemma 6.2.1.12. Let $q : \mathcal{D} \to \mathcal{E}$ be a Cartesian fibration of simplicial sets. To show that the functor $\Phi(q) : \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ commutes with small limits, it will suffice to show that the functor $\Phi_0 : \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}} \to \mathcal{C}\mathrm{at}_\infty$ appearing in Construction 6.2.1.7 has the same property (since $\Phi(q)$ is obtained from Φ_q by precomposition with an equiavalence of ∞-categories). By definition, the functor Φ_0 can be obtained by composing the functor $f : \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta)^\circ_{/\mathcal{E}} \to \mathrm{N}(\mathcal{S}\mathrm{et}^+_\Delta)^\circ_{/\mathcal{E}}$ of Lemma 6.2.1.14 with the functor $G_0 : \mathrm{N}(\mathcal{S}\mathrm{et}^+_\Delta)^\circ_{/S} \to (\mathcal{S}\mathrm{et}^+_\Delta)^\circ$ induced by the right adjoint to the left Quillen functor $K \mapsto K \times S^\sharp$ from $\mathcal{S}\mathrm{et}^+_\Delta$ to $(\mathcal{S}\mathrm{et}^+_\Delta)_{/S}$. Since the functor G_0 preserves small limits, it follows from Lemma 6.2.1.14 that the functor Φ_0 also preserves small limits. □

6.2.2 Application: Quasi-Coherent Sheaves

We now consider Construction 6.2.1.7 (or, rather, of the variant of Remark 6.2.1.11) in a concrete example.

Definition 6.2.2.1. Let $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ denote the ∞-category whose objects are pairs (A, M), where A is a connective $\mathbb{E}_\infty$-ring and M is an A-module spectrum. Let q :

$\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod} \to \mathrm{CAlg}^{\mathrm{cn}}$ denote the projection onto the first factor, so that q is a coCartesian fibration. We let $\mathrm{QCoh} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the functor $\Phi'(q)$ obtained by applying Remark 6.2.1.11 to q; here $\widehat{\mathcal{S}}$ denotes the ∞-category of spaces which are not necessarily small, and $\widehat{\mathcal{C}\mathrm{at}}_\infty$ is defined similarly.

If $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is any functor, we will refer to $\mathrm{QCoh}(X) \in \widehat{\mathcal{C}\mathrm{at}}_\infty$ as the *∞-category of quasi-coherent sheaves on X*.

Remark 6.2.2.2. There is no real need to restrict to connective $\mathbb{E}_\infty$-rings in Definition 6.2.2.1. Using exactly the same procedure, we can associate to any functor $X : \mathrm{CAlg} \to \widehat{\mathcal{S}}$ an ∞-category $\mathrm{QCoh}'(X)$ of quasi-coherent sheaves on X. In some sense, this definition is strictly more general than that of Definition 6.2.2.1: if $X_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is any functor, then there is a canonical equivalence of ∞-categories $\mathrm{QCoh}(X_0) \simeq \mathrm{QCoh}'(X)$, where $X : \mathrm{CAlg} \to \widehat{\mathcal{S}}$ is a left Kan extension of X_0. However, for most of our applications it will be convenient to consider functors which are defined only on the full subcategory $\mathrm{CAlg}^{\mathrm{cn}} \subseteq \mathrm{CAlg}$ spanned by the connective $\mathbb{E}_\infty$-rings.

Notation 6.2.2.3. Let R be a connective $\mathbb{E}_\infty$-ring. We let $\operatorname{Spec} R : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \subseteq \widehat{\mathcal{S}}$ denote the functor corepresented by R, given by the formula $(\operatorname{Spec} R)(A) = \mathrm{Map}_{\mathrm{CAlg}}(R, A)$. We will sometimes refer to $\operatorname{Spec} R$ as the *spectrum of R*.

Remark 6.2.2.4. If R is a connective $\mathbb{E}_\infty$-ring, then we can identify $\operatorname{Spec} R$ with the functor represented by the affine spectral Deligne-Mumford stack $\operatorname{Sp\acute{e}t} R$. In other words, the fully faithful embedding $h : \mathrm{SpDM} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ carries $\operatorname{Sp\acute{e}t} R$ to $\operatorname{Spec} R$.

Remark 6.2.2.5. Lemma 6.2.1.13 implies that the composition of the Yoneda embedding $\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}}$ with the functor $\mathrm{QCoh} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classifies the coCartesian fibration $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod} \to \mathrm{CAlg}^{\mathrm{cn}}$. In other words, for every $\mathbb{E}_\infty$-ring R we have an equivalence of ∞-categories $\mathrm{QCoh}(\operatorname{Spec} R) \simeq \mathrm{Mod}_R$, which depends functorially on R.

Notation 6.2.2.6. If $f : X \to X'$ is a natural transformation between functors $X, X' : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, then f determines a functor $\mathrm{QCoh}(X') \to \mathrm{QCoh}(X)$. We will denote this functor by f^*, and refer to it as the functor given by *pullback along f*.

Remark 6.2.2.7. By construction, if $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is a functor classifying a left fibration $\mathcal{C} \to \mathrm{CAlg}$, then the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on F can be identified with the ∞-category $\mathrm{Fun}^{\mathrm{ccart}}_{\mathrm{CAlg}}(\mathcal{C}, \mathrm{Mod})$ of Construction 6.2.1.7. More informally, we can think of an object $\mathscr{F} \in \mathrm{QCoh}(X)$ as a functor which assigns to every connective $\mathbb{E}_\infty$-ring R and every point $\eta \in X(R)$ (encoded by an object $\widetilde{R} \in \mathcal{C}$ lifting R) an R-module $\mathscr{F}(\eta) \in \mathrm{Mod}_R$. These modules are required to depend functorially on R in the following strong sense: if $\phi : R \to R'$ is a map of connective $\mathbb{E}_\infty$-rings and $\eta' \in X(R')$ is the image of

η under ϕ (so that we have a morphism $\widetilde{f} : \widetilde{R} \to \widetilde{R}'$ in $\mathcal{C}$), then we obtain a q-coCartesian morphism $\mathscr{F}(\eta) \to \mathscr{F}(\eta')$ in Mod, which we can view as an equivalence of R'-modules $R' \otimes_R \mathscr{F}(\eta) \xrightarrow{\sim} \mathscr{F}(\eta')$.

Note that we can identify with a choice of point $\eta \in X(R)$ with a natural transformation of functors $\operatorname{Spec} R \to X$. Then $\mathscr{F}(\eta)$ can be identified with the the image of $\mathscr{F}$ under the composition $\operatorname{QCoh}(X) \xrightarrow{\eta^*} \operatorname{QCoh}(\operatorname{Spec} R) \simeq \operatorname{Mod}_R$, where the equivalence $\operatorname{QCoh}(\operatorname{Spec} R) \simeq \operatorname{Mod}_R$ is supplied by Remark 6.2.2.5. Motivated by this observation, we will sometimes denote the R-module $\mathscr{F}(\eta)$ by $\eta^* \mathscr{F}$.

6.2.3 Formal Properties of $\operatorname{QCoh}(X)$

Let $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ denote the full subcategory of $\operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ spanned by those functors which are sheaves with respect to the fpqc topology of Variant B.6.1.7.

Proposition 6.2.3.1. *Let* $L : \operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}}) \to \widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ *denote a left adjoint to the inclusion functor. Then:*

(a) *The canonical map* $\operatorname{QCoh} \circ L \to \operatorname{QCoh}$ *is an equivalence of functors from* $\operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}}$ *to* $\widehat{\mathcal{C}\mathrm{at}}_\infty$.

(b) *The restriction* $\operatorname{QCoh}|_{(\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}})^{\mathrm{op}}} : \widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ *preserves small limits.*

Proof. Since the functor QCoh preserves limits (Lemma 6.2.1.12), Proposition 1.3.1.7 implies that the functor QCoh factors through L if and only if the composition of QCoh with the Yoneda embedding $\operatorname{CAlg}^{\mathrm{cn}} \xrightarrow{j} \operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}}$ is a $\widehat{\mathcal{C}\mathrm{at}}_\infty$-valued sheaf on $\operatorname{CAlg}^{\mathrm{cn}}$. Assertion ($a$) now follows from Corollary D.6.3.3, and (b) is an immediate consequence of (a). □

Remark 6.2.3.2. We can restate Proposition 6.2.3.1 as follows: if $X : \operatorname{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is any functor and X' is the sheafification of F with respect to the fpqc topology, then the pullback map $\operatorname{QCoh}(X') \to \operatorname{QCoh}(X)$ is an equivalence of ∞-categories.

Remark 6.2.3.3. We can strengthen Proposition 6.2.3.1 slightly: the ∞-category $\operatorname{QCoh}(X)$ of quasi-coherent sheaves on a functor $X : \operatorname{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ depends only on the *hypercompletion* of the sheaf $L(X) \in \widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$.

We now summarize some formal properties enjoyed by ∞-categories of the form $\operatorname{QCoh}(X)$:

Proposition 6.2.3.4. (1) *For every functor* $X : \operatorname{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, *the* ∞*-category* $\operatorname{QCoh}(X)$ *is stable and admits small colimits.*

(2) *For every natural transformation morphism* $\alpha : X \to X'$ *in* $\operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, *the pullback functor* $\alpha^* : \operatorname{QCoh}(X') \to \operatorname{QCoh}(X)$ *preserves small colimits. In particular, it is exact.*

(3) *Suppose that* $X \in \widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ *belongs to the smallest full subcategory of* $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ *which is closed under small colimits and contains the essential image of the Yoneda embedding. Then the* ∞*-category* $\mathrm{QCoh}(X)$ *is presentable.*

Proof. Let $\mathcal{C}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ spanned by those ∞-categories which are stable and admit small colimits, and those functors which preserve small colimits. Then $\mathcal{C}$ admits limits, and the inclusion $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}\mathrm{at}}_\infty$ preserves limits. Since the coCartesian fibration $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod} \to \mathrm{CAlg}^{\mathrm{cn}}$ is classified by a functor $\chi : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}$, it follows from Proposition 6.2.1.9 that QCoh factors through $\mathcal{C}$. This proves (1) and (2). To prove (3), we let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by the presentable ∞-categories. Using Proposition HTT.5.5.3.13 , we deduce that $\mathcal{C}_0$ is stable under small limits in $\mathcal{C}$, so that $\mathrm{QCoh}^{-1}\mathcal{C}_0$ is stable under small colimits in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$. It therefore suffices to observe that $\mathrm{QCoh}(X)$ is presentable whenever X is corepresented by a connective $\mathbb{E}_\infty$-ring R: this follows from the equivalence $\mathrm{QCoh}(X) \simeq \mathrm{Mod}_R$ of Remark 6.2.2.5. □

6.2.4 Comparison with the Geometric Definition

We now show that when we restrict our attention to functors which are (representable by) spectral Deligne-Mumford stacks, Definition 6.2.2.1 recovers the theory of quasi-coherent sheaves developed in Chapter 2.

Proposition 6.2.4.1. *Let* X *be a spectral Deligne-Mumford stack and let* $h_{\mathsf{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ *denote the functor represented by* X *(given by the formula* $h_{\mathsf{X}}(A) = \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, A, \mathsf{X})$*). Then there is a canonical equivalence of* ∞*-categories* $\mathrm{QCoh}(\mathsf{X}) \simeq \mathrm{QCoh}(h_{\mathsf{X}})$*, where* $\mathrm{QCoh}(\mathsf{X})$ *is given by Definition 2.2.2.1 and* $\mathrm{QCoh}(h_{\mathsf{X}})$ *by Definition 6.2.2.1.*

Proof. To avoid confusion, let us temporarily denote the ∞-category of quasi-coherent sheaves on a spectral Deligne-Mumford stack X by $\mathrm{QCoh}'(\mathsf{X})$, so that the construction $\mathsf{X} \mapsto \mathrm{QCoh}'(\mathsf{X})$ determines a functor $\mathrm{QCoh}' : \mathrm{SpDM}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. Let $h : \mathrm{SpDM} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ be the fully faithful embedding of Proposition 1.6.4.2. Lemma 6.2.1.13 shows that the functors QCoh' and $\mathrm{QCoh} \circ h$ are equivalent when restricted to the full subcategory $\mathrm{Aff} \subseteq \mathrm{SpDM}$ spanned by the affine spectral Deligne-Mumford stacks. Since the functor h is fully faithful, Proposition 6.2.1.9 implies that $\mathrm{QCoh} \circ h$ is a right Kan extension of its restriction to Aff, so we obtain a natural transformation of functors $\alpha : \mathrm{QCoh}' \to \mathrm{QCoh} \circ h$ (which is an equivalence when restricted to affine spectral Deligne-Mumford stacks). We will complete the proof by showing that α is an equivalence: that is, for every spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, the induced map $\alpha(\mathsf{X}) : \mathrm{QCoh}'(\mathsf{X}) \to \mathrm{QCoh}(h_{\mathsf{X}})$ is an equivalence.

For each object $U \in \mathcal{X}$, let X_U denote the spectral Deligne-Mumford stack given by $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$, so that α determines a functor $\alpha(\mathsf{X}_U) : \mathrm{QCoh}'(\mathsf{X}_U) \to \mathrm{QCoh}(h_{\mathsf{X}_U})$. Let $\mathcal{X}_0$ denote the full subcategory of $\mathcal{X}$ spanned by those objects U for which $\alpha(\mathsf{X}_U)$ is an equivalence. We will complete the proof by showing that $\mathcal{X}_0 = \mathcal{X}$. By construction, $\mathcal{X}_0$

contains every affine object $U \in \mathcal{X}$. By virtue of Proposition 1.4.7.9, it will suffice to show that $\mathcal{X}_0$ is closed under small colimits in $\mathcal{X}$. To prove this, suppose we are given a small diagram $\{U_\alpha\}$ in the ∞-category $\mathcal{X}_0$ having a colimit $U \in \mathcal{X}$. We then have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}'(\mathsf{X}_U) & \xrightarrow{\alpha(\mathsf{X}_U)} & \mathrm{QCoh}(h_{\mathsf{X}_U}) \\ \downarrow & & \downarrow \phi \\ \varprojlim \mathrm{QCoh}'(\mathsf{X}_{U_\alpha}) & \longrightarrow & \varprojlim \mathrm{QCoh}(h_{\mathsf{X}_{U_\alpha}}). \end{array}$$

The lower horizontal map is an equivalence by virtue of our assumption that each U_α belongs to $\mathcal{X}_0$, and the left vertical map is an equivalence by the proof of Proposition 2.2.4.1. It will therefore suffice to show that the functor ϕ is an equivalence. Using Lemma **??**, we see that X_U can be identified with the colimit of the diagram $\{\mathsf{X}_{U_\alpha}\}$ in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\text{ét}}$ of étale sheaves. It follows that $L\,\mathsf{X}_U$ can be identified with the colimit of the diagram $\{L\,\mathsf{X}_{U_\alpha}\}$ in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$, where $L : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}}) \to \widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ is a left adjoint to the inclusion. The desired result now follows from Proposition 6.2.3.1. □

6.2.5 Local Properties of Quasi-Coherent Sheaves

Many of the local properties of quasi-coherent sheaves discussed in §2.8 make sense in the context of quasi-coherent sheaves on an arbitrary functor.

Definition 6.2.5.1. Let P be a condition on pairs (A, M), where A is a connective $\mathbb{E}_\infty$-ring and M is an A-module. We will say that P is *stable under base change* if, whenever a pair (A, M) has the property P and $f : A \to B$ is a map of connective $\mathbb{E}_\infty$-rings, the pair $(B, B \otimes_A M)$ also has the property P.

Proposition 6.2.5.2. *The following conditions on a pair* $(A, M) \in \mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ *are stable under base change:*

(1) *The condition that M is n-connective (when regarded as a spectrum), where n is a fixed integer.*

(2) *The condition that M is almost connective: that is, M is $(-n)$-connective for $n \gg 0$.*

(3) *The condition that M is has* Tor*-amplitude $\leqslant n$, where n is a fixed integer.*

(4) *The condition that M is flat.*

(5) *The condition that M is perfect to order n over A, where n is a fixed integer.*

(6) *The condition that M is almost perfect over A.*

(7) *The condition that M is perfect over A.*

(8) *The condition that M is locally free of finite rank over A.*

Proof. Assertions (1), (2), and (7) are obvious. Assertion (3) follows from Lemma 2.8.4.3 and assertion (5) from Proposition 2.7.3.1. Assertions (4) and (6) are immediate consequences of (1), (3), and (5), and assertion (8) follows from (4) and (7). □

Definition 6.2.5.3. Let P be a condition on objects $(A, M) \in \mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ which is stable under base change and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. We will say that an object $\mathscr{F} \in \mathrm{QCoh}(X)$ *has the property* P if, for every connective $\mathbb{E}_\infty$-ring R and every point $\eta \in X(R)$, the pair $(R, \mathscr{F}(\eta))$ has the property P, where $\mathscr{F}(\eta) \in \mathrm{Mod}_R$ is the R-module of Remark 6.2.2.7.

Remark 6.2.5.4. Let P be a condition on pairs $(A, M) \in \mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ which is stable under base change and let $\alpha : X \to X'$ be a natural transformation between functors $X, X' : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. If $\mathscr{F} \in \mathrm{QCoh}(X')$ has the property P, then $\alpha^* \mathscr{F} \in \mathrm{QCoh}(X)$ has the property P.

Example 6.2.5.5. Let R be a connective $\mathbb{E}_\infty$-ring and let $\operatorname{Spec} R : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor corepresented by R, so that Remark 6.2.2.5 gives an equivalence of ∞-categories $\theta : \mathrm{QCoh}(\operatorname{Spec} R) \simeq \mathrm{Mod}_R$. If P is a property of objects of $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ which is stable under base change, then an object $\mathscr{F} \in \mathrm{QCoh}(\operatorname{Spec} R)$ has the property P if and only if the pair $(R, \theta(\mathscr{F}))$ has the property P.

Remark 6.2.5.6. Let $\alpha : X \to X'$ be a natural transformation between functors $X, X' : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, and suppose that α induces an equivalence after sheafification with respect to the flat topology. Proposition 6.2.3.1 implies that the pullback functor $\alpha^* : \mathrm{QCoh}(X') \to \mathrm{QCoh}(X)$ is an equivalence of ∞-categories. If P is a property of objects of $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ which is stable under base change and $\mathscr{F} \in \mathrm{QCoh}(X')$ has the property P, then the pullback $\alpha^* \mathscr{F} \in \mathrm{QCoh}(X)$ has the property P (Remark 6.2.5.4). The converse holds provided that P is local with respect to the flat topology, in the sense of Definition 2.8.4.1.

Example 6.2.5.7. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor and let $\mathscr{F} \in \mathrm{QCoh}(X)$. We say that $\mathscr{F}$ is *connective* if, for every point $\eta \in X(R)$, the R-module $\mathscr{F}(\eta)$ is connective. We let $\mathrm{QCoh}(X)^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{QCoh}(X)$ spanned by the connective quasi-coherent sheaves on X. It is clear that $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is closed under small colimits and extensions in $\mathrm{QCoh}(X)$, and is therefore a prestable ∞-category in the sense of Definition C.1.2.1.

Remark 6.2.5.8. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor which satisfies condition (3) of Proposition 6.2.3.4. Arguing as in the proof of Proposition 6.2.3.4, we deduce that the ∞-category $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is presentable. It follows from Proposition HA.1.4.4.11 that the pair of subcategories $(\mathrm{QCoh}(X)^{\mathrm{cn}}, \mathrm{QCoh}(X)_{\leqslant 0})$ determines an accessible t-structure on $\mathrm{QCoh}(X)$, where $\mathrm{QCoh}(X)_{\leqslant 0}$ denotes the full subcategory of $\mathrm{QCoh}(X)$ spanned by those objects $\mathscr{G}$ for which the mapping space $\mathrm{Map}_{\mathrm{QCoh}(X)}(\Sigma \mathscr{F}, \mathscr{G})$ is contractible for all $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$.

Warning 6.2.5.9. In the situation of Remark 6.2.5.8, there is no obvious way to test that whether an object $\mathscr{F} \in \mathrm{QCoh}(X)$ belongs to $\mathrm{QCoh}(X)_{\leqslant 0}$. For example, the inclusion $\mathscr{F} \in \mathrm{QCoh}(X)$ does *not* imply that $\mathscr{F}(\eta) \in (\mathrm{Mod}_R)_{\leqslant 0}$ when $\eta \in X(R)$, since the property of being 0-truncated is not stable under base change. In general, the t-structure of Remark 6.2.5.8 is generally not compatible with filtered colimits, and the prestable ∞-category $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is not a Grothendieck prestable ∞-category (see Definition C.1.4.2). To guarantee that the t-structure of Remark 6.2.5.8 is well-behaved, we typically need to impose some additional assumptions on X (like the assumption that X is representable by a spectral Deligne-Mumford stack, or admits a flat covering by a spectral Deligne-Mumford stack).

Proposition 6.2.5.10. *Let X be a spectral Deligne-Mumford stack, let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by X, and let $\theta : \mathrm{QCoh}(\mathsf{X}) \simeq \mathrm{QCoh}(X)$ be the equivalence supplied by the proof of Proposition 6.2.4.1. Let P be a property of objects of $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ which is stable under base change and local for the flat topology. An object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ has the property P if and only if $\theta(\mathscr{F}) \in \mathrm{QCoh}(F)$ has the property P.*

Proof. The "only if" direction is obvious. For the converse, let us suppose that $\mathscr{F}$ has the property P. Let A be a connective $\mathbb{E}_\infty$-ring and let $f : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$ be a map of spectral Deligne-Mumford stacks; we wish to show that the pair $(A, f^* \mathscr{F})$ has the property P. In verifying this, we are free to replace X by any open substack through which f factors; we may therefore assume without loss of generality that X is quasi-compact. Choose an étale surjection $u : \mathsf{U} \to \mathsf{X}$, where $\mathsf{U} \simeq \mathrm{Sp\acute{e}t}\, R$ is affine. We can then choose a faithfully flat étale map $A \to A'$ such that the composite map $\mathrm{Sp\acute{e}t}\, A' \to \mathrm{Sp\acute{e}t}\, A \xrightarrow{f} \mathsf{X}$ factors through U. Since P is local for the flat topology, we may replace A by A' and thereby reduce to the case where f factors through U. Then $f^* \mathscr{F} \simeq A \otimes_R u^* \mathscr{F}$. Since P is stable under base change, we are reduced to proving that the pair $(R, u^* \mathscr{F})$ has the property P, which follows from our assumption that $\mathscr{F}$ has the property P. □

6.2.6 Tensor Products of Quasi-Coherent Sheaves

Let $\mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ denote the ∞-category of (not necessarily small) symmetric monoidal ∞-categories. We have an evident forgetful functor $\theta : \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which preserves limits. The functor $R \mapsto \mathrm{Mod}_R$ factors as a composition

$$\mathrm{CAlg}^{\mathrm{cn}} \xrightarrow{U} \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \xrightarrow{\theta} \widehat{\mathcal{C}\mathrm{at}}_\infty,$$

where U assigns to each connective $\mathbb{E}_\infty$-ring the symmetric monoidal ∞-category $\mathrm{Mod}_R^{\otimes}$ (see §HA.4.5.3). Let $\mathrm{QCoh}^{\otimes} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ be a right Kan extension of U

along the Yoneda embedding $(\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}}$. Then the functor $\mathrm{QCoh}^{\otimes}$ assigns to each functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ a symmetric monoidal ∞-category $\mathrm{QCoh}(X)^{\otimes}$, whose underyling ∞-category can be identified with $\mathrm{QCoh}(X)$. We can describe the situation more informally by saying that for every functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the ∞-category $\mathrm{QCoh}(X)$ admits a symmetric monoidal structure. Unwinding the definitions, we see that the tensor product on $\mathrm{QCoh}(X)$ is given pointwise: that is, it is described by the formula

$$(\mathscr{F} \otimes \mathscr{F}')(\eta) \simeq \mathscr{F}(\eta) \otimes_R \mathscr{F}'(\eta)$$

for $\eta \in X(R)$. It follows that the tensor product on $\mathrm{QCoh}(X)$ preserves small colimits separately in each variable.

Notation 6.2.6.1. For any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, we let $\mathscr{O}_X$ denote the unit object of $\mathrm{QCoh}(X)$ (with respect to the symmetric monoidal structure defined above). More informally, $\mathscr{O}_X$ assigns to each point $\eta \in X(R)$ the spectrum R, regarded as a module over itself. We will refer to $\mathscr{O}_X$ as the *structure sheaf* of the functor X.

Proposition 6.2.6.2. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor, and let $\mathscr{F} \in \mathrm{QCoh}(X)$. The following conditions are equivalent:*

(1) *The quasi-coherent sheaf $\mathscr{F}$ is perfect.*

(2) *The quasi-coherent sheaf is a dualizable object of the symmetric monoidal ∞-category $\mathrm{QCoh}(X)$.*

Proof. Using Proposition HA.4.6.1.11 , we can reduce to the case where X is corepresentable by a connective $\mathbb{E}_\infty$-ring R. In this case, we can identify $\mathscr{F}$ with an R-module M. We wish to show that M is a dualizable object of Mod_R if and only if M is perfect. The collection of dualizable objects of Mod_R forms a stable subcategory which is closed under retracts. Since $R \in \mathrm{Mod}_R$ is dualizable, it follows that every perfect object of Mod_R is dualizable. Conversely, suppose that M admits a dual $M^\vee$. Then the functor $N \mapsto \mathrm{Map}_{\mathrm{Mod}_R}(M, N)$ is given by $N \mapsto \Omega^\infty(M^\vee \otimes_R N)$, and therefore commutes with filtered colimits. It follows that M is a compact object of Mod_R, and therefore perfect (Proposition HA.7.2.4.2). □

Note that for any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the full subcategory $\mathrm{QCoh}(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}(X)$ contains $\mathscr{O}_X$ and is closed under tensor products, and therefore inherits the structure of a symmetric monoidal ∞-category.

Proposition 6.2.6.3. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor and let $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$. Then $\mathscr{F}$ is a dualizable object of $\mathrm{QCoh}(X)^{\mathrm{cn}}$ if and only if $\mathscr{F}$ is locally free of finite rank.*

Proof. Combine Propositions HA.4.6.1.11 and 2.9.1.5. □

6.3 Relative Representability

Let X and Y be functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. In this section, we will study natural transformations $f : X \to Y$ which are *relatively representable*, meaning that the fiber of f over any representable functor $\operatorname{Spec} R$ is representable by a spectral Deligne-Mumford stack (Definition 6.3.2.1).

6.3.1 Étale-Local Properties of Morphisms

We begin by introducing some terminology.

Definition 6.3.1.1. Let P be a property of morphisms $f : \mathsf{X} \to \mathsf{Y}$ between spectral Deligne-Mumford stacks. We will say that f is *local on the target with respect to the étale topology* if the following conditions are satisfied:

(1) Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y} \end{array}$$

where g is étale. If f has the property P, then f' also has the property P.

(2) Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks, and suppose we are given a surjective étale morphism $\amalg_\alpha \mathsf{Y}_\alpha \to \mathsf{Y}$. If each of the induced maps $\mathsf{Y}_\alpha \times_{\mathsf{Y}} \mathsf{X} \to \mathsf{Y}_\alpha$ has the property P, then f has the property P.

Remark 6.3.1.2. Let P be a property of morphisms between spectral Deligne-Mumford stacks which is local on the target with respect to the étale topology. Then a morphism $f : \mathsf{X} \to \mathsf{Y}$ has the property P if and only if, for every étale map $u : \operatorname{Spét} R \to \mathsf{Y}$, the pullback $\operatorname{Spét} R \times_{\mathsf{Y}} \mathsf{X} \to \operatorname{Spét} R$ has the property P.

Remark 6.3.1.3. Suppose we are given a property P_0 of morphisms of spectral Deligne-Mumford stacks having the form $\mathsf{Z} \to \operatorname{Spét} R$. Let $f : \mathsf{X} \to \mathsf{Y}$ be an arbitrary morphism of spectral Deligne-Mumford stacks. We will say that f *locally has the property* P_0 if, for every étale map $\operatorname{Spét} R \to \mathsf{Y}$, the induced map $\operatorname{Spét} R \times_{\mathsf{Y}} \mathsf{X} \to \operatorname{Spét} R$ has the property P_0. Suppose that P_0 satisfies the following conditions:

(i) Let $f : \mathsf{Z} \to \operatorname{Spét} R$ be a map of spectral Deligne-Mumford stacks and $u : R \to R'$ an étale morphism of $\mathbb{E}_\infty$-rings. If f has the property P_0, then the induced map $\operatorname{Spét} R' \times_{\operatorname{Spét} R} \mathsf{Z} \to \operatorname{Spét} R$ has the property P_0. The converse holds if u is faithfully flat.

(*ii*) If we are given a finite collection of morphisms $\{\mathsf{Z}_i \to \mathrm{Sp\acute{e}t}\, R_i\}$ having the property P_0, then the induced map $\amalg \mathsf{Z}_i \to \mathrm{Sp\acute{e}t}(\prod R_i)$ has the property P_0.

Then the condition that a morphism f locally has the property P_0 is local on the target with respect to the étale topology, in the sense of Definition 6.3.1.1. Moreover, a morphism $f : \mathsf{Z} \to \mathrm{Sp\acute{e}t}\, R$ satisfies this condition if and only if f has the property P_0 (this follows immediately from (*i*)). Combining this observation with Remark 6.3.1.2, we obtain a bijective correspondence between the following:

(*a*) Properties P of arbitrary morphisms $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks, which are local on the target with respect to the étale topology.

(*b*) Properties P_0 of morphisms of the form $f : \mathsf{Z} \to \mathrm{Sp\acute{e}t}\, R$ which satisfy conditions (*i*) and (*ii*).

Example 6.3.1.4. The condition that a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks is étale is local on the target with respect to the étale topology.

Example 6.3.1.5. The condition that a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks is an equivalence is local on the target with respect to the étale topology.

Example 6.3.1.6. The condition that a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks is an open immersion is local on the target with respect to the étale topology. This follows from Examples 6.3.1.4 and 6.3.1.5, since f is an open immersion if and only if f is étale and the diagonal map $\mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ is an equivalence.

Example 6.3.1.7. The condition that a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks is surjective is local on the target with respect to the étale topology (see Proposition 3.5.5.4).

Example 6.3.1.8. The condition that a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks is flat is local on the target with respect to the étale topology.

Example 6.3.1.9. For $0 \leqslant n \leqslant \infty$, the condition that a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks is n-quasi-compact is local on the target with respect to the étale topology (Proposition 2.3.3.1).

Example 6.3.1.10. For every integer $n \geqslant 0$, the condition that a map of spectral Deligne-Mumford stacks $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ induce an equivalence $(\mathcal{X}, \tau_{\leqslant n} \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \tau_{\leqslant n} \mathcal{O}_{\mathcal{Y}})$ is local on the target with respect to the étale topology.

Definition 6.3.1.11. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks and let $n \geqslant -2$ be an integer. We will say that f is a *relative n-stack* if, for every discrete commutative ring R, the induced map

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X}) \to \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{Y})$$

has n-truncated homotopy fibers. We will say that f is a *relative spectral algebraic space* if it is a relative 0-stack.

Example 6.3.1.12. When $n = -2$, a map of spectral Deligne-Mumford stacks $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a relative Deligne-Mumford n-stack if and only if, for every discrete commutative ring R, the map

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, (\mathcal{X}, \mathcal{O}_{\mathcal{X}})) \to \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}))$$

is a homotopy equivalence. This is equivalent to the requirement that f induces an equivalence $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathcal{O}_{\mathcal{Y}})$.

Remark 6.3.1.13. The condition that a map $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a relative Deligne-Mumford n-stack depends only on the underlying map of 0-truncated spectral Deligne-Mumford stacks $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathcal{O}_{\mathcal{Y}})$.

Remark 6.3.1.14. If $n \geqslant 0$, then a morphism $f : \mathsf{X} \to \mathsf{Y}$ is a relative Deligne-Mumford n-stack if and only if, for every discrete commutative ring R and every map $u : \mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$, the pullback $\mathrm{Sp\acute{e}t}\, R \times_{\mathsf{Y}} \mathsf{X}$ is a spectral Deligne-Mumford n-stack (Definition 1.6.8.1). Using Remark 6.3.1.13, we see that this is equivalent to assertion that for every connective $\mathbb{E}_\infty$-ring R and every map $u : \mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$, the pullback $\mathrm{Sp\acute{e}t}\, R \times_{\mathsf{Y}} \mathsf{X}$ is a spectral Deligne-Mumford n-stack.

Proposition 6.3.1.15. *Let $n \geqslant -2$ be an integer. The condition that a map $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks be a relative spectral Deligne-Mumford n-stack is local on the target with respect to the étale topology.*

Proof. The proof proceeds by induction on n. When $n = -2$, the desired result follows from Examples 6.2.2.7 and 6.3.1.9. If $n > -2$, we observe that $f : \mathsf{X} \to \mathsf{Y}$ is a relative spectral Deligne-Mumford n-stack if and only if the diagonal map $\mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ is a relative Deligne-Mumford $(n-1)$-stack. □

Proposition 6.3.1.16. *The condition that a map $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks be affine is local on the target with respect to the étale topology.*

Proof. Using Remark 6.3.1.3, we are reduced to verifying the following assertion:

(∗) Let $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ be a map of spectral Deligne-Mumford stacks, and suppose there exists a faithfully flat étale morphism $R \to R^0$ such that the fiber product $\mathsf{X}_0 = \mathrm{Sp\acute{e}t}\, R^0 \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ is affine. Then X is affine.

To prove (∗), let $R^\bullet$ be the Čech nerve of the map $R \to R^0$ (in the ∞-category $\mathrm{CAlg}^{\mathrm{op}}$). For each $n \geqslant 0$, the fiber product $\mathrm{Sp\acute{e}t}\, R^n \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ is an affine spectral Deligne-Mumford stack, of the form Spét A^n for some $\mathbb{E}_\infty$-ring A^n. Let A denote the totalization of the cosimplicial $\mathbb{E}_\infty$-ring $A^\bullet$. It follows from Theorem D.6.3.1 that $A^\bullet$ is the Čech nerve of the morphism $A \to A^0 \simeq R^0 \otimes_R A$. Write $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$, so that the simplicial spectral Deligne-Mumford stack $\mathsf{X}_\bullet$ corresponds to a simplicial object $U_\bullet$ in $\mathcal{X}$, whose geometric realization is a final object $\mathbf{1} \in \mathcal{X}$. Then we have a chain of equivalences

$$\mathcal{O}_\mathcal{X}(\mathbf{1}) = \mathcal{O}_\mathcal{X}(|U_\bullet|) \simeq \varprojlim \mathcal{O}_\mathcal{X}(U_\bullet) \simeq \varprojlim A^\bullet \simeq A.$$

The composite equivalence determines a map $\theta : \mathsf{X} \to \text{Spét}\, A$. The map θ is an equivalence, since it can be obtained as the geometric realization of an equivalence of simplicial spectral Deligne-Mumford stacks $\mathsf{X}_\bullet \simeq \text{Spét}\, A^\bullet$. This proves that X is affine, as desired. □

Proposition 6.3.1.17. *The condition that a morphism of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ be quasi-affine is local on the target with respect to the étale topology.*

Proof. Using Remark 6.3.1.3, we are reduced to proving the following:

(∗) Let $f : \mathsf{X} \to \text{Spét}\, R$ be a map of spectral Deligne-Mumford stacks, and suppose there exists a faithfully flat étale morphism $R \to R^0$ such that the fiber product $\mathsf{X}_0 = \text{Spét}\, R^0 \times_{\text{Spét}\, R} \mathsf{X}$ is quasi-affine. Then X is quasi-affine.

Write $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$. We first claim that the pushforward $f_* \mathcal{O}_\mathcal{X}$ is a quasi-coherent sheaf on Spét R. This assertion can be tested locally with respect to the étale topology on Spét R, and therefore follows from Corollary 2.5.4.6. We can identify $f_* \mathcal{O}_\mathcal{X}$ with an $\mathbb{E}_\infty$-algebra over R. Let A denote the connective cover of this $\mathbb{E}_\infty$-algebra. The map $A \to f_* \mathcal{O}_\mathcal{X}$ classifies a map of spectral Deligne-Mumford stacks $g : \mathsf{X} \to \text{Spét}\, A$. We claim that g is a quasi-compact open immersion. Since this assertion is local on the target with respect to the étale topology (Examples 6.3.1.6 and 6.3.1.9), we may replace R by R' and thereby reduce to the case where X is quasi-affine. In this case, the desired result follows from Proposition 2.4.1.3 (see the proof of Proposition 2.4.2.3). □

6.3.2 Representable Morphisms of Functors

Recall that the ∞-category SpDM of spectral Deligne-Mumford stacks can be identified with a full subcategory of $\text{Fun}(\text{CAlg}^{\text{cn}}, \mathcal{S})$ (the identification is given by carrying a spectral Deligne-Mumford stack X to the functor given informally by the formula $R \mapsto \text{Map}_{\text{SpDM}}(\text{Spét}\, R, \mathsf{X})$). Many of the properties of morphisms considered above can be generalized to the setting of natural transformations between functors $X, Y : \text{CAlg}^{\text{cn}} \to \mathcal{S}$.

Definition 6.3.2.1. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \text{CAlg}^{\text{cn}} \to \widehat{\mathcal{S}}$. We will say that f is *representable* if, for every connective $\mathbb{E}_\infty$-ring R and every natural transformation $\text{Spec}\, R \to Y$ (corresponding to a choice of point $\eta \in Y(R)$), the fiber product $X \times_Y \operatorname{Spec} R$ is representable by a spectral Deligne-Mumford stack (here $\operatorname{Spec} R$ denotes the functor corepresented by R).

Proposition 6.3.2.2. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Assume that Y is representable by a spectral Deligne-Mumford stack Y. Then f is representable (in the sense of Definition 6.3.2.1) if and only if X is representable by a spectral Deligne-Mumford stack.*

The proof of Proposition 6.3.2.2 will require the following general observation about sheaves.

Proof of Proposition 6.3.2.2. Let $\widehat{\mathcal{S}\mathrm{hv}}_{\text{ét}}$ be the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ spanned by those functors which are sheaves with respect to the étale topology. Since the ∞-category of spectral Deligne-Mumford stacks admits fiber products, it is clear that if X is representable by a spectral Deligne-Mumford stack, then f is representable. To prove the converse, write $\mathsf{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. For every object $U \in \mathcal{Y}$, let Y_U denote the functor represented by the spectral Deligne-Mumford stack $\mathsf{Y}_U = (\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}}|_U)$, and let $X_U = X \times_Y Y_U$. Let us say that U is *good* if X_U is representable by a spectral Deligne-Mumford stack X_U. Assuming that f is representable by spectral Deligne-Mumford stacks, we will show that every object $U \in \mathcal{Y}$. Our assumption immediately implies that every affine object $U \in \mathcal{Y}$ is good. It will therefore suffice to show that the collection of good objects of $\mathcal{Y}$ is closed under small colimits (Lemma **??**). To this end, suppose we are given a diagram of object $\{U_\alpha\}$ in $\mathcal{Y}$ having a colimit U, and that each X_{U_α} is representable by a spectral Deligne-Mumford stack X_{U_α}. Note that for every morphism $U_\alpha \to U_\beta$ in our diagram, the induced map $\mathsf{X}_{U_\alpha} \to \mathsf{X}_{U_\beta}$ is étale(since it is a pullback of the étale morphism $\mathsf{Y}_{U_\alpha} \to \mathsf{Y}_{U_\beta}$). It follows from Proposition **??** that the diagram $\{\mathsf{X}_{U_\alpha}\}$ has a colimit X_U in the ∞-category SpDM. Moreover, X_U represents a functor $F : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ which is the colimit of the diagram $\{X_{U_\alpha}\}$ in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\text{ét}}$ spanned by the étale sheaves (Lemma **??**). To prove that U is good, it will suffice to show that $F \simeq X_U$: that is, that X_U is the colimit of the diagram $\{X_{U_\alpha}\}$ in $\widehat{\mathcal{S}\mathrm{hv}}_{\text{ét}}$. Since colimits in $\widehat{\mathcal{S}\mathrm{hv}}_{\text{ét}}$ are universal, we are reduced to proving the following pair of assertions:

(a) The functor X is a sheaf with respect to the étale topology.

(b) The functor Y_U is a colimit of the diagram $\{Y_{U_\alpha}\}$ in $\widehat{\mathcal{S}\mathrm{hv}}_{\text{ét}}$.

Assertion (a) follows from Lemma D.4.3.2, and assertion (b) follows from Lemma **??**. □

Corollary 6.3.2.3. *Suppose we are given natural transformations $X \xrightarrow{f} Y \xrightarrow{g} Z$ of functors $X, Y, Z : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, and assume that g is representable. Then f is representable if and only if $g \circ f$ is representable.*

Proof. Without loss of generality, we may assume that Z is corepresentable by a connective $\mathbb{E}_\infty$-ring R. Then Y is representable by a spectral Deligne-Mumford stack Y. The desired equivalence now follows immediately from Proposition 6.3.2.2. □

6.3.3 Properties Stable Under Base Change

We now consider properties of morphisms $f : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which can be tested "fiberwise": that is, after base change along an arbitrary map $\mathrm{Spec}\, R \to Y$.

Definition 6.3.3.1. Let P be a property of morphisms of spectral Deligne-Mumford stacks. We will say that P is *stable under base change* if, for every pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y} \end{array}$$

such that f has the property P, the morphism f' also has the property P.

Remark 6.3.3.2. Let P be a property of morphisms of spectral Deligne-Mumford stacks which is local on the target with respect to the étale topology. Then P is stable under base change if and only if, for every pullback diagram

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Sp\acute{e}t}\, R' & \longrightarrow & \mathrm{Sp\acute{e}t}\, R \end{array}$$

such that f has the property P, the morphism f' also has the property P.

Definition 6.3.3.3. Let P be a property of morphisms of spectral Deligne-Mumford stacks which is local on the target with respect to the étale topology and stable under base change. Let $f : X \to Y$ be a representable morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. We will say that f *has the property* P if, for every connective $\mathbb{E}_\infty$-ring R and every natural transformation $\mathrm{Spec}\, R \to Y$ (determined by a point $\eta \in Y(R)$), the fiber product $\mathrm{Spec}\, R \times_Y X$ is representable by a spectral Deligne-Mumford stack X_η such that the induced map $\mathsf{X}_\eta \to \mathrm{Sp\acute{e}t}\, R$ has the property P.

Remark 6.3.3.4. In the situation of Definition 6.3.3.3, the natural transformation $f : X \to Y$ has the property P if and only if the following apparently stronger condition holds: for every pullback diagram of functors

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y, \end{array}$$

if Y' is representable by a spectral Deligne-Mumford stack Y (so that X' is representable by a spectral Deligne-Mumford stack X, by virtue of Proposition 6.3.2.2), the induced map $\mathsf{X} \to \mathsf{Y}$ has the property P.

Remark 6.3.3.5. Let P a property of morphisms of spectral Deligne-Mumford stacks which is local on the target with respect to the étale topology and stable under base change. Let $\phi : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks, let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functors represented by X and Y, and let $f : X \to Y$ be the natural transformation determined by ϕ. Then f has the property P if and only if ϕ has the property P.

Example 6.3.3.6. The following conditions on a morphism of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ are local for the étale topology and stable under base change:

(1) The condition that f is étale.

(2) The condition that f is an equivalence.

(3) The condition that f is an open immersion.

(4) The condition that f is flat.

(5) The condition that f is a relative Deligne-Mumford n-stack, where $n \geqslant -2$ is some fixed integer (see Definition 6.3.1.11).

(6) The condition that $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ induces an equivalence $(\mathcal{X}, \tau_{\leqslant n} \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \tau_{\leqslant n} \mathcal{O}_{\mathcal{Y}})$, where $n \geqslant 0$ is some fixed integer.

(7) The condition that f is a relative spectral algebraic space.

(8) The condition that f is surjective.

(9) The condition that f is affine.

(10) The condition that f is quasi-affine.

(11) The condition that f is quasi-compact.

(12) The condition that f is quasi-separated.

(13) The condition that f is locally almost of finite presentation.

(14) The condition that f is proper.

Consequently, we make make sense of each of these conditions for an arbitrary representable morphism $f : X \to Y$ between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$.

Definition 6.3.3.7. Let $f : X \to Y$ be a representable morphism of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will say that f is *faithfully flat* if it is flat and surjective (in the sense of Example 6.3.3.6).

Remark 6.3.3.8. Let $f : X \to Y$ be a representable natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Assume that Y is a sheaf with respect to the fpqc topology. Then X is also a sheaf with respect to the fpqc topology (see Proposition **??**). Moreover, if f is faithfully flat and quasi-compact, then it is an effective epimorphism of fpqc sheaves.

Remark 6.3.3.9. Let P be a property of morphisms of spectral Deligne-Mumford stacks which is local on the target with respect to the étale topology and stable under base change. Let $f : X \to Y$ be a natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, and suppose that Y is given as the colimit of a diagram $\{Y_\alpha\}$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. Then f is representable and has the property P if and only if each of the induced maps $X \times_Y Y_\alpha \to Y_\alpha$ satisfies the same conditions. The "only if" direction is obvious, and the converse follows from the observation that every map map $\mathrm{Sp\acute{e}t}\, A \to Y$ factors through some Y_α.

6.3.4 Direct Image Functors

Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. In §6.2, we introduced ∞-categories $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y)$ of quasi-coherent sheaves on X and Y and the pullback functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$. Under mild hypotheses, the functor f^* admits a right adjoint.

Proposition 6.3.4.1. *Let $f : X \to Y$ be a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which is quasi-compact, quasi-separated relative spectral algebraic space. Then:*

(a) *The functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ admits a right adjoint $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$.*

(b) *For every pullback diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, the associated diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(Y) & \xrightarrow{f^*} & \mathrm{QCoh}(X) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ \mathrm{QCoh}(Y') & \xrightarrow{f'^*} & \mathrm{QCoh}(X') \end{array}$$

is right adjointable: that is, the Beck-Chevalley transformation $g^* f_* \to f'_* g'^*$ is an equivalence of functors from $\mathrm{QCoh}(X)$ *to* $\mathrm{QCoh}(Y')$*.*

Proof. Write Y as the colimit of a diagram $q : S \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, where each $q(s) \simeq \operatorname{Spec} A_s$ is affine. Our hypothesis on f guarantees that each of the fiber products $X \times_Y \operatorname{Spec} A_s$ is representable by a spectral algebraic space X_s which is quasi-compact and quasi-separated. Every edge $s \to s'$ in S determines a pullback diagram

$$\begin{array}{ccc} \mathsf{X}_s & \longrightarrow & \mathsf{X}_{s'} \\ \downarrow & & \downarrow \\ \operatorname{Sp\acute{e}t} A_s & \longrightarrow & \operatorname{Sp\acute{e}t} A_{s'}. \end{array}$$

Using Corollary **??**, we deduce that the associated diagram of pullback functors

$$\begin{array}{ccc} \mathrm{QCoh}(\operatorname{Spec} A_{s'}) & \longrightarrow & \mathrm{QCoh}(\mathsf{X}_{s'}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\operatorname{Spec} A_s) & \longrightarrow & \mathrm{QCoh}(\mathsf{X}_s) \end{array}$$

is right adjointable. Since $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(\mathsf{X}_s)$ and $\mathrm{QCoh}(Y) \simeq \varprojlim \mathrm{QCoh}(\operatorname{Spec} A_s)$, Corollary HA.4.7.4.18 implies the following:

(i) The functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ admits a right adjoint.

(ii) For each $s \in S$, the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(Y) & \xrightarrow{f^*} & \mathrm{QCoh}(X) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\operatorname{Sp\acute{e}t} A_s) & \xrightarrow{f_s^*} & \mathrm{QCoh}(\mathsf{X}_s) \end{array}$$

is right adjointable.

This proves (a). Moreover, we can assume that every morphism $\operatorname{Spec} A \to Y$ appears as a map $q(s) \to Y$ for some $s \in S$, so that (ii) implies that (b) is satisfied whenever Y' is affine. To prove (b) in general, consider a pullback square σ :

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

and let $\mathscr{F} \in \mathrm{QCoh}(X)$; we wish to show that the Beck-Chevalley map $\theta : g^* f_* \mathscr{F} \to f'_* g'^* \mathscr{F}$ is an equivalence. To prove this, it will suffice to show that for every map $h : \operatorname{Spec} A \to Y'$, the pullback $h^*(\theta)$ is an equivalence in $\mathrm{QCoh}(\operatorname{Spec} A)$. Extending σ to a rectangular diagram

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} A & \xrightarrow{h} & Y' & \xrightarrow{g} & Y \end{array}$$

where both squares are pullbacks, we see that $h^*(\theta)$ fits into a commutative diagram

$$\begin{array}{ccc} & h^* f'_* g'^* \mathscr{F} & \\ {\scriptstyle h^*(\theta)} \nearrow & & \searrow {\scriptstyle \theta'} \\ h^* g^* f_* \mathscr{F} & \xrightarrow{\theta''} & f''_* h'^* g'^* \mathscr{F}, \end{array}$$

where θ' and θ'' are equivalences by virtue of the fact that (b) holds in the special case where Y' is corepresentable. □

Warning 6.3.4.2. The existence of the right adjoint f_* follows from Corollary HTT.5.5.2.9 if $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y)$ are presentable. However, it is generally a poorly behaved construction (and is not compatible with base change) without additional hypotheses on f (such as those which appear in the statement of Proposition 6.3.4.1).

Corollary 6.3.4.3 (Projection Formula)**.** *Let $f : X \to Y$ be a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which is quasi-compact, quasi-separated relative spectral algebraic space. Then:*

(1) *The direct image functor $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ preserves small colimits.*

(2) *For every pair of objects $\mathscr{F} \in \mathrm{QCoh}(X)$ and $\mathscr{G} \in \mathrm{QCoh}(Y)$, the canonical map $\mathscr{F} \otimes f_* \mathscr{G} \to f_*(f^* \mathscr{F} \otimes \mathscr{G})$ is an equivalence.*

Proof. Using assertion (b) of Proposition 6.3.4.1, we can reduce to the case where $Y = \operatorname{Spec} R$ is affine. In this case, assertion (1) follows from Corollary 3.4.2.2 and assertion (2) from Remark 3.4.2.6. □

Corollary 6.3.4.4. *Let $f : X \to Y$ be a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which is representable, proper, and locally almost of finite presentation. Then the functor $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ of Proposition 6.3.4.1 carries almost perfect objects of $\mathrm{QCoh}(X)$ to almost perfect objects of $\mathrm{QCoh}(Y)$.*

Proof. Combine Proposition 6.3.4.1 with Theorem 5.6.0.2. □

Proposition 6.3.4.5. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor and let $\mathrm{Aff}_{/X}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ spanned by those morphisms $f : Y \to X$ which are affine (in the sense of Example 6.3.3.6). Then the construction $Y \mapsto f_* \mathscr{O}_Y$ induces an equivalence of ∞-categories $\mathrm{Aff}_{/X} \to \mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}})$.*

Proof. Writing X as a colimit of representable functors, we can reduce to the case where $X = \operatorname{Spec} A$ is corepresentable, in which case the desired result follows from Proposition 2.5.1.2. □

In the situation of Proposition 6.3.4.1, the direct image functor $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is lax symmetric monoidal (Proposition 2.5.5.1) and therefore admits a canonical factorization $\mathrm{QCoh}(X) \to \mathrm{Mod}_{f_* \mathscr{O}_X}(\mathrm{QCoh}(Y)) \to \mathrm{QCoh}(Y)$ (Corollary 2.5.5.3).

Proposition 6.3.4.6. *Let $f : X \to Y$ be a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which is representable and quasi-affine. Then the induced map $\mathrm{QCoh}(X) \to \mathrm{Mod}_{f_* \mathcal{O}_X}(\mathrm{QCoh}(Y))$ is an equivalence of ∞-categories.*

Proof. When Y is a corepresentable functor, the desired result follows from Corollaries 2.5.4.6 and **??**. In general, we can write Y as the colimit of a diagram $q : S \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, where each $q(s) \simeq \operatorname{Sp\acute{e}t} A_s$ is affine. Since f is representable and quasi-affine, each of the fiber products $X \times_Y q(s)$ is representable by a quasi-affine spectral algebraic space X_s. Using Proposition 6.3.4.1, we deduce that $f_* \mathcal{O}_X$ is a quasi-coherent sheaf on Y whose restriction to each $\operatorname{Spec} A_s$ is given by $B_s = (f_s)_* \mathscr{O}_s$, where $\mathscr{O}_s$ denotes the structure sheaf of X_s. It follows that the functor $\mathrm{QCoh}(X) \to \mathrm{Mod}_{f_* \mathscr{O}_X}(\mathrm{QCoh}(Y))$ is given by a limit of equivalences $\mathrm{QCoh}(\mathsf{X}_s) \to \mathrm{Mod}_{B_s}(\mathrm{QCoh}(\operatorname{Sp\acute{e}t} A_s))$, and is therefore an equivalence. □

Corollary 6.3.4.7. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} X & \xleftarrow{f} & Y \\ \updownarrow{\scriptstyle g} & & \uparrow{\scriptstyle g'} \\ X' & \xleftarrow{f'} & Y' \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, where f is representable and quasi-affine. Assume that $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(X')$ are presentable. Then the diagram of symmetric monoidal ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \longrightarrow & \mathrm{QCoh}(Y) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(X') & \longrightarrow & \mathrm{QCoh}(Y') \end{array}$$

is a pushout square in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

6.4 Grothendieck Duality

Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and suppose that, for every map $\operatorname{Sp\acute{e}t} R \to \mathsf{Y}$, the fiber $\mathsf{X}_R = \operatorname{Sp\acute{e}t} R \times_{\mathsf{Y}} \mathsf{X}$ is a quasi-compact, quasi-separated spectral algebraic space. It follows from Corollary 3.4.2.2 that the direct image functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ preserves small colimits. Using the adjoint functor theorem (Corollary HTT.5.5.2.9), we deduce that the functor f_* admits a right adjoint. We will be interested in the following special case:

Definition 6.4.0.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation. We let $f^! : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ denote a right adjoint to the direct image functor f_*. We refer to $f^!$ as the *exceptional inverse image functor* associated to f.

Warning 6.4.0.2. To guarantee that the direct image functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ admits a right adjoint, we need much weaker assumptions than those of Definition 6.4.0.1. However, we will denote the right adjoint of f_* by $f^!$ only when f is proper and locally almost of finite presentation (see Remark **??**).

Example 6.4.0.3. Let $\phi : A \to B$ be a finite morphism of $\mathbb{E}_\infty$-rings. Then ϕ induces a morphism $f : \operatorname{Sp\acute{e}t} B \to \operatorname{Sp\acute{e}t} A$ which is proper (Proposition 5.2.1.1) and locally almost of finite presentation (Corollary 5.2.2.2). The functor $f^! : \mathrm{Mod}_A \simeq \mathrm{QCoh}(\operatorname{Sp\acute{e}t} A) \to \mathrm{QCoh}(\operatorname{Sp\acute{e}t} B) \simeq \mathrm{Mod}_B$ is right adjoint to the restriction of scalars functor $\mathrm{Mod}_B \to \mathrm{Mod}_A$. At the level of spectra, it is given by the construction $M \mapsto \underline{\mathrm{Map}}_A(B, M)$.

Example 6.4.0.4. Let A be a commutative ring and let $a \in A$ be a regular element, so that the sequence of A-modules $0 \to A \xrightarrow{a} A \to A/(a) \to 0$ is exact. Let $f : \operatorname{Sp\acute{e}t} A/(a) \to \operatorname{Sp\acute{e}t} A$ be the associated closed immersion. Then the functor $f^! : \mathrm{Mod}_A \to \mathrm{Mod}_{A/(a)}$ is given by

$$M \mapsto \underline{\mathrm{Map}}_A(A/(a), M) \simeq \mathrm{fib}(M \xrightarrow{a} M) \simeq \Omega((A/(a)) \otimes_A M).$$

In particular, there is an equivalence of functors $f^! \simeq \Omega f^*$. Beware that this equivalence depends on the choice of a (and not only on the closed immersion f).

Remark 6.4.0.5. The definition exceptional inverse image functor $f^! : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ admits a natural generalization to non-proper morphisms $f : \mathsf{X} \to \mathsf{Y}$ (which are still assumed to be locally almost of finite presentation). However, the definition requires some care (in the case where f is not proper, the functor $f^!$ is not right adjoint to the direct image functor f_*), so we will confine our attention in this book to the case where f is proper.

The goal of this section is to study the functor $f^!$ of Definition 6.4.0.1 and to establish some of its basic formal properties. Our principal results can be summarized as follows:

(*a*) Let $f : \mathsf{X} \to \mathsf{Y}$ be any morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation. When restricted to *truncated* quasi-coherent sheaves, the exceptional inverse image functor $f^!$ is compatible with base change along morphisms $g : \mathsf{Y}' \to \mathsf{Y}$ which are of finite Tor-amplitude (Proposition 6.4.1.4). In particular, it is compatible with étale base change: that is, it is "local" on Y.

(*b*) If X and Y are locally Noetherian, then the functor $f^! : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ carries truncated coherent sheaves on Y to truncated coherent sheaves on X (Proposition 6.4.3.4).

(*c*) Suppose that $f : \mathsf{X} \to \mathsf{Y}$ is proper, locally almost of finite presentation, and locally of finite Tor-amplitude. Then there is a canonical equivalence $f^! \mathscr{F} \simeq f^* \mathscr{F} \otimes \omega_{\mathsf{X}/\mathsf{Y}}$ for $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ (Corollary 6.4.2.7), where $\omega_{\mathsf{X}/\mathsf{Y}}$ is the *relative dualizing sheaf* of $\mathscr{F}$ (Definition 6.4.2.4).

(*d*) Under the hypotheses of (*c*), the relative dualizing sheaf $\omega_{\mathsf{X}/\mathsf{Y}}$ is almost perfect and of finite Tor-amplitude over X (Proposition 6.4.4.1). Moreover, the construction $\mathscr{F} \mapsto f_*(\mathscr{F} \otimes \omega_{\mathsf{X}/\mathsf{Y}})$ determines a *left* adjoint to the pullback functor $f^* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ (Proposition 6.4.5.3).

6.4.1 The Exceptional Inverse Image Functor

Our first goal is to show that the exceptional inverse image functor of Definition 6.4.0.1 behaves well with respect to base change, under some mild additional assumptions.

Notation 6.4.1.1. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack. Recall that a quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is *locally truncated* if, for every affine $U \in \mathcal{X}$, the object $\mathscr{F}(U) \in \mathrm{Mod}_{\mathscr{O}_{\mathsf{X}}(U)}$ is truncated (if X is quasi-compact, this is equivalent to the requirement that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})_{\leqslant n}$ for some $n \gg 0$). We let $\mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}}$ denote the full subcategory of $\mathrm{QCoh}(\mathsf{X})$ spanned by those objects which are locally truncated.

Remark 6.4.1.2. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. If f is locally of finite Tor-amplitude, then the pullback functor $f^* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ carries $\mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}}$ into $\mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}}$).

Remark 6.4.1.3. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation. If Y is affine, then Corollary 3.4.2.3 guarantees that there exists an integer n for which the direct image functor f_* carries $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$ into $\mathrm{QCoh}(\mathsf{Y})_{\geqslant -n}$. It follows that the exceptional inverse image functor $f^!$ carries $\mathrm{QCoh}(\mathsf{Y})_{\leqslant 0}$ into $\mathrm{QCoh}(\mathsf{X})_{\leqslant n}$. In particular, $f^!$ carries $\mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}}$ into $\mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}}$.

Proposition 6.4.1.4. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation. Then:*

(1) *The exceptional inverse image functor* $f^! : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ *carries* $\mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}}$ *into* $\mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}}$.

(2) *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{g'} & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y}, \end{array}$$

where g *is locally of finite* Tor*-amplitude. Then the commutative diagram of* ∞*-categories*

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}} & \xrightarrow{f_*} & \mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}} \\ \downarrow{\scriptstyle g'^*} & & \downarrow{\scriptstyle g^*} \\ \mathrm{QCoh}(\mathsf{X}')_{\mathrm{ltr}} & \xrightarrow{f'_*} & \mathrm{QCoh}(\mathsf{Y}')_{\mathrm{ltr}} \end{array}$$

is right adjointable. In other words, for every truncated quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}}$*, the canonical map* $g'^* f^! \mathscr{F} \to f'^! g^* \mathscr{F}$ *is an equivalence in* $\mathrm{QCoh}(\mathsf{X}')$.

The proof of Proposition 6.4.1.4 will require some preliminaries. We begin by analyzing the case where Y is affine.

Lemma 6.4.1.5. *Let* $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, A$ *be a morphism of spectral algebraic spaces which is proper and locally almost of finite presentation. Then the exceptional inverse image functor* $f^! : \mathrm{Mod}_A \simeq \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A) \to \mathrm{QCoh}(\mathsf{X})$ *commutes with filtered colimits when restricted to* $(\mathrm{Mod}_A)_{\leqslant 0}$.

Proof. According to Proposition 9.6.1.1, the ∞-category $\mathrm{QCoh}(\mathsf{X})$ is compactly generated and an object of $\mathrm{QCoh}(\mathsf{X})$ is compact if and only if it is perfect. It will therefore suffice to show that for every perfect object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, the construction

$$(M \in (\mathrm{Mod}_A)_{\leqslant 0}) \mapsto \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, f^! M) \to \mathrm{Map}_{\mathrm{Mod}_A}(\Gamma(\mathsf{X}; \mathscr{F}); M)$$

commutes with filtered colimits. To prove this, it suffices to observe that $\Gamma(\mathsf{X}; \mathscr{F})$ is almost perfect when regarded as an A-module, by virtue of Theorem 5.6.0.2. □

Remark 6.4.1.6. Lemma 6.4.1.5 is essentially a reformulation of Theorem 5.6.0.2.

Construction 6.4.1.7. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation. Suppose we are given quasi-coherent sheaves $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathsf{Y})$. Then the projection formula of Remark 3.4.2.6 supplies an equivalence $\theta : (f_* f^! \mathscr{F}) \otimes \mathscr{G} \to f_*(f^! \mathscr{F} \otimes f^* \mathscr{G})$. The inverse equivalence θ^{-1} then determines a morphism $\theta' : f^! \mathscr{F} \otimes f^* \mathscr{G} \to f^!((f_* f^! \mathscr{F}) \otimes \mathscr{G})$. Composing with the counit map $f_* f^! \mathscr{F} \to \mathscr{F}$, we obtain a natural map $\rho_{\mathscr{F},\mathscr{G}} : f^! \mathscr{F} \otimes f^* \mathscr{G} \to f^!(\mathscr{F} \otimes \mathscr{G})$, depending naturally on $\mathscr{F}$ and $\mathscr{G}$.

Lemma 6.4.1.8. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation. Suppose that Y is affine. Let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ be truncated and let $\mathscr{G} \in \mathrm{QCoh}(\mathsf{Y})$ be of finite* Tor*-amplitude. Then the map $\rho_{\mathscr{F},\mathscr{G}} : f^! \mathscr{F} \otimes f^* \mathscr{G} \to f^!(\mathscr{F} \otimes \mathscr{G})$ of Construction 6.4.1.7 is an equivalence.*

Proof. Without loss of generality, we may assume that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})_{\leqslant 0}$ and that $\mathscr{G}$ has Tor-amplitude $\leqslant 0$. Using Proposition 9.6.7.1, we can write $\mathscr{G}$ as the colimit of a filtered diagram $\{\mathscr{G}_\alpha\}$, where each $\mathscr{G}_\alpha \in \mathrm{QCoh}(\mathsf{Y})$ is perfect and of Tor-amplitude $\leqslant 0$. Then each tensor product $\mathscr{F} \otimes \mathscr{G}_\alpha$ is 0-truncated. Using Lemma 6.4.1.5, we can identify $\rho_{\mathscr{F},\mathscr{G}}$ with the colimit of the diagram $\{\rho_{\mathscr{F},\mathscr{G}_\alpha}\}$ in the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{QCoh}(\mathsf{X}))$. We may therefore replace $\mathscr{G}$ by $\mathscr{G}_\alpha$ and thereby reduce to the case where $\mathscr{G}$ is perfect.

Let us regard $\mathscr{F}$ as fixed. Let $\mathcal{C} \subseteq \mathrm{QCoh}(\mathsf{Y})$ be the full subcategory of $\mathrm{QCoh}(\mathsf{Y})$ spanned by those objects $\mathscr{G}$ for which the morphism $\rho_{\mathscr{F},\mathscr{G}}$ is an equivalence. We will complete the proof by showing that every perfect object of $\mathrm{QCoh}(\mathsf{Y})$ belongs to $\mathcal{C}$. Since the construction $\mathscr{G} \mapsto \rho_{\mathscr{F},\mathscr{G}}$ is exact, the ∞-category $\mathcal{C}$ is a stable subcategory of $\mathrm{QCoh}(\mathsf{Y})$ which is closed under retracts. By virtue of our assumption that Y is affine, it will suffice to show that $\mathcal{C}$ contains the structure sheaf $\mathscr{O}_{\mathsf{Y}}$, which follows immediately from the construction of $\rho_{\mathscr{F},\mathscr{G}}$. □

Proof of Proposition 6.4.1.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation. Write $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}})$. For each $U \in \mathcal{Y}$, set $\mathsf{Y}_U = (\mathcal{Y}_{/U}, \mathscr{O}_{\mathsf{Y}}|_U)$ and $\mathsf{X}_U = \mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}_U$, and let $f_U : \mathsf{X}_U \to \mathsf{Y}_U$ denote the projection onto the second factor. Let us say that $U \in \mathcal{Y}$ is *good* if it satisfies the following condition:

- $(*)$ For every morphism $h : \mathrm{Sp\acute{e}t}\, A \to \mathsf{Y}_U$ which is locally of finite Tor-amplitude, the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X}_U)_{\mathrm{ltr}} & \xrightarrow{f_{U*}} & \mathrm{QCoh}(\mathsf{Y}_U)_{\mathrm{ltr}} \\ \downarrow & & \downarrow h^* \\ \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A \times_{\mathsf{Y}_U} \mathsf{X}_U)_{\mathrm{ltr}} & \longrightarrow & \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A)_{\mathrm{ltr}} \end{array}$$

  is right adjointable.

Note that $(*)$ implies in particular that the direct image functor $f_{U*} : \mathrm{QCoh}(\mathsf{X}_U)_{\mathrm{ltr}} \to \mathrm{QCoh}(\mathsf{Y}_U)_{\mathrm{ltr}}$ admits a right adjoint, which we will denote by $f_U^{\dagger}$.

Our first step is to prove the following:

(*a*) If $V \to U$ is a morphism of good objects of $\mathcal{Y}$, then the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X}_U)_{\mathrm{ltr}} & \xrightarrow{f_{U*}} & \mathrm{QCoh}(\mathsf{Y}_U)_{\mathrm{ltr}} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{X}_V)_{\mathrm{ltr}} & \xrightarrow{f_{V*}} & \mathrm{QCoh}(\mathsf{Y}_V)_{\mathrm{ltr}} \end{array}$$

is right adjointable.

Let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y}_U)$ be truncated; we wish to show that the canonical map $\xi : (f_U^\dagger \mathscr{F})|_{\mathsf{X}_V} \to f_V^\dagger(\mathscr{F}|_{\mathsf{Y}_V})$ is an equivalence. To prove this, it will suffice to show that for every map $W \to V$ where $W \in \mathcal{Y}$ is affine, the map ξ becomes an equivalence after restriction to X_W. This follows from our assumption that U and V are good (applying condition $(*)$ to the maps $\mathsf{Y}_W \to \mathsf{Y}_V$ and $\mathsf{Y}_W \to \mathsf{Y}_U$).

We next prove:

(*b*) The collection of good objects of $\mathcal{Y}$ is closed under small colimits.

To prove (*b*), suppose that $\{U_\alpha\}$ is a small diagram of good objects of $\mathcal{Y}$ having colimit $U \in \mathcal{Y}$. Combining (*a*) with Corollary HA.4.7.4.18 , we deduce that the functor $f_{U*} : \mathrm{QCoh}(\mathsf{X}_U)_{\mathrm{ltr}} \to \mathrm{QCoh}(\mathsf{Y}_U)_{\mathrm{ltr}}$ admits a right adjoint $f_U^\dagger$ and that the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X}_U)_{\mathrm{ltr}} & \xrightarrow{f_{U*}} & \mathrm{QCoh}(\mathsf{Y}_U)_{\mathrm{ltr}} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{X}_{U_\alpha})_{\mathrm{ltr}} & \xrightarrow{f_{V*}} & \mathrm{QCoh}(\mathsf{Y}_{U_\alpha})_{\mathrm{ltr}} \end{array}$$

is right adjointable for each index α. To complete the proof of (*b*), we must show that for each $u : \mathrm{Spét}\, A \to \mathsf{Y}_U$, the diagram appearing in condition $(*)$ is right adjointable. Using Lemma 6.4.1.8, we see that this assertion can be tested locally with respect to the étale topology on $\mathrm{Spét}\, A$, so we may assume that u factors through Y_{U_α} for some index α. In this case, the desired result follows from our assumption that U_α is good.

It follows from Lemma 6.4.1.8 that every affine object $U \in \mathcal{Y}$ is good. Combining (*b*) with Proposition 1.4.7.9, we deduce that every object of $\mathcal{Y}$ is good. In particular, the final object of $\mathcal{Y}$ is good. That is, we have the following:

$(*')$ For every morphism $h : \mathrm{Spét}\, A \to \mathsf{Y}$ which is locally of finite Tor-amplitude, the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}} & \xrightarrow{f_*} & \mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}} \\ \downarrow & & \downarrow h^* \\ \mathrm{QCoh}(\mathrm{Spét}\, A \times_{\mathsf{Y}} \mathsf{X})_{\mathrm{ltr}} & \longrightarrow & \mathrm{QCoh}(\mathrm{Spét}\, A)_{\mathrm{ltr}} \end{array}$$

is right adjointable.

In particular, the functor $f_* : \mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}} \to \mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}}$ admits a right adjoint $f^\dagger : \mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}} \to \mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}}$. We next prove:

(c) For each $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}}$, the counit map $v : f_* f^\dagger \mathscr{F} \to \mathscr{F}$ induces an equivalence $f^\dagger \mathscr{F} \to f^! \mathscr{F}$. In other words, $f^\dagger$ coincides with the restriction of $f^!$ to $\mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}}$.

To prove (c), it will suffice to show that for each object $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$ (not necessarily locally truncated), composition with v induces a homotopy equivalence $\rho : \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{G}, f^\dagger \mathscr{F}) \to \mathrm{Map}_{\mathrm{QCoh}(\mathsf{Y})}(f_* \mathscr{G}, \mathscr{F})$. More generally, for each $U \in \mathcal{Y}$, composition with v induces a map $\rho_U : \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X}_U)}(\mathscr{G}|_{\mathsf{X}_U}, (f^\dagger \mathscr{F})|_{\mathsf{X}_U}) \to \mathrm{Map}_{\mathrm{QCoh}(\mathsf{Y}_U)}((f_* \mathscr{G})_{\mathsf{Y}_U}, \mathscr{F}|_{\mathsf{Y}_U})$. Note that the collection of those objects $U \in \mathcal{Y}$ for which ρ_U is an equivalence is closed under small colimits. By virtue of Proposition 1.4.7.9, it will suffice to show that ρ_U is an equivalence when $U \in \mathcal{Y}$ is affine. Using (a), we can replace Y by Y_U and thereby reduce to the case where Y is affine. In this case, Remark 6.4.1.3 implies that $f^! \mathscr{F}$ is locally truncated, so that assertion (c) is clear.

Assertion (1) of Proposition 6.4.1.4 is an immediate consequence of (c). To prove (2), suppose we are given a pullback square

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{g'} & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y}, \end{array}$$

where g is locally of finite Tor-amplitude. We wish to show that for each $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}}$, the canonical map $u : g'^* f^! \mathscr{F} \to f'^! g^* \mathscr{F}$ is an equivalence. Fix an étale morphism $h : \mathrm{Spét}\, A \to \mathsf{Y}'$; we will show that $h^*(u)$ is an equivalence. This follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}} & \xrightarrow{f_*} & \mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}} \\ \downarrow{\scriptstyle g'^*} & & \downarrow{\scriptstyle g^*} \\ \mathrm{QCoh}(\mathsf{X}')_{\mathrm{ltr}} & \xrightarrow{f'_*} & \mathrm{QCoh}(\mathsf{Y}')_{\mathrm{ltr}} \\ \downarrow & & \downarrow{\scriptstyle h^*} \\ \mathrm{QCoh}(\mathrm{Spét}\, A \times_{\mathsf{Y}'} \mathsf{X}')_{\mathrm{ltr}} & \longrightarrow & \mathrm{QCoh}(\mathrm{Spét}\, A)_{\mathrm{ltr}}, \end{array}$$

since the lower square and outer rectangle are right adjointable by virtue of $(*')$. $\square$

Using Proposition 6.4.1.4, we immediately deduce "global" analogues of Lemmas 6.4.1.5 and 6.4.1.8:

Corollary 6.4.1.9. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation. Then:*

(1) *The exceptional inverse image functor* $f^! : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ *commutes with filtered colimits when restricted to* $\mathrm{QCoh}(\mathsf{Y})_{\leqslant 0}$.

(2) *If* $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ *is truncated and* $\mathscr{G} \in \mathrm{QCoh}(\mathsf{Y})$ *is of finite* Tor*-amplitude, then the map* $\rho_{\mathscr{F},\mathscr{G}} : f^! \mathscr{F} \otimes f^* \mathscr{G} \to f^!(\mathscr{F} \otimes \mathscr{G})$ *of Construction 6.4.1.7 is an equivalence.*

Proof. Using Proposition 6.4.1.4, we can reduce to the case where Y is affine, in which case the desired results follow from Lemmas 6.4.1.5 and 6.4.1.8. □

6.4.2 The Relative Dualizing Sheaf

When restricted to morphisms of finite Tor-amplitude, the exceptional inverse image functor of Definition 6.4.0.1 is particularly well-behaved. In particular, we will see that it is compatible with base change, and therefore makes sense more generally for morphisms of (not necessarily representable) functors.

Proposition 6.4.2.1. *Let* $f : X \to Y$ *be a natural transformation between functors* $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. *Assume that* f *is representable, proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude. Then:*

(a) *The direct image functor* $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ *admits a right adjoint* $f^! : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$.

(b) *For every pullback diagram of functors*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y, \end{array}$$

the commutative diagram of ∞*-categories*

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{f_*} & \mathrm{QCoh}(Y) \\ \downarrow{\scriptstyle g'^*} & & \downarrow{\scriptstyle g^*} \\ \mathrm{QCoh}(X') & \xrightarrow{f'_*} & \mathrm{QCoh}(Y') \end{array}$$

(supplied by Proposition 6.3.4.1) is right adjointable. In other words, for every quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(Y)$*, the canonical map* $g'^* f^! \mathscr{F} \to f'^! g^* \mathscr{F}$ *is an equivalence in* $\mathrm{QCoh}(X')$.

As in the proof of Proposition 6.4.1.4, we begin with an analysis of the affine case.

Lemma 6.4.2.2. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude. Assume that Y is affine. Then:*

(1) *The exceptional inverse image functor $f^! : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ commutes with small colimits.*

(2) *For every pair of objects $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathsf{Y})$, the map $\rho_{\mathscr{F},\mathscr{G}} : f^! \mathscr{F} \otimes f^* \mathscr{G} \to f^!(\mathscr{F} \otimes \mathscr{G})$ of Construction 6.4.1.7 is an equivalence in $\mathrm{QCoh}(\mathsf{X})$.*

Remark 6.4.2.3. In the statement of Lemma 6.4.2.2, the assumption that Y is affine is superfluous; see Corollary 6.4.2.7 below.

Proof of Lemma 6.4.2.2. Note that X is a quasi-compact, quasi-separated spectral algebraic space. It follows that $\mathrm{QCoh}(\mathsf{X})$ is compactly generated, and that an object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is compact if and only if it is perfect (Proposition 9.6.1.1). Using Theorem 6.1.3.2, we deduce that the direct image functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ carries compact objects to compact objects. Applying Proposition HTT.5.5.7.2 , we deduce that the exceptional inverse image functor $f^! : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ commutes with filtered colimits, and therefore with all small colimits (since it is an exact functor of stable ∞-categories); this completes the proof of (1).

To prove (2), let us regard $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ as fixed, and let $\mathcal{C} \subseteq \mathrm{QCoh}(\mathsf{Y})$ be the full subcategory spanned by those objects $\mathscr{G}$ for which the map $\rho_{\mathscr{F},\mathscr{G}}$ is an equivalence. Using (1), we see that $\mathcal{C}$ is a stable subcategory of $\mathrm{QCoh}(\mathsf{Y})$ which is closed under small colimits. Since Y is affine, to show that $\mathcal{C} = \mathrm{QCoh}(\mathsf{Y})$ it will suffice to show that $\mathcal{C}$ contains the structure sheaf $\mathscr{O}_{\mathsf{Y}}$, which follows immediately from the definition given in Construction 6.4.1.7. □

Proof of Proposition 6.4.2.1. We proceed as in the proof of Proposition 6.3.4.1. Let $f : X \to Y$ be a morphism of functors which is representable, proper, locally almost of finite presentation, and locally of finite Tor-amplitude. Write Y as the colimit of a diagram $q : S \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, where each $q(s) \simeq \mathrm{Spec}\, A_s$ is affine. Our hypothesis on f guarantees that each of the fiber products $X \times_Y \mathrm{Spec}\, A_s$ is representable by a spectral algebraic space X_s which is proper, locally almost of finite presentation, and locally of finite Tor-amplitude over A_s. Every edge $s \to s'$ in S determines a pullback diagram

$$\begin{array}{ccc} \mathsf{X}_s & \longrightarrow & \mathsf{X}_{s'} \\ \downarrow{\scriptstyle f_s} & & \downarrow{\scriptstyle f_{s'}} \\ \mathrm{Sp\acute{e}t}\, A_s & \longrightarrow & \mathrm{Sp\acute{e}t}\, A_{s'}. \end{array}$$

Using Lemma 6.4.2.2, we deduce that the associated diagram of pullback functors

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X}_{s'}) & \xrightarrow{f_{s*}} & \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A_{s'}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{X}_{s}) & \xrightarrow{f_{s*}} & \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A_{s}) \end{array}$$

is right adjointable. Since $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(\mathsf{X}_s)$ and $\mathrm{QCoh}(Y) \simeq \varprojlim \mathrm{QCoh}(\mathrm{Spec}\, A_s)$, Corollary HA.4.7.4.18 implies the following:

(i) The direct image functor $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ admits a right adjoint $f^!$.

(ii) For each $s \in S$, the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{f_{*}} & \mathrm{QCoh}(Y) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{X}_{s}) & \xrightarrow{f_{s*}} & \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A_{s}) \end{array}$$

is right adjointable.

This proves (a). Moreover, we can assume that every morphism $\mathrm{Spec}\, A \to Y$ appears as a map $q(s) \to Y$ for some $s \in S$, so that (ii) shows that (b) is satisfied whenever Y' is affine. To prove (b) in general, consider a pullback square σ :

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

and let $\mathscr{F} \in \mathrm{QCoh}(Y)$; we wish to show that the Beck-Chevalley map $\theta : g'^* f^! \mathscr{F} \to f'^! g^* \mathscr{F}$ is an equivalence in $\mathrm{QCoh}(X')$. To prove this, it will suffice to show that for every map $h : \mathrm{Spec}\, A \to Y'$, the image of θ in $\mathrm{QCoh}(X' \times_{Y'} \mathrm{Spec}\, A)$ is an equivalence. Extending σ to a rectangular diagram

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}\, A & \xrightarrow{h} & Y' & \xrightarrow{g} & Y \end{array}$$

where both squares are pullbacks, we see that $h'^*(\theta)$ fits into a commutative diagram

$$\begin{array}{ccccc} & & h'^* f'^! g^* \mathscr{F} & & \\ & \nearrow^{h'^*(\theta)} & & \searrow^{\theta'} & \\ h'^* g'^* f^! \mathscr{F} & & \xrightarrow{\theta''} & & f''^! h^* g^* \mathscr{F}, \end{array}$$

where θ' and θ'' are equivalences by virtue of the fact that (b) holds in the special case where Y' is corepresentable. $\square$

Definition 6.4.2.4. Let $f : X \to Y$ be a morphism of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which is representable, proper, locally almost of finite presentation, and locally of finite Tor-amplitude. We let $\omega_{X/Y}$ denote the quasi-coherent sheaf $f^! \mathscr{O}_Y \in \mathrm{QCoh}(X)$. We will refer to $\omega_{X/Y}$ as the *relative dualizing sheaf* of the morphism f.

Variant 6.4.2.5. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation. In this case, we let $\omega_{\mathsf{X}/\mathsf{Y}}$ denote the quasi-coherent sheaf $f^! \mathscr{O}_{\mathsf{Y}} \in \mathrm{QCoh}(\mathsf{X})$. We will refer to $\omega_{\mathsf{X}/\mathsf{Y}}$ as the *relative dualizing sheaf* of the morphism f. Note that if f is locally of finite Tor-amplitude, then this definition agrees with Definition 6.4.2.4 (where we abuse notation by identifying X and Y with the corresponding representable functors).

Remark 6.4.2.6. Suppose we are given a pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, where f is representable, proper, locally almost of finite presentation, and locally of finite Tor-amplitude. In this case, Proposition 6.4.1.4 supplies a canonical equivalence $g^* \omega_{X/Y} \simeq \omega_{X'/Y'}$.

If we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{g} & \mathsf{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g'} & \mathsf{Y}. \end{array}$$

where f is proper and locally almost of finite presentation, then there is a canonical map $\rho : g^* \omega_{\mathsf{X}/\mathsf{Y}} \to \omega_{\mathsf{X}'/\mathsf{Y}'}$. This map need not be an equivalence. However, it is an equivalence if either f or g' has finite Tor-amplitude (Propositions 6.4.2.1 and 6.4.1.4).

Corollary 6.4.2.7. *Let $f : X \to Y$ be a morphism between functors $X, Y \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which is representable, proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude. Then, for any quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(Y)$, we have a canonical equivalence $\omega_{X/Y} \otimes f^* \mathscr{F} \simeq f^! \mathscr{F}$ in the ∞-category* $\mathrm{QCoh}(X)$.

Proof. Arguing as in Construction 6.4.1.7, we have a canonical map

$$\rho : \omega_{X/Y} \otimes f^* \mathscr{F} \simeq (f^! \mathscr{O}_Y) \otimes (f^* \mathscr{F}) \to f^!(\mathscr{O}_Y \otimes \mathscr{F}) \simeq f^! \mathscr{F}.$$

We claim that this map is an equivalence. By virtue of Proposition 6.4.2.1, the assertion is local on Y. We may therefore assume that Y is affine, in which case the desired result is a special case of Lemma 6.4.2.2. □

Corollary 6.4.2.8. *[Transitivity] Let $f : X \to Y$ and $g : Y \to Z$ be morphisms between functors $X, Y, Z \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which are representable, proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude. Then there is a canonical equivalence $\omega_{X/Z} \simeq \omega_{X/Y} \otimes \omega_{Y/Z}$ in the ∞-category* $\mathrm{QCoh}(X)$.

Example 6.4.2.9. Let A be a commutative ring, let $\mathsf{Y} = \mathrm{Spét}\, A$, and let X denote the closed subspace of Y given by the vanishing locus of an ideal $I \subseteq A$. If I is generated by a regular sequence $a_1, a_2, \ldots, a_n \in A$, then Example 6.4.0.4 supplies an equivalence $\omega_{\mathsf{X}/\mathsf{Y}} \simeq \Sigma^{-n} \mathscr{O}_{\mathsf{X}}$ (which depends on the choice of regular sequence $a_1, \ldots, a_n$).

6.4.3 Preservation of Coherence

Our next goal is to establish a finiteness property enjoyed by the exceptional inverse image functor $f^!$ of Definition 6.4.0.1. First, we need to introduce a bit of terminology.

Definition 6.4.3.1. Let X be a locally Noetherian spectral Deligne-Mumford stack. We will say that an object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is *coherent* if, for every integer n, the truncation $\tau_{\geqslant n} \mathscr{F}$ is almost perfect. We let $\mathrm{Coh}(\mathsf{X})$ denote the full subcategory of $\mathrm{QCoh}(\mathsf{X})$ spanned by the coherent sheaves.

Example 6.4.3.2. Let $\mathsf{X} = \mathrm{Spét}\, A$ be an affine spectral algebraic space and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ be the quasi-coherent sheaf corresponding to the A-module $M = \Gamma(\mathsf{X}; \mathscr{F})$. Then $\mathscr{F}$ is coherent if and only if each homotopy group $\pi_n M$ is finitely generated as a module over $\pi_0 A$.

Warning 6.4.3.3. In Definition 6.4.3.1, we impose no boundedness conditions on $\mathscr{F}$.

Proposition 6.4.3.4. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a proper morphism between locally Noetherian spectral Deligne-Mumford stacks and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$. If $\mathscr{F}$ is coherent and locally truncated, then $f^! \mathscr{F}$ is coherent and locally truncated.*

The proof of Proposition 6.4.3.4 will require some preliminaries.

Lemma 6.4.3.5. *Let A be a Noetherian $\mathbb{E}_\infty$-ring, let M be an almost perfect A-module, and let N be a truncated A-module whose homotopy groups are finitely generated over $\pi_0 A$. Then the groups $\mathrm{Ext}^i_A(M, N)$ are finitely generated modules over $\pi_0 A$.*

Proof. Replacing M and N by suitable suspensions, we can assume that N is 0-truncated and that $i = 0$. Choose an integer m such that M is m-connective. We proceed by descending induction on m. Note that if $m > 0$, then $\mathrm{Ext}^0_A(M, N) \simeq 0$ and there is nothing to prove. Otherwise, the assumption that M is almost perfect guarantees the existence of a fiber sequence $\Sigma^m A^k \to M \to M'$, where M' is $(m+1)$-connective. We have an exact sequence of $\pi_0 A$-modules $\mathrm{Ext}^0_A(M', N) \to \mathrm{Ext}^0_A(M, N) \to \mathrm{Ext}^0_A(\Sigma^m A^k, N)$. The inductive hypothesis implies that $\mathrm{Ext}^0_A(M', N)$ is finitely generated over $\pi_0 A$, and we have a canonical isomorphism $\mathrm{Ext}^0_A(\Sigma^m A^k, N) \simeq (\pi_m N)^k$. It follows that $\mathrm{Ext}^0_A(M, N)$ is finitely generated over $\pi_0 A$. □

Corollary 6.4.3.6. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a finite morphism of locally Noetherian spectral Deligne-Mumford stacks. Then the functor $f^! : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ carries locally truncated coherent objects of $\mathrm{QCoh}(\mathsf{Y})$ to locally truncated coherent objects of $\mathrm{QCoh}(\mathsf{X})$.*

Proof. By virtue of Proposition 6.4.1.4, the assertion is local on Y; we may therefore assume without loss of generality that $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, A$ is affine. In this case, the desired result follows immediately from Lemma 6.4.3.5. □

Lemma 6.4.3.7. *Let $f : A \to B$ be a map of Noetherian $\mathbb{E}_\infty$-rings, Suppose that the induced map $\pi_0 A \to \pi_0 B$ is a surjection of commutative rings whose kernel $I \subseteq \pi_0 A$ is nilpotent. Let K be a truncated A-module, and suppose that the homotopy groups $\pi_i \underline{\mathrm{Map}}_A(B, K)$ are finitely generated modules over $\pi_0 B$. Then the homotopy groups $\pi_i K$ are finitely generated modules over $\pi_0 A$.*

Proof. We may assume without loss of generality that K is 0-truncated. We prove that the homotopy groups $\pi_{-n} K$ are finitely generated over $\pi_0 A$ using induction on n, the case $n < 0$ being trivial. For each integer $k \geqslant 1$, let $M(k)$ denote the submodule of $\pi_0 K$ consisting of elements which are annihilated by I^k. Since $\pi_0 A$ is Noetherian, there exists a finite set of generators $x_1, \ldots, x_n$ for the ideal I. Multiplication by the elements x_i determines a map $M(k) \to M(k-1)^n$, which fits into an exact sequence $0 \to M(1) \to M(k) \to M(k-1)^n$ Note that $M(1) \simeq \pi_0 \underline{\mathrm{Map}}_A(B, K)$ is finitely generated over $\pi_0 A$. It follows by induction on k that each $M(k)$ is finitely generated over $\pi_0 A$. Since the ideal I is nilpotent, we have $M(k) \simeq \pi_0 K$ for $k \gg 0$, so that $\pi_0 K$ is finitely generated over $\pi_0 A$. This completes the proof when $n = 0$. If $n > 0$, we apply Lemma 6.4.3.5 to deduce that the homotopy groups of $\underline{\mathrm{Map}}_A(B, \pi_0 K)$ are finitely generated over $\pi_0 A$, and therefore finitely generated over $\pi_0 B$. Let $K' = \Sigma(\tau_{\leqslant -1} K)$, so that we have a fiber sequence of A-modules $\pi_0 K \to K \to \Sigma^{-1} K'$. It follows that the homotopy groups of $\pi_0 \underline{\mathrm{Map}}_A(B, K')$ are finitely generated over $\pi_0 B$. Applying the inductive hypothesis, we deduce that $\pi_{-n} K \simeq \pi_{1-n} K'$ is finitely generated over $\pi_0 A$, as desired. □

Corollary 6.4.3.8. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a locally Noetherian spectral Deligne-Mumford stack, let $\mathsf{X}_0 = (\mathcal{X}, \pi_0 \mathscr{O}_{\mathsf{X}})$ be the 0-truncation of X, and let $i : \mathsf{X}_0 \to \mathsf{X}$ denote the associated* *closed immersion. If* $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ *is locally truncated and* $i^! \mathscr{F} \in \mathrm{QCoh}(\mathsf{X}_0)$ *is coherent, then* $\mathscr{F}$ *is coherent.*

Proof. Using Proposition 6.4.1.4, we can reduce to the case where X is affine, in which case the desired result follows from Lemma 6.4.3.7. □

Proof of Proposition 6.4.3.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be a proper morphism of locally Noetherian spectral Deligne-Mumford stacks and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ be truncated and coherent. Then $f^! \mathscr{F}$ is locally truncated by virtue of Proposition 6.4.1.4; we wish to show that it is also coherent. Let X_0 and Y_0 denote the 0-truncations of X and Y, respectively, so that we have a commutative diagram

$$\begin{array}{ccc} \mathsf{X}_0 & \xrightarrow{i} & \mathsf{X} \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}_0 & \xrightarrow{i'} & \mathsf{Y} \end{array}$$

where the horizontal maps are closed immersions. To show that $f^! \mathscr{F}$ is locally truncated, it will suffice (by virtue of Corollary 6.4.3.8) to show that $i^! f^! \mathscr{F} \simeq f_0^! i'^! \mathscr{F}$ is coherent. It follows from Corollary 6.4.3.6 that $i'^! \mathscr{F} \in \mathrm{QCoh}(\mathsf{Y}_0)$ is truncated and coherent. We may therefore replace f by f_0 and thereby reduce to the case where X and Y are 0-truncated.

Using Proposition 6.4.1.4, we see that the assertion that $f^! \mathscr{F}$ is coherent can be tested locally with respect to the étale topology of Y. We may therefore assume without loss of generality that $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, A$ for some Noetherian commutative ring A. Choose an étale surjection $u : \mathsf{U} \to \mathsf{X}$ where $\mathsf{U} \simeq \mathrm{Sp\acute{e}t}\, B$ is affine; we will show that $u^* f^! \mathscr{F}$ is coherent. Since the morphism f is of finite type, the commutative ring B is finitely generated as an A-algebra. We can therefore choose a surjection of A-algebras $\phi : A[x_1, \ldots, x_n] \to B$. Set $\mathsf{Y}' = \mathrm{Sp\acute{e}t}\, A[x_1, \ldots, x_n]$ and form a pullback square

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{g'} & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y}. \end{array}$$

Amalgamating u and ϕ, we obtain a map $\overline{u} : \mathsf{U} \to \mathsf{X}'$. By construction, $f' \circ \overline{u}$ is a closed immersion. Since f' is separated, it follows that $\overline{u}$ is a closed immersion.

Locally on X', the morphism $\overline{u}$ exhibits U as the closed substack of X' given by the vanishing locus of a regular sequence (of length n). It follows that the morphism $\overline{u}$ has finite Tor-amplitude. Moreover, Example 6.4.2.9 (and Proposition 6.4.1.4) imply that relative dualizing sheaf $\omega_{\mathsf{U}/\mathsf{X}'}$ has the form $\Sigma^{-n} \mathscr{L}$, where $\mathscr{L}$ is a line bundle on U. Consequently, to show that $u^* f^! \mathscr{F}$ is coherent, it will suffice to show that the tensor product $\omega_{\mathsf{U}/\mathsf{X}'} \otimes u^* f^! \mathscr{F}$

is coherent Using Corollary 6.4.2.7 and Proposition 6.4.1.4, we obtain equivalences

$$\begin{aligned} \omega_{\mathsf{U}/\mathsf{X}'} \otimes u^* f^! \mathscr{F} &\simeq \omega_{\mathsf{U}/\mathsf{X}'} \otimes \overline{u}^* g'^* f^! \mathscr{F} \\ &\simeq \overline{u}^! g'^* f^! \mathscr{F} \\ &\simeq \overline{u}^! f'^! g^* \mathscr{F}. \end{aligned}$$

We may therefore replace Y by Y', $\mathscr{F}$ by $g^* \mathscr{F}$, and f by the map $(f' \circ \overline{u}) : \mathsf{U} \to \mathsf{Y}'$ determined by ϕ, and thereby reduce to proving Proposition 6.4.3.4 in the special case where f is a closed immersion. In this case, the desired result follows from Corollary 6.4.3.6. □

6.4.4 Finiteness Properties of $\omega_{X/Y}$

We now apply Proposition 6.4.3.4 to study the finiteness properties of relative dualizing sheaves. Our goal is to prove the following:

Proposition 6.4.4.1. *Let $f : X \to Y$ be a morphism of functors which is representable, proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude. Then the relative dualizing sheaf $\omega_{X/Y} \in \mathrm{QCoh}(X)$ is almost perfect.*

The proof of Proposition 6.4.4.1 will require some preliminaries.

Lemma 6.4.4.2. *Let X be a spectral algebraic space which is quasi-compact and separated, let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, and let n be an integer. Suppose that, for every perfect object $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$ having* Tor*-amplitude $\leqslant 0$, the groups $\mathrm{Ext}^i_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{G}, \mathscr{F})$ vanish for $i > n$. Then $\mathscr{F}$ is $(-n)$-connective.*

Proof. Let $f : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$ be an étale morphism; we wish to show that $\Gamma(\mathrm{Sp\acute{e}t}\, A; f^* \mathscr{F}) \in \mathrm{Mod}_A$ is $(-n)$-connective. Note that we have canonical equivalences $\Gamma(\mathrm{Sp\acute{e}t}\, A; f^* \mathscr{F}) \simeq \Gamma(\mathsf{X}; f_* f^* \mathscr{F}) \simeq \Gamma(\mathsf{X}; (f_* \mathscr{O}_{\mathrm{Sp\acute{e}t}\, A}) \otimes \mathscr{F})$. Since f is affine, the direct image $f_* \mathscr{O}_{\mathrm{Sp\acute{e}t}\, A} \in \mathrm{QCoh}(\mathsf{X})$ is connective. Using Proposition 9.6.1.2, we can write $f_* \mathscr{O}_{\mathrm{Sp\acute{e}t}\, A}$ as the colimit of a filtered diagram $\{\mathscr{G}_\alpha\}$, where each $\mathscr{G}_\alpha \in \mathrm{QCoh}(\mathsf{X})$ is connective and almost perfect. Since the global sections functor $\Gamma(\mathsf{X}; \bullet)$ commutes with filtered colimits, we are reduced to showing that each of the spectra $\Gamma(\mathsf{X}; \mathscr{G}_\alpha \otimes \mathscr{F})$ is $(-n)$-connective: in other words, that the homotopy groups

$$\pi_{-i} \Gamma(\mathsf{X}; \mathscr{G}_\alpha \otimes \mathscr{F}) \simeq \mathrm{Ext}^i_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{G}_\alpha^\vee, \mathscr{F})$$

vanish for $i > n$. This follows from our hypothesis on $\mathscr{F}$, since each $\mathscr{G}_\alpha^\vee$ has Tor-amplitude $\leqslant 0$. □

Lemma 6.4.4.3. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper, locally almost of finite presentation, and of* Tor*-amplitude $\leqslant n$. Then the relative dualizing sheaf $\omega_{\mathsf{X}/\mathsf{Y}}$ is $(-n)$-connective.*

Proof. By virtue of Proposition 6.4.2.1, the assertion is local on Y; we may therefore assume without loss of generality that $\mathsf{Y} \simeq \operatorname{Sp\acute{e}t} A$ is affine. Using Lemma 6.4.4.2, we are reduced to showing that if $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$ is perfect of Tor-amplitude $\leqslant 0$, then the groups $\operatorname{Ext}^i_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{G}, \omega_{\mathsf{X}/\mathsf{Y}})$ vanish for $i > n$. Set $N = \Gamma(\mathsf{X}; \mathscr{G})$, so that N is a perfect A-module (Theorem 6.1.3.2) and we have a canonical isomorphism $\operatorname{Ext}^i_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{G}, \omega_{\mathsf{X}/\mathsf{Y}}) \simeq \operatorname{Ext}^i_A(N, A) \simeq \pi_{-i} N^\vee$. It will therefore suffice to show that $N^\vee$ is $(-n)$-connective, or equivalently that N has Tor-amplitude $\leqslant n$ as an A-module. Let $M \in \mathrm{Mod}^\heartsuit_A$, and let $\mathscr{F} = f^* M \in \mathrm{QCoh}(\mathsf{X})$ be the associated quasi-coherent sheaf on X. Then we have a canonical equivalence $M \otimes_A N \simeq \Gamma(\mathsf{X}; \mathscr{F} \otimes \mathscr{G})$. Since f has Tor-amplitude $\leqslant n$, the quasi-coherent sheaf $\mathscr{F}$ is n-truncated. Since $\mathscr{G}$ has Tor-amplitude $\leqslant 0$, the tensor product $\mathscr{F} \otimes \mathscr{G}$ is n-truncated, so that the A-module $M \otimes_A N \simeq \Gamma(\mathsf{X}; \mathscr{F} \otimes \mathscr{G}$ is also n-truncated. Allowing M to vary, we conclude that N has Tor-amplitude $\leqslant n$ as desired. □

Proof of Proposition 6.4.4.1. Assume that $f : X \to Y$ is representable proper, locally almost of finite presentation, and locally of finite Tor-amplitude; we wish to show that $\omega_{X/Y}$ is almost perfect. By virtue of Remark 6.4.2.6, the assertion is local on Y; we may therefore assume without loss of generality that $Y \simeq \operatorname{Spec} A$ is affine. Then X is representable by a spectral algebraic space X which is proper over A. In particular, X is quasi-compact, so there exists an integer n for which f has Tor-amplitude $\leqslant n$. Using Lemma 6.4.4.3, we deduce that $\omega_{X/Y}$ is $(-n)$-connective. Set $\mathsf{X}_0 = \mathsf{X} \times_{\operatorname{Sp\acute{e}t} A} \operatorname{Sp\acute{e}t}(\pi_0 A)$, and let $i : \mathsf{X}_0 \to \mathsf{X}$ be the evident closed immersion. To complete the proof that $\omega_{X/Y}$ is almost perfect, it will suffice to show that $i^* \omega_{X/Y}$ is almost perfect (Proposition 2.7.3.2). Using Proposition 6.4.2.1, we can replace f by the projection map $\mathsf{X}_0 \to \operatorname{Sp\acute{e}t}(\pi_0 A)$ and thereby reduce to the case where A is discrete.

Write A as the colimit of a filtered diagram $\{A_\alpha\}$, where each A_α is a finitely generated commutative ring. Using Theorem 4.4.2.2, we deduce that there exists an index α, a finitely n-presented morphism $f_\alpha : \mathsf{X}_\alpha \to \operatorname{Sp\acute{e}t} A_\alpha$, and an identification of X with the n-truncation of $\mathsf{X}_\alpha \times_{\operatorname{Sp\acute{e}t} A_\alpha} \operatorname{Sp\acute{e}t} A$. Enlarging α if necessary, we can assume that f_α is proper (Proposition 5.5.4.1) and of Tor-amplitude $\leqslant n$ (Proposition 6.1.6.1), so that f is a pullback of f_α. Applying Remark 6.4.2.6 again, we can replace f by f_α and thereby reduce to the case where A is Noetherian. In this case, Corollary 6.4.3.6 implies that $\omega_{X/Y} \in \mathrm{QCoh}(X)$ is coherent. In particular, the truncation $\tau_{\geqslant -n} \omega_{X/Y}$ is almost perfect. Since $\omega_{X/Y}$ is $(-n)$-connective (Lemma 6.4.4.2), we conclude that $\omega_{X/Y}$ is almost perfect. □

Warning 6.4.4.4. In the situation of Proposition 6.4.4.1, the relative dualizing sheaf $\omega_{X/Y}$ need not be perfect: in other words, it need not be of finite Tor-amplitude over X. However, it is always locally of finite Tor-amplitude over Y. In other words, for every commutative

diagram

$$\begin{array}{ccc} \operatorname{Spec} B & \xrightarrow{g} & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} A & \xrightarrow{g'} & Y \end{array}$$

for which the induced map $\operatorname{Spec} B \to \operatorname{Spec} A \times_Y X$ is étale, the pullback $g^* \omega_{X/Y} \in \operatorname{QCoh}(\operatorname{Spec} B) \simeq \operatorname{Mod}_B$ has finite Tor-amplitude when regarded as an A-module. To prove this, we may replace Y by $\operatorname{Spec} A$ and thereby reduce to the case where g' is an equivalence. Then there exists an integer n for which the exceptional inverse image functor $f^! : \operatorname{Mod}_A \simeq \operatorname{QCoh}(Y) \to \operatorname{QCoh}(X)$ carries $(\operatorname{Mod}_A)_{\leqslant 0}$ into $\operatorname{QCoh}(X)_{\leqslant n}$ (see Remark 6.4.1.3). Using Corollary 6.4.2.7, we deduce that $\omega_{X/Y} \otimes f^* M \simeq f^! M$ is n-truncated for every discrete A-module M, so that $\omega_{X/Y}$ has Tor-amplitude $\leqslant n$ over A.

6.4.5 The Functor f_+

Let $f : X \to Y$ be a morphism of functors $X, Y : \operatorname{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ which is representable, proper, locally almost of finite presentation, and locally of finite Tor-amplitude. Let $\omega_{X/Y}$ denote the relative dualizing sheaf of f. According to Corollary 6.4.2.7, the construction

$$(\mathscr{F} \in \operatorname{QCoh}(Y)) \mapsto (\omega_{\mathsf{X}/\mathsf{Y}} \otimes f^* \mathscr{F} \in \operatorname{QCoh}(X))$$

determines a right adjoint to the direct image functor f_*. In this section, we will discuss a closely related (but formally dual) feature of the sheaf $\omega_{X/Y}$: it can be used to construct a *left* adjoint to the pullback functor f^*.

Construction 6.4.5.1. Let $f : X \to Y$ be a morphism of functors $X, Y : \operatorname{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ which is representable, proper, locally almost of finite presentation, and locally of finite Tor-amplitude. We let $f_+ : \operatorname{QCoh}(X) \to \operatorname{QCoh}(Y)$ denote the functor given by the formula $f_+ \mathscr{F} = f_*(\omega_{X/Y} \otimes \mathscr{F})$. Let $v_0 : f_* \omega_{X/Y} = f_* f^! \mathscr{O}_Y \to \mathscr{O}_Y$ denote the counit for the adjunction between f_* and $f^!$. For each object $\mathscr{F} \in \operatorname{QCoh}(Y)$, v_0 induces a map

$$f_+ f^* \mathscr{F} = f_*(\omega_{X/Y} \otimes f^* \mathscr{F}) \simeq (f_* \omega_{X/Y}) \otimes \mathscr{F} \xrightarrow{\mathrm{id} \otimes v_0} \mathscr{F}.$$

This construction depends functorially on $\mathscr{F}$, and therefore determines a natural transformation of functors $f_+ f^* \to \mathrm{id}_{\operatorname{QCoh}(Y)}$.

Remark 6.4.5.2. Let $f : X \to Y$ be as in Construction 6.4.5.1. Then:

- (a) If $\mathscr{F} \in \operatorname{QCoh}(\mathsf{X})$ is locally of finite Tor-amplitude, then $f_+ \mathscr{F} \in \operatorname{QCoh}(Y)$ is locally of finite Tor-amplitude.

- (b) If $\mathscr{F} \in \operatorname{QCoh}(\mathsf{X})$ is almost perfect, then $f_+ \mathscr{F} \in \operatorname{QCoh}(Y)$ is almost perfect.

(c) If $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is perfect, then $f_+ \mathscr{F} \in \mathrm{QCoh}(Y)$ is perfect.

Assertion (a) follows from Proposition 6.4.4.1 and Theorem 5.6.0.2, assertion (b) follows from Warning 6.4.4.4, and assertion (c) is an immediate consequence of (a) and (b).

Proposition 6.4.5.3. *Let $f : X \to Y$ as in Construction 6.4.5.1. Then the natural transformation $v : f_+ f^* \to \mathrm{id}_{\mathrm{QCoh}(Y)}$ of Construction 6.4.5.1 exhibits the functor f_+ as a left adjoint of the functor f^*.*

Proof. Let $\mathscr{F} \in \mathrm{QCoh}(X)$ and $\mathscr{G} \in \mathrm{QCoh}(Y)$; we wish to show that the composite map

$$\rho_{\mathscr{F},\mathscr{G}} : \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{F}, f^* \mathscr{G}) \to \mathrm{Map}_{\mathrm{QCoh}(Y)}(f_+ \mathscr{F}, f_+ f^* \mathscr{G}) \xrightarrow{v} \mathrm{Map}_{\mathrm{QCoh}(Y)}(f_+ \mathscr{F}, \mathscr{G})$$

is a homotopy equivalence. Writing Y as a colimit of representable functors, we can reduce to the case where $Y = \mathrm{Spec}\, A$ is affine. In this case, X is representable by a spectral algebraic space X which is quasi-compact and quasi-separated. Applying Proposition 9.6.1.1, we can write $\mathscr{F}$ as the colimit of a filtered diagram $\{\mathscr{F}_\alpha\}$, where each $\mathscr{F}_\alpha \in \mathrm{QCoh}(\mathsf{X})$ is perfect (Proposition 9.6.1.1). It follows that $\rho_{\mathscr{F},\mathscr{G}}$ can be identified with the limit of the diagram $\{\rho_{\mathscr{F}_\alpha,\mathscr{G}}\}$. It will therefore suffice to show that each $\rho_{\mathscr{F}_\alpha,\mathscr{G}}$ is a homotopy equivalence. Replacing $\mathscr{F}$ by $\mathscr{F}_\alpha$, we may reduce to the case where $\mathscr{F}$ is perfect.

The assumption that $\mathscr{F}$ is perfect guarantees that $f_+ \mathscr{F}$ is also perfect (Remark 6.4.5.2). In this case, we can write $\rho_{\mathscr{F},\mathscr{G}} = \Omega^\infty \overline{\rho_{\mathscr{F},\mathscr{G}}}$, where $\overline{\rho}_{\mathscr{F},\mathscr{G}}$ denotes the map of spectra given by the composition

$$\Gamma(\mathsf{X}; \mathscr{F}^\vee \otimes f^* \mathscr{G}) \xrightarrow{\Gamma} (\mathsf{Y}; (f_+ \mathscr{F})^\vee \otimes (f_+ f^* \mathscr{G})) \xrightarrow{v} \Gamma(\mathrm{Sp\acute{e}t}\, A; (f_+ \mathscr{F})^\vee \otimes \mathscr{G}).$$

Let us regard $\mathscr{F}$ as fixed, and let $\mathcal{C}$ denote the full subcategory of $\mathrm{QCoh}(Y)$ spanned by those objects $\mathscr{G}$ for which the map $\overline{\rho}_{\mathscr{F},\mathscr{G}}$ is an equivalence. Then $\mathcal{C}$ is a stable subcategory of $\mathrm{QCoh}(Y)$. Our assumption that $\mathscr{F}$ is perfect guarantees that $f_+ \mathscr{F}$ is also perfect (Remark 6.4.5.2), so that the ∞-category $\mathcal{C}$ is closed under filtered colimits. Consequently, to show that $\mathcal{C} = \mathrm{QCoh}(Y)$, it will suffice to show that $\mathcal{C}$ contains the structure sheaf $\mathscr{O}_Y$ (by virtue of our assumption that Y is affine). We may therefore assume without loss of generality that $\mathscr{G} = \mathscr{O}_Y$. In this case, $\overline{\rho}_{\mathscr{F},\mathscr{G}}$ is a morphism of perfect A-modules, whose dual can be identified with a morphism $\psi : f_+ \mathscr{F} \to (f_* \mathscr{F}^\vee)^\vee$ in the ∞-category $\mathrm{QCoh}(Y)$. To complete the proof, it will suffice to show that for every object $\mathscr{H} \in \mathrm{QCoh}(Y)$, composition with ψ induces a homotopy equivalence $\psi_{\mathscr{H}} : \mathrm{Map}_{\mathrm{QCoh}(Y)}(\mathscr{H}, f_+ \mathscr{F}) \to \mathrm{Map}_{\mathrm{QCoh}(Y)}(\mathscr{H}, (f_* \mathscr{F}^\vee)^\vee)$. We now complete the proof by observing that $\psi_{\mathscr{H}}$ can be identified with the composition of homotopy equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{QCoh}(Y)}(\mathscr{H}, f_+ \mathscr{F}) &= \mathrm{Map}_{\mathrm{QCoh}(Y)}(\mathscr{H}, f_*(\omega_{X/Y} \otimes \mathscr{F})) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(f^* \mathscr{H}, \omega_{X/Y} \otimes \mathscr{F}) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(X)}((f^* \mathscr{H}) \otimes \mathscr{F}^\vee, \omega_{X/Y}) \\
&= \mathrm{Map}_{\mathrm{QCoh}(X)}((f^* \mathscr{H}) \otimes \mathscr{F}^\vee, f^! \mathscr{O}_Y) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(Y)}(f_*(f^* \mathscr{H} \otimes \mathscr{F}^\vee), \mathscr{O}_Y) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(Y)}(\mathscr{H} \otimes f_* \mathscr{F}^\vee, \mathscr{O}_Y) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(Y)}(\mathscr{H}, (f_* \mathscr{F}^\vee)^\vee).
\end{aligned}$$

□

We now study the behavior of the functor f_+ with respect to base change. With an eye toward future applications, we work in a slightly different setting:

Proposition 6.4.5.4. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a map between quasi-compact, quasi-separated spectral algebraic spaces.*

(1) *Suppose that the pushforward functor f_* carries perfect objects of $\mathrm{QCoh}(\mathsf{X})$ to perfect objects of $\mathrm{QCoh}(\mathsf{Y})$. Then the pullback functor $f^* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ admits a left adjoint $f_+ : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$, whose restriction to $\mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}$ is given by $f_+(\mathscr{F}) = (f_* \mathscr{F}^\vee)^\vee$.*

(2) *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc}
\mathsf{X}' & \xrightarrow{g'} & \mathsf{X} \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
\mathsf{Y}' & \xrightarrow{g} & \mathsf{Y},
\end{array}$$

where f and f' satisfy the assumptions of (1). Then the diagram of ∞-categories

$$\begin{array}{ccc}
\mathrm{QCoh}(\mathsf{Y}) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathsf{X}) \\
\downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\
\mathrm{QCoh}(\mathsf{Y}') & \xrightarrow{f'^*} & \mathrm{QCoh}(\mathsf{X}')
\end{array}$$

is left adjointable. In other words, the canonical natural transformation $f'_+ \circ g'^* \to g^* \circ f_+$ is an equivalence of functors from $\mathrm{QCoh}(\mathsf{X})$ to $\mathrm{QCoh}(\mathsf{Y}')$.

Remark 6.4.5.5. The hypotheses of Proposition 6.4.5.4 are satisfied when f is proper, locally almost of finite presentation, and locally of finite Tor-amplitude (Theorem 6.1.3.2); in this case, Proposition 6.4.5.3 guarantees that the functor f_+ appearing in Proposition 6.4.5.4 agrees with the functor appearing in Construction 6.4.5.1. We will later see that the converse holds under some mild additional hypotheses (see Theorem 11.1.4.1); our proof of this will make use of Proposition 6.4.5.4.

Proof of Proposition 6.4.5.4. We first prove (1). The collection of those objects $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ for which the functor $\mathscr{G} \mapsto \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, f^* \mathscr{G})$ is corepresentable by an object of $\mathrm{QCoh}(\mathsf{Y})$ is closed under small colimits in $\mathrm{QCoh}(\mathsf{X})$. Consequently, to verify the existence of f_+, it will suffice to show prove the corepresentability in the case where $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is perfect. In this case, we have canonical homotopy equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, f^* \mathscr{G}) &\simeq \Omega^\infty \Gamma(\mathsf{X}; \mathscr{F}^\vee \otimes f^* \mathscr{G}) \\ &\simeq \Omega^\infty \Gamma(\mathsf{Y}; f_*(\mathscr{F}^\vee \otimes f^* \mathscr{G})) \\ &\simeq \Omega^\infty \Gamma(\mathsf{Y}; (f_* \mathscr{F}^\vee) \otimes \mathscr{G}) \\ &\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{Y})}((f_* \mathscr{F}^\vee)^\vee, \mathscr{G}). \end{aligned}$$

We now prove (2). Since the functors f_+, f'_+, g'^*, and g^* preserve filtered colimits, it suffices to show that the base change map $f'_+ g'^* \mathscr{F} \to g^* f_+ \mathscr{F}$ is an equivalence when $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is perfect, in which case the desired result follows immediately from the description of f_+ and f'_+ on compact objects. □

Remark 6.4.5.6. Let R be a connective $\mathbb{E}_\infty$-ring, let $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ exhibit X as a quasi-compact quasi-separated spectral algebraic space over R, and suppose that the pushforward functor f_* carries $\mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}$ into $\mathrm{Mod}_R^{\mathrm{perf}}$. Regard $\mathrm{QCoh}(\mathsf{X})$ as tensored over the ∞-category Mod_R. For every object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ and every R-module M, we have a canonical map

$$\gamma_M : f_+(M \otimes_R \mathscr{F}) \to f_+(M \otimes_R (f^* f_+ \mathscr{F})) \simeq f_+ f^*(M \otimes_R f_+ \mathscr{F}) \to M \otimes_R f_+ \mathscr{F}.$$

The map γ_M is clearly an equivalence when $M = R$. As functors of M, both sides are exact and preserve small colimits. It follows that γ_M is an equivalence for all $M \in \mathrm{Mod}_R$.

Remark 6.4.5.7. Suppose we are given a pullback diagram

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{g'} & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Sp\acute{e}t}\, R' & \xrightarrow{g} & \mathrm{Sp\acute{e}t}\, R, \end{array}$$

where the pushforward functors preserve perfect objects. Proposition 6.4.5.4 supplies an equivalence $g^* f_+ \simeq f'_+ g'^*$, which is adjoint to a natural transformation $\alpha : f_+ g'_* \to g_* f'_+$. We claim that the natural transformation α is also an equivalence. To prove this, we first observe that the collection of objects $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}')$ such that α induces an equivalence $\alpha_{\mathscr{F}} : f_+ g'_* \mathscr{F} \to g_* f'_+ \mathscr{F}$ is closed under small colimits in $\mathrm{QCoh}(\mathsf{X}')$. Since $\mathrm{QCoh}(\mathsf{X}') \simeq \mathrm{Mod}_{R'}(\mathrm{QCoh}(\mathsf{X}))$ is generated under small colimits by the essential image of the pullback functor g'^*, it will suffice to show that $\alpha_{\mathscr{F}}$ is an equivalence when $\mathscr{F} = g'^* \mathscr{G}$, for some $\mathscr{G} \in \mathsf{X}$. In this case, we can use Proposition 6.4.5.4 to identify $\alpha_{\mathscr{F}}$ with the canonical map

$$f_+(R' \otimes_R \mathscr{F}) \simeq f_+(g'_* g'^* \mathscr{F}) \to g_* g^* f_+ \mathscr{F} \simeq R' \otimes_R f_+ \mathscr{F},$$

appearing in Remark 6.4.5.6.

6.5 Digression: Injective Dimension of Quasi-Coherent Sheaves

Our goal in this section is to review some general facts about injective dimension of quasi-coherent sheaves on Noetherian spectral algebraic spaces which will be needed for our discussion of absolute dualizing sheaves in §6.6.

6.5.1 Injective Dimension

We begin by introducing some terminology.

Definition 6.5.1.1. Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ which is right complete and compatible with filtered colimits. Let n be an integer. We will say that an object $C \in \mathcal{C}$ has *injective dimension* $\leqslant n$ if, for every object $D \in \mathcal{C}^{\heartsuit}$, the abelian groups $\mathrm{Ext}^i_{\mathcal{C}}(D, C)$ vanish for $i > n$.

Example 6.5.1.2. Let $\mathcal{C}$ be as in Definition 6.5.1.1. Then an object $C \in \mathcal{C}$ is injective (in the sense of Definition C.5.7.2) if and only if it belongs to $\mathcal{C}_{\leqslant 0}$ and has injective dimension $\leqslant 0$.

Remark 6.5.1.3. Let $\mathcal{C}$ be as in Definition 6.5.1.1. Then an object $C \in \mathcal{C}$ has injective dimension $\leqslant n$ if and only if the suspension $\Sigma^k C$ has injective dimension $\leqslant n - k$.

Remark 6.5.1.4. Let $\mathcal{C}$ be as in Definition 6.5.1.1 and suppose we are given a fiber sequence $C' \to C \to C''$ in $\mathcal{C}$. Using the exactness of the sequence $\mathrm{Ext}^i_{\mathcal{C}}(\bullet, C') \to \mathrm{Ext}^i_{\mathcal{C}}(\bullet, C) \to \mathrm{Ext}^i_{\mathcal{C}}(\bullet, C'')$, we deduce the following:

- If C' and C'' have injective dimension $\leqslant n$, then C has injective dimension $\leqslant n$.

Combining this observation with Remark 6.5.1.3, we also have:

- If C has injective dimension $\leqslant n$ and C'' has injective dimension $\leqslant n-1$, then C' has injective dimension $\leqslant n$.
- If C has injective dimension $\leqslant n$ and C' has injective dimension $\leqslant n+1$, then C'' has injective dimension $\leqslant n$.

Remark 6.5.1.5. Let $\mathcal{C}$ be as in Definition 6.5.1.1 and let C be an object of $\mathcal{C}$. Then C has injective dimension $\leqslant n$ if and only if the following condition is satisfied:

$(*)$ For every object $D \in \mathcal{C}_{\leqslant m}$, the groups $\mathrm{Ext}^i_{\mathcal{C}}(D, C)$ vanish for $i > n + m$.

The "if" direction is obvious. To prove the converse, we may assume (replacing C by a suspension if necessary) that $i \leqslant 0$. Assume that C has injective dimension $\leqslant n$ and that $D \in \mathcal{C}_{\leqslant m}$; we will show that the mapping pace $\mathrm{Map}_{\mathcal{C}}(D, C)$ is $(-m-n)$-connective. Since the t-structure on $\mathcal{C}$ is right complete, we can identify $\mathrm{Map}_{\mathcal{C}}(D, C)$ with the limit of a tower of spaces $\{\mathrm{Map}_{\mathcal{C}}(\tau_{\geqslant k} D, C)\}_{k \in \mathbf{Z}}$. Note that when $k > m$, the truncation $\tau_{\geqslant k} D$ vanishes, so the space $\mathrm{Map}_{\mathcal{C}}(\tau_{\geqslant k} D, C)$ is contractible. It will therefore suffice to show that for $k \leqslant m$, the transition map $\rho : \mathrm{Map}_{\mathcal{C}}(\tau_{\geqslant k} D, C) \to \mathrm{Map}_{\mathcal{C}}(\tau_{\geqslant k+1} D, C)$ has $(-m-n)$-connective homotopy fibers. This is clear, since each nonempty homotopy fiber of ρ is equivalent to $\mathrm{Map}_{\mathcal{C}}(\Sigma^k(\pi_k D), C)$, whose sth homotopy group is given by $\mathrm{Ext}^{-s-k}_{\mathcal{C}}(\pi_{-k} D, C)$ and therefore vanishes for $s < -m-n \leqslant -k-n$ (by virtue of our assumption that C has injective dimension $\leqslant n$).

Example 6.5.1.6. Let $\mathcal{C}$ be as in Definition 6.5.1.1, let $C \in \mathcal{C}$ be an object, and suppose we are given a morphism $u : C \to Q$ where $Q \in \mathcal{C}$ is injective. Using Remark 6.5.1.4, we deduce that if $n > 0$, then C has injective dimension $\leqslant n$ if and only if $\mathrm{cofib}(u)$ has injective dimension $\leqslant n-1$.

6.5.2 The Case of $\mathrm{QCoh}(\mathsf{X})$

We will be primarily interested in studying the notion of injective dimension for truncated objects $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, where X is a spectral algebraic space. To ensure that this notion is well-behaved, we will need some mild assumptions.

Definition 6.5.2.1. Let X be a spectral algebraic space. We will say that X is *Noetherian* if it is locally Noetherian (Definition 2.8.1.4), quasi-compact, and quasi-separated.

Proposition 6.5.2.2. *Let $f : \mathsf{X} \to \mathsf{Y}$ be an étale morphism between Noetherian spectral algebraic spaces and let $\mathscr{F}$ be a truncated object of* $\mathrm{QCoh}(\mathsf{Y})$*. Then:*

(a) If $\mathscr{F}$ has injective dimension $\leqslant n$, then $f^* \mathscr{F}$ has injective dimension $\leqslant n$.

(b) *If f is surjective and $f^* \mathscr{F}$ has injective dimension $\leqslant n$, then $\mathscr{F}$ has injective dimension $\leqslant n$.*

Proof. Replacing $\mathscr{F}$ by a suitable suspension $\Sigma^k(\mathscr{F})$ (and replacing n by $n-k$), we may assume that $\mathscr{F}$ belongs to $\mathrm{QCoh}(\mathsf{Y})_{\leqslant 0}$. Note that if $n < 0$, then $\mathscr{F}$ has injective dimension $\leqslant n$ if and only if $\mathscr{F} \simeq 0$ (and similarly for $f^* \mathscr{F}$), so that assertions (a) and (b) are obvious. The proof in general proceeds by induction on n. In the case $n = 0$, the desired result follows from Theorem 10.5.3.4, Remark 10.5.3.3, and Example 6.5.1.2. Suppose that $n > 0$, and choose a morphism $u : \mathscr{F} \to \mathscr{G}$ in $\mathrm{QCoh}(\mathsf{Y})$ which exhibits $\mathscr{G}$ as an injective hull of $\mathscr{F}$ (see Example C.5.7.9). Then $f^* \mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$ is injective (by virtue of our inductive hypothesis). Our inductive hypothesis also implies that if $\mathrm{cofib}(u)$ has injective dimension $\leqslant n-1$, then $f^* \mathrm{cofib}(u)$ has injective dimension $\leqslant n-1$; moreover, the converse holds if f is surjective. The desired result now follows from Example 6.5.1.6. □

Remark 6.5.2.3. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper and locally almost of finite presentation and let $f^! : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ be the exceptional inverse image functor of Definition 6.4.0.1. If $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ has injective dimension $\leqslant n$, then $f^! \mathscr{F}$ in $\mathrm{QCoh}(\mathsf{X})$ has injective dimension $\leqslant n$: this follows immediately from the characterization of injective dimension supplied by Remark 6.5.1.5 (and the left t-exactness of the direct image functor f_*).

Using Proposition 6.5.2.2, we can often reduce questions about the injective dimension of sheaves $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ to the case where X is affine. In this case, the following observation is convenient:

Proposition 6.5.2.4. *Let A be a Noetherian $\mathbb{E}_\infty$-ring, let K be a truncated A-module, and let n be an integer. The following conditions are equivalent:*

(a) *The A-module K has injective dimension $\leqslant n$.*

(b) *For every prime ideal $\mathfrak{p} \subseteq \pi_0 A$, the groups $\mathrm{Ext}^i_A((\pi_0 A)/\mathfrak{p}, K)$ vanish for $i > n$.*

Proof. The implication (a) ⇒ (b) is immediate. To prove the converse, we may replace K by a suitable desuspension to reduce to the case where $K \in (\mathrm{Mod}_A)_{\leqslant 0}$. In this case, we proceed by induction on n. We first treat the case $n = 0$. Suppose that condition (b) is satisfied, and let $M \in \mathrm{Mod}_A^{\heartsuit}$ be finitely generated as a module over $\pi_0 A$. Then M admits a finite filtration by $(\pi_0 A)$-modules of the form $(\pi_0 A)/\mathfrak{p}$, so condition (b) implies that $\mathrm{Ext}^i_A(M, K) \simeq 0$ for $i > 0$. Applying Proposition C.6.10.1, we deduce that K is an injective object of Mod_A.

Now suppose that $n > 0$. Choose a map $\alpha : K \to Q$ which exhibits Q as an injective hull of K (Example C.5.7.9). For every prime ideal $\mathfrak{p} \subseteq \pi_0 A$, we have short exact sequences

$$\mathrm{Ext}^i_A((\pi_0 A)/\mathfrak{p}, Q) \to \mathrm{Ext}^i_A((\pi_0 A)/\mathfrak{p}, \mathrm{cofib}(\alpha)) \to \mathrm{Ext}^{i+1}_A((\pi_0 A)/\mathfrak{p}, M).$$

It follows from assumption (b) that the groups $\mathrm{Ext}^i_A((\pi_0 A)/\mathfrak{p}, \mathrm{cofib}(\alpha))$ vanish for $i \geqslant n$. Applying our inductive hypothesis, we deduce that $\mathrm{cofib}(\alpha)$ has injective dimension $\leqslant n-1$. Using the fiber sequence $K \to Q \to \mathrm{cofib}(\alpha)$, we conclude that K has injective dimension $\leqslant n$. □

Proposition 6.5.2.5. *Let X be a Noetherian spectral algebraic space and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$. If $\mathscr{F}$ is truncated and of injective dimension $\leqslant m$ and $\mathscr{O}_{\mathsf{X}}$ is n-truncated, then $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})_{\geqslant -m-n}$.*

Proof. Replacing $\mathscr{F}$ by a suitable desuspension, we may assume that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})_{\leqslant 0}$. We now proceed by induction on m. Suppose first that $m = 0$: that is, $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is injective; we wish to prove that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})_{\geqslant -n}$. By virtue of Proposition 6.5.2.2, this assertion can be tested locally on X. We may therefore assume without loss of generality that $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ for some n-truncated Noetherian $\mathbb{E}_\infty$-ring. In this case, the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \simeq \mathrm{Mod}_A^{\mathrm{cn}}$ is n-complicial (Example C.5.3.5), so that $\mathscr{F}$ is $(-n)$-connective by virtue of Proposition C.5.7.11.

We now carry out the inductive step. Assume that $m > 0$ and that $\mathscr{F}$ has injective dimension $\leqslant m$. Let $\alpha : \mathscr{F} \to \mathscr{G}$ be a morphism in $\mathrm{QCoh}(\mathsf{X})_{\leqslant 0}$ which exhibits $\mathscr{G}$ as an injective hull of $\mathscr{F}$ (Example C.5.7.9). Then $\mathrm{cofib}(\alpha)$ has injective dimension $\leqslant m-1$. Applying our inductive hypothesis, we deduce that $\mathscr{G}$ and $\mathrm{cofib}(\alpha)$ belong to $\mathrm{QCoh}(\mathsf{X})_{\geqslant 1-m-n}$. The fiber sequence $\mathscr{F} \to \mathscr{G} \to \mathrm{cofib}(\alpha)$ then shows that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})_{\geqslant -m-n}$, as desired. □

6.5.3 Internal Mapping Sheaves

Let $(\mathcal{X}, \mathscr{O})$ be a spectrally ringed ∞-topos. Then the symmetric monoidal ∞-category $\mathrm{Mod}_{\mathscr{O}}$ is closed: that is, for every pair of objects $\mathscr{F}, \mathscr{G} \in \mathrm{Mod}_{\mathscr{O}}$, there exists an object $\underline{\mathrm{Map}}_{\mathscr{O}}(\mathscr{F}, \mathscr{G}) \in \mathrm{Mod}_{\mathscr{O}}$ with the following universal property: for every object $\mathscr{E} \in \mathrm{Mod}_{\mathscr{O}}$, there is a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}}}(\mathscr{E}, \underline{\mathrm{Map}}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}}}(\mathscr{E} \otimes_{\mathscr{O}} \mathscr{F}, \mathscr{G}).$$

We now discuss some circumstances in which the construction $(\mathscr{F}, \mathscr{G}) \mapsto \underline{\mathrm{Map}}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})$ preserves quasi-coherence, in the case where X is a spectral Deligne-Mumford stack. We begin with some general observations.

Remark 6.5.3.1. Let $(\mathcal{X}, \mathscr{O})$ be a spectrally ringed ∞-topos and let $\mathscr{F}, \mathscr{G} \in \mathrm{Mod}_{\mathscr{O}}$. For each $U \in \mathcal{X}$, we have a canonical homotopy equivalence $\Omega^\infty \underline{\mathrm{Map}}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})(U) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}|_U}}(\mathscr{F}|_U, \mathscr{G}|_U)$ (this follows immediately from the universal property of $\underline{\mathrm{Map}}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})$, applied in the special case where $\mathscr{E}$ is the the image of $\mathscr{O}|_U$ under the left adjoint to the restriction functor $\mathrm{Mod}_{\mathscr{O}} \to \mathrm{Mod}_{\mathscr{O}|_U}$).

Variant 6.5.3.2. If A is an $\mathbb{E}_\infty$-ring, then the symmetric monoidal ∞-category Mod_A is closed. Given objects $M, N \in \mathrm{Mod}_A$ we let $\underline{\mathrm{Map}}_A(M,N) \in \mathrm{Mod}_A$ denote a classifying object for morphisms from M to N in the ∞-category Mod_A: that is, $\underline{\mathrm{Map}}_A(M,N)$ is an A-module with the following universal property: there exists a map $e : \underline{\mathrm{Map}}_A(M,N) \otimes_A M \to N$ such that, for every A-module M', composition with e induces a homotopy equivalence $\mathrm{Map}_{\mathrm{Mod}_A}(M', \underline{\mathrm{Map}}_A(M,N)) \to \mathrm{Map}_{\mathrm{Mod}_A}(M' \otimes_A M, N)$. Note that this is really a special case of the construction $(\mathscr{F}, \mathscr{G}) \mapsto \underline{\mathrm{Map}}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})$, where we take the ∞-topos $\mathcal{X}$ to be the ∞-category $\mathcal{S}$ of spaces.

Example 6.5.3.3. Let $(\mathcal{X}, \mathscr{O})$ be a spectrally ringed ∞-topos and let $A = \Gamma(\mathcal{X}; \mathscr{O})$ be the $\mathbb{E}_\infty$-ring of global sections of $\mathscr{O}$. For every pair of objects $\mathscr{F}, \mathscr{G} \in \mathrm{Mod}_{\mathscr{O}}$, the global sections functor $\Gamma(\mathcal{X}; \bullet) : \mathrm{Mod}_{\mathscr{O}} \mapsto \mathrm{Mod}_A$ induces a map

$$\theta_{\mathscr{F},\mathscr{G}} : \Gamma(\mathcal{X}; \underline{\mathrm{Map}}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})) \mapsto \underline{\mathrm{Map}}_A(\Gamma(\mathcal{X}; \mathscr{F}), \Gamma(\mathcal{X}; \mathscr{G})).$$

The map $\theta_{\mathscr{F},\mathscr{G}}$ is an equivalence in either of the following cases:

(a) The sheaf $\mathscr{F}$ coincides with $\mathscr{O}$.

(b) The pair $\mathsf{X} = (\mathcal{X}, \mathscr{O})$ is an affine spectral Deligne-Mumford stack and the sheaf $\mathscr{F}$ is quasi-coherent.

Assertion (a) follows easily from Remark 6.5.3.1. To prove (b), we observe that for fixed $\mathscr{G}$, the construction $\mathscr{F} \mapsto \theta_{\mathscr{F},\mathscr{G}}$ carries colimits in $\mathrm{Mod}_{\mathscr{O}}$ to limits in $\mathrm{Fun}(\Delta^1, \mathrm{Mod}_A)$. Consequently, the objects $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}}$ for which the morphism $\theta_{\mathscr{F},\mathscr{G}}$ is an equivalence span a stable subcategory of $\mathrm{Mod}_{\mathscr{O}}$ which is closed under colimits. This subcategory contains $\mathscr{O}$ by (a), and therefore contains all quasi-coherent sheaves on X.

Remark 6.5.3.4. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O})$ be a spectral Deligne-Mumford stack and let $\mathscr{F}, \mathscr{G} \in \mathrm{Mod}_{\mathscr{O}}$. Assume that $\mathscr{F}$ is quasi-coherent. Combining Proposition 2.2.4.3 with Example 6.5.3.3, we deduce that $\underline{\mathrm{Map}}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})$ is quasi-coherent if and only if the following condition is satisfied:

($*$) For every morphism $U \to V$ in $\mathcal{X}$, where U and V are affine, the natural map

$$\mathscr{O}(U) \otimes_{\mathscr{O}(V)} \underline{\mathrm{Map}}_{\mathscr{O}(V)}(\mathscr{F}(V), \mathscr{G}(V)) \to \underline{\mathrm{Map}}_{\mathscr{O}(U)}(\mathscr{F}(U), \mathscr{G}(U))$$

is an equivalence of $\mathscr{O}(U)$-modules.

If $\mathscr{G}$ is also quasi-coherent, then we can rephrase ($*$) as follows:

($*'$) For every morphism $U \to V$ in $\mathcal{X}$, where U and V are affine, the natural map

$$\mathscr{O}(U) \otimes_{\mathscr{O}(U)} \underline{\mathrm{Map}}_{\mathscr{O}(V)}(\mathscr{F}(V), \mathscr{G}(V)) \to \underline{\mathrm{Map}}_{\mathscr{O}(V)}(\mathscr{F}(V), \mathscr{O}(U) \otimes_{\mathscr{O}(V)} \mathscr{G}(V))$$

is an equivalence of $\mathscr{O}(U)$-modules.

Proposition 6.5.3.5. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a proper morphism between Noetherian spectral Deligne-Mumford stacks. Then the direct image functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ carries $\mathrm{Coh}(\mathsf{X})$ into $\mathrm{Coh}(\mathsf{Y})$.*

Proof. We first note that f is locally almost of finite presentation (Remark 4.2.0.4). We wish to show that for every integer n, the truncation $\tau_{\geqslant n} f_* \mathscr{F}$ is almost perfect. Using Corollary 3.4.2.3, we can replace $\mathscr{F}$ by $\tau_{\geqslant -m} \mathscr{F}$ for some $m \gg 0$. In this case, $\mathscr{F}$ is almost perfect, so that $f_* \mathscr{F}$ is also almost perfect (Theorem 5.6.0.2). Since Y is locally Noetherian, it follows that the truncation $\tau_{\geqslant n} f_* \mathscr{F}$ is also almost perfect. □

Proposition 6.5.3.6. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack and let $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$. Then $\underline{\mathrm{Map}}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})$ is quasi-coherent under any of the following additional assumptions:*

(a) The quasi-coherent sheaf $\mathscr{F}$ is perfect.

(b) The quasi-coherent sheaf $\mathscr{F}$ is almost perfect and $\mathscr{G}$ is truncated.

(c) The spectral Deligne-Mumford stack X is a Noetherian spectral algebraic space, each truncation $\tau_{\geqslant -n} \mathscr{F}$ is almost perfect, and $\mathscr{G}$ is truncated and of finite injective dimension.

The proof of Proposition 6.5.3.6 rests on the following:

Lemma 6.5.3.7. *Let A be a connective $\mathbb{E}_\infty$-ring, and suppose we are given modules $M, N, P \in \mathrm{Mod}_A$. Assume that M is almost perfect, that N is truncated, and that P has finite* Tor*-amplitude. Then the canonical map*

$$\theta_M : P \otimes_A \underline{\mathrm{Map}}_A(M, N) \to \underline{\mathrm{Map}}_A(M, P \otimes_A N)$$

is an equivalence.

Proof. Replacing N and P by suitable suspensions, we may assume without loss of generality that that N is 0-truncated and that P has Tor-amplitude $\leqslant 0$. Fix an integer n. Since M is almost perfect, we can choose a fiber sequence $M' \to M \to M''$ where M' is perfect and M'' is n-connective (Corollary 2.7.2.2). We then have a fiber sequence of A-modules $\mathrm{fib}(\theta_{M''}) \xrightarrow{\alpha} \mathrm{fib}(\theta_M) \to \mathrm{fib}(\theta_{M'})$. Since M' is perfect, the map $\theta_{M'}$ is an equivalence. It follows that $\mathrm{fib}(\theta_{M'}) \simeq 0$ and therefore α is an equivalence. Since N is 0-truncated and P has Tor-amplitude $\leqslant 0$, the tensor product $P \otimes_A N$ is also 0-truncated. Applying Remark 9.5.3.1, we deduce that $\underline{\mathrm{Map}}_A(M'', N)$ and $\underline{\mathrm{Map}}_A(M'', P \otimes_A N)$ are $(-n)$-truncated. Using our assumption that P has Tor-amplitude $\leqslant 0$, we conclude that $P \otimes_A \underline{\mathrm{Map}}_A(M'', N)$ is $(-n)$-truncated. The map $\theta_{M''}$ has $(-n)$-truncated domain and codomain, so that $\mathrm{fib}(\theta_{M''})$ is also $(-n)$-truncated. Since α is an equivalence, it follows that $\mathrm{fib}(\theta_M)$ is $(-n)$-truncated. Since n is arbitrary, we conclude that $\mathrm{fib}(\theta_M) \simeq 0$, so that θ_M is an equivalence as desired. □

Proof of Proposition 6.5.3.6. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack and let $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$. We will show that if any of the conditions (a), (b), or (c) is satisfied, then the pair $(\mathscr{F}, \mathscr{G})$ satisfies hypothesis $(*')$ of Remark 6.5.3.4. Fix a morphism $U \to V$ between affine objects of $\mathcal{X}$, so that we have equivalences $(\mathcal{X}_{/V}, \mathcal{O}|_V) \simeq \operatorname{Sp\acute{e}t} A$ and $(\mathcal{X}_{/U}, \mathcal{O}|_U) \simeq \operatorname{Sp\acute{e}t} B$ for some $\mathbb{E}_\infty$-rings A and B. Set $M = \mathscr{F}(V)$ and $N = \mathscr{G}(V)$; we wish to show that the canonical map $\rho : B \otimes_A \underline{\mathrm{Map}}_A(M, N) \to \underline{\mathrm{Map}}_A(M, B \otimes_A N)$ is an equivalence. This is trivial in case (a), and follows from Proposition 6.5.3.6 in case (b). To handle case (c), we may assume without loss of generality that $\mathscr{G}$ has injective dimension $\leqslant 0$. Using Proposition 6.5.2.2, we deduce that N has injective dimension $\leqslant 0$ as an object of the ∞-category Mod_A and that $B \otimes_A N$ has injective dimension $\leqslant 0$ as an object of the ∞-category Mod_B (hence also as an object of the ∞-category Mod_A, since B is flat over A). For each integer n, we have a commutative diagram

$$\begin{array}{ccc} B \otimes_A \underline{\mathrm{Map}}_A(M, N) & \xrightarrow{\rho} & \underline{\mathrm{Map}}_A(M, B \otimes_A N) \\ & & \downarrow \\ B \otimes_A \underline{\mathrm{Map}}_A(\tau_{\geqslant -n} M, N) & \xrightarrow{\rho'} & \underline{\mathrm{Map}}_A(\tau_{\geqslant -n} M, B \otimes_A N). \end{array}$$

The map ρ' is an equivalence by virtue of (b), and the vertical maps induce isomorphisms on π_n. It follows that ρ induces an isomorphism $\pi_n(B \otimes_A \underline{\mathrm{Map}}_A(M, N)) \to \pi_n(\underline{\mathrm{Map}}_A(M, B \otimes_A N))$. Since n was arbitrary, the morphism ρ is an equivalence as desired. □

Remark 6.5.3.8. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack. Then $\mathrm{QCoh}(\mathsf{X})$ is a presentable ∞-category and the tensor product $\otimes : \mathrm{QCoh}(\mathsf{X}) \times \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X})$ preserves small colimits separately in each variable. It follows that the symmetric monoidal structure on $\mathrm{QCoh}(\mathsf{X})$ is closed: that is, for every pair of objects $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$, there exists an object $\underline{\mathrm{Map}}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \mathscr{G}) \in \mathrm{QCoh}(\mathsf{X})$ with the following universal property: for every *quasi-coherent* sheaf $\mathscr{E} \in \mathrm{QCoh}(\mathsf{X})$, there is a canonical homotopy equivalence $\mathrm{Map}_{\mathrm{Mod}_\mathcal{O}}(\mathscr{E}, \underline{\mathrm{Map}}_\mathcal{O}(\mathscr{F}, \mathscr{G})) \simeq \mathrm{Map}_{\mathrm{Mod}_\mathcal{O}}(\mathscr{E} \otimes_\mathcal{O} \mathscr{F}, \mathscr{G})$. It follows immediately from the definitions that there is a canonical map $\theta : \underline{\mathrm{Map}}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \mathscr{G}) \to \underline{\mathrm{Map}}_\mathcal{O}(\mathscr{F}, \mathscr{G})$, which exhibits $\underline{\mathrm{Map}}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \mathscr{G})$ as universal among quasi-coherent sheaves $\mathscr{E}$ equipped with a map $\mathscr{E} \to \underline{\mathrm{Map}}_\mathcal{O}(\mathscr{F}, \mathscr{G})$. In particular, the morphism θ is an equivalence if and only if $\underline{\mathrm{Map}}_\mathcal{O}(\mathscr{F}, \mathscr{G})$ is quasi-coherent. Beware that if θ is not an equivalence, then $\underline{\mathrm{Map}}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \mathscr{G})$ is badly behaved in general: for example, it need not be compatible with étale base change.

6.5.4 !-Fibers and Injective Dimension

For later use, we record a facts about the relationship between the Krull dimension of Noetherian spectral algebraic space X and the injective dimension of coherent sheaves on X.

Notation 6.5.4.1. Let X be a Noetherian spectral algebraic space, let x be a point of the underlying topological space $|\mathsf{X}|$. Let $\kappa(x)$ denote the residue field of X at the point x and let $\mathsf{X}_0 \subseteq \mathsf{X}$ be the schematic image of the associated map $i : \operatorname{Sp\acute{e}t} \kappa(x) \to |\mathsf{X}|$ (that is, the reduced closed subspace of X corresponding to the closed subset $\overline{\{x\}} \subseteq |\mathsf{X}|$), so that the map i factors as a composition $\operatorname{Sp\acute{e}t} \kappa(x) \xrightarrow{i'} \mathsf{X}_0 \xrightarrow{i''} \mathsf{X}$. If $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}}$, we let $\mathscr{F}^!_x$ denote the image of $\mathscr{F}$ under the composite functor

$$\mathrm{QCoh}(\mathsf{X}) \xrightarrow{i''^!} \mathrm{QCoh}(\mathsf{X}_0) \xrightarrow{i'^*} \mathrm{QCoh}(\operatorname{Sp\acute{e}t} \kappa(x)) \simeq \mathrm{Mod}_{\kappa(x)} .$$

We will refer to $\mathscr{F}^!_x$ as the *!-fiber of $\mathscr{F}$ at the point x*. In the special case where $\mathscr{F} = \mathscr{O}_{\mathsf{X}}$, we will denote $\mathscr{F}^!_x$ by $\mathscr{O}^!_{X,x}$.

Our next result summarizes some of the formal properties of the formation of !-fibers:

Proposition 6.5.4.2. *Let X be a Noetherian spectral algebraic space, let $x \in |\mathsf{X}|$, and let $U : \mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}} \to \mathrm{Mod}_\kappa$ be the functor given by $\mathscr{F} \mapsto \mathscr{F}^!_x$. Then:*

(1) *The functor U commutes with filtered colimits when restricted to $\mathrm{QCoh}(\mathsf{X})_{\leqslant 0}$.*

(2) *The functor U is left t-exact.*

(3) *If $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}}$ is coherent, then $\mathscr{F}^!_x$ is coherent: that is, each homotopy group $\pi_i \mathscr{F}^!_x$ is finite-dimensional when regarded as a vector space over $\kappa(x)$.*

Proof. Assertion (1) follows from Proposition 6.4.1.4, assertion (2) follows from the observation that the map $i' : \operatorname{Sp\acute{e}t} \kappa(x) \to \mathsf{X}_0$ appearing in Notation 6.5.4.1 is flat, and assertion (3) follows from Proposition 6.4.3.4. □

Remark 6.5.4.3. Let X be a Noetherian spectral algebraic space, let $x \in |\mathsf{X}|$, and let $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$. If $\mathscr{F}$ is truncated and $\mathscr{G}$ has finite Tor-amplitude, then Corollary 6.4.1.9 supplies a canonical equivalence $\mathscr{F}^!_x \otimes_{\kappa(x)} \mathscr{G}_x \simeq (\mathscr{F} \otimes \mathscr{G})^!_x$, where $\mathscr{G}_x$ denotes the fiber of $\mathscr{G}$ at x (that is, the image of $\mathscr{G}$ under the pullback functor $\mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\operatorname{Sp\acute{e}t} \kappa(x)) \simeq \mathrm{Mod}_{\kappa(x)}$).

Remark 6.5.4.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism between Noetherian spectral algebraic spaces which is locally of finite Tor-amplitude. Let x be a point of $|\mathsf{X}|$, let $y = f(x)$ be the image of x in $|\mathsf{Y}|$, and let X_y denote the fiber product $\mathsf{X} \times_{\mathsf{Y}} \operatorname{Sp\acute{e}t} \kappa(y)$. Then, for every object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})_{\mathrm{ltr}}$, we have a canonical equivalence $(f^* \mathscr{F})^!_x \simeq \mathscr{F}^!_y \otimes_{\kappa(y)} \mathscr{O}^!_{\mathsf{X}_y, x}$ in $\mathrm{Mod}_{\kappa(x)}$. This follows by applying Proposition 6.4.1.4 to the upper square and bottom right square of the commutative diagram

$$\begin{array}{ccccc}
\operatorname{Sp\acute{e}t}\kappa(x) & \longrightarrow & \mathsf{X}_0 \times_{\mathsf{Y}_0} \operatorname{Sp\acute{e}t}\kappa(y) & \longrightarrow & \mathsf{X}_0 \\
 & & \downarrow & & \downarrow \\
 & & \mathsf{X}_y & \longrightarrow & \mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}_0 & \longrightarrow & \mathsf{X} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \operatorname{Sp\acute{e}t}\kappa(y) & \longrightarrow & \mathsf{Y}_0 & \longrightarrow & \mathsf{Y};
\end{array}$$

here X_0 and Y_0 denote the schematic images of $\operatorname{Sp\acute{e}t}\kappa(x)$ and $\operatorname{Sp\acute{e}t}\kappa(y)$ in X and Y, respectively.

Example 6.5.4.5. Let $f : \mathsf{X} \to \mathsf{Y}$ be a flat morphism of Noetherian spectral algebraic spaces, let $x \in |\mathsf{X}|$ be a point having image $y = f(x) \in |\mathsf{Y}|$, and suppose that the diagram

$$\begin{array}{ccc}
\operatorname{Sp\acute{e}t}\kappa(x) & \longrightarrow & \mathsf{X} \\
\downarrow & & \downarrow \\
\operatorname{Sp\acute{e}t}\kappa(y) & \longrightarrow & \mathsf{Y}
\end{array}$$

is a pullback square. Then, for every object $\mathscr{F} \in \operatorname{QCoh}(\mathsf{Y})_{\mathrm{ltr}}$, Remark 6.5.4.4 supplies an equivalence $(f^* \mathscr{F})^!_x \simeq \mathscr{F}^!_y \otimes_{\kappa(y)} \kappa(x)$.

Example 6.5.4.6. Let $\mathsf{X} = \operatorname{Sp\acute{e}t} A$, where A is a Noetherian commutative ring, and let $x \in |\mathsf{X}| = |\operatorname{Spec} A|$ be a point corresponding to a prime ideal $\mathfrak{p} \subseteq A$ for which $A_{\mathfrak{p}}$ is a regular Noetherian ring of Krull dimension d. Then $\mathscr{O}^!_{\mathsf{X},x} \simeq \Sigma^{-d}\kappa(x)$.

Lemma 6.5.4.7. *Let X be Noetherian spectral algebraic space, let $x \in |\mathsf{X}|$, and let $\mathscr{F} \in \operatorname{Coh}(\mathsf{X})$. Suppose that $\mathscr{F}$ is n-truncated and that the coherent sheaf $\pi_n \mathscr{F}$ does not vanish at x. Then $\pi_m \mathscr{F}^!_x \neq 0$ for some $m \geqslant n - \operatorname{ht}(x)$, where $\operatorname{ht}(x)$ denotes the height of x (see Definition 3.7.6.1).*

Proof. Using Example 6.5.4.5, we can reduce to the case where $\mathsf{X} = \operatorname{Sp\acute{e}t} A$ where A is a local Noetherian $\mathbb{E}_\infty$-ring and x is the closed point of $|\operatorname{Spec} A|$. Set $d = \operatorname{ht}(x)$, and let $a_1, \ldots, a_d \in \pi_0 A$ be a system of parameters for $\pi_0 A$. For $1 \leqslant i \leqslant d$, let $N(i) \in \operatorname{Mod}_A$ denote the cofiber of the multiplication map $a_i : A \to A$. Set $M = \Gamma(\mathsf{X}; \mathscr{F})$ and set $K = M \otimes_A N(1) \otimes_A \cdots \otimes_A N(d)$. Note that each $N(i)$ has Tor-amplitude $\leqslant 1$, so that K is $(n+d)$-truncated. Applying Nakayama's lemma repeatedly, we deduce that $\pi_n K \neq 0$. Let s be the largest integer for which $\pi_s K$ is nonzero. Then $\pi_s K$ is an Artinian module over $\pi_0 A$, so there exists a nonzero map $\rho : \kappa(x) \to \pi_s K$. Set

$$N_\kappa(i) = \kappa(x) \otimes_A N(i) \simeq \operatorname{cofib}(a_i : \kappa(x) \to \kappa(x)) \simeq \kappa(x) \oplus \Sigma\kappa(x)$$

Then we can identify ρ with a nonzero element of

$$\begin{aligned}\pi_s \underline{\mathrm{Map}}_A(\kappa(x), K) &\simeq \pi_s(\mathscr{F}^!_x \otimes_{\kappa(x)} N_\kappa(1) \otimes_{\kappa(x)} \cdots \otimes_{\kappa(x)} N_\kappa(d)) \\ &\simeq \pi_s \bigoplus_{J \subseteq \{1,\ldots,d\}} \Sigma^{|J|} \mathscr{F}^!_x \\ &\simeq \bigoplus_{J \subseteq \{1,\ldots,d\}} \pi_{s-|J|} \mathscr{F}^!_x .\end{aligned}$$

It follows that $\pi_m \mathscr{F}^!_x$ does not vanish for some $m \geqslant s - d \geqslant n - d = n - \mathrm{ht}(x)$. □

Proposition 6.5.4.8. *Let* X *be a Noetherian spectral algebraic space, let* $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ *be truncated and coherent, and let* $x \in |\mathsf{X}|$. *The following conditions are equivalent:*

- (a) *The stalk of the sheaf* $\mathscr{F}$ *vanishes at the point* x.
- (b) *The* !*-fiber* $\mathscr{F}^!_x$ *vanishes.*

Proof. The implication $(a) \Rightarrow (b)$ is obvious, and the converse follows from Lemma 6.5.4.7. □

Proposition 6.5.4.9. *Let* X *be a Noetherian spectral algebraic space, let* $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})_{\mathrm{ltr}}$, *and let* n *be an integer. The following conditions are equivalent:*

1. *The sheaf* $\mathscr{F}$ *has injective dimension* $\leqslant n$.
2. *For every point* $x \in |\mathsf{X}|$, *the homotopy groups* $\pi_i \mathscr{F}^!_x$ *vanish for* $i < -n$.

Proof. Suppose first that condition (1) is satisfied. Let $x \in |\mathsf{X}|$ and let X_0 be the schematic image of the map $\mathrm{Sp\acute{e}t}\, \kappa(x) \to \mathsf{X}$. Let $i : \mathsf{X}_0 \hookrightarrow \mathsf{X}$ be the tautological closed immersion. Since the direct image functor i_* is t-exact, the sheaf $i^! \mathscr{F} \in \mathrm{QCoh}(\mathsf{X}_0)$ has injective dimension $\leqslant n$. The structure sheaf of X_0 is 0-truncated, so Proposition 6.5.2.5 implies that $i^! \mathscr{F} \in \mathrm{QCoh}(\mathsf{X}_0)_{\geqslant -n}$. Restricting to the generic point of X_0, we deduce that $\mathscr{F}^!_x$ is $(-n)$-connective.

Now suppose that condition (2) is satisfied; we wish to show that $\mathscr{F}$ has injective dimension $\leqslant n$. Using Proposition 6.5.2.2 and Remark 6.5.4.4, we can work locally on X and thereby reduce to the case where $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ is affine. In this case, we can identify $\mathscr{F}$ with the truncated A-module $M = \Gamma(\mathsf{X}; \mathscr{F})$. Let us say that an ideal $I \subseteq \pi_0 A$ is *good* if the module $\underline{\mathrm{Map}}_A((\pi_0 A)/I, M)$ is $(-n)$-connective. To prove (1), it will suffice to show that every prime ideal $\mathfrak{p} \subset \pi_0 A$ is good (Proposition 6.5.2.4). Suppose otherwise: then, by virtue of our assumption that A is Noetherian, there exists an ideal $I \subseteq \pi_0 A$ which is maximal among those ideals which are not good. Replacing A by $(\pi_0 A)/I$ and K by $\underline{\mathrm{Map}}_A((\pi_0 A)/I, M)$, we may assume that A is discrete and that $I = (0)$. If A is not an integral domain, then the maximality of I implies that every prime ideal of A is good, so that M has injective dimension $\geqslant -n$ by virtue of Proposition 6.5.2.4, which contradicts our assumption that $I = (0)$ is not good. We may therefore assume without loss of generality that A is an integral domain and that every nonzero ideal of A is good.

Since $I = (0)$ is not good, the module M is not $(-n)$-truncated. We may therefore choose a nonzero element $u \in \pi_m M$ for $m < -n$. Let K denote the fraction field of A. Applying assumption (2) (in the case where x is the generic point of X) we deduce that $K \otimes_A (\pi_m M) \simeq \pi_m \mathscr{F}^!_x \simeq 0$. It follows u is annihilated by some nonzero element $a \in A$. The exactness of the sequence $\pi_m \underline{\mathrm{Map}}_A(A/(a), M) \to \pi_m M \xrightarrow{a} \pi_m M$ then shows that $\pi_m \underline{\mathrm{Map}}_A(A/(a), M) \neq 0$, contradicting our assumption that the nonzero ideal (a) is good. □

Corollary 6.5.4.10 (Serre)**.** *Let A be a regular Noetherian ring of Krull dimension $\leqslant n$. Then A has injective dimension $\leqslant n$ (when regarded as an object of* Mod_A*).*

Proof. Combine Proposition 6.5.4.9 with Example 6.5.4.6. □

6.6 Absolute Dualizing Sheaves

In his work on the duality theory of coherent sheaves, Grothendieck introduced the notion of a *dualizing complex* on a Noetherian scheme X (see [94]). Our goal in this section is to adapt the theory of dualizing complexes to the setting of spectral algebraic geometry. If X is a Noetherian spectral algebraic space, we define the notion of a *dualizing sheaf* $\omega_{\mathsf{X}} \in \mathrm{QCoh}(\mathsf{X})$ (Definition 6.6.1.1). Specializing to the case where $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ is affine and using the equivalence $\mathrm{QCoh}(\mathsf{X}) \simeq \mathrm{Mod}_A$, we obtain the notion of a *dualizing module* for A. Our principal results can be summarized as follows:

- Let X be a Noetherian spectral algebraic space and let $\mathrm{Coh}(\mathsf{X}) \subseteq \mathrm{QCoh}(\mathsf{X})$ denote the full subcategory spanned by those objects having coherent homotopy (Definition **??**). If X admits a dualizing sheaf ω_{X}, then the construction $\mathscr{F} \mapsto \underline{\mathrm{Map}}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \omega_{\mathsf{X}})$ determines an equivalence of the ∞-category $\mathrm{Coh}(\mathsf{X})$ with its opposite (Theorem 6.6.1.8).

- If X is a Noetherian spectral algebraic space which admits a dualizing sheaf ω_{X}, then ω_{X} is unique up to tensor product with an invertible object of $\mathrm{QCoh}(\mathsf{X})$ (Proposition 6.6.2.1).

- Let $f : \mathsf{X} \to \mathsf{Y}$ be a proper morphism between Noetherian spectral algebraic spaces. Then the functor $f^! : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ of Definition 6.4.0.1 carries dualizing sheaves on Y to dualizing sheaves on X (Proposition 6.6.3.1). In particular, if Y admits a dualizing sheaf, then so does X.

- A Noetherian $\mathbb{E}_\infty$-ring A admits a dualizing module if and only if the commutative ring $\pi_0 A$ admits a dualizing module (Theorem 6.6.4.1).

- Let $\phi : A \to B$ be a morphism of Noetherian $\mathbb{E}_\infty$-rings which is almost of finite presentation. If A admits a dualizing module, then so does B (Theorem 6.6.4.3).

Warning 6.6.0.1. In the setting of classical algebraic geometry, the term *dualizing complex* is commonly used to refer to what we call a *dualizing sheaf* (or *dualizing module*). Our choice of terminology is consistent with our convention of using the term *quasi-coherent sheaf* to refer to any object of the stable ∞-category $\mathrm{QCoh}(\mathsf{X})$, as opposed to objects which belong to the heart $\mathrm{QCoh}(\mathsf{X})^\heartsuit$. Note also that the term "complex" is potentially misleading: if X is not 0-truncated, then the stable ∞-category $\mathrm{QCoh}(\mathsf{X})$ cannot be recovered as the derived ∞-category of its heart; consequently, it is unreasonable to expect a dualizing sheaf ω_{X} to admit an explicit representative as a chain complex.

6.6.1 Dualizing Sheaves

We begin by introducing the central definition of this section:

Definition 6.6.1.1. Let X be a Noetherian spectral algebraic space. We will say that an object $\omega_{\mathsf{X}} \in \mathrm{QCoh}(\mathsf{X})$ is a *dualizing sheaf* if it satisfies the following conditions:

(1) The sheaf ω_{X} is truncated (that is, the homotopy sheaves $\pi_n \omega_{\mathsf{X}}$ vanish for $n \gg 0$).

(2) The sheaf ω_{X} is coherent: that is, each truncation $\tau_{\geqslant -n}\omega_{\mathsf{X}}$ is almost perfect.

(3) The sheaf ω_{X} has finite injective dimension (Definition 6.5.1.1).

(4) The unit map $\mathscr{O}_{\mathsf{X}} \to \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\omega_{\mathsf{X}}, \omega_{\mathsf{X}})$ is an equivalence.

If A is a Noetherian $\mathbb{E}_\infty$-ring, we will say that an object $K \in \mathrm{Mod}_A$ is a *dualizing module* if it is the image of a dualizing sheaf under the equivalence $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A) \simeq \mathrm{Mod}_A$.

Remark 6.6.1.2. In the situation of Definition 6.6.1.1, conditions (1), (2) and (3) guarantee that the sheaf $\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\omega_{\mathsf{X}}, \omega_{\mathsf{X}})$ is quasi-coherent (Proposition 6.5.3.6).

Remark 6.6.1.3. Let $f : \mathsf{X} \to \mathsf{Y}$ be an étale morphism between Noetherian spectral algebraic spaces and let $\omega_{\mathsf{Y}} \in \mathrm{QCoh}(\mathsf{Y})$. Then:

- If ω_{Y} is a dualizing sheaf for Y, then $f^*\omega_{\mathsf{Y}}$ is a dualizing sheaf for X.

- If f is surjective and $f^*\omega_{\mathsf{Y}}$ is a dualizing sheaf for Y, then ω_{Y} is a dualizing sheaf for Y.

In fact, the corresponding assertions hold for each of the properties (1) through (4) which appear in Definition 6.6.1.1. For properties (1), (2) and (4) this is obvious, and for (3) it follows from Proposition 6.5.2.2.

Remark 6.6.1.4. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a Noetherian spectral algebraic space. Using Remark 6.6.1.3, we see that an object $\omega_{\mathsf{X}} \in \mathrm{QCoh}(\mathsf{X})$ is a dualizing module if and only if $\omega_{\mathsf{X}}(U)$ is a dualizing module for $\mathscr{O}_{\mathsf{X}}(U)$ for every affine object $U \in \mathcal{X}$.

Remark 6.6.1.5. In the situation of Definition 6.6.1.1, condition (4) can be rephrased as follows:

(4′) For every object $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathsf{X}}}$, the canonical map $\mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}_{\mathsf{X}}}}(\mathscr{F}, \mathscr{O}_{\mathsf{X}}) \to \mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}_{\mathsf{X}}}}(\mathscr{F} \otimes \omega_{\mathsf{X}}, \omega_{\mathsf{X}})$ is a homotopy equivalence.

(4″) For every object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, the canonical map $\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \mathscr{O}_{\mathsf{X}}) \to \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F} \otimes \omega_{\mathsf{X}}, \omega_{\mathsf{X}})$ is a homotopy equivalence.

The implications $(4) \Leftrightarrow (4') \Rightarrow (4'')$ are tautological, and the implication $(4'') \Rightarrow (4')$ follows from the quasi-coherence of $\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\omega_{\mathsf{X}}, \omega_{\mathsf{X}})$ (see Remarks 6.5.3.8 and 6.6.1.2).

Construction 6.6.1.6 (Grothendieck Duality)**.** Let X be a Noetherian spectral algebraic space and suppose that we have fixed a dualizing sheaf $\omega_{\mathsf{X}} \in \mathrm{QCoh}(\mathsf{X})$. For each $\mathscr{F} \in \mathrm{Coh}(\mathsf{X})$, we let $\mathbb{D}(\mathscr{F})$ denote the sheaf given by $\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}, \omega_{\mathsf{X}})$ (note that the quasi-coherence of $\mathbb{D}(\mathscr{F})$ follows from Proposition 6.5.3.6). We will refer to $\mathbb{D}(\mathscr{F})$ as the *Grothendieck dual* of $\mathscr{F}$.

Warning 6.6.1.7. The Grothendieck dual $\mathbb{D}(\mathscr{F})$ depends on a choice of dualizing sheaf $\omega_{\mathsf{X}} \in \mathrm{QCoh}(\mathsf{X})$.

The terminology of Definition 6.6.1.1 is motivated by the following observation:

Theorem 6.6.1.8. *Let X be a Noetherian spectral algebraic space and let ω_{X} be a dualizing sheaf on X. Then:*

(1) *The construction $\mathscr{F} \mapsto \mathbb{D}(\mathscr{F})$ induces a contravariant equivalence of* $\mathrm{Coh}(\mathsf{X})$ *with itself.*

(2) *For $\mathscr{F} \in \mathrm{Coh}(\mathsf{X})$, the canonical map $\mathscr{F} \to \mathbb{D}(\mathbb{D}(\mathscr{F}))$ is an equivalence.*

Proof. We first show that if $\mathscr{F} \in \mathrm{Coh}(\mathsf{X})$, then $\mathbb{D}(\mathscr{F}) \in \mathrm{Coh}(\mathsf{X})$. This assertion is local on X, so we may assume that $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ is affine. Set $M = \Gamma(\mathsf{X}; \mathscr{F}) \in \mathrm{Mod}_A$ and $K = \Gamma(\mathsf{X}; \omega_{\mathsf{X}}) \in \mathrm{Mod}_A$. It follows from Proposition 6.5.3.6 that $\mathbb{D}(\mathscr{F})$ is quasi-coherent, and is therefore determined by the A-module $\Gamma(\mathsf{X}; \mathbb{D}(\mathscr{F})) \simeq \underline{\mathrm{Map}}_A(M, K)$. We wish to show that each homotopy group $\pi_i \underline{\mathrm{Map}}_A(M, K) = \mathrm{Ext}_A^{-i}(M, K)$ is finitely generated as a module over $\pi_0 A$. Replacing M by a suitable suspension, we can assume that $i = 0$.

Choose an integer n such that $K \in \mathrm{Mod}_A$ has injective dimension $\leqslant n$, and choose a fiber sequence $M' \to M \to M''$ where M' is $(-n)$-connective and M'' is $(-n-1)$-truncated. We have an exact sequence $\mathrm{Ext}_A^0(M'', K) \to \mathrm{Ext}_A^0(M, K) \to \mathrm{Ext}_A^0(M', K)$, where the first term vanishes by virtue of Remark Remark 6.5.1.5. Since the commutative ring $\pi_0 A$ is Noetherian, it will suffice to show that $\mathrm{Ext}_A^0(M', K)$ is finitely generated as a module over $\pi_0 A$. Replacing M by M', we may reduce to the case where there exists an integer k such that M is k-connective. We now proceed by descending induction on k. If $k \gg 0$, then $\mathrm{Ext}_A^0(M, K) \simeq 0$ (since K is truncated). To carry out the inductive step, choose a map $\alpha : \Sigma^k A^a \to M$ which induces a surjection $(\pi_0 A)^a \to \pi_k M$ and set $N = \mathrm{cofib}(\alpha)$. Then N is $(k+1)$-connective, so that $\mathrm{Ext}_A^0(N, K)$ is finitely generated by our inductive hypothesis. Using the exact sequence $\mathrm{Ext}_A^0(N, K) \to \mathrm{Ext}_A^0(M, K) \to (\pi_k K)^a$, we deduce that $\mathrm{Ext}_A^0(M, K)$ is also finitely generated as a module over $\pi_0 A$ as desired.

To complete the proof of (1), it will suffice to prove (2) (so that the duality functor $\mathscr{F} \mapsto \mathbb{D}(\mathscr{F})$ is homotopy inverse to itself). Let $\mathscr{F} \in \mathrm{Coh}(\mathsf{X})$; we wish to show that the biduality map $u_{\mathscr{F}} : \mathscr{F} \to \mathbb{D}(\mathbb{D}(\mathscr{F}))$ is an equivalence. Once again, the assertion is local, so that we may assume that $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ is affine so that $\mathscr{F}$ is determined by the module $M = \Gamma(\mathsf{X}; \mathscr{F}) \in \mathrm{Mod}_A$. We wish to show that for every integer i, the map $u_{\mathscr{F}}$ induces an isomorphism $\pi_i M \to \pi_i \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K)$. Replacing M by a suspension if necessary, we may again suppose that $i = 0$.

Choose an integer m such that K is m-truncated. Then for every k-truncated A-module N, the dual $\underline{\mathrm{Map}}_A(N, K)$ is $(-n-k)$-connective (Remark 6.5.1.5), so that the double dual $\underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K)$ is $(m+n+k)$-truncated. Choose k so that both k and $m+n+k$ are negative. Then the fiber sequence $\tau_{\geqslant k+1} M \xrightarrow{\rho} M \to \tau_{\leqslant k} M$ gives rise to a fiber sequence

$$\underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(\tau_{\geqslant k+1} M, K), K) \xrightarrow{\rho'} \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K) \to \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(\tau_{\leqslant k} M, K), K),$$

where ρ and ρ' both induce isomorphisms on π_0. We may therefore replace M by $\tau_{\geqslant k+1} M$ and thereby reduce to showing that the map $u_{\mathscr{F}}$ is an equivalence whenever M is almost connective. Replacing M by a suitable suspension, we are reduced to proving the following:

$(*)$ Let $M \in \mathrm{Mod}_A$ be connective and almost perfect. Then the biduality map $\theta_M : M \to \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K)$ is an equivalence.

To prove $(*)$, we show that the morphism θ_M is p-connective for every integer p. Since M is connective, the dual $\underline{\mathrm{Map}}_A(M, K)$ is m-truncated, so that $\underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K)$ is $(-n-m)$-connective. Our claim therefore follows automatically if $p < 0, -n-m$. We proceed in general using induction on p. Since $\pi_0 M$ is finitely generated as a $\pi_0 A$-module, we can choose a fiber sequence $N \to A^a \to M$ where $N \in \mathrm{Mod}_A$ is connective and almost perfect. We therefore have a fiber sequence $\mathrm{fib}(\theta_N) \to \mathrm{fib}(\theta_A)^a \to \mathrm{fib}(\theta_M)$. The definition of a dualizing complex guarantees that u_A is an equivalence, so we obtain an equivalence $\mathrm{fib}(\theta_M) \simeq \Sigma\,\mathrm{fib}(\theta_N)$. The inductive hypothesis implies that $\Sigma\,\mathrm{fib}(\theta_N)$ is $(p-1)$-connective, so that $\mathrm{fib}(\theta_M)$ is p-connective as desired. □

In the situation of Theorem 6.6.1.8, there is a close relationship between properties of a sheaf $\mathscr{F} \in \mathrm{Coh}(\mathsf{X})$ and properties of its Grothendieck dual $\mathbb{D}(\mathscr{F})$:

Proposition 6.6.1.9. *Let X be a Noetherian spectral algebraic space, let ω_{X} be a dualizing sheaf on X, and let $\mathscr{F} \in \mathrm{Coh}(\mathsf{X})$. Then:*

(a) *The sheaf $\mathscr{F}$ is almost perfect if and only if $\mathbb{D}(\mathscr{F})$ is truncated.*

(b) *If $\mathscr{F}$ has finite* Tor*-amplitude, then the dual $\mathbb{D}(\mathscr{F})$ has finite injective dimension.*

(c) *The sheaf $\mathscr{F}$ is perfect if and only if $\mathbb{D}(\mathscr{F})$ is truncated and of finite injective dimension.*

Lemma 6.6.1.10. *Let X be a Noetherian spectral algebraic space, let $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$, and assume that $\mathscr{F}$ is coherent.*

(1) *If $\mathscr{F}$ is m-connective and $\mathscr{G}$ is n-truncated, then $\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}, \mathscr{G})$ is $(n-m)$-truncated.*

(2) *If $\mathscr{F}$ has* Tor*-amplitude $\leqslant m$ and $\mathscr{G}$ has injective dimension $\leqslant n$, then $\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}, \mathscr{G})$ has injective dimension $\leqslant n+m$.*

Proof. For each object $\mathscr{H} \in \mathrm{QCoh}(\mathsf{X})$, we have canonical isomorphisms

$$\mathrm{Ext}^*_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{H}, \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}, \mathscr{G})) \simeq \mathrm{Ext}^*_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F} \otimes \mathscr{H}, \mathscr{G}).$$

If the hypothesis of (1) is satisfied and $\mathscr{H}$ is connective, then $\mathscr{F} \otimes \mathscr{H}$ is m-connective, so that our assumption that $\mathscr{G}$ is n-truncated guarantees that the groups $\mathrm{Ext}^*_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F} \otimes \mathscr{H}, \mathscr{G})$ vanish for $* < -n+m$, so that $\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}, \mathscr{G})$ is $(n-m)$-truncated. If the hypothesis of (2) is satisfied and $\mathscr{H}$ is 0-truncated, then the tensor product $\mathscr{F} \otimes \mathscr{H}$ is m-truncated. Our assumption that $\mathscr{G}$ has injective dimension $\leqslant n$ guarantees that the groups $\mathrm{Ext}^*_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F} \otimes \mathscr{H}, \mathscr{G})$ vanish for $* > n+m$, so that $\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}, \mathscr{G})$ has injective dimension $\leqslant n+m$. □

Proof of Proposition 6.6.1.9. Let X be a Noetherian spectral algebraic space which admits a dualizing sheaf ω_{X}. Replacing ω_{X} by a shift if necessary, we may assume that ω_{X} is 0-truncated and has injective dimension $\leqslant n$, for some integer n. If $\mathscr{F}$ is almost perfect, then it is $(-m)$-connective for some $m \gg 0$, so that $\mathbb{D}(\mathscr{F})$ is m-truncated by virtue of Lemma 6.6.1.10. Conversely, suppose that $\mathbb{D}(\mathscr{F})$ is k-truncated for some integer k. We claim that $\mathbb{D}(\mathbb{D}(\mathscr{F}))$ is $(-k-n)$-connective, so that $\mathscr{F}$ is $(-k-n)$-connective (by virtue of Theorem

6.6.1.8). To prove this, we can work locally (Proposition 6.5.3.6) and thereby reduce to the case where $\mathsf{X} \simeq \text{Spét}\, A$ is affine. In this case, it suffices to show that the homotopy groups

$$\pi_{-i}\Gamma(\mathsf{X}; \mathbb{D}(\mathbb{D}(\mathscr{F}))) \simeq \text{Ext}^i_{\text{QCoh}(\mathsf{X})}(\mathbb{D}(\mathscr{F}), \omega_{\mathsf{X}})$$

vanish for $i > k + n$, which follows from our assumption that $\mathbb{D}(\mathscr{F})$ is k-truncated and ω_{X} has injective dimension $\leqslant n$ (see Remark 6.5.1.5). This completes the proof of (a).

If $\mathscr{F}$ has Tor-amplitude $\leqslant k$, then Lemma 6.6.1.10 guarantees that $\mathbb{D}(\mathscr{F})$ has injective dimension $\leqslant k + n$, which proves (b).

If $\mathscr{F}$ is perfect, then it is both almost perfect and of finite Tor-amplitude, so that $\mathbb{D}(\mathscr{F})$ is truncated and of finite injective dimension by virtue of (a) and (b). We will complete the proof by establishing the converse. Assume that $\mathbb{D}(\mathscr{F})$ is truncated and of finite injective dimension; we wish to show that $\mathscr{F}$ is perfect. This assertion can be tested locally on X (see Propositions 6.5.2.2 and 6.5.3.6), so we may assume without loss of generality that $\mathsf{X} \simeq \text{Spét}\, A$ is affine.

Choose an integer k such that $\mathbb{D}(\mathscr{F})$ has injective dimension $\leqslant k$. Applying Corollary 2.7.2.2, we can choose a fiber sequence $\mathscr{F}' \to \mathscr{F} \xrightarrow{\alpha} \mathscr{F}''$ in $\text{QCoh}(\mathsf{X})$, where $\mathscr{F}'$ is perfect and $\mathscr{F}''$ is $(k+1)$-connective. Lemma 6.6.1.10 then implies that the Grothendieck dual $\mathbf{D}(\mathscr{F}'')$ is $(-k-1)$-truncated. Invoking our assumption that $\mathbb{D}(\mathscr{F})$ has injective dimension $\leqslant k$ (and Remark 6.5.1.5), we conclude that the map $\mathbb{D}(\alpha) : \mathbb{D}(\mathscr{F}'') \to \mathbb{D}(\mathscr{F})$ is nullhomotopic. Theorem 6.6.1.8 implies that α is nullhomotopic, so that $\mathscr{F}$ is equivalent to a direct summand of $\mathscr{F}'$ and is therefore perfect, as desired. □

Let X be a Noetherian spectral algebraic space with dualizing sheaf ω_{X}. If $\mathscr{F} \in \text{Coh}(\mathsf{X})$ is perfect, then the Grothendieck dual $\mathbb{D}(\mathscr{F})$ is given by the tensor product $\mathscr{F}^\vee \otimes \omega_{\mathsf{X}}$, where $\mathscr{F}^\vee$ is the $\mathscr{O}_{\mathsf{X}}$-linear dual of $\mathscr{F}$. Note that the construction $\mathscr{F} \mapsto \mathscr{F}^\vee$ induces an equivalence of the ∞-category $\text{QCoh}(\mathsf{X})^{\text{perf}}$ with its opposite. Combining this observation with Theorem 6.6.1.8 and Proposition 6.6.1.9, we obtain the following:

Corollary 6.6.1.11. *Let X be a Noetherian spectral algebraic space with dualizing sheaf ω_{X}. Then the construction $\mathscr{F} \mapsto \mathscr{F} \otimes \omega_{\mathsf{X}}$ determines a fully faithful embedding $\text{QCoh}(\mathsf{X})^{\text{perf}} \to \text{Coh}(\mathsf{X})$, whose essential image is spanned by those objects $\mathscr{G} \in \text{Coh}(\mathsf{X})$ which are truncated and of finite injective dimension.*

6.6.2 Uniqueness of Dualizing Sheaves

If X is a Noetherian spectral algebraic space which admits a dualizing sheaf ω_{X}, then ω_{X} is essentially unique.

Proposition 6.6.2.1. *Let X be a Noetherian spectral algebraic space and let ω_{X} be a dualizing sheaf for X. Then an arbitrary object $\omega'_{\mathsf{X}} \in \text{QCoh}(\mathsf{X})$ is a dualizing sheaf if and only if there exists an equivalence $\omega'_{\mathsf{X}} \simeq \omega_{\mathsf{X}} \otimes \mathscr{L}$, where $\mathscr{L}$ is an invertible object of $\text{QCoh}(\mathsf{X})$.*

Remark 6.6.2.2. In the situation of Proposition 6.6.2.1, the invertible sheaf $\mathscr{L} \in \mathrm{QCoh}(\mathsf{X})$ is uniquely determined up to equivalence: it can be recovered as $\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\omega_{\mathsf{X}}, \omega'_{\mathsf{X}})$. In fact, we have the following more precise assertion: if X admits a dualizing sheaf ω_{X}, then the functor $\mathscr{L} \mapsto \omega_{\mathsf{X}} \otimes \mathscr{L}$ determines an equivalence from the full subcategory of $\mathrm{QCoh}(\mathsf{X})$ spanned by the invertible objects to the full subcategory of $\mathrm{QCoh}(\mathsf{X})$ spanned by the dualizing sheaves (this follows by combining Proposition 6.6.2.1 with Corollary 6.6.1.11).

Lemma 6.6.2.3. *Let A be a Noetherian $\mathbb{E}_\infty$-ring, let K be a dualizing module for A, and let Q be an R-module of finite injective dimension. For every almost perfect R-module M, the canonical map*

$$f_M : M \otimes_A \underline{\mathrm{Map}}_A(K, Q) \simeq \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K) \otimes_R \underline{\mathrm{Map}}_A(K, Q) \to \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), Q)$$

is an equivalence.

Proof. Replacing M by a shift, we may assume without loss of generality that M is connective. Let K be m-truncated and let Q have injective dimension $\leqslant n$. Remark 6.5.1.5 implies that $\underline{\mathrm{Map}}_A(K, Q)$ is $(-n-m)$-connective, so that the tensor product $M \otimes_A \underline{\mathrm{Map}}_A(K, Q)$ is $(-n-m)$-connective. Similarly, the connectivity of M implies that $\underline{\mathrm{Map}}_A(M, K)$ is m-truncated, so that $\underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), Q)$ is also $(-n-m)$-connective. It follows that the morphism f_M is $(-n-m-1)$-connective. We prove that f_M is k-connective for every integer k, using induction on k. Since $\pi_0 M$ is a finitely generated module over $\pi_0 A$, we can choose a fiber sequence $M' \to A^a \to M$, where M' is connective. We therefore obtain a fiber sequence $\mathrm{fib}(f_{M'}) \to \mathrm{fib}(f_A)^a \to \mathrm{fib}(f_M)$. It follows immediately from the definitions that f_A is an equivalence, so that $\mathrm{fib}(f_M) \simeq \Sigma \, \mathrm{fib}(f_{M'})$. The inductive hypothesis implies that $\mathrm{fib}(f_{M'})$ is $(k-1)$-conncetive, so that $\mathrm{fib}(f_M)$ is k-connective as desired. □

Proof of Proposition 6.6.2.1. It follows immediately from the definitions that if ω_{X} is a dualizing sheaf on X and $\omega'_{\mathsf{X}} \simeq \omega_{\mathsf{X}} \otimes \mathscr{L}$ for some invertible object $\mathscr{L} \in \mathrm{QCoh}(\mathsf{X})$, then ω'_{X} is also a dualizing sheaf on X. To prove the converse, suppose that $\omega_{\mathsf{X}}, \omega'_{\mathsf{X}} \in \mathrm{QCoh}(\mathsf{X})$ are both dualizing sheaves. Set $\mathscr{L} = \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\omega_{\mathsf{X}}, \omega'_{\mathsf{X}})$. It follows from Lemma 6.6.2.3 that if $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is almost perfect, then the canonical map

$$\mathscr{F} \otimes \mathscr{L} \to \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}, \omega_{\mathsf{X}}), \omega'_{\mathsf{X}})$$

is an equivalence. Taking $\mathscr{F} = \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}'}}(\omega'_{\mathsf{X}}, \omega_{\mathsf{X}})$ (which is almost perfect by Proposition 6.6.1.9), we obtain an equivalence $\mathscr{F} \otimes \mathscr{L} \simeq \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\omega'_{\mathsf{X}}, \omega'_{\mathsf{X}}) \simeq \mathscr{O}_{\mathsf{X}}$, so that $\mathscr{L}$ is an invertible object of $\mathrm{QCoh}(\mathsf{X})$. To complete the proof, it will suffice to show that the canonical map $\omega_{\mathsf{X}} \otimes \mathscr{L} \to \omega'_{\mathsf{X}}$ is an equivalence. This is a consequence of the following more general assertion (applied in the case $\mathscr{G} = \omega_{\mathsf{X}}$):

$(*)$ For every object $\mathscr{G} \in \mathrm{Coh}(\mathsf{X})$, the canonical map $u_{\mathscr{G}} : \mathscr{G} \otimes \mathscr{L} \to \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{G}, \omega_{\mathsf{X}}), \omega'_{\mathsf{X}})$ is an equivalence.

To prove $(*)$, it will suffice to show that $u_{\mathscr{G}}$ induces an isomorphism of homotopy sheaves

$$\pi_i(\mathscr{G} \otimes \mathscr{L}) \to \pi_i \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{G}, \omega_{\mathsf{X}}), \omega'_{\mathsf{X}})$$

for every integer i. Replacing $\mathscr{G}$ by a shift, we may suppose that $i = 0$. Let $\mathscr{F}$ be defined as above, so that we can identify the domain of $u_{\mathscr{G}}$ with $\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}, \mathscr{G})$. Choose an integer a such that $\mathscr{F}$ is a-connective. For every integer k, the spectrum $\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}, \tau_{\leqslant k}\mathscr{G})$ is $(a+k)$-truncated. Let ω_{X} have injective dimension $\leqslant n$, so that $\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\tau_{\leqslant k}\mathscr{G}, \omega_{\mathsf{X}})$ is $(-n-k)$-connective (Remark 6.5.1.5). Choose m such that ω'_{X} is m-truncated, so that $\underline{\mathrm{Map}}_{\omega_{\mathsf{X}}}(\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\tau_{\leqslant k}\mathscr{G}, \omega_{\mathsf{X}}), \omega'_{\mathsf{X}})$ is $(k+m+n)$-truncated. It follows that if $k < -a, -m-n$, then the maps

$$\pi_0((\tau_{\geqslant k}\mathscr{G}) \otimes \mathscr{L}) \to \pi_0(\mathscr{G} \otimes \mathscr{L})$$

$$\pi_0 \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\tau_{\leqslant k}\mathscr{G}, \omega_{\mathsf{X}}), \omega'_{\mathsf{X}}) \to \pi_0 \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{G}, \omega_{\mathsf{X}}), \omega'_{\mathsf{X}})$$

are isomorphisms. We may therefore replace $\mathscr{G}$ with $\tau_{\geqslant k}\mathscr{G}$, in which case the desired result follows from Lemma 6.6.2.3. □

6.6.3 Base Change of Dualizing Sheaves

If $f : \mathsf{X} \to \mathsf{Y}$ is a proper morphism of Noetherian spectral algebraic spaces, then there is a close relationship between dualizing sheaves on X and dualizing sheaves on Y:

Proposition 6.6.3.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a proper morphism of Noetherian spectral algebraic spaces, and let $f^! : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ be the exceptional inverse image functor of Definition 6.4.0.1. If ω_{Y} is a dualizing sheaf on Y, then $f^!\omega_{\mathsf{Y}}$ is a dualizing sheaf on X. In particular, if Y admits a dualizing sheaf, then so does X.*

Proof. Set $\omega_{\mathsf{X}} = f^!\omega_{\mathsf{Y}}$. It follows from Proposition 6.4.3.4 that ω_{X} is truncated and coherent, and it follows from Remark 6.5.2.3 that ω_{X} has finite injective dimension. To complete the proof, it will suffice to show that the canonical map $\mathscr{O}_{\mathsf{X}} \to \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\omega_{\mathsf{X}}, \omega_{\mathsf{X}})$ is an equivalence. Equivalently, we must show that for each object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, tensor product with ω_{X} induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \mathscr{O}_{\mathsf{X}}) \to \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F} \otimes \omega_{\mathsf{X}}, \omega_{\mathsf{X}}) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{Y})}(f_*(\mathscr{F} \otimes \omega_{\mathsf{X}}), \omega_{\mathsf{Y}}).$$

Writing $\mathscr{F}$ as a filtered colimit of perfect objects of $\mathrm{QCoh}(\mathsf{X})$ (Proposition 9.6.1.1), we can assume that $\mathscr{F}$ is perfect. Let $\mathscr{F}^\vee = \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}, \mathscr{O}_{\mathsf{X}})$. We then have a canonical equivalence

$$\begin{aligned} f_*(\mathscr{F} \otimes \omega_{\mathsf{X}}) &\simeq f_* \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}^\vee, \omega_{\mathsf{X}}) \\ &\simeq f_* \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{F}^\vee, f^!\omega_{\mathsf{Y}}) \\ &\simeq \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{Y}}}(f_* \mathscr{F}^\vee, \omega_{\mathsf{Y}}). \end{aligned}$$

Under this equivalence, the map θ is obtained from the biduality map $\theta' : f_* \mathscr{F}^\vee \to \underline{\mathrm{Map}}_{\mathscr{O}_\mathsf{Y}}(\underline{\mathrm{Map}}_{\mathscr{O}_\mathsf{Y}}(f_* \mathscr{F}^\vee, \omega_\mathsf{Y})\omega_\mathsf{Y})$ by passing to global sections. It follows from Theorem 5.6.0.2 that $f_* \mathscr{F}^\vee \in \mathrm{QCoh}(\mathsf{Y})$ is almost perfect, so that θ' is an equivalence by virtue of Theorem 6.6.1.8. □

Corollary 6.6.3.2. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of Noetherian spectral algebraic spaces which is proper and locally of finite* Tor*-amplitude. Suppose that Y admits a dualizing sheaf ω_Y. Then the tensor product $\omega_{\mathsf{X}/\mathsf{Y}} \otimes f^* \omega_\mathsf{Y}$ is a dualizing sheaf for X.*

Proof. Combine Proposition 6.6.3.1 with Corollary 6.4.2.7. □

In the special case where X and Y are affine, Proposition 6.6.3.1 specializes to the following:

Corollary 6.6.3.3. *Let A be a Noetherian $\mathbb{E}_\infty$-ring which admits a dualizing module K. Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings which exhibits B as an almost perfect A-module. Then $K' = \underline{\mathrm{Map}}_A(B, K)$ is a dualizing module for B.*

Corollary 6.6.3.4. *Let A be a Noetherian $\mathbb{E}_\infty$-ring. If A admits a dualizing module, then the commutative ring $\pi_0 A$ admits a dualizing module.*

Corollary 6.6.3.5. *Let X be a Noetherian spectral algebraic space and let $\omega \in \mathrm{QCoh}(\mathsf{X})$ be a dualizing sheaf. Then, for each $x \in |\mathsf{X}|$ with residue field $\kappa(x)$, the !-fiber $\omega_x^!$ is an invertible object of $\mathrm{Mod}_{\kappa(x)}$ (that is, it is equivalent to some suspension $\Sigma^d \kappa(x)$).*

Proof. Using Remark 6.6.1.3, we can reduce to the case where $\mathsf{X} = \mathrm{Spét}\, A$ is affine, so $M = \Gamma(\mathsf{X}; \omega)$ is a dualizing module for A and the point x corresponds to some prime ideal $\mathfrak{p} \subseteq A$. Using Corollary 6.6.3.3, we may replace A by $(\pi_0 A)/\mathfrak{p}$ and M by $\underline{\mathrm{Map}}_A((\pi_0 A)/\mathfrak{p}, M)$ and further reduce to the case where A is an integral domain and $\mathfrak{p} = (0)$. Let $K = \kappa(x)$ be the fraction field of A, so that we can identify $\omega_x^!$ with the tensor product $K \otimes_A M$. Our assumption that M is a dualizing module guarantee that the homotopy groups $\pi_n M$ are finitely generated modules over A which vanish for $n \gg 0$, and Proposition 6.5.2.5 guarantees that they also vanish for $n \ll 0$. Using Lemma **??**, we deduce that the canonical map

$$\theta : K \otimes_A \underline{\mathrm{Map}}_A(M, M) \to \underline{\mathrm{Map}}_A(M, K \otimes_A M) \simeq \underline{\mathrm{Map}}_K(K \otimes_A M, K \otimes_A M)$$

is an equivalence. Since M is a dualizing module for A, we can identify the domain of θ with K. It follows that $K \otimes_A M$ is an indecomposable K-module (that is, it is nonzero and cannot be nontrivially decomposed as a direct sum). Since K is a field, it follows that $\omega_x^! \simeq K \otimes_A M$ an invertible object of Mod_K. □

6.6.4 Reduction to Commutative Algebra

Our next goal is to establish a converse of Corollary 6.6.3.4:

Theorem 6.6.4.1. *Let A be a Noetherian $\mathbb{E}_\infty$-ring. If $\pi_0 A$ admits a dualizing module, then A admits a dualizing module.*

Example 6.6.4.2. Let S denote the sphere spectrum. Since $\pi_0 S \simeq \mathbf{Z}$ admits a dualizing module, S also admits a dualizing module. In fact, we can describe this dualizing module explicitly. Let $I \in \mathrm{Sp}$ be the Brown-Comenetz dual of the sphere spectrum (see Example HA.**??**). Then I is an injective object of Sp, which is characterized up to equivalence by the formula

$$\pi_n \underline{\mathrm{Map}}_S(M, I) \simeq \mathrm{Hom}(\pi_{-n} M, \mathbf{Q}/\mathbf{Z})$$

for every integer n and every spectrum M. In particular, we have

$$\pi_n I \simeq \begin{cases} 0 & \text{if } n > 0 \\ \mathbf{Q}/\mathbf{Z} & \text{if } n = 0 \\ \mathrm{Hom}(\pi_{-n} S, \mathbf{Q}/\mathbf{Z}) & \text{if } n < 0. \end{cases}$$

Let $\mathbf{Q}$ denote the field of rational numbers, which we regard as a discrete spectrum. The map of abelian groups $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ induces a map of spectra $\alpha : \mathbf{Q} \to I$. We let K denote the fiber of α. Then

$$\underline{\mathrm{Map}}_S(\mathbf{Z}, K) \simeq \mathrm{fib}(\underline{\mathrm{Map}}_S(\mathbf{Z}, \mathbf{Q}) \to \underline{\mathrm{Map}}_S(\mathbf{Z}, I)) \simeq \mathrm{fib}(\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}) \simeq \mathbf{Z}$$

is a dualizing module for $\mathbf{Z}$. It follows from Proposition 6.6.4.6 that K is a dualizing module for S. The spectrum K is often called the *Anderson dual* of the sphere spectrum. Its homotopy groups are given by

$$\pi_n K \simeq \begin{cases} 0 & \text{if } n > 0 \\ \mathbf{Z} & \text{if } n = 0 \\ 0 & \text{if } n = -1 \\ \mathrm{Hom}(\pi_k S, \mathbf{Q}/\mathbf{Z}) & \text{if } n = -k-1, k > 0. \end{cases}$$

Before turning the proof of Theorem 6.6.4.1, let us note the following consequence:

Theorem 6.6.4.3. *Let A be a Noetherian $\mathbb{E}_\infty$-ring, and let $B \in \mathrm{CAlg}_A^{\mathrm{cn}}$ be almost of finite presentation over A. If A admits a dualizing module, then B admits a dualizing module.*

Remark 6.6.4.4. Since the sphere spectrum S admits a dualizing module (Example 6.6.4.2), it follows from Theorem 6.6.4.3 that every $\mathbb{E}_\infty$-ring B which is almost of finite presentation over S also admits a dualizing module. In particular, every connective $\mathbb{E}_\infty$-ring R can be written as a filtered colimit of Noetherian $\mathbb{E}_\infty$-rings which admit dualizing modules. This fact can be quite useful in combination with the Noetherian approximation techniques of Chapter **??**.

Proof of Theorem 6.6.4.3. It follows from Theorem HA.7.2.4.31 that B is Noetherian. We wish to show that B admits a dualizing module. Using Corollary 6.6.3.4 and Theorem 6.6.4.1, we can replace A and B by $\pi_0 A$ and $\pi_0 B$, and thereby reduce to the case where B is a commutative ring which is finitely presented as an algebra over the Noetherian commutative ring A. Choose a surjection $A[x_1, \ldots, x_n] \rightarrow B$. To prove that A admits a dualizing module, it will suffice to show that $A[x_1, \ldots, x_n]$ admits a dualizing module (Proposition 6.6.3.3), or equivalently that the affine space $\operatorname{Sp\acute{e}t} A[x_1, \ldots, x_n]$ admits a dualizing sheaf. Note that there exists an open immersion $j : \operatorname{Sp\acute{e}t} A[x_1, \ldots, x_n] \hookrightarrow \mathbf{P}^n_A$. By virtue of Remark 6.6.1.3, it will suffice to show that $\mathbf{P}^n_A$ admits a dualizing sheaf. This follows from Proposition 6.6.4.6, since the projection map $\mathbf{P}^n_A \rightarrow \operatorname{Sp\acute{e}t} A$ is proper (Corollary 5.4.3.4). □

Remark 6.6.4.5. In the proof of Theorem 6.6.4.3, we can be more precise: if A is a Noetherian commutative ring and K is a dualizing module for A, then $K[x_1, \ldots, x_n] = K \otimes_A A[x_1, \ldots, x_n]$ is a dualizing module for $A[x_1, \ldots, x_n]$.

Our proof of Theorem 6.6.4.1 is based on the following recognition criterion:

Proposition 6.6.4.6. *Let $f : A \rightarrow B$ be a map of Noetherian $\mathbb{E}_\infty$-rings. Suppose that the induced map $\pi_0 A \rightarrow \pi_0 B$ is a surjection of commutative rings whose kernel $I \subseteq \pi_0 A$ is nilpotent. Let K be a truncated A-module, and suppose that $\underline{\operatorname{Map}}_A(B, K)$ is a dualizing module for B. Then K is a dualizing module for A.*

Proof. Let $f : A \rightarrow B$ be a morphism of Noetherian $\mathbb{E}_\infty$-rings which exhibits $\pi_0 B$ as the quotient of $\pi_0 A$ by a nilpotent ideal $I \subseteq \pi_0 A$, and suppose that $K \in \operatorname{Mod}_A$ has the property that $f^\dagger K = \underline{\operatorname{Map}}_K(B, A)$ is a dualizing module for B. It follows from Lemma 6.4.3.7 that the homotopy groups of $\pi_i K$ are finitely generated over $\pi_0 A$. Choose $n \gg 0$ such that $\underline{\operatorname{Map}}_A(B, K)$ has injective dimension $\leqslant n$ as a B-module. We claim that K has injective dimension $\leqslant n$ as an A-module. Let M be a discrete A-module; we wish to prove that the groups $\operatorname{Ext}^i_A(M, K)$ vanish for $i > n$. Since I is nilpotent, the module M is annihilated by I^k for some integer $k \geqslant 1$. We proceed by induction on k. We have an exact sequence $0 \rightarrow IM \rightarrow M \rightarrow M/IM \rightarrow 0$ of discrete $\pi_0 A$-modules, giving rise to short exact sequences $\operatorname{Ext}^i_A(M/IM, K) \rightarrow \operatorname{Ext}^i_A(M, K) \rightarrow \operatorname{Ext}^i_A(IM, K)$. The groups $\operatorname{Ext}^i_A(IM, K)$ vanish for $i > n$ by the inductive hypothesis. The quotient M/IM has the structure of a module over $\pi_0 B$, so that $\operatorname{Ext}^i_A(IM, K) \simeq \operatorname{Ext}^i_B(IM, \underline{\operatorname{Map}}_A(B, K))$ vanishes since $\underline{\operatorname{Map}}_A(B, K)$ has injective dimension $\leqslant n$. It follows that $\operatorname{Ext}^i_A(M, K)$ vanishes, as desired.

To complete the proof, it will suffice to show that the biduality map $A \to \underline{\mathrm{Map}}_A(K, K)$ is an equivalence. We will prove more generally that for every almost perfect A-module M, the canonical map $u_M : M \to \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K)$ is an equivalence. For this, it suffices to show that u_M induces an isomorphism $\pi_i M \to \pi_i \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K)$ for every integer i. Replacing M by a shift, we can assume that $i = 0$. Choose an integer m such that K is m-truncated. For every integer k, the module $\underline{\mathrm{Map}}_A(\tau_{\geqslant k} M, K)$ is $(m-k)$-truncated, so that $\underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(\tau_{\geqslant k} M, K), K)$ is $(k-n-m)$-connective (Remark 6.5.1.5). If $k > n+m$, it follows that the canonical map

$$\pi_0 \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K) \to \pi_0 \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(\tau_{\leqslant k-1} M, K), K)$$

is an isomorphism. Assuming also that k is positive (so that $\pi_0 M \simeq \pi_0 \tau_{\leqslant k-1} M$), we may replace M by $\tau_{\leqslant k-1} M$ and thereby reduce to the case where M is truncated. It will therefore suffice to show that u_M is an equivalence whenever M is truncated and almost perfect. In this case, M is a successive extension of A-modules which are concentrated in a single degree. It will therefore suffice to show that u_M is an equivalence when M is discrete A-module which is finitely generated over $\pi_0 A$. Since I is nilpotent, we can write M as a successive extension of discrete A-modules which are annihilated by I. We may therefore assume that M is annihilated by I, and therefore admits the structure of a B-module. In this case, we have

$$\underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K) \simeq \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_B(M, K'), K) \simeq \underline{\mathrm{Map}}_B(\underline{\mathrm{Map}}_B(M, K'), K'),$$

where $K' = \underline{\mathrm{Map}}_A(B, K)$. The assertion that u_M is an equivalence now follows from Theorem 6.6.1.8. □

Notation 6.6.4.7. For every connective $\mathbb{E}_\infty$-ring A, let $(\mathrm{Mod}_A)_{<\infty}$ denote the full subcategory of Mod_A spanned by the truncated A-modules (that is, $(\mathrm{Mod}_A)_{<\infty} = \bigcup_n (\mathrm{Mod}_A)_{\leqslant n}$).

If $f : A \to B$ be a morphism fo connective $\mathbb{E}_\infty$-rings, we let $f^\dagger : \mathrm{Mod}_A \to \mathrm{Mod}_B$ denote the right adjoint to the forgetful functor $\mathrm{Mod}_B \to \mathrm{Mod}_A$, given by $f^\dagger M = \underline{\mathrm{Map}}_A(B, M)$. Note that the functor $f^\dagger : \mathrm{Mod}_A \to \mathrm{Mod}_B$ carries $(\mathrm{Mod}_A)_{<\infty}$ into $(\mathrm{Mod}_B)_{<\infty}$.

Lemma 6.6.4.8. *Suppose we are given a pullback diagram of connective $\mathbb{E}_\infty$-rings τ :*

$$\begin{array}{ccc} A' & \xrightarrow{f'} & A \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ B' & \xrightarrow{f} & B, \end{array}$$

where f and g induce surjections $\pi_0 A \to \pi_0 B$, $\pi_0 B' \to \pi_0 B$. Then the induced diagram σ :

$$\begin{array}{ccc} (\mathrm{Mod}_{A'})_{<\infty} & \xrightarrow{f'^{\dagger}} & (\mathrm{Mod}_{A})_{<\infty} \\ \downarrow{\scriptstyle g'^{\dagger}} & & \downarrow{\scriptstyle g^{\dagger}} \\ (\mathrm{Mod}_{B'})_{<\infty} & \xrightarrow{f^{\dagger}} & (\mathrm{Mod}_{B})_{<\infty} \end{array}$$

is a pullback square of ∞-categories.

Proof. Let $\mathcal{C}$ denote the fiber product $(\mathrm{Mod}_A)_{<\infty} \times_{(\mathrm{Mod}_B)_{<\infty}} (\mathrm{Mod}_{B'})_{<\infty}$. Unwinding the definitions, we can identify the objects of $\mathcal{C}$ with triples (M, N, α), where M is a truncated A-module, N is a truncated B'-module, and $\alpha : \underline{\mathrm{Map}}_A(B, M) \to \underline{\mathrm{Map}}_{B'}(B, N)$ is an equivalence of B'-modules. The diagram σ determines a functor $G : (\mathrm{Mod}_{A'})_{<\infty} \to \mathcal{C}$; we wish to prove that G is an equivalence. We note that G has a left adjoint F, given on objects by the formula $F(M, N, \alpha) = M \amalg_{\underline{\mathrm{Map}}_A(B,M)} N$. We first prove that the counit map $v : F \circ G \to \mathrm{id}$ is an equivalence from $(\mathrm{Mod}_{A'})_{<\infty}$ to itself. Unwinding the definitions, we must show that if M is a truncated A'-module, then the diagram

$$\begin{array}{ccc} \underline{\mathrm{Map}}_{A'}(B, M) & \longrightarrow & \underline{\mathrm{Map}}_{A'}(A, M) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Map}}_{A'}(B', M) & \longrightarrow & \underline{\mathrm{Map}}_{A'}(A', M) \end{array}$$

is a pushout square of A'-modules. This follows from our assumption that τ is a pullback square.

Since v is an equivalence, we deduce that the functor G is fully faithful. To complete the proof, it will suffice to show that F is conservative. Since F is an exact functor between stable ∞-categories, it will suffice to show that if (M, N, α) is an object of $\mathcal{C}$ which is annihilated by F, then M and N are both trivial. Suppose otherwise. Then there exists a smallest integer n such that $\pi_i M \simeq \pi_i N \simeq 0$ for $i > n$. Then $\pi_n M$ and $\pi_n N$ cannot both vanish; without loss of generality, we may assume that $\pi_n N \neq 0$. We have an exact sequence

$$0 \to \pi_n \underline{\mathrm{Map}}_A(B, M) \to \pi_n M \oplus \pi_n N \to \pi_n F(M, N, \alpha).$$

Since $\pi_n F(M, N, \alpha)$ vanishes, we deduce that the map $\pi_n \underline{\mathrm{Map}}_A(B, M) \to \pi_n M \oplus \pi_n N$ is an isomorphism. This contradicts the nontriviality of $\pi_n N$, since the map $\pi_n \underline{\mathrm{Map}}_A(B, M) \simeq \pi_n M$ is injective (because the homotopy groups $\pi_i M$ vanish for $i > n$). □

Lemma 6.6.4.9. *Let B be a Noetherian $\mathbb{E}_\infty$-ring, and let A be a square-zero extension of B by a connective, almost perfect B-module M. Suppose that B admits a dualizing module K. Then there exists a dualizing module K' for A and an equivalence $K \simeq \underline{\mathrm{Map}}_A(B, K')$.*

Proof. We will show that there exists a truncated A-module K' and an equivalence $K \simeq \underline{\mathrm{Map}}_A(B, K')$. Then K' is automatically a dualizing module for A, by Proposition 6.6.4.6. We have a pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \eta \\ B & \xrightarrow{\eta_0} & B \oplus \Sigma M. \end{array}$$

By virtue of Lemma 6.6.4.8, it will suffice to show that $\eta^\dagger K$ and $\eta_0^\dagger K$ are equivalent as modules over $B \oplus \Sigma M$. Both $\eta^\dagger K$ and $\eta_0^\dagger K$ are dualizing modules for $B \oplus \Sigma M$ (Proposition 6.6.3.1). It follows that there exists an invertible module P for $B \oplus \Sigma M$ and an equivalence $\eta_0^\dagger K \simeq \eta^\dagger K \otimes_{B \oplus \Sigma M} P$. To complete the proof, it will suffice to show that P is trivial. Let $p : B \oplus \Sigma M \to B$ denote the projection map. Then

$$K \simeq p^\dagger \eta_0^\dagger K \simeq p^\dagger(\eta^\dagger K \otimes_{B \oplus \Sigma M} P) \simeq (p^\dagger \eta^\dagger K) \otimes_{B \oplus \Sigma M} P \simeq K \otimes_{B \oplus \Sigma M} P.$$

Invoking Remark 6.6.2.2, we deduce that $P \otimes_{B \oplus \Sigma M} B$ is equivalent to B (as an B-module). In particular, there exists an isomorphism

$$\pi_0 P \simeq \pi_0(P \otimes_{B \oplus \Sigma M} B) \simeq \pi_0 B.$$

Lifting the unit element of $\pi_0 B$ under such an isomorphism, we obtain an element $e \in \pi_0 P$, which determines a map $\gamma : B \oplus \Sigma M \to P$ of $(B \oplus \Sigma M)$-modules. Then $\mathrm{fib}(\gamma) \otimes_{B \oplus \Sigma M} B$ vanishes, so that $\mathrm{fib}(\gamma) \otimes_{B \oplus \Sigma M} N \simeq 0$ whenever N admits the structure of a B-module. SInce $B \oplus \Sigma M$ can be obtained as an extension of two $(B \oplus \Sigma M)$-modules which admit B-module structures, we deduce htat

$$\mathrm{fib}(\gamma) \simeq \mathrm{fib}(\gamma) \otimes_{B \oplus \Sigma M} (B \oplus \Sigma M)$$

vanishes, so that γ is an equivalence and $P \simeq B \oplus \Sigma M$ as desired. □

Proof of Theorem 6.6.4.1. Let A be a Noetherian $\mathbb{E}_\infty$-ring, and let $K(0)$ be a dualizing module for $\pi_0 A$. Without loss of generality, we may assume that $K(0)$ is 0-truncated. We will show that there exists a 0-truncated A-module K and an equivalence $K(0) \simeq \underline{\mathrm{Map}}_A(\pi_0 A, K)$. It will then follow from Proposition 6.6.4.6 that K is a dualizing module for A.

Since each truncation $\tau_{\leq n+1} A$ is a square-zero extension of $\tau_{\leq n} A$, Lemma 6.6.4.9 allows us to choose a sequence $K(n)$ of dualizing modules for $\tau_{\leq n} A$, together with equivalences

$$K(n) \simeq \underline{\mathrm{Map}}_{\tau_{\leq n+1} A}(\tau_{\leq n} A, K(n+1)).$$

It then follows by induction on n that each $K(n)$ is 0-truncated. Moreover, we have canonical fiber sequences

$$K(n-1) \xrightarrow{\beta_n} K(n) \to \underline{\mathrm{Map}}_{\tau_{\leq n+1} A}(\Sigma^n(\pi_n A), K(n)),$$

so that $\mathrm{cofib}(\beta_n)$ is $(-n)$-truncated for every integer n. Let $K = \varinjlim_n K(n)$, where the colimit is taken in the ∞-category Mod_A. Then K is a 0-truncated R-module, and we have a canonical map of $\pi_0 A$-modules $\alpha : K(0) \to K$. We will complete the proof by showing that α induces an equivalence $e : K(0) \to \underline{\mathrm{Map}}_A(\pi_0 A, K)$. Fix an integer $n \geqslant 0$, so that e is given by the composition

$$K(0) \simeq \underline{\mathrm{Map}}_{\tau_{\leqslant n} A}(\pi_0 A, K(n)) \xrightarrow{e'} \underline{\mathrm{Map}}_{\tau_{\leqslant n} A}(\pi_0 A, \underline{\mathrm{Map}}_A(\tau_{\leqslant n} A, K)) \simeq \underline{\mathrm{Map}}_A(\pi_0 A, K).$$

Here e' is induced by the map $f : K(n) \to \underline{\mathrm{Map}}_A(\tau_{\leqslant n} A, K)$. Let $f' : \underline{\mathrm{Map}}_A(\tau_{\leqslant n} A, K) \to K$ be the canonical map, so that $\mathrm{cofib}(f') \simeq \underline{\mathrm{Map}}_A(\tau_{\geqslant n+1} A, K)$ is $(-n-1)$-truncated. Since each of the maps β_m has $(-m)$-truncated cofiber, we deduce that $\mathrm{cofib}(f' \circ f)$ is $(-n-1)$-truncated. It follows that $\mathrm{cofib}(f)$ is $(-n-1)$-truncated, so that $\mathrm{cofib}(e') \simeq \mathrm{cofib}(e)$ is $(-n-1)$-truncated. Since we can choose n to be arbitrarily large, we conclude that e is an equivalence. □

6.6.5 Gorenstein Spectral Algebraic Spaces

Let X be a Noetherian spectral algebraic space. Then the structure sheaf $\mathscr{O}_{\mathsf{X}}$ automatically satisfies conditions (2) and (4) of Definition 6.6.1.1: that is, it is a coherent sheaf for which the unit map $\mathscr{O}_{\mathsf{X}} \to \underline{\mathrm{Map}}_{\mathscr{O}_{\mathsf{X}}}(\mathscr{O}_{\mathsf{X}}, \mathscr{O}_{\mathsf{X}}$ is an equivalence. This motivates the following:

Definition 6.6.5.1. Let X be a spectral algebraic space. We say that X is *Gorenstein* if X is Noetherian and the structure sheaf $\mathscr{O}_{\mathsf{X}}$ is a dualizing sheaf for X (see Definition 6.6.1.1). We say that a connective $\mathbb{E}_\infty$-ring A is *Gorenstein* if the spectral algebraic space $\mathrm{Spét}\, A$ is Gorenstein.

Remark 6.6.5.2. Gorenstein conditions in the setting of (not necessarily commutative) differential graded algebras have been investigated by a number of authors (see, for example, [63] and [66]). Gorenstein conditions on ring spectra are studied Dywer-Greenlees-Iyengar ([55]), though the context differs from the one we consider here (the reference [**?**] formulates a *local* version of Gorenstein duality, for ring spectra which need not be connective or commutative).

Remark 6.6.5.3. Unwinding the definitions, we see that a Noetherian spectral algebraic space X is Gorenstein if and only if the structure sheaf $\mathscr{O}_{\mathsf{X}}$ is truncated and of finite injective dimension.

Example 6.6.5.4. Let A be a regular Noetherian ring of finite Krull dimension. Then A is Gorenstein (Corollary 6.5.4.10).

Remark 6.6.5.5. Let X be a Noetherian spectral algebraic space. Using Proposition 6.5.4.9, we deduce that X is Gorenstein if and only if there exists integers $m, n \gg 0$ which satisfy the following conditions:

(a) The structure sheaf $\mathscr{O}_{\mathsf{X}}$ is m-truncated.

(b) For each point $x \in |\mathsf{X}|$ having residue field $\kappa(x)$, the !-fiber $\mathscr{O}^!_{\mathsf{X},x} \in \mathrm{Mod}_{\kappa(x)}$ is $(-n)$-connective.

If these conditions are satisfied, then Corollary 6.6.3.5 guarantees that for each $x \in |\mathsf{X}|$, there is an equivalence $\mathscr{O}^!_{\mathsf{X},x} \simeq \Sigma^{d(x)}\kappa(x)$ for some $-n \leqslant d(x) \leqslant m$. Note that Lemma 6.5.4.7 guarantees that $d(x) \geqslant -\mathrm{ht}(x)$, where $\mathrm{ht}(x)$ denotes the height of the point $x \in |\mathsf{X}|$.

Remark 6.6.5.6. Let X be a quasi-compact spectral algebraic space. The condition that X is Gorenstein can be tested locally (with respect to the étale topology) on X. More precisely, suppose that $f : \mathsf{U} \to \mathsf{X}$ is an étale morphism of spectral algebraic spaces, where U is quasi-compact (and therefore also Noetherian). If X is Gorenstein, then U is also Gorenstein; the converse holds if f is surjective (this is an immediate consequence of Remark 6.6.1.3).

Warning 6.6.5.7. In the setting of classical algebraic geometry, Definition 6.6.5.1 is not standard. Condition (a) of Remark 6.6.5.5 is automatic in classical algebraic geometry, but many authors replace (b) by the following *a priori* weaker condition:

(b') For each point $x \in |\mathsf{X}|$, the !-fiber $\mathscr{O}^!_{\mathsf{X},x}$ is an invertible object of $\mathrm{Mod}_{\kappa(x)}$.

Note that if Lemma 6.5.4.7 then guarantees that $\mathscr{O}^!_{\mathsf{X},x} \simeq \Sigma^d\kappa(x)$ for some $d \geqslant -\mathrm{ht}(x)$, where $\mathrm{ht}(x)$ denotes the height of the point $x \in |\mathsf{X}|$. It follows that if $|\mathsf{X}|$ has finite Krull dimension, then conditions (b) and (b') are equivalent.

The Gorenstein property is geometric, in the following sense:

Proposition 6.6.5.8. *Let κ be a field, let $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\,\kappa$ be a morphism between Noetherian spectral algebraic spaces which is of finite type, and let κ' be an extension field of κ. Then X is Gorenstein if and only if $\mathsf{X}' = \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathrm{Sp\acute{e}t}\,\kappa'$ is Gorenstein.*

Proof. Without loss of generality, we may assume that the structure sheaf $\mathscr{O}_{\mathsf{X}}$ is truncated (so that $\mathscr{O}_{\mathsf{X}'}$ is also truncated). Let $q : \mathsf{X}' \to \mathsf{X}$ be the projection map. Since the pullback functor q^* is t-exact, the direct image functor q_* carries objects of $\mathrm{QCoh}(\mathsf{X}')$ of injective dimension $\leqslant n$ to objects of $\mathrm{QCoh}(\mathsf{X})$ of injective dimension $\leqslant n$. Consequently, if X' is Gorenstein, then $q_* \mathscr{O}_{\mathsf{X}'}$ has finite injective dimension. It then follows that $\mathscr{O}_{\mathsf{X}}$ has finite injective dimension (since it is a direct summand of $q_* \mathscr{O}_{\mathsf{X}'}$), so that X is also Gorenstein.

For the converse, assume that X is Gorenstein; we wish to show that X' is also Gorenstein. Without loss of generality, we may assume that $\mathsf{X} = \mathrm{Sp\acute{e}t}\,A$ is affine (Remark 6.6.5.6), so that $\mathsf{X}' \simeq \mathrm{Sp\acute{e}t}\,A'$ for $A' = \kappa' \otimes_\kappa A$. Choose an integer d such that $\pi_0 A$ is generated as a κ-algebra by d elements. Our assumption that X is Gorenstein guarantees that there exists an integer $n \gg 0$ such that $\mathscr{O}^!_{\mathsf{X},x}$ is $(-n)$-connective for each point $x \in |\mathsf{X}|$ (Remark 6.6.5.5). For each point $x' \in |\mathsf{X}'|$ having image $x \in |\mathsf{X}|$, let $\mathsf{X}'_x = \mathrm{Sp\acute{e}t}\,\kappa(x) \times_{\mathsf{X}} \mathsf{X}'$. Note that $\kappa(x)$ is a finitely generated field extension of κ, and can therefore be identified with the fraction field of a reduced closed subvariety $\mathsf{Y} \subseteq \mathbf{P}^m_\kappa$ for some $m \gg 0$. Let $\omega = \omega_{\mathsf{Y}/\operatorname{Sp\acute{e}t}\kappa}$ denote the relative dualizing sheaf of the projection map $\mathsf{Y} \to \operatorname{Sp\acute{e}t}\kappa$, so that $\omega \in \operatorname{QCoh}(\mathsf{Y})$ is also an (absolute) dualizing sheaf (Proposition 6.6.3.1). Set $\mathsf{Y}' = \mathsf{Y} \times_{\mathbf{P}^m_\kappa} \mathbf{P}^m_{\kappa'}$ and let $g : \mathsf{Y}' \to \mathsf{Y}$ be the projection map, so that $g^*\omega \simeq \omega_{\mathsf{Y}'/\operatorname{Sp\acute{e}t}\kappa'}$ is a dualizing sheaf for Y' (Proposition 6.6.3.1 and Remark 6.4.2.6). It follows from Corollary 6.6.3.5 that the restriction of ω to the generic point $\operatorname{Sp\acute{e}t}\kappa(x)$ of Y is an invertible object of $\operatorname{QCoh}(\operatorname{Mod}_{\kappa(x)})$, so $(g^*\omega)|_{\mathsf{X}'_x}$ is an invertible object of $|\mathsf{X}'_x|$. Since $g^*\omega$ is a dualizing sheaf on Y', it has finite injective dimension; the criterion of Proposition 6.5.4.9 then shows that $(g^*\omega)|_{\mathsf{X}'_x} \in \operatorname{QCoh}(\mathsf{X}'_x)$ is also of finite injective dimension, so that X'_x is Gorenstein. Since X'_x has Krull dimension $\leqslant d$, the !-fiber $\mathscr{O}^!_{\mathsf{X}'_x, x'}$ for each point $x' \in |\mathsf{X}'|$ lying over x. Remark 6.5.4.4 then supplies an equivalence $\mathscr{O}^!_{\mathsf{X}',x'} \simeq \mathscr{O}^!_{\mathsf{X},x} \otimes_{\kappa(x)} \mathscr{O}^!_{\mathsf{X}'_x,x'}$, which shows that $\mathscr{O}^!_{\mathsf{X}',x'}$ is $(-n-d)$-connective. Using the criterion of Remark 6.6.5.5, we deduce that X' is Gorenstein. □

6.6.6 Gorenstein Morphisms

We now introduce a relative version of Definition **??**.

Definition 6.6.6.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. We will say that f is *Gorenstein* if the following conditions are satisfied:

(*a*) The morphism f is locally almost of finite presentation and locally of finite Tor-amplitude.

(*b*) For every field κ and every morphism $\eta : \operatorname{Sp\acute{e}t}\kappa \to \mathsf{Y}$, the fiber product $\mathsf{X}_\eta = \mathsf{X} \times_{\mathsf{Y}} \operatorname{Sp\acute{e}t}\kappa$ is locally Gorenstein: that is, for every étale morphism $\operatorname{Sp\acute{e}t} A \to \mathsf{X}_\eta$, the $\mathbb{E}_\infty$-ring A is Gorenstein (Definition 6.6.5.1).

Warning 6.6.6.2. In Definition 6.6.6.1, we do *not* require that the morphism $f : \mathsf{X} \to \mathsf{Y}$ is flat: an assumption of flatness would exclude many examples of interest (for example, quasi-smooth morphisms in the setting of derived algebraic geometry; see Example **??**).

Remark 6.6.6.3. In the situation of Definition 6.6.6.1, suppose that Y is a quasi-separated spectral algebraic space. Then it suffices to check the following *a priori* weaker version of condition (*b*):

(*b'*) For every point $y \in |\mathsf{Y}|$ with residue field $\kappa(y)$, the fiber product $\mathsf{X}_y = \operatorname{Sp\acute{e}t}\kappa(y) \times_{\mathsf{Y}} \mathsf{X}$ is Gorenstein.

This follows from Proposition 6.6.5.8.

Proposition 6.6.6.4. *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y}. \end{array}$$

If f is Gorenstein, so is f'. The converse holds if g is a flat covering.

Proof. The first assertion follows immediately from Propositions 4.2.1.6 and 6.1.2.2. For the converse, assume that g is flat cover and that f' is Gorenstein. Using Propositions 4.2.1.5 and 6.1.2.2, we deduce that f is locally of finite Tor-amplitude and locally almost of finite presentation. To complete the proof that f is Gorenstein, it will suffice to show that for every field κ and every morphism $\eta : \operatorname{Sp\acute{e}t} \kappa \to \mathsf{Y}$, the fiber product $\mathsf{X}_\eta = \operatorname{Sp\acute{e}t} \kappa \times_{\mathsf{Y}} \mathsf{X}$ is locally Gorenstein. Using Proposition 6.6.5.8, we can enlarge the field κ and thereby reduce to the case where η factors through g, in which case the desired result follows from our assumption that f' is Gorenstein. □

Proposition 6.6.6.5. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of Noetherian spectral algebraic spaces. Assume that f is locally of finite type, that f is of finite* Tor*-amplitude, and that $\mathscr{O}_{\mathsf{Y}}$ is truncated. If Y is Gorenstein and the morphism f is Gorenstein, then X is Gorenstein. The converse holds if f is surjective.*

Proof. Suppose first that Y is Gorenstein and that f is Gorenstein; we wish to prove that X is Gorenstein. Without loss of generality, we may suppose that $\mathsf{Y} \simeq \operatorname{Sp\acute{e}t} A$ and $\mathsf{X} = \operatorname{Sp\acute{e}t} B$ are affine. Choose integers $a, b \geqslant 0$ such that A is a-truncated and f has Tor-amplitude $\leqslant b$, so that B is $(a+b)$-truncated. Since f is locally of finite type, we can choose a finite collection of elements $b_1, \ldots, b_d \in \pi_0 B$ which generate $\pi_0 B$ as an algebra over $\pi_0 A$. Since Y is Gorenstein, there exists an integer $n \gg 0$ such that $\mathscr{O}^!_{\mathsf{Y},y}$ is $(-n)$-connective for each $y \in |\mathsf{Y}|$. Let $x \in |\mathsf{X}|$, let $\kappa(x)$ denote the residue field of x. Set $y = f(x) \in |\mathsf{Y}|$ and $\mathsf{X}_y = \operatorname{Sp\acute{e}t} \kappa(x) \times_{\mathsf{Y}} \mathsf{X}$. Since $\mathsf{X}_y \simeq \operatorname{Sp\acute{e}t}(\kappa \otimes_A B)$ has Krull dimension $\leqslant d$, our assumption that f is Gorenstein guarantees that $\mathscr{O}^!_{\mathsf{Y}_y,x}$ is $(-d)$-connective (see Remark 6.6.5.5). It follows Using Remark 6.5.4.4, we obtain a canonical equivalence $\mathscr{O}^!_{\mathsf{X},x} \simeq \mathscr{O}^!_{\mathsf{Y},y} \otimes_{\kappa(y)} \mathscr{O}^!_{\mathsf{X}_y,x}$, so that $\mathscr{O}^!_{\mathsf{X},x}$ is $(-n-d)$-connective. Applying Remark 6.6.5.5, we deduce that X is Gorenstein.

Now suppose that f is surjective and that X is Gorenstein. Choose an integer m such that $\mathscr{O}^!_{\mathsf{X},x}$ is $(-m)$-connective for every point $x \in |\mathsf{X}|$. For each $y \in |\mathsf{Y}|$, we can choose some $x \in |\mathsf{X}|$ lying over y. Then $\mathscr{O}^!_{\mathsf{X}_y,x}$ is nonzero and b-truncated (Proposition 6.5.4.8). Using the equivalence $\mathscr{O}^!_{\mathsf{X},x} \simeq \mathscr{O}^!_{\mathsf{Y},y} \otimes_{\kappa(y)} \mathscr{O}^!_{\mathsf{X}_y,x}$ of Remark 6.5.4.4, we deduce that $\mathscr{O}^!_{\mathsf{Y},y}$ is $(-m-b)$-connective. Applying Remark 6.6.5.5, we conclude that Y is Gorenstein. Similarly, since $\mathscr{O}^!_{\mathsf{Y},y}$ is nonzero and a-truncated, we deduce that $\mathscr{O}^!_{\mathsf{Y}_y,x}$ is $(-m-a)$-connective for each $x \in |\mathsf{X}|$ lying over y, so that the fiber product $\mathsf{X} \times_{\mathsf{Y}} \operatorname{Sp\acute{e}t} \kappa(y)$ is Gorenstein (Remark 6.6.5.5). Using Remark 6.6.6.3, we conclude that f is Gorenstein. □

Corollary 6.6.6.6. *Let $f : \mathsf{X} \to \mathsf{Y}$ and $g : \mathsf{Y} \to \mathsf{Z}$ be morphisms of spectral Deligne-Mumford stacks. If f and g are Gorenstein, then the composition $g \circ f$ is Gorenstein.*

Proof. It follows from Proposition 4.2.3.3 that $(g \circ f)$ is locally almost of finite presentation and from Proposition 6.1.2.5 that $(g \circ f)$ is locally of finite Tor-amplitude. To complete the proof, it will suffice to show that for every field κ and every map $\eta : \operatorname{Spét} \kappa \to \mathsf{Z}$, the fiber product $\mathsf{X}_\eta = \operatorname{Spét} \kappa \times_{\mathsf{Z}} \mathsf{X}$ is Gorenstein. Set $\mathsf{Y}_\eta = \operatorname{Spét} \kappa \times_{\mathsf{Z}} \mathsf{Y}$. Since g is Gorenstein, the fiber Y_η is Gorenstein. The projection map $f_\eta : \mathsf{X}_\eta \to \mathsf{Y}_\eta$ is a pullback of f and is therefore Gorenstein by virtue of Proposition 6.6.6.4. Applying Proposition 6.6.6.4, we deduce that X_η is also Gorenstein. □

In the case of a proper morphism, Definition 6.6.6.1 admits a reformulation in terms of the relative dualizing complex:

Proposition 6.6.6.7. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude. The following conditions are equivalent:*

(a) The morphism f is Gorenstein, in the sense of Definition 6.6.6.1.

(b) The relative dualizing sheaf $\omega_{\mathsf{X}/\mathsf{Y}}$ is an invertible object of $\operatorname{QCoh}(\mathsf{X})$*.*

The proof of Proposition 6.6.6.7 will require the following:

Lemma 6.6.6.8. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and let $\mathscr{F} \in \operatorname{QCoh}(\mathsf{X})$. Assume that:*

(i) The quasi-coherent sheaf $\mathscr{F}$ is almost perfect.

(ii) The morphism f has finite Tor*-amplitude and the quasi-coherent sheaf $\mathscr{F}$ has finite* Tor*-amplitude over Y (see Warning 6.4.4.4).*

(iii) For every field κ and every map $\eta : \operatorname{Spét} \kappa \to \mathsf{Y}$, the restriction of $\mathscr{F}$ to the fiber $\mathsf{X}_\eta = \operatorname{Spét} \kappa \times_{\mathsf{Y}} \mathsf{X}$ is an invertible object of $\operatorname{QCoh}(\mathsf{X}_\eta)$*.*

Then $\mathscr{F}$ is an invertible object of $\operatorname{QCoh}(\mathsf{X})$*.*

Proof. The assertion is local on X. We may therefore assume without loss of generality, we may assume that $\mathsf{Y} \simeq \operatorname{Spét} A$ and $\mathsf{X} \simeq \operatorname{Spét} B$ are affine. Set $M = \Gamma(\mathsf{X}; \mathscr{F})$; we wish to show that M is an invertible object of Mod_A. Fix a point $x \in |\mathsf{X}| \simeq |\operatorname{Spec} A|$ and let $\kappa(x)$ denote the residue field of A at x. It follows from (iii) that $\kappa(x) \times_B M$ is an invertible $\kappa(x)$-module. Replacing M by a suitable suspension, we may assume that $\kappa(x) \otimes_B M \simeq \kappa(x)$. Using our assumption that M is almost perfect, we can assume (after replacing X by an open neighborhood of x if necessary) that M is connective (Proposition 2.7.4.1). Choose an element $u \in \pi_0 M$ having nonzero image in $\pi_0(\kappa(x) \otimes_A M)$, so that u determines an A-module map $\alpha : B \to M$. We will show that, after replacing X by a smaller neighborhood of x if necessary, we can assume that α is an equivalence. To prove this, we are free to replace B by $\pi_0 B$, the $\mathbb{E}_\infty$-ring A by the tensor product $(\pi_0 B) \otimes_B A$, and the module M by the tensor product $(\pi_0 B) \otimes_B M$. In this case, assumption (ii) guarantees that $\mathrm{cofib}(\alpha)$ is n-truncated for some $n \gg 0$. Since $\kappa(x) \otimes_B \mathrm{cofib}(\alpha) \simeq 0$, we can arrange (after replacing X by a smaller neighborhood of x) that $\mathrm{cofib}(\alpha)$ is $(n+1)$-connective (Proposition 2.7.4.1), so that $\mathrm{cofib}(\alpha) \simeq 0$. It follows that α is an equivalence, so that M is an invertible object of Mod_A. □

Proof of Proposition 6.6.6.7. Suppose first that (a) is satisfied. It follows from Proposition 6.4.4.1 and Warning 6.4.4.4 that $\omega_{\mathsf{X}/\mathsf{Y}}$ is almost perfect and of finite Tor-amplitude over Y. For every map $\eta : \mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Y}$, set $\mathsf{X}_\eta = \mathrm{Sp\acute{e}t}\,\kappa \times_{\mathsf{Y}} \mathsf{X}$. Remark 6.4.2.6 supplies an equivalence $\omega_{\mathsf{X}_\eta/\mathrm{Sp\acute{e}t}\,\kappa} \simeq \omega_{\mathsf{X}/\mathsf{Y}}|_{\mathsf{X}_\eta}$. Proposition 6.6.3.1 guarantees that $\omega_{\mathsf{X}_\eta/\mathrm{Sp\acute{e}t}\,\kappa}$ is a dualizing sheaf for X_η, and is therefore invertible by virtue of assumption (a). Using Lemma 6.6.6.8, we deduce that $\omega_{\mathsf{X}/\mathsf{Y}}$ is invertible.

Now suppose that (b) is satisfied. We wish to show that for every field κ and every map $\eta : \mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Y}$ as above, the fiber $\mathsf{X}_\eta = \mathrm{Sp\acute{e}t}\,\kappa \times_{\mathsf{Y}} \mathsf{X}$ is Gorenstein. Assumption (b) guarantees that $\omega_{\mathsf{X}_\eta/\mathrm{Sp\acute{e}t}\,\kappa} \simeq \omega_{\mathsf{X}/\mathsf{Y}}|_{\mathsf{X}_\eta}$ is an invertible object of $\mathrm{QCoh}(\mathsf{X}_\eta)$. Since $\omega_{\mathsf{X}_\eta/\mathrm{Sp\acute{e}t}\,\kappa}$ is a dualizing object sheaf for X_η (Proposition 6.6.3.1), Proposition 6.6.2.1 guarantees that $\mathscr{O}_{\mathsf{X}_\eta}$ is also a dualizing sheaf for X_η, so that X_η is Gorenstein as desired. □

Chapter 7

Nilpotent, Local, and Complete Modules

Let X be a spectral Deligne-Mumford stack and let $j : \mathsf{U} \to \mathsf{X}$ be a quasi-compact open immersion, corresponding to an open subset $U \subseteq |\mathsf{X}|$. Then the pushforward functor $j_* : \mathrm{QCoh}(\mathsf{U}) \to \mathrm{QCoh}(\mathsf{X})$ is fully faithful. Roughly speaking, we can think of the ∞-category $\mathrm{QCoh}(\mathsf{X})$ as built from two parts: sheaves coming from the open set U (that is, those sheaves which lie in the essential image of j_*), and sheaves coming from the closed subset $K = |\mathsf{X}| - U$. In this section, we will develop some language which allows us to articulate this idea more precisely.

We begin in §7.1 by studying support conditions on quasi-coherent sheaves. We say that a sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is *supported* on the closed set $K \subseteq |\mathsf{X}|$ if the pullback $j^* \mathscr{F} \in \mathrm{QCoh}(\mathsf{U})$ vanishes. The quasi-coherent sheaves which are supported on K form a full subcategory $\mathrm{QCoh}_K(\mathsf{X}) \subseteq \mathrm{QCoh}(\mathsf{X})$. We will be particularly interested in the case where $\mathsf{X} = \mathrm{Sp\acute{e}t}\, R$ is affine. In this case, we can identify $K \subseteq |\mathsf{X}| \simeq |\mathrm{Spec}\, R|$ with the vanishing locus of an ideal $I \subseteq \pi_0 R$. A quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is supported on K if and only if the R-module $M = \Gamma(\mathsf{X}; \mathscr{F})$ is *I-nilpotent*: that is, if and only the action of each element $x \in I$ is locally nilpotent on $\pi_* M$. We let $\mathrm{Mod}_R^{\mathrm{Nil}(I)}$ denote the full subcategory of Mod_R spanned by the I-nilpotent R-modules, so that the equivalence of ∞-category $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R) \simeq \mathrm{Mod}_R$ restricts to an equivalence $\mathrm{QCoh}_K(\mathrm{Sp\acute{e}t}\, R) \simeq \mathrm{Mod}_R^{\mathrm{Nil}(I)}$.

We will say that an R-module M is *I-local* if, for every I-nilpotent R-module N, the mapping space $\mathrm{Map}_{\mathrm{Mod}_R}(N, M)$ is contractible. The collection of I-local R-modules span a full subcategory $\mathrm{Mod}_R^{\mathrm{Loc}(I)} \subseteq \mathrm{Mod}_R$. In §7.2, we will show that the pair of subcategories $(\mathrm{Mod}_R^{\mathrm{Nil}(I)}, \mathrm{Mod}_R^{\mathrm{Loc}(I)})$ determine a *semi-orthogonal decomposition* of the stable ∞-category Mod_R (Definition 7.2.0.1): in other words, every R-module M fits into an essentially unique fiber sequence $\Gamma_I M \to M \to L_I M$, where $\Gamma_I M$ is I-nilpotent and $L_I M$ is I-local. This is essentially a formal consequence of the observation that the collection of I-nilpotent objects of Mod_R is closed under colimits: modulo set-theoretic technicalities, one can construct $\Gamma_I M$ as the colimit $\varinjlim_{N \to M} N$, where N ranges over all I-nilpotent objects of $(\mathrm{Mod}_R)_{/M}$. Alternatively, when the ideal I is finitely generated, then the construction $M \mapsto L_I M$ can be identified with the localization functor $\mathscr{F} \mapsto j_* j^* \mathscr{F}$, where $j : \mathsf{U} \hookrightarrow \mathrm{Spét}\, R$ is the open immersion complementary to the vanishing locus of I.

We will say that an R-module M is *I-complete* if, for every I-local R-module N, the mapping space $\mathrm{Map}_{\mathrm{Mod}_R}(N, M)$ is contractible. If the ideal I is finitely generated, then the pair of subcategories $(\mathrm{Mod}_R^{\mathrm{Loc}(I)}, \mathrm{Mod}_R^{\mathrm{Cpl}(I)})$ determine another semi-orthogonal decomposition of the stable ∞-category Mod_R: in other words, every R-module M fits into an essentially unique fiber sequence $M' \to M \to M^{\wedge}_I$, where M' is I-local and $M^{\wedge}_I$ is I-complete. We will refer to the R-module $M^{\wedge}_I$ as the *I-completion* of M. In §7.3, we will make a detailed study of the construction $M \mapsto M^{\wedge}_I$ and its relationship with the classical I-adic completion functor $M \mapsto \varprojlim \{M/I^n M\}$ on the category $\mathrm{Mod}_R^{\heartsuit}$ of discrete R-modules.

Remark 7.0.0.1. By definition, and R-module M is I-nilpotent if and only if, for each $x \in M$, the colimit of the sequence

$$M \xrightarrow{x} M \xrightarrow{x} M \to \cdots$$

vanishes. Similarly, an R-module M is I-complete if and only if, for each $x \in I$, the inverse limit of the tower

$$\cdots \to M \xrightarrow{x} M \xrightarrow{x} M$$

vanishes. The conditions of I-nilpotence and I-completeness are two different ways of making precise the idea that an R-module M should be concentrated on the vanishing locus of the ideal I. In fact, the ∞-categories of I-nilpotent and I-complete R-modules are canonically equivalent (Proposition 7.3.1.7), though they do not coincide as subcategories of Mod_R.

The semi-orthogonal decompositions $(\mathrm{Mod}_R^{\mathrm{Nil}(I)}, \mathrm{Mod}_R^{\mathrm{Loc}(I)})$ and $(\mathrm{Mod}_R^{\mathrm{Loc}(I)}, \mathrm{Mod}_R^{\mathrm{Cpl}(I)})$ of Mod_R can be viewed as "linear" incarnations of a geometric idea: namely, that the affine spectral Deligne-Mumford stack $\mathsf{X} = \mathrm{Spét}\, R$ is obtained by "gluing" a formal neighborhood of the closed set $K \subseteq |\mathsf{X}|$ to the open substack $\mathsf{U} \subseteq \mathsf{X}$ complementary to K. In §7.4, we will make this idea more precise by establishing an ∞-categorical Beauville-Laszlo theorem (Theorem 7.4.0.1): if we take $\widehat{R} = R^{\wedge}_I$ to be the I-completion of R and $\widehat{\mathsf{U}}$ the inverse image of U in $\mathrm{Spét}\, \widehat{R}$, then the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{Spét}\, R) & \longrightarrow & \mathrm{QCoh}(\mathsf{U}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathrm{Spét}\, \widehat{R}) & \longrightarrow & \mathrm{QCoh}(\widehat{\mathsf{U}}) \end{array}$$

is a pullback square.

Though our main interest in this section is in studying R-modules, many of the notions we introduce in this section (such as I-nilpotence, I-locality, and I-completeness) make sense in greater generality. With an eye toward later applications, we frame most of our definitions in the context of R-linear ∞-categories, where R is an $\mathbb{E}_2$-ring. For a review of the relevant definitions, we refer the reader to Appendix D

Contents

7.1 Nilpotence and Support Conditions

Let X be a topological space, let $Y \subseteq X$ be a closed subset, and let $\mathscr{F}$ be a sheaf of abelian groups on X. We will say that $\mathscr{F}$ is *supported on* Y if the restriction $\mathscr{F}|_U \simeq 0$, where $U \subseteq X$ is the complement of Y. If $X = \operatorname{Spec} R$ is an affine scheme, Y is the vanishing locus of an ideal $I \subseteq R$, and $\mathscr{F}$ is the quasi-coherent sheaf associated to an R-module $M = \mathscr{F}(X)$, then $\mathscr{F}$ is supported on Y if and only if each element $x \in I$ determines a locally nilpotent map $M \xrightarrow{x} M$: that is, if and only if the localization $M[x^{-1}]$ vanishes for $x \in I$. Our goal in this section is to study an analogous condition in the setting of spectral algebraic geometry.

7.1.1 I-Nilpotent Objects

We begin with some general remarks. In what follows, we will assume that the reader is familiar with the language of R-linear prestable ∞-categories developed in Appendix D.

Definition 7.1.1.1. Let R be a connective $\mathbb{E}_2$-ring, let $x \in \pi_0 R$, and let $\mathcal{C}$ be a prestable R-linear ∞-category (see Definition D.1.2.1). For each object $C \in \mathcal{C}$, we let $C[x^{-1}]$ denote the relative tensor product $R[x^{-1}] \otimes_R C$ (which we will regard either as an object of $\mathcal{C}$ or of $R[x^{-1}] \otimes_R \mathcal{C} = \mathrm{LMod}_{R[x^{-1}]}(\mathcal{C})$, depending on the context). We will say that C is *x-nilpotent* if the localization $C[x^{-1}]$ vanishes.

Example 7.1.1.2. We will be primarily interested in the special case of Definition 7.1.1.1 where $\mathcal{C} = \mathrm{LMod}_R$ is the ∞-category of left R-modules. In this case, an object $M \in \mathcal{C}$ is x-nilpotent if and only if the action of x on $\pi_* M$ is locally nilpotent: that is, if and only if for each $y \in \pi_k M$, there exists an integer $n \gg 0$ such that $x^n y = 0$ (in the abelian group $\pi_k M$).

Remark 7.1.1.3. Let R be a connective $\mathbb{E}_2$-ring and let M be a left R-module. Then the canonical maps

$$R[x^{-1}] \otimes_R M \to R[x^{-1}] \otimes_R M \otimes_R R[x^{-1}] \leftarrow M \otimes_R R[x^{-1}]$$

are equivalences. Consequently, the localization $M[x^{-1}]$ can be obtained from M by tensoring with $R[x^{-1}]$ on either the left or the right.

Remark 7.1.1.4. Let R be a connective $\mathbb{E}_2$-ring, let $x \in \pi_0 R$, and let $\mathcal{C}$ be a prestable R-linear ∞-category. If $C \in \mathcal{C}$ is x-nilpotent, then the tensor product $M \otimes_R C$ vanishes for any left R-module M for which the map $M \xrightarrow{x} M$ is an equivalence. In this case, the unit map $M \to M[x^{-1}]$ is also an equivalence, so we have

$$M \otimes_R C \simeq M[x^{-1}] \otimes_R C \simeq M \otimes_R R[x^{-1}] \otimes_R C \simeq M \otimes_R C[x^{-1}] \simeq 0.$$

Proposition 7.1.1.5. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $C \in \mathcal{C}$ be an object. Then the set $I = \{x \in \pi_0 R : C \text{ is } x\text{-nilpotent } \}$ is a radical ideal in the commutative ring $\pi_0 R$.*

Proof. For $x, y \in \pi_0 R$, we have $C[(xy)^{-1}] \simeq (C[x^{-1}])[y^{-1}]$, so that $x \in I \Rightarrow xy \in I$. We next show that I is closed under addition. Choose elements $x, y \in \pi_0 R$, so that we have a cofiber sequence of connective R-modules

$$R[(x+y)^{-1}] \to R[(x+y)^{-1}, x^{-1}] \oplus R[(x+y)^{-1}, y^{-1}] \to R[(x+y)^{-1}, x^{-1}, y^{-1}],$$

hence a cofiber sequence

$$C[(x+y)^{-1}] \to C[x^{-1}][(x+y)^{-1}] \oplus C[y^{-1}][(x+y)^{-1}] \to C[x^{-1}][y^{-1}][(x+y)^{-1}].$$

If x and y belong to I, then the second terms in this fiber sequence vanish, so that the first does as well. This completes the proof that I is an ideal. We conclude by observing that for each $x \in \pi_0 R$, we have $C[x^{-1}] \simeq C[(x^n)^{-1}]$ for any $n > 0$, so that $x \in I$ if and only if $x^n \in I$. □

Definition 7.1.1.6. Let R be a connective $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $I \subseteq \pi_0 R$ be an ideal. We say that an object $C \in \mathcal{C}$ is *I-nilpotent* if it is x-nilpotent for each $x \in I$ (in other words, if I is contained in the radical ideal of Proposition 7.1.1.5). We let $\mathcal{C}^{\mathrm{Nil}(I)}$ denote the full subcategory of $\mathcal{C}$ spanned by the I-nilpotent objects.

Example 7.1.1.7. Let R be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 R$ be a finitely generated ideal. Then a left R-module M is I-nilpotent if and only if every element of $\pi_* M$ is annihilated by some power of I.

Remark 7.1.1.8. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable R-linear ∞-category. The condition that an object $C \in \mathcal{C}$ is I-nilpotent depends only on the nilradical of I (Proposition 7.1.1.5). In other words, it depends only on the closed subset of $|\operatorname{Spec} \pi_0 R|$ given by the vanishing locus of I.

Remark 7.1.1.9. Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be an R-linear prestable ∞-category, and let $I \subseteq \pi_0 R$ be the sum of a collection of ideals $I_\alpha \subseteq \pi_0 R$. Then an object $C \in \mathcal{C}$ is I-nilpotent if and only if C is I_α-nilpotent for each α.

Remark 7.1.1.10. Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_2$-rings, let $I \subseteq \pi_0 R$ be an ideal, and let I' denote the ideal of $\pi_0 R'$ generated by the image of I. Let $\mathcal{C}$ be a R'-linear prestable ∞-category, which we regard as an R-linear prestable ∞-category by restriction of scalars. Then an object $C \in \mathcal{C}$ is I'-nilpotent if and only if it is I-nilpotent.

Remark 7.1.1.11. If we restrict our attention to the case where $\mathcal{C}$ is stable, then Definition 7.1.1.6 makes sense without the assumption that R is connective (see Variant D.1.5.1). However, this does not really result in any additional generality: by virtue of Remark 7.1.1.10, the I-nilpotence of an object $C \in \mathcal{C}$ does not depend on whether we regard $\mathcal{C}$ as an R-linear ∞-category or as a $(\tau_{\geq 0} R)$-linear ∞-category.

Proposition 7.1.1.12. *Let R be a connective $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $I \subseteq \pi_0 R$ be an ideal. Then:*

- (a) *The ∞-category $\mathcal{C}^{\mathrm{Nil}(I)}$ is presentable and is closed under colimits and extensions in $\mathcal{C}$, and is therefore a Grothendieck prestable ∞-category.*
- (b) *For every left R-module M, the construction $C \mapsto M \otimes_R C$ carries I-nilpotent objects of $\mathcal{C}$ to I-nilpotent objects of $\mathcal{C}$. Consequently, $\mathcal{C}^{\mathrm{Nil}(I)}$ inherits the structure of an R-linear prestable ∞-category.*
- (c) *If $\mathcal{C}$ is stable, then $\mathcal{C}^{\mathrm{Nil}(I)}$ is stable.*
- (d) *The full subcategory $\mathcal{C}^{\mathrm{Nil}(I)}$ is closed under finite limits. Consequently, the inclusion functor $\mathcal{C}^{\mathrm{Nil}(I)} \hookrightarrow \mathcal{C}$ is left exact.*
- (e) *If $\mathcal{C}$ is compactly generated and I is finitely generated, then $\mathcal{C}^{\mathrm{Nil}(I)}$ is also compactly generated. Moreover, the inclusion $\mathcal{C}^{\mathrm{Nil}(I)} \hookrightarrow \mathcal{C}$ preserves compact objects.*

Proof. Assertion (a) is obvious, and assertion (b) follows from the calculation

$$\begin{aligned}(M \otimes_R C)[x^{-1}] &\simeq R[x^{-1}] \otimes_R M \otimes_R C \\ &\simeq M[x^{-1}] \otimes_R C \\ &\simeq M \otimes_R R[x^{-1}] \otimes_R C \\ &\simeq M \otimes_R C[x^{-1}];\end{aligned}$$

see Remark 7.1.1.3. Assertion (c) is an immediate consequence of (a), since the collection of I-nilpotent objects of $\mathcal{C}$ is closed under desuspension. To prove (d), it suffices to show that for each $x \in \pi_0 R$, the collection of x-nilpotent objects of $\mathcal{C}$ is closed under finite limits. This follows from the fact that filtered colimits in $\mathcal{C}$ are left exact (so that the construction $C \mapsto C[x^{-1}] = \varinjlim(C \xrightarrow{x} C \xrightarrow{x} C \xrightarrow{x} \cdots)$ commutes with finite limits).

We now prove (e). Assume that $\mathcal{C}$ is compactly generated; we will prove that $\mathcal{C}^{\mathrm{Nil}(I)}$ is compactly generated by verifying the criterion of Corollary C.6.3.3. Let C be a nonzero I-nilpotent object of $\mathcal{C}$; we wish to show that there exists a nonzero map $C_0 \to C$, where C_0 is a compact object of $\mathcal{C}^{\mathrm{Nil}(I)}$. Applying the criterion of Corollary C.6.3.3 to the ∞-category $\mathcal{C}$, we deduce that there exists a nonzero map $\rho : C' \to C$, where C' is a compact object of $\mathcal{C}$. For each $x \in I$, the localization $C[x^{-1}]$ vanishes so that the composite map $C' \xrightarrow{\rho} C \to C[x^{-1}]$ is nullhomotopic. Realizing $C[x^{-1}]$ as the colimit of the sequence

$$C \xrightarrow{x} C \xrightarrow{x} C \xrightarrow{x} \cdots,$$

we deduce that the composite map $C' \xrightarrow{\rho} C \xrightarrow{x^k} C$ vanishes for $k \gg 0$. Using the commutativity of the diagram

$$\begin{array}{ccc} C' & \xrightarrow{\rho} & C \\ \downarrow{x^k} & & \downarrow{x^k} \\ C' & \xrightarrow{\rho} & C, \end{array}$$

we deduce that ρ factors through the cofiber $\mathrm{cofib}(x^k : C' \to C')$ We may therefore replace C' by $\mathrm{cofib}(x^k : C' \to C')$ and thereby reduce to the case where $C'[x^{-1}] \simeq 0$. Applying this procedure repeatedly, we can reduce to the case where $C'[x_i^{-1}] \simeq 0$ for some finite sequence $x_1, \ldots, x_n \in I$ which generates the ideal I. Then C' is I-nilpotent (Proposition 7.1.1.5), so that $\rho : C' \to C$ is a nonzero morphism in $\mathcal{C}^{\mathrm{Nil}(I)}$ whose domain is compact.

The preceding argument shows that every nonzero object $C \in \mathcal{C}^{\mathrm{Nil}(I)}$ admits a nonzero map where $C' \in \mathcal{D}$ is compact as an object of $\mathcal{C}$. The collection of such objects is closed under finite colimits and extensions, and therefore forms a set of compact generators for $\mathcal{C}^{\mathrm{Nil}(I)}$ (see Proposition C.6.3.1 and Corollary C.6.3.3). It follows immediately that $\mathcal{C}^{\mathrm{Nil}(I)}$ is compactly generated and that the inclusion $\mathcal{C}^{\mathrm{Nil}(I)} \hookrightarrow \mathcal{C}$ preserves compact objects, as desired. □

Variant 7.1.1.13. Let R be a connective $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be an abelian R-linear ∞-category (see Definition D.1.4.1), and let $I \subseteq \pi_0 R$ be an ideal. Then $\mathcal{C}^{\mathrm{Nil}(I)}$ is also an R-linear abelian category.

Example 7.1.1.14. Let R be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 R$ be an ideal. Then the ∞-category $\mathrm{LMod}_R^{\mathrm{Nil}(I)}$ is bitensored over LMod_R. More precisely, if $M \in \mathrm{LMod}_R$ is I-nilpotent and $N \in \mathrm{LMod}_R$ is arbitrary, then $M \otimes_R N$ and $N \otimes_R M$ are I-nilpotent.

Remark 7.1.1.15. Let R be a connective $\mathbb{E}_2$-ring, and let $F : \mathcal{C} \to \mathcal{D}$ be an R-linear functor between prestable R-linear ∞-categories. For every ideal $I \subseteq \pi_0 R$, F restricts to an R-linear functor $F^{\mathrm{Nil}(I)} : \mathcal{C}^{\mathrm{Nil}(I)} \to \mathcal{D}^{\mathrm{Nil}(I)}$. It follows from assertion (d) of Proposition 7.1.1.12 that if F is left exact, then $F^{\mathrm{Nil}(I)}$ is left exact.

Remark 7.1.1.16. Let R be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be an R-linear prestable ∞-category. For each $x \in \pi_0 R$ and each $C \in \mathcal{C}$, we have canonical isomorphisms $\pi_n C[x^{-1}] \simeq (\pi_n C)[x^{-1}]$ in the abelian category $\mathcal{C}^{\heartsuit}$. Consequently, if $\mathcal{C}$ is separated and $I \subseteq \pi_0 R$ is an ideal, then an object $C \in \mathcal{C}$ is I-nilpotent if and only if $\pi_n C \in \mathcal{C}^{\heartsuit}$ is I-nilpotent for each $n \geqslant 0$. In this case, the prestable ∞-category $\mathcal{C}^{\mathrm{Nil}(I)}$ is also separated. If $\mathcal{C}$ is complete, the $\mathcal{C}^{\mathrm{Nil}(I)}$ is also complete.

Remark 7.1.1.17. Let R be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be an R-linear stable ∞-category equipped with a t-structure that is right complete and compatible with filtered colimits (so that $\mathcal{C}_{\geqslant 0}$ is a prestable R-linear ∞-category and $\mathcal{C} \simeq \mathrm{Sp}(\mathcal{C}_{\geqslant 0})$). For every ideal $I \subseteq \pi_0 R$, the following conditions on an object $C \in \mathcal{C}$ are equivalent:

- (a) The object $C \in \mathcal{C}$ is I-nilpotent.
- (b) For every integer n, the truncations $\tau_{\geqslant n} C$ and $\tau_{\leqslant n} C$ are I-nilpotent.
- (c) For every integer n, the truncation $\tau_{\geqslant n} C$ is I-nilpotent.

It follows that the pair $(\mathcal{C}^{\mathrm{Nil}(I)}_{\geqslant 0}, \mathcal{C}^{\mathrm{Nil}(I)}_{\leqslant 0}) = (\mathcal{C}^{\mathrm{Nil}(I)} \cap \mathcal{C}_{\geqslant 0}, \mathcal{C}^{\mathrm{Nil}(I)} \cap \mathcal{C}_{\leqslant 0})$ is a right compete t-structure on $\mathcal{C}^{\mathrm{Nil}(I)}$ which is compatible with filtered colimits. In particular, we have $\mathcal{C}^{\mathrm{Nil}(I)} \simeq \mathrm{Sp}(\mathcal{C}^{\mathrm{Nil}(I)}_{\geqslant 0})$.

If the t-structure on $\mathcal{C}$ is separated (meaning that $\bigcap \mathcal{C}_{\geqslant n}$ consists only of zero objects of $\mathcal{C}$), then conditions (a), (b), and (c) are equivalent to the following *a priori* weaker condition:

- (d) For each $n \in \mathbf{Z}$, the object $\pi_n C \in \mathcal{C}^{\heartsuit}$ is I-nilpotent.

Remark 7.1.1.18. Let R be connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 R$ be an ideal. Let us say that a prestable R-linear ∞-category $\mathcal{C}$ is *I-nilpotent* if $\mathcal{C} = \mathcal{C}^{\mathrm{Nil}(I)}$: that is, if every object $C \in \mathcal{C}$ is I-nilpotent. Let $\mathrm{LinCat}^{\mathrm{PSt}}_R$ denote the ∞-category of prestable R-linear ∞-categories (Definition D.1.4.1), and let $\mathrm{LinCat}^{\mathrm{Nil}(I)}_R$ denote the full category of $\mathrm{LinCat}^{\mathrm{PSt}}_R$ spanned by the I-nilpotent prestable R-linear ∞-categories. It follows from Remark 7.1.1.15 that if $\mathcal{C}$ is I-nilpotent, then any R-linear functor $F : \mathcal{C} \to \mathcal{D}$ factors through the full subcategory $\mathcal{D}^{\mathrm{Nil}(I)}$. Consequently, we can view the construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{Nil}(I)}$ as a right adjoint to the inclusion $\mathrm{LinCat}^{\mathrm{Nil}(I)}_R \hookrightarrow \mathrm{LinCat}^{\mathrm{PSt}}_R$.

7.1.2 Local Cohomology

Let R be a commutative ring containing an ideal $I \subseteq R$. Every (discrete) R-module M contains a largest I-nilpotent submodule $M_0 \subseteq M$, given by $M_0 = \{x \in M : (\forall y \in I)[y^n x = 0 \text{ for } n \gg 0\}$. We now discuss a generalization of this construction.

Definition 7.1.2.1. Let R be a connective $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be an ideal, and let $\mathcal{C}$ be a prestable R-linear ∞-category. We let $\Gamma_I : \mathcal{C} \to \mathcal{C}^{\mathrm{Nil}(I)}$ denote a right adjoint to the inclusion functor $\iota : \mathcal{C}^{\mathrm{Nil}(I)} \hookrightarrow \mathcal{C}$ (note that the existence of Γ_I is guaranteed by Corollary HTT.5.5.2.9 , since ι is a colimit preserving functor between presentable ∞-categories).

Warning 7.1.2.2. The notation of Definition 7.1.2.1 is potentially ambiguous: the functor Γ_I depends not only on the ideal I, but also on the prestable R-linear ∞-category $\mathcal{C}$. When the ∞-category $\mathcal{C}$ is not clear from the context, we will denote the functor Γ_I by $\Gamma^{\mathcal{C}}_I$.

Our next goal is to show that the operation Γ_I is reasonably well-behaved when the ideal I is finitely generated. We begin by treating the case where I is a principal ideal:

Proposition 7.1.2.3. *Let R be a connective $\mathbb{E}_2$-ring, let x be an element of $\pi_0 R$, and let $\mathcal{C}$ be a prestable R-linear ∞-category. For every object $C \in \mathcal{C}$, we have a fiber sequence (depending functorially on C) $\Gamma_{(x)} C \to C \to C[x^{-1}]$.*

Proof. Let $u : C \to C[x^{-1}]$ denote the canonical map (obtained from the localization map $R \to R[x^{-1}]$ by tensoring with C). Since filtered colimits in $\mathcal{C}$ are left exact, we have $\mathrm{fib}(u)[x^{-1}] = \mathrm{fib}(C[x^{-1}] \to (C[x^{-1}])[x^{-1}]) \simeq 0$: that is, the fiber $\mathrm{fib}(u)$ is (x)-nilpotent. For any (x)-nilpotent object $D \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(D, C[x^{-1}]) \simeq \mathrm{Map}_{\mathrm{LMod}_{R[x^{-1}]}(\mathcal{C})}(D[x^{-1}], C[x^{-1}])$ is contractible. It follows that the map $\mathrm{Map}_{\mathcal{C}}(D, \mathrm{fib}(u)) \to \mathrm{Map}_{\mathcal{C}}(D, C)$ is a homotopy equivalence, so that $\mathrm{fib}(u) \simeq \Gamma_{(x)} C$. □

Remark 7.1.2.4. Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $I, J \subseteq \pi_0 R$ be ideals. It follows from Remark 7.1.1.9 that we have $\mathcal{C}^{\mathrm{Nil}(I+J)} = (\mathcal{C}^{\mathrm{Nil}(I)})^{\mathrm{Nil}(J)}$. Consequently, the inclusion $\mathcal{C}^{\mathrm{Nil}(I+J)} \hookrightarrow \mathcal{C}$ factors as a composition $(\mathcal{C}^{\mathrm{Nil}(I)})^{\mathrm{Nil}(J)} \hookrightarrow \mathcal{C}^{\mathrm{Nil}(I)} \hookrightarrow \mathcal{C}$. It follows that the functor $\Gamma^{\mathcal{C}}_{I+J} : \mathcal{C} \to \mathcal{C}^{\mathrm{Nil}(I+J)}$ decomposes as compositions

$$\mathcal{C} \xrightarrow{\Gamma^{\mathcal{C}}_I} \mathcal{C}^{\mathrm{Nil}(I)} \xrightarrow{\Gamma^{\mathcal{C}^{\mathrm{Nil}(I)}}_J} (\mathcal{C}^{\mathrm{Nil}(I)})^{\mathrm{Nil}(J)} = \mathcal{C}^{\mathrm{Nil}(I+J)}.$$

Corollary 7.1.2.5. *Let R be a connective $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $\mathcal{C}$ be a prestable R-linear ∞-category. Then the functor $\Gamma_I : \mathcal{C} \to \mathcal{C}^{\mathrm{Nil}(I)}$ commutes with filtered colimits.*

Proof. Using Remark 7.1.2.4, we can reduce to the case where I is generated by a single element, in which case the desired result follows from Proposition 7.1.2.3. □

Remark 7.1.2.6. In the situation of Corollary 7.1.2.5, suppose that the ∞-category $\mathcal{C}$ is stable. Then the full subcategory $\mathcal{C}^{\mathrm{Nil}(I)} \subseteq \mathcal{C}$ is also stable (Proposition 7.1.1.12). Consequently, the assumption that I is finitely generated implies that we can regard $\Gamma_I : \mathcal{C} \to \mathcal{C}^{\mathrm{Nil}(I)}$ as an R-linear functor (Remark D.1.5.3).

Corollary 7.1.2.7. *Let R be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 R$ be a finitely generated ideal. Then there exists a left R-module V such that the functor $M \mapsto M \otimes_R V$ is right adjoint to the inclusion $\mathrm{LMod}_R^{\mathrm{Nil}(I)} \hookrightarrow \mathrm{LMod}_R$.*

Proof. Using Remark 7.1.2.6, we see that the functor $\Gamma_I : \mathrm{LMod}_R \to \mathrm{LMod}_R^{\mathrm{Nil}(I)}$ is given by $M \mapsto M \otimes_R \Gamma_I(R)$. □

Corollary 7.1.2.8. *Let R be a connective $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $F : \mathcal{C} \to \mathcal{D}$ be an R-linear functor between prestable R-linear ∞-categories. If F is compact (Definition C.3.4.2), then the induced map $F^{\mathrm{Nil}(I)} : \mathcal{C}^{\mathrm{Nil}(I)} \to \mathcal{D}^{\mathrm{Nil}(I)}$ is also compact.*

Proof. The commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{Nil}(I)} & \longrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle F^{\mathrm{Nil}(I)}} & & \downarrow{\scriptstyle F} \\ \mathcal{D}^{\mathrm{Nil}(I)} & \longrightarrow & \mathcal{D} \end{array}$$

yields a commutative diagram of right adjoints

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{Nil}(I)} & \xleftarrow{\Gamma_I^{\mathcal{C}}} & \mathcal{C} \\ \uparrow{\scriptstyle G'} & & \uparrow{\scriptstyle G} \\ \mathcal{D}^{\mathrm{Nil}(I)} & \xleftarrow{\Gamma_I^{\mathcal{D}}} & \mathcal{D} . \end{array}$$

Since F is compact, the functor G commutes with filtered colimits. Using Corollary 7.1.2.5, we deduce that the functor $(\Gamma_I^{\mathcal{C}} \circ G \simeq G' \circ \Gamma_I^{\mathcal{D}}) : \mathcal{D} \to \mathcal{C}^{\mathrm{Nil}(I)}$ commutes with filtered colimits. Restricting to the full subcategory $\mathcal{D}^{\mathrm{Nil}(I)} \subseteq \mathcal{D}$, we conclude that G' commutes with filtered colimits, so that $F^{\mathrm{Nil}(I)}$ is compact. □

Remark 7.1.2.9. In the special case where $\mathcal{C}$ is compactly generated, Corollaries 7.1.2.5 and 7.1.2.8 can also be deduced from part (e) of Proposition 7.1.1.12 (see Proposition HTT.5.5.7.2).

Corollary 7.1.2.10. *Let R be a connective $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $F : \mathcal{C} \to \mathcal{D}$ be an R-linear functor between prestable R-linear ∞-categories. If F is left exact, then the diagram*

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{Nil}(I)} & \longrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle F^{\mathrm{Nil}(I)}} & & \downarrow{\scriptstyle F} \\ \mathcal{D}^{\mathrm{Nil}(I)} & \longrightarrow & \mathcal{D} \end{array}$$

is right adjointable. In particular, we have a canonical equivalence $F^{\mathrm{Nil}(I)} \circ \Gamma_I^{\mathcal{C}} \simeq \Gamma_I^{\mathcal{D}} \circ F$.

Proof. Using Remarks 7.1.2.4 and 7.1.1.15, we can reduce to the case where $I = (x)$ is generated by a single element. In this case, the desired result follows from the explicit description of the functors $\Gamma_I^{\mathcal{C}}$ and $\Gamma_I^{\mathcal{D}}$ given by Proposition 7.1.2.3. □

Corollary 7.1.2.11. *Let R be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 R$ be a finitely generated ideal. For any prestable R-linear ∞-category $\mathcal{C}$, the canonical map*

$$\theta : (\mathrm{LMod}_R^{\mathrm{cn}})^{\mathrm{Nil}(I)} \otimes_R \mathcal{C} \to \mathrm{LMod}_R^{\mathrm{cn}} \otimes_R \mathcal{C} \simeq \mathcal{C}$$

is a fully faithful embedding, whose essential image is the full subcategory $\mathcal{C}^{\mathrm{Nil}(I)} \subseteq \mathcal{C}$.

Proof. Note that if $M \in \mathrm{LMod}_R^{\mathrm{cn}}$ is I-nilpotent, then the tensor product $M \otimes_R C$ is I-nilpotent for each $C \in \mathcal{C}$. It follows that the functor θ factors as a composition

$$(\mathrm{LMod}_R^{\mathrm{cn}})^{\mathrm{Nil}(I)} \otimes_R \mathcal{C} \xrightarrow{\theta_0} \mathcal{C}^{\mathrm{Nil}(I)} \hookrightarrow \mathcal{C}$$

Since the inclusion $(\mathrm{LMod}_R^{\mathrm{cn}})^{\mathrm{Nil}(I)} \hookrightarrow \mathrm{LMod}_R^{\mathrm{cn}}$ is left exact (Proposition 7.1.1.12), Proposition C.4.4.1 shows that the functor θ is left exact. Since $\mathcal{C}^{\mathrm{Nil}(I)}$ is closed under finite limits in $\mathcal{C}$ (Proposition 7.1.1.12) we conclude that θ_0 is left exact. It follows that the induced map of stabilizations

$$\mathrm{Sp}(\theta_0) : \mathrm{Sp}((\mathrm{LMod}_R^{\mathrm{cn}})^{\mathrm{Nil}(I)} \otimes_R \mathcal{C}) \to \mathrm{Sp}(\mathcal{C}^{\mathrm{Nil}(I)})$$

is t-exact. Consequently, to show that θ_0 is an equivalence of ∞-categories, it will suffice to show that $\mathrm{Sp}(\theta_0)$ is an equivalence of ∞-categories. For this, we can replace $\mathcal{C}$ by $\mathrm{Sp}(\mathcal{C})$ and thereby reduce to the case where $\mathcal{C}$ is stable, and we can identify $\mathrm{Sp}(\theta)$ with the canonical map

$$\mathrm{LMod}_R^{\mathrm{Nil}(I)} \otimes_R \mathcal{C} \to \mathrm{LMod}_R \otimes_R \mathcal{C} \simeq \mathcal{C}\,.$$

Let us regard the inclusion $\iota : \mathrm{LMod}_R^{\mathrm{Nil}(I)} \hookrightarrow \mathrm{LMod}_R$ and the functor $\Gamma_I : \mathrm{LMod}_R \to \mathrm{LMod}_R^{\mathrm{Nil}(I)}$ of Definition 7.1.2.1 as linear with respect to the *right* actions of LMod_R on LMod_R and $\mathrm{LMod}_R^{\mathrm{Nil}(I)}$ (Remark 7.1.2.6). It follows that Γ_I induces a functor

$$\phi : \mathrm{Sp}(\mathcal{C}) \simeq \mathrm{LMod}_R \otimes_R \mathrm{Sp}(\mathcal{C}) \to \mathrm{LMod}_R^{\mathrm{Nil}(I)} \otimes_R \mathrm{Sp}(\mathcal{C})$$

which we can regard as a right adjoint to $\mathrm{Sp}(\theta)$. Since the unit map $\mathrm{id}_{\mathrm{LMod}_R^{\mathrm{Nil}(I)}} \to \Gamma_I \circ \iota$ is an equivalence, it follows that the unit map $\mathrm{id} \to \phi \circ \mathrm{Sp}(\theta)$ is also an equivalence: that is, the functor $\mathrm{Sp}(\theta)$ is fully faithful. It will therefore suffice to show that the essential image of $\mathrm{Sp}(\theta)$ coincides with $\mathcal{C}^{\mathrm{Nil}(I)}$. Equivalently, we wish to show that an object $C \in \mathcal{C}$ is I-nilpotent if and only if it can be obtained as a colimit of objects of the form $M \otimes_R C'$, where $C' \in \mathcal{C}$ is arbitrary and $M \in \mathrm{LMod}_R$ is I-nilpotent. The "only if" direction is clear. To prove the converse, we first note that the construction $M \mapsto M \otimes_R C$ determines an R-linear functor $\mathrm{LMod}_R \to \mathcal{C}$. If C is I-nilpotent, then Corollary 7.1.2.10 supplies an equivalence

$$C \simeq \Gamma_I^{\mathcal{C}}(C) = \Gamma_I^{\mathcal{C}}(R \otimes_R C) \simeq \Gamma_I^{\mathrm{LMod}_R}(R) \otimes_R C,$$

where $\Gamma_I^{\mathrm{LMod}_R}(R)$ is I-nilpotent □

Remark 7.1.2.12. In the statements of Proposition 7.1.2.3 (and Corollaries 7.1.2.5, 7.1.2.7, 7.1.2.8, 7.1.2.10, and 7.1.2.11), the hypothesis that I is finitely generated can be weakened: it is sufficient to assume that there exists a finitely generated ideal $J \subseteq I$ having the same radical as I (see Remark 7.1.1.8).

7.1.3 The Locally Noetherian Case

We now specialize Definition 7.1.1.6 to the locally Noetherian case.

Proposition 7.1.3.1. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{A}$ be an abelian R-linear ∞-category, and let $I \subseteq \pi_0 R$ be an ideal. If $\mathcal{A}$ is locally Noetherian, then $\mathcal{A}^{\mathrm{Nil}(I)}$ is locally Noetherian.*

Proof. It follows from Variant 7.1.1.13 that $\mathcal{A}^{\mathrm{Nil}(I)}$ is an abelian R-linear ∞-category. If $\mathcal{A}$ is locally Noetherian, then every object $X \in \mathcal{A}^{\mathrm{Nil}(I)}$ can be written as a filtered colimit of Noetherian subobjects $X' \subseteq X$. Since X is I-nilpotent, any such subobject is also I-nilpotent, and is therefore a Noetherian object of $\mathcal{A}^{\mathrm{Nil}(I)}$. □

Corollary 7.1.3.2. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be an prestable R-linear ∞-category, and let $I \subseteq \pi_0 R$ be an ideal. If $\mathcal{C}$ is locally Noetherian (see Definition C.6.9.1), then $\mathcal{C}^{\mathrm{Nil}(I)}$ is locally Noetherian.*

Proof. Proposition 7.1.3.1 shows that the heart of $\mathcal{C}^{\mathrm{Nil}(I)}$ is locally Noetherian. Moreover, an object $X \in \mathcal{C}^{\mathrm{Nil}(I)\heartsuit}$ is Noetherian if and only if it is Noetherian when regarded as an object of $\mathcal{C}^{\heartsuit}$ (since the partially ordered set of isomorphism classes of subobjects of X does not depend on whether we view X as an object of $\mathcal{C}^{\heartsuit}$ or $\mathcal{C}^{\mathrm{Nil}(I)\heartsuit}$). In this case, our assumption that $\mathcal{C}$ is locally Noetherian guarantees that X is a compact object of $\tau_{\leqslant n}\mathcal{C}$ for each $n \geqslant 0$, and therefore also compact when viewed as an object of the ∞-category $\tau_{\leqslant n}\mathcal{C}^{\mathrm{Nil}(I)}$. □

Remark 7.1.3.3. Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $I \subseteq \pi_0 R$ be a finitely generated ideal. It follows immediately from the definitions that the functor $\Gamma_I : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})^{\mathrm{Nil}(I)}$ carries injective objects of $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ to injective objects of $\mathrm{Sp}(\mathcal{C})^{\mathrm{Nil}(I)}_{\leqslant 0}$.

Proposition 7.1.3.4. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $I \subseteq \pi_0 R$ be an ideal. Suppose that I is finitely generated and that $\mathcal{C}$ is locally Noetherian. Then the inclusion map $\mathrm{Sp}(\mathcal{C}^{\mathrm{Nil}(I)}) \simeq \mathrm{Sp}(\mathcal{C})^{\mathrm{Nil}(I)} \hookrightarrow \mathrm{Sp}(\mathcal{C})$ carries injective objects of $\mathrm{Sp}(\mathcal{C}^{\mathrm{Nil}(I)})$ to injective objects of $\mathrm{Sp}(\mathcal{C})$ (see .Definition C.5.7.2).*

Proof. Proceeding by induction on the number of generators of I (and using Remark 7.1.2.4), we can reduce to the case where I is generated by a single element. Let Q be an injective object of $\mathrm{Sp}(\mathcal{C}^{\mathrm{Nil}(I)})_{\leqslant 0}$; we wish to show that Q is also injective when viewed as an object of $\mathrm{Sp}(\mathcal{C})$. By virtue of Proposition C.6.10.1, it will suffice to show that the group $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X, Q)$ vanishes for $n > 0$ when X is a Noetherian object of $\mathcal{C}^{\heartsuit}$. Let X' denote the kernel of the map $u : X \to X[x^{-1}]$, and let X'' be the image of u. We then have an exact sequence $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X'', Q) \to \mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X, Q) \to \mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X', Q)$, where the third term vanishes since

X' belongs to the heart of $\mathcal{C}^{\mathrm{Nil}(I)}$. We may therefore replace X by X'' and thereby reduce to the case where the multiplication map $x : X \to X$ is a monomorphism.

Since $\mathcal{C}$ is locally Noetherian, the object X is compact in $\tau_{\leqslant n}\mathcal{C}$, so the construction

$$Y \mapsto \mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X, Y) \simeq \pi_0 \mathrm{Map}_{\mathcal{C}}(X, \Omega^{\infty - n} Y)$$

commutes with filtered colimits when restricted to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. In particular, the vanishing of $Q[x^{-1}]$ guarantees that the action of x on $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X, Q)$ is locally nilpotent. It follows that every element of $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X, Q)$ lies in the image of the natural map $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(\mathrm{cofib}(x^k : X \to X), Q) \to \mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X, Q)$ for $k \gg 0$. We conclude by observing that the abelian group $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(\mathrm{cofib}(x^k : X \to X), Q)$ vanishes, since $\mathrm{cofib}(x^k : X \to X)$ is I-nilpotent and Q is an injective object of $\mathrm{Sp}(\mathcal{C}^{\mathrm{Nil}(I)})_{\leqslant 0}$. □

Remark 7.1.3.5. Let R be a commutative ring, let $I \subseteq R$ be a finitely generated ideal, and let $\mathcal{A}$ be an abelian R-linear category which is locally Noetherian. Combining Proposition 7.1.3.4 with Theorem C.5.7.4, we deduce that the inclusion functor $\mathcal{A}^{\mathrm{Nil}(I)} \hookrightarrow \mathcal{A}$ carries injective objects to injective objects.

7.1.4 The Case of a Derived ∞-Category

Let $\mathcal{A}$ be a Grothendieck abelian category, and let $\mathcal{D}(\mathcal{A})$ be its derived ∞-category (see §HA.**??**). In what follows, we will abuse notation by identifying $\mathcal{A}$ with the full subcategory $\mathcal{D}(\mathcal{A})^{\heartsuit} \subseteq \mathcal{D}(\mathcal{A})$. If $\mathcal{A}$ is equipped with an action of a connective $\mathbb{E}_\infty$-ring R, then $\mathcal{D}(\mathcal{A})$ inherits an action of R (Example D.1.3.9). If $I \subseteq \pi_0 R$ is an ideal, we will say that an object $X \in \mathcal{A}$ is *I-nilpotent* if it is I-nilpotent when regarded as an object of $\mathcal{D}(\mathcal{A})$. Let $\mathcal{A}^{\mathrm{Nil}(I)}$ denote the full subcategory of $\mathcal{A}$ spanned by the I-nilpotent objects. Then $\mathcal{A}^{\mathrm{Nil}(I)} \simeq \mathcal{D}(\mathcal{A})^{\mathrm{Nil}(I)}_{\geqslant 0} \cap \mathcal{D}(\mathcal{A})^{\mathrm{Nil}(I)}_{\leqslant 0}$ is the heart of t-structure $(\mathcal{D}(\mathcal{A})^{\mathrm{Nil}(I)}_{\geqslant 0}, \mathcal{D}(\mathcal{A})^{\mathrm{Nil}(I)}_{\leqslant 0})$ of Remark 7.1.1.17. In particular, $\mathcal{A}^{\mathrm{Nil}(I)}$ is a Grothendieck abelian category which is stable under extensions in $\mathcal{A}$, and the inclusion $\mathcal{A}^{\mathrm{Nil}(I)} \hookrightarrow \mathcal{A}$ is an exact functor. Using Theorem C.5.4.9, we see that the inclusion $\mathcal{A}^{\mathrm{Nil}(I)} \hookrightarrow \mathcal{D}(\mathcal{A})^{\mathrm{Nil}(I)}$ extends to a t-exact functor $\mathcal{D}(\mathcal{A}^{\mathrm{Nil}(I)}) \to \mathcal{D}(\mathcal{A})^{\mathrm{Nil}(I)}$.

Theorem 7.1.4.1. *Let R be a commutative ring, let $I \subseteq R$ be an ideal, and let $\mathcal{A}$ be an abelian R-linear ∞-category. Suppose that $\mathcal{A}$ is locally Noetherian and that I is finitely generated. Then the map $\theta : \mathcal{D}(\mathcal{A}^{\mathrm{Nil}(I)}) \to \mathcal{D}(\mathcal{A})^{\mathrm{Nil}(I)}$ described above is an equivalence of ∞-categories.*

Remark 7.1.4.2. In the statement of Theorem 7.1.4.1, the assumption that R is discrete is irrelevant: the same conclusion is valid for any connective $\mathbb{E}_2$-ring R (see Example D.1.3.6).

We are primarily interested in the following special case of Theorem 7.1.4.1:

Corollary 7.1.4.3. *Let R be a Noetherian commutative ring, let $I \subseteq R$ be an ideal, and let $\mathrm{Mod}_R^{\heartsuit\,\mathrm{Nil}(I)}$ denote the abelian category of discrete, I-nilpotent R-modules. Then the inclusion $\mathrm{Mod}_R^{\heartsuit\,\mathrm{Nil}(I)} \hookrightarrow \mathrm{Mod}_R^{\mathrm{Nil}(I)}$ extends to a t-exact equivalence of ∞-categories $\mathcal{D}(\mathrm{Mod}_R^{\heartsuit\,\mathrm{Nil}(I)}) \to \mathrm{Mod}_R^{\mathrm{Nil}(I)}$.*

The proof of Theorem 7.1.4.1 will require some preliminaries.

Proposition 7.1.4.4. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, let $n \geqslant 0$ be an integer, and let $I \subseteq \pi_0 R$ be a finitely generated ideal. Then:*

(a) If $\mathcal{C}$ is locally Noetherian and weakly n-complicial, then $\mathcal{C}^{\mathrm{Nil}(I)}$ is weakly n-complicial.

(b) If $\mathcal{C}$ is locally Noetherian and n-complicial, then $\mathcal{C}^{\mathrm{Nil}(I)}$ is n-complicial.

Proof. We first prove (a). Assume that $\mathcal{C}$ is locally Noetherian and weakly n-complicial; we wish to show that $\mathcal{C}^{\mathrm{Nil}(I)}$ is weakly n-complicial. By virtue of Proposition C.5.7.11, it will suffice to show that every injective object $Q \in \mathrm{Sp}(\mathcal{C}^{\mathrm{Nil}(I)})$ belongs to $\mathrm{Sp}(\mathcal{C}^{\mathrm{Nil}(I)})_{\geqslant -n}$. Our assumption that $\mathcal{C}$ is locally Noetherian guarantees that Q is also injective when viewed as an object of $\mathrm{Sp}(\mathcal{C})$ (Proposition 7.1.3.4). The desired result now follows from Proposition C.5.7.11, since $\mathcal{C}$ is assumed to be weakly n-complicial.

We now prove (b). Assume that $\mathcal{C}$ is locally Noetherian and n-complicial; we wish to show that $\mathcal{C}^{\mathrm{Nil}(I)}$ is also n-complicial. Proceeding by induction on the number of generators of I, we can reduce to the case where $I = (x)$ is a principal ideal. Fix an object $X \in \mathcal{C}^{\mathrm{Nil}(I)}$; we wish to prove that there exists a map $f : Y \to X$ which induces an epimorphism on π_0, where Y is an n-truncated object of $\mathcal{C}^{\mathrm{Nil}(I)}$. Writing X as a filtered colimit of objects of the form $\mathrm{fib}(x^n : X \to X)$, we can reduce to the case where the map $x^n : X \to X$ is nullhomotopic. Our assumption that $\mathcal{C}$ is n-complicial guarantees that we can find a map $g : Z \to X$ which induces an epimorphism on π_0, where Z is an n-truncated object of $\mathcal{C}$. Then g factors as a composition $Z \to Z' \xrightarrow{g'} X$, where $Z' = \mathrm{cofib}(x^n : Z \to Z)$. Since Z is n-truncated, the object Z' is $(n+1)$-truncated. We now apply (a) to deduce the existence of an n-truncated object $Y \in \mathcal{C}^{\mathrm{Nil}(I)}$ and a morphism $h : Y \to Z'$ which induces an epimorphism on π_0. We complete the proof by setting $f = g' \circ h$. □

Proof of Theorem 7.1.4.1. Combine Proposition 7.1.4.4 with Remark C.5.4.11.

□

Remark 7.1.4.5. In the situation of Theorem 7.1.4.1, it follows from Remark 7.1.3.3 that the functor $\Gamma_I|_{\mathcal{D}^+(\mathcal{A})} : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{A}^{\mathrm{Nil}(I)})$ can be identified with the right derived functor of the left exact functor of abelian categories $\Gamma_I^{\heartsuit} : \mathcal{A} \to \mathcal{A}^{\mathrm{Nil}(I)}$, in the sense of Example HA.1.3.3.4 .

Example 7.1.4.6. Let R be a Noetherian commutative ring and let $I \subseteq R$ be an ideal. Applying Remark 7.1.4.5 to the abelian category $\mathcal{A} = \mathrm{Mod}_R^{\heartsuit}$, we see that the functor $\Gamma_I : \mathrm{Mod}_R \to \mathrm{Mod}_R^{\mathrm{Nil}(I)}$ can be identified with the right derived functor of $\Gamma_I^{\heartsuit} : \mathrm{Mod}_R^{\heartsuit} \to \mathrm{Mod}_R^{\heartsuit \, \mathrm{Nil}(I)}$ given by $\Gamma_I^{\heartsuit} M = \{x \in M : I^n x = 0 \text{ for } n \gg 0\} \subseteq M$. In other words, Definition 7.1.2.1 reproduces Grothendieck's theory of local cohomology (see, for example, [**?**]).

7.1.5 Supports of Quasi-Coherent Sheaves

Let R be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be an ideal, and let M be an R-module. We now observe that the condition M is I-nilpotent (Definition 7.1.1.6) has a geometric interpretation: it is equivalent to the requirement that the quasi-coherent sheaf associated to M is supported on the vanishing locus of I (Proposition 7.1.5.3).

Definition 7.1.5.1. Let X be a spectral Deligne-Mumford stack, let $K \subseteq |\mathsf{X}|$ be a closed subset, and let $j : \mathsf{U} \to \mathsf{X}$ be the complementary open immersion. We will say that a quasi-coherent sheaf $\mathscr{F} \in \mathsf{X}$ is *supported on* K if $j^* \mathscr{F}$ is a zero object of $\mathrm{QCoh}(\mathsf{U})$. We let $\mathrm{QCoh}_K(\mathsf{X})$ denote the full subcategory of $\mathrm{QCoh}(\mathsf{X})$ spanned by those quasi-coherent sheaves which are supported on K.

Remark 7.1.5.2. Let X be a spectral Deligne-Mumford stack, let $K \subseteq |\mathsf{X}|$ be a closed subset, and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$. Then $\mathscr{F}$ is supported on K if and only if each of the homotopy sheaves $\pi_i \mathscr{F}$ is supported on K. It follows that the full subcategories

$$\mathrm{QCoh}_K(\mathsf{X})_{\geqslant 0} = \mathrm{QCoh}_K(\mathsf{X}) \cap \mathrm{QCoh}(\mathsf{X})_{\geqslant 0} \qquad \mathrm{QCoh}_K(\mathsf{X})_{\leqslant 0} = \mathrm{QCoh}_K(\mathsf{X}) \cap \mathrm{QCoh}(\mathsf{X})_{\leqslant 0}$$

determine a t-structure on $\mathrm{QCoh}_K(\mathsf{X})$.

In the case where X is affine, Definition 7.1.5.1 reduces to Definition 7.1.1.6:

Proposition 7.1.5.3. *Let $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ be an affine spectral Deligne-Mumford stack, let $I \subseteq \pi_0 A$ be an ideal, and let $K \subseteq |\mathsf{X}| \simeq |\mathrm{Spec}\, A|$ be the vanishing locus of I. Then the equivalence of ∞-categories $\mathrm{QCoh}(\mathsf{X}) \simeq \mathrm{Mod}_A$ restricts to an equivalence of ∞-categories $\mathrm{QCoh}_K(\mathsf{X}) \simeq \mathrm{Mod}_A^{\mathrm{Nil}(I)}$.*

Proof. Let U be the open substack of X complementary to K. Note that U can be written as a union of open substacks of the form $\mathrm{Sp\acute{e}t}\, A[x^{-1}]$, where $x \in I$. Let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ be a quasi-coherent sheaf with image $M \in \mathrm{Mod}_A$. Then $\mathscr{F} \in \mathrm{QCoh}_K(\mathsf{X})$ if and only if $A[x^{-1}] \otimes_A M \simeq 0$ for each $x \in I$: that is, if and only if the action of each $x \in I$ is locally nilpotent on $\pi_* M$. □

Definition 7.1.5.4. Let X be a spectral Deligne-Mumford stack and let $\mathscr{F}$ be a quasi-coherent sheaf on X. The collection of closed subsets $K \subseteq |\mathsf{X}|$ such that $\mathscr{F}$ is supported on

K is closed under intersections. It follows that there is a smallest closed subset $K \subseteq |\mathsf{X}|$ such that $\mathscr{F} \in \mathrm{QCoh}_K(\mathsf{X})$. We will refer to the set K as the *support of* $\mathscr{F}$ and denote it by $\mathrm{Supp}(\mathscr{F})$.

In general, the support $\mathrm{Supp}(\mathscr{F})$ of a quasi-coherent sheaf is not a very well-behaved invariant: for example, it is not stable under base change. However, the situation is better if we assume that $\mathscr{F}$ is perfect.

Proposition 7.1.5.5. *Let* X *be a spectral Deligne-Mumford stack, and let* $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ *be perfect. Then:*

- (a) *The support* $\mathrm{Supp}(\mathscr{F})$ *is cocompact. That is, if* U *denotes the open substack of* X *complementary to* $\mathrm{Supp}(\mathscr{F})$*, then the open immersion* $\mathsf{U} \hookrightarrow \mathsf{X}$ *is quasi-compact.*

- (b) *For every field* κ *and every map* $\eta : \mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{X}$*, the resulting point of* $|\mathsf{X}|$ *belongs to* $\mathrm{Supp}(\mathscr{F})$ *if and only if* $\eta^* \mathscr{F} \neq 0$*.*

Corollary 7.1.5.6. *Let* X *be a spectral Deligne-Mumford stack, and let* $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$ *be perfect. Then* $\mathrm{Supp}(\mathscr{F} \otimes \mathscr{G}) = \mathrm{Supp}(\mathscr{F}) \cap \mathrm{Supp}(\mathscr{G})$*. In particular, if* $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ *is perfect and* $\mathrm{End}(\mathscr{F}) \simeq \mathscr{F} \otimes \mathscr{F}^\vee \in \mathrm{Alg}(\mathrm{QCoh}(\mathsf{X}))$ *classifies endomorphisms of* $\mathscr{F}$*, then* $\mathrm{Supp}(\mathrm{End}(\mathscr{F})) = \mathrm{Supp}(\mathscr{F})$*.*

Corollary 7.1.5.7. *Let* $f : \mathsf{X} \to \mathsf{Y}$ *be a morphism of spectral Deligne-Mumford stacks, and let* $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})$ *be perfect. Then* $\mathrm{Supp}(f^* \mathscr{F}) \subseteq |\mathsf{X}|$ *is the inverse image of* $\mathrm{Supp}(\mathscr{F}) \subseteq |\mathsf{Y}|$ *under the induced map* $|f| : |\mathsf{X}| \to |\mathsf{Y}|$*.*

Proof of Proposition 7.1.5.5. We first claim that there exists a unique closed subset $K \subseteq |\mathsf{X}|$ with the following property:

- (b') For every field κ and every map $\eta : \mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{X}$, the resulting point of $|\mathsf{X}|$ belongs to K if and only if $\eta^* \mathscr{F} \neq 0$.

Moreover, we claim that K is cocompact.

To prove these assertions, we can work locally on X and thereby reduce to the case where $\mathsf{X} = \mathrm{Sp\acute{e}t}\,A$ for some connective $\mathbb{E}_\infty$-ring A, and that $\mathscr{F}$ corresponds to perfect A-module M under the equivalence $\mathrm{QCoh}(\mathsf{X}) \simeq \mathrm{Mod}_A$. Replacing A by $\pi_0 A$ and M by $(\pi_0 A) \otimes_A M$, we may assume that A is discrete. Write A as a union of finitely generated subrings A_α. Using Lemma HA.7.3.5.13 , we can write $M = A \otimes_{A_\alpha} M_\alpha$, for some index α and some perfect A_α-module M_α. We may therefore replace A by A_α, and thereby reduce to the case where A is a finitely generated discrete commutative ring. In particular, A is Noetherian.

Since M is an almost perfect A-module, each homotopy group $\pi_n M$ is a finitely generated as a discrete module over A (Proposition HA.7.2.4.17). Since A is discrete, every A-module of finite Tor-amplitude (and therefore every perfect A-module) is m-truncated for $m \gg 0$. It follows that $\pi_* M$ is a finitely generated A-module. Let $I \subseteq A$ be the annihilator of $\pi_* M$. Then I determines a closed subset $K \subseteq |\operatorname{Spec} A| \simeq |\mathsf{X}|$ (which is automatically cocompact, since A is Noetherian). We claim that K satisfies (b').

Suppose first that we are given a field κ and a ring homomorphism $\phi : A \to \kappa$ for which the induced map $|\operatorname{Sp\acute{e}t} \kappa| \to |\mathsf{X}|$ does not factor through K. Then I contains an element a such that $\phi(a) \neq 0$. Since $a \in I$, we have $M[a^{-1}] \simeq 0$. Then $\kappa \otimes_A M \simeq \kappa \otimes_{A[a^{-1}]} M[a^{-1}] \simeq 0$.

Conversely, suppose that $\phi : A \to \kappa$ is a ring homomorphism such that $\phi(I) = 0$. We wish to prove that $\kappa \otimes_A M \neq 0$. Let $\mathfrak{p} = \ker(\phi)$. Since $\phi(I) = 0$, the localization $(\pi_* M)_{\mathfrak{p}}$ does not vanish. Let n be the smallest integer such that $(\pi_n M)_{\mathfrak{p}}$ is nonzero. Then the direct sum $\bigoplus_{m<n} \pi_m M$ is a finitely generated A-module whose tensor product with $A_{\mathfrak{p}}$ vanishes. It follows that there exists an element $a \notin \mathfrak{p}$ such that $\pi_m M[a^{-1}] \simeq 0$ for $m < n$. Replacing A by $A[a^{-1}]$ and M by $M[a^{-1}]$, we may assume that $\pi_m M \simeq 0$ for $m < n$. In this case, Corollary HA.7.2.1.23 supplies a canonical isomorphism $\pi_n(\kappa \otimes_A M) \simeq \operatorname{Tor}_0^A(\kappa, \pi_n M)$. Since $(\pi_n M)_{\mathfrak{p}}$ is a nonzero and finitely generated over $A_{\mathfrak{p}}$, Nakayama's lemma implies that $\operatorname{Tor}_0^A(\kappa, \pi_n M) \neq 0$, so that $\kappa \otimes_A M \neq 0$. This completes the proof of (b').

To complete the proof of Proposition 7.1.5.5, it will suffice to show that $K = \operatorname{Supp}(\mathscr{F})$. The inclusion $K \subseteq \operatorname{Supp}(\mathscr{F})$ follows immediately from (b'). To prove the reverse inclusion, it will suffice to show that if U is the open substack of X complementary to K, then $\mathscr{F}|_{\mathsf{U}} \simeq 0$. Replacing X by U, we are reduced to proving that $K = \emptyset$ implies $\mathscr{F} \simeq 0$. The desired conclusion $\mathscr{F} \simeq 0$ can be tested locally on X, so we may assume as above that $\mathsf{X} \simeq \operatorname{Sp\acute{e}t} A$ is affine and that $\mathscr{F}$ corresponds to a perfect A-module M. Suppose that M is nonzero. Since M is perfect, there exists a smallest integer n such that $\pi_n M \neq 0$. For every prime ideal $\mathfrak{p} \subseteq \pi_0 A$, let $\kappa(\mathfrak{p})$ denote the residue field of $\pi_0 A$ at $\mathfrak{p}$. Since $\operatorname{Supp}(M) = \emptyset$, Corollary HA.7.2.1.23 supplies an isomorphism $\operatorname{Tor}_0^{\pi_0 A}(\kappa(\mathfrak{p}), \pi_n M) \simeq \pi_n(\kappa(\mathfrak{p}) \otimes_A M) \simeq 0$ for every prime ideal $\mathfrak{p} \subseteq \pi_0 A$. Since $\pi_n M$ is a finitely generated module over $\pi_0 A$, Nakayama's lemma implies that the localization $(\pi_n M)_{\mathfrak{p}}$ vanishes. Since $\mathfrak{p}$ is arbitrary, we deduce that $\pi_n M \simeq 0$, contrary to our assumption that $M \neq 0$. $\square$

7.2 Semi-Orthogonal Decompositions

Let X be a topological space, let $Y \subseteq X$ be a closed subset, and let $U = X - Y$ denote the complement of Y. Let $i : Y \hookrightarrow X$ and $j : U \hookrightarrow X$ be the corresponding closed and open immersions. Then the direct image functors

$$i_* : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(Y) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X) \qquad j_* : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(U) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)$$

are fully faithful. Moreover, for every spectrum-valued sheaf $\mathscr{F}$ on X, we have a canonical fiber sequence $i_* i^! \mathscr{F} \to \mathscr{F} \to j_* j^* \mathscr{F}$, where $i^!$ denotes the right adjoint to i_*. The following definition allows us to describe the situation axiomatically:

Definition 7.2.0.1. Let $\mathcal{C}$ be a stable ∞-category. We will say that a pair $(\mathcal{C}_+, \mathcal{C}_-)$ of full subcategories $\mathcal{C}_+, \mathcal{C}_- \subseteq \mathcal{C}$ is a *semi-orthogonal decomposition* of $\mathcal{C}$ is the following axioms are satisfied:

(a) The full subcategories $\mathcal{C}_+$ and $\mathcal{C}_-$ are closed under finite limits and colimits (in particular, they are stable).

(b) For every object $C \in \mathcal{C}_+$ and every object $D \in \mathcal{C}_-$, the mapping space $\mathrm{Map}_{\mathcal{C}}(C, D)$ is contractible.

(c) Every object $C \in \mathcal{C}$, there exists a fiber sequence $C_+ \to C \to C_-$, where $C_+ \in \mathcal{C}_+$ and $C_- \in \mathcal{C}_-$.

Remark 7.2.0.2. Let $\mathcal{C}$ be a stable ∞-category. Unwinding the definitions, we see that a semi-orthogonal decomposition of $\mathcal{C}$ is a t-structure $(\mathcal{C}_+, \mathcal{C}_-)$ on $\mathcal{C}$ such that $\mathcal{C}_+$ and $\mathcal{C}_-$ are stable subcategories of $\mathcal{C}$ (see Definition HA.1.2.1.1). It follows that for each object $C \in \mathcal{C}$, the fiber sequence $C_+ \to C \to C_-$ whose existence is asserted by part (c) of Definition 7.2.0.1 is essentially unique, and depends functorially on C: the construction $C \mapsto C_+$ is right adjoint to the inclusion $\mathcal{C}_+ \hookrightarrow \mathcal{C}$, and the construction $C \mapsto C_-$ is left adjoint to the inclusion $\mathcal{C}_- \hookrightarrow \mathcal{C}$.

Example 7.2.0.3. Let X be a topological space, let $i : Y \hookrightarrow X$ be a closed immersion, and let $j : U \hookrightarrow X$ be the complementary open immersion. Then the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)$ admits a semiorthogonal decomposition $(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)_+, \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)_-)$, where $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)_+$ is the essential image of the fully faithful embedding $i_* : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(Y) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)$ and $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)_-$ is the essential image of the fully faithful embedding $j_* : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(U) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(X)$.

Our goal in this section is to review the theory of semi-orthogonal decompositions and to describe some examples which arise naturally in algebraic geometry.

7.2.1 Semi-Orthogonal Decompositions of Stable ∞-Categories

We begin with a few general categorical remarks.

Definition 7.2.1.1. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D} \subseteq \mathcal{C}$ be a subcategory. We define full subcategories ${}^{\perp}\mathcal{D} \subseteq \mathcal{C} \supseteq \mathcal{D}^{\perp}$ as follows:

- An object $X \in \mathcal{C}$ belongs to ${}^{\perp}\mathcal{D}$ if and only if, for every object $Y \in \mathcal{D}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is contractible.
- An object $Y \in \mathcal{C}$ belongs to $\mathcal{D}^{\perp}$ if and only if, for every object $X \in \mathcal{D}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is contractible.

We will refer to ${}^{\perp}\mathcal{D}$ as the *left orthogonal* to the full subcategory $\mathcal{D} \subseteq \mathcal{C}$, and to $\mathcal{D}^{\perp}$ as the *right orthogonal* to $\mathcal{D} \subseteq \mathcal{C}$.

Example 7.2.1.2. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$. Then for every integer n, we have ${}^{\perp}\mathcal{C}_{\geqslant n} = \mathcal{C}_{\leqslant n-1}$ and $\mathcal{C}_{\leqslant n}^{\perp} = \mathcal{C}_{\geqslant n+1}$. In particular, if $(\mathcal{C}_{+}, \mathcal{C}_{-})$ is a semi-orthogonal decomposition of $\mathcal{C}$, then $\mathcal{C}_{-} = {}^{\perp}\mathcal{C}_{+}$ and $\mathcal{C}_{+} = \mathcal{C}_{-}^{\perp}$.

Remark 7.2.1.3. Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{D}$ be a stable subcategory of $\mathcal{C}$. Then the left and right orthogonals ${}^{\perp}\mathcal{D}$ and $\mathcal{D}^{\perp}$ are stable subcategories of $\mathcal{C}$, and we have evident inclusions ${}^{\perp}(\mathcal{D}^{\perp}) \supseteq \mathcal{D} \subseteq ({}^{\perp}\mathcal{D})^{\perp}$. Moreover, the subcategory $\mathcal{D}^{\perp}$ is closed under all limits which exist in $\mathcal{C}$, and the subcategory ${}^{\perp}\mathcal{D}$ is closed under all colimits which exist in $\mathcal{C}$.

Proposition 7.2.1.4. *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{D} \subseteq \mathcal{C}$ be a stable subcategory which is closed under equivalence. Then the following conditions are equivalent:*

(1) *There exists a semi-orthogonal decomposition $(\mathcal{C}_{+}, \mathcal{C}_{-})$ of $\mathcal{C}$ with $\mathcal{C}_{+} = \mathcal{D}$.*

(2) *There exists a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ on $\mathcal{C}$ with $\mathcal{C}_{\geqslant 0} = \mathcal{D}$.*

(3) *The inclusion functor $\iota : \mathcal{D} \to \mathcal{C}$ admits a right adjoint.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are clear. Conversely, suppose that (3) is satisfied. Set $\mathcal{C}_{+} = \mathcal{D}$ and $\mathcal{C}_{-} = \mathcal{D}^{\perp}$; we claim that $(\mathcal{C}_{+}, \mathcal{C}_{-})$ is a semi-orthogonal decomposition of $\mathcal{C}$. Note that $\mathcal{C}_{+}$ is a stable subcategory of $\mathcal{C}$ by assumption and that $\mathcal{C}_{-}$ is a stable subcategory of $\mathcal{C}$ by virtue of Remark 7.2.1.3. The definition of $\mathcal{C}_{-}$ guarantees that $\operatorname{Map}_{\mathcal{C}}(C_{+}, C_{-})$ is contractible for all $C_{+} \in \mathcal{C}_{+}$, $C_{-} \in \mathcal{C}_{-}$. To complete the proof, it will suffice to show that for every object $C \in \mathcal{C}$, there exists a cofiber sequence $C_{+} \xrightarrow{\alpha} C \to C_{-}$ where $C_{+} \in \mathcal{C}_{+}$ and $C_{-} \in \mathcal{C}_{-}$. Assumption (3) guarantees that we can choose such a fiber sequence for which α exhibit C_{+} as a $\mathcal{D}$-colocalization of C: that is, such that α induces a homotopy equivalence $\operatorname{Map}_{\mathcal{C}}(D, C') \to \operatorname{Map}_{\mathcal{C}}(D, C)$ for each $D \in \mathcal{D}$. It follows that $\operatorname{fib}(\alpha) \in \mathcal{D}^{\perp} = \mathcal{C}_{-}$, so that $C_{-} = \Sigma \operatorname{fib}(\alpha)$ also belong to $\mathcal{C}_{-}$. □

Remark 7.2.1.5. In the situation of Proposition 7.2.1.4, the t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ is uniquely determined by the requirement $\mathcal{C}_{\geqslant 0} = \mathcal{D}$ (and is therefore automatically a semi-orthogonal decomposition).

Remark 7.2.1.6. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$. Using Proposition 7.2.1.4, we see that the following conditions are equivalent:

(i) The t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ is a semi-orthogonal decomposition of $\mathcal{C}$.

(ii) The full subcategory $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$ is closed under desuspension.

(iii) The full subcategory $\mathcal{C}_{\leqslant 0} \subseteq \mathcal{C}$ is closed under suspension.

Not that conditions (*ii*) and (*iii*) are both equivalent to the requirement that the heart $\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0}$ contains only zero objects of $\mathcal{C}$.

Replacing $\mathcal{C}$ by its opposite ∞-category, we obtain the following dual version of Proposition 7.2.1.4:

Corollary 7.2.1.7. *Let $\mathcal{C}$ be a stable ∞-category, let $\mathcal{D} \subseteq \mathcal{C}$ be a stable subcategory which is closed under equivalence. Then the following conditions are equivalent:*

(1) *There exists a semi-orthogonal decomposition $(\mathcal{C}_+, \mathcal{C}_-)$ of $\mathcal{C}$ with $\mathcal{C}_- = \mathcal{D}$.*

(2) *There exists a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ on $\mathcal{C}$ with $\mathcal{C}_{\leqslant 0} = \mathcal{D}$.*

(3) *The inclusion functor $\iota : \mathcal{D} \to \mathcal{C}$ admits a left adjoint.*

Corollary 7.2.1.8. *Let $\mathcal{C}$ be a stable ∞-category, let $\mathcal{D}$ be a stable subcategory of $\mathcal{C}$, and let $\iota : \mathcal{D} \to \mathcal{C}$ be the inclusion functor. If ι admits a right adjoint, then the inclusion $\mathcal{D} \subseteq {}^{\perp}(\mathcal{D}^{\perp})$ is an equivalence of ∞-categories. If ι admits a left adjoint, then the inclusion $\mathcal{D} \subseteq ({}^{\perp}\mathcal{D})^{\perp}$ is an equivalence of ∞-categories.*

Proof. Without loss of generality, we may assume that $\mathcal{D}$ coincides with the essential image of ι. By symmetry, it will suffice to prove the first assertion. Using Proposition 7.2.1.4, we see that there exists a semi-orthogonal decomposition $(\mathcal{C}_+, \mathcal{C}_-)$ of $\mathcal{C}$ with $\mathcal{C}_+ = \mathcal{D}$, so that $\mathcal{C}_- = \mathcal{D}^{\perp}$. It follows that the inclusion $\mathcal{D}^{\perp} \hookrightarrow \mathcal{C}$ admits a left adjoint. Applying Corollary 7.2.1.7, we deduce that there exists a semi-orthogonal decompositions $(\mathcal{C}'_+, \mathcal{C}'_-)$ of $\mathcal{C}$ with $\mathcal{C}'_- = \mathcal{D}^{\perp}$, so that $\mathcal{C}'_+ = {}^{\perp}(\mathcal{D}^{\perp})$. Since $\mathcal{C}'_- = \mathcal{C}_-$, we conclude that $\mathcal{D} = \mathcal{C}_+ = \mathcal{C}'_+ = {}^{\perp}(\mathcal{D}^{\perp})$. □

Example 7.2.1.9. Let $\mathcal{C}$ be a presentable stable ∞-category, let $\mathcal{D} \subseteq \mathcal{C}$ be a stable subcategory which is also presentable, and let $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ denote the inclusion map. Then:

(a) If $\mathcal{D}$ is closed under small colimits in $\mathcal{C}$, then Corollary HTT.5.5.2.9 implies that ι admits a left adjoint, so that $(\mathcal{D}, \mathcal{D}^{\perp})$ is a semi-orthogonal decomposition of $\mathcal{C}$ and $\mathcal{D} = {}^{\perp}(\mathcal{D}^{\perp})$.

(b) Suppose there exists a regular cardinal κ such that $\mathcal{D}$ is closed under small limits and under κ-filtered colimits in $\mathcal{C}$. Then Corollary HTT.5.5.2.9 implies that ι admits a right adjoint, so that $({}^{\perp}\mathcal{D}, \mathcal{D})$ is a semi-orthogonal decomposition of $\mathcal{C}$ and $\mathcal{D} = ({}^{\perp}\mathcal{D})^{\perp}$.

Proposition 7.2.1.10. *Let $\mathcal{C}$ be a stable ∞-category, let $\mathcal{D} \subseteq \mathcal{C}$ be a stable subcategory, and suppose that the inclusion map $\iota : \mathcal{D} \to \mathcal{C}$ admits both a left and a right adjoint. Let $F : \mathcal{C} \to \mathcal{D}^{\perp}$ be a left adjoint to the inclusion (which exhibits by virtue of Proposition 7.2.1.4). Then F induces an equivalence of ∞-categories ${}^{\perp}\mathcal{D} \to \mathcal{D}^{\perp}$.*

Proof. Let $F_0 : {}^{\perp}\mathcal{D} \to \mathcal{D}^{\perp}$ be the restriction of F. Corollary 7.2.1.7 implies that the inclusion ${}^{\perp}\mathcal{D} \hookrightarrow \mathcal{C}$ admits a right adjoint G. Let G_0 denote the restriction of G to $\mathcal{D}^{\perp}$. Then G_0 is given by the composition $\mathcal{D}^{\perp} \hookrightarrow \mathcal{C} \xrightarrow{G} {}^{\perp}\mathcal{D}$, and is therefore right adjoint to F_0. We claim that G_0 is homotopy inverse to F_0. To prove this, we will show that the unit map $u : \mathrm{id} \to G_0 \circ F_0$ is an equivalence of functors from ${}^{\perp}\mathcal{D}$ to itself; the proof that the counit map $v : F_0 \circ G_0 \to \mathrm{id}$ is an equivalence follows by the same argument. Fix an object $C \in {}^{\perp}\mathcal{D}$; we wish to show that the canonical map $\alpha : C \to G(F(C))$ is an equivalence. Unwinding the definitions, we see that α is obtained by composing a homotopy inverse to the equivalence $G(C) \to C$ with the canonical map $\alpha' : G(C) \to G(F(C))$. We are therefore reduced to proving that α' is an equivalence. By construction, we have a fiber sequence $D \to C \to F(C)$, where $D \in \mathcal{D}$. Since G is an exact functor, we are reduced to proving that $G(D) \simeq 0$, which follows immediately from the definitions. □

7.2.2 Example: Quasi-Coherent Sheaves on the Projective Line

Our next result describes an elementary example of a semi-orthogonal decomposition in algebraic geometry:

Theorem 7.2.2.1. *Let R be a connective $\mathbb{E}_\infty$-ring and let $q : \mathbf{P}^1_R \to \operatorname{Sp\acute{e}t} R$ be the projection map. Then:*

(1) *For every integer n, the construction $\mathscr{F} \mapsto \mathscr{O}(n) \otimes q^* \mathscr{F}$ induces a fully faithful embedding $\mathrm{QCoh}(\operatorname{Sp\acute{e}t} R) \to \mathrm{QCoh}(\mathbf{P}^1_R)$, whose essential image we will denote by $\mathcal{E}(n)$.*

(2) *For every integer n, the pair $(\mathcal{E}(n), \mathcal{E}(n-1))$ is a semi-orthogonal decomposition of the ∞-category $\mathrm{QCoh}(\mathbf{P}^1_R)$.*

The proof of Theorem 7.2.2.1 will require a brief digression. Fix a connective $\mathbb{E}_\infty$-ring R, and consider the projective space $\mathbf{P}^n_R$ of dimension n over R. Construction 5.4.2.5 provides canonical maps $x_0, x_1, \ldots, x_n : \mathscr{O}(0) \to \mathscr{O}(1)$ in $\mathrm{QCoh}(\mathbf{P}^n_R)$. Taking the tensor product of these maps, we obtain a cubical diagram

$$P([n]) \to \mathrm{QCoh}(\mathbf{P}^n_R) \qquad J \mapsto \bigotimes_{j \in J} \mathscr{O}(1) \simeq \mathscr{O}(|J|),$$

where $P([n])$ denotes the collection of all subsets of $[n]$.

Lemma 7.2.2.2. *For every connective $\mathbb{E}_\infty$-ring R and every $n \geqslant 0$, the functor $P([n]) \to \mathrm{QCoh}(\mathbf{P}^n_R)$ described above is a colimit diagram: that is, it classifies an equivalence $\rho : \varinjlim_{J \subsetneq [n]} \mathscr{O}(|J|) \to \mathscr{O}(n+1)$ in the ∞-category $\mathrm{QCoh}(\mathbf{P}^n_R)$.*

Proof. It will suffice to show that ρ is an equivalence after pulling back along each of the open immersions $\phi_I : \mathrm{Sp\acute{e}t}\, R[M_I] \to \mathbf{P}^n_R$ appearing in Proposition 5.4.1.7. This is clear, since the map $\phi_I^* \mathscr{O}(0) \xrightarrow{x_i} \phi_I^* \mathscr{O}(1)$ is an equivalence for any $i \in I$. □

Specializing Lemma 7.2.2.2 to the case $n = 1$, we deduce the existence of a pushout diagram

$$\begin{array}{ccc} \mathscr{O}(0) & \longrightarrow & \mathscr{O}(1) \\ \downarrow & & \downarrow \\ \mathscr{O}(1) & \longrightarrow & \mathscr{O}(2) \end{array}$$

in the ∞-category $\mathrm{QCoh}(\mathbf{P}^1_R)$. Tensoring with $\mathscr{O}(m)$, we obtain the following:

Lemma 7.2.2.3. *For every integer m, there exists a fiber sequence $\mathscr{O}(m) \to \mathscr{O}(m+1) \oplus \mathscr{O}(m+1) \to \mathscr{O}(m+2)$ in the ∞-category $\mathrm{QCoh}(\mathbf{P}^1_R)$.*

Proof of Theorem 7.2.2.1. To prove (1), we may assume without loss of generality that $n = 0$ (since the construction $\mathscr{G} \mapsto \mathcal{O}(n) \otimes \mathscr{G}$ induces an equivalence from $\mathrm{QCoh}(\mathbf{P}^1_R)$ to itself). In this case, we wish to show that the pullback functor $q^* : \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R) \to \mathrm{QCoh}(\mathbf{P}^1_R)$ is fully faithful: that is, that the unit map $u_{\mathscr{F}} : \mathscr{F} \to q_* q^* \mathscr{F}$ is an equivalence for each $\mathscr{F} \in \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R)$. Note that the collection of those objects $\mathscr{F} \in \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R) \simeq \mathrm{Mod}_R$ for which $u_{\mathscr{F}}$ is an equivalence is closed under colimits and desuspensions. It will therefore suffice to show that $u_{\mathscr{F}}$ is an equivalence when $\mathscr{F}$ is the structure sheaf of $\mathrm{Sp\acute{e}t}\, R$, which follows from Theorem 5.4.2.6.

We now prove (2). Once again, it will suffice to treat the case $n = 0$. We will show that the pair of subcategories $(\mathcal{E}(0), \mathcal{E}(-1))$ satisfies conditions (a), (b) and (c) of Definition 7.2.0.1:

- (a) Since the functors $\mathscr{F} \mapsto q^* \mathscr{F}$ and $\mathscr{F} \mapsto \mathscr{O}(-1) \otimes q^* \mathscr{F}$ are fully faithful and exact, their essential images $\mathcal{E}(0)$ and $\mathcal{E}(-1)$ are stable subcategories of $\mathrm{QCoh}(\mathbf{P}^1_R)$.

- (b) Let $\mathscr{F}$ and $\mathscr{G}$ be objects of $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R)$; we wish to prove that the mapping space

$$\begin{aligned} \mathrm{Map}_{\mathrm{QCoh}(\mathbf{P}^1_R)}(q^* \mathscr{F}, \mathscr{O}(-1) \otimes q^* \mathscr{G}) &\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R)}(\mathscr{F}, q_*(\mathscr{O}(-1) \otimes q^* \mathscr{G})) \\ &\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R)}(\mathscr{F}, (q_* \mathscr{O}(-1)) \otimes \mathscr{G}) \end{aligned}$$

  is contractible. This follows from the vanishing of $q_* \mathscr{O}(-1)$ (Theorem 5.4.2.6).

- (c) Let $\mathscr{F}$ be an object of $\mathrm{QCoh}(\mathbf{P}^1_R)$, and form a cofiber sequence $q^* q_* \mathscr{F} \to \mathscr{F} \to \mathscr{G}$. We claim that $\mathscr{G}$ belongs to $\mathcal{E}(-1)$: that is, that the counit map $v : q^* q_*(\mathscr{O}(1) \otimes \mathscr{G}) \to \mathscr{O}(1) \otimes \mathscr{G}$ is an equivalence. Note that the domain of v belongs to $\mathcal{E}(0)$ and therefore to ${}^{\perp}\mathcal{E}(1)$ (by virtue of (b)), and that the codomain of v belongs to ${}^{\perp}\mathcal{E}(1)$ since $\mathscr{G} \in {}^{\perp}\mathcal{E}(0)$

by construction. It follows that $\mathrm{cofib}(v) \in {}^{\perp}\mathcal{E}(1)$, and we also have $\mathrm{cofib}(v) \in {}^{\perp}\mathcal{E}(0)$. In particular, we have

$$\mathrm{Ext}^*_{\mathrm{QCoh}(\mathbf{P}^1_R)}(\mathscr{O}(0), \mathrm{cofib}(v)) \simeq 0 \simeq \mathrm{Ext}^*_{\mathrm{QCoh}(\mathbf{P}^1_R)}(\mathscr{O}(1), \mathrm{cofib}(v)).$$

It follows from Lemma 7.2.2.3 the groups $\mathrm{Ext}^*_{\mathrm{QCoh}(\mathbf{P}^1_R)}(\mathscr{O}(n), \mathrm{cofib}(v))$ vanish for every integer n. It follows from Lemma 5.6.2.2 that there exists a morphism $\alpha : \mathscr{H} \to \mathrm{cofib}(v)$ which induces an epimorphism $\pi_* \mathscr{H} \to \pi_* \mathrm{cofib}(v)$, where $\mathscr{H}$ can be written as a direct sum of sheaves of the form $\Sigma^m \mathscr{O}(n)$. Then the map α is automatically nullhomotopic, so $\pi_* \mathrm{cofib}(v) \simeq 0$ and therefore v is an equivalence as desired.

□

Remark 7.2.2.4. In the situation of Theorem 7.2.2.1, each of the full subcategories $\mathcal{E}(n) \subseteq \mathrm{QCoh}(\mathbf{P}^1_R)$ is canonically equivalent to $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R) \simeq \mathrm{Mod}_R$. Moreover, we have ${}^{\perp}\mathcal{E}(n) = \mathcal{E}(n+1)$ and $\mathcal{E}(n)^{\perp} = \mathcal{E}(n-1)$.

7.2.3 Example: Closed and Open Subspaces

We now consider an algebro-geometric version of Example 7.2.0.3.

Proposition 7.2.3.1. *Let $j : \mathsf{U} \to \mathsf{X}$ be a quasi-compact open immersion of spectral Deligne-Mumford stacks, and let $K \subseteq |\mathsf{X}|$ denote the complement of the image of j. Then:*

(1) *The pushforward functor $j_* : \mathrm{QCoh}(\mathsf{U}) \to \mathrm{QCoh}(\mathsf{X})$ is fully faithful. Let us denote its essential image by $\mathcal{E} \subseteq \mathrm{QCoh}(\mathsf{X})$.*

(2) *The left orthogonal ${}^{\perp}\mathcal{E}$ coincides with $\mathrm{QCoh}_K(\mathsf{X})$.*

(3) *The pair $(\mathrm{QCoh}_K(\mathsf{X}), \mathcal{E})$ is a semi-orthogonal decomposition of $\mathrm{QCoh}(\mathsf{X})$.*

(4) *The inclusion $\iota : \mathrm{QCoh}_K(\mathsf{X}) \hookrightarrow \mathrm{QCoh}(\mathsf{X})$ admits a left adjoint.*

(5) *The ∞-category $\mathcal{E}$ coincides with the right orthogonal $\mathrm{QCoh}_K(\mathsf{X})^{\perp}$.*

Proof. Since the morphism j is quasi-affine, it is is relatively scalloped (Example 2.5.3.3). Applying Proposition 2.5.4.5 to the pullback diagram

$$\begin{array}{ccc} \mathsf{U} & \longrightarrow & \mathsf{U} \\ \downarrow & & \downarrow j \\ \mathsf{U} & \xrightarrow{j} & \mathsf{X}, \end{array}$$

we deduce that the counit map $j^* j_* \to \mathrm{id}_{\mathrm{QCoh}(\mathsf{U})}$ is an equivalence, so that j_* is fully faithful. This proves (1). To prove (2), we note that a quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ belongs to ${}^{\perp}\mathcal{E}$ if and only if the mapping space

$$\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, j_* \mathscr{G}) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{U})}(j^* \mathscr{F}, \mathscr{G})$$

is contractible for every object $\mathscr{G} \in \mathrm{QCoh}(\mathsf{U})$, which is equivalent to the vanishing of $j^* \mathscr{F}$. Assertion (3) now follows from Corollary 7.2.1.7, and (4) and (5) are formal consequences of (3) (see Corollary 7.2.1.8). □

7.2.4 Example: Nilpotent and Local Objects

In the special case where $\mathsf{X} = \mathrm{Spét}\, R$ is affine, then the full subcategory $\mathrm{QCoh}_K(\mathsf{X}) \subseteq \mathrm{QCoh}(\mathsf{X})$ appearing in Proposition 7.2.3.1 can be identified with the subcategory $\mathrm{Mod}_R^{\mathrm{Nil}(I)} \subseteq \mathrm{Mod}_R$, where $I \subseteq \pi_0 R$ is an ideal defining the closed subset $K \subseteq |\mathrm{Spec}\, R|$ (Proposition 7.1.5.3). We now consider a more general situation:

Definition 7.2.4.1. Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be an ideal, and let $\mathcal{C}$ be a stable R-linear ∞-category. We will say that an object $C \in \mathcal{C}$ is *I-local* if, for every I-nilpotent object $D \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(D, C)$ is contractible. We let $\mathcal{C}^{\mathrm{Loc}(I)}$ denote the full subcategory of $\mathcal{C}$ spanned by the I-local objects.

Remark 7.2.4.2. In the situation of Definition 7.2.4.1, the ∞-category $\mathcal{C}^{\mathrm{Loc}(I)}$ is the right orthogonal of the full subcategory $\mathcal{C}^{\mathrm{Nil}(I)} \subseteq \mathcal{C}$ of Definition 7.1.1.6. In particular, $\mathcal{C}^{\mathrm{Loc}(I)}$ is a stable subcategory of $\mathcal{C}$ which is closed under small limits (Remark 7.2.1.3).

Example 7.2.4.3. Let R be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let U be the open substack of $\mathrm{Spét}\, A$ complementary to the vanishing locus of I. Using Proposition 7.2.3.1, we deduce that the global sections functor $\Gamma : \mathrm{QCoh}(\mathsf{U}) \to \mathrm{Mod}_A$ is a fully faithful embedding, whose essential image is the full subcategory $\mathrm{Mod}_A^{\mathrm{Loc}(I)} \subseteq \mathrm{Mod}_A$.

Combining Proposition 7.1.1.12 with Example 7.2.1.9, we obtain the following:

Proposition 7.2.4.4. *Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. For every ideal $I \subseteq \pi_0 R$, the pair $(\mathcal{C}^{\mathrm{Nil}(I)}, \mathcal{C}^{\mathrm{Loc}(I)})$ is a semi-orthogonal decomposition of $\mathcal{C}$.*

Example 7.2.4.5. Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a stable R-linear ∞-category, and let $I, J \subseteq \pi_0 R$ be ideals satisfying $I + J = \pi_0 R$. Then Remark 7.1.2.4 shows that $(\mathcal{C}^{\mathrm{Nil}(I)})^{\mathrm{Nil}(J)} = \mathcal{C}^{\mathrm{Nil}(I+J)}$ contains only zero objects of $\mathcal{C}$. It follows that $(\mathcal{C}^{\mathrm{Nil}(I)})^{\mathrm{Loc}(J)} = \mathcal{C}^{\mathrm{Nil}(I)}$. In other words, every I-nilpotent object of $\mathcal{C}$ is J-local.

Notation 7.2.4.6. In the situation of Proposition 7.2.4.4, the inclusion functor $\mathcal{C}^{\mathrm{Loc}(I)} \to \mathcal{C}$ admits a left adjoint, which we will denote by $L_I : \mathcal{C} \to \mathcal{C}^{\mathrm{Loc}(I)}$. If we regard L_I as a functor from $\mathcal{C}$ to itself, then it fits into a canonical fiber sequence $\Gamma_I \to \mathrm{id}_{\mathcal{C}} \to L_I$, where $\Gamma_I : \mathcal{C} \to \mathcal{C}^{\mathrm{Nil}(I)} \subseteq \mathcal{C}$ is as in Definition 7.1.2.1.

Remark 7.2.4.7. Let $\phi : R \to R'$ be a morphism of $\mathbb{E}_2$-rings, let $I \subseteq \pi_0 R$ be an ideal, and let $I' \subseteq \pi_0 R'$ be the ideal generated by the image of I. Suppose that $\mathcal{C}$ is a stable R'-linear ∞-category, which we can also regard as a stable R-linear ∞-category (by restriction of scalars). Then $\mathcal{C}^{\mathrm{Nil}(I)} = \mathcal{C}^{\mathrm{Nil}(I')}$ (Remark 7.1.1.10), so $\mathcal{C}^{\mathrm{Loc}(I)} = \mathcal{C}^{\mathrm{Loc}(I')}$. In particular, the notion of I-local object does not change if we replace an $\mathbb{E}_2$-ring R by its connective cover $\tau_{\geqslant 0} R$.

Remark 7.2.4.8. Let R be a commutative ring, let $I \subseteq R$ be a finitely generated ideal, and let $\mathcal{A}$ be a locally Noetherian abelian category equipped with an action of R. It follows from Remark 7.1.3.3 that the functor $\Gamma_I : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})^{\mathrm{Nil}(I)} \subseteq \mathcal{D}(\mathcal{A})$ carries injective objects $\mathcal{A}$ to injective objects of $\mathcal{A}^{\mathrm{Nil}(I)}$. Consequently, if $Q \in \mathcal{A}$ is injective, then the canonical map $\Gamma_I^{\heartsuit} Q = \pi_0 \Gamma_I Q \to \Gamma_I Q$ is an equivalence, where $\Gamma_I^{\heartsuit} : \mathcal{A} \to \mathcal{A}^{\mathrm{Nil}(I)}$ denotes a right adjoint to the inclusion (so that we can identify $\Gamma_I^{\heartsuit} X$ with the largest I-nilpotent subobject of X, for any object $X \in \mathcal{A}$). It follows that the functor L_I also carries injective objects of $\mathcal{A}$ to the heart $\mathcal{D}(\mathcal{A})$: that is, $L_I : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is derived from the left exact functor of abelian categories $X \mapsto \pi_0 L_I(X)$.

We will be interested primarily in the case where the ideal I is finitely generated.

Proposition 7.2.4.9. *Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $\mathcal{C}$ be a stable R-linear ∞-category. Then:*

(1) *The functor L_I of Notation 7.2.4.6 preserves small colimits, when viewed as a functor from $\mathcal{C}$ to itself.*

(2) *The full subcategory $\mathcal{C}^{\mathrm{Loc}(I)} \subseteq \mathcal{C}$ is closed under small colimits.*

(3) *For every left R-module M and every I-local object $C \in \mathcal{C}$, the tensor product $M \otimes_R C$ is I-local. Consequently, $\mathcal{C}^{\mathrm{Loc}(I)}$ inherits the structure of a stable R-linear ∞-category (for which the inclusion $\mathcal{C}^{\mathrm{Loc}(I)} \hookrightarrow \mathcal{C}$ is R-linear).*

(4) *Let $F : \mathcal{C} \to \mathcal{D}$ be an R-linear functor between stable R-linear ∞-categories. Then F carries I-local objects of $\mathcal{C}$ to I-local objects of $\mathcal{D}$.*

Remark 7.2.4.10. In the situation of Proposition 7.2.4.9, we can regard $\mathcal{C}^{\mathrm{Loc}(I)}$ as an R-linear ∞-category even if the ideal I is not finitely generated. However, in general it is not true that the inclusion $\mathcal{C}^{\mathrm{Loc}(I)} \hookrightarrow \mathcal{C}$ is an R-linear functor (since it need not commute with filtered colimits). However, we can always regard $L_I : \mathcal{C} \to \mathcal{C}^{\mathrm{Loc}(I)}$ as an R-linear functor.

Proof of Proposition 7.2.4.9. Assertion (1) follows from Corollary 7.1.2.5 and the fiber sequence $\Gamma_I \to \mathrm{id}_{\mathcal{C}} \to L_I$ of Notation 7.2.4.6. Assertion (2) follows immediately from (1), and assertion (3) follows from (2): if $C \in \mathcal{C}$ is I-local, then the collection of those left R-modules M such that $M \otimes_R C$ is I-local contains R and is closed under colimits and desuspensions, and therefore contains all left R-modules. To prove (4), we note that an object $C \in \mathcal{C}$ is I-local if and only if $\Gamma_I C \simeq 0$. In this case, Corollary 7.1.2.10 implies that $\Gamma_I F(C) \simeq F(\Gamma_I C) \simeq 0$, so that $F(C)$ is an I-local object of $\mathcal{D}$. □

Remark 7.2.4.11. Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $\mathcal{C}$ be a stable R-linear ∞-category. Proposition 7.2.4.9 implies that the inclusion $\mathcal{C}^{\mathrm{Loc}(I)} \hookrightarrow \mathcal{C}$ commutes with filtered colimits. Applying Proposition HTT.5.5.7.2 , we deduce that the functor $L_I : \mathcal{C} \to \mathcal{C}^{\mathrm{Loc}(I)}$ carries compact objects of $\mathcal{C}$ to compact objects of $\mathcal{C}^{\mathrm{Loc}(I)}$. In particular, if $\mathcal{C}$ is compactly generated, then $\mathcal{C}^{\mathrm{Loc}(I)}$ is also compactly generated.

Corollary 7.2.4.12. *Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let M and N be left R-modules. If either M or N is I-local, then the tensor product $M \otimes_R N$ is I-local.*

Proof. The case where N is I-nilpotent follows from Proposition 7.2.4.9, and the other case follows by symmetry. □

7.2.5 Spectral Decompositions of Injective Objects

Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be an additive R-linear ∞-category. We can think of the action of R on $\mathcal{C}$ as providing a kind of spectral decomposition for the ∞-category $\mathcal{C}$, breaking $\mathcal{C}$ into pieces parametrized by the topological space $|\operatorname{Spec} R|$. We now make this heuristic a bit more precise.

Definition 7.2.5.1. Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let C be an object of $\mathcal{C}$. We will say that C is *centered* if, for every element $x \in \pi_0 R$, one of the following conditions holds:

- (i) The object C is (x)-nilpotent: that is, the localization $C[x^{-1}]$ vanishes.
- (ii) The object C is (x)-local: that is, the canonical map $C \to C[x^{-1}]$ is an equivalence.

Warning 7.2.5.2. In the situation of Definition 7.2.5.1, the condition that an object $C \in \mathcal{C}$ be centered depends on the chosen action of R on $\mathcal{C}$.

Remark 7.2.5.3. In the situation of Definition 7.2.5.1, the zero object $0 \in \mathcal{C}$ is always centered: it satisfies *both* conditions (i) and (ii), for any element $x \in \pi_0 R$. This is a somewhat degenerate situation: note that a nonzero object $C \in \mathcal{C}$ cannot satisfy simultaneously satisfy conditions (i) and (ii).

Example 7.2.5.4. In the situation of Definition 7.2.5.1, suppose that R is a field. Then every object $C \in \mathcal{C}$ is centered: it is (x)-nilpotent for $x = 0$ and (x)-local for $x \neq 0$.

Remark 7.2.5.5. Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category and let C be a nonzero object of $\mathcal{C}$. Set $\mathfrak{p} = \{x \in \pi_0 R : C[x^{-1}] \simeq 0\}$. It follows from Proposition 7.1.1.5 that $\mathfrak{p}$ is an ideal of $\pi_0 R$. If C is centered, then the complement of $\mathfrak{p}$ is the set $\{x \in \pi_0 R : C \simeq C[x^{-1}]\}$, which is closed under multiplication. It follows that $\mathfrak{p}$ is a prime ideal of $\pi_0 R$.

Motivated by Remark 7.2.5.5, we introduce the following slight refinement of Definition 7.2.5.1:

Definition 7.2.5.6. Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category. Suppose we are given an object $C \in \mathcal{C}$ and a prime ideal $\mathfrak{p} \subseteq \pi_0 R$. We will say that C is *centered at* $\mathfrak{p}$ if C is (x)-nilpotent for $x \in \mathfrak{p}$ and (x)-local for $x \notin \mathfrak{p}$. We let $\mathcal{C}[\mathfrak{p}]$ denote the full subcategory of $\mathcal{C}$ spanned by those objects of $\mathcal{C}$ which are centered at $\mathfrak{p}$.

Remark 7.2.5.7. In the situation of Definition 7.2.5.6, suppose that $C \in \mathcal{C}$ is nonzero. Then C is centered (in the sense of Definition 7.2.5.1) if and only if it is centered at $\mathfrak{p}$, for some $\mathfrak{p} \in |\operatorname{Spec} R|$ (in the sense of Definition 7.2.5.6). Moreover, the prime ideal $\mathfrak{p}$ is uniquely determined (Remark 7.2.5.5).

The zero object $0 \in \mathcal{C}$ is centered at every point $\mathfrak{p} \in |\operatorname{Spec} R|$.

Remark 7.2.5.8. Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $\mathfrak{p} \in |\operatorname{Spec} R|$. Then the full subcategory $\mathcal{C}[\mathfrak{p}] \subseteq \mathcal{C}$ is closed under small colimits, finite limits, and extensions. Moreover, it follows from Proposition HTT.5.4.7.10 that $\mathcal{C}[\mathfrak{p}]$ is accessible. It follows that $\mathcal{C}[\mathfrak{p}]$ is also an R-linear Grothendieck prestable ∞-category.

Example 7.2.5.9. Let $R = \mathbf{C}[x]$ and let V be a discrete R-module, which we identify with a complex vector space equipped with a $\mathbf{C}$-linear endomorphism $x : V \to V$. Let $\mathfrak{p}$ be a closed point of $|\operatorname{Spec} R|$ corresponding to the prime ideal $(x - \lambda)$, for some complex number λ. Then V is centered at $\mathfrak{p}$ if and only if multiplication by $(x - \lambda)$ is locally nilpotent on V: in other words, if and only if every element $v \in V$ is a generalized eigenvector for x with eigenvalue λ.

If V is a finite-dimensional vector space over $\mathbf{C}$, then V admits an (essentially unique) decomposition $V \simeq \bigoplus V_\alpha$ into submodules which are centered at closed points of $|\operatorname{Spec} R|$. However, such a decomposition need not exist if V is not finite dimensional (for example, the module $V = R$ does not contain any nonzero submodules which are centered).

Remark 7.2.5.10. In the situation of Definition 7.2.5.6, suppose that we are given object $C \in \mathcal{C}[\mathfrak{p}]$ and $D \in \mathcal{C}[\mathfrak{q}]$. If $\mathfrak{p} \subsetneq \mathfrak{q}$, then we can choose an element $x \in \mathfrak{p} - \mathfrak{q}$ for which C is (x)-nilpotent and D is (x)-local. It follows that the mapping space $\operatorname{Map}_{\mathcal{C}}(C, D)$ is contractible.

Proposition 7.2.5.11. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let Q be an indecomposable injective object of $\mathrm{Sp}(\mathcal{C})$. If $\mathcal{C}$ is locally Noetherian, then Q is centered.*

Proof. Fix an object $x \in \pi_0 R$, let $I \subseteq \pi_0 R$ denote the principal ideal generated by (x), and set $Q' = \Gamma_I(Q)$. Since the inclusion $\mathrm{Sp}(\mathcal{C}^{\mathrm{Nil}(I)}) \hookrightarrow \mathrm{Sp}(\mathcal{C})$ is t-exact, Q' is an injective object of $\mathrm{Sp}(\mathcal{C}^{\mathrm{Nil}(I)})$. Applying Proposition 7.1.3.4, we conclude that Q' is an injective object of $\mathrm{Sp}(\mathcal{C})$. Since $Q[x^{-1}] \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$, we have $\mathrm{Ext}^1_{\mathrm{Sp}(\mathcal{C})}(Q[x^{-1}], Q') \simeq 0$. It follows that the fiber sequence $Q' \to Q \to Q[x^{-1}]$ splits: that is, Q' is a direct summand of Q. Since Q is indecomposable, it follows that either $Q' \simeq Q$ (in which case Q is (x)-nilpotent) or $Q' \simeq 0$ (in which case Q is (x)-local). □

Corollary 7.2.5.12. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let Q be an injective object of $\mathrm{Sp}(\mathcal{C})$. If $\mathcal{C}$ is locally Noetherian, then there exists a decomposition*

$$Q \simeq \bigoplus_{\mathfrak{p} \in |\operatorname{Spec} R|} Q(\mathfrak{p}),$$

where each $Q(\mathfrak{p})$ is an injective object of $\mathrm{Sp}(\mathcal{C})$ which is centered at $\mathfrak{p}$.

Proof. Combine Propositions 7.2.5.11 and C.6.10.6. □

Corollary 7.2.5.13. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let Q be an injective object of $\mathrm{Sp}(\mathcal{C})$. Let $I \subseteq \pi_0 R$ be a finitely generated ideal. If $\mathcal{C}$ is locally Noetherian, then the fiber sequence $\Gamma_I(Q) \to Q \to L_I Q$ splits. In particular, both $\Gamma_I(Q)$ and $L_I Q$ are injective objects of $\mathrm{Sp}(\mathcal{C})$.*

Proof. Using Corollary 7.2.5.12, we can assume that Q is centered at some point $\mathfrak{p} \in |\operatorname{Spec} R|$. In this case, we either have $\Gamma_I(Q) \simeq 0$ (if $I \subsetneq \mathfrak{p}$) or $L_I Q \simeq 0$ (if $I \subseteq \mathfrak{p}$). □

Corollary 7.2.5.14. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $f : X \to Q$ be a morphism in $\mathrm{Sp}(\mathcal{C})$ which exhibits Q as an injective hull of X (see Example C.5.7.9). If X is centered at a point $\mathfrak{p} \in |\operatorname{Spec} R|$, then Q is also centered at $\mathfrak{p}$.*

Proof. Fix an element $x \in \pi_0 R$ and let $I = (x)$ be the principal ideal generated by x. If $x \in \mathfrak{p}$, then X is (x)-nilpotent, so the map f factors as a composition $X \to \Gamma_I Q \xrightarrow{f'} Q$. It follows from Corollary 7.2.5.13 that the map f' exhibits $\Gamma_I Q$ as a direct summand of Q. Since $\pi_0 Q$ is an essential extension of $\pi_0 X$, the map f' must be an equivalence, so that Q is also (x)-nilpotent.

If $x \notin \mathfrak{p}$, then X is I-local. In this case, $\pi_0(\Gamma_I Q)$ is a subobject of $\pi_0 Q$ whose intersection with $\pi_0 X$ vanishes. Since $\pi_0 Q$ is an essential extension of $\pi_0 X$, it follows that $\pi_0(\Gamma_I Q) \simeq 0$. Using Theorem C.5.7.4, we conclude that $\Gamma_I Q \simeq 0$, so that Q is I-local. □

Example 7.2.5.15. Let R be a Noetherian $\mathbb{E}_\infty$-ring. For every prime ideal $\mathfrak{p} \subseteq \pi_0 R$, let $Q_\mathfrak{p} \in \mathrm{Mod}_R$ be an injective hull of the residue field $\kappa(\mathfrak{p})$. According to Example C.6.10.7, every indecomposable injective R-module Q has the form $Q_\mathfrak{p}$, for some $\mathfrak{p} \subseteq \pi_0 R$. It follows from Corollary 7.2.5.14 that $Q_\mathfrak{p}$ is centered at $\mathfrak{p}$.

Consequently, in the case $\mathcal{C} = \mathrm{Mod}_R^{\mathrm{cn}}$ we obtain a stronger version of Proposition 7.2.5.11: for every point $\mathfrak{p} \in |\operatorname{Spec} R|$ there is a *unique* indecomposable injective object of $\mathrm{Sp}(\mathcal{C})$ (up to equivalence) which is centered at $\mathfrak{p}$.

Lemma 7.2.5.16. *Let R be a connective $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be an ideal, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $C \in \mathcal{C}$ be an I-nilpotent object. Suppose we are given elements $x, y \in \pi_0 R$ such that $x \equiv y \operatorname{Mod} I$. Then C is (x)-local if and only if it is (y)-local.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & C[x^{-1}] \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta'} \\ C[y^{-1}] & \xrightarrow{\alpha'} & C[(xy)^{-1}]. \end{array}$$

We wish to show that α is an equivalence if and only if β is an equivalence. To prove this, it will suffice to show that α' and β' are equivalences. We will show that α' is an equivalence (the proof that β' is an equivalence is similar). For this, we note that $C[y^{-1}]$ is an $I[y^{-1}]$-nilpotent object of $\mathrm{LMod}_{R[y^{-1}]}(\mathcal{C})$, so that multiplication by x is invertible on $C[y^{-1}]$ by virtue of Example 7.2.4.5. □

Proposition 7.2.5.17. *Let $\phi : R \to R'$ be an étale morphism of connective $\mathbb{E}_2$-rings let $\mathfrak{p}' \subseteq \pi_0 R'$ be a prime ideal, and let $\mathfrak{p} = \phi^{-1}\mathfrak{p}' \subseteq \pi_0 R$. Let $\mathcal{C}$ be a prestable R-linear ∞-category and set $\mathcal{C}' = R' \otimes_R \mathcal{C}$. If ϕ induces an isomorphism of residue fields $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}')$, then restriction of scalars induces an equivalence of ∞-categories $\mathcal{C}'[\mathfrak{p}'] \to \mathcal{C}[\mathfrak{p}]$.*

Proof. Replacing R' by a localization $R'[b^{-1}]$ for suitably chosen $b \notin \mathfrak{p}'$, we can arrange that $\mathfrak{p}'$ is the only prime ideal of R' lying over $\mathfrak{p}$. In this case, the canonical map $\kappa(\mathfrak{p}) \to (\pi_0 R') \otimes_{\pi_0 R} \kappa(\mathfrak{p})$ is an isomorphism. Using a direct limit argument, we deduce that there exists a finitely generated ideal $I \subseteq \mathfrak{p}$ and an element $a \in (\pi_0 R) - \mathfrak{p}$ for which the induced map $\xi : (\pi_0 R)/I[a^{-1}] \to (\pi_0 R')/I[a^{-1}]$ is also an isomorphism. Replacing R by $R[a^{-1}]$ (and R' by $R'[a^{-1}]$), we can reduce to the case where we have an excision square of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{U}' & \longrightarrow & \operatorname{Sp\acute{e}t} R' \\ \downarrow & & \downarrow \\ \mathsf{U} & \longrightarrow & \operatorname{Sp\acute{e}t} R, \end{array}$$

so that restriction of scalars induces an equivalence of ∞-categories $\rho : \mathcal{C}'^{\mathrm{Nil}(I)} \to \mathcal{C}^{\mathrm{Nil}(I)}$. To complete the proof, it will suffice to show that if C is an I-nilpotent object of $\mathcal{C}'$ and $\rho(C)$ is centered at $\mathfrak{p}$, then C is centered at $\mathfrak{p}'$. Note that since $\mathfrak{p}'$ is generated by the image of $\mathfrak{p}$, our assumption that $\rho(C)$ is $\mathfrak{p}$-nilpotent guarantees that C is $\mathfrak{p}'$-nilpotent. To complete the proof, it will suffice to show that C is (x)-local for each $x \in (\pi_0 R') - \mathfrak{p}'$. Since ξ is an isomorphism, we can find an element $y \in \pi_0 R$ such that $x \equiv \phi(y) \operatorname{Mod} \mathfrak{p}'$. In this case, our assumption that $\rho(C)$ is centered at $\mathfrak{p}$ guarantees that the object $\rho(C) \in \mathcal{C}$ is (y)-local, or equivalently that $C \in \mathcal{C}'$ is $(\phi(y))$-local. Applying Lemma 7.2.5.16, we deduce that C is also (x)-local, as desired. □

Remark 7.2.5.18. In the situation of Proposition 7.2.5.17, suppose that $Q \in \mathrm{Sp}(\mathcal{C})$ is an injective object centered at $\mathfrak{p}$. Let $\theta : \mathrm{Sp}(\mathcal{C}') \to \mathrm{Sp}(\mathcal{C})$ be the restriction of scalars functor, so that we can write $Q = \theta(Q')$ for some object $Q' \in \mathrm{Sp}(\mathcal{C}')$ centered at $\mathfrak{p}'$. We claim that Q' is an injective object of $\mathrm{Sp}(\mathcal{C}')$. To prove this, choose a morphism $f : Q' \to Q''$ in $\mathrm{Sp}(\mathcal{C}')$ which exhibits Q'' as an injective hull of Q'. Then Q'' is also centered at $\mathfrak{p}$ (Corollary **??**). It follows from Proposition **??** that the induced map $\theta(f) : Q \simeq \theta(Q') \to \theta(Q'')$ exhibits $\pi_0\theta(Q')$ as an essential extension of $\pi_0\theta(Q'')$. Moreover, the object $\theta(Q'') \in \mathrm{Sp}(\mathcal{C})$ is injective (since the functor θ is right adjoint to the t-exact functor $X \mapsto R' \otimes_R X$). It follows that $\theta(f)$ exhibits $\theta(Q'')$ as an injective hull of Q in the ∞-category $\mathrm{Sp}(\mathcal{C})$. Since Q is injective, the map $\theta(f)$ must be an equivalence (see Example C.5.7.9). Applying Proposition 7.2.5.17, we deduce that f is an equivalence, so that $Q' \in \mathrm{Sp}(\mathcal{C}')$ is injective as desired.

Corollary 7.2.5.19. *Let $\phi : A \to B$ be an étale morphism of connective $\mathbb{E}_\infty$-rings, let $\mathcal{C}$ be a prestable A-linear ∞-category, and let Q be an injective object of $\mathrm{Sp}(\mathcal{C})$. If $\mathcal{C}$ is locally Noetherian, then $B \otimes_A Q$ is an injective object of $\mathrm{Sp}(B \otimes_A \mathcal{C}) \simeq \mathrm{LMod}_B(\mathrm{Sp}(\mathcal{C}))$.*

Proof. Suppose first that $\phi : A \to B$ is a finite étale morphism. In this case, there exists a B-linear equivalence $B \simeq B^\vee$, where $B^\vee$ denotes the A-linear dual of B. It follows that the extension of scalars functor $F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{LMod}_B(\mathrm{Sp}(\mathcal{C}))$ is *right* adjoint to the restriction of scalars functor $G : \mathrm{LMod}_B(\mathrm{Sp}(\mathcal{C})) \to \mathrm{Sp}(\mathcal{C})$. Since G is t-exact, it follows that functor F carries injective objects to injective objects.

We now treat the general case. Using Corollaries 7.2.5.12 and C.6.10.3, we can reduce to the case where Q is centered at some point $\mathfrak{p} \in |\operatorname{Spec} A|$. Using Corollary B.3.4.5, we can choose an étale morphism $A \to A'$ and a prime ideal $\mathfrak{p}' \in |\operatorname{Spec} A'|$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) \simeq \kappa(\mathfrak{p}')$ and the tensor product $B' = A' \otimes_A B$ factors as a Cartesian product $B'_0 \times B'_1$, where B'_0 is finite étale over A' and $B'_1 \otimes_{A'} \kappa(\mathfrak{p}') \simeq 0$. Using Proposition 7.2.5.17, we can lift Q to an object $Q' \in \mathrm{Sp}(A' \otimes_A \mathcal{C})$ which is centered at $\mathfrak{p}'$. Moreover, the object Q' is injective (Remark 7.2.5.18). Then $B \otimes_A Q$ is the image of $B' \otimes_{A'} Q'$ under the restriction of scalars functor $\theta : \mathrm{LMod}_{B'}(\mathrm{Sp}(\mathcal{C})) \to \mathrm{LMod}_B(\mathrm{Sp}(\mathcal{C}))$. Since B' is flat over B, the extension of scalars functor $\mathrm{LMod}_B(\mathrm{Sp}(\mathcal{C})) \to \mathrm{LMod}_{B'}(\mathrm{Sp}(\mathcal{C}))$ is t-exact, so the functor θ carries injective objects to injective objects. We will complete the proof by showing that $B' \otimes_{A'} Q' \in \mathrm{LMod}_{B'}(\mathrm{Sp}(\mathcal{C}))$ is injective. Equivalently, we will show that $B'_0 \otimes_{A'} Q'$ is an injective object of $\mathrm{LMod}_{B'_0}(\mathrm{Sp}(\mathcal{C}))$ and that $B'_1 \otimes_{A'} Q'$ is an injective object of $\mathrm{LMod}_{B'_1}(\mathrm{Sp}(\mathcal{C}))$. In the former case, this follows from the first part of the proof (since B'_0 is finite étale over A'); in the latter, it follows from our assumption that Q' is centered at $\mathfrak{p}'$ (so that $B'_1 \otimes_{A'} Q' \simeq 0$). □

We close by proving a converse to Corollary 7.2.5.19, which implies that injectivity can be tested locally with respect to the étale topology:

Proposition 7.2.5.20. *Let $\phi : A \to B$ be a faithfully flat étale morphism of connective $\mathbb{E}_\infty$-rings, let $\mathcal{C}$ be a prestable A-linear ∞-category, and let Q be an object of $\mathrm{Sp}(\mathcal{C})$. If $\mathcal{C}$ is locally Noetherian and $B \otimes_A Q$ is an injective object of $\mathrm{Sp}(B \otimes_A \mathcal{C}) \simeq \mathrm{LMod}_B(\mathrm{Sp}(\mathcal{C}))$, then Q is an injective object of $\mathrm{Sp}(\mathcal{C})$.*

Proof. Using Propositions B.1.1.3 and 4.6.1.2, we can reduce to the case where A and B are Noetherian. Let $I \subseteq \pi_0 A$ be an ideal which is maximal with respect to the property that $L_I Q \in \mathrm{Sp}(\mathcal{C})$ is injective. Since the collection of injective objects of $\mathrm{Sp}(\mathcal{C})$ is closed under extensions, to show that $Q \in \mathrm{Sp}(\mathcal{C})$ is injective, it will suffice to show that $\Gamma_I Q$ is injective. We may therefore replace Q by $\Gamma_I Q$ (note that this does not injure our hypothesis that $B \otimes_A Q \in \mathrm{LMod}_B(\mathrm{Sp}(\mathcal{C}))$ is injective, by virtue of Corollary 7.2.5.13) and thereby reduce to the case where Q is I-nilpotent. If $I = \pi_0 A$, then $Q \simeq 0$ and there is nothing to prove. Otherwise, choose $\mathfrak{p} \subseteq \pi_0 A$ which is minimal among prime ideals containing I. Using Corollary B.3.4.5, we can choose an étale morphism $A \to A'$ and a prime ideal $\mathfrak{p}' \in |\operatorname{Spec} A'|$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) \simeq \kappa(\mathfrak{p}')$ and the tensor product $B' = A' \otimes_A B$ factors as a Cartesian product $B'_0 \times B'_1$, where B'_0 is finite étale over A' and $B'_1 \otimes_{A'} \kappa(\mathfrak{p}') \simeq 0$. Replacing A' by a localization if necessary, we may assume that B'_0 is faithfully flat over A' and that the map of affine schemes $j : \operatorname{Spec}(\pi_0 A')/I(\pi_0 A') \to \operatorname{Spec} \pi_0 A/I(\pi_0 A)$ is an open immersion.

Combining our assumption that $B \otimes_A Q$ is an injective object of $\mathrm{LMod}_B(\mathrm{Sp}(\mathcal{C}))$ with Corollary 7.2.5.19, we deduce that $B'_0 \otimes_A Q$ is an injective object of $\mathrm{LMod}_{B'_0}(\mathrm{Sp}(\mathcal{C}))$. Since the forgetful functor $\mathrm{LMod}_{B'_0}(\mathrm{Sp}(\mathcal{C})) \to \mathrm{LMod}_{A'}(\mathrm{Sp}(\mathcal{C}))$ admits a t-exact left adjoint, it follows that $B'_0 \otimes_A Q$ is an injective object of $\mathrm{LMod}_{A'}(\mathrm{Sp}(\mathcal{C}))$. Since B'_0 is faithfully flat and finite étale over A', the unit map $A' \to B'_0$ admits a left homotopy inverse in the ∞-category $\mathrm{Mod}_{A'}$. It follows that $A' \otimes_A Q$ is a retract of $B' \otimes_A Q$ in the ∞-category $\mathrm{LMod}_{A'}(\mathrm{Sp}(\mathcal{C}))$, so that $A' \otimes_A Q$ is also an injective object of $\mathrm{LMod}_{A'}(\mathrm{Sp}(\mathcal{C}))$. The forgetful functor $\mathrm{LMod}_{A'}(\mathrm{Sp}(\mathcal{C})) \to \mathrm{Sp}(\mathcal{C})$ admits a t-exact left adjoint, and therefore carries injective objects to injective objects. It follows that $A' \otimes_A Q$ is an injective object of $\mathrm{Sp}(\mathcal{C})$. Since Q is I-nilpotent and the map j is an open immersion, we can identify $A' \otimes_A Q$ with the localization $L_{I'} Q$, where $I' \supseteq I$ is an ideal in $\pi_0 A$ whose vanishing locus is complementary to the image of j. The injectivity of $L_{I'}Q$ now contradicts our maximality assumption on I. □

7.3 Completions of Modules

Let R be a commutative ring, let $I \subseteq R$ be an ideal, and let M be a (discrete) R-module. We let $\mathrm{Cpl}(M; I)$ denote the inverse limit of the tower

$$\cdots \to M/I^4M \to M/I^3M \to M/I^2M \to M/IM,$$

formed in the abelian category $\mathrm{Mod}_R^{\heartsuit}$ of discrete R-modules. We will refer to $\mathrm{Cpl}(M; I)$ as the *I-adic completion of M*. We say that M is *I-adically complete* if the canonical map $M \to \mathrm{Cpl}(M; I)$ is an isomorphism.

If the ring R is Noetherian and we restrict our attention to finitely generated R-modules, then the I-adic completion $M \mapsto \mathrm{Cpl}(M; I)$ has excellent formal properties: for example, it is an exact functor of M. If we are given a short exact sequence of R-modules $0 \to M' \to M \to M'' \to 0$, then we obtain a short exact sequence

$$0 \to K \to \mathrm{Cpl}(M; I) \to \mathrm{Cpl}(M''; I) \to 0,$$

where K denotes the completion of M' with respect to topology given by the descending sequence of submodules $\{M' \cap I^n M\}_{n \geqslant 0}$. If R is Noetherian and M is finitely generated, then the Artin-Rees lemma allows us to identify K with the I-adic completion of M'. However, this fails in general: even when $R = \mathbf{Z}$ and $I = (p)$ for some prime number p, the theory of I-adic completions can exhibit some bad behavior:

Example 7.3.0.1. The p-adic completion functor on abelian groups is not left exact. For example, let $M = \mathbf{Q}_p / \mathbf{Z}_p$, and let M' be the submodule of M generated by $\frac{1}{p} \in \mathbf{Q}_p$ (so that M' is a cyclic group of order p). Then the p-adic completion of M vanishes (since $pM = M$), but the p-adic completion of M' is isomorphic to M'. Consequently, the inclusion $M' \hookrightarrow M$ does not induce an injection of p-adic completions.

Example 7.3.0.2. The p-adic completion functor on abelian groups is not right exact. To see this, consider the exact sequence

$$0 \to \bigoplus_{n \geqslant 0} p^n \mathbf{Z} \xrightarrow{\beta} \bigoplus_{n \geqslant 0} \mathbf{Z} \xrightarrow{\alpha} \bigoplus_{n \geqslant 0} \mathbf{Z} / p^n \mathbf{Z} \to 0.$$

Then:

- The p-adic completion of $\bigoplus_{n \geqslant 0} \mathbf{Z}$ can be identified with the submodule $N \subseteq \prod_{n \geqslant 0} \mathbf{Z}_p$ consisting of those sequences $\{\lambda_n \in \mathbf{Z}_p\}_{n \geqslant 0}$ which converge p-adically to zero.

- The p-adic completion of $\bigoplus_{n \geqslant 0} p^n \mathbf{Z}$ can be identified with the submodule $N' \subseteq \prod_{n \geqslant 0} \mathbf{Z}_p$ consisting of those sequences $\{\lambda_n \in \mathbf{Z}_p\}_{n \geqslant 0}$ for which each λ_n is divisible by p^n and the sequence $p^{-n}\lambda_n$ converges p-adically to zero.

- The p-adic completion of $\bigoplus_{n \geqslant 0} \mathbf{Z}/p^n \mathbf{Z}$ can be identified with the image N'' of N in the product $\prod_{n \geqslant 0} \mathbf{Z}/p^n \mathbf{Z}$.

- The p-adic completion of α is surjection whose kernel is given by the product $\prod_{n \geqslant 0} p^n \mathbf{Z}_p \subseteq N$.

Consequently, the sequence of I-adic completions $0 \to N' \xrightarrow{\widehat{\beta}} N \xrightarrow{\widehat{\alpha}} N'' \to 0$ is not exact in the middle: that is, we have $\operatorname{im}(\widehat{\beta}) \subsetneq \ker(\widehat{\alpha})$.

Example 7.3.0.3. The collection of p-adically complete abelian groups is not closed under the formation of cokernels. For example, if $\widehat{\beta} : N' \hookrightarrow N$ is as in Example 7.3.0.3, then the cokernel of $\widehat{\beta}$ is not p-adically complete (the p-adic completion of $\operatorname{coker}(\widehat{\beta})$ can be identified with N'').

Example 7.3.0.4. The collection of p-adically complete abelian groups is not closed under extensions. For example, in the situation of Example 7.3.0.1, we have a short exact sequence of abelian groups

$$0 \to \ker(\widehat{\alpha})/\operatorname{im}(\widehat{\beta}) \to \operatorname{coker}(\widehat{\beta}) \to N'' \to 0$$

where the outer terms are p-adically complete, but the middle term is not.

When working with modules which are not finitely generated or rings which are not Noetherian, it is often convenient to consider a different notion of completion.

Definition 7.3.0.5. Let R be a commutative ring, let M be a (discrete) R-module, and let $I \subseteq R$ be a finitely generated ideal. We will say that M is *I-complete* if, for every $x \in I$, we have $\operatorname{Ext}^0_R(R[x^{-1}], M) \simeq 0 \simeq \operatorname{Ext}^1_R(R[x^{-1}], M)$.

Remark 7.3.0.6. The vanishing conditions $\operatorname{Ext}^0_R(R[x^{-1}], M) \simeq 0 \simeq \operatorname{Ext}^1_R(R[x^{-1}], M)$ are equivalent to the requirement that every short exact sequence

$$0 \to M \to N \to R[x^{-1}] \to 0$$

admits a unique splitting. Note that the groups $\operatorname{Ext}^i_R(R[x^{-1}], M)$ automatically vanish for $i > 1$, since $R[x^{-1}]$ is an R-module of projective dimension $\leqslant 1$.

Remark 7.3.0.7. In the situation of Definition 7.3.0.5, if $I = (x_1, \ldots, x_n)$, then to verify that an R-module M is I-complete it suffices to have $\operatorname{Ext}^0_R(R[x_i^{-1}], M) \simeq 0 \simeq \operatorname{Ext}^1_R(R[x_i^{-1}], M)$ for $1 \leqslant i \leqslant n$ (see Remark 7.3.4.2).

We now summarize some of the pleasant features of Definition 7.3.0.5:

- Let R be a commutative ring, let $I \subseteq R$ be a finitely generated ideal, and let $\mathcal{A} \subseteq \mathrm{Mod}_R^{\heartsuit}$ be the full subcategory of spanned by the I-complete R-modules. Then $\mathcal{A}$ is closed under the formation of kernels, cokernels, and extensions in $\mathrm{Mod}_R^{\heartsuit}$. In particular, $\mathcal{A}$ is an abelian category, and the inclusion $\mathcal{A} \hookrightarrow \mathrm{Mod}_R^{\heartsuit}$ is an exact functor (Corollary 7.3.4.6).

- Every I-adically complete R-module M is I-complete. Conversely, if M is I-complete and I-adically separated (meaning that $\bigcap_{n \geqslant 0} I^n M = \{0\}$), then M is I-adically complete (Corollary 7.3.6.3). However, in general there can exist R-modules M which are I-complete but not I-adically separated (and therefore not I-adically complete): for example, the module $\mathrm{coker}(\widehat{\beta})$ appearing in Example 7.3.0.3.

- The inclusion $\mathcal{A} \hookrightarrow \mathrm{Mod}_R^{\heartsuit}$ admits a left adjoint $L : \mathrm{Mod}_R^{\heartsuit} \to \mathcal{A}$ (Remark 7.3.4.7). For any R-module M, there is a canonical epimorphism $LM \to \mathrm{Cpl}(M; I)$ (Corollary 7.3.6.2), which is an isomorphism in many cases (for example, if M is a Noetherian R-module; see Proposition 7.3.6.1).

Since the inclusion $\mathcal{A} \hookrightarrow \mathrm{Mod}_R^{\heartsuit}$ is an exact functor, it follows formally that that the functor $L : \mathrm{Mod}_R^{\heartsuit} \to \mathcal{A}$ is right exact (unlike the formation of I-adic completions; see Example 7.3.0.2). However, it is usually not left exact. To obtain a more complete picture, it is convenient to work at the derived level: that is, to replace the abelian category $\mathrm{Mod}_R^{\heartsuit}$ of discrete R-modules by the ∞-category Mod_R of all R-modules. In this section, we will introduce a full subcategory $\mathrm{Mod}_R^{\mathrm{Cpl}(I)} \subseteq \mathrm{Mod}_R$ whose objects we will refer to as *I-complete R-modules* (Definition 7.3.1.1), whose intersection with $\mathrm{Mod}_R^{\heartsuit}$ consists of those discrete R-modules which are I-complete in the sense of Definition 7.3.0.5 (Corollary 7.3.4.6). We will see inclusion functor $\mathrm{Mod}_R^{\mathrm{Cpl}(I)} \hookrightarrow \mathrm{Mod}_R$ admits a left adjoint, which we will refer to as *I-completion* and denote by $M \mapsto M_I^{\wedge}$ (Notation 7.3.1.5). The localization functor $L : \mathrm{Mod}_R^{\heartsuit} \to \mathcal{A}$ is then given by the construction $M \mapsto \pi_0 M_I^{\wedge}$ (Remark 7.3.4.7). When R is Noetherian, we can be more precise: the stable ∞-category $\mathrm{Mod}_R^{\mathrm{Cpl}(I)}$ agrees (up to right completion) with the derived ∞-category of the abelian category $\mathcal{A}$ (Proposition 7.3.7.3), and the I-completion functor $M \mapsto M_I^{\wedge}$ agrees with the left derived functor of $L : \mathrm{Mod}_R^{\heartsuit} \to \mathcal{A}$ (Corollary 7.3.7.5).

7.3.1 Completeness

We begin by discussing completions in a very general setting.

Definition 7.3.1.1. Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a stable R-linear ∞-category, and let $I \subseteq \pi_0 R$ be an ideal. We will say that an object $C \in \mathcal{C}$ is *I-complete* if, for every I-local object $D \in \mathcal{C}$ (Definition 7.2.4.1), the mapping space $\mathrm{Map}_{\mathcal{C}}(D, C)$ is contractible. We let $\mathcal{C}^{\mathrm{Cpl}(I)}$ denote the full subcategory of $\mathcal{C}$ spanned by the I-complete objects.

Remark 7.3.1.2. In the situation of Definition 7.3.1.1, suppose that $I, J \subseteq \pi_0 R$ are two ideals having the same radical. Then an object $C \in \mathcal{C}$ is I-complete if and only if it is J-complete.

Remark 7.3.1.3. Let $\phi : R \to R'$ be a morphism of $\mathbb{E}_2$-rings, let $\mathcal{C}$ be a stable R'-linear ∞-category. Let $I \subseteq \pi_0 R$ be an ideal and let $I' \subseteq \pi_0 R'$ be the ideal generated by the image of I. Then an object $C \in \mathcal{C}$ is I'-complete if and only if it is I-complete (where we regard $\mathcal{C}$ as a stable R-linear ∞-category via restriction of scalars).

We will generally be interested in Definition 7.3.1.1 only in the special case where I is finitely generated (or at least contains a finitely generated ideal with the same radical). In this case, we have the following:

Proposition 7.3.1.4. *Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a stable R-linear ∞-category, and let $I \subseteq \pi_0 R$ be a finitely generated ideal. Then the pair $(\mathcal{C}^{\mathrm{Loc}(I)}, \mathcal{C}^{\mathrm{Cpl}(I)})$ is a semi-orthogonal decomposition of $\mathcal{C}$. In particular, every object $C \in \mathcal{C}$ fits into an (essentially unique) fiber sequence $C' \to C \to C''$, where C' is I-local and C'' is I-complete.*

Proof. Our assumption that I is finitely generated implies that $\mathcal{C}^{\mathrm{Loc}(I)}$ is closed under small colimits in $\mathcal{C}$ (Proposition 7.2.4.9). The desired result now follows from Example 7.2.1.9. □

Notation 7.3.1.5. Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $\mathcal{C}$ be a stable R-linear ∞-category. It follows from Proposition 7.3.1.4 that the inclusion $\mathcal{C}^{\mathrm{Cpl}(I)} \hookrightarrow \mathcal{C}$ admits a left adjoint. We will refer to this left adjoint as the *I-completion functor*, and denote it by $C \mapsto C^{\wedge}_I$.

Remark 7.3.1.6. Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $\mathcal{C}$ be a stable R-linear ∞-category. For each object $C \in \mathcal{C}$, the diagram σ_C :

$$\begin{array}{ccc} C & \longrightarrow & C^{\wedge}_I \\ \downarrow & & \downarrow \\ L_I(C) & \longrightarrow & L_I(C^{\wedge}_I) \end{array}$$

is a pullback square. To see this, we observe that the fibers of the vertical maps are I-nilpotent, while the fibers of the horizontal maps are I-local. It follows that the total homotopy fiber of σ_C is both I-nilpotent and I-local, and therefore vanishes.

Proposition 7.3.1.7. *Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a stable R-linear ∞-category, and let $I \subseteq \pi_0 R$ be a finitely generated ideal. Then the functors $C \mapsto C^{\wedge}_I$ and $C \mapsto \Gamma_I C$ induce mutually inverse equivalences $\mathcal{C}^{\mathrm{Nil}(I)} \underset{\Gamma_I}{\rightleftarrows} \mathcal{C}^{\mathrm{Cpl}(I)}$.*

Proof. Combine Propositions 7.2.1.10 and 7.3.1.4. □

Remark 7.3.1.8. In the situation of Proposition 7.3.1.7, we can regard $\mathcal{C}^{\mathrm{Nil}(I)}$ as a stable R-linear ∞-category (Proposition 7.1.1.12). Consequently, there is an essentially unique action of R on the ∞-category $\mathcal{C}^{\mathrm{Cpl}(I)}$ for which the equivalence of Proposition 7.3.1.7 is R-linear. Note that the composition of this equivalence with the I-completion functor $\mathcal{C} \to \mathcal{C}^{\mathrm{Cpl}(I)}$ can be identified with the functor $\Gamma_I : \mathcal{C} \to \mathcal{C}^{\mathrm{Nil}(I)}$, which is R-linear by virtue of Remark 7.1.2.6. It follows that we can regard the I-completion functor $\mathcal{C} \to \mathcal{C}^{\mathrm{Cpl}(I)}$ as an R-linear functor.

Warning 7.3.1.9. In the situation of Proposition 7.3.1.7, the inclusion $\mathcal{C}^{\mathrm{Cpl}(I)} \hookrightarrow \mathcal{C}$ is usually not an R-linear functor (where we regard $\mathcal{C}^{\mathrm{Cpl}(I)}$ as a stable R-linear ∞-category as in Remark 7.3.1.8), because it need not commute with filtered colimits.

Warning 7.3.1.10. Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $F : \mathcal{C} \to \mathcal{D}$ be an R-linear functor between stable R-linear ∞-categories. Then the functor F need not carry I-complete objects of $\mathcal{C}$ to I-complete objects of $\mathcal{D}$. However, the functor F admits a right adjoint G which carries I-complete objects of $\mathcal{D}$ to I-complete objects of $\mathcal{C}$ (since F carries I-local objects of $\mathcal{C}$ to I-local objects of $\mathcal{D}$; see Proposition 7.2.4.9).

7.3.2 The Case of a Principal Ideal

We now study the I-completion functor $C \mapsto C^{\wedge}_I$ in the situation where the ideal I is generated by a single element. In this case, we have the following dual version of Proposition 7.1.2.3.

Proposition 7.3.2.1. *Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a stable R-linear ∞-category, and let $x \in \pi_0 R$ be an element. For any object $C \in \mathcal{C}$, let $T(C)$ denote the limit of the tower*

$$\cdots \xrightarrow{x} C \xrightarrow{x} C \xrightarrow{x} C \xrightarrow{x} C.$$

Then the (x)-completion of C can be identified with the cofiber of the canonical map $\theta : T(C) \to C$.

Proof. For any object $D \in \mathcal{C}$, we can identify the mapping space $\mathrm{Map}_{\mathcal{C}}(D, T(C))$ with the homotopy limit of the tower of spaces

$$\cdots \xrightarrow{x} \mathrm{Map}_{\mathcal{C}}(D, C) \xrightarrow{x} \mathrm{Map}_{\mathcal{C}}(D, C) \xrightarrow{x} \mathrm{Map}_{\mathcal{C}}(D, C) \xrightarrow{x} \mathrm{Map}_{\mathcal{C}}(D, C).$$

If D is (x)-local, then each of the transition maps in this diagram is a homotopy equivalence. It follows that composition with θ induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(D, T(C)) \to \mathrm{Map}_{\mathcal{C}}(D, C)$, so that $\mathrm{Map}_{\mathcal{C}}(D, \mathrm{fib}(\theta))$ is contractible. Allowing D to vary, we deduce that

$\mathrm{fib}(\theta)$ is (x)-complete, so that $\mathrm{cofib}(\theta) = \Sigma \mathrm{fib}(\theta)$ is also (x)-complete. For any (x)-complete object $E \in \mathcal{C}$, we have a canonical fiber sequence

$$\mathrm{Map}_{\mathcal{C}}(\mathrm{cofib}(\theta), E) \xrightarrow{\alpha} \mathrm{Map}_{\mathcal{C}}(C, E) \to \mathrm{Map}_{\mathcal{C}}(T(C), E).$$

It is not difficult to see that $T(C)$ is (x)-local, so the third term in this fiber sequence is contractible and therefore α is a homotopy equivalence. It follows that the natural map $C \to \mathrm{cofib}(\theta)$ exhibits $\mathrm{cofib}(\theta)$ as an (x)-completion of C. □

Corollary 7.3.2.2. *Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a stable R-linear ∞-category, and let $x \in \pi_0 R$. The following conditions on an object $C \in \mathcal{C}$ are equivalent:*

(1) *The module C is (x)-complete.*

(2) *The limit of the tower*
$$\cdots \xrightarrow{x} C \xrightarrow{x} C \xrightarrow{x} C \xrightarrow{x} C.$$
vanishes.

Corollary 7.3.2.3. *Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a stable R-linear ∞-category, let $I \subseteq \pi_0 R$ be an ideal, and let $x \in \pi_0 R$. Then the (x)-completion functor $C \mapsto C^{\wedge}_{(x)}$ carries I-complete objects of $\mathcal{C}$ to I-complete objects of $\mathcal{C}$.*

Proof. Combine the description of the functor $C \mapsto C^{\wedge}_{(x)}$ given by Proposition 7.3.2.1 with the observation that $\mathcal{C}^{\mathrm{Cpl}(I)}$ is closed under limits in $\mathcal{C}$. □

We now study the exactness properties of the (x)-completion functor in the case $\mathcal{C} = \mathrm{LMod}_R$.

Corollary 7.3.2.4. *Let R be an $\mathbb{E}_2$-ring, let $x \in \pi_0 R$, and let M be a left R-module.*

(1) *If M is connective, then $M^{\wedge}_{(x)}$ is connective.*

(2) *If $M \in (\mathrm{LMod}_R)_{\leqslant 0}$, then $M^{\wedge}_{(x)}$ belongs to $(\mathrm{LMod}_R)_{\leqslant 1}$.*

Proof. Let $T(M)$ be as in the statement of Proposition 7.3.2.1, so that we have an exact sequence $\pi_k M \to \pi_k M^{\wedge}_{(x)} \to \pi_{k-1} T(M)$. In case (1), the desired result follows from the observation that $\pi_{k-1} T(M) \simeq 0$ for $k < 0$. In case (2), we observe instead that $\pi_{k-1} T(M) \simeq 0$ for $k > 1$. □

7.3.3 The Case of a Finitely Generated Ideal

We now show that many of the preceding results can be generalized from principal ideals to finitely generated ideals. Our starting point is the following observation:

Lemma 7.3.3.1. *Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a stable R-linear ∞-category, and suppose we are given a pair of ideals $I, J \subseteq \pi_0 R$. Assume that either I or J is finitely generated. Then the full subcategory $\mathcal{C}^{\mathrm{Loc}(I+J)}$ is generated under extensions by the full subcategories $\mathcal{C}^{\mathrm{Loc}(I)}, \mathcal{C}^{\mathrm{Loc}(J)} \subseteq \mathcal{C}^{\mathrm{Loc}(I+J)}$.*

Proof. Suppose that I is finitely generated and let C be an object of $\mathcal{C}^{\mathrm{Loc}(I+J)}$. Using Proposition 7.2.4.4, we can choose a fiber sequence $C' \to C \to C''$ where C' is J-nilpotent and C'' is J-local. Then C'' is also $(I+J)$-local, that C' is $(I+J)$-local. Consider the fiber sequence $\Gamma_I C' \xrightarrow{\alpha} C' \xrightarrow{L}_I C'$. Since I is finitely generated, the functor Γ_I is R-linear (Remark 7.1.2.6) and therefore carries J-nilpotent objects of $\mathcal{C}$ to J-nilpotent objects of $\mathcal{C}$ (Proposition 7.1.1.12). Consequently, $\Gamma_I C'$ is both I-nilpotent and J-nilpotent, and is therefore $(I+J)$-nilpotent (Remark 7.1.1.9). Since C' is $(I+J)$-local, the morphism α is nullhomotopic. It follows that C' is a direct summand of $L_I C'$ and is therefore I-local. It follows that C can be written as an extension of a J-local object of $\mathcal{C}$ by an I-local object of $\mathcal{C}$. □

Proposition 7.3.3.2. *Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a stable R-linear ∞-category, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $I' \subseteq \pi_0 R$ be the ideal generated by I together with an element $x \in \pi_0 R$. For each object $C \in \mathcal{C}$, the composite map*

$$C \xrightarrow{\alpha} M_I^{\wedge} \xrightarrow{\beta} (M_I^{\wedge})_{(x)}^{\wedge}$$

exhibits $(C_I^{\wedge})_{(x)}^{\wedge}$ as an I'-completion of M.

Proof. It is clear that $(C_I^{\wedge})_{(x)}^{\wedge}$ is (x)-complete, and Corollary 7.3.2.3 shows that it is also I-complete. Using Lemma 7.3.3.1 we deduce that $(C_I^{\wedge})_{(x)}^{\wedge}$ is I'-complete. It will therefore suffice to show that the fiber of $\beta \circ \alpha$ is I'-local. This follows because the fibers $\mathrm{fib}(\alpha)$ and $\mathrm{fib}(\beta)$ are both I'-local: in fact, $\mathrm{fib}(\alpha)$ is I-local and $\mathrm{fib}(\beta)$ is (x)-local. □

Corollary 7.3.3.3. *Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a stable R-linear ∞-category, and let $I \subseteq \pi_0 R$ be a finitely generated ideal. The following conditions on an object $C \in \mathcal{C}$ are equivalent:*

(1) *The object C is I-complete.*

(2) *For each $x \in I$, the object C is (x)-complete.*

(3) *There exists a set of generators $x_1, \ldots, x_n$ for the ideal I such that C is (x_i)-complete for $1 \leqslant i \leqslant n$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. We prove that (3) $\Rightarrow$ (1). Assume that (3) is satisfied. For $0 \leqslant i \leqslant n$, let $I(i)$ be the ideal generated by $x_1, \ldots, x_i$. We prove that C is $I(i)$-complete by induction on i, the case $i = 0$ being trivial. Assume that $i < n$ and that M is $I(i)$-complete. Then the map $\alpha : M \to M^{\wedge}_{I(i)}$ is an equivalence. Since M is x_{i+1}-complete, the map $\beta : M \to M^{\wedge}_{(x_{i+1})}$ is also an equivalence. Using Proposition 7.3.3.2, we deduce that the map $M \to M^{\wedge}_{I(i+1)}$ is an equivalence, so that M is $I(i+1)$-complete. □

Corollary 7.3.3.4. *Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $F : \mathcal{C} \to \mathcal{D}$ be an R-linear functor between stable R-linear ∞-categories. Suppose that F commutes with sequential inverse limits. Then:*

(a) *The functor F carries I-complete objects of $\mathcal{C}$ to I-complete object of $\mathcal{D}$.*

(b) *The functor F commutes with I-completion: that is, for every object $C \in \mathcal{C}$, the canonical map $F(C) \to F(C^{\wedge}_I)$ exhibits $F(C^{\wedge}_I)$ as an I-completion of $F(C)$.*

Proof. To prove (a), we can use Corollary 7.3.3.3 to reduce to the case where I is generated by a single element, in which case the desired result follows from the criterion of Corollary 7.3.2.2. To prove (b), let $C \in \mathcal{C}$ be an object and let $\alpha : C \to C^{\wedge}_I$ be the canonical map. Then $\mathrm{fib}(\alpha)$ is an I-local object of $\mathcal{C}$, so that $F(\mathrm{fib}(\alpha)) \simeq \mathrm{fib}(F(\alpha))$ is an I-local object of $\mathcal{D}$ by virtue of Proposition 7.2.4.9. Since the codomain of $F(C^{\wedge}_I)$ is I-complete, it follows that $F(\alpha)$ exhibits $F(C^{\wedge}_I)$ as an I-completion of C. □

Corollary 7.3.3.5. *Let $\phi : R \to R'$ be a morphism of $\mathbb{E}_2$-rings, let $\mathcal{C}$ be a stable R-linear ∞-category, and let $\mathcal{C}' = R' \otimes_R \mathcal{C} = \mathrm{LMod}_{R'}(\mathcal{C})$ be the stable R'-linear ∞-category obtained from $\mathcal{C}$ by extension of scalars. Let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $I' \subseteq \pi_0 R'$ be the ideal generated by the image of I. Then:*

(1) *An object $C \in \mathcal{C}'$ is I'-complete if and only if its image under the forgetful functor $\rho : \mathcal{C}' \to \mathcal{C}$ is I-complete.*

(2) *For each object $C \in \mathcal{C}'$, the canonical map $C \to C^{\wedge}_{I'}$ exhibits $\rho(C^{\wedge}_{I'})$ as an I-completion of $\rho(C)$.*

Proof. Assertion (2) follows from Corollary 7.3.3.4 and Remark 7.3.1.3. Assertion (1) is a consequence of (2), since the functor ρ is conservative. □

Corollary 7.3.3.6. *Let $\phi : R \to R'$ be a morphism of $\mathbb{E}_2$-rings, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $I' \subseteq \pi_0 R'$ be the ideal generated by the image of I. Then:*

(1) *A left R'-module M is I'-complete if and only if it is I-complete (when regarded as an R-module).*

(2) *For every left R'-module M, the canonical map $M \to M^{\wedge}_{I'}$ exhibits $M^{\wedge}_{I'}$ as an I-completion of M (when regarded as a morphism of R-modules).*

Example 7.3.3.7. Let $\phi : R \to R'$ be a morphism of $\mathbb{E}_2$-rings and let M be a left R'-module. Then, when regarded as an R-module, M is complete with respect to any finitely generated ideal $I \subseteq \pi_0 R$ which is annihilated by ϕ.

7.3.4 Completeness and Homotopy Groups

We now show that the completeness of modules over an $\mathbb{E}_2$-ring can be tested at the level of homotopy groups:

Theorem 7.3.4.1. *Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let M be a left R-module. The following conditions are equivalent:*

(a) *The left R-module M is I-complete, in the sense of Definition 7.3.1.1.*

(b) *For every integer k, the homotopy group $\pi_k M$ is I-complete when regarded as a discrete module over the commutative ring $A = \pi_0 R$, in the sense of Definition 7.3.0.5. That is, for each $x \in I$, we have $\mathrm{Ext}^0_A(A[x^{-1}], \pi_k M) \simeq 0 \simeq \mathrm{Ext}^1_A(A[x^{-1}], \pi_k M)$.*

Proof. Fix an element $x \in \pi_0 R$, and let $T(M)$ denote the limit of the tower

$$\cdots \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M$$

as in Proposition 7.3.2.1. For each integer k, let D_k denote the tower of abelian groups

$$\cdots \xrightarrow{x} \pi_k M \xrightarrow{x} \pi_k M \xrightarrow{x} \pi_k M \xrightarrow{x} \pi_k M,$$

so that we have Milnor exact sequences

$$0 \to \varprojlim{}^1 D_{k+1} \to \pi_k T(M) \to \varprojlim{}^0 D_k \to 0.$$

Writing $A[x^{-1}]$ as the direct limit of the sequence $A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} \cdots$, we obtain canonical isomorphisms

$$\varprojlim{}^1 D_{k+1} \simeq \mathrm{Ext}^1_A(A[x^{-1}], \pi_{k+1} M) \qquad \varprojlim{}^0 D_k \simeq \mathrm{Ext}^0_A(A[x^{-1}], \pi_k M).$$

Consequently, the groups $\mathrm{Ext}^i_A(A[x^{-1}], \pi_k M)$ vanish for all $k \in \mathbf{Z}$ and $i \in \{0, 1\}$ if and only if $T(M) \simeq 0$: that is, if and only if M is (x)-complete (Corollary 7.3.2.2). The equivalence of (a) and (b) now follows from Corollary 7.3.3.3. □

Remark 7.3.4.2. In the situation of Theorem 7.3.4.1, if $x_1, \ldots, x_n \in I$ is a set of generators for I, then it suffices to verify condition (b) in the special case $x = x_i$ (Corollary 7.3.3.3).

Corollary 7.3.4.3. *Let R be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 R$ be a finitely generated ideal. If $M \in \mathrm{LMod}_R$ is I-complete, then the truncations $\tau_{\geqslant n} M$ and $\tau_{\leqslant n} M$ are I-complete for every integer n.*

Proposition 7.3.4.4. *Let R be a connective $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and set*

$$(\mathrm{LMod}_R^{\mathrm{Cpl}(I)})_{\leqslant 0} = (\mathrm{LMod}_R)_{\leqslant 0} \cap \mathrm{LMod}_R^{\mathrm{Cpl}(I)} \quad (\mathrm{LMod}_R^{\mathrm{Cpl}(I)})_{\geqslant 0} = (\mathrm{LMod}_R)_{\geqslant 0} \cap \mathrm{LMod}_R^{\mathrm{Cpl}(I)}.$$

Then:

(1) *The pair $((\mathrm{LMod}_R^{\mathrm{Cpl}(I)})_{\geqslant 0}, (\mathrm{LMod}_R^{\mathrm{Cpl}(I)})_{\leqslant 0})$ determines a t-structure on the stable ∞-category $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ which is both left and right complete.*

(2) *The inclusion functor $\mathrm{LMod}_R^{\mathrm{Cpl}(I)} \hookrightarrow \mathrm{LMod}_R$ is t-exact.*

(3) *The I-completion functor $\mathrm{LMod}_R \to \mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ is right t-exact.*

(4) *If I is generated by n elements, then the I-completion functor $\mathrm{LMod}_R \to \mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ carries $(\mathrm{LMod}_R)_{\leqslant 0}$ into $(\mathrm{LMod}_R^{\mathrm{Cpl}(I)})_{\leqslant n}$.*

Proof. Assertion (1) follows immediately from Corollary 7.3.4.3 (and the left and right completeness for the standard t-structure on LMod_R), assertion (2) immediate from the definition, and assertion (3) is a formal consequence of (2). To prove (4), choose a set of generators $x_1, \ldots, x_n$ for the ideal I. Proposition 7.3.3.2 implies that the I-completion functor can be obtained by composing the (x_i)-completion functors for $1 \leqslant i \leqslant n$. We can therefore reduce to the case where $I = (x)$ is generated by a single element, in which case the desired result follows from Corollary 7.3.2.4.

□

Warning 7.3.4.5. In the situation of Proposition 7.3.4.4, the t-structure on $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ is usually not compatible with filtered colimits. For example, suppose that $R = \mathbf{Z}$ and set $I = (p)$ for some prime number p. The direct system of abelian groups

$$(p^{-1}\mathbf{Z})/\mathbf{Z} \hookrightarrow (p^{-2}\mathbf{Z})/\mathbf{Z} \hookrightarrow (p^{-3}\mathbf{Z})/\mathbf{Z} \hookrightarrow (p^{-4}\mathbf{Z})/\mathbf{Z} \hookrightarrow \cdots$$

can be regarded as a diagram in the heart of $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{Cpl}(I)}$. However, the colimit of this diagram in $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{Cpl}(I)}$ is given by the I-completion of $\mathbf{Z}[p^{-1}]/\mathbf{Z} \simeq \mathbf{Q}_p/\mathbf{Z}_p$, which is the suspension of $\mathbf{Z}_p \in \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{Cpl}(p)}$.

Corollary 7.3.4.6. *Let R be a commutative ring, let $I \subseteq R$ be a finitely generated ideal, and let M be a discrete R-module. The following conditions are equivalent:*

(a) *The R-module M is I-complete in the sense of Definition 7.3.1.1: that is, the mapping space* $\mathrm{Map}_{\mathrm{Mod}_R}(N, M)$ *is contractible whenever* $N \in \mathrm{Mod}_R$ *is I-local.*

(b) *The R-module M is I-complete in the sense of Definition 7.3.0.5: that is, we have* $\mathrm{Ext}^0_R(R[x^{-1}], M) \simeq 0 \simeq \mathrm{Ext}^1_R(R[x^{-1}], M)$ *for every* $x \in I$.

Moreover, the collection of those R-modules M which satisfy these conditions span an abelian subcategory $\mathcal{A} \subseteq \mathrm{Mod}_R^{\heartsuit}$ *which is closed under kernels, cokernels, and extensions.*

Remark 7.3.4.7. In the situation of Corollary 7.3.4.6, the inclusion $\mathcal{A} \hookrightarrow \mathrm{Mod}_R^{\heartsuit}$ admits a left adjoint, given by the construction $M \mapsto \pi_0 M_I^{\wedge}$.

Proposition 7.3.4.8. *Let R be a connective $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $x \in \pi_0 R$ be an element whose image in $(\pi_0 R)/I$ is invertible. If M is an I-complete left R-module, then multiplication by x induces an equivalence from M to itself.*

Proof. Using Proposition 7.3.1.7, we can write $M = N_I^{\wedge}$ for some I-nilpotent left R-module N. It will now that the map $x : N \to N$ induces an isomorphism on each homotopy group, which follows from Example 7.2.4.5. □

Corollary 7.3.4.9. *Let R be an $\mathbb{E}_2$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 R$, and let $x \in \pi_0 R$ be an element whose image in $(\pi_0 R)/I$ is invertible. Then x is invertible in $\pi_0 R$.*

Remark 7.3.4.10. We can rephrase Corollary 7.3.4.9 as follows: if R is an $\mathbb{E}_2$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 R$, then I is contained in the Jacobson radical of $\pi_0 R$.

7.3.5 Monoidal Structures

Let R be an $\mathbb{E}_2$-ring. We now show that if $I \subseteq \pi_0 R$ is a finitely generated ideal, then the ∞-category $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ inherits the structure of a monoidal ∞-category.

Proposition 7.3.5.1. *Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $\alpha : M \to M'$ be a morphism of left R-modules which induces an equivalence $M_I^{\wedge} \simeq M_I'^{\wedge}$. Then, for any left R-module N, the induced maps*

$$(M \otimes_A N)_I^{\wedge} \to (M' \otimes_A N)_I^{\wedge} \qquad (N \otimes_A M)_I^{\wedge} \to (N \otimes_A M')_I^{\wedge}$$

are equivalences.

Proof. Note that α induces an equivalence of I-completions if and only if the fiber $\mathrm{fib}(\alpha)$ is I-local. Consequently, we can view Proposition 7.3.5.1 as a reformulation of Corollary 7.2.4.12. □

Combining Proposition 7.3.5.1 with Proposition HA.2.2.1.9 , we obtain the following:

Corollary 7.3.5.2. *Let R be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 R$ be a finitely generated ideal. Then there is an essentially unique monoidal structure on the ∞-category $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ for which the I-completion functor $M \mapsto M_I^\wedge$ is monoidal.*

Warning 7.3.5.3. In the situation of Corollary 7.3.5.2, the inclusion functor $\mathrm{LMod}_R^{\mathrm{Cpl}(I)} \hookrightarrow \mathrm{LMod}_R$ is generally not monoidal (however, it has the structure of a lax monoidal functor, since it is right adjoint to the monoidal completion functor $\mathrm{LMod}_R^{\mathrm{Cpl}(I)} \to \mathrm{LMod}_R$). To emphasize that the tensor product on $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ differs from the tensor product on LMod_R, it is convenient to denote the former by $(M, N) \mapsto M \widehat{\otimes}_R N$. Concretely, it is given by the formula $M \widehat{\otimes}_R N = (M \otimes_R N)_I^\wedge$. Similarly, the unit object of $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ is given by $R_I^\wedge$.

Remark 7.3.5.4. In the situation of Corollary 7.3.5.2, the monoidal I-completion functor $\mathrm{LMod}_R \to \mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ determines an action of R on the stable ∞-category $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$. It is not difficult to see that this action of R agrees with the action given in Remark 7.3.1.8.

Remark 7.3.5.5. In the situation of Corollary 7.3.5.2, the I-completion functor $\mathrm{LMod}_R \to \mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ can be regarded as morphism between associative algebra objects of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. Consequently, it induces a functor

$$\mathrm{LMod}_{\mathrm{LMod}_R^{\mathrm{Cpl}(I)}}(\mathcal{P}\mathrm{r}^{\mathrm{St}}) \to \mathrm{LMod}_{\mathrm{LMod}_R}(\mathcal{P}\mathrm{r}^{\mathrm{St}}) \simeq \mathrm{LinCat}_R^{\mathrm{St}}.$$

It is not difficult to see that this functor is a fully faithful embedding, whose essential image is spanned by those stable R-linear ∞-categories $\mathcal{C}$ which are I-nilpotent in the sense of Remark 7.1.1.18. In particular, if $\mathcal{C}$ is a stable R-linear ∞-category which is I-nilpotent, then we can regard $\mathcal{C}$ as an ∞-category which is left-tensored over $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$.

Variant 7.3.5.6. Let R be an $\mathbb{E}_{k+1}$-ring for $1 \leqslant k \leqslant \infty$, so that the ∞-category LMod_R inherits the structure of an $\mathbb{E}_k$-monoidal ∞-category. Let $I \subseteq \pi_0 R$ be a finitely generated ideal. Using Proposition 7.3.5.1, we see that $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ inherits the structure of an $\mathbb{E}_k$-monoidal ∞-category for which the I-completion functor $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ is $\mathbb{E}_k$-monoidal. In particular, if R is an $\mathbb{E}_\infty$-ring, then we can regard $\mathrm{Mod}_R^{\mathrm{Cpl}(I)}$ as a symmetric monoidal ∞-category (with respect to the I-completed tensor product $\widehat{\otimes}_R$ of Warning 7.3.5.3) and the construction $M \mapsto M_I^\wedge$ as a symmetric monoidal functor $\mathrm{Mod}_R \to \mathrm{Mod}_R^{\mathrm{Cpl}(I)}$.

Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ a finitely generated ideal, and let M be a left R-module. Using Proposition 7.3.5.1, we see that the canonical map $M \simeq R \otimes_R M \to R_I^\wedge \otimes_R M$ induces an equivalence of I-completions. We therefore obtain a canonical map $R_I^\wedge \otimes_R M \to M_I^\wedge$.

Proposition 7.3.5.7. *Let R be a connective $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let M be an almost perfect R-module. Then the canonical map $R_I^\wedge \otimes_R M \to M_I^\wedge$ is an equivalence. In particular, if R is I-complete, then M is I-complete.*

Proof. Fix an integer n; we will show that the map $\phi_M : R^{\wedge}_I \otimes_R M \to M^{\wedge}_I$ is n-connective. Since M is almost perfect, there exists a perfect R-module N and an n-connective map $N \to M$. We have a commutative diagram

$$\begin{array}{ccc} R^{\wedge}_I \otimes_R N & \xrightarrow{\phi_N} & N^{\wedge}_I \\ \downarrow & & \downarrow \\ R^{\wedge}_I \otimes_R M & \xrightarrow{\phi_M} & M^{\wedge}_I . \end{array}$$

Since the I-completion functor is right t-exact (Proposition 7.3.4.4), the vertical maps in this diagram are n-connective. It will therefore suffice to show that the map ϕ_N is n-connective. Let $\mathcal{C} \subseteq \mathrm{LMod}_R$ be the full subcategory spanned by those objects N for which ϕ_N is an equivalence. Then $\mathcal{C}$ is a stable subcategory which is closed under the formation of retracts. Consequently, to show that $\mathcal{C}$ contains all perfect R-modules, it suffices to show that $\mathcal{C}$ contains R, which is clear. □

7.3.6 Comparison with I-adic Completions

Let R be a commutative ring and let M be a discrete R-module. If $I \subseteq R$ is a finitely generated ideal, we let $\mathrm{Cpl}(M; I)$ denote the I-adic completion of M: that is, the limit of the tower

$$\cdots \to M/I^4M \to M/I^3M \to M/I^2M \to M/IM.$$

Here we can form the limit either in the abelian category $\mathrm{Mod}^{\heartsuit}_R$ of discrete R-modules or in the larger ∞-category Mod_R: the result is the same, since the group $\varprojlim^1\{M/I^nM\}$ vanishes by virtue of the surjectivity of the transition maps $M/I^{n+1}M \to M/I^nM$. Note that each quotient M/I^nM can be regarded as a module over the ring R/I^n and is therefore complete with respect to the ideal I^n (Example 7.3.3.7), hence also with respect to I. It follows that the I-adic completion $\mathrm{Cpl}(M; I)$ is I-complete (in the sense of Definition 7.3.1.1), so that the canonical map $M \to \mathrm{Cpl}(M; I)$ admits an essentially unique factorization $M \to M^{\wedge}_I \xrightarrow{\alpha} \mathrm{Cpl}(M; I)$. The map α is generally not an equivalence. However, it is an equivalence under some mild finiteness assumptions:

Proposition 7.3.6.1. *Let R be a connective $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let M be a discrete R-module. Assume that M is Noetherian: that is, that every submodule of M is finitely generated. For every set S, the canonical map $M'^{\wedge}_I \to \mathrm{Cpl}(M'; I)$ is an equivalence, where $M' = \bigoplus_{\beta \in S} M$. In particular, we have $M^{\wedge}_I \simeq \mathrm{Cpl}(M; I)$*

Proof. We proceed by induction on the minimal number of generators of I. If $I = (0)$ there is nothing to prove. Otherwise, we may assume that $I = J + (x)$ for some $x \in \pi_0 A$ and that $M'^{\wedge}_J = \mathrm{Cpl}(M'; J)$. Using Proposition 7.3.3.2, we deduce that $M'^{\wedge}_I \simeq (\mathrm{Cpl}(M'; J))^{\wedge}_{(x)}$.

For $m, n \geqslant 0$, we let $X_{m,n}$ denote the cofiber of the map $M'/J^m M' \to M'/J^m M'$ given by multiplication by x^n. Then $\pi_i X_{m,n}$ vanishes for $i \notin \{0,1\}$, and Proposition 7.3.2.1 implies that $M'^{\wedge}_I \simeq \varprojlim \{X_{m,n}\}$. It follows that there is a canonical isomorphism $\pi_1 M'^{\wedge}_I \simeq \varprojlim \{\pi_1 X_{m,n}\}$ and a short exact sequence

$$0 \to \varprojlim{}^1 \{\pi_1 X_{m,n}\} \to \pi_0 M'^{\wedge}_I \to \varprojlim \{\pi_0 X_{m,n}\} \to 0.$$

To complete the proof, it will suffice to show that $\varprojlim \{\pi_1 X_{m,n}\} \simeq \varprojlim{}^1 \{\pi_1 X_{m,n}\} \simeq 0$. In fact, we claim that $\{\pi_1 X_{m,n}\}_{m,n \geqslant 0}$ is trivial as a pro-object in the category of abelian groups. To prove this, it suffices to show that for each $m, n \geqslant 0$, there exists $n' \geqslant n$ such that the induced map $\pi_1 X_{m,n'} \to \pi_1 X_{m,n}$ is zero. For each $k \geqslant 0$, let $Y(k) = \{y \in M/I^m M : x^k y = 0\}$. Since M is Noetherian, the quotient $M/I^m M$ is also Noetherian, so the ascending chain of submodules

$$0 = Y(0) \subseteq Y(1) \subseteq Y(2) \subseteq \cdots$$

must eventually stabilize. It follows that there exists $k \geqslant 0$ such that if $y \in M/I^m M$ is annihilated by x^{k+1}, then it is annihilated by x^k. It follows that the map $\pi_1 X_{m,n+k} \to \pi_1 X_{m,n}$ is zero, as desired. □

Corollary 7.3.6.2. *Let R be a commutative ring, let $I \subseteq R$ be a finitely generated ideal, and let M be a discrete R-module. Then the canonical map $M^{\wedge}_I \to \mathrm{Cpl}(M; I)$ induces a surjection $\pi_0 M^{\wedge}_I \to \mathrm{Cpl}(M; I)$.*

Proof. Choose a finite set $(x_1, \ldots, x_n)$ of generators for I. Replacing R by the subring generated by the $\{x_i\}$, we can reduce to the case where R is Noetherian. Choose a surjection $\alpha : P \to M$, where P is a free R-module. We then have a commutative diagram

$$\begin{array}{ccc} \pi_0 P^{\wedge}_I & \longrightarrow & \pi_0 M^{\wedge}_I \\ \downarrow & & \downarrow \\ \mathrm{Cpl}(P; I) & \longrightarrow & \mathrm{Cpl}(M; I). \end{array}$$

The left vertical map is an isomorphism by virtue of Proposition 7.3.6.1 and the lower horizontal map is surjective because α is surjective. It follows from a diagram chase that the right vertical map is also surjective. □

Corollary 7.3.6.3. *Let R be a commutative ring, let $I \subseteq R$ be a finitely generated ideal, and let M be a discrete R-module. The following conditions are equivalent:*

- (*a*) *The module M is I-adically complete (that is, the canonical map $M \to \mathrm{Cpl}(M; I)$ is an isomorphism).*
- (*b*) *The module M is I-complete and I-adically separated (that is, the canonical map $M \to \mathrm{Cpl}(M; I)$ is injective).*

Proof. The implication $(a) \Rightarrow (b)$ is obvious. Conversely, suppose that (b) is satisfied. Then the canonical map $M \to \mathrm{Cpl}(M; I)$ is an injection by virtue of our assumption that M is I-adically separated, and a surjection by virtue of Corollary 7.3.6.2. □

Corollary 7.3.6.4. *Let R be a commutative ring and let $I \subseteq R$ be a finitely generated ideal. If R is I-complete, then the canonical map $\epsilon : R \to \mathrm{Cpl}(R; I) = \varprojlim R/I^n$ is a surjection and the kernel $\ker(\epsilon)$ is nilpotent.*

Proof. The surjectivity of ϵ follows from Corollary 7.3.6.2. Write $I = (x_1, \ldots, x_n)$. We will complete the proof by showing that every element $t \in \ker(\epsilon)$ satisfies $t^{n+1} = 0$. For $1 \leqslant i \leqslant n$ and $m \geqslant 0$, let $K_{i,m}$ denote the cofiber of the map $x_i^m : R \to R$, and set $K_m = \bigotimes_{1 \leqslant i \leqslant n} K_{i,m}$ (that is, K_m is the Koszul complex of the sequence $(x_1^m, \ldots, x_n^m)$). Note that each homotopy group of K_m is annihilated by some power of I, and is therefore annihilated by t. It follows that, for each $k \geqslant 0$, multiplication by t annihilates the Pro-object $\{\pi_k K_m\}_{m \geqslant 0}$, so that the map $\{\tau_{\geqslant k} K_m\}_{m \geqslant 0} \xrightarrow{t} \{\tau_{\geqslant k} K_m\}_{m \geqslant 0}$ factors through the tower $\{\tau_{\geqslant k+1} K_m\}_{m \geqslant 0}$. Invoking this observation repeatedly, we deduce that each of the maps $t^s : \{K_m\}_{m \geqslant 0} \to \{K_m\}_{m \geqslant 0}$ factors through the tower $\{\tau_{\geqslant s} K_m\}_{m \geqslant 0}$, which vanishes for $s > n$. It follows that multiplication by t^{n+1} annihilates the Pro-object $\{K_m\}_{m \geqslant 0}$ and therefore annihilates its inverse limit $\varprojlim K_m \simeq R$. □

Corollary 7.3.6.5. *Let R be a commutative ring which is I-complete for some finitely generated ideal $I \subseteq R$. Then the pair (R, I) is Henselian.*

Proof. Let $J = \bigcap_n I^n$ be the kernel of the map $R \to \varprojlim R/I^n$. Then J is a nilpotent ideal (Corollary 7.3.6.4), so the pair (R, J) is Henselian (Corollary B.3.1.5). We may therefore replace R by R/J (Corollary B.3.3.3) and thereby reduce to the case where $R \simeq \varprojlim R/I^n$ is I-adically complete in the classical sense, in which case the desired result follows from Hensel's lemma (Proposition B.3.1.4). □

Corollary 7.3.6.6. *Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $M \in \mathrm{LMod}_R$. Assume that each of the homotopy groups $\pi_n M$ is Noetherian when regarded as a (discrete) module over $\pi_0 R$. Then, for each integer n, there is a canonical isomorphism $\pi_n M_I^{\wedge} \simeq \mathrm{Cpl}(\pi_n M; I)$.*

Proof. Using Corollary 7.3.3.6, we can reduce to the case where R is connective. Then each homotopy group $\pi_n M$ can be regarded as a module over R. Using Corollary 7.3.3.6 and Proposition 7.3.6.1, we deduce that the completion $(\pi_n M)_I^{\wedge}$ can be identified with $\mathrm{Cpl}(\pi_n M; I)$. In particular, $(\pi_n M)_I^{\wedge}$ is a discrete R-module. We will complete the proof by producing isomorphisms $\pi_n(M_I^{\wedge}) \simeq (\pi_n M)_I^{\wedge}$. We have a fiber sequence of R-modules $\tau_{\geqslant n+1} M \to M \to \tau_{\leqslant n} M$, hence a fiber sequence of I-complete R-modules $(\tau_{\geqslant n+1} M)_I^{\wedge} \to M_I^{\wedge} \to (\tau_{\leqslant n} M)_I^{\wedge}$. Since the functor of I-completion is right t-exact, the associated long exact sequence of homotopy groups gives an isomorphism $\pi_n M^{\wedge}_I \simeq \pi_n(\tau_{\leqslant n} M)^{\wedge}_I$. We may therefore replace M by $\tau_{\leqslant n} M$ and thereby reduce to the case where M is n-truncated. Let $N = \tau_{\leqslant n-1} M$, so that we have a fiber sequence $\Sigma^n(\pi_n M) \to M \to N$, hence a fiber sequence of I-completions $\Sigma^n(\pi_n M)^{\wedge}_I \to M^{\wedge}_I \to N^{\wedge}_I$. Using the associated long exact sequence, we are reduced to proving that $N^{\wedge}_I$ is $(n-1)$-truncated. We first prove by descending induction on k that $(\tau_{\geqslant k} N)^{\wedge}_I$ is $(n-1)$-truncated. For $k \geqslant n$, there is nothing to prove. Assume therefore that $k < n$. $(\tau_{\geqslant k+1} N)^{\wedge}_I \to (\tau_{\geqslant k} N)^{\wedge}_I \to \Sigma^k(\pi_k N)^{\wedge}_I$. The inductive hypothesis implies that $(\tau_{\geqslant k+1} N)^{\wedge}_I$ is $(n-1)$-truncated, and Proposition 7.3.6.1 implies that $(\pi_k N)^{\wedge}_I$ is discrete. It follows that $(\tau_{\geqslant k} N)^{\wedge}_I$ is $(n-1)$-truncated. We have a fiber sequence $(\tau_{\geqslant k} N)^{\wedge}_I \to N^{\wedge}_I \to (\tau_{\leqslant k-1} N)^{\wedge}_I$. For $k \ll 0$, Proposition 7.3.4.4 implies that $(\tau_{\leqslant k-1} N)^{\wedge}_I$ is $(n-1)$-truncated, so that $N^{\wedge}_I$ is $(n-1)$-truncated as desired. □

Remark 7.3.6.7. Let R be a Noetherian $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be an ideal, and suppose that $\pi_0 R$ is I-adically complete: that is, that the canonical map $\pi_0 R \to \varprojlim_n (\pi_0 R)/I^n$ is an isomorphism of commutative rings. Proposition 7.3.6.1 implies that $\pi_0 R$ is I-complete. Using Propositions HA.7.2.4.17 and 7.3.5.7, we deduce that every finitely generated discrete module over $\pi_0 R$ is I-complete. It then follows from Theorem 7.3.4.1 that R is I-complete.

Remark 7.3.6.8. Let R be a Noetherian $\mathbb{E}_\infty$-ring and suppose we are given a pair of ideals $I \subseteq J \subseteq \pi_0(R)$. If R is I-complete and $\pi_0(R)/I$ is (J/I)-complete, then R is J-complete. To prove this, it will suffice to show that every finitely generated $\pi_0(R)$-module M is J-complete. Our assumption that R is I-complete guarantees that M is I-complete (Proposition 7.3.6.1) and therefore I-adically complete (Corollary 7.3.6.6). Realizing M as the limit of the tower $M/I^k M$, we are reduced to showing that each $M/I^k M$ is J-complete. Proceeding by induction on k, we are reduced to showing that $I^{k-1}M/I^k M$ is J-complete. This follows from Proposition 7.3.6.1, since $\pi_0(R)/I$ is (J/I)-complete and $I^{k-1}M/I^k M$ is almost perfect as a module over $\pi_0(R)/I$.

Corollary 7.3.6.9. *Let R be a Noetherian $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be an ideal. Then the completion $R^{\wedge}_I$ is flat over R.*

Proof. Let M be a discrete R-module; we wish to show that $R^{\wedge}_I \otimes_R M$ is discrete. Since the construction $M \mapsto R^{\wedge}_I \otimes_R M$ commutes with filtered colimits, we can assume that M is finitely presented (when regarded as a module over the commutative ring $\pi_0 R$). In this case, M is almost perfect as an R-module (Proposition HA.7.2.4.17), so that $R^{\wedge}_I \otimes_R M$ can be identified with the I-completion $M^{\wedge}_I$ (Proposition 7.3.5.7). The discreteness of $M^{\wedge}_I$ now follows from Proposition 7.3.6.1. □

Corollary 7.3.6.10 (Krull Intersection Theorem)**.** *Let R be a local Noetherian ring and let $I \subsetneq R$ be an ideal. Then R is I-adically separated: that is, the $\bigcap_{n \geqslant 0} I^n = \{0\}$.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of R, so that $I \subseteq \mathfrak{m}$. The canonical map $R \to R/\mathfrak{m}$ factors through R/I, and therefore also through $R^{\wedge}_I$. It follows that the flat map $\phi : R \to R^{\wedge}_I$ of Corollary 7.3.6.9 is faithfully flat, and is therefore injective. It follows from Proposition 7.3.6.1 that we can identify $R^{\wedge}_I$ with the I-adic completion $\varprojlim\{R/I^n\}$, so that $\bigcap_{n \geqslant 0} I^n = \ker(\phi) = (0)$. □

7.3.7 Complete Modules as a Derived Category

Let R be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 R$ be a finitely generated ideal. We let $\mathrm{LMod}_R^{\heartsuit\,\mathrm{Cpl}(I)}$ denote the full subcategory of LMod_R spanned by those R-modules M which are both discrete and I-complete. Then $\mathrm{LMod}_R^{\heartsuit\,\mathrm{Cpl}(I)}$ is the heart of the t-structure on $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ described in Proposition 7.3.4.4. In particular, it is an abelian category.

Remark 7.3.7.1. In the situation described above, the inclusion functor $\mathrm{LMod}_R^{\mathrm{Cpl}(I)} \hookrightarrow \mathrm{LMod}_R$ is t-exact and therefore restricts to an exact functor $\mathrm{LMod}_R^{\heartsuit\,\mathrm{Cpl}(I)} \hookrightarrow \mathrm{LMod}_R^{\heartsuit}$. In particular, if $\alpha : M \to N$ is a morphism of discrete I-complete R-modules, then α is a monomorphism (epimorphism) in the abelian category $\mathrm{LMod}_R^{\heartsuit\,\mathrm{Cpl}(I)}$ if and only if it is injective (surjective) when regarded as a homomorphism of abelian groups.

Proposition 7.3.7.2. *Let R be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 R$ be a finitely generated ideal. Then the abelian category $\mathrm{LMod}_R^{\heartsuit\,\mathrm{Cpl}(I)}$ has enough projective objects.*

Proof. Let M be a left R-module which is discrete and I-complete. Choose a morphism $\alpha : P \to M$ in LMod_R which is surjective on π_0, where P is a direct sum of copies of R. Since M is I-complete, the morphism α factors as a composition $P \to P^{\wedge}_I \xrightarrow{\beta} M$. Since M is discrete, the morphism β factors as a composition $P^{\wedge}_I \to \pi_0 P^{\wedge}_I \xrightarrow{\gamma} M$. Note that $\pi_0 P^{\wedge}_I$ is discrete and I-complete (Corollary 7.3.4.3), so we can regard γ as an epimorphism in the abelian category $\mathrm{LMod}_R^{\heartsuit\,\mathrm{Cpl}(I)}$ (Remark 7.3.7.1). We will complete the proof by showing that $\pi_0 P^{\wedge}_I$ is a projective object of $\mathrm{LMod}_R^{\heartsuit\,\mathrm{Cpl}(I)}$. Let $u : N' \to N$ be an epimorphism in $\mathrm{LMod}_R^{\heartsuit\,\mathrm{Cpl}(I)}$; we wish to show that the left vertical map in the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{LMod}_R}(\pi_0 P^{\wedge}_I, N') & \longrightarrow & \mathrm{Map}_{\mathrm{LMod}_R}(P, N') \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{LMod}_R}(\pi_0 P^{\wedge}_I, N) & \longrightarrow & \mathrm{Map}_{\mathrm{LMod}_R}(P, N) \end{array}$$

is surjective on connected components. Since N and N' are discrete and I-complete, the horizontal maps are homotopy equivalences. It will therefore suffice to show that the right vertical map is surjective on connected components. This is clear: the right vertical map can be identified with a direct product of copies of the underlying map $N' \to N$, which is a surjection of abelian groups (Remark 7.3.7.1). □

In the situation of Proposition 7.3.7.2, the existence of enough projective objects in the abelian category $\mathrm{LMod}_R^{\heartsuit\,\mathrm{Cpl}(I)}$ allows us to consider the derived ∞-category $\mathcal{D}^-(\mathrm{LMod}_R^{\heartsuit\,\mathrm{Cpl}(I)})$. Since the t-structure on $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ is left complete, Theorem HA.1.3.3.2 and Remark HA.1.3.3.6 imply that the inclusion map $\mathrm{LMod}_R^{\heartsuit\,\mathrm{Cpl}(I)} \hookrightarrow \mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ admits an essentially unique extension to a t-exact functor $F : \mathcal{D}^-(\mathrm{LMod}_R^{\heartsuit\,\mathrm{Cpl}(I)}) \to \mathrm{LMod}_R^{\mathrm{Cpl}(I)}$.

Proposition 7.3.7.3. *Let R be a Noetherian commutative ring and let $I \subseteq R$ be an ideal. Then the functor $F : \mathcal{D}^-(\mathrm{Mod}_R^{\heartsuit\,\mathrm{Cpl}(I)}) \to \mathrm{Mod}_R^{\mathrm{Cpl}(I)}$ is a fully faithful embedding whose essential image is the full subcategory $\bigcup_n (\mathrm{Mod}_R^{\mathrm{Cpl}(I)})_{\geqslant -n}$ consisting of right-bounded objects of $\mathrm{Mod}_R^{\mathrm{Cpl}(I)}$.*

Proof. By virtue of Proposition HA.1.3.3.7 , it will suffice to show that the abelian groups $\mathrm{Ext}^i_R(X, Y)$ vanish for $i > 0$ if X and Y are objects of $\mathrm{Mod}_R^{\heartsuit\,\mathrm{Cpl}(I)}$ with X projective. The proof of Proposition 7.3.7.2 shows that there exists an epimorphism $\pi_0 P^{\wedge}_I \to X$, where P is a direct sum of copies of R. Since X is projective, this epimorphism splits. Since R is Noetherian, Proposition 7.3.6.1 implies that $P^{\wedge}_I$ is discrete. It follows that X is a direct summand of $P^{\wedge}_I$. It will therefore suffice to show that the groups $\mathrm{Ext}^i_R(P^{\wedge}_I, Y) \simeq \mathrm{Ext}^i_R(P, Y)$ vanish for $i > 0$. This is clear, since Y is discrete and P is a direct sum of copies of R. □

Warning 7.3.7.4. In the situation of Proposition 7.3.7.3, the abelian category $\mathrm{Mod}_R^{\heartsuit\,\mathrm{Cpl}(I)}$ is usually not a Grothendieck abelian category.

Corollary 7.3.7.5. *Let R be a Noetherian commutative ring, let $I \subseteq R$ be an ideal, and let $\rho : \mathrm{Mod}_R^{\heartsuit} \to \mathrm{Mod}_R^{\heartsuit\,\mathrm{Cpl}(I)}$ be the functor of abelian categories given by $\rho(M) = \pi_0 M^{\wedge}_I$. Then:*

(1) *The functor ρ preserves small colimits. In particular, it is right exact.*

(2) *If M is a projective R-module, there is a canonical isomorphism $\rho(M) \simeq \mathrm{Cpl}(M; I)$.*

(3) *The diagram of ∞-categories*

$$\begin{array}{ccc} \mathcal{D}^-(\mathrm{Mod}_R^{\heartsuit}) & \xrightarrow{L\rho} & \mathcal{D}^-(\mathrm{Mod}_R^{\heartsuit\,\mathrm{Cpl}(I)}) \\ \downarrow & & \downarrow \\ \mathrm{Mod}_R & \longrightarrow & \mathrm{Mod}_R^{\mathrm{Cpl}(I)} \end{array}$$

*commutes up to canonical homotopy, where the vertical maps are the full faithful embeddings provided by Proposition 7.3.7.3, $L\rho$ denotes the left derived functor of ρ (see Example **??**), and the bottom horizontal map is given by I-completion.*

Remark 7.3.7.6. Let R be a Noetherian commutative ring, let M be a discrete R-module, and choose a projective resolution

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M.$$

It follows from Corollary 7.3.7.5 that for any ideal $I \subseteq R$, the completion $M^{\wedge}_I$ is represented by the chain complex of R-modules

$$\cdots \to \mathrm{Cpl}(P_3; I) \to \mathrm{Cpl}(P_2; I) \to \mathrm{Cpl}(P_1; I) \to \mathrm{Cpl}(P_0; I).$$

Proof of Corollary 7.3.7.5. Assertion (1) follows from the right t-exactness of the functor $M \mapsto M^{\wedge}_I$ and assertion (2) follows from Proposition 7.3.6.1. Assertion (3) follows from (2) and Theorem HA.1.3.3.2 . □

7.3.8 Complete Noetherian Rings

We conclude this section by reviewing a few standard facts about the completion of Noetherian rings.

Proposition 7.3.8.1. *Let R be a commutative ring and let $I \subseteq R$ be a finitely generated ideal. Suppose that R is I-adically complete: that is, that the canonical map $R \to \varprojlim\{R/I^n\}$ is an isomorphism. If R/I is Noetherian, then R is Noetherian.*

Proof. For each $n \geqslant 0$, set $A_n = I^n/I^{n+1}$, and let A denote the graded ring $\bigoplus A_n$. Choose a finite set of generators $x_1, \ldots, x_k$ for the ideal I, and let $\overline{x}_1, \ldots, \overline{x}_k$ denote their images in $A_1 = I/I^2$. The elements $\overline{x}_i$ generate A as an algebra over $A_0 = R/I$. It follows from the Hilbert basis theorem that A is Noetherian.

Let $J \subseteq R$ be an arbitrary ideal; we wish to show that J is finitely generated. For each $n \geqslant 0$, set $J_n = (J \cap I^n)/(J \cap I^{n+1})$, which we regard as a submodule of A_n. The direct sum $\bigoplus_{n \geqslant 0} J_n$ is an ideal in the commutative ring A. Since A is Noetherian, this ideal is finitely generated. Choose a finite set of homogeneous generators $\overline{y}_1, \ldots, \overline{y}_m \in \bigoplus_{n \geqslant 0} J_n$, where $\overline{y}_i \in J_{d_i}$. For $1 \leqslant i \leqslant m$, let y_i denote a lift of $\overline{y}_i$ to $J \cap I^n$. We claim that the elements $y_1, \ldots, y_m \in J$ generate the ideal J.

Let $d = \max\{d_i\}$. We will prove the following:

- $(*)$ For each $z \in J \cap I^n$, we can find coefficients $c_i \in R$ such that $c_i \in I^{n-d}$ if $n > d$, and $z - \sum_{1 \leqslant i \leqslant m} c_i y_i$ belongs to I^{n+1}.

To prove $(*)$, we let $\overline{z}$ denote the image of z in J_n. Since the elements $\overline{y}_i$ generate $\bigoplus J_n$ as an A-module, we can write $\overline{z} = \sum \overline{c_i y_i}$ for some homogeneous elements $\overline{c}_i \in A$ of degree $n - d_i$. For $1 \leqslant i \leqslant m$, choose $c_i \in I^{n-d_i}$ to be any lift of $\overline{c_i}$; then the elements c_i have the desired property.

Now let $z \in J$ be an arbitrary element. We will define a sequence of elements $z_0, z_1, \ldots, \in J$ such that $z - z_q \in I^q$. Set $z_0 = 0$. Assuming that z_q has been defined, we apply $(*)$ to write

$$z - z_q \equiv \sum_{1 \leqslant i \leqslant m} c_{i,q} y_i \pmod{I^{q+1}}$$

where $c_{i,q} \in I^{q-d}$ for $q \geqslant d$. Now set $z_{q+1} = z - z_q - \sum_{1 \leqslant i \leqslant m} c_{i,q} y_i$. For each $1 \leqslant i \leqslant m$, the sum $\sum_{q \geqslant 0} c_{i,q}$ converges I-adically to a unique element $c_i \in R$. We now observe that $z = \sum c_i y_i$ belongs to the ideal generated by the elements y_i, as desired. □

Corollary 7.3.8.2. *Let R be a Noetherian ring, let $I \subseteq R$ be an ideal, and let $\mathrm{Cpl}(R; I) = \varprojlim \{R/I^n\}$ denote the I-adic completion of R. Then $\mathrm{Cpl}(R; I)$ is Noetherian.*

Proof. For each integer n, let J_n denote the ideal of $\mathrm{Cpl}(R; I)$ given by $\varprojlim I^n / I^{n+m}$, so that the canonical map $\phi : R \to \mathrm{Cpl}(R; I)$ induces isomorphisms $R/I^n \to \mathrm{Cpl}(R; I)/J_n$ for each $n \geqslant 0$. It follows that the canonical map $\mathrm{Cpl}(R; I) \to \varprojlim \mathrm{Cpl}(R; I)/J_n$ is an isomorphism. We will show that $J_n = I^n \mathrm{Cpl}(R; I)$ for each $n \geqslant 0$. Assuming this, we deduce that J_1 is finitely generated, that $\mathrm{Cpl}(R; I)$ is J_1-adically complete, and that $\mathrm{Cpl}(R; I)/J_1 \simeq R/I$ is Noetherian. It then follows from Proposition 7.3.8.1 that $\mathrm{Cpl}(R; I)$ is Noetherian.

Choose a finite set of generators $x_1, \ldots, x_k$ for the ideal I^n and an arbitrary $z \in J_n$, given by a compatible sequence of elements $\{\overline{z}_m \in I^n / I^{n+m}\}_{m \geqslant 0}$. Lift each $\overline{z_m}$ to an element $z_m \in I^n$. Then $z_{m+1} - z_m \in I^{n+m}$, so we can write

$$z_{m+1} = z_m + \sum_{1 \leqslant i \leqslant k} c_{m,i} x_i$$

for some $c_{m,i} \in I^m$. For $1 \leqslant i \leqslant k$, the residue classes of the partial sums $\{\sum_{j \leqslant m} c_{j,i}\}_{m \geqslant 0}$ determine an element $c_i \in \mathrm{Cpl}(R; I)$. Then $z = \phi(z_0) + \sum c_i \phi(x_i)$, so that z belongs to the ideal $I^n \mathrm{Cpl}(R; I)$. □

Corollary 7.3.8.3. *Let R be a Noetherian $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be an ideal. Then the completion $R^\wedge_I$ is a Noetherian $\mathbb{E}_\infty$-ring.*

Proof. Corollary 7.3.6.6 implies that $\pi_0 R^\wedge_I$ is the I-adic completion of the Noetherian commutative ring $\pi_0 R$, and therefore a Noetherian commutative ring (Corollary 7.3.8.2). To complete the proof, it will suffice to show that each $\pi_k R^\wedge_I$ is a finitely generated module over $\pi_0 R^\wedge_I$. Since $A \to A^\wedge_I$ is flat (Corollary 7.3.6.9), we have a canonical isomorphism

$$\pi_k R^\wedge_I \simeq \mathrm{Tor}_0^{\pi_0 R}(\pi_0 R^\wedge_I, \pi_k R).$$

It will therefore suffice to show that $\pi_k R$ is a finitely generated module over $\pi_0 R$, which follows from our assumption that A is Noetherian. □

7.4 The Beauville-Laszlo Theorem

Let A be a commutative ring containing an element x and let $\widehat{A} = \varprojlim A/x^n A$ denote the (x)-adic completion of A. We then have a diagram of commutative rings

$$\begin{array}{ccc} A & \longrightarrow & \widehat{A} \\ \downarrow & & \downarrow \\ A[x^{-1}] & \longrightarrow & \widehat{A}[x^{-1}], \end{array}$$

which determines a diagram of categories σ :

$$\begin{array}{ccc} \mathrm{Mod}^{\mathrm{lf}}_A & \longrightarrow & \mathrm{Mod}^{\mathrm{lf}}_{\widehat{A}} \\ \downarrow & & \downarrow \\ \mathrm{Mod}^{\mathrm{lf}}_{A[x^{-1}]} & \longrightarrow & \mathrm{Mod}^{\mathrm{lf}}_{\widehat{A}[x^{-1}]}; \end{array}$$

here $\mathrm{Mod}^{\mathrm{lf}}_R$ denotes the category whose objects are projective R-modules of finite rank. The *Beauville-Laszlo theorem* asserts that if x is not a zero-divisor in A, then the diagram σ is a (homotopy) pullback diagram of categories (see [**?**]). In other words, the data of a projective A-module of finite rank is equivalent to the data of a triple (M_0, M_1, α), where M_0 and M_1 are projective modules of finite rank over $A[x^{-1}]$ and $\widehat{A}$, respectively, and $\alpha : \widehat{A}[x^{-1}] \otimes_{A[x^{-1}]} M_0 \simeq M_1[x^{-1}]$ is an isomorphism of $\widehat{A}[x^{-1}]$-modules. Heuristically, the Beauville-Laszlo theorem is one expression of the idea that the diagram of affine schemes

$$\begin{array}{ccc} \mathrm{Spec}\, A & \longleftarrow & \mathrm{Spec}\, \widehat{A} \\ \uparrow & & \uparrow \\ \mathrm{Spec}\, A[x^{-1}] & \longleftarrow & \mathrm{Spec}\, \widehat{A}[x^{-1}] \end{array}$$

exhibits $X = \mathrm{Spec}\, R$ as obtained from the open subscheme $U = \mathrm{Spec}\, R[x^{-1}]$ by "gluing in" a formal neighborhood of the divisor $D = X - U$.

Our goal in this section is to establish the following variant of the Beauville-Laszlo theorem:

Theorem 7.4.0.1. *Let $\phi : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and let U denote the quasi-compact open substack of* $\mathrm{Sp\acute{e}t}\, A$ *which is complementary to the vanishing locus of I, and form a pullback diagram*

$$\begin{array}{ccc} \mathsf{V} & \longrightarrow & \mathsf{U} \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}\, B & \longrightarrow & \mathrm{Sp\acute{e}t}\, A. \end{array}$$

If ϕ induces an equivalence of I-completions $A^{\wedge}_{I} \to B^{\wedge}_{I}$, then the diagrams of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A) & \longrightarrow & \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, B) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{U}) & \longrightarrow & \mathrm{QCoh}(\mathsf{V}) \end{array}$$

is a pullback square.

Example 7.4.0.2. Let A be a connective $\mathbb{E}_{\infty}$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, let U be the open substack of $\mathrm{Sp\acute{e}t}\, A$ complementary to the vanishing locus of I, and form pullback diagrams

$$\begin{array}{ccc} \mathsf{V} & \longrightarrow & \mathsf{U} \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}\, A^{\wedge}_{I} & \longrightarrow & \mathrm{Sp\acute{e}t}\, A. \end{array}$$

Then Theorem 7.4.0.1 implies that the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A) & \longrightarrow & \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A^{\wedge}_{I}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{U}) & \longrightarrow & \mathrm{QCoh}(\mathsf{V}) \end{array}$$

is a pullback square.

7.4.1 Proof of the Beauville-Laszlo Theorem

Theorem 7.4.0.1 is an immediate consequence of the following more general result (together with the description of the ∞-category of quasi-coherent sheaves on a quasi-affine spectral Deligne-Mumford stack given by Proposition 2.4.1.4):

Proposition 7.4.1.1. *Let R be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 R$ be a finitely generated ideal. Let $\phi : A \to B$ be a morphism of $\mathbb{E}_1$-algebras over R, and suppose that ϕ induces an equivalence of I-completions $A^{\wedge}_{I} \to B^{\wedge}_{I}$. Then, for every stable R-linear ∞-category $\mathcal{C}$, the diagram*

$$\begin{array}{ccc} \mathrm{LMod}_A(\mathcal{C}) & \longrightarrow & \mathrm{LMod}_B(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{LMod}_{L_I(A)}(\mathcal{C}) & \longrightarrow & \mathrm{LMod}_{L_I(B)}(\mathcal{C}) \end{array}$$

is a pullback square of ∞-categories.

Lemma 7.4.1.2. *Let R be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $\phi : A \to B$ be a morphism of $\mathbb{E}_1$-algebras over R which induces an equivalence $A^{\wedge}_I \to B^{\wedge}_I$. Let $\mathcal{C}$ be a stable R-linear ∞-category. Then, for every object $M \in \mathrm{LMod}_A(\mathcal{C})$, the diagram σ :*

$$\begin{array}{ccc} M & \longrightarrow & B \otimes_A M \\ \downarrow & & \downarrow \\ L_I M & \longrightarrow & L_I(B \otimes_A M) \end{array}$$

is a pullback square.

Proof. Let K denote the fiber of the map

$$M \to (B \otimes_A M) \times_{L_I(B \otimes_A M)} L_I(M).$$

Then $K^{\wedge}_I \simeq 0$ (since the horizontal maps in the diagram σ are equivalences after I-completion) and $L_I(K) \simeq 0$ (since the vertical maps in the diagram σ are equivalences after applying the functor L_I). It follows that $K \simeq 0$, so that σ is a pullback diagram as desired. □

Proof of Proposition 7.4.1.1. Form a pullback diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathrm{LMod}_B(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{LMod}_{L_I(A)}(\mathcal{C}) & \longrightarrow & \mathrm{LMod}_{L_I(B)}(\mathcal{C}), \end{array}$$

and let us identify objects of $\mathcal{D}$ with triples (P, Q, α), where P is a left B-module object of $\mathcal{C}$, Q is a left $L_I(A)$-module object of $\mathcal{C}$, and $\alpha : L_I(B) \otimes_B P \to L_I(B) \otimes_{L_I(A)} Q$ is an equivalence. Let F denote the canonical map from $\mathrm{LMod}_A(\mathcal{C})$ to $\mathcal{D}$, given on objects by

$$M \mapsto (B \otimes_A M, L_I(A) \otimes_A N, \mathrm{id}_{L_I(B) \otimes_A M}).$$

Then F admits a right adjoint G, given by $G(P, Q, \alpha) = P \times_{L_I B \otimes_{L_I A} Q} Q$. We first claim that F is fully faithful: that is, that the unit map $u_M : M \to (G \circ F)(M)$ is an equivalence for each $M \in \mathrm{LMod}_A(\mathcal{C})$. This follows immediately from Lemma 7.4.1.2. To complete the proof that F is an equivalence of ∞-categories, it will suffice to show that the functor G is conservative. Since G is an exact functor between stable ∞-categories, we are reduced to proving that if $G(P, Q, \alpha) \simeq 0$, then $P \simeq Q \simeq 0$. Note that for any M in LMod_R, we have a canonical equivalence $L_I M \simeq L_I R \otimes_R M$. In particular, the canonical map $B \to L_I B$ is an equivalence after tensoring with $L_I R$. It follows that for every object $P \in \mathrm{LMod}_B(\mathcal{C})$, the canonical map $P \to L_I(B) \otimes_B P$ becomes equivalence (in $\mathcal{C}$) after tensoring with $L_I R$. We conclude that the projection map $G(P, Q, \alpha) \to Q$ becomes an equivalence after tensoring with $L_I(R)$. We then have

$$\begin{aligned} Q &\simeq L_I(A) \otimes_{L_I(A)} Q \\ &\simeq (L_I R \otimes_R L_I(A)) \otimes_{L_I(A)} Q \\ &\simeq L_I R \otimes_R Q \\ &\simeq L_I R \otimes_R G(P, Q, \alpha) \\ &\simeq 0. \end{aligned}$$

It follows that the projection map $G(P, Q, \alpha) \to P$ is an equivalence, so that $P \simeq 0$ as well. □

Remark 7.4.1.3. In the situation of Proposition 7.4.1.1, suppose that the R-linear ∞-category $\mathcal{C}$ is compactly generated. For every object $R' \in \mathrm{Alg}_R$, let $\mathrm{LMod}_{R'}(\mathcal{C})^c$ denote the full subcategory of $\mathrm{LMod}_{R'}(\mathcal{C})$ spanned by the compact objects, so that we have a commutative diagram of ∞-categories τ :

$$\begin{array}{ccc} \mathrm{LMod}_A(\mathcal{C})^c & \longrightarrow & \mathrm{LMod}_B(\mathcal{C})^c \\ \downarrow & & \downarrow \\ \mathrm{LMod}_{L_I(A)}(\mathcal{C})^c & \longrightarrow & \mathrm{LMod}_{L_I(B)}(\mathcal{C})^c \end{array}$$

Then τ is a pullback square. That is, the functor F appearing in the proof of Proposition 7.4.1.1 restricts to an equivalence of ∞-categories

$$F^c : \mathrm{LMod}_A(\mathcal{C})^c \to \mathrm{LMod}_B(\mathcal{C})^c \times_{\mathrm{LMod}_{L_I(B)}(\mathcal{C})^c} \mathrm{LMod}_{L_I(A)}(\mathcal{C})^c.$$

It follows immediately from Proposition 7.4.1.1 that F^c is fully faithful. The essential surjectivity follows from the observation that every object of the fiber product

$$\mathrm{LMod}_B(\mathcal{C})^c \times_{\mathrm{LMod}_{L_I(B)}(\mathcal{C})^c} \mathrm{LMod}_{L_I(A)}(\mathcal{C})^c$$

is compact when viewed as an object of $\mathrm{LMod}_B(\mathcal{C}) \times_{\mathrm{LMod}_{L_I(B)}(\mathcal{C})} \mathrm{LMod}_{L_I(A)}(\mathcal{C})$.

7.4.2 The Case of Vector Bundles

To recover the classical Beauville-Laszlo theorem, we would like to know that the pullback diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A) & \longrightarrow & \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, B) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{U}) & \longrightarrow & \mathrm{QCoh}(\mathsf{V}) \end{array}$$

remains a pullback square if we restrict our attention to the full subcategories spanned by the locally free sheaves. This is a consequence of the following:

Proposition 7.4.2.1. *Let P be one of the following properties of quasi-coherent sheaves:*

(a) *The property of being perfect.*

(b) *The property of being perfect with* Tor*-amplitude $\leqslant n$.*

(c) *The property of being perfect and n-connective.*

(d) *The property of being locally free of finite rank.*

For every spectral Deligne-Mumford stack X, let $\mathrm{QCoh}(\mathsf{X})^P$ denote the full subcategory of $\mathrm{QCoh}(\mathsf{X})$ spanned by those quasi-coherent sheaves having the property P. Let $\phi : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and define

$$\begin{array}{ccc} \mathsf{V} & \longrightarrow & \mathsf{U} \\ \downarrow & & \downarrow u \\ \mathrm{Sp\acute{e}t}\, B & \xrightarrow{v} & \mathrm{Sp\acute{e}t}\, A. \end{array}$$

as in Theorem 7.4.0.1. If ϕ induces an equivalence $A^{\wedge}_I \to B^{\wedge}_I$, then the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A)^P & \longrightarrow & \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, B)^P \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{U})^P & \longrightarrow & \mathrm{QCoh}(\mathsf{V})^P \end{array}$$

is a pullback square.

Remark 7.4.2.2. In the situation of Proposition 7.4.2.1, suppose that A is Noetherian and that $B = A^{\wedge}_I$. Then the map $\phi : A \to B$ is flat (Corollary 7.3.6.9), so that the induced map

$$\mathrm{Sp\acute{e}t}\, B \amalg \mathsf{U} \to \mathrm{Sp\acute{e}t}\, A$$

is a flat covering (see Definition 2.8.3.1). It follows that the conclusion of Proposition 7.4.2.1 is valid for any property P which is local with respect to the flat topology.

Proof of Proposition 7.4.2.1. By virtue of Theorem 7.4.0.1, it will suffice to show that if $\mathscr{F} \in \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A)$ is a quasi-coherent sheaf such that $u^* \mathscr{F}$ and $v^* \mathscr{F}$ have the property P, then $\mathscr{F}$ has the property P. We have four cases to consider.

(a) Suppose that $u^* \mathscr{F}$ and $v^* \mathscr{F}$ are perfect. Then $u^* \mathscr{F}$ and $v^* \mathscr{F}$ are compact objects of $\mathrm{QCoh}(\mathsf{U})$ and $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, B)$ (Theorem **??**), so that $\mathscr{F}$ is a compact object of $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A)$ (Remark 7.4.1.3) and is therefore perfect.

(*b*) Suppose that $u^* \mathscr{F}$ and $v^* \mathscr{F}$ are perfect with Tor-amplitude $\leqslant n$, for some integer n. It follows from (*a*) that $\mathscr{F}$ is perfect. We wish to show that $\mathscr{F}$ has Tor-amplitude $\leqslant n$, for some integer n. Replacing $\mathscr{F}$ by a shift if necessary, we may suppose that $\mathscr{F}$ is connective. Using Corollary 6.1.4.7, we are reduced to proving that $\eta^* \mathscr{F}$ is n-truncated, for any map $\eta : \mathrm{Sp\acute{e}t}\, \kappa \to \mathrm{Sp\acute{e}t}\, A$ where κ is a field. Then η determines a map of commutative rings $\pi_0 A \to \kappa$, whose kernel is a prime ideal $\mathfrak{p} \subseteq \pi_0 A$. If $I \subseteq \mathfrak{p}$, then η factors through v, and the desired result follows from our assumption that $v^* \mathscr{F}$ has Tor-amplitude $\leqslant n$. If $I \nsubseteq \mathfrak{p}$, then η factors through u, and the desired result follows from our assumption that $u^* \mathscr{F}$ has Tor-amplitude $\leqslant n$.

(*c*) Suppose that $u^* \mathscr{F}$ and $v^* \mathscr{F}$ are perfect and n-connective. It follows from (*a*) that $\mathscr{F}$ is perfect, and therefore a dualizable object of $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A)$. Let $\mathscr{F}^\vee$ denote its dual. Then $u^* \mathscr{F}^\vee$ and $v^* \mathscr{F}^\vee$ have Tor-amplitude $\leqslant -n$, so that (*b*) implies that $\mathscr{F}^\vee$ has Tor-amplitude $\leqslant -n$, and therefore $\mathscr{F}$ is n-connective.

(*d*) Suppose that $u^* \mathscr{F}$ and $v^* \mathscr{F}$ are locally free of finite rank. It follows from (*b*) and (*c*) that $\mathscr{F}$ is flat and perfect, hence locally free of finite rank.

□

We conclude by recording a concrete consequence of Remark 7.4.2.2, which we will need in §**??**:

Corollary 7.4.2.3. *Let A be a Noetherian commutative ring containing an element a and let $\widehat{A}$ denote the completion of A with respect to the principal ideal (a). Suppose we are given a discrete $A[a^{-1}]$-module N and a discrete $\widehat{A}$-module $\widehat{M}$, together with a map $\alpha : N \simeq \widehat{M}[a^{-1}]$ which induces an isomorphism $\overline{\alpha} : \widehat{A}[a^{-1}] \otimes_{A[a^{-1}]} N \to \widehat{M}[a^{-1}]$. Then the canonical map $\mu : N \oplus \widehat{M} \to \widehat{M}[a^{-1}]$ is surjective. Moreover, if N and $\widehat{M}$ are finitely generated over $A[a^{-1}]$ and $\widehat{A}$, respectively, then* $\ker(\mu)$ *is a finitely generated A-module.*

Proof. Using Theorem 7.4.0.1, we deduce that there exists an A-module M equipped with equivalences $\widehat{M} \simeq \widehat{A} \otimes_A M$ and $N \simeq M[a^{-1}]$, compatible with the map α. Tensoring M with the fiber sequence $A \to \widehat{A} \oplus A[a^{-1}] \to \widehat{A}[a^{-1}]$, we obtain a fiber sequence $M \to \widehat{M} \oplus N \xrightarrow{\mu} \widehat{M}[a^{-1}]$. Since the property of being discrete is local with respect to the flat topology (Proposition 2.8.4.2), we deduce that M is discrete (Remark 7.4.2.2), so that μ is surjective and $M \simeq \ker(\mu)$. Now suppose that N and $\widehat{M}$ are finitely generated over $A[a^{-1}]$ and $\widehat{A}$, respectively. Since the property of being perfect to order 0 is local with respect to the flat topology (Proposition 2.8.4.2), we deduce that M is perfect to order 0 (Remark 7.4.2.2) and therefore finitely generated as a module over A. □

Chapter 8

Formal Spectral Algebraic Geometry

In Chapter 7, we studied the operation of completing a module M over an $\mathbb{E}_\infty$-ring R with respect to a finitely generated ideal $I \subseteq \pi_0 R$. In this chapter, we will study a geometric counterpart of this operation. Given a spectral Deligne-Mumford stack X and a (cocompact) closed subset $K \subseteq |\mathsf{X}|$, we will associate a new object $\mathsf{X}^\wedge_K$ which we call the *formal completion of* X *along* K (Definition 8.1.6.1). The formal completion $\mathsf{X}^\wedge_K$ can be considered from (at least) two different perspectives:

(*a*) If X is given as a spectrally ringed ∞-topos $(\mathcal{X}, \mathscr{O}_\mathsf{X})$, then we can identify $\mathsf{X}^\wedge_K$ with a spectrally ringed ringed ∞-topos $(\mathcal{X}_K, \mathscr{O}^\wedge_\mathsf{X})$, where $\mathcal{X}_K$ denotes the closed subtopos of $\mathcal{X}$ determined by K and $\mathscr{O}^\wedge_\mathsf{X}$ is obtained from $\mathscr{O}_\mathsf{X}$ by a suitable completion construction.

(*b*) If we identify X with its functor of points $h_\mathsf{X} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, then we can identify the formal completion $\mathsf{X}^\wedge_K$ with a subfunctor of h_X: namely, the functor which assigns to each connective $\mathbb{E}_\infty$-ring R the summand $\mathsf{X}^\wedge_K(R) \subseteq \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X})$ consisting of those maps $\phi : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$ for which the underlying map of topological spaces $|\operatorname{Spec} R| \to |\mathsf{X}|$ factors through K.

The formal completion $\mathsf{X}^\wedge_K$ is generally not a spectral Deligne-Mumford stack. However, it is a prototypical example of a more general type of geometric object called a *formal spectral Deligne-Mumford stack*, which we introduce in §8.1 (Definition 8.1.3.1). Following the same outline as in Chapter 1, we begin from the "geometric" perspective (defining a formal spectral Deligne-Mumford to be a spectrally ringed ∞-topos $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_\mathfrak{X})$ satisfying certain local requiremets) and then establish the equivalence with the "functorial" perspective (by showing that a formal spectral Deligne-Mumford stack $\mathfrak{X}$ is determined by its functor of points $h_\mathfrak{X} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$: see Theorem 8.1.5.1).

In Chapter 6, we defined the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on an arbitrary functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ (see Definition 6.2.2.1). In the special case where X is representable by a spectral Deligne-Mumford stack X, we established an equivalence of $\mathrm{QCoh}(X)$ with the ∞-category $\mathrm{QCoh}(\mathsf{X}) \subseteq \mathrm{Mod}_{\mathscr{O}_{\mathsf{X}}}$ of quasi-coherent $\mathscr{O}_{\mathsf{X}}$-modules (Proposition 6.2.4.1). Our main goal in this first half of this chapter is to prove a variant of this result for *formal* spectral Deligne-Mumford stacks. In §8.2, we associate to each formal spectral Deligne-Mumford stack $\mathfrak{X}$ a full subcategory $\mathrm{QCoh}(\mathfrak{X}) \subseteq \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ of *quasi-coherent* sheaves of $\mathscr{O}_{\mathfrak{X}}$-modules (Definition 8.2.4.7); this can be regarded as a globalization of the theory of complete modules developed in Chapter 7. In §8.3, we show that if $X = h_{\mathfrak{X}}$ is the functor represented by a formal spectral Deligne-Mumford stack $\mathfrak{X}$, then there is a canonical restriction map $\mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(X)$ which is an equivalence when restricted to connective quasi-coherent sheaves (Theorem 8.3.4.4).

The formal completion construction $\mathsf{X} \mapsto \mathsf{X}^{\wedge}_{K}$ is easiest to analyze in the case where $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ is affine. The hypothesis that $K \subseteq |\mathsf{X}|$ is cocompact guarantees that it can be described as the vanishing locus of a finitely generated ideal $I \subseteq \pi_0 A$. In this case, we will denote the formal completion $\mathsf{X}^{\wedge}_{K}$ by $\mathrm{Spf}\, A$ and refer to it as the *formal spectrum of* A. In this case, the ∞-category $\mathrm{QCoh}(\mathrm{Spf}\, A)$ can be identified with the ∞-category $\mathrm{Mod}_A^{\mathrm{Cpl}(I)}$ of I-complete A-modules (Corollary 8.2.4.15). If A itself is I-complete, then this restricts to an equivalence $\mathrm{QCoh}(\mathrm{Spf}\, A)^{\mathrm{aperf}} \simeq \mathrm{Mod}_A^{\mathrm{aperf}}$ of almost perfect quasi-coherent sheaves on $\mathrm{Spf}\, A$ with almost perfect modules over A. In §8.5, we apply the results of Chapter 5 to prove a much stronger result: if A is I-complete and $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, A$ is a morphism which is proper and almost of finite presentation, then the restriction functor $\mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathrm{Spf}\, A \times_{\mathrm{Sp\acute{e}t}\, A} \mathsf{X})$ induces an equivalence on almost perfect objects (Theorem 8.5.0.3). This can be regarded as a derived version of the classical Grothendieck existence theorem, and has many useful consequences: for example, it implies that the formal completion construction $\mathsf{X} \mapsto \mathrm{Spf}\, A \times_{\mathrm{Sp\acute{e}t}\, A} \mathsf{X}$ is fully faithful when restricted to spectral algebraic spaces which are proper and locally almost of finite presentation over A (Corollary 8.5.3.4).

One pleasant feature of the present framework is that the Grothendieck existence theorem and its corollaries (such as the theorem on formal functions and the formal GAGA principle) do not require any Noetherian hypotheses on A. However, the price we pay for working with non-Noetherian rings is that we get statements only at the "derived" level (that is, at the level of ∞-categories), rather than at the level of abelian categories. For a general spectral Deligne-Mumford stack X, the ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$ does not have a (useful) t-structure. However, the situation is better if X is locally Noetherian: in this case, the full subcategory $\mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}} \subseteq \mathrm{QCoh}(\mathsf{X})$ is closed under truncations, and therefore inherits a t-structure from $\mathrm{QCoh}(\mathsf{X})$. In §8.4, we show that an analogous situation holds in the setting of formal spectral algebraic geometry. More precisely, we introduce a notion of locally Noetherian formal spectral Deligne-Mumford stack (Definition 8.4.2.1) and show that, if $\mathfrak{X}$ is locally Noetherian, then the ∞-category $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ of almost perfect quasi-coherent sheaves on $\mathfrak{X}$ is closed under truncations (in the larger ∞-category $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$) and therefore inherits a t-structure (Corollary 8.4.2.4). Moreover, the heart of $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ can be identified with the abelian category of coherent sheaves on the underlying classical formal stack (for a precise statement, see 8.4.3.5 and Example 8.4.3.6). Specializing to the case where A is a Noetherian $\mathbb{E}_\infty$-ring which is I-complete for some $I \subseteq \pi_0 A$ and $\mathfrak{X} = \mathrm{Spf}\, A \times_{\mathrm{Sp\acute{e}t}\, A} \mathsf{X}$ for some X which is proper and almost of finite presentation over A, the equivalence $\mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}} \simeq \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ of Theorem 8.5.0.3 is t-exact (Proposition 8.4.2.10) and therefore induces equivalence of abelian categories, from which one can extract the classical Grothendieck existence theorem (Theorem 8.5.0.1).

Remark 8.0.0.1. In non-Noetherian situations, the failure of the Grothendieck existence theorem at the level of abelian categories is only to be expected, since the operation of I-completion $M \mapsto M^\wedge_I$ does not have good exactness properties.

In classical algebraic geometry, one very useful consequence of the Grothendieck existence theorem (or, more precisely, of Grothendieck's theorem on formal functions) is Zariski's connectedness theorem: if $f : X \to Z$ is a proper morphism of Noetherian schemes, then f admits a *Stein factorization* $X \xrightarrow{g} Y \xrightarrow{h} Z$ where h is a finite morphism and g has connected (geometric) fibers. In §8.7, we discuss Stein factorizations and show that an analogous assertion holds in spectral algebraic geometry (Theorem **??**). We also discuss a related construction which we call the *reduced Stein factorization* (Definition 8.7.3.2). The theory of reduced Stein factorizations will require some basic facts about geometrically reduced morphisms of spectral Deligne-Mumford stacks, which we discuss in §8.6.

Contents

8.1 Formal Spectral Deligne-Mumford Stacks

Let us begin with a brief review of the theory of the classical theory of (locally Noetherian) formal schemes.

Construction 8.1.0.1 (The Formal Spectrum)**.** Let A be a Noetherian commutative ring and let $I \subseteq A$ be an ideal. The *formal spectrum of* A (with respect to I) is the ringed space $(|\operatorname{Spf} A|, \mathscr{O}_{|\operatorname{Spf} A|})$ given as follows:

- The underlying topological space $|\operatorname{Spf} A|$ is the closed subset of the Zariski spectrum $|\operatorname{Spec} A|$ given by the vanishing locus of the ideal I: that is, the collection of those prime ideals $\mathfrak{p} \subseteq A$ which contain I. We endow $|\operatorname{Spf} A|$ with the subspace topology, so that it has a basis of open sets of the form $U_x = \{\mathfrak{p} \subseteq A : I \subseteq \mathfrak{p}, x \notin \mathfrak{p}\}$ for $x \in X$.

- The structure sheaf $\mathscr{O}_{|\operatorname{Spf} A|}$ is given on basic open sets by the formula $\mathscr{O}_{|\operatorname{Spf} A|}(U_x) = \varprojlim (A/I^n)[x^{-1}]$. In other words, $\mathscr{O}_{|\operatorname{Spf} A|}$ is the inverse limit of the structure sheaves of the affine schemes $\operatorname{Spec} A/I^n$, each of which has underlying topological space $|\operatorname{Spf} A|$.

Remark 8.1.0.2. In the setting of Construction 8.1.0.1, it is not necessary to assume that A is I-adically complete. However, it is harmless to add this assumption: replacing A by its I-adic completion $\varprojlim A/I^n$ does not change the ringed space $(|\operatorname{Spf} A|, \mathscr{O}_{|\operatorname{Spf} A|})$.

Definition 8.1.0.3. Let $(X, \mathscr{O}_X)$ be a ringed space. We say that $(X, \mathscr{O}_X)$ is a *locally Noetherian formal scheme* if there exists a covering of X by open subsets U_α such that each of the ringed spaces $(U_\alpha, \mathscr{O}_X|_{U_\alpha})$ has the form $(|\operatorname{Spf} A_\alpha|, \mathscr{O}_{|\operatorname{Spf} A_\alpha|})$, for some Noetherian ring A_α (and some ideal $I_\alpha \subseteq A_\alpha$).

Our goal in this section is to introduce a variant of Definition 8.1.0.3 to the setting of spectral algebraic geometry. We begin in §8.1.1 by introducing the *formal spectrum* $\operatorname{Spf} A$ of an $\mathbb{E}_\infty$-ring A with respect to an ideal $I \subseteq \pi_0 A$ (Construction 8.1.1.10). Our definition is loosely modeled on Construction 8.1.0.1, with a few important differences:

(a) In Construction 8.1.1.10, we allow A to be a connective $\mathbb{E}_\infty$-ring (rather than an ordinary commutative ring).

(b) In Construction 8.1.1.10, we do not require A to be Noetherian (however, we do require the ideal $I \subseteq \pi_0 A$ to be finitely generated).

(c) In place of the topological space $|\operatorname{Spf} A|$ (which is a closed subset of the Zariski spectrum $|\operatorname{Spec} A|$), we consider an ∞-topos $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A$ (which is a closed subtopos of the ∞-topos $\mathcal{S}\mathrm{hv}^{\text{ét}}_A$). In other words, we work locally with respect to the étale topology, rather than the Zariski topology.

Warning 8.1.0.4. In Construction 8.1.0.1, the hypothesis that A is Noetherian is not necessary. However, when applied to non-Noetherian rings, Construction 8.1.0.1 yields a notion of formal spectrum which is not compatible with Construction 8.1.1.10. Roughly speaking, this is because Construction 8.1.0.1 involves the I-adic completion functor $A \mapsto \varprojlim A/I^n$, while Construction 8.1.1.10 involves the I-completion functor $A \mapsto A^\wedge_I$ of Notation 7.3.1.5. We refer the reader to §7.3 for a comparison of these two notions of completion (and some indication of why the latter might be preferable in non-Noetherian contexts).

In §8.1.3, we define a notion of *formal spectral Deligne-Mumford stack* (Definition 8.1.3.1). Here we take our cue from Definition 8.1.0.3: a formal spectral Deligne-Mumford stack is a spectrally ringed ∞-topos $\mathfrak{X}$ which is locally of the form $\operatorname{Spf} A$, for some connective $\mathbb{E}_\infty$-ring A (and some finitely generated ideal $I \subseteq \pi_0 A$).

The collection of formal spectral Deligne-Mumford stacks can be organized into an ∞-category fSpDM, which contains the ∞-category SpDM of spectral Deligne-Mumford stacks as a full subcategory. An underlying theme of this section is that an object $\mathfrak{X} \in \mathrm{fSpDM}$ can be "built from" the ordinary spectral Deligne-Mumford stacks which map to $\mathfrak{X}$. In §8.1.3, we will give a precise articulation of this idea in the case where $\mathfrak{X} = \operatorname{Spf} A$ is a formal spectrum: in this case, we can identify $\mathfrak{X}$ with the direct limit $\varinjlim \operatorname{Spét} B$, where B ranges over all connective A-algebras for which the ideal $I(\pi_0 B)$ is nilpotent; here the direct limit is formed in the ∞-category of locally spectrally ringed ∞-topoi (Proposition 8.1.2.1). In §8.1.5, we use this result to show that a formal spectral Deligne-Mumford stack $\mathfrak{X}$ is determined by its "functor of points" $R \mapsto \mathrm{Map}_{\mathrm{fSpDM}}(\operatorname{Spét} R, \mathfrak{X})$ (Theorem 8.1.5.1).

For our purposes, the theory of formal spectral Deligne-Mumford stacks is primarily of interest as a tool for proving results about ordinary spectral Deligne-Mumford stacks. In §8.1.6, we will define the *formal completion* $\mathsf{X}^\wedge_K$ of a spectral Deligne-Mumford stack X along a (cocompact) closed subset $K \subseteq |\mathsf{X}|$ (Definition 8.1.6.1). Roughly speaking, $\mathsf{X}^\wedge_K$ is a formal spectral Deligne-Mumford stack which encodes the behavior of X "infinitesimally close" to K. This language will play an essential role when we discuss the Grothendieck existence theorem in §8.5.

8.1.1 Adic $\mathbb{E}_\infty$-Rings and Formal Spectra

We start by reviewing some terminology from commutative algebra.

Definition 8.1.1.1. Let A be a commutative ring. Every ideal $I \subseteq A$ determines a topology on A, which has a basis of open sets of the form $x + I^n \subseteq A$ where $x \in A$ and $n \geqslant 0$. We will refer to this topology as the *I-adic topology*.

If A is a topological commutative ring, we say that an ideal $I \subseteq A$ is an *ideal of definition* if the topology on A coincides with the I-adic topology.

An *adic ring* is a commutative ring A equipped with a topology which admits a finitely generated ideal of definition. We let $\mathrm{CAlg}^\heartsuit_{\mathrm{ad}}$ denote the category whose objects are adic rings and whose morphisms are continuous ring homomorphisms.

Warning 8.1.1.2. Definition 8.1.1.1 is not standard: many authors allow topological commutative rings which admit an ideal of definition $I \subseteq A$, without imposing the requirement that I is finitely generated. Additionally, it is common to also require that A is I-adically complete (that is, that the canonical map $A \to \varprojlim A/I^n$ is an isomorphism), which we do not assume.

Remark 8.1.1.3. Let A and A' be adic rings having finitely generated ideals of definition $I \subseteq A$ and $I' \subseteq A'$. Then a ring homomorphism $\phi : A \to A'$ is continuous if and only if $\phi(I^n) \subseteq I'$ for some $n \gg 0$. Equivalently, ϕ is continuous if and only if the induced map $|\operatorname{Spec} A'| \to |\operatorname{Spec} A|$ carries the vanishing locus of I' to the vanishing locus of I.

Remark 8.1.1.4. Let A be a commutative ring. For any finitely generated ideal $I \subseteq A$, we can regard A as an adic ring by equipping it with the I-adic topology. Note that if $I, J \subseteq A$ are finitely generated ideals, then the I-adic topology and J-adic topology coincide if and only if $I^n \subseteq J$ and $J^n \subseteq I$ for $n \gg 0$. Equivalently, the I-adic topology and J-adic topology coincide if and only if I and J have the same vanishing locus in the Zariski spectrum $|\operatorname{Spec} A|$.

Definition 8.1.1.5. An *adic* $\mathbb{E}_\infty$*-ring* is a connective $\mathbb{E}_\infty$-ring together with a topology on the commutative ring $\pi_0 A$, which makes $\pi_0 A$ into an adic ring (in the sense of Definition 8.1.1.1). We let $\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}$ denote the fiber product $\mathrm{CAlg}^{\heartsuit}_{\mathrm{ad}} \times_{\mathrm{CAlg}^{\heartsuit}} \mathrm{CAlg}^{\mathrm{cn}}$. We will refer to $\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}$ as the ∞*-category of adic* $\mathbb{E}_\infty$*-rings*.

Remark 8.1.1.6. Put more informally: the objects of $\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}$ are adic $\mathbb{E}_\infty$-rings, and the morphisms in $\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}$ are morphisms $\phi : A \to A'$ of $\mathbb{E}_\infty$-rings for which the underlying ring homomorphism $\pi_0 A \to \pi_0 A'$ is continuous.

Remark 8.1.1.7. Let A be a connective $\mathbb{E}_\infty$-ring. Using Remark 8.1.1.4, we see that there is a canonical bijection between the following data:

- Topologies on $\pi_0 A$ which admit a finitely generated ideal of definition.
- Closed subsets $X \subseteq |\operatorname{Spec} A|$ with quasi-compact complement.

By means of this bijection, we can identify adic $\mathbb{E}_\infty$-rings with pairs (A, X), where A is a connective $\mathbb{E}_\infty$-ring and $X \subseteq |\operatorname{Spec} A|$ is a closed set with quasi-compact complement. Under this identification, a morphism of adic $\mathbb{E}_\infty$-rings from (A, X) to (A', X') is a morphism of $\mathbb{E}_\infty$-rings $\phi : A \to A'$ for which the induced map $|\operatorname{Spec} A'| \to |\operatorname{Spec} A|$ carries X' into X.

Notation 8.1.1.8. Let A be an adic $\mathbb{E}_\infty$-ring and let $\mathcal{S}\mathrm{hv}^{\mathrm{\acute{e}t}}_A$ denote the ∞-category of functors $\mathscr{F} : \mathrm{CAlg}^{\mathrm{\acute{e}t}}_A \to \mathcal{S}$ which are sheaves with respect to the étale topology. Let $X \subseteq |\operatorname{Spec} A|$ be the vanishing locus of an ideal of definition of $\pi_0 A$. We let $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A$ denote the closed subtopos of $\mathcal{S}\mathrm{hv}^{\mathrm{\acute{e}t}}_A$ corresponding to the closed subset $X \subseteq |\operatorname{Spec} A|$. More precisely, we let $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A$ denote the full subcategory of $\mathcal{S}\mathrm{hv}^{\mathrm{\acute{e}t}}_A$ spanned by those sheaves $\mathscr{F}$ having the property that $\mathscr{F}(B)$ is contractible whenever $X \times_{|\operatorname{Spec} A|} |\operatorname{Spec} B|$ is empty.

Remark 8.1.1.9. Let A be an adic $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition. Using Proposition 3.1.4.1, we obtain a canonical equivalence of ∞-categories $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A \simeq \mathcal{S}\mathrm{hv}^{\mathrm{\acute{e}t}}_{(\pi_0 A)/I}$.

Construction 8.1.1.10. Let A be an adic $\mathbb{E}_\infty$-ring and let $\mathscr{O}_{\mathrm{Sp\acute{e}t}\, A}$ be the structure sheaf of the étale spectrum Spét A, which we identify with the inclusion functor $\mathrm{CAlg}_A^{\acute{e}t} \hookrightarrow \mathrm{CAlg}_A$. Let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition. We let $\mathscr{O}_{\mathrm{Spf}\, A} : \mathrm{CAlg}_A^{\acute{e}t} \to \mathrm{CAlg}_A$ denote the composition of $\mathscr{O}_{\mathrm{Sp\acute{e}t}\, A}$ with the I-completion functor, given by $B \mapsto B_I^\wedge$. Note that $\mathscr{O}_{\mathrm{Spf}\, A}$ is still a sheaf with respect to the étale topology (since the formation of I-completions preserves small limits). Moreover, $\mathscr{O}_{\mathrm{Spf}\, A}(B) = B_I^\wedge$ vanishes whenever I generates the unit ideal of $\pi_0 B$. It follows that we can regard $\mathscr{O}_{\mathrm{Spf}\, A}$ as a CAlg_A-valued sheaf on the closed subtopos $\mathcal{S}\mathrm{hv}_A^{\mathrm{ad}} \subseteq \mathcal{S}\mathrm{hv}_A^{\acute{e}t}$. We let Spf A denote the spectrally ringed ∞-topos $(\mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}, \mathscr{O}_{\mathrm{Spf}\, A})$. We will refer to Spf A as the *formal spectrum of A*.

Example 8.1.1.11. In the situation of Construction 8.1.1.10, suppose that I is nilpotent: that is, the commutative ring $\pi_0 A$ is equipped with the discrete topology. Then the formal spectrum Spf A is equivalent to the étale spectrum Spét A.

Remark 8.1.1.12. Let A be an adic $\mathbb{E}_\infty$-ring. Then the inclusion functor $\mathcal{S}\mathrm{hv}_A^{\mathrm{ad}} \hookrightarrow \mathcal{S}\mathrm{hv}_A^{\acute{e}t}$ admits a (left exact) left adjoint $i^* : \mathcal{S}\mathrm{hv}_A^{\acute{e}t} \to \mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}$. We will say that an object $U \in \mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}$ is *affine* if it has the form $i^*\overline{U}$, where $\overline{U} \in \mathcal{S}\mathrm{hv}_A^{\acute{e}t}$ is affine (in other words, $\overline{U}$ is corepresentable by some object $B \in \mathrm{CAlg}_A^{\acute{e}t}$).

Note that $U \in \mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}$ is affine if and only if its image under the equivalence $\mathcal{S}\mathrm{hv}_A^{\mathrm{ad}} \simeq \mathcal{S}\mathrm{hv}_{(\pi_0 A)/I}^{\acute{e}t}$ of Remark 8.1.1.9 is affine.

Proposition 8.1.1.13. *Let A be an adic $\mathbb{E}_\infty$-ring and let* Spf $A = (\mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}, \mathscr{O}_{\mathrm{Spf}\, A})$ *be its formal spectrum (Construction 8.1.1.10). Then the structure sheaf $\mathscr{O}_{\mathrm{Spf}\, A}$ is connective and strictly Henselian.*

The proof of Proposition 8.1.1.13 will require an elementary observation about the formation of completions:

Lemma 8.1.1.14. *Let $R \to R'$ be an étale morphism of connective $\mathbb{E}_\infty$-rings and let S be a connective $\mathbb{E}_\infty$-algebra over R which is I-complete for some finitely generated ideal $I \subseteq \pi_0 S$. Then the canonical map $\rho : \mathrm{Map}_{\mathrm{CAlg}_R}(R', S) \to \mathrm{Map}_{\mathrm{CAlg}_R}(R', (\pi_0 S)/I)$ is a homotopy equivalence.*

Proof. Without loss of generality we may assume that S is discrete. Proceeding by induction on the number of generators of I, we can assume that $I = (x)$ is a principal ideal. For each $n \geqslant 1$, let S_n denote the tensor product $S \otimes_{\mathbf{Z}[x]} \mathbf{Z}[x]/(x^n)$. Then we have a canonical isomorphism $S/I \simeq \pi_0 S_1$, and the map ρ factors as a composition

$$\mathrm{Map}_{\mathrm{CAlg}_R}(R', S) \xrightarrow{\rho'} \varprojlim_{n \geqslant 1} \mathrm{Map}_{\mathrm{CAlg}_R}(R', S_n) \xrightarrow{\rho''} \mathrm{Map}_{\mathrm{CAlg}_R}(R', S_1) \xrightarrow{\rho'''} \mathrm{Map}_{\mathrm{CAlg}_R}(R', S/I).$$

The map ρ' is a homotopy equivalence by virtue of our assumption that S is I-complete, the map ρ'' is a homotopy equivalence since each S_{n+1} is a square-zero extension of S_n, and the map ρ''' is an equivalence since S_1 is a square-zero extension of S/I. $\square$

Proof of Proposition 8.1.1.13. Let A be an adic $\mathbb{E}_\infty$-ring. Note that if $U \in \mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}$ is affine (in the sense of Remark 8.1.1.12), then $\mathscr{O}_{\mathrm{Spf}\, A}(U) \simeq B^\wedge_I$ for some étale A-algebra B and is therefore connective. Since $\mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}$ is generated under small colimits by affine objects, it follows that $\mathscr{O}_{\mathrm{Spf}\, A}$ is connective. We now argue that $\mathscr{O}_{\mathrm{Spf}\, A}$ is strictly Henselian. Let R be a commutative ring, let $\{R \to R_\alpha\}$ be a finite collection of étale maps for which the map $R \to \prod R_\alpha$ is faithfully flat, and suppose we are given a map $\rho : R \to (\pi_0\, \mathscr{O}_{\mathrm{Spf}\, A})(U)$ for some object $U \in \mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}$. We wish to show that there exists a covering $\{U_\alpha \to U\}$ of U such that each of the composite maps $R \to (\pi_0\, \mathscr{O}_{\mathrm{Spf}\, A})(U) \to (\pi_0\, \mathscr{O}_{\mathrm{Spf}\, A}(U_\alpha)$ factors through R_α. Without loss of generality, we may assume that R is a finitely generated commutative ring. The desired conclusion is local on U, so we may further and that ρ is given by a map $\overline{\rho} : R \to \pi_0(\mathscr{O}_{\mathrm{Spf}\, A}(U))$. Let $i^* : \mathcal{S}\mathrm{hv}_A^{\text{ét}} \to \mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}$ be a left adjoint to the inclusion. Localizing further, we may assume that U is affine: that is, we have $U \simeq i^* h^B$, where h^B denotes the functor corepresented by some étale A-algebra B. In this case, we can regard $\overline{\rho}$ as a ring homomorphism $R \to \pi_0(B^\wedge_I)$. For each index α, set $C_\alpha = R_\alpha \otimes_R (\pi_0 B)/I(\pi_0 B)$. Using the classification of étale morphisms (Proposition B.1.1.3), we can write each C_α as a tensor product $B_\alpha \otimes_B (\pi_0 B)/I(\pi_0 B)$, where B_α is étale over B. Setting $U_\alpha = i^* B_\alpha$, we obtain a covering $\{U_\alpha \to U\}$ in the ∞-topos $\mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}$. We will complete the proof by showing that each of the composite maps $R \xrightarrow{\overline{\rho}} \pi_0(\mathscr{O}_{\mathrm{Spf}\, A}(U)) \to \pi_0(\mathscr{O}_{\mathrm{Spf}\, A}(U_\alpha)) \simeq \pi_0(B_\alpha)^\wedge_I$ factors through R_α. Set $S = \pi_0(B_\alpha)^\wedge_I$. Then S is I-complete (Theorem 7.3.4.1), so Lemma 8.1.1.14 implies that the map $\mathrm{Hom}_R(R_\alpha, S) \to \mathrm{Hom}_R(R_\alpha, S/IS)$ is bijective. We conclude by observing that the codomain of this map is nonempty (since S/IS is isomorphic to C_α). □

8.1.2 Functoriality of the Formal Spectrum

Let A be an adic $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition I. Roughly speaking, we can think of the formal spectrum Spf A as the "formal completion" of Spét A along the vanishing locus $K \subseteq |\operatorname{Spec} A|$ of the ideal I. We now articulate this idea more precisely by showing that Spf A can be identified with the colimit $\varinjlim \text{Spét}\, B$, where B ranges over all connective A-algebras for which the underlying map $|\operatorname{Spec} B| \to |\operatorname{Spec} A|$ factors through K (we will revisit the same idea in §8.1.6).

Proposition 8.1.2.1. *Let $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ denote the ∞-category introduced in Definition 1.4.2.1, whose objects are spectrally ringed ∞-topoi $(\mathcal{X}, \mathscr{O})$, where $\mathscr{O}$ is strictly Henselian. The étale spectrum functor* Spét : $(\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ *admits a left Kan extension along the inclusion* $(\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}} \hookrightarrow (\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}})^{\mathrm{op}}$. *This left Kan extension is given on objects by the construction $A \mapsto$ Spf A.*

The proof of Proposition 8.1.2.1 will require some preliminaries.

Lemma 8.1.2.2. *Let A be an adic $\mathbb{E}_\infty$-ring having a finitely generated ideal of definition $I \subseteq \pi_0 A$. Then there exists a tower $\cdots \to A_4 \to A_3 \to A_2 \to A_1$ in the ∞-category CAlg_A with the following properties:*

(a) Each A_i is a connective $\mathbb{E}_\infty$-algebra over A, and each of the maps $A_{i+1} \to A_i$ induces a surjection $\pi_0 A_{i+1} \to \pi_0 A_i$ with nilpotent kernel.

(b) For every connective $\mathbb{E}_\infty$-ring B, the canonical map $\varinjlim_n \mathrm{Map}_{\mathrm{CAlg}}(A_n, B) \to \mathrm{Map}_{\mathrm{CAlg}}(A, B)$ induces a homotopy equivalence of $\varinjlim_n \mathrm{Map}_{\mathrm{CAlg}}(A_n, B)$ with the summand of $\mathrm{Map}_{\mathrm{CAlg}}(A, B)$ consisting of those maps $\phi : A \to B$ which annihilate some power of the ideal I.

(c) Each of the $\mathbb{E}_\infty$-rings A_n is almost perfect when regarded as an A-module.

Proof. Choose any element $x \in \pi_0 A$. We will construct a tower of $\mathbb{E}_\infty$-algebras over A

$$\cdots \to A(x)_3 \to A(x)_2 \to A(x)_1$$

having the following properties:

(a_x) Each $A(x)_i$ is a connective $\mathbb{E}_\infty$-ring, and each of the maps $A(x)_{i+1} \to A(x)_i$ determines a surjection $\pi_0 A(x)_{i+1} \to \pi_0 A(x)_i$ with nilpotent kernel.

(b_x) That is, for every connective $\mathbb{E}_\infty$-ring B, the canonical map $\varinjlim_n \mathrm{Map}_{\mathrm{CAlg}}(A(x)_n, B) \to \mathrm{Map}_{\mathrm{CAlg}}(A, B)$ induces a homotopy equivalence of $\varinjlim_n \mathrm{Map}_{\mathrm{CAlg}}(A(x)_n, B)$ with the summand of $\mathrm{Map}_{\mathrm{CAlg}}(A, B)$ consisting of those maps $\phi : A \to B$ which annihilate some power of x.

(c_x) Each of $\mathbb{E}_\infty$-rings $A(x)_n$ is almost perfect when regarded as an A-module.

Assuming that this can be done, choose a finite set of generators $x_1, \ldots, x_k$ for the ideal I. Setting $A_n = A(x_1)_n \otimes_A A(x_2)_n \otimes_A \cdots \otimes_A A(x_k)_n$, we obtain a tower of $\mathbb{E}_\infty$-algebras over A satisfying conditions (a), (b), and (c).

It remains to construct the tower $\{A(x)_n\}$. For each integer $n > 0$, let $A\{t_n\}$ denote a free $\mathbb{E}_\infty$-algebra over A on one generator t_n. We have A-algebra morphisms $\alpha_n : A\{t_n\} \to A$ and $\beta_n : A\{t_n\} \to A$, determined uniquely up to homotopy by the requirements that $t_n \mapsto x^n \in \pi_0 A$ and $t_n \mapsto 0 \in \pi_0 A$. Moreover, we have maps $\gamma_n : A\{t_n\} \to A\{t_{n-1}\}$ determined up to homotopy by the requirement that $t_n \mapsto x t_{n-1} \in \pi_0 A\{t_{n-1}\}$. For each $n \geqslant 0$, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}} & A \\ \uparrow{\scriptstyle \alpha_n} & & \uparrow{\scriptstyle \alpha_{n-1}} \\ A\{t_n\} & \xrightarrow{\gamma_n} & A\{t_{n-1}\} \\ \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \beta_{n-1}} \\ A & \xrightarrow{\mathrm{id}} & A \end{array}$$

commutes up to homotopy and can therefore be lifted to a commutative diagram in CAlg_A (since it is indexed by a partially ordered set whose nerve has dimension $\leqslant 2$).

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\mathrm{id}} & A & \xrightarrow{\mathrm{id}} & A \\
\uparrow & & \uparrow{\scriptstyle \alpha_3} & & \uparrow{\scriptstyle \alpha_2} & & \uparrow{\scriptstyle \alpha_1} \\
\cdots & \longrightarrow & A\{t_3\} & \xrightarrow{\gamma_3} & A\{t_2\} & \xrightarrow{\gamma_2} & A\{t_1\} \\
\downarrow & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \beta_2} & & {\scriptstyle \beta_1}\downarrow \\
\cdots & \longrightarrow & A & \xrightarrow{\mathrm{id}} & A & \xrightarrow{\mathrm{id}} & A.
\end{array}$$

For each $n > 0$, let $A(x)_n$ denote the colimit of the nth column of this diagram, so that we have a tower

$$\cdots \rightarrow A(x)_3 \rightarrow A(x)_2 \rightarrow A(x)_1$$

where $A(x)_n \simeq A \otimes_{A\{t_n\}} A$ is the A-algebra obtained by freely "coning off" the element $x^n \in \pi_0 A$. In particular, we have $\pi_0 A(x)_n \simeq (\pi_0 A)/(x^n)$, thereby verifying condition (a_x). To verify (c_x), it will suffice to show that A is almost perfect when regarded as an $A\{t_n\}$-module via β. For this, it suffices to show that the sphere spectrum is almost perfect when regarded as an $S\{t_n\}$-module via the map of $\mathbb{E}_\infty$-rings $S\{t_n\} \rightarrow S$ given by $t_n \mapsto 0 \in \pi_0 S$. Since $S\{t_n\}$ is Noetherian (Proposition HA.7.2.4.31), this is equivalent to the assertion that each homotopy group $\pi_k S$ is finitely generated as a module over the commutative ring $\pi_0(S\{t_n\}) \simeq \mathbf{Z}[t_n]$ (Proposition HA.7.2.4.17). This is clear, since the stable homotopy groups of spheres are finitely generated abelian groups.

To verify (b_x), we note that if $\phi : A \rightarrow B$ is a map of connective $\mathbb{E}_\infty$-rings, then the homotopy fiber of the map $\varinjlim_n \mathrm{Map}_{\mathrm{CAlg}}(A(x)_n, B) \rightarrow \mathrm{Map}_{\mathrm{CAlg}}(A, B)$ over the point ϕ is given by a sequential colimit $\varinjlim_n P_n$, where each P_n can be identified with a space of paths in $\Omega^\infty B$ joining the base point to a suitably chosen representative for the image of x^n in $\pi_0 B$. Let $y \in \pi_0 B$ be the image of x under ϕ. If y is not nilpotent, then each P_n is empty. Assume otherwise; we wish to show that $P_\infty = \varinjlim P_n$ is contractible. Note that if P_n contains some point p_n, the we have canonical isomorphisms $\pi_k(P_n, p_n) \simeq \pi_{k+1} B$. For $m \geqslant n$, let p_m denote the image of p_n in P_m, and let p_∞ denote the image of p_n in P_∞. Note that the induced map

$$\pi_{k+1} B \simeq \pi_k(P_n, p_n) \rightarrow \pi_k(P_m, p_m) \rightarrow \pi_{k+1} B$$

is given by multiplication by y^{m-n}. Since y is nilpotent, this map is trivial for $m \gg n$. It follows that $\pi_k(P_\infty, p_\infty) \simeq \varinjlim \pi_k(P_m, p_m)$ is trivial. Since p_n was chosen arbitrarily, we conclude that P_∞ is contractible as desired. □

Lemma 8.1.2.3. *Let A be an adic $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition, and let $\{A_n\}_{n \geqslant 1}$ be as in Lemma 8.1.2.2. For every connective A-module M, the canonical map $M \rightarrow \varprojlim(A_n \otimes_A M)$ exhibits $\varprojlim(A_n \otimes_A M)$ as an I-completion of M.*

Proof. For each R-module M, set $U(M) = \varprojlim(A_n \otimes_A M)$. Note that each $A_n \otimes_A M$ is I-complete (since I generates a nilpotent ideal in A_n), so that $U(M)$ is I-complete. Consequently, the canonical map $M \to U(M)$ factors as a composition $M \to M^{\wedge}_I \xrightarrow{\beta_M} U(M)$. We wish to show that β_M is an equivalence when M is connective.

Choose an element $x \in I$, and let $C(x^n)$ denote the cofiber of the map of A-modules $A \to A$ given by multiplication by x^n. Since $\mathrm{fib}(\beta_M)$ is I-complete, we have

$$\mathrm{fib}(\beta_M) \simeq \varprojlim \mathrm{fib}(\beta_M) \otimes_A C(x^n);$$

it will therefore suffice to show that each tensor product $\mathrm{fib}(\beta_M) \otimes_A C(x^n)$ vanishes. Since $C(x^n)$ can be obtained as a successive extension of n copies of $C(x)$, we may suppose that $n = 1$. Note that $\mathrm{fib}(\beta_M) \otimes_A C(x) \simeq \mathrm{fib}(\beta_{M \otimes_A C(x)})$. Consequently, to show that β_M is an equivalence, it suffices to show that $\beta_{M \otimes_A C(x)}$ is an equivalence.

Choose generators $x_1, \ldots, x_n \in I$ for the ideal I. Using the above argument repeatedly, we are reduced to proving that β_N is an equivalence when $N = M \otimes_A C(x_1) \otimes_A C(x_2) \otimes \cdots \otimes_A C(x_n)$. For $1 \leqslant i \leqslant n$, we observe that N can be obtained as a successive extension of 2^{n-1} (shifted) copies of $M \otimes_A C(x_i)$. Since the homotopy groups of $M \otimes_A C(x_i)$ are annihilated by multiplication by x_i^2, we conclude that each of the homotopy groups of N is annihilated by multiplication by $x_i^{2^n}$. We are therefore reduced to proving the following special case of Lemma 8.1.2.3:

$(*)$ Let M be a connective A-module, and suppose that there exists an integer k such that each homotopy group $\pi_i M$ is annihilated by the ideal $I^k \subseteq \pi_0 A$. Then $\beta_M : M^{\wedge}_I \to U(M)$ is an equivalence.

To prove $(*)$, it suffices to show that for every integer $j \geqslant 0$, the map $\pi_j M^{\wedge}_I \to \pi_j U(M)$ is an isomorphism of abelian groups. Both $M^{\wedge}_I$ and $U(M)$ are right t-exact functors of M. We may therefore replace M by $\tau_{\leqslant j} M$ and thereby reduce to proving $(*)$ under the additional assumption that M is p-truncated for some integer p. We now proceed by induction on p. If $p < 0$, then $M \simeq 0$ and there is nothing to prove. Otherwise, we have a map of fiber sequences

$$\begin{array}{ccccc} \Sigma^p(\pi_p M)^{\wedge}_I & \longrightarrow & M^{\wedge}_I & \longrightarrow & (\tau_{\leqslant p-1} M)^{\wedge}_I \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^p U(\pi_p M) & \longrightarrow & U(M) & \longrightarrow & U(\tau_{\leqslant p-1} M) \end{array}$$

where the right vertical map is an equivalence by our inductive hypothesis. We may therefore replace M by $\pi_p M$ and thereby reduce to the case where M is discrete. In this case, M has the structure of a module over the discrete A-algebra $R = (\pi_0 A)/I^k$.

For $n \geqslant 1$, set $R_n = A_n \otimes_A R$. Since I generates a nilpotent ideal in R, the tower of R-algebras $\{R_n\}_{n \geqslant 1}$ is equivalent (as a Pro-object of $\mathrm{CAlg}^{\mathrm{cn}}_R$) to the constant diagram taking the value R. It follows that we can identify β_M with the canonical map

$$M \simeq M_I^\wedge \to U(M) \simeq \varprojlim \{A_n \otimes_A M\} \simeq \varprojlim \{R_n \otimes_R M\} \simeq R \otimes_R M,$$

which is evidently an equivalence. □

Proof of Proposition 8.1.2.1. Let A be an adic $\mathbb{E}_\infty$-ring. We will show that the functor $\mathrm{Spét} : (\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ admits a left Kan extension to $(\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}})^{\mathrm{op}}$ at A, whose value at A can be identified with $\mathrm{Spf}\, A$. Let $\{A_n\}_{n>0}$ be a tower of A-algebras which satisfies the requirements of Lemma 8.1.2.2, so that the construction $n \mapsto A_n$ determines a right cofinal map

$$\mathrm{N}(\mathbf{Z}^{\mathrm{op}}_{>0}) \to \mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}} (\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}})_{A/}.$$

It will therefore suffice to show that we can identify $\mathrm{Spf}\, A$ with a colimit of the diagram $\{\mathrm{Spét}\, A_n\}_{n>0}$ in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$.

Since each of the transition maps $A_{n+1} \to A_n$ induces a surjective ring homomorphism $\pi_0 A_{n+1} \to \pi_0 A_n$ with nilpotent kernel, the spectrally ringed ∞-topoi $\{\mathrm{Spét}\, A_n\}_{n>0}$ all have the same underlying ∞-topos, which we can identify with $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A$ (see Remark 8.1.1.9). Let us use this identification to view each structure sheaf $\mathscr{O}_{\mathrm{Spét}\, A_n}$ as a CAlg-valued sheaf on $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A$, given by the functor

$$\mathrm{CAlg}^{\text{ét}}_A \to \mathrm{CAlg} \qquad B \mapsto (A_n \otimes_A B).$$

Applying Lemma 8.1.2.3, we see that $\mathscr{O}_{\mathrm{Spf}\, A}$ can be identified with the limit of the tower $\{\mathscr{O}_{\mathrm{Spét}\, A_n}\}$ in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A)$. It follows that we can identify $\mathrm{Spf}\, A$ with the colimit $\varinjlim \mathrm{Spét}\, A_n$ in the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ of *all* spectrally ringed ∞-topoi. To show that it is also a colimit of the diagram $\{\mathrm{Spét}\, A_n\}_{n>0}$ in the subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$, we must verify the following:

(∗) Let $\mathsf{X} = (\mathcal{X}, \mathscr{O})$ be an object of $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$. Then a morphism of spectrally ringed ∞-topoi $f : \mathrm{Spf}\, A \to \mathsf{X}$ is local if and only if each of the composite maps $\mathrm{Spét}\, A_n \to \mathrm{Spf}\, A \xrightarrow{f} \mathsf{X}$ is local.

To prove this, it suffices to show that each of the maps $\mathscr{O}_{\mathrm{Spf}\, A} \to \mathscr{O}_{\mathrm{Spét}\, A_n}$ is local (see Remark 1.2.1.7), which follows immediately from Lemma 8.1.1.14. □

Remark 8.1.2.4. Let A be an adic $\mathbb{E}_\infty$-ring with a finitely generated ideal of definition $I \subseteq \pi_0 A$, and let $\widehat{A}$ denote the I-completion $A^\wedge_I$. Choose a tower $\{A_n\}_{n>0}$ satisfying the requirements of Lemma 8.1.2.2 for A. The fiber of the unit map $u : A \to \widehat{A}$ is I-local and each A_n is I-nilpotent, so the tensor product $\mathrm{fib}(u) \otimes_A A_n$ vanishes. It follows that u induces equivalences $A_n \to \widehat{A} \otimes_A A_n$, from which we deduce that the tower $\{A_n\}_{n>0}$ also satisfies the requirements of Lemma 8.1.2.2 for $\widehat{A}$ (which we regard as an adic $\mathbb{E}_\infty$-ring by equipping $\pi_0 \widehat{A}$ with the $\widehat{I}$-adic topology for $\widehat{I} = I(\pi_0 \widehat{A})$).

It follows from Proposition 8.1.2.1 that u induces an equivalence of formal spectra $\mathrm{Spf}(u) : \mathrm{Spf}\,\widehat{A} \to \mathrm{Spf}\,A$ (both the domain and codomain of $\mathrm{Spf}(u)$ can be identified with the colimit of the diagram $\{\mathrm{Sp\acute{e}t}\,A_n\}_{n>0}$ in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$).

8.1.3 Formal Spectral Deligne-Mumford Stacks

We are now ready to introduce our main objects of interest.

Definition 8.1.3.1. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ be a spectrally ringed ∞-topos. We will say that $\mathfrak{X}$ is an *affine formal spectral Deligne-Mumford stack* if there exists an equivalence $\mathfrak{X} \simeq \mathrm{Spf}\,A$, where A is an adic $\mathbb{E}_\infty$-ring.

More generally, we will say that $\mathfrak{X}$ is a *formal spectral Deligne-Mumford stack* if there exists a covering of $\mathcal{X}$ by objects $\{U_\alpha \in \mathcal{X}\}$ such that each $(\mathcal{X}_{/U_\alpha}, \mathcal{O}_{\mathfrak{X}}|_{U_\alpha})$ is an affine formal spectral Deligne-Mumford stack. We let fSpDM denote the full subcategory of $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ spanned the formal spectral Deligne-Mumford stacks.

Any spectral Deligne-Mumford stack X is a formal spectral Deligne-Mumford stack: this follows from Example 8.1.1.11. Moreover, X is affine as a formal spectral Deligne-Mumford stack if and only if it is affine as a spectral Deligne-Mumford stack. This is an immediate consequence of the following:

Proposition 8.1.3.2. *Let A be an adic $\mathbb{E}_\infty$-ring with finitely generated ideal of definition $I \subseteq \pi_0 A$. The following conditions are equivalent:*

(1) *The ideal I is nilpotent.*

(2) *The canonical map* $\mathrm{Spf}\,A \to \mathrm{Sp\acute{e}t}\,A$ *is an equivalence of spectrally ringed ∞-topoi.*

(3) *The formal spectrum* $\mathrm{Spf}\,A$ *is a spectral Deligne-Mumford stack.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are immediate. We will complete the proof by showing that (3) ⇒ (1). Note that $\mathrm{Spf}\,A$ is a quasi-compact, quasi-separated spectral algebraic space (the quasi-compactness and quasi-separatedness follow from the fact that the underlying ∞-topos of $\mathrm{Spf}\,A$ is $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A$, and the fact that $\mathrm{Spf}\,A$ is a spectral algebraic space follows from Proposition 8.1.5.2). It follows that the global sections functor $\Gamma : \mathrm{QCoh}(\mathrm{Spf}\,A) \to \mathrm{Mod}_A$ commutes with filtered colimits. Every element $x \in I$ determines a global section of $\mathcal{O}_{\mathrm{Spf}\,A}$ which is nowhere invertible. It follows that $\mathcal{O}_{\mathrm{Spf}\,A}[x^{-1}] \simeq 0$, so that the unit element $1 \in \pi_0 A$ has vanishing image in $\Gamma(\mathrm{Spf}\,A; \mathcal{O}_{\mathrm{Spf}\,A}[x^{-1}]) \simeq \Gamma(\mathrm{Spf}\,A; \mathcal{O}_{\mathrm{Spf}\,A})[x^{-1}] = A[x^{-1}]$. It follows that x is nilpotent. Allowing x to vary (and invoking our assumption that I is finitely generated), we conclude that I is nilpotent. □

The condition of being a formal spectral Deligne-Mumford stack is local:

Proposition 8.1.3.3. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ be a spectrally ringed ∞-topos. Then:*

(*a*) *If $\mathfrak{X}$ is a formal spectral Deligne-Mumford stack, then $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathcal{O}_{\mathfrak{X}}|_U)$ is also a formal spectral Deligne-Mumford stack for any object $U \in \mathcal{X}$.*

(*b*) *If there exists a covering $\{U_\alpha\}$ of $\mathcal{X}$ such that each $\mathfrak{X}_{U_\alpha}$ is a formal spectral Deligne-Mumford stack, then $\mathfrak{X}$ is a formal spectral Deligne-Mumford stack.*

Proof. Assertion (b) follows immediately from the definitions. To prove (a), let us assume that $\mathfrak{X}$ is a formal spectral Deligne-Mumford stack and choose any object $U \in \mathcal{X}$; we wish to show that $\mathfrak{X}_U$ is also a formal spectral Deligne-Mumford stack. By virtue of (b), this can be tested locally on $\mathfrak{X}$. We may therefore assume that $\mathfrak{X}$ is affine: that is, there exists an equivalence $\mathfrak{X} \simeq \operatorname{Spf} A$ for some adic $\mathbb{E}_\infty$-ring A with a finitely generated ideal of definition $I \subseteq \pi_0 A$. Let us identify $\mathcal{X}$ with the ∞-topos $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A$. Then $\mathcal{X}$ is generated under small colimits by objects which are affine (in the sense of Remark 8.1.1.12). We may therefore assume without loss of generality that U is affine: that is, it is the sheafification of the functor corepresented by some étale A-algebra $B \in \mathrm{CAlg}^{\text{ét}}_A$. In this case, we have an equivalence $\mathfrak{X}_U \simeq \operatorname{Spf} B$ (where we regard B as an adic $\mathbb{E}_\infty$-ring by taking $I(\pi_0 B)$ as an ideal of definition). $\square$

Definition 8.1.3.4. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack. We will say that an object $U \in \mathcal{X}$ is *affine* if $(\mathcal{X}_{/U}, \mathcal{O}_{\mathfrak{X}}|_U)$ is an affine formal spectral Deligne-Mumford stack.

Example 8.1.3.5. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack. Then an object $U \in \mathcal{X}$ is affine in the sense of Definition 8.1.3.4 if and only if it is affine in the sense of Definition 1.4.7.8: this follows from Proposition 8.1.3.2.

Proposition 8.1.3.6. *Let A be an adic $\mathbb{E}_\infty$-ring and consider the formal spectrum* $\operatorname{Spf} A = (\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A, \mathcal{O}_{\operatorname{Spf} A})$*. Then an object $U \in \mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A$ is affine in the sense of Definition 8.1.3.4 if and only if it is affine in the sense of Remark 8.1.1.12: that is, if and only if it the pullback of an affine object of $\mathcal{S}\mathrm{hv}^{\text{ét}}_A$.*

Proof. The "if" direction follows immediately from the definitions. To prove the converse, suppose that $\mathfrak{X} = ((\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A)_{/U}, \mathcal{O}_{\operatorname{Spf} A}|_U)$ is equivalent to $\operatorname{Spf} B$ for some adic $\mathbb{E}_\infty$-ring B. Without loss of generality, we may assume that B is complete (Remark 8.1.2.4) so that the projection map $\mathfrak{X} \to \operatorname{Spf} A$ is determined by a morphism of adic $\mathbb{E}_\infty$-rings $f : A \to B$. Let $I \subseteq \pi_0 A$ and $J \subseteq \pi_0 B$ be ideals of definition, and set $A_0 = (\pi_0 A)/I$. Let U' denote the image of U under the equivalence of ∞-topoi $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A \simeq \mathcal{S}\mathrm{hv}^{\text{ét}}_{A_0}$ of Example 8.1.1.9; we wish to show that U' is affine. Set $\mathsf{U} = ((\mathcal{S}\mathrm{hv}^{\text{ét}}_{A_0})_{/U'}, \mathcal{O}_{\operatorname{Spét} A_0}|_{U'})$, so that U is a spectral

Deligne-Mumford stack which fits into a pullback diagram

$$\begin{array}{ccc} \mathsf{U} & \longrightarrow & \operatorname{Spf} B \\ \downarrow & & \downarrow \\ \operatorname{Sp\acute{e}t} A_0 & \longrightarrow & \operatorname{Spf} A \end{array}$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$. Set $B_0 = A_0 \otimes_A B$, and regard B_0 as an adic $\mathbb{E}_\infty$-ring by equipping $\pi_0 B_0$ with the J-adic topology. It follows from Proposition 8.1.5.2 that U and $\operatorname{Spf} B_0$ represent the same functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and are therefore equivalent as spectrally ringed ∞-topoi (Theorem 8.1.5.1). Invoking Example 8.1.3.5, we deduce that $U' \in (\mathcal{S}\mathrm{hv}^{\acute{e}t}_{A_0})$ is affine, as desired. □

We have the following analogue of Proposition 1.4.7.9:

Proposition 8.1.3.7. *Let $(\mathcal{X}, \mathcal{O})$ be an formal spectral Deligne-Mumford stack, and let $\mathcal{X}_0$ be the full subcategory of $\mathcal{X}$ spanned by the affine objects. Then $\mathcal{X}$ is generated by $\mathcal{X}_0$ under small colimits (in other words, $\mathcal{X}$ is the smallest full subcategory of itself which contains $\mathcal{X}_0$ and is closed under small colimits).*

Proof. Let $\mathcal{X}_1$ be the smallest full subcategory of $\mathcal{X}$ which contains $\mathcal{X}_0$ and is closed under small colimits. Fix an object $X \in \mathcal{X}$; we wish to show that $X \in \mathcal{X}_1$. Our assumption that $(\mathcal{X}, \mathcal{O})$ is a formal spectral Deligne-Mumford stack guarantees that we can choose an effective epimorphism $u : U \to X$, where each U is a coproduct of objects belonging to $\mathcal{X}_0$. Let $U_\bullet$ be the Čech nerve of u. Since $X \simeq |U_\bullet|$, it will suffice to show that each U_n belongs to $\mathcal{X}_1$. Choose a projection map $v : U_n \to U$. Since U is a coproduct of objects belonging to $\mathcal{X}_0$, we can write U_n as a coproduct of objects V which admit a map $V \to W$, for $W \in \mathcal{X}_0$. We may therefore replace $\mathcal{X}$ by $\mathcal{X}_{/W}$ and thereby reduce to the case where $(\mathcal{X}, \mathcal{O})$ is affine, in which case the desired result is obvious (Proposition 8.1.3.6). □

8.1.4 The Reduction of a Formal Spectral Deligne-Mumford Stack

Let $(X, \mathcal{O}_X)$ be a scheme. Then we can associate to X a reduced scheme X^{red} having the same underlying topological space, whose structure sheaf is given by $\mathcal{O}^{\mathrm{red}}_X = \mathcal{O}_X / \mathcal{I}$, where $\mathcal{I} \subseteq \mathcal{O}_X$ is the subsheaf of (locally) nilpotent sections of $\mathcal{O}_X$. The scheme X^{red} is then universal among reduced schemes equipped with a map to X. We now generalize this construction to the setting of formal spectral Deligne-Mumford stacks:

Definition 8.1.4.1. Let $f : \mathsf{X}_0 \to \mathfrak{X}$ be a morphism of formal spectral Deligne-Mumford stacks. We will say that f *exhibits* X_0 *as a reduction of* $\mathfrak{X}$ if the following pair of conditions is satisfied:

(a) The formal spectral Deligne-Mumford stack X_0 is a reduced spectral Deligne-Mumford stack.

(b) For every reduced spectral Deligne-Mumford stack Y, composition with f induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Y}, \mathsf{X}_0) \to \mathrm{Map}_{\mathrm{fSpDM}}(\mathsf{Y}, \mathfrak{X}).$$

Remark 8.1.4.2. It follows immediately from the definitions that if a formal spectral Deligne-Mumford stack $\mathfrak{X}$ admits a reduction X_0, then X_0 is uniquely determined (up to equivalence) and depends functorially on $\mathfrak{X}$. We will indicate this dependence by denoting X_0 by $\mathfrak{X}^{\mathrm{red}}$.

Remark 8.1.4.3. In the situation of Definition 8.1.4.1, it suffices to verify condition (b) in the special case where Y is affine.

We now show that every formal spectral Deligne-Mumford stack $\mathfrak{X}$ admit a reduction:

Proposition 8.1.4.4. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then there exists a morphism $f : \mathfrak{X}^{\mathrm{red}} \to \mathfrak{X}$ which exhibit $\mathfrak{X}^{\mathrm{red}}$ as a reduction of $\mathfrak{X}$. Moreover, the morphism f is an equivalence of the underlying ∞-topoi.*

Example 8.1.4.5. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack. Then $\mathsf{X}^{\mathrm{red}}$ is given by $(\mathcal{X}, \mathscr{O}_{\mathsf{X}}^{\mathrm{red}})$, where $\mathscr{O}_{\mathsf{X}}^{\mathrm{red}}$ denotes the quotient of $\pi_0 \mathscr{O}_{\mathsf{X}}$ by the subsheaf $\mathcal{I} \subseteq \pi_0 \mathscr{O}_{\mathsf{X}}$ of (locally) nilpotent sections of $\pi_0 \mathscr{O}_{\mathsf{X}}$.

We begin by treating the affine case:

Lemma 8.1.4.6. *Let A be an adic $\mathbb{E}_\infty$-ring with finitely generated ideal of definition $I \subseteq \pi_0(A)$, and set $R = (\pi_0 A)/I$. Then the canonical map $f : \mathrm{Sp\acute{e}t}(R^{\mathrm{red}}) \to \mathrm{Spf}(A)$ exhibits $\mathrm{Sp\acute{e}t}(R^{\mathrm{red}})$ as a reduction of the affine formal spectral Deligne-Mumford stack $\mathrm{Spf}(A)$.*

Proof. Since $\mathrm{Sp\acute{e}t}(R^{\mathrm{red}})$ is reduced, it will suffice to show that f satisfies condition (b) of Definition 8.1.4.1: that is, that the canonical map

$$\theta : \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Y}, \mathrm{Sp\acute{e}t}(R^{\mathrm{red}})) \to \mathrm{Map}_{\mathrm{fSpDM}}(\mathsf{Y}, \mathrm{Spf}(A))$$

is a homotopy equivalence for every reduced spectral Deligne-Mumford stack Y. Using Remark 8.1.4.3, we can reduce to the case $\mathsf{Y} = \mathrm{Sp\acute{e}t}(B)$, where B is a reduced commutative ring. In this case, we can identify θ with the composition

$$\mathrm{Map}_{\mathrm{CAlg}}(R^{\mathrm{red}}, B) \xrightarrow{\theta'} \mathrm{Map}_{\mathrm{CAlg}}(R, B) \xrightarrow{\theta''} \mathrm{Map}_{\mathrm{CAlg}_{\mathrm{ad}}^{\heartsuit}}(\pi_0 A, B) \xrightarrow{\theta'''} \mathrm{Map}_{\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cn}}}(A, B).$$

The map θ' is a homotopy equivalence since B is reduced and the map θ''' is a homotopy equivalence since B is discrete. To prove that θ'' is a homotopy equivalence, we must show that a ring homomorphism $\rho : \pi_0 A \to B$ which satisfies $\rho(I^n) = 0$ for $n \gg 0$ must annihilate the ideal I itself. This is clear: each element $x \in I$ satisfies $\rho(x)^n = \rho(x^n) \in \rho(I^n) = \{0\}$, so that $\rho(x) = 0$ by virtue of our assumption that B is reduced. □

Lemma 8.1.4.7. *Suppose we are given a pullback diagram of formal spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}_0 & \longrightarrow & \mathsf{Y}_0 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \mathfrak{X} & \xrightarrow{h} & \mathfrak{Y}, \end{array}$$

where h is étale. If g exhibits Y_0 as a reduction of $\mathfrak{Y}$, then f exhibits X_0 as a reduction of $\mathfrak{X}$.

Proof. Since h is étale, the morphism $\mathsf{X}_0 \to \mathsf{Y}_0$ is also étale. Using the assumption that Y_0 is a reduced spectral Deligne-Mumford stack, we deduce that X_0 is also a reduced spectral Deligne-Mumford stack. If Z is any reduced spectral Deligne-Mumford stack, we have a pullback diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Z}, \mathsf{X}_0) & \longrightarrow & \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Z}, \mathsf{Y}_0) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Z}, \mathfrak{X}) & \longrightarrow & \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Z}, \mathfrak{Y}). \end{array}$$

Our hypothesis guarantees that the right vertical map is a homotopy equivalence, so the left vertical map is a homotopy equivalence as well. □

Proof of Proposition 8.1.4.4. Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack. For each object $U \in \mathcal{X}$, let $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathfrak{X}}|_U)$. Let us say that U is *good* if the formal spectral Deligne-Mumford stack $\mathfrak{X}_U$ admits a reduction $\mathfrak{X}_U^{\mathrm{red}}$. It follows from Lemma 8.1.4.6 that every affine object of $\mathcal{X}$ is good. We claim that every object of $\mathcal{X}$ is good. By virtue of Proposition 8.1.3.7, it will suffice to show that the collection of good objects of $\mathcal{X}$ is closed under colimits. To prove this, suppose we are given a diagram $\{U_\alpha\}$ in $\mathcal{X}$, having colimit U. We then have a diagram of reduced spectral Deligne-Mumford stacks $\{\mathfrak{X}_{U_\alpha}^{\mathrm{red}}\}$, where the transition maps are étale (Lemma 8.1.4.7). This diagram therefore has a colimit $\mathfrak{X}_U^{\mathrm{red}}$ in the ∞-category of spectral Deligne-Mumford stacks. The universal property of the colimit yields a map $f : \mathfrak{X}_U^{\mathrm{red}} \to \mathfrak{X}_U$. We claim that f exhibits $\mathfrak{X}_U^{\mathrm{red}}$ as a reduction of $\mathfrak{X}_U$. It is immediate from the construction that $\mathfrak{X}_U^{\mathrm{red}}$ is reduced. It will therefore suffice to show that f satisfies condition (b) of Definition 8.1.4.1: that is, for every reduced spectral Deligne-Mumford stack Y and every morphism $g : \mathsf{Y} \to \mathfrak{X}_U$, the homotopy fiber $F = \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Y}, \mathfrak{X}_U^{\mathrm{red}} \times_{\mathrm{Map}_{\mathrm{fSpDM}}(\mathsf{Y}, \mathfrak{X}_U)} \{g\}$ is contractible. This assertion can be tested locally on Y; we may therefore assume without loss of generality that the map g factors through some $g_\alpha : \mathsf{Y} \to \mathfrak{X}_{U_\alpha}$. Note that the diagram of formal spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathfrak{X}^{\mathrm{red}}_{U_\alpha} & \longrightarrow & \mathfrak{X}^{\mathrm{red}}_{U} \\ \downarrow & & \downarrow \\ \mathfrak{X}_{U_\alpha} & \longrightarrow & \mathfrak{X}_U \end{array}$$

is a pullback square, so we can identify F with the homotopy fiber $\mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Y}, \mathfrak{X}^{\mathrm{red}}_{U_\alpha} \times_{\mathrm{Map}_{\mathrm{fSpDM}}(\mathsf{Y}, \mathfrak{X}_{U_\alpha})} \{g_\alpha\}$. The contractibility of F now follows from our assumption that $\mathfrak{X}^{\mathrm{red}}_{U_\alpha}$ is a reduction of $\mathfrak{X}_{U_\alpha}$. This completes the proof that every object of $\mathcal{X}$ is good. In particular, the final object of $\mathcal{X}$ is good, so that $\mathfrak{X}$ admits a reduction $\mathfrak{X}^{\mathrm{red}}$. To complete the proof, we must show that the natural map $\mathfrak{X}^{\mathrm{red}} \to \mathfrak{X}$ induces an equivalence at the level of underlying ∞-topoi. By virtue of Lemma 8.1.4.7, this assertion can be tested locally on $\mathfrak{X}$. We may therefore assume without loss of generality that $\mathfrak{X} = \mathrm{Spf}(A)$ for some adic $\mathbb{E}_\infty$-ring A. Let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition, and set $R = (\pi_0 A)/I$. Then we can identify the underlying ∞-topos of $\mathfrak{X}$ with the closed subtopos $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A \subseteq \mathcal{S}\mathrm{hv}^{\text{ét}}_A$ given by the vanishing locus of I, and Lemma 8.1.4.6 allows us to identify the underlying ∞-topos of $\mathfrak{X}^{\mathrm{red}}$ with $\mathcal{S}\mathrm{hv}^{\text{ét}}_{R^{\mathrm{red}}}$. The equivalence of these ∞-topoi follows from Proposition 3.1.4.1 (and its proof). □

8.1.5 The Functor of Points

Recall that if $\mathfrak{X}$ is a locally spectrally ringed ∞-topos, then $\mathfrak{X}$ represents a functor $h_{\mathfrak{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, given by the formula $h_{\mathfrak{X}}(R) = \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\, R, \mathfrak{X})$ (Definition 1.6.4.1). Our next goal is to prove the following:

Theorem 8.1.5.1. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then, for every connective $\mathbb{E}_\infty$-ring R, the space $h_{\mathfrak{X}}(R) = \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\, R, \mathfrak{X})$ is essentially small. Moreover, the construction $\mathfrak{X} \mapsto h_{\mathfrak{X}}$ determines a fully faithful embedding* $\mathrm{fSpDM} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$.

We will give the proof of Theorem 8.1.5.1 at the end of this section. Our first step is to analyze the case of an affine formal spectral Deligne-Mumford stack.

Proposition 8.1.5.2. *Let A be an adic $\mathbb{E}_\infty$-ring with a finitely generated ideal of definition $I \subseteq \pi_0 A$, let R be an arbitrary $\mathbb{E}_\infty$-ring, and let*

$$\theta : \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\, R, \mathrm{Spf}\, A) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\, R, \mathrm{Sp\acute{e}t}\, A) \simeq \mathrm{Map}_{\mathrm{CAlg}}(A, R)$$

by given by composition with the evident map $\iota : \mathrm{Spf}\, A \to \mathrm{Sp\acute{e}t}\, A$. Then θ induces a homotopy equivalence from $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\, R, \mathrm{Spf}\, A)$ to the summand of $\mathrm{Map}_{\mathrm{CAlg}}(A, R)$ spanned by those morphisms $\phi : A \to R$ which annihilate some power of the ideal I.

Proof. Fix a morphism of $\mathbb{E}_\infty$-rings $\phi : A \to R$ and let X_ϕ denote the homotopy fiber of θ over the point ϕ. Note that if ϕ does not annihilate a power of I, then the induced map of topological spaces $|\operatorname{Spec} R| \to |\operatorname{Spec} A|$ does not factor through the vanishing locus of I, so that X_ϕ is empty. We will complete the proof by showing that if ϕ *does* annihilate a power of I, then the space X_ϕ is contractible. Let $f : \operatorname{Spét} R \to \operatorname{Spét} A$ be the morphism of spectral Deligne-Mumford stacks determined by ϕ. Our assumption that ϕ annihilates some power of I guarantees that the underlying geometric morphism $f_* : \mathcal{S}\mathrm{hv}^{\text{ét}}_R \to \mathcal{S}\mathrm{hv}^{\text{ét}}_A$ factors through the closed subtopos $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A \subseteq \mathcal{S}\mathrm{hv}^{\text{ét}}_A$. Note that any lift of f to a map of spectrally ringed ∞-topoi $\overline{f} : \operatorname{Spét} R \to \operatorname{Spf} A$ is automatically local (Lemma 8.1.1.14). It follows that we can identify X_ϕ with the mapping space $\operatorname{Map}_{\operatorname{Fun}(\mathrm{CAlg}^{\text{ét}}_A, \mathrm{CAlg}_A)_{\mathscr{O}_{\operatorname{Spét} A}/}}(\mathscr{O}_{\operatorname{Spf} A}, f_* \mathscr{O}_{\operatorname{Spét} R})$. To show that this space is contractible, it will suffice to show that the direct image $f_* \mathscr{O}_{\operatorname{Spét} R}$ takes values in I-complete A-modules: that is, that the tensor product $R \otimes_A B$ is I-complete for any étale R-algebra B. This is clear, since I generates a nilpotent ideal in $\pi_0 R$. □

Definition 8.1.5.3. Let A be an adic $\mathbb{E}_\infty$-ring. We will say that A is *complete* if it is I-complete, where $I \subseteq \pi_0 A$ is an ideal of definition.

Corollary 8.1.5.4. *Let A and B be adic $\mathbb{E}_\infty$-rings. If B is complete, then the canonical map* $\operatorname{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}}(A, B) \to \operatorname{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\operatorname{Spf} B, \operatorname{Spf} A)$ *is a homotopy equivalence.*

Proof. Let $\{B_n\}_{n>0}$ be a tower of $\mathbb{E}_\infty$-algebras over B satisfying the requirements of Lemma 8.1.2.2. We then have a commutative diagram

$$\begin{array}{ccc} \operatorname{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}}(A, B) & \longrightarrow & \operatorname{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\operatorname{Spf} B, \operatorname{Spf} A) \\ \downarrow & & \downarrow \\ \varprojlim \operatorname{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}}(A, B_n) & \longrightarrow & \varprojlim \operatorname{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\operatorname{Spét} B_n, \operatorname{Spf} A). \end{array}$$

The left vertical map is a homotopy equivalence by virtue of our assumption that B is complete (see Lemma 8.1.2.3), the right vertical map is a homotopy equivalence by virtue of Proposition 8.1.2.1, and the bottom horizontal map is a homotopy equivalence by virtue of Proposition 8.1.5.2. □

Proof of Theorem 8.1.5.1. Let $\widehat{\mathcal{S}\mathrm{hv}}$ denote the full subcategory of $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ spanned by those functors which are sheaves with respect to the étale topology, and let $\mathcal{S}\mathrm{hv}$ denote the full subcategory spanned by those sheaves $\mathscr{F}$ such that $\mathscr{F}(R)$ is essentially small for each $R \in \mathrm{CAlg}^{\mathrm{cn}}$. Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack. For each $U \in \mathcal{X}$, set $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathfrak{X}}|_U)$. Then $\mathfrak{X}_U$ is also a formal spectral Deligne-Mumford stack (Proposition 8.1.3.3), so the structure sheaf $\mathscr{O}_{\mathfrak{X}|_U}$ is strictly Henselian (Proposition 8.1.1.13). It follows that the construction $U \mapsto h_{\mathfrak{X}_U}$ determines a functor $\mathcal{X} \to \widehat{\mathcal{S}\mathrm{hv}}$ which commutes with small colimits. Note that if $\mathfrak{X}_U$ is affine, then $h_{\mathfrak{X}_U}$ belongs to the subcategory $\mathcal{S}\mathrm{hv} \subseteq \widehat{\mathcal{S}\mathrm{hv}}$. Since $\mathcal{S}\mathrm{hv}$ is closed under small colimits in $\widehat{\mathcal{S}\mathrm{hv}}$, it follows from Proposition 8.1.3.7 that $h_{\mathfrak{X}} \in \mathcal{S}\mathrm{hv}$.

Now suppose that we are given another formal spectral Deligne-Mumford stack $\mathfrak{Y}$. We wish to show that the canonical map $\theta : \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}(\mathfrak{X}, \mathfrak{Y}) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}}(h_{\mathfrak{X}}, h_{\mathfrak{Y}})$ is a homotopy equivalence. In fact, we will show more generally that the canonical map $\theta_U : \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}(\mathfrak{X}_U, \mathfrak{Y}) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}}(h_{\mathfrak{X}_U}, h_{\mathfrak{Y}})$ is an equivalence for each $U \in \mathcal{X}$. Note that the collection of objects $U \in \mathcal{X}$ for which θ_U is a homotopy equivalence is closed under small colimits. Using Proposition 8.1.3.7, we can reduce to the case where $\mathfrak{X} = \mathrm{Spf}\, A$ is affine. Let $\{A_n\}_{n>0}$ be a tower of $\mathbb{E}_\infty$-algebras over A which satisfies the hypotheses of Lemma 8.1.2.2. We then have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}(\mathrm{Spf}\, A, \mathfrak{Y}) & \xrightarrow{\theta} & \mathrm{Map}_{\mathcal{S}\mathrm{hv}}(h_{\mathfrak{X}}, h_{\mathfrak{Y}}) \\
\downarrow & & \downarrow \\
\varprojlim \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\, A_n, \mathfrak{Y}) & \longrightarrow & \varprojlim \mathrm{Map}_{\mathcal{S}\mathrm{hv}}(h_{\mathrm{Sp\acute{e}t}\, A_n}, h_{\mathfrak{Y}}).
\end{array}$$

The vertical maps are homotopy equivalences by virtue of Propositions 8.1.5.2 and 8.1.2.1. We are therefore reduced to showing that the lower horizontal map is an equivalence, which follows from Yoneda's lemma. □

8.1.6 Example: Formal Completions

Motivated by Proposition 8.1.5.2, let us introduce the following definition:

Definition 8.1.6.1. Let X be a spectral Deligne-Mumford stack and let $K \subseteq |\mathsf{X}|$ be a cocompact closed subset. We will say that a morphism of formal spectral Deligne-Mumford stacks $i : \mathfrak{X} \to \mathsf{X}$ *exhibits* $\mathfrak{X}$ *as a formal completion of* X *along* K if the following universal property is satisfied: for every connective $\mathbb{E}_\infty$-ring R, composition with i induces a homotopy equivalence from $h_{\mathfrak{X}}(R)$ to the summand of $h_{\mathsf{X}}(R)$ spanned by those maps $f : \mathrm{Sp\acute{e}t}\, R \to |\mathsf{X}|$ for which the underlying map of topological spaces $|\mathrm{Spec}\, R| \to |\mathsf{X}|$ factors through K.

Remark 8.1.6.2. In the situation of Definition 8.1.6.1, the formal spectral Deligne-Mumford stack $\mathfrak{X}$ depends only on X and the closed subset $K \subseteq |\mathsf{X}|$. We will indicate this dependence by writing $\mathfrak{X} = \mathsf{X}^{\wedge}_K$.

Remark 8.1.6.3. Let X be a spectral Deligne-Mumford stack, let $K \subseteq |\mathsf{X}|$ be a cocompact closed subset, and let $\mathsf{X}^{\wedge}_K$ be the formal completion of X along K. Then, for every spectral Deligne-Mumford stack Y, the canonical map $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}(\mathsf{Y}, \mathsf{X}^{\wedge}_K) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}(\mathsf{Y}, \mathsf{X})$ induces a homotopy equivalence from $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}(\mathsf{Y}, \mathsf{X}^{\wedge}_K)$ to the summand of $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}(\mathsf{Y}, \mathsf{X})$ spanned by those maps $f : \mathsf{Y} \to \mathsf{X}$ for which the underlying map of topological spaces $|\mathsf{Y}| \to |\mathsf{X}|$ factors through K.

Example 8.1.6.4. Let A be a connective $\mathbb{E}_\infty$-ring and let $K \subseteq |\operatorname{Spec} A|$ be a cocompact closed subset. Then we can regard A as an adic $\mathbb{E}_\infty$-ring (by equipping $\pi_0 A$ with the I-adic topology, for some finitely generated ideal I with vanishing locus K). Using Proposition 8.1.5.2, we can identify the formal spectrum $\operatorname{Spf} A$ with the formal completion of $\operatorname{Spét} A$ along K.

Example 8.1.6.5. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathsf{X})$ be a spectral Deligne-Mumford stack and let $f : \mathfrak{X} = (\mathcal{X}', \mathcal{O}_\mathfrak{X}) \to \mathsf{X}$ be a morphism of formal spectral Deligne-Mumford stacks which exhibits $\mathfrak{X}$ as a formal completion of X along a cocompact closed subset $K \subseteq |\mathsf{X}|$. For every object $U \in \mathcal{X}$, the induced map $f_U : \mathfrak{X}_U = (\mathcal{X}'_{/f^*U}, \mathcal{O}_\mathfrak{X}|_{f^*U}) \to (\mathcal{X}_{/U}, \mathcal{O}_\mathsf{X}|_U) = \mathsf{X}_U$ exhibits $\mathfrak{X}_U$ as a formal completion of X_U along the inverse image of K.

Proposition 8.1.6.6. *Let X be a spectral Deligne-Mumford stack and $K \subseteq |\mathsf{X}|$ be a cocompact closed subset. Then there exists a map of formal Deligne-Mumford stacks $i : \mathfrak{X} \to \mathsf{X}$ which exhibits $\mathfrak{X}$ as a formal completion of X along K.*

Proof. Write $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathsf{X})$. For each object $U \in \mathcal{X}$, set $\mathsf{X}_U = (\mathcal{X}_{/U}, \mathcal{O}_\mathsf{X}|_U)$ and let $K_U \subseteq |\mathsf{X}_U|$ be the inverse image of K. Let us say that U is *good* if there exists a formal completion of X_U along K_U. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the full subcategory of $\mathcal{X}$ spanned by the good objects. Note that every affine object of $\mathcal{X}$ is good (Example 8.1.6.4). To complete the proof, it will suffice to show that $\mathcal{X}_0$ is closed under small colimits in $\mathcal{X}$ (Proposition 8.1.3.7). Let $\{U_\alpha\}_{\alpha \in I}$ be a small diagram in $\mathcal{X}_0$ having colimit $U \in \mathcal{X}$. The construction $(\alpha \in I) \mapsto (\mathsf{X}_{U_\alpha})^\wedge_{K_\alpha}$ determines a diagram of formal spectral Deligne-Mumford stacks in which the transition maps are étale (as morphisms of spectrally ringed ∞-topoi; see Example 8.1.6.5). Let $\mathfrak{X}_U$ denote the colimit of this diagram in $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$, so that we have an étale surjection $\amalg_\alpha (\mathsf{X}_{U_\alpha})^\wedge_{K_\alpha} \to \mathfrak{X}_U$ (see Proposition 21.4.6.4). It follows that $\mathfrak{X}_U$ is also a formal spectral Deligne-Mumford stack (Proposition 8.1.3.3). Moreover, the functor $h_{\mathfrak{X}_U}$ represented by $\mathfrak{X}_U$ can be identified with the colimit $\varinjlim h_{(\mathsf{X}_{U_\alpha})^\wedge_{K_\alpha}}$ in the ∞-category $\mathcal{S}\mathrm{hv} \subseteq \operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ of functors which are sheaves with respect to the étale topology, from which it is easy to see that $\mathfrak{X}_U$ is a formal completion of X_U along K_U. □

Remark 8.1.6.7. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathsf{X})$ be a spectral Deligne-Mumford stack, let $K \subseteq |\mathsf{X}|$ be a cocompact closed subset, and let $\mathsf{X}^\wedge_K$ be a formal completion of X along K. Then the underlying ∞-topos of X can be identified with the closed subtopos of $\mathcal{X}$ corresponding to K. To prove this, we can use Example 8.1.6.5 to reduce to the case where $\mathsf{X} = \operatorname{Spét} A$ is affine, in which case the desired result follows from Example 8.1.6.4 (and the construction of $\operatorname{Spf} A$).

8.1.7 Fiber Products

We close this section by establishing an analogue of Proposition 1.4.11.1:

Proposition 8.1.7.1. *The ∞-category* fSpDM *of formal spectral Deligne-Mumford stacks admits finite limits.*

Warning 8.1.7.2. The inclusion functor $\mathrm{fSpDM} \hookrightarrow \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ does not preserve finite limits: the formation of fiber products in the setting of formal spectral Deligne-Mumford stacks requires the formation of "completed" tensor products at the level of $\mathbb{E}_\infty$-rings.

We begin by establishing a simple special case of Proposition 8.1.7.1. First, we observe that the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}$ of adic $\mathbb{E}_\infty$-rings admits pushouts: given a diagram of adic $\mathbb{E}_\infty$-rings $A \leftarrow R \rightarrow B$ (with finitely generated ideals of definition $I \subseteq \pi_0 A$ and $J \subseteq \pi_0 B$), the tensor product $A \otimes_R B$ inherits the structure of an adic $\mathbb{E}_\infty$-ring (where we equip $\pi_0(A \otimes_R B)$ with the K-adic topology, where $K \subseteq \pi_0(A \otimes_R B)$ is the ideal generated by the images of I and J).

Lemma 8.1.7.3. *The construction $A \mapsto \mathrm{Spf}\, A$ carries pushouts diagrams of adic $\mathbb{E}_\infty$-rings to pullback diagrams of formal spectral Deligne-Mumford stacks.*

Proof. Suppose we are given a pushout diagram of adic $\mathbb{E}_\infty$-rings σ :

$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \otimes_R B. \end{array}$$

We wish to show that the associated diagram of formal spectra τ :

$$\begin{array}{ccc} \mathrm{Spf}\, R & \longleftarrow & \mathrm{Spf}\, A \\ \uparrow & & \uparrow \\ \mathrm{Spf}\, B & \longleftarrow & \mathrm{Spf}\, A \otimes_R B \end{array}$$

is a pullback square. By virtue of Theorem 8.1.5.1, it will suffice to show that τ is a pullback square in the larger ∞-category of functors $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Using Proposition 8.1.5.2, we are reduced to showing that for any connective $\mathbb{E}_\infty$-ring C (which we regard as an adic $\mathbb{E}_\infty$-ring by equipping $\pi_0 C$ with the discrete topology), the diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}}(R, C) & \longleftarrow & \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}}(A, C) \\ \uparrow & & \uparrow \\ \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}}(B, C) & \longleftarrow & \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}}(A \otimes_R B, C) \end{array}$$

is a pullback square, which follows immediately from our assumption that σ is a pullback square. □

Proof of Proposition 8.1.7.1. It is clear that the ∞-category fSpDM admits a final object (given by Spét S, where S is the sphere spectrum). We will complete the proof by showing that every diagram of formal spectral Deligne-Mumford stacks $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xleftarrow{g} \mathfrak{Z}$ admits a pullback in fSpDM. Write $\mathfrak{Y} = (\mathcal{Y}, \mathscr{O}_{\mathfrak{Y}})$. For each object $U \in \mathcal{Y}$, set $\mathfrak{Y}_U = (\mathcal{Y}_{/U}, \mathscr{O}_{\mathfrak{Y}}|_U)$, $\mathfrak{X}_U = (\mathcal{X}_{/f^*U}, \mathscr{O}_{\mathfrak{X}}|_{f^*U})$, and $\mathfrak{Z}_U = (\mathcal{Z}_{/g^*U}, \mathscr{O}_{\mathfrak{Z}}|_{g^*U})$. Let us say that $U \in \mathcal{Y}$ is *good* if the fiber product $\mathfrak{X}_U \times_{\mathfrak{Y}_U} \mathfrak{Z}_U$ is representable by a formal spectral Deligne-Mumford stack $\mathfrak{W}_U$. Note that if $\{U_\alpha\}$ is a small diagram of good objects of $\mathcal{Y}$ having colimit U, then $\{\mathfrak{W}_{U_\alpha}\}$ is a diagram of formal spectral Deligne-Mumford stacks with étale transition maps. This diagram has some colimit $\mathfrak{W}_U \in \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ (Proposition 21.4.6.4). The evident map $\amalg_\alpha \mathfrak{W}_{U_\alpha} \to \mathfrak{W}_U$ is an étale surjection, so Proposition 8.1.3.3 guarantees that $\mathfrak{W}_U$ is also a formal spectral Deligne-Mumford stack. If we let $\mathcal{S}\mathrm{hv} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ denote the ∞-category of functors which are sheaves with respect to the étale topology, then $h_{\mathfrak{W}_U}$ is the colimit of the diagram $\{h_{\mathfrak{W}_{U_\alpha}}\}$ in the ∞-category $\mathcal{S}\mathrm{hv}$. Applying the same argument to $h_{\mathfrak{X}_U}$, $h_{\mathfrak{Y}_U}$, and $h_{\mathfrak{Z}_U}$ (and using the fact that colimits are universal in the ∞-category $\mathcal{S}\mathrm{hv}$) we conclude that the diagram of functors

$$\begin{array}{ccc} h_{\mathfrak{W}_U} & \longrightarrow & h_{\mathfrak{X}_U} \\ \downarrow & & \downarrow \\ h_{\mathfrak{Z}_U} & \longrightarrow & h_{\mathfrak{Y}_U} \end{array}$$

is a pullback square. Theorem 8.1.5.1 implies we can identify $\mathfrak{W}_U$ with a fiber product $\mathfrak{X}_U \times_{\mathfrak{Y}_U} \mathfrak{Z}_U$ in the ∞-category fSpDM, so that U is good.

The preceding argument shows that the collection of good objects of $\mathcal{Y}$ is closed under small colimits. By virtue of Proposition 8.1.3.7, to show that all objects of $\mathcal{Y}$ are good, it will suffice to show that all affine objects of $\mathcal{Y}$ are good. We are therefore reduced to proving Proposition 8.1.7.1 in the special case where $\mathfrak{Y} \simeq \mathrm{Spf}\, R$ is affine. Using a simliar argument, we may assume that $\mathfrak{X} \simeq \mathrm{Spf}\, A$ and $\mathfrak{Z} \simeq \mathrm{Spf}\, B$ are affine. Without loss of generality, we may also assume that A and B are complete (Remark 8.1.2.4), so that f and g are obtained from morphisms of adic $\mathbb{E}_\infty$-rings $A \leftarrow R \rightarrow B$ (Corollary 8.1.5.4). In this case, the desired result follows from Lemma 8.1.7.3. □

8.2 Quasi-Coherent Sheaves on Formal Stacks

Let A be an adic $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition. In Chapter 7, we saw that the stable ∞-category Mod_A admits a pair of semi-orthogonal decompositions

$$(\mathrm{Mod}_A^{\mathrm{Nil}(I)}, \mathrm{Mod}_A^{\mathrm{Loc}(I)}) \qquad (\mathrm{Mod}_A^{\mathrm{Loc}(I)}, \mathrm{Mod}_A^{\mathrm{Cpl}(I)}),$$

where $\mathrm{Mod}_A^{\mathrm{Nil}(I)}, \mathrm{Mod}_A^{\mathrm{Loc}(I)}, \mathrm{Mod}_A^{\mathrm{Cpl}(I)} \subseteq \mathrm{Mod}_A$ denote the full subcategories spanned by the I-nilpotent, I-local, and I-complete A-modules, respectively. In this section, we will develop an analogous picture in the setting of sheaves on a formal spectral Deligne-Mumford stack $\mathfrak{X}$. Our principal results can be summarized as follows:

- In §8.2.3, we introduce the notion of a *weakly quasi-coherent sheaf* on a formal spectral Deligne-Mumford stack $\mathfrak{X}$ (Definition 8.2.3.1). The weakly quasi-coherent sheaves span a full subcategory $\mathrm{WCoh}(\mathfrak{X}) \subseteq \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ with good closure properties (for example, stability under small colimits and tensor products).

- The stable ∞-category $\mathrm{WCoh}(\mathfrak{X})$ admits a pair of semi-orthogonal decompositions $(\mathrm{NilCoh}(\mathfrak{X}), \mathrm{WCoh}^{\circ}(\mathfrak{X}))$ and $(\mathrm{WCoh}^{\circ}(\mathfrak{X}), \mathrm{QCoh}(\mathfrak{X}))$ (Corollaries 8.2.3.7 and 8.2.4.10). We refer to $\mathrm{NilCoh}(\mathfrak{X})$ and $\mathrm{QCoh}(\mathfrak{X})$ as the ∞-categories of *nilcoherent* and *quasi-coherent* sheaves on $\mathfrak{X}$, respectively. Moreover, the ∞-categories $\mathrm{NilCoh}(\mathfrak{X})$ and $\mathrm{QCoh}(\mathfrak{X})$ are canonically equivalent to one another (Corollary 8.2.4.12), though they are usually distinct as subcategories of $\mathrm{WCoh}(\mathfrak{X}) \subseteq \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$.

- In the special case where $\mathfrak{X} = \operatorname{Spf} A$ for some adic $\mathbb{E}_{\infty}$-ring A with finitely generated ideal of definition $I \subseteq \pi_0 A$, passage to global sections induces equivalences of ∞-categories

$$\Gamma(\mathfrak{X}; \bullet) : \mathrm{NilCoh}(\mathfrak{X}) \to \mathrm{Mod}_A^{\mathrm{Nil}(I)} \qquad \Gamma(\mathfrak{X}; \bullet) : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Mod}_A^{\mathrm{Cpl}(I)}$$

  (see Proposition 8.2.1.3 and Corollary 8.2.4.15).

- In the special case where $\mathfrak{X}$ is an ordinary spectral Deligne-Mumford stack, we have $\mathrm{NilCoh}(\mathfrak{X}) = \mathrm{WCoh}(\mathfrak{X}) = \mathrm{QCoh}(\mathfrak{X})$ (and all three coincide with the full subcategory $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ spanned by those sheaves which are quasi-coherent in the sense of Definition 2.2.2.1.

Warning 8.2.0.4. In the special case where $\mathfrak{X} = \operatorname{Spf} A$ is affine, the ∞-categories $\mathrm{WCoh}(\mathfrak{X})$ and $\mathrm{WCoh}^{\circ}(\mathfrak{X})$ are typically much larger than Mod_A and $\mathrm{Mod}_A^{\mathrm{Loc}(I)}$, respectively: see Proposition 8.3.2.2.

8.2.1 Nilcoherent Sheaves

We begin by globalizing the notion of I-nilpotent module over an adic $\mathbb{E}_{\infty}$-ring A.

Definition 8.2.1.1. Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. We will say that $\mathscr{F}$ is *nilcoherent* if it satisfies the following pair of conditions:

(*a*) For each affine object $U \in \mathcal{X}$, the spectrum $\mathscr{F}(U)$ is I-nilpotent when regarded as a module over $\mathscr{O}_{\mathfrak{X}}(U)$, where I is an ideal of definition of $\mathscr{O}_{\mathfrak{X}}(U)$.

(*b*) For every morphism $U \to V$ between affine objects of $\mathcal{X}$, the induced map $\mathscr{O}_{\mathfrak{X}}(U) \otimes_{\mathscr{O}_{\mathfrak{X}}(V)} \mathscr{F}(V) \to \mathscr{F}(U)$ is an equivalence.

We let $\mathrm{NilCoh}(\mathfrak{X})$ denote the full subcategory of $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ spanned by the nilcoherent objects.

Example 8.2.1.2. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack. Then every object $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathsf{X}}}$ satisfies condition (a) of Definition 8.2.1.1 (see Example 8.1.3.5), and condition (b) is equivalent to the requirement that $\mathscr{F}$ is quasi-coherent (Proposition 2.2.4.3). We therefore have $\mathrm{NilCoh}(\mathsf{X}) = \mathrm{QCoh}(\mathsf{X})$.

We begin by analyzing Definition 8.2.1.1 in the affine case.

Proposition 8.2.1.3. *Let A be an adic $\mathbb{E}_\infty$-ring with a finitely generated ideal of definition $I \subseteq \pi_0 A$. Then the global sections functor $\Gamma(\mathrm{Spf}\, A; \bullet)$ induces an equivalence of ∞-categories $\Gamma : \mathrm{NilCoh}(\mathrm{Spf}\, A) \to \mathrm{Mod}_A^{I-nil}$.*

Proof. We begin by explicitly constructing a homotopy inverse to the functor Γ. Let us identify $\mathscr{O}_{\mathrm{Spf}\, A}$ with the commutative algebra object of $\mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathrm{Sp})$ given by $B \mapsto B_I^\wedge$. For every A-module M, we let $\widetilde{M}$ denote the $\mathscr{O}_{\mathrm{Spf}\, A}$-module object of $\mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathrm{Sp})$ given by $B \mapsto B_I^\wedge \otimes_A M$.

Note that if M is I-nilpotent, then we can identify $\widetilde{M}$ with the functor $B \mapsto B \otimes_A M$: that is, with the quasi-coherent sheaf on $\mathrm{Spét}\, A$ given by the construction of §2.2.1. In this case, $\widetilde{M}$ is a sheaf for the étale topology on $\mathrm{CAlg}_A^{\text{ét}}$, which vanishes when I generates the unit ideal of $\pi_0 B$ (by virtue of our assumption that M is I-nilpotent). We can therefore regard $\widetilde{M}$ as a $\mathscr{O}_{\mathrm{Spf}\, A}$-module object of the ∞-category $\mathrm{Sp}(\mathcal{S}\mathrm{hv}_A^{\mathrm{ad}})$. Using the description of affine objects of $\mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}$ supplied by Proposition 8.1.3.6, we immediately deduce that $\widetilde{M}$ is nilcoherent. Consequently, we can regard the construction $M \mapsto \widetilde{M}$ as a functor $F : \mathrm{Mod}_A^{I-nil} \to \mathrm{NilCoh}(\mathrm{Spf}\, A)$. It follows immediately from the definitions that F is left adjoint to the global sections functor Γ, and that the unit map $M \to \Gamma(\mathrm{Spf}\, A; F(M))$ is equivalence for $M \in \mathrm{Mod}_A^{I-nil}$. To complete the proof, it suffices to observe that the functor Γ is conservative, which is an immediate consequence of condition (b) of Definition 8.2.1.1. □

Remark 8.2.1.4. In the situation of Proposition 8.2.1.3, the functor F is t-exact when viewed as a functor from Mod_A^{I-nil} to $\mathrm{Mod}_{\mathscr{O}_{\mathrm{Spf}\, A}}$. It follows that the subcategory $\mathrm{NilCoh}(\mathrm{Spf}\, A) \subseteq \mathrm{Mod}_{\mathscr{O}_{\mathrm{Spf}\, A}}$ is closed under the formation of truncations. In particular, the stable ∞-category $\mathrm{NilCoh}(\mathrm{Spf}\, A)$ inherits a t-structure from $\mathrm{Mod}_{\mathscr{O}_{\mathrm{Spf}\, A}}$, and the equivalence $\mathrm{NilCoh}(\mathrm{Spf}\, A) \simeq \mathrm{Mod}_A^{I-nil}$ is t-exact.

The condition of nilcoherence can be tested locally:

Proposition 8.2.1.5. *Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. For each object $U \in \mathcal{X}$, set $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathfrak{X}}|_U)$. Then:*

- (a) *The sheaf $\mathscr{F}$ is nilcoherent if and only if $\mathscr{F}|_U \, \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}|_U}$ is nilcoherent for each affine object $U \in \mathcal{X}$.*

(b) *If $\mathscr{F}$ is nilcoherent, then $\mathscr{F}|_U \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}|_U}$ is nilcoherent for every object $U \in \mathcal{X}$.*

(c) *If there exists a covering of $\mathcal{X}$ by objects $\{U_\alpha\}$ for which each restriction $\mathscr{F}|_{U_\alpha} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}|_{U_\alpha}}$ is nilcoherent, then $\mathscr{F}$ is nilcoherent.*

Proof. Assertion (a) follows immediately from the definitions, and (b) is an immediate consequence of (a). To prove (c), we can use (a) to reduce to the case where $\mathfrak{X}$ has the form $\mathrm{Spf}\, A$, for some adic $\mathbb{E}_\infty$-ring A. Let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition and let $\{U_\alpha\}$ be a covering of $\mathcal{X}$ for which each restriction $\mathscr{F}|_{U_\alpha}$ is nilcoherent. Without loss of generality, we may assume that each U_α is affine, corresponding to some étale A-algebra B_α. Let $i : \mathrm{Spf}\, A \to \mathrm{Spét}\, A$ be the canonical map and let $i_* \mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathrm{Spét}\, A}}$ be the direct image of $\mathscr{F}$. Using Proposition 8.2.1.3, we deduce that $(i_* \mathscr{F})|_{\mathrm{Spét}\, B_\alpha}$ is quasi-coherent and supported on the vanishing locus of $I(\pi_0 B)$. Moreover, it follows immediately from the definition that $(i_* \mathscr{F})|_{\mathsf{V}}$ vanishes, where V is the open substack of $\mathrm{Spét}\, A$ complementary to the vanishing locus of I. Since $\mathrm{Spét}\, A$ is covered by $\{\mathrm{Spét}\, B_\alpha\}$ and V, it follows that $i_* \mathscr{F}$ is quasi-coherent and supported on the vanishing locus of I: that is, it is the quasi-coherent sheaf associated to some I-nilpotent A-module M. It follows that $\mathscr{F} \simeq \widetilde{M} \in \mathrm{NilCoh}(\mathrm{Spf}\, A)$, where $\widetilde{M}$ is defined as in the proof of Proposition 8.2.1.3. □

Corollary 8.2.1.6. *Let X be a spectral Deligne-Mumford stack, let $K \subseteq |\mathsf{X}|$ be a cocompact closed subset, and let $i : \mathfrak{X} \to \mathsf{X}$ be a morphism of formal spectral Deligne-Mumford stacks which exhibits $\mathfrak{X}$ as a formal completion of X along K. Then the direct image functor i_* induces an equivalence of ∞-categories $\mathrm{NilCoh}(\mathfrak{X}) \to \mathrm{QCoh}_K(\mathsf{X})$, where $\mathrm{QCoh}_K(\mathsf{X})$ is the ∞-category of quasi-coherent sheaves on X which are supported on K.*

Proof. By virtue of Proposition 8.2.1.5, we can work locally on X and thereby reduce to the case where X is affine. In this case, the desired result is a reformulation of Proposition 8.2.1.3. □

Proposition 8.2.1.7. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F}, \mathscr{G} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. If $\mathscr{F}$ and $\mathscr{G}$ are nilcoherent, then the tensor product $\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}$ is nilcoherent.*

Proof. Using Proposition 8.2.1.5, we can reduce to the case where $\mathfrak{X}$ is affine. Write $\mathfrak{X} \simeq \mathrm{Spf}\, A$ for some adic $\mathbb{E}_\infty$-ring A with finitely generated ideal of definition $I \subseteq \pi_0 A$. The construction $M \mapsto \widetilde{M}$ appearing in the proof of Proposition 8.2.1.3 determines a nonunital symmetric monoidal functor $\mathrm{Mod}_A^{\mathrm{Nil}(I)} \to \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. Since this functor is fully faithful, its essential image is closed under tensor products. □

We now establish a generalization of Proposition 2.2.4.1:

Proposition 8.2.1.8. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then:*

(1) *The ∞-category $\mathrm{NilCoh}(\mathfrak{X})$ is closed under small colimits in $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$.*

(2) *The* ∞*-category* $\mathrm{NilCoh}(\mathfrak{X})$ *is stable.*

(3) *The* ∞*-category* $\mathrm{NilCoh}(\mathfrak{X})$ *is presentable.*

Proof. To prove (1), we can use Proposition 8.2.1.5 to reduce to the case where $\mathfrak{X} = \operatorname{Spf} A$ is affine. In this case, the proof of Proposition 8.2.1.3 shows that $\mathrm{NilCoh}(\mathfrak{X})$ can be identified with the essential image of a fully faithful functor $F : \mathrm{Mod}_A^{\mathrm{Nil}(I)} \to \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ which preserves small colimits, and is therefore closed under small colimits.

Write $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$. For each object $U \in \mathcal{X}$, set $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathfrak{X}}|_U)$. Using Proposition 8.2.1.5, we see that the construction $U \mapsto \mathrm{NilCoh}(\mathsf{X}_U)$ determines a functor $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which carries colimits in $\mathcal{X}$ to limits in $\widehat{\mathcal{C}\mathrm{at}}_\infty$. Using (1), we see that each of the ∞-categories $\mathrm{NilCoh}(\mathsf{X}_U)$ admits small colimits and that each of the transition maps $\mathrm{NilCoh}(\mathfrak{X}_U) \to \mathrm{NilCoh}(\mathfrak{X}_V)$ preserves small colimits. Using Proposition HTT.5.5.3.13 and Theorem HA.**??**, we see that the collection of those objects $U \in \mathcal{X}$ for which $\mathrm{NilCoh}(\mathfrak{X}_U)$ is stable and presentable is closed under small colimits in $\mathcal{X}$. We may therefore use Proposition 8.1.3.7 to reduce the proofs of (2) and (3) to the case where $\mathfrak{X}$ is affine, in which case the desired result follows from Proposition 8.2.1.3. □

8.2.2 Approximate Units for $\mathrm{NilCoh}(\mathfrak{X})$

Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. The structure sheaf $\mathscr{O}_{\mathfrak{X}}$ is not necessarily nilcoherent: in fact, it is nilcoherent if and only if $\mathfrak{X}$ is a spectral Deligne-Mumford stack. However, it can be closely approximated by nilcoherent sheaves.

Definition 8.2.2.1. Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack. A *approximate unit* for $\mathrm{NilCoh}(\mathfrak{X})$ is a pair $(\mathscr{O}_{\mathfrak{X}}^{\approx}, \alpha)$ where $\mathscr{O}_{\mathfrak{X}}^{\approx}$ is a nilcoherent sheaf on $\mathfrak{X}$ and $\alpha : \mathscr{O}_{\mathfrak{X}}^{\approx} \to \mathscr{O}_{\mathfrak{X}}$ is a morphism with the following property:

($*$) For every object $\mathscr{F} \in \mathrm{NilCoh}(\mathfrak{X})$, the map α induces an equivalence

$$\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}}^{\approx} \xrightarrow{\mathrm{id} \otimes \alpha} \mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}} \simeq \mathscr{F}.$$

Remark 8.2.2.2. In the situation of Definition 8.2.2.1, we will often abuse terminology by simply referring to the object $\mathscr{O}_{\mathfrak{X}}^{\approx}$ as an *approximate unit* for $\mathrm{NilCoh}(\mathfrak{X})$, or say that α *exhibits* $\mathscr{O}_{\mathfrak{X}}^{\approx}$ *as an approximate unit for* $\mathrm{NilCoh}(\mathfrak{X})$.

Example 8.2.2.3. Let A be an adic $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 A$ be an ideal of definition. Then we can choose a fiber sequence of A-modules $\Gamma_I(A) \xrightarrow{u} A \to L_I(A)$ where $\Gamma_I(A)$ is I-nilpotent and $L_I(A)$ is I-local. For each A-module M, let $\widetilde{M} \in \mathrm{Mod}_{\mathscr{O}_{\operatorname{Spf} A}}$ be as in the proof of Proposition 8.2.1.3, and set $\mathscr{O}_{\operatorname{Spf} A}^{\approx} = \widetilde{\Gamma_I(A)}$. Then u induces a map

$$\alpha : \mathscr{O}_{\operatorname{Spf} A}^{\approx} = \widetilde{\Gamma_I(A)} \to \widetilde{A} = \mathscr{O}_{\operatorname{Spf} A}.$$

For any A-module M, we have a canonical fiber sequence $\Gamma_I(A) \otimes_A M \xrightarrow{u_M} A \otimes_A M \to L_I(A) \otimes_A M$. If M is I-nilpotent, then the third term vanishes, so that u_M induces an equivalence $\Gamma_I(A) \otimes_A M \to M$. It follows that α induces an equivalence $\mathscr{O}^{\approx}_{\operatorname{Spf} A} \otimes_{\mathscr{O}_{\operatorname{Spf} A}} \widetilde{M} \to \widetilde{M}$, so that $(\mathscr{O}^{\approx}_{\operatorname{Spf} A}, \alpha)$ is an approximate unit for $\operatorname{NilCoh}(\operatorname{Spf} A)$.

Example 8.2.2.4. Let X be a spectral Deligne-Mumford stack. Then a morphism $\alpha : \mathscr{O}^{\approx}_{\mathsf{X}} \to \mathscr{O}_{\mathsf{X}}$ is an approximate unit for $\operatorname{NilCoh}(\mathsf{X}) = \operatorname{QCoh}(\mathsf{X})$ if and only if α is an equivalence.

We now prove show that approximate units always exist and are essentially unique:

Proposition 8.2.2.5. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack and let $\mathcal{C}$ denote the full subcategory of $\operatorname{NilCoh}(\mathfrak{X}) \times_{\operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}}} (\operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}})_{/\mathscr{O}_{\mathfrak{X}}}$ spanned by those objects which are approximate units for $\operatorname{NilCoh}(\mathfrak{X})$. Then $\mathcal{C}$ is a contractible Kan complex. In particular, there exists an approximate unit for $\operatorname{NilCoh}(\mathfrak{X})$.*

Proof. Assume first that $\mathcal{C}$ is nonempty: that is, there exists an approximate unit $\alpha_0 : \mathscr{O}^{\approx}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}$. For every object $\mathscr{F} \in \mathcal{C}$, we have a commutative diagram $\sigma_{\mathscr{F}}$:

$$\begin{array}{ccc} \mathscr{O}^{\approx}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F} & \longrightarrow & \mathscr{O}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F} \\ \downarrow & & \downarrow \\ \mathscr{O}^{\approx}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}} & \longrightarrow & \mathscr{O}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathsf{X}}} \mathscr{O}_{\mathfrak{X}}, \end{array}$$

where the left vertical map and upper horizontal map are equivalences. The diagram $\sigma_{\mathscr{F}}$ then determines an equivalence of $\mathscr{O}^{\approx}_{\mathfrak{X}}$ with $\mathscr{F}$ in the ∞-category $(\operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}/}})_{/\mathscr{O}_{\mathfrak{X}}}$. This equivalence depends functorially on $\mathscr{F}$: that is, the construction $\sigma_{\mathscr{F}}$ determines an equivalence from the identity functor $\operatorname{id}_{\mathcal{C}}$ to the constant functor with the value $\mathscr{O}^{\approx}_{\mathfrak{X}}$, so that $\mathcal{C}$ is a contractible Kan complex as desired.

We now treat the general case. Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack. For each object $U \in \mathcal{X}$, set $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathfrak{X}}|_U)$. Let us say that a morphism $\alpha : \mathscr{O}^{\approx} \to \mathscr{O}_{\mathfrak{X}}|_U$ in $\operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}|_U}$ is a *universal approximate unit* if, for every map $V \to U$ in $\mathcal{X}$, the induced map $\mathscr{O}^{\approx}|_V \to \mathscr{O}_{\mathfrak{X}}|_V$ is an approximate unit for $\operatorname{NilCoh}(\mathfrak{X}_V)$. Let $\mathcal{C}_U$ denote the full subcategory of $(\operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}|_U})_{/\mathscr{O}_{\mathfrak{X}}|_U}$ spanned by the universal approximate units. Then the construction $U \mapsto \mathcal{C}_U$ carries colimits in $\mathcal{X}$ to limits in $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$, and Lemma **??** guarantees that $\mathcal{C}_U$ is either empty or a contractible Kan complex for each $U \in \mathcal{X}$. It follows that the collection of objects $U \in \mathcal{X}$ for which $\mathcal{C}_U$ is nonempty is closed under small colimits. By virtue of Proposition 8.1.3.7, it will suffice to show that $\mathcal{C}_U$ is nonempty in the special case where U is affine. This follows from Example 8.2.2.3 (note that if A is an adic $\mathbb{E}_{\infty}$-ring with finitely generated ideal of definition I, then the approximate unit $\Gamma_I(A)$ of Example 8.2.2.3 is a universal approximate unit, since $\Gamma_I(A) \otimes_A B \simeq \Gamma_{I\pi_0 B}(B)$ for any $\mathbb{E}_{\infty}$-algebra B over A). □

Remark 8.2.2.6. It follows from the proof of Proposition 8.2.2.5 that every approximate unit for $\mathfrak{X}$ is a universal approximate unit: that is, it remains an approximate unit after pullback along any étale morphism.

8.2.3 Weakly Quasi-Coherent Sheaves

We now introduce an enlargement of the ∞-category of nilcoherent sheaves.

Definition 8.2.3.1. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. We will say that a sheaf $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ is *weakly quasi-coherent* if $\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}}^{\approx}$ is nilcoherent. Here $\mathscr{O}_{\mathfrak{X}}^{\approx}$ denotes an approximate unit for $\mathrm{NilCoh}(\mathfrak{X})$. We let $\mathrm{WCoh}(\mathfrak{X})$ denote the full subcategory of $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ spanned by the weakly quasi-coherent sheaves.

Example 8.2.3.2. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. If $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ is nilcoherent, then it is weakly quasi-coherent (in this case, the tensor product $\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}}^{\approx}$ is equivalent to $\mathscr{F}$).

Example 8.2.3.3. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the structure sheaf $\mathscr{O}_{\mathfrak{X}}$ is weakly quasi-coherent (the nilcoherence of the tensor product $\mathscr{O}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}}^{\approx} \simeq \mathscr{O}_{\mathfrak{X}^{\approx}}$ is part of the definition of approximate unit).

Example 8.2.3.4. Let X be a spectral Deligne-Mumford stack. Then a sheaf $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathsf{X}}}$ is weakly quasi-coherent if and only if it is quasi-coherent (combine Examples 8.2.1.2 and 8.2.2.4).

Remark 8.2.3.5. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the ∞-category $\mathrm{WCoh}(\mathfrak{X})$ is closed under small colimits and desuspensions in $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. In particular, it is a stable ∞-category. It is also a presentable ∞-category: this follows from Propositions 8.2.1.8 and HTT.5.5.3.13 .

Proposition 8.2.3.6. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the inclusion* $\mathrm{NilCoh}(\mathfrak{X}) \hookrightarrow \mathrm{WCoh}(\mathfrak{X})$ *admits a right adjoint, given by the construction* $\mathscr{F} \mapsto \mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F}$.

Proof. Let $\alpha : \mathscr{O}_{\mathfrak{X}}^{\approx} \to \mathscr{O}_{\mathfrak{X}}$ exhibit $\mathscr{O}_{\mathfrak{X}}^{\approx}$ as an approximate unit for $\mathscr{O}_{\mathfrak{X}}$ and let $L : \mathrm{WCoh}(\mathfrak{X}) \to \mathrm{NilCoh}(\mathfrak{X}) \subseteq \mathrm{WCoh}(\mathfrak{X})$ be the functor given by tensor product with $\mathscr{O}_{\mathfrak{X}}^{\approx}$. Then α induces a natural transformation of functors $u : L \to \mathrm{id}$. Note that the essential image of L coincides with $\mathrm{NilCoh}(\mathfrak{X})$ (since L is equivalent to the identity when restricted to $\mathrm{NilCoh}(\mathfrak{X})$). By virtue of Proposition HTT.5.2.7.4 , it will suffice to prove the following:

(a) For each object $\mathscr{F} \in \mathrm{WCoh}(\mathfrak{X})$, the map $u(L\mathscr{F}) : L(L\mathscr{F}) \to L\mathscr{F}$ is an equivalence.

(b) For each object $\mathscr{F} \in \mathrm{WCoh}(\mathfrak{X})$, the map $L(u(\mathscr{F})) : L(L\mathscr{F}) \to L\mathscr{F}$ is an equivalence.

Assertion (a) follows from the definition of an approximate unit (since $L\mathscr{F} \in \mathrm{NilCoh}(\mathfrak{X})$). Assertion (b) follows from (a) together with the observation that $L(u\mathscr{F}))$ is equivalent to the composition of $u(L\mathscr{F})$ with the automorphism of

$$L(L\mathscr{F}) \simeq \mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F}$$

given by permuting the two factors of $\mathscr{O}_{\mathfrak{X}}^{\approx}$. □

Corollary 8.2.3.7. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the ∞-category* $\mathrm{WCoh}(\mathfrak{X})$ *admits a semi-orthogonal decomposition* $(\mathrm{NilCoh}(\mathfrak{X}), \mathrm{WCoh}^{\circ}(\mathfrak{X}))$*, where* $\mathrm{WCoh}^{\circ}(\mathfrak{X})$ *is the full subcategory of* $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ *spanned by those objects $\mathscr{F}$ such that $\mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F}$ vanishes.*

Proof. Combine Propositions 8.2.3.6 and 7.2.1.4. □

Remark 8.2.3.8. Corollary 8.2.3.7 implies in particular that each weakly quasi-coherent sheaf $\mathscr{F} \in \mathrm{WCoh}(\mathfrak{X})$ fits into an essentially unique fiber sequence $\mathscr{F}' \xrightarrow{u} \mathscr{F} \rightarrow \mathscr{F}''$, where $\mathscr{F}' \in \mathrm{NilCoh}(\mathfrak{X})$ and $\mathscr{F}'' \in \mathrm{WCoh}^{\circ}(\mathfrak{X})$. Here $\mathscr{F}'$ can be identified with the tensor product $\mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F}$, and u is obtained by tensoring the canonical map $\mathscr{O}_{\mathfrak{X}}^{\approx} \rightarrow \mathscr{O}_{\mathfrak{X}}$ with the identity on $\mathscr{F}$.

The condition of weak quasi-coherence can be tested locally:

Proposition 8.2.3.9. *Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. For each object $U \in \mathcal{X}$, set $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathfrak{X}}|_U)$. Then:*

(1) *If $\mathscr{F}$ is weakly quasi-coherent, then $\mathscr{F}|_U \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}|_U}$ is weakly quasi-coherent for every object $U \in \mathcal{X}$.*

(2) *If there exists a covering of $\mathcal{X}$ by objects $\{U_\alpha\}$ for which each restriction $\mathscr{F}|_{U_\alpha} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}|_{U_\alpha}}$ is weakly quasi-coherent, then $\mathscr{F}$ is weakly quasi-coherent.*

Proof. Combine Proposition 8.2.1.5 with Remark 8.2.2.6. □

Proposition 8.2.3.10. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then* $\mathrm{WCoh}(\mathfrak{X})$ *is a symmetric monoidal subcategory of* $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$*: that is, it contains the unit object and is closed under tensor products. In particular, there is an essentially unique symmetric monoidal structure on* $\mathrm{WCoh}(\mathfrak{X})$ *for which the inclusion functor* $\mathrm{WCoh}(\mathfrak{X}) \hookrightarrow \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ *is symmetric monoidal.*

Proof. The first assertion follows from Example 8.2.3.3. To complete the proof, we must show that if $\mathscr{F}, \mathscr{G} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ are weakly quasi-coherent, then the tensor product $\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}$ is weakly quasi-coherent. Let $\mathscr{O}_{\mathfrak{X}}^{\approx}$ be an approximate unit for $\mathrm{NilCoh}(\mathfrak{X})$. We then have

$$\begin{aligned}
\mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G} &\simeq (\mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}}^{\approx}) \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G} \\
&\simeq (\mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F}) \otimes_{\mathscr{O}_{\mathfrak{X}}} (\mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}).
\end{aligned}$$

Our assumption that $\mathscr{F}$ and $\mathscr{G}$ are weakly quasi-coherent implies that the tensor products $\mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F}$ and $\mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}$ are nilcoherent. Applying Proposition 8.2.1.7, we conclude that $\mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}$ is also nilcoherent, so that $\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}$ is weakly quasi-coherent. □

Remark 8.2.3.11. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the full subcategory $\mathrm{NilCoh}(\mathfrak{X}) \subseteq \mathrm{WCoh}(\mathfrak{X})$ is closed under tensor products (but usually does not contain the unit object). It follows that $\mathrm{NilCoh}(\mathfrak{X})$ inherits the structure of a *nonunital* symmetric monoidal functor, which is determined uniquely by the requirement that the inclusion $\iota : \mathrm{NilCoh}(\mathfrak{X}) \hookrightarrow \mathrm{WCoh}(\mathfrak{X})$ be a nonunital symmetric monoidal functor. This nonunital symmetric monoidal structure on $\mathrm{NilCoh}(\mathfrak{X})$ can be promoted (in an essentially unique way) to a symmetric monoidal structure, with unit object given by $\mathscr{O}_{\mathfrak{X}}^{\approx}$ (see Corollary HA.5.4.4.7). However, the inclusion ι is usually not a symmetric monoidal functor.

We have the following criterion for weak quasi-coherence:

Proposition 8.2.3.12. *Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. Then $\mathscr{F}$ is weakly quasi-coherent if and only if it satisfies the following condition:*

- $(*)$ *For every morphism $U \to V$ between affine objects of $\mathcal{X}$, the induced map $\mathscr{O}_{\mathfrak{X}}(U) \otimes_{\mathscr{O}_{\mathfrak{X}}(V)} \mathscr{F}(V) \to \mathscr{F}(U)$ has I_U-local fiber, where $I_U \subseteq \pi_0 \mathscr{O}_{\mathfrak{X}}(U)$ is a finitely generated ideal of definition.*

Remark 8.2.3.13. In the situation of Proposition 8.2.3.12, if $\mathscr{F}$ satisfies condition (a) of Definition 8.2.1.1, then $(*)$ is equivalent to condition (b) of Definition 8.2.1.1. From this point of view, the ∞-category $\mathrm{WCoh}(\mathfrak{X})$ appears as a natural enlargement of $\mathrm{NilCoh}(\mathfrak{X})$ (where we relax the nilpotence requirements on the local sections of $\mathscr{F}$ which are imposed by Definition 8.2.1.1).

Proof of Proposition 8.2.3.12. Using Proposition 8.2.3.9, we can reduce to the case where $\mathfrak{X}$ is affine. Write $\mathfrak{X} = \mathrm{Spf}\, A$, where A is a complete adic $\mathbb{E}_\infty$-ring. Let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition. Let us identify $\mathscr{O}_{\mathfrak{X}}$ with the functor

$$\mathrm{CAlg}_A^{\text{ét}} \to \mathrm{CAlg}_A \qquad B \mapsto B_I^{\wedge},$$

and view $\mathscr{F}$ as a $\mathscr{O}_{\mathfrak{X}}$-module object of the presheaf ∞-category $\mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathrm{Sp})$. Set $M = \Gamma(\mathfrak{X}; \mathscr{F}) \in \mathrm{Mod}_A$. Then we can rephrase condition $(*)$ as follows:

- $(*')$ For every étale A-algebra B, the canonical map $B_I^{\wedge} \otimes_A M \to \mathscr{F}(B)$ has a fiber which is I-local.

Suppose first that $(*')$ is satisfied; we will show that $\mathscr{F}$ is weakly quasi-coherent. Note that we can identify the approximate unit $\mathscr{O}_{\mathfrak{X}}^{\approx}$ with the $\mathscr{O}_{\mathfrak{X}}$-module object of $\mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathrm{Sp})$

given by the construction $B \mapsto \Gamma_I(B^{\wedge}_I) \simeq \Gamma_I(B)$ (see Example 8.2.2.3). It follows that the tensor product $\mathscr{O}^{\approx}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F}$ is obtained by sheafifying the functor $\mathscr{F}^{\circ} : \mathrm{CAlg}^{\text{ét}}_B \to \mathrm{Sp}$ given by $\mathscr{F}^{\circ}(B) = \Gamma_I(B^{\wedge}_I) \otimes_{B^{\wedge}_I} \mathscr{F}(B)$. It follows from $(*')$ that the functor $\mathscr{F}^{\circ}$ is also given the formula

$$\mathscr{F}^{\circ}(B) = \Gamma_I(B^{\wedge}_I) \otimes_{B^{\wedge}_I} (B^{\wedge}_I \otimes_A M) \simeq \Gamma_I(B^{\wedge}_I) \otimes_A M \simeq B^{\wedge}_I \otimes_A \Gamma_I(M).$$

This functor is already a sheaf, which is the image of $\Gamma_I(M)$ under the equivalence $\mathrm{Mod}^{\mathrm{Nil}(I)}_A \simeq \mathrm{NilCoh}(\mathfrak{X})$ supplied by Proposition 8.2.1.3.

We now prove the converse. Suppose that $\mathscr{F}$ is weakly quasi-coherent; we wish to show that condition $(*')$ is satisfied. Choose a collection of elements $x_1, \ldots, x_n \in \pi_0 A$ which generate the ideal I. For $1 \leqslant i \leqslant n$, we note that $\mathscr{F}$ satisfies condition $(*')$ if and only if the cofiber of the map $x_i : \mathscr{F} \to \mathscr{F}$ also satisfies condition $(*')$. We may therefore replace $\mathscr{F}$ by $\mathrm{cofib}(x_i : \mathscr{F} \to \mathscr{F})$, and thereby reduce to the case where $\mathscr{F}(B)$ is (x_i)-nilpotent for each $B \in \mathrm{CAlg}^{\text{ét}}_A$. Applying this argument for $1 \leqslant i \leqslant n$, we can reduce to the case where $\mathscr{F}(B)$ is I-nilpotent for each $B \in \mathrm{CAlg}^{\text{ét}}_A$. In this case, the canonical map $\mathscr{F}^{\circ} \to \mathscr{F}$ is an equivalence of $\mathscr{O}_{\mathfrak{X}}$-module objects of $\mathrm{Fun}(\mathrm{CAlg}^{\text{ét}}_A, \mathrm{Sp})$. It follows that $\mathscr{F} \simeq \mathscr{O}^{\approx}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F}$, so our assumption that $\mathscr{F}$ is weakly quasi-coherent guarantees that it is nilcoherent. In this case, the desired result follows immediately from the definitions. □

8.2.4 Quasi-Coherent Sheaves

We are now almost ready to introduce the main objects of interest in this section.

Construction 8.2.4.1 (Completions of Weakly Quasi-Coherent Sheaves)**.** Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack and let $\alpha : \mathscr{O}^{\approx}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}$ be an approximate unit for $\mathfrak{X}$ (Proposition 8.2.2.5). For every weakly quasi-coherent sheaf $\mathscr{F} \in \mathrm{NilCoh}(\mathfrak{X})$, we let $\mathscr{F}^{\wedge}$ denote the sheaf $\underline{\mathrm{Map}}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{O}^{\approx}_{\mathfrak{X}}, \mathscr{F}) \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ (see §6.5.3). We will refer to $\mathscr{F}^{\wedge}$ as the *completion* of $\mathscr{F}$. Note that α induces a canonical map $\alpha_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^{\wedge}$.

Remark 8.2.4.2. Construction 8.2.4.1 is compatible with localization: if $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$, then for each $U \in \mathcal{X}$ and $\mathscr{F} \in \mathrm{NilCoh}(\mathfrak{X})$ we a canonical equivalence $\mathscr{F}^{\wedge}|_U \simeq (\mathscr{F}|_U)^{\wedge}$ (see Remark 6.5.3.1). In particular, there is a canonical homotopy equivalence

$$\Omega^{\infty} \mathscr{F}^{\wedge}(U) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}|_U}}(\mathscr{O}^{\approx}_{\mathfrak{X}}|_U, \mathscr{F}|_U).$$

Remark 8.2.4.3. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{WCoh}(\mathfrak{X})$. Using Remark 8.2.4.2, we see that the canonical map $\mathscr{O}^{\approx}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F} \to \mathscr{F}$ induces an equivalence of completions $(\mathscr{O}^{\approx}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F})^{\wedge} \to \mathscr{F}^{\wedge}$.

Example 8.2.4.4. Let X be a spectral Deligne-Mumford stack. Then the structure sheaf $\mathscr{O}_{\mathsf{X}}$ is an approximate unit for $\mathrm{NilCoh}(\mathsf{X})$, so the completion functor $\mathscr{F} \mapsto \mathscr{F}^{\wedge}$ is equivalent to the identity.

We begin by analyzing the completion functor in the affine case.

Proposition 8.2.4.5. *Let A be an adic $\mathbb{E}_\infty$-ring, let $\mathscr{F} \in \mathrm{NilCoh}(\operatorname{Spf} A)$, and let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition. Then the canonical map $\mathscr{F} \to \mathscr{F}^\wedge$ exhibits $\mathscr{F}^\wedge$ as an I-completion of $\mathscr{F}$ (in the A-linear stable ∞-category $\mathrm{Mod}_{\mathscr{O}_{\operatorname{Spf} A}}$). In other words, for every object $U \in \mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}$, the canonical map $u : \mathscr{F}(U) \to \mathscr{F}^\wedge(U)$ exhibits $\mathscr{F}^\wedge(U)$ as an I-completion of $\mathscr{F}(U)$.*

Proof. The assertion is local on U (since the formation of I-completions commutes with small limits). We may therefore assume without loss of generality that U is affine. Using Remark 8.2.4.2, we can further reduce to the case where U is the final object of $\mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}$. In this case, we have canonical equivalences

$$\begin{aligned} \mathscr{F}^\wedge(U) &\simeq \Gamma(\operatorname{Spf} A; \underline{\mathrm{Map}}_{\mathscr{O}_{\operatorname{Spf} A}}(\mathscr{O}^{\approx}_{\operatorname{Spf} A}, \mathscr{F})) \\ &\simeq \underline{\mathrm{Map}}_A(\Gamma(\operatorname{Spf} A; \mathscr{O}^{\approx}_{\operatorname{Spf} A}), \Gamma(\operatorname{Spf} A; \mathscr{F})) \\ &\simeq \underline{\mathrm{Map}}_A(\Gamma_I(A), M). \end{aligned}$$

If $N \in \mathrm{Mod}_A$ is I-local, we obtain $\mathrm{Map}_{\mathrm{Mod}_A}(N, \mathscr{F}^\wedge(U)) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(N \otimes_A \Gamma_I(A), M) \simeq 0$, since $N \otimes_A \Gamma_I(A) \simeq \Gamma_I(N) \simeq 0$. It follows that $\mathscr{F}^\wedge(U)$ is I-complete. To complete the proof, it will suffice to show that the fiber $\mathrm{fib}(u) \simeq \underline{\mathrm{Map}}_A(L_I(A), M)$ is I-local. This is clear: if N' is I-nilpotent, then the mapping space

$$\mathrm{Map}_{\mathrm{Mod}_A}(N', \underline{\mathrm{Map}}_A(L_I(A), M)) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(N' \otimes_A L_I(A), M) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(L_I(N'), M)$$

is contractible (since $L_I(N') \simeq 0$). □

Corollary 8.2.4.6. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{WCoh}(\mathfrak{X})$ be a weakly quasi-coherent sheaf. Then the completion $\mathscr{F}^\wedge$ is also weakly quasi-coherent. Moreover, the canonical map $\mathscr{F} \to \mathscr{F}^\wedge$ induces an equivalence $\mathscr{O}^{\approx}_{\mathfrak{X}} \otimes_{\mathscr{O}_\mathfrak{X}} \mathscr{F} \to \mathscr{O}^{\approx}_{\mathfrak{X}} \otimes_{\mathscr{O}_\mathfrak{X}} \mathscr{F}^\wedge$.*

Proof. By virtue of Remark 8.2.4.3, we can assume without loss of generality that $\mathscr{F}$ is nilcoherent. Using Remark 8.2.4.2 and Proposition 8.2.3.9, we can further assume that $\mathfrak{X}$ is affine. In this case, the desired result follows from Proposition 8.2.4.5. □

Definition 8.2.4.7. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. We will say that a sheaf $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_\mathfrak{X}}$ is *quasi-coherent* if it belongs to the essential image of the completion functor $\mathrm{WCoh}(\mathfrak{X}) \to \mathrm{WCoh}(\mathfrak{X})$. We let $\mathrm{QCoh}(\mathfrak{X})$ denote the full subcategory of $\mathrm{Mod}_{\mathscr{O}_\mathfrak{X}}$ spanned by the quasi-coherent sheaves.

Example 8.2.4.8. Let X be a spectral Deligne-Mumford stack. Then a sheaf $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_\mathsf{X}}$ is quasi-coherent in the sense of Definition 8.2.4.7 if and only if it is quasi-coherent in the sense of Definition 2.2.2.1 (see Examples 8.2.3.4 and 8.2.4.4).

Proposition 8.2.4.9. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the construction $\mathscr{F} \mapsto \mathscr{F}^{\wedge}$ is left adjoint to the inclusion* $\mathrm{QCoh}(\mathfrak{X}) \hookrightarrow \mathrm{WCoh}(\mathfrak{X})$.

Proof. Let $\mathscr{F}$ be an object of $\mathrm{WCoh}(\mathfrak{X})$; we claim that the canonical map $\alpha : \mathscr{F} \to \mathscr{F}^{\wedge}$ exhibits $\mathscr{F}^{\wedge}$ as a $\mathrm{QCoh}(\mathfrak{X})$-localization of $\mathscr{F}$. Since $\mathscr{F}^{\wedge}$ is quasi-coherent, it will suffice to show that for each object $\mathscr{G} \in \mathrm{QCoh}(\mathfrak{X})$, composition with α induces a homotopy equivalence $\rho : \mathrm{Map}_{\mathrm{WCoh}(\mathfrak{X})}(\mathscr{F}^{\wedge}, \mathscr{G}) \to \mathrm{Map}_{\mathrm{WCoh}(\mathfrak{X})}(\mathscr{F}, \mathscr{G})$. Writing $\mathscr{G} = \mathscr{G}_0^{\wedge}$ for some $\mathscr{G}_0 \in \mathrm{WCoh}(\mathfrak{X})$, we can identify ρ with the natural map

$$\mathrm{Map}_{\mathrm{WCoh}(\mathfrak{X})}(\mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F}^{\wedge}, \mathscr{G}_0) \to \mathrm{Map}_{\mathrm{WCoh}(\mathfrak{X})}(\mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F}, \mathscr{G}_0),$$

which is a homotopy equivalence by virtue of Corollary 8.2.4.6. □

Corollary 8.2.4.10. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the ∞-category* $\mathrm{WCoh}(\mathfrak{X})$ *admits a semi-orthogonal decomposition* $(\mathrm{WCoh}^{\circ}(\mathfrak{X}), \mathrm{QCoh}(\mathfrak{X}))$*, where* $\mathrm{WCoh}^{\circ}(\mathfrak{X})$ *is defined as in Corollary 8.2.3.7.*

Remark 8.2.4.11. Corollary 8.2.4.10 asserts that every weakly quasi-coherent sheaf $\mathscr{F}$ fits into an essentially unique fiber sequence $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$, where $\mathscr{F}' \in \mathrm{WCoh}^{\circ}(\mathfrak{X})$ and $\mathscr{F}''$ is quasi-coherent. Here we can identify $\mathscr{F}''$ with the completion of $\mathscr{F}$ (in the sense of Construction 8.2.4.1).

Proof of Corollary 8.2.4.10. By virtue of Propositions 8.2.4.9 and 7.2.1.4, it will suffice to show that a weakly quasi-coherent sheaf $\mathscr{F}$ belongs to $\mathrm{WCoh}^{\circ}(\mathfrak{X})$ if and only if the completion $\mathscr{F}^{\wedge}$ vanishes. The "if" direction follows from Corollary 8.2.4.6, and the "only if" direction from Remark 8.2.4.3. □

Corollary 8.2.4.12. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the construction $\mathscr{F} \mapsto \mathscr{F}^{\wedge}$ induces an equivalence of ∞-categories* $\mathrm{NilCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X})$. *The inverse equivalence is given by the construction $\mathscr{F} \mapsto \mathscr{O}_{\mathfrak{X}}^{\approx} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{F}$.*

Proof. Combine Corollary 8.2.4.10 with Proposition 7.2.1.10. □

Corollary 8.2.4.13. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the ∞-category* $\mathrm{QCoh}(\mathfrak{X})$ *of quasi-coherent sheaves on $\mathfrak{X}$ is stable and presentable.*

Proof. Combine Corollary 8.2.4.12 with Proposition 8.2.1.8. □

Warning 8.2.4.14. In the situation of Corollary 8.2.4.13, the ∞-category $\mathrm{QCoh}(\mathfrak{X})$ need not be closed under colimits in $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. However, it is closed under *limits* in the ∞-category $\mathrm{WCoh}(\mathfrak{X})$ (which is in turn closed under colimits in $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$).

Corollary 8.2.4.15. *Let A be an adic $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition. Then the global sections functor $\Gamma(\operatorname{Spf} A; \bullet) : \operatorname{Mod}_{\mathscr{O}_{\operatorname{Spf} A}} \to \operatorname{Mod}_A$ induces an equivalence of ∞-categories $\operatorname{QCoh}(\operatorname{Spf} A) \to \operatorname{Mod}_A^{\operatorname{Cpl}(I)}$.*

Proof. It follows from Lemma 8.2.4.5 that we have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \operatorname{NilCoh}(\operatorname{Spf} A) & \longrightarrow & \operatorname{QCoh}(\operatorname{Spf} A) \\ \downarrow \Gamma & & \downarrow \Gamma \\ \operatorname{Mod}_A^{\operatorname{Nil}(I)} & \longrightarrow & \operatorname{Mod}_A^{\operatorname{Cpl}(I)} \end{array}$$

where the horizontal maps are given by I-adic completion and the vertical maps are given by passage to global sections. The upper horizontal map is an equivalence by Corollary 8.2.4.12, the left vertical map is an equivalence by Proposition 8.2.1.3, and the bottom horizontal map is an equivalence by Proposition 7.3.1.7. It follows that the right vertical map is also an equivalence of ∞-categories. □

Remark 8.2.4.16. Let A be as in Corollary 8.2.4.15, let $\mathscr{F} \in \operatorname{QCoh}(\operatorname{Spf} A)$, and let $M = \Gamma(\operatorname{Spf} A; \mathscr{F})$ be the corresponding I-complete A-module. It follows from Corollary 8.2.4.15 that $\mathscr{F}$ is determined by M, up to canonical equivalence. Moreover, the proof of Corollary 8.2.4.15 gives an explicit recipe for reconstructing $\mathscr{F}$: as a functor from $\operatorname{CAlg}_A^{\text{ét}}$ to the ∞-category Mod_A, the sheaf $\mathscr{F}$ is given by the formula

$$\mathscr{F}(B) = (B \otimes_A \Gamma_I(M))^\wedge_I \simeq (B \otimes_A M)^\wedge_I .$$

The condition of quasi-coherence can be tested locally:

Proposition 8.2.4.17. *Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. For each object $U \in \mathcal{X}$, set $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathfrak{X}}|_U)$. Then:*

(1) *If $\mathscr{F}$ is quasi-coherent, then $\mathscr{F}|_U \in \operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}|_U}$ is quasi-coherent for every object $U \in \mathcal{X}$.*

(2) *If there exists a covering of $\mathcal{X}$ by objects $\{U_\alpha\}$ for which each restriction $\mathscr{F}|_{U_\alpha} \in \operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}|_{U_\alpha}}$ is quasi-coherent, then $\mathscr{F}$ is quasi-coherent.*

Proof. Combine Proposition 8.2.3.9 with Remark 8.2.4.2. □

The class of quasi-coherent sheaves admits the following concrete characterization (compare with Definition 8.2.1.1):

Proposition 8.2.4.18. *Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. Then $\mathscr{F}$ is quasi-coherent if and only if the following conditions are satisfied:*

(*a*) *For every affine object* $U \in \mathcal{X}$*, the* $\mathscr{O}_{\mathfrak{X}}(U)$*-module* $\mathscr{F}(U)$ *is* I*-complete, where* I *is an ideal of definition of* $\mathscr{O}_{\mathfrak{X}}(U)$*.*

(*b*) *For every morphism* $U \to V$ *between affine objects of* $\mathcal{X}$*, the induced map* $\mathscr{O}_{\mathfrak{X}}(U) \otimes_{\mathscr{O}_{\mathfrak{X}}(V)} \mathscr{F}(V) \to \mathscr{F}(U)$ *exhibits* $\mathscr{F}(U)$ *as an* I*-completion of* $\mathscr{O}_{\mathfrak{X}}(U) \otimes_{\mathscr{O}_{\mathfrak{X}}(V)} \mathscr{F}(V)$*, where* I *is an ideal of definition of* $\mathscr{O}_{\mathfrak{X}}(U)$*.*

Proof. By virtue of Proposition 8.2.4.17, we can assume without loss of generality that $\mathfrak{X} = \operatorname{Spf} A$ is affine. If $\mathscr{F}$ is quasi-coherent, then assertions (*a*) and (*b*) follow immediately from the description given in Remark 8.2.4.16. Conversely, suppose that conditions (*a*) and (*b*) are satisfied and set $M = \Gamma(\operatorname{Spf} A; \mathscr{F})$. Then M is I-complete (where $I \subseteq \pi_0 A$ is a finitely generated ideal of definition). It follows from (*b*) that, when regarded as a sheaf on the ∞-category $\mathrm{CAlg}^{\text{ét}}_A$, the functor $\mathscr{F}$ is given by the formula $\mathscr{F}(B) = (B^{\wedge}_I \otimes_{A^{\wedge}_I} M)^{\wedge}_I \simeq (B \otimes_A \Gamma_I(M))^{\wedge}_I$, so that $\mathscr{F}$ is quasi-coherent by virtue of Proposition 8.2.4.5. □

Remark 8.2.4.19. In the situation of Proposition 8.2.4.18, if $\mathscr{F}$ satisfies condition (*a*), then condition (*b*) can restated as follows:

(b') The sheaf $\mathscr{F}$ is weakly quasi-coherent.

This follows from the criterion for weak quasi-coherence supplied by Proposition 8.2.3.12.

Corollary 8.2.4.20. *Let* $\mathfrak{X}$ *be a formal spectral Deligne-Mumford stack. Then the structure sheaf* $\mathscr{O}_{\mathfrak{X}}$ *is quasi-coherent.*

We close this section with a few remarks about tensor products of quasi-coherent sheaves.

Proposition 8.2.4.21. *Let* $\mathfrak{X}$ *be a formal spectral Deligne-Mumford stack. Then the completion functor*

$$\mathrm{WCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X}) \qquad \mathscr{F} \mapsto \mathscr{F}^{\wedge}$$

of Construction 8.2.4.1 is compatible (in the sense of Definition HA.2.2.1.6) with the symmetric monoidal structure on the ∞*-category* $\mathrm{WCoh}(\mathfrak{X})$ *described in Proposition 8.2.3.10.*

Proof. Using Corollary 8.2.4.10, we are reduced to showing that if $\mathscr{F} \in \mathrm{WCoh}^{\circ}(\mathfrak{X})$ and $\mathscr{G} \in \mathrm{WCoh}(\mathfrak{X})$, then the tensor product $\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}$ belongs to $\mathrm{WCoh}^{\circ}(\mathfrak{X})$. This follows immediately from the definition of the ∞-category $\mathrm{WCoh}^{\circ}(\mathfrak{X})$ (see Corollary 8.2.3.7). □

Corollary 8.2.4.22. *Let* $\mathfrak{X}$ *be a formal spectral Deligne-Mumford stack. Then there exists an essentially unique symmetric monoidal structure on the* ∞*-category* $\mathrm{QCoh}(\mathfrak{X})$ *for which the completion functor* $\mathrm{WCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X})$ *is symmetric monoidal.*

Proof. Combine Propositions 8.2.4.21 and HA.2.2.1.9. □

Notation 8.2.4.23. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. We will generally regard the ∞-category $\mathrm{QCoh}(\mathfrak{X})$ as equipped with the symmetric monoidal structure described in Corollary 8.2.4.22. We denote the tensor product on $\mathrm{QCoh}(\mathfrak{X})$ by $\widehat{\otimes} : \mathrm{QCoh}(\mathfrak{X}) \times \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X})$ to distinguish it from the tensor product on the larger ∞-categories $\mathrm{WCoh}(\mathfrak{X}) \subseteq \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. Concretely, it is given by the formula $\mathscr{F} \widehat{\otimes} \mathscr{G} = (\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G})^{\wedge}$. Moreover, the unit object of $\mathrm{QCoh}(\mathfrak{X})$ is the structure sheaf $\mathscr{O}_{\mathfrak{X}}$ (see Corollary 8.2.4.20).

Remark 8.2.4.24. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the equivalence $\mathrm{NilCoh}(\mathfrak{X}) \simeq \mathrm{QCoh}(\mathfrak{X})$ of Corollary 8.2.4.12 can be promoted to an equivalence of symmetric monoidal ∞-categories, where $\mathrm{QCoh}(\mathfrak{X})$ is equipped with the symmetric monoidal structure of Corollary 8.2.4.22 and $\mathrm{NilCoh}(\mathfrak{X})$ is equipped with the symmetric monoidal structure of Remark 8.2.3.11.

8.2.5 Connectivity Conditions

We now study the behavior of sheaves on formal spectral Deligne-Mumford stacks with respect to truncation. We begin with the nilcoherent case.

Proposition 8.2.5.1. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then:*

(a) *If $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ is nilcoherent, then the truncations $\tau_{\geqslant n} \mathscr{F}$ and $\tau_{\leqslant n} \mathscr{F}$ are nilcoherent for all $n \in \mathbf{Z}$.*

(b) *Define full subcategories $\mathrm{NilCoh}(\mathfrak{X})_{\geqslant 0}, \mathrm{NilCoh}(\mathfrak{X})_{\leqslant 0} \subseteq \mathrm{NilCoh}(\mathfrak{X})$ by the formulae*

$$\mathrm{NilCoh}(\mathfrak{X})_{\geqslant 0} = \mathrm{NilCoh}(\mathfrak{X}) \cap (\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}})_{\geqslant 0} \qquad \mathrm{NilCoh}(\mathfrak{X})_{\leqslant 0} = \mathrm{NilCoh}(\mathfrak{X}) \cap (\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}})_{\leqslant 0}.$$

Then the pair $(\mathrm{NilCoh}(\mathfrak{X})_{\geqslant 0}, \mathrm{NilCoh}(\mathfrak{X})_{\leqslant 0})$ determines a t-structure on $\mathrm{NilCoh}(\mathfrak{X})$.

(c) *The t-structure of (b) is left and right complete and compatible with filtered colimits.*

Proof. To prove (a), we can use Proposition 8.2.1.5 to reduce to the case where $\mathfrak{X}$ is affine, in which case the desired result follows from Remark 8.2.1.4. Assertion (b) follows formally from (a). To prove (c), we can again reduce to the case where $\mathfrak{X} \simeq \mathrm{Spf}\, A$ is affine. In this case, Proposition **??** supplies a t-exact equivalence $\mathrm{NilCoh}(\mathfrak{X}) \simeq \mathrm{Mod}_A^{\mathrm{Nil}(I)}$ where $I \subseteq \pi_0 A$ is a finitely generated ideal of definition (see Remark 7.1.1.17). □

Remark 8.2.5.2. Let X be a spectral Deligne-Mumford stack, let $K \subseteq |\mathsf{X}|$ be a cocompact closed subset, and let $i_* : \mathrm{NilCoh}(\mathsf{X}^{\wedge}_K) \to \mathrm{QCoh}_K(\mathsf{X})$ be the equivalence of Corollary 8.2.1.6. Then i_* is t-exact (where we regard $\mathrm{NilCoh}(\mathsf{X}^{\wedge}_K)$ as equipped with the t-structure of Proposition 8.2.5.1 and $\mathrm{QCoh}_K(\mathsf{X})$ with the t-structure of Remark 7.1.5.2).

Proposition 8.2.5.3. *Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack, let $\mathscr{F} \in \mathrm{WCoh}(\mathfrak{X})$ be weakly quasi-coherent, and let $\mathscr{F}^{\wedge} \in \mathrm{QCoh}(\mathfrak{X})$ be its completion. Suppose that $\mathscr{F}$ is connective (when regarded as a sheaf of spectra on $\mathfrak{X}$). Then, for each affine object $U \in \mathcal{X}$, the spectrum $\mathscr{F}^{\wedge}(U)$ is connective.*

Proof. Without loss of generality, we may assume that U is the final object of $\mathcal{X}$, so that $\mathfrak{X} \simeq \operatorname{Spf} A$ for some adic $\mathbb{E}_\infty$-ring A. Let us regard $\mathrm{WCoh}(\mathfrak{X})$ as a stable A-linear ∞-category. Let $I \subseteq \pi_0 A$ be an ideal of definition which is generated by finitely many elements $x_1, \ldots, x_n \in \pi_0 A$. Set $M = \Gamma(\operatorname{Spf} A; \mathscr{F}^{\wedge})$; we wish to show that M is connective.

Fix $1 \leqslant i \leqslant n$. For $m \geqslant 0$, let M_m denote the cofiber of the map $x_i^m : M \to M$. Since M is I-complete, we have an equivalence $M \simeq \varprojlim M_m$. Consequently, to prove that M is connective, it will suffice to show that each M_m is connective and that each of the transition maps $\pi_0 M_{m+1} \to \pi_0 M_m$ is surjective. Using the fiber sequence $M_1 \xrightarrow{x_i^m} M_{m+1} \to M_m$, we are reduced to showing that M_1 is connective. We may therefore replace $\mathscr{F}$ by the cofiber of the map $x_i : \mathscr{F} \to \mathscr{F}$ and thereby reduce to the case where $\mathscr{F}$ is (x_i)-nilpotent. Applying this argument for $1 \leqslant i \leqslant n$, we can assume that $\mathscr{F}$ is I-nilpotent, hence nilcoherent (Corollary 8.3.2.3). Using Remark 8.2.1.4, we deduce that $\Gamma(\mathfrak{X}; \mathscr{F})$ is connective, so that the completion $M \simeq \Gamma(\mathfrak{X}; \mathscr{F})^{\wedge}_I$ is also connective (Proposition 7.3.4.4). □

Corollary 8.2.5.4. *Let A be an adic $\mathbb{E}_\infty$-ring and let $\mathscr{F} \in \mathrm{QCoh}(\operatorname{Spf} A)$. The following conditions are equivalent:*

(1) *The A-module $\Gamma(\operatorname{Spf} A; \mathscr{F})$ is connective.*

(2) *Write $\operatorname{Spf} A = (\mathcal{X}, \mathscr{O}_{\operatorname{Spf} A})$. For each affine object $U \in \mathcal{X}$, the spectrum $\mathscr{F}(U)$ is connective.*

(3) *The sheaf $\mathscr{F}$ is connective (when viewed as a sheaf of spectra on $\mathcal{X}$).*

Proof. Let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition and set $M = \Gamma(\operatorname{Spf} A; \mathscr{F})$. Using Example 8.1.3.5 and Remark 8.2.4.16, we see that for every affine object $U \in \mathcal{X}$, we can write $\mathscr{F}(U) = (B \otimes_A M)^{\wedge}_I$ for some étale A-algebra B. Since the I-completion functor is right t-exact (Proposition 7.3.4.4), it follows that (1) ⇒ (2). The implication (2) ⇒ (3) follows immediately from the definitions (since $\pi_n \mathscr{F}$ is the sheafification of the functor $U \mapsto \pi_n(\mathscr{F}(U))$), and the implication (3) ⇒ (1) follows from Proposition 8.2.5.3. □

Corollary 8.2.5.5. *Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$ be a quasi-coherent sheaf. The following conditions are equivalent:*

(1) *The sheaf $\mathscr{F}$ is connective when viewed as a* Sp*-valued sheaf on $\mathcal{X}$: that is, the homotopy sheaves $\pi_n \mathscr{F}$ vanish for $n < 0$.*

(2) *For each affine object $U \in \mathcal{X}$, the spectrum $\mathscr{F}(U)$ is connective.*

Definition 8.2.5.6. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. We will say that a quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$ is *connective* if it satisfies the equivalent conditions of Corollary 8.2.5.5. We let $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{QCoh}(\mathfrak{X})$ spanned by the connective objects.

Variant 8.2.5.7. Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack. We say that a quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$ is *almost connective* if, for every affine object $U \in \mathcal{X}$, the spectrum $\mathscr{F}(U)$ is almost connective (that is, it is $(-n)$-connective for $n \gg 0$). We let $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{acn}}$ denote the full subcategory of $\mathrm{QCoh}(\mathfrak{X})$ spanned by the almost connective objects.

Remark 8.2.5.8. Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$. If $\Sigma^n \mathscr{F}$ is connective for some $n \gg 0$, then $\mathscr{F}$ is almost connective. The converse holds if $\mathcal{X}$ is quasi-compact.

Remark 8.2.5.9. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack and let $\mathrm{WCoh}(\mathfrak{X})^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{WCoh}(\mathfrak{X})$ spanned by those objects which are connective when viewed as Sp-valued sheaves. It follows from Proposition 8.2.5.3 that the completion functor $\mathrm{WCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X})$ of Construction 8.2.4.1 carries connective objects of $\mathrm{WCoh}(\mathfrak{X})$ to connective objects of $\mathrm{QCoh}(\mathfrak{X})$.

Proposition 8.2.5.10. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then:*

(1) *The ∞-category $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}$ contains the unit object of $\mathrm{QCoh}(\mathfrak{X})$ and is closed under the completed tensor product functor $\widehat{\otimes} : \mathrm{QCoh}(\mathfrak{X}) \times \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X})$.*

(2) *The ∞-category $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}$ is closed under colimits and extensions in $\mathrm{QCoh}(\mathfrak{X})$.*

(3) *The ∞-category $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}$ is presentable.*

Proof. Note that the ∞-category $\mathrm{WCoh}(\mathfrak{X})^{\mathrm{cn}}$ is closed under small colimits and tensor products (since the analogous assertions hold in the larger ∞-category $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$). Assertions (1) and (2) now follow from Remark 8.2.5.9. To prove (3), it will suffice to show that the ∞-category $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}$ is accessible, which follows by applying Proposition HTT.5.4.6.6 to the pullback diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}} & \longrightarrow & \mathrm{QCoh}(\mathfrak{X}) \\ \downarrow & & \downarrow \\ \mathrm{Mod}^{\mathrm{cn}}_{\mathscr{O}_{\mathfrak{X}}} & \longrightarrow & \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}} . \end{array}$$

□

Corollary 8.2.5.11. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the ∞-category $\mathrm{QCoh}(\mathfrak{X})$ admits an accessible t-structure $(\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}, \mathrm{QCoh}'(\mathfrak{X}))$.*

Proof. Combine Propositions 8.2.5.10 and HA.1.4.4.11. □

Warning 8.2.5.12. The t-structure $(\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}, \mathrm{QCoh}'(\mathfrak{X}))$ of Corollary 8.2.5.11 is generally not well-behaved: for example, it is not compatible with filtered colimits. Moreover, the inclusion functor $\mathrm{QCoh}(\mathfrak{X}) \hookrightarrow \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ is not t-exact (though it is right t-exact). These difficulties can often be circumvented by restricting attention to almost perfect sheaves in the locally Noetherian setting: see §8.4 for further discussion.

Warning 8.2.5.13. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the equivalence $\mathrm{NilCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X})$ of Corollary 8.2.4.12 is not t-exact, where we endow $\mathrm{NilCoh}(\mathfrak{X})$ with the t-structure of Proposition 8.2.5.1 and $\mathrm{QCoh}(\mathfrak{X})$ with the t-structure of Corollary 8.2.5.11. However, it is right t-exact (by virtue of Remark 8.2.5.9).

8.3 Direct and Inverse Images

In §8.2, we introduced the ∞-category $\mathrm{QCoh}(\mathfrak{X})$ of quasi-coherent sheaves on a formal spectral Deligne-Mumford stack $\mathfrak{X}$. In this section, we study the behavior of quasi-coherent sheaves as $\mathfrak{X}$ varies. If $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of formal spectral Deligne-Mumford stacks, then the pullback functor $f^* : \mathrm{Mod}_{\mathscr{O}_{\mathfrak{Y}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ does not necessarily preserve quasi-coherence. However, in §8.3.2 we show that f^* does preserve *weak* quasi-coherence (Proposition 8.3.2.1). Using this observation, we define a *completed* pullback functor $f^{\star} : \mathrm{QCoh}(\mathfrak{Y}) \to \mathrm{QCoh}(\mathfrak{X})$ (Remark 8.3.2.10). Under mild hypotheses, we show that the direct image functor f_* preserves quasi-coherence, and therefore restricts to a functor $f_* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{Y})$ which is right adjoint to $f^{\star}$ (Proposition 8.3.3.1).

Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by $\mathfrak{X}$. In §8.3.4, we use the completed pullback functors to define a canonical map $\mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(X)$, where $\mathrm{QCoh}(X)$ is the ∞-category of quasi-coherent sheaves on X introduced in Definition 6.2.2.1. Our main result asserts that this functor is an equivalence when restricted to connective (or almost connective) quasi-coherent sheaves (Theorem 8.3.4.4 and Corollary 8.3.4.6). In §**??**, we exploit this equivalence to study various finiteness conditions on quasi-coherent sheaves, which will play an important role in §8.5.

8.3.1 Digression: Representable Morphisms

In §6.3.2, we introduced the notion of a representable morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We now specialize to the setting where X and Y are (representable by) formal spectral Deligne-Mumford stacks.

Definition 8.3.1.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. We will say that f is *representable* (or *representable by spectral Deligne-Mumford* *stacks*) if, for every pullback diagram of formal spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}_0 & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathsf{Y}_0 & \longrightarrow & \mathfrak{Y} \end{array}$$

where Y_0 is a spectral Deligne-Mumford stack, X_0 is also a spectral Deligne-Mumford stack.

Remark 8.3.1.2. In the situation of Definition 8.3.1.1, the condition that X_0 is a spectral Deligne-Mumford stack can be tested locally on X_0 (with respect to the étale topology). Consequently, to show that a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is representable, it suffices to verify the following *a priori* weaker condition:

($*$) For every map $\operatorname{Sp\acute{e}t} R \to \mathfrak{Y}$, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} \operatorname{Sp\acute{e}t} R$ is a spectral Deligne-Mumford stack.

Remark 8.3.1.3. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. Then f is representable (in the sense of Definition 8.3.1.1) if and only if the map of functors $h_{\mathfrak{X}} \to h_{\mathfrak{Y}}$ is representable (in the sense of Definition 6.3.2.1): this follows immediately from Remark 8.3.1.2.

Remark 8.3.1.4. Let $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{Z}$ be morphisms of formal spectral Deligne-Mumford stacks. If f and g are representable, then $(g \circ f) : \mathfrak{X} \to \mathfrak{Z}$ is representable.

Remark 8.3.1.5. Suppose we are given a pullback diagram of formal spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}. \end{array}$$

If f is representable, then so is f'. The converse holds if g is an étale surjection.

Remark 8.3.1.6. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an étale morphism of formal spectral Deligne-Mumford stacks. Then f is representable.

We now specialize to the case of *affine* formal spectral Deligne-Mumford stacks.

Proposition 8.3.1.7. *Let $f : A \to B$ be a morphism of adic $\mathbb{E}_\infty$-rings and let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition. The following conditions are equivalent:*

(*a*) *The ideal $I(\pi_0 B) \subseteq \pi_0 B$ is an ideal of definition for B.*

(*b*) *The map $\operatorname{Spf} f : \operatorname{Spf} B \to \operatorname{Spf} A$ is representable (Definition 8.3.1.1).*

Remark 8.3.1.8. It follows from Proposition 8.3.1.7 that condition (a) is independent of the choice of ideal of definition $I \subseteq \pi_0 A$.

Proof of Proposition 8.3.1.7. Suppose first that (a) is satisfied; we will show that the map $\operatorname{Spf} B \to \operatorname{Spf} A$ satisfies condition $(*)$ of Remark 8.3.1.2. Choose a connective $\mathbb{E}_\infty$-ring R and a map $g : \operatorname{Sp\acute{e}t} R \to \operatorname{Spf} A$, which we can identify with a morphism of $\mathbb{E}_\infty$-rings $A \to R$ which annihilates some power of I (Proposition 8.1.5.2). Using Lemma 8.1.7.3 and assumption (a), we can identify the fiber product $(\operatorname{Sp\acute{e}t} R) \times_{\operatorname{Spf} A} \operatorname{Spf} B$ with $\operatorname{Spf}(B \otimes_A R)$, where we endow $\pi_0(B \otimes_A R)$ with the I-adic topology. Since I generates a nilpotent ideal in R, this topology coincides with the discrete topology, so that $\operatorname{Spf}(B \otimes_A R) \simeq \operatorname{Sp\acute{e}t}(B \otimes_A R)$ is an (affine) spectral Deligne-Mumford stack.

Now suppose that (b) is satisfied. Set $R = (\pi_0 A)/I$. Assumption (b) guarantees that the fiber product $(\operatorname{Sp\acute{e}t} R) \times_{\operatorname{Spf} A} \operatorname{Spf} B \simeq \operatorname{Spf}(B \otimes_A R)$ is a spectral Deligne-Mumford stack. Let $J \subseteq \pi_0 B$ be a finitely generated ideal of definition. Then the J-adic topology on $\pi_0(B \otimes_A R) \simeq (\pi_0 B)/I(\pi_0 B)$ coincides with the discrete topology. It follows that $J^n \subseteq I(\pi_0 B)$ for $n \gg 0$, so that $I(\pi_0 B)$ is also an ideal of definition for $\pi_0 B$. □

Corollary 8.3.1.9. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. The following conditions are equivalent:*

(1) *The morphism f is representable.*

(2) *For every commutative diagram*

$$\begin{array}{ccc} \operatorname{Spf} B & \longrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \operatorname{Spf} A & \longrightarrow & \mathfrak{Y} \end{array}$$

where the horizontal maps are étale and f' is induced by a morphism of adic $\mathbb{E}_\infty$-rings $A \to B$, if $I \subseteq \pi_0 A$ is a finitely generated ideal of definition, then $I(\pi_0 B)$ is an ideal of definition for B.

Proof. Suppose first that (1) is satisfied and consider a diagram

$$\begin{array}{ccc} \operatorname{Spf} B & \longrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \operatorname{Spf} A & \longrightarrow & \mathfrak{Y} \end{array}$$

as in (2). Then f' can be written as a composition $\operatorname{Spf} B \to (\mathsf{X} \times_{\mathsf{Y}} \operatorname{Spf} A) \to \operatorname{Spf} A$, where the first map is étale (and therefore representable by Remark 8.3.1.6) and the second map is representable by Remark 8.3.1.5. Applying Remark 8.3.1.4, we see that f' is representable, so that $I(\pi_0 B)$ is an ideal of definition for $\pi_0 B$ by virtue of Proposition 8.3.1.7.

Now suppose that (2) is satisfied; we wish to show that f is representable. Using Remark 8.3.1.5, we can reduce to the case where $\mathfrak{Y} = \operatorname{Spf} A$ is affine. We wish to show that for any morphism $g : \mathsf{Y}_0 \to \mathfrak{Y}$, where Y_0 is a spectral Deligne-Mumford stack, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} \mathsf{Y}_0$ is also a spectral Deligne-Mumford stack. Choose an étale surjection $\amalg \operatorname{Spf} B_\alpha \to \mathfrak{X}$. Then we have an étale surjection $\amalg \operatorname{Spf} B_\alpha \times_{\operatorname{Spf} A} \mathsf{Y}_0 \to \mathfrak{X} \times_{\mathfrak{Y}} \mathsf{Y}_0$. It will therefore suffice to show that each fiber product $\operatorname{Spf} B_\alpha \times_{\operatorname{Spf} A} \mathsf{Y}_0$ is a spectral Deligne-Mumford stack. Without loss of generality, we may assume that each B_α is complete (Remark 8.1.2.4), so that the map $\operatorname{Spf} B_\alpha \to \operatorname{Spf} A$ is induced by a morphism of adic $\mathbb{E}_\infty$-rings $A \to B_\alpha$ (Corollary **??**). The desired result now follows from assumption (2) together with Proposition 8.3.1.7. □

Corollary 8.3.1.10. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. If $\mathfrak{X}$ is a spectral Deligne-Mumford stack, then f is representable.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} \operatorname{Spf} B & \longrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \operatorname{Spf} A & \longrightarrow & \mathfrak{Y} \end{array}$$

as in Corollary 8.3.1.9. Let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition. Our assumption that $\mathfrak{X}$ is a spectral Deligne-Mumford stack guarantees that the topology on $\pi_0 B$ is discrete, so the map $A \to B$ annihilates some power of I. It follows that $I(\pi_0 B)$ is a nilpotent ideal (and therefore an ideal of definition for $\pi_0 B$). □

Corollary 8.3.1.11. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the diagonal map $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable.*

Proof. Let $g : \mathsf{X} \to \mathfrak{X} \times \mathfrak{X}$ be a morphism of formal spectral Deligne-Mumford stacks, where X is a spectral Deligne-Mumford stack. Let Y denote the fiber product $\mathsf{X} \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X}$. We wish to show that Y is a spectral Deligne-Mumford stack. Identifying g with a pair of maps $g_0, g_1 : \mathsf{X} \to \mathfrak{X}$, we have a pullback diagram

$$\begin{array}{ccc} \mathsf{Y} & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow{\scriptstyle g_0} \\ \mathsf{X} & \xrightarrow{g_1} & \mathfrak{X}. \end{array}$$

The desired result now follows from the fact that g_0 is representable (Corollary 8.3.1.10). □

Proposition 8.3.1.12. *Let $f : X \to Y$ be a representable morphism in the ∞-category* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. *If Y is representable by a formal spectral Deligne-Mumford stack, then X is representable by a formal spectral Deligne-Mumford stack.*

Proof. Let $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathfrak{Y}})$ be a formal spectral Deligne-Mumford stack which represents the functor Y. For each object $V \in \mathcal{Y}$, let $\mathfrak{Y}_V = (\mathcal{Y}_{/V}, \mathcal{O}_{\mathfrak{Y}}|_V)$ and let Y_V be the functor represented by $\mathfrak{Y}_U$. Let us say that $V \in \mathcal{Y}$ is *good* if the fiber product $X \times_Y Y_V$ is representable by a formal spectral Deligne-Mumford stack. The collection of good objects of $\mathcal{Y}$ is closed under colimits. Consequently, it will suffice to show that every affine object $U \in \mathcal{Y}$ is good (Proposition 8.1.3.7). We may therefore reduce to the case where $\mathfrak{Y}$ is affine.

Choose an equivalence $\mathfrak{Y} \simeq \mathrm{Spf}\, A$, where A is an adic $\mathbb{E}_\infty$-ring with finitely generated ideal of definition $I \subseteq \pi_0 A$. Let $\{A_n\}_{n>0}$ be a tower of A-algebras satisfying the requirements of Lemma 8.1.2.2. Using our assumption that f is representable, we deduce that each fiber product $X_n = X \times_Y \mathrm{Spec}\, A_n$ is representable by a spectral Deligne-Mumford stack $\mathsf{X}_n = (\mathcal{X}_n, \mathcal{O}_n)$. These spectral Deligne-Mumford stacks fit into pullback diagrams

$$\begin{array}{ccc} \mathsf{X}_{n+1} & \longrightarrow & \mathsf{X}_n \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}\, A_{n+1} & \longrightarrow & \mathrm{Sp\acute{e}t}\, A_n. \end{array}$$

Since each of the maps $\pi_0 A_{n+1} \to \pi_0 A_n$ is a surjection with nilpotent kernel, the underlying geometric morphism $\mathcal{X}_{n+1} \to \mathcal{X}_n$ is an equivalence of ∞-topoi. We may therefore identify each $\mathcal{X}_n$ with a single ∞-topos $\mathcal{X}$. Set $\mathcal{O}_{\mathfrak{X}} = \varprojlim \mathcal{O}_n \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ and define $\mathfrak{X} = (\mathcal{X}, \widehat{\mathcal{O}})$. To complete the proof, it will suffice to show verify the following:

(a) The spectrally ringed ∞-topos $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ is a formal spectral Deligne-Mumford stack.

(b) The functor X is represented by $\mathfrak{X}$.

We begin by proving (a). Fix an object $U \in \mathcal{X}$ such that $(\mathcal{X}_{/U}, \mathcal{O}_1|_U)$ is affine. Then $(\mathcal{X}_{/U}, \mathcal{O}_n|_U)$ is affine for all $n > 0$ (see Lemma 8.3.3.2). Write $(\mathcal{X}_{/U}, \mathcal{O}_n|_U) \simeq \mathrm{Sp\acute{e}t}\, B_n$ for some A_n-algebra B_n. Let B be the image of the tower $\{B_n\}_{n>0}$ under the (symmetric monoidal) equivalence of ∞-categories $\varprojlim \mathrm{Mod}^{\mathrm{cn}}_{A_n} \simeq (\mathrm{Mod}^{\mathrm{Cpl}(I)}_A)^{\mathrm{cn}}$ supplied by Theorem 8.3.4.4, so that we have equivalences $B_n \simeq A_n \otimes_A B$. Let us regard B as an adic $\mathbb{E}_\infty$-ring with ideal of definition $I(\pi_0 B)$. Note that the tower $\{B_n\}$ satisfies the requirements of Lemma 8.1.2.2 for B, so the proof of Proposition 8.1.2.1 yields an equivalence $(\mathcal{X}_{/U}, \mathcal{O}_{\mathfrak{X}}|_U) \simeq \mathrm{Spf}\, B$. Allowing U to vary, we deduce that $\mathfrak{X}$ is a formal spectral Deligne-Mumford stack.

To prove (b), let $X' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by $\mathfrak{X}$. For each $n > 0$, the projection map $\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_n$ determines a natural transformation of functors $X_n \to X'$.

These maps are compatible as n varies, and give rise to a map $\alpha : X = \varinjlim_n X_n \to X'$. For each object $U \in \mathcal{X}$, let X'_U denote the functor represented by $(\mathcal{X}_{/U}, \mathscr{O}_{\mathfrak{X}}|_U)$ and set $X_U = X'_U \times_{X'} X$. Using Proposition 8.1.5.2, we deduce that the projection map $X_U \to X'_U$ is an equivalence whenever U is affine. Since the functors X and X' are both sheaves with respect to the étale topology and the map $\amalg_U X'_U \to X'$ is an effective epimorphism with respect to the étale topology (where the coproduct is taken over all affine objects $U \in \mathcal{X}$), it follows that α is an equivalence, so that X is representable by $\mathfrak{X}$. □

8.3.2 Inverse Images of Quasi-Coherent Sheaves

We now discuss the functorial dependence of the ∞-categories $\mathrm{NilCoh}(\mathfrak{X})$, $\mathrm{WCoh}(\mathfrak{X})$, and $\mathrm{QCoh}(\mathfrak{X})$ on the formal spectral Deligne-Mumford stack $\mathfrak{X}$. Recall that any morphism of spectrally ringed ∞-topoi $f : \mathfrak{X} \to \mathfrak{Y}$ determines a pair of adjoint functors

$$\mathrm{Mod}_{\mathscr{O}_{\mathfrak{Y}}} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}} .$$

Proposition 8.3.2.1. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. Then the functor $f^* : \mathrm{Mod}_{\mathscr{O}_{\mathfrak{Y}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ carries $\mathrm{WCoh}(\mathfrak{Y})$ to $\mathrm{WCoh}(\mathfrak{X})$.*

The proof of Proposition 8.3.2.1 will require some preliminaries.

Proposition 8.3.2.2. *Let A be an adic $\mathbb{E}_\infty$-ring with finitely generated ideal of definition $I \subseteq \pi_0 A$, and regard $\mathrm{Mod}_{\mathscr{O}_{\mathrm{Spf}\, A}}$ as a stable A-linear ∞-category. Let $\mathscr{F}$ be a $\mathscr{O}_{\mathrm{Spf}\, A}$-module. Then $\mathscr{F}$ is I-local if and only if it belongs to the full subcategory $\mathrm{WCoh}^\circ(\mathrm{Spf}\, A) \subseteq \mathrm{Mod}_{\mathscr{O}_{\mathrm{Spf}\, A}}$ appearing in Corollary 8.2.3.7.*

Proof. Without loss of generality, we may assume that A is complete. Let us identify $\mathscr{O}_{\mathfrak{X}}$ with the functor

$$\mathrm{CAlg}_A^{\text{ét}} \to \mathrm{CAlg}_A \qquad B \mapsto B_I^\wedge,$$

and view $\mathscr{F}$ as a $\mathscr{O}_{\mathfrak{X}}$-module object of the presheaf ∞-category $\mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathrm{Sp})$. Define $\mathscr{F}^\circ : \mathrm{CAlg}_A^{\text{ét}} \to \mathrm{Sp}$ as in the proof of Proposition 8.2.3.12, so that $\mathscr{F}^\circ(B) \simeq \Gamma_I(B_I^\wedge) \otimes_{B_I^\wedge} \mathscr{F}(B)$. Then $\mathscr{O}^{\approx}_{\mathrm{Spf}\, A} \otimes_{\mathscr{O}_{\mathrm{Spf}\, A}} \mathscr{F}$ can be identified with the sheafification of $\mathscr{F}^\circ$. If $\mathscr{F}$ is I-local, then $\mathscr{F}^\circ \simeq 0$, so that $\mathscr{O}^{\approx}_{\mathrm{Spf}\, A} \otimes_{\mathscr{O}_{\mathrm{Spf}\, A}} \mathscr{F} \simeq 0$ and $\mathscr{F} \in \mathrm{WCoh}^\circ(\mathrm{Spf}\, A)$, as desired. Conversely, suppose that $\mathscr{F} \in \mathrm{WCoh}^\circ(\mathrm{Spf}\, A)$. Then $\mathscr{F}$ is weakly coherent. The proof of Proposition 8.2.3.12 shows that $\mathscr{F}^\circ$ is already a sheaf, so we have $\mathscr{F}^\circ \simeq \mathscr{O}^{\approx}_{\mathrm{Spf}\, A} \otimes_{\mathscr{O}_{\mathrm{Spf}\, A}} \mathscr{F} \simeq 0$. It follows that $\mathscr{F}^\circ(B) \simeq 0$ for all $B \in \mathrm{CAlg}_A^{\text{ét}}$, which implies that $\mathscr{F}(B)$ is I-local. Allowing B to vary, we conclude that $\mathscr{F}$ is I-local. □

Corollary 8.3.2.3. *Let A be an adic $\mathbb{E}_\infty$-ring with finitely generated ideal of definition $I \subseteq \pi_0 A$, and regard $\mathrm{WCoh}(\mathrm{Spf}\, A)$ as a stable A-linear ∞-category. Then:*

(*a*) *An object* $\mathscr{F} \in \mathrm{WCoh}(\mathrm{Spf}\, A)$ *is nilcoherent if and only if it is* I*-nilpotent.*

(*b*) *An object* $\mathscr{F} \in \mathrm{WCoh}(\mathrm{Spf}\, A)$ *is quasi-coherent if and only if it is* I*-complete.*

Proof. Combine Proposition 8.3.2.2 with Corollaries 8.2.3.7 and 8.2.4.10. □

Corollary 8.3.2.4. *Let* $f : \mathfrak{X} \to \mathfrak{Y}$ *be a morphism of formal spectral Deligne-Mumford stacks. Then the functor* $f^* : \mathrm{Mod}_{\mathscr{O}_{\mathfrak{Y}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ *carries* $\mathrm{WCoh}^{\circ}(\mathfrak{Y})$ *to* $\mathrm{WCoh}^{\circ}(\mathfrak{X})$*.*

Proof. Without loss of generality, we can assume that $\mathfrak{X} \simeq \mathrm{Spf}\, B$ and $\mathfrak{Y} \simeq \mathrm{Spf}\, A$ are affine, and that f is induced by a morphism of adic $\mathbb{E}_\infty$-rings. In this case, we can regard the pullback $f^* : \mathrm{Mod}_{\mathscr{O}_{\mathfrak{Y}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ as an A-linear functor. Let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition. Then f^* carries I-local objects of $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{Y}}}$ to I-local objects of $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$, which are then also J-local for any finitely generated ideal of definition $J \subseteq \pi_0 B$. The desired result now follows from the criterion of Proposition 8.3.2.2. □

Proof of Proposition 8.3.2.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks and let $\mathscr{F} \in \mathrm{WCoh}(\mathfrak{Y})$; we wish to show that $f^* \mathscr{F}$ is weakly quasi-coherent. Using Corollary 8.2.3.7, we obtain a fiber sequence $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ where $\mathscr{F}'$ is nilcoherent and $\mathscr{F}'' \in \mathrm{WCoh}^{\circ}(\mathfrak{Y})$. It follows from Proposition 8.3.2.2 that $f^* \mathscr{F}''$ is weakly quasi-coherent. It will therefore suffice to show that $f^* \mathscr{F}'$ is weakly quasi-coherent. We may therefore replace $\mathscr{F}$ by $\mathscr{F}'$ and thereby reduce to the case where $\mathscr{F}$ is nilcoherent.

Working locally on $\mathfrak{Y}$, we may further assume that $\mathfrak{Y} \simeq \mathrm{Spf}\, A$ for some adic $\mathbb{E}_\infty$-ring A. Let us regard $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{Y}}}$ and $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ as A-linear stable ∞-categories and f^* as an A-linear functor. Set $M = \Gamma(\mathfrak{Y}; \mathscr{F})$. Our assumption that $\mathscr{F}$ is nilcoherent guarantees that the canonical map $M \otimes_A \mathscr{O}_{\mathfrak{Y}} \to \mathscr{F}$ is an equivalence (see the proof of Proposition 8.2.1.3). We therefore obtain equivalences

$$f^* \mathscr{F} \simeq M \otimes_A f^* \mathscr{O}_{\mathfrak{Y}} \simeq M \otimes_A \mathscr{O}_{\mathfrak{X}}.$$

The full subcategory $\mathrm{WCoh}(\mathfrak{X}) \subseteq \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ is closed under small colimits and desuspensions (Remark 8.2.3.5), and is therefore an A-linear subcategory of $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. The desired result now follows from the weak quasi-coherence of the structure sheaf $\mathscr{O}_{\mathfrak{X}}$ (Example 8.2.3.3). □

In the case of a representable morphism, Proposition 8.3.2.1 admits the following refinement:

Proposition 8.3.2.5. *Let* $f : \mathfrak{X} \to \mathfrak{Y}$ *be a representable morphism of formal spectral Deligne-Mumford stacks. Then the pullback functor* $f^* : \mathrm{Mod}_{\mathscr{O}_{\mathfrak{Y}}} \to \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ *carries nilcoherent sheaves on* $\mathfrak{Y}$ *to nilcoherent sheaves on* $\mathfrak{X}$*.*

Proof. Working locally on $\mathfrak{X}$ and $\mathfrak{Y}$, we can assume that $\mathfrak{Y} = \operatorname{Spf} A$ and $\mathfrak{X} = \operatorname{Spf} B$ are affine. Moreover, we may assume without loss of generality that B is complete (Remark 8.1.2.4), so that f is induced by a morphism of adic $\mathbb{E}_\infty$-rings $A \to B$. Let us regard $\operatorname{Mod}_{\mathscr{O}_\mathfrak{Y}}$ and $\operatorname{Mod}_{\mathscr{O}_\mathfrak{X}}$ as stable A-linear ∞-categories (so that the pullback functor f^* is A-linear). Let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition. Let $\mathscr{F} \in \operatorname{NilCoh}(\mathfrak{Y})$ and set $M = \Gamma(\mathfrak{Y}; \mathscr{F})$, so that M is an I-nilpotent A-module and $\mathscr{F} \simeq M \otimes_A \mathscr{O}_\mathfrak{Y}$.

$$f^* \mathscr{F} \simeq M \otimes_A (f^* \mathscr{O}_\mathfrak{Y}) \simeq M_B \otimes_B (f^* \mathscr{O}_\mathfrak{X}),$$

where $M_B = M \otimes_A B$. Note that $M \otimes_A B$ is $I(\pi_0 B)$-nilpotent. Our representability assumption guarantees that $I(\pi_0 B)$ is an ideal of definition for $\pi_0 B$ (Proposition 8.3.1.7), so that $f^* \mathscr{F}$ is also nilcoherent. □

In the case where $f : \mathfrak{X} \to \mathfrak{Y}$ is not representable, the conclusion of Proposition 8.3.2.5 generally does not hold without some additional assumptions. For example, we have the following:

Proposition 8.3.2.6. *Let* X *be a spectral Deligne-Mumford stack, let* $K \subseteq |\mathsf{X}|$ *be a cocompact closed subset, and let* $i : \mathsf{X}^\wedge_K \to \mathsf{X}$ *exhibit* $\mathsf{X}^\wedge_K$ *as a formal completion of* X *along* K*. If* $\mathscr{F} \in \operatorname{QCoh}(\mathsf{X})$ *is supported on* K*, then* $i^* \mathscr{F}$ *is nilcoherent.*

Proof. The assertion is local, so we may assume without loss of generality that $\mathsf{X} \simeq \operatorname{Spét} A$ is affine and that K is the vanishing locus of some finitely generated ideal $I \subseteq \pi_0 A$. In this case, our assumption that $\mathscr{F}$ is supported on K guarantees that $M = \Gamma(\mathsf{X}; \mathscr{F})$ is an I-nilpotent A-module. We then have $\mathscr{F} \simeq M \otimes_A \mathscr{O}_\mathsf{X}$, so that $f^* \mathscr{F} = M \otimes_A \mathscr{O}_{\mathsf{X}^\wedge_K}$ is nilcoherent as desired (see the proof of Proposition 8.2.1.3). □

Remark 8.3.2.7. In the situation of Proposition 8.3.2.6, the pullback functor $i^* : \operatorname{QCoh}_K(\mathsf{X}) \to \operatorname{NilCoh}(\mathfrak{X}^\wedge_K)$ is homotopy inverse to the equivalence $i_* : \operatorname{NilCoh}(\mathfrak{X}^\wedge_K) \to \operatorname{QCoh}_K(\mathsf{X})$ of Corollary 8.2.1.6.

We now consider the behavior of quasi-coherent sheaves with respect to pullback. Here, the situation is a bit more subtle: if $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of formal spectral Deligne-Mumford stacks, then the pullback functor $f^* : \operatorname{WCoh}(\mathfrak{Y}) \to \operatorname{WCoh}(\mathfrak{X})$ need not carry quasi-coherent sheaves to quasi-coherent sheaves. Nevertheless, there is a well-defined *completed* pullback functor $f^\star : \operatorname{QCoh}(\mathfrak{Y}) \to \operatorname{QCoh}(\mathfrak{X})$.

Notation 8.3.2.8. Let $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{Mod}}$ be the ∞-category of Notation 2.2.1.3, whose objects are triples $(\mathcal{X}, \mathscr{O}, \mathscr{F})$ where $\mathcal{X}$ is an ∞-topos, $\mathscr{O}$ is a strictly Henselian sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and $\mathscr{F}$ is a $\mathscr{O}$-module. We let $\mathrm{fSpDM}_{\mathrm{WCoh}}$ denote the full subcategory of $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{Mod}}$ spanned by those triples $(\mathcal{X}, \mathscr{O}, \mathscr{F})$ where $(\mathcal{X}, \mathscr{O})$ is a formal spectral Deligne-Mumford stack and $\mathscr{F}$ is weakly quasi-coherent. It follows from Proposition 8.3.2.1 that the forgetful functor $(\mathcal{X}, \mathcal{O}, \mathscr{F}) \mapsto (\mathcal{X}, \mathcal{O})$ determines a Cartesian fibration $\mathrm{fSpDM}_{\mathrm{WCoh}} \to \mathrm{fSpDM}$, whose fiber over a formal spectral Deligne-Mumford stack $\mathfrak{X}$ can be identified with the ∞-category $\mathrm{WCoh}(\mathfrak{X})^{\mathrm{op}}$. We let $\mathrm{fSpDM}_{\mathrm{QCoh}}$ denote the full subcategory of $\mathrm{fSpDM}_{\mathrm{WCoh}}$ spanned by those triples $(\mathcal{X}, \mathcal{O}, \mathscr{F})$ where $\mathscr{F}$ is quasi-coherent.

Proposition 8.3.2.9. *The forgetful functor* $q : \mathrm{fSpDM}_{\mathrm{QCoh}} \to \mathrm{fSpDM}$ *is a Cartesian fibration.*

Remark 8.3.2.10. We can summarize the contents of Proposition 8.3.2.9 more informally as follows: the construction $\mathfrak{X} \mapsto \mathrm{QCoh}(\mathfrak{X})$ can be regarded as a functor $(\mathrm{fSpDM})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$, which assigns to each morphism $f : \mathfrak{X} \to \mathfrak{Y}$ a functor $f^{\star} : \mathrm{QCoh}(\mathfrak{Y}) \to \mathrm{QCoh}(\mathfrak{X})$. We will see below that the functor $f^{\star}$ is given on objects by the formula $f^{\star} \mathscr{F} = (f^* \mathscr{F})^{\wedge}$. We will refer to $f^{\star}$ as the *completed pullback functor* associated to f.

Proof of Proposition 8.3.2.9. Let $f : \mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathfrak{Y}}) \to (\mathcal{Z}, \mathcal{O}_{\mathfrak{Z}}) = \mathfrak{Z}$ be a morphism of formal spectral Deligne-Mumford stacks and let $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{Z})$. Then $f^* \mathscr{F}$ is weakly quasi-coherent, so its completion $\mathscr{F}' = (f^* \mathscr{F})^{\wedge}$ can be regarded as an object of $\mathrm{QCoh}(\mathfrak{Y})$. The composite map $u : \mathscr{F} \to f_* f^* \mathscr{F} \to f_* \mathscr{F}'$ determines a lift of f to a morphism $\overline{f} : (\mathcal{Y}, \mathcal{O}_{\mathfrak{Y}}, \mathscr{F}') \to (\mathcal{Z}, \mathcal{O}_{\mathfrak{Z}}, \mathscr{F})$ in the ∞-category $\mathrm{fSpDM}_{\mathrm{QCoh}}$. We claim that $\overline{f}$ is q-Cartesian. To prove this, we must show that for every morphism of formal spectral Deligne-Mumford stacks $g : \mathfrak{X} \to \mathfrak{Y}$ and every object $\mathscr{G} \in \mathrm{QCoh}(\mathfrak{X})$, composition with u induces a homotopy equivalence

$$\rho : \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{Y})}(\mathscr{F}', g_* \mathscr{G}) \to \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{Z})}(\mathscr{F}, (f \circ g)_* \mathscr{G}).$$

Unwinding the definitions, we can identify ρ with the first map appearing in the fiber sequence

$$\mathrm{Map}_{\mathrm{WCoh}(\mathfrak{X})}(g^* \mathscr{F}', \mathscr{G}) \to \mathrm{Map}_{\mathrm{WCoh}(\mathfrak{X})}((f \circ g)^* \mathscr{F}, \mathscr{G}) \to \mathrm{Map}_{\mathrm{WCoh}(\mathfrak{X})}(g^* \mathrm{fib}(f^* \mathscr{F} \to \mathscr{F}'), \mathscr{G}).$$

To prove that ρ is a homotopy equivalence, it will suffice to show that the mapping space $\mathrm{Map}_{\mathrm{WCoh}(\mathfrak{X})}(g^* \mathrm{fib}(f^* \mathscr{F} \to \mathscr{F}'), \mathscr{G})$ is contractible. By virtue of our assumption that $\mathscr{G}$ is quasi-coherent, we are reduced to showing that $g^* \mathrm{fib}(f^* \mathscr{F} \to \mathscr{F}')$ belongs to $\mathrm{WCoh}^{\circ}(\mathfrak{X})$ (Corollary 8.2.4.10). This follows from Proposition 8.3.2.2, since $\mathrm{fib}(f^* \mathscr{F} \to \mathscr{F}')$ belongs to $\mathrm{WCoh}^{\circ}(\mathfrak{Y})$ (again by virtue of Corollary 8.2.4.10). □

Remark 8.3.2.11. The inclusion functor $\mathrm{fSpDM}_{\mathrm{QCoh}} \hookrightarrow \mathrm{fSpDM}_{\mathrm{WCoh}}$ admits a right adjoint relative to fSpDM. In particular, for every morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of formal spectral Deligne-Mumford stacks, the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{WCoh}(\mathfrak{Y}) & \xrightarrow{f^*} & \mathrm{WCoh}(\mathfrak{X}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathfrak{Y}) & \xrightarrow{f^{\star}} & \mathrm{QCoh}(\mathfrak{X}) \end{array}$$

commutes up to canonical equivalence, where the vertical maps are given by the completion functors of Construction 8.2.4.1. It follows that we can regard $f^\star : \mathrm{QCoh}(\mathfrak{Y}) \to \mathrm{QCoh}(\mathfrak{X})$ as a symmetric monoidal functor, where $\mathrm{QCoh}(\mathfrak{X})$ and $\mathrm{QCoh}(\mathfrak{Y})$ are equipped with the symmetric monoidal structure described in Corollary 8.2.4.22.

Remark 8.3.2.12. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. If $\mathfrak{X}$ is a spectral Deligne-Mumford stack, then the completion functor $\mathrm{WCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X})$ is equivalent to the identity. Consequently, the completed pullback functor $f^\star$ of Remark 8.3.2.10 coincides with the usual pullback functor f^* (restricted to the ∞-category $\mathrm{QCoh}(\mathfrak{Y})$).

Remark 8.3.2.13. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. Then the completed pullback functor $f^\star : \mathrm{QCoh}(\mathfrak{Y}) \to \mathrm{QCoh}(\mathfrak{X})$ is right t-exact: that is, it carries $\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}$ into $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}$. This follows from Remark 8.2.5.9, since the pullback functor $f^* : \mathrm{WCoh}(\mathfrak{Y}) \to \mathrm{WCoh}(\mathfrak{X})$ carries $\mathrm{WCoh}(\mathfrak{Y})^{\mathrm{cn}}$ into $\mathrm{WCoh}(\mathfrak{X})^{\mathrm{cn}}$.

8.3.3 Direct Images of Quasi-Coherent Sheaves

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. Then f determines a direct image functor $f_* : \mathrm{Mod}_{\mathscr{O}_\mathfrak{X}} \to \mathrm{Mod}_{\mathscr{O}_\mathfrak{Y}}$. In good cases, this functor preserves quasi-coherence:

Proposition 8.3.3.1. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. Assume that f is representable by quasi-compact, quasi-separated spectral algebraic spaces: that is, for every map $\mathrm{Sp\acute{e}t}\, R \to \mathfrak{Y}$, the fiber product $\mathrm{Sp\acute{e}t}\, R \times_\mathfrak{Y} \mathfrak{X}$ is a quasi-compact, quasi-separated spectral algebraic space. Then the direct image functor $f_* : \mathrm{Mod}_{\mathscr{O}_\mathfrak{X}} \to \mathrm{Mod}_{\mathscr{O}_\mathfrak{Y}}$ carries $\mathrm{QCoh}(\mathfrak{X})$ into $\mathrm{QCoh}(\mathfrak{Y})$.*

The proof depends on the following:

Lemma 8.3.3.2. *Let A be an adic $\mathbb{E}_\infty$-ring with a finitely generated ideal of definition $I \subseteq \pi_0 A$, and let $f : \mathfrak{X} \to \mathrm{Spf}\, A$ be a representable morphism of formal spectral Deligne-Mumford stacks. The following conditions are equivalent:*

- (a) *The formal spectral Deligne-Mumford stack $\mathfrak{X}$ is affine.*
- (b) *The spectral Deligne-Mumford stack $\mathfrak{X} \times_{\mathrm{Spf}\, A} \mathrm{Sp\acute{e}t}(\pi_0 A/I)$ is affine.*

Proof. The implication $(a) \Rightarrow (b)$ is clear. Suppose that (b) is satisfied. Applying Lemma 17.1.3.7 repeatedly, we deduce the following stronger version of (b):

- (b') Let $J \subseteq \pi_0 A$ be an ideal containing I^n for some $n \gg 0$. Then the spectral Deligne-Mumford stack $\mathfrak{X} \times_{\mathrm{Spf}\, A} \mathrm{Sp\acute{e}t}(\pi_0 A/J)$ is affine.

Let $\{A_n\}_{n>0}$ be a tower of A-algebras satisfying the requirements of Lemma 8.1.2.2. Combining (b') with Corollary **??**, we deduce that $\mathfrak{X} \times_{\operatorname{Spf} A} \operatorname{Spét} A_n$ is affine for $n > 0$. Write $\mathfrak{X} \times_{\operatorname{Spf} A} \operatorname{Spét} A_n \simeq \operatorname{Spét} B_n$ for some $B_n \in \mathrm{CAlg}^{\mathrm{cn}}_{A_n}$. Then we can regard $\{B_n\}_{n>0}$ as a commutative algebra object of $\varprojlim \mathrm{Mod}^{\mathrm{cn}}_{A_n}$. Let $B \in \mathrm{CAlg}^{\mathrm{cn}}_A$ denote the image of the tower $\{B_n\}_{n>0}$ under the (symmetric monoidal) equivalence $\varprojlim \mathrm{Mod}^{\mathrm{cn}}_{A_n} \simeq (\mathrm{Mod}^{\mathrm{Cpl}(I)}_A)^{\mathrm{cn}}$ supplied by Proposition 8.3.4.4. Let us regard B as an adic $\mathbb{E}_\infty$-ring with ideal of definition $I(\pi_0 B)$. Using Proposition 8.1.5.2, we deduce that $\mathfrak{X}$ and $\operatorname{Spf} B$ represent the same functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and are therefore equivalent (Theorem 8.1.5.1). □

Lemma 8.3.3.3. *Let $\phi : A \to B$ be a morphism of adic $\mathbb{E}_\infty$-rings and let $f : \operatorname{Spf} B \to \operatorname{Spf} A$ be the induced map of formal spectra. Assume that f is representable. Then the direct image functor $f_* : \mathrm{Mod}_{\mathscr{O}_{\operatorname{Spf} B}} \to \mathrm{Mod}_{\mathscr{O}_{\operatorname{Spf} A}}$ carries $\mathrm{QCoh}(\operatorname{Spf} B)$ to $\mathrm{QCoh}(\operatorname{Spf} A)$.*

Proof. Let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition, so that $I(\pi_0 B)$ is an ideal of definition for B. Let $\mathscr{F} \in \mathrm{QCoh}(\operatorname{Spf} B)$ and set $M = \Gamma(\operatorname{Spf} B; \mathscr{F}) \in \mathrm{Mod}_B$. As a spectrum-valued functor on $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_B$, we can identify $\mathscr{F}$ with the functor given by $B' \mapsto (M \otimes_B B')^\wedge_I$ (see Remark 8.2.4.16). It follows that $f_* \mathscr{F}$ can be identified with the functor $\mathrm{CAlg}^{\text{ét}}_A \to \mathrm{Sp}$ given by $A' \mapsto (M \otimes_B (B \otimes_A A')^\wedge_I \simeq (M \otimes_A A')^\wedge_I$, and is therefore quasi-coherent (Remark 8.2.4.16). □

Proof of Proposition 8.3.3.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks which is representable by quasi-compact, quasi-separated spectral algebraic spaces and let $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$; we wish to show that $f_* \mathscr{F}$ is quasi-coherent. This can be tested locally on $\mathfrak{Y}$ (Proposition 8.2.4.17), so we may assume without loss of generality that $\mathfrak{Y} = \operatorname{Spf} A$ is affine. Let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition and set $R = (\pi_0 A)/I$. Let $\mathcal{X}$ be the underlying ∞-topos of $\mathfrak{X}$, and for each object $U \in \mathcal{X}$ set $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathfrak{X}}|_U)$ and let $f_U : \mathfrak{X}_U \to \mathfrak{Y}$ be the canonical map. Let us say that U is *good* if $f_{U*}(\mathscr{F}|_U) \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{Y}}}$ is quasi-coherent. The collection of good objects of $\mathcal{X}$ contains all affine objects (Lemma 8.3.3.3) and is closed under finite colimits. Write $\operatorname{Spét} R \times_{\mathfrak{Y}} \mathfrak{X} \simeq (\mathcal{X}, \mathscr{O}_0)$. Note that an object $U \in \mathcal{X}$ is affine if and only if the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathscr{O}_0|_U)$ is affine. Since $(\mathcal{X}, \mathscr{O}_0)$ is a quasi-compact, quasi-separated spectral algebraic space, Combining Proposition 2.5.3.5 with Theorem 3.4.2.1, we deduce that the final object of $\mathcal{X}$ is good, so that $f_* \mathscr{F}$ is quasi-coherent. □

Remark 8.3.3.4. In the situation of Proposition 8.3.3.1, the direct image functor $f_* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{Y})$ can be regarded as a right adjoint to the completed pullback functor $f^\star : \mathrm{QCoh}(\mathfrak{Y}) \to \mathrm{QCoh}(\mathfrak{X})$ of Remark 8.3.2.10.

Remark 8.3.3.5. In the situation of Proposition 8.3.3.1, if $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$ is almost connective, then $f_* \mathscr{F} \in \mathrm{QCoh}(\mathfrak{Y})$ is almost connective.

Proposition 8.3.3.6. *Suppose we are given a pullback diagram of formal spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}. \end{array}$$

Assume that f is representable by quasi-compact, quasi-separated spectral algebraic spaces (so that f' has the same property). Then the diagram of completed pullback functors

$$\begin{array}{ccc} \mathrm{QCoh}(\mathfrak{Y}) & \xrightarrow{f^\star} & \mathrm{QCoh}(\mathfrak{X}) \\ \downarrow{\scriptstyle g^\star} & & \downarrow{\scriptstyle g'^\star} \\ \mathrm{QCoh}(\mathfrak{Y}') & \xrightarrow{g'^\star} & \mathrm{QCoh}(\mathfrak{X}') \end{array}$$

is right adjointable. In other words, for every quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$, the Beck-Chevalley map $\rho : g^\star f_* \mathscr{F} \to f'_* g'^\star \mathscr{F}$ is an equivalence.

Proof. The assertion is local on both $\mathfrak{Y}$ and $\mathfrak{Y}'$, so we can assume that $\mathfrak{Y} \simeq \mathrm{Spf}\, A$ and $\mathfrak{Y}' \simeq \mathrm{Spf}\, A'$ are affine. Write $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$. For each object $U \in \mathcal{X}$, set $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathfrak{X}}|_U)$ and $\mathfrak{X}'_U = \mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{X}_U$, so that we have a pullback diagram

$$\begin{array}{ccc} \mathfrak{X}'_U & \xrightarrow{g'_U} & \mathfrak{X}_U \\ \downarrow{\scriptstyle f'_U} & & \downarrow{\scriptstyle f_U} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}. \end{array}$$

Let us say that U is *good* if the induced map $g^\star f_{U*} \mathscr{F}|_U \to f'_{U*} g'^\star_U \mathscr{F}|_U$ is an equivalence. To complete the proof, it will suffice to show that the final object of $\mathcal{X}$ is good. Note that the collection of good objects of $\mathcal{X}$ is closed under pushouts. Arguing as in the proof of Proposition 8.3.3.1, we are reduced to showing that every affine object of $\mathcal{X}$ is good. We are therefore reduced to proving Proposition 8.3.3.6 in the special case where $\mathfrak{X} \simeq \mathrm{Spf}\, B$ is also affine. Let $I' \subseteq \pi_0 A'$ be a finitely generated ideal of definition, so that $\mathfrak{X}' \simeq \mathrm{Spf}\, B'$, where $B' = A' \otimes_A B$ is regarded as an adic $\mathbb{E}_\infty$-ring with ideal of definition $I'(\pi_0 B')$. It now suffices to observe that the domain and codomain of ρ can be identified with the quasi-coherent sheaf on $\mathrm{Spf}\, A'$ associated to the I'-completion of $A' \otimes_A \Gamma(\mathrm{Spf}\, B; \mathscr{F}) \simeq B' \otimes_B \Gamma(\mathrm{Spf}\, B; \mathscr{F})$. □

8.3.4 Quasi-Coherent Sheaves on Functors

In §6.2, we introduced the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on an arbitrary functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ (Definition 6.2.2.1). In the special case where X is representable by a spectral Deligne-Mumford stack X, we have a canonical equivalence of ∞-categories $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(\mathsf{X})$ (Proposition 6.2.4.1). We now prove a similar (but slightly weaker) result in the setting of formal spectral Deligne-Mumford stacks.

Construction 8.3.4.1. Let fSpDM denote the ∞-category of formal spectral Deligne-Mumford stacks. It follows from Proposition 8.3.2.9 that we can regard the construction $\mathfrak{X} \to \mathrm{QCoh}(\mathfrak{X})$ as a functor $\mathrm{QCoh} : (\mathrm{fSpDM})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. Moreover, the composite functor

$$\mathrm{CAlg}^{\mathrm{cn}} \xrightarrow{\mathrm{Sp\acute{e}t}} (\mathrm{fSpDM})^{\mathrm{op}} \xrightarrow{\mathrm{NilCoh}} \widehat{\mathcal{C}\mathrm{at}}_\infty$$

can be identified with the functor $A \mapsto \mathrm{Mod}_A$ (see Example 8.2.4.8) Note that this composite functor admits a right Kan extension along the fully faithful embedding $\mathrm{Sp\acute{e}t} : \mathrm{CAlg}^{\mathrm{cn}} \to (\mathrm{fSpDM})^{\mathrm{op}}$, given by the construction $\mathfrak{X} \mapsto \mathrm{QCoh}(h_{\mathfrak{X}})$ of Definition 6.2.2.1; here $h_{\mathfrak{X}}$ denotes the functor represented by $\mathfrak{X}$. We therefore obtain a canonical map $\Theta_{\mathfrak{X}} : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(h_{\mathfrak{X}})$, which depends functorially on $\mathfrak{X}$ and is an equivalence when $\mathfrak{X} = \mathrm{Sp\acute{e}t}\, A$ for some connective $\mathbb{E}_\infty$-ring A.

Remark 8.3.4.2. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the ∞-category $\mathrm{QCoh}(h_{\mathfrak{X}})$ can be described informally as follows: an object of $\mathrm{QCoh}(h_{\mathfrak{X}})$ is rule which assigns to each map $\eta : \mathrm{Sp\acute{e}t}\, R \to \mathfrak{X}$ an R-module M_η, which is functorial in the sense that if $\eta' : \mathrm{Sp\acute{e}t}\, R' \to \mathfrak{X}$ is given by a composition $\mathrm{Sp\acute{e}t}\, R' \to \mathrm{Sp\acute{e}t}\, R \xrightarrow{\eta} \mathfrak{X}$, then we have a canonical equivalence $M_{\eta'} \times R' \otimes_R M_\eta$. We can describe the functor $\Theta_{\mathfrak{X}}$ of Construction 8.3.4.1 as follows: if $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$ is a quasi-coherent sheaf on $\mathfrak{X}$, then $\Theta_{\mathfrak{X}}(\mathscr{F})$ is the quasi-coherent sheaf on $h_{\mathfrak{X}}$ given by $\Theta_{\mathfrak{X}}(\mathscr{F})_\eta = \Gamma(\mathrm{Sp\acute{e}t}\, R; \eta^* \mathscr{F})$.

Remark 8.3.4.3. In the special case where X is a spectral Deligne-Mumford stack, the functor Θ_{X} coincides with the equivalence $\mathrm{QCoh}(\mathsf{X}) \simeq \mathrm{QCoh}(h_{\mathsf{X}})$ of Proposition 6.2.4.1.

We can now formulate our main result:

Theorem 8.3.4.4. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the functor $\Theta_{\mathfrak{X}} : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(h_{\mathfrak{X}})$ induces an equivalence of ∞-categories $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}} \to \mathrm{QCoh}(h_{\mathfrak{X}})^{\mathrm{cn}}$.*

Remark 8.3.4.5. We do not know if the connectivity hypotheses appearing in Theorem 8.3.4.4 can be dropped: that is, we do not know if the map $\Theta_{\mathfrak{X}}$ is itself an equivalence of ∞-categories. However, the analogous assertion holds in the setting of (formal) derived algebraic geometry. In particular, one can show that the map $\Theta_{\mathfrak{X}}$ is an equivalence if there exists a map $\mathfrak{X} \to \mathrm{Sp\acute{e}t}\, \mathbf{Q}$.

Proof of Theorem 8.3.4.4. Without loss of generality, we may assume that $\mathfrak{X}$ is affine. Choose an equivalence $\mathfrak{X} \simeq \mathrm{Spf}\, A$, where A is an adic $\mathbb{E}_\infty$-ring. Let $I \subseteq \pi_0 A$ be a finitely generated idea l of definition and let $\{A_n\}_{n>0}$ be a tower of $\mathbb{E}_\infty$-algebras over A which satisfies the requirements of Lemma 8.1.2.2. We then have an equivalence of functors $h_{\mathfrak{X}} \simeq \varinjlim \operatorname{Spec} A_n$, hence an equivalence of ∞-categories $\operatorname{QCoh}(h_{\mathfrak{X}}) \simeq \varprojlim \operatorname{Mod}_{A_n}$. Corollary 8.2.4.15 supplies an equivalence of ∞-categories $\operatorname{QCoh}(\mathfrak{X}) \simeq \operatorname{Mod}_A^{\operatorname{Cpl}(I)}$. Under these equivalences, we can identify $\Theta_{\mathfrak{X}}$ with the functor $F : \operatorname{Mod}_A^{\operatorname{Cpl}(I)} \to \varprojlim \operatorname{Mod}_{A_n}$ given on objects by $M \mapsto \{A_n \otimes_A M\}_{n>0}$. This functor admits a right adjoint G, which carries a compatible system $\{M_n \in \operatorname{Mod}_{A_n}\}_{n \geqslant 0}$ to the limit $G(\{M_n\}) = \varprojlim_n M_n \in \operatorname{Mod}_A$. By assumption each of the maps $A_{n+1} \to A_n$ is surjective on π_0. It follows that if each M_k is connective, then each of the maps $M_{n+1} \to A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$ is surjective on π_0. In particular, $G(\{M_n\})$ is a connective A-module and each of the maps $G(\{M_n\}) \to M_n$ is surjective on π_0. Consequently, the restrictions of F and G yield a pair of adjoint functors

$$\operatorname{Mod}_A^{\operatorname{Cpl}(I)} \cap \operatorname{Mod}_A^{\operatorname{cn}} \underset{G^{\operatorname{cn}}}{\overset{F^{\operatorname{cn}}}{\rightleftarrows}} \varprojlim \operatorname{Mod}_{A_n}^{\operatorname{cn}}.$$

We wish to show that F^{cn} and G^{cn} are mutually inverse equivalences.

It follows from Lemma 8.1.2.3 that the unit map $\operatorname{id} \to G^{\operatorname{cn}} \circ F^{\operatorname{cn}}$ is an equivalence: that is, the functor F^{cn} induces a fully faithful embedding $\operatorname{Mod}_A^{\operatorname{Cpl}(I)} \cap \operatorname{Mod}_A^{\operatorname{cn}} \to \varprojlim \operatorname{Mod}_{A_n}^{\operatorname{cn}}$. To complete the proof, it will therefore suffice to show that the functor G^{cn} is conservative. Since G is an exact functor between stable ∞-categories, it will suffice to show that if $\{M_n\}$ is an object of $\varprojlim \operatorname{Mod}_{A_n}^{\operatorname{cn}}$ satisfying $G(\{M_n\}) \simeq 0$, then each $M_n \in \operatorname{Mod}_{A_n}$ vanishes. We prove by induction k that $\pi_i M_n \simeq 0$ for $i \leqslant k$. When $k = 0$, this follows from our observation that each of the maps $\pi_0 G(\{M_n\}) \to \pi_0 M_n$ is surjective. If $k > 0$, the inductive hypothesis implies that $\{M_n\}$ is the k-fold suspension of an object $\{N_n\} \in \varprojlim \operatorname{Mod}_{A_n}^{\operatorname{cn}}$. Then $G(\{N_n\}) \simeq 0$ and we can apply the inductive hypothesis to deduce that $\pi_k M_n \simeq \pi_0 N_n \simeq 0$. □

Corollary 8.3.4.6. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then the functor $\Theta_{\mathfrak{X}} : \operatorname{QCoh}(\mathfrak{X}) \to \operatorname{QCoh}(h_{\mathfrak{X}})$ induces an equivalence of ∞-categories $F : \operatorname{QCoh}(\mathfrak{X})^{\operatorname{acn}} \to \operatorname{QCoh}(h_{\mathfrak{X}})^{\operatorname{acn}}$.*

Proof. The assertion is local on $\mathfrak{X}$, so we may assume without loss of generality that $\mathfrak{X}$ is affine. In this case, we can write F as a filtered colimit of functors $F_{\geqslant -n} : \Sigma^{-n} \operatorname{QCoh}(\mathfrak{X})^{\operatorname{cn}} \to \Sigma^{-n} \operatorname{QCoh}(h_{\mathfrak{X}})^{\operatorname{cn}}$, each of which is an equivalence by virtue of Theorem 8.3.4.4. □

Under mild hypotheses, the equivalence of Theorem 8.3.4.4 is compatible with the formation of direct images:

Proposition 8.3.4.7. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. Assume that f is representable by quasi-compact, quasi-separated spectral algebraic spaces (see Proposition 8.3.3.1). Let $X, Y : \operatorname{CAlg}^{\operatorname{cn}} \to \mathcal{S}$ denote the functors represented by $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, so that f determines a natural transformation $F : X \to Y$. Then the* *commutative diagram of ∞-categories*

$$\begin{array}{ccc} \mathrm{QCoh}(\mathfrak{Y}) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathfrak{X}) \\ \downarrow{\scriptstyle \Theta_{\mathfrak{Y}}} & & \downarrow{\scriptstyle \Theta_{\mathfrak{X}}} \\ \mathrm{QCoh}(Y) & \xrightarrow{F^*} & \mathrm{QCoh}(X) \end{array}$$

is right adjointable. In other words, for every quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$, the canonical map $\Theta_{\mathfrak{Y}}(f_* \mathscr{F}) \to F_*(\Theta_{\mathfrak{X}} \mathscr{F})$ is an equivalence in $\mathrm{QCoh}(X)$.

Proof. Apply Proposition 8.3.3.6. □

8.3.5 Finiteness Conditions on Quasi-Coherent Sheaves

We close this section by showing that the equivalence of Corollary 8.3.4.4 is compatible with several natural finiteness conditions on quasi-coherent sheaves.

Definition 8.3.5.1. Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}$. Then:

- We say that $\mathscr{F}$ is *perfect to order n* if, for every affine object $U \in \mathcal{X}$, the spectrum $\mathscr{F}(U)$ is perfect to order n when regarded as a module over $\mathscr{O}_{\mathfrak{X}}(U)$.
- We say that $\mathscr{F}$ is *almost perfect* if, for every affine object $U \in \mathcal{X}$, the spectrum $\mathscr{F}(U)$ is almost perfect when regarded as a module over $\mathscr{O}_{\mathfrak{X}}(U)$.
- We say that $\mathscr{F}$ is *perfect* if, for every affine object $U \in \mathcal{X}$, the spectrum $\mathscr{F}(U)$ is perfect when regarded as a module over $\mathscr{O}_{\mathfrak{X}}(U)$.
- We say that $\mathscr{F}$ is *locally free of finite rank* if, for every affine object $U \in \mathcal{X}$, the spectrum $\mathscr{F}(U)$ is locally free of finite rank when regarded as a module over $\mathscr{O}_{\mathfrak{X}}(U)$.

Our main result can be formulated as follows:

Theorem 8.3.5.2. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$ be a quasi-coherent sheaf on $\mathfrak{X}$. Assume that $\mathscr{F}$ is almost connective. Then the following conditions are equivalent:*

(1) *The sheaf $\mathscr{F}$ is perfect to order n (almost perfect, perfect, locally free of finite rank), in the sense of Definition 8.3.5.1.*

(2) *For every morphism $f : \mathsf{X} \to \mathfrak{X}$, where X is a spectral Deligne-Mumford stack, the pullback $f^* \mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is perfect to order n (almost perfect, perfect, locally free of finite rank), in the sense of Definition 2.8.4.4.*

(3) *The image of* $\mathscr{F}$ *under the equivalence* $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{acn}} \to \mathrm{QCoh}(h_{\mathfrak{X}})^{\mathrm{acn}}$ *of Corollary 8.3.4.6 is perfect to order* n *(almost perfect, perfect, locally free of finite rank), in the sense of Definition 6.2.5.3).*

Corollary 8.3.5.3. *Let* $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ *be a formal spectral Deligne-Mumford stack and let* $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$. *Then:*

(a) *If* $\mathscr{F}$ *is perfect to order* n *(almost perfect, perfect, locally free of finite rank), then* $\mathscr{F}|_U \in \mathrm{QCoh}((\mathcal{X}_{/U}, \mathcal{O}_{\mathfrak{X}}|_U))$ *is also perfect to order* n *(almost perfect, perfect, locally free of finite rank).*

(b) *If* $\mathcal{X}$ *admits a covering by objects* $\{U_\alpha\}$ *such that each restriction* $\mathscr{F}|_{U_\alpha} \in \mathrm{QCoh}((\mathcal{X}_{/U_\alpha}, \mathcal{O}_{\mathfrak{X}}|_{U_\alpha}))$ *is perfect to order* n *(almost perfect, perfect, locally free of finite rank), then* $\mathscr{F}$ *is also perfect to order* n *(almost perfect, perfect, locally free of finite rank).*

Proof. Assertion (a) follows immediately from the definitions. To prove (b), we can use Theorem 8.3.5.2 to reduce to the case where $\mathfrak{X}$ is a spectral Deligne-Mumford stack, in which case the desired result follows from Propositions 2.8.4.7 and 2.9.1.4. □

The proof of Theorem 8.3.5.2 will require some preliminaries. We begin with a somewhat technical assertion.

Lemma 8.3.5.4. *Suppose we are given a tower of connective* $\mathbb{E}_\infty$*-rings*

$$\cdots \to A_3 \to A_2 \to A_1$$

having limit A, *where each of the maps* $\pi_0 A_{i+1} \to \pi_0 A_i$ *is surjection whose kernel is a nilpotent ideal of* $\pi_0 A_{i+1}$. *For every integer* $i > 0$, *suppose we are given a connective* A_i*-module* M_i, *and for* $i > 1$ *a map of* A_{i-1}*-modules* $\phi_i : A_{i-1} \otimes_{A_i} M_i \to M_{i-1}$. *Let* $n \geqslant 0$ *be an integer. Suppose that each of the spectra* $\mathrm{fib}(\phi_i)$ *is* n*-connective, and that* M_1 *is perfect to order* $(n-1)$ *if* $n > 0$. *Then:*

(1) *If* $n > 0$, *then* $M = \varprojlim M_i$ *is perfect to order* $(n-1)$, *when regarded as an* A*-module.*

(2) *For every integer* i, *let* $\psi_i : A_i \otimes_A M \to M_i$ *be the canonical map. Then* $\mathrm{fib}(\psi_i)$ *is* n*-connective.*

Proof. Since each M_i is connective and each of the maps $\pi_0 M_{i+1} \to \pi_0 M_i$ is surjective, we deduce that M is connective and that each of the maps $\pi_0 M \to \pi_0 M_i$ is surjective. This proves (2) in the case $n = 0$ (and condition (1) is automatic). We handle the general case using induction on n. Assume that $n > 0$. Then $\pi_0 M_1$ is finitely generated as a module over $\pi_0 A_1$. We may therefore choose finitely many elements $x_1, \ldots, x_k \in \pi_0 M$ whose images generate $\pi_0 M_1$. The elements x_i determine a map of A-modules $A^k \to M$, which in turn determines a compatible family of A_i-module maps $\theta_i : A_i^k \to M_i$. We claim that each of the maps θ_i is surjective on connected components. This holds by hypothesis when $i = 0$. If $i > 0$, then the image of θ_i generates $\pi_0 M_i / J \pi_0 M_i \simeq \pi_0 M_{i-1}$, where J denotes the kernel of $\pi_0 A_i \to \pi_0 A_{i-1}$, and therefore generates $\pi_0 M_i$ by Nakayama's lemma (since J is a nilpotent ideal).

For $i \geqslant 0$, form a fiber sequence $N_i \to A_i^k \to M_i$, so that each N_i is connective. Note that if $n \geqslant 2$, then N_1 is perfect to order $n-2$ as an A_1-module. Moreover, we have maps $\phi'_i : A_{i-1} \otimes_{A_i} N_i \to N_{i-1}$ such that $\mathrm{fib}(\phi'_i) \simeq \Sigma^{-1} \mathrm{fib}(\phi_i)$ is $(n-1)$-connective for each i. Let $N = \varprojlim N_i$. Applying our inductive hypothesis, we deduce that each of the maps $\psi'_i : A_i \otimes_A N \to N_i$ is $(n-1)$-connective. This proves (2), since $\mathrm{fib}(\psi_i) \simeq \Sigma \mathrm{fib}(\psi'_i)$. Note that N is connective, and is perfect to order $n-2$ if $n \geqslant 2$. Using the fiber sequence $N \to A^k \to M$, we deduce that M is perfect to order $n-1$, which proves (1). □

Proposition 8.3.5.5. *Let A be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 A$, let M be an A-module which is I-complete and almost connective, and let n be an integer. The following conditions are equivalent:*

(1) *The A-module M is perfect to order n.*

(2) *Set $R = (\pi_0 A)/I$. Then the R-module $R \otimes_A M$ is perfect to order n.*

(3) *For every morphism of connective $\mathbb{E}_\infty$-rings $f : A \to B$ which annihilates some power of I, the B-module $B \otimes_A M$ is perfect to order n.*

Proof. The implication (1) ⇒ (2) follows from Proposition 2.7.3.1 and the implication (2) ⇒ (3) from Proposition 2.7.3.2. We will complete the proof by showing that (3) implies (1). Replacing M by $\Sigma^k M$ (and n by $n+k$) for $k \gg 0$ and thereby reduce to the case where M is connective and $n \geqslant 0$. Choose a tower of A-algebras $\cdots \to A_3 \to A_2 \to A_1$ satisfying the requirements of Lemma 8.1.2.2. Since A and M are connective and I-complete, Lemma 8.1.2.3 supplies equivalences

$$A \simeq \varprojlim A_i \qquad M \simeq \varprojlim A_i \otimes_A M$$

(see Lemma 8.1.2.3). By virtue of Lemma 8.3.5.4, to show that M is perfect to order n, it will suffice to show that $A_1 \otimes_R A$ is perfect to order n, which follows immediately from assumption (3). □

Corollary 8.3.5.6. *Let A be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 A$ and let M be an A-module which is almost connective. The following conditions are equivalent:*

(1) *The A-module M is almost perfect.*

(2) *Set $R = (\pi_0 A)/I$. Then M is I-complete and the R-module $R \otimes_A M$ is almost perfect.*

(3) *The A-module M is I-complete, and for every morphism of connective $\mathbb{E}_\infty$-rings $f : A \to B$ which annihilates some power of I, the B-module $B \otimes_A M$ is almost perfect.*

Proof. Combine Propositions 7.3.5.7 and 8.3.5.5. □

Proposition 8.3.5.7. *Let A be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 A$ and let $M \in \mathrm{Mod}_A$ be almost connective. The following conditions are equivalent:*

(1) *The A-module M is locally free of finite rank.*

(2) *The A-module M is I-complete and the tensor product $R \otimes_A M$ is locally free of finite rank, where $R = (\pi_0 A)/I$.*

(3) *The A-module M is I-complete and, for every morphism of connective $\mathbb{E}_\infty$-rings $f : A \to B$ which annihilates some power of I, the B-module $B \otimes_A M$ is locally free of finite rank.*

Proof. The implication (1) $\Rightarrow$ (2) is follows from Proposition 7.3.5.7. If (2) is satisfied, then the tensor product $R \otimes_A M$ is connective, almost perfect, and of Tor-amplitude $\leqslant 0$. If $f : A \to B$ is as in (3), then $B \otimes_A M$ is likewise connective, almost perfect (Corollary 8.3.5.6), and of Tor-amplitude $\leqslant 0$ (Proposition 2.7.3.2). It follows that $B \otimes_A M$ is locally free of finite rank (Corollary 2.9.1.3), which proves (3).

We now complete the proof by showing that (3) implies (1). Assume that (3) is satisfied. Using Theorem 8.3.4.4 and Corollary 8.3.4.6, we deduce that M is connective. Corollary 8.3.5.6 shows that M is almost perfect, so the homotopy group $\pi_0 M$ is finitely presented as a module over $\pi_0 A$. We may therefore choose a map $u : A^n \to M$ which induces a surjection $\pi_0 A^n \to \pi_0 M$. To prove (1), it will suffice to show that u admits a section. For this, it suffices to show that the map

$$\phi : \mathrm{Map}_{\mathrm{Mod}_A}(M, A^n) \to \mathrm{Map}_{\mathrm{Mod}_A}(M, M)$$

is surjective on π_0. Letting K denote the cofiber of u, we are reduced to proving that the mapping space $\mathrm{Map}_{\mathrm{Mod}_A}(M, K)$ is connected. Choose a tower of $\mathbb{E}_\infty$-algebras

$$\cdots \to A_3 \to A_2 \to A_1$$

satisfying the requirements of Lemma 8.1.2.2. By virtue of Lemma 8.1.2.3, we can recover $K \simeq K^\wedge_I$ as the limit of the tower $\{A_i \otimes_A K\}$. Then $\mathrm{Map}_{\mathrm{Mod}_A}(M, K)$ is the limit of the tower $\mathrm{Map}_{\mathrm{Mod}_A}(M, A_i \times_A K)$. It will therefore suffice to prove the following:

(a) Each of the mapping spaces $\mathrm{Map}_{\mathrm{Mod}_A}(M, A_i \otimes_A K)$ is connected.

(b) Each of the maps $\psi_i : \mathrm{Map}_{\mathrm{Mod}_A}(M, A_i \otimes_A K) \to \mathrm{Map}_{\mathrm{Mod}_A}(M, A_{i-1} \otimes_R K)$ induces a surjection of fundamental groups.

Note that K is 1-connective, so that $A_i \otimes_A K$ is a 1-connective module over A_i. We have a homotopy equivalence $\mathrm{Map}_{\mathrm{Mod}_A}(M, A_i \otimes_A K) \simeq \mathrm{Map}_{\mathrm{Mod}_{A_i}}(A_i \otimes_A M, A_i \otimes_A K)$. Consequently, assertion (a) follows immediately from assumption (2). To prove (b), we note that the homotopy fiber of ψ_i (over the base point) can be identified with $\mathrm{Map}_{\mathrm{Mod}_{A_i}}(A_i \otimes_A M, J \otimes_A K)$, where $J = \mathrm{fib}(A_i \to A_{i-1})$. Since J is connective, $J \otimes_A K$ is 1-connective, and the desired result follows from the projectivity of $A_i \otimes_A M$. □

Corollary 8.3.5.8. *Let A be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 A$, let $M \in \mathrm{Mod}_A$ be almost perfect, and let n be an integer. The following conditions are equivalent:*

(1) *As an A-module, M has* Tor-*amplitude $\leqslant n$.*

(2) *The tensor product $R \otimes_A M$ has* Tor-*amplitude $\leqslant n$ over R, where $R = (\pi_0 A)/I$.*

(3) *For every morphism of connective $\mathbb{E}_\infty$-rings $f : A \to B$ which annihilates some power of I, the B-module $B \otimes_A M$ has* Tor-*amplitude $\leqslant n$.*

Proof. Choose k such that $M \in (\mathrm{Mod}_R)_{\geqslant -k}$. Replacing M by $\Sigma^k M$ and n by $n + k$, we may reduce to the case where M is connective. The implication (1) ⇒ (2) is obvious, and the implication (2) ⇒ (3) follows from Proposition 2.7.3.2. We will prove the implication (3) ⇒ (1) using induction on n. When $n = 0$, the desired result follows from Propositions 8.3.5.7 and HA.7.2.4.20 . If $n > 0$, we can choose a fiber sequence

$$N \to R^m \to M,$$

where N is connective. Then $f^* N$ has Tor-amplitude $\leqslant n - 1$, so the inductive hypothesis implies that N has Tor-amplitude $\leqslant n$. Using Proposition HA.7.2.4.23 , we deduce that M has Tor-amplitude $\leqslant n$. □

Corollary 8.3.5.9. *Let A be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 A$ and let $M \in \mathrm{Mod}_A$ be almost connective. The following conditions are equivalent:*

(1) *The A-module M is perfect.*

(2) *Set $R = (\pi_0 A)/I$. Then M is I-complete and the R-module $R \otimes_A M$ is perfect.*

(3) *The A-module M is I-complete, and for every morphism of connective $\mathbb{E}_\infty$-rings $f : A \to B$ which annihilates some power of I, the B-module $B \otimes_A M$ is perfect.*

Proof. Combine Corollaries 8.3.5.8 and 8.3.5.6 with the criterion of Proposition HA.7.2.4.23. □

Proof of Theorem 8.3.5.2. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})^{\mathrm{acn}}$. We first show that (1) ⇒ (2). Assume that $\mathscr{F}$ is perfect to order n (almost perfect, perfect, locally free of finite rank) and let $f : \mathsf{X} \to \mathfrak{X}$ be a morphism where X is a spectral Deligne-Mumford stack; we wish to show that $f^* \mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is also perfect to order n (almost perfect, perfect, locally free of finite rank). By virtue of Proposition 2.8.4.7, this can be tested locally on X. We may therefore assume without loss of generality that $\mathsf{X} = \mathrm{Sp\acute{e}t}\, B$ is affine and that f factors as a composition $\mathrm{Sp\acute{e}t}\, B \xrightarrow{f'} \mathrm{Spf}\, A \to \mathfrak{X}$, where $A = \mathcal{O}_{\mathfrak{X}}(U)$ for some affine object $U \in \mathcal{X}$. In this case, the image of $f^* \mathscr{F}$ under the equivalence $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, B) \simeq \mathrm{Mod}_B$ can be identified with the tensor product $B \otimes_A \mathscr{F}(U)$. Assumption (1) guarantees that $\mathscr{F}(U)$ is perfect to order n (almost perfect, perfect, locally free of finite rank) as an A-module, so that $B \otimes_A \mathscr{F}(U)$ is perfect to order n (almost perfect, perfect, locally free of finite rank) as a B-module.

The implication (2) ⇒ (3) follows immediately from the definitions. We will complete the proof by showing that (3) implies (1). Assume that the image of $\mathscr{F}$ under the equivalence $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{acn}} \simeq \mathrm{QCoh}(h_{\mathfrak{X}})^{\mathrm{acn}}$ is perfect to order n (almost perfect, perfect, locally free of finite rank), and let $U \in \mathcal{X}$ be affine. Set $A = \mathcal{O}_{\mathfrak{X}}(U)$, let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition, and set $R = (\pi_0 A)/I$. We have an evident map $f : \mathrm{Sp\acute{e}t}\, R \to \mathrm{Spf}\, A \to \mathfrak{X}$, and assumption (3) guarantees that $f^* \mathscr{F} \in \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R) \simeq \mathrm{Mod}_R$ is perfect to order n (almost perfect, perfect, locally free of finite rank). It follows that $R \otimes_A \mathscr{F}(U)$ is perfect to order n (almost perfect, perfect, locally free of finite rank). Applying Proposition 8.3.5.5 (Corollary 8.3.5.6, Corollary 8.3.5.9, Proposition 8.3.5.7), we deduce that $\mathscr{F}(U)$ is perfect to order n (almost perfect, perfect, locally free of finite rank) as an A-module. Allowing U to vary, we conclude that $\mathscr{F}$ satisfies condition (1). □

Corollary 8.3.5.10. *Let A be a complete adic $\mathbb{E}_\infty$-ring and let $\mathscr{F} \in \mathrm{QCoh}(\mathrm{Spf}\, A)$. The following conditions are equivalent:*

(1) *The quasi-coherent sheaf $\mathscr{F}$ is perfect to order n (almost perfect, perfect, locally free of finite rank).*

(2) *The A-module $\Gamma(\mathrm{Spf}\, A; \mathscr{F})$ is perfect to order n (almost perfect, perfect, locally free of finite rank).*

Proof. The implication (1) ⇒ (2) follows immediately from the definitions (note that $A \simeq \Gamma(\mathrm{Spf}\, A : \mathcal{O}_{\mathrm{Spf}\, A})$ by virtue of our assumption that A is complete). To prove the converse, we note that (2) implies that $\mathscr{F}$ is almost connective (Corollary 8.2.5.4). Using (2) together with Proposition 8.3.5.5 (Corollary 8.3.5.6, Corollary 8.3.5.9, Proposition 8.3.5.7), we deduce that the image of $\mathscr{F}$ under the equivalence $\mathrm{QCoh}(\mathrm{Spf}\, A)^{\mathrm{acn}} \simeq \mathrm{QCoh}(h_{\mathrm{Spf}\, A})^{\mathrm{acn}}$

is perfect to order n (almost perfect, perfect, locally free of finite rank). Assertion (1) now follows from Theorem 8.3.5.2. □

Warning 8.3.5.11. In the statement of Corollary 8.3.5.10, the assumption that A is complete is essential (note that replacing A by its completion does not change the formal spectrum $\mathrm{Spf}\, A$, but can drastically alter the finiteness properties enjoyed by the A-module $M = \Gamma(\mathrm{Spf}\, A; \mathscr{F})$).

8.4 The Noetherian Case

In §8.2, we introduced the stable ∞-category of quasi-coherent sheaves $\mathrm{QCoh}(\mathfrak{X})$ on a formal spectral Deligne-Mumford stack $\mathfrak{X}$ (Definition 8.2.4.7). In the special case where $\mathfrak{X} = \mathrm{Spf}\, A$ for some adic $\mathbb{E}_\infty$-ring A, there is a canonical equivalence of ∞-categories

$$\Gamma(\mathrm{Spf}\, A; \bullet) : \mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{Mod}_A^{\mathrm{Cpl}(I)},$$

where $I \subseteq \pi_0 A$ is a finitely generated ideal of definition. Note that the full subcategory $\mathrm{Mod}_A^{\mathrm{Cpl}(I)} \subseteq \mathrm{Mod}_A$ of I-complete A-modules is stable under truncation (Corollary 7.3.4.3), and therefore inherits a t-structure from Mod_A. A similar assertion holds in the non-affine setting: for any formal spectral Deligne-Mumford stack $\mathfrak{X}$, the ∞-category $\mathrm{QCoh}(\mathfrak{X})$ comes equipped with a t-structure $(\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}, \mathrm{QCoh}'(\mathfrak{X})$ (see Corollary 8.2.5.11), where $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}$ denotes the full subcategory of $\mathrm{QCoh}(\mathfrak{X})$ spanned by the *connective* quasi-coherent sheaves on $\mathfrak{X}$ (Definition 8.2.5.6). However, this observation is generally not very useful: we do not know a concrete description of the subcategory $\mathrm{QCoh}'(\mathfrak{X}) \subseteq \mathrm{QCoh}(\mathfrak{X})$ when $\mathfrak{X}$ is not affine (beware that the requirement that a quasi-coherent sheaf $\mathscr{F}$ belongs to $\mathrm{QCoh}'(\mathfrak{X})$ is not local for the étale topology). In this section, we will show that the situation improves considerably if we are willing to impose appropriate finiteness conditions on $\mathfrak{X}$ (by demanding that $\mathfrak{X}$ is *locally Noetherian*; see Definition **??**) and on the quasi-coherent sheaves that we consider (by restricting our attention to the full subcategory $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ of quasi-coherent sheaves which are *almost perfect*, in the sense of Definition 8.3.5.1). Our main results can be summarized as follows:

- If $\mathfrak{X}$ is a locally Noetherian formal spectral Deligne-Mumford stack, then the ∞-category $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ admits a t-structure $(\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\geqslant 0}, \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\leqslant 0})$, where $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\geqslant 0} = \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}} \cap \mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}$ denotes the ∞-category of quasi-coherent sheaves on $\mathfrak{X}$ which are connective and almost perfect (Corollary 8.4.2.4). Moreover, this t-structure is compatible with étale localization.

- If $\mathfrak{X}$ is a locally Noetherian formal spectral Deligne-Mumford stack, then the heart $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}\,\heartsuit} \subseteq \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ admits a concrete description: it is equivalent to the abelian category of finitely 0-presented sheaves on the functor $h_{\mathfrak{X}}$ represented by $\mathfrak{X}$ (see Proposition 8.4.3.5).

8.4.1 Almost Perfect Sheaves

We begin with some general remarks which do not require any Noetherian hypotheses.

Notation 8.4.1.1. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. We let $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ denote the full subcategory of $\mathrm{QCoh}(\mathfrak{X})$ spanned by those quasi-coherent sheaves which are almost perfect, in the sense of Definition 8.3.5.1.

If $\mathfrak{X} = \operatorname{Spf} A$ is affine and A is complete, then Corollaries 8.3.5.10 and 8.2.4.15 supply an equivalence of ∞-categories $\Gamma(\mathfrak{X}; \bullet) : \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}} \simeq \mathrm{Mod}_A^{\mathrm{aperf}}$. This functor admits an explicit homotopy inverse:

Proposition 8.4.1.2. *Let A be a complete adic $\mathbb{E}_\infty$-ring, set $\mathfrak{X} = \operatorname{Spf} A$, and regard $\mathrm{Mod}_{\mathscr{O}_{\operatorname{Spf} A}}$ as a stable A-linear ∞-category. Then the construction $M \mapsto M \otimes_A \mathscr{O}_{\mathfrak{X}}$ induces an equivalence of ∞-categories $\mathrm{Mod}_A^{\mathrm{aperf}} \to \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}} \subseteq \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$, which is homotopy inverse to the equivalence $\Gamma(\mathfrak{X}; \bullet) : \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}} \to \mathrm{Mod}_A^{\mathrm{aperf}}$.*

Proof. Let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition. Let us regard $\mathscr{O}_{\mathfrak{X}}$ as a commutative algebra object of the stable presheaf ∞-category $\mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathrm{Sp})$, given by the formula $\mathscr{O}_{\mathfrak{X}}(B) = B_I^{\wedge}$. We can then identify $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ with a full subcategory of the ∞-category of $\mathscr{O}_{\mathfrak{X}}$-module objects of $\mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathrm{Sp})$. The equivalence $\Gamma(\mathfrak{X}; \bullet) : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Mod}_A^{\mathrm{Cpl}(I)}$ of Corollary 8.2.4.15 admits a homotopy inverse ρ, which carries an A-module M to the sheaf given by $B \mapsto (M \otimes_A B)_I^{\wedge} \simeq (M \otimes_A B_I^{\wedge})_I^{\wedge}$ (see Remark 8.2.4.16). If M is almost perfect as an A-module, then $M \otimes_A B_I^{\wedge}$ is almost perfect as a $B_I^{\wedge}$-module and is therefore I-complete (Proposition 7.3.5.7). It follows that we can identify $\rho(M)$ with the tensor product $M \otimes_A \mathscr{O}_{\mathfrak{X}}$ (in the ∞-category $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}(\mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathrm{Sp}))$, and therefore also in its localization $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$). The desired result now follows from Corollary 8.2.4.15. □

Corollary 8.4.1.3. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack and let $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$. Then the tensor product $\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}$ (formed in the ∞-category $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$) also belongs to $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$.*

Proof. The assertion is local on $\mathfrak{X}$, so we may assume without loss of generality that $\mathfrak{X} = \operatorname{Spf} A$ for some complete adic $\mathbb{E}_\infty$-ring A. Let us regard $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ as a stable A-linear ∞-category. Proposition 8.4.1.2 supplies equivalences

$$\mathscr{F} \simeq M \otimes_A \mathscr{O}_{\mathfrak{X}} \qquad \mathscr{G} \simeq N \otimes_A \mathscr{O}_{\mathfrak{X}}$$

for $M, N \in \mathrm{Mod}_A^{\mathrm{aperf}}$. We then have

$$\begin{aligned} \mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G} &\simeq (M \otimes_A \mathscr{O}_{\mathfrak{X}}) \otimes_{\mathscr{O}_{\mathfrak{X}}} (N \otimes_A \mathscr{O}_{\mathfrak{X}}) \\ &\simeq (M \otimes_A N) \otimes_A \mathscr{O}_{\mathfrak{X}}. \end{aligned}$$

The desired result now follows from Proposition 8.4.1.2, since the tensor product $M \otimes_A N$ is an almost perfect A-module. □

It follows from Corollary 8.4.1.3 that the completed tensor product of Notation 8.2.4.23 coincides with the usual tensor product:

Corollary 8.4.1.4. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack, and let $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$. Then the canonical map $\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G} \to \mathscr{F} \widehat{\otimes} \mathscr{G}$ is an equivalence in $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$.*

Corollary 8.4.1.5. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ is a symmetric monoidal subcategory of $\mathrm{QCoh}(\mathfrak{X})$ (that is, it contains the unit object and is closed under the completed tensor product functor $\widehat{\otimes}$). Moreover, the lax symmetric monoidal inclusion functor $\mathrm{QCoh}(\mathfrak{X}) \hookrightarrow \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ is symmetric monoidal when restricted to $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$.*

We can similar reasoning to the pullback functor associated to a map $f : \mathfrak{X} \to \mathfrak{Y}$ of formal spectral Deligne-Mumford stacks.

Corollary 8.4.1.6. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. If $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ is quasi-coherent and almost perfect, then the pullback $f^* \mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ is quasi-coherent and almost perfect.*

Proof. The desired conclusion can be tested locally on $\mathfrak{X}$. We may therefore assume without loss of generality that $\mathfrak{X}$ and $\mathfrak{Y}$ are affine. Write $\mathfrak{Y} = \mathrm{Spf}\, A$ and $\mathfrak{X} = \mathrm{Spf}\, B$, where A and B are complete adic $\mathbb{E}_\infty$-rings. Then f is induced by a morphism of adic $\mathbb{E}_\infty$-rings $\phi : A \to B$. Let us regard $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{Y}}}$ and $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ as stable A-linear and B-linear ∞-categories, respectively. Using Proposition 8.4.1.2, we can write $\mathscr{F} = M \otimes_A \mathscr{O}_{\mathfrak{Y}}$. We then have an equivalence

$$\begin{aligned} f^* \mathscr{F} &\simeq f^*(M \otimes_A \mathscr{O}_{\mathfrak{Y}}) \\ &\simeq M \otimes_A f^* \mathscr{O}_{\mathfrak{Y}} \\ &\simeq M \otimes_A \mathscr{O}_{\mathfrak{X}} \\ &\simeq (M \otimes_A B) \otimes_B \mathscr{O}_{\mathfrak{X}}. \end{aligned}$$

Since $M \otimes_A B$ is an almost perfect B-module (Proposition 2.7.3.1), it follows from Proposition 8.4.1.2 that $f^* \mathscr{F}$ is quasi-coherent and almost perfect. □

Corollary 8.4.1.7. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. If $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{Y})$ is almost perfect, then the completed pullback $f^{\star}\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$ is also almost perfect. Moreover, the canonical map $f^{*}\mathscr{F} \to f^{\star}\mathscr{F}$ is an equivalence in $\mathrm{WCoh}(\mathfrak{X})$.*

Corollary 8.4.1.8. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a representable morphism of formal spectral Deligne-Mumford stacks. If $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{Y})$ is nilcoherent and almost perfect, then $f^{\star}\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$ is nilcoherent and almost perfect.*

Proof. Combine Corollary 8.4.1.7 with Proposition 8.3.2.5. □

Proposition 8.4.1.9. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. Suppose that f is representable, proper, and locally almost of finite presentation (that is, for every map $\mathrm{Sp\acute{e}t}\, R \to \mathfrak{Y}$), the fiber product $\mathrm{Sp\acute{e}t}\, R \times_{\mathfrak{Y}} \mathfrak{X}$ is a spectral algebraic space which is proper and locally almost of finite presentation over R). If $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})$ is almost perfect, then $f_{*}\mathscr{F} \in \mathrm{QCoh}(\mathfrak{Y})$ is almost perfect.*

Proof. Remark 8.3.3.5 implies that $f_{*}\mathscr{F}$ is almost connective. Using Proposition 8.3.3.6 and Theorem 8.3.5.2, we can reduce to the case where $\mathfrak{X}$ and $\mathfrak{Y}$ are spectral Deligne-Mumford stacks, in which case the desired result follows from Theorem 5.6.0.2. □

8.4.2 Locally Noetherian Formal Spectral Deligne-Mumford Stacks

We now generalize Definition 2.8.1.4 to the setting of formal spectral Deligne-Mumford stacks.

Definition 8.4.2.1. Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack. We will say that $\mathfrak{X}$ is *locally Noetherian* if, for every affine object $U \in \mathcal{X}$, the $\mathbb{E}_{\infty}$-ring $\mathscr{O}_{\mathfrak{X}}(U)$ is Noetherian.

Proposition 8.4.2.2. *Let A be a complete adic $\mathbb{E}_{\infty}$-ring. Then the formal spectrum $\mathrm{Spf}\, A$ is locally Noetherian if and only if A is Noetherian.*

Proof. It follows immediately from the definitions that if $\mathrm{Spf}\, A$ is locally Noetherian, then $A \simeq \Gamma(\mathrm{Spf}\, A; \mathscr{O}_{\mathrm{Spf}\, A})$ is locally Noetherian. For the converse, suppose that A is Noetherian, and let $U \in \mathcal{S}\mathrm{hv}_{A}^{\mathrm{ad}}$ be affine. It follows from Proposition 8.1.3.6 that we have an equivalence $\mathscr{O}_{\mathrm{Spf}\, A}(U) \simeq B_{I}^{\wedge}$, where B is an étale A-algebra and $I \subseteq \pi_0 A$ is a finitely generated ideal of definition. Theorem HA.7.2.4.31 implies that B is Noetherian, so that $B_{I}^{\wedge}$ is also Noetherian by virtue of Corollary 7.3.8.3. □

Proposition 8.4.2.3. *Let $\mathfrak{X}$ be a locally Noetherian formal spectral Deligne-Mumford stack and let $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$. Suppose that $\mathscr{F}$ is quasi-coherent and almost perfect. Then the truncations $\tau_{\geq n}\mathscr{F}$ and $\tau_{\leq n}\mathscr{F}$ are also quasi-coherent and almost perfect.*

Proof. The assertion is local on $\mathfrak{X}$. We may therefore assume without loss of generality that $\mathfrak{X} = \operatorname{Spf} A$ for some complete adic $\mathbb{E}_\infty$-ring A. Our assumption that $\mathfrak{X}$ is locally Noetherian guarantees that A is Noetherian. It follows that the subcategory $\operatorname{Mod}_A^{\mathrm{aperf}} \subseteq \operatorname{Mod}_A$ of almost perfect A-modules is closed under truncation, and therefore inherits a t-structure from Mod_A. Proposition 8.4.1.2 implies that the construction $M \mapsto M \otimes_A \mathscr{O}_{\mathfrak{X}}$ determines a fully faithful embedding $F : \operatorname{Mod}_A^{\mathrm{aperf}} \to \operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}}$, whose essential image is the full subcategory $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$. Consequently, to show that $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ is closed under truncations in $\operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}}$, it will suffice to show that the functor F is t-exact. It follows immediately from the definitions that F is right t-exact. To verify left t-exactness, we must show that if $M \in (\operatorname{Mod}_A^{\mathrm{aperf}})_{\leqslant 0}$, then the sheaf $\mathscr{F} = M \otimes_A \mathscr{O}_{\mathfrak{X}}$ is 0-truncated. In fact, we claim that $\mathscr{F}(U)$ is 0-truncated for every affine object $U \in \mathcal{S}\mathrm{hv}_A^{\mathrm{ad}}$. To prove this, it suffices to show that $M \otimes_A B_I^\wedge$ is 0-truncated for every étale A-algebra B, where $I \subseteq \pi_0 A$ is a finitely generated ideal of definition. This is clear: our assumption that A is Noetherian guarantees that B is Noetherian (Theorem HA.7.2.4.31), so that $B_I^\wedge$ is flat over B (Corollary 7.3.6.9) and therefore also flat over A. □

Corollary 8.4.2.4. *Let $\mathfrak{X}$ be a locally Noetherian formal spectral Deligne-Mumford stack. Then the ∞-category $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ is equipped with a t-structure $(\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\geqslant 0}, \operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\leqslant 0})$, where $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\geqslant 0} = \operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}} \cap (\operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}})_{\geqslant 0}$ and $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\leqslant 0} = \operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}} \cap (\operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}})_{\leqslant 0}$.*

Remark 8.4.2.5. Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a locally Noetherian formal spectral Deligne-Mumford stack. For each affine object $U \in \mathcal{X}$, the construction $\mathscr{F} \mapsto \mathscr{F}(U)$ determines a t-exact functor $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}} \to \operatorname{Mod}^{\mathrm{aperf}}_{\mathscr{O}_{\mathfrak{X}}(U)}$ (left t-exactness is clear from the definition, and right t-exactness follows from the proof of Proposition 8.4.2.3). We can therefore describe the t-structure of Corollary 8.4.2.4 as follows:

- A sheaf $\mathscr{F} \in \operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ belongs to $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\geqslant 0}$ if and only if $\mathscr{F}(U)$ is connective, for each affine object $U \in \mathcal{X}$
- A sheaf $\mathscr{F} \in \operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ belongs to $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\leqslant 0}$ if and only if $\mathscr{F}(U)$ is 0-truncated, for each affine object $U \in \mathcal{X}$.

Remark 8.4.2.6. Let $\mathfrak{X}$ be a locally Noetherian formal spectral Deligne-Mumford stack. Then the t-structure of Corollary 8.4.2.4 is left complete. To prove this, we can work locally on $\mathfrak{X}$ and thereby reduce to the case where $\mathfrak{X} \simeq \operatorname{Spf} A$, for some complete adic $\mathbb{E}_\infty$-ring A. In this case, Proposition 8.4.1.2 supplies a t-exact equivalence $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}} \simeq \operatorname{Mod}_A^{\mathrm{aperf}}$, and the t-structure on $\operatorname{Mod}_A^{\mathrm{aperf}}$ is evidently left complete (see Proposition HA.7.2.4.17).

Beware that the t-structure on $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ is never right complete (except in trivial cases). However, it is right bounded (see §HA.1.2.1) when $\mathfrak{X}$ is quasi-compact.

Proposition 8.4.2.7. *Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack. Then:*

(1) *If $\mathfrak{X}$ is locally Noetherian, then $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathfrak{X}}|_U)$ is locally Noetherian for each object $U \in \mathcal{X}$.*

(2) *If there exists a collection of objects $\{U_\alpha\}$ which cover $\mathcal{X}$ such that each $\mathfrak{X}_{U_\alpha}$ is locally Noetherian, then $\mathfrak{X}$ is locally Noetherian.*

Proof. Assertion (1) follows immediately from the definitions. To prove (2), suppose that there exists a covering $\{U_\alpha\}$ such that each $\mathfrak{X}_{U_\alpha}$ is locally Noetherian. Using Proposition 8.4.2.3, we deduce the following:

($*$) Let $\mathscr{F} \in \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$. Then, for any integer n, the truncations $\tau_{\geqslant n} \mathscr{F}$ and $\tau_{\leqslant n} \mathscr{F}$ also belong to $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$.

Let $V \in \mathcal{X}$ be any affine object. We wish to show that $\mathscr{O}_{\mathfrak{X}}(V)$ is Noetherian.

Replacing $\mathfrak{X}$ by $\mathfrak{X}_V$, we can reduce to the case where $\mathfrak{X} \simeq \operatorname{Spf} A$ for some complete adic $\mathbb{E}_\infty$-ring A. It follows from Proposition 8.4.1.2 that the construction $M \mapsto M \otimes_A \mathscr{O}_{\mathfrak{X}}$ determines an equivalence $F : \mathrm{Mod}_A^{\mathrm{aperf}} \to \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$. Combining this observation with ($*$), we deduce that the ∞-category $\mathrm{Mod}_A^{\mathrm{aperf}}$ admits a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ for which the functor F is t-exact. Note that an object $M \in \mathrm{Mod}_A^{\mathrm{aperf}}$ belongs to $\mathcal{C}_{\geqslant 0}$ if and only if M is connective (Corollary 8.2.5.4). In particular, every object $M \in \mathrm{Mod}_A^{\mathrm{aperf}}$ fits into a fiber sequence $M' \to M \to M''$ where M' is n-connective and $M'' \in \mathcal{C}_{\leqslant n-1}$. Then $F(M'')$ is a $(n-1)$-truncated object of $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}$, so that $M'' \simeq \Gamma(\mathfrak{X}; F(M''))$ is also $(n-1)$-truncated. We therefore have $M' \simeq \tau_{\geqslant n} M$ and $M'' \simeq \tau_{\leqslant n-1} M$. Allowing M to vary, we deduce that $\mathrm{Mod}_A^{\mathrm{aperf}}$ is closed under truncations (when regarded as a subcategory of the ∞-category Mod_A), so that the $\mathbb{E}_\infty$-ring A is coherent (Proposition HA.7.2.4.18).

To complete the proof, it will suffice to show that the commutative ring $\pi_0 A$ is Noetherian. Let $J \subseteq \pi_0 A$ be an ideal. Write J as a union of finitely generated ideals $J_\beta \subseteq J$. Since A is coherent, each J_α is almost perfect when regarded as an A-module (Proposition HA.7.2.4.17). It follows that $\{F(J_\beta)\}$ can be regarded as a filtered diagram of subobjects of $\pi_0 \mathscr{O}_{\mathfrak{X}}$ in the abelian category $\mathrm{Mod}_{\mathscr{O}_{\mathfrak{X}}}^{\heartsuit}$. Choose a covering of $\mathcal{X}$ by affine objects $\{U_i\}_{1 \leqslant i \leqslant n}$ such that each $\mathscr{O}_{\mathfrak{X}}(U_i)$ is Noetherian. For $1 \leqslant i \leqslant n$, we can regard $\{F(J_\beta)(U_i)\}$ as a filtered diagram of ideals in the Noetherian commutative ring $(\pi_0 \mathscr{O}_{\mathfrak{X}})(U_i) \simeq \pi_0 \mathscr{O}_{\mathfrak{X}}(U_i)$. Each of these diagrams must stabilize, so the diagram $\{F(J_\beta)\}$ must also stablize. Since F is an equivalence of ∞-categories, we conclude that the diagram of ideals $\{J_\beta\}$ stabilizes: that is, the ideal J is itself finitely generated. Allowing J to vary, we deduce that the commutative ring $\pi_0 A$ is Noetherian as desired. □

Corollary 8.4.2.8. *Let X be a locally Noetherian spectral Deligne-Mumford stack and let $K \subseteq |\mathsf{X}|$ be a closed subset. Then the formal completion $\mathsf{X}^{\wedge}_K$ is a locally Noetherian formal spectral Deligne-Mumford stack.*

Proof. The assertion is local on X, so we may assume without loss of generality that $\mathsf{X} = \operatorname{Spét} A$ for some Noetherian $\mathbb{E}_\infty$-ring A. Then K can be identified with the vanishing locus of a (finitely generated) ideal $I \subseteq \pi_0 A$. Example 8.1.6.4 and Remark 8.1.2.4 then supply equivalences $\mathsf{X}^\wedge_K \simeq \operatorname{Spf} A \simeq \operatorname{Spf} A^\wedge_I$. By virtue of Proposition 8.4.2.2, it will suffice to show that $A^\wedge_I$ is Noetherian, which follows from Corollary 7.3.8.3. □

Example 8.4.2.9. Let X be a spectral Deligne-Mumford stack. Then X is locally Noetherian in the sense of Definition 8.4.2.1 if and only if it is locally Noetherian in the sense of Definition 2.8.1.4.

Proposition 8.4.2.10. *Let X be a locally Noetherian spectral Deligne-Mumford stack, let $K \subseteq |\mathsf{X}|$ be a closed subset, and let $i : \mathsf{X}^\wedge_K \to \mathsf{X}$ be a morphism of formal spectral Deligne-Mumford stacks which exhibits $\mathsf{X}^\wedge_K$ as a formal completion of X along K. Then the pullback functor $i^* : \operatorname{QCoh}(\mathsf{X})^{\mathrm{aperf}} \to \operatorname{QCoh}(\mathsf{X}^\wedge_K)^{\mathrm{aperf}}$ is t-exact.*

Proof. Without loss of generality, we may assume that $\mathsf{X} = \operatorname{Spét} A$ is affine and that K is the vanishing locus of a finitely generated ideal $I \subseteq \pi_0 A$. Using Proposition 8.4.1.2, we can identify i^* with the functor $\operatorname{Mod}_A^{\mathrm{aperf}} \to \operatorname{Mod}_{A^\wedge_I}^{\mathrm{aperf}}$ given by extension of scalars along the canonical map $\phi : A \to A^\wedge_I$. It now suffices to observe that ϕ is flat (Corollary 7.3.6.9). □

8.4.3 The Heart of $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$

Let $\mathfrak{X}$ be a locally Noetherian formal spectral Deligne-Mumford stack, and regard $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ as equipped with the t-structure described in Corollary 8.4.2.4. Our next goal is to describe the heart of $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$. More generally, we will describe the intersection $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{cn}} \cap \operatorname{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\leqslant n}$, for every integer $n \geqslant 0$. First, we need to introduce a variant of Definition 6.2.2.1.

Notation 8.4.3.1. Fix an integer $n \geqslant 0$. For every connective $\mathbb{E}_\infty$-ring A, let $\operatorname{Mod}_A^{n-fp}$ denote the full subcategory of Mod_A spanned by those A-modules which are connective and finitely n-presented (see Definition 2.7.1.1): in other words, the ∞-category of compact objects of $\tau_{\leqslant n} \operatorname{Mod}_A^{\mathrm{cn}}$.

Let $\operatorname{Mod} = \operatorname{Mod}(\mathrm{Sp})$ denote the ∞-category of pairs (A, M), where A is an $\mathbb{E}_\infty$-ring and M is an A-module spectrum, and let $\operatorname{Mod}^{n-fp}$ denote the full subcategory of Mod spanned by those pairs (A, M) where A is connective and $M \in \operatorname{Mod}_A^{n-fp}$. The construction $(A, M) \mapsto A$ determines a coCartesian fibration $q : \operatorname{Mod}^{n-fp} \to \operatorname{CAlg}^{\mathrm{cn}}$ which is classified by a functor $\operatorname{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$ given by $A \mapsto \operatorname{Mod}_A^{n-fp}$. If $f : A \to B$ is a map of connective $\mathbb{E}_\infty$-rings, then the induced functor $\operatorname{Mod}_A^{n-fp} \to \operatorname{Mod}_B^{n-fp}$ is given by the construction $M \mapsto \tau_{\leqslant n}(B \otimes_A M)$.

Let $\operatorname{QCoh}^{n-fp} : \operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the functor $\Phi'(q)$ obtained by applying Remark 6.2.1.11 to the coCartesian fibration q; here $\widehat{\mathcal{S}}$ denotes the ∞-category of spaces which are not necessarily small, and $\widehat{\mathcal{C}\mathrm{at}}_\infty$ is defined similarly.

Example 8.4.3.2. By virtue of Proposition 6.2.1.9, we can regard the construction $X \mapsto \mathrm{QCoh}^{n-fp}(X)$ as a right Kan extension of the construction $A \mapsto \mathrm{Mod}_A^{n-fp}$ along the Yoneda embedding $\mathrm{Spec} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}}$. In particular, for every connective $\mathbb{E}_\infty$-ring A, we have a canonical equivalence $\mathrm{QCoh}^{n-fp}(\mathrm{Spec}\, A) \simeq \mathrm{Mod}_A^{n-fp}$.

Example 8.4.3.3. Let X be a spectral Deligne-Mumford stack representing a functor $h_{\mathsf{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Arguing as in the proof of Proposition 6.2.4.1, we see that there is a canonical equivalence $\mathrm{QCoh}^{n-fp}(\mathsf{X}) \simeq \mathrm{QCoh}^{n-fp}(h_{\mathsf{X}})$, where the left hand side is defined by Construction 4.5.2.2 and the right hand side is defined in Notation 8.4.3.1.

Remark 8.4.3.4. Let $X \in \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor classifying a left fibration $\mathcal{C} \to \mathrm{CAlg}^{\mathrm{cn}}$. We then have a canonical equivalence of ∞-categories $\mathrm{QCoh}^{n-fp}(X) \simeq \mathrm{Fun}^{\mathrm{ccart}}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathcal{C}, \mathrm{Mod}^{n-fp})$. More informally, we can view an object $\mathscr{F} \in \mathrm{QCoh}(X)$ as a rule which assigns to each point $\eta \in X(A)$ an A-module $\mathscr{F}(\eta) \in \mathrm{Mod}_A^{n-fp}$, which depends functorially on A in the following sense: for every morphism of connective $\mathbb{E}_\infty$-rings $\phi : A \to A'$ carrying η to a point $\eta' \in X(A')$, there is a canonical equivalence of A'-modules $\tau_{\leqslant n}(A' \otimes_A \mathscr{F}(\eta)) \xrightarrow{\sim} \mathscr{F}(\eta')$.

For any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, we have an evident functor $\tau_{\leqslant n} : \mathrm{QCoh}(X)^{\mathrm{cn,aperf}} \to \mathrm{QCoh}(X)^{n-fp}$, which is given pointwise by the construction $(\tau_{\leqslant n}\mathscr{F})(\eta) \simeq \tau_{\leqslant n}(\mathscr{F}(\eta)) \in \mathrm{Mod}_A^{n-fp}$ for $\eta \in X(A)$. Our main result is the following:

Proposition 8.4.3.5. *Let $\mathfrak{X}$ be a locally Noetherian formal spectral Deligne-Mumford stack representing a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. For every integer $n \geqslant 0$, the composite functor*

$$\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}} \cap \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\leqslant n} \hookrightarrow \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\geqslant 0} \simeq \mathrm{QCoh}(X)^{\mathrm{aperf}} \xrightarrow{\tau_{\leqslant n}} \mathrm{QCoh}^{n-fp}(X)$$

is an equivalence of ∞-categories.

Example 8.4.3.6. Let $\mathfrak{X}$ be a locally Noetherian formal spectral Deligne-Mumford stack. Applying Proposition 8.4.3.5 in the case $n = 0$, we obtain an equivalence of categories $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}\heartsuit} \xrightarrow{\sim} \mathrm{QCoh}^{0-fp}(X)$. Unwinding the definitions, we see that an object $\mathscr{F} \in \mathrm{QCoh}^{0-fp}(X)$ can be identified with a rule which does the following:

- (a) To each morphism $\eta_A : \mathrm{Sp\acute{e}t}\, A \to \mathfrak{X}$, $\mathscr{F}$ assigns a finitely presented (discrete) $(\pi_0 A)$-module $\mathscr{F}(\eta_A)$.
- (b) To every commutative diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}\, B & \xrightarrow{\alpha} & \mathrm{Sp\acute{e}t}\, A \\ & {\scriptstyle \eta_B} \searrow \quad \swarrow {\scriptstyle \eta_A} & \\ & \mathfrak{X}, & \end{array}$$

  the object $\mathscr{F}$ assigns a homomorphism of discrete A-modules $u_\alpha : \mathscr{F}(\eta_A) \to \mathscr{F}(\eta_B)$, which induces an isomorphism $\pi_0(B \otimes_A \mathscr{F}(\eta_A)) \simeq \eta_B$ (and depends only on the homotopy class of α).

(c) The homomorphisms u_α appearing in (b) have the transitivity property $u_{\alpha\circ\beta} = u_\beta \circ u_\alpha$. In particular, when $B = A$ and $\eta_B = \eta_A$, we have $u_{\mathrm{id}_{\operatorname{Sp\acute{e}t} A}} = \mathrm{id}_{\mathscr{F}(\eta_A)}$.

Moreover, it suffices to specify the data of (a) and (b) (and to verify the condition (c)) for maps $\eta_A : \operatorname{Sp\acute{e}t} A \to \mathsf{X}$ where A is a commutative ring (since the abelian category of discrete modules over an arbitrary connective $\mathbb{E}_\infty$-ring A is equivalent to the abelian category of discrete modules over the commutative ring $\pi_0 A$).

Warning 8.4.3.7. In the situation of Example **??**, the existence of an equivalence of categories $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}\,\heartsuit} \xrightarrow{\sim} \mathrm{QCoh}^{0-fp}(X)$ implies that the category $\mathrm{QCoh}^{0-fp}(X)$ is abelian. This is not obvious from the definition: for a general functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, the category $\mathrm{QCoh}^{0-fp}(X)$ is additive, but need not be abelian. For example, if $X = \operatorname{Spec} R$ for some commutative ring R, then $\mathrm{QCoh}^{0-fp}(X)$ can be identified with the category of finitely presented R-modules (Example 8.4.3.2), which is abelian if and only if R is coherent.

8.4.4 The Proof of Proposition 8.4.3.5

The proof of Proposition 8.4.3.5 will require some preliminaries.

Lemma 8.4.4.1. *Let R be a Noetherian $\mathbb{E}_\infty$-ring, and let M be a connective R-module. If M is perfect to order n, then $\tau_{\leqslant n} M$ is almost perfect.*

Proof. According to Remark HA.7.2.4.19 , it will suffice to prove that $\pi_i M$ is a finitely generated module over $\pi_0 R$ for $0 \leqslant i \leqslant n$. We proceed by induction on n. When $n = 0$, the result is obvious. Assume therefore that $n > 0$. Then there exists a fiber sequence $N \to R^k \to M$ where N is connective and perfect to order $(n-1)$. For $i \leqslant n$, we have an exact sequence $(\pi_i R)^k \to \pi_i M \to \pi_{i-1} N$ of modules over $\pi_0 R$. Since $\pi_{i-1} N$ is finitely generated by the inductive hypothesis and $(\pi_i R)^k$ is finitely generated (by virtue of our assumption that R is Noetherian), we conclude that $\pi_i M$ is finitely generated, as desired. □

Lemma 8.4.4.2. *Let R be a Noetherian commutative ring, let $I \subseteq R$ be an ideal, and let M and N be discrete R-modules. Assume that N is I-nilpotent and that M is finitely generated. Then every class $\eta \in \mathrm{Ext}^p_R(M, N)$ vanishes when restricted to $\mathrm{Ext}^p_R(I^m M, N)$ for $m \gg 0$.*

Proof. We proceed by induction on p. If $p = 0$, the result is obvious. Otherwise, choose an injective map $u : N \to Q$, where Q is an injective R-module. Let $Q_0 \subseteq Q$ be the submodule consisting of elements which are annihilated by I^k for $k \gg 0$. Using Remark 7.1.3.5, we deduce that Q_0 is also an injective R-module. Since N is I-nilpotent, the map u factors through Q_0, so we have an exact sequence of I-nilpotent R-modules

$$0 \to N \to Q_0 \to N' \to 0.$$

Since $p > 0$, we have $\mathrm{Ext}^p_R(M, Q_0) \simeq 0$, so the boundary map $\partial : \mathrm{Ext}^{p-1}_R(M, N') \to \mathrm{Ext}^p_R(M, N)$ is surjective. Write $\eta = \partial(\overline{\eta})$ for some class $\overline{\eta} \in \mathrm{Ext}^{p-1}_R(M, N')$. Applying the inductive hypothesis, we deduce that $\overline{\eta}$ has trivial image in $\mathrm{Ext}^{p-1}_R(I^m M, N')$ for $m \gg 0$. It follows that the image of η in $\mathrm{Ext}^p_R(M, N)$ vanishes as well.

□

Lemma 8.4.4.3. *Let R be a Noetherian commutative ring and let M be a finitely generated discrete R-module. Let $I \subseteq R$ be an ideal, and choose a tower*

$$\cdots \to A_4 \to A_3 \to A_2 \to A_1$$

of $\mathbb{E}_\infty$-algebras over R satisfying the requirements of Lemma 8.1.2.2. For every integer $n \geqslant 0$, the canonical map

$$\theta : \{\tau_{\leqslant n} A_i \otimes_R M\}_{i>0} \to \{\pi_0(A_i \otimes_R M)\}_{i>0} \simeq \{M/I^j M\}_{j \geqslant 0}$$

is an equivalence of Pro-*objects of the ∞-category* Mod_R.

Proof. Let $\mathcal{C}$ be denote the full subcategory of Mod_R spanned by those objects which are connective, almost perfect, n-truncated, and I-nilpotent. Then the domain and codomain of θ can be identified with Pro-objects of $\mathcal{C}$. It will therefore suffice to show that θ induces a homotopy equivalence

$$\alpha_N : \varinjlim_{j>0} \mathrm{Map}_{\mathrm{Mod}_R}(M/I^j M, N) \to \varinjlim_{i>0} \mathrm{Map}_{\mathrm{Mod}_R}(\tau_{\leqslant n}(A_i \otimes_R M), N)$$

for every object $N \in \mathcal{C}$. Since N is n-truncated, we can identify the codomain of α with $\varinjlim_{i>0} \mathrm{Map}_{\mathrm{Mod}_R}(A_i \otimes_R M, N)$. The collection of those objects $N \in \mathcal{C}$ for which α_N is a homotopy equivalence is closed under extensions; we may therefore suppose that $N = \Sigma^m N_0$, where N_0 is a finitely generated discrete R-module and $0 \leqslant m \leqslant n$. Since N is I-nilpotent, N_0 is a module over the quotient ring R/I^k for $k \gg 0$. It follows that the codomain of α_N can be rewritten as $\varinjlim \mathrm{Map}_{\mathrm{Mod}_{R/I^k}}((R/I^k \otimes_R A_i) \otimes_R M, N)$. The projection map $\mathrm{Spf}\, R \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, R/I^k \to \mathrm{Sp\acute{e}t}\, R/I^k$ is an equivalence, so the tower $\{R/I^k \otimes_R A_i\}$ is equivalent to R/I^k in the ∞-category $\mathrm{Pro}(\mathrm{CAlg})$. We can therefore identify the codomain of α_N with $\mathrm{Map}_{\mathrm{Mod}_{R/I^k}}(R/I^k \otimes_R M, N) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(M, N)$. To prove that α_N is a homotopy equivalence, it will suffice to show that the direct limit $\varinjlim_{j \geqslant 0} \mathrm{Map}_{\mathrm{Mod}_R}(I^j M, N)$ vanishes. For this, it suffices to show for every integer p, the abelian group $\varinjlim_{j \geqslant 0} \mathrm{Ext}^p_R(I^j M, N_0)$ vanishes. This follows immediately from Lemma 8.4.4.2. □

Remark 8.4.4.4. Let Ab denote the category of abelian groups, and Pro(Ab) the category of Pro-objects of Ab. Let R be a commutative ring and $I \subseteq R$ an ideal. To any discrete

R-module M, we can associate an object of Pro(Ab), represented by the inverse system $\{M/I^n M\}_{n \geqslant 0}$. Given an exact sequence of discrete R-modules

$$0 \to M' \xrightarrow{\phi} M \to M'' \to 0,$$

we obtain an exact sequence of Pro-objects

$$0 \to \{M'/\phi^{-1}(I^n M)\}_{n \geqslant 0} \to \{M/I^n M\}_{n \geqslant 0} \to \{M''/I^n M''\}_{n \geqslant 0} \to 0.$$

If R is Noetherian and M is a finitely generated R-module, then the Artin-Rees lemma allows us to identify the term on the left side with the Pro-abelian group $\{M'/I^n M'\}_{n \geqslant 0}$. It follows that we have an exact sequence

$$0 \to \{M'/I^n M'\}_{n \geqslant 0} \to \{M/I^n M\}_{n \geqslant 0} \to \{M''/I^n M''\}_{n \geqslant 0} \to 0$$

in the abelian category Pro(Ab). We can summarize the above discussion as follows: if R is a Noetherian commutative ring and $I \subseteq R$ is an ideal, then the construction $M \mapsto \{M/I^n M\}_{n \geqslant 0}$ determines an exact functor from the category of finitely generated R-modules to the category Pro(Ab).

Lemma 8.4.4.5. *Let R be a Noetherian $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and choose a tower of R-algebras*

$$\cdots \to A_3 \to A_2 \to A_1$$

satisfying the requirements of Lemma 8.1.2.2. Let M be an almost perfect R-module. For every integer n, the canonical map

$$\theta_n^M : \{\pi_n(A_i \otimes_R M)\}_{i>0} \to \{\mathrm{Tor}_0^{\pi_0 R}(\pi_0 A_i, \pi_n M)\}_{i>0} \simeq \{(\pi_n M)/I^j(\pi_n M)\}_{j>0}$$

is an isomorphism in the category Pro(Ab) *of* Pro-*abelian groups.*

Proof. Let us say that an R-module M is *n-good* if the map θ_n^M is an isomorphism, and that M is *good* if it is n-good for every integer n. Note that M is n-good if and only if the truncation $\tau_{\leqslant n} M$ is n-good. Consequently, to prove that every almost perfect R-module M is good, it will suffice to treat the case where M is truncated.

Suppose we are given a fiber sequence of R-modules $M' \to M \to M''$. We then obtain a commutative diagram

$$\begin{array}{ccc}
\{\pi_{n+1}(A_i \otimes_R M'')\}_{i>0} & \xrightarrow{\theta^{M''}_{n+1}} & \{(\pi_{n+1}M'')/I^j(\pi_{n+1}M'')\}_{j>0} \\
\downarrow & & \downarrow \\
\{\pi_n(A_i \otimes_R M')\}_{i>0} & \xrightarrow{\theta^{M'}_{n}} & \{(\pi_n M')/I^j(\pi_n M')\}_{j>0} \\
\downarrow & & \downarrow \\
\{\pi_n(A_i \otimes_R M)\}_{i>0} & \xrightarrow{\theta^{M}_{n}} & \{(\pi_n M)/I^j(\pi_n M)\}_{j>0} \\
\downarrow & & \downarrow \\
\{\pi_n(A_i \otimes_R M'')\}_{i>0} & \xrightarrow{\theta^{M''}_{n}} & \{(\pi_n M'')/I^j(\pi_n M'')\}_{j>0} \\
\downarrow & & \downarrow \\
\{\pi_{n-1}(A_i \otimes_R M')\}_{i>0} & \xrightarrow{\theta^{M'}_{n-1}} & \{(\pi_{n-1}M')/I^j(\pi_{n-1}M')\}_{j>0}
\end{array}$$

in the category Pro(Ab). The left column is obviously exact. If M, M', and M'' are almost perfect, then Remark 8.4.4.4 shows that the right column is also exact. Applying the five lemma, we deduce that if M' and M'' are good, then M is also good. Consequently, the collection of almost perfect good R-modules is closed under extensions. To prove that every truncated almost perfect R-module M is good, it will suffice to treat the case where M is discrete. In this case, we can regard M as a module over the discrete commutative ring $\pi_0 R$. Replacing R by $\pi_0 R$ (and the tower $\{A_i\}_{i>0}$ with $\{\pi_0 R \otimes_R A_i\}_{i>0}$), we can assume that R is also discrete. In this case, the desired result follows immediately from Lemma 8.4.4.3. □

Lemma 8.4.4.6. *Let R be a Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to a ideal $I \subseteq \pi_0 R$ and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by the formal spectra* $\mathrm{Spf}\, R$ *(given by $X(B) = \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}}(R, B)$). For every integer $n \geqslant 0$, the canonical map $f : \mathrm{Mod}_R^{n-fp} \to \mathrm{QCoh}^{n-fp}(X)$ is an equivalence of ∞-categories.*

Proof. Choose a tower of R-algebras

$$\cdots \to A_3 \to A_2 \to A_1$$

satisfying the requirements of Lemma 8.1.2.2, so that $R \simeq \varprojlim A_i$ and $\mathrm{Spf}\, R \simeq \varinjlim \mathrm{Sp\acute{e}t}\, A_i$. Then the ∞-category $\mathrm{QCoh}^{n-fp}(X)$ can be identified with the limit of the tower of ∞-categories $\{\mathrm{Mod}_{A_i}^{n-fp}\}_{i>0}$. The functor f is given by the restriction of a functor $F : (\mathrm{Mod}_R)_{\leqslant n} \to \varprojlim (\mathrm{Mod}_{A_i})_{\leqslant n}$. The functor F admits a right adjoint G, which carries a compatible family of n-truncated A_i-modules $\{M_i\}$ to the limit $\varprojlim M_i$. If each M_i is connective, then the maps

$$\pi_0 M_i \to \mathrm{Tor}_0^{\pi_0 A_i}(\pi_0 A_{i-1}, \pi_0 M_i) \simeq \pi_0 M_{i-1}$$

are surjective, so that $G\{M_i\} = \varprojlim M_i$ is also connective. If, in addition, each M_i is almost perfect, then Lemma 8.3.5.4 implies that $G\{M_i\}$ is perfect to order n. Since $G\{M_i\}$ is n-truncated, we conclude that $G\{M_i\}$ is almost perfect (Lemma 8.4.4.1). It follows that the functor G restricts to a functor $g : \varprojlim \mathrm{Mod}_{A_i}^{n-fp} \to \mathrm{Mod}_R^{n-fp}$, so we have an adjunction

$$\mathrm{Mod}_R^{n-fp} \underset{g}{\overset{f}{\rightleftarrows}} \varprojlim \mathrm{Mod}_{A_i}^{n-fp}.$$

It follows immediately from Lemma 8.3.5.4 that the counit map $f \circ g \to \mathrm{id}$ is an equivalence. We wish to prove that the unit map $\mathrm{id} \to g \circ f$ is also an equivalence. In other words, we wish to show that if $M \in \mathrm{Mod}_R^{n-fp}$, then the map $u_M : M \to \varprojlim \tau_{\leqslant n}(A_i \otimes_R M)$ is an equivalence. Let K denote the fiber of u and note that K is n-truncated. The proof of Theorem 8.3.4.4 shows that $M \simeq \varprojlim (A_i \otimes_R M)$, so that $K \simeq \varprojlim \tau_{\geqslant n+1}(A_i \otimes_R M)$. It follows that K is n-connective and that $\pi_n K \simeq \varprojlim^1 \pi_{n+1}(A_i \otimes_R M)$. It will therefore suffice to show that the abelian group $\varprojlim^1 \pi_{n+1}(A_i \otimes_R M)$ vanishes. This follows from the observation that the inverse system $\{\pi_{n+1}(A_i \otimes_R M)\}_{i \geqslant 0}$ is trivial as an object of $\mathrm{Pro}(\mathrm{Ab})$, because $\pi_{n+1} M \simeq 0$ (Lemma 8.4.4.5). □

Proof of Proposition 8.4.3.5. Let $\mathfrak{X}$ be a locally Noetherian formal spectral Deligne-Mumford stack, let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by $\mathfrak{X}$, and let $n \geqslant 0$ be an integer; we wish to show that the composite functor

$$\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}} \cap \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\leqslant n} \hookrightarrow \mathrm{QCoh}(\mathsf{X}^{\wedge}_K)^{\mathrm{aperf}}_{\geqslant 0} \xrightarrow{\tau_{\leqslant n}} \mathrm{QCoh}^{n-fp}(X)$$

is an equivalence of ∞-categories. The assertion is local on $\mathfrak{X}$. We may therefore reduce to the case where $\mathfrak{X} = \mathrm{Spf}\, A$ for some complete adic $\mathbb{E}_\infty$-ring A, in which case the desired result follows from Lemma 8.4.4.6. □

8.5 The Grothendieck Existence Theorem

Let R be a commutative ring, let X be an R-scheme, and let $I \subseteq R$ be an ideal. For each $n \geqslant 0$, let X_n denote the fiber product $\mathrm{Spec}\, R/I^n \times_{\mathrm{Spec}\, R} X$ (formed in the category of schemes). Each X_n can be regarded as a closed subscheme of X, so we restriction functors

$$\mathrm{QCoh}(X)^{\heartsuit} \to \mathrm{QCoh}(X_n)^{\heartsuit} \qquad \mathscr{F} \mapsto \mathscr{F}|_{X_n}.$$

These restriction functors are compatible as n varies, and therefore define a restriction functor $\mathrm{QCoh}(X)^{\heartsuit} \to \varprojlim_{n \geqslant 0} \mathrm{QCoh}(X_n)^{\heartsuit}$. A fundamental theorem of Grothendieck (see Theorem 5.1.4 and Corollary 5.1.6 of [89]) asserts that, if we impose appropriate finiteness conditions, then this restriction functor becomes an equivalence of categories:

Theorem 8.5.0.1 (Grothendieck Existence Theorem)**.** *Let R be a Noetherian ring which is complete with respect to an ideal I, let X be a proper R-scheme, and for $m \geqslant 0$ define $X_m = X \times_{\operatorname{Spec} R} \operatorname{Spec}(R/I^m)$ as above. Let $\operatorname{Coh}(X)$ and $\operatorname{Coh}(X_m)$ denote the abelian categories of coherent sheaves on X and X_m, respectively. Then the canonical map $\operatorname{Coh}(X) \to \varprojlim\{\operatorname{Coh}(X_m)\}$ is an equivalence of categories.*

Remark 8.5.0.2. In the language of formal schemes, Theorem 8.5.0.1 asserts that the category of coherent sheaves on X is equivalent to the category of coherent sheaves on the formal scheme $X^\wedge$ obtained by completing X along the inverse image of the closed subset $|\operatorname{Spec} R/I| \subseteq |\operatorname{Spec} R|$.

Our goal in this section is to prove an analogue of Theorem 8.5.0.1 in the setting of spectral algebraic geometry. Our result can be stated as follows:

Theorem 8.5.0.3. *[Derived Grothendieck Existence Theorem] Let R be an $\mathbb{E}_\infty$-ring which is I-complete for some finitely generated ideal $I \subseteq \pi_0 R$, let X be a spectral algebraic space which is proper and locally almost of finite presentation over R, and let $\mathsf{X}^\wedge = \operatorname{Spf} R \times_{\operatorname{Spec} R} \mathsf{X}$ denote the formal completion of X along the vanishing locus of I. Then the restriction functor $\operatorname{QCoh}(\mathsf{X})^{\mathrm{aperf}} \to \operatorname{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$ is an equivalence of ∞-categories.*

The classical Grothendieck existence theorem is an immediate consequence of Theorem 8.5.0.3:

Proof of Theorem 8.5.0.1. Let R be a Noetherian ring which is complete with respect to an ideal I and let X be a proper R-scheme (or, more generally, a proper algebraic space over R). Let us identify X with a 0-truncated spectral algebraic space X over R (which is automatically locally almost of finite presentation over R in the sense of spectral algebraic geometry: see Remark 4.2.0.4). Choose a tower of $\mathbb{E}_\infty$-algebras $\cdots \to A_3 \to A_2 \to A_1$ satisfying the requirements of Lemma 8.1.2.2, so that each of the commutative rings $\pi_0 A_n$ can be identified with the quotient R/J_n for some ideal $J_n \subseteq R$. Let $\mathsf{X}^\wedge$ denote the formal completion $\mathsf{X} \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Spf} R$. Theorem 8.5.0.3 implies that the restriction functor $\operatorname{QCoh}(\mathsf{X})^{\mathrm{aperf}} \to \operatorname{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$ is an equivalence of ∞-categories. Proposition 8.4.2.10 implies that this restriction functor is t-exact, where we regard $\operatorname{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$ as endowed with the t-structure of Corollary 8.4.2.4. We therefore obtain an equivalence of hearts $\operatorname{QCoh}(\mathsf{X})^{\mathrm{aperf}\heartsuit} \to \operatorname{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}\heartsuit}$. Using the description of these hearts supplied by Proposition 8.4.3.5, we obtain an equivalence of categories

$$\operatorname{QCoh}^{0-fp}(\mathsf{X}) \to \operatorname{QCoh}^{0-fp}(\mathsf{X}^\wedge) \simeq \varprojlim \operatorname{QCoh}^{0-fp}(\mathsf{X} \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Sp\acute{e}t} A_n).$$

We now observe that the left hand side of this equivalence can be identified with the abelian category $\operatorname{Coh}(X)$ of coherent sheaves on X, while the right hand side can be identified with the limit $\varprojlim \{\mathrm{Coh}(X_n)\}_{n>0}$, where X_n denotes the fiber product $X \times_{\operatorname{Spec} R} \operatorname{Spec}(R/J_n)$ formed in the category of schemes (which is the underlying scheme of the spectral algebraic space $\mathsf{X} \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Sp\acute{e}t} A_n$). Theorem 8.5.0.1 now follows by observing that the sequences of ideals $\{J_n\}_{n>0}$ and $\{I^m\}_{m \geqslant 0}$ are mutually cofinal. □

Remark 8.5.0.4. One feature which distinguishes Theorem 8.5.0.3 from its classical analogue is that it does need any Noetherian hypotheses on the $\mathbb{E}_\infty$-ring R. However, this generality comes at a price: Theorem 8.5.0.3 asserts the existence of an equivalence of ∞-categories, but does not *a priori* tell us anything at the level of abelian categories: in general the ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$ is not closed under truncation, and therefore does not inherit a t-structure from $\mathrm{QCoh}(\mathsf{X})$.

Remark 8.5.0.5. In the situation of Theorem 8.5.0.3, an object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$ is perfect if and only if the restriction $\mathscr{F}|_{\mathsf{X}^\wedge} \in \mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$ is perfect. The "only if" direction is obvious. Conversely, if $\mathscr{F}|_{\mathsf{X}^\wedge}$ is perfect, then it is dualizable as an object of $\mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$, so that (by virtue of Theorem 8.5.0.3) the sheaf $\mathscr{F}$ is dualizable as an object of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$. It then follows that $\mathscr{F}$ is dualizable as an object of $\mathrm{QCoh}(\mathsf{X})$, and is therefore perfect by virtue of Proposition 6.2.6.2.

8.5.1 Full Faithfulness

As a first step towards the proof of Theorem 8.5.0.3, we show that the restriction functor $\mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}} \to \mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$ is fully faithful (Proposition 8.5.1.2). As we will see in a moment, this is essentially a formal consequence of the direct image theorem for proper morphisms (Theorem 5.6.0.2).

Lemma 8.5.1.1 (Derived Theorem on Formal Functions)**.** *Let R be a connective $\mathbb{E}_\infty$-ring which is I-complete for some finitely generated ideal $I \subseteq \pi_0 R$. Let X be a spectral algebraic space which is proper and locally almost of finite presentation over $\operatorname{Sp\acute{e}t} R$, let $\mathsf{X}^\wedge = \operatorname{Spf} R \times_{\operatorname{Sp\acute{e}t} R} \mathsf{X}$ be the formal completion of X along the vanishing locus of I, and let $f : \mathsf{X}^\wedge \hookrightarrow \mathsf{X}$ denote the canonical map. If $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is almost perfect, then the restriction map $\theta : \Gamma(\mathsf{X}; \mathscr{F}) \to \Gamma(\mathsf{X}^\wedge; f^* \mathscr{F})$ is an equivalence of spectra.*

Proof. Using Proposition 8.3.3.6, we can replace $\mathscr{F}$ by $f_* \mathscr{F}$ and thereby reduce to the case where $\mathsf{X} = \operatorname{Sp\acute{e}t} R$ (note that $f_* \mathscr{F} \in \mathrm{QCoh}(\operatorname{Sp\acute{e}t} R)$ is almost perfect by virtue of Theorem 5.6.0.2). In this case, Lemma 8.1.2.3 implies that the map θ exhibits $\Gamma(\mathsf{X}^\wedge; f^* \mathscr{F})$ as an I-completion of $\Gamma(\mathsf{X}; \mathscr{F})$. We conclude by observing that $\Gamma(\mathsf{X}; \mathscr{F})$ is already I-complete: this follows from Proposition 7.3.5.7, since the $\mathbb{E}_\infty$-ring R is assumed to be I-complete. □

Proposition 8.5.1.2. *Let R be a connective $\mathbb{E}_\infty$-ring which is I-complete for some finitely generated ideal $I \subseteq \pi_0 R$. Let X be a spectral algebraic space which is proper and locally almost* *of finite presentation over* Spét R*, and let* $\mathsf{X}^\wedge = \operatorname{Spf} R \times_{\operatorname{Sp\acute{e}t} R} \mathsf{X}$ *denote the formal completion of* X *along the closed substack determined by* I*, and let* $f : \mathsf{X}^\wedge \to \mathsf{X}$ *be the canonical map. Let* $\mathscr{F}, \mathscr{G} \in \operatorname{QCoh}(\mathsf{X})$*, and assume that* $\mathscr{G}$ *is almost perfect. Then the canonical map*

$$\operatorname{Map}_{\operatorname{QCoh}(\mathsf{X})}(\mathscr{F}, \mathscr{G}) \to \operatorname{Map}_{\operatorname{QCoh}(\mathsf{X}^\wedge)}(f^\star \mathscr{F}, f^\star \mathscr{G})$$

is a homotopy equivalence.

Corollary 8.5.1.3. *In the situation of Proposition 8.5.1.2, the complted pullback functor* $f^\star : \operatorname{QCoh}(\mathsf{X}) \to \operatorname{QCoh}(\mathsf{X}^\wedge)$ *is fully faithful when restricted to the full subcategory* $\operatorname{QCoh}(\mathsf{X})^{\mathrm{aperf}} \subseteq \operatorname{QCoh}(\mathsf{X})$ *spanned by the almost perfect objects.*

Proof of Proposition 8.5.1.2. Let us first regard $\mathscr{G}$ as fixed, and regard the morphism

$$\theta_{\mathscr{F}} : \operatorname{Map}_{\operatorname{QCoh}(\mathsf{X})}(\mathscr{F}, \mathscr{G}) \to \operatorname{Map}_{\operatorname{QCoh}(\mathsf{X}^\wedge)}(f^\star \mathscr{F}, f^\star \mathscr{G})$$

as a functor of $\mathscr{F}$. This functor carries colimits in $\operatorname{QCoh}(\mathsf{X})$ to limits in $\operatorname{Fun}(\Delta^1, \mathcal{S})$. Consequently, the collection of those objects $\mathscr{F} \in \operatorname{QCoh}(\mathsf{X})$ for which $\theta_{\mathscr{F}}$ is a homotopy equivalence is closed under colimits. Using Theorem **??**, we are reduced to proving that $\theta_{\mathscr{F}}$ is an equivalence in the special case where $\mathscr{F}$ is perfect. In this case, $\mathscr{F}$ is a dualizable object of $\operatorname{QCoh}(\mathsf{X})$; let us denote its dual by $\mathscr{F}^\vee$. Replacing $\mathscr{G}$ by $\mathscr{F}^\vee \otimes \mathscr{G}$, we can reduce to the case where $\mathscr{F} = \mathscr{O}_{\mathsf{X}}$. In this case, we can identify $\theta_{\mathscr{F}}$ with the restriction map $\Gamma(\mathsf{X}; \mathscr{G}) \to \Gamma(\mathsf{X}^\wedge; f^\star \mathscr{G})$, which is an equivalence by virtue of Lemma 8.5.1.1. □

If the $\mathbb{E}_\infty$-ring R appearing in Proposition 8.5.1.2 is not Noetherian, then we generally do not have good t-structures on the ∞-categories $\operatorname{QCoh}(\mathsf{X})^{\mathrm{aperf}}$ and $\operatorname{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$. However, we nevertheless have the following exactness property (compare with Proposition 8.4.2.10):

Proposition 8.5.1.4. *Let* R *be a connective* $\mathbb{E}_\infty$*-ring which is* I*-complete for some finitely generated ideal* $I \subseteq \pi_0 R$*. Let* X *be a spectral algebraic space which is proper and locally almost of finite presentation over* Spét R*, let* $\mathscr{F} \in \operatorname{QCoh}(\mathsf{X})^{\mathrm{aperf}}$*, and let* $\mathsf{X}^\wedge$ *denote the formal completion* $\operatorname{Spf} R \times_{\operatorname{Sp\acute{e}t} R} \mathsf{X}$*. Then* $\mathscr{F}$ *is connective if and only if* $\mathscr{F}|_{\mathsf{X}^\wedge}$ *is connective.*

Proof. The "only if" direction is trivial. To prove the converse, let us suppose that $\mathscr{F}$ is not connective. Since X is quasi-compact, there exists some smallest integer $n < 0$ such that $\pi_n \mathscr{F}$ does not vanish. Let $K \subseteq |\mathsf{X}|$ be the support of X. Then $K \neq \emptyset$. Since f is proper, the image $f(K)$ is a nonempty closed subset of $|\operatorname{Spec} R|$. Let y be a closed point of $|\operatorname{Spec} R|$ which is contained in $f(K)$, and let $x \in |\mathsf{X}|$ be a point lying over y. Then we can choose an étale map $u : \operatorname{Sp\acute{e}t} A \to \mathsf{X}$ such that x lifts to a point $\overline{x} \in |\operatorname{Spec} A|$. Let κ denote the residue field of $\pi_0 A$ at the point $\overline{x}$. Since $\mathscr{F}$ is n-connective and almost perfect, $\pi_n u^* \mathscr{F}$ is a finitely generated module over $\pi_0 A$. Using Nakayama's lemma and our assumption $x \in \operatorname{Supp}(\mathscr{F})$, we deduce that $\pi_n(\kappa \otimes_A u^* \mathscr{F}) \neq 0$. Note that Remark 7.3.4.10 implies that the composite map $\operatorname{Sp\acute{e}t} \kappa \to \operatorname{Sp\acute{e}t} A \xrightarrow{u} \mathsf{X}$ factors through $\mathsf{X}^\wedge$, so that $\mathscr{F}|_{\mathsf{X}^\wedge}$ is not connective. □

8.5.2 The Grothendieck Existence Theorem

Our goal in this section is to give the proof of Theorem 8.5.0.3. We begin by treating the projective case. First, we note the following slight variant of Theorem 7.2.2.1:

Lemma 8.5.2.1. *Let R be a connective $\mathbb{E}_\infty$-ring, let $n \geqslant 0$, and let $q : \mathbf{P}^n_R \to \operatorname{Spét} R$ be the projection map. Suppose that $\mathscr{F} \in \operatorname{QCoh}(\mathbf{P}^n_R)$ is a quasi-coherent sheaf having the property that the direct images $q_*(\mathscr{F} \otimes \mathscr{O}(i))$ vanish for $0 \leqslant i \leqslant n$. Then $\mathscr{F} \simeq 0$.*

Proof. We will show that $\pi_m \mathscr{F} \simeq 0$ for every integer m. Replacing $\mathscr{F}$ by $\Sigma^{-m}\mathscr{F}$, we may assume that $m = 0$. Using Lemma 5.6.2.2, we can choose a map $\gamma : \bigoplus_\alpha \mathscr{O}(d_\alpha) \to \mathscr{F}$ which induces an epimorphism on π_0. It will therefore suffice to show that γ is nullhomotopic. In fact, we claim that the mapping space $\operatorname{Map}_{\operatorname{QCoh}(\mathbf{P}^n_R)}(\mathscr{O}(d), \mathscr{F}) \simeq \operatorname{Map}_{\operatorname{Mod}_R}(R, q_*(\mathscr{F} \otimes \mathscr{O}(-d)))$ is contractible for every integer d. To prove this, let S be the collection of integers i for which $q_*(\mathscr{F} \otimes \mathscr{O}(-d))$ vanishes. By hypothesis, S contains the set $\{0, -1, \ldots, -n\}$. Lemma 7.2.2.2 implies that whenever S contains the set $\{k, k-1, \ldots, k-n\}$, it also contains $k+1$ and $k-n-1$. An easy induction now shows that S contains all integers. □

Proposition 8.5.2.2. *Let R be an $\mathbb{E}_\infty$-ring which is I-complete for some finitely generated ideal $I \subseteq \pi_0 R$, let X be a spectral algebraic space which is equipped with a map $f : \mathsf{X} \to \mathbf{P}^n_R$ which is finite and locally almost of finite presentation, and let $\mathsf{X}^\wedge = \operatorname{Spf} R \times_{\operatorname{Spét} R} \mathsf{X}$ denote the formal completion of X along the vanishing locus of I. Then the restriction functor $\theta : \operatorname{QCoh}(\mathsf{X})^{\mathrm{aperf}} \to \operatorname{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$ is an equivalence of ∞-categories.*

Proof. Let $\mathscr{A}$ denote the direct image $f_* \mathscr{O}_{\mathsf{X}}$, which we regard as a commutative algebra object of $\operatorname{QCoh}(\mathbf{P}^n_R)$. Our assumption that f is finite and locally almost of finite presentation guarantees that $\mathscr{A} \in \operatorname{QCoh}(\mathbf{P}^n_R)$ is almost perfect (Corollary 5.2.2.2). Let us abuse notation by identifying $\mathscr{A}$ with its image under the restriction functor $\operatorname{QCoh}(\mathbf{P}^n_R)^{\mathrm{aperf}} \to \operatorname{QCoh}(\mathbf{P}^n_R \times_{\operatorname{Spét} R} \operatorname{Spf} R)^{\mathrm{aperf}}$. Since f is affine, it follows from Propositions 2.5.6.1, 5.6.1.1, and Corollary 8.3.4.6 that we have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \operatorname{QCoh}(\mathsf{X})^{\mathrm{aperf}} & \xrightarrow{\theta} & \operatorname{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}} \\ \downarrow & & \downarrow \\ \operatorname{Mod}_{\mathscr{A}}(\operatorname{QCoh}(\mathbf{P}^n_R)^{\mathrm{aperf}}) & \longrightarrow & \operatorname{Mod}_{\mathscr{A}}(\operatorname{QCoh}(\mathbf{P}^n_R \times_{\operatorname{Spec} R} \operatorname{Spf} R)^{\mathrm{aperf}}) \end{array}$$

where the vertical maps are equivalences. We may therefore replace X by $\mathbf{P}^n_R$ and thereby reduce to the case where X is a projective space over R.

Let $q : \mathsf{X} \to \operatorname{Spét} R$ denote the projection map and let $\widehat{q} : \mathsf{X}^\wedge \to \operatorname{Spf} R$ denote its restriction to $\mathsf{X}^\wedge$. Consider the following assertion for $m \geqslant -1$:

$(*_m)$ If $\mathscr{F} \in \operatorname{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$ has the property that $\widehat{q}_*(\mathscr{F} \otimes \theta(\mathscr{O}(i)) \in \operatorname{QCoh}(\operatorname{Spf} R)^{\mathrm{aperf}}$ vanishes for $0 \leqslant i \leqslant m$, then $\mathscr{F}$ belongs to the essential image of θ.

Note that assertion $(*_{-1})$ is equivalent to Proposition 8.5.2.2 (since the restriction functor θ is already known to be fully faithful by virtue of Corollary 8.5.1.3). We will prove assertion that $(*_m)$ holds for all $m \leqslant n$ using descending induction on m. The case $m = n$ follows from Lemma 8.5.2.1. To carry out the inductive step, let us assume that $(*_m)$ is satisfied, and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$ satisfy the hypotheses of $(*_{m-1})$. Set $\mathscr{G} = \widehat{q}_*(\mathscr{F} \otimes \mathscr{O}(m)) \in \mathrm{QCoh}(\operatorname{Spf} R)^{\mathrm{aperf}}$ and form a cofiber sequence $\widehat{q}^\star \mathscr{G} \otimes \mathscr{O}(-m) \xrightarrow{\alpha} \mathscr{F} \to \mathscr{F}'$. It follows from Theorem 5.4.2.6 (and the vanishing of $\widehat{q}_*(\mathscr{F} \otimes \mathscr{O}(i))$ for $0 \leqslant i < m$) that α induces an equivalence $\widehat{q}_*(\widehat{q}^\star \mathscr{G} \otimes \mathscr{O}(i-m)) \to \widehat{q}_*(\mathscr{F} \otimes \mathscr{O}(i))$ for $0 \leqslant i \leqslant m$. Applying our inductive hypothesis to $\mathscr{F}'$, we deduce that $\mathscr{F}'$ belongs to the essential image of θ. It will therefore suffice to show that $\mathscr{G}$ belongs to the essential image of θ, which follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\operatorname{Sp\acute{e}t} R)^{\mathrm{aperf}} & \longrightarrow & \mathrm{QCoh}(\operatorname{Spf} R)^{\mathrm{aperf}} \\ \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle \widehat{q}^\star} \\ \mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}} & \xrightarrow{\theta} & \mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}} \end{array}$$

(note that the upper horizontal map is an equivalence of ∞-categories by virtue of Corollary **??**). □

We now prove the general form of the Grothendieck existence theorem.

Proof of Theorem 8.5.0.3. Let R be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 R$, let $q : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ be a morphism of spectral algebraic spaces which is proper and locally almost of finite presentation, and let $\mathsf{X}^\wedge = \operatorname{Spf} R \times_{\operatorname{Sp\acute{e}t} R} \mathsf{X}$ denote the formal completion of X along the vanishing locus of I. We wish to show that the restriction functor $\theta : \mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}} \to \mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$ is an equivalence of ∞-categories. Note that we have already seen that θ is fully faithful (Corollary 8.5.1.3).

Applying Chow's Lemma (Theorem **??** and Remarks 5.5.0.3 and 5.5.0.4), we can choose a finite sequence of closed immersions

$$\varnothing = \mathsf{Y}_0 \to \mathsf{Y}_1 \to \mathsf{Y}_2 \to \cdots \to \mathsf{Y}_n \to \mathsf{X}$$

and closed immersions $h_i : \widetilde{\mathsf{Y}}_i \to \mathsf{Y}_i \times_{\operatorname{Sp\acute{e}t} R} \mathbf{P}^{d_i}_R$ which are almost of finite presentation satisfying the following conditions:

(i) The closed immersion $\mathsf{Y}_n \to \mathsf{X}$ induces a homeomorphism $|\mathsf{Y}_n| \to |\mathsf{X}|$.

(ii) The projection maps $\widetilde{\mathsf{Y}}_i \to \mathbf{P}^{d_i}_R$ are finite (and locally almost of finite presentation).

(iii) The projection maps $h'_i : \widetilde{\mathsf{Y}}_i \to \mathsf{Y}_i$ induce equivalences $\widetilde{\mathsf{Y}}_i \times_{\mathsf{Y}_i} \mathsf{U}_i \to \mathsf{U}_i$, where U_i is the open substack of Y_i complementary to Y_{i-1}.

For $0 \leqslant i \leqslant n$, let $Y_i \subseteq |\mathsf{X}|$ denote the image of $|\mathsf{Y}_i|$. We will prove the following:

$(*_i)$ Let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$ and let $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$ be supported on Y_i. Then the tensor product $\mathscr{F} \otimes \theta(\mathscr{G})$ belongs to the essential image of θ.

Note that $(*_n)$ implies Theorem 8.5.0.3 (since (i) implies we can take $\mathscr{G} = \mathscr{O}_{\mathsf{X}}$) and that $(*_0)$ is trivial. To complete the proof, it will suffice to show that if $1 \leqslant i \leqslant n$ and $(*_{i-1})$ is satisfied, then $(*_i)$ is also satisfied.

Let v denote the closed immersion $\mathsf{Y}_i \to \mathsf{X}$. We first establish the following special case of $(*_i)$:

$(*')$ For every object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$, the tensor product $\mathscr{F} \otimes \theta(v_* \mathscr{O}_{\mathsf{Y}_i})$ belongs to the essential image of θ.

To prove $(*')$, let $\mathscr{G}$ denote the fiber of the unit map $\mathscr{O}_{\mathsf{Y}_i} \to h'_{i*} \mathscr{O}_{\tilde{\mathsf{Y}}_i}$, so that we have a fiber sequence

$$\mathscr{F} \otimes \theta(v_* \mathscr{G}) \to \mathscr{F} \otimes \theta(v_* \mathscr{O}_{\mathsf{Y}_i}) \to \mathscr{F} \times \theta((v \circ h'_i)_* \mathscr{O}_{\tilde{\mathsf{Y}}_i}).$$

Assumption (iii) guarantees that $v_* \mathscr{G}$ is supported on Y_{i-1}, so that $\mathscr{F} \otimes \theta(v_* \mathscr{G})$ belongs to the essential image of θ by virtue of our inductive hypothesis $(*_{i-1})$. Let $\widetilde{\mathsf{Y}}_i^\wedge$ denote the formal completion $\widetilde{\mathsf{Y}}_i \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Spf}\, R$, so that $(v \circ h'_i)$ induces a map $\widehat{u} : \widetilde{\mathsf{Y}}_i^\wedge \to \mathsf{X}^\wedge$. The projection formula (Corollary 6.3.4.3) then supplies an equivalence $\mathscr{F} \times \theta((v \circ h'_i)_* \mathscr{O}_{\tilde{\mathsf{Y}}_i}) \simeq \widehat{u}_* \widehat{u}^\star \mathscr{F}$. Using the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\widetilde{\mathsf{Y}}_i)^{\mathrm{aperf}} & \longrightarrow & \mathrm{QCoh}(\widetilde{\mathsf{Y}}_i^\wedge)^{\mathrm{aperf}} \\ \downarrow {\scriptstyle (v \circ h'_i)_*} & & \downarrow {\scriptstyle \widehat{u}_*} \\ \mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}} & \xrightarrow{\theta} & \mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}} \end{array}$$

and the observation that the upper horizontal map is an equivalence (by virtue of (ii) and Proposition 8.5.2.2), we see that $\widehat{u}_* \widehat{u}^\star \mathscr{F}$ belongs to the essential image of θ. Since the essential image of θ is closed under extensions, we deduce that $\mathscr{F} \otimes \theta(v_* \mathscr{O}_{\mathsf{Y}_i})$ is contained in the essential image of θ. This competes the proof of $(*')$.

We now return to the proof of $(*_i)$. Let us henceforth regard $\mathscr{F}$ as a fixed object of $\mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$, which we assume to be connective. Let $\mathcal{I}$ denote the fiber of the unit map $\mathscr{O}_{\mathsf{X}} \to v_* \mathscr{O}_{\mathsf{Y}_i}$. Fix an integer $m \geqslant 0$. It follows from $(*')$ that the cofiber of each of the natural maps

$$\mathscr{F} \otimes \theta(\mathcal{I}^{\otimes m}) \to \mathscr{F} \otimes \theta(\mathcal{I}^{\otimes m-1}) \to \cdots \to \mathscr{F} \otimes \theta(\mathcal{I}) \to \mathscr{F}$$

belongs to the essential image of θ. Since the essential image of θ is closed under extensions, we can choose a cofiber sequence $\mathscr{F} \otimes \theta(\mathcal{I}^{\otimes m}) \to \mathscr{F} \to \theta(\mathscr{F}_m)$ for some objects $\mathscr{F}_m \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$ (which are then automatically connective, by virtue of Proposition 8.5.1.4). To complete the proof, it will suffice to verify the following:

(a) For every connective object $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$ which is supported on Y_i, the towers $\{\mathscr{F}_m \otimes \mathscr{G}\}_{m \geqslant 0}$ and $\{\theta(\mathscr{F}_m \otimes \mathscr{G})\}_{m \geqslant 0}$ admit limits in the ∞-categories $\mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}} \cap \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ and $\mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}} \cap \mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{cn}}$, respectively.

(b) Let $\mathscr{G}$ be as in (a). Then the canonical maps

$$\mathscr{F} \otimes \theta(\mathscr{G}) \rightarrow \varprojlim_m \theta(\mathscr{F}_m \otimes \mathscr{G}) \leftarrow \theta(\varprojlim_m (\mathscr{F}_m \otimes \mathscr{G}))$$

are equivalences in the ∞-category $\mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$.

Note that we have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}} \cap \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} & \longrightarrow & \mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}} \cap \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \\ \downarrow & & \downarrow \\ \varprojlim_k \mathrm{QCoh}^{k-fp}(\mathsf{X}) & \longrightarrow & \varprojlim_k \mathrm{QCoh}^{k-fp}(\mathsf{X}^\wedge), \end{array}$$

where the vertical maps are equivalences of ∞-categories (see Notation 8.4.3.1). Consequently, in order to prove (a) and (b), it will suffice to verify the following analogous assertions, for each $k \geqslant 0$:

(a_k) For every connective object $\mathscr{G} \in \mathrm{QCoh}_{Y_i}(\mathsf{X})^{\mathrm{aperf}}$, the diagrams $\{\tau_{\leqslant k}(\mathscr{F}_m \otimes \mathscr{G})\}_{m \geqslant 0}$ and $\{\theta_k(\tau_{\leqslant k}(\mathscr{F}_m \otimes \mathscr{G}))\}_{m \geqslant 0}$ both admit limits in the ∞-categories $\mathrm{QCoh}^{k-fp}(\mathsf{X})$ and $\mathrm{QCoh}^{k-fp}(\mathsf{X}^\wedge)$, respectively. Here $\theta_k : \mathrm{QCoh}^{k-fp}(\mathsf{X}) \rightarrow \mathrm{QCoh}^{k-fp}(\mathsf{X}^\wedge)$ denotes the restriction map.

(b_k) Let $\mathscr{G}$ be as in (a_k). Then the canonical maps

$$\tau_{\leqslant k}(\mathscr{F} \otimes \theta(\mathscr{G})) \xrightarrow{\alpha} \varprojlim_m \theta_k(\tau_{\leqslant k}(\mathscr{F}_m \otimes \mathscr{G})) \leftarrow \theta_k(\varprojlim_m \tau_{\leqslant k}(\mathscr{F}_m \otimes \mathscr{G}))$$

are equivalences in the ∞-category $\mathrm{QCoh}^{k-fp}(\mathsf{X}^\wedge)$.

Since $\mathscr{G}$ is almost perfect and is supported on Y_i, it follows from Proposition **??** that there exists an integer $t \gg 0$ for which the composite map $\mathcal{I}^{\otimes t} \otimes \mathscr{G} \xrightarrow{e} \mathscr{G} \rightarrow \tau_{\leqslant k} \mathscr{G}$ is nullhomotopic. It follows that $\tau_{\leqslant k} \mathscr{G}$ is retract of $\tau_{\leqslant k}(\mathrm{cofib}(e))$. We may therefore replace $\mathscr{G}$ by $\mathrm{cofib}(e)$ and thereby reduce to the case where $\mathscr{G}$ can be written as a finite extension of objects belonging to the essential image of the functor $v_* : \mathrm{QCoh}(\mathsf{Y}_i)^{\mathrm{aperf}} \rightarrow \mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$. Consequently, it will suffice to prove the following:

(a') For every connective object $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$ which can be written as a finite extension of objects belonging to the essential image of v_*, the towers the towers $\{\mathscr{F}_m \otimes \mathscr{G}\}_{m \geqslant 0}$ and $\{\theta(\mathscr{F}_m \otimes \mathscr{G}))\}_{m \geqslant 0}$ are constant when regarded as Pro-objects of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$ and $\mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$, respectively.

(b') Let $\mathscr{G}$ be as in (a'). Then the canonical maps

$$\mathscr{F} \otimes \theta(\mathscr{G}) \xrightarrow{\alpha} \varprojlim_m \theta(\mathscr{F}_m \otimes \mathscr{G}) \xleftarrow{\beta} \theta(\varprojlim_m (\mathscr{F}_m \otimes \mathscr{G}))$$

are equivalences in the ∞-category $\mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$ (note that both limits are well-defined by virtue of (a')).

Note that the collection of those objects $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$ for which the conclusions of (a') and (b') are satisfied is closed under extensions. We may therefore assume that $\mathscr{G} = v_* \mathscr{G}_0$ for some object $\mathscr{G}_0 \in \mathrm{QCoh}(\mathsf{Y}_i)^{\mathrm{aperf}}$. We then have a fiber sequence of towers in $\mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{aperf}}$

$$\{\mathscr{F} \otimes \theta(\mathcal{I}^{\otimes m} \otimes v_* \mathscr{G}_0)\}_{m \geqslant 0} \to \{\mathscr{F} \otimes \theta(v_* \mathscr{G}_0)\}_{m \geqslant 0} \to \{\theta(\mathscr{F}_m \otimes v_* \mathscr{G}_0)\}_{m \geqslant 0}.$$

Here the tower on the left is trivial as a Pro-object (since it has vanishing transition maps) and the tower in the middle is constant already as a diagram, so the tower on the right as constant as Pro-object. This shows that α is an equivalence and, since the functor θ is fully faithful (Corollary 8.5.1.3), that the tower the tower $\{\mathscr{F}_m \otimes v_* \mathscr{G}_0\}_{m \geqslant 0}$ is constant as a Pro-object of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{aperf}}$. It follows formally that the tower $\{\theta(\mathscr{F}_m \otimes v_* \mathscr{G}_0)\}_{m \geqslant 0}$ is also a constant Pro-object and that the comparison map β is an equivalence. □

8.5.3 The Formal GAGA Theorem

Let R be a Noetherian commutative ring which is complete with respect to an ideal $I \subseteq R$. One of the most useful consequences of the classical Grothendieck existence theorem is that a proper R-scheme X is determined by its completion along the vanishing locus of I: that is, by the direct system of closed subschemes $\{\operatorname{Spec} R/I^n \times_{\operatorname{Spec} R} X\}_{n \geqslant 0}$. In the setting of spectral algebraic geometry, we have the following parallel result:

Theorem 8.5.3.1. *Let R be an $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 R$, let X be a spectral algebraic space which is proper and locally almost of finite presentation over R, and let $\mathsf{X}^\wedge = \operatorname{Spf} R \times_{\operatorname{Sp\acute{e}t} R} \mathsf{X}$ denote the formal completion of X along the vanishing locus of I. Let Y be a spectral algebraic space which is quasi-separated. Then the restriction map $\mathrm{Map}_{\mathrm{SpDM}}(\mathsf{X}, \mathsf{Y}) \to \mathrm{Map}_{\mathrm{fSpDM}}(\mathsf{X}^\wedge, \mathsf{Y})$ is a homotopy equivalence.*

Proof. Writing Y as a union of quasi-compact open substacks (and using the quasi-compactness of X and $\mathsf{X}^\wedge$), we can reduce to the case where Y is quasi-compact. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{X}, \mathsf{Y}) & \longrightarrow & \mathrm{Map}_{\mathrm{fSpDM}}(\mathsf{X}^\wedge, \mathsf{Y}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathsf{Y})^{\mathrm{perf}}, \mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}) & \longrightarrow & \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathsf{Y})^{\mathrm{perf}}, \mathrm{QCoh}(\mathsf{X}^\wedge)^{\mathrm{perf}}). \end{array}$$

It follows from Corollary 9.6.4.2 and Theorem 8.1.5.1 that the vertical maps are fully faithful embeddings, and that their essential images are the full subcategories of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathsf{Y})^{\mathrm{perf}}, \mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}})$ and $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathsf{Y})^{\mathrm{perf}}, \mathrm{QCoh}(\mathsf{X}^{\wedge})^{\mathrm{perf}})$ spanned by those symmetric monoidal functors which are exact. To complete the proof, it suffices to observe that the restriction map $\mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}} \to \mathrm{QCoh}(\mathsf{X}^{\wedge})^{\mathrm{perf}}$ is an equivalence of ∞-categories (by virtue of Theorem 8.5.0.3 and Remark 8.5.0.5). □

Remark 8.5.3.2. The conclusion of Theorem 8.5.3.1 is valid for many algebro-geometric objects Y other than quasi-separated algebraic spaces; see, for example, Corollary 9.5.5.3.

Corollary 8.5.3.3. *Let R be an $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal ideal $I \subseteq \pi_0 R$. Let X and Y be spectral algebraic spaces over R, and let $\mathsf{X}^{\wedge}$ and $\mathsf{Y}^{\wedge}$ denote their formal completions along the vanishing locus of I. Assume that X is proper and locally almost of finite presentation over R and that Y is quasi-separated. Then the restriction map*

$$\mathrm{Map}_{\mathrm{SpDM}_{/\operatorname{Sp\acute{e}t} R}}(\mathsf{X}, \mathsf{Y}) \to \mathrm{Map}_{\mathrm{fSpDM}_{/\operatorname{Spf} R}}(\mathsf{X}^{\wedge}, \mathsf{Y}^{\wedge})$$

is a homotopy equivalence.

Corollary 8.5.3.4 (Formal GAGA Theorem)**.** *Let R be an $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 R$, and let $\mathcal{C}$ be the full subcategory of $\mathrm{SpDM}_{/\operatorname{Sp\acute{e}t} R}$ spanned by those spectral Deligne-Mumford stacks which are proper and locally almost of finite presentation over R. Then the construction $\mathsf{X} \mapsto \operatorname{Spf} R \times_{\operatorname{Sp\acute{e}t} R} \mathsf{X}$ determines a fully faithful embedding $\mathcal{C} \to \mathrm{fSpDM}_{/\operatorname{Spf} R}$.*

Remark 8.5.3.5. In the setting of spectral algebraic geometry, Theorem 8.5.3.1 (and Corollaries 8.5.3.3 and 8.5.3.4) do not require any Noetherian hypotheses on R. However, if R is not Noetherian, then the formation of I-completions generally does not have good exactness properties. Consequently, even if R is an ordinary commutative ring and X is an ordinary algebraic space over R, the formal completion $\mathsf{X}^{\wedge}$ which appears in Theorem 8.5.3.1 might differ from its classical analogue (which is simply the direct limit of the closed subspaces $\operatorname{Spec} R/I^n \times_{\operatorname{Spec} R} \mathsf{X} \subseteq \mathsf{X}$).

Remark 8.5.3.6. Let R be an adic $\mathbb{E}_\infty$-ring and let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack equipped with a map $f : \mathfrak{X} \to \operatorname{Spf} R$. We will say that $\mathfrak{X}$ is *algebraizable* if there exists a pullback square

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow \\ \operatorname{Spf} R & \longrightarrow & \operatorname{Sp\acute{e}t} R, \end{array}$$

where X is a spectral algebraic space which is proper and locally almost of finite presentation over R. It follows from Corollary 8.5.3.4 that if such a diagram exists, then it is uniquely determined (up to a contractible space of choices).

8.6 Digression: Geometrically Reduced Morphisms

Let $f : X \to Y$ be a morphism of schemes. Recall that f is said to be *geometrically reduced* if f is flat and the geometric fibers of f are reduced. This definition extends to the setting of spectral algebraic geometry without essential change:

Definition 8.6.0.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. We will say that f is *geometrically reduced* if f is flat and, for every pullback diagram

$$\begin{array}{ccc} \mathsf{X}_\kappa & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow f \\ \operatorname{Sp\acute{e}t} \kappa & \longrightarrow & \mathsf{Y} \end{array}$$

where κ is a field, the fiber X_κ is reduced (Definition 3.1.6.1).

Our goal in this section is to establish some elementary properties of geometrically reduced morphisms in spectral algebraic geometry (most of which are immediate consequences of the analogous facts in classical algebraic geometry).

8.6.1 Geometrically Reduced Algebras over a Field

We begin by reviewing the classical theory of geometrically reduced algebras over a field.

Definition 8.6.1.1. Let κ be a field and let $A \in \mathrm{CAlg}_\kappa^\heartsuit$ be a commutative algebra over κ. We will say that A is *geometrically reduced over* κ if, for every extension field κ' of κ, the tensor product $A \otimes_\kappa \kappa'$ is reduced.

Remark 8.6.1.2. Let κ be a field and let $A \in \mathrm{CAlg}_\kappa^\heartsuit$ be geometrically reduced over κ. Then every localization of A is geometrically reduced over κ.

Remark 8.6.1.3. Let κ be a field and let $A \in \mathrm{CAlg}_\kappa^\heartsuit$. If A is geometrically reduced over κ, then every κ-subalgebra of A is geometrically reduced over κ. Conversely, if every finitely generated κ-subalgebra of A is geometrically reduced over κ, then A is geometrically reduced over κ.

Lemma 8.6.1.4. *Let κ' be a finitely generated extension of a field κ, and suppose that the module of Kähler differentials $\Omega_{\kappa'/\kappa}$ vanishes. Then κ' is a finite separable extension of κ.*

Proof. Suppose first that κ' is generated, as a field extension of κ, by a single element $x \in \kappa'$. If x is trancendental over κ, then κ' is the fraction field of $\kappa[x]$, so that dx is a nonzero element of $\Omega_{\kappa'/\kappa}$. We may therefore assume that x is algebraic over κ, so that we can write $\kappa' = \kappa[x]/(f(x))$ for some nonzero polynomial $f(x)$. Then $\Omega_{\kappa'/\kappa}$ can be identified with the cokernel of the map $\kappa' \xrightarrow{f'(x)} \kappa'$, where f' denotes the derivative of f with respect to x. The vanishing of $\Omega_{\kappa'/\kappa}$ then implies that $f'(x) \neq 0$, so that κ' is a separable extension of κ.

We now handle the general case. Suppose that κ' is generated, as a field extension of κ, by a sequence of elements $x_1, \ldots, x_n \in \kappa'$. We proceed by induction on n, the case $n = 0$ being trivial. To handle the inductive step, let $\kappa'_0 \subseteq \kappa'$ be the subfield generated by κ together with the elements $x_1, \ldots, x_{n-1}$. Since $\Omega_{\kappa'/\kappa'_0}$ is a quotient of $\Omega_{\kappa'/\kappa}$, we deduce that $\Omega_{\kappa'/\kappa'_0}$ vanishes. The first part of the proof shows that κ' is a finite separable extension of κ'_0. It follows that the canonical map $\kappa' \otimes_{\kappa'_0} \Omega_{\kappa'_0/\kappa} \to \Omega_{\kappa'/\kappa}$ is an isomorphism. Our vanishing assumption then gives $\Omega_{\kappa'_0/\kappa} \simeq 0$, so that κ'_0 is a finite separable extension of κ by our inductive hypothesis. By transitivity, we conclude that κ' is a finite separable extension of κ. □

Proposition 8.6.1.5. *Let κ' be an extension of a field κ. The following conditions are equivalent:*

- (a) *The field κ' is geometrically reduced over κ.*
- (b) *Either κ has characteristic zero, or κ has characteristic $p > 0$ and the tensor product $\kappa^{\frac{1}{p^n}} \otimes_\kappa \kappa'$ is reduced for all $n > 0$ (here $\kappa^{\frac{1}{p^n}}$ denotes a copy of the field κ, regarded as a κ-algebra by the nth power of the Frobenius map).*

Moreover, if κ' is a finitely generated field extension of κ, then both these conditions are equivalent to the following:

- (c) *The field κ' is a finite separable extension of a free extension $\kappa(x_1, \ldots, x_m)$ for some $m \geqslant 0$.*

Proof. Writing κ' as a union of finitely generated subextensions (and using Remark 8.6.1.3), we can assume that κ' is finitely generated as an extension field of κ. The implication $(a) \Rightarrow (b)$ is obvious. If (c) is satisfied, then κ' is étale over $\kappa(x_1, \ldots, x_m)$ for some $m \geqslant 0$. Writing $\kappa(x_1, \ldots, x_m)$ as a direct limit of localizations of $\kappa[x_1, \ldots, x_n]$ and using the structure theory of étale morphisms (Proposition B.1.1.3), we can choose an étale $\kappa[x_1, \ldots, x_n]$-algebra A and an isomorphism $\kappa' \simeq A \otimes_{\kappa[x_1,\ldots,x_n]} \kappa(x_1, \ldots, x_n)$. By virtue of Remark 8.6.1.2, it will suffice to show that A is geometrically reduced over κ. Note that for every field extension κ'' of κ, the tensor product $A \otimes_\kappa \kappa''$ is étale over the polynomial ring $\kappa''[x_1, \ldots, x_m]$, and is therefore reduced. This shows that $(c) \Rightarrow (a)$.

We now complete the proof by showing that (b) implies (c). Assume that (b) is satisfied and let $\Omega_{\kappa'/\kappa}$ denote the module of Kähler differentials of κ' over κ. Since κ' is a finitely generated field extension of κ, $\Omega_{\kappa'/\kappa}$ is finite-dimensional as a vector space over κ. We can therefore choose elements $x_1, \ldots, x_m \in \kappa'$ such that $\{dx_i \in \Omega_{\kappa'/\kappa}\}_{1 \leqslant i \leqslant m}$ forms a basis for $\Omega_{\kappa'/\kappa}$. Let $\overline{\kappa}$ denote the subfield of κ' generated by κ and the elements $\{x_i\}_{1 \leqslant i \leqslant m}$. Since the elements dx_i generate $\Omega_{\kappa'/\kappa}$, it follows that $\Omega_{\kappa'/\overline{\kappa}} \simeq 0$. Applying Lemma 8.6.1.4, we deduce that κ' is a finite separable extension of $\overline{\kappa}$. To complete the proof, it will suffice to show that $\overline{\kappa} \simeq \kappa(x_1, \ldots, x_m)$: that is, that the elements $x_1, \ldots, x_m$ are algebraically independent over κ. Assume otherwise: then there exists a nonzero polynomial $f(t_1, \ldots, t_m) \in \kappa[t_1, \ldots, t_m]$ satisfying $f(x_1, \ldots, x_m) = 0$. Let us assume that the we have chosen f so that its degree is minimal. Note that we have

$$0 = df(x_1, \ldots, x_m) = \sum \frac{\partial f}{\partial t_i}(x_1, \ldots, x_m) dx_i \in \Omega_{\kappa'/\kappa},$$

so that each of the polynomials $\frac{\partial f}{\partial t_i}$ vanishes on $(x_1, \ldots, x_m)$. It follows from the minimality of f that the polynomials $\frac{\partial f}{\partial t_i}$ vanish identically. Consequently, the field κ must have characteristic $p > 0$ and we can write $f(t_1, \ldots, t_m) = g(t_1^{p^n}, \ldots, t_m^{p^n})$ for some $n > 0$. Choose n as large as possible, so that g has at least one nonvanishing partial derivative. Let $h \in \kappa^{\frac{1}{p^n}}[t_1, \ldots, t_m]$ be the polynomial obtained from g by taking the p^nth root of each coefficient. We then have $0 = f(x_1, \ldots, x_m) = h(x_1, \ldots, x_m)^{p^n}$ in the tensor product $R = \kappa^{\frac{1}{p^n}} \otimes_\kappa \kappa'$. Since R is reduced, it follows that $h(x_1, \ldots, x_m) = 0$. As above, we compute

$$0 = dh(x_1, \ldots, x_m) = \sum \frac{\partial h}{\partial t_i}(x_1, \ldots, x_m) dx_i \in \Omega_{R/\kappa^{\frac{1}{p^n}}}.$$

Since the elements dx_i form a basis for $\Omega_{\kappa'/\kappa}$ as a κ'-module, they also form a basis for $\Omega_{R/\kappa^{\frac{1}{p^n}}}$ as an R-module. It follows that $\frac{\partial h}{\partial t_i}(x_1, \ldots, x_m)$ vanishes in R for each i. By assumption, there exists an index i such that $h_i = \frac{\partial h}{\partial t_i}$ is not identically zero. Then the polynomial $h_i(t_1, \ldots, t_n)^{p^n}$ has coefficients in κ and vanishes when evaluated on $x_1, \ldots, x_n$, contradicting the minimality of the degree of f. □

Remark 8.6.1.6. Let κ be a field and let A be a κ-algebra. Then:

(a) If κ has characteristic zero and A is reduced, then it is geometrically reduced over κ.

(b) If κ has characteristic $p > 0$ and $\kappa' \otimes_\kappa A$ is reduced for every finite purely inseparable extension κ' of κ, then A is geometrically reduced over κ.

To see this, we can use Remark 8.6.1.3 to reduce to the case where A is finitely generated over κ. Then A is a reduced Noetherian ring, so we have an injective map $A \to \prod_{1 \leqslant i \leqslant n} A_{\mathfrak{p}_i}$ where $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ are the minimal prime ideals of A. Using Remark 8.6.1.3 again, it will suffice to show that (a) and (b) hold for each of the localizations $A_{\mathfrak{p}_i}$, which is a finitely generated extension field of κ. In this case, the desired result follows from Proposition 8.6.1.5.

Corollary 8.6.1.7. *Let κ be a field, let A be a finitely generated algebra over κ, and let $\mathfrak{p}$ be a prime ideal in A. The following conditions are equivalent:*

(a) *The localization $A_{\mathfrak{p}}$ is geometrically reduced over κ.*

(b) *There exists elements $a, b \in A$ such that $a \notin \mathfrak{p}$, b is not a zero-divisor in $A[a^{-1}]$, and the localization $A[b^{-1}]$ factors as a product $\prod_{1 \leqslant i \leqslant n} A_i$, where each A_i admits an étale morphism $\kappa[x_1, \ldots, x_m] \to A_i$ (for some $m \geqslant 0$).*

Proof. Suppose that condition (b) is satisfied, and let κ' be an extension field of κ. Then $\kappa' \otimes_{\kappa} A[b^{-1}]$ can be identified with a subring of the product $\prod_{1 \leqslant i \leqslant n} \kappa' \otimes_{\kappa} A_i$, each factor of which is an étale algebra over some polynomial ring $\kappa'[x_1, \ldots, x_m]$. It follows that $\kappa' \otimes_{\kappa} A[(ab)^{-1}]$ is a localization of a subring of a reduced ring and therefore reduced. Since b is not a zero-divisor in $A[a^{-1}]$, we conclude that $\kappa' \otimes_{\kappa} A[a^{-1}]$ is also reduced. Consequently, $A[a^{-1}]$ is geometrically reduced over κ, so the further localization $A_{\mathfrak{p}}$ is geometrically reduced over κ by Remark 8.6.1.2. This shows that (b) implies (a).

We now show that (a) implies (b). Assume that $A_{\mathfrak{p}}$ is geometrically reduced over κ. Let $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ be the minimal prime ideals of A which are contained in $\mathfrak{p}$. Then each localization $A_{\mathfrak{q}_i}$ is a field K_i which is geometrically reduced over κ. Using Proposition 8.6.1.5, we can assume that each K_i is a finite separable extension of $\kappa(x_{i,1}, \ldots, x_{i,m_i})$ for some elements $x_{i,1}, \ldots, x_{i,m_i} \in K_i$ which are algebraically independent over κ. Choose an element $b \in A - \bigcup \mathfrak{q}_i$ such that each $x_{i,j}$ belongs to the image A_i of the natural map $A[b^{-1}] \to K_i$. We then have $K_i \simeq A_i \otimes_{\kappa[x_{i,1}, \ldots, x_{i,m_i}]} \kappa(x_{i,1}, \ldots, x_{i,m_i})$. Since K_i is étale over $\kappa(x_{i,1}, \ldots, x_{i,m_i})$, we can assume (after changing b if necessary) that each A_i is étale over $\kappa[x_{i,1}, \ldots, x_{i,m_i}]$ and that the natural map $A[b^{-1}] \to \prod A_i$ is an isomorphism. Let $I = \{x \in A : bx = 0\}$. Since $A_{\mathfrak{p}}$ is reduced and b does not belong to any $\mathfrak{q}_i$, the image of b in $A_{\mathfrak{p}}$ is not a zero-divisor. It follows that $I_{\mathfrak{p}} \simeq 0$. Since A is Noetherian, the ideal I is finitely generated. We can therefore choose $a \in A - \mathfrak{p}$ such that $I[a^{-1}] \simeq 0$, so that the image of b in $A[a^{-1}]$ is not a zero-divisor. □

8.6.2 Geometrically Reduced Morphisms of Spectral Deligne-Mumford Stacks

We now summarize some of the formal properties of Definition 8.6.0.1.

Proposition 8.6.2.1. *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \xrightarrow{g} & \mathsf{Y}. \end{array}$$

If f is geometrically reduced, then f' is geometrically reduced. The converse holds if g is a flat covering (Definition 2.8.3.1).

Proof. The first assertion is immediate from the definitions. To prove the second, let us assume that g is a flat covering and quasi-compact and that f' is geometrically reduced. Then f' is flat, so f is flat. Suppose we are given a pullback diagram

$$\begin{array}{ccc} \mathsf{X}_\kappa & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow f \\ \mathrm{Sp\acute{e}t}\,\kappa & \xrightarrow{\eta} & \mathsf{Y}, \end{array}$$

where κ is a field. We wish to show that X_κ is reduced. Choose an étale map $\mathrm{Sp\acute{e}t}\,A \to \mathsf{X}_\kappa$; we will show that the $\mathbb{E}_\infty$-ring A is reduced. Since g is surjective, there exists an extension field κ' of κ such that η factors through g. In this case, we can obtain an étale map

$$\mathrm{Sp\acute{e}t}(\kappa' \otimes_\kappa A) \to \mathrm{Sp\acute{e}t}\,\kappa' \times_{\mathsf{Y}'} \mathsf{X}',$$

so our assumption that f' is geometrically reduced guarantees that the $\mathbb{E}_\infty$-ring $\kappa' \otimes_\kappa A$ is reduced. In particular, the homotopy groups $\pi_n(\kappa' \otimes_\kappa A) \simeq \kappa' \otimes_\kappa \pi_n A$ vanish for $n > 0$, so that A is discrete. Since the map $A \to \kappa' \otimes_\kappa A$ is injective, it follows that A is reduced. □

Proposition 8.6.2.2. *Let $\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z}$ be morphisms of spectral Deligne-Mumford stacks. If f is étale and g is geometrically reduced, then $g \circ f$ is geometrically reduced. If f is an étale surjection and $g \circ f$ is geometrically reduced, then g is geometrically reduced.*

Proof. Suppose first that f is étale and g is geometrically reduced. Then g is flat, so $g \circ f$ is flat. Moreover, for any field κ and any map $\mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Z}$, we have an étale map

$$\mathsf{X} \times_{\mathsf{Z}} \mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Y} \times_{\mathsf{Z}} \mathrm{Sp\acute{e}t}\,\kappa.$$

Since the codomain of this map is reduced, so is the domain.

Now suppose that f is an étale surjection and that $g \circ f$ is geometrically reduced. Then $g \circ f$ is flat (Proposition 2.8.2.4) and for every map $\mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Z}$ as above, we obtain an étale surjection $\mathsf{X} \times_{\mathsf{Z}} \mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Y} \times_{\mathsf{Z}} \mathrm{Sp\acute{e}t}\,\kappa$ whose domain is reduced. Since the property of being reduced is local for the étale topology (Remark 3.1.6.2) we deduce that the codomain is also reduced. □

Corollary 8.6.2.3. *Any étale morphism of spectral Deligne-Mumford stacks is geometrically reduced.*

Definition 8.6.2.4. Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. We will say that ϕ is *geometrically reduced* if the map of spectral Deligne-Mumford stacks $\mathrm{Sp\acute{e}t}\,B \to \mathrm{Sp\acute{e}t}\,A$ is geometrically reduced. In other words, ϕ is geometrically reduced if it is flat and, for every morphism of $\mathbb{E}_\infty$-rings $A \to \kappa$ where κ is a field, the commutative ring $B_\kappa = B \otimes_A \kappa$ is reduced.

Remark 8.6.2.5. Let $\phi : A \to B$ be a flat morphism of connective $\mathbb{E}_\infty$-rings. Then ϕ is geometrically reduced if and only if, for every morphism of commutative rings $\pi_0 A \to \kappa$, the tensor product $B_\kappa = B \otimes_A \kappa$ is reduced.

Remark 8.6.2.6. In the case where A is a field, Definition 8.6.2.4 specializes to Definition 8.6.1.1.

Warning 8.6.2.7. To verify that a morphism $\phi : A \to B$ is geometrically reduced, it is not enough to show that the tensor product $B_\kappa = B \otimes_A \kappa$ is reduced for every residue field κ of A. However, it *is* sufficient to restrict attention to those fields κ which occur as finite purely inseparable extension fields of residue fields of A: this follows from Remark 8.6.1.6.

We now study the closure of geometrically reduced morphisms under composition.

Proposition 8.6.2.8. *Let A be a reduced Noetherian ring and let $\phi : A \to B$ be a geometrically reduced morphism. Then B is a reduced commutative ring.*

Proof. Since ϕ is flat, the $\mathbb{E}_\infty$-ring B is discrete. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be the minimal prime ideals of A. The assumption that A is reduced implies that each of the localizations $A_{\mathfrak{p}_i}$ is a field and that the localization map $A \to \prod_{1 \leqslant i \leqslant n} A_{\mathfrak{p}_i}$ is injective. Since B is flat over A, the induced map $B \to \prod_{1 \leqslant i \leqslant n} (A_{\mathfrak{p}_i} \otimes_A B)$ is injective. Our assumption that ϕ is geometrically reduced implies that each tensor product $A_{\mathfrak{p}_i} \otimes_A B$ is a reduced ring, so that B is also a reduced ring. □

Proposition 8.6.2.9. *Let $\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z}$ be morphisms of spectral Deligne-Mumford stacks. If f and g are geometrically reduced and g is locally of finite type, then the composition $(g \circ f)$ is geometrically reduced.*

Proof. Since f and g are flat, the composition $g \circ f$ is flat. Fix a field κ and a map $\operatorname{Spét} \kappa \to \mathsf{Z}$, so that f induces a map $f_\kappa : \mathsf{X} \times_{\mathsf{Z}} \operatorname{Spét} \kappa \to \mathsf{Y} \times_{\mathsf{Z}} \operatorname{Spét} \kappa$. We wish to show that the domain of f_κ is reduced. Choose an étale map $\operatorname{Spét} B \xrightarrow{\eta} \mathsf{X} \times_{\mathsf{Z}} \operatorname{Spét} \kappa$; we wish to show that the $\mathbb{E}_\infty$-ring B is reduced. This property is local with respect to the étale topology on B; we may therefore assume without loss of generality that the composite map $f_\kappa \circ \eta$ factors through an étale morphism $\operatorname{Spét} A \to \mathsf{Y} \times_{\mathsf{Z}} \operatorname{Spét} \kappa$. Since g is geometrically reduced and of finite type, the $\mathbb{E}_\infty$-ring A is reduced and finitely generated as a κ-algebra. In particular, A is Noetherian. Our assumption that f is geometrically reduced implies that the map $A \to B$ is geometrically reduced, so that B is reduced by virtue of Proposition 8.6.2.8. □

8.6.3 The Geometrically Reduced Locus

Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Since the property of being geometrically reduces is local on the domain with respect to the étale topology

(Proposition 8.6.2.2), there exists a largest open substack $\mathsf{U} \subseteq \mathsf{X}$ for which the restriction $f|_{\mathsf{U}} : \mathsf{U} \to \mathsf{Y}$ is geometrically reduced. Let us refer to U as the *geometrically reduced locus* of f. Note that if we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}, \end{array}$$

then we can identify $\mathsf{U} \times_{\mathsf{X}} \mathsf{X}'$ with an open substack of X', and the restriction of f' to $\mathsf{U} \times_{\mathsf{X}} \mathsf{X}'$ is geometrically reduced (Proposition 8.6.2.1). Consequently, the fiber product $\mathsf{U} \times_{\mathsf{X}} \mathsf{X}'$ is contained in the geometrically reduced locus for the map f'. Under some mild assumptions, we can say more:

Proposition 8.6.3.1 (Universality of the Geometrically Reduced Locus)**.** *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y} \end{array}$$

where f and f' are flat and locally almost of finite presentation. If $\mathsf{U} \subseteq \mathsf{X}$ is the geometrically reduced locus of f, then $\mathsf{U} \times_{\mathsf{X}} \mathsf{X}'$ is the geometrically reduced locus of f'.

Proof. Using Proposition 8.6.2.1, we can reduce to the case where $\mathsf{Y} \simeq \operatorname{Sp\acute{e}t} A$ and $\mathsf{Y}' \simeq \operatorname{Sp\acute{e}t} A'$ are affine. Using Proposition 8.6.2.2, we can further assume that $\mathsf{X} \simeq \operatorname{Sp\acute{e}t} B$ is affine, so that $\mathsf{X}' \simeq \operatorname{Sp\acute{e}t} B'$ for $B' = A' \otimes_A B$. Using Remark 8.6.2.5 we can replace A and A' by the commutative rings $\pi_0 A$ and $\pi_0 A'$ and thereby reduce to the case where A and A' are discrete. Then B is discrete and is finitely presented as a commutative algebra over A. We can therefore choose a finitely generated subalgebra $A_0 \subseteq A$ and a finitely presented A_0-algebra B_0 such that $B \simeq \tau_{\leqslant 0}(A \otimes_{A_0} B_0)$. Enlarging A_0 if necessary, we may assume that B_0 is flat over A_0 (Proposition 6.1.6.1). For each point $x \in |\operatorname{Spec} B_0|$, let $f_0(x) \in |\operatorname{Spec} A_0|$ denote its image and let $\kappa(f_0(x))$ denote the residue field of A_0 at the point $f_0(x)$. Set

$$V = \{\mathfrak{p} \in |\operatorname{Spec} B_0| : \kappa(f(x)) \otimes_{A_0} B_{0\mathfrak{p}} \text{ is geometrically reduced over } \kappa(f_0(x)) \}.$$

Unwinding the definitions, we see that U can be defined (as a subset of $|\operatorname{Spec} B|$) as the largest open set which is contained in $V \times_{|\operatorname{Spec} B_0|} |\operatorname{Spec} B|$, and the geometrically reduced locus of f' can be identified (as a subset of $|\operatorname{Spec} B'|$) with the largest open set which is contained in $V \times_{|\operatorname{Spec} B_0|} |\operatorname{Spec} B'|$. Consequently, to complete the proof, it will suffice to show that V is open in $|\operatorname{Spec} B_0|$. We will prove this by verifying the hypotheses of Lemma 6.1.5.3:

(a) The set V is stable under generalization. Suppose that x and x' are points of $|\operatorname{Spec} B_0|$ such that $x \in V$ and x belongs to the closure of $\{x'\}$; we wish to show that x' belongs to V. Using Proposition 5.3.2.2, we can choose a discrete valuation ring R and a map $\rho : \operatorname{Spét} R \to \operatorname{Spét} B_0$ which carries the closed point of $|\operatorname{Spec} R|$ to x and the generic point of $|\operatorname{Spec} R|$ to x'. Let t be a uniformizer of R and set $\overline{R} = R \otimes_{A_0} B_0$. Then ρ determines an R-algebra homomorphism $\epsilon : \overline{R} \to R$ such that $\mathfrak{p} = \ker(\epsilon)$ is a prime ideal of $\overline{R}$ lying over x' and $\mathfrak{q} = \epsilon^{-1}(t)$ is a prime ideal of $\overline{R}$ lying over x. To show that x' belongs to V, we must show that the localization $\overline{R}_{\mathfrak{p}}$ is geometrically reduced as an algebra over the fraction field K of R. Equivalently, we must show that $\overline{R}_{\mathfrak{p}} \otimes_K K'$ is reduced, for every finite purely inseparable extension K' of K (see Remark 8.6.1.6). Replacing R by a localization of the integral closure of R in K' (which is a Dedekind ring, by virtue of the Krull-Akizuki theorem; see Theorem 5.3.2.1), we can are reduced to showing that $\overline{R}_{\mathfrak{p}}$ is reduced. In fact, we claim that the localization $\overline{R}_{\mathfrak{q}}$ is reduced. To prove this, let I denote the nilradical of $\overline{R}_{\mathfrak{q}}$. Choose any element $a \in I$, so that $a^n = 0$ for $n \gg 0$. Our assumption that x belongs to V guarantees that the quotient $\overline{R}_{\mathfrak{q}}/t\overline{R}_{\mathfrak{q}}$ is reduced, so a belongs to the kernel of the reduction map $\overline{R}_{\mathfrak{q}} \to \overline{R}_{\mathfrak{q}}/t\overline{R}_{\mathfrak{q}}$. We can therefore write $a = tb$ for some $b \in \overline{R}_{\mathfrak{q}}$. We then have $t^n b^n = a^n = 0$. Since $\overline{R}$ is flat over R, it follows that $b^n = 0$. We therefore have $b \in I$, so that $a \in tI$. Since a was chosen arbitrarily, we conclude that $I \subseteq tI$. Iterating this observation, we obtain inclusions $I \subseteq tI \subseteq t^2 I \subseteq t^3 I \subseteq \cdots$, so that $I \subseteq \bigcap_{m \geqslant 0} (t^m)$. Since $\overline{R}_{\mathfrak{q}}$ is a Noetherian local ring, it follows from the Krull intersection theorem (Corollary 7.3.6.10) that $I = 0$, so that $\overline{R}_{\mathfrak{q}}$ is reduced.

(b) For every point $x \in V$, there exists an open subset of the closure $\overline{\{x\}}$ which is contained in V. Let us identify x with a prime ideal $\mathfrak{p} \subseteq B_0$, and let $\mathfrak{q} \subseteq A_0$ be the inverse image of $\mathfrak{p}$. Replacing A_0 by $A_0/\mathfrak{q}$, we can assume that A_0 is an integral domain and that $\mathfrak{q} = 0$. Let K be the fraction field of A_0. Our assumption that $x \in V$ implies that the localization $B_{0\mathfrak{p}}$ is geometrically reduced over K. Set $B_K = K \otimes_{A_0} B_0$. Applying Corollary 8.6.1.7, we may assume (after replacing B_0 by a localization if necessary) that there exists an element $b \in B_K$ which is not a zero-divisor such that $B_K[b^{-1}]$ factors as a product $\prod_{1 \leqslant i \leqslant n} B_{Ki}$, where each B_{Ki} is étale over a polynomial ring $K[x_1, \ldots, x_{m_i}]$. Replacing A_0 by a suitable localization, we may assume that b lies in B_0 (which we identify with a subring of B_K) and that $B_0[b^{-1}] \simeq \prod_{1 \leqslant i \leqslant n} B_{0i}$, where each B_{0i} is étale over a polynomial ring $A_0[x_1, \ldots, x_{m_i}]$. Let M denote the quotient B_0/bB_0. Using Lemma 6.1.5.4, we may assume (after passing to a suitable localization of A_0) that M is free as an A_0-module. It follows that, for every residue field κ of A_0, the exact sequence $0 \to B_0 \xrightarrow{b} B_0 \to M$ remains exact after tensoring with κ: that is, that the image of b not a zero-divisor in $\kappa \otimes_{A_0} B_0$. Applying the criterion of Corollary 8.6.1.7, we deduce that $\kappa \otimes_{A_0} B_0$ is geometrically reduced over κ, so that

$V = |\operatorname{Spec} B_0|$.

□

8.6.4 Geometric Connectivity

We now restrict our attention to *proper* geometrically reduced morphisms. In this case, we have the following:

Proposition 8.6.4.1. *Let $f : \mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper, locally almost of finite presentation, and geometrically reduced. The following conditions are equivalent:*

(1) *The morphism f is geometrically connected: that is, for every pullback square*

$$\begin{array}{ccc} \mathsf{X}_\kappa & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow f \\ \operatorname{Sp\acute{e}t} \kappa & \longrightarrow & \mathsf{Y} \end{array}$$

where κ is an algebraically closed field, the topological space $|\mathsf{X}_\kappa|$ is connected.

(2) *Let $u : \mathcal{O}_{\mathcal{Y}} \to f_* \mathcal{O}_{\mathcal{X}}$ be the unit map. Then $\operatorname{cofib}(u) \in \operatorname{QCoh}(\mathsf{Y})$ has Tor-amplitude $\leqslant -1$.*

The proof of Proposition 8.6.4.1 will require some algebraic preliminaries.

Lemma 8.6.4.2. *Let R be a commutative ring and let M be an R-module which is perfect to order 1. Let $\mathfrak{p} \subseteq R$ be a prime ideal and let κ denote the residue field at $\mathfrak{p}$. Then the following conditions are equivalent:*

(a) *There exists an element $a \in R$ which does not belong to $\mathfrak{p}$ for which the localization $M[a^{-1}] \in \operatorname{Mod}_{R[a^{-1}]}$ is 1-split (see Definition* **??***).*

(b) *The vector space $\pi_1(\kappa \otimes_R M)$ vanishes.*

Proof. The implication $(a) \Rightarrow (b)$ is obvious. Conversely, suppose that (b) is satisfied. Since M is perfect to order 1, it is k-connective for every sufficiently small integer k. We proceed by descending induction on k, beginning with the case $k = 1$. If M is 1-connective, then we have $\pi_1(\kappa \otimes_R M) \simeq \operatorname{Tor}_0^R(\kappa, \pi_1 M)$. Our assumption that M is perfect to order 1 guarantees that $\pi_1 M$ is a finitely generated R-module. Consequently, if condition (ii) is satisfied then Nakayama's lemma guarantees that $\pi_1 M[a^{-1}] \simeq 0$ for some $a \notin \mathfrak{p}$, so that $M[a^{-1}]$ is 2-connective and condition (a) follows.

To carry out the inductive step, assume that $k \leqslant 0$ and that M is k-connective. Then our assumption that M is perfect to order 1 guarantees that $\pi_k M$ is a finitely presented R-module, and the k-connectivity of M supplies an isomorphism $\pi_k(\kappa \otimes_R M) \simeq \mathrm{Tor}_0^R(\kappa, \pi_k M)$. Choose a collection of elements $x_1, \ldots, x_m \in \pi_k M$ whose images form a basis for $\mathrm{Tor}_0^R(\kappa, \pi_k M)$ as a vector space over κ. It follows from Nakayama's lemma that, after localizing at some element $a \in R - \mathfrak{p}$, we may assume that $x_1, \ldots, x_m$ generate $\pi_k M$ as an R-module, and therefore determine a map $u : \Sigma^k R^m \to M$ whose cofiber is $(k+1)$-connective. We have a short exact sequence

$$\pi_1(\kappa \otimes_R M) \to \pi_1(\kappa \otimes_R \mathrm{cofib}(u)) \to \pi_0(\kappa \otimes_R \Sigma^k R^m) \xrightarrow{\rho} \pi_0(\kappa \otimes_R M)$$

where the third term vanishes for $k < 0$ and ρ is injective when $k = 0$. Consequently, the cofiber $\mathrm{cofib}(u)$ also satisfies condition (ii). Applying the inductive hypothesis, we deduce that $\mathrm{cofib}(u)$ satisfies condition (i). Localizing if necessary, we may assume that $\mathrm{cofib}(u)$ splits as a direct sum $M' \oplus M''$, where M' is 2-connective and M'' is perfect of Tor-amplitude $\leqslant 1$. Then M can be described as the fiber of a map $\theta : M' \oplus M'' \to \Sigma^{k+1} R^m$. Since the codomain of θ is 1-truncated, the restriction $\theta|_{M'}$ is nullhomotopic, so that $M \simeq M' \oplus \mathrm{fib}(\theta|_{M''})$ is 1-split as desired. □

Corollary 8.6.4.3. *Let R be a connective $\mathbb{E}_\infty$-ring and let M be an almost perfect R-module. The following conditions are equivalent:*

- (a) *The module M is perfect of* Tor-*amplitude* $\leqslant 0$.
- (b) *For every maximal ideal $\mathfrak{m}$ of $\pi_0 R$ with residue field $\kappa = (\pi_0 R)/\mathfrak{m}$, the homotopy groups $\pi_n(\kappa \otimes_R M)$ vanish for $n > 0$.*

Proof. The implication $(a) \Rightarrow (b)$ is trivial. We will show that $(b) \Rightarrow (a)$. Using Proposition 2.7.3.2, we can reduce to the case where R is discrete. Since the condition of being perfect of Tor-amplitude $\leqslant 0$ can be tested locally with respect to the étale topology (Proposition 2.8.4.2), we can use condition (b) and Lemma 8.6.4.2 to reduce to the case where M splits as a direct sum $M' \oplus M''$, where M' is perfect of Tor-amplitude $\leqslant 0$ and M'' is 2-connective. We will complete the proof by showing that $M'' \simeq 0$. Assume otherwise: then there exists some smallest integer $n \geqslant 2$ such that $\pi_n M'' \neq 0$. Since M'' is a direct summand of M, it is almost perfect as an R-module. Consequently, the first nonvanishing homotopy group $\pi_n M''$ is a finitely generated R-module. For each maximal ideal $\mathfrak{m} \subseteq R$, the quotient $(\pi_n M'')/\mathfrak{m}(\pi_n M'')$ appears as a direct summand of $\pi_n(\kappa \otimes_R M)$ and therefore vanishes. Invoking Nakayama's lemma, we deduce that $\pi_n M'' \simeq 0$ and obtain a contradiction. □

Proof of Proposition 8.6.4.1. Let $f : \mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y}) = \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper, locally almost of finite presentation, and geometrically reduced, and let $u : \mathscr{O}_{\mathcal{Y}} \to f_* \mathscr{O}_{\mathcal{X}}$ be the unit map. We wish to show that f has connected geometric fibers if and only if the cofiber $\mathrm{cofib}(u)$ has Tor-amplitude $\leqslant -1$. Both assertions are local on Y, so we may assume without loss of generality that $\mathsf{Y} \simeq \mathrm{Spét}\, R$ is affine. It follows from Theorem 5.6.0.2 that the cofiber $\mathrm{cofib}(u)$ is almost perfect. Using Corollary 8.6.4.3, we can reformulate (2) as follows:

($2'$) For every map $R \to \kappa$, where κ is an algebraically closed field, the homotopy groups $\pi_n(\kappa \otimes_R \mathrm{cofib}(u))$ vanish for $n \geqslant 0$.

Consequently, to prove the equivalence of (1) and ($2'$), there is no loss of generality in assuming that $R = \kappa$ is an algebraically closed field. In this case, X is a reduced spectral algebraic space, so that $\pi_n f_* \mathscr{O}_{\mathcal{X}}$ vanishes for $n > 0$. Consequently, assertion ($2'$) is equivalent to the statement that the unit map $e : \kappa \to \pi_0 f_* \mathscr{O}_{\mathcal{X}} = \mathrm{H}^0(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$ is an isomorphism. If this condition is satisfied, the $\mathrm{H}^0(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$ contains exactly two idempotent elements and therefore the topological space $|\mathsf{X}|$ is connected. For the converse, we observe that $\mathrm{H}^0(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$ is finite-dimensional as a vector space over κ. The connectedness of $|\mathsf{X}|$ implies that $\mathrm{H}^0(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$ is a local Artinian ring. Since f is geometrically reduced, this Artinian ring is reduced and is therefore a finite extension of κ. Since κ is algebraically closed, it follows that e is an isomorphism. □

8.7 Application: Stein Factorizations

Let $f : X \to Z$ be a morphism of Noetherian schemes. Then f admits an essentially unique factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g induces an isomorphism $\mathscr{O}_Y \to g_* \mathscr{O}_X$ (where g_* denotes the pushforward in the *abelian* category of quasi-coherent sheaves) and h is affine: the scheme Y can be described as the spectrum of the quasi-coherent sheaf of algebras $f_* \mathscr{O}_X$ on Z. If the morphism f is proper, then one can say more:

(*i*) The direct image $f_* \mathscr{O}_X$ is a *coherent* sheaf of algebras on Z.

(*ii*) The morphism h is finite.

(*iii*) The scheme Y is Noetherian.

(*iv*) The morphism g has connected fibers.

Assertion (*i*) is a special case of Theorem 5.6.0.1, the implications (*i*) ⇒ (*ii*) ⇒ (*iii*) are obvious, and assertion (*iv*) is Zariski's connectedness theorem. In this case, we refer to the pair (g, h) as the *Stein factorization* of the morphism f.

Stein factorizations are a very useful tool in algebraic geometry. However, one must take care that the formation of Stein factorizations is not compatible with base change: given a pullback diagram of Noetherian schemes

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{u} & Z, \end{array}$$

the associated comparison map $\alpha : u^* f_* \mathscr{O}_X \to f'_* \mathscr{O}_{X'}$ need not be an isomorphism in general (Warning 8.7.1.8). However, the situation is somewhat better if we make the additional assumption that f is geometrically reduced (Definition ??): in this case, the scheme Y appearing in the Stein factorization of f is étale over Z and is compatible with arbitrary base change (Remark 8.7.3.4).

Our goal in this section is to discuss Stein factorizations in the setting of spectral algebraic geometry. Here the situation is a bit more complicated. There are (at least) two generalizations of the Stein factorization:

(a) If $f : \mathsf{X} \to \mathsf{Z}$ is a quasi-compact, quasi-separated morphism of spectral algebraic spaces, then f admits an essentially unique factorization

$$\mathsf{X} \xrightarrow{g} \mathsf{Y} \xrightarrow{h} \mathsf{Z}$$

where h is affine and g exhibits the structure sheaf of Y as the connective cover of the (derived) direct image $f_* \mathscr{O}_{\mathsf{X}}$ (Theorem 8.7.1.5). If X and Z are locally Noetherian and f is proper, then we can use Theorem 5.6.0.2 to see that Y is locally Noetherian and the morphism h is finite, and we can use Theorem 8.5.0.3 to show that the geometric fibers of g are connected, just as in classical algebraic geometry.

(b) If $f : \mathsf{X} \to \mathsf{Z}$ is a morphism of spectral algebraic spaces which is proper, locally almost of finite presentation, and geometrically reduced (Definition 8.6.0.1), then f admits an essentially unique factorization $\mathsf{X} \xrightarrow{g'} \mathsf{Y}' \xrightarrow{h'} \mathsf{Z}$, where h' is finite étale and the geometric fibers of g' are connected (Theorem 8.7.3.1).

However, in the setting of spectral algebraic geometry, these factorizations are generally distinct from one another. To distinguish them, we will refer to (a) as the *Stein factorization of f* (Definition 8.7.1.4) and to (b) as the *reduced Stein factorization of f* (Definition 8.7.3.2).

8.7.1 Stein Factorizations

We begin with some general observations.

Proposition 8.7.1.1. *Suppose we are given a commutative diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X} & \longrightarrow & \mathsf{Z}' \\ \downarrow{\scriptstyle g} & \nearrow & \downarrow{\scriptstyle h} \\ \mathsf{Y} & \longrightarrow & \mathsf{Z} \end{array}$$

satisfying the following conditions:

(*a*) *The morphism h is affine.*

(*b*) *The unit map $\mathscr{O}_{\mathcal{Y}} \to f_* \mathscr{O}_{\mathcal{X}}$ exhibits $\mathscr{O}_{\mathcal{Y}}$ as a connective cover of $g_* \mathscr{O}_{\mathcal{X}}$.*

Then the mapping space $\mathrm{Map}_{\mathrm{SpDM}_{\mathsf{X}//\mathsf{Z}}}(\mathsf{Y}, \mathsf{Z}')$ *(which parametrizes dotted arrows as indicated in the diagram above) is contractible.*

Proof. The assertion can be tested locally with respect to the étale topology of Z. We may therefore assume without loss of generality that $\mathsf{Z} \simeq \mathrm{Spét}\, A$ is affine. Since h is affine, it follows that $\mathsf{Z}' \simeq \mathrm{Spét}\, B$ is affine. Using the universal properties of $\mathrm{Spét}\, A$ and $\mathrm{Spét}\, B$, we obtain a fiber sequence

$$\mathrm{Map}_{\mathrm{SpDM}_{\mathsf{X}//\mathsf{Z}}}(\mathsf{Y}, \mathsf{Z}') \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, \Gamma(\mathcal{Y}; \mathscr{O}_{\mathcal{Y}})) \xrightarrow{\rho} \mathrm{Map}_{\mathrm{CAlg}_A}(B; \Gamma(\mathcal{X}, \mathscr{O}_{\mathcal{X}})).$$

It will therefore suffice to show that ρ is a homotopy equivalence. Since B is connective, this follows immediately from (b) (which guarantees that the unit map $\Gamma(\mathcal{Y}; \mathscr{O}_{\mathcal{Y}}) \to \Gamma(\mathcal{X}; \mathscr{O}_{\mathcal{X}})$ induces an equivalence of connective covers). □

Proposition 8.7.1.2. *Let $f : \mathsf{X} \to \mathsf{Z}$ be a morphism of spectral Deligne-Mumford stacks. Assume that f is a quasi-compact, quasi-separated relative algebraic space. Then f factors as a composition $\mathsf{X} \xrightarrow{g} \mathsf{Y} \xrightarrow{h} \mathsf{Z}$, where h is affine and g satisfies condition* (b) *of Proposition 8.7.1.1.*

Proof. Let $\mathscr{O}_{\mathsf{X}}$ denote the structure sheaf of X. It follows from Corollary 3.4.2.2 that the direct image $f_* \mathscr{O}_{\mathcal{X}} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{Z}))$ is quasi-coherent. Then $\mathscr{A} = \tau_{\geq 0} f_* \mathscr{O}_{\mathcal{X}}$ is a connective quasi-coherent commutative algebra object of $\mathrm{QCoh}(\mathsf{Z})$. Let Y denote the spectrum of $\mathscr{A}$ relative to Z (see Notation **??**). Then Y is equipped with affine map $h : \mathsf{Y} \to \mathsf{Z}$ (and is therefore a spectral Deligne-Mumford stack), and the canonical map $f^* \mathscr{A} \to f^* f_* \mathscr{O}_{\mathcal{X}} \to \mathscr{O}_{\mathcal{X}}$ in $\mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X}))$ classifies a morphism $g : \mathsf{X} \to \mathsf{Y}$ in $\mathrm{SpDM}_{/\mathsf{Z}}$. Let $\mathscr{O}_{\mathsf{Y}}$ denote the structure sheaf of Y; we wish to show that the unit map $u : \mathscr{O}_{\mathsf{Y}} \to g_* \mathscr{O}_{\mathsf{X}}$ exhibits $\mathscr{O}_{\mathsf{Y}}$ as a connective cover of $g_* \mathscr{O}_{\mathsf{X}}$. Since h is affine, the direct image functor $h_* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{Z})$ is conservative and t-exact. It will therefore suffice to show that the map $h_*(u) : h_* \mathscr{O}_{\mathsf{Y}} \to h_* g_* \mathscr{O}_{\mathsf{X}} \simeq f_* \mathscr{O}_{\mathsf{X}}$ exhibits $h_* \mathscr{O}_{\mathsf{Y}}$ as the connective cover of $f_* \mathscr{O}_{\mathsf{X}}$, which follows immediately from the definition of Y. □

Remark 8.7.1.3. In the situation of Proposition 8.7.1.2, the morphism $g : \mathsf{X} \to \mathsf{Y}$ is also a quasi-compact, quasi-separated relative algebraic space.

Definition 8.7.1.4. Let $f : \mathsf{X} \to \mathsf{Z}$ be a morphism of spectral Deligne-Mumford stacks which is a quasi-compact, quasi-separated relative algebraic space. We will say that a commutative diagram

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{f} & \mathsf{Z} \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & \mathsf{Y} & \end{array}$$

is a *Stein factorization of* f if g and h satisfy the requirements of Proposition 8.7.1.2.

Theorem 8.7.1.5. *Let* SpDM° *denote the subcategory of* SpDM *whose objects are spectral Deligne-Mumford stacks and whose morphisms are quasi-compact, quasi-separated relative algebraic spaces. Let* S^L *denote the collection of morphisms* $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ *in* SpDM° *which exhibit* $\mathcal{O}_{\mathcal{Y}}$ *as a connective cover of* $f_* \mathcal{O}_{\mathsf{X}}$*, and let* S^R *denote the collection of affine morphisms in* SpDM°*. Then the pair* (S^L, S^R) *is a factorization system on the* ∞*-category* SpDM° *(see Definition HTT.5.2.8.8).*

Proof. Combine Propositions 8.7.1.1 and 8.7.1.2 (and Remark 8.7.1.3). □

Remark 8.7.1.6. It follows from Theorem 8.7.1.5 that every quasi-compact, quasi-separated relative algebraic space $f : \mathsf{X} \to \mathsf{Z}$ admits a Stein factorization $\mathsf{X} \xrightarrow{g} \mathsf{Y} \xrightarrow{h} \mathsf{Z}$, which is uniquely determined up to a contractible space of choices (see Proposition HTT.5.2.8.17).

Proposition 8.7.1.7. *Suppose we are given a diagram of spectal Deligne-Mumford stacks* σ :

$$\begin{array}{ccccc} \mathsf{X}' & \xrightarrow{g'} & \mathsf{Y}' & \xrightarrow{h'} & \mathsf{Z}' \\ \downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle q} \\ \mathsf{X} & \xrightarrow{g} & \mathsf{Y} & \xrightarrow{h} & \mathsf{Z}, \end{array}$$

where all morphisms are quasi-compact, quasi-separated relative algebraic spaces, both squares are pullbacks, and the pair (g, h) *is a Stein factorization of* $f = h \circ g$*. If* q *is flat, then the paif* (g', h') *is a Stein factorization of* $f' = h' \circ g'$*.*

Warning 8.7.1.8. Proposition 8.7.1.7 is usually not true if we do not assume that q is flat. In other words, the formation of Stein factorizations is compatible with flat base change, but not with arbitrary base change.

Proof of Proposition 8.7.1.7. It is clear that h' is affine. It will therefore suffice to show that the unit map $e' : \mathcal{O}_{\mathsf{Y}'} \to g'_* \mathcal{O}_{\mathsf{X}'}$ exhibits $\mathcal{O}_{\mathsf{Y}'}$ as a connective cover of $g'_* \mathcal{O}_{\mathsf{X}'}$. Since the structure sheaf $\mathcal{O}_{\mathsf{Y}'}$ is connective, this is equivalent to the requirement that $\mathrm{cofib}(e')$ belongs to $\mathrm{QCoh}(\mathsf{Y}')_{\leqslant -1}$. Because both squares in the diagram σ are pullbacks, we can identify $\mathrm{cofib}(e')$ with $r^* \mathrm{cofib}(e)$, where $e : \mathscr{O}_{\mathsf{Y}} \to g_* \mathscr{O}_{\mathsf{X}}$ is the unit map determined by g. Since (g, h) is a Stein factorization of $f = h \circ g$, the cofiber $\mathrm{cofib}(e)$ belongs to $\mathrm{QCoh}(\mathsf{Y})_{\leqslant -1}$. The desired result now follows from the observation that the pullback functor $r^* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{Y}')$ is t-exact (by virtue of our assumption that q is flat). □

8.7.2 The Noetherian Case

Let $f : \mathsf{X} \to \mathsf{Z}$ be a morphism of quasi-compact, quasi-separated spectral algebraic spaces. It follows from Proposition 8.7.1.2 that f admits a Stein factorization $\mathsf{X} \xrightarrow{g} \mathsf{Y} \xrightarrow{h} \mathsf{Z}$. However, the spectral algebraic space Y which appears in this factorization is generally a somewhat wild object, which one should not expect to have any good finiteness properties. The situation is better if we assume that f is a proper morphism whose domain and codomain are locally Noetherian:

Theorem 8.7.2.1. *Let $f : \mathsf{X} \to \mathsf{Z}$ be a proper morphism between locally Noetherian spectral Deligne-Mumford stacks, and let $\mathsf{X} \xrightarrow{g} \mathsf{Y} \xrightarrow{h} \mathsf{Z}$ be a Stein factorization of X. Then:*

- (a) *The morphism h is finite.*
- (b) *The spectral Deligne-Mumford stack Y is locally Noetherian.*
- (c) *The morphism g is proper.*

Proof. Since X and Z are locally Noetherian, it follows from Remark 4.2.0.4 that f is locally almost of finite presentation. Applying Theorem 5.6.0.2, we deduce that the direct image $f_* \mathscr{O}_{\mathsf{X}}$ is almost perfect. Since Z is locally Noetherian, it follows that the truncation $\tau_{\geqslant 0} f_* \mathscr{O}_{\mathsf{X}} \simeq h_* \mathscr{O}_{\mathsf{Y}}$ is almost perfect. Applying Corollary 5.2.2.2, we deduce that h is finite and locally almost of finite presentation. This proves (a), and (b) follows from Remark 4.2.0.4. Assertion (c) now follows from Proposition 5.1.4.1. □

In the situation of Theorem 8.7.2.1, the factorization $\mathsf{X} \xrightarrow{\mathsf{Y}} \xrightarrow{\mathsf{Z}}$ enjoys several pleasant features:

Theorem 8.7.2.2 (Zariski Connectedness Theorem)**.** *Let $f : \mathsf{X} \to \mathsf{Z}$ be a proper morphism between locally Noetherian spectral Deligne-Mumford stacks, and let $\mathsf{X} \xrightarrow{g} \mathsf{Y} \xrightarrow{h} \mathsf{Z}$ be a Stein factorization of f. Then the morphism g is geometrically connected. That is, for every field κ and every map $\eta : \mathrm{Sp\acute{e}t}\, \kappa \to \mathsf{Y}$, the fiber product $\mathsf{X} \times_{\mathsf{Y}} \mathrm{Sp\acute{e}t}\, \kappa$ is connected.*

Proof. Without loss of generality, we may replace Z by Y and thereby reduce to the case where (f, id) is the Stein factorization of f: that is, f exhibits $\mathscr{O}_{\mathsf{Z}}$ as the connective cover of $f_* \mathscr{O}_{\mathsf{X}}$. In this case, we wish to deduce that for every field κ and every map $\eta : \mathrm{Sp\acute{e}t}\, \kappa \to \mathsf{Z}$, the fiber $\mathrm{Sp\acute{e}t}\,\kappa \times_{\mathsf{Z}} \mathsf{X}$ is connected. Replacing κ by an extension field if necessary, we may assume that η factors through an étale map $q : \mathrm{Sp\acute{e}t}\, A \to \mathsf{Z}$. Using Proposition 8.7.1.7, we can replace Z by $\mathrm{Sp\acute{e}t}\, A$ and thereby reduce to the case where $\mathsf{Z} = \mathrm{Sp\acute{e}t}\, A$ is affine. The map η is then classified by a homomorphism of commutative rings $\phi : \pi_0 A \to \kappa$. Write κ as a filtered colimit $\varinjlim \kappa_\alpha$, where each κ_α is the fraction field of a finitely generated $\pi_0 A$-algebra. Using Proposition 4.3.5.5, we see that in order to prove that $\mathsf{X} \times_{\mathsf{Z}} \mathrm{Sp\acute{e}t}\,\kappa$ is connected, it will suffice to show that each fiber product $\mathsf{X} \times_{\mathsf{Z}} \mathrm{Sp\acute{e}t}\,\kappa_\alpha$ is connected. We may therefore replace κ by κ_α and reduce to the case where κ is the fraction field of the subring $R \subseteq \kappa$ generated by the elements $\{x_i\}$ and the image of $\pi_0 A$.

Let $A[\mathbf{Z}^n_{\geqslant 0}]$ denote the monoid algebra over A on the commutative monoid $\mathbf{Z}^n_{\geqslant 0}$ (see Notation 5.4.1.1), so we can write $\pi_0 A[Z^n_{\geqslant 0}] \simeq (\pi_0 A)[X_1, \ldots, X_n]$. It follows that ϕ admits a unique extension to a ring homomorphism $\overline{\phi} : \pi_0 A[\mathbf{Z}^n_{\geqslant 0}] \to \kappa$ satisfying $\overline{\phi}(X_i) = x_i$. The ring homomorphism $\overline{\phi}$ classifies a map $\overline{\eta} : \mathrm{Sp\acute{e}t}\,\kappa \to \mathrm{Sp\acute{e}t}\, A[\mathbf{Z}^n_{\geqslant 0}]$. Since $A[\mathbf{Z}^n_{\geqslant 0}]$ is flat over A, we can use Proposition 8.7.1.7 to replace A by $A[\mathbf{Z}^n_{\geqslant 0}]$ and η by $\overline{\eta}$ and thereby reduce to the case where κ is the fraction field of the image of the map ϕ: in other words, ϕ exhibits κ as the residue field of $\pi_0 A$ as some prime ideal $\mathfrak{p} \subseteq \pi_0 A$. Let $A^\wedge$ denote the $\mathfrak{q}$-completion of the localization $A_{\mathfrak{q}}$. Since $A_{\mathfrak{q}}$ is Noetherian, $A^\wedge$ is flat over A (Corollary 7.3.6.9). Using Proposition 8.7.1.7 again, we can replace A by $A^\wedge$ and thereby reduce to the case where $\pi_0 A$ is a complete local Noetherian ring with residue field $\mathfrak{q}$.

Set $\mathsf{X}_0 = \mathsf{X} \times_{\mathsf{Z}} \mathrm{Sp\acute{e}t}\,\kappa$. Then X_0 is a proper algebraic space over κ, so the topological space $|\,\mathsf{X}_0\,|$ is Noetherian (Remark 3.6.3.5). It follows that we can write X_0 as a disjoint union of finitely many connected components $\{\mathsf{K}_i\}_{1 \leqslant i \leqslant m}$. Let $\mathsf{X}^\wedge$ denote the formal completion of X along the vanishing locus of $\mathfrak{q}$, so that the decomposition $\mathsf{X}_0 \simeq \amalg \mathsf{K}_i$ induces a decomposition of $\mathsf{X}^\wedge$ as a disjoint union $\amalg \mathsf{K}_i^\wedge$. Lemma 8.5.1.1 implies that we can identify A with the connective cover of $\Gamma(\mathsf{X}^\wedge; \mathscr{O}_{\mathsf{X}^\wedge})$. Consequently, the decomposition $\mathsf{X}^\wedge \simeq \amalg \mathsf{K}_i^\wedge$ determines a direct product decomposition $A \simeq \prod_{1 \leqslant i \leqslant m} A_i$. Since A is local and each factor A_i is nonzero, we must have $m = 1$: that is, X_0 is connected. □

Theorem 8.7.2.3. *Let $f : \mathsf{X} \to \mathsf{Z}$ be a proper morphism between locally Noetherian, quasi-compact, quasi-separated spectral algebraic spaces, let $\mathsf{X} \xrightarrow{g} \mathsf{Y} \xrightarrow{h} \mathsf{Z}$ be a Stein factorization of f, and let $x \in |\,\mathsf{X}\,|$ be a point. The following conditions are equivalent:*

- *(a) The point x is isolated in the fiber $|\,\mathsf{X}\,| \times_{|\,\mathsf{Z}\,|} \{x\}$.*
- *(b) There exists an open substack $\mathsf{U} \subseteq \mathsf{Y}$ which contains the image of x, for which the projection map $\mathsf{X} \times_{\mathsf{Y}} \mathsf{U} \to \mathsf{U}$ is an equivalence.*

Proof. Since the morphism h is finite, the topological space $|\,\mathsf{Y}\,| \times_{|\,\mathsf{Z}\,|} \{x\}$ is discrete (and consists of finitely many points). Consequently, x is isolated in the fiber $|\,\mathsf{X}\,| \times_{|\,\mathsf{Z}\,|} \{x\}$ if and only if it is isolated in the fiber $|\,\mathsf{X}\,| \times_{|\,\mathsf{Y}\,|} \{x\}$. We may therefore replace Z by Y and thereby reduce to the case where h is an equivalence: that if, the morphism f exhibits $\mathscr{O}_{\mathsf{Z}}$ as a connective cover of $f_* \mathscr{O}_{\mathsf{X}}$. Take U to be the largest open substack of Y for which the projection map $\mathsf{X} \times_{\mathsf{Z}} \mathsf{U} \to \mathsf{U}$ is an equivalence. We wish to show that $|\mathsf{U}|$ contains the image of x. This assertion is local on Z: we may therefore assume without loss of generality that $\mathsf{Z} \simeq \operatorname{Sp\acute{e}t} R$ is affine, so we can identify the image of x in $|\mathsf{Z}| \simeq |\operatorname{Spec} A|$ with a prime ideal $\mathfrak{p} \subseteq \pi_0 R$. To show that this point is contained in $|\mathsf{U}|$, we must show that the projection map $f_a : \mathsf{X} \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Sp\acute{e}t} R[a^{-1}] \to \operatorname{Sp\acute{e}t} R[a^{-1}]$ is an equivalence for some $a \notin \mathfrak{p}$. Note that this is equivalent to the requirement that f_a is affine (since f_a satisfies condition (b) of Proposition 8.7.1.1). Using Proposition 4.6.1.1 and Remark 2.4.4.2, we are reduced to showing that the map $\mathsf{X} \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Sp\acute{e}t} R_{\mathfrak{p}} \to \operatorname{Sp\acute{e}t} R_{\mathfrak{p}}$ is affine. Consequently, we may replace R by $R_{\mathfrak{p}}$ and thereby reduce to the case where R is a local $\mathbb{E}_\infty$-ring with maximal ideal $\mathfrak{p}$; in this case, we will show that f is an equivalence. Let $R^\wedge$ denote the completion of R with respect to $\mathfrak{p}$. Since R is Noetherian, $R^\wedge$ is faithfully flat over R (Corollary 7.3.6.9). Using Corollary 2.8.3.4 and Proposition 8.7.1.7, we can replace R by $R^\wedge$ and thereby reduce to the case where R is $\mathfrak{p}$-complete.

Let κ denote the residue field of R, let $\mathsf{X}_0 = \operatorname{Sp\acute{e}t} \kappa \times_{\operatorname{Sp\acute{e}t} R} \mathsf{X}$, and let $\mathsf{X}^\wedge$ denote the formal spectral Deligne-Mumford stack given by $\operatorname{Spf} R^\wedge \times_{\operatorname{Spec} R} \mathsf{X}$. Since x is isolated, the topological space $|\mathsf{X}_0|$ has a single point, so that X_0 is affine. Using Lemma 8.3.3.2, we deduce that $\mathsf{X}^\wedge$ is affine. Write $\mathsf{X}^\wedge = \operatorname{Spf} A$ for some adic $\mathbb{E}_\infty$-ring A. Using Lemma 8.5.1.1, we deduce that $\Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}}) \simeq \Gamma(\mathsf{X}^\wedge; \mathscr{O}_{\mathsf{X}^\wedge}) \simeq A$ is connective. It follows that the unit map $R \to A$ is an equivalence, so that the projection $\mathsf{X}^\wedge \to \operatorname{Spf} R$ is an equivalence. Applying Corollary 8.5.3.4, we deduce that the projection map $\mathsf{X} \to \operatorname{Sp\acute{e}t} R$ is also an equivalence, as desired.

□

8.7.3 Reduced Stein Factorizations

We now show that all every proper geometrically reduced morphism which is locally almost of finite presentation can be obtained by combining Corollary 8.6.2.3 with Proposition 8.6.4.1. More precisely, we have the following result:

Theorem 8.7.3.1. *Let $\mathcal{C}$ denote the subcategory of* SpDM *spanned by all objects and those morphisms $f : \mathsf{X} \to \mathsf{Z}$ which are proper, geometrically reduced, and locally almost of finite presentation. Then $\mathcal{C}$ admits a factorization system (S^L, S^R), where S^L consists of those morphisms f which are geometrically connected (see Proposition 8.6.4.1) and S^R consists of those morphisms which are finite étale. In other words:*

(*a*) *For every commutative diagram*

$$\begin{array}{ccc} \mathsf{X} & \longrightarrow & \mathsf{Y} \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ \mathsf{Z} & \longrightarrow & \mathsf{Z}' \end{array}$$

in $\mathcal{C}$ *where* f *is geometrically connected and* g *is finite étale, the space* $\mathrm{Map}_{\mathcal{C}_{\mathsf{X}//\mathsf{Z}'}}(\mathsf{Z},\mathsf{Y})$ *(parametrizing dotted arrows which render the preceding diagram commutative) is contractible.*

(*b*) *Every morphism* $f : \mathsf{X} \to \mathsf{Z}$ *in* $\mathcal{C}$ *admits a factorization* $\mathsf{X} \xrightarrow{f'} \mathsf{Y} \xrightarrow{f''} \mathsf{Z}$*, where* f' *is geometrically connected and* f'' *is finite étale.*

Definition 8.7.3.2. Let $f : \mathsf{X} \to \mathsf{Z}$ be a morphism of spectral Deligne-Mumford stacks which is proper, geometrically reduced, and locally almost of finite presentation. We will refer to the factorization $\mathsf{X} \xrightarrow{f'} \mathsf{Y} \xrightarrow{f''} \mathsf{Z}$ whose existence is asserted by part (*b*) of Theorem 8.7.3.1 as the *reduced Stein factorization* of f. It follows from part (*a*) of Theorem 8.7.3.1 (and Proposition HTT.5.2.8.17) that the reduced Stein factorization is well-defined up to a contractible space of choices.

Warning 8.7.3.3. Let $f : \mathsf{X} \to \mathsf{Z}$ be a morphism of spectral Deligne-Mumford stacks which is proper, geometrically reduced, and locally almost of finite presentation, and let $\mathsf{X} \xrightarrow{f'} \mathsf{Y} \xrightarrow{f''} \mathsf{Z}$ be the Stein factorization of f (Definition 8.7.1.4). If $\mathscr{O}_{\mathsf{Z}}$ is 0-truncated, then we will prove in a moment that (f', f'') is also a reduced Stein factorization of f (Proposition 8.7.4.2). However, if $\mathscr{O}_{\mathsf{Z}}$ is not 0-truncated, then (f', f'') is usually *not* a reduced Stein factorization of f. Roughly speaking, the problem is that direct image of the homotopy sheaves $\pi_n \mathscr{O}_{\mathsf{X}}$ make "contributions" to the algebra $\tau_{\geqslant 0} f_* \mathscr{O}_{\mathsf{X}}$ in all degrees $\leqslant n$, but to obtain a spectral Deligne-Mumford stack which is étale over Z we will need to discard the "contributions" in degrees $< n$.

Remark 8.7.3.4 (Functoriality of Reduced Stein Factorizations)**.** Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks σ :

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{f'} & \mathsf{Z}' \\ \downarrow & & \downarrow \\ \mathsf{X} & \xrightarrow{f} & \mathsf{Z}, \end{array}$$

where the horizontal maps are proper, geometrically reduced, and locally almost of finite presentation. Then f admits a reduced Stein factorization $\mathsf{X} \xrightarrow{g} \mathsf{Y} \xrightarrow{h} \mathsf{Z}$. Define $\mathsf{Y}' = \mathsf{Y} \times_{\mathsf{Z}} \mathsf{Z}'$, so that we can extend σ to a diagram

$$\begin{array}{ccccc} \mathsf{X}' & \xrightarrow{g'} & \mathsf{Y}' & \xrightarrow{h'} & \mathsf{Z}' \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{X} & \xrightarrow{g} & \mathsf{Y} & \xrightarrow{h} & \mathsf{Z}. \end{array}$$

The map h' is defined as a pullback to h, and is therefore finite étale. Since the right square and the outer rectangle are pullbacks, it follows that the left square is a pullback as well. In particular, g' is a pullback of g and is therefore proper, geometrically reduced, geometrically connected, and locally almost of finite presentation. It follows that (g', h') is the reduced Stein factorization of f'.

8.7.4 The Proof of Theorem 8.7.3.1

We begin by proving part (a) of Theorem 8.7.3.1. Note that by replacing Y by the fiber product $\mathsf{Y} \times_{\mathsf{Z}'} \mathsf{Z}$, we may reduce to the case where $\mathsf{Z}' = \mathsf{Z}$. In this case, the desired result is a consequence of the following more general assertion:

Proposition 8.7.4.1. *Let $f : \mathsf{X} \to \mathsf{Z}$ and $g : \mathsf{Y} \to \mathsf{Z}$ be morphisms of spectral Deligne-Mumford stacks. Assume that f is proper and geometrically connected (that is, it satisfies condition* (1) *of Proposition 8.6.4.1) and that g is finite étale. Then composition with f induces a homotopy equivalence*

$$\theta : \mathrm{Map}_{\mathrm{SpDM}_{/\mathsf{Z}}}(\mathsf{Z}, \mathsf{Y}) \to \mathrm{Map}_{\mathrm{SpDM}_{/\mathsf{Z}}}(\mathsf{X}, \mathsf{Y}).$$

Proof. The assertion is local on Z. We may therefore assume without loss of generality that Z is affine and that g exhibits Y as a disjoint union of n copies of Z (see Lemma B.7.6.3). In this case, we can identify the domain of θ with the set of all decompositions of the topological space $|\mathsf{Z}|$ into n open sets, and the codomain of θ with the set of all decompositions of the topological space $|\mathsf{X}|$ into n open sets. To prove that θ is bijective, it will suffice to show that the construction $U \mapsto f^{-1}U$ induces a bijection from the set of closed and open subsets of $|\mathsf{Z}|$ to the set of closed and open subsets of $|\mathsf{X}|$. Since f is geometrically connected, the underlying map of topological spaces $|\mathsf{X}| \to |\mathsf{Z}|$ is surjective: consequently, an open set $U \subseteq |\mathsf{Z}|$ is determined by its inverse image $f^{-1}U$. To complete the proof, we must show that if $V \subseteq |\mathsf{X}|$ is a closed and open subset, then V has the form $f^{-1}U$ where $U \subseteq |\mathsf{Z}|$ is a closed an open subset. Since f is proper, the images $f(V)$ and $f(|\mathsf{X}| - V)$ are closed subsets of $|\mathsf{Z}|$. The surjectivity of f implies that these subsets cover $|\mathsf{Z}|$. If $f(V) \cap f(|\mathsf{X}| - V)$ were nonempty, then we could choose a geometric point $\mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{Z}$ of the intersection, in which case V and $|\mathsf{X}| - V$ supply a nontrivial decomposition of the topological space $|\mathsf{X} \times_{\mathsf{Z}} \mathrm{Sp\acute{e}t}\,\kappa|$, contradicting our assumption that f is geometrically reduced. It follows that $U = f(V)$ is closed and open in $|\mathsf{Z}|$. We have $V \subseteq f^{-1}U$ by definition, and the reverse inclusion follows from

$$|\mathsf{X}| - V \subseteq f^{-1}f(|\mathsf{X}| - V))f^{-1}(|\mathsf{Z}| - U) = \mathsf{X} - f^{-1}U.$$

□

Proposition 8.7.4.2. *Let $f : \mathsf{X} \to \mathsf{Z}$ be a morphism of 0-truncated spectral Deligne-Mumford stacks which is proper, geometrically reduced, and locally almost of finite presentation, and let $\mathsf{X} \xrightarrow{f'} \mathsf{Y} \xrightarrow{f''} \mathsf{Z}$ be a reduced Stein factorization of f (Definition 8.7.1.4). Then (f', f'') is also a reduced Stein factorization of f: that is, the morphism f'' is finite étale and the morphism f' satisfies the equivalent conditions of Proposition 8.6.4.1.*

Proof. Using Proposition 8.7.1.7, we see that the desired conclusions can be tested locally with respect to the étale topology on X. We may therefore assume without loss of generality that $\mathsf{Z} \simeq \operatorname{Spét} R$ for some commutative ring R. Using Theorem 4.4.2.2, Proposition 5.5.4.1, and Proposition 6.1.6.1, we can choose a finitely generated subring $R_0 \subseteq R$ and a pullback square

$$\begin{array}{ccc} \mathsf{X} & \longrightarrow & \mathsf{X}_0 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} \\ \operatorname{Spét} R & \longrightarrow & \operatorname{Spét} R_0 \end{array}$$

where f_0 is proper, flat, and locally almost of finite presentation. Let $K \subseteq |\mathsf{X}_0|$ be the complement of the geometrically reduced locus of f_0. Since f_0 is proper, the image $f_0(K)$ is a closed subset of $|\operatorname{Spec} R_0|$ and is therefore defined by a finitely generated ideal $I \subseteq R_0$. Since the map f is geometrically reduced, Proposition 8.6.3.1 guarantees that $f^{-1}K \subseteq |\mathsf{X}|$ is empty. Consequently, IR is the unit ideal in R, so we have an equation $1 = x_1 y_1 + \cdots + x_k y_k$ in the ring R where each x_i belongs to I. Enlarging R_0 if necessary, we may assume that each y_i belong to I, so that $I = R_0$ and the map f_0 is geometrically reduced. Regard the pushforward $f_{0*} \mathscr{O}_{\mathsf{X}_0}$ as an object of Mod_{R_0}, so that we have a canonical equivalence $f_* \mathscr{O}_{\mathsf{X}} \simeq R \otimes_{R_0} f_{0*} \mathscr{O}_{\mathsf{X}_0}$. We therefore have a fiber sequence

$$R \otimes_{R_0} (\tau_{\geqslant 0} f_{0*} \mathscr{O}_{\mathsf{X}_0}) \to f_* \mathscr{O}_{\mathcal{X}} \to R \otimes_{R_0} (\tau_{\leqslant -1} f_{0*} \mathscr{O}_{\mathsf{X}_0}).$$

Note that the first term of this sequence is connective. Consequently, if $\tau_{\leqslant -1} f_{0*}(\mathscr{O}_{\mathsf{X}_0})$ has Tor-amplitude $\leqslant -1$, then then the third term of this fiber sequence belongs to $(\operatorname{Mod}_R)_{\leqslant -1}$, so that we obtain equivalences

$$R \otimes_{R_0} (\tau_{\geqslant 0} f_{0*} \mathscr{O}_{\mathsf{X}_0}) \simeq \tau_{\geqslant 0} f_* \mathscr{O}_{\mathsf{X}} \qquad R \otimes_{R_0} (\tau_{\leqslant -1} f_{0*} \mathscr{O}_{\mathsf{X}_0}) \simeq \tau_{\leqslant -1} f_* \mathscr{O}_{\mathsf{X}}.$$

Consequently, to show that $\tau_{\geqslant 0} f_* \mathscr{O}_{\mathsf{X}}$ is finite étale over R and that $\tau_{\leqslant -1} f_* \mathscr{O}_{\mathsf{X}}$ has Tor-amplitude $\leqslant -1$, it will suffice to show that $\tau_{\geqslant 0} f_{0*} \mathscr{O}_{\mathsf{X}_0}$ is finite étale over R_0 and that

$\tau_{\leqslant -1} f_{0*} \mathscr{O}_{\mathsf{X}_0}$ has Tor-amplitude $\leqslant -1$ over R_0. We may therefore replace f by f_0 and thereby reduce to the case where the commutative ring R is Noetherian.

Applying Theorem 8.7.2.1, we deduce that $A = \pi_0 f_* \mathscr{O}_{\mathsf{X}}$ is finitely generated as an R-module, and therefore also as an R-algebra. Consequently, to show that A is finite étale over R, it will suffice to show that the relative cotangent complex $L_{A/R}$ vanishes (Lemma B.1.3.3). Since A is locally almost of finite presentation over R (Theorem HA.7.2.4.31), the relative cotangent complex $L_{A/R}$ is almost perfect as an A-module, and therefore also as an R-module (since A is finite over R). Note that Theorem 8.7.2.1 also guarantees that the truncation $\tau_{\leqslant -1} f_* \mathscr{O}_{\mathsf{X}}$ is almost perfect. Using Corollary 2.7.4.4 and Corollary 8.6.4.3, we see that it will suffice to prove verify the following conditions for every ring homomorphism $\rho : R \to \kappa$, where κ is a field:

(i) The tensor product $\kappa \otimes_R L_{A/R}$ vanishes.

(ii) The homotopy groups $\pi_n(\kappa \otimes_R \tau_{\leqslant -1} f_* \mathscr{O}_{\mathsf{X}})$ vanish for $n \geqslant 0$.

In fact, it suffices to verify (i) and (ii) in the special case where κ is a finite extension of some residue field of R (or even in the special case where κ itself is a residue field of R). Set $A_0 = \pi_0(\kappa \otimes_R f_* \mathscr{O}_{\mathcal{X}})$, so that A_0 is the ring of functions on fiber $\mathsf{X}_0 = \mathsf{X} \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Sp\acute{e}t} \kappa$. Then A_0 is a finite κ-algebra. Since f is geometrically reduced, A_0 is a product of finite separable extensions of κ. Replacing κ by a finite extension if necessary, we may assume that there exists an isomorphism $\alpha_0 : \kappa^m \simeq A_0$ for some integer m, so that the fiber X_0 decomposes as a disjoint union of connected components $\amalg_{1 \leqslant i \leqslant m} \mathsf{X}_{0,i}$.

Note that to verify (i) and (ii), we are free to extend scalars along any *flat* map $v : R \to R'$ for which ρ factors through v. In particular, we may replace R by a polynomial ring $R[x_1, \ldots, x_k]$ to ensure that κ is a residue field of R (rather than a finite extension therefof). Set $\mathfrak{m} = \ker(\rho)$. Replacing R by the completion of the local ring $R_{\mathfrak{m}}$, we may assume that R is a complete local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let $\mathsf{X}^{\wedge}$ denote the formal completion of X along the vanishing locus of $\mathfrak{m}$. Using Theorem 8.5.0.3, we can identify $A = \pi_0 f_* \mathscr{O}_{\mathsf{X}}$ with the endomorphism ring $\operatorname{Ext}^0_{\operatorname{QCoh}(\mathsf{X}^{\wedge}_{\mathfrak{m}})}(\mathscr{O}_{\mathsf{X}^{\wedge}}, \mathscr{O}_{\mathsf{X}^{\wedge}})$. The decomposition $\mathsf{X}_0 \simeq \amalg_{1 \leqslant i \leqslant m} \mathsf{X}_{0,i}$ lifts to a decomposition of formal spectral Deligne-Mumford stacks $\mathsf{X}^{\wedge} \simeq \amalg_{1 \leqslant i \leqslant m} \mathsf{X}^{\wedge}_i$, so that the endomorphism ring A admits a corresponding decomposition $A \simeq \prod_{1 \leqslant i \leqslant m} A_i$. Each A_i is an R-algebra, so that α_0 lifts to a map of R-algebras $\alpha : R^m \to A$. Composing α with the natural map $A \to f_* \mathscr{O}_{\mathsf{X}}$, we obtain a fiber sequence of R-modules

$$R^m \xrightarrow{\alpha} f_* \mathscr{O}_{\mathsf{X}} \to M$$

for some perfect R-module M. By construction, $\kappa \otimes_R M$ is the cofiber of the map $\kappa^m \simeq A_0 \to \kappa \otimes_R f_* \mathscr{O}_{\mathsf{X}})$ and therefore belongs to $(\operatorname{Mod}_\kappa)_{\leqslant -1}$. Applying Corollary 8.6.4.3, we deduce that M has Tor-amplitude $\leqslant -1$. In particular, since R is discrete, we have $M \in (\operatorname{Mod}_R)_{\leqslant -1}$, so that α exhibits R^m as the connective cover of $f_* \mathscr{O}_{\mathsf{X}}$ and M as the truncation $\tau_{\leqslant -1} f_* \mathscr{O}_{\mathsf{X}}$. Assertions (i) and (ii) now follow immediately. □

Proof of Theorem 8.7.3.1. Part (a) follows immediately from Proposition 8.7.4.1. We will prove (b). Let $f : \mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}}) \to (\mathcal{Z}, \mathscr{O}_{\mathsf{Z}}) = \mathsf{Z}$ be a morphism of spectral Deligne-Mumford stacks which is proper, geometrically reduced, and locally almost of finite presentation. We wish to show that f admits a reduced Stein factorization. Consider the induced map of 0-truncated spectral Deligne-Mumford stacks

$$f_0 : \mathsf{X}_0 = (\mathcal{X}, \pi_0 \mathscr{O}_{\mathsf{X}}) \to (\mathcal{Z}, \pi_0 \mathscr{O}_{\mathsf{Z}}) = \mathsf{Z}.$$

Using Proposition 8.7.4.2, we deduce that f_0 admits a reduced Stein factorization $\mathsf{X}_0 \xrightarrow{f_0'} \mathsf{Y}_0 \xrightarrow{f_0''} \mathsf{Z}_0$. Then f_0'' is étale, so we can write $\mathsf{Y}_0 = (\mathcal{Z}_{/U}, (\pi_0 \mathscr{O}_{\mathsf{Z}})|_U)$ for some object $U \in \mathcal{Z}$. The map f_0' determines a point $\eta \in \Gamma(\mathsf{X}; f^* U)$, which in turn determines a map $f' : \mathsf{X} \to (\mathcal{Z}_{/U}, \mathscr{O}_{\mathsf{Z}}|_U)$. The map f' fits into a commutative diagram

$$\begin{array}{ccccc} \mathsf{X}_0 & \xrightarrow{f_0'} & (\mathcal{Z}_{/U}, \pi_0 \mathscr{O}_{\mathsf{Z}}|_U) & \xrightarrow{f_0''} & \mathsf{Z}_0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{X} & \xrightarrow{f'} & (\mathcal{Z}_{/U}, \mathscr{O}_{\mathsf{Z}}|_U) & \xrightarrow{f''} & \mathsf{Z} \end{array}$$

where the outer rectangle is a pullback (by virtue of our assumption that f is flat) and the right square is a pullback by inspection. It follows that the left square is a pullback as well. Since f_0' is geometrically reduced, it follows from Remark 8.6.2.5 that the map f' is geometrically reduced. The map f'' is finite étale by construction, so that f' is proper by Proposition 5.1.4.1 and locally almost of finite presentation by Proposition 4.2.3.3. Consequently, the pair (f', f'') is a reduced Stein factorization of f. □

Part III

Tannaka Reconstruction and Quasi-Coherent Stacks

Let X be a scheme and let $\Gamma(X; \mathscr{O}_X)$ denote the ring of regular functions on X. We can then ask to what extent the scheme X is determined by the commutative ring $\Gamma(X; \mathscr{O}_X)$. More precisely, we ask the following:

Question 8.7.0.1. Let A be a commutative ring. Every morphism of schemes $f : \operatorname{Spec} A \to X$ determines a ring homomorphism $f^* : \Gamma(X; \mathscr{O}_X) \to A$, given by pullback along f. The construction $f \mapsto f^*$ determines a map ρ_A from the set $X(A)$ of A-valued points of X to the set of ring homomorphisms $\operatorname{Hom}(\Gamma(X; \mathscr{O}_X), A)$. When is the function ρ_A a bijection?

Question 8.7.0.2. The construction $\mathscr{F} \mapsto \Gamma(X; \mathscr{F})$ determines a functor $\Gamma(X; \bullet)$ from the abelian category of quasi-coherent sheaves on X to the abelian category of $\Gamma(X; \mathscr{O}_X)$-modules. When is the functor $\Gamma(X; \bullet)$ an equivalence of categories?

Roughly speaking, the answers each of these questions is "only if X is affine." More precisely, we have the following:

- (i) A scheme X is affine if and only if the map ρ_A of Question 8.7.0.1 is a bijection for every commutative ring A (this is an immediate consequence of the universal property characterizing the affine scheme $\operatorname{Spec} \Gamma(X; \mathscr{O}_X)$).
- (ii) If X is affine, then the functor $\Gamma(X; \bullet)$ of Question 8.7.0.2 is an equivalence of categories. The converse holds if X is quasi-compact and separated (this is one formulation Serre's affineness criterion; see Proposition 11.4.3.1 for a related statement).

Our goal in Part III is to study *categorified* versions of Questions 8.7.0.1 and 8.7.0.2, informed by the following heuristic dictionary:

Remark 8.7.0.3.

Classical Notion	Categorified Notion
Regular Function on X	Quasi-Coherent Sheaf on X
Commutative Ring	Symmetric monoidal ∞-category
Ring homomorphism	Symmetric monoidal functor
$\Gamma(X; \mathscr{O}_X)$	$\operatorname{QCoh}(X)$
Quasi-coherent sheaf on X	Quasi-coherent stack on X
$\Gamma(X; \mathscr{O}_X)$-Modules	∞-Category Tensored over $\operatorname{QCoh}(X)$

The central theme of Part III is that the categorified analogues of Questions 8.7.0.1 and 8.7.0.2 have affirmative answers for a much larger class of geometric objects X: roughly speaking, we do not need to assume that X is affine, only that the diagonal map $\delta : X \to X \times X$ is affine (or, in some cases, quasi-affine).

Let us begin considering the analogue of Question 8.7.0.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor (for example, the functor represented by a scheme or some other algebro-geometric object). Let $\mathrm{QCoh}(X)$ denote the ∞-category of quasi-coherent sheaves on X (Definition 6.2.2.1). Given a connective $\mathbb{E}_\infty$-ring A and a point $\eta \in X(A)$, we can regard the pullback functor $\eta^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\operatorname{Spec} A) \simeq \mathrm{Mod}_A$ as an object of the ∞-category $\mathrm{LFun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{Mod}_A)$ of symmetric monoidal functors from $\mathrm{QCoh}(X)$ to Mod_A which preserve small colimits. The construction $\eta \mapsto \eta^*$ determines a functor $\theta : X(A) \to \mathrm{LFun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{Mod}_A)$, which we can regard as a categorified analogue of the map ρ_A appearing in Question 8.7.0.1. In Chapter 9, we will show that the functor θ is often very close to being an equivalence:

- If X is a quasi-compact, quasi-separated spectral algebraic space, then the functor θ is always an equivalence (Theorem 9.6.0.1).
- If X is a quasi-geometric stack (see Definition 9.1.0.1; roughly speaking, this means that X has quasi-affine diagonal), then the functor θ is always fully faithful (Theorem 9.2.0.2).
- For quasi-geometric stacks satisfying various auxiliary conditions, we can explicitly describe the essential image of θ (see Theorems 9.3.0.3, 9.3.5.1, 9.3.7.1, 9.4.1.4, 9.4.4.7, 9.5.4.1, 9.5.4.2, and 9.7.2.1).
- If we restrict our attention to the case where X is a classical algebro-geometric object, then there is analogous way to recover X from the *abelian* category $\mathrm{QCoh}(X)^{\heartsuit}$ of quasi-coherent sheaves on X (Theorem **??**). In the special case where $X = BG$ is the classifying stack of an affine group scheme G, this recovers the classical theory of Tannaka duality for the group G (see Theorem 9.0.0.1).

The bulk of Chapter 10 is devoted studying the categorified analogue of Question 8.7.0.2. Our first step is to introduce the notion of a *quasi-coherent stack* on a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ (see Construction 10.1.1.1). Roughly speaking, a *stable quasi-coherent stack* $\mathcal{C}$ on X is a rule which assigns to each point $\eta \in X(R)$ an R-linear stable ∞-category $\mathcal{C}_\eta$, in the sense of Appendix D. The collection of stable quasi-coherent stacks on X form an ∞-category $\mathrm{QStk}^{\mathrm{St}}(X)$, which we can regard as a categorified analogue of the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X. To each object $\mathcal{C} \in \mathrm{QStk}^{\mathrm{St}}(X)$ a stable ∞-category $\mathrm{QCoh}(X; \mathcal{C})$ of *quasi-coherent sheaves on X with values in $\mathcal{C}$*, which is endowed with an action of the symmetric monoidal ∞-category $\mathrm{QCoh}(X)$. Under mild assumptions on X, the construction

$\mathcal{C} \mapsto \mathrm{QCoh}(X;\mathcal{C})$ detemrines a functor $\mathrm{QCoh}(X;\bullet) : \mathrm{QStk}^{\mathrm{St}}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$, which we regard as an analogue of the functor $\Gamma(X;\bullet)$ appearing in Question 8.7.0.2. One of our main results asserts that if X is a quasi-compact, quasi-separated algebraic space, then the functor $\mathrm{QCoh}(X;\bullet)$ is an equivalence of ∞-categories. In fact, we establish a more general result for *prestable* quasi-coherent stacks (Theorem 10.2.0.2), and slightly weaker results for more general functors X (see Theorems 10.2.0.1, 10.4.2.3, and 10.6.2.1).

The results of Chapters 9 and 10 suggest in particular that any spectral algebraic space X which is quasi-compact and quasi-separated can be viewed as the "spectrum" of the ∞-category $\mathrm{QCoh}(\mathsf{X})$, which can regard as a commutative algebra object in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ of presentable stable ∞-categories. In Chapter 11, we consider what happens if we neglect the algebra structure on $\mathrm{QCoh}(\mathsf{X})$. More precisely, we ask the following:

Question 8.7.0.4. Let R be a connective $\mathbb{E}_\infty$-ring and let $q : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ be a morphism of spectral algebraic spaces. To what extent is X controlled by the stable R-linear ∞-category $\mathrm{QCoh}(\mathsf{X})$ (without its symmetric monoidal structure)?

Question 8.7.0.4 leads naturally to the subject of (derived) *non-commutative geometry*. Although one generally needs the symmetric monoidal structure on $\mathrm{QCoh}(\mathsf{X})$ in order to recover X itself, some of the most important geometric conditions on X can be formulated directly in terms of $\mathrm{QCoh}(\mathsf{X})$ (while neglecting its tensor structure). Following ideas of Kontsevich, we introduce the notions of *smooth* and *proper* stable R-linear ∞-categories, and show that the smoothness and properness of $\mathrm{QCoh}(\mathsf{X})$ are closely related to the corresponding geometric properties of the morphism $q : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ (see Theorems 11.1.4.1, 11.3.6.1, and 11.4.0.3).

Chapter 9

Tannaka Duality

Let G be an affine algebraic group over a field κ. For every commutative κ-algebra A, let $G(A)$ denote the group of A-valued points of G. Let $\mathrm{Rep}(G)$ denote the abelian category of finite-dimensional representations of G. By definition, for each $V \in \mathrm{Rep}(G)$, we can equip the tensor product $A \otimes_{\kappa} V$ with an action of the group $G(A)$. This action depends functorially on V: consequently, we can view the group $G(A)$ as acting on the forgetful functor $e : \mathrm{Rep}(G) \to \mathrm{Mod}_A^{\mathrm{lf}}$ given by $e(V) = A \otimes_{\kappa} V$; here $\mathrm{Mod}_A^{\mathrm{lf}}$ denotes the category of projective A-modules of finite rank. The theory of Tannaka duality (in its algebro-geometric incarnation) asserts that this action can be used to recover the group $G(A)$:

Theorem 9.0.0.1 (Tannaka Duality)**.** *Let G be an affine algebraic group over a field κ, let A be a commutative κ-algebra, and let $e : \mathrm{Rep}(G) \to \mathrm{Mod}_A^{\heartsuit}$ be the functor given by $e(V) = A \otimes_{\kappa} V$. Then the the above construction induces an isomorphism $G(A) \to \mathrm{Aut}^{\otimes}(e)$; here $\mathrm{Aut}^{\otimes}(e)$ denotes the automorphism group of e in the category of (symmetric) monoidal functors from $\mathrm{Rep}(G)$ to $\mathrm{Mod}_A^{\mathrm{lf}}$.*

For a proof, we refer the reader to [48] (see also Corollaries 9.2.2.2 and 9.3.7.5).

It is often convenient to interpret Theorem 9.0.0.1 as a statement about the classifying stack of the algebraic group G. For every commutative κ-algebra A, let $\mathrm{Nil}()G(A)$ denote the category of G-torsors on the affine scheme $\mathrm{Spec}\, A$ (which we require to be locally trivial with respect to the fpqc topology). The construction $A \mapsto \mathrm{Nil}()G(A)$ determines a stack (with respect to the fpqc topology) on the category of commutative κ-algebras, which we denote by BG and refer to as the *classifying stack* of G. We can then identify $\mathrm{Rep}(G)$ with the category of quasi-coherent sheaves on BG which are locally free of finite rank.

If A is a commutative ring, then the datum of a map $\epsilon : \mathrm{Spec}\, A \to BG$ is equivalent to the data of a pair $(\phi, \mathcal{P})$, where $\phi : \kappa \to A$ is a ring homomorphism and $\mathcal{P}$ is a G-torsor on $\mathrm{Spec}\, A$. For every commutative κ-algebra A, the trivial G-torsor on $\mathrm{Spec}\, A$ is classified by a map $\epsilon_0 : \mathrm{Spec}\, A \to BG$. The fiber functor e appearing in Theorem 9.0.0.1 is given by pullback along ϵ_0, and we can identify $G(A)$ with the automorphism group of ϵ_0 (equivalently, the automorphism group of the trivial G-torsor on $\operatorname{Spec} A$). Theorem 9.0.0.1 can then be regarded as a special case of the following assertion:

Theorem 9.0.0.2. *Let G be an affine algebraic group over a field κ and let A be a commutative ring Then the construction*

$$(\epsilon : \operatorname{Spec} A \to BG) \mapsto (\epsilon^* : \operatorname{Rep}(G) \to \operatorname{Mod}_A^{\heartsuit})$$

determines a fully faithful embedding from the category of A-valued points of BG to the category of symmetric monoidal functors from $\operatorname{Rep}(G)$ to $\operatorname{Mod}_A^{\heartsuit}$.

Our goal in this chapter is to discuss two questions which are inspired by Theorem 9.0.0.2:

Question 9.0.0.3. For which algebro-geometric objects X can one expect some analogue of Theorem 9.0.0.2, asserting that a map $f : \operatorname{Spec} A \to X$ can be recovered from the pullback functor $f^* : \operatorname{QCoh}(X) \to \operatorname{Mod}_A$ on quasi-coherent sheaves (or some variant thereof)?

Question 9.0.0.4. How can one characterize those symmetric monoidal functors $F : \operatorname{QCoh}(X) \to \operatorname{Mod}_A$ which are of the form f^*, for some map $f : \operatorname{Spec} A \to X$?

Let us now outline the contents of this chapter. We begin in §9.1 by introducing the notion of a *quasi-geometric stack*. Roughly speaking, a quasi-geometric stack is an object which arises as the quotient of an affine spectral algebraic space $\operatorname{Spec} A$ by the action of a groupoid which is flat and quasi-affine (see Definition 9.1.0.1 and Remark 9.1.1.6). The class of quasi-geometric stacks includes all quasi-compact, quasi-separated spectral algebraic spaces (Corollary 9.1.4.6), and also the classifying stack of any affine group scheme over a field (see Example 9.1.1.7). Our motivation for introducing the notion of quasi-geometric stack is that this notion provides an answer to Question 9.0.0.3. In §9.2, we will show that if X is a quasi-geometric stack and A is a connective $\mathbb{E}_\infty$-ring, then the construction $(f : \operatorname{Spec} A \to X) \mapsto (f^* : \operatorname{QCoh}(X) \to \operatorname{Mod}_A)$ determines a fully faithful embedding Θ from the space $X(A)$ of A-valued points of X to the ∞-category $\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(X), \operatorname{Mod}_A)$ of symmetric monoidal functors from $\operatorname{QCoh}(X)$ to Mod_A (Proposition 9.2.2.1). Moreover, we also address Question 9.0.0.4 by characterizing the essential image of Θ: according to Theorem 9.2.0.2, it consists of those symmetric monoidal functors F which are right t-exact and participate in an adjunction $\operatorname{QCoh}(X) \underset{G}{\overset{F}{\rightleftarrows}} \operatorname{Mod}_A$ of colimit-preserving functors which satisfies a projection formula.

Theorem 9.2.0.2 can be regarded as an ∞-categorical avatar of the classical theory of Tannaka duality, and is the basic prototype of the sort of result we are interested in. The remainder of this chapter is devoted to studying special classes of quasi-geometric stacks for which stronger results are available:

- In §9.3, we study the class of *geometric stacks*: quasi-geometric stacks X for which the diagonal map $\delta : X \to X \times X$ is affine. For geometric stacks, we give a different characterization of the essential image of Θ: it consists of those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_A$ which preserve small colimits, connective objects, and flat objects (Theorem 9.3.0.3).

- In §9.4, we study the class of *perfect stacks*: quasi-geometric stacks X for which the ∞-category $\mathrm{QCoh}(X)$ has "enough" perfect objects (see also [23]). For perfect stacks, the essential image of Θ admits an even simpler characterization: it consists of those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_A$ which preserve small colimits and connective objects (this was proven by Bhatt and Halpern-Leistner in [27]; we include a proof here as Corollary 9.4.4.7);

- In §9.5, we study *locally Noetherian* stacks: quasi-geometric stacks X for which there exists a faithfully flat map $\mathrm{Spec}\, A \to X$, where A is Noetherian. For locally Noetherian geometric stacks, the essential image of Θ again admits a simple description: it again consists of those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_A$ which preserve small colimits and connective objects (Theorem 9.5.4.1; moreover, we prove a slightly weaker statement in the quasi-geometric case as Theorem 9.5.4.2).

- In §9.6, we consider the case where X is a quasi-compact, quasi-separated algebraic space. In this case, we have a further simplification: the essential image of Θ consists of all symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_A$ which preserve small colimits (Theorem 9.6.0.1).

- In §9.7, we describe the relationship between the ∞-categorical version of Tannaka duality studied in this section (which is formulated in terms of ∞-categories of quasi-coherent sheaves) and its incarnation in classical algebraic geometry (which is formulated in terms of abelian categories of quasi-coherent sheaves). We say that a quasi-geometric stack X is 0-*truncated* if there exists a faithfully flat map $\mathrm{Spec}\, R \to X$, where R is a commutative ring (Definition 9.1.6.2). Any 0-truncated quasi-geometric stack X can be regarded as a *classical* algebro-geometric object, in the sense that it can be recovered from knowledge of its A-valued points where A ranges over ordinary commutative rings (Proposition 9.1.6.9). In §9.7, we show that a 0-truncated geometric stack X can also be recovered from its abelian category of quasi-coherent sheaves $\mathrm{QCoh}(X)^{\heartsuit}$ (Theorem 9.7.3.2). In fact, one can even recover X from the full subcategory $\mathrm{QCoh}(X)^{\flat} \subseteq \mathrm{QCoh}(X)^{\heartsuit}$ spanned by the flat quasi-coherent sheaves on X (Theorem 9.7.2.1).

Remark 9.0.0.5. Many variations on the theme of this chapter have been explored in the literature. For further discussion, we refer the reader to Bhatt ([26]), Bhatt-Halpern-Leistner

([27]). Brandenburg-Chirvasitu ([33]), Fukuyama-Iwanari ([70]), Hall-Rydh ([92]), Iwanari ([102]), Lurie ([137]), Schäppi ([178], [180], [181], [182]), and Wallbridge ([221]).

Contents

9.1 Quasi-Geometric Stacks

Our goal in this section is to set the stage for our discussion of Tannaka duality by introducing the class of algebro-geometric objects that we are interested in:

Definition 9.1.0.1. A *quasi-geometric stack* is a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ which satisfies the following conditions:

(a) The functor X satisfies descent with respect to the fpqc topology.

(b) The diagonal map $\delta : X \to X \times X$ is quasi-affine (in the sense of Example 6.3.3.6).

(c) There exists a connective $\mathbb{E}_\infty$-ring A and a faithfully flat morphism $f : \operatorname{Spec} A \to X$ (see Definition 6.3.3.7).

More informally, a quasi-geometric stack is an object which can be obtained as the quotient (with respect to the fpqc topology) of an affine spectral scheme $\operatorname{Spec} A$ by the action of a groupoid $G \underset{q}{\overset{p}{\rightrightarrows}} \operatorname{Sp\acute{e}t} A$ which is flat and quasi-affine (see Remark 9.1.1.6). As we will see in §9.2, the hypotheses of Definition 9.1.0.1 guarantee that the functor X can be functorially recovered from the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X (Theorem 9.2.0.2). Moreover, these hypotheses are not terribly restrictive: the class of quasi-geometric stacks includes most algebro-geometric objects which arise in practice:

(a) Any quasi-compact, quasi-separated spectral algebraic space is a quasi-geometric stack (when identified with its functor of points): see Corollary 9.1.4.6.

(b) If G is an affine group scheme over a field κ, then the classifying stack BG can be regarded as a quasi-geometric stack; this recovers the classical setting for Tannaka duality (see Corollary 9.2.2.2).

(c) More generally, any quasi-compact Artin stack with quasi-affine diagonal (in the sense of classical algebraic geometry) can be regarded as a quasi-geometric stack.

(d) The class of quasi-geometric stacks includes some "infinite-dimensional" algebraic stacks which arise naturally in the study of chromatic homotopy theory, such as the moduli stack of 1-dimensional formal groups and its homotopy-theoretic analogue (see Examples **??** and 9.3.1.8).

Remark 9.1.0.2. In Definition 9.1.0.1, we allow the functor X to take values in the ∞-category $\widehat{\mathcal{S}}$ of spaces which are not necessarily small. This is motivated by a set-theoretic technicality: the class of functors $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is not closed under sheafification with respect to the fpqc topology. However, we will see later that if X is a quasi-geometric stack, then the space $X(R)$ is essentially small for every connective $\mathbb{E}_\infty$-ring R (Proposition 9.2.4.2). Consequently, every quasi-geometric stack X can be regarded as a functor from $\mathrm{CAlg}^{\mathrm{cn}}$ to the ∞-category $\mathcal{S}$ of (small) spaces, so we can replace the ∞-category $\widehat{\mathcal{S}}$ by $\mathcal{S}$ in Definition 9.1.0.1 with no essential change.

9.1.1 Examples of Quasi-Geometric Stacks

Our first goal is to show that quasi-geometric stacks exist in great abundance. We begin with some general remarks about the axiomatics of Definition 9.1.0.1:

Proposition 9.1.1.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. The following conditions are equivalent:*

(b) *The diagonal map $X \to X \times X$ is quasi-affine.*

(b′) *For every pair of connective $\mathbb{E}_\infty$-rings A and B equipped with maps $\operatorname{Spec} A \to X \leftarrow \operatorname{Spec} B$, the fiber product $\operatorname{Spec} A \times_X \operatorname{Spec} B$ is quasi-affine (that is, it is representable by a quasi-affine spectral Deligne-Mumford stack).*

(b″) *For every connective $\mathbb{E}_\infty$-ring A and every morphism $f : \operatorname{Spec} A \to X$, the morphism f is quasi-affine.*

Proof. The equivalence $(b') \Leftrightarrow (b'')$ is clear. To show that $(b') \Rightarrow (b)$, choose any morphism $\operatorname{Spec} A \to X \times X$, corresponding to a pair of maps $f, g : \operatorname{Spec} A \to X$. We have a pullback

diagram

$$\begin{array}{ccc} \operatorname{Spec} A \times_{X \times X} X & \longrightarrow & \operatorname{Spec} A \times_X \operatorname{Spec} A \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} A \times \operatorname{Spec} A \end{array}$$

where the bottom horizontal map is affine, so the upper horizontal map is affine as well. Condition (b') implies that the codomain of the upper horizontal map is quasi-affine, so that the domain is quasi-affine by virtue of Lemma 2.5.7.2. The implication $(b) \Rightarrow (b')$ follows from the observation that for any pair of morphisms $\operatorname{Spec} A \to X \leftarrow \operatorname{Spec} B$, we have an equivalence

$$\operatorname{Spec} A \times_X \operatorname{Spec} B \simeq \operatorname{Spec}(A \otimes B) \times_{X \times X} X.$$

□

It follows from Proposition 9.1.1.1 that if $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is a functor with quasi-affine diagonal, then *any* morphism $\operatorname{Spec} A \to X$ is quasi-affine. We will need the following converse:

Proposition 9.1.1.2. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor which is a sheaf for the fpqc topology and suppose that there exists a morphism $f : \operatorname{Spec} A \to X$ which is representable and faithfully flat. If f is quasi-affine, then the diagonal map $\delta : X \to X \times X$ is quasi-affine.*

Lemma 9.1.1.3. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ of fpqc sheaves. If f' is quasi-affine and g is an effective epimorphism of fpqc sheaves, then f is quasi-affine.

Proof. Choose any map $\eta : \operatorname{Spec} R \to Y$; we wish to show that the fiber product $Z = \operatorname{Spec} R \times_Y X$ is quasi-affine. Since g is an fpqc surjection, we can choose a faithfully flat map $R \to R^0$ such that $\eta|_{\operatorname{Spec} R^0}$ factors through Y'. Let $R^\bullet$ denote the Čech nerve of the faithfully flat map $R \to R^0$ (formed in the ∞-category $\mathrm{CAlg}^{\mathrm{op}}$) and set $Z_\bullet = \operatorname{Spec} R^\bullet \times_Y X$. By construction, each of the maps $\operatorname{Spec} R^m \to Y$ factors through Y', so that each Z_m is quasi-affine by virtue of our assumption on f'. Using Proposition 2.4.3.2, we can write $Z_\bullet = \operatorname{Spec} R^\bullet \times_{\operatorname{Spec} R} Z'$ for some quasi-affine map $Z' \to \operatorname{Spec} R$. Then both Z and Z' can be identified with the geometric realization $|Z_\bullet|$ (formed in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$), so that Z is quasi-affine as desired. □

Proof of Proposition 9.1.1.2. By virtue of Proposition 9.1.1.1, it will suffice to show that every morphism $g : \operatorname{Spec} B \to X$ is quasi-affine. Using Lemma 9.1.1.3, we are reduced to showing that the projection map $\operatorname{Spec} A \times_X \operatorname{Spec} B \to \operatorname{Spec} A$ is quasi-affine. This is equivalent to the quasi-affineness of the fiber product $\operatorname{Spec} A \times_X \operatorname{Spec} B$, which follows from our assumption that $f : \operatorname{Spec} A \to X$ is quasi-affine. □

Proposition 9.1.1.4. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor which satisfies descent for the fpqc topology, and suppose there exists a morphism $f : X_0 \to X$ satisfying the following conditions:*

(a) *The functor $X_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is a quasi-geometric stack.*

(b) *The morphism f is representable, quasi-affine, and faithfully flat.*

Then X is a quasi-geometric stack.

Proof. Since X_0 is a quasi-geometric stack, we can choose a connective $\mathbb{E}_\infty$-ring A and a faithfully flat morphism $g : \operatorname{Spec} A \to X_0$ (which is automatically representable and quasi-affine, by virtue of Remark 9.1.1.1). Replacing f by $f \circ g$, we can assume that $X_0 = \operatorname{Spec} A$ is a corepresentable functor. Then the diagonal $\delta : X \to X \times X$ is quasi-affine by virtue of Proposition 9.1.1.2, so that X is a quasi-geometric stack. □

Corollary 9.1.1.5. *Let $X_\bullet$ be a simplicial object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ satisfying the following conditions:*

(a) *The functor $X_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is a quasi-geometric stack.*

(b) *The functor $X_\bullet$ is a groupoid object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$.*

(c) *The face map $d_0 : X_1 \to X_0$ is representable, quasi-affine, and faithfully flat.*

Then the geometric realization $X = |X_\bullet|$ (formed in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ of fpqc sheaves) is a quasi-geometric stack.

Proof. We have a pullback diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{d_0} & X_0 \\ \downarrow{\scriptstyle d_1} & & \downarrow \\ X_0 & \longrightarrow & X \end{array}$$

where the vertical maps are effective epimorphisms in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$. It follows from (c) and Lemma 9.1.1.3 that the map $X_0 \to X$ is quasi-affine and faithfully flat, so that X is a quasi-geometric stack by virtue of Proposition 9.1.1.4. □

Remark 9.1.1.6 (Existence of Atlases)**.** Every quasi-geometric stack X can be written as a geometric realization $|X_\bullet|$, where $X_\bullet$ satisfies the hypotheses of Corollary 9.1.1.5 and X_0 is affine: to see this, we can take $X_\bullet$ to be the Čech nerve of any faithfully flat map $\operatorname{Spec} A \to X$.

Example 9.1.1.7 (The Classifying Stack of a Group Scheme)**.** Let R be a commutative ring and let G be a flat quasi-affine group scheme over R. For each $n \geqslant 0$, let $X_n : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by the R-scheme $G \times_{\operatorname{Spec} R} G \times_{\operatorname{Spec} R} \cdots \times_{\operatorname{Spec} R} G$ (where the factor G appears n times). Using the group structure on G, we can regard the construction $[n] \mapsto X_n$ as a simplicial object $X_\bullet$ of the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$. We let BG denote the geometric realization $|X_\bullet|$ in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$. It follows from Corollary 9.1.1.5 that BG is a quasi-geometric stack, which we will refer to as the *classifying stack* of G.

Remark 9.1.1.8. Let R be a commutative ring and let G be a flat quasi-affine group scheme over R. For every commutative ring A, we can identify $BG(A)$ with the classifying space for the groupoid of pairs $(\phi, \mathcal{P})$, where $\phi : R \to A$ is a ring homomorphism and $\mathcal{P}$ is a G-torsor on the affine scheme $\operatorname{Spec} A$ (which is locally trivial with respect to the fpqc topology).

9.1.2 Quasi-Geometric Morphisms

Definitions 9.1.0.1 and 9.3.0.1 can be relativized:

Definition 9.1.2.1. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. We will say that f is *quasi-geometric* if, for every connective $\mathbb{E}_\infty$-ring A and every morphism $\operatorname{Spec} A \to Y$, the fiber product $\operatorname{Spec} A \times_Y X$ is a quasi-geometric stack (geometric stack).

Proposition 9.1.2.2. *Suppose we are given a pullback square*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$. If f is quasi-geometric, then f' is also quasi-geometric. The converse holds if g is an effective epimorphism in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$.

Proof. The first assertion is immediate. To prove the second, suppose we are given a map $\operatorname{Spec} R \to Y$ and set $Z = \operatorname{Spec} R \times_Y X$; we wish to show that Z is a quasi-geometric stack. Since g is an effective epimorphism, we can choose a faithfully flat map $R \to R'$ for which the composite map $\operatorname{Spec} R' \to \operatorname{Spec} R \to Y$ factors through Y'. If f' is quasi-geometric, then the fiber product $Z' = \operatorname{Spec} R' \times_{\operatorname{Spec} R} Z \simeq \operatorname{Spec} R' \times_{Y'} X'$ is a quasi-geometric stack. Since the projection map $Z' \to Z$ is affine and faithfully flat, it follows from Proposition 9.1.1.4 that Z is also a quasi-geometric stack. □

Proposition 9.1.2.3. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. If Y is a quasi-geometric stack and f is quasi-geometric, then X is a quasi-geometric stack.*

Proof. Choose a connective $\mathbb{E}_\infty$-ring A and a faithfully flat morphism $u : \operatorname{Spec} A \to Y$. Set $X_0 = \operatorname{Spec} A \times_Y X$. Since Y is quasi-geometric, the morphism u is quasi-affine. It follows that the projection map $u' : X_0 \to X$ is quasi-affine and faithfully flat. Our assumption that f is quasi-geometric guarantees that X_0 is a quasi-geometric stack, so that X is also a quasi-geometric stack by virtue of Proposition 9.1.1.4. □

Corollary 9.1.2.4. *Let $f : X \to Y$ and $g : Y \to Z$ be quasi-geometric morphisms in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. Then $g \circ f : X \to Z$ is also quasi-geometric.*

Corollary 9.1.2.5. *Let Y be a quasi-geometric stack and let $f : X \to Y$ be a natural transformation in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. If X is a quasi-geometric stack, then the morphism f is quasi-geometric.*

Proof. Suppose we are given a map $u : \operatorname{Spec} A \to Y$; we wish to show that the fiber product $\operatorname{Spec} A \times_Y X$ is a quasi-geometric stack. Since Y is quasi-geometric, the morphism u is quasi-affine, so the projection map $\operatorname{Spec} A \times_Y X \to X$ is quasi-affine (and therefore also quasi-geometric). Since X is a quasi-geometric stack, Proposition 9.1.2.3 implies that $\operatorname{Spec} A \times_Y X$ is a quasi-geometric stack. □

Corollary 9.1.2.6. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. *Assume that Y is a quasi-geometric stack. If X and Y' are quasi-geometric stacks, then X' is also a quasi-geometric stack.*

Proof. It follows from Corollary 9.1.2.5 that the morphism f is quasi-geometric, so that f' is also quasi-geometric. Since Y' is a quasi-geometric stack (geometric stack), Proposition 9.1.2.3 implies that X' is also a quasi-geometric stack. □

9.1.3 Quasi-Coherent Sheaves on a Quasi-Geometric Stack

We begin by showing that if $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is a quasi-geometric stack, then the ∞-category of quasi-coherent sheaves $\mathrm{QCoh}(X)$ is well-behaved. Our main result can be expressed most efficiently using the formalism of prestable ∞-categories developed in §C:

Proposition 9.1.3.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a quasi-geometric stack. Then $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is a complete Grothendieck prestable ∞-category. Moreover, the inclusion $\mathrm{QCoh}(X)^{\mathrm{cn}} \hookrightarrow \mathrm{QCoh}(X)$ exhibits $\mathrm{QCoh}(X)$ as a stabilization of $\mathrm{QCoh}(X)^{\mathrm{cn}}$.*

Proof. Choose a faithfully flat map $f : X_0 \to X$, where $X_0 = \operatorname{Spec} A$ for some connective $\mathbb{E}_\infty$-ring A. Let $X_\bullet$ denote the Čech nerve of f. Since f is faithfully flat, we can identify X with the geometric realization of $X_\bullet$ in the ∞-category of fpqc sheaves. Using Proposition 6.2.3.1 (and the right cofinality of the inclusion map $\mathbf{\Delta}_s \hookrightarrow \mathbf{\Delta}$), we obtain equivalences

$$\mathrm{QCoh}(X) \simeq \varprojlim_{[n] \in \mathbf{\Delta}_s} \mathrm{QCoh}(X_n) \qquad \mathrm{QCoh}(X)^{\mathrm{cn}} \simeq \varprojlim_{[n] \in \mathbf{\Delta}_s} \mathrm{QCoh}(X_n)^{\mathrm{cn}}.$$

Since the diagonal of X is quasi-affine, each X_n is (representable by) a quasi-affine spectral Deligne-Mumford stack. It follows that each $\mathrm{QCoh}(X_n)^{\mathrm{cn}}$ is a complete Grothendieck prestable ∞-category and that each of the inclusion maps $\mathrm{QCoh}(X_n)^{\mathrm{cn}} \hookrightarrow \mathrm{QCoh}(X_n)$ exhibits $\mathrm{QCoh}(X_n)$ as a stabilization of $\mathrm{QCoh}(X_n)^{\mathrm{cn}}$ (Proposition 2.2.5.4). Moreover, for every *injective* map $[m] \to [n]$ of finite linearly ordered sets, the associated transition map $X_n \to X_m$ is flat (since it is a composition of pullbacks of the flat morphism f), so that the associated pullback functor $\mathrm{QCoh}(X_m)^{\mathrm{cn}} \to \mathrm{QCoh}(X_n)^{\mathrm{cn}}$ is left exact. Applying Proposition C.3.2.4 to the semisimplicial ∞-category of $\mathrm{QCoh}(X_\bullet)^{\mathrm{cn}}$, we deduce that $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is a Grothendieck prestable ∞-category, which is automatically complete (see Proposition C.3.6.3). The final assertion now follows from Corollary C.3.2.5. □

Using Proposition C.1.4.1, we can reformulate Proposition 9.1.3.1 as follows:

Corollary 9.1.3.2. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a quasi-geometric stack. Then:*

(1) *The stable ∞-category $\mathrm{QCoh}(X)$ is presentable.*

(2) *There exists a t-structure $(\mathrm{QCoh}(X)_{\geqslant 0}, \mathrm{QCoh}(X)_{\leqslant 0})$ on $\mathrm{QCoh}(X)$, where $\mathrm{QCoh}(X)_{\geqslant 0} = \mathrm{QCoh}(X)^{\mathrm{cn}}$ is the full subcategory of $\mathrm{QCoh}(X)$ spanned by the connective objects.*

(3) *The t-structure on $\mathrm{QCoh}(X)$ is compatible with filtered colimits: that is, $\mathrm{QCoh}(X)_{\leqslant 0}$ is closed under filtered colimits in $\mathrm{QCoh}(X)$.*

(4) *The t-structure on $\mathrm{QCoh}(X)$ is both right and left complete.*

Remark 9.1.3.3. To prove Proposition 9.1.3.1 (or the equivalent Corollary 9.1.3.2), one does not need the full strength of the assertion that X is a quasi-geometric stack: it is enough to assume that there exists a morphism $u : X_0 \to X$, where u is representable and faithfully flat and X_0 is (representable by) a spectral Deligne-Mumford stack. However, for the moment we are primarily interested in the case of quasi-geometric stacks (we will study some other variants in Part VIII).

Remark 9.1.3.4. Let X be a quasi-geometric stack and let $f : \operatorname{Spec} A \to X$ be a faithfully flat map. The proof of Proposition 9.1.3.1 shows that the pullback functor $f^* : \mathrm{QCoh}(X)^{\mathrm{cn}} \to \mathrm{QCoh}(\operatorname{Spec} A)^{\mathrm{cn}} \simeq \mathrm{Mod}_A^{\mathrm{cn}}$ is left exact and conservative. Equivalently, the pullback functor $f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\operatorname{Spec} A) \simeq \mathrm{Mod}_A$ is t-exact and conservative. In particular, a quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(X)$ belongs to $\mathrm{QCoh}(X)_{\leqslant 0}$ if and only if $f^* \mathscr{F} \in \mathrm{Mod}_A$ belongs to $(\mathrm{Mod}_A)_{\leqslant 0}$.

We now consider a slight variant of Remark 9.1.3.4:

Proposition 9.1.3.5. *Let P be a property of pairs $(A, M) \in \mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}(\mathrm{Sp})$ which is stable under base change and local with respect to the flat topology. Let X be a quasi-geometric stack, and choose a faithfully flat morphism $f : \operatorname{Spec} A \to X$. Let $\mathscr{F} \in \mathrm{QCoh}(X)$ and set $M = f^* \mathscr{F} \in \mathrm{QCoh}(\operatorname{Spec} A) \simeq \mathrm{Mod}_A$. Then $\mathscr{F}$ has the property P (in the sense of Definition 6.2.5.3) if and only if the pair (A, M) has the property P.*

Proof. The "only if" assertion is obvious. Conversely, suppose that the pair (A, M) has the property P. Let B be a connective $\mathbb{E}_\infty$-ring, $g : \operatorname{Spec} B \to X$ be an arbitrary morphism, and $N = g^* \mathscr{F} \in \mathrm{QCoh}(\operatorname{Spec} B) \simeq \mathrm{Mod}_B$. We wish to show that the pair (B, N) has the property P. Choose a flat surjection $\operatorname{Spec} C \to \operatorname{Spec} A \times_X \operatorname{Spec} B$. Since f is faithfully flat, it follows that C is faithfully flat over B. Since P is local with respect to the flat topology, we may B by C and thereby reduce to the case where the map g factors through f, in which case the desired result follows from our assumption that P is stable under base change. □

9.1.4 Quasi-Geometric Deligne-Mumford Stacks

We now consider a variant of Definition 9.1.2.1 in the setting of spectral Deligne-Mumford stacks.

Definition 9.1.4.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. We will say that f is *quasi-geometric* if it is quasi-compact and the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ is quasi-affine.

We will say that a spectral Deligne-Mumford stack X is *quasi-geometric* if the map $\mathsf{X} \to \operatorname{Spét} S$ is quasi-geometric; here S denotes the sphere spectrum.

Remark 9.1.4.2. The condition that a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks be quasi-geometric is local on the target with respect to the étale topology and stable under base change. This follows immediately from the corresponding assertions for quasi-affine morphisms (see Example 6.3.3.6).

Let X be a spectral Deligne-Mumford stack and let $h_{\mathsf{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by X. It follows immediately from the definitions that if the functor h_{X} is quasi-geometric (in the sense of Definition 9.1.0.1), then the spectral Deligne-Mumford stack X is quasi-geometric (in the sense of Definition 9.1.4.1). It is not immediately obvious that the converse holds: in general, the functor h_{X} represented by a spectral Deligne-Mumford stack X need not be a sheaf with respect to the fpqc topology. However, this problem does not arise if X is quasi-geometric:

Proposition 9.1.4.3. *Let X be a quasi-geometric spectral Deligne-Mumford stack, and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by X. Then X is a hypercomplete sheaf with respect to the fpqc topology on $\mathrm{CAlg}^{\mathrm{cn}}$.*

Proof. Let $\widehat{\mathcal{S}\mathrm{hv}}^{\mathrm{hyp}}_{\mathrm{fpqc}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ spanned by those functors which are hypercomplete sheaves with respect to the flat topology. The inclusion $\widehat{\mathcal{S}\mathrm{hv}}^{\mathrm{hyp}}_{\mathrm{fpqc}} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ admits a left exact left adjoint, which we will denote by L. Let $Y = LX \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. We would like to show that the unit map $X \to Y$ is an equivalence. We first claim that for every connective $\mathbb{E}_\infty$-ring R, the map $X(R) \to Y(R)$ is (-1)-truncated: that is, it exhibits $X(R)$ as a summand of $Y(R)$. To prove this, it will suffice to show that for any pair of points $x, x' \in X(R)$, the induced map $\theta : \{x\} \times_{X(R)} \{x'\} \to \{x\} \times_{Y(R)} \{x'\}$ is a homotopy equivalence. We note that x and x' determine a pair of maps from $\mathrm{Sp\acute{e}t}\, R$ to X. Let X' denote the fiber product $\mathrm{Sp\acute{e}t}\, R \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, R$ and let X' be the functor represented by X'. To prove that θ is a homotopy equivalence, it will suffice to show that the canonical map $\beta : X' \to \mathrm{Spec}\, R \times_Y \mathrm{Spec}\, R$ is an equivalence. Since the functor L is left exact, β induces an equivalence $LX' \simeq \mathrm{Spec}\, R \times_Y \mathrm{Spec}\, R$. It will therefore suffice to show that X' is a hypercomplete sheaf with respect to the flat topology. This follows from Proposition 2.4.3.1, since our hypothesis that X is quasi-geometric guarantees that X' is quasi-affine.

Note that X and Y are both sheaves with respect to the étale topology on $\mathrm{CAlg}^{\mathrm{cn}}$. To complete the proof that the unit map $X \to Y$ is an equivalence, it will suffice to show that it is an effective epimorphism with respect to the étale topology. Choose a point $\eta \in Y(A)$ for some connective $\mathbb{E}_\infty$-ring A. For every morphism of connective $\mathbb{E}_\infty$-rings $A \to B$, let η_B denote the image of η in $Y(B)$. We wish to prove that there exists a faithfully flat étale map $A \to B$ such that η_B belongs to the essential image of the map $X(B) \to Y(B)$.

Since $Y = LX$, there exists finite collection of flat maps $A \to B_\alpha$ such that the induced map $A \to \prod_\alpha B_\alpha$ is faithfully flat and each η_{B_α} belongs to the essential image of the map $X(B_\alpha) \to Y(B_\alpha)$. Let $B^0 = \prod_\alpha B_\alpha$, and let $B^\bullet$ be the cosimplicial object of $\mathrm{CAlg}^{\mathrm{cn}}$ given by the Čech nerve of the map $A \to B^0$. For every integer n, the point η_{B^n} belongs to the essential image of the fully faithful embedding $X(B^n) \to Y(B^n)$, and therefore classifies a morphism of spectral Deligne-Mumford stacks $\phi_n : \mathrm{Sp\acute{e}t}\, B^n \to \mathsf{X}$.

Choose an étale surjection $\mathsf{U} \to \mathsf{X}$, where U is affine. For each $n \geqslant 0$, let $\mathsf{V}_n = \mathsf{U} \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, B^n$. Since X is quasi-geometric, each V_n is a quasi-affine spectral Deligne-Mumford stack over B^n. Using Proposition 2.4.3.2, we deduce that there exists a quasi-affine spectral Deligne-Mumford stack V and a map $\mathsf{V} \to \mathrm{Sp\acute{e}t}\, A$ such that $\mathsf{V} \times_{\mathrm{Sp\acute{e}t}\, A} \mathrm{Sp\acute{e}t}\, B^\bullet \simeq \mathsf{V}_\bullet$. Since $\mathsf{U} \to \mathsf{X}$ is surjective, the map $\mathsf{V}_0 \to \mathrm{Sp\acute{e}t}\, B^0$ is surjective. Because B^0 is faithfully flat over A, the composite map $\mathsf{V}_0 \to \mathrm{Sp\acute{e}t}\, B^0 \to \mathrm{Sp\acute{e}t}\, A$ is surjective. It follows that $\mathsf{V} \to \mathrm{Sp\acute{e}t}\, A$ is surjective. Using Proposition 2.8.3.3, we deduce that $\mathsf{V} \to \mathrm{Sp\acute{e}t}\, A$ is étale. Choose an étale surjection $\mathrm{Sp\acute{e}t}\, A' \to \mathsf{V}$, so that A' is a faithfully flat étale A-algebra. By construction, the point $\eta_{A'} \in Y(A')$ lifts to $X(A')$. □

Corollary 9.1.4.4. *Let X be a spectral Deligne-Mumford stack and let $h_{\mathsf{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by X. Then X is quasi-geometric (in the sense of Definition 9.1.4.1) if and only if the functor h_{X} is quasi-geometric (in the sense of Definition 9.1.0.1).*

Corollary 9.1.4.5. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Then f is quasi-geometric if and only if the induced map $h_{\mathsf{X}} \to h_{\mathsf{Y}}$ is quasi-geometric.*

Corollary 9.1.4.6. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is represented by a quasi-compact, quasi-separated spectral algebraic space. Then X is a quasi-geometric stack.*

Proof. Combine Corollary 9.1.4.4 with Proposition 3.4.1.3. □

9.1.5 Compact Objects of $\mathrm{QCoh}(X)$

Let X be a quasi-geometric stack and let $\mathscr{F} \in \mathrm{QCoh}(X)$ be a quasi-coherent sheaf on X. We now study the relationship between two natural finiteness conditions on $\mathscr{F}$: the condition that $\mathscr{F}$ is perfect (which can be tested locally on X), and the condition that $\mathscr{F}$ is a compact object of $\mathrm{QCoh}(X)$ (which is an *a priori* global condition).

Proposition 9.1.5.1. *Let X be a quasi-geometric stack and let $\mathscr{F} \in \mathrm{QCoh}(X)$. The following conditions are equivalent:*

- (a) *The sheaf $\mathscr{F} \in \mathrm{QCoh}(X)$ is almost perfect.*
- (b) *For every integer n, the functor $\mathscr{G} \mapsto \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G})$ commutes with filtered colimits when restricted to $\mathrm{QCoh}(X)_{\leqslant n}$.*

Proof. We first show that $(a) \Rightarrow (b)$. Assume that $\mathscr{F}$ is almost perfect. Replacing $\mathscr{F}$ by a suspension if necessary, we may assume that $\mathscr{F}$ is connective. Fix an integer $n \geqslant 0$ and let $\lambda : \mathrm{QCoh}(X)_{\leqslant n} \to \mathcal{S}_{\leqslant n}$ be the functor given by $\lambda(\mathscr{G}) = \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G})$; we wish to show that λ commutes with filtered colimits. Choose a faithfully flat Choose a faithfully flat map $u : X_0 \to X$, where $X_0 \simeq \operatorname{Spec} A$ is affine. Let $X_\bullet$ denote the Čech nerve of u and for each $k \geqslant 0$ let $u_k : X_k \to X$ denote the projection map. Then the construction $\mathscr{G} \mapsto \operatorname{Map}_{\operatorname{QCoh}(X_k)}(u_k^* \mathscr{F}, u_k^* \mathscr{G})$ determines a functor $\lambda^k : \operatorname{QCoh}(X)_{\leqslant n} \to \mathcal{S}_{\leqslant n}$, and we can identify λ with the totalization of the cosimplicial functor $\lambda^\bullet$. Since $\mathcal{S}_{\leqslant n}$ is equivalent to an $(n+1)$-category, the totalization $\lambda \simeq \operatorname{Tot}(\lambda^\bullet)$ can be identified with the partial totalization $\operatorname{Tot}^{n+1}(\lambda^\bullet)$, which is a finite limit of the functors λ^k. It will therefore suffice to show that each λ^k commutes with filtered colimits. We may therefore replace X by X_k and thereby reduce to the case where X_k is quasi-affine. In particular, we may assume that X has affine diagonal. Repeating the above argument, we can reduce to the case where $X \simeq \operatorname{Spec} B$ is affine, so that the equivalence $\operatorname{QCoh}(X) \simeq \operatorname{Mod}_B$ carries $\mathscr{F}$ to an almost perfect B-module M. In this case, the desired result follows immediately from the definitions.

We now show that $(b) \Rightarrow (a)$. Assume that $\mathscr{F}$ satisfies condition (b); we wish to show that $\mathscr{F}$ is almost perfect. This is equivalent to the assertion that $u^* \mathscr{F} \in \operatorname{QCoh}(X_0) \simeq \operatorname{Mod}_A$ is almost perfect: that is, that for every integer n, the functor

$$(\operatorname{Mod}_A)_{\leqslant n} \simeq \operatorname{QCoh}(X_0)_{\leqslant n} \xrightarrow{\operatorname{Map}_{\operatorname{QCoh}(X_0)}(u^* \mathscr{F}, \bullet)} \mathcal{S}$$

commutes with filtered colimits. Unwinding the definnitions, we see that this functor can be rewritten as a composition

$$(\operatorname{Mod}_A)_{\leqslant n} \simeq \operatorname{QCoh}(X_0)_{\leqslant n} \xrightarrow{u_*} \operatorname{QCoh}(X)_{\leqslant n} \xrightarrow{\operatorname{Map}_{\operatorname{QCoh}(X)}(\mathscr{F}, \bullet)} \mathcal{S}.$$

The desired result now follows from (b) and the fact that u_* commutes with filtered colimits (see Corollary 6.3.4.3). □

The analogue of Proposition 9.1.5.1 for perfect objects is more subtle.

Proposition 9.1.5.2. *Let X be a quasi-geometric stack and let $\mathscr{F}$ be a compact object of* $\operatorname{QCoh}(X)$*. Then $\mathscr{F}$ is perfect.*

Proof. Choose a map $f : \operatorname{Spec} R \to X$; we wish to show that $f^* \mathscr{F} \in \operatorname{QCoh}(\operatorname{Spec} R) \simeq \operatorname{Mod}_R$ is perfect. Since X is quasi-geometric, the morphism f is quasi-affine. It follows from Proposition 6.3.4.1 and Corollary 6.3.4.3 that the pullback functor $f^* : \operatorname{QCoh}(X) \to \operatorname{QCoh}(U)$ admits a right adjoint that preserves small colimits. It follows that the functor f^* carries compact objects of $\operatorname{QCoh}(X)$ to compact objects of $\operatorname{QCoh}(\operatorname{Spec} R)$ (Proposition HTT.5.5.7.2), so that $f^* \mathscr{F}$ is perfect as desired. □

The converse of Proposition 9.1.5.2 is false in general.

Proposition 9.1.5.3. *Let X be a quasi-geometric stack. The following conditions are equivalent:*

(1) *The quasi-geometric stack X has finite cohomological dimension. That is, there exists an integer $n \gg 0$ for which the global sections functor $\mathscr{F} \mapsto \Gamma(X; \mathscr{F})$ carries* $\mathrm{QCoh}(X)_{\geqslant 0}$ *into* $\mathrm{Sp}_{\geqslant -n}$.

(2) *The structure sheaf $\mathscr{O}_X \in \mathrm{QCoh}(X)$ is compact.*

(3) *The global sections functor $\Gamma(X; \bullet) : \mathrm{QCoh}(X) \to \mathrm{Sp}$ commutes with small colimits.*

(4) *Every perfect object $\mathscr{F} \in \mathrm{QCoh}(X)$ is compact.*

(5) *An object $\mathscr{F} \in \mathrm{QCoh}(X)$ is compact if and only if it is perfect.*

Remark 9.1.5.4. Let κ be a field and let X be an Artin stack of finite type over κ, in the sense of classical algebraic geometry (see [129]). If the diagonal map $X \to X \times_{\operatorname{Spec} \kappa} X$ is quasi-affine, then we can identify X with a quasi-geometric stack (by Kan extending along the inclusion $\mathrm{CAlg}^{\heartsuit} \hookrightarrow \mathrm{CAlg}^{\mathrm{cn}}$ and then sheafifying with respect to the fpqc topology). In this case, we see a sharp dichotomy:

- If the field κ has characteristic zero, then X always satisfies the equivalent conditions of Proposition 9.1.5.3 (this is proven by Drinfeld and Gaitsgory; see [51]).
- If the field κ has characteristic $p > 0$, then it is very rare for X to satisfy the conditions of Proposition 9.1.5.3 (at least when X exhibits "stacky" behavior) . For example, if X is the classifying stack of the finite group $\mathbf{Z}/p\,\mathbf{Z}$, then $\mathscr{O}_X$ is not a compact object of $\mathrm{QCoh}(X)$. If $X = B\mathbf{G}_a$ is the classifying stack of the additive group, then the ∞-category $\mathrm{QCoh}(X)$ does not contain any nonzero compact objects at all (see [93]).

Proof of Proposition 9.1.5.3. We first show that (1) $\Rightarrow$ (2). Assume that there exists an integer $n \gg 0$ for which the functor $\mathscr{F} \mapsto \Gamma(X; \mathscr{F})$ carries $\mathrm{QCoh}(X)_{\geqslant 0}$ into $\mathrm{Sp}_{\geqslant -n}$. We wish to show that the functor $\mathscr{F} \mapsto \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{O}_X, \mathscr{F})$ commutes with filtered colimits. Equivalently, we wish to show that for every integer $k \geqslant 0$, the construction

$$\mathscr{F} \mapsto \pi_k \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{O}_X, \mathscr{F}) = \pi_k \Gamma(X; \mathscr{F})$$

commutes with filtered colimits. Using the exactness of the sequence

$$\pi_k \Gamma(X; \tau_{\geqslant n+k+1} \mathscr{F}) \to \pi_k \Gamma(X; \mathscr{F}) \to \pi_k \Gamma(X; \tau_{\leqslant n+k} \mathscr{F}) \to \pi_{k-1} \Gamma(X; \tau_{\geqslant n+k+1} \mathscr{F})$$

(where the outer terms vanish by virtue of assumption (1)), we are reduced to showing that the construction

$$\mathscr{F} \mapsto \pi_k \Gamma(X; \tau_{\leqslant n+k} \mathscr{F}) \simeq \pi_k \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{O}_X, \tau_{\leqslant n+k} \mathscr{F})$$

commutes with filtered colimits, which follows from Proposition 9.1.5.1 (since $\mathscr{O}_X \in \mathrm{QCoh}(X)$ is almost perfect).

We now show that (2) ⇒ (3). The global sections functor $\Gamma : \mathrm{QCoh}(X) \to \mathrm{Sp}$ is an exact functor of stable ∞-categories, and therefore commutes with small colimits if and only if it commutes with small filtered colimits: that is, if and only if for each integer n the composite functor

$$\mathrm{QCoh}(X) \xrightarrow{\Gamma} \mathrm{Sp} \xrightarrow{\Omega^{\infty-n}} \mathcal{S}$$

preserves small filtered colimits. Unwinding the definitions, we see that this functor is given by $\mathscr{F} \mapsto \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{O}_X, \Sigma^n \mathscr{F})$, which commutes with filtered colimits if $\mathscr{O}_X$ is compact.

We next show that (3) implies (4). If $\mathscr{F} \in \mathrm{QCoh}(X)$ is perfect, then it is a dualizable object of $\mathrm{QCoh}(X)$ (Proposition 6.2.6.2). It follows that for each $\mathscr{G} \in \mathrm{QCoh}(X)$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}) \simeq \Omega^{\infty}\Gamma(X; \mathscr{F}^{\vee} \otimes \mathscr{G}).$$

Consequently, if the global sections functor Γ commutes with small colimits, then $\mathscr{F}$ is compact.

The implication (4) ⇒ (5) follows from Proposition 9.1.5.2. We now complete the proof by showing that (5) ⇒ (1). Assume, for a contradiction, that X does not have finite cohomological dimension. It follows that for every integer $n \geqslant 0$, we can choose an n-connective object $\mathscr{F}_n \in \mathrm{QCoh}(X)_{\geqslant n}$ and a map $\eta_n : \mathscr{O}_X \to \mathscr{F}_n$ which is not nullhomotopy. Since the t-structure on $\mathrm{QCoh}(X)$ is left complete (Corollary 9.1.3.2), it follows that the canonical map $\bigoplus_{n \geqslant 0} \mathscr{F}_n \to \prod_{n \geqslant 0} \mathscr{F}_n$ is an equivalence. Consequently, the maps $\{\eta_n\}_{n \geqslant 0}$ determine a map $\eta : \mathscr{O}_X \to \bigoplus_{n \geqslant 0} \mathscr{F}_n$. If (5) is satisfied, then $\mathscr{O}_X$ is compact, so that η factors through some finite sum $\bigoplus_{0 \leqslant n \leqslant N} \mathscr{F}_n$. This guarantees that η_n is nullhomotopic for $n > N$, contrary to our assumption. □

Corollary 9.1.5.5. *Let* X *be a quasi-compact, quasi-separated spectral algebraic space. Then an object* $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ *is compact if and only if it is perfect.*

Proof. Combine Corollary 3.4.2.2 with Proposition 9.1.5.3. □

Proposition 9.1.5.6. *Suppose we are given a morphism* $f : X \to Y$ *in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, *where* X *is a quasi-geometric stack and* $Y \simeq \mathrm{Spec}\, R$ *is affine. Assume that the structure sheaf* $\mathscr{O}_X$ *is a compact object of* $\mathrm{QCoh}(X)$. *Then:*

(a) *The pullback functor* $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ *admits a right adjoint* f_*.

(b) *The functor* f_* *commutes with small colimits.*

(c) *For every pair of sheaves* $\mathscr{F} \in \mathrm{QCoh}(Y)$ *and* $\mathscr{G} \in \mathrm{QCoh}(\mathrm{Spec}\, X)$, *the canonical map*

$$\theta_{\mathscr{F},\mathscr{G}} : \mathscr{F} \otimes f_* \mathscr{G} \to f_*(f^* \mathscr{F} \otimes \mathscr{G})$$

is an equivalence in $\mathrm{QCoh}(Y)$.

Proof. Since the ∞-categories $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y)$ are presentable (Corollary 9.1.3.2) and the pullback functor f^* preserves small colimits, assertion (a) follows from the adjoint functor theorem (Corollary HTT.5.5.2.9). To prove (b), we note that the composite functor

$$\mathrm{QCoh}(X) \xrightarrow{f_*} \mathrm{QCoh}(Y) \xrightarrow{\Gamma(Y;\bullet)} \mathrm{Sp}$$

can be identified with the global sections functor $\Gamma(X;\bullet)$, and therefore commutes with filtered colimits by virtue of Proposition 9.1.5.3 (and our assumption that $\mathscr{O}_X$ is compact). Since Y is affine the functor $\Gamma(Y;\bullet)$ is conservative and preserves small colimits, so the functor f_* must preserve small colimits as well. This proves (b). To prove (c), we note that if $\mathscr{G}$ is fixed, then the collection of those objects $\mathscr{F} \in \mathrm{QCoh}(Y)$ for which $\theta_{\mathscr{F},\mathscr{G}}$ is an equivalence is a stable subcategory of $\mathrm{QCoh}(Y)$ which is closed under small colimits (by virtue of (b)). To show that this subcategory coincides with $\mathrm{QCoh}(Y)$, it will suffice (because of our assumption that Y is affine) to show that it contains the structure sheaf $\mathscr{O}_Y$. This is clear, since $\theta_{\mathscr{O}_Y,\mathscr{G}}$ is equivalent to the identity map from $f_*\mathscr{G}$ to itself. □

We now consider a variant of Proposition 9.1.5.6 where Y is not assumed to be affine.

Proposition 9.1.5.7. *Let $f : X \to Y$ be a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which satisfies the following condition:*

- $(*)$ *For every connective $\mathbb{E}_\infty$-ring R and every point $\eta \in Y(R)$, the fiber product $X_\eta = \mathrm{Spec}\, R \times_Y X$ is a quasi-geometric stack and the structure sheaf $\mathscr{O}_{X_\eta}$ is a compact object of $\mathrm{QCoh}(X_\eta)$.*

Then:

1. *The pullback functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ admits a right adjoint $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$.*

2. *The functor f_* commutes with small colimits.*

3. *For every pullback diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, the associated diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(Y) & \xrightarrow{f^*} & \mathrm{QCoh}(X) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ \mathrm{QCoh}(Y') & \xrightarrow{f'^*} & \mathrm{QCoh}(X') \end{array}$$

is right adjointable: in other words, the Beck-Chevalley transformation $g^* f_* \to f'_* g'^*$ *is an equivalence of functors from* $\mathrm{QCoh}(X)$ *to* $\mathrm{QCoh}(Y')$.

(4) *For every pair of sheaves* $\mathscr{F} \in \mathrm{QCoh}(Y)$ *and* $\mathscr{G} \in \mathrm{QCoh}(\mathrm{Spec}\, X)$*, the canonical map*

$$\theta_{\mathscr{F},\mathscr{G}} : \mathscr{F} \otimes f_* \mathscr{G} \to f_*(f^* \mathscr{F} \otimes \mathscr{G})$$

is an equivalence in $\mathrm{QCoh}(Y)$.

Proof. We proceed as in the proof of Proposition 6.3.4.1. Write Y as the colimit of a diagram $q : S \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, where each $Y_s = q(s)$ is affine. Hypothesis $(*)$ guarantees that each of the fiber products $X_s = X \times_Y Y_s$ is a quasi-geometric stack for which $\mathscr{O}_{X_s}$ is a compact object of $\mathrm{QCoh}(X_s)$. Every edge $s \to s'$ in S determines a pullback diagram

$$\begin{array}{ccc} X_s & \longrightarrow & X_{s'} \\ \downarrow & & \downarrow \\ Y_s & \longrightarrow & Y_{s'}. \end{array}$$

Using part (c) of Proposition 9.1.5.6, we deduce that the associated diagram of pullback functors

$$\begin{array}{ccc} \mathrm{QCoh}(Y_{s'}) & \longrightarrow & \mathrm{QCoh}(X_{s'}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(Y_s) & \longrightarrow & \mathrm{QCoh}(X_s) \end{array}$$

is right adjointable. Using the equivalences $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(X_s)$ and $\mathrm{QCoh}(Y) \simeq \varprojlim \mathrm{QCoh}(Y_s)$, Corollary HA.4.7.4.18 implies the following:

(i) The functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ admits a right adjoint.

(ii) For each $s \in S$, the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(Y) & \xrightarrow{f^*} & \mathrm{QCoh}(X) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(Y_s) & \xrightarrow{f_s^*} & \mathrm{QCoh}(X_s) \end{array}$$

is right adjointable.

This proves (1). Moreover, we can assume that every morphism $\mathrm{Spec}\, A \to Y$ appears as a map $Y_s \to Y$ for some $s \in S$, so that (ii) implies that (3) is satisfied whenever Y' is affine. To prove (3) in general, consider a pullback square σ :

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

and let $\mathscr{F} \in \mathrm{QCoh}(X)$; we wish to show that the Beck-Chevalley map $\theta : g^* f_* \mathscr{F} \to f'_* g'^* \mathscr{F}$ is an equivalence. To prove this, it will suffice to show that for every map $h : \operatorname{Spec} A \to Y'$, the pullback $h^*(\theta)$ is an equivalence in $\mathrm{QCoh}(\operatorname{Spec} A)$. Extending σ to a rectangular diagram

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} A & \xrightarrow{h} & Y' & \xrightarrow{g} & Y \end{array}$$

where both squares are pullbacks, we see that $h^*(\theta)$ fits into a commutative diagram

$$\begin{array}{ccc} & h^* f'_* g'^* \mathscr{F} & \\ {\scriptstyle h^*(\theta)} \nearrow & & \searrow {\scriptstyle \theta'} \\ h^* g^* f_* \mathscr{F} & \xrightarrow{\theta''} & f''_* h'^* g'^* \mathscr{F}, \end{array}$$

where θ' and θ'' are equivalences by virtue of the fact that (2) holds in the special case where Y' is affine. This completes the proof of (3).

To prove (2) and (4), we can use (3) to reduce to the special case where Y is affine. In this case, the desired result follows from Proposition 9.1.5.6. □

Proposition 9.1.5.8. *Let $f : X \to Y$ be a morphism of quasi-geometric stacks. If $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$, then f satisfies condition $(*)$ of Proposition 9.1.5.7.*

Proof. Let R be a connective $\mathbb{E}_\infty$-ring, let $\eta \in Y(R)$, and let $X_R = X \times_Y \operatorname{Spec} R$. We then have a pullback diagram

$$\begin{array}{ccc} X_R & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \xrightarrow{g} & Y. \end{array}$$

Since Y is quasi-geometric, the map g is quasi-affine. It follows that g' is also quasi-affine, so the pullback functor g'^* admits a right adjoint $g'_* : \mathrm{QCoh}(X_R) \to \mathrm{QCoh}(X)$ which preserves small colimits (Corollary 6.3.4.3). The global sections functor $\Gamma(X_R; \bullet)$ factors as a composition

$$\mathrm{QCoh}(X_R) \xrightarrow{g'_*} \mathrm{QCoh}(X) \xrightarrow{\Gamma(X;\bullet)} \mathrm{Sp},$$

and therefore also preserves small colimits. Applying Proposition 9.1.5.3, we deduce that $\mathscr{O}_{X_R}$ is a compact object of $\mathrm{QCoh}(X_R)$. □

9.1.6 Truncations of Quasi-Geometric Stacks

Let R be a commutative ring and let G be a quasi-affine group scheme over R. By our definition, the classifying stack BG of Example 9.1.1.7 is a functor from the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-rings to the ∞-category $\widehat{\mathcal{S}}$ of spaces. However, the classifying stack BG is really a *classical* algebro-geometric object. Our goal in this section is to make this idea more precise, and to show that no information is lost by restricting the domain of BG to the full subcategory $\mathrm{CAlg}^{\heartsuit} \subseteq \mathrm{CAlg}^{\mathrm{cn}}$ spanned by the ordinary commutative rings (which is described in Remark 9.1.1.8).

Proposition 9.1.6.1. *Let X be a quasi-geometric stack and let $n \geqslant 0$ be an integer. The following conditions are equivalent:*

(a) *For every flat morphism $f : \operatorname{Spec} A \to X$, the $\mathbb{E}_\infty$-ring A is n-truncated.*

(b) *There exists a faithfully flat morphism $\operatorname{Spec} A \to X$, where the $\mathbb{E}_\infty$-ring A is n-truncated.*

Proof. The implication $(a) \Rightarrow (b)$ is immediate. Conversely, suppose that (b) is satisfied, so that there exists a faithfully flat map $f : \operatorname{Spec} A \to X$ where A is n-truncated. Let $g : \operatorname{Spec} B \to X$ be any flat map, and form a pullback diagram

$$\begin{array}{ccc} Y & \xrightarrow{f'} & \operatorname{Spec} B \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \operatorname{Spec} A & \xrightarrow{f} & X. \end{array}$$

Since X is a quasi-geometric stack, the functor Y is (representable by) a quasi-affine spectral Deligne-Mumford stack. Since g is flat, the morphism g' is flat and therefore Y is n-truncated (by virtue of our assumption that A is n-truncated). Since f' is faithfully flat, it follows that $\operatorname{Spec} B$ is n-truncated: that is, the $\mathbb{E}_\infty$-ring B is n-truncated. This shows that $(b) \Rightarrow (a)$. $\square$

Definition 9.1.6.2. Let X be a quasi-geometric stack and let n be a nonnegative integer. We will say that X is *n-truncated* if it satisfies the equivalent conditions of Proposition 9.1.6.1.

Example 9.1.6.3. Let R be a commutative ring, let G be a flat quasi-affine group scheme over R, and let BG denote the classifying stack of G (Example 9.1.1.7). Then BG is a 0-truncated quasi-geometric stack.

Example 9.1.6.4. Let X be a quasi-geometric Deligne-Mumford stack and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by X. Then X is n-truncated (in the sense of Definition 1.4.6.1) if and only if X is n-truncated (in the sense of Definition **??**).

Definition 9.1.6.5. Let $f : X \to Y$ be a morphism of quasi-geometric stacks and let $n \geqslant 0$ be an integer. We will say that the morphism f *exhibits X as an n-truncation of Y* if the following conditions are satisfied:

(a) The quasi-geometric stack X is n-truncated.

(b) For every n-truncated quasi-geometric stack X', composition with f induces a homotopy equivalence $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\widehat{\mathcal{S}})}(X', X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\widehat{\mathcal{S}})}(X', Y)$.

Let Y be a quasi-geometric stack. It follows immediately from Definition 9.1.6.5 that if there exists a morphism $f : X \to Y$ which exhibits X as the n-truncation of Y, then the quasi-geometric stack X (and the morphism f) are uniquely determined up to equivalence. For existence, we have the following result:

Proposition 9.1.6.6. *Let $f : X \to Y$ be a morphism of quasi-geometric stacks. Suppose that f is affine and the unit map $\mathscr{O}_Y \to f_* \mathscr{O}_X$ exhibits $f_* \mathscr{O}_X$ as an n-truncation of $\mathscr{O}_Y$ in the ∞-category* $\mathrm{QCoh}(Y)$*. Then f exhibits X as an n-truncation of Y, in the sense of Definition 9.1.6.5.*

Proof. Choose a faithfully flat map $g : Y_0 \to Y$, where $Y_0 \simeq \operatorname{Spec} A$ is affine. Form a pullback diagram of quasi-geometric stacks

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y. \end{array}$$

Since f is affine, the map f_0 is affine, so that $X_0 \simeq \operatorname{Spec} B$ is affine. Since f exhibits $f_* \mathscr{O}_X$ as an n-truncation of $\mathscr{O}_Y$ and the map g is flat, it follows that f_0 exhibits $f_{0*} \mathscr{O}_{X_0}$ as an n-truncation of $\mathscr{O}_{Y_0}$ in $\mathrm{QCoh}(Y_0)$: in other words, we can identify B with the truncation $\tau_{\leqslant n} A$. The map g' is a pullback of g and therefore faithfully flat. Since B is n-truncated, it follows that the quasi-geometric stack X is n-truncated (Proposition 9.1.6.1).

Let X' be an n-truncated quasi-geometric stack; we wish to show that composition with f induces a homotopy equivalence $\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(X', X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(X', Y)$. Since the morphism f is affine, we can identify the homotopy fiber of θ over a point $h \in \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(X', Y)$ with the mapping space $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(X')}(h^* f_* \mathscr{O}_X, \mathscr{O}_{X'})$. Note that the unit map $u : \mathscr{O}_{X'} \simeq h^* \mathscr{O}_Y \to h^* f_* \mathscr{O}_X$ induces an equivalence of n-truncations. Since $\mathscr{O}_{X'}$ is n-truncated, it follows that composition with u induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(X'))}(h^* f_* \mathscr{O}_X, \mathscr{O}_{X'}) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(X'))}(\mathscr{O}_{X'}, \mathscr{O}_{X'}),$$

so that the mapping space $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(X'))}(h^* f_* \mathscr{O}_X, \mathscr{O}_{X'})$ is contractible as desired. $\square$

Corollary 9.1.6.7. *Let Y be a quasi-geometric stack and let $n \geqslant 0$ be an integer. Then there exists a morphism $f : X \to Y$ of quasi-geometric stacks which exhibits X as an n-truncation of Y. Moreover, f is affine.*

Proof. Combine Propositions 9.1.6.6 and 6.3.4.5. □

Corollary 9.1.6.8. *Let $\mathcal{C} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ denote the full subcategory spanned by the quasi-geometric stacks, let $n \geqslant 0$ be an integer, and let $\mathcal{C}_n \subseteq \mathcal{C}$ denote the full subcategory spanned by the n-truncated quasi-geometric stacks. Then the inclusion $\mathcal{C}_n \hookrightarrow \mathcal{C}$ admits a right adjoint (which assigns to each quasi-geometric stack its n-truncation).*

We now observe that an n-truncated quasi-geometric stack X is determined by its values on n-truncated connective $\mathbb{E}_\infty$-rings.

Proposition 9.1.6.9. *Let X be a quasi-geometric stack which is n-truncated for some $n \geqslant 0$, and let $X' : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a left Kan extension of $X|_{\tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}}$. Then the canonical map $X' \to X$ exhibits X as a sheafification of X' with respect to the fpqc topology.*

Proof. Choose a faithfully flat map $f : X_0 \to X$, where X_0 is affine. Let $X_\bullet$ denote the Čech nerve of f. For each $k \geqslant 0$, let X'_k denote a left Kan extension of $X|_{\tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}}$. We then have a commutative diagram

$$\begin{array}{ccc} |X'_\bullet| & \longrightarrow & |X_\bullet| \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X, \end{array}$$

where the geometric realizations are formed in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. We wish to show that the lower horizontal map induces an equivalence after sheafification with respect to the fpqc topology. Since the vertical maps induce equivalences after sheafification for the fpqc topology, it will suffice to prove that the upper horizontal map induces an equivalence after fpqc sheafification. In other words, it will suffice to show that the conclusion of Proposition 9.1.6.9 after replacing X by X_k, for each $k \geqslant 0$. We may therefore assume without loss of generality that X is (representable by) a quasi-affine spectral algebraic space. In this case, the diagonal of X is affine. Repeating the above argument, we can reduce to the case where $X \simeq \operatorname{Spec} R$ for some connective $\mathbb{E}_\infty$-ring R. In this case, our assumption that X is n-truncated guarantees that R is n-truncated, so that the map $X' \to X$ is an equivalence. □

Corollary 9.1.6.10. *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be functors, let $n \geqslant 0$ be an integer, and let $X_0, Y_0 : \tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ denote the restrictions of X and Y. If X is an n-truncated* *quasi-geometric stack and Y is a sheaf with respect to the fpqc topology, then the restriction map*

$$\phi : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(X, Y) \to \mathrm{Map}_{\mathrm{Fun}(\tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(X_0, Y_0).$$

Proof. Let X' be as in Proposition 9.1.6.9. We then have a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(X, Y) & & \\ & \nearrow^{\psi} & & \searrow^{\phi} & \\ \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(X', Y) & & \xrightarrow{\phi \circ \psi} & & \mathrm{Map}_{\mathrm{Fun}(\tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(X_0, Y_0). \end{array}$$

The map ψ is a homotopy equivalence by virtue of our assumption that Y is a sheaf for the fpqc topology, and the composite map $\phi \circ \psi$ is a homotopy equivalence because X' is defined as a left Kan extension of X_0. It follows that the map ϕ is a homotopy equivalence as well. □

Corollary 9.1.6.11. *Let $n \geqslant 0$ and let $\mathcal{C}_n \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ denote the full subcategory spanned by the n-truncated quasi-geometric stacks. Then the restriction functor $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}}) \to \mathrm{Fun}(\tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ is fully faithful when restricted to $\mathcal{C}_n$.*

9.2 Tannaka Duality for Quasi-Geometric Stacks

Our goal in this section is to address the following general question:

Question 9.2.0.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. To what extent can X be recovered from the ∞-category $\mathrm{QCoh}(Y)$ of quasi-coherent sheaves on Y?

Our main result addresses Question 9.2.0.1 in the special case where X is a quasi-geometric stack (see Definition 9.1.0.1):

Theorem 9.2.0.2 (Tannaka Duality for Quasi-Geometric Stacks)**.** *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be functors. Assume that X is a quasi-geometric stack and that Y is (representable by) a quasi-compact, quasi-separated spectral algebraic space. Let $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$ denote the ∞-category of symmetric monoidal functors from $\mathrm{QCoh}(X)$ to $\mathrm{QCoh}(Y)$. Then the construction $(f : Y \to X) \mapsto (f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y))$ determines a fully faithful embedding*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$$

whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which satisfy the following additional conditions:

- (a) *The functor F admits a right adjoint G (equivalently, the functor F preserves small colimits: see Corollary HTT.5.5.2.9).*

(b) *The functor F carries connective objects of* $\mathrm{QCoh}(X)$ *to connective objects of* $\mathrm{QCoh}(Y)$ *(equivalently, the functor* $G : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ *is left t-exact).*

(c) *The functor G preserves small colimits.*

(d) *For every pair of objects* $\mathscr{F} \in \mathrm{QCoh}(X)$ *and* $\mathscr{G} \in \mathrm{QCoh}(Y)$*, the canonical map*

$$\mathscr{F} \otimes G(\mathscr{G}) \to G(F(\mathscr{F}) \otimes \mathscr{G})$$

is an equivalence in $\mathrm{QCoh}(X)$ *(in other words, the adjoint functors* $\mathrm{QCoh}(X) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{QCoh}(Y)$ *satisfy a projection formula).*

9.2.1 Digression on Quasi-Affine Morphisms

Let $\widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the ∞-category of (not necessarily small) ∞-categories. We let $\widehat{\mathcal{C}\mathrm{at}}_\infty^L$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ whose objects are ∞-categories which admit small colimits and whose morphisms are functors which preserve small colimits. We will regard $\widehat{\mathcal{C}\mathrm{at}}_\infty^L$ as equipped with the symmetric monoidal described in §HA.4.8.1 : given a pair of objects $\mathcal{C}$ and $\mathcal{D}$ in $\widehat{\mathcal{C}\mathrm{at}}_\infty^L$, the tensor product $\mathcal{C} \otimes \mathcal{D}$ is universal among objects $\mathcal{E} \in \widehat{\mathcal{C}\mathrm{at}}_\infty^L$ for which there exists a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserves small colimits separately in each variable. Note that we can identify commutative algebra objects of $\widehat{\mathcal{C}\mathrm{at}}_\infty^L$ with symmetric monoidal ∞-categories $\mathcal{C}$ for which $\mathcal{C}$ admits small colimits and the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable.

For every functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, let QAff_X be the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/X}$ spanned by the quasi-affine morphisms $Y \to X$. Our proof of Theorem 9.2.0.2 will make use of the following:

Proposition 9.2.1.1. *Let* $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ *be a functor. Then the construction* $(f : Y \to X) \mapsto \mathrm{QCoh}(Y)$ *determines a fully faithful embedding* $\mathrm{QAff}_X \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty^L)_{\mathrm{QCoh}(X)/}$.

Lemma 9.2.1.2. *Suppose we are given morphisms* $f : Y \to X$, $g : Z \to X$ *in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. *Assume that f is quasi-affine, and let* $\mathscr{A} = f_* \mathscr{O}_Y \in \mathrm{CAlg}(\mathrm{QCoh}(X))$. *Then the canonical map*

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/X}}(Z, Y) \to \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(Z))}(g^* \mathscr{A}, \mathscr{O}_Z)$$

is a homotopy equivalence.

Proof. When regarded as functors of Z, both the domain and codomain of θ carry colimits of functors to limits of spaces. Writing Z as a colimit of corepresentable functors, we may reduce to the case where Z is corepresentable by a connective $\mathbb{E}_\infty$-ring R. Replacing X by Z and Y by the fiber product $Y \times_X Z$, we may reduce to the case where Y is representable by a quasi-affine spectral Deligne-Mumford stack Y equipped with a map $\mathsf{Y} \to \mathrm{Sp\acute{e}t}\, R$. In this case, the desired result is a consequence of Proposition ??. □

Proof of Proposition 9.2.1.1. For every quasi-affine morphism $f : Y \to X$, let $\mathscr{A}_Y = f_* \mathscr{O}_X \in \mathrm{CAlg}(\mathrm{QCoh}(X))$. Proposition 6.3.4.6 supplies an equivalence of ∞-categories $\mathrm{QCoh}(Y) \simeq \mathrm{Mod}_{\mathscr{A}_Y}(\mathrm{QCoh}(X))$. If we are given another quasi-affine morphism $g : Z \to X$, then Corollary HA.4.8.5.21 supplies a homotopy equivalence

$$\begin{aligned} \mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}^L_\infty)_{\mathrm{QCoh}(X)/}}(\mathrm{QCoh}(Y), \mathrm{QCoh}(Z)) &\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(X))}(\mathscr{A}_Y, \mathscr{A}_Z) \\ &\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(Z))}(g^* \mathscr{A}_Y, \mathscr{O}_Z). \end{aligned}$$

The desired result now follows from Lemma 9.2.1.2. □

Remark 9.2.1.3. Suppose we are given quasi-affine morphisms $f : Y \to X$ and $g : Z \to X$ in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. Proposition 9.2.1.1 implies that if $F : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(Y)$ is a $\mathrm{QCoh}(X)$-linear symmetric monoidal functor which preserves small colimits, then F is given by pullback along some map $h : Y \to Z$ such that $h \circ f \simeq g$. Moreover, if we are given two such maps $h, h' : Y \to Z$, then any $\mathrm{QCoh}(X)$-linear symmetric monoidal equivalence α between h^* and h'^* can be lifted to a homotopy between h and h'. In fact, the assumption that α is an equivalence is superfluous: any $\mathrm{QCoh}(X)$-linear symmetric monoidal functor from h^* to h'^* is automatically an equivalence (Remark HA.4.8.5.9).

9.2.2 Weak Tannaka Duality

We are now ready to prove a preliminary version of Theorem 9.2.0.2.

Proposition 9.2.2.1. *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be functors, and suppose that the diagonal map $\delta : X \to X \times X$ is quasi-affine. Then the canonical map*

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$$

is fully faithful. Here $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$ denotes the ∞-category of symmetric monoidal functors from $\mathrm{QCoh}(X)$ to $\mathrm{QCoh}(Y)$. In particular, for any pair of maps $f, f' : Y \to X$, then any $\mathrm{QCoh}(X)$-linear symmetric monoidal natural transformation $\alpha : f^* \to f'^*$ is an equivalence.

Before giving the proof of Proposition 9.2.2.1, let us explain its relationship to the classical theory of Tannaka duality.

Let κ be a field, let G be an affine group scheme over κ, and let BG denote the classifying stack of G (Example 9.1.1.7). In the situation of Notation **??**, the classifying stack BG is equipped with a canonical base point $\epsilon : \mathrm{Spec}\, \kappa \to BG$. For every commutative κ-algebra A, let ϵ_A denote the composite map $\mathrm{Spec}\, A \to \mathrm{Spec}\, R \xrightarrow{\epsilon} BG$. Since the classifying space BG has quasi-affine diagonal, Proposition 9.2.2.1 guarantees that the natural map

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(\mathrm{Spec}\, A, BG) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(BG), \mathrm{Mod}_A)$$

is fully faithful. In particular, we obtain a homotopy equivalence

$$\{\epsilon_A\} \times_{\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(\operatorname{Spec} A, BG)} \{\epsilon_A\} \to \mathrm{Map}_{\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(BG), \mathrm{Mod}_A)}(\epsilon_A^*, \epsilon_A^*).$$

Unwinding the definitions, we see that the domain of this homotopy equivalence can be identified with the set $G(A)$ of A-valued points of G. We therefore obtain the following:

Corollary 9.2.2.2 (Classical Tannaka Duality, Derived Version)**.** *Let G be an affine group scheme over a field κ and let A be a commutative κ-algebra. Then the canonical map $G(A) \to \mathrm{Map}_{\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(BG), \mathrm{Mod}_A)}(\epsilon_A^*, \epsilon_A^*)$ is a homotopy equivalence. More informally, we can identify $G(A)$ with the group of symmetric monoidal automorphisms of the functor $\epsilon_A^* : \mathrm{QCoh}(BG) \to \mathrm{Mod}_A$ (moreover, every symmetric monoidal endomorphism of ϵ_A^* is also an isomorphism).*

Remark 9.2.2.3. Corollary 9.2.2.2 is very similar Theorem 9.0.0.1. Unwinding the definitions, we can identify the category $\mathrm{Rep}(G)$ of finite-dimensional representations of G with the full subcategory of $\mathrm{QCoh}(BG)$ spanned by those quasi-coherent sheaves which are locally free of finite rank. For any commutative κ-algebra A, the functor ϵ_A^* restricts to a functor $e : \mathrm{Rep}(G) \to \mathrm{Mod}_A^{\mathrm{lf}}$, where $\mathrm{Mod}_A^{\mathrm{lf}}$ denotes the category of projective A-modules of finite rank. Let $\mathrm{Aut}^{\otimes}(\epsilon_A^*)$ denote the space of equivalences from ϵ_A^* with itself as a symmetric monoidal functor of ∞-categories, and let $\mathrm{Aut}^{\otimes}(e)$ denote the set of isomorphisms of e with itself as a symmetric monoidal functor of ordinary categories. We then have a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Aut}^{\otimes}(\epsilon_A^*) & & \\ & \nearrow & & \searrow & \\ G(A) & & \longrightarrow & & \mathrm{Aut}^{\otimes}(e). \end{array}$$

The classical version of Tannaka duality asserts that the bottom horizontal map is an isomorphism (Theorem 9.0.0.1), while our derived version of Tannaka duality asserts that the diagonal map on the left is a homotopy equivalence. To see that these assertions are equivalent, it would suffice to show that the diagonal map on the right is also a homotopy equivalence: that is, that the datum of a symmetric monoidal automorphism of $\epsilon_A^* : \mathrm{QCoh}(BG) \to \mathrm{Mod}_A$ is equivalent to the datum of a symmetric monoidal automorphism of $\epsilon_A^*|_{\mathrm{Rep}(G)}$. We will discuss this comparison in §9.7; see Corollary 9.3.7.5.

Remark 9.2.2.4. It follows from Proposition 9.2.2.1 that many of the hypotheses of Corollary 9.2.2.2 are superfluous:

(*a*) We do not need to assume that the map $\epsilon_A : \operatorname{Spec} A \to BG$ factors through the base point $\epsilon : \operatorname{Spec} \kappa \to BG$. The proof of Corollary 9.2.2.2 shows more generally that for any pair of maps $u, v : \operatorname{Spec} A \to BG$, we obtain a homotopy equivalence

$$\{u\} \times_{BG(A)} \{v\} \simeq \mathrm{Map}_{\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(BG), \mathrm{Mod}_A)}(u^*, v^*).$$

Here u and v classify G-torsors $\mathcal{P}_u$ and $\mathcal{P}_v$ over $\operatorname{Spec} A$, and we can identify the fiber product $\{u\} \times_{BG(A)} \{v\}$ with the space of isomorphisms between $\mathcal{P}_u$ and $\mathcal{P}_v$.

(*b*) We do not need to assume that A is a commutative ring: Corollary 9.2.2.2 is valid more generally if A is a connective $\mathbb{E}_\infty$-algebra over κ.

(*c*) We do not need to assume that κ is a field: Corollary 9.2.2.2 remains valid if κ is an arbitrary connective $\mathbb{E}_\infty$-ring, provided that G is flat over κ.

(*d*) We do not need to assume that G is affine: Corollary 9.2.2.2 is valid more generally if G is quasi-affine.

Note however that for the additional generality described in (*b*), (*c*), and (*d*), it is necessary to work in the derived setting (rather than at the level of abelian categories, as in the statement of Theorem 9.0.0.1).

Remark 9.2.2.5. We will be primarily interested in the special case of Proposition 9.2.2.1 where X is a quasi-geometric stack (in which case we can describe the essential image of the embedding $\operatorname{Map}_{\operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \hookrightarrow \operatorname{Fun}^{\otimes}(\operatorname{QCoh}(X), \operatorname{QCoh}(Y))$ using Theorem 9.2.0.2). However, there are algebro-geometric objects X of interest which are not quasi-geometric but nonetheless satisfy the hypotheses of Proposition 9.2.2.1: for example, quasi-separated spectral algebraic spaces which are not quasi-compact (see Warning **??**).

Proof of Proposition 9.2.2.1. Fix a functor $X : \operatorname{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ for which the diagonal $\delta : X \to X \times X$ is quasi-affine. The constructions

$$Y \mapsto \operatorname{Map}_{\operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \qquad Y \mapsto \operatorname{Fun}^{\otimes}(\operatorname{QCoh}(X), \operatorname{QCoh}(Y))$$

both carry colimits in $\operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ to limits of ∞-categories. Writing Y as a colimit of corepresentable functors, we may reduce to the case where $Y = \operatorname{Spec} R$ for some connective $\mathbb{E}_\infty$-ring R. Since δ is quasi-affine, any morphism $Y \to X$ is automatically quasi-affine (Proposition 9.1.1.1). Choose a pair of maps $f, g : Y \to X$, which we will identify with objects $Y_f, Y_g \in \operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/X}$. Remark 9.2.1.3 implies that every symmetric monoidal transformation from f^* to g^* is an equivalence. We have a commutative diagram

$$\begin{array}{ccc} \operatorname{Map}_{\operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/X}}(Y_f, Y_g) & \longrightarrow & \operatorname{Map}_{\operatorname{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty^L)_{\operatorname{QCoh}(X)/}}(\operatorname{QCoh}(Y), \operatorname{QCoh}(Y)) \\ \downarrow & & \downarrow \\ \operatorname{Map}_{\operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, Y) & \longrightarrow & \operatorname{Map}_{\operatorname{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty^L)_{\mathrm{Sp}/}}(\operatorname{QCoh}(Y), \operatorname{QCoh}(Y)). \end{array}$$

We wish to show that this diagram induces a homotopy equivalence between the homotopy fibers of the vertical maps (taken over the points id_Y and $\mathrm{id}_{\operatorname{QCoh}(Y)}$). To prove this, it suffices to show that the horizontal maps are equivalences. This follows from Proposition 9.2.1.1, since the maps $f, g : Y \to X$ and the projection $Y \to *$ are all quasi-affine (here $*$ denotes the final object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$). □

9.2.3 The Proof of Theorem 9.2.0.2

We now turn to the proof of Theorem 9.2.0.2. Suppose that we are given functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, where X is a quasi-geometric stack and Y is (representable by) a quasi-compact, quasi-separated spectral algebraic space. Suppose we are given a morphism $f : Y \to X$. Note that any morphism $\mathrm{Spec}\, R \to X$ is quasi-affine (Proposition 9.1.1.1), so the fiber product $\mathrm{Spec}\, R \times_X Y$ is quasi-affine over Y and is therefore also (representable by) a quasi-compact, quasi-separated spectral algebraic space. It follows that the pullback functor f^* admits a right adjoint $f_* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ (Proposition 6.3.4.1) which preserves small colimits and satisfies the projection formula (Corollary 6.3.4.3); moreover, the functor f_* will be left t-exact (since f^* preserves connective objects). Using Proposition 9.2.2.1, we deduce that the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$$

is a fully faithful embedding whose image consists of functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which satisfy the hypotheses of Theorem 9.2.0.2. To complete the proof, we must verify the converse:

Proposition 9.2.3.1. *Let X be a quasi-geometric stack, let Y be a quasi-compact, quasi-separated spectral algebraic space, and let $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ be a symmetric monoidal functor satisfying the following conditions:*

(a) *The functor F admits a right adjoint G.*

(b) *The functor F is right t-exact (so that G is left t-exact).*

(c) *The functor G preserves small colimits.*

(d) *For every pair of objects $\mathscr{F} \in \mathrm{QCoh}(X)$ and $\mathscr{G} \in \mathrm{QCoh}(Y)$, the canonical map*

$$\mathscr{F} \otimes G(\mathscr{G}) \to G(F(\mathscr{F}) \otimes \mathscr{G})$$

is an equivalence in $\mathrm{QCoh}(X)$.

Then there exists a map $f : Y \to X$ and a symmetric monoidal equivalence $F \simeq f^*$.

Proof. For every morphism $g : Y' \to Y$ of quasi-compact, quasi-separated spectral algebraic spaces, let F_g denote the composite functor $\mathrm{QCoh}(X) \xrightarrow{F} \mathrm{QCoh}(Y) \xrightarrow{g^*} \mathrm{QCoh}(Y')$. Note that if F satisfies conditions (a) through (d), then so does F_g (the functor F_g admits a right adjoint given by the composition $G \circ g_*$, which will preserve small colimits and satisfy the projection formula by virtue of Corollary 3.4.2.2 and Remark 3.4.2.6). In this case, let $\chi(Y')$ denote the homotopy fiber of the map $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y', X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y'))$ over the point F_g. It follows from Proposition 9.2.2.1 that the space $\chi(Y')$ is either empty or contractible. We wish to show that $\chi(Y)$ is nonempty. Since X is a sheaf with respect to the fpqc topology, the construction $Y' \mapsto \chi(Y')$ satisfies fpqc descent. Consequently, it will suffice to show that there exists a faithfully flat morphism $g : Y' \to Y$ such that $\chi(Y')$ is nonempty. In particular, we may assume without loss of generality that $Y = \mathrm{Spec}\, R$ is affine.

Since X is quasi-geometric, we can choose a faithfully flat morphism $u : U \to X$, where U is affine. Set $\mathscr{A} = u_* \mathscr{O}_U \in \mathrm{CAlg}(\mathrm{QCoh}(X))$. The key point is to prove the following:

$(*)$ There exists a faithfully flat, quasi-affine morphism $v : Y' \to Y$ and an equivalence $F(\mathscr{A}) \simeq v_* \mathscr{O}_{Y'}$ in $\mathrm{CAlg}(\mathrm{QCoh}(Y))$.

Assume for the moment that $(*)$ is satisfied. Then Proposition 6.3.4.6 supplies (symmetric monoidal) equivalences

$$\mathrm{QCoh}(U) \simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{QCoh}(X))$$

$$\mathrm{QCoh}(Y') \simeq \mathrm{Mod}_{v_* \mathscr{O}_{Y'}}(\mathrm{QCoh}(Y)) \simeq \mathrm{Mod}_{F(\mathscr{A})}(\mathrm{QCoh}(Y)).$$

Consequently, F induces a symmetric monoidal functor $F' : \mathrm{QCoh}(U) \to \mathrm{QCoh}(Y')$ which fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{F} & \mathrm{QCoh}(Y) \\ \downarrow{u^*} & & \downarrow{v^*} \\ \mathrm{QCoh}(U) & \xrightarrow{F'} & \mathrm{QCoh}(Y'). \end{array}$$

Since U is affine and Y' is quasi-affine, Proposition 9.2.1.1 implies the existence of a symmetric monoidal equivalence $F' \simeq f'^*$ for some map $f' : Y' \to U$. It follows that there is a symmetric monoidal equivalence $F_v \simeq (u \circ f')^*$, so that $\chi(Y') \neq \emptyset$ and therefore $\chi(Y) \neq \emptyset$ (since Y' is faithfully flat over Y).

We now prove $(*)$. Write $Y = \mathrm{Spec}\, R$ for some connective $\mathbb{E}_\infty$-ring R, so that we can identify $F(\mathscr{A})$ with an $\mathbb{E}_\infty$-algebra $A \in \mathrm{CAlg}_R$. Set $B = F(\tau_{\geq 0} \mathscr{A}) \in \mathrm{CAlg}_R$, so that assumption (b) guarantees that B is connective. We first claim that there is an equivalence $F(\mathscr{A}) \simeq v_* \mathscr{O}_{Y'}$ for some quasi-affine morphism $v : Y' \to Y$. By virtue of Theorem 2.6.0.2, it will suffice to verify the following:

(i) There exists an integer $n \gg 0$ such that A is $(-n)$-connective.

(ii) The $\mathbb{E}_\infty$-ring A is a compact idempotent object of CAlg_B.

Note first that $U \times_X U$ is quasi-affine, so that the spectrum

$$\Gamma(U \times_X U; \mathscr{O}_{U \times_X U}) \simeq \Gamma(U; u^* u_* \mathscr{O}_U)$$

is $(-n)$-connective for $n \gg 0$. Since U is affine, it follows that $u^* u_* \mathscr{O}_U$ is $(-n)$-connective as an object of $\mathrm{QCoh}(U)$. Since the map u is faithfully flat, it follows from Proposition 9.1.3.5 that $\mathscr{A} = u_* \mathscr{O}_U$ is a $(-n)$-connective object of $\mathrm{QCoh}(X)$ for $n \gg 0$, so that assertion (i) follows from our assumption (b).

We now prove (ii). Note that the projection map $q : U \times_X U \to U$ (onto either factor) is quasi-affine, so that $u^* \mathscr{A} \simeq q_* \mathscr{O}_{U \times_X U}$ satisfies the equivalent conditions of Theorem 2.6.0.2. In particular, $u^* \mathscr{A}$ is an idempotent algebra over its connective cover $\tau_{\geqslant 0} u^* \mathscr{A}$. That is, the multiplication map

$$u^* \mathscr{A} \otimes_{\tau_{\geqslant 0} u^* \mathscr{A}} u^* \mathscr{A} \to u^* \mathscr{A}$$

is an equivalence. Since u is faithfully flat, it follows that the multiplication map $\mathscr{A} \otimes_{\tau_{\geqslant 0} \mathscr{A}} \mathscr{A} \to \mathscr{A}$ is an equivalence in $\mathrm{QCoh}(X)$. Applying the functor F, we deduce that the multiplication $A \otimes_B A \to A$ is an equivalence, so that A is an idempotent object of CAlg_B.

We now complete the proof of (2) by showing that A is a compact object of CAlg_B. Suppose we are given a filtered diagram $\{B_\alpha\}$ in CAlg_B; we wish to show that the canonical map

$$\varinjlim \mathrm{Map}_{\mathrm{CAlg}_B}(A, B_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_B}(A, \varinjlim B_\alpha)$$

is a homotopy equivalence. Set $\mathcal{C} = \mathrm{CAlg}_{\tau_{\geqslant 0} \mathscr{A}}(\mathrm{QCoh}(X))$. Unwinding the definitions, we wish to show that the composite map

$$\varinjlim \mathrm{Map}_{\mathcal{C}}(\mathscr{A}, G(B_\alpha)) \to \mathrm{Map}_{\mathcal{C}}(\mathscr{A}, \varinjlim G(B_\alpha)) \to \mathrm{Map}_{\mathcal{C}}(\mathscr{A}, G(\varinjlim B_\alpha))$$

is a homotopy equivalence. The map on the right is a homotopy equivalence by virtue of our assumption (c). It will therefore suffice to show that $\mathscr{A}$ is a compact object of $\mathcal{C}$. Note that for any object $\mathscr{A}' \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(\mathscr{A}, \mathscr{A}')$ is either empty (if the unit map $e : \mathscr{A}' \to \mathscr{A} \otimes_{\tau_{\geqslant 0} \mathscr{A}} \mathscr{A}'$ is not an equivalence) or contractible. Note that since u is faithfully flat, the map e is an equivalence if and only if $u^*(e)$ is an equivalence. Consequently, to show that $\mathscr{A}$ is a compact object of $\mathcal{C}$, it will suffice to show that $u^* \mathscr{A}$ is a compact object of $\mathrm{CAlg}_{\tau_{\geqslant 0} u^* \mathscr{A}}(\mathrm{QCoh}(U))$, which is a consequence of Theorem 2.6.0.2. This completes our verification of (i) and (ii), which establishes the existence of a quasi-affine morphism $v : Y' \to Y$ and an equivalence $v_* \mathscr{O}_{Y'} \simeq F(\mathscr{A})$.

We now show that v is flat. Since Y is affine, it will suffice to show that for each $\mathscr{F} \in \mathrm{QCoh}(Y)^{\heartsuit}$, the pullback $v^* \mathscr{F}$ belongs to $\mathrm{QCoh}(Y')_{\leqslant 0}$. Since Y' is quasi-affine, this is equivalent to the statement that the spectrum $\Gamma(Y'; v^* \mathscr{F})$ is 0-truncated. We have equivalences

$$\begin{aligned}\Gamma(Y'; v^* \mathscr{F}) &\simeq \Gamma(Y; v_* v^* \mathscr{F}) \\ &\simeq \Gamma(Y; v_* \mathscr{O}_{Y'} \otimes \mathscr{F}) \\ &\simeq \Gamma(Y; F(\mathscr{A}) \otimes \mathscr{F}) \\ &\simeq \Gamma(X; G(F(\mathscr{A}) \otimes \mathscr{F})) \\ &\xleftarrow{\gamma} \Gamma(X; \mathscr{A} \otimes G(\mathscr{F})) \\ &\simeq \Gamma(X; u_* \mathscr{O}_U \otimes G(\mathscr{F})) \\ &\simeq \Gamma(X; u_* u^* G(\mathscr{F})) \\ &\simeq \Gamma(U; u^* G(\mathscr{F}));\end{aligned}$$

here the map γ is an equivalence by virtue of assumption (d). The spectrum $\Gamma(U; u^* G(\mathscr{F}))$ is 0-truncated because $\mathscr{F}$ is 0-truncated, the functor G is left t-exact (by virtue of (b)), and the functor u^* is t-exact. This completes the proof that v is flat.

It remains to show that v is faithfully flat. For this, it will suffice to show that for $\mathscr{F} \in \mathrm{QCoh}(Y)^{\heartsuit}$ as above, the canonical map $\rho : \Gamma(Y; \mathscr{F}) \to \Gamma(Y'; v^* \mathscr{F})$ induces an injection on π_0 (in this case, the vanishing of $v^* \mathscr{F}$ guarantees the vanishing of $\pi_0 \Gamma(Y; \mathscr{F})$, hence also the vanishing of $\mathscr{F}$ since Y is affine and $\mathscr{F}$ is discrete). Using the preceding calculation, we can identify ρ with the canonical map $\Gamma(X; G(\mathscr{F})) \to \Gamma(U; u^* G(\mathscr{F}))$, which is injective on π_0 since $G(\mathscr{F}) \in \mathrm{QCoh}(X)$ is 0-truncated and the map u is faithfully flat. □

9.2.4 Some Consequences of Tannaka Duality

We now collect a few easy consequences of Theorem 9.2.0.2.

Proposition 9.2.4.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a quasi-geometric stack. Then X is a hypercomplete sheaf with respect to the fpqc topology on $\mathrm{CAlg}^{\mathrm{cn}}$.*

Proof. Our assumption that X is quasi-geometric guarantees that X is a sheaf with respect to the fpqc topology. To show that X is hypercomplete, it will suffice (by virtue of Proposition D.6.7.4) to show that for every connective $\mathbb{E}_\infty$-ring R and every pair of points $\eta, \eta' \in X(R)$, the functor $Y : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ given by $Y(A) = \{\eta\} \times_{X(A)} \{\eta'\}$ is a hypercomplete sheaf with respect to the fpqc topology. For each $A \in \mathrm{CAlg}^{\mathrm{cn}}_R$, let $\eta_A, \eta'_A : \operatorname{Spec} A \to X$ denote the maps determined by η and η'. By virtue of Proposition 9.2.2.1, the functor Y is given by the formula $Y(A) = \mathrm{Map}_{\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{Mod}_A)}(\eta_A^*, \eta_A'^*)$. Since the construction $A \mapsto \mathrm{Mod}_A$ determines a hypercomplete sheaf (with values in the ∞-category $\mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ of symmetric monoidal ∞-categories) by virtue of Corollary D.6.3.3, it follows that Y is hypercomplete.

□

Proposition 9.2.4.2. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a quasi-geometric stack. Then, for every connective $\mathbb{E}_\infty$-ring A, the space $X(A)$ is essentially small.*

Proof. Proposition 9.1.3.1 guarantees that $\mathrm{QCoh}(X)$ is a presentable ∞-category. Consequently, if τ is a sufficiently large regular cardinal, then there is an equivalence $\mathrm{QCoh}(X) \simeq \mathrm{Ind}_\tau(\mathrm{QCoh}(X)^\tau)$, where $\mathrm{QCoh}(X)^\tau$ denotes the full subcategory of $\mathrm{QCoh}(X)$ spanned by the τ-compact objects. Enlarging τ if necessary, we may assume that the full subcategory $\mathrm{QCoh}(X)^\tau$ contains the unit object $\mathscr{O}_X$ and is closed under tensor products. It follows that $\mathrm{QCoh}(X)^\tau$ inherits the structure of a symmetric monoidal ∞-category. Let Mod^τ_A denote the full subcategory of Mod_A spanned by the τ-compact objects, which we also regard as a symmetric monoidal ∞-category. Proposition HA.4.8.1.10 guarantees that $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)_\tau, \mathrm{Mod}^\tau_A)$ can be identified with the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{Mod}_A)$ spanned by those symmetric monoidal functors which preserve τ-filtered colimits and τ-compact objects. Note that these conditions are automatically satisfied for any functor $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_A$ which admits a right adjoint that preserves small colimits (see Proposition HTT.5.5.7.2). Consequently, Theorem 9.2.0.2 supplies a fully faithful embedding

$$X(A) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(\mathrm{Spec}\, A, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^\tau, \mathrm{Mod}^\tau_A).$$

Since the codomain of this embedding is essentially small, it follows that $X(A)$ is essentially small. □

Proposition 9.2.4.3. *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be quasi-geometric stacks and let $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ be a t-exact symmetric monoidal equivalence of ∞-categories. Then we have $F \simeq f^*$ for some equivalence $f : Y \xrightarrow{\sim} X$ (uniquely determined up to equivalence, by virtue of Proposition 9.2.2.1).*

Proof. We claim that for every functor $Z : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ and every natural transformation $g : Z \to Y$, the composite functor $\mathrm{QCoh}(X) \xrightarrow{F} \mathrm{QCoh}(Y) \xrightarrow{g^*} \mathrm{QCoh}(Z)$ is equivalent to f^*_Z, for some map $f_Z : Z \to X$. Applying this result in the case $Z = Y$, we deduce that $F \simeq f^*$ for some map $f : Y \to X$, in which case Theorem 9.2.0.2 shows that f induces a homotopy equivalence $Y(R) \to X(R)$ for every connective $\mathbb{E}_\infty$-ring R, and is therefore an equivalence. To prove the existence of f_Z, we can assume (by virtue of the uniqueness provided by Proposition 9.2.2.1) write Z as a colimit of corepresentable functors and thereby reduce to the case where Z is corepresentable. In this case, it will suffice to show that the functor $g^* \circ F$ satisfies criteria (a) through (d) of Theorem 9.2.0.2. This is clear, since F is a t-exact equivalence of ∞-categories and g^* satisfies conditions (a) through (d) of Theorem 9.2.0.2. □

Proposition 9.2.4.4. *Let $f : A \to B$ be a flat morphism of connective $\mathbb{E}_\infty$-rings and let $I \subseteq \pi_0 A$ be a finitely generated ideal. Let $U \subseteq \mathrm{Spec}\, A$ be the open subfunctor complementary* *to the vanishing locus of I (so that $U(R) \subseteq \mathrm{Map}_{\mathrm{CAlg}}(A, R)$ is the subspace consisting of those maps $A \to R$ which exhibit R as an I-local A-module, for each $R \in \mathrm{CAlg}^{\mathrm{cn}}$), and define $V = U \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, B \subseteq \mathrm{Spec}\, B$ similarly. Suppose that f an equivalence of I-completions $A^{\wedge}_{I} \to B^{\wedge}_{I}$. Then, for every quasi-geometric stack X, the diagram σ :*

$$\begin{array}{ccc} X(A) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(U, X) \\ \downarrow & & \downarrow \\ X(B) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(V, X) \end{array}$$

is a pullback square.

Remark 9.2.4.5. Let A be a Noetherian $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 A$ be an ideal. Then the canonical map $A \to A^{\wedge}_{I}$ satisfies the hypotheses of Proposition 9.2.4.4 (see Corollary 7.3.6.9).

Proof of Proposition 9.2.4.4. It follows from Theorem 7.4.0.1 that the diagram of pullback functors σ :

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{Spec}\, A) & \longrightarrow & \mathrm{QCoh}(U) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathrm{Spec}\, B) & \longrightarrow & \mathrm{QCoh}(V) \end{array}$$

is a pullback square of (symmetric monoidal) ∞-categories. We therefore obtain a pullback diagram

$$\begin{array}{ccc} \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(\mathrm{Spec}\, A)) & \longrightarrow & \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(U)) \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(\mathrm{Spec}\, B)) & \longrightarrow & \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(V)). \end{array}$$

Combining this observation with Proposition 9.2.2.1, we immediately deduce that the natural map

$$X(A) \to X(B) \times_{\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(V, X)} \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(U, X)$$

is fully faithful. To verify essential surjectivity, it will suffice to show that if $F_A : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\mathrm{Spec}\, A)$ is a symmetric monoidal functor having the property that the induced maps

$$F_B : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\mathrm{Spec}\, B)$$

$$F_U : \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$$

$$F_V : \mathrm{QCoh}(X) \to \mathrm{QCoh}(V)$$

satisfy conditions (a) through (d) of Theorem 9.2.0.2, then F_A also satisfies conditions (a) through (d) of Theorem 9.2.0.2. We consider each of these conditions in turn:

(a) Suppose that the functors F_B, F_U, and F_V admit right adjoints

$$G_B : \mathrm{QCoh}(\operatorname{Spec} B) \to \mathrm{QCoh}(X)$$

$$G_U : \mathrm{QCoh}(\operatorname{Spec} B) \to \mathrm{QCoh}(X)$$

$$G_V : \mathrm{QCoh}(V) \to \mathrm{QCoh}(X).$$

Then F_B, F_U, and F_V preserve small colimits. Since σ is a pullback square, it follows that F_A preserves small colimits, and therefore admits a right adjoint G_A : $\mathrm{QCoh}(\operatorname{Spec} A) \to \mathrm{QCoh}(X)$ by virtue of the adjoint functor theorem (Corollary HTT.5.5.2.9). However, we do not need to appeal to the adjoint functor theorem: the right adjoint G_A can be given directly by the formula

$$G_A(\mathscr{F}) \simeq G_B(\mathscr{F}|_{\operatorname{Spec} B}) \times_{G_V(\mathscr{F}|_V)} G_U(\mathscr{F}|_U).$$

(b) Because f is flat, the maps $\operatorname{Spec} B \to \operatorname{Spec} A \leftarrow U$ comprise a flat covering. Consequently, an object $\mathscr{F} \in \mathrm{QCoh}(\operatorname{Spec} A)$ is connective if and only if $\mathscr{F}|_{\operatorname{Spec} B}$ and $\mathscr{F}|_U$ are connective. Since F_B and F_U are right t-exact, it follows that F_A is also right t-exact.

(c) The functors G_B, G_V, and G_U preserve small colimits. Using the canonical identification $G_A(\mathscr{F}) \simeq G_B(\mathscr{F}|_{\operatorname{Spec} B}) \times_{G_V(\mathscr{F}|_V} G_U(\mathscr{F}|_U)$, we deduce that the functor G_A also preserves small colimits.

(d) Suppose we are given a pair of objects $\mathscr{F} \in \mathrm{QCoh}(X)$ and $\mathscr{G} \in \mathrm{QCoh}(\operatorname{Spec} A)$. We wish to show that the canonical map

$$\theta_A : \mathscr{F} \otimes G_A(\mathscr{G}) \to G_A(F_A(\mathscr{F}) \otimes \mathscr{G})$$

is an equivalence in $\mathrm{QCoh}(X)$. Unwinding the definitions, we see that there is a pullback diagram

$$\begin{array}{ccc} \theta_A & \longrightarrow & \theta_U \\ \downarrow & & \downarrow \\ \theta_B & \longrightarrow & \theta_V \end{array}$$

in the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{QCoh}(X))$, where

$$\theta_B : \mathscr{F} \otimes G_B(\mathscr{G}|_{\operatorname{Spec} B}) \to G_B(F_B(\mathscr{F}) \otimes \mathscr{G}|_{\operatorname{Spec} B})$$

$$\theta_V : \mathscr{F} \otimes G_V(\mathscr{G}|_V) \to G_V(F_V(\mathscr{F}) \otimes \mathscr{G}|_V)$$

$$\theta_U : \mathscr{F} \otimes G_U(\mathscr{G}|_U) \to G_U(F_U(\mathscr{F}) \otimes \mathscr{G}|_U)$$

are the equivalences appearing in the projection formula for F_B, F_V, and F_U, respectively. It follows that θ_A is an equivalence, as desired.

□

Example 9.2.4.6. Let $f : A \to B$ be a morphism of Noetherian $\mathbb{E}_\infty$-rings and let $x \in \pi_0 A$ be an element for which f induces an equivalence $A^{\wedge}_{(x)} \to B^{\wedge}_{(x)}$. Then, for every quasi-geometric stack $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, the diagram

$$\begin{array}{ccc} X(A) & \longrightarrow & X(B) \\ \downarrow & & \downarrow \\ X(A[x^{-1}]) & \longrightarrow & X(B[x^{-1}]) \end{array}$$

is a pullback square.

9.3 Geometric Stacks

In §9.1, we introduced the notion of a *quasi-geometric stack* $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. The main result of §9.2 implies that a quasi-geometric stack X can be functorially recovered from the ∞-category $\mathrm{QCoh}(X)$. More precisely, Theorem 9.2.0.2 asserts that if Y is a quasi-compact, quasi-separated algebraic space, then the datum of a map $Y \to X$ is equivalent to the datum of a symmetric monoidal, right t-exact functor $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which admits a right adjoint G that preserves colimits and satisfies a projection formula. However, for many applications, this characterization is inconvenient: one would prefer to have a more direct description involving only properties of the functor F (which can be tested locally on Y), rather than its right adjoint G (which depends on the global structure of Y). In this section, we will study a special class of quasi-geometric stacks for which such a description is possible:

Definition 9.3.0.1. A *geometric stack* is a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which satisfies the following conditions:

(a) The functor X satisfies descent with respect to the fpqc topology.

(b) The diagonal map $\delta : X \to X \times X$ is affine.

(c) There exists a connective $\mathbb{E}_\infty$-ring A and a faithfully flat morphism $f : \operatorname{Spec} A \to X$.

Remark 9.3.0.2. A geometric stack X can be defined as a quasi-geometric stack (in the sense of Definition 9.1.0.1) for which the diagonal map $\delta : X \to X \times X$ is affine (see Remark 9.1.0.2). It follows from Proposition 9.2.4.1 that a geometric stack X is automatically a hypercomplete sheaf with respect to the fpqc topology).

The main result of this section is the following variant of Theorem 9.2.0.2:

Theorem 9.3.0.3. *[Tannaka Duality for Geometric Stacks] Let* $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ *be functors, and suppose that* X *is a geometric stack. Then the construction* $(f : Y \to X) \mapsto (f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y))$ *determines a fully faithful embedding*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)),$$

whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ *which satisfy the following conditions:*

(a) *The functor* F *preserves small colimits.*

(b) *The functor* F *carries connective objects of* $\mathrm{QCoh}(X)$ *to connective objects of* $\mathrm{QCoh}(Y)$.

(c) *The functor* F *carries flat objects of* $\mathrm{QCoh}(X)$ *to flat objects of* $\mathrm{QCoh}(Y)$.

9.3.1 Examples of Geometric Stacks

We begin with some general remarks concerning Definition **??**. First, we need an analogue of Lemma 9.1.1.3:

Lemma 9.3.1.1. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

in the ∞*-category* $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ *of fpqc sheaves. If* f' *is affine and* g *is an effective epimorphism of fpqc sheaves, then* f *is affine.*

Proof. Choose any map $\eta : \operatorname{Spec} R \to Y$; we wish to show that the fiber product $Z = \operatorname{Spec} R \times_Y X$ is affine. Since g is an fpqc surjection, we can choose a faithfully flat map $R \to R^0$ such that $\eta|_{\operatorname{Spec} R^0}$ factors through Y'. Let $R^\bullet$ denote the Čech nerve of the faithfully flat map $R \to R^0$ (formed in the ∞-category $\mathrm{CAlg}^{\mathrm{op}}$) and set $Z_\bullet = \operatorname{Spec} R^\bullet \times_Y X$. By construction, each of the maps $\operatorname{Spec} R^m \to Y$ factors through Y', so we can write $Z_m \simeq \operatorname{Spec} A^\bullet$ for some cosimplicial object $A^\bullet$ of $\mathrm{CAlg}^{\mathrm{cn}}$. Using Theorem D.6.3.5, we deduce that $A^\bullet \simeq A \otimes_R R^\bullet$ for some connective $\mathbb{E}_\infty$-algebra A over R. Then

$$Z \simeq |Z_\bullet| \simeq |\operatorname{Spec} A^\bullet| \simeq \operatorname{Spec} A \times_{\operatorname{Spec} R} |\operatorname{Spec} R^\bullet| \simeq \operatorname{Spec} A$$

is affine, as desired. □

Proposition 9.3.1.2. *Let* $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ *be a quasi-geometric stack. The following conditions are equivalent:*

(1) *The diagonal map* $\delta : X \to X \times X$ *is affine (that is,* X *is a geometric stack).*

(2) *For every pair of morphisms* $\operatorname{Spec} A \to X \leftarrow \operatorname{Spec} B$*, the fiber product* $\operatorname{Spec} A \times_X \operatorname{Spec} B$ *is affine.*

(3) *Every morphism* $\operatorname{Spec} A \to X$ *is affine.*

(4) *There exists a faithfully flat affine morphism* $f : \operatorname{Spec} A \to X$*.*

Proof. The implication (1) ⇒ (2) follows from the existence of a pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} A \times_X \operatorname{Spec} B & \longrightarrow & \operatorname{Spec}(A \otimes B) \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X, \end{array}$$

and the equivalence (2) ⇔ (3) is tautological. The implication (2) ⇒ (1) follows from the observation that every map $\operatorname{Spec} A \to X \times X$ determines a pullback square

$$\begin{array}{ccc} \operatorname{Spec} A \times_{X \times X} X & \longrightarrow & \operatorname{Spec} A \times_X \operatorname{Spec} A \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} A \times \operatorname{Spec} A. \end{array}$$

The implication (3) ⇒ (4) follows from our assumption that X is quasi-geometric (which guarantees the existence of a faithfully flat quasi-affine morphism $\operatorname{Spec} A \to X$). We will complete the proof by showing that (4) ⇒ (3). Let $f : \operatorname{Spec} A \to X$ be a faithfully flat affine morphism and let $g : \operatorname{Spec} B \to X$ be arbitrary; we claim that g is affine. Form a pullback square

$$\begin{array}{ccc} Y & \longrightarrow & \operatorname{Spec} A \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} B & \xrightarrow{g} & X. \end{array}$$

By virtue of Lemma 9.3.1.1, it will suffice to show that g' is affine. This is clear, since our assumption that f is affine guarantees that Y is affine. □

Proposition 9.3.1.3. *Let* $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ *be a functor which satisfies descent for the fpqc topology, and suppose there exists a morphism* $f : X_0 \to X$ *satisfying the following conditions:*

(*a*) *The functor* $X_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ *is a geometric stack.*

(*b*) *The morphism* f *is representable, affine, and faithfully flat.*

Then X is a geometric stack.

Proof. The functor X is a quasi-geometric stack by virtue of Proposition 9.1.1.4. Choose a faithfully flat map $g : \operatorname{Spec} A \to X_0$. Condition (a) implies that g is affine (Proposition 9.3.1.2). It follows from (b) that the composition $g \circ f$ is affine, so that X is a geometric stack by virtue of Proposition 9.3.1.2. □

Corollary 9.3.1.4. *Let $X_\bullet$ be a simplicial object of $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ satisfying the following conditions:*

(a) *The functor $X_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is a geometric stack.*

(b) *The functor $X_\bullet$ is a groupoid object of $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$.*

(c) *The face map $d_0 : X_1 \to X_0$ is representable, affine, and faithfully flat.*

Then the geometric realization $X = |X_\bullet|$ (formed in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ of fpqc sheaves) is a geometric stack.

Proof. We have a pullback diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{d_0} & X_0 \\ \downarrow{\scriptstyle d_1} & & \downarrow \\ X_0 & \longrightarrow & X \end{array}$$

where the vertical maps are effective epimorphisms in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$. It follows from (c) and Lemma 9.3.1.1 that the map $X_0 \to X$ is affine and faithfully flat, so that X is a geometric stack by virtue of Proposition 9.3.1.3. □

Remark 9.3.1.5 (Existence of Atlases)**.** Every geometric stack X can be written as a geometric realization $|X_\bullet|$, where $X_\bullet$ satisfies the hypotheses of Corollary 9.3.1.4 and X_0 is affine: to see this, we can take $X_\bullet$ to be the Čech nerve of any faithfully flat map $\operatorname{Spec} A \to X$.

Example 9.3.1.6 (Classifying Stacks)**.** Let R be a commutative ring and let G be a flat affine group scheme over R. Then the classifying stack BG (see Example 9.1.1.7) is a geometric stack.

Example 9.3.1.7 (The Moduli Stack of Formal Groups)**.** Let R be a commutative ring. We define the category $\mathcal{FGL}(R)$ of *formal group laws over R* as follows:

- An object of $\mathcal{FGL}(R)$ is a (commutative, 1-dimensional) formal group law over R: that is, a power series $f(x, y) \in R[[x, y]]$ satisfying the identities

$$f(x,y) = f(y,x) \qquad f(x,0) = x \qquad f(x, f(y,z)) = f(f(x,y), z).$$

- If f and f' are formal group laws over R, then a morphism from f to f' in $\mathcal{FGL}(R)$ is an invertible formal power series $g(x) \in R[[x]]$ satisfying $f(g(x), g(y)) = g(f'(x,y))$.

For each $n \geqslant 0$, let $\mathrm{N}(\mathcal{FGL}(R))_n$ denote the set of n-simplices of the nerve of the category $\mathcal{FGL}(R)$: that is, the collection of $(n+1)$-tuples of formal group laws $f_0, \ldots, f_n \in R[[x,y]]$ together with invertible power series $g_1, \ldots, g_n \in R[[x]]$ satisfying $f_i(g_i(x), g_i(y)) = g_i(f_{i+1}(x,y))$. The construction $R \mapsto \mathrm{N}(\mathcal{FGL}(R))_n$ determines a functor from the category of commutative rings to the category of sets. It is not difficult to show that this functor is corepresentable: that is, there exists a commutative ring L^n and a bijection $\mathrm{N}(\mathcal{FGL}(R))_n \simeq \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit}(L^n, R)$. Let $\operatorname{Spec} L^n$ denote the functor corepresented by L^n on the ∞-category of connective $\mathbb{E}_\infty$-rings, so that we can regard $\operatorname{Spec} L^\bullet$ as a simplicial object of $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$. It is not difficult to show that $\operatorname{Spec} L^\bullet$ satisfies the hypotheses of Corollary 9.3.1.4, so that the geometric realization $|\operatorname{Spec} L^\bullet|$ (formed in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$) is a geometric stack. We will denote this geometric stack by $\mathcal{FG}$ and refer to it as the *moduli stack of formal groups*. We will refer to the commutative ring L^0 as the *Lazard ring*: by a theorem of Lazard, it is isomorphic to a polynomial ring $\mathbf{Z}[v_1, v_2, \ldots]$ on infinitely many variables. Concretely, the geometric stack $\mathcal{FG}$ can be described as the quotient of the affine scheme $\operatorname{Spec} L^0$ by the action of the affine group scheme $\operatorname{Spec} \mathbf{Z}[a_1^{\pm 1}, a_2, a_3, \ldots]$ parametrizing invertible formal power series $g(x) = \sum_{n>0} a_n x^n$. For every commutative ring R, one can identify $\mathcal{FG}(R)$ with (the nerve of) the groupoid of (commutative, 1-dimensional) formal groups $\mathbf{G}$ over R. This category can be regarded as a slight enlargement of the category $\mathcal{FGL}(R)$ (more precisely, there is a fully faithful embedding $\mathcal{FGL}(R) \hookrightarrow \mathcal{FG}(R)$ whose essential image consists of those formal groups $\mathbf{G}$ over R whose Lie algebra is isomorphic to R).

Example 9.3.1.8 (The Derived Moduli Stack of Formal Groups)**.** Let MP denote the $\mathbb{E}_\infty$-ring of *periodic complex bordism*: that is, the Thom spectrum of the tautological virtual complex vector bundle on the classifying space $\mathbf{Z} \times \mathrm{BU}$ for complex K-theory. We let $\mathrm{MP}^\bullet$ denote the cosimplicial $\mathbb{E}_\infty$-ring given by the Čech nerve (in the ∞-category $\mathrm{CAlg}^{\mathrm{op}}$) of the unit map $S \to \mathrm{MP}$, so that MP^n can be identified with the $(n+1)$-fold smash power of MP. A celebrated theorem of Quillen supplies an isomorphism $L^0 \xrightarrow{\sim} \pi_0 \mathrm{MP}$, where L^0 is the Lazard ring of Example 9.3.1.7 (see [169]). A slight elaboration of Quillen's theorem establishes an isomorphism of cosimplicial commutative rings $L^\bullet \xrightarrow{\sim} \pi_0 \mathrm{MP}^\bullet$.

The cosimplicial $\mathbb{E}_\infty$-ring $\mathrm{MP}^\bullet$ does not quite fit into the framework of this section, because the $\mathbb{E}_\infty$-rings MP^n are not connective (in fact, each MP^n even periodic: that is, we have noncanonical isomorphisms $\pi_* \mathrm{MP}^n \simeq (\pi_0 \mathrm{MP}^n)[u^{\pm 1}]$ for some $u \in \pi_2 \mathrm{MP}^n$). We can correct this by passing to connective covers: one can show that the spectrum $\operatorname{Spec} \tau_{\geqslant 0} \mathrm{MP}^\bullet$ is a simplicial object of $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ which satisfies the hypotheses of Corollary 9.3.1.4, so that the geometric realization $|\operatorname{Spec} \tau_{\geqslant 0} \mathrm{MP}^\bullet|$ (formed in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$) is a geometric stack, which we will denote by $\mathcal{FG}^{\mathrm{der}}$. The geometric stack $\mathcal{FG}^{\mathrm{der}}$ can be regarded as a "derived version" of the moduli stack of formal groups $\mathcal{FG}$ defined in Example 9.3.1.7: the isomorphisms $L^{\bullet} \simeq \pi_0 \, \mathrm{MP}^{\bullet}$ induce a morphism of geometric stacks $\mathcal{FG} \to \mathcal{FG}^{\mathrm{der}}$ which exhibit $\mathcal{FG}$ as the 0-truncation of $\mathcal{FG}^{\mathrm{der}}$, in the sense of Definition **??**.

Remark 9.3.1.9 (The Adams-Novikov Spectral Sequence)**.** Let E be a spectrum, and let $\mathrm{MP}^{\bullet}$ be the cosimplicial $\mathbb{E}_{\infty}$-ring appearing in Example 9.3.1.8. The construction $[n] \mapsto E \otimes \mathrm{MP}^n$ determines an object $\mathscr{F}_E \in \mathrm{Tot}(\mathrm{Mod}_{\tau_{\geqslant 0} \mathrm{MP}^{\bullet}}) \simeq \mathrm{QCoh}(\mathcal{FG}^{\mathrm{der}})$. Moreover, we have a canonical map $E \to \mathrm{Tot}(E \otimes \mathrm{MP}^{\bullet}) \simeq \Gamma(\mathcal{FG}^{\mathrm{der}}; \mathscr{F}_E)$, which is an equivalence in many cases (for example, it is an equivalence whenever E is connective). For every integer n, we can regard $\pi_n \mathscr{F}_E$ as an object of the abelian category $\mathrm{QCoh}(\mathcal{FG}^{\mathrm{der}})^{\heartsuit} \simeq \mathrm{QCoh}(\mathcal{FG})^{\heartsuit}$. We will denote the homotopy groups $\pi_* \Gamma(\mathcal{FG}; \pi_n \mathscr{F}_E)$ by $\mathrm{H}^{-*}(\mathcal{FG}; \pi_n \mathscr{F}_E)$, and refer to them as the *cohomology groups of $\mathcal{FG}$ with coefficients in $\mathscr{F}_E$*. The cosimplicial spectrum $E \otimes \mathrm{MP}^{\bullet}$ determines a Bousfield-Kan spectral sequence $\{E_r^{s,t}, d_r\}_{r \geqslant 2}$, whose second page is given by $E_2^{s,t} = \mathrm{H}^{-t}(\mathcal{FG}; \pi_s \mathscr{F}_E)$ which, in good cases, converges to $\pi_{s+t} E$. This spectral sequence is known as the *Adams-Novikov spectral sequence*.

Example 9.3.1.10 (Closure Under Pullbacks)**.** Suppose we are given a pullback diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$, where X and Y' are geometric stacks and Y is a quasi-geometric stack. Then X' is a geometric stack. To prove this, we first note that X' is a quasi-geometric stack (Corollary 9.1.2.6). It will therefore suffice to show that the diagonal map $\delta_{X'} : X' \to X' \times X'$ is affine. The map $\delta_{X'}$ factors as a composition $X' \xrightarrow{u} X' \times_X X' \xrightarrow{v} X' \times X'$, where u is a pullback of the relative diagonal map $u_0 : Y' \to Y' \times_Y Y'$ and v is a pullback of the diagonal map $\delta_X : X \to X \times X$. The diagonal δ_X is affine by virtue of our assumption that X is geometric, so it will suffice to show that u_0 is affine. For this, we note that u_0 factors as a composition

$$Y' \xrightarrow{u_0'} (Y' \times_Y Y') \times_{Y' \times Y'} Y' \xrightarrow{u_0''} (Y' \times_Y Y')$$

where u_0'' is a pullback of the diagonal $\delta_{Y'} : Y \to Y \times Y$ and is therefore affine (since Y is geometric). We are therefore reduced to showing that u_0' is affine, which follows from the observation that u_0' is a pullback of the relative diagonal $Y \to Y \times_{Y \times Y} Y$ of the diagonal map $\delta_Y : Y \to Y \times Y$. Since Y is quasi-geometric, the map δ_Y is quasi-affine, so its relative diagonal is affine as desired.

9.3.2 Tannaka Duality for Geometric Stacks

Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be functors, and suppose that X is a geometric stack. Using Proposition 9.2.2.1, we deduce that the construction

$$(f : Y \to X) \mapsto (f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y))$$

determines a fully faithful embedding $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(Y, X) \to \mathrm{Fun}^{\otimes}(X, Y)$. To prove Theorem 9.3.0.3, we must show that the essential image of this fully faithful embedding is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which preserve small colimits, connective objects, and flat objects. Writing Y as a colimit of corepresentable functors, we can assume without loss of generality that $Y \simeq \mathrm{Spec}\, A$ for some connective $\mathbb{E}_\infty$-ring A. We are therefore reduced to proving the following:

Proposition 9.3.2.1. *Let X be a geometric stack, let A be a connective $\mathbb{E}_\infty$-ring, and let $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_A$ be a symmetric monoidal functor which preserves small colimits and connective objects. The following conditions are equivalent:*

(1) *There exists a point $\eta \in X(A)$ and a symmetric monoidal equivalence $F \simeq \eta^*$.*

(2) *The functor F carries flat objects of $\mathrm{QCoh}(X)$ to flat objects of Mod_A.*

(3) *For every faithfully flat morphism $g : X_0 \to X$ where X_0 is affine, the A-algebra $F(g_* \mathscr{O}_{X_0})$ is faithfully flat.*

(4) *There exists a faithfully flat morphism $g : X_0 \to X$ where X_0 is affine and the A-algebra $F(g_* \mathscr{O}_{X_0})$ is faithfully flat.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. We next show that (2) implies (3). Let $g : X_0 \to X$ be a faithfully flat morphism, where X_0 is affine. Set $\mathscr{B} = g_* \mathscr{O}_{X_0} \in \mathrm{CAlg}(\mathrm{QCoh}(X))$. Since X is geometric, the morphism g is affine, so that $\mathscr{B}$ is a faithfully flat commutative algebra object of $\mathrm{QCoh}(X)$. Using Lemma D.4.4.3, we see that the cofiber of the unit map $e : \mathscr{O}_X \to \mathscr{B}$ is flat. Assumption (2) then implies that $F(\mathrm{cofib}(e)) \in \mathrm{Mod}_A$ is flat, so that $F(\mathscr{B})$ is faithfully flat over A (by virtue of Lemma D.4.4.3).

The implication (3) $\Rightarrow$ (4) follows immediately from the existence of a faithfully flat morphism $g : X_0 \to X$, where X_0 is affine (which follows from our assumption that X is geometric). We will complete the proof by showing that (4) implies (1). Let $g : X_0 \to X$ be a faithfully flat where X_0 is affine, and set $\mathscr{B} = g_* \mathscr{O}_{X_0}$ as above. Since F preserves small colimits, it follows from the adjoint functor theorem (Corollary HTT.5.5.2.9) that F admits a right adjoint G. By virtue of Theorem 9.2.0.2, it will suffice to prove the following:

(a) The functor G preserves small colimits.

(b) For every quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(X)$ and every object $M \in \mathrm{Mod}_A$, the canonical map $G(M) \otimes \mathscr{F} \to G(M \otimes_A F(\mathscr{F}))$ is an equivalence in $\mathrm{QCoh}(X)$.

Set $A^0 = F(\mathscr{B})$. Since F is exact and preserves flatness, it follows that the cofiber of the unit map $A \to A^0$ is a flat A-module: that is, A^0 is faithfully flat over A (Lemma D.4.4.3). Let $\mathscr{B}^\bullet$ denote the Čech nerve of the unit map $\mathscr{O}_X \to \mathscr{B}$ (computed the ∞-category $\mathrm{CAlg}(\mathrm{QCoh}(X))^{\mathrm{op}}$) and set $A^\bullet = F(\mathscr{B}^\bullet)$. Let $X_\bullet$ be the Čech nerve of g, so that we have an equivalence of cosimplicial ∞-categories $\mathrm{QCoh}(X_\bullet) \simeq \mathrm{Mod}_{\mathscr{B}^\bullet}(\mathrm{QCoh}(X))$. Since X is geometric, the map each of the functors Xk is corepresentable: that is, we can write $X_\bullet \simeq \operatorname{Spec} B^\bullet$ for some cosimplicial connective $\mathbb{E}_\infty$-ring $B^\bullet$.

The functor F determines a natural transformation of cosimplicial ∞-categories

$$F^\bullet \, \mathrm{Mod}_{B^\bullet} \simeq \mathrm{QCoh}(X_\bullet) \simeq \mathrm{Mod}_{\mathscr{B}^\bullet}(\mathrm{QCoh}(X)) \to \mathrm{Mod}_{A^\bullet}(\mathrm{Mod}_A) \simeq \mathrm{Mod}_{A^\bullet}.$$

Using Lemma D.3.5.6, we deduce that for every map $[m] \to [n]$ in $\mathbf{\Delta}$, the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(X_m) & \xrightarrow{F^m} & \mathrm{Mod}_{A^m} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(X_n) & \xrightarrow{F^n} & \mathrm{Mod}_{A^n} \end{array}$$

is right adjointable; moreover, the right adjoint to $F^n : \mathrm{QCoh}(X_n) \to \mathrm{Mod}_{A^n}$ can be identified with the forgetful functor $G_n : \mathrm{Mod}_{A^n} \to \mathrm{Mod}_{B^n}$ induced by a map of $\mathbb{E}_\infty$-rings $B^n \to A^n$. In particular, G_n commutes with small colimits.

Since $X \simeq |X_\bullet|$ in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$, we get a canonical equivalence $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(X_\bullet)$. Corollary D.6.3.3 guarantees that $\mathrm{Mod}_A \simeq \varprojlim \mathrm{Mod}_{A^\bullet}$, so that the functor F can be identified with the limit of the functors F^n. It follows from Corollary HA.4.7.4.18 that each of the diagrams

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{F} & \mathrm{Mod}_{A} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(X_n) & \xrightarrow{F^n} & \mathrm{Mod}_{A^n} \end{array}$$

is right adjointable.

To prove that $G : \mathrm{Mod}_A \to \mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(X_\bullet)$ preserves small colimits, it suffices to show that each of the composite functors $\mathrm{Mod}_A \to \mathrm{QCoh}(X) \to \mathrm{QCoh}(X_n)$ preserves small colimits. By the above, this is equivalent to the assertion that the composite functor

$$\mathrm{Mod}_A \to \mathrm{Mod}_{A^n} \xrightarrow{G_n} \mathrm{QCoh}(X_n) \simeq \mathrm{Mod}_{B^n} \qquad (M \mapsto G_n(A^n \otimes_A M))$$

preserves small colimits, which is clear (since G_n preserves small colimits). This completes the proof of (a).

It remains to verify that the functor $G : \mathrm{Mod}_A \to \mathrm{QCoh}(X)$ satisfies (b). Fix an A-module M and a quasi-coherent sheaf $\mathscr{F}$ on X; we wish to show that the canonical map $\alpha : G(M) \otimes \mathscr{F} \to G(M \otimes_A F(\mathscr{F}))$ is an equivalence in $\mathrm{QCoh}(X)$. Since the pullback functor $g^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X_0) \simeq \mathrm{Mod}_B$ is conservative, it will suffice to show that the induced map

$$g^*(\alpha) : g^* G(M) \otimes g^* \mathscr{F} \to g^* G(M \otimes_A F(\mathscr{F}))$$

is an equivalence of B-modules. This map fits into a commutative diagram

$$\begin{array}{ccc} g^*G(M) \otimes g^* \mathscr{F} & \xrightarrow{g^*(\alpha)} & g^*G(M \otimes_A F(\mathscr{F})) \\ \downarrow & & \downarrow \\ G_0(A^0 \otimes_A M) \otimes g^* \mathscr{F} & \longrightarrow & G_0(A^0 \otimes_A M \otimes F(\mathscr{F})) \end{array}$$

where the vertical maps are equivalences. We are therefore reduced to showing that the bottom horizontal map is an equivalence, which follows from the projection formula for the adjunction $\mathrm{Mod}_B \underset{G^0}{\overset{F^0}{\rightleftarrows}} \mathrm{Mod}_{A^0}$. □

9.3.3 The Resolution Property

Let X be a geometric stack, and let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be some other functor According to Theorem 9.3.0.3, we can identify maps $f : Y \to X$ with symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which preserve small colimits, connective objects, and flat objects. There are some cases in which the last condition follows from the first two. For example, if $\mathscr{F} \in \mathrm{QCoh}(X)$ is locally free of finite rank, then $\mathscr{F}$ is dualizable as an object of $\mathrm{QCoh}(X)^{\mathrm{cn}}$ (Proposition 6.2.6.3). If F is symmetric monoidal and preserves connective objects, it follows that $F(\mathscr{F})$ is dualizable as an object of $\mathrm{QCoh}(Y)^{\mathrm{cn}}$, hence locally free of finite rank (and in particular flat). To fully exploit this observation, it will be useful to restrict our attention to geometric stacks which have "enough" sheaves which are locally free of finite rank.

Notation 9.3.3.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be an arbitrary functor. We let $\mathrm{Vect}(X)$ denote the full subcategory of $\mathrm{QCoh}(X)$ spanned by those quasi-coherent sheaves $\mathscr{F}$ which are locally free of finite rank (equivalently, $\mathrm{Vect}(X)$ is the full subcategory of $\mathrm{QCoh}(X)^{\mathrm{cn}}$ spanned by the dualizable objects: see Proposition 6.2.6.3).

Definition 9.3.3.2. Let X be a geometric stack. We will say that X *has the resolution property* if, for every truncated object $\mathscr{F} \in \mathrm{QCoh}(X)$, the following condition is satisfied:

($*$) There exists a collection of objects $\{\mathscr{E}_\alpha\}_{\alpha \in I}$ of $\mathrm{Vect}(X)$ and a collection of maps $\rho_\alpha : \mathscr{E}_\alpha \to \mathscr{F}$ for which the induced map $\bigoplus_{\alpha \in I} \pi_0 \mathscr{E}_\alpha \to \pi_0 \mathscr{F}$ is an epimorphism in the abelian category $\mathrm{QCoh}(X)^\heartsuit$.

Remark 9.3.3.3. In the situation of Definition 9.3.3.2, it suffices to verify condition ($*$) when $\mathscr{F}$ is connective.

Remark 9.3.3.4. Let X be a 0-truncated geometric stack. In this case, we can identify the $\mathrm{QCoh}(X)$ with the completed derived ∞-category of the abelian category $\mathrm{QCoh}(X)^\heartsuit$

(Proposition **??**). Using this identification, we can represented each truncated object $\mathscr{F} \in \mathrm{QCoh}(X)$ by a chain complex

$$\cdots \to \mathcal{I}_2 \xrightarrow{d_2} \mathcal{I}_1 \xrightarrow{d_1} \mathcal{I}_0 \xrightarrow{d_0} \mathcal{I}_{-1} \xrightarrow{d_{-1}} \mathcal{I}_{-2} \to \cdots$$

of injective objects of $\mathrm{QCoh}(X)^{\heartsuit}$, where $\mathcal{I}_n \simeq 0$ for $n \gg 0$. It follows that there exists a map $\ker(d_0) \to \mathscr{F}$ which is an epimorphism on π_0, where $\ker(d_0) \in \mathrm{QCoh}(X)^{\heartsuit}$. Consequently, in the situation of Definition 9.3.3.2, it suffices to verify condition $(*)$ in the special case where $\mathscr{F} \in \mathrm{QCoh}(X)^{\heartsuit}$.

Remark 9.3.3.5. Let X be a geometric stack and suppose that X is locally Noetherian (see Definition 9.5.1.1). Using Proposition 9.5.2.3 and Corollary **??**, we see that every object $\mathscr{F} \in \mathrm{QCoh}(X)_{\leqslant n}$ can be written as the colimit of a filtered diagram $\{\mathscr{F}_\beta\}$, where each $\mathscr{F}_\beta \in \mathrm{QCoh}(X)_{\leqslant n}$ is almost perfect. Consequently, to verify assertion $(*)$ of Definition 9.3.3.2 for X, it suffices to verify $(*)$ for each $\mathscr{F}_\beta$. Note that since $\mathscr{F}_\beta$ is almost perfect, a map $\bigoplus_{\alpha \in I} \mathscr{E}_\alpha \to \mathscr{F}_\beta$ induces an epimorphism on π_0 if and only if there exists a finite subset $I_0 \subseteq I$ for which the composite map

$$\bigoplus_{\alpha \in I_0} \mathscr{E}_\alpha \to \bigoplus_{\alpha \in I} \mathscr{E}_\alpha \to \mathscr{F}_\beta$$

is an epimorphism on π_0. It follows that X has the resolution property if and only if every truncated object $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{aperf}}$ satisfies the following condition:

$(*')$ There exists a map $\rho : \mathscr{E} \to \mathscr{F}$, where $\mathscr{E} \in \mathrm{Vect}(X)$ and the induced map $\pi_0 \mathscr{E} \to \pi_0 \mathscr{F}$ is an epimorphism in the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$.

Remark 9.3.3.6. Let X be a geometric stack which is 0-truncated and locally Noetherian. Combining Remarks 9.3.3.3, 9.3.3.4, and 9.3.3.5, we see that X has the resolution property if and only if, for every object $\mathscr{F} \in \mathrm{Coh}(X)^{\heartsuit}$ (Notation 9.5.2.1), there exists an epimorphism $\mathscr{E} \to \mathscr{F}$ with $\mathscr{E} \in \mathrm{Vect}(X)$. In the setting of Artin stacks, this condition has been studied by many authors; see for example [217].

Proposition 9.3.3.7. *Let X be a geometric stack, and suppose that the structure sheaf $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$. The following conditions are equivalent:*

(1) *The geometric stack X has the resolution property (Definition 9.3.3.2).*

(2) *The full subcategory $\mathrm{Vect}(X) \subseteq \mathrm{QCoh}(X)^{\mathrm{cn}}$ is a generating subcategory for $\mathrm{QCoh}(X)^{\mathrm{cn}}$, in the sense of Definition C.2.1.1. In other words, for every object $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$, there exists a collection of objects $\{\mathscr{E}_\alpha\}_{\alpha \in I}$ of $\mathrm{Vect}(X)$ and a map $\bigoplus_{\alpha \in I} \mathscr{E}_\alpha \to \mathscr{F}$ which is an epimorphism on π_0.*

Proof. The implication (2) $\Rightarrow$ (1) is immediate (and does not require the assumption that $\mathscr{O}_X$ is compact). To prove the converse, we note that the assumption that $\mathscr{O}_X$ is compact guarantees that there exists an integer $n \gg 0$ such that the functor $\mathscr{G} \mapsto \Gamma(X; \mathscr{G})$ carries $\mathrm{QCoh}(X)_{\geqslant 0}$ into $\mathrm{Sp}_{\geqslant -n}$ (Proposition 9.1.5.3). Let $\mathscr{F}$ be an arbitrary object of $\mathrm{QCoh}(X)$. Using assumption (1), we deduce the existence of objects $\mathscr{E}_\alpha \in \mathrm{Vect}(X)$ and maps $\rho_\alpha : \mathscr{E}_\alpha \to \tau_{\leqslant n} \mathscr{F}$ which induce an epimorphism $\bigoplus \pi_0 \mathscr{E}_\alpha \to \pi_0 \mathscr{F}$. Using the exactness of the sequences

$$\pi_0 \Gamma(X; \mathscr{E}_\alpha^\vee \otimes \mathscr{F}) \to \pi_0 \Gamma(X; \mathscr{E}_\alpha^\vee \otimes \tau_{\leqslant n} \mathscr{F}) \to \pi_{-1} \Gamma(X; \mathscr{E}_\alpha^\vee \otimes \tau_{\geqslant n+1} \mathscr{F})$$

(whose third terms vanishes, by virtue of our assumption on n), we see that each ρ_α can be lifted to a map $\overline{\rho}_\alpha : \mathscr{E}_\alpha \to \mathscr{F}$. We conclude by observing that the induced map $\bigoplus_{\alpha \in I} \mathscr{E}_\alpha \to \mathscr{F}$ is also surjective on π_0. □

Proposition 9.3.3.8. *Let $f : X \to Y$ be a morphism of geometric stacks. If f is quasi-affine and Y has the resolution property, then X has the resolution property.*

Proof. Let $\mathscr{F}$ be a truncated object of $\mathrm{QCoh}(X)$. Then $f_* \mathscr{F}$ is a truncated object of $\mathrm{QCoh}(Y)$. Since Y has the resolution property, we can choose objects $\mathscr{E}_\alpha \in \mathrm{Vect}(Y)$ and a map $\rho : \bigoplus_\alpha \mathscr{E}_\alpha \to f_* \mathscr{F}$ which induces an epimorphism on π_0. We can then identify ρ with a map $\rho' : \bigoplus_\alpha f^* \mathscr{E}_\alpha \to \mathscr{F}$ in $\mathrm{QCoh}(X)$. Since f is quasi-affine, it follows that ρ' is also an epimorphism on π_0. □

Corollary 9.3.3.9. *Let X be a geometric stack. If X has the resolution property, then the n-truncation of X also has the resolution property, for any $n \geqslant 0$.*

9.3.4 The Adams Condition

Let X be a geometric stack. It is somewhat difficult to verify that X has the resolution property by proceeding directly from the definition: *a priori*, it would require us to "resolve" every truncated quasi-coherent sheaf $\mathscr{F}$ on X by vector bundles. In this section, we will show that (under mild hypotheses) the condition that X has the resolution property admits an alternative formulation which is often much easier to verify.

Remark 9.3.4.1. For 0-truncated geometric stacks, the results of this section are due to Schäppi. We will follow the presentation of [178] with some minor modifications.

Definition 9.3.4.2. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a geometric stack. We will say that X *satisfies the Adams condition* if there exists a faithfully flat map $f : U \to X$ where U is affine and the direct image $f_* \mathscr{O}_U$ can be written as the colimit of a filtered diagram $\{\mathscr{F}_\alpha\}$, where each $\mathscr{F}_\alpha \in \mathrm{QCoh}(X)$ is locally free of finite rank.

Remark 9.3.4.3. Definition 9.3.4.2 can be regarded as a "derived" version of the notion of Adams stack introduced in [80]; it specializes to the definition given in [80] in the case when X is 0-truncated (that is, when there exists a faithfully flat morphism $\operatorname{Spec} A \to X$, where A is a commutative ring).

Remark 9.3.4.4. Let $f : X \to Y$ be a morphism of geometric stacks. If f is affine and Y satisfies the Adams condition, then X satisfies the Adams condition. To see this, suppose that $g : U \to Y$ is a faithfully flat morphism, where $g_* \mathscr{O}_U$ can be written as the colimit of a filtered diagram $\{\mathscr{F}_\alpha\}$ in $\operatorname{QCoh}(Y)$ with each $\mathscr{F}_\alpha$ locally free of finite rank. Form a pullback diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y. \end{array}$$

Our assumption that f is affine guarantees that V is affine, and the projection formula allows us to identify $g'_* \mathscr{O}_V$ with the colimit of the diagram $\{f^* \mathscr{F}_\alpha\}$ in $\operatorname{QCoh}(X)$. Since g' is faithfully flat and each $f^* \mathscr{F}_\alpha$ is locally free of finite rank, it follows that X satisfies the Adams condition.

Remark 9.3.4.5. Let X be a geometric stack and let $n \geqslant 0$ be an integer. If X satisfies the Adams condition, then the n-truncation X' of X satisfies the Adams condition: this follows from Remark 9.3.4.4, since the canonical map $X' \to X$ is affine (see Remark **??**).

The terminology of Definition 9.3.4.2 is motivated by the following example, which based on an observation of Adams:

Example 9.3.4.6 (The Derived Moduli Stack of Formal Groups)**.** Let $\mathcal{FG}^{\mathrm{der}}$ denote the derived moduli stack of formal groups (Example 9.3.1.8). Then $\mathcal{FG}^{\mathrm{der}}$ satisfies the Adams condition. To see this, let MP denote the periodic complex bordism spectrum, given as the Thom spectrum of the (universal) virtual complex vector bundle ξ over the space $\mathbf{Z} \times \mathrm{BU}$. By construction, there exists a faithfully flat map $f : U \to \mathcal{FG}^{\mathrm{der}}$, where $U = \operatorname{Spec} \tau_{\geqslant 0} \mathrm{MP}$ is affine. One can show that the direct image $f_* \mathscr{O}_U \in \operatorname{QCoh}(\mathcal{FG}^{\mathrm{der}})$ is the truncation $\tau_{\geqslant 0} \mathscr{F}_{\mathrm{MP}}$, where $\mathscr{F}_{\mathrm{MP}} \in \operatorname{QCoh}(\mathcal{FG}^{\mathrm{der}})$ is the quasi-coherent sheaf appearing in Remark 9.3.1.9. The space $\mathbf{Z} \times \mathrm{BU}$ can be written as a filtered homotopy colimit $\varinjlim F_\alpha$, where each F_α is a disjoint union of Grassmann manifolds $\mathrm{Gr}_{p,q} = \{V \subseteq \mathbf{C}^{p+q} : \dim(V) = p\}$. For each index α, let E_α denote the Thom spectrum of $\xi|_{F_\alpha}$, so that we have equivalences

$$\varinjlim E_\alpha \simeq \mathrm{MP} \qquad \varinjlim \mathscr{F}_{E_\alpha} \simeq \mathscr{F}_{\mathrm{MP}} .$$

To prove the desired result, it suffices to observe that each $\tau_{\geqslant 0} \mathscr{F}_{E_\alpha} \in \operatorname{QCoh}(\mathcal{FG}^{\mathrm{der}})$ is locally free of finite rank. This is a consequence of the fact that each $\mathrm{Gr}_{p,q}$ admits a cell decomposition which uses only cells of even dimension (for example, the Bruhat decomposition of $\mathrm{Gr}_{p,q}$) together with the fact that the homotopy groups of MP are concentrated in even degrees.

Example 9.3.4.7 (The Moduli Stack of Formal Groups)**.** Let $\mathcal{FG}$ denote the moduli stack of formal groups (Example 9.3.1.7). Then $\mathcal{FG}$ satisfies the Adams condition. This follows from Example 9.3.4.6 and Remark 9.3.4.5, since $\mathcal{FG}$ can be identified with the 0-truncation of $\mathcal{FG}^{\mathrm{der}}$. For a more direct proof (which does not appeal to Quillen's work on the relationship between formal group laws and complex bordism) we refer the reader to [80].

The Adams condition of Definition 9.3.4.2 is linked to the resolution property of Definition 9.3.3.2 by the following result:

Theorem 9.3.4.8. *Let X be a geometric stack. Assume either that $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$ or that X is n-truncated for some $n \geqslant 0$. Then X has the resolution property if and only if X satisfies the Adams condition.*

Remark 9.3.4.9. When X is 0-truncated, Theorem 9.3.4.8 appears in [178].

Theorem 9.3.4.8 is a consequence of the following more precise assertions:

Proposition 9.3.4.10. *Let X be a geometric stack. If X satisfies the Adams condition, then X has the resolution property.*

Proposition 9.3.4.11. *Let X be a geometric stack. Assume either that $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$ or that X is n-truncated for some $n \geqslant 0$. Then, for any flat map $f : U \to X$ where U is affine, the direct image $f_* \mathscr{O}_U$ can be written as the colimit of a filtered diagram in $\mathrm{Vect}(X)$. In particular, X satisfies the Adams condition.*

Remark 9.3.4.12. It follows from Propositions 9.3.4.10 and 9.3.4.11 that, if $\mathscr{O}_X$ is compact or X is n-truncated, then the statement of the Adams condition appearing in Definition 9.3.4.2 does not depend on the choice of the faithfully flat map $f : U \to X$ (so long as U is affine).

Proof of Proposition 9.3.4.10. Let X be a geometric stack. If X satisfies the Adams condition, then we can choose a there exists a faithfully flat morphism $f : U \to X$ with U affine and an equivalence of $f_* \mathscr{O}_U$ with the colimit of a filtered diagram $\{\mathscr{E}_\alpha\}$, where each $\mathscr{E}_\alpha$ belongs to $\mathrm{Vect}(X)$. We wish to prove that X has the resolution property. Let $\mathscr{F}$ be a truncated object of $\mathrm{QCoh}(X)$. Since U is affine, we can choose a map $\bigoplus_{\beta \in I} \mathscr{O}_U \xrightarrow{u} f^* \mathscr{F}$ which induces an epimorphism on π_0. For each $\beta \in I$, we can identify the restriction of u to the corresponding summand of $\bigoplus_{\beta \in I} \mathscr{O}_U$ with a map $u_\beta : \mathscr{O}_X \to f_* f^* \mathscr{F} \simeq \mathscr{F} \otimes f_* \mathscr{O}_U \simeq \varinjlim \mathscr{F} \otimes \mathscr{E}_\alpha$. Using Proposition 9.1.5.1 (and our assumption that $\mathscr{F}$ is truncated), we deduce that u_β factors as a composition $\mathscr{O}_X \xrightarrow{u'_\beta} \mathscr{F} \otimes \mathscr{E}_\alpha \to \mathscr{F} \otimes f_* \mathscr{O}_U$ for some index α (which might depend on β). Then u'_β classifies a map $v_\beta : \mathscr{E}_\alpha^\vee \to \mathscr{F}$. Amalgamating these, we obtain a map $v : \bigoplus \mathscr{E}_\alpha^\vee \to \mathscr{F}$. By construction, the map u factors through $f^*(v)$. Since u induces an epimorphism on π_0, it follows that $f^*(v)$ also induces an epimorphism on π_0. The map f is faithfully flat, so that v must also induce an epimorphism on π_0. Allowing $\mathscr{F}$ to vary, we deduce that X has the resolution property. □

The proof of Proposition 9.3.4.11 is more involved. We first establish the following:

Lemma 9.3.4.13. *Let X be a geometric stack. Assume that X has the resolution property and satisfies one of the following additional conditions:*

- (i) *The structure sheaf $\mathscr{O}_X$ is a compact object of* $\mathrm{QCoh}(X)$.
- (ii) *There exists an integer $n \geqslant 0$ such that X is n-truncated.*

Let $f : U \to X$ be a flat morphism, where U is affine. Then the ∞-category $\mathrm{Vect}(X) \times_{\mathrm{QCoh}(X)} \mathrm{QCoh}(X)_{/f_* \mathscr{O}_U}$ *is filtered.*

Proof. Suppose we are given a finite simplicial set K and a diagram $q : K \to \mathrm{Vect}(X) \times_{\mathrm{QCoh}(X)} \mathrm{QCoh}(X)_{/f_* \mathscr{O}_U}$; we wish to show that q can be extended to a map $\overline{q} : K^{\rhd} \to \mathrm{Vect}(X) \times_{\mathrm{QCoh}(X)} \mathrm{QCoh}(X)_{/f_* \mathscr{O}_U}$. For each vertex $\alpha \in K$, let us view $q(\alpha)$ as an object $\mathscr{E}_\alpha \in \mathrm{Vect}(X)$ equipped with a map $u_\alpha : \mathscr{E}_\alpha \to f_* \mathscr{O}_U$, which we will identify with a map $u'_\alpha : f^* \mathscr{E}_\alpha \to \mathscr{O}_U$. Amalgamating the maps u'_α, we obtain a map $u' : \varinjlim_\alpha f^* \mathscr{E}_\alpha \to \mathscr{O}_U$, which we can identify with a map $u'^\vee : \mathscr{O}_U \to \varprojlim_\alpha f^* \mathscr{E}_\alpha^\vee \simeq f^* \varprojlim_\alpha \mathscr{E}_\alpha^\vee$. We claim that there exists a collection of objects $\{\mathscr{G}_\beta\}_{\beta \in I}$ in $\mathrm{Vect}(X)$ and a map $v : \bigoplus \mathscr{G}_\beta \to \varprojlim \mathscr{E}_\alpha^\vee$ in $\mathrm{QCoh}(X)$ which induces an epimorphism on π_0. If condition (ii) is satisfied, then there exists an integer $n \geqslant 0$ such that X is n-truncated, so that each $\mathscr{E}_\alpha^\vee \in \mathrm{Vect}(X)$ is also n-truncated and therefore $\varprojlim_\alpha \mathscr{E}_\alpha^\vee$ is an n-truncated object of $\mathrm{QCoh}(X)$. In this case, the existence of v follows from our assumption that X has the resolution property. If condition (i) is satisfied, the existence of v follows from Proposition 9.3.3.7.

Since U is affine, the structure sheaf $\mathscr{O}_U$ is a compact projective object of $\mathrm{QCoh}(U)^{\mathrm{cn}}$. It follows that we can choose a finite subset $I_0 \subseteq I$ for which the map $u'^\vee$ factors as a composition $\mathscr{O}_U \xrightarrow{w} \bigoplus_{\beta \in I_0} f^* \mathscr{G}_\beta \xrightarrow{f^* v} f^* \varprojlim_\alpha \mathscr{F}_\alpha^\vee$. Set $\mathscr{G} = \bigoplus_{\beta \in I_0} \mathscr{G}_\beta$, so that $\mathscr{G} \in \mathrm{QCoh}(X)$ is locally free of finite rank. Then $v_0 = v|_{\mathscr{G}}$ determines a morphism $v_0^\vee : \varinjlim_\alpha \mathscr{E}_\alpha \to \mathscr{G}^\vee$ in the ∞-category $\mathrm{QCoh}(X)$ for which the map u' factors as a composition

$$f^* \varinjlim_\alpha \mathscr{E}_\alpha \xrightarrow{f^* v_0^\vee} f^* \mathscr{G}^\vee \xrightarrow{w^\vee} \mathscr{O}_U;$$

a choice of such factorization can be identified with the desired extension $\overline{q}$. □

Proof of Proposition 9.3.4.11. Let X be a geometric stack satisfying the hypotheses of Lemma 9.3.4.13 and let $f : U \to X$ be a flat morphism, where U is affine. We wish to show that $f_* \mathscr{O}_U$ can be written as as the colimit of a filtered diagram in $\mathrm{Vect}(X)$. Let $\mathcal{C} = \mathrm{Fun}^\pi(\mathrm{Vect}(X)^{\mathrm{op}}, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Vect}(X)^{\mathrm{op}}, \mathcal{S})$ spanned by those functors which preserve finite products. The construction $(\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}) \mapsto$

$\mathrm{Map}_{\mathrm{QCoh}(X)}(\bullet, \mathscr{F})$ determines a left exact functor of Grothendieck prestable ∞-categories $G : \mathrm{QCoh}(X)^{\mathrm{cn}} \to \mathcal{C}$. The functor G admits a left adjoint $F : \mathcal{C} \to \mathrm{QCoh}(X)^{\mathrm{cn}}$ (which can be characterized abstractly as the essentially unique extension of the inclusion $\mathrm{Vect}(X) \hookrightarrow \mathrm{QCoh}(X)^{\mathrm{cn}}$ which commutes with sifted colimits). Unwinding the definitions, we see that for each object $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$, the counit map $v_{\mathscr{F}} : (F \circ G)(\mathscr{F}) \to \mathscr{F}$ can be identified with the natural map

$$\varinjlim_{\mathscr{E} \in \mathrm{Vect}(X) \times_{\mathrm{QCoh}(X)} \mathrm{QCoh}(X)_{/\mathscr{F}}} \mathscr{E} \to \mathscr{F}.$$

If $\mathscr{F} = f_* \mathscr{O}_U$, then Lemma 9.3.4.13 implies that the domain of $v_{\mathscr{F}}$ is the colimit of a filtered diagram taking values in $\mathrm{Vect}(X)$. It will therefore suffice to show that $v_{\mathscr{F}}$ is an equivalence. We consider two cases:

- (i) Suppose that $\mathscr{O}_X$ is compact. In this case, our assumption that X has the resolution property guarantees that $\mathrm{Vect}(X)$ is a generating subcategory of $\mathrm{QCoh}(X)^{\mathrm{cn}}$, in the sense of Definition C.2.1.1 (see Proposition 9.3.3.7). Since $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is separated, the ∞-categorical Gabriel-Popescu theorem (Theorem C.2.1.6) implies that G is fully faithful, so that $v_{\mathscr{F}}$ is an equivalence for *every* object of $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$.

- (ii) Suppose that X is n-truncated. Then every object $\mathscr{E} \in \mathrm{Vect}(X)$ is n-truncated. Combining our assumption that X has the resolution property with Propositions C.2.5.2 and C.2.5.3, we deduce that F is left exact and that G is fully faithful when restricted to n-truncated objects. In particular, for every n-truncated object $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$, the counit map $v_{\mathscr{F}} : (F \circ G)(\mathscr{F}) \to \mathscr{F}$ is an equivalence. We now conclude by observing that our assumption that X is n-truncated guarantees that $f_* \mathscr{O}_U \in \mathrm{QCoh}(X)^{\mathrm{cn}}$ is n-truncated.

□

9.3.5 Consequences for Tannaka Duality

In the presence of the resolution property, Tannaka duality takes a particularly simple form.

Theorem 9.3.5.1 (Tannaka Duality for Adams Stacks)**.** *Let X be a geometric stack, and suppose that the* 0*-truncation of X has the resolution property. Then the construction* $(f : Y \to X) \mapsto (f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y))$ *determines a fully faithful embedding*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)),$$

whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ *which preserve small colimits and connective objects.*

Remark 9.3.5.2. In the situation of Theorem **??**, the hypothesis that the 0-truncation of X has the resolution property is satisfied whenever X itself has the resolution property (Corollary 9.3.3.9).

Remark 9.3.5.3. Theorem 9.3.5.1 is a "derived" analogue of a result of Schäppi (see [178]), which we will discuss in §9.7; see Theorem **??**.

Proof of Theorem 9.3.5.1. Choose a faithfully flat map $f : U \to X$, where U is affine. By virtue of Theorem 9.3.0.3, it will suffice to show that if $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is a symmetric monoidal functor which preserves small colimits and connective objects, then $F(\mathscr{O}_U) \in \mathrm{CAlg}(\mathrm{QCoh}(Y))$ is faithfully flat. To verify this condition, we may assume without loss of generality that $Y \simeq \operatorname{Spec} A$ is affine. Let us abuse notation by identifying F with a symmetric monoidal functor $\mathrm{QCoh}(X) \to \mathrm{Mod}_A$. Set $B = F(\pi_0 \mathscr{O}_X)$. Since F is right t-exact, the canonical map $A \simeq F(\mathscr{O}_X) \to F(\pi_0 \mathscr{O}_X) = B$ has 1-connective fiber. In particular, it induces an isomorphism of commutative rings $\pi_0 A \to \pi_0 B$. It follows that an A-algebra A' is faithfully flat if and only if the tensor product $B \otimes_A A'$ is faithfully flat over B. Note that F induces a functor $F' : \mathrm{Mod}_{\pi_0 \mathscr{O}_X}(\mathrm{QCoh}(X)) \to \mathrm{Mod}_B$ which fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{F} & \mathrm{Mod}_A \\ \downarrow & & \downarrow{\scriptstyle B\otimes_A} \\ \mathrm{Mod}_{\pi_0 \mathscr{O}_X}(\mathrm{QCoh}(X)) & \xrightarrow{F'} & \mathrm{Mod}_B . \end{array}$$

Here we can identify $\mathrm{Mod}_{\pi_0 \mathscr{O}_X}(\mathrm{QCoh}(X))$ with the ∞-category of quasi-coherent sheaves on the 0-truncation X_0 of X. Replacing X by X_0 and F by F', we are reduced to proving Theorem 9.3.5.1 in the special case where X is 0-truncated. In this case, Proposition 9.3.4.11 implies that we can write $f_* \mathscr{O}_U$ can be written as the colimit of a diagram $\{\mathscr{E}_\alpha\}_{\alpha \in I}$ indexed by a filtered partially ordered set I, where each $\mathscr{E}_\alpha$ belongs to $\mathrm{Vect}(X)$. Our assumption that X is 0-truncated implies that each $\mathscr{E}_\alpha$ belongs to the heart of $\mathrm{QCoh}(X)$. Since $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)^\heartsuit$ (Proposition 9.1.5.1), we may assume that the unit map $e : \mathscr{O}_X \to f_* \mathscr{O}_U$ factors as a composition $\mathscr{O}_X \xrightarrow{e_\alpha} \mathscr{E}_\alpha \to f_* \mathscr{O}_X$ for some $\alpha \in I$. For $\beta \geqslant \alpha$, let e_β denote the composition of e_α with the transition map $\mathscr{E}_\alpha \to \mathscr{E}_\beta$, so that we have an equivalence $\mathrm{cofib}(e) \simeq \varinjlim_{\beta \geqslant \alpha} \mathrm{cofib}(e_\beta)$. By virtue of Lemma D.4.4.3, to prove that $F(f_* \mathscr{O}_U)$ is faithfully flat, it will suffice to show that $\mathrm{cofib}\, F(e) \simeq \varinjlim_{\beta \geqslant \alpha} F(\mathrm{cofib}(e_\beta)) \in \mathrm{Mod}_A$ is flat. Since the collection of flat objects of Mod_A is closed under filtered colimits, it will suffice to show that each $F(\mathrm{cofib}(e_\beta) \in \mathrm{Mod}_A$ is flat. In fact, we will prove that each $F(\mathrm{cofib}(e_\beta))$ is locally free of finite rank: that is, that it is a dualizable object of $\mathrm{Mod}_A^{\mathrm{cn}}$ (see Proposition 6.2.6.3). By assumption, the functor F is symmetric monoidal and preserves connective objects; it will therefore suffice to show that each $\mathrm{cofib}(e_\beta)$ is a dualizable object of $\mathrm{QCoh}(X)^{\mathrm{cn}}$.

Note that the composite map $f^* \mathscr{O}_X \xrightarrow{f^* e_\beta} f^* \mathscr{E}_\alpha \to f^* f_* \mathscr{O}_U \to \mathscr{O}_U$ is an equivalence, so that $f^* e_\alpha$ admits a left homotopy inverse. It follows that $\mathrm{cofib}(f^* e_\alpha) \simeq f^* \mathrm{cofib}(e_\beta)$ is a direct summand of $f^* \mathscr{E}_\alpha$, and is therefore locally free of finite rank. Since the map f is faithfully flat, it follows that $\mathrm{cofib}(e_\beta)$ is locally free of finite rank, and therefore dualizable as an object of $\mathrm{QCoh}(X)^{\mathrm{cn}}$ by virtue of Proposition 6.2.6.3. □

9.3.6 Example: Classifying Stacks over Dedekind Rings

The following observation furnishes many examples of geometric stacks with the resolution property:

Proposition 9.3.6.1. *Let X be a geometric stack. Suppose that there exists a faithfully flat morphism $q : \operatorname{Spec} R \to X$, where R is a Dedekind ring. Then the inclusion $\mathrm{Vect}(X) \hookrightarrow \mathrm{QCoh}(X)$ induces an equivalence of ∞-categories $\mathrm{Ind}(\mathrm{Vect}(X)) \simeq \mathrm{QCoh}(X)^\flat$; here $\mathrm{QCoh}(X)^\flat$ denotes the full subcategory of $\mathrm{QCoh}(X)$ spanned by the flat objects.*

Corollary 9.3.6.2. *Let X be a geometric stack. If there exists a faithfully flat morphism $\operatorname{Spec} R \to X$, where R is a Dedekind ring, then X has the resolution property.*

Proof. Combine Propositions 9.3.6.1 and 9.3.4.10. □

Example 9.3.6.3. Let R be a Dedekind ring and let G be a flat affine group scheme over R. Then the classifying stack BG (see Example 9.1.1.7) has the resolution property.

Proof of Proposition 9.3.6.1. Let $U : \mathrm{QCoh}(X)^\heartsuit \to \mathrm{QCoh}(X)^\heartsuit$ denote the functor given by $q_* q^*$. For any object $\mathscr{G} \in \mathrm{QCoh}(X)^\heartsuit$, we can identify $\mathscr{G}$ with the totalization of the cosimplicial object $U^{\bullet+1}\mathscr{G}$. For each $\mathscr{F} \in \mathrm{QCoh}(X)^\heartsuit$, we have a canonical bijection

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{QCoh}(X)^\heartsuit}(\mathscr{F}, \mathscr{G}) &\simeq \mathrm{Tot}(\mathrm{Hom}_{\mathrm{QCoh}(X)^\heartsuit}(\mathscr{F}, U^{\bullet+1}\mathscr{G}) \\
&\simeq \ker(\mathrm{Hom}_{\mathrm{QCoh}(X)^\heartsuit}(\mathscr{F}, U\mathscr{G}) \to \mathrm{Hom}_{\mathrm{QCoh}(X)^\heartsuit}(\mathscr{F}, U^2\mathscr{G})) \\
&\simeq \ker(\mathrm{Hom}_{\mathrm{Mod}_R^\heartsuit}(q^*\mathscr{F}, q^*\mathscr{G}) \to \mathrm{Hom}_{\mathrm{Mod}_R^\heartsuit}(q^*\mathscr{F}, q^* q_* q^*\mathscr{G})).
\end{aligned}$$

If $\mathscr{F}$ is locally free of finite rank, then $q^*\mathscr{F}$ is compact as an object of $\mathrm{Mod}_R^\heartsuit$, so that the constructions

$$\mathscr{G} \mapsto \mathrm{Hom}_{\mathrm{Mod}_R^\heartsuit}(q^*\mathscr{F}, q^*\mathscr{G}) \qquad \mathscr{G} \mapsto \mathrm{Hom}_{\mathrm{Mod}_R^\heartsuit}(q^*\mathscr{F}, q^* q_* q^*\mathscr{G})$$

commute with filtered colimits. It follows that the construction $\mathscr{G} \mapsto \mathrm{Hom}_{\mathrm{QCoh}(X)^\heartsuit}(\mathscr{F}, \mathscr{G})$ also commutes with filtered colimits: that is, $\mathscr{F}$ is a compact object of the abelian category $\mathrm{QCoh}(X)^\heartsuit$, and therefore also as an object of $\mathrm{QCoh}(X)^\flat$. It follows that the construction $\theta : \mathrm{Ind}(\mathrm{Vect}(X)) \to \mathrm{QCoh}(X)^\flat$ is fully faithful.

We now prove that θ is essentially surjective. Let $\mathscr{F} \in \mathrm{QCoh}(X)^{\flat}$, so that $M = q^* \mathscr{F}$ is a flat R-module. Write M as a union $\bigcup M_\alpha$, where each M_α is a finitely generated submodule of M. For each index α, set $\mathscr{F}_\alpha = \mathscr{F} \times_{q_* M} q_* M_\alpha$, where the fiber product is formed in the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$. Since $\mathrm{QCoh}(X)^{\heartsuit}$ is a Grothendieck abelian category, the formation of filtered colimits in $\mathrm{QCoh}(X)^{\heartsuit}$ commutes with finite limits. We therefore have an equivalence $\mathscr{F} \simeq \varinjlim \mathscr{F}_\alpha$. To complete the proof, it will suffice to show that each $\mathscr{F}_\alpha$ is locally free of finite rank. In other words, we wish to show that $q^* \mathscr{F}_\alpha$ is locally free of finite rank when regarded as an R-module. Note that the projection map $\mathscr{F}_\alpha \to \mathscr{F}$ induces a monomorphism $q^* \mathscr{F}_\alpha \to q^* \mathscr{F} \simeq M$ which factors through M_α, so that $q^* \mathscr{F}_\alpha$ is a submodule of M_α. Since M_α is finitely generated and R is Noetherian, it follows that $q^* \mathscr{F}_\alpha$ is also finitely generated as an R-module. The flatness $\mathscr{F}$ guarantees that $M = q^* \mathscr{F}$ is a torsion free R-module, so that the submodule $q^* \mathscr{F}_\alpha$ is also torsion-free and therefore a locally free R-module (by virtue of our assumption that R is a Dedekind ring). □

9.3.7 Restriction to Vector Bundles

Let X be a geometric stack. It follows from Proposition 9.2.2.1 that an A-valued point $\eta \in X(A)$ is determined (up to contractible ambiguity) by the symmetric monoidal pullback functor $\eta^* : \mathrm{QCoh}(X) \to \mathrm{Mod}_A$. In fact, we can often make do with less: if X has "enough" vector bundles, then η is determined by the functor $\eta^*|_{\mathrm{Vect}(X)}$.

Theorem 9.3.7.1 (Tannaka Duality, Vector Bundle Version)**.** *Let X be a geometric stack which has the resolution property. Assume either that X is n-truncated for some integer $n \geqslant 0$ or that $\mathscr{O}_X$ is a compact object of* $\mathrm{QCoh}(X)$*. Then, for any functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, the construction $(f : Y \to X) \mapsto (f^* : \mathrm{Vect}(X) \to \mathrm{Vect}(Y))$ determines a fully faithful embedding*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{Vect}(X), \mathrm{Vect}(Y))$$

whose essential image is spanned by those symmetric monoidal functors $f : \mathrm{Vect}(X) \to \mathrm{Vect}(Y)$ which satisfy the following condition:

(∗) *The functor f preserves zero objects and, for every diagram σ :*

$$\begin{array}{ccc} \mathscr{E}' & \longrightarrow & \mathscr{E} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{E}'' \end{array}$$

in $\mathrm{Vect}(X)$ *which is a cofiber sequence* $\mathrm{QCoh}(X)^{\mathrm{cn}}$*, the image $f(\sigma)$ is a cofiber sequence in* $\mathrm{QCoh}(Y)^{\mathrm{cn}}$*.*

Remark 9.3.7.2. In the special case where X is 0-truncated, Theorem **??** was proven by Schäppi in [178].

Before giving the proof of Theorem 9.3.7.1, let us describe some of its consequences.

Corollary 9.3.7.3. *Let R be a Dedekind ring, let G be a flat affine group scheme over R, and let $\mathrm{Rep}(G)$ denote the category of algebraic representations of G which are projective R-modules of finite rank (which we identify with the full subcategory $\mathrm{Vect}(BG) \subseteq \mathrm{QCoh}(BG)$). Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be an arbitrary functor. Then the canonical map*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, BG) \to \mathrm{Fun}^{\otimes}(\mathrm{Rep}(G), \mathrm{Vect}(Y))$$

is a fully faithful embedding, whose essential image is the full subcategory spanned by those symmetric monoidal functors $F : \mathrm{Rep}(G) \to \mathrm{Vect}(Y)$ which carry exact sequences

$$0 \to V' \to V \to V'' \to 0$$

in $\mathrm{Rep}(G)$ to fiber sequences in $\mathrm{QCoh}(Y)$.

Proof. Combine Theorem 9.3.7.1 with Proposition 9.3.6.1. □

Warning 9.3.7.4. In the situation of Corollary 9.3.7.3, the exactness condition on the functor $F : \mathrm{Rep}(G) \to \mathrm{Vect}(Y)$ is not automatic. For example, suppose that $Y = \mathrm{Spec}\, \mathbf{C}[x, y]$ is an affine space of dimension 2, and that $U \subseteq Y$ is the open subset obtained from Y by deleting the origin. Then the restriction map $\mathrm{Vect}(Y) \to \mathrm{Vect}(U)$ is an equivalence of symmetric monoidal categories. However, there can exist G-torsors $\mathcal{P}$ on U which do not extend to G-torsors on Y. For example, if $G = \mathbf{G}_a$ denotes the additive group (regarded as a group scheme over $\mathbf{C}$, say), then every G-torsor on Y is trivial (since Y is affine), while G-torsors on U are classified up to isomorphism by the cohomology group $\mathrm{H}^1(U; \mathscr{O}_U)$ (which is an infinite-dimensional vector space over $\mathbf{C}$). Nontrivial elements of $\mathrm{H}^1(U; \mathscr{O}_U)$ are classified by symmetric monoidal functors $\mathrm{Rep}(G) \to \mathrm{Vect}(U) \simeq \mathrm{Vect}(Y)$ which carry exact sequences in $\mathrm{Rep}(G)$ to diagrams $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ in $\mathrm{Vect}(Y)$ which are not necessarily exact (but are exact when restricted to U).

From Corollary **??**, we immediately deduce a slightly stronger version of Theorem 9.0.0.1:

Corollary 9.3.7.5. *Let R be a Dedekind ring, let G be a flat affine group scheme over R, let A be a commutative R-algebra, and let $e : \mathrm{Rep}(G) \to \mathrm{Mod}_A^{\mathrm{lf}}$ be the symmetric monoidal functor given by $V \mapsto A \otimes_R V$. Then the canonical map $G(A) \to \mathrm{Aut}^{\otimes}(e)$ is an isomorphism, where $\mathrm{Aut}^{\otimes}(e)$ denotes the automorphism group of e in the category $\mathrm{Fun}^{\otimes}(\mathrm{Rep}(G), \mathrm{Mod}_A^{\mathrm{lf}})$ of symmetric monoidal functors from $\mathrm{Rep}(G)$ to $\mathrm{Mod}_A^{\mathrm{lf}}$.*

Corollary 9.3.7.6. *Let X be a geometric stack which has the resolution property. Assume either that X is n-truncated for some integer $n \geqslant 0$ or that $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$. Let R be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely* *generated ideal I, let Y be a spectral algebraic spaces which is proper and almost of finite presentation over R, and let $\mathsf{Y}^\wedge = \mathsf{Y} \times_{\operatorname{Spec} R} \operatorname{Spf} R$ be the formal completion of Y along the vanishing locus of I. Let us abuse notation by regarding Y and $\mathsf{Y}^\wedge$ with the functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ that they represent. Then the canonical map*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathsf{Y}, X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathsf{Y}^\wedge, X)$$

is a homotopy equivalence.

Proof. By virtue of Theorem 9.3.7.1, it will suffice to show that the restriction map $\mathrm{Vect}(\mathsf{Y}) \to \mathrm{Vect}(\mathsf{Y}^\wedge)$ is an equivalence of ∞-categories. Note that $\mathrm{Vect}(\mathsf{Y})$ and $\mathrm{Vect}(\mathsf{Y}^\wedge)$ can be identified with the full subcategories of $\mathrm{QCoh}(\mathsf{Y})^{\mathrm{aperf},\mathrm{cn}}$ and $\mathrm{QCoh}(\mathsf{Y}^\wedge)^{\mathrm{aperf},\mathrm{cn}}$ spanned by the dualizable objects. We conclude by observing that Theorem 8.5.0.3 (together with Theorems 8.3.4.4 and 8.3.5.2) and Proposition 8.5.1.4 imply that the restriction functor $\mathrm{QCoh}(\mathsf{Y})^{\mathrm{aperf},\mathrm{cn}} \to \mathrm{QCoh}(\mathsf{Y}^\wedge)^{\mathrm{aperf},\mathrm{cn}}$ is an equivalence of ∞-categories. □

The proof of Theorem 9.3.7.1 is based on the following:

Proposition 9.3.7.7. *Let X be a geometric stack which has the resolution property. Assume either that X is n-truncated for some integer $n \geqslant 0$ or that $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$. Let $G : \mathrm{QCoh}(X)^{\mathrm{cn}} \to \mathrm{Fun}^{\pi}(\mathrm{Vect}(X)^{\mathrm{op}}, \mathcal{S})$ be the functor given by $G(\mathscr{F})(\mathscr{E}) = \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{E}, \mathscr{F})$. Then, for every integer $m \geqslant 0$, the functor G restricts to a fully faithful embedding*

$$\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant m} \hookrightarrow \mathrm{Fun}^{\pi}(\mathrm{Vect}(X)^{\mathrm{op}}, \tau_{\leqslant m} \mathcal{S})$$

whose essential image is spanned by those functors $\lambda : \mathrm{Vect}(X)^{\mathrm{op}} \to \tau_{\leqslant m} \mathcal{S}$ which satisfy the following additional condition:

($\star$) *The space $\lambda(0)$ is contractible and, for every diagram σ :*

$$\begin{array}{ccc} \mathscr{E}' & \longrightarrow & \mathscr{E} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{E}'' \end{array}$$

in $\mathrm{Vect}(X)$ which is a cofiber sequence $\mathrm{QCoh}(X)^{\mathrm{cn}}$, the diagram $\lambda(\sigma)$ is a pullback square in $\mathcal{S}$.

Proof. Set $\mathcal{C} = \mathrm{Fun}^{\pi}(\mathrm{Vect}(X)^{\mathrm{op}}, \mathcal{S})$, and let $F : \mathcal{C} \to \mathrm{QCoh}(X)^{\mathrm{cn}}$ be left adjoint to G. Using our assumption that X has the resolution property (together with Proposition 9.3.3.7), we see that $\mathrm{Vect}(X) \hookrightarrow \mathrm{QCoh}(X)^{\mathrm{cn}}$ satisfies the hypothesis $(*_k)$ of Propositions C.2.5.2 and C.2.5.3, where we take $k = \infty$ if $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$, or any finite $k \geqslant n$ if X is n-truncated. We therefore obtain the following:

(i) The functor $F : \mathcal{C} \to \mathrm{QCoh}(X)^{\mathrm{cn}}$ is left exact.

(ii) The functor G is fully faithful when restricted to $\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant m}$, for any $m \geqslant 0$. (moreover, if $\mathscr{O}_X$ is compact, then the functor G itself is fully faithful).

Let $\mathcal{C}_0 \subseteq \mathcal{C}$ denote the full subcategory spanned by the m-truncated objects functors which satisfy condition $(\star)$, and let $\mathcal{C}_1$ be the essential image of $G|_{\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant m}}$. We wish to show that $\mathcal{C}_0 = \mathcal{C}_1$. The inclusion $\mathcal{C}_1 \subseteq \mathcal{C}_0$ follows immediately from the definition. To prove the reverse inclusion, we observe that for every object $C \in \mathcal{C}_0$, the unit map $u : C \to (G \circ F)(C)$ is m-truncated (since the domain and codomain of u are m-truncated), the codomain of u belongs to $\mathcal{C}_1$, and $F(u)$ is an equivalence (since the functor G is fully faithful). It will therefore suffice to prove the following:

(a_k) Let $f : C \to D$ be a k-truncated morphism between objects of $\mathcal{C}_0$. If $F(f)$ is an equivalence, then f is an equivalence.

Our proof of (a_k) proceeds by induction on k. If $k = -2$, then the assumption that f is k-truncated guarantees that f is an equivalence, so there is nothing to prove. Let us therefore assume that $k \geqslant -1$. Our assumption that f is k-truncated implies that the diagonal map $\delta : C \to C \times_D C$ is $(k-1)$-truncated. Using (i), we deduce that $F(\delta)$ is an equivalence, so that our inductive hypothesis guarantees that δ is an equivalence. It follows that the morphism $f : C \to D$ is (-1)-truncated: that is, it induces a homotopy equivalence $C(\mathscr{E}) \to D(\mathscr{E})$ for each object $\mathscr{E} \in \mathrm{Vect}(X)$. To complete the proof that f is an equivalence, it will suffice to show that each point $\eta \in D(\mathscr{E})$ can be lifted to a point of $C(\mathscr{E})$. To prove this, we can replace D by $\mathscr{E}$ and C by the fiber product $C \times_D \mathscr{E}$ and thereby reduce to the case where η classifies an equivalence $\mathscr{E} \to D$ in the ∞-category $\mathcal{C}$.

Choose a collection of objects $\{\mathscr{E}_i\}_{i \in I}$ in $\mathrm{Vect}(X)$ and morphisms $\phi_i : \mathscr{E}_i \to C$ which induce an epimorphism $\pi_0(\bigoplus \mathscr{E}_i) \to \pi_0 C$ in the abelian category $\mathcal{C}^{\heartsuit}$. Since $F(f)$ is an equivalence, it follows that the composite map

$$\psi : \bigoplus_{i \in I} \mathscr{E}_i \xrightarrow{F(\phi_i)} F(C) \xrightarrow{F(f)} F(D) = \mathscr{E}$$

induces an epimorphism on π_0 in the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$. We may therefore choose a finite subset $I_0 \subseteq I$ for which $\psi|_{\bigoplus_{i \in I_0} \mathscr{E}_i}$ induces an epimorphism on π_0. Set $\overline{\mathscr{E}} = \bigoplus_{i \in I_0} \mathscr{E}_i$. Amalgamating the maps $\{\phi_i\}_{i \in I_0}$, we obtain a morphism $\phi : \overline{\mathscr{E}} \to \mathscr{E}$ in $\mathrm{Vect}(X)$ which induces an epimorphism on π_0, so that the fiber $\mathrm{fib}(\phi)$ belongs to $\mathrm{Vect}(X)$. Since $C, D \in \mathcal{C}_0$, we have a commutative diagram of fiber sequences

$$\begin{array}{ccccc} C(\mathscr{E}) & \longrightarrow & C(\overline{\mathscr{E}}) & \longrightarrow & C(\mathrm{fib}(\phi)) \\ \downarrow & & \downarrow & & \downarrow \\ D(\mathscr{E}) & \longrightarrow & D(\overline{\mathscr{E}}) & \longrightarrow & D(\mathrm{fib}(\phi)) \end{array}$$

where the vertical maps are induced by f and therefore have (-1)-truncated homotopy fibers. By construction, the image of η in $D(\overline{\mathscr{E}})$ can be lifted to a point of $C(\overline{\mathscr{E}})$, so that η can be lifted to a point of $C(\mathscr{E})$ as desired. □

Proof of Theorem 9.3.7.1. Let X be a geometric stack with the resolution property. If $\mathcal{D}$ is an ∞-category which admits small colimits, we let $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Vect}(X), \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{Vect}(X), \mathcal{D})$ spanned by those symmetric monoidal functors f which satisfy the following additional condition:

$(*_{\mathcal{D}})$ The functor f preserves initial objects and, for every diagram σ :

$$\begin{array}{ccc} \mathscr{E}' & \longrightarrow & \mathscr{E} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{E}'' \end{array}$$

in $\mathrm{Vect}(X)$ which is a cofiber sequence $\mathrm{QCoh}(X)^{\mathrm{cn}}$, the image $f(\sigma)$ is a cofiber sequence in $\mathrm{QCoh}(Y)^{\mathrm{cn}}$.

Note that when $\mathcal{D} = \mathrm{QCoh}(Y)^{\mathrm{cn}}$, then any symmetric monoidal functor $f : \mathrm{Vect}(X) \to \mathcal{D}$ automatically factors through the full subcategory $\mathrm{Vect}(Y) \subseteq \mathrm{QCoh}(Y)^{\mathrm{cn}}$.

Assume now that either X is n-truncated for some integer n, or that the structure sheaf $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$. We wish to show that the canonical map

$$\rho : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X) \to \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Vect}(X), \mathrm{QCoh}(Y)^{\mathrm{cn}})$$

is an equivalence of ∞-categories. Without loss of generality, we may assume that $Y = \operatorname{Spec} R$ is affine. Using Theorem 9.3.5.1, we can identify the domain of ρ with the ∞-category $\mathrm{LFun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}, \mathrm{QCoh}(Y)^{\mathrm{cn}})$ of colimit-preserving symmetric monoidal functors from $\mathrm{QCoh}(X)^{\mathrm{cn}}$ to $\mathrm{QCoh}(Y)^{\mathrm{cn}}$. In this case, we can identify ρ with the limit of a tower of forgetful functors

$$\rho_m : \mathrm{LFun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}, \mathrm{QCoh}(Y)^{\mathrm{cn}}_{\leqslant m}) \to \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Vect}(X), \mathrm{QCoh}(Y)^{\mathrm{cn}}_{\leqslant m}).$$

It will therefore suffice to show that each ρ_m is an equivalence of ∞-categories.

Let $\mathcal{C} = \mathcal{P}_{\Sigma}(\mathrm{Vect}(X))$ be the full subcategory $\mathrm{Fun}^{\pi}(\mathrm{Vect}(X)^{\mathrm{op}}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathrm{Vect}(X)^{\mathrm{op}}, \mathcal{S})$ spanned by those functors which preserve finite products. Let us abuse notation by identifying $\mathrm{Vect}(X)$ with its essential image under the Yoneda embedding $\mathrm{Vect}(X) \to \mathcal{C}$. Using the results of §**??**, we see that if $\mathcal{D}$ is any presentable symmetric monoidal ∞-category (whose tensor product preserves small colimits separately in each variable), then the restriction functor $\theta : \mathrm{LFun}^{\otimes}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\otimes}(\mathrm{Vect}(X), \mathcal{D})$ is fully faithful, and its essential image is spanned by those symmetric monoidal functors $\mathrm{Vect}(X) \to \mathcal{D}$ which preserve finite coproducts. In particular, the inclusion $\mathrm{Vect}(X) \hookrightarrow \mathrm{QCoh}(X)^{\mathrm{cn}}$ determines an object $F : \mathrm{LFun}^{\otimes}(\mathcal{C}, \mathrm{QCoh}(X)^{\mathrm{cn}})$. Let $\mathrm{LFun}^{\otimes}_0(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{LFun}^{\otimes}(\mathcal{C}, \mathcal{D})$ spanned by those functors which satisfy condition $(*_{\mathcal{D}})$, so that θ restricts to an equivalence of ∞-categories $\mathrm{LFun}^{\otimes}_0(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Vect}(X), \mathcal{D})$. Under this equivalence, we see can identify ρ_m with the functor

$$\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}, \mathrm{QCoh}(Y)^{\mathrm{cn}}_{\leqslant m}) \to \mathrm{Fun}^{\otimes}_0(\mathcal{C}, \mathrm{QCoh}(Y)^{\mathrm{cn}}_{\leqslant m})$$

given by composition with F. It follows from Proposition 9.3.7.7 that this functor is an equivalence of ∞-categories. □

9.4 Perfect Stacks

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a quasi-geometric stack. For every connective $\mathbb{E}_\infty$-ring R, Theorem 9.2.0.2 supplies a fully faithful embedding $X(R) \hookrightarrow \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{Mod}_R)$, whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_R$ which satisfy the following conditions:

(1) The functor F preserves small colimits and is right t-exact.

(2) The functor F admits a right adjoint G which preserves small colimits and satisfies a projection formula.

In this section, we will show that for a large class of quasi-geometric stacks X, condition (2) is a formal consequence of (1). To get a sense of the ideas involved, note that if the stable ∞-category $\mathrm{QCoh}(X)$ is compactly generated, then a colimit-preserving functor $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_R$ preserves compact objects if and only if it has a right adjoint G which preserves filtered colimits (hence all colimits, since G is an exact functor between stable ∞-categories). Consequently, to simplify the hypotheses of Theorem 9.2.0.2, it is convenient to assume that the ∞-category $\mathrm{QCoh}(X)$ is compactly generated. Following [23], we will say that a quasi-geometric stack X is *perfect* if the canonical map $\mathrm{Ind}(\mathrm{QCoh}(X)^{\mathrm{perf}}) \to \mathrm{QCoh}(X)$ is an equivalence of ∞-categories (see Definition 9.4.4.1). A result of Bhatt and Halpern-Leistner asserts that if X is perfect, then a symmetric monoidal functor $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is given by pullback along a map $f : Y \to X$ if and only if F preserves small colimits and connective objects (Corollary 9.4.4.7). In this section, we will introduce several *a priori* weaker conditions on X which guarantee the same result, which are related as follows:

$$\begin{array}{c} X \text{ is perfect} \\ \Downarrow \\ X \text{ is weakly perfect} \\ \Downarrow \\ \mathrm{QCoh}(X) \text{ is dualizable} \\ \Downarrow \\ X \text{ has the diagonal approximation property.} \end{array}$$

Remark 9.4.0.1. We do not know which (if any) of the above implications are reversible.

9.4.1 The Diagonal Approximation Property

Let $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ whose objects are presentable stable ∞-categories and whose morphisms are functors which preserve small colimits. We will regard $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ as equipped with the symmetric monoidal structure described in §HA.4.8.2 : if $\mathcal{C}$ and $\mathcal{D}$ are presentable stable ∞-categories, then the tensor product $\mathcal{C} \otimes \mathcal{D}$ is universal among presentable stable ∞-categories $\mathcal{E}$ for which there exists a map $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserves small colimits separately in each variable.

Construction 9.4.1.1. Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be quasi-geometric stacks, and consider the projection maps $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$. For every pair of quasi-coherent sheaves $\mathscr{F} \in \mathrm{QCoh}(X)$ and $\mathscr{G} \in \mathrm{QCoh}(Y)$, we let $\mathscr{F} \boxtimes \mathscr{G}$ denote the tensor product $p^* \mathscr{F} \otimes q^* \mathscr{G} \in \mathrm{QCoh}(X \times Y)$. We will refer to $\mathscr{F} \boxtimes \mathscr{G}$ as the *external tensor product of* $\mathscr{F}$ *and* $\mathscr{G}$. The construction $(\mathscr{F}, \mathscr{G}) \mapsto \mathscr{F} \boxtimes \mathscr{G}$ determines a functor $\boxtimes : \mathrm{QCoh}(X) \times \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X \times Y)$ which preserves colimits separately in each variable, and is therefore classified by a colimit-preserving functor $e_{X,Y} : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X \times Y)$.

Definition 9.4.1.2. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a quasi-geometric stack. We will say that X has the *diagonal approximation property* if the functor $e_{X,X} : \mathrm{QCoh}(X) \times \mathrm{QCoh}(X) \to \mathrm{QCoh}(X \times X)$ of Construction 9.4.1.1 is an equivalence of ∞-categories.

Remark 9.4.1.3. Let X be a quasi-geometric stack and let $\delta : X \to X \times X$ denote the diagonal map. The terminology of Definition 9.4.1.2 is motivated by the observation that if X has the diagonal approximation property, then $\delta_* \mathscr{O}_X$ belongs to the essential image of the functor $e_{X,X} : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \to \mathrm{QCoh}(X \times X)$ (in other words, the structure sheaf of the diagonal of X can be "built from" quasi-coherent sheaves which are arise as pullbacks along the projection maps $X \leftarrow X \times X \to X$). We will see later that under a mild additional hypothesis, the converse holds: that is, if $\delta_* \mathscr{O}_X$ belongs to the essential image of $e_{X,X}$, then $e_{X,X}$ is an equivalence of ∞-categories (Corollary 9.4.3.2).

Our main interest in Definition 9.4.1.2 is due to the following:

Theorem 9.4.1.4 (Tannaka Duality, Diagonal Approximation Version)**.** *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be functors. Assume that X is a quasi-geometric stack with the diagonal approximation property. Then the construction $(f : Y \to X) \mapsto (f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y))$ determines a fully faithful embedding*

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$$

whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which preserve small colimits and connective objects.

Proof. Writing Y as a (not necessarily small) colimit of corepresentable functors, we can reduce to the case where $Y = \mathrm{Spec}\, R$ for some connective $\mathbb{E}_\infty$-ring R. In this case, Theorem 9.2.0.2 implies that the functor θ is a fully faithful embedding whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which preserve small colimits, connective objects, and admit a right adjoint G which preserves small colimits and satisfies the projection formula. To prove Theorem 9.4.1.4, it will suffice to verify the following:

($*$) Let $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ be a symmetric monoidal functor which preserves small colimits. Then F admits a right adjoint $G : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ which also preserves small colimits. Moreover, the functor G is automatically $\mathrm{QCoh}(X)$-linear (that is, G satisfies the projection formula).

To prove ($*$), we note that the functor F factors as a composition

$$\mathrm{QCoh}(X) \xrightarrow{F'} \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y) \xrightarrow{F''} \mathrm{QCoh}(Y),$$

where F' is obtained by tensoring the identity functor $\mathrm{id}_{\mathrm{QCoh}(X)}$ with the unit map $e : \mathrm{Sp} \to \mathrm{QCoh}(Y)$ classifying the structure sheaf $\mathscr{O}_Y$, while F'' classifies the functor $\mathrm{QCoh}(X) \times \mathrm{QCoh}(Y) \to \mathrm{QCoh}(Y)$ given by $(\mathscr{F}, \mathscr{G}) \mapsto F(\mathscr{F}) \otimes \mathscr{G}$. Let us regard $\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y)$ as a $\mathrm{QCoh}(X)$-module object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ by allowing $\mathrm{QCoh}(X)$ to act on the first factor (so that $\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y)$ is the free $\mathrm{QCoh}(X)$-module generated by $\mathrm{QCoh}(Y)$). To prove ($*$), it will suffice to verify the following:

($*'$) The functor $F' : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y)$ admits a right adjoint $G' : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ which preserves small colimits and is $\mathrm{QCoh}(X)$-linear.

$(*'')$ The functor $F'' : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y) \to \mathrm{QCoh}(Y)$ admits a right adjoint $G'' : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y)$ which preserves small colimits and is $\mathrm{QCoh}(X)$-linear.

To prove $(*')$, we note that the unit map $e : \mathrm{Sp} \to \mathrm{QCoh}(Y)$ admits a right adjoint $\Gamma : \mathrm{QCoh}(Y) \to \mathrm{Sp}$. Since Y is affine, the global sections functor Γ preserves small colimits, and can therefore be identified with a morphism in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. Tensoring Γ with the identity functor $\mathrm{id}_{\mathrm{QCoh}(X)}$, we obtain a $\mathrm{QCoh}(X)$-linear functor $G' : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$, which is easily seen to be a left adjoint to F'.

The proof of $(*'')$ is similar. Let $\mathcal{C}$ denote the tensor product $\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X)$ and let $m : \mathcal{C} \to \mathrm{QCoh}(X)$ classify the tensor product functor $\otimes : \mathrm{QCoh}(X) \times \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$. Note that the functor m factors as a composition

$$\mathcal{C} = \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \xrightarrow{e_{X,X}} \mathrm{QCoh}(X \times X) \xrightarrow{\delta^*} \mathrm{QCoh}(X),$$

where $e_{X,X}$ is the functor given in Construction 9.4.1.1 and $\delta : X \to X \times X$ is the diagonal. Since X is quasi-geometric, the map δ is quasi-affine. It follows functor δ^* has a right adjoint given by the direct image functor $\delta_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X \times X)$ (Proposition 6.3.4.1), which is colimit-preserving and $\mathrm{QCoh}(X \times X)$-linear by virtue of Corollary 6.3.4.3. Because X has the diagonal approximation property, the functor $e_{X,X}$ is an equivalence of ∞-categories. It follows that the functor $m : \mathcal{C} \to \mathrm{QCoh}(X)$ admits a right adjoint $m' : \mathrm{QCoh}(X) \to \mathcal{C}$ which preserves small colimits and is $\mathcal{C}$-linear. Then m' induces a colimit-preserving, $\mathcal{C}$-linear functor

$$\begin{aligned} \mathrm{QCoh}(Y) &\simeq \mathrm{QCoh}(X) \otimes_{\mathcal{C}} (\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y)) \\ &\xrightarrow{m' \otimes \mathrm{id}} \mathcal{C} \otimes_{\mathcal{C}} (\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y)) \\ &\simeq \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y), \end{aligned}$$

which is easily seen to be right adjoint to F''. □

9.4.2 Dualizability

We now record some observations relating the diagonal approximation property of Definition 9.4.1.2 to the duality theory of presentable stable ∞-categories.

Proposition 9.4.2.1. *Let X be a quasi-geometric stack with the diagonal approximation property. Then the ∞-category $\mathrm{QCoh}(X)$ is smooth when regarded as an algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ (in the sense of Definition HA.4.6.4.13). That is, $\mathrm{QCoh}(X)$ is dualizable when viewed as a module over $\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X)$.*

Proof. Let $\delta : X \to X \times X$ be the diagonal map, and let $\mathscr{A} = \delta_* \mathscr{O}_X$. Since δ is quasi-affine, it induces an equivalence $\mathrm{QCoh}(X) \simeq \mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(X \times X))$ (Proposition 6.3.4.6).

Consequently, $\mathrm{QCoh}(X)$ is dualizable (in fact, it is self-dual) when viewed as a module over $\mathrm{QCoh}(X \times X)$ (see Remark HA.4.8.4.8). If X has the diagonal approximation property, it follows that $\mathrm{QCoh}(X)$ is also dualizable when viewed as a module over $\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X)$. □

Corollary 9.4.2.2. *Let X be a quasi-coherent stack and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$ be a presentable stable ∞-category equipped with an action of $\mathrm{QCoh}(X)$. If X has the diagonal approximation property and $\mathcal{C}$ is dualizable as an object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, then it is also dualizable as an object of $\mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$.*

Corollary 9.4.2.3. *Suppose we are given a pullback diagram of quasi-geometric stacks*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y. \end{array}$$

If Y has the diagonal approximation property and $\mathrm{QCoh}(Y')$ is a dualizable object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, then the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X') & \longleftarrow & \mathrm{QCoh}(X) \\ \uparrow & & \uparrow \\ \mathrm{QCoh}(Y') & \longleftarrow & \mathrm{QCoh}(Y) \end{array}$$

is a pushout square in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$: that is, the canonical map $\theta : \mathrm{QCoh}(Y') \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(X) \to \mathrm{QCoh}(X')$ is an equivalence of ∞-categories.

Proof. Choose a faithfully flat map $f : U_0 \to X$, where U_0 is affine. Let $U_\bullet$ be the Čech nerve of f. Then X can be identified with the geometric realization $|U_\bullet|$ in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$, and X' can be identified with the geometric realization $|Y' \times_Y U_\bullet|$ in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$. In particular, we have equivalences

$$\mathrm{QCoh}(X') \simeq \mathrm{Tot}\,\mathrm{QCoh}(Y' \times_Y U_\bullet) \qquad \mathrm{QCoh}(X) \simeq \mathrm{Tot}\,\mathrm{QCoh}(U_\bullet).$$

Using these equivalences, we can identify θ with the composition

$$\begin{aligned} \mathrm{QCoh}(Y') \otimes_{\mathrm{QCoh}(Y)} \mathrm{Tot}(\mathrm{QCoh}(U_\bullet)) &\xrightarrow{\theta'} \mathrm{Tot}(\mathrm{QCoh}(Y') \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(U_\bullet)) \\ &\xrightarrow{\theta''} \mathrm{Tot}(\mathrm{QCoh}(Y' \times_Y U_\bullet)). \end{aligned}$$

Since $\mathrm{QCoh}(Y)$ is dualizable as an object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, it is also dualizable as a module over $\mathrm{QCoh}(Y)$ (Corollary 9.4.2.2). Consequently, the construction $\mathcal{C} \mapsto \mathrm{QCoh}(Y) \otimes_{\mathrm{QCoh}(Y')} \mathcal{C}$ commutes with limits, so that θ' is an equivalence of ∞-categories. To show that θ'' is an equivalence of ∞-categories, it will suffice to show that the natural map $\mathrm{QCoh}(Y') \otimes_{\mathrm{QCoh}(Y)}$

$\mathrm{QCoh}(U_k) \to \mathrm{QCoh}(Y' \otimes_Y U_k)$ is an equivalence for each $k \geqslant 0$. In other words, to prove Corollary 9.4.2.3 we are free to replace X by U_k, and thereby reduce to the case where X is quasi-affine. In this case, our assumption that Y is quasi-geometric guarantees that the morphism f is quasi-affine. Set $\mathscr{A} = f_* \mathscr{O}_X$, so that Proposition 6.3.4.6 supplies an equivalence $\mathrm{QCoh}(X) \to \mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(Y))$. Similarly, using Proposition 6.3.4.6 and Corollary 6.3.4.3 we obtain a compatible equivalence $\mathrm{QCoh}(X') \simeq \mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(Y'))$. Using these equivalences, we can identify the functor θ with the natural map

$$\mathrm{QCoh}(Y') \otimes_{\mathrm{QCoh}(Y)} \mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(Y)) \to \mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(Y')),$$

which is an equivalence of ∞-categories by virtue of Theorem HA.4.8.4.6. □

Applying Corollary 9.4.2.3 in the special case where $Y = \operatorname{Spec} S$, we obtain the following:

Corollary 9.4.2.4. *Let X and Y be quasi-geometric stacks. If $\mathrm{QCoh}(X)$ is a dualizable object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, then the map $e_{X,Y} : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X \times Y)$ of Construction 9.4.1.1 is an equivalence of ∞-categories.*

Corollary 9.4.2.5. *Let X and Y be quasi-geometric stacks. If $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y)$ are dualizable objects of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, then $\mathrm{QCoh}(X \times Y)$ is a dualizable object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$.*

Corollary 9.4.2.6. *Let X be a quasi-geometric stack. If $\mathrm{QCoh}(X)$ is a dualizable object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, then X has the diagonal approximation property.*

Corollary 9.4.2.7. *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be functors. Assume that X is a quasi-geometric stack and that $\mathrm{QCoh}(X)$ is a dualizable object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. Then the construction $(f : Y \to X) \mapsto (f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y))$ determines a fully faithful embedding*

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$$

whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which preserve small colimits and connective objects.

Proof. Combine Corollary 9.4.2.6 with Theorem 9.4.1.4. □

9.4.3 Weakly Perfect Stacks

One can ask if Proposition 9.4.2.4 has a converse: if X is a quasi-geometric stack with the diagonal approximation property, then is the ∞-category $\mathrm{QCoh}(X)$ dualizable as an object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$? We next show that, under a mild additional hypothesis, the diagonal approximation property ensures not only that $\mathrm{QCoh}(X)$ is dualizable, but that it is (canonically) self-dual (Proposition 9.4.3.1).

Proposition 9.4.3.1. *Let X be a quasi-geometric stack which satisfies the following conditions:*

(1) *The structure sheaf $\mathscr{O}_X$ is a compact object of* $\mathrm{QCoh}(X)$*.*

(2) *Let $\delta : X \to X \times X$ denote the diagonal map. Then $\delta_* \mathscr{O}_X$ belongs to the essential image of the functor $e_{X,X} : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ of Construction 9.4.1.1.*

Let $m : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ classify the tensor product on $\mathrm{QCoh}(X)$. Then the composite functor

$$e : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \xrightarrow{m} \mathrm{QCoh}(X) \xrightarrow{\Gamma(X;\bullet)} \mathrm{Sp}$$

is a duality datum in the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. In particular, $\mathrm{QCoh}(X)$ is a dualizable object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$.

Corollary 9.4.3.2. *Let X be a quasi-geometric stack and suppose that $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$. Then the following conditions are equivalent:*

(a) *The ∞-category $\mathrm{QCoh}(X)$ is a dualizable object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$.*

(b) *The quasi-geometric stack X has the diagonal approximation property.*

(c) *The object $\delta_* \mathscr{O}_X$ belongs to the essential image of the functor $e_{X,X} : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \to \mathrm{QCoh}(X \times X)$.*

Proof. The implication $(a) \to (b)$ follows from Proposition 9.4.2.4, the implication $(b) \Rightarrow (c)$ is tautological, and the implication $(c) \Rightarrow (a)$ follows from Proposition 9.4.3.1. □

Definition 9.4.3.3. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is a *weakly perfect stack* if it satisfies the equivalent conditions of Corollary 9.4.3.2: that is, if X is a quasi-geometric stack with the diagonal approximation property and the structure sheaf $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$.

Proof of Proposition 9.4.3.1. Let X be a quasi-geometric stack and let $C \in \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X)$ satisfy $e_{X,X}(C) \simeq \delta_* \mathscr{O}_X$. The object C classifies a colimit-preserving functor $c : \mathrm{Sp} \to \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X)$. We will show that c is a coevaluation map compatible with the functor $\Gamma(X; \bullet) \circ m$: in other words, that the composition

$$\begin{aligned} \mathrm{QCoh}(X) &\xrightarrow{c \otimes \mathrm{id}} \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \\ &\xrightarrow{\mathrm{id} \otimes m} \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \\ &\xrightarrow{\mathrm{id} \otimes \Gamma} \mathrm{QCoh}(X). \end{aligned}$$

determines a functor $\lambda : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ which is homotopic to the identity map.

Let S denote the sphere spectrum and let $p : X \to \operatorname{Spec} S$ denote the projection map, so that we have a pullback diagram of quasi-geometric stacks σ :

$$\begin{array}{ccc} X \times X & \xrightarrow{\pi_1} & X \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{p} & \operatorname{Spec} S. \end{array}$$

The construction $(\mathscr{F}, \mathscr{G}) \mapsto \mathscr{F} \otimes \pi_2^* \mathscr{G}$ determines a functor $\operatorname{QCoh}(X \times X) \times \operatorname{QCoh}(X) \to \operatorname{QCoh}(X \times X)$ which preserves small colimits separately in each variable, and is therefore classified by a functor $M : \operatorname{QCoh}(X \times X) \otimes \operatorname{QCoh}(X) \to \operatorname{QCoh}(X \times X)$. Consider the diagram of ∞-categories

$$\begin{array}{ccc} \operatorname{QCoh}(X) & \xrightarrow{\delta_* \mathscr{O}_X \otimes \operatorname{id}} & \\ \downarrow & & \searrow \\ \operatorname{QCoh}(X) \times \operatorname{QCoh}(X) \otimes \operatorname{QCoh}(X) & \xrightarrow{e_{X,X} \otimes \operatorname{id}} & \operatorname{QCoh}(X \times X) \otimes \operatorname{QCoh}(X) \\ \downarrow{\scriptstyle \operatorname{id} \otimes m} & & \downarrow{\scriptstyle M} \\ \operatorname{QCoh}(X) \otimes \operatorname{QCoh}(X) & \xrightarrow{e_{X,X}} & \operatorname{QCoh}(X \times X) \\ \downarrow{\scriptstyle \operatorname{id} \otimes \Gamma(X; \bullet)} & \swarrow{\scriptstyle \pi_{1*}} & \\ \operatorname{QCoh}(X). & & \end{array}$$

We claim that it commutes up to canonical homotopy:

- The upper triangle commutes by the definition of c.
- The middle square commutes, since the composition in either direction classifies the functor $\operatorname{QCoh}(X) \times \operatorname{QCoh}(X) \times \operatorname{QCoh}(X) \to \operatorname{QCoh}(X \times X)$ given on objects by $(\mathscr{F}, \mathscr{G}, \mathscr{H}) \mapsto \pi_1^* \mathscr{F} \otimes \pi_2^* \mathscr{G} \otimes \pi_2^* \mathscr{H}$.
- The commutativity of the lower triangle follows from the calculation

$$\begin{aligned} \mathscr{F} \otimes p^* p_* \mathscr{G} &\simeq \mathscr{F} \otimes \pi_{1*} \pi_2^* \mathscr{G} \\ &\simeq \pi_{1*}(\pi_1^* \mathscr{F} \otimes \pi_2^* \mathscr{G}) \end{aligned}$$

for $\mathscr{F}, \mathscr{G} \in \operatorname{QCoh}(X)$, where the equivalences are obtained by applying Proposition 9.1.5.7 to the pullback square σ.

It follows that the functor $\lambda : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ is given by the construction

$$\begin{aligned}\lambda(\mathscr{F}) &= \pi_{1*}(\delta_* \mathscr{O}_X \otimes \pi_2^* \mathscr{F}) \\ &\simeq \pi_{1*}(\delta_*(\mathscr{O}_X \otimes \delta^* \pi_2^* \mathscr{F}) \\ &\simeq \pi_{1*}(\delta_*(\mathscr{O}_X \otimes \mathscr{F}) \\ &\simeq \pi_{1*}\delta_*(\mathscr{F}) \\ &\simeq \mathscr{F},\end{aligned}$$

where the calculation uses the fact that $\pi_1 \circ \delta$ and $\pi_2 \circ \delta$ are both homotopic to the identity map id_X, together with the projection formula for the quasi-affine map δ (Corollary 6.3.4.3). □

Corollary 9.4.3.4. *Let X be a weakly perfect stack. Then the global sections functor $\Gamma : \mathrm{QCoh}(X) \to \mathrm{Sp}$ exhibits $\mathrm{QCoh}(X)$ as a Frobenius algebra object of the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ (see Definition HA.4.6.5.1).*

Corollary 9.4.3.5. *Let X be a weakly perfect stack and suppose we are given a duality datum $e : \mathcal{C} \otimes_{\mathrm{QCoh}(X)} \mathcal{D} \to \mathrm{QCoh}(X)$ in the symmetric monoidal ∞-category $\mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$. Then the composite functor*

$$\mathcal{C} \otimes \mathcal{D} \to \mathcal{C} \otimes_{\mathrm{QCoh}(X)} \mathcal{D} \xrightarrow{e} \mathrm{QCoh}(X) \xrightarrow{\Gamma(X;\bullet)} \mathrm{Sp}$$

is a duality datum in $\mathcal{P}\mathrm{r}^{\mathrm{St}}$.

Proof. Combine Corollary 9.4.3.4 with Corollary HA.4.6.5.14. □

In the situation of Corollary 9.4.3.5, the ∞-category $\mathrm{QCoh}(X)$ is also a smooth algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ (Proposition 9.4.2.1). We therefore have the following converse to Corollary 9.4.3.5 (which is a more precise version of Corollary 9.4.2.2); see Remark HA.4.6.5.15:

Corollary 9.4.3.6. *Let X be a weakly perfect stack and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$. Suppose that $e : \mathcal{C} \otimes \mathcal{D} \to \mathrm{Sp}$ is a duality datum in $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. Then $\mathcal{D}$ can be equipped with an action of $\mathrm{QCoh}(X)$ for which the functor e factors as a composition*

$$\mathcal{C} \otimes \mathcal{D} \to \mathcal{C} \otimes_{\mathrm{QCoh}(X)} \mathcal{D} \xrightarrow{\overline{e}} \mathrm{QCoh}(X) \xrightarrow{\Gamma(X;\bullet)} \mathrm{Sp},$$

where $\overline{e}$ is a duality datum in $\mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$.

Remark 9.4.3.7. We can summarize Corollaries **??** and 9.4.3.6 more informally as follows: if $\mathrm{QCoh}(X)$ is a weakly perfect stack, then the forgetful functor $\mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{St}}) \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$ is compatible with duality.

Corollary 9.4.3.8. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

of quasi-geometric stacks, where Y is weakly perfect. Then:

(*a*) *If either* $\mathrm{QCoh}(X)$ *or* $\mathrm{QCoh}(Y')$ *is dualizable as an object of* $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, *then the canonical map* $\mathrm{QCoh}(Y') \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(X) \to \mathrm{QCoh}(X')$ *is an equivalence of ∞-categories.*

(*b*) *If both* $\mathrm{QCoh}(X)$ *and* $\mathrm{QCoh}(Y')$ *are dualizable as objects of* $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, *then* $\mathrm{QCoh}(X')$ *is also dualizable as an object of* $\mathcal{P}\mathrm{r}^{\mathrm{St}}$.

(*c*) *If X and Y' are weakly perfect, then X' is weakly perfect.*

Proof. Assertion (*a*) is a special case of Corollary 9.4.2.3. To prove (*b*), we first note that if $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y')$ are dualizable as objects of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, then they are also dualizable as modules over $\mathrm{QCoh}(Y)$ (Corollary 9.4.2.2). It follows from (*a*) that $\mathrm{QCoh}(X') \simeq \mathrm{QCoh}(Y') \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(X)$ is also dualizable as a module over $\mathrm{QCoh}(Y)$. Using Corollary 9.4.3.5, we deduce that $\mathrm{QCoh}(X')$ is dualizable as an object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$.

We now prove (*c*). Assume that X and Y' are weakly perfect. Since $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$ and Y is quasi-geometric, Proposition 9.1.5.8 implies that f satisfies condition $(*)$ of Proposition 9.1.5.7. It follows that f' satisfies the same condition, so that the pullback functor f'^* admits a right adjoint $f_* : \mathrm{QCoh}(X') \to \mathrm{QCoh}(Y')$ which preserves small colimits. The global sections functor $\Gamma(X'; \bullet)$ factors as a composition

$$\mathrm{QCoh}(X') \xrightarrow{f_*} \mathrm{QCoh}(Y') \xrightarrow{\Gamma(Y'; \bullet)} \mathrm{Sp},$$

so our assumption that $\mathscr{O}_Y$ is compact guarantees that $\Gamma(X'; \bullet)$ commutes with small colimits. Combining this with assertion (*b*), we deduce that X' is weakly perfect. □

9.4.4 Perfect Stacks

We now specialize to the study of quasi-geometric stacks X for which $\mathrm{QCoh}(X)$ has "enough" compact objects.

Definition 9.4.4.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is a *perfect stack* if it satisfies the following conditions:

(*a*) The functor X is a quasi-geometric stack (Definition 9.1.0.1).

(*b*) The structure sheaf $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$.

(c) Every quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(X)$ can be obtained as the colimit of a filtered diagram $\{\mathscr{F}_\alpha\}$, where each $\mathscr{F}_\alpha$ is a perfect object of $\mathrm{QCoh}(X)$.

Remark 9.4.4.2. Definition 9.4.4.1 is a slight variant of the notion of perfect stack introduced in [23]. There are three differences:

- For a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ to be a perfect stack in the sense of Definition 9.4.4.1, we require the existence of a faithfully flat map $\operatorname{Spec} A \to X$ for some connective $\mathbb{E}_\infty$-ring A; in [23] this requirement does not appear.
- The definition of a perfect stack X given in [23] requires that the diagonal map $\delta : X \to X \times X$ is affine, while Definition 9.4.4.1 requires only that it is quasi-affine.
- Definition 9.4.4.1 requires a perfect stack to satisfy descent with respect to the fpqc topology, while [23] requires descent only for the étale topology (though the difference is immaterial if we are interested only in properties of the ∞-category $\mathrm{QCoh}(X)$; see Proposition 6.2.3.1).

Example 9.4.4.3. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be (the functor represented by) a quasi-compact, quasi-separated algebraic space. In §9.6, we will show that X is a perfect stack (Proposition 9.6.1.1).

Example 9.4.4.4. Let G be an affine algebraic group defined over a field κ of characteristic zero. Then the classifying stack BG is perfect. More generally, if X is quasi-projective κ-scheme equipped with an action of G, then the (stack-theoretic) quotient X/G is a perfect stack (see [23]).

We now consider some equivalent formulations of Definition 9.4.4.1:

Proposition 9.4.4.5. *Let X be a quasi-geometric stack. The following conditions are equivalent:*

(1) *The quasi-geometric stack X is perfect.*

(2) *The ∞-category $\mathrm{QCoh}(X)$ is compactly generated and the structure sheaf $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$.*

(3) *The inclusion $\mathrm{QCoh}(X)^{\mathrm{perf}} \hookrightarrow \mathrm{QCoh}(X)$ extends to an equivalence of ∞-categories $\mathrm{Ind}(\mathrm{QCoh}(X)^{\mathrm{perf}}) \to \mathrm{QCoh}(X)$.*

Proof. Note that conditions (1), (2), and (3) all guarantee that $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$. It then follows that an object $\mathscr{F} \in \mathrm{QCoh}(X)$ is compact if and only if it is perfect (Proposition 9.1.5.3), from which the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow immediately. □

Corollary 9.4.4.6. *Let X be a perfect stack. Then X is weakly perfect. In particular, X has the diagonal approximation property.*

Proof. Since X is perfect, the ∞-category $\mathrm{QCoh}(X)$ is compactly generated (Proposition 9.4.4.5) and is therefore dualizable as an object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ (Proposition D.7.2.3). The desired result now follows from Corollary 9.4.3.2. □

Combining Corollary 9.4.4.6 with Theorem 9.4.1.4, we obtain the following result of Bhatt and Halpern-Leistner (see [27]):

Corollary 9.4.4.7 (Tannaka Duality, Perfect Stack Version)**.** *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be functors. If X is perfect, then the construction $(f : Y \to X) \mapsto (f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y))$ determines a fully faithful embedding*

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$$

whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which preserve small colimits and connective objects.

Remark 9.4.4.8. In the situation of Corollary 9.4.4.7, let $\mathrm{LFun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$ denote the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$ spanned by those symmetric monoidal functors which preserve small colimits and let $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{QCoh}(X)^{\mathrm{perf}}, \mathrm{QCoh}(Y)^{\mathrm{perf}})$ denotes the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{perf}}, \mathrm{QCoh}(Y)^{\mathrm{perf}})$ spanned by the symmetric monoidal functors which are exact. Since X is a perfect stack, composition with the inclusion $\mathrm{QCoh}(X)^{\mathrm{perf}} \hookrightarrow \mathrm{QCoh}(X)$ induces an equivalence of ∞-categories $\mathrm{LFun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)) \to \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{QCoh}(X)^{\mathrm{perf}}, \mathrm{QCoh}(Y)^{\mathrm{perf}})$. However, it is not obvious how to describe the essential image of $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X)$ under this equivalence, since the full subcategory $\mathrm{QCoh}(X)^{\mathrm{perf}} \subseteq \mathrm{QCoh}(X)$ need not be closed under truncations.

We conclude by observing a pleasant stability property of the class of perfect stacks:

Proposition 9.4.4.9. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. If X and Y' are perfect stacks and Y is weakly perfect, then X' is perfect.

Proof. It follows from Corollary 9.4.3.8 that X' is weakly perfect and that the induced map

$$\mathrm{QCoh}(Y') \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(X) \to \mathrm{QCoh}(X')$$

is an equivalence of ∞-categories. Combining this with the assumption that X and Y' are perfect, we deduce that $\mathrm{QCoh}(X)$ is generated under colimits by the essential image of the composite functor

$$\mathrm{QCoh}(Y')^{\mathrm{perf}} \times \mathrm{QCoh}(X)^{\mathrm{perf}} \to \mathrm{QCoh}(Y') \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(X) \to \mathrm{QCoh}(X').$$

Since this composite functor factors through $\mathrm{QCoh}(X')^{\mathrm{perf}}$, it follows that X' is perfect. □

9.5 Locally Noetherian Stacks

Let X be a geometric stack and let R be an $\mathbb{E}_\infty$-ring. According to Theorem 9.3.0.3, we can identify R-valued points of X with symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_R$ which preserve small colimits, connective objects, and flat objects. In §9.4, we proved that if X is perfect, then the third condition is superfluous: any symmetric monoidal functor $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_R$ which preserves small colimits and connective objects automatically preserves flat objects (Corollary 9.4.4.7). The assumption X is perfect is a kind of finiteness hypothesis: it guarantees that the ∞-category $\mathrm{QCoh}(X)$ has "enough" perfect objects. In this section, we will study a different finiteness hypothesis: the condition that X is *locally Noetherian* (Definition 9.5.1.1). Our main result asserts that if X is locally Noetherian geometric stack, then Tannaka duality again takes a particularly simple form: every symmetric monoidal functor $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_R$ which preserves small colimits and connective objects automatically preserves flat objects, and therefore arises from an essentially unique R-valued point of X (Theorem 9.5.4.1). We also prove a slightly weaker result in the case where X is quasi-geometric (Theorem 9.5.4.2).

9.5.1 Noetherian and Locally Noetherian Quasi-Geometric Stacks

We now introduce some finiteness conditions on quasi-geometric stacks.

Definition 9.5.1.1. Let X be a quasi-geometric stack. We will say that X is *locally Noetherian* if there exists a faithfully flat map $\operatorname{Spec} R \to X$, where R is a Noetherian $\mathbb{E}_\infty$-ring.

Let X be a quasi-geometric stack. The condition that X is locally Noetherian guarantees that there *exists* a faithfully flat map $\operatorname{Spec} R \to X$, where R is Noetherian. However, it does not follow that for *every* faithfully flat map $f : \operatorname{Spec} R \to X$, the $\mathbb{E}_\infty$-ring R is Noetherian (this condition is essentially never satisfied). However, we can do better if we assume some finiteness conditions on f:

Proposition 9.5.1.2. *Let X be a locally Noetherian quasi-geometric stack, and let $f : \operatorname{Spec} R \to X$ be a morphism which locally almost of finite presentation. Then R is Noetherian.*

Proof. Since X is locally Noetherian, we can choose a faithfully flat map $g : \operatorname{Spec} A \to X$, where A is Noetherian. Form a pullback diagram

$$\begin{array}{ccc} Y & \xrightarrow{g'} & \operatorname{Spec} R \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} A & \xrightarrow{g} & X. \end{array}$$

Then Y is (representable by) a quasi-affine spectral algebraic space. Choose an étale surjection $\operatorname{Spec} B \to Y$. Since f is locally almost of finite presentation, the $\mathbb{E}_\infty$-ring B is almost of finite presentation over A and is therefore Noetherian (Proposition HA.7.2.4.31). The map g is faithfully flat, so B is faithfully flat over R. Applying Lemma 2.8.1.6, we deduce that R is Noetherian. □

Corollary 9.5.1.3. *Let X be a quasi-geometric stack and suppose that there exists a morphism $f : \operatorname{Spec} A \to X$ which is faithfully flat and locally almost of finite presentation. Then the following conditions are equivalent:*

- (a) *For every morphism $\operatorname{Spec} B \to X$ which is locally almost of finite presentation, the $\mathbb{E}_\infty$-ring B is Noetherian.*
- (b) *The $\mathbb{E}_\infty$-ring A is Noetherian.*
- (c) *The quasi-geometric stack X is locally Noetherian.*

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are tautological, and the implication $(c) \Rightarrow (a)$ follows from Proposition 9.5.1.2. □

Definition 9.5.1.4. We will say that a quasi-geometric stack X is *Noetherian* if it satisfies the equivalent conditions of Corollary 9.5.1.3 (that is, if there exists a morphism $f : \operatorname{Spec} A \to X$ which is faithfully flat and locally almost of finite presentation, where A is Noetherian).

Example 9.5.1.5. Let R be a Noetherian commutative ring and let G be a flat quasi-affine group scheme over R. Then the classifying stack BG (see Example 9.1.1.7) is locally Noetherian. If G is of finite presentation over R, then BG is Noetherian.

Example 9.5.1.6. Let $\mathcal{FG}$ be the moduli stack of formal groups (see Example 9.3.1.7). Then $\mathcal{FG}$ is not locally Noetherian. For every prime number p and every integer $n \geqslant 1$, the open substack $\mathcal{FG}_{\leqslant n} \subseteq \mathcal{FG} \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} \mathbf{Z}_{(p)}$ classifying formal groups of height $\leqslant n$ (over p-local rings) is locally Noetherian but not Noetherian.

9.5.2 Noetherian Hypotheses and Quasi-Coherent Sheaves

We now show that if X is a locally Noetherian quasi-geometric stack, then the Grothendieck prestable ∞-category $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is locally Noetherian (in the sense of Definition C.6.9.1). We begin by isolating an appropriate finiteness condition on objects of the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$.

Notation 9.5.2.1. Let X be a locally Noetherian quasi-geometric stack. We let $\mathrm{Coh}(X)^{\heartsuit}$ denote the full subcategory of $\mathrm{QCoh}(X)^{\heartsuit}$ spanned by those object $\mathscr{F} \in \mathrm{QCoh}(X)^{\heartsuit}$ which are almost perfect.

Remark 9.5.2.2. Let X be a locally Noetherian quasi-geometric stack and let $f : \operatorname{Spec} A \to X$ be a faithfully flat surjection, where A is a Noetherian $\mathbb{E}_\infty$-ring. Then the pullback functor f^* determines an exact functor of abelian categories $\mathrm{QCoh}(X)^{\heartsuit} \to \mathrm{QCoh}(\operatorname{Spec} A)^{\heartsuit} \simeq \mathrm{Mod}_A^{\heartsuit}$, and an object $\mathscr{F} \in \mathrm{QCoh}(X)^{\heartsuit}$ belongs to $\mathrm{Coh}(X)^{\heartsuit}$ if and only if $f^* \mathscr{F}$ is finitely generated when regarded as an A-module (Propositions 9.1.3.5 and HA.7.2.4.17). It follows that $\mathrm{Coh}(X)^{\heartsuit}$ is an abelian subcategory of $\mathrm{QCoh}(X)^{\heartsuit}$ which is closed under extensions, subobjects, and quotient objects.

Proposition 9.5.2.3. *Let X be a locally Noetherian quasi-geometric stack. Then the prestable ∞-category $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is locally Noetherian (see Definition C.6.9.1). Moreover, an object $\mathscr{F} \in \mathrm{QCoh}(X)^{\heartsuit}$ is Noetherian if and only if it belongs to $\mathrm{Coh}(X)^{\heartsuit}$.*

Proof. Choose a faithfully flat map $f : \operatorname{Spec} A \to X$, where A is a Noetherian $\mathbb{E}_\infty$-ring. We first claim that each object $\mathscr{F} \in \mathrm{Coh}(X)^{\heartsuit}$ is a Noetherian object of $\mathrm{QCoh}(X)^{\heartsuit}$. To prove this, it will suffice to show that every ascending sequence of subobjects

$$\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \cdots$$

of $\mathscr{F}$ (in the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$) is eventually constant. Since the pullback functor $f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\operatorname{Spec} A) \simeq \mathrm{Mod}_A$ is t-exact, we can identify $\{f^* \mathscr{F}_n\}_{n \geqslant 0}$ as an ascending sequence of subobjects of $f^* \mathscr{F} \in \mathrm{Mod}_A^{\heartsuit}$. Our assumption that $\mathscr{F} \in \mathrm{Coh}(X)^{\heartsuit}$ guarantees that $f^* \mathscr{F}$ is finitely generated as a module over $\pi_0 A$. Since $\pi_0 A$ is a Noetherian ring, the chain of submodules $\{f^* \mathscr{F}_n\}_{n \geqslant 0}$ is eventually constant. Since the functor f^* is conservative, it follows that the ascending chain $\{\mathscr{F}_n\}_{n \geqslant 0}$ is also eventually constant.

We now prove that every object $\mathscr{F} \in \mathrm{QCoh}(X)^{\heartsuit}$ can be obtained as a filtered colimit of Noetherian subobjects of $\mathscr{F}$, using an argument of Deligne. Let us identify $\mathscr{G} = f^* \mathscr{F}$ with a (discrete) module over the ring $\pi_0 A$. Write $\mathscr{G}$ as a filtered colimit $\varinjlim \mathscr{G}_\beta$, where each $\mathscr{G}_\beta$ is subobject of $\mathscr{G}$ which corresponds to a finitely generated module over $\pi_0 A$. Since the direct image functor f_* is left t-exact, we can identify each truncation $\tau_{\geqslant 0} f_* \mathscr{G}_\beta$ with a subobject of $\tau_{\geqslant 0} f_* \mathscr{G}$. For each index β, let $\mathscr{F}_\beta$ denote the subobject of $\mathscr{F}$ given by the fiber product $\mathscr{F} \times_{\tau_{\geqslant 0} f_* \mathscr{G}} \tau_{\geqslant 0} f_* \mathscr{G}_\beta$. Since the functor f_* preserves small colimits, we have $f_* \mathscr{G} \simeq \varinjlim f_* \mathscr{G}_\beta$. The t-structure on $\mathrm{QCoh}(X)$ is compatible with filtered colimits (Corollary 9.1.3.2), so we have $\tau_{\geqslant 0} f_* \mathscr{G} \simeq \varinjlim \tau_{\geqslant 0} f_* \mathscr{G}_\beta$ and we can therefore identify $\mathscr{F}$ with the colimit of the diagram $\{\mathscr{F}_\beta\}$. It will therefore suffice to show that each $\mathscr{F}_\beta$ belongs to $\mathrm{Coh}(X)^\heartsuit$. Equivalently, we must show that the pullback $f^* \mathscr{F}_\beta$ is finitely generated when regarded as a (discrete) module over the commutative ring $\pi_0 A$. By construction, we have a commutative diagram

$$\begin{array}{ccc} f^* \mathscr{F}_\beta & \longrightarrow & f^* \mathscr{F} \\ \downarrow & & \downarrow \\ f^* \tau_{\geqslant 0} f_* \mathscr{G}_\beta & \longrightarrow & f^* \tau_{\geqslant 0} f_* \mathscr{G} \\ \downarrow & & \downarrow \\ \mathscr{G}_\beta & \longrightarrow & \mathscr{G} \end{array}$$

in the abelian category $\mathrm{QCoh}(\operatorname{Spec} A)^\heartsuit$. The upper horizontal map in this diagram is a monomorphism and the right vertical composition is an isomorphism. It follows by a diagram chase that the left vertical composition is a monomorphism, so that we can regard $f^* \mathscr{F}_\beta$ as a subobject of $\mathscr{G}_\beta$. By construction, $\mathscr{G}_\beta$ corresponds to a finitely generated module over $\pi_0 A$. Since the commutative ring $\pi_0 A$ is Noetherian, it follows that $f^* \mathscr{F}_\beta$ also corresponds to a finitely generated module over $\pi_0 A$, so that $\mathscr{F}_\beta$ belongs to $\mathrm{Coh}(X)^\heartsuit$ as desired (Remark 9.5.2.2).

Note that if $\mathscr{F} \in \mathrm{QCoh}(X)^\heartsuit$ is Noetherian, then writing $\mathscr{F} \simeq \varinjlim \mathscr{F}_\beta$ as above we must have $\mathscr{F} \simeq \mathscr{F}_\beta$ for some index β, so that $\mathscr{F} \in \mathrm{Coh}(X)^\heartsuit$. Combining this with the first part of the argument, we see that an object $\mathscr{F} \in \mathrm{QCoh}(X)^\heartsuit$ is Noetherian if and only if it belongs to $\mathrm{Coh}(X)^\heartsuit$. Consequently, the above argument shows that each object $\mathscr{F} \in \mathrm{QCoh}(X)^\heartsuit$ is a filtered colimit of Noetherian subobjects of $\mathscr{F}$, so that the abelian category $\mathrm{QCoh}(X)^\heartsuit$ is locally Noetherian. To complete the proof that $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is locally Noetherian, it will suffice to show that every object $\mathscr{F} \in \mathrm{Coh}(X)^\heartsuit$ is a compact object of $\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}$ for each $n \geqslant 0$. In other words, we wish to show that the construction $\mathscr{G} \mapsto \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G})$ determines a functor $U : \mathrm{QCoh}(X) \to \mathcal{S}$ which commutes with filtered colimits when restricted to $\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}$.

Let $X_\bullet$ denote the Čech nerve of f, so that each X_m is (representable by) a quasi-affine spectral Deligne-Mumford stack. For each $m \geqslant 0$, let $f_m : X_m \to X$ denote the projection map, and let $U^m : \mathrm{QCoh}(X) \to \mathcal{S}$ denote the functor given by $\mathscr{G} \mapsto \mathrm{Map}_{\mathrm{QCoh}(X_m)}(f_m^* \mathscr{F}, f_m^* \mathscr{G})$. Since f is faithfully flat, we have $\mathrm{QCoh}(X) \simeq \mathrm{Tot}(\mathrm{QCoh}(X_\bullet))$, so we can identify U with the functor $\mathrm{Tot}(U^\bullet) = \varprojlim_{[m] \in \mathbf{\Delta}} U^m \simeq \varprojlim_{[m] \in \mathbf{\Delta}_+} U^m$. Note that, when restricted to $\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}$, each of the functors U^m takes values in $\tau_{\leqslant n} \mathcal{S}$. It follows that the restriction map $\mathrm{Tot}(U^\bullet) \to \varprojlim_{[m] \in \mathbf{\Delta}_{+, \leqslant n+1}} U^m$ is an equivalence when restricted to $\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}$. We may therefore identify $U|_{\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}}$ with the finite limit

$\varprojlim_{[m]\in\mathbf{\Delta}_{+,\leqslant n+1}} U^m|_{\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}}$. It will therefore suffice to show that each of the functors U^m commutes with filtered colimits when restricted to $\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}$. This is clear, since X_m is (representable by) a quasi-affine spectral Deligne-Mumford stack and $f_m^* \mathscr{F} \in \mathrm{QCoh}(X_m)$ is almost perfect. □

9.5.3 Digression: Internal Hom-Sheaves

Let X be a quasi-geometric stack. Then the ∞-category $\mathrm{QCoh}(X)$ is presentable (Corollary 9.1.3.2) and the tensor product $\otimes : \mathrm{QCoh}(X) \times \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ preserves small colimits separately in each variable. It follows that the symmetric monoidal structure on $\mathrm{QCoh}(X)$ is *closed*: that is, for every pair of quasi-coherent sheaves $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(X)$, we can define a new object $\underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}) \in \mathrm{QCoh}(X)$ with the following universal property: there is an evaluation map $e : \mathscr{F} \otimes \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}) \to \mathscr{G}$ such that, for every object $\mathscr{H} \in \mathrm{QCoh}(X)$, the composite map

$$\begin{aligned} \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{H}, \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G})) &\to \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{F} \otimes \mathscr{H}, \mathscr{F} \otimes \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G})) \\ &\xrightarrow{e} \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{F} \otimes \mathscr{H}, \mathscr{G}) \end{aligned}$$

is a homotopy equivalence.

Remark 9.5.3.1. Let X be a quasi-geometric stack and suppose we are given quasi-coherent sheaves $\mathscr{F} \in \mathrm{QCoh}(X)_{\geqslant 0}$ and $\mathscr{G} \in \mathrm{QCoh}(X)_{\leqslant 0}$. Then $\underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}) \in \mathrm{QCoh}(X)_{\leqslant 0}$. To prove this, it suffices to observe that for $\mathscr{H} \in \mathrm{QCoh}(X)_{\geqslant 1}$, the mapping space

$$\mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{H}, \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G})) \simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{F} \otimes \mathscr{H}, \mathscr{G})$$

is contractible.

Remark 9.5.3.2. In the special case where $X = \operatorname{Spec} A$ is affine, we can identify $\mathscr{F}$ and $\mathscr{G}$ with A-modules M and N. In this case, we can identify the mapping object $\underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G})$ with an A-module which we will denote by $\underline{\mathrm{Map}}_A(M, N)$.

Suppose we are given a morphism $f : Y \to X$ of quasi-geometric stacks, and quasi-coherent sheaves $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(X)$. Applying the pullback functor f^* to the evaluation map e, we obtain a map

$$f^*(e) : f^* \mathscr{F} \otimes f^* \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}) \to f^* \mathscr{G},$$

which is classified by a morphism $\rho : f^* \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}) \to \underline{\mathrm{Map}}_{\mathrm{QCoh}(Y)}(f^* \mathscr{F}, f^* \mathscr{G})$. In general, the morphism ρ need not be an equivalence (in other words, the construction $(\mathscr{F}, \mathscr{G}) \mapsto \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G})$ is not "local" on X). However, we have the following:

Proposition 9.5.3.3. *Let $f : Y \to X$ be a morphism of quasi-geometric stacks which is quasi-affine and flat, let $\mathscr{F} \in \mathrm{QCoh}(X)$ be almost perfect, and let $\mathscr{G} \in \mathrm{QCoh}(X)_{\leqslant 0}$. Then the canonical map*

$$\rho : f^* \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}) \to \underline{\mathrm{Map}}_{\mathrm{QCoh}(Y)}(f^* \mathscr{F}, f^* \mathscr{G})$$

is an equivalence.

Remark 9.5.3.4. In the statement of Proposition 9.5.3.3, the hypothesis that f is quasi-affine is not really needed. However, it will be satisfied in our case of interest, and will slightly simplify the proof.

Remark 9.5.3.5. For a closely related result in a different context, see §6.5.3.

Proof of Proposition 9.5.3.3. Replacing $\mathscr{F}$ by a suspension if necessary, we may assume that $\mathscr{F}$ is connective. Choose a faithfully flat map $u : X_0 \to X$, where X_0 is affine. Let $X_\bullet$ denote the Čech nerve of u_0 and for $n \geqslant 0$ let $u_n : X_n \to X$ denote the evident map. Set $\mathscr{G}^n = u_{n*} u_n^* \mathscr{G}$, so that $\mathscr{G}^\bullet$ is a cosimplicial object of $\mathrm{QCoh}(X)_{\leqslant 0}$ and we have a canonical equivalence $\mathscr{G} \simeq \mathrm{Tot}(\mathscr{G}^\bullet)$. We can therefore identify $\underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G})$ with the totalization $\mathrm{Tot}(\underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}^\bullet))$. Since the map f is flat, the pullback functor f^* preserves totalizations of cosimplicial objects of $\mathrm{QCoh}(X)_{\leqslant 0}$. Using Remark 9.5.3.1, we obtain equivalences

$$f^* \mathscr{G} \simeq \mathrm{Tot}(f^* \mathscr{G}^\bullet) \qquad f^* \mathrm{Tot}(\underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}^\bullet)) \simeq \mathrm{Tot}(f^* \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}^\bullet)).$$

Using these equivalences, we can identify ρ with the totalization of a cosimplicial map

$$\rho^\bullet : f^* \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}^\bullet) \to \underline{\mathrm{Map}}_{\mathrm{QCoh}(Y)}(f^* \mathscr{F}, f^* \mathscr{G}^\bullet).$$

We may therefore replace $\mathscr{G}$ by $\mathscr{G}^n$ and thereby reduce to the case where $\mathscr{G} \simeq u_* \mathscr{G}'$ for some $\mathscr{G}' \in \mathrm{QCoh}(X_0)_{\leqslant 0}$. Form a pullback diagram

$$\begin{array}{ccc} Y_0 & \xrightarrow{f_0} & X_0 \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ Y & \xrightarrow{f} & X \end{array}$$

so that we have equivalences

$$\underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}) \simeq u_* \underline{\mathrm{Map}}_{\mathrm{QCoh}(X_0)}(u^* \mathscr{F}, \mathscr{G}')$$

$$\underline{\mathrm{Map}}_{\mathrm{QCoh}(Y)}(f^* \mathscr{F}, f^* \mathscr{G}) \simeq v_* \underline{\mathrm{Map}}_{\mathrm{QCoh}(Y_0)}(f_0^* u^* \mathscr{F}, f_0^* \mathscr{G}).$$

Combining these equivalences with the projection formula of Corollary 6.3.4.3, we can identify ρ with the direct image under v of the map

$$\rho_0 : f_0^* \underline{\mathrm{Map}}_{\mathrm{QCoh}(X_0)}(u^* \mathscr{F}, \mathscr{G}') \to \underline{\mathrm{Map}}_{\mathrm{QCoh}(Y_0)}(f_0^* u^* \mathscr{F}, f_0^* \mathscr{G}').$$

Replacing $f : Y \to X$ by $f_0 : Y_0 \to X_0$, we may thereby reduce to the special case where $X = \operatorname{Spec} A$ is affine.

Let us identify $\mathrm{QCoh}(X)$ with the ∞-category Mod_A. Set $B = \Gamma(Y; \mathscr{O}_Y)$. Since f is quasi-affine, the global sections functor $\Gamma(Y; \bullet)$ induces an equivalence of ∞-categories $\mathrm{QCoh}(Y) \to \mathrm{Mod}_B$ (Proposition 2.4.1.4). Under this equivalence, the construction $\mathscr{H} \mapsto \Gamma(Y; f^* \mathscr{H})$ corresponds to the extension of scalars functor $M \mapsto B \otimes_A M$. Since f is flat, the construction $\mathscr{H} \mapsto \Gamma(Y; f^* \mathscr{H})$ is left t-exact, so that B has Tor-amplitude $\leqslant 0$ over A. Set $M = \Gamma(X; \mathscr{F})$ and $N = \Gamma(X; \mathscr{G})$. Unwinding the definitions, we are reduced to showing that the canonical map

$$B \otimes_A \underline{\mathrm{Map}}_A(M, N) \to \underline{\mathrm{Map}}_A(M, B \otimes_A N) \simeq \underline{\mathrm{Map}}_B(B \otimes_A M, B \otimes_A N)$$

is an equivalence of B-modules, which is a special case of Lemma 6.5.3.7. □

Corollary 9.5.3.6. *Let X be a quasi-geometric stack and let $\mathscr{F}, \mathscr{G}, \mathscr{H} \in \mathrm{QCoh}(X)$. Suppose that $\mathscr{F}$ is almost perfect, $\mathscr{G}$ is 0-truncated, and $\mathscr{H}$ has* Tor*-amplitude $\leqslant 0$. Then the canonical map*

$$\theta : \mathscr{H} \otimes \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}) \to \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{H} \otimes \mathscr{G})$$

is an equivalence.

Proof. Using Proposition 9.5.3.3, we can reduce to the case where X is affine, in which case Corollary 9.5.3.6 is a reformulation of Lemma 6.5.3.7. □

9.5.4 Tannaka Duality in the Locally Noetherian Case

We consider Tannaka duality in the locally Noetherian context. Our main results can be stated as follows:

Theorem 9.5.4.1 (Tannaka Duality for Locally Noetherian Geometric Stacks)**.** *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be functors and suppose that X is a locally Noetherian geometric stack. Then the construction $(f : Y \to X) \mapsto (f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y))$ determines a fully faithful embedding*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)),$$

whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which preserve small colimits and connective objects.

For quasi-geometric stacks, we have the following slightly weaker result:

Theorem 9.5.4.2 (Tannaka Duality for Locally Noetherian Quasi-Geometric Stacks)**.** *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be functors and suppose that X is a locally Noetherian quasi-geometric stack. Then the construction $(f : Y \to X) \mapsto (f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y))$ determines a fully faithful embedding*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)),$$

whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which preserve small colimits, connective objects, and almost perfect objects.

Remark 9.5.4.3. In the case where X is assumed to be Noetherian, Theorem 9.5.4.2 was proven by by Bhatt and Halpern-Leistner in [27]; our proof will follow a similar strategy.

Remark 9.5.4.4. In the statements of Theorems 9.5.4.1 and 9.5.4.2, we can replace the assumption that X is locally Noetherian by the weaker assumption that the 0-truncation of X is locally Noetherian, or that there exists an affine map from the 0-truncation of X to a locally Noetherian quasi-geometric stack. The last condition is automatically satisfied for a large class of algebraic stacks (see [177]).

The main step in the proof of Theorems 9.5.4.1 and 9.5.4.2 is contained in the following:

Lemma 9.5.4.5. *Let X be a 0-truncated quasi-geometric stack, let R be a connective $\mathbb{E}_\infty$-ring, and let $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_R$ be a functor which preserves small colimits and connective objects. Let $\alpha : \mathscr{F} \to \mathscr{F}'$ be a morphism in $\mathrm{QCoh}(X)$, where $\mathscr{F}$ is almost perfect and $\mathscr{F}'$ has* Tor*-amplitude $\leqslant 0$. Then, for every object $M \in (\mathrm{Mod}_R)_{\leqslant 0}$, the induced map $\alpha_M : M \otimes_R F(\mathscr{F}) \to M \otimes_R F(\mathscr{F}')$ factors through an object of $(\mathrm{Mod}_R)_{\leqslant 0}$.*

Proof. Set $\mathscr{F}^\vee = \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{O}_X)$. Since $\mathscr{F}$ is almost perfect, $\mathscr{O}_X$ is 0-truncated, and $\mathscr{F}'$ has Tor-amplitude $\leqslant 0$, it follows from Corollary 9.5.3.6 that the canonical map

$$\rho : \mathscr{F}^\vee \otimes \mathscr{F}' \to \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{F}')$$

is an equivalence. Let $\mathscr{G} = \tau_{\geqslant 0} \mathscr{F}^\vee$ denote the connective cover of $\mathscr{F}^\vee$. Since $\mathscr{F}'$ has Tor-amplitude $\leqslant 0$, the natural map $\mathscr{G} \otimes \mathscr{F}' \to \mathscr{F}^\vee \otimes \mathscr{F}'$ induces an equivalence on connective covers. It follows that we can choose a map $c : \mathscr{O}_X \to \mathscr{G} \otimes \mathscr{F}'$ which is determined (uniquely up to homotopy) by the requirement that the composition

$$\mathscr{O}_X \xrightarrow{c} \mathscr{G} \otimes \mathscr{F}' \to \mathscr{F}^\vee \otimes \mathscr{F}' \xrightarrow{\rho} \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{F}')$$

classifies α. The tautological pairing $\mathscr{F} \otimes \mathscr{F}^\vee \to \mathscr{O}_X$ determines an "evaluation map" $e : \mathscr{F} \otimes \mathscr{G} \to \mathscr{O}_X$. Unwinding the definitions, we see that the composite map

$$\mathscr{F} \xrightarrow{\mathrm{id} \otimes c} \mathscr{F} \otimes \mathscr{G} \otimes \mathscr{F}' \xrightarrow{e \otimes \mathrm{id}} \mathscr{F}'$$

is homotopic to α.

Set $N = \underline{\mathrm{Map}}_R(F(\mathscr{G}), M)$. The evaluation morphism e determines a map $e_M : M \otimes_R F(\mathscr{F}) \otimes_R F(\mathscr{G}) \to M$, which is classified by a morphism of R-modules $\beta : M \otimes_R F(\mathscr{F}) \to N$. Let γ denote the composite map

$$N \xrightarrow{\mathrm{id} \otimes F(c)} N \otimes_R F(\mathscr{G}) \otimes_R F(\mathscr{F}') \to M \otimes_R F(\mathscr{F}').$$

Using the fact that $(e \otimes \mathrm{id}) \circ (\mathrm{id} \otimes c)$ is homotopic to α, we deduce that the composition $\gamma \circ \beta$ is homotopic to α_M. Because functor F preserves connective objects, so that R-module $F(\mathscr{G})$ is connective. Since $M \in (\mathrm{Mod}_R)_{\leqslant 0}$, it follows that $N \in (\mathrm{Mod}_R)_{\leqslant 0}$, so that (β, γ) provides the desired factorization of α. □

Proposition 9.5.4.6. *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be functors and let $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ be a symmetric monoidal functor which preserves small colimits and connective objects. If X is a locally Noetherian quasi-geometric stack and $\mathscr{F} \in \mathrm{QCoh}(X)$ has* Tor*-amplitude $\leqslant 0$, then $F(\mathscr{F}) \in \mathrm{QCoh}(Y)$ has* Tor*-amplitude $\leqslant 0$.*

Proof. Without loss of generality we may assume that $Y = \operatorname{Spec} R$ is affine, in which case we can identify $\mathrm{QCoh}(Y)$ with the ∞-category Mod_R. Let M be a discrete R-module; we wish to show that $M \otimes_R F(\mathscr{F})$ belongs to $(\mathrm{Mod}_R)_{\leqslant 0}$. Set $R' = F(\pi_0 \mathscr{O}_X)$. Since the functor F preserves connective objects, the unit map $R = F(\mathscr{O}_X) \to F(\pi_0 \mathscr{O}_X) \to R'$ has 1-connective fiber and therefore induces an isomorphism on π_0. Since M is discrete, it can be regarded as an R'-module in an essentially unique way. Let $v : X' \to X$ exhibit X' as a 0-truncation of X. The functor F then induces a symmetric monoidal functor

$$F' : \mathrm{QCoh}(X') \simeq \mathrm{Mod}_{\pi_0 \mathscr{O}_X}(\mathrm{QCoh}(X)) \xrightarrow{F} \mathrm{Mod}_{R'}$$

which preserves small colimits and connective objects. We have a canonical equivalence $M \otimes_R F(\mathscr{F}) \simeq M \otimes_{R'} F'(v^* \mathscr{F})$. Replacing X by X', F by F', and $\mathscr{F}$ by $v^* \mathscr{F}$, we may reduce to the case where X is 0-truncated. In this case, the assumption that $\mathscr{F}$ has Tor-amplitude $\leqslant 0$ guarantees that it belongs to $\mathrm{QCoh}(X)_{\leqslant 0}$. Our assumption that X is locally Noetherian guarantees that the prestable ∞-category $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is locally Noetherian (Proposition 9.5.2.3). Applying Corollary **??**, we deduce that $\mathscr{F}$ can be written as the colimit of a filtered diagram $\{\mathscr{F}_\alpha\}$ where each $\mathscr{F}_\alpha$ is almost perfect (and 0-truncated, but we will not need this). Since the functor F preserves colimits, we obtain an equivalence $M \otimes_R F(\mathscr{F}) \simeq \varinjlim (M \otimes_R F(\mathscr{F}_\alpha))$. In particular, any element of $\pi_n(M \otimes_R F(\mathscr{F}))$ lies in the image of the natural map $\pi_n(M \otimes_R F(\mathscr{F}_\alpha)) \to \pi_n(M \otimes_R F(\mathscr{F}))$ for some index α. These maps vanish for $n > 0$ by virtue of Lemma 9.5.4.5, so that $M \otimes_R F(\mathscr{F})$ belongs to $(\mathrm{Mod}_R)_{\leqslant 0}$ as desired. □

Corollary 9.5.4.7. *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be functors and let $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ be a symmetric monoidal functor which preserves small colimits and connective objects. If X is a locally Noetherian quasi-geometric stack, then F carries flat objects to flat objects.*

Proof. If $\mathscr{F} \in \mathrm{QCoh}(X)$ is flat, then $\mathscr{F}$ is connective and of Tor-amplitude $\leqslant 0$. It follows from Proposition 9.5.4.6 (and our assumption that F preserves connective objects) that $F(\mathscr{F})$ is connective and of Tor-amplitude $\leqslant 0$, and is therefore flat. □

Proof of Theorem 9.5.4.1. Combine Theorem 9.3.0.3 with Corollary 9.5.4.7. □

Remark 9.5.4.8. The proof of Theorem 9.5.4.1 did not require full strength of the assumption that X is locally Noetherian: only that every discrete object of $\mathrm{QCoh}(X)$ can be written as a filtered colimit of almost perfect objects of $\mathrm{QCoh}(X)$. This condition is also satisfied when X is perfect, which gives an alternative proof of Corollary 9.4.4.7 (at least when X has affine diagonal).

Proof of Theorem 9.5.4.2. We begin as in the proof of Theorem 9.2.0.2. Let X be a locally Noetherian quasi-geometric stack and let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be arbitrary. Note that the map $\theta_Y : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$ is automatically fully faithful (Proposition 9.2.2.1). It will therefore suffice to show that if $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is a symmetric monoidal functor which preserves small colimits, connective objects, and almost perfect objects, then there exists a symmetric monoidal equivalence $F \simeq f^*$ for some map $f : Y \to X$; the map f is then uniquely determined (up to a contractible space of choices). Writing Y as a colimit of corepresentable functors, we can assume that $Y = \operatorname{Spec} R$ for some connective $\mathbb{E}_\infty$-ring R.

Since X is quasi-geometric, we can choose a faithfully flat morphism $u : U \to X$, where U is affine (in fact, we can assume that U is corepresentable by a Noetherian $\mathbb{E}_\infty$-ring, but we will not need this). Set $\mathscr{A} = u_* \mathscr{O}_U \in \mathrm{CAlg}(\mathrm{QCoh}(X))$. As in the proof of Theorem 9.2.0.2, the main step is to establish the following:

$(*)$ There exists a faithfully flat, quasi-affine morphism $v : Y' \to Y$ and an equivalence $F(\mathscr{A}) \simeq v_* \mathscr{O}_{Y'}$ in $\mathrm{CAlg}(\mathrm{QCoh}(Y))$.

Assuming $(*)$, we can apply Proposition 6.3.4.6 to obtain symmetric monoidal equivalences

$$\mathrm{QCoh}(U) \simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{QCoh}(X))$$

$$\mathrm{QCoh}(Y') \simeq \mathrm{Mod}_{v_* \mathscr{O}_{Y'}}(\mathrm{QCoh}(Y)) \simeq \mathrm{Mod}_{F(\mathscr{A})}(\mathrm{QCoh}(Y)).$$

Consequently, F induces a symmetric monoidal functor $F' : \mathrm{QCoh}(U) \to \mathrm{QCoh}(Y')$ which fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{F} & \mathrm{QCoh}(Y) \\ \downarrow u^* & & \downarrow v^* \\ \mathrm{QCoh}(U) & \xrightarrow{F'} & \mathrm{QCoh}(Y'). \end{array}$$

Since U is affine and Y' is quasi-affine, Proposition 9.2.1.1 implies the existence of a symmetric monoidal equivalence $F' \simeq f'^*$ for some map $f' : Y' \to U$. It follows that $v^* \circ F$ belongs to the essential image of the fully faithful embedding $\theta_{Y'}$. Since v is faithfully flat, it will then follow that F belongs to the essential image of θ_Y, as desired.

We now prove $(*)$. Using Proposition 6.3.4.5, we can write $\tau_{\geqslant 0} \mathscr{A}$ as the direct image $\overline{u}_* \mathscr{O}_{\overline{U}}$ for some affine morphism of quasi-geometric stacks $\overline{u} : \overline{U} \to X$ (beware that $\overline{U}$ is not affine). The natural map $\tau_{\geqslant 0} \mathscr{A} \hookrightarrow \mathscr{A}$ then determines a map $j : U \to \overline{U}$. Since X is a quasi-geometric stack, the map u is quasi-affine, so that j is a quasi-compact open immersion. Let $\mathcal{I} \subseteq \pi_0 \mathscr{A}$ be the ideal sheaf whose vanishing locus is the reduced closed substack of $\overline{U}$ complementary to j. Since X is locally Noetherian, we can write $\mathcal{I}$ as the colimit of a filtered diagram $\{\mathscr{F}_\alpha\}$ in $\mathrm{Coh}(X)^\heartsuit$ (Proposition 9.5.2.3). For each index α, the we have a map $e_\alpha : \mathscr{F}_\alpha \to \mathcal{I}$ whose image generates an ideal sheaf $\mathcal{I}_\alpha \subseteq \mathcal{I}$, whose vanishing locus is complementary to an open subfunctor $U_\alpha \subseteq U$. Note that $U = \bigcup_\alpha U_\alpha$. Since j is quasi-compact, we can choose an index α such that $U_\alpha = U$. Set $\mathscr{F} = \mathscr{F}_\alpha$ and $e = e_\alpha$, so that we have a map $e : \mathscr{F} \to \mathcal{I}$ in $\mathrm{QCoh}(X)^\heartsuit$.

Set $A = F(\tau_{\geqslant 0} \mathscr{A}) \in \mathrm{CAlg}_R^{\mathrm{cn}}$ and set $\overline{Y}' = \operatorname{Spec} A$. Note that e determines a map

$$F(\mathscr{F}) \xrightarrow{F(e)} F(\mathcal{I}) \to F(\pi_0 \mathscr{A}).$$

Since F is right t-exact, the projection map $F(\tau_{\geqslant 0} \mathscr{A}) \to F(\pi_0 \mathscr{A})$ induces an isomorphism on π_0, so that $F(e)$ induces a map of discrete R-modules $\lambda : \pi_0 F(\mathscr{F}) \to \pi_0 A$. Let $J \subseteq \pi_0 A$ be the ideal generated by the image of λ. The object $\mathscr{F} \in \mathrm{QCoh}(X)$ is connective and almost perfect, so that the R-module $F(\mathscr{F})$ is connective and almost perfect. It follows that $\pi_0 F(\mathscr{F})$ is a finitely presented R-module, so that the ideal J is finitely generated. Let $Y' \subseteq \overline{Y}'$ denote the open substack complementary to the vanishing locus of J. The affine projection map $\overline{v} : \overline{Y}' \to Y = \operatorname{Spec} R$ then restricts to a quasi-affine map $v : Y' \to \operatorname{Spec} R$. We will deduce $(*)$ from the following:

$(*')$ There is a canonical equivalence $F(\mathscr{A}) \simeq v_* \mathscr{O}_{Y'}$ in the ∞-category $\mathrm{CAlg}(\mathrm{QCoh}(Y)) \simeq \mathrm{CAlg}_R$.

By construction, we can identify $v_* \mathscr{O}_{Y'}$ with the J-localization of the $\mathbb{E}_\infty$-ring $A = F(\tau_{\geqslant 0} \mathscr{A})$. Set $M = \mathrm{fib}(A \to F(\mathscr{A})) \simeq F(\Sigma(\tau_{\leqslant -1} \mathscr{A}))$, so that we have a canonical fiber sequence $M \to A \to F(\mathscr{A})$. To prove $(*')$, it will suffice to show that this fiber sequence exhibits $F(\mathscr{A})$ as a J-localization of A. This is equivalent to the following pair of assertions:

(i) The R-module M is J-nilpotent.

(ii) The R-module $F(\mathscr{A})$ is J-local.

We first prove (i). Note that we can write M as a successive extension of shifted copies of objects of the form $F(\pi_m \mathscr{A})$, where $m < 0$. It will therefore suffice to show that $F(\pi_m \mathscr{A}) \in \mathrm{Mod}_R$ is J-nilpotent for $m < 0$. Using Proposition 9.5.2.3, we can write $\pi_m \mathscr{A}$ as the colimit of a filtered diagram $\{\mathscr{G}_\beta\}$ in the abelian category $\mathrm{QCoh}(X)^\heartsuit$. For each index β, let $\mathscr{G}_\beta^+$ denote the $(\pi_0 \mathscr{A})$-submodule of $\pi_m \mathscr{A}$ generated by $\mathscr{G}_\beta$. Under the equivalence $\mathrm{QCoh}(\overline{U}) \simeq \mathrm{Mod}_{\tau_{\geqslant 0} \mathscr{A}}(\mathrm{QCoh}(X))$, we can identify $\mathscr{G}_\beta^+$ with an object of $\mathrm{QCoh}(\overline{U})^\heartsuit$ which is perfect to order 0 and whose image in $\mathrm{QCoh}(U)$ vanishes (since $m < 0$). It follows that there exists an integer $n \gg 0$ such that the composite map

$$\mathscr{F}^{\otimes n} \otimes \mathscr{G}_\beta^+ \xrightarrow{e^n \otimes \mathrm{id}} (\pi_0 \mathscr{A}) \otimes \mathscr{G}_\beta \to \mathscr{G}_\beta^+$$

is nullhomotopic. Applying the functor F, we deduce that the map $F(\mathscr{F}^{\otimes n}) \otimes F(\mathscr{G}_\beta^+) \to F(\mathscr{G}_\beta^+)$ is nullhomotopic, so that the graded $(\pi_0 R)$-module $\pi_* F(\mathscr{G}_\beta^+)$ is annihilated by the ideal J^n and therefore $F(\mathscr{G}_\beta^+)$ is J-nilpotent. Writing $F(\pi_m \mathscr{A}) \simeq \varinjlim_\beta F(\mathscr{G}_\beta^+)$, we deduce that $F(\pi_m \mathscr{A})$ is J-nilpotent for $m < 0$, which completes the proof of (i).

We now prove (ii). Let us identify the map $e : \mathscr{F} \to \pi_0 \mathscr{A} = \overline{u}_*(\pi_0 \mathscr{O}_{\overline{U}})$ with a map $e' : \overline{u}^* \mathscr{F} \to \pi_0 \mathscr{O}_{\overline{U}}$. By construction, the induced map $e'|_U : u^* \mathscr{F} \to \pi_0 \mathscr{O}_U$ induces an epimorphism on π_0. Since U is affine, it follows that we can choose a map $s : \mathscr{O}_U \to u^* \mathscr{F}$ whose composition with $e'|_U$ is the identity section of $\pi_0 \mathscr{O}_U$. The direct image of s along u can be identified with a morphism $\mathscr{A} \to \mathscr{A} \otimes \mathscr{F}$ of $\mathscr{A}$-module objects of $\mathrm{QCoh}(X)$, and therefore determines a morphism of R-modules $F(s) : F(\mathscr{A}) \to F(\mathscr{A}) \otimes_R F(\mathscr{F})$. By construction, the map $F(s)$ fits into a commutative diagram

$$\begin{array}{ccc} & F(\mathscr{A}) & \\ {}^{F(s)}\swarrow & & \searrow \\ F(\mathscr{A}) \otimes_R F(\mathscr{F}) & \longrightarrow & F(\mathscr{A}) \otimes_A \pi_0 A. \end{array}$$

It follows that the composite map $F(\mathscr{A}) \to F(\mathscr{A}) \otimes_A (\pi_0 A) \to F(\mathscr{A}) \otimes_A (\pi_0 A/J)$ is nullhomotopic. Since this map is a morphism of $\mathbb{E}_\infty$-algebras, we conclude that $F(\mathscr{A}) \otimes_A (\pi_0 A/J)$. Since $\mathscr{A}$ is almost connective (Proposition 9.1.3.5) and the functor F is right t-exact, the A-algebra $F(\mathscr{A})$ is almost connective. Applying Lemma **??**, we deduce that $F(\mathscr{A})$ is J-local, as desired. This completes the proof of (ii), and therefore also the proof of $(*')$.

To complete the proof of $(*)$, we must show that the map $v : Y' \to Y$ is faithfully flat. We first claim that v is flat. Since Y is affine, it will suffice to show that for each

$\mathscr{G} \in \mathrm{QCoh}(Y)^{\heartsuit}$, the pullback $v^* \mathscr{G}$ belongs to $\mathrm{QCoh}(Y')_{\leqslant 0}$. Since Y' is quasi-affine, this is equivalent to the statement that the spectrum $\Gamma(Y'; v^* \mathscr{G})$ is 0-truncated. We now compute

$$\begin{aligned} \Gamma(Y'; v^* \mathscr{G}) &\simeq \Gamma(Y; v_* v^* \mathscr{G}) \\ &\simeq \Gamma(Y; v_* \mathscr{O}_{Y'} \otimes \mathscr{G}) \\ &\simeq \Gamma(Y; F(\mathscr{A}) \otimes \mathscr{G}). \end{aligned}$$

It now suffices to observe that $F(\mathscr{A}) \otimes \mathscr{G}$ is 0-truncated, since $\mathscr{G}$ is 0-truncated by assumption that $F(\mathscr{A})$ has Tor-amplitude $\leqslant 0$ by virtue of Proposition 9.5.4.6.

It remains to show that v is faithfully flat. For this, it will suffice to show that for $\mathscr{G} \in \mathrm{QCoh}(Y)^{\heartsuit}$ as above, the canonical map $\rho : \Gamma(Y; \mathscr{G}) \to \Gamma(Y'; v^* \mathscr{G})$ induces an injection on π_0 (in this case, the vanishing of $v^* \mathscr{G}$ guarantees the vanishing of $\pi_0 \Gamma(Y; \mathscr{G})$, hence also the vanishing of $\mathscr{G}$ since Y is affine and $\mathscr{G}$ is discrete). Using the preceding calculation, we can identify ρ with the canonical map $\Gamma(Y; \mathscr{G}) \to \Gamma(Y; F(\mathscr{A}) \otimes \mathscr{G})$, whose cofiber is given by $\Gamma(Y; F(\mathrm{cofib}(\mathscr{O}_X \to \mathscr{A})) \otimes \mathscr{G})$. It will therefore suffice to show that the group $\pi_1 \Gamma(Y; F(\mathrm{cofib}(\mathscr{O}_X \to \mathscr{A})) \otimes \mathscr{G})$ vanishes. This is clear: the tensor product $F(\mathrm{cofib}(\mathscr{O}_X \to \mathscr{A})) \otimes \mathscr{G}$ is 0-truncated, since $\mathscr{G}$ is 0-truncated by assumption and $F(\mathrm{cofib}(\mathscr{O}_X \to \mathscr{A}))$ has Tor-amplitude $\leqslant 0$ by virtue of Proposition 9.5.4.6. □

9.5.5 Almost Perfect Sheaves

Let X be a locally Noetherian quasi-geometric stack. It follows from Proposition 9.5.2.3 that the ∞-category $\mathrm{QCoh}(X)$ contains many almost perfect objects. We now show that the Tannaka duality results of §9.5.4 can be reformulated in terms of almost perfect sheaves.

Theorem 9.5.5.1. *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be functors and suppose that X is a locally Noetherian geometric stack. Then the construction $(f : Y \to X) \mapsto (f^* : \mathrm{QCoh}(X)^{\mathrm{aperf}} \to \mathrm{QCoh}(Y)^{\mathrm{aperf}})$ determines a fully faithful embedding*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{aperf}}, \mathrm{QCoh}(Y)^{\mathrm{aperf}}),$$

whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X)^{\mathrm{aperf}} \to \mathrm{QCoh}(Y)^{\mathrm{aperf}}$ which are exact and preserve connective objects.

Remark 9.5.5.2. Under the slightly stronger assumption that X is Noetherian, Theorem 9.5.5.1 was proven by Bhatt and Halpern-Leistner in [27].

Corollary 9.5.5.3. *Let R be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal I, let Y be a spectral algebraic spaces which is proper and almost of finite presentation over R, and let $\mathsf{Y}^{\wedge} = \mathsf{Y} \times_{\mathrm{Spec}\, R} \mathrm{Spf}\, R$ be the formal completion of Y*

along the vanishing locus of I, which we regard as a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Let X be a locally Noetherian quasi-geometric stack. Then the restriction map

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(\mathsf{Y}, X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(\mathsf{Y}^{\wedge}, X)$$

is a homotopy equivalence.

Proof. It follows from Theorem 8.5.0.3 (together with Corollary 8.3.4.6 and Theorem 8.3.5.2) that the restriction functor $\mathrm{QCoh}(\mathsf{Y})^{\mathrm{aperf}} \to \mathrm{QCoh}(\mathsf{Y}^{\wedge})^{\mathrm{aperf}}$ is an equivalence of ∞-categories, and from Proposition 8.5.1.4 that an object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})^{\mathrm{aperf}}$ is connective if and only if $\mathscr{F}|_{\mathsf{Y}^{\wedge}}$ is connective. The desired result now follows from Theorem 9.5.5.1. □

Proof of Theorem 9.5.5.1. Without loss of generality, we may assume that $Y = \mathrm{Spec}\, R$ is affine. Using Theorem 9.5.4.2, we can identify $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(Y, X)$ with the full subcategory

$$\mathrm{Fun}_0^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)) \subseteq \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$$

spanned by those symmetric monoidal functors which preserve small colimits, connective objects, and almost perfect objects. Let $\mathrm{Fun}_0^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{aperf}}, \mathrm{QCoh}(Y))$ denote the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{aperf}}, \mathrm{QCoh}(Y))$ spanned by those symmetric monoidal functors which preserve finite colimits, connective objects, and almost perfect objects. We wish to show that the restriction map

$$\theta : \mathrm{Fun}_0^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)) \to \mathrm{Fun}_0^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{aperf}}, \mathrm{QCoh}(Y))$$

is an equivalence of ∞-categories. To prove this, we may assume without loss of generality that $Y = \mathrm{Spec}\, R$ is affine.

Since X is locally Noetherian, the full subcategory $\mathrm{QCoh}(X)^{\mathrm{aperf}} \subseteq \mathrm{QCoh}(X)$ is stable under truncation, and therefore inherits a t-structure $(\mathrm{QCoh}(X)^{\mathrm{aperf}}_{\geqslant 0}, \mathrm{QCoh}(X)^{\mathrm{aperf}}_{\leqslant 0})$ where $\mathrm{QCoh}(X)^{\mathrm{aperf}}_{\geqslant 0} = \mathrm{QCoh}(X)^{\mathrm{aperf}} \cap \mathrm{QCoh}(X)_{\geqslant 0}$ and $\mathrm{QCoh}(X)^{\mathrm{aperf}}_{\leqslant 0} = \mathrm{QCoh}(X)^{\mathrm{aperf}} \cap \mathrm{QCoh}(X)_{\leqslant 0}$. Note that every almost perfect object of $\mathrm{QCoh}(X)$ is $(-n)$-truncated for $n \gg 0$, so that the t-structure $(\mathrm{QCoh}(X)^{\mathrm{aperf}}_{\geqslant 0}, \mathrm{QCoh}(X)^{\mathrm{aperf}}_{\leqslant 0})$ is right-bounded. Set $\mathcal{C} = \mathrm{Ind}(\mathrm{QCoh}(X)^{\mathrm{aperf}})$, so that $\mathcal{C}$ inherits a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ with $\mathcal{C}_{\geqslant 0} \simeq \mathrm{Ind}(\mathrm{QCoh}(X)^{\mathrm{aperf}}_{\geqslant 0})$ and $\mathcal{C}_{\leqslant 0} \simeq \mathrm{Ind}(\mathrm{QCoh}(X)^{\mathrm{aperf}}_{\leqslant 0})$. Moreover, the t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ is right complete and compatible with filtered colimits (Lemma C.2.4.3).

Let $\mathcal{D}$ be any symmetric monoidal ∞-category which admits small colimits, for which the tensor product $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$ preserves small colimits separately in each variable. Using Corollary HA.4.8.1.14 , we see that composition with the Yoneda embedding $\mathrm{QCoh}(X)^{\mathrm{aperf}} \hookrightarrow \mathcal{C}$ induces an equivalence of ∞-categories

$$\mathrm{LFun}^{\otimes}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\otimes}_{\mathrm{rex}}(\mathrm{QCoh}(X)^{\mathrm{aperf}}, \mathcal{D}),$$

where $\mathrm{LFun}^{\otimes}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ spanned by those symmetric monoidal functors which preserve small colimits, and $\mathrm{Fun}^{\otimes}_{\mathrm{rex}}(\mathrm{QCoh}(X)^{\mathrm{aperf}}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}^{\otimes}_{\mathrm{rex}}(\mathrm{QCoh}(X)^{\mathrm{aperf}}, \mathcal{D})$ spanned by those functors which preserve finite colimits. Applying this observation with $\mathcal{D} = \mathrm{QCoh}(X)$ (and using the fact that the t-structure on $\mathrm{QCoh}(X)$ is compatible with filtered colimits), we see that the inclusion $\mathrm{QCoh}(X)^{\mathrm{aperf}} \hookrightarrow \mathrm{QCoh}(X)$ admits an essentially unique extension $\lambda : \mathcal{C} \to \mathrm{QCoh}(X)$ which commutes with small colimits; moreover, λ is t-exact. Similarly, taking $\mathcal{D} = \mathrm{QCoh}(Y)$, we obtain an equivalence of ∞-categories $\mathrm{Fun}^{\otimes}_{0}(\mathrm{QCoh}(X)^{\mathrm{aperf}}, \mathrm{QCoh}(Y)) \simeq \mathrm{Fun}^{\otimes}_{0}(\mathcal{C}, \mathrm{QCoh}(Y))$, where $\mathrm{Fun}^{\otimes}_{0}(\mathcal{C}, \mathrm{QCoh}(Y))$ denotes the full subcategory of $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathrm{QCoh}(Y))$ spanned by those symmetric monoidal functors which are right t-exact and preserve small colimits. Under this equivalence, we can identify θ with the map $\mathrm{Fun}^{\otimes}_{0}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)) \to \mathrm{Fun}^{\otimes}_{0}(\mathcal{C}, \mathrm{QCoh}(Y))$ given by precomposition with λ.

Let $\mathrm{Fun}^{\otimes}_{0}(\mathrm{QCoh}(X)^{\mathrm{cn}}, \mathrm{QCoh}(Y)^{\mathrm{cn}}) \subseteq \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}, \mathrm{QCoh}(Y)^{\mathrm{cn}})$ be the full subcategory spanned by those symmetric monoidal functors which preserve small colimits, and define $\mathrm{Fun}^{\otimes}_{0}(\mathcal{C}_{\geqslant 0}, \mathrm{QCoh}(Y)^{\mathrm{cn}})$ similarly. Using Proposition C.3.1.1 (together with the observation that the construction $\mathcal{E} \mapsto \mathrm{Sp}(\mathcal{E})$ determines a symmetric monoidal functor $\mathsf{Groth}_{\infty} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$), we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}^{\otimes}_{0}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)) & \xrightarrow{\theta} & \mathrm{Fun}^{\otimes}_{0}(\mathcal{C}, \mathrm{QCoh}(Y)) \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{\otimes}_{0}(\mathrm{QCoh}(X)^{\mathrm{cn}}, \mathrm{QCoh}(Y)^{\mathrm{cn}}) & \xrightarrow{\theta'} & \mathrm{Fun}^{\otimes}_{0}(\mathcal{C}_{\geqslant 0}, \mathrm{QCoh}(Y)^{\mathrm{cn}}) \end{array}$$

where the vertical maps are equivalences. We will complete the proof by showing that θ' is an equivalence of ∞-categories. Since the Grothendieck prestable ∞-category $\mathrm{QCoh}(Y)^{\mathrm{cn}} \simeq \mathrm{Mod}^{\mathrm{cn}}_{R}$ is complete, it will suffice to show that the functor $\lambda : \mathcal{C}_{\geqslant 0} \to \mathrm{QCoh}(X)^{\mathrm{cn}}$ exhibits $\mathrm{QCoh}(X)^{\mathrm{cn}}$ as the completion of $\mathcal{C}_{\geqslant 0}$, in the sense of Proposition C.3.6.3. In other words, it will suffice to show that for each $n \geqslant 0$, the functor λ induces an equivalence of ∞-categories $\tau_{\leqslant n} \mathcal{C}_{\geqslant 0} \to \tau_{\leqslant n} \mathrm{QCoh}(X)^{\mathrm{cn}}$. Note that the domain of this functor can be identified with $\mathrm{Ind}(\tau_{\leqslant n} \mathrm{QCoh}(X)^{\mathrm{aperf}}_{\geqslant 0})$. The desired result now follows from Corollary **??** (since the Grothendieck prestable ∞-category $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is locally Noetherian by virtue of Proposition 9.5.2.3. □

9.6 Tannaka Duality for Spectral Algebraic Spaces

Let X be a spectral algebraic space which is quasi-compact and quasi-separated. Throughout this section, we will abuse notation by not distinguishing between X and the functor $h_{\mathsf{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ represented by X. With this abuse, we can regard X as a quasi-geometric stack (Corollary 9.1.4.6). It follows from Theorem 9.2.0.2 that X can be recovered from the symmetric monoidal ∞-category $\mathrm{QCoh}(\mathsf{X})$ of quasi-coherent sheaves on X, together with its t-structure. In this section, we will show that even the t-structure is unnecessary, thanks to the following refinement of Theorem 9.2.0.2:

Theorem 9.6.0.1 (Tannaka Duality for Spectral Algebraic Spaces)**.** *Let X and Y be spectral Deligne-Mumford stacks. Assume that X is a quasi-compact, quasi-separated spectral algebraic space. Then the construction $(f : \mathsf{Y} \to \mathsf{X}) \mapsto (f^* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y}))$ induces a fully faithful embedding $\mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Y}, \mathsf{X}) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathsf{X}), \mathrm{QCoh}(\mathsf{Y}))$, whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ which preserve small colimits.*

Corollary 9.6.0.2. *Let $\mathcal{C} \subseteq \mathrm{SpDM}$ denote the full subcategory spanned by the quasi-compact, quasi-separated algebraic spaces. Then the construction $\mathsf{X} \mapsto \mathrm{QCoh}(\mathsf{X})$ determines a fully faithful embedding $\mathcal{C}^{\mathrm{op}} \to \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.*

9.6.1 Compact Generation of $\mathrm{QCoh}(\mathsf{X})$

Our first step in the proof of Theorem 9.6.0.1 is to establish the following:

Proposition 9.6.1.1. *Let X be a quasi-compact, quasi-separated spectral algebraic space. Then the ∞-category $\mathrm{QCoh}(\mathsf{X})$ is compactly generated. Moreover, an object of $\mathrm{QCoh}(\mathsf{X})$ is compact if and only if it is perfect. In other words, X is a perfect stack (Definition 9.4.4.1).*

Combining Proposition 9.6.1.1 with Corollary 9.4.4.7, we immediately obtain the following weaker version of Theorem 9.6.0.1:

$(*)$ If X is a quasi-compact, quasi-separated spectral algebraic space and Y is an arbitrary spectral Deligne-Mumford stack, then the construction $(f : \mathsf{Y} \to \mathsf{X}) \mapsto (f^* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y}))$ determines an equivalence from $\mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Y}, \mathsf{X})$ to the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathsf{X}), \mathrm{QCoh}(\mathsf{Y}))$ spanned by those symmetric monoidal functors which are right t-exact and preserve small colimits.

To deduce Theorem 9.6.0.1 from $(*)$, we will need to show that the hypothesis of right t-exactness is automatically satisfied. To prove this, we will need the following variant of Proposition 9.6.1.1:

Proposition 9.6.1.2. *Let X be a quasi-compact, quasi-separated spectral algebraic space, and let $\mathcal{C} \subseteq \mathrm{QCoh}(\mathsf{X})$ denote the full subcategory spanned by those quasi-coherent sheaves which are perfect and connective. Then the inclusion $\mathcal{C} \hookrightarrow \mathrm{QCoh}(\mathsf{X})$ extends to an equivalence of ∞-categories $\mathrm{Ind}(\mathcal{C}) \simeq \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$.*

Before giving the proof of Proposition 9.6.1.2, let us describe some of its consequences.

Corollary 9.6.1.3. *Let* X *be a quasi-compact, quasi-separated spectral algebraic space, and let* $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ *be connective and* n*-truncated. Then* $\mathscr{F}$ *can be written as a filtered colimit* $\varinjlim \mathscr{F}_\alpha$*, where each* $\mathscr{F}_\alpha$ *is connective and finitely* n*-presented.*

Proof. Using Proposition 9.6.1.2, we can write $\mathscr{F}$ as a filtered colimit $\varinjlim_\alpha \mathscr{G}_\alpha$, where each $\mathscr{G}_\alpha$ is connective and perfect. Since $\mathscr{F}$ is n-truncated, we obtain $\mathscr{F} \simeq \tau_{\leqslant n} \mathscr{F} \simeq \varinjlim_\alpha \tau_{\leqslant n} \mathscr{G}_\alpha$, where each $\tau_{\leqslant n} \mathscr{G}_\alpha$ is connective and finitely n-presented. □

Corollary 9.6.1.4. *Let* $f : \mathsf{X} \to \mathsf{Y}$ *be a morphism of spectral Deligne-Mumford stacks which is proper, locally almost of finite presentation, and of relative dimension* $\leqslant d$*. Then the pushforward functor* $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ *carries* $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$ *into* $\mathrm{QCoh}(\mathsf{Y})_{\geqslant -d}$*.*

Proof. Let $\mathscr{F}$ be a connective quasi-coherent sheaf on X; we wish to show that $f_* \mathscr{F}$ is $(-d)$-connective. This assertion is local on Y; we may therefore assume without loss of generality that $\mathsf{Y} \simeq \mathrm{Sp\acute{e}t}\, A$ for some connective $\mathbb{E}_\infty$-ring A. Under this assumption, X is a quasi-compact quasi-separated spectral algebraic space. Using Proposition 9.6.1.2, we can write $\mathscr{F}$ as the colimit of a filtered diagram $\{\mathscr{F}_\alpha\}$ in $\mathrm{QCoh}(\mathsf{X})$, where each $\mathscr{F}_\alpha$ is perfect and connective. Since the functor f_* commutes with filtered colimits and the full subcategory $\mathrm{QCoh}(\mathsf{Y})_{\geqslant -d} \subseteq \mathrm{QCoh}(\mathsf{Y})$ is closed under filtered colimits, it will suffice to show that each pushforward $f_* \mathscr{F}_\alpha$ belongs to $\mathrm{QCoh}(\mathsf{Y})_{\geqslant -d}$. We may therefore replace $\mathscr{F}$ by $\mathscr{F}_\alpha$ and thereby reduce to the case where $\mathscr{F}$ is perfect. In this case, $f_* \mathscr{F} \in \mathrm{QCoh}(\mathsf{Y}) \simeq \mathrm{Mod}_A$ is almost perfect (Theorem 5.6.0.2). Consequently, to prove that $f_* \mathscr{F}$ is n-connective, it will suffice to show that for each residue field κ of A, the tensor product $\kappa \otimes_A f_* \mathscr{F}$ is n-connective (Corollary 2.7.4.3). We may therefore replace A by κ and thereby reduce to the case where $A = \kappa$ is a field. In this case, our assumption that f has relative dimension $\leqslant d$ guarantees that X is a locally Noetherian spectral algebraic space of Krull dimension $\leqslant d$ (Proposition 3.7.6.2), so that the $f_* \mathscr{F} \simeq \Gamma(\mathsf{X}; \mathscr{F})$ is $(-d)$-connective by virtue of Theorem 3.7.0.2. □

Warning 9.6.1.5. Corollary 9.6.1.4 is not true if f is not assumed to be proper. For example, every open immersion $j : \mathsf{U} \hookrightarrow \mathsf{X}$ has relative dimension $\leqslant 0$, but the pushforward functor j_* is generally not right t-exact unless j is affine.

Proof of Proposition 9.6.1.1 from Proposition 9.6.1.2. Let X be a spectral algebraic space which is quasi-compact and quasi-separated and let $\mathcal{C}$ denote the full subcategory of $\mathrm{QCoh}(\mathsf{X})$ spanned by the perfect objects. It follows from Corollary 9.1.5.5 that the inclusion $\mathcal{C} \hookrightarrow \mathrm{QCoh}(\mathsf{X})$ extends to a fully faithful embedding $\theta : \mathrm{Ind}(\mathcal{C}) \to \mathrm{QCoh}(\mathsf{X})$, whose essential image is closed under small colimits. Let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$; we wish to show that $\mathscr{F}$ belongs to the essential image of θ. Since the t-structure on $\mathrm{QCoh}(\mathsf{X})$ is right-complete, we can write $\mathscr{F}$ as a colimit of the sequence

$$\cdots \to \tau_{\geqslant 0} \mathscr{F} \to \tau_{\geqslant -1} \mathscr{F} \to \tau_{\geqslant -2} \mathscr{F} \to \cdots .$$

It therefore suffices to show that each truncation $\tau_{\geqslant -n}\mathscr{F}$ belongs to the essential image of θ, which follows immediately from Proposition 9.6.1.2. □

Corollary 9.6.1.6. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be a quasi-compact, quasi-separated spectral algebraic space over R. For every map of connective $\mathbb{E}_\infty$-rings $R \to A$, let X_A denote the fiber product $\mathrm{Sp\acute{e}t}\, A \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$, and let $\mathrm{QCoh}(\mathsf{X}_A)^{\mathrm{perf}}$ denote the full subcategory of $\mathrm{QCoh}(\mathsf{X}_A)$ spanned by those quasi-coherent sheaves which are perfect. Then construction $A \mapsto \mathrm{QCoh}(\mathsf{X}_A)^{\mathrm{perf}}$ determines a functor $\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$ which preserves small filtered colimits.*

Proof. Since X is perfect (Proposition 9.6.1.1), Corollary 9.4.2.3 supplies equivalences

$$\mathrm{QCoh}(\mathsf{X}_A) \simeq \mathrm{Mod}_A \otimes_{\mathrm{Mod}_R} \mathrm{QCoh}(\mathsf{X}).$$

Using Lemma HA.7.3.5.12 (and that the forgetful functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Alg}$ preserves filtered colimits), we deduce that the construction $A \mapsto \mathrm{QCoh}(\mathsf{X}_A)$ determines a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ which preserves small filtered colimits. Using Corollary 9.1.5.5, we can identify $\mathrm{QCoh}(\mathsf{X}_A)^{\mathrm{perf}}$ with the full subcategory of $\mathrm{QCoh}(\mathsf{X}_A)$ spanned by the compact objects. The desired result now follows from Lemma HA.7.3.5.11 . □

9.6.2 The Proof of Proposition 9.6.1.2

If X is a spectral algebraic space, then $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is a Grothendieck prestable ∞-category. We wish to show that if X is quasi-compact and quasi-separated, then the ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is compactly generated (note that an object of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is compact if and only if it is perfect, by virtue of Corollary 9.1.5.5 and Lemma C.6.1.3). By virtue of Corollary C.6.3.3, Proposition 9.6.1.2 can be reformulated as follows:

Proposition 9.6.2.1. *Let X be a quasi-compact, quasi-separated spectral algebraic space and let $\mathscr{F}$ be a nonzero connective object of $\mathrm{QCoh}(\mathsf{X})$. Then there exists a connective perfect object $\mathscr{F}_0 \in \mathrm{QCoh}(\mathsf{X})$ and a morphism $\mathscr{F}_0 \to \mathscr{F}$ which is not nullhomotopic.*

The proof of Proposition 9.6.2.1 will require some preliminaries.

Lemma 9.6.2.2. *Let A be a connective $\mathbb{E}_\infty$-ring, let $I = (a_1, \ldots, a_d)$ be a finitely generated ideal in $\pi_0 A$, let M be an I-nilpotent A-module, and let $K \subseteq \pi_0 M$ be a finitely generated $(\pi_0 A)$-submodule. Then there exists a connective perfect I-nilpotent A-module P and a map $\beta : P \to M$ which induces a surjection from $\pi_0 P$ onto K. Moreover, if Q is an A-module of* Tor*-amplitude $\leqslant 0$ and we are given an element $\eta \in \pi_n(M \otimes_A Q)$ for $n \geqslant 0$, then we can arrange that η lifts to a class $\eta_P \in \pi_n(P \otimes_A Q)$.*

Proof. We will prove that for each $0 \leqslant i \leqslant d$, we can choose a connective perfect A-module P_i which is $(a_1, \ldots, a_i)$-nilpotent, and a map $\beta_i : P_i \to M$ which induces a surjection from $\pi_0 P_i$ on K, and for which η is the image of some class $\eta_i \in \pi_n(P_i \otimes_A Q)$.

We first treat the case $i = 0$. Write $\tau_{\geqslant 0} M$ as a colimit $\varinjlim N_\alpha$ indexed by a filtered partially ordered set, where each N_α is perfect and connective. Since K is finitely generated, we can choose α for which the image of the map $\pi_0 N_\alpha \to \pi_0 M$ contains K. The condition that Q has Tor-amplitude $\leqslant 0$ implies that the map $\pi_n(\tau_{\geqslant 0} M \otimes_A Q) \to \pi_n(M \otimes_A Q)$ is bijective. Enlarging α if necessary, we can arrange that η belongs to the image of the map $\pi_n(N_\alpha \otimes_A Q) \to \pi_n(M \otimes_A Q)$. We can therefore take $P_0 = N_\alpha$.

To carry out the inductive step, assume that $\beta_{i-1} : P_{i-1} \to M$ has been constructed. Since P_{i-1} is perfect, we can identify β_{i-1} with an element of $\pi_0(P_{i-1}^\vee \otimes_A M)$, where $P_{i-1}^\vee$ denotes the A-linear dual of P_{i-1}. Since M is I-nilpotent, the tensor product $\pi_0(P_{i-1}^\vee \otimes_A M)$ is I-nilpotent. It follows that β_{i-1} is annihilated by a_i^e for $e \gg 0$. Consequently, the map β_{i-1} factors as a composition $P_{i-1} \to P_i \xrightarrow{\beta_i} M$, where $P_i = \mathrm{cofib}(a_i^e : P_{i-1} \to P_{i-1}$. We can then take η_i to be the image of η_{i-1}. □

Lemma 9.6.2.3. *Let A be a connective $\mathbb{E}_\infty$-ring, let $I = (a_1, \ldots, a_d)$ be a finitely generated ideal in $\pi_0 A$, and let M be a perfect A-module for which each localization $M[a_i^{-1}]$ is connective. Then there exists a morphism of A-modules $\alpha : M' \to M$, where M' is perfect and connective and* $\mathrm{cofib}(\alpha)$ *is I-nilpotent.*

Moreover, if Q is an A-module of Tor*-amplitude $\leqslant 0$ and $\eta \in \pi_0(M \otimes_A Q)$, then we choose α so that η lifts to an element $\eta' \in \pi_0(M' \otimes_A Q)$.*

Proof. Since M is perfect, we can choose an integer n such that M is $(-n)$-connective. We proceed by induction on n. If $n \geqslant 0$, then M is connected and we can take $\alpha = \mathrm{id}_M$. Assume therefore that $n < 0$. Then $\pi_{-n} M$ is a finitely generated module over $\pi_0 A$, and each of the localizations $(\pi_{-n} M)[a_i^{-1}]$ vanishes. Replacing the a_i by suitable powers, we may assume without loss of generality that $\pi_{-n} M$ is annihilated by each a_i. Let N denote the tensor product of the A-modules $\mathrm{cofib}(a_i : A \to A)$, so that $\pi_1(N \otimes_A (\pi_{-n} M))$ is a direct sum of finitely many copies of $(\pi_{-n} M)$, and therefore a finitely generated module over $\pi_0 A$.

Let η_N denote the image of η in $\pi_0(N \otimes_A M \otimes_A Q)$. We have an exact sequence

$$\pi_{1-n}(N \otimes_A \tau_{\geqslant 1-n} M) \xrightarrow{\phi} \pi_{1-n}(N \otimes_A M) \to \pi_1(N \otimes_A (\pi_{-n} M)) \to 0.$$

We can therefore choose a finitely generated submodule $K \subseteq \pi_{1-n}(N \otimes_A M)$ such that $K + \mathrm{im}(\phi) = \pi_{1-n}(N \otimes_A M)$. Applying Lemma 9.6.2.2, we can choose a connective perfect A-module P which is I-nilpotent and a map $\beta : \Sigma^{1-n} P \to N \otimes_A M$ which induces a surjection $\pi_0 P \to K \subseteq \pi_{1-n}(N \otimes_A M)$, and for which η_N lifts to an element $\overline{\eta}_N \in \pi_0(\Sigma^{1-n} P \otimes_A Q)$.

Form a pullback diagram

$$\begin{array}{ccc} M'' & \longrightarrow & \Sigma^{1-n} P \\ \downarrow & & \downarrow \beta \\ M & \longrightarrow & N \otimes_A M. \end{array}$$

Using the exactness of the sequence

$$\pi_0(M'' \otimes_A Q) \to \pi_0(M \otimes_A Q) \oplus \pi_0(\Sigma^{1-n} P \otimes_A Q) \to \pi_0(N \otimes_A M \otimes_A Q),$$

we deduce that there exists an element $\eta'' \in \pi_0(M'' \otimes_A Q)$ lifting η.

It is clear from the construction that M'' is $(-n)$-connective, and we have an exact sequence

$$\pi_0 P \oplus \pi_{1-n} M \xrightarrow{u} \pi_{1-n}(N \otimes_A M) \to \pi_{-n} M'' \to \pi_{-n} M \xrightarrow{v} \pi_{-n}(N \otimes_A V).$$

By construction, u is surjective and v is bijective, so that $\pi_{-n} M'' \simeq 0$. Since $N \otimes_A M$ and P are I-nilpotent, the map $M'' \to M$ induces an equivalence $M''[a_i^{-1}] \simeq M[a_i^{-1}]$ for $1 \leqslant i \leqslant d$. Applying the inductive hypothesis, we conclude that there is a connective perfect A-module M' with a map $M' \to M''$ which induces equivalences $M'[a_i^{-1}] \simeq M''[a_i^{-1}]$ for $1 \leqslant i \leqslant d$, and for which $\eta'' \in (M'' \otimes_A Q)$ can be lifted to an element $\eta' \in (M' \otimes_A Q)$. The composite map $M' \to M'' \to M$ has the desired properties. □

Lemma 9.6.2.4. *Let A be a connective $\mathbb{E}_\infty$-ring, let $I = (a_1, \ldots, a_d)$ be a finitely generated ideal in $\pi_0 A$, and let M be a perfect A-module for which each localization $M[a_i^{-1}]$ is connective. Then there exists a morphism of A-modules $\alpha : M' \to M$, where M' is perfect and connective and* $\mathrm{cofib}(\alpha)$ *is I-nilpotent.*

Proof. Since M is perfect, we can choose an integer n such that M is $(-n)$-connective. We proceed by induction on n. If $n \geqslant 0$, then M is connected and we can take $\alpha = \mathrm{id}_M$. Assume therefore that $n < 0$. Then $\pi_{-n} M$ is a finitely generated module over $\pi_0 A$, and each of the localizations $(\pi_{-n} M)[a_i^{-1}]$ vanishes. Replacing the a_i by suitable powers, we may assume without loss of generality that $\pi_{-n} M$ is annihilated by each a_i. Let N denote the tensor product of the A-modules $\mathrm{cofib}(a_i : A \to A)$, so that $\pi_1(N \otimes_A (\pi_{-n} M))$ is a direct sum of finitely many copies of $(\pi_{-n} M)$, and therefore a finitely generated module over $\pi_0 A$. We have an exact sequence

$$\pi_{1-n}(N \otimes_A \tau_{\geqslant 1-n} M) \xrightarrow{\phi} \pi_{1-n}(N \otimes_A M) \to \pi_1(N \otimes_A (\pi_{-n} M)) \to 0.$$

We can therefore choose a finitely generated submodule $K \subseteq \pi_{1-n}(N \otimes_A M)$ such that $K + \mathrm{im}(\phi) = \pi_{1-n}(N \otimes_A M)$. Applying Lemma 9.6.2.2, we can choose a connective perfect

A-module P which is I-nilpotent and a map $\beta : \Sigma^{1-n} P \to N \otimes_A M$ which induces a surjection $\pi_0 P \to K \subseteq \pi_{1-n}(N \otimes_A M)$. Form a pullback diagram

$$\begin{array}{ccc} M'' & \longrightarrow & \Sigma^{1-n} P \\ \downarrow & & \downarrow \beta \\ M & \longrightarrow & N \otimes_A M. \end{array}$$

It is clear from the construction that M'' is $(-n)$-connective, and we have an exact sequence

$$\pi_0 P \oplus \pi_{1-n} M \xrightarrow{u} \pi_{1-n}(N \otimes_A M) \to \pi_{-n} M'' \to \pi_{-n} M \xrightarrow{v} \pi_{-n}(N \otimes_A V).$$

By construction, u is surjective and v is bijective, so that $\pi_{-n} M'' \simeq 0$. Since $N \otimes_A M$ and P are I-nilpotent, the map $M'' \to M$ induces an equivalence $M''[a_i^{-1}] \simeq M[a_i^{-1}]$ for $1 \leqslant i \leqslant d$. Applying the inductive hypothesis, we conclude that there is a connective perfect A-module M' with a map $M' \to M''$ which induces equivalences $M'[a_i^{-1}] \simeq M''[a_i^{-1}]$ for $1 \leqslant i \leqslant d$. The composite map $M' \to M'' \to M$ has the desired properties. □

Proof of Proposition 9.6.2.1. Let X be a quasi-compact, quasi-separated algebraic space and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ be nonzero and connective. Using Theorem 3.4.2.1, we can choose a scallop decomposition

$$\varnothing \simeq \mathsf{U}_0 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_m \simeq \mathsf{X}.$$

For $0 \leqslant i \leqslant m$, let $\mathscr{F}_i$ denote the restriction of $\mathscr{F}$ to U_i. Let i_0 be the smallest integer such that $\mathscr{F}_{i_0} \neq 0$. Then there exists an integer $n \geqslant 0$ for which $\pi_n \mathscr{F}_{i_0} \neq 0$. Replacing $\mathscr{F}$ by $\Omega^n \mathscr{F}$, we can assume that $\pi_0 \mathscr{F}_{i_0} \neq 0$, while $\mathscr{F}_i \simeq 0$ for $i < i_0$. We will complete the proof by showing the following:

- $(*)$ For $i_0 \leqslant i \leqslant m$, there exists a perfect connective object $\mathscr{G}_i \in \mathrm{QCoh}(\mathsf{U}_i)$ and a nonzero morphism $\beta_i : \mathscr{G}_i \to \mathscr{F}_i$.

The proof proceeds by induction on i. Choose an excision square

$$\begin{array}{ccc} \mathsf{V} & \xrightarrow{f} & \mathrm{Sp\acute{e}t}\, A \\ \downarrow g & & \downarrow \\ \mathsf{U}_{i-1} & \longrightarrow & \mathsf{U}_i, \end{array}$$

so that V is the quasi-compact open substack of $\mathrm{Sp\acute{e}t}\, A$ complementary to the vanishing locus of a finitely generated ideal $I = (a_1, \ldots, a_d) \subseteq \pi_0 A$. Let $M \in \mathrm{Mod}_A \simeq \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A)$ denote the connective A-module given by the pullback of $\mathscr{F}_i$.

We begin by treating the base case $i = i_0$. In this case, we have $\mathscr{F}_{i-1} \simeq 0$ so that the module M is I-nilpotent. By construction, we have $\pi_0 M \neq 0$. Then Lemma 9.6.2.2

guarantees the existence of a connective perfect I-nilpotent A-module P and a nonzero map $q : P \to M$. It follows from Nisnevich descent that we can identify the ∞-category of I-nilpotent A-modules with the full subcategory of $\mathrm{QCoh}(\mathsf{U}_i)$ consisting of objects which vanish in $\mathrm{QCoh}(\mathsf{U}_{i-1})$, so that q determines a map $\beta_i : \mathscr{G}_i \to \mathscr{F}_i$ having the desired properties.

We now carry out the inductive step. Assume that $i > i_0$ and that $\beta_{i-1} : \mathscr{G}_{i-1} \to \mathscr{F}_{i-1}$ has already been constructed. Replacing $\mathscr{G}_{i-1}$ by $\mathscr{G}_{i-1} \oplus \Sigma \mathscr{G}_{i-1}$ if necessary, we may assume that $g^* \mathscr{G}_{i-1} \simeq f^* N$ for some perfect object $N \in \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A)$ (Lemma **??**). In this case, we can identify $g^* \beta_{i-1}$ with a map $v_0 : f^* N \to f^* M$.

Let $u : M \to f_* f^* M$ denote the unit map. Then $\mathrm{fib}(u)$ is I-nilpotent, and can therefore be written as a filtered colimit of perfect I-nilpotent A-modules K_α. For each index α, let M_α denote the cofiber of the map $K_\alpha \to N$, so that $f_* f^* M \simeq \varinjlim M_\alpha$. The map v_0 can be identified with a map $N \to f_* f^* M \simeq \varinjlim M_\alpha$. Since N is perfect, this map factors as a composition $N \xrightarrow{p} M_\alpha \to f_* f^* M$. Form a pullback diagram

$$\begin{array}{ccc} N' & \longrightarrow & N \\ \downarrow & & \downarrow \\ M & \longrightarrow & M_\alpha, \end{array}$$

so that we have a fiber sequence $K_\alpha \to N' \to N$. Replacing N by N', we can reduce to the case where v_0 lifts to a map $v : N \to M$. Applying Lemma 9.6.2.4, we may further reduce to the case where N is connective. Using Nisnevich descent, we conclude that v and β_{i-1} can be amalgamated to a morphism $\beta_i : \mathscr{G}_i \to \mathscr{F}_i$ in $\mathrm{QCoh}(\mathsf{U}_i)$ having the desired properties. □

9.6.3 Generation of $\mathrm{QCoh}(\mathsf{X})$ by a Single Object

Let X be a quasi-compact, quasi-separated spectral algebraic space. Proposition 9.6.1.1 asserts that the ∞-category $\mathrm{QCoh}(\mathsf{X})$ is compactly generated. In fact, we can say a bit more: it can be generated (as a presentable stable ∞-category) by a single compact object. This is a consequence of the following:

Proposition 9.6.3.1. *Let X be a quasi-compact, quasi-separated spectral algebraic space. Then there exists an object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}$ with the following property:*

- $(*)$ *Let $\mathcal{C}$ be the smallest full subcategory of $\mathrm{QCoh}(\mathsf{X})$ which contains $\mathscr{F}$ and is closed under colimits and extensions. Then $\mathrm{QCoh}(\mathsf{X})_{\geq 0} \subseteq \mathcal{C}$.*

Corollary 9.6.3.2. *Let X be a quasi-compact, quasi-separated spectral algebraic space. Then the ∞-category $\mathrm{QCoh}(\mathsf{X})$ has a compact generator. More precisely, there exists an object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}$ such that, for every nonzero object $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$, the graded abelian group $\mathrm{Ext}^*_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \mathscr{G})$ is nonzero.*

Corollary 9.6.3.3. *Let* X *be a quasi-compact, quasi-separated spectral algebraic space. Then there exists an* $\mathbb{E}_1$*-ring* A *and an equivalence of* ∞*-categories* $\mathrm{QCoh}(\mathsf{X}) \simeq \mathrm{LMod}_A$.

Proof. Combine Corollary 9.6.3.2 with Theorem HA.7.1.2.1 . □

Proof of Proposition 9.6.3.1. Let X be a quasi-compact, quasi-separated spectral algebraic space. Using Theorem 3.4.2.1, we can choose a scallop decomposition

$$\varnothing = \mathsf{U}_0 \hookrightarrow \mathsf{U}_1 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n = \mathsf{X}.$$

For $0 \leqslant i \leqslant n$, let $K_i \subseteq |\mathsf{X}|$ denote the closed subset complementary to the image of the open immersion $\mathsf{U}_i \hookrightarrow \mathsf{X}$. We will prove, by descending induction on i, that there exists objects $\mathscr{F}_i \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}$ and integers c_i satisfying the following version of $(*)$:

- $(*_i)$ Let $\mathcal{C}_{i+1}$ be the smallest full subcategory of $\mathrm{QCoh}(\mathsf{X})$ which contains $\mathscr{F}_i$ and is closed under colimits and extensions. Then $\mathrm{QCoh}_{K_i}(\mathsf{X})_{\geqslant c_i} \subseteq \mathcal{C}$.

When $i = n$, this is evident (since $K_n = \varnothing$, we can take $\mathscr{F}_n = 0$), and when $i = 0$ it implies the desired result (take $\mathscr{F} = \Sigma^{-c_0}\mathscr{F}_0$). We now carry out the inductive step. Suppose that $i < n$, and that we are given a sheaf $\mathscr{F}_{i+1} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}$ and an integer c_{i+1} satisfying $(*_{i+1})$. Choose an excision square

$$\begin{array}{ccc} \mathsf{V} & \xrightarrow{j} & \mathrm{Sp\acute{e}t}\, A \\ \downarrow & & \downarrow g \\ \mathsf{U}_i & \longrightarrow & \mathsf{U}_{i+1} \end{array}$$

where j is a quasi-compact open immersion. In what follows, we will abuse notation by identifying quasi-coherent sheaves on $\mathrm{Sp\acute{e}t}\, A$ with their images under the equivalence of ∞-categories $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A) \simeq \mathrm{Mod}_A$. Let $Y \subseteq |\operatorname{Spec} A|$ denote the complement of the image of j, so that Y is the vanishing locus of a finitely generated ideal $(x_1, \ldots, x_d) \subseteq \pi_0 A$. For $1 \leqslant k \leqslant d$, let M_k denote the fiber of the map $x_k : A \to A$, and set $M = \bigotimes_{1 \leqslant k \leqslant d} M_k$, so that M is a perfect A-module whose restriction to V vanishes. We will identify M with the corresponding quasi-coherent sheaf on $\mathrm{Sp\acute{e}t}\, A$, so that $g_* M$ is a perfect object of $\mathrm{QCoh}(\mathsf{U}_{i+1})$. The proof of Theorem **??** shows that the direct sum $g_* M \oplus \Sigma g_* M$ can be written as the restriction to U_{i+1} of some object $\mathscr{F}' \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}$.

Let g' denote the composition of g with the inclusion map $\mathsf{U}_{i+1} \hookrightarrow \mathsf{X}$. Proposition 2.5.4.4 implies that there exists an integer n such that the pushforward functor g'_* carries $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A)_{\geqslant 0}$ into $\mathrm{QCoh}(\mathsf{X})_{\geqslant -n}$ (if X is separated, then g' is an affine morphism and we can take $n = 0$). Let $\mathscr{F}''$ denote the cofiber of the canonical map $\mathscr{F}' \to g'_* M \oplus \Sigma g'_* M$. Since $\mathscr{F}'$ is perfect and $g'_* \oplus \Sigma g'_* M \in \mathrm{QCoh}(\mathsf{X})_{\geqslant -n}$, the sheaf $\mathscr{F}''$ belongs to $\mathcal{S}\mathrm{hv}_{K_{i+1}}(\mathsf{X})_{\geqslant -m}$ for some integer m. Choose an integer $c_i \geqslant 0$ such that $c_i > c_{i+1} + n$ and $c_i \geqslant c_{i+1} + m$, and set $\mathscr{F}_i = \mathscr{F}_{i+1} \oplus \mathscr{F}$. We will prove that $(\mathscr{F}_i, c_i)$ satisfies $(*)$.

Fir an object $\mathscr{G} \in \mathrm{QCoh}_{K_i}(\mathsf{X})_{\geqslant c_i}$; we wish to show that $\mathscr{G} \in \mathcal{C}_i$. We have a canonical fiber sequence $\mathscr{G}' \to \mathscr{G} \to g'_* g'^* \mathscr{G}$. Our choice of n guarantees that $g'_* g'^* \mathscr{G} \in \mathrm{QCoh}(\mathsf{X})_{\geqslant c_i - n} \subseteq \mathrm{QCoh}(\mathsf{X})_{\geqslant c_{i+1}+1}$, so that $\mathscr{G}' \in \mathrm{QCoh}_{K_{i+1}}(\mathsf{X})_{\geqslant c_{i+1}} \subseteq \mathcal{C}_{i+1} \subseteq \mathcal{C}_i$. Since $\mathcal{C}_i$ is closed under extensions, we are reduced to proving that $g'_* g'^* \mathscr{G} \in \mathcal{C}_i$. This follows from the following more general claim:

$(*')$ For every object $N \in \mathrm{QCoh}_Y(\mathrm{Spét}\, A)_{\geqslant c_i}$, the direct image $g'_* N$ belongs to $\mathcal{C}_i$.

We will prove the following more general claim, using descending induction on $k \leqslant d$:

$(*'_k)$ For every objcet $N \in \mathrm{QCoh}_Y(\mathrm{Spét}\, A)_{\geqslant c_i}$, the direct image $g'_*(N \otimes_A M_1 \otimes_A \cdots \otimes_A M_k)$ belongs to $\mathcal{C}_i$.

Assume that $k < d$ and that $(*'_{k+1})$ has been verified. Since N is supported on Y, the action of x_{k+1} on $\pi_* N$ is locally nilpotent. It follows that we can write N as a filtered colimit of the A-modules $\mathrm{fib}(x^t_{k+1} : N \to N)$. Since $\mathcal{C}_i$ is closed under colimits, we are reduced to proving that $g'_* \Sigma^{-k}(\mathrm{fib}(x^t_{k+1} : N \to N) \otimes_A M_1 \otimes_A \cdots \otimes_A M_k) \in \mathcal{C}_i$ for each $t \geqslant 0$. Since $\mathcal{C}_i$ is closed under extensions, we can reduce to the case where $t = 1$, which follows from $(*'_{k+1})$.

It remains to prove $(*'_d)$. Here, we prove the following more general claim:

$(*'')$ For every object $N \in \mathrm{QCoh}(\mathrm{Spét}\, A)_{\geqslant c_i}$, the direct image $g'_*(N \otimes_A M_1 \otimes_A \cdots \otimes_A M_d)$ belongs to $\mathcal{C}_i$.

Since $\mathrm{QCoh}(\mathrm{Spét}\, A)_{\geqslant c_i}$ is generated under small colimits by the B-module $\Sigma^{c_i} A$, it suffices to prove $(*'')$ in the case $N = \Sigma^{c_i} A$. That is, we are reduced to proving that $\Sigma^{c_i} g'_* M \in \mathcal{C}_i$. Since $\mathcal{C}_i$ is closed under retracts, it will suffice to show that $\Sigma^{c_i} g'_* M \oplus \Sigma^{c_i+1} g'_* M$ belongs to $\mathcal{C}_i$. We have a fiber sequence

$$\Sigma^{c_i} \mathscr{F}' \to \Sigma^{c_i} g'_* M \oplus \Sigma^{c_i+1} g'_* M \to \Sigma^{c_i} \mathscr{F}''.$$

Here $\Sigma^{c_i} \mathscr{F}' \in \mathcal{C}_i$ since $c_i \geqslant 0$, and

$$\Sigma^{c_i} \mathscr{F}'' \in \mathrm{QCoh}_{K_{i+1}}(\mathsf{X})_{\geqslant c_i - m} \subseteq \mathrm{QCoh}_{K_{i+1}}(\mathsf{X})_{\geqslant c_{i+1}} \subseteq \mathcal{C}_{i+1} \subseteq \mathcal{C}_i .$$

Since $\mathcal{C}_i$ is closed under extensions, we conclude that $\Sigma^{c_i} g'_* M \oplus \Sigma^{c_i+1} g'_* M \in \mathcal{C}_i$ as desired. □

9.6.4 The Proof of Theorem 9.6.0.1

Let X and Y be spectral Deligne-Mumford stacks, where X is a quasi-compact, quasi-separated algebraic space. Let

$$\theta : \mathrm{Map}_{\mathrm{SpDM}}(\mathsf{Y}, \mathsf{X}) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathsf{X}), \mathrm{QCoh}(\mathsf{Y}))$$

denote the functor given by $(f : \mathsf{Y} \to \mathsf{X}) \mapsto (f^* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y}))$. Using Proposition 9.6.1.1 and Corollary 9.4.4.7, we deduce that θ is a fully faithful embedding whose essential image consists of those symmetric monoidal functors $F : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ which satisfy the following conditions:

- (a) The functor F preserves small colimits.
- (b) The functor F carries connective objects of $\mathrm{QCoh}(\mathsf{X})$ to connective objects of $\mathrm{QCoh}(\mathsf{Y})$.

To prove Theorem 9.6.0.1, it will suffice to show that any symmetric monoidal functor $F : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ which satisfies (a) automatically satisfies (b) as well. Note that condition (b) can be tested locally on Y; we may therefore assume without loss of generality that $\mathsf{Y} \simeq \mathrm{Sp\acute{e}t}\, R$ is affine.

Since F is a symmetric monoidal functor, it carries dualizable objects of $\mathrm{QCoh}(\mathsf{X})$ to dualizable objects of $\mathrm{QCoh}(\mathsf{Y}) \simeq \mathrm{Mod}_R$. Consequently, for every perfect object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, the R-module $F(\mathscr{F})$ is perfect.

Let $\mathscr{F}_0 \in \mathrm{QCoh}(\mathsf{X})$ be a perfect object which satisfies the requirements of Proposition 9.6.3.1. Then $F(\mathscr{F}_0) \in \mathrm{Mod}_R$ is perfect. It follows that there exists an integer n such that $F(\mathscr{F}_0)$ is $(-n)$-connective. Let $\mathcal{C} \subseteq \mathrm{QCoh}(\mathsf{X})$ denote the full subcategory spanned by those objects $\mathscr{F}$ such that $F(\mathscr{F})$ is $(-n)$-connective. Since F preserves small colimits, the full subcategory $\mathcal{C}$ is closed under colimits and extensions, and therefore contains $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$. This proves the following weaker version of (b):

- (b') There exists an integer $n \gg 0$ such that the functor F carries $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$ into $\mathrm{QCoh}(\mathsf{Y})_{\geqslant -n}$.

We now claim that F satisfies (b). Fix a connective object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$; we wish to show that $F(\mathscr{F})$ is connective. Using (a) and Proposition 9.6.1.2, we can assume that $\mathscr{F}$ is perfect. Then $F(\mathscr{F})$ is perfect. Suppose that $F(\mathscr{F})$ is not connective: then there exists some largest integer m such that $\pi_{-m}F(\mathscr{F}) \neq 0$. Since $\pi_{-m}F(\mathscr{F})$ is a finitely generated module over $\pi_0 R$, it follows from Nakayama's lemma that there exists some residue field κ of R for which $\pi_{-m}(\kappa \otimes_R F(\mathscr{F})) \neq 0$. Replacing $\mathscr{F}$ by $\mathscr{F}^{\otimes(n+1)}$, we can arrange that $m > n$, which contradicts (b'). This completes the proof of Theorem 9.6.0.1.

Corollary 9.6.4.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism between quasi-compact, quasi-separated spectral algebraic spaces. If the pullback functor $f^* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ is an equivalence of ∞-categories, then f is an equivalence.*

Corollary 9.6.4.2. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is representable by a quasi-compact, quasi-separated spectral algebraic space. Then for every functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the canonical map*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{perf}}, \mathrm{QCoh}(Y)^{\mathrm{perf}})$$

is a fully faithful embedding, whose essential image is spanned by the exact symmetric monoidal functors from $\mathrm{QCoh}(X)^{\mathrm{perf}}$ *to* $\mathrm{QCoh}(Y)^{\mathrm{perf}}$*.*

Proof. The assertion is compatible with colimits in Y; we may therefore assume without loss of generality that Y is representable by an affine spectral Deligne-Mumford stack $\mathrm{Sp\acute{e}t}\, R$. Note that every symmetric monoidal functor $\mathrm{QCoh}(X)^{\mathrm{perf}} \to \mathrm{QCoh}(Y)$ preserves dualizable objects and therefore takes values in the full subcategory $\mathrm{QCoh}(Y)^{\mathrm{perf}} \subseteq \mathrm{QCoh}(X)$. It will therefore suffice to show that the canonical map

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{perf}}, \mathrm{QCoh}(Y))$$

is a fully faithful embedding, whose essential image is spanned by the exact functors. Note that θ factors as a composition

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \xrightarrow{\theta'} \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)) \xrightarrow{\theta''} \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{perf}}, \mathrm{QCoh}(Y)).$$

Theorem 9.6.0.1 implies that θ' is a fully faithful embedding, whose essential image $\mathcal{C}$ is spanned by those symmetric monoidal functors from $\mathrm{QCoh}(X)$ to $\mathrm{QCoh}(Y)$ which preserve small colimits. It follows from Proposition 9.6.1.1 that the restriction $\theta''|_{\mathcal{C}}$ is fully faithful, and that its essential image is the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{perf}}, \mathrm{QCoh}(Y))$ spanned by those symmetric monoidal functors which preserve finite colimits. □

9.6.5 Application: Serre's Criterion for Affineness

We now use Tannaka duality to establish the following converse to Proposition **??**:

Proposition 9.6.5.1 (Serre's Affineness Criterion)**.** *Let* X *be a quasi-compact, quasi-separated spectral algebraic space. Then* X *is affine if and only if the global sections functor* $\Gamma(\mathsf{X}; \bullet) : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Sp}$ *is t-exact.*

Proof. If $\mathsf{X} \simeq \mathrm{Sp\acute{e}t}\, A$ is affine, then the global sections functor $\Gamma(\mathsf{X}; \bullet)$ can be identified with the forgetful functor $\mathrm{Mod}_A(\mathrm{Sp}) \to \mathrm{Sp}$ and is therefore t-exact. Conversely, suppose that $\Gamma(\mathsf{X}; \bullet)$ is t-exact and set $A = \Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}})$, so that A is a connective $\mathbb{E}_\infty$-ring. The identity map $A \to \Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}})$ determines a map $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, A$ which satisfies the following condition:

- $(*)$ The pushforward functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A) \simeq \mathrm{Mod}_A$ is t-exact and the unit map $A \to f_* \mathscr{O}_{\mathsf{X}}$ is an equivalence.

We wish to show that condition $(*)$ implies that f is an equivalence. By virtue of Corollary 9.6.4.1, this is equivalent to the requirement that the pullback functor $f^* : \mathrm{Mod}_A \simeq \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A) \to \mathrm{QCoh}(\mathsf{X})$ is an equivalence of ∞-categories. By construction, A is the endomorphism ring of the compact object $\mathscr{O}_{\mathsf{X}} \in \mathrm{QCoh}(\mathsf{X})$. Using Theorem HA.7.1.2.1 and Remark **??**, it will suffice to show that the structure sheaf $\mathscr{O}_{\mathsf{X}}$ generates the stable ∞-category $\mathrm{QCoh}(\mathsf{X})$: that is, that the global sections functor $\Gamma(\mathsf{X}; \bullet) : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Sp}$ is conservative.

Our proof proceeds in several steps.

(a) Suppose first that A is a field κ. Then $\Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}}) \neq 0$, so X is nonempty. Since X is a quasi-compact, quasi-separated spectral algebraic space, it admits a scallop decomposition (Theorem 3.4.2.1). In particular, there a quasi-compact open substack $\mathsf{U} \subsetneq \mathsf{X}$ which fits into an excision square

$$\begin{array}{ccc} \mathsf{V} & \xrightarrow{j} & \operatorname{Spét} B \\ \downarrow & & \downarrow \\ \mathsf{U} & \longrightarrow & \mathsf{X}. \end{array}$$

The image of j is a quasi-compact open subset of the Zariski spectrum $|\operatorname{Spec} B|$, whose complement $K \subseteq |\operatorname{Spec} B|$ can be identified with the vanishing locus of a finitely generated ideal $I \subseteq (\pi_0 B)$. Let us identify the quotient $(\pi_0 B)/I^2$ with a quasi-coherent sheaf on $\operatorname{Spét} B$ whose restriction to V vanishes. Since the diagram of ∞-categories

$$\begin{array}{ccc} \operatorname{QCoh}(\mathsf{X}) & \longrightarrow & \operatorname{QCoh}(\mathsf{U}) \\ \downarrow & & \downarrow \\ \operatorname{QCoh}(\operatorname{Spét} B) & \longrightarrow & \operatorname{QCoh}(\mathsf{V}) \end{array}$$

is a pullback square, the quotient $(\pi_0 B)/I^2$ admits an essentially unique lift to an object $\mathscr{F} \in \operatorname{QCoh}(\mathsf{X})^{\heartsuit}$ satisfying $\mathscr{F}|_{\mathsf{U}} \simeq 0$. Note that the unit map $\pi_0 \mathscr{O}_{\mathsf{X}} \to \mathscr{F}$ is an epimorphism in $\operatorname{QCoh}(\mathsf{X})^{\heartsuit}$. Since the global sections functor $\Gamma(\mathsf{X}; \bullet)$ is t-exact, the induced map

$$\kappa \simeq \pi_0 \Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}}) \simeq \Gamma(\mathsf{X}; \pi_0 \mathscr{O}_{\mathsf{X}}) \to \Gamma(\mathsf{X}; \mathscr{F}) \simeq (\pi_0 B)/I^2$$

is surjective. The assumption $\mathsf{U} \neq \mathsf{X}$ guarantees that $(\pi_0 B)/I \neq 0$. Using the fact that κ is a field, we deduce that the surjections $\kappa \to (\pi_0 B)/I^2 \to (\pi_0 B)/I \neq 0$ are also injective: that is, we have $I^2 = I$. Since I is finitely generated, it follows that I is generated by an idempotent element $e \in \pi_0 B$. We therefore have $\operatorname{Spét} B \simeq (\operatorname{Spét} B[e^{-1}]) \amalg \mathsf{V}$ and therefore $\mathsf{X} \simeq \mathsf{U} \amalg \operatorname{Spét} B[e^{-1}]$. Because $\Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}}) \simeq \kappa$ is a field, the spectral algebraic space X must be connected: we therefore have $\mathsf{U} \simeq \emptyset$ and $\mathsf{X} \simeq \operatorname{Spét} B[e^{-1}]$ is affine as desired.

(b) Suppose that A is a valuation ring and the map $f : \mathsf{X} \to \operatorname{Spét} A$ admits a section $s : \operatorname{Spét} A \to \mathsf{X}$. Note that for every residue field κ of A, the induced map

$$f_\kappa : \mathsf{X} \times_{\operatorname{Spét} A} \operatorname{Spét} \kappa \to \operatorname{Spét} \kappa$$

also satisfies condition $(*)$. It follows from (a) that f_κ is an equivalence. Consequently, the map f induces a bijection $|f| : |\mathsf{X}| \to |\operatorname{Spét} A|$ between the underlying topological spaces of X and $\operatorname{Spét} A$. Since f admits a section, the map $|f|$ is a homeomorphism.

Since X is a quasi-compact, quasi-separated spectral algebraic space, it admits a scallop decomposition

$$\emptyset \simeq \mathsf{U}_0 \hookrightarrow \mathsf{U}_1 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n = \mathsf{X}\,.$$

We will prove that f is an equivalence using induction on n, the case $n = 0$ being trivial. Assume that $n > 0$, so that U_{n-1} determines a quasi-compact open subset of the topological space $|\mathsf{X}|$. Since $|f|$ is a homeomorphism, it follows that U_{n-1} can be identified with the inverse image of a quasi-compact open substack $\mathsf{W} \subseteq \operatorname{Spét} A$. Since A is a valuation ring, we can write $\mathsf{W} \simeq \operatorname{Spét} A[t^{-1}]$ for some $t \in A$. Applying the inductive hypothesis to the map $f_0 : \mathsf{U}_{n-1} \to \mathsf{W}$, we deduce that f_0 is an equivalence. In particular, the pushforward functor $f_{0*} : \mathrm{QCoh}(\mathsf{U}_{n-1}) \to \mathsf{W}$ is conservative. Consequently, if $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ satisfies $f_* \mathscr{F} \simeq 0$, then the restriction $\mathscr{F}|_{\mathsf{U}} = 0$. Since the inclusion $j : \mathsf{U}_{n-1} \to \mathsf{X}$ fits into an excision square σ :

$$\begin{array}{ccc} \mathsf{V} & \longrightarrow & \operatorname{Spét} B \\ \downarrow & & \downarrow g \\ \mathsf{U}_{n-1} & \xrightarrow{j} & \mathsf{X}, \end{array}$$

it follows that we can write $\mathscr{F} = g_* \mathscr{G}$, where $\mathscr{G} \in \mathrm{QCoh}(\operatorname{Spét} B)$ is a quasi-coherent sheaf satisfying $\mathscr{G}|_{\mathsf{V}} \simeq 0$. Since $\operatorname{Spét} B$ is affine, the vanishing of $\Gamma(\mathsf{X}; \mathscr{F}) \simeq \Gamma(\operatorname{Spét} B; \mathscr{G})$ guarantees that $\mathscr{G} \simeq 0$, so that $\mathscr{F} \simeq g_* \mathscr{G} \simeq 0$.

(c) We next claim that any map of quasi-compact quasi-separated algebraic spaces $f : \mathsf{X} \to \operatorname{Spét} A$ satisfying condition $(*)$ is separated. To prove this, we will verify that f satisfies the valuative criterion for separatedness (Corollary **??**). Let $V \in \mathrm{CAlg}_A$ be a valuation ring with residue field K suppose we are given a pair of maps $g_0, g_1 : \operatorname{Spét} V \to \mathsf{X}$ of spectral Deligne-Mumford stacks over A. We claim that g_0 and g_1 are homotopic to one another. To prove this, we can replace A by V and X by the fiber product $\operatorname{Spét} V \times_{\operatorname{Spét} A} \mathsf{X}$ and thereby reduce to the case where $A = V$. In this case, the map f admits a section (given by either g_0 or g_1), so that f is an equivalence by virtue of (b).

(d) We now handle the general case. Let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ satisfy $\Gamma(\mathsf{X}; \mathscr{F}) \simeq 0$; we wish to prove that $\mathscr{F} \simeq 0$. Since $\Gamma(\mathsf{X}; \bullet)$ is t-exact, we have $\Gamma(\mathsf{X}; \pi_n \mathscr{F}) \simeq \pi_n \Gamma(\mathsf{X}; \mathscr{F}) \simeq 0$ for each integer n. We may therefore replace $\mathscr{F}$ by $\pi_n \mathscr{F}$ and thereby reduce to the case where $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\heartsuit}$.

Fix a map $\eta : \operatorname{Spét} B \to \mathsf{X}$ and let $\mathcal{B} \in \mathrm{QCoh}(\mathsf{X})$ denote the direct image of the structure sheaf of $\operatorname{Spét} B$. Note that each element $b \in \pi_0 B$ determines a map $\phi_b : \mathscr{O}_{\mathsf{X}} \to \mathcal{B}$. We claim that the induced map $\bigoplus_{b \in \pi_0 B} \mathscr{O}_{\mathsf{X}} \to \mathcal{B}$ induces an epimorphism on π_0 (in the abelian category $\mathrm{QCoh}(\mathsf{X})^{\heartsuit}$). To prove this, it will suffice to show that for every map $\eta' : \operatorname{Spét} B' \to \mathsf{X}$, the induced map $\bigoplus_{b \in \pi_0 B} \pi_0 \eta'^* \mathscr{O}_{\mathsf{X}} \to \pi_0 \eta'^* \mathcal{B}$ is a surjection of

$(\pi_0 B')$-modules. Write $\operatorname{Sp\acute{e}t} B \times_{\mathsf{X}} \operatorname{Sp\acute{e}t} B' = \operatorname{Sp\acute{e}t} R$. Unwinding the definitions, we wish to show that $\pi_0 R$ is generated (as a module over $\pi_0 B'$) by the elements of B: that is, that the canonical map $\pi_0(B \otimes_A B') \to \pi_0 R$ is surjective. This is equivalent to the statement that the map

$$\operatorname{Sp\acute{e}t} B \times_{\mathsf{X}} \operatorname{Sp\acute{e}t} B' \to \operatorname{Sp\acute{e}t} B \times_{\operatorname{Sp\acute{e}t} A} \operatorname{Sp\acute{e}t} B'$$

is a closed immersion, which follows from the separatedness of X over $\operatorname{Sp\acute{e}t} A$.

It follows that the map ϕ_b induce an epimorphism

$$\bigoplus_{b \in \pi_0 B} \mathscr{F} \simeq \bigoplus_{b \in \pi_0 B} \pi_0(\mathscr{O}_{\mathsf{X}} \otimes \mathscr{F}) \to \pi_0(\mathcal{B} \otimes \mathscr{F}) \simeq \pi_0(\eta_* \eta^* \mathscr{F})$$

in the abelian category $\operatorname{QCoh}(\mathsf{X})^{\heartsuit}$. Using the t-exactness of the global sections functor $\Gamma(\mathsf{X}; \bullet)$, we obtain a surjection

$$0 \simeq \bigoplus_{b \in \pi_0 B} \Gamma(\mathsf{X}; \mathscr{F}) \to \Gamma(\mathsf{X}; \pi_0 \eta_* \eta^* \mathscr{F}) \simeq \pi_0 \Gamma(\operatorname{Sp\acute{e}t} B; \eta^* \mathscr{F}).$$

Since $\operatorname{Sp\acute{e}t} B$ is affine, this guarantees that $\pi_0 \eta^* \mathscr{F}$ vanishes. Since the map η was chosen arbitrarily, we conclude that $\pi_0 \mathscr{F} \simeq 0$ and therefore $\mathscr{F} \simeq 0$, as desired.

□

9.6.6 Application: A Criterion for Quasi-Affineness

We now discuss a variant of Proposition 9.6.5.1, which characterizes the class of quasi-affine spectral Deligne-Mumford stacks.

Proposition 9.6.6.1. *Let X be a quasi-compact, quasi-separated spectral algebraic space. The following conditions are equivalent:*

(1) *The spectral algebraic space X is quasi-affine.*

(2) *The structure sheaf $\mathscr{O}_{\mathsf{X}}$ generates the t-structure on $\operatorname{QCoh}(\mathsf{X})$, in the sense of Definition C.2.1.1. That is, for every quasi-coherent sheaf $\mathscr{F} \in \operatorname{QCoh}(\mathsf{X})$, there exists a map $\bigoplus \mathscr{O}_{\mathsf{X}} \to \mathscr{F}$ which is an epimorphism on π_0.*

Proof. Suppose first that (1) is satisfied, so there exists a quasi-compact open immersion $j : \mathsf{X} \hookrightarrow \operatorname{Sp\acute{e}t} A$ for some connective $\mathbb{E}_\infty$-ring A. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Choose a collection of elements $f_i \in \pi_0 \Gamma(\mathsf{X}; \mathscr{F})$ which generate $\pi_0 \Gamma(\mathsf{X}; \mathscr{F})$ as a module over A. The elements f_i classify a map $\bigoplus A \to \Gamma(\mathsf{X}; \mathscr{F})$ in the ∞-category Mod_A which is an epimorphism on π_0. Applying the functor $j^* : \operatorname{Mod}_A \simeq \operatorname{QCoh}(\operatorname{Sp\acute{e}t} A) \to \mathsf{X}$, we obtain a map $\bigoplus \mathscr{O}_{\mathsf{X}} \to j^* \Gamma(\mathsf{X}; \mathscr{F}) \simeq \mathscr{F}$ which is an epimorphism on π_0. Allowing $\mathscr{F}$ to range over all quasi-coherent sheaves on X, we deduce that condition (2) is satisfied.

Now suppose that (2) is satisfied. Choose a morphism of $\mathbb{E}_\infty$-rings $\rho : A \to \Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}})$ which exhibits A as a connective cover of $\Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}})$, so that ρ classifies a morphism of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \operatorname{Spét} A$. It follows from assumption (2) and Theorem **??** that the pullback functor $f^* : \operatorname{Mod}_A \simeq \operatorname{QCoh}(\operatorname{Spét} A) \to \operatorname{QCoh}(\mathsf{X})$ is t-exact and that its right adjoint $f_* : \operatorname{QCoh}(\mathsf{X}) \to \operatorname{Mod}_A$ is fully faithful. It follows that $\operatorname{QCoh}(\mathsf{X})$ is idempotent as an A-linear ∞-category: that is, the multiplication map $\operatorname{QCoh}(\mathsf{X}) \otimes_A \operatorname{QCoh}(\mathsf{X}) \to \operatorname{QCoh}(\mathsf{X})$ is an equivalence of ∞-categories. Combining this observation with Corollaries 9.4.2.3 and 9.6.4.1, we deduce that the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times_{\operatorname{Spét} A} \mathsf{X}$ is an equivalence: in other words, the map f is a monomorphism (in the ∞-category of spectral algebraic spaces). It is also flat (since the pullback functor f^* is t-exact). We will complete the proof by establishing the following:

$(*)$ Let $f : \mathsf{X} \to \operatorname{Spét} A$ be a flat monomorphism between quasi-compact, quasi-separated spectral algebraic spaces which exhibits A as the connective cover of $\Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}})$. Then f is an open immersion.

To prove $(*)$, we first apply Theorem 3.4.2.1 to choose a scallop decomposition

$$\emptyset \simeq \mathsf{U}_0 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n \simeq \mathsf{X}$$

of X. We will prove that each of the maps $f|_{\mathsf{U}_i}$ is an open immersion. The proof proceeds by induction on i, the case $i = 0$ being trivial. To carry out the inductive step, let us assume that $i < n$ and that we have established that $f|_{\mathsf{U}_i}$ is an open immersion; we wish to prove that $f|_{\mathsf{U}_{i+1}}$ is an open immersion. Let $\mathfrak{p}$ be a prime ideal of the commutative ring $\pi_0 A$ which belongs to the image of the map $|\mathsf{U}_{i+1}| \to |\operatorname{Spét} A| \simeq |\operatorname{Spec} A|$; we will show that there exists an open neighborhood of $\mathfrak{p}$ over which the map $f|_{\mathsf{U}_{i+1}}$ is an equivalence. Let $A_{\mathfrak{p}}$ denote the localization of A at $\mathfrak{p}$. The projection morphism $f_{\mathfrak{p}} : \operatorname{Spét} A_{\mathfrak{p}} \times_{\operatorname{Spét} A} \mathsf{X} \to \operatorname{Spét} A_{\mathfrak{p}}$ is a flat map whose image contains the closed point of $|\operatorname{Spec} A_{\mathfrak{p}}|$. It is therefore faithfully flat. Since f is a monomorphism, the pullback of $f_{\mathfrak{p}}$ along the faithfully flat map $f_{\mathfrak{p}}$ is an equivalence. Applying Proposition 2.8.3.3, we deduce that $f_{\mathfrak{p}}$ is an equivalence. In particular, the restriction of $f_{\mathfrak{p}}$ to the fiber product $\operatorname{Spét} A_{\mathfrak{p}} \times_{\operatorname{Spét} A} \mathsf{U}_{i+1}$ is an open immersion. Since the image of this open immersion contains the closed point of $|\operatorname{Spec} A_{\mathfrak{p}}|$, it is an equivalence: that is, the inclusion map

$$\operatorname{Spét} A_{\mathfrak{p}} \times_{\operatorname{Spét} A} \mathsf{U}_{i+1} \hookrightarrow \operatorname{Spét} A_{\mathfrak{p}} \times_{\operatorname{Spét} A} \mathsf{X}$$

is an equivalence.

Let Z be the (reduced) closed substack of X complementary to the open immersion $\mathsf{U}_{i+1} \hookrightarrow \mathsf{X}$, so that $\operatorname{Spét} A_{\mathfrak{p}} \times_{\operatorname{Spét} A} \mathsf{Z} \simeq \emptyset$. Since Z is quasi-compact, it follows that there exists an element $a \in \pi_0 A$ which does not belong to $\mathfrak{p}$ such that $\operatorname{Sp\acute{e}t} A[a^{-1}] \times_{\operatorname{Sp\acute{e}t} A} \mathsf{Z} \simeq \emptyset$. Replacing A by $A[a^{-1}]$, we may assume that $\mathsf{Z} = \emptyset$ and therefore $\mathsf{X} = \mathsf{U}_{i+1}$. It follows that there exists an excision square σ :

$$\begin{array}{ccc} \mathsf{V} & \longrightarrow & \operatorname{Sp\acute{e}t} B \\ \downarrow & & \downarrow \\ \mathsf{U}_i & \longrightarrow & \mathsf{X}. \end{array}$$

Let us regard B as an $\mathbb{E}_\infty$-algebra over A, and let $B_{\mathfrak{p}}$ denote the tensor product $A_{\mathfrak{p}} \otimes_A B$. Since $f_{\mathfrak{p}}$ is an equivalence, we see that σ determines another excision square

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} A_{\mathfrak{p}} \times_{\operatorname{Sp\acute{e}t} A} \mathsf{V} & \longrightarrow & \operatorname{Sp\acute{e}t} B_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ \operatorname{Sp\acute{e}t} A_{\mathfrak{p}} \times_{\operatorname{Sp\acute{e}t} A} \mathsf{U}_i & \longrightarrow & \operatorname{Sp\acute{e}t} A_{\mathfrak{p}} \end{array}$$

In particular, the canonical map $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is étale. Using the structure theory of étale morphisms (Proposition B.1.1.3), we can assume (replacing A by a localization $A[a^{-1}]$ if necessary) that there exists an étale A-algebra B' and a map $\phi : B' \to B$ which induces an equivalence after localization at $\mathfrak{p}$. We can therefore extend σ to a commutative diagram $\overline{\sigma}$:

$$\begin{array}{ccccc} \mathsf{V} & \longrightarrow & \operatorname{Sp\acute{e}t} B & \longrightarrow & \operatorname{Sp\acute{e}t} B' \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{U}_i & \longrightarrow & \mathsf{X} & \longrightarrow & \operatorname{Sp\acute{e}t} A. \end{array}$$

The outer rectangle in $\overline{\sigma}$ becomes an excision square after localization at $\mathfrak{p}$. We can therefore replace A by a localization $A[a^{-1}]$ for some $a \in (\pi_0 A) - \mathfrak{p}$ and thereby reduce to the case where the outer rectangle in $\overline{\sigma}$ is itself an excision square. Let $\mathscr{O}_{\mathcal{V}}$ denote the structure sheaf of V. Passing to global functors, we obtain a commutative diagram

$$\begin{array}{ccccc} \Gamma(\mathsf{V}; \mathscr{O}_{\mathcal{V}}) & \longleftarrow & B & \xleftarrow{\phi} & B' \\ \uparrow & & \uparrow & & \uparrow \\ \Gamma(\mathsf{U}_i; \mathscr{O}_{\mathcal{X}}|_{\mathsf{U}_i}) & \longleftarrow & \Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}}) & \xleftarrow{\rho} & A \end{array}$$

where the left square and the outer rectangle are pullback squares. It follows that the right square is a pullback as well. Consequently, it induces a homotopy equivalence of spectra $\operatorname{cofib}(\rho) \simeq \operatorname{cofib}(\phi)$. Since ρ exhibits A as a connective cover of $\Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}})$, the cofiber $\operatorname{cofib}(\rho)$ belongs to $\mathrm{Sp}_{\leqslant -1}$. But ϕ is a morphism of connective spectra, so $\operatorname{cofib}(\phi)$ is connective. It follows that $\operatorname{cofib}(\phi) \simeq 0$, so that ϕ is an equivalence. Since the right square in the diagram $\overline{\sigma}$ is a pushout, we conclude that the map $\mathsf{X} \to \operatorname{Sp\acute{e}t} A$ is also an equivalence, as desired. $\square$

9.6.7 Perfect Approximation for Sheaves of Tor-Amplitude $\leqslant 0$

We close this section by proving a dual version of Proposition 9.6.1.2:

Proposition 9.6.7.1. *Let* X *be a quasi-compact, quasi-separated spectral algebraic space. Let* $\mathcal{C} \subseteq \mathrm{QCoh}(\mathsf{X})$ *denote the full subcategory spanned by those quasi-coherent sheaves* $\mathscr{F}$ *for which* $\mathscr{F}$ *is perfect and of* Tor*-amplitude* $\leqslant 0$*. Then the inclusion* $\mathcal{C} \hookrightarrow \mathrm{QCoh}(\mathsf{X})$ *extends to a fully faithful embedding* $\mathrm{Ind}(\mathcal{C}) \to \mathrm{QCoh}(\mathsf{X})$*, whose essential image is spanned by those quasi-coherent sheaves which have* Tor*-amplitude* $\leqslant 0$*.*

Lemma 9.6.7.2. *Let* X *be a quasi-compact, quasi-separated spectral algebraic space, and let* $\alpha : \mathscr{F}_0 \to \mathscr{F}$ *be a morphism in* $\mathrm{QCoh}(\mathsf{X})$*, where* $\mathscr{F}_0$ *is perfect and* $\mathscr{F}$ *has* Tor*-amplitude* $\leqslant 0$*. Then* α *factors as a composition* $\mathscr{F}_0 \to \mathscr{F}'_0 \to \mathscr{F}$*, where* $\mathscr{F}'_0$ *is perfect and of* Tor*-amplitude* $\leqslant 0$*.*

Proof. Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, and identify α with a point $\eta \in \pi_0 \Gamma(\mathsf{X}; \mathscr{F}_0^{\vee} \otimes \mathscr{F})$. For each object $U \in \mathcal{X}$, set $\mathsf{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathcal{X}}|_U)$, and let η_U denote the image of η in $\pi_0 \Gamma(\mathsf{X}_U; (\mathscr{F}_0^{\vee} \otimes \mathscr{F})|_U)$. Using Theorem 3.4.2.1, we can choose a scallop decomposition

$$\varnothing \simeq U_0 \hookrightarrow U_1 \hookrightarrow \cdots \hookrightarrow U_m = \mathsf{X}.$$

We will prove the following:

$(*_i)$ There exists a connective perfect object $\mathscr{G}_i \in \mathrm{QCoh}(\mathsf{X}_{U_i})$ equipped with a map $\mathscr{G}_i \to \mathscr{F}_0^{\vee}|_{U_i}$ for which η_{U_i} can be lifted to an element of $\pi_0 \Gamma(\mathsf{X}_{U_i}; \mathscr{G}_i \otimes \mathscr{F}|_{U_i})$.

Assuming that $(*_m)$ is satisfied, we complete the proof by taking $\mathscr{F}'_0 = \mathscr{G}_m^{\vee}$. The proof of $(*_i)$ proceeds by induction on i, the case $i = 0$ being trivial. Assume that $i > 0$ and that $(*_{i-1})$ is satisfied. Choose an excision square

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U_{i-1} & \longrightarrow & U_i, \end{array}$$

where V is affine and W is quasi-compact. Write $\mathsf{X}_V = \mathrm{Spét}\, A$, so that X_W is the quasi-compact open substack of $\mathrm{Spét}\, A$ complementary to the vanishing locus of a finitely generated ideal $I \subseteq \pi_0 A$. Let $\mathscr{G}_W$ denote the image of $\mathscr{G}_{i-1}$ in $\mathrm{QCoh}(\mathsf{X}_W)$. Replacing $\mathscr{G}_{i-1}$ by $\mathscr{G}_{i-1} \oplus \Sigma \mathscr{G}_{i-1}$, we can assume that $\mathscr{G}_W$ can be extended to a perfect object $M \in \mathrm{Mod}_A \simeq \mathrm{QCoh}(\mathsf{X}_V)$ (Lemma **??**).

Let N be the perfect A-module given by the restriction $\mathscr{F}_0^{\vee}|_V$, and let $N' \in \mathrm{Mod}_A$ denote the direct image of $\mathscr{F}_0^{\vee}|_W$. We have a canonical map $N \to N'$, whose cofiber is I-nilpotent. We may therefore write $\mathrm{cofib}(N \to N')$ as a filtered colimit $\varinjlim N''_{\gamma}$, where each

N''_γ is perfect and I-nilpotent. The map $\mathscr{G}_{i-1} \to \mathscr{F}_0^\vee|_{U_{i-1}}$ determines a map of A-modules $\beta : M \to N'$. Since M is perfect, the composite map $M \xrightarrow{\beta} N' \to \varinjlim N''_\gamma$ factors through some N''_γ. Replacing M by $\mathrm{fib}(M \to N''_\gamma)$, we may suppose that β factors through a map $\overline{\beta} : M \to N$.

Let M' denote the direct image of $\mathscr{G}_W$ in $\mathrm{QCoh}(\mathsf{X}_V)$, so that we have a canonical map $v : M \to N \times_{N'} M'$ such that $\mathrm{cofib}(v)$ is I-nilpotent. We may therefore write $\mathrm{cofib}(v) \simeq \varinjlim L_\beta$, where each L_β is perfect and I-nilpotent. Let $Q \in \mathrm{Mod}_A \simeq \mathrm{QCoh}(\mathsf{X}_V)$ denote the inverse image of $\mathscr{F}$. Together, the points η_{i-1} and η_V determine a point $\eta' \in \pi_0((N \times_{N'} M') \otimes_A Q)$. Let η'' denote the image of η' in $\pi_0(\mathrm{cofib}(v) \otimes_A Q)$. Then η'' can be lifted to a point of $\pi_0(L_\beta \otimes_A Q)$ for some index β. Replacing M by the fiber product $(N \times_{N'} M') \times_{\mathrm{cofib}(v)} L_\beta$, we can assume that η' lifts to an element $\overline{\eta}' \in \pi_0(M \otimes_A Q)$. Using Lemma 9.6.2.3, we can reduce further to the case where M is connective. In this case, M and $\mathscr{G}_{i-1}$ can be amalgamated to a connective perfect object $\mathscr{G}_i \in \mathrm{QCoh}(\mathsf{X}_{U_i})$, and $\overline{\eta}'$ and η_{i-1} can be amalgamated to an element $\eta_i \in \pi_0\Gamma(\mathsf{X}_{U_i}; \mathscr{G}_i \otimes \mathscr{F}|_{U_i})$. □

Proof of Proposition 9.6.7.1. Let X be a quasi-compact, quasi-separated spectral algebraic space and let $\mathcal{C} \subseteq \mathrm{QCoh}(\mathsf{X})$ be the full subcategory spanned by those objects which are perfect and of Tor-amplitude $\leqslant 0$. The inclusion $\mathcal{C} \hookrightarrow \mathrm{QCoh}(\mathsf{X})$ extends to a functor $\rho : \mathrm{Ind}(\mathcal{C}) \to \mathrm{QCoh}(\mathsf{X})$. Since every object of $\mathcal{C}$ is compact in $\mathrm{QCoh}(\mathsf{X})$ (Corollary 9.1.5.5), the functor ρ is fully faithful. Let us denote its essential image by $\mathcal{C}' \subseteq \mathrm{QCoh}(\mathsf{X})$. Since the collection of quasi-coherent sheaves of Tor-amplitude $\leqslant 0$ is closed under filtered colimits, we conclude that each object of $\mathcal{C}'$ has Tor-amplitude $\leqslant 0$. We wish to prove the converse.

For each object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, let $\mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}_{/\mathscr{F}}$ denote the full subcategory of $\mathrm{QCoh}(\mathsf{X})_{/\mathscr{F}}$ spanned by those maps $\mathscr{F}_0 \to \mathscr{F}$ where $\mathscr{F}_0$ is perfect. Let $U(\mathscr{F})$ denote a colimit of the projection map $\mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}_{/\mathscr{F}} \to \mathrm{QCoh}(\mathsf{X})$, so that we have a canonical map $u_{\mathscr{F}} : U(\mathscr{F}) \to \mathscr{F}$. If $\mathscr{F}$ is perfect, then $\mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}_{/\mathscr{F}}$ has a final object, and the map $u_{\mathscr{F}}$ is an equivalence. More generally, the fact that each perfect object of $\mathrm{QCoh}(\mathsf{X})$ is compact (Corollary 9.1.5.5) implies that the functor $\mathscr{F} \mapsto U(\mathscr{F})$ preserves filtered colimits, so that the collection of those objects $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ for which $u_{\mathscr{F}}$ is an equivalence is closed under filtered colimits. Applying Proposition 9.6.1.1, we see that $u_{\mathscr{F}}$ is an equivalence for every object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$.

Suppose now that $\mathscr{F}$ has Tor-amplitude $\leqslant 0$. Let $\mathcal{E} \subseteq \mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}_{/\mathscr{F}}$ denote the full subcategory spanned by those morphisms $\mathscr{F}_0 \to \mathscr{F}$ where $\mathscr{F}_0$ is perfect and of Tor-amplitude $\leqslant 0$. If K is a finite simplicial set and we are given a diagram $q : K \to \mathcal{E}$, then K admits a colimit in the ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}_{/\mathscr{F}}$, which we can identify with a morphism $\mathscr{F}_0 \to \mathscr{F}$ in $\mathrm{QCoh}(\mathsf{X})$. Applying Lemma 9.6.7.2, we can factor α as a composition $\mathscr{F}_0 \to \mathscr{F}'_0 \to \mathscr{F}$, where $\mathscr{F}'_0$ is perfect and of Tor-amplitude $\leqslant 0$, and therefore determines an object of $\mathcal{E}$. It follows that $\mathcal{E}_{q/}$ is nonempty. Allowing q to vary over all finite diagrams, we conclude that the ∞-category $\mathcal{E}$ is filtered. It follows that the colimit of the composite map $\mathcal{E} \xrightarrow{\iota} \mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}_{/\mathscr{F}} \to \mathrm{QCoh}(\mathsf{X})$ belongs to the ∞-category $\mathcal{C}'$. Consequently, to prove that $\mathscr{F}$ belongs to $\mathcal{C}'$, it will suffice to show that the inclusion ι is left cofinal.

Fix an object $\mathscr{F}_0 \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}_{/\mathscr{F}}$. We wish to prove that the ∞-category

$$\mathcal{E}_{\mathscr{F}_0/} = \mathcal{E} \times_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}_{/\mathscr{F}}} (\mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}_{/\mathscr{F}})_{\mathscr{F}_0/}$$

is weakly contractible. In fact, we claim that $\mathcal{E}_{\mathscr{F}_0/}$ is filtered. Let $q : K \to \mathcal{E}_{\mathscr{F}_0/}$ be a diagram indexed by a finite simplicial set. Arguing as above, we see that q admits a colimit in the ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}_{/\mathscr{F}}$, which we can identify with a map $\beta : \mathscr{F}'_0 \to \mathscr{F}$. Applying Lemma 9.6.7.2, we deduce that β factors as a composition $\mathscr{F}'_0 \to \mathscr{F}''_0 \to \mathscr{F}$, where $\mathscr{F}''_0$ is perfect and of Tor-amplitude $\leqslant 0$. It follows that $(\mathcal{E}_{\mathscr{F}_0/})_{q/}$ is nonempty, as desired. □

9.7 Tannaka Duality for Abelian Categories

Let X be a quasi-geometric stack. It follows from Theorem 9.2.0.2 that X can be recovered, up to canonical equivalence, from the symmetric monoidal ∞-category $\mathrm{QCoh}(X)$ (together with its t-structure). In this section, we study an analogous problem in a more concrete setting: to what extent is X determined by the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$? Note that the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$ depends only the the 0-truncation of X (see Remark **??**), so at best we could only hope to recover the 0-truncation of X. In the positive direction, we have the following:

Theorem 9.7.0.1. *Let X be a 0-truncated geometric stack. For any commutative ring R, the construction*

$$(\eta \in X(R)) \mapsto (\tau_{\leqslant 0}\eta^* : \mathrm{QCoh}(X)^{\heartsuit} \to \mathrm{Mod}_R^{\heartsuit})$$

induces a fully faithful embedding $X(R) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\heartsuit}, \mathrm{Mod}_R^{\heartsuit})$ whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X)^{\heartsuit} \to \mathrm{Mod}_R^{\heartsuit}$ with the following properties:

(1) *The functor F preserves small colimits. In other words, F is right exact and commutes with arbitrary direct sums.*

(2) *The functor F carries flat quasi-coherent sheaves on X to flat R-modules.*

(3) *For every exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of flat quasi-coherent sheaves on X, the sequence of R-modules*

$$0 \to F(\mathscr{F}') \to F(\mathscr{F}) \to F(\mathscr{F}'') \to 0$$

is also exact.

It is possible to prove Theorem **??** by imitating the proof of Theorem 9.3.0.3 at the level of abelian categories (for a proof along these lines, we refer the reader to [137]). We will give a different proof in §9.7.3 (see Theorem 9.7.3.2 and Example 9.7.3.4), which proceeds by reducing Theorem **??** to Theorem 9.3.0.3. Though more circuitous than the direct approach, our method has the virtue of yielding several other results of independent interest (for example, we will show how to recover an arbitrary geometric stack X from the full subcateegory $\mathrm{QCoh}(X)^\flat$ of flat quasi-coherent sheaves on X; see Theorem 9.7.2.1).

9.7.1 Exact Functors on Flat Sheaves

For every geometric stack X, we let $\mathrm{QCoh}(X)^\flat$ denote the full subcategory of $\mathrm{QCoh}(X)$ spanned by the flat quasi-coherent sheaves on X. According to Theorem 9.3.0.3, an arbitrary map $f : Y \to X$ is determined by the symmetric monoidal functor $f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$. In fact, we will show in a moment that f can be recovered from the restriction $f^*|_{\mathrm{QCoh}(X)^\flat} : \mathrm{QCoh}(X)^\flat \to \mathrm{QCoh}(Y)^\flat$. First, let us single out an important property enjoyed by functors of this form.

Proposition 9.7.1.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a geometric stack, let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be an arbitrary functor, and let $F : \mathrm{QCoh}(X)^\flat \to \mathrm{QCoh}(Y)^\flat$ be a symmetric monoidal functor which preserves filtered colimits. Then the following conditions are equivalent:*

(1) *The functor F preserves finite direct sums and carries faithfully flat commutative algebra objects of $\mathrm{QCoh}(X)$ to faithfully flat commutative algebra objects of $\mathrm{QCoh}(Y)$.*

(2) *The functor F preserves zero objects, and for every fiber sequence σ:*

$$\begin{array}{ccc} \mathscr{F}' & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{F}'' \end{array}$$

in $\mathrm{QCoh}(X)$ where $\mathscr{F}'$, $\mathscr{F}$, and $\mathscr{F}''$ are flat, the image $F(\sigma)$ is a fiber sequence in $\mathrm{QCoh}(Y)$.

(3) *The functor F preserves finite direct sums, and for every fiber sequence σ :*

$$\begin{array}{ccc} \mathscr{O}_X & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{F}'' \end{array}$$

in $\mathrm{QCoh}(X)$ where $\mathscr{F}$ and $\mathscr{F}''$ are flat, the image $F(\sigma)$ is a fiber sequence in $\mathrm{QCoh}(Y)$.

Remark 9.7.1.2. Condition (2) of Proposition 9.7.1.1 is automatically satisfied if F extends to a right exact functor $\mathrm{QCoh}(X)^{\mathrm{cn}} \to \mathrm{QCoh}(Y)^{\mathrm{cn}}$ (or an exact functor $\mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$).

Motivated by Remark 9.7.1.2, we introduce the following terminology:

Definition 9.7.1.3. Let X be a geometric stack, let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be an arbitrary functor, and let $F : \mathrm{QCoh}(X)^{\flat} \to \mathrm{QCoh}(Y)^{\flat}$ be a symmetric monoidal functor which preserves filtered colimits. We will say that F is *exact* if it satisfies the equivalent conditions of Proposition 9.7.1.1.

Proof of Proposition 9.7.1.1. Let us assume that the functor F preserves zero objects. Note that for every pair of objects $\mathscr{F}', \mathscr{F}'' \in \mathrm{QCoh}(X)^{\flat}$, we have a fiber sequence σ :

$$\begin{array}{ccc} \mathscr{F}' & \longrightarrow & \mathscr{F}' \oplus \mathscr{F}'' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{F}'', \end{array}$$

and that $F(\sigma)$ is a fiber sequence in $\mathrm{QCoh}(Y)$ if and only if F carries $F(\mathscr{F}' \oplus \mathscr{F}'')$ to a direct sum of $F(\mathscr{F}')$ and $F(\mathscr{F}'')$. It follows immediately that $(2) \Rightarrow (3)$. The implication $(3) \Rightarrow (1)$ follows from Lemma D.4.4.3. We will complete the proof by showing that $(1) \Rightarrow (2)$. Suppose that condition (1) is satisfied, and that σ :

$$\begin{array}{ccc} \mathscr{F}' & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{F}'' \end{array}$$

is a pullback diagram in $\mathrm{QCoh}(X)$ consisting of flat objects. We wish to show that $F(\sigma)$ is a pullback square in $\mathrm{QCoh}(Y)$. Since X is geometric, there exists a faithfully flat map $\pi : \mathrm{Spec}\, A \to X$ for some connective $\mathbb{E}_\infty$-ring A. Set $\mathscr{A} = f_* \mathscr{O}_{\mathrm{Spec}\, A} \in \mathrm{CAlg}(\mathrm{QCoh}(X))$, so that $\mathscr{A}$ is a faithfully flat commutative algebra object of $\mathrm{QCoh}(X)$. It follows from (1) that $F(\mathscr{A})$ is a faithfully flat object of $\mathrm{QCoh}(Y)$. It will therefore suffice to show that $F(\sigma)$ becomes a pullback square after tensoring with $F(\mathscr{A})$. Since F is a symmetric monoidal functor, we are reduce to proving that $F(\pi_* \pi^* \sigma)$ is a pullback square in $\mathrm{QCoh}(Y)$.

Let us abuse notation by identifying $\mathrm{QCoh}(\mathrm{Spec}\, A)$ with Mod_A, so that $\pi^* \sigma$ can be identified with a fiber sequence of flat A-modules $M' \to M \to M''$. Using Theorem HA.7.2.2.15 , we can write M'' as a filtered colimit $\varinjlim M''_\alpha$, where each M''_α is a free A-module. For each index α, set $M_\alpha = M \times_{M''} M''_\alpha$, so that we have a fiber sequence $\tau_\alpha : M' \to M_\alpha \to M''_\alpha$. Since M' is connective and M''_α is free, this sequence splits. It follows that $\pi_* \tau_\alpha$ is a split fiber sequence in $\mathrm{QCoh}(X)$. Since F preserves finite direct sums, we conclude that $F(\pi_* \tau_\alpha)$ is a (split) fiber sequence in $\mathrm{QCoh}(Y)$. Because F preserves filtered colimits, the diagram $F(\pi_* \pi^* \sigma) \simeq \varinjlim F(\pi_* \tau_\alpha)$ is also a fiber sequence in $\mathrm{QCoh}(Y)$. □

9.7.2 Recovering X from $\mathrm{QCoh}(X)^{\flat}$

We can now formulate our main result.

Theorem 9.7.2.1 (Tannaka Duality, Flat Sheaf Version)**.** *Let X be a geometric stack and let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be an arbitrary functor. Then the construction $(f : Y \to X) \mapsto (f^* : \mathrm{QCoh}(X)^{\flat} \to \mathrm{QCoh}(Y)^{\flat})$ determines a fully faithful embedding*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(Y,X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\flat}, \mathrm{QCoh}(Y)^{\flat}),$$

whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X)^{\flat} \to \mathrm{QCoh}(Y)^{\flat}$ which preserve filtered colimits and are exact (in the sense of Definition 9.7.1.3).

We will deduce both Theorem 9.7.2.1 and Theorem **??** from the following result, whose proof we defer until §**??**:

Theorem 9.7.2.2. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a geometric stack, let $0 \leqslant d \leqslant \infty$, and let R be a connective $\mathbb{E}_{\infty}$-ring which is d-truncated. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}, (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d})$ spanned by those symmetric monoidal functors which preserve small colimits and restrict to an exact functor $\mathrm{QCoh}(X)^{\flat} \to \mathrm{Mod}_R^{\flat}$ (in the sense of Definition 9.7.1.3). Then the restriction functor $\mathcal{C} \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\flat}, (\mathrm{Mod}_R^{\flat})_{\leqslant d})$ is a fully faithful embedding. Its essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X)^{\flat} \to \mathrm{Mod}_R^{\flat}$ which are exact and preserve filtered colimits.*

Corollary 9.7.2.3. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a geometric stack, let R be a connective $\mathbb{E}_{\infty}$-ring, and let $\mathcal{E}$ denote the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{Mod}_R)$ spanned by those symmetric monoidal functors which preserve small colimits, flat objects, and connective objects. Then the restriction functor $\mathcal{E} \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\flat}, \mathrm{Mod}_R^{\flat})$ is a fully faithful embedding. Its essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X)^{\flat} \to \mathrm{Mod}_R^{\flat}$ which are exact and preserve filtered colimits.*

Proof. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category of presentable ∞-categories. We regard $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ as equipped with the symmetric monoidal structure described in §HA.4.8.1 . Let $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ denote the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ spanned by the presentable stable ∞-categories, so that $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ inherits a symmetric monoidal structure. The inclusion $\mathcal{P}\mathrm{r}^{\mathrm{St}} \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$ admits a symmetric monoidal left adjoint, given by $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$. Consequently, if $\mathcal{C}$ and $\mathcal{C}'$ are presentable symmetric monoidal ∞-categories for which the tensor product functors

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \otimes : \mathcal{C}' \times \mathcal{C}' \to \mathcal{C}'$$

preserve small colimits separately in each variable, and $\mathcal{C}'$ is stable, then we have a canonical homotopy equivalence $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})}(\mathcal{C}, \mathcal{C}') \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{St}})}(\mathrm{Sp}(\mathcal{C}), \mathcal{C}')$. Replacing $\mathcal{C}'$ by $\mathrm{Fun}(K, \mathcal{C}')$ and allowing K to vary, we obtain an equivalence

$$\mathrm{Fun}_0^{\otimes}(\mathcal{C}, \mathcal{C}') \simeq \mathrm{Fun}_0^{\otimes}(\mathrm{Sp}(\mathcal{C}), \mathcal{C}'),$$

where $\mathrm{Fun}_0^{\otimes}(\mathcal{C}, \mathcal{C}')$ denotes the full subcategory of $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{C}')$ spanned by those symmetric monoidal functors which preserve small colimits, and $\mathrm{Fun}_0^{\otimes}(\mathrm{Sp}(\mathcal{C}), \mathcal{C}')$ is defined similarly. Taking $\mathcal{C} = \mathrm{QCoh}(X)^{\mathrm{cn}}$, we observe that $\mathrm{Sp}(\mathcal{C})$ can be identified with $\mathrm{QCoh}(X)$ (Proposition 9.1.3.1). It follows that the restriction functor

$$\mathrm{Fun}_0^{\otimes}(\mathrm{QCoh}(X), \mathrm{Mod}_R) \to \mathrm{Fun}_0^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}, \mathrm{Mod}_R)$$

is an equivalence of ∞-categories. Under this equivalence, $\mathcal{E}$ can be identified with the full subcategory of $\mathrm{Fun}_0^{\otimes}(\mathrm{QCoh}(X), \mathrm{Mod}_R)$ spanned by those functors which preserve connective and flat objects. The desired result now follows from Theorem 9.7.2.2 (applied in the case $d = \infty$). □

Proof of Theorem 9.7.2.1. Writing Y as a colimit of corepresentable functors, we can reduce to the case where $Y = \mathrm{Spec}\, R$ is affine. Let $\mathcal{E} \subseteq \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{Mod}_R)$ be defined as in Corollary 9.7.2.3. $\theta : X(R) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\flat}, \mathrm{Mod}_R^{\flat})$ factors as a composition

$$X(R) \xrightarrow{\theta'} \mathcal{E} \xrightarrow{\theta''} \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\flat}, \mathrm{Mod}_R^{\flat}).$$

Corollary 9.7.2.3 implies that θ'' is a fully faithful embedding whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X)^{\flat} \to \mathrm{Mod}_R^{\flat}$ which are exact and preserve filtered colimits. We are therefore reduced to showing that θ' is an equivalence of ∞-categories, which follows from Theorem 9.3.0.3. □

9.7.3 The Truncated Case

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a geometric stack. For every connective $\mathbb{E}_\infty$-ring R, Theorem 9.3.0.3 allows us to recover the space $X(R)$ from the symmetric monoidal ∞-category $\mathrm{QCoh}(X)$. We now show that if R is n-truncated, we can recover $X(R)$ from the full subcategory $\mathrm{QCoh}(X)^{\mathrm{cn}}$ spanned by the n-truncated objects.

Remark 9.7.3.1. If X is a geometric stack, then the symmetric monoidal structure on $\mathrm{QCoh}(X)$ is compatible with its t-structure. It follows that for each $n \geqslant 0$, the ∞-category $\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}$ inherits the structure of a symmetric monoidal ∞-category, with tensor product given by $(\mathscr{F}, \mathscr{G}) \mapsto \tau_{\leqslant n}(\mathscr{F} \otimes \mathscr{G})$.

Theorem 9.7.3.2. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a geometric stack, let $n \geqslant 0$ be an integer, let R be a connective $\mathbb{E}_\infty$-ring which is n-truncated, and let $\theta : X(R) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}, (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant n})$ be the map which carries a point $\eta \in X(R)$ to the functor $\mathscr{F} \mapsto \tau_{\leqslant n}\eta^* \mathscr{F}$. Then θ is a fully faithful embedding, whose essential image is spanned by those symmetric monoidal functors $F : (\mathrm{QCoh}(X)^{\mathrm{cn}})_{\leqslant n} \to (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant n}$ with the following properties:*

(a) *The functor F preserves small colimits.*

(b) *The construction* $\mathscr{F} \mapsto F(\tau_{\leqslant n}\mathscr{F})$ *determines a functor from* $\mathrm{QCoh}(X)^{\flat}$ *to* $\mathrm{Mod}_R^{\flat}$ *which is exact (in the sense of Definition 9.7.1.3).*

Remark 9.7.3.3. In the situation of Theorem 9.7.3.2, suppose that X is an n-truncated geometric stack. Then every object $\mathscr{F} \in \mathrm{QCoh}(X)^{\flat}$ is n-truncated. We can therefore rephrase condition (b) as follows:

(b') The construction $\mathscr{F} \mapsto F(\mathscr{F})$ determines a functor from $\mathrm{QCoh}(X)^{\flat}$ to $\mathrm{Mod}_R^{\flat}$ which is exact (in the sense of Definition 9.7.1.3).

Example 9.7.3.4. Let X be a 0-truncated geometric stack. Applying Theorem 9.7.3.2 (and Remark 9.7.3.3) in the case $n = 0$, we deduce that for every commutative ring, the natural map $X(R) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\heartsuit}, \mathrm{Mod}_R^{\heartsuit})$ is a fully faithful embedding, whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X)^{\heartsuit} \to \mathrm{Mod}_R^{\heartsuit}$ which preserve small colimits, flat objects, and exact sequences of flat objects. We can therefore regard Theorem 9.7.3.2 as a generalization of Theorem **??**.

Proof of Theorem 9.7.3.2. Let $\mathcal{C} \subseteq \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}, (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant n})$ be the full subcategory spanned by those functors which satisfy conditions (a) and (b). It is clear that θ factors through $\mathcal{C}$.

Let $\mathrm{Fun}_0^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}, (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant n}) \subseteq \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}, (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant n})$ be the full subcategory spanned by those functors which preserve small colimits, and define the ∞-category $\mathrm{Fun}_0^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}, (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant n})$ similarly. Since $(\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant n}$ is equivalent to an $(n+1)$-category, Remark HA.4.8.2.17 implies that composition with the truncation map $\tau_{\leqslant n} : \mathrm{QCoh}(X)^{\mathrm{cn}} \to (\mathrm{QCoh}(X)^{\mathrm{cn}})_{\leqslant n}$ induces an equivalence of ∞-categories

$$\mathrm{Fun}_0^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant n}, (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant n}) \to \mathrm{Fun}_0^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}, (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant n}).$$

This induces an equivalence of $\mathcal{C}$ with the full subcategory $\mathcal{C}' \subseteq \mathrm{Fun}_0^{\otimes}(\mathrm{QCoh}(X)^{\mathrm{cn}}, (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant n})$ spanned by those functors which restrict to exact functors $\mathrm{QCoh}(X)^{\flat} \to \mathrm{Mod}_R^{\flat}$. Using Theorem 9.7.2.2 (and the theory of faithfully flat descent), we see that that the restriction map $\mathcal{C}' \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\flat}, \mathrm{Mod}_R^{\flat})$ is a fully faithful embedding, whose essential image is the full subcategory $\mathcal{C}'' \subseteq \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\flat}, \mathrm{Mod}_R^{\flat})$ spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X)^{\flat} \to \mathrm{Mod}_R^{\flat}$ which are exact and preserve filtered colimits. We have a commutative diagram

$$\begin{array}{ccccc} & & & & \mathcal{C} \\ & \overset{\theta}{\nearrow} & & \nearrow \sim & \\ X(R) & \longrightarrow & \mathcal{C}' & & \\ & \underset{\theta'}{\searrow} & & \searrow \sim & \\ & & & & \mathcal{C}''. \end{array}$$

It will therefore suffice to show that θ' is an equivalence of ∞-categories, which follows from Theorem 9.7.2.1. □

9.7.4 The Proof of Theorem 9.7.2.2

We now provide a proof of Theorem 9.7.2.2. Let X be a geometric stack, let $0 \leqslant d \leqslant \infty$, and let R be an $\mathbb{E}_\infty$-ring which is connective and d-truncated. Roughly speaking, Theorem 9.7.2.2 asserts that a well-behaved symmetric monoidal functor $F : \mathrm{QCoh}(X)^{\mathrm{cn}} \to (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ can be reconstructed from the restriction $F^\flat = F|_{\mathrm{QCoh}(X)^\flat}$. Since the details of this reconstruction process are somewhat involved, let us first sketch the basic idea. Suppose that $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$, and that we wish to describe $F(\mathscr{F})$ using only the functor $F^\flat$. If X is affine, then $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is freely generated under sifted colimits by the full subcategory $\mathrm{Vect}(X) \subseteq \mathrm{QCoh}(X)^{\mathrm{cn}}$ spanned by the locally free sheaves of finite rank. It follows that any functor $F : \mathrm{QCoh}(X)^{\mathrm{cn}} \to (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ which preserves small colimits (or even small sifted colimits) can be recovered from its restriction to $\mathrm{Vect}(X) \subseteq \mathrm{QCoh}(X)^\flat \subseteq \mathrm{QCoh}(X)^{\mathrm{cn}}$. To handle the general case, we will choose a faithfully flat map $\phi : U_0 \to X$, where U_0 is affine. Setting $\mathscr{A} = \phi_* \mathscr{O}_{U_0}$, we see that F determines a symmetric monoidal functor

$$F_0 : \mathrm{QCoh}(U_0)^{\mathrm{cn}} \simeq \mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(X)^{\mathrm{cn}}) \xrightarrow{F} (\mathrm{Mod}_{F(\mathscr{A})}^{\mathrm{cn}})_{\leqslant d}.$$

Since U_0 is affine, the functor F_0 is determined by its restriction $F_0^\flat = F_0|_{\mathrm{QCoh}(U_0)^\flat}$, and therefore by $F^\flat$. We can then recover the functor F from F_0 using faithfully flat descent (which involves making an analogous argument for the iterated fiber products $U_0 \times_X U_0 \times_X \cdots \times_X U_0$). Our implementation of this strategy will be complicated by the fact that we need to recover F from $F^\flat$ not only as a functor, but as a *symmetric monoidal* functor. In what follows, we will assume that the reader is familiar with the language of symmetric monoidal ∞-categories developed in [139].

Notation 9.7.4.1. For each $n \geqslant 0$, we let $\langle n \rangle$ denote the finite pointed set $\{1, 2, \ldots, n, *\}$ (with base point $*$). Let $\mathcal{F}\mathrm{in}_*$ denote the category whose objects are pointed sets of the form $\langle n \rangle$, and whose morphisms are maps $\alpha : \langle m \rangle \to \langle n \rangle$ satisfying $\alpha(*) = *$. Recall that if $\mathcal{E}$ is an ∞-category which admits finite products, then a *commutative monoid object* of $\mathcal{E}$ is a functor $G : \mathcal{F}\mathrm{in}_* \to \mathcal{E}$ which satisfies the "Segal condition" that the natural map $G(\langle n \rangle) \to G(\langle 1 \rangle)^n$ is an equivalence. In what follows, it will be convenient to identify symmetric monoidal ∞-categories with commutative monoid objects of the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$ of (not necessarily small) ∞-categories.

Let X be a geometric stack, and choose a faithfully flat morphism $\phi : U_0 \to X$, where U_0 is affine. Let $U_\bullet$ denote the Čech nerve of ϕ. We will regard $U_\bullet$ as an augmented simplicial object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ with $U_{-1} = X$. For each $n \geqslant -1$, we will regard $\mathrm{QCoh}(U_n)^{\mathrm{cn}}$ as a symmetric monoidal ∞-category: that is, as an object of the ∞-category

$\mathrm{Mon}_{\mathrm{Comm}}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \subseteq \mathrm{Fun}(\mathcal{F}\mathrm{in}_*, \widehat{\mathcal{C}\mathrm{at}}_\infty)$ spanned by the commutative monoid objects. The construction $[n] \mapsto \mathrm{QCoh}(U_n)^{\mathrm{cn}}$ determines a functor

$$\mathbf{\Delta}_+ \to \mathrm{Mon}_{\mathrm{Comm}}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \subseteq \mathrm{Fun}(\mathcal{F}\mathrm{in}_*, \widehat{\mathcal{C}\mathrm{at}}_\infty),$$

which we can identify with a map $\mathbf{\Delta}_+ \times \mathcal{F}\mathrm{in}_* \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classifying a coCartesian fibration of ∞-categories $p : \mathcal{M}_+ \to \mathbf{\Delta}_+ \times \mathcal{F}\mathrm{in}_*$. We can describe the ∞-category $\mathcal{M}_+$ more informally as follows:

- An object of $\mathcal{M}_+$ is a tuple $([m], \mathscr{F}_1, \ldots, \mathscr{F}_n)$, where $m \geqslant -1$ and each $\mathscr{F}_i$ is an object of $\mathrm{QCoh}(U_m)^{\mathrm{cn}}$.
- A morphism from $([m], \mathscr{F}_1, \ldots, \mathscr{F}_n)$ to $([m'], \mathscr{F}'_1, \ldots, \mathscr{F}'_{n'})$ in $\mathcal{M}_+$ consists of a nondecreasing map $\alpha : [m] \to [m']$, a map of pointed finite sets $\beta : \langle n \rangle \to \langle n' \rangle$, and a collection of maps $\{q^* \bigoplus_{\beta(i)=j} \mathscr{F}_i \to \mathscr{F}'_j\}_{1 \leqslant j \leqslant n'}$ in $\mathrm{QCoh}(U_{m'})$, where $q : U_{m'} \to U_m$ denotes the map determined by α.

Let $\mathcal{M}_+^\flat$ denote the full subcategory of $\mathcal{M}_+$ spanned by those objects $([m], \mathscr{F}_1, \ldots, \mathscr{F}_n)$ where each $\mathscr{F}_i$ is flat (as a quasi-coherent sheaf on U_m). The symmetric monoidal structure on $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is encoded by a coCartesian fibration $p_{-1} : \mathrm{QCoh}(X)^{\mathrm{cn}\otimes} \to \mathcal{F}\mathrm{in}_*$, where we can identify $\mathrm{QCoh}(X)^{\mathrm{cn}\otimes}$ with the full subcategory of $\mathcal{M}_+$ spanned by objects of the form $([-1], \mathscr{F}_1, \ldots, \mathscr{F}_n)$ Similarly, the symmetric monoidal structure on $\mathrm{QCoh}(X)^\flat$ is encoded by a coCartesian fibration $\mathrm{QCoh}(X)^{\flat\otimes} \to \mathcal{F}\mathrm{in}_*$, where we can identify $\mathrm{QCoh}(X)^{\flat\otimes}$ with the full subcategory of $\mathcal{M}_+^\flat$ spanned by objects of the form $([-1], \mathscr{F}_1, \ldots, \mathscr{F}_n)$.

Notation 9.7.4.2. Let $0 \leqslant d \leqslant \infty$ and let R be a connective $\mathbb{E}_\infty$-ring which is d-truncated. We let $\mathcal{N}$ denote the ∞-category $(\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$. The symmetric monoidal structure on $\mathcal{N}$ is encoded by a coCartesian fibration of ∞-categories $r : \mathcal{N}^\otimes \to \mathcal{F}\mathrm{in}_*$. More concretely, the objects of $\mathcal{N}^\otimes$ can be identified with finite sequences $(M_1, \ldots, M_n)$, where each M_i is a connective d-truncated R-module; the functor r is given on objects by $r(M_1, \ldots, M_n) = \langle n \rangle$.

We will identify the ∞-category $\mathrm{Fun}^\otimes(\mathrm{QCoh}(X)^{\mathrm{cn}}, (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d})$ of symmetric monoidal functors from $\mathrm{QCoh}(X)^{\mathrm{cn}}$ to $(\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ with the full subcategory of $\mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathrm{QCoh}(X)^{\mathrm{cn}\otimes}, \mathcal{N}^\otimes)$ spanned by those maps $F : \mathrm{QCoh}(X)^{\mathrm{cn}\otimes} \to \mathcal{N}^\otimes$ which carry p_{-1}-coCartesian morphisms of $\mathrm{QCoh}(X)^{\mathrm{cn}\otimes}$ to r-coCartesian morphisms of $\mathcal{N}^\otimes$. Similarly, let us identify the ∞-category $\mathrm{Fun}^\otimes(\mathrm{QCoh}(X)^\flat, \mathrm{Mod}_R^\flat)$ with a full subcategory of $\mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathrm{QCoh}(X)^{\flat\otimes}, \mathcal{N}^\otimes)$.

Let $\mathcal{C} \subseteq \mathrm{Fun}^\otimes(\mathrm{QCoh}(X)^{\mathrm{cn}}, (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d})$ be the full subcategory spanned by those symmetric monoidal functors which the underlying functor $\mathrm{QCoh}(X)^{\mathrm{cn}} \to (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ preserves small colimits and carries $\mathrm{QCoh}(X)^\flat$ into $\mathrm{Mod}_R^\flat$. Let $\mathcal{C}^\flat$ denote the full subcategory of $\mathrm{Fun}^\otimes(\mathrm{QCoh}(X)^\flat, (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d})$ spanned by those objects for which the underlying functor $\mathrm{QCoh}(X)^\flat \to (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ factors through $\mathrm{Mod}_R^\flat$, preserves small filtered colimits, and is exact in the sense of Definition 9.7.1.3. To prove Theorem 9.7.2.2, we must show that the evident restriction functor $\mathcal{C} \to \mathcal{C}^\flat$ is an equivalence of ∞-categories.

Notation 9.7.4.3. Let $\mathcal{M} = \mathcal{M}_+ \times_{\mathbf{\Delta}_+} \mathbf{\Delta}$, and let $\mathcal{M}^\flat = \mathcal{M}_+^\flat \cap \mathcal{M}$. Suppose we are given a map $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}^\flat, \mathcal{N}^\otimes)$ whose restriction to $\{[m]\} \times_{\mathbf{\Delta}} \mathcal{M}^\flat$ is a lax symmetric monoidal functor, for each $m \geq 0$. For each $m \geq 0$, we can regard $\mathscr{O}_{U_m}$ as a commutative algebra object of $\mathrm{QCoh}(U_m)^\flat$, so that $F(\mathscr{O}_{U_m})$ can be regarded as a commutative algebra object $A_F^m \in \mathrm{CAlg}(\mathcal{N}) \simeq \tau_{\leq d}\,\mathrm{CAlg}_R^{\mathrm{cn}}$. The construction $[m] \mapsto A_F^m$ determines a functor $\mathbf{\Delta} \to \tau_{\leq d}\,\mathrm{CAlg}_R^{\mathrm{cn}}$, which we will denote by $A_F^\bullet$. If $\overline{F} \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}, \mathcal{N}^\otimes)$ is a map whose restriction to $\{[m]\} \times_{\mathbf{\Delta}} \mathcal{M}$ is lax symmetric monoidal for each $m \geq 0$, then we let $A_{\overline{F}}^\bullet$ denote the cosimplicial object $A_F^\bullet$, where $F = \overline{F}|_{\mathcal{M}^\flat}$.

Notation 9.7.4.4. We let $\mathcal{D}$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}, \mathcal{N}^\otimes)$ spanned by those functors $F : \mathcal{M} \to \mathcal{N}^\otimes$ which satisfy the following conditions:

- (a) For each $m \geq 0$, the restriction of F to $\{[m]\} \times_{\mathbf{\Delta}} \mathcal{M}$ is a lax symmetric monoidal functor (that is, F carries inert morphisms in $\{[m]\} \times_{\mathbf{\Delta}} \mathcal{M}$ to inert morphisms in $\mathcal{N}^\otimes$).
- (b) Let $q : \mathcal{M} \to \mathbf{\Delta}$ denote the projection map. Then F carries q-Cartesian morphisms in $\mathcal{M}$ to equivalences in $\mathcal{N}^\otimes$.
- (c) Let $m \geq 0$, and let $F(m) : \mathrm{QCoh}(U_m) \to (\mathrm{Mod}_R^{\mathrm{cn}})_{\leq d}$ denote the restriction of F to $\mathrm{QCoh}(U_m)^{\mathrm{cn}} \simeq \{([m], \langle 1 \rangle)\} \times_{\mathbf{\Delta} \times \mathcal{F}\mathrm{in}_*} \mathcal{M}$. Then $F(m)$ preserves small colimits.
- (d) The cosimplicial object $A_F^\bullet$ of $\mathrm{CAlg}_R^{\mathrm{cn}}$ is 0-coskeletal: that is, for each $m \geq 0$, the inclusions $\{i\} \hookrightarrow [m]$ induce an equivalence $\bigotimes_{0 \leq i \leq m} A_F^0 \to A_F^m$. Moreover, A_F^0 is faithfully flat over R.

Similarly, we let $\mathcal{D}^\flat$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}^\flat, \mathcal{N}^\otimes)$ spanned by those functors $F : \mathcal{M}^\flat \to \mathcal{N}^\otimes$ which satisfy the following analogous conditions:

- (a') For each $m \geq 0$, the restriction of F to $\{[m]\} \times_{\mathbf{\Delta}} \mathcal{M}^\flat$ is a lax symmetric monoidal functor from $\mathrm{QCoh}(U_m)^\flat$ to $\mathrm{Mod}_R^\flat$.
- (b') Let $q^\flat : \mathcal{M}^\flat \to \mathbf{\Delta}$ denote the projection map. Then F carries $q^\flat$-Cartesian morphisms in $\mathcal{M}^\flat$ to equivalences in $\mathcal{N}^\otimes$.
- (c') Let $m \geq 0$, and let $F(m) : \mathrm{QCoh}(U_m)^\flat \to \mathrm{Mod}_R^\flat$ denote the restriction of F to $\mathrm{QCoh}(U_m)^\flat \simeq \{([m], \langle 1 \rangle)\} \times_{\mathbf{\Delta} \times \mathcal{F}\mathrm{in}_*} \mathcal{M}$. Then $F(m)$ preserves small filtered colimits and finite direct sums.
- (d') The cosimplicial object $A_F^\bullet$ of $\mathrm{CAlg}_R^{\mathrm{cn}}$ is 0-coskeletal: that is, for each $m \geq 0$, the inclusions $\{i\} \hookrightarrow [m]$ induce an equivalence $\bigotimes_{0 \leq i \leq m} A_F^0 \to A_F^m$. Moreover, A_F^0 is faithfully flat over R.

We will deduce Theorem 9.7.2.2 from the following assertions:

Lemma 9.7.4.5. *Let $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}_+, \mathcal{N}^{\otimes})$. The following conditions are equivalent:*

- (i) *The restriction $F|_{\mathrm{QCoh}(X)^{\mathrm{cn}\otimes}}$ belongs to $\mathcal{C}$, and F is an r-left Kan extension of $F|_{\mathrm{QCoh}(X)^{\mathrm{cn}\otimes}}$.*
- (ii) *The restriction $F|_{\mathcal{M}}$ belongs to $\mathcal{D}$, and F is an r-right Kan extension of $F|_{\mathrm{QCoh}(X)^{\mathrm{cn}\otimes}}$.*

Moreover, every functor $F_0 \in \mathcal{C}$ admits an r-left Kan extension $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}_+, \mathcal{N}^{\otimes})$, and every functor $F_1 \in \mathcal{D}$ admits an r-right Kan extension $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}_+, \mathcal{N}^{\otimes})$.

Lemma 9.7.4.6. *Let $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}^{\flat}_+, \mathcal{N}^{\otimes})$. The following conditions are equivalent:*

- (i') *The restriction $F|_{\mathrm{QCoh}(X)^{\flat\otimes}}$ belongs to $\mathcal{C}$, and F is an r-left Kan extension of $F|_{\mathrm{QCoh}(X)^{\flat\otimes}}$.*
- (ii') *The restriction $F|_{\mathcal{M}^{\flat}}$ belongs to $\mathcal{D}$, and F is an r-right Kan extension of $F|_{\mathrm{QCoh}(X)^{\flat\otimes}}$.*

Moreover, every functor $F_0 \in \mathcal{C}^{\flat}$ admits an r-left Kan extension $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}^{\flat}_+, \mathcal{N}^{\otimes})$, and every functor $F_1 \in \mathcal{D}^{\flat}$ admits an r-right Kan extension $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}^{\flat}_+, \mathcal{N}^{\otimes})$.

Lemma 9.7.4.7. *A functor $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}, \mathcal{N}^{\otimes})$ belongs to $\mathcal{D}$ if and only if $F|_{\mathcal{M}^{\flat}}$ belongs to $\mathcal{D}^{\flat}$ and and F is an r-left Kan extension of $F|_{\mathcal{M}^{\flat}}$*

Lemma 9.7.4.8. *Every functor $F_0 \in \mathcal{D}^{\flat}$ admits an r-left Kan extension $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}, \mathcal{N}^{\otimes})$.*

Proof of Theorem 9.7.2.2. Let $\mathcal{E}$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}_+, \mathcal{N}^{\otimes})$ spanned by those functors which satisfy conditions (i) and (ii) of Lemma 9.7.4.5, and let $\mathcal{E}^{\flat}$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}^{\flat}_+, \mathcal{N}^{\otimes})$ spanned by those functors which satisfy conditions (i') and (ii') of Lemma 9.7.4.6. We have a commutative diagram of ∞-categories and restriction maps

$$\begin{array}{ccccc} \mathcal{C} & \longleftarrow & \mathcal{E} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}^{\flat} & \longleftarrow & \mathcal{E}^{\flat} & \longrightarrow & \mathcal{D}^{\flat} . \end{array}$$

Using Lemmas 9.7.4.5 and 9.7.4.6 together with Proposition HTT.4.3.2.15 , we deduce that the horizontal maps in this diagram are equivalences. Consequently, to complete the proof, it will suffice to show that the restriction map $\mathcal{D} \to \mathcal{D}^{\flat}$ is an equivalence. This follows from Lemmas 9.7.4.7 and 9.7.4.8, together with Proposition HTT.4.3.2.15 . □

Proof of Lemma 9.7.4.8. Let $F_0 \in \mathcal{D}^{\flat}$; we wish to show that F admits an r-left Kan extension $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}, \mathcal{N}^{\otimes})$. Let $M = ([m], \mathscr{F}_1, \ldots, \mathscr{F}_n)$ be an object of $\mathcal{M}$, and let $\mathcal{M}^{\flat}_{/M}$ denote the fiber product $\mathcal{M}^{\flat} \times_{\mathcal{M}} \mathcal{M}_{/M}$. According to Lemma HTT.4.3.2.13 , it will suffice to show that the lifting problem

$$\begin{array}{ccc} \mathcal{M}^{\flat}_{/M} & \xrightarrow{F_0|_{\mathcal{M}^{\flat}_{/M}}} & \mathcal{N}^{\otimes} \\ \downarrow & \nearrow & \downarrow \\ (\mathcal{M}^{\flat}_{/M})^{\triangleleft} & \longrightarrow & \mathcal{F}\mathrm{in}_* \end{array}$$

admits a solution, where the dotted arrow is an r-colimit diagram. Let $\mathcal{X}$ denote the fiber product

$$(\mathcal{M}^{\flat}_{/M}) \times_{\mathbf{\Delta}_{/[m]} \times \mathcal{F}\mathrm{in}_{*/\langle n \rangle}} \{(\mathrm{id}_{[m]}, \mathrm{id}_{\langle n \rangle})\}.$$

Note that the inclusion $\mathcal{X} \hookrightarrow \mathcal{M}^{\flat}_{/M}$ admits a left adjoint, and is therefore left cofinal. Let f denote the composite map $\mathcal{X} \to \mathcal{M}^{\flat}_{/M} \to \mathcal{M}^{\flat} \xrightarrow{F_0} \mathcal{N}^{\otimes}$. We are therefore reduced to proving that f can be extended to an r-colimit diagram $\mathcal{X}^{\triangleleft} \to \mathcal{N}^{\otimes}_{\langle n \rangle}$.

Unwinding the definitions, we can identify $\mathcal{X}$ with the product $\prod_{1 \leqslant i \leqslant n} \mathrm{QCoh}(U_m)^{\flat}_{/\mathscr{F}_i}$, where $\mathrm{QCoh}(U_m)^{\flat}_{/\mathscr{F}_i}$ denotes the full subcategory of $\mathrm{QCoh}(U_m)_{/\mathscr{F}_i}$ spanned by those maps $\mathscr{F}'_i \to \mathscr{F}_i$ where $\mathscr{F}'_i \in \mathrm{QCoh}(U_m)$ is flat. Let $\mathrm{Vect}(U_m)_{/\mathscr{F}_i}$ be the full subcategory of $\mathrm{QCoh}(U_m)_{/\mathscr{F}_i}$ spanned by those maps $\mathscr{F}'_i \to \mathscr{F}_i$ where $\mathscr{F}'_i$ is locally free of finite rank, so that we can identify the product $\prod_{1 \leqslant i \leqslant n} \mathrm{Vect}(U_m)_{/\mathscr{F}_i}$ with a full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$. We first claim that f is an r-left Kan extension of $f_0 = f|_{\mathcal{X}_0}$. To prove this, fix an object $M' \in \mathcal{X}$, corresponding to a collection of maps $\mathscr{F}'_i \to \mathscr{F}_i$ in $\mathrm{QCoh}(U_m)$ where each $\mathscr{F}'_i$ is flat, and let $\mathcal{Y} = \mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_{/M'}$. Then f induces a map $g : \mathcal{Y}^{\triangleleft} \to \mathcal{N}^{\otimes}_{\langle n \rangle} \subseteq \mathcal{N}^{\otimes}$; we wish to prove that g is an r-colimit diagram. Note that $\mathcal{Y}$ can be identified with the product $\prod_{1 \leqslant i \leqslant n} \mathrm{Vect}(U_m)_{/\mathscr{F}'_i}$. In particular, the ∞-category $\mathcal{Y}$ admits finite coproducts, and is therefore sifted. Since the tensor product on $(\mathrm{Mod}^{\mathrm{cn}}_R)_{\leqslant d}$ preserves small colimits separately in each variable, every map of pointed finite sets $\langle a \rangle \to \langle b \rangle$ induces a functor $\mathcal{N}^{\otimes}_{\langle a \rangle} \to \mathcal{N}^{\otimes}_{\langle b \rangle}$ which preserves small sifted colimits. Using Proposition HTT.4.3.1.10 , we see that g is an r-colimit diagram in $\mathcal{N}^{\otimes}$ if and only if it is a colimit diagram in the ∞-category $\mathcal{N}^{\otimes}_{\langle n \rangle} \simeq (\mathrm{Mod}^{\mathrm{cn}}_R)^n_{\leqslant d}$. Since F_0 satisfies condition (a) of Notation 9.7.4.4, we are reduced to proving that for $1 \leqslant i \leqslant n$, the composite map

$$\mathcal{Y}^{\triangleleft} \to \mathrm{Vect}(U_m)^{\triangleleft}_{/\mathscr{F}'_i} \xrightarrow{g_i} (\mathrm{Mod}^{\mathrm{cn}}_R)_{\leqslant d}$$

is a colimit diagram, where g_i is induced by the map $F_0(m) : \mathrm{QCoh}(U_m) \to \mathrm{Mod}^{\flat}_R$ appearing in assertion (c'). Since $\mathrm{Vect}(U_m)_{/\mathscr{F}'_j}$ is weakly contractible for $j \neq i$, we are reduced to proving that each g_i is a colimit diagram. In fact, we claim that the functor $F_0(m)$ is a left Kan extension of its restriction to $\mathrm{Vect}(U_m)$. This follows from Lemma HTT.5.3.5.8 , since $\mathrm{QCoh}(U_m)^{\flat} \simeq \mathrm{Ind}(\mathrm{Vect}(U_m))$ by Proposition HA.7.2.2.15 and the functor $F_0(m)$ preserves small filtered colimits by virtue of our assumption that F_0 satisfies (c').

Since f is an r-left Kan extension of f_0, we are reduced to proving that f_0 can be extended to an r-colimit diagram $\mathcal{X}_0^{\triangleleft} \to \mathcal{N}^{\otimes}_{\langle n \rangle}$ (Proposition HTT.4.3.2.8). Using Proposition

HTT.4.3.1.10 , we are reduced to proving that f_0 admits a colimit in the ∞-category $\mathcal{N}^{\otimes}_{\langle n \rangle}$. This is clear, since $\mathcal{N}^{\otimes}_{\langle n \rangle} \simeq (\mathrm{Mod}_R^{\mathrm{cn}})^n_{\leqslant d}$ admits small colimits. □

Our proof of Lemma 9.7.4.8 immediately yields the following variant of Lemma 9.7.4.7:

Lemma 9.7.4.9. *Let $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}, \mathcal{N}^{\otimes})$ be a functor such that $F|_{\mathcal{M}^\flat}$ belongs to $\mathcal{D}^\flat$. Then F is an r-left Kan extension of $F|_{\mathcal{M}^\flat}$ if and only if, for every object $([m], \mathscr{F}_1, \ldots, \mathscr{F}_n) \in \mathcal{M}$, the canonical map*

$$\varinjlim_{\mathscr{F}'_i \in \mathrm{Vect}(U_m)_{/\mathscr{F}_i}} F([m], \mathscr{F}'_1, \ldots, \mathscr{F}'_n) \to F([m], \mathscr{F}_1, \ldots, \mathscr{F}_n)$$

is an equivalence in $\mathcal{N}^{\otimes}_{\langle n \rangle} \simeq (\mathrm{Mod}_R^{\mathrm{cn}})^n_{\leqslant d}$.

Proof of Lemma 9.7.4.7. Let $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}, \mathcal{N}^{\otimes})$ be a functor such that $F|_{\mathcal{M}^\flat}$ belongs to $\mathcal{D}^\flat$. We will show that $F \in \mathcal{D}$ if and only if F satisfies the criterion of Lemma 9.7.4.9. We first establish the "only if" direction. Assume that F satisfies conditions (a) through (d) of Notation 9.7.4.4, and let $([m], \mathscr{F}_1, \ldots, \mathscr{F}_n) \in \mathcal{M}$; we wish to show that the canonical map

$$\varinjlim_{\mathscr{F}'_i \in \mathrm{Vect}(U_m)_{/\mathscr{F}_i}} F([m], \mathscr{F}'_1, \ldots, \mathscr{F}'_n) \to F([m], \mathscr{F}_1, \ldots, \mathscr{F}_n)$$

is an equivalence in $(\mathrm{Mod}_R^{\mathrm{cn}})^n_{\leqslant d}$. Using (a) and the weak contractibility of the simplicial sets $\mathrm{Vect}(U_m)_{/\mathscr{F}_i}$, we may reduce to the case $n = 1$. In this case, we are reduced to proving that the functor $F(m) : \mathrm{QCoh}(U_m)^{\mathrm{cn}} \to (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ appearing in (c) is a left Kan extension of its restriction to $\mathrm{Vect}(U_m)$. Since $\mathrm{Vect}(U_m)$ is a subcategory of compact projective generators for $\mathrm{QCoh}(U_m)^{\mathrm{cn}}$, it suffices to note that $F(m)$ preserves small sifted colimits (see Proposition HTT.5.5.8.15 and its proof), which follows from assumption (c).

We now establish the "if" direction. Let $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}, \mathcal{N}^{\otimes})$ be such that $F_0 = F|_{\mathcal{M}^\flat}$ belongs to $\mathcal{D}^\flat$, and assume that F satisfies the criterion of Lemma 9.7.4.9. We wish to prove that $F \in \mathcal{D}$. For this, we must verify that F satisfies conditions (a) through (d) of Notation 9.7.4.4:

(a) Let $\alpha : \langle n \rangle \to \langle n' \rangle$ be an inert morphism in $\mathcal{F}\mathrm{in}_*$, let $m \geqslant 0$, and let $\overline{\alpha}$ be an inert morphism in $\{[m]\} \times_{\mathbf{\Delta}} \mathcal{M} \simeq \mathrm{QCoh}(U_m)^{\otimes}$ lying over α; we wish to show that $F(\overline{\alpha})$ is an inert morphism in $\mathcal{N}^{\otimes}$. Without loss of generality, $\overline{\alpha}$ is given by

$$([m], \mathscr{F}_1, \ldots, \mathscr{F}_n) \to ([m], \mathscr{F}_{\alpha^{-1}(1)}, \ldots, \mathscr{F}_{\alpha^{-1}(n')}).$$

Since F satisfies the criterion of $(2')$, we can identify $F(\alpha)$ with the canonical map

$$\varinjlim_{\mathscr{F}'_i \in \mathrm{Vect}(U_m)_{/\mathscr{F}_i}} F_0([m], \mathscr{F}'_1, \ldots, \mathscr{F}'_n) \to \varinjlim_{\mathscr{F}'_i \in \mathrm{Vect}(U_m)_{/\mathscr{F}_i}} F_0([m], \mathscr{F}'_{\alpha^{-1}(1)}, \ldots, \mathscr{F}'_{\alpha^{-1}(n')}),$$

which is an inert morphism by virtue of our assumption that F_0 satisfies condition (a').

(c) Let $m \geqslant 0$, and let $F(m) : \mathrm{QCoh}(U_m)^{\mathrm{cn}} \to (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ be the map induced by F. We wish to show that $F(m)$ preserves small colimits. By assumption, $F(m)$ is a left Kan extension of its restriction to $\mathrm{Vect}(U_m)$. Using Proposition HTT.5.5.8.15 , we are reduced to proving that $F(m)|_{\mathrm{Vect}(U_m)}$ preserves finite coproducts, which follows from our assumption that F_0 satisfies (c').

(b) Let $\beta : [m] \to [m']$ be a morphism in $\mathbf{\Delta}$, and let $\psi : U_{m'} \to U_m$ be the induced map in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. We wish to show that, for every n-tuple of objects $\mathscr{F}_1, \ldots, \mathscr{F}_n \in \mathrm{QCoh}(U_{m'})^{\mathrm{cn}}$, the canonical map

$$F([m], \psi_* \mathscr{F}_1, \ldots, \psi_* \mathscr{F}_n) \to F([m'], \mathscr{F}_1, \ldots, \mathscr{F}_n)$$

is an equivalence of R-modules. Since F satisfies (a), we can reduce to the case where $n = 1$. That is, we must show that for $\mathscr{F} \in \mathrm{QCoh}(U'_m)^{\mathrm{cn}}$, the canonical map $\xi_{\mathscr{F}} : F([m], \psi_* \mathscr{F}) \to F([m'], \mathscr{F})$ is an equivalence of R-modules. Since F satisfies (b), the collection of those $\mathscr{F}$ for which $\xi_{\mathscr{F}}$ is an equivalence is closed under small colimits in $\mathrm{QCoh}(U_m)^{\mathrm{cn}}$. It will therefore suffice to show that $\xi_{\mathscr{F}}$ is an equivalence when $\mathscr{F} = \mathscr{O}_{U_m}$, which follows from our assumption that F_0 satisfies (b').

(d) By definition, we have $A^\bullet_F = A^\bullet_{F_0}$, so that F satisfies condition (d) if and only if F_0 satisfies condition (d').

□

Proofs of Lemmas 9.7.4.5 and 9.7.4.6. We will give the proof of Lemma 9.7.4.5. The proof of Lemma 9.7.4.6 is similar, requiring only minor changes of notation. We begin by observing that $\mathrm{QCoh}(X)^{\mathrm{cn}\otimes}$ is a colocalization of $\mathcal{M}_+$: that is, the inclusion $\mathrm{QCoh}(X)^{\mathrm{cn}\otimes} \hookrightarrow \mathcal{M}_+$ admits a left adjoint ν, given on objects by

$$([m], \mathscr{F}_1, \ldots, \mathscr{F}_n) \mapsto ([-1], \phi_{m*} \mathscr{F}_1, \ldots, \phi_{m*} \mathscr{F}_n),$$

where $\phi_m : U_m \to X$ denotes the projection map. Moreover, for every object $M \in \mathcal{M}_+$, the image in $\mathcal{F}\mathrm{in}_*$ of the canonical map $\nu(M) \to M$ is an equivalence in $\mathcal{F}\mathrm{in}_*$. We may therefore assume without loss of generality that ν is the identity on $\mathrm{QCoh}(X)^{\otimes}$, and that the image of each map $\nu(M) \to M$ is an identity morphism in $\mathcal{F}\mathrm{in}_*$. It follows that every functor $F_0 \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathrm{QCoh}(X)^{\mathrm{cn}\otimes}, \mathcal{N}^{\otimes})$ admits an r-left Kan extension $F \in \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}_+, \mathcal{N}^{\otimes})$, given by $F = F_0 \circ \nu$. To show that (i) implies (ii), we first show that if $F_0 \in \mathcal{C}$, then F satisfies conditions (a) through (d) of Notation 9.7.4.4:

(a) The assumption that $F_0 \in \mathcal{C}$ implies that F_0 can be identified with a symmetric monoidal functor from $\mathrm{QCoh}(X)^{\mathrm{cn}}$ to $(\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$. To prove that the restriction of F to $\{[m]\} \times_{\mathbf{\Delta}_+} \mathcal{M}_+$ is lax symmetric monoidal, it suffices to observe that the restriction of ν to $\{[m]\} \times_{\mathbf{\Delta}_+} \mathcal{M}_+$ is a lax symmetric monoidal functor (which is clear from the description of ν given above).

(*b*) Let α be a q-Cartesian morphism in $\mathcal{M}$. We wish to prove that $F(\alpha)$ is an equivalence in $\mathcal{N}^{\otimes}$. To prove this, it suffices to observe that $\mu(\alpha)$ is an equivalence in $\mathrm{QCoh}(X)^{\mathrm{cn},\otimes}$.

(*c*) Let $m \geqslant 0$ and let $F(m) : \mathrm{QCoh}(U_m)^{\mathrm{cn}} \to (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ be the restriction of F. We wish to prove that $F(m)$ preserves small colimits. This follows from the observation that $F(m)$ is given by the composition of functors

$$\mathrm{QCoh}(U_m)^{\mathrm{cn}} \xrightarrow{\phi_{m*}} \mathrm{QCoh}(X)^{\mathrm{cn}} \xrightarrow{F_0} (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d},$$

each of which preserves small colimits.

(*d*) Let $\mathscr{A} = \phi_* \mathscr{O}_{U_0} \in \mathrm{CAlg}(\mathrm{QCoh}(X))$. Then $\phi_{m*} \mathscr{O}_{U_m} \simeq \mathscr{A}^{\otimes m+1}$ for each $m \geqslant 0$. Since F_0 is a symmetric monoidal functor, we obtain a canonical equivalence $A_F^m \simeq F_0(\mathscr{A})^{\otimes m+1}$. It follows immediately that $A_F^{\bullet}$ is 0-coskeletal. Moreover, since $\phi : U_0 \to X$ is faithfully flat, $\mathscr{A}$ is faithfully flat. Since F_0 restricts to an exact symmetric monoidal functor $\mathrm{QCoh}(X)^{\flat} \to \mathrm{Mod}_R^{\flat}$, it follows that A_F^0 is faithfully flat.

To complete the proof that (*i*) implies (*ii*), we show that if $F_0 \in \mathcal{C}$, then F is an r-right Kan extension of $F|_{\mathcal{M}}$. To prove this, it suffice to show that for each object $M = ([-1], \mathscr{F}_1, \ldots, \mathscr{F}_n) \in \mathrm{QCoh}(X)^{\mathrm{cn}\otimes} \subseteq \mathcal{M}_+$, the composite map

$$(\mathcal{M} \times_{\mathcal{M}_+} (\mathcal{M}_+)_{M/})^{\triangleleft} \to \mathcal{M}_+ \xrightarrow{F} \mathcal{N}^{\otimes}$$

is an r-limit diagram. Note that the construction $[m] \mapsto ([m], \phi_m^* \mathscr{F}_1, \ldots, \phi_m^* \mathscr{F}_n)$ determines a right cofinal map $\mathbf{\Delta} \to (\mathcal{M} \times_{\mathcal{M}_+} (\mathcal{M}_+)_{M/})$. We are therefore reduced to proving that $F(M)$ is an r-limit of the diagram $[m] \mapsto F([m], \phi_m^* \mathscr{F}_1, \ldots, \phi_m^* \mathscr{F}_n)$. Using Corollary HTT.4.3.1.15 , we see that this is equivalent to the requirement that $F(M)$ is a limit of the diagram $[m] \mapsto F([m], \phi_m^* \mathscr{F}_1, \ldots, \phi_m^* \mathscr{F}_n)$ in $\mathcal{N}^{\otimes}_{\langle n \rangle}$. Equivalently, we must show that for $1 \leqslant i \leqslant n$, the map F exhibits $F_0(\mathscr{F}_i)$ as a totalization of the simplicial object $[m] \mapsto F([m], \phi_m^* \mathscr{F}_i)$. Invoking the definition of F, we are reduced to proving that $F_0(\mathscr{F}_i)$ is a limit of the diagram $[m] \mapsto F_0(\phi_{m*}\phi_m^* \mathscr{F}_i)$ in the ∞-category $(\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$. In fact, we claim that $F_0(\mathscr{F}_i)$ is a limit of the diagram $[m] \mapsto F_0(\phi_{m*}\phi_m^* \mathscr{F}_i)$ in the larger ∞-category Mod_R. To prove this, let $\mathscr{A}^{\bullet}$ denote the cosimplicial object of $\mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}})$ given by $\mathscr{A}^m = \phi_{m*} \mathscr{O}_{U_m}$, and let $R' = F_0(\mathscr{A}^0)$. Since $F_0 \in \mathcal{C}$, R' is a faithfully flat R-module, and we can identify $F_0(\mathscr{A}^m)$ with the $(m+1)$st tensor power of R' over R. We then have canonical equivalences

$$F_0(\phi_{m*}\phi_m^* \mathscr{F}_i) \simeq F_0(\mathscr{A}^m \otimes \mathscr{F}_i) \simeq R'^{\otimes(m+1)} \otimes_R F(\mathscr{F}_i).$$

The desired result now follows from Corollary D.6.3.4. This completes the proof that $(i) \Rightarrow (ii)$.

Now suppose that $G \in \mathcal{D} \subseteq \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}, \mathcal{N}^{\otimes})$. For each $m \geqslant 0$, let $G_{[m]} : \mathrm{QCoh}(U_m)^{\otimes} \to \mathcal{N}^{\otimes}$ denote the restriction of G to $\{[m]\} \times_{\mathbf{\Delta}} \mathcal{M}$. Using the above arguments, we see that G admits an r-right Kan extension $\overline{G} \subseteq \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{M}_+, \mathcal{N}^{\otimes})$, given on objects of $\mathrm{QCoh}(X)^{\mathrm{cn}\otimes} \subseteq \mathcal{M}_+$ by the formula

$$\overline{G}([-1], \mathscr{F}_1, \ldots, \mathscr{F}_n) = \varprojlim_{[m] \in \mathbf{\Delta}} G([m], \phi_m^* \mathscr{F}_1, \ldots, \phi_m^* \mathscr{F}_n).$$

To prove that (ii) implies (i), we first show that $\overline{G}$ is an r-left Kan extension of $\overline{G}|_{\mathrm{QCoh}(X)^{\mathrm{cn}\otimes}}$.

Note that each $G_{[m]}$ is a lax symmetric monoidal functor from $\mathrm{QCoh}(U_m)$ to $(\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ carrying the unit object $\mathscr{O}_{U_m}$ to the commutative R-algebra $A_G^m \in \mathrm{CAlg}_R^{\mathrm{cn}}$, and therefore induces a lax symmetric monoidal functor $\widetilde{G}_{[m]} : \mathrm{QCoh}(U_m)^{\mathrm{cn}} \to (\mathrm{Mod}_{A_G^m}^{\mathrm{cn}})_{\leqslant d}$. We claim that $\widetilde{G}_{[m]}$ is actually a symmetric monoidal functor. It is clear that $\widetilde{G}_{[m]}$ preserves unit objects; it will therefore suffice to show that for every pair of objects $\mathscr{F}, \mathscr{F}' \in \mathrm{QCoh}(U_m)^{\mathrm{cn}}$, the canonical map $\widetilde{G}_{[m]}(\mathscr{F}) \otimes_{A_G^m} \widetilde{G}_{[m]}(\mathscr{F}') \to \widetilde{G}_{[m]}(\mathscr{F} \otimes \mathscr{F}')$ exhibits $\widetilde{G}_{[m]}(\mathscr{F} \otimes \mathscr{F}')$ as a d-truncation of $\widetilde{G}_{[m]}(\mathscr{F}) \otimes_R \widetilde{G}_{[m]}(\mathscr{F}')$. If we regard $\mathscr{F}$ as fixed, the collection of those objects $\mathscr{F}' \in \mathrm{QCoh}(U_m)^{\mathrm{cn}}$ which satisfy this condition is closed under small colimits. Since U_m is corepresented by a connective $\mathbb{E}_\infty$-ring, the ∞-category $\mathrm{QCoh}(U_m)^{\mathrm{cn}}$ is generated under small colimits by the unit object $\mathscr{O}_{U_m}$. We may therefore reduce to the case where $\mathscr{F}' = \mathscr{O}_{U_m}$. Similarly, we can reduce to the case where $\mathscr{F} = \mathscr{O}_{U_m}$, in which case the result is obvious.

Fix an object $M = ([m], \mathscr{F}_1, \ldots, \mathscr{F}_n) \in \mathcal{M}$, so that $\nu(M) = ([-1], \phi_{m*} \mathscr{F}_1, \ldots, \phi_{m*} \mathscr{F}_n)$. We wish to show that the map $\gamma : \overline{G}(\nu(M)) \to \overline{G}(M) = G(M)$ is an equivalence. Unwinding the definitions, we see that the left hand side is given by

$$\overline{G}(\nu(M)) \simeq \varprojlim_{[m'] \in \mathbf{\Delta}} G([m'], \phi_{m'}^* \phi_{m*} \mathscr{F}_1, \ldots, \phi_{m'}^* \phi_{m*} \mathscr{F}_n).$$

Using condition (a) of Notation 9.7.4.4, we can reduce to the case where $n = 1$. Set $\mathscr{F} = \mathscr{F}_1$. For each $m' \geqslant 0$, let $\psi_{m'}$ denote the projection map from $U_m \times_X U_{m'} \simeq U_{m+m'+1}$ onto U_m. Using condition (b), we can rewrite $\overline{G}(\nu(M))$ as the limit

$$\begin{aligned} \varprojlim_{[m'] \in \mathbf{\Delta}} G([m], \psi_{m'*} \psi_{m'}^* \mathscr{F}) &\simeq \varprojlim_{[m'] \in \mathbf{\Delta}} G([m], \mathscr{F} \otimes \psi_{m'*} \psi_{m'}^* \mathscr{O}_{U_m}) \\ &\simeq \varprojlim_{[m'] \in \mathbf{\Delta}} G([m], \mathscr{F} \otimes (\phi_m^* \mathscr{A})^{\otimes m'+1}) \\ &\simeq \varprojlim_{[m'] \in \mathbf{\Delta}} (G([m], \mathscr{F}) \otimes_{A_G^m} B^{\otimes m+1} \end{aligned}$$

where $B = G([m], \phi_m^* \mathscr{A})$, and the limit (and tensor power) are computed in the ∞-category $(\mathrm{Mod}_{A_G^m}^{\mathrm{cn}})_{\leqslant d}$. Since ϕ is faithfully flat, $\phi_m^* \mathscr{A}$ is a faithfully flat commutative algebra object of $\mathrm{QCoh}(U_m)$. It follows that the cofiber of the unit map $\mathscr{O}_{U_m} \to \phi_m^* \mathscr{A}$ is flat (Lemma D.4.4.3), and can therefore be written as a filtered colimit of finite direct sums of copies of $\mathscr{O}_{U_m}$ (Theorem HA.7.2.2.15). Since $G_{[m]}$ preserves filtered colimits and finite direct sums, we conclude that B is a flat A_G^m-module. We now observe that γ is a left homotopy inverse of the canonical map $\gamma' : G([m], \mathscr{F}) \to \varprojlim_{[m'] \in \mathbf{\Delta}} G([m], \mathscr{F}) \otimes_{A_G^m} B^{\otimes m+1}$. Corollary D.6.3.4 impies that γ' is an equivalence, so that γ is also an equivalence.

Let $g : \mathrm{QCoh}(X)^{\mathrm{cn}\otimes} \to \mathcal{N}^{\otimes}$ be the restriction of $\overline{G}$ to $\mathrm{QCoh}(X)^{\mathrm{cn}\otimes}$. We will complete the proof by showing that $g \in \mathcal{C}$. The description of g given above shows that it is a lax symmetric monoidal functor. Let g' denote the composition $\mathrm{QCoh}(X)^{\mathrm{cn}\otimes} \xrightarrow{\phi_0^*} \mathrm{QCoh}(U_0)^{\mathrm{cn}\otimes} \xrightarrow{G_{[0]}} \mathcal{N}^{\otimes}$, so that g' is also a lax symmetric monoidal functor, with $g'(\mathscr{O}_X) = A_G^0$. We have an evident (lax symmetric monoidal) natural transformation $g \to g'$. In particular, for each $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$, we obtain a canonical map

$$\theta_{\mathscr{F}} : g(\mathscr{F}) \otimes_R A_G^0 \to g'(\mathscr{F}) \otimes_R g'(\mathscr{O}_X) \to g'(\mathscr{F} \times \mathscr{O}_X) \simeq g'(\mathscr{F}).$$

We will prove:

$(*)$ For each $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$, the map $\theta_{\mathscr{F}}$ is an equivalence.

Assuming $(*)$ for the moment, we show that the functor g belongs to $\mathcal{C}$. First, we claim that g is actually a symmetric monoidal functor: that is, that the canonical map

$$g(\mathscr{F}_1) \otimes_R \cdots \otimes_R g(\mathscr{F}_n) \to g(\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n)$$

exhibits $g(\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n)$ as a d-truncation of $g(\mathscr{F}_1) \otimes_R \cdots \otimes_R g(\mathscr{F}_n)$ for every sequence of objects $\mathscr{F}_1, \ldots, \mathscr{F}_n \in \mathrm{QCoh}(X)^{\mathrm{cn}}$. Since A_G^0 is faithfully flat over R, it will suffice to prove that the induced map

$$g(\mathscr{F}_1) \otimes_R \cdots \otimes_R g(\mathscr{F}_n) \otimes_R A_G^0 \to g(\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n) \otimes_R A_G^0$$

exhibits $g(\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n) \otimes_R A_G^0$ as a d-truncation of $g(\mathscr{F}_1 \otimes_R \cdots \otimes_R g(\mathscr{F}_n) \otimes_R A_G^0$. Using $(*)$, we are reduced to proving that the map

$$g'(\mathscr{F}_1) \otimes_{A_G^0} \cdots \otimes g'(\mathscr{F}_n) \to g'(\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n)$$

exhibits $g'(\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n)$ as a d-truncation of $g'(\mathscr{F}_1) \otimes_{A_G^0} \cdots \otimes_{A_G^0} g'(\mathscr{F}_n)$. In other words, we are reduced to proving that g' induces a symmetric monoidal functor from $\mathrm{QCoh}(X)^{\mathrm{cn}}$ to $(\mathrm{Mod}_{A_G^0}^{\mathrm{cn}})_{\leqslant d}$ is a symmetric monoidal functor. This is clear, since the functors ϕ_0^* and $\widetilde{G}_{[0]}$ are symmetric monoidal.

We next claim that the functor $\mathrm{QCoh}(X)^{\mathrm{cn}} \to (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ determined by g preserves small colimits. Since A_G^0 is faithfully flat over R, it suffices to show that the functor

$\mathscr{F} \mapsto g(\mathscr{F}) \otimes_R A_G^0$ preserves small colimits. Using $(*)$, we are reduced to proving that the functor $\mathscr{F} \mapsto g'(\mathscr{F})$ preserves small colimits. This is clear, since the pullback functor ϕ_0^* preserves small colimits, and $G_{[0]} : \mathrm{QCoh}(U_0)^{\mathrm{cn}} \to (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ preserves small colimits by virtue of our assumption that G satisfies condition (c) of Notation 9.7.4.4.

Finally, we claim that the functor g carries flat objects of $\mathrm{QCoh}(X)^{\mathrm{cn}}$ to flat R-modules (the exactness of the restriction $g|_{\mathrm{QCoh}(X)^{\flat}}$ is then automatic, by virtue of Remark 9.7.1.2). Since A_G^0 is faithfully flat over R, it will suffice to show that $g(\mathscr{F}) \otimes_R A_G^0$ is flat for each $\mathscr{F} \in \mathrm{QCoh}(X)^{\flat}$ (Proposition 2.8.4.2). Since ϕ_0^* carries $\mathrm{QCoh}(X)^{\flat}$ into $\mathrm{QCoh}(U_0)^{\flat}$, it will suffice to show that the functor $G_{[0]}$ carries $\mathrm{QCoh}(U_0)^{\flat}$ into $\mathrm{Mod}_R^{\flat}$. Condition (c) implies that $G_{[0]} : \mathrm{QCoh}(U_0)^{\mathrm{cn}} \to (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ preserves finite direct sums and filtered colimits. Since $\mathrm{QCoh}(U_0)^{\flat}$ is generated by $\mathscr{O}_{U_0}$ under finite direct sums and filtered colimits (Theorem HA.7.2.2.15), we are reduced to proving that $G_{[0]}(\mathscr{O}_{U_0}) = A_G^0$ is a flat R-module, which follows from our assumption that G satisfies condition (d).

It now remains only to prove $(*)$. Let us regard $A_G^\bullet$ as defining a map $\mathbf{\Delta} \to \mathrm{CAlg}(\mathcal{N})$, let $\mathcal{Z}$ denote the fiber product $\mathbf{\Delta} \times_{\mathrm{CAlg}(\mathcal{N})} \mathrm{Mod}(\mathcal{N})$. Let $\Gamma = \mathrm{Fun}_{\mathbf{\Delta}}(\mathbf{\Delta}, \mathcal{Z})$ denote the ∞-category of sections of the projection map $v : \mathcal{Z} \to \mathbf{\Delta}$. We have an evident functor $\iota : (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d} \to \Gamma$, which carries an object $M \in (\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$ to the section of v given by $[m] \mapsto M \otimes_R A_G^m$. It follows from Corollary D.6.3.4 that the functor ι is fully faithful, and that its essential image is the full subcategory $\Gamma_0 \subseteq \Gamma$ spanned by those sections which carry each morphism in $\mathbf{\Delta}$ to a v-coCartesian morphism in $\mathcal{Z}$. Note that the functor ι admits a right adjoint ξ, which carries a section $M^\bullet : \mathbf{\Delta} \to \mathcal{Z}$ to the limit of the underlying diagram in $(\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant d}$.

Every object $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$ determines an object $M^\bullet \in \Gamma$, given by $M^m = \widetilde{G}_{[m]}(\phi_m^* \mathscr{F})$. Note that we have canonical equivalences $g(\mathscr{F}) \simeq \xi(M^\bullet)$ and $g'(\mathscr{F}) \simeq M^0$, which we can use to identify $\theta_{\mathscr{F}} : g(\mathscr{F}) \otimes_R A_G^0 \to g'(\mathscr{F})$ with the map induced by the counit $\iota(\xi(M^\bullet)) \to M^\bullet$. We will prove $(*)$ by showing that this counit map is an equivalence: that is, that $M^\bullet$ belongs to the full subcategory $\Gamma_0 \subseteq \Gamma$. For this, we must show that for every map $[m] \to [n]$ in $\mathbf{\Delta}$, the induced map $G_{[m]}(\phi_m^* \mathscr{F}) \otimes_{A_G^m} A_G^n \to G_{[n]}(\phi_n^* \mathscr{F})$ is an equivalence. This is a consequence of the following more general claim: for every object $\mathscr{F}' \in \mathrm{QCoh}(U_m)^{\mathrm{cn}})$, the canonical map $G_{[m]}(\mathscr{F}') \otimes_{A_G^m} A_G^n \to G_{[n]}(\gamma^* \mathscr{F}')$ is an equivalence, where $\gamma : U_n \to U_m$ denotes the projection map. Since the functors $G_{[m]}$ and $G_{[n]}$ preserve small colimits, the collection of those objects $\mathscr{F}' \in \mathrm{QCoh}(U_m)$ which satisfy this condition is closed under small colimits in $\mathrm{QCoh}(U_m)^{\mathrm{cn}}$. We may therefore reduce to the case where $\mathscr{F}' = \mathscr{O}_{U_m}$, in which case the result is obvious. $\square$

Chapter 10

Quasi-Coherent Stacks

Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack. According to Proposition 6.2.4.1, there are (at least) two different ways to define the notion of a quasi-coherent sheaf on X:

(*a*) A quasi-coherent sheaf $\mathscr{F}$ on X can be viewed as a spectrum object of $\mathcal{X}$ which is equipped with an action of the structure sheaf $\mathscr{O}_{\mathsf{X}}$ and satisfies certain local conditions.

(*b*) A quasi-coherent sheaf $\mathscr{F}$ on X can be viewed as a rule which assigns to each R-point $\eta : \operatorname{Spét} R \to \mathsf{X}$ an R-module $\mathscr{F}(\eta) \in \operatorname{Mod}_R$, depending functorially on the pair (R, η).

The equivalence of these two definitions relies on the fact that the construction $R \mapsto \operatorname{Mod}_R$ is a sheaf with respect to the étale topology. In Appendix D, we prove a categorified version of this result. For every connective $\mathbb{E}_\infty$-ring R, let $\operatorname{LinCat}_R^{\mathrm{PSt}}$ denote the ∞-category of R-linear prestable ∞-categories (Definition D.1.4.1). According to Theorem D.4.1.2, the construction $R \mapsto \operatorname{LinCat}_R^{\mathrm{PSt}}$ is also a sheaf for the étale topology. In §10.1, we will apply Theorem D.4.1.2 to develop a "global" version of the theory of prestable R-linear ∞-categories. To every spectral Deligne-Mumford stack X, we associate an ∞-category $\operatorname{QStk}^{\mathrm{PSt}}(\mathsf{X})$, which we call the ∞-category of (prestable) *quasi-coherent stacks* on X (Definition 10.1.2.1). The definition is suggested by (*b*): in essence, a quasi-coherent stack on X is a rule which assigns to each R-point $\eta : \operatorname{Spét} R \to \mathsf{X}$ an R-linear ∞-category $\mathcal{C}_\eta \in \operatorname{LinCat}_R$, depending functorially on the pair (R, η).

Let X be a spectral Deligne-Mumford stack and let $R = \Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}})$ denote the $\mathbb{E}_\infty$-ring of global functions on X. Then the construction $\mathscr{F} \mapsto \Gamma(\mathsf{X}; \mathscr{F})$ determines a functor $\Gamma : \operatorname{QCoh}(\mathsf{X}) \to \operatorname{Mod}_R$. If X is affine, then this functor is an equivalence of ∞-categories. In §10.2, we will study the analogous construction for quasi-coherent stacks. To every prestable quasi-coherent stack $\mathcal{C}$ on X, we will associate a Grothendieck prestable ∞-category $\operatorname{QCoh}(\mathsf{X}; \mathcal{C})$, whose objects we refer to as *quasi-coherent sheaves on* X *with values in* $\mathcal{C}$ (Construction 10.1.7.1). The ∞-category $\operatorname{QCoh}(\mathsf{X}; \mathcal{C})$ is naturally tensored over the

∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ of connective quasi-coherent sheaves on X, and the construction $\mathcal{C} \mapsto \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ can be viewed as a "global sections functor"

$$\mathrm{QCoh}(\mathsf{X}; \bullet) : \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \to \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_{\infty}).$$

In the special case where $\mathsf{X} = \mathrm{Sp\acute{e}t}\, R$ is affine, it follows easily from the definitions that this functor is an equivalence of ∞-categories: both the domain and codomain can be identified with the ∞-category $\mathrm{LinCat}_R^{\mathrm{PSt}}$ of prestable R-linear ∞-categories studied in Appendix D. The main result of §10.2 asserts that this is a much more general phenomenon: the functor $\mathrm{QCoh}(\mathsf{X}; \bullet)$ is an equivalence of ∞-categories whenever X is a quasi-compact, quasi-separated spectral algebraic space (Theorem 10.2.0.2).

To develop the theory of quasi-coherent stacks on X, the assumption that X is a spectral Deligne-Mumford stack is not essential: the same definition makes sense for an arbitrary functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. For example, we could take X to be the classifying stack BG, where G is a flat group scheme over a commutative ring A (see Example 9.1.1.7). In this case, we would like to say that a prestable quasi-coherent stack on X can be viewed as a prestable A-linear ∞-category equipped with an "action" of the group scheme G. However, we encounter a technical obstacle: Theorem D.4.1.2 guarantees that the construction $R \mapsto \mathrm{LinCat}_R^{\mathrm{PSt}}$ satisfies descent for the étale topology, but we do not know if the analogous statement holds for the flat topology. To address this difficulty, it is convenient to restrict our attention to prestable quasi-coherent stacks $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$ which are *complete* in the sense that the prestable R-linear ∞-category $\mathcal{C}_\eta$ is complete for each point $\eta \in X(R)$. The collection of complete prestable quasi-coherent stacks on X forms a full subcategory $\mathrm{QStk}^{\mathrm{comp}}(X) \subseteq \mathrm{QStk}^{\mathrm{PSt}}(X)$ which we will study in §10.4. Our main result is another local-to-global principle: if X is a geometric stack, then the global sections functor $\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{comp}}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_{\infty})$ is a fully faithful embedding, whose essential image admits admits a simple description (Theorem 10.4.2.3).

The definition of a complete quasi-coherent stack illustrates a general paradigm: given any property P of Grothendieck prestable ∞-categories, we say that a prestable quasi-coherent stack $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$ *has the property* P if, for every connective $\mathbb{E}_\infty$-ring R and every point $\eta \in X(R)$, the ∞-category $\mathcal{C}_\eta$ has the property P. In §10.3 and §10.5, we will illustrate this paradigm in several examples, choosing P from among the properties introduced in Appendix C. Our principal results again take the form of local-to-global principles, asserting that under certain conditions a quasi-coherent stack $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$ has the property P if and only if the ∞-category $\mathrm{QCoh}(X; \mathcal{C})$ has the property P. For example, if X is a quasi-compact, quasi-separated spectral algebraic space, we show that $\mathcal{C}$ is compactly generated if and only if $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is compactly generated (see Theorem 10.3.2.1).

Remark 10.0.0.1. Compact generation is a rather mild finiteness condition in the setting of prestable ∞-categories. In Chapter 11, we will study more sophisticated finiteness conditions on quasi-coherent stacks, whose local-to-global principles are more intimately tied to the algebro-geometric properties of X.

Throughout this chapter, we will concentrate primarily on the study of *prestable* quasi-coherent stacks on a functor X, which associate to each point $\eta \in X(R)$ a prestable R-linear ∞-category. However, there is an entirely parallel story in a more familiar setting: we define an *abelian quasi-coherent stack* on X to be a rule which assigns to each point $\eta \in X(R)$ an R-linear Grothendieck abelian category $\mathcal{C}_\eta$, depending functorially on the pair (R, η). The collection of abelian quasi-coherent stacks can be organized into a 2-category which we denote by $\mathrm{QStk}^{\mathrm{Ab}}(X)$ (see Definition 10.1.2.1). In §10.6, we will establish a local-to-global principle for abelian quasi-coherent stacks (Theorem 10.6.2.1) and study the relationship between the abelian and prestable theories (the abelian theory is in some sense subsumed by the prestable theory: see Theorem 10.6.6.1).

Remark 10.0.0.2. The study of abelian quasi-coherent stacks really belongs to classical algebraic geometry, rather than spectral algebraic geometry: the notion of R-linear abelian category depends only on the commutative ring $\pi_0 R$ (see Example D.1.3.6), and similarly the notion of abelian quasi-coherent stack on a functor X depends only the restriction of X to the subcategory $\mathrm{CAlg}^{\heartsuit} \subseteq \mathrm{CAlg}^{\mathrm{cn}}$ (see Remark 10.1.2.6).

Contents

10.1 Sheaves of ∞-Categories

Let R be a connective $\mathbb{E}_\infty$-ring. We define an *additive R-linear ∞-category* to be a presentable ∞-category $\mathcal{C}$ which is tensored over $\mathrm{Mod}_R^{\mathrm{cn}}$ and for which the action $\otimes : \mathrm{Mod}_R^{\mathrm{cn}} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable (see Definition D.1.2.1). The collection of all additive R-linear ∞-categories can be organized into an ∞-category $\mathrm{LinCat}_R^{\mathrm{Add}} = \mathrm{Mod}_{\mathrm{Mod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. In Appendix D, we introduced full subcategories

$$\mathrm{LinCat}_R^{\mathrm{St}}, \mathrm{LinCat}_R^{\mathrm{PSt}}, \mathrm{LinCat}_R^{\mathrm{Ab}} \subseteq \mathrm{LinCat}_R^{\mathrm{Add}},$$

whose objects are the stable R-linear ∞-categories, prestable R-linear ∞-categories, and abelian R-linear ∞-categories, respectively (Definition D.1.4.1). Our goal in this section is to study global versions of these constructions, where the $\mathbb{E}_\infty$-ring R is replaced by an arbitrary functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. We will primarily be interested in the cases where X is (representable by) a spectral Deligne-Mumford stack, or where X is quasi-geometric (in the sense of Definition 9.1.0.1).

10.1.1 Quasi-Coherent Stacks on a Functor

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. The ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X can be described informally as the inverse limit $\varprojlim_{\eta \in X(R)} \mathrm{Mod}_R$, taken over all R-valued points η of X (see Definition 6.2.2.1). We now consider a variant of this definition, where we replace R-modules by additive R-linear ∞-categories.

Construction 10.1.1.1. Let $\mathrm{LinCat}^{\mathrm{Add}}$ denote the ∞-category $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})} \mathrm{Mod}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ whose objects are pairs $(R, \mathcal{C})$, where R is a connective $\mathbb{E}_\infty$-ring and $\mathcal{C}$ is an additive R-linear ∞-category (Definition D.1.2.1). We let

$$\mathrm{QStk}^{\mathrm{Add}} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$$

denote the functor obtained by applying Construction 6.2.1.7 to the projection map $q : \mathrm{LinCat}^{\mathrm{Add}} \to \mathrm{CAlg}^{\mathrm{cn}}$. Given a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, we will refer to the $\mathrm{QStk}^{\mathrm{Add}}(X)$ as the *∞-category of additive quasi-coherent stacks on X*.

Remark 10.1.1.2. According to Lemma 6.2.1.13, the functor $\mathrm{QStk}^{\mathrm{Add}} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ is characterized (up to equivalence) by the following properties:

(*i*) The composition of $\mathrm{QStk}^{\mathrm{Add}}$ with the Yoneda embedding $\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}}$ classifies the coCartesian fibration $\mathrm{LinCat}^{\mathrm{Add}} \to \mathrm{CAlg}^{\mathrm{cn}}$. More informally, for every connective $\mathbb{E}_\infty$-ring R, we have a canonical equivalence $\mathrm{QStk}^{\mathrm{Add}}(\operatorname{Spec} R) \simeq \mathrm{LinCat}_R^{\mathrm{Add}}$.

(*ii*) The functor $\mathrm{QStk}^{\mathrm{Add}}$ preserves limits. Consequently, for every functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the canonical map

$$\mathrm{QStk}^{\mathrm{Add}}(X) \to \varprojlim_{\eta \in X(R)} \mathrm{QStk}^{\mathrm{Add}}(\operatorname{Spec} R) \simeq \varprojlim_{\eta \in X(R)} \mathrm{LinCat}_R^{\mathrm{Add}}$$

is an equivalence of ∞-categories.

Remark 10.1.1.3. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor which classifies a left fibration $p : \mathrm{CAlg}_X^{\mathrm{cn}} \to \mathrm{CAlg}^{\mathrm{cn}}$ (so that an object of $\mathrm{CAlg}_X^{\mathrm{cn}}$ can be identified with a pair (R, η), where R is a connective $\mathbb{E}_\infty$-ring and $\eta \in X(R)$ is an R-valued point of X). Then the ∞-category $\mathrm{QStk}^{\mathrm{Add}}(X)$ of additive quasi-coherent stacks on X can be identified with the full subcategory of $\mathrm{Fun}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathrm{CAlg}_X^{\mathrm{cn}}, \mathrm{LinCat}^{\mathrm{Add}})$ spanned by those commutative diagrams

$$\begin{array}{ccc} \mathrm{CAlg}_X^{\mathrm{cn}} & \longrightarrow & \mathrm{LinCat}^{\mathrm{Add}} \\ & \searrow \quad \swarrow_q & \\ & \mathrm{CAlg}^{\mathrm{cn}} & \end{array}$$

where the horizontal map carries p-coCartesian morphisms of $\mathrm{CAlg}_X^{\mathrm{cn}}$ to q-coCartesian morphisms of $\mathrm{LinCat}^{\mathrm{Add}}$. More informally, we can think of an object $\mathcal{C} \in \mathrm{QStk}^{\mathrm{Add}}(X)$ as a rule which assigns to each point $\eta \in X(A)$ an additive A-linear ∞-category $\mathcal{C}_\eta$ which is functorial in the following sense: if $f : A \to B$ is a map of connective $\mathbb{E}_\infty$-rings and η has image $f_!(\eta) \in X(B)$, then there is a canonical equivalence

$$\mathcal{C}_{f_!(\eta)} \simeq B \otimes_A \mathcal{C}_\eta = \mathrm{LMod}_B(\mathcal{C}_\eta)$$

of additive B-linear ∞-categories.

10.1.2 Stable, Prestable, and Abelian Quasi-Coherent Stacks

The theory of additive R-linear ∞-categories is not particularly well-behaved from an algebro-geometric point of view: for example, the construction $R \mapsto \mathrm{LinCat}_R^{\mathrm{Add}}$ does not even satisfy descent with respect to the Zariski topology. Consequently, the functor $X \mapsto \mathrm{QStk}^{\mathrm{Add}}(X)$ of Construction 10.1.1.1 has poor formal properties. However, we can remedy this by restricting our attention to additive quasi-coherent stacks which satisfy some further conditions.

Definition 10.1.2.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{Add}}(X)$ be an additive quasi-coherent stack. For each point $\eta \in X(R)$, let $\mathcal{C}_\eta \in \mathrm{LinCat}_R^{\mathrm{Add}}$ be defined as in Remark 10.1.1.3. Then:

- We will say that $\mathcal{C}$ is a *stable quasi-coherent stack* if, for each connective $\mathbb{E}_\infty$-ring R and each point $\eta \in X(R)$, the ∞-category $\mathcal{C}_\eta$ is stable. We let $\mathrm{QStk}^{\mathrm{St}}(X)$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{Add}}(X)$ spanned by the stable quasi-coherent stacks.

- We will say that $\mathcal{C}$ is a *prestable quasi-coherent stack* if, for each connective $\mathbb{E}_\infty$-ring R and each point $\eta \in X(R)$, the ∞-category $\mathcal{C}_\eta$ is a Grothendieck prestable ∞-category. Let $\mathrm{QStk}^{\mathrm{PSt}}(X)$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{Add}}(X)$ spanned by the prestable quasi-coherent stacks.

- We will say that $\mathcal{C}$ is an *abelian quasi-coherent stack* if, for each connective $\mathbb{E}_\infty$-ring R and each point $\eta \in X(R)$, the ∞-category $\mathcal{C}_\eta$ is a Grothendieck abelian category. Let $\mathrm{QStk}^{\mathrm{Ab}}(X)$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{Add}}(X)$ spanned by the abelian quasi-coherent stacks.

Remark 10.1.2.2. Every presentable stable ∞-category is a Grothendieck prestable ∞-category. Consequently, for any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, we have inclusions $\mathrm{QStk}^{\mathrm{St}}(X) \subseteq \mathrm{QStk}^{\mathrm{PSt}}(X) \subseteq \mathrm{QStk}^{\mathrm{Add}}(X)$

Notation 10.1.2.3 (Pullback of Quasi-Coherent Stacks)**.** If $f : X \to Y$ is a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, we let f^* denote the induced functor $\mathrm{QStk}^{\mathrm{Add}}(Y) \to \mathrm{QStk}^{\mathrm{Add}}(X)$, given concretely by the formula $(f^* \mathcal{C})_\eta = \mathcal{C}_{f(\eta)}$ for each point $\eta \in X(R)$. Note that f^* restricts to functors

$$\mathrm{QStk}^{\mathrm{St}}(Y) \to \mathrm{QStk}^{\mathrm{St}}(X) \qquad \mathrm{QStk}^{\mathrm{PSt}}(Y) \to \mathrm{QStk}^{\mathrm{PSt}}(X) \qquad \mathrm{QStk}^{\mathrm{Ab}}(Y) \to \mathrm{QStk}^{\mathrm{Ab}}(X),$$

which we will also denote by f^*.

Remark 10.1.2.4. The constructions $X \mapsto \mathrm{QStk}^{\mathrm{St}}(X), \mathrm{QStk}^{\mathrm{PSt}}(X), \mathrm{QStk}^{\mathrm{Ab}}(X)$ determine functors

$$\mathrm{QStk}^{\mathrm{St}}, \mathrm{QStk}^{\mathrm{PSt}}, \mathrm{QStk}^{\mathrm{Ab}} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty.$$

These functors are right Kan extensions of the constructions

$$R \mapsto \mathrm{LinCat}_R^{\mathrm{St}} \qquad R \mapsto \mathrm{LinCat}_R^{\mathrm{PSt}} \qquad R \mapsto \mathrm{LinCat}_R^{\mathrm{Ab}}$$

along the Yoneda embedding $j : \mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}}$.

Example 10.1.2.5 (The Affine Case)**.** Let R be connective $\mathbb{E}_\infty$-ring. Then we have canonical equivalences

$$\mathrm{QStk}^{\mathrm{Add}}(\operatorname{Spec} R) \simeq \mathrm{LinCat}_R^{\mathrm{Add}} \qquad \mathrm{QStk}^{\mathrm{St}}(\operatorname{Spec} R) \simeq \mathrm{LinCat}_R^{\mathrm{St}}$$

$$\mathrm{QStk}^{\mathrm{PSt}}(\operatorname{Spec} R) \simeq \mathrm{LinCat}_R^{\mathrm{PSt}} \qquad \mathrm{QStk}^{\mathrm{Ab}}(\operatorname{Spec} R) \simeq \mathrm{LinCat}_R^{\mathrm{Ab}}.$$

We will generally abuse terminology by using these equivalences to identify (additive, stable, prestable, abelian) quasi-coherent stacks on $\operatorname{Spec} R$ with the corresponding (additive, stable, prestable, abelian) R-linear ∞-categories.

Remark 10.1.2.6. For every connective $\mathbb{E}_\infty$-ring R, extension of scalars induces an equivalence of ∞-categories $\mathrm{LinCat}_R^{\mathrm{Ab}} \simeq \mathrm{LinCat}_{\pi_0 R}^{\mathrm{Ab}}$ (see Remark D.1.4.6). In other words, the functor $R \mapsto \mathrm{LinCat}_R^{\mathrm{Ab}}$ is a right Kan extension of its restriction to discrete $\mathbb{E}_\infty$-rings. It follows that for any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the ∞-category $\mathrm{QStk}^{\mathrm{Ab}}(X)$ of abelian quasi-coherent stacks on X depends only on the restriction of X to the full subcategory of $\mathrm{CAlg}^{\mathrm{cn}}$ spanned by the discrete $\mathbb{E}_\infty$-rings. In particular, if $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford stack with 0-truncation $\mathsf{X}_0 = (\mathcal{X}, \pi_0 \mathscr{O}_{\mathcal{X}})$, then the pullback map $\mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X}) \to \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X}_0)$ is an equivalence of ∞-categories.

Construction 10.1.2.7 (Stabilization of Quasi-Coherent Stacks)**.** For every connective $\mathbb{E}_\infty$-ring R, the construction $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ determines a stabilization functor $\mathrm{LinCat}_R^{\mathrm{PSt}} \to \mathrm{LinCat}_R^{\mathrm{St}}$. These functors depend functorially on R and can therefore be extended to a functor $\mathrm{QStk}^{\mathrm{PSt}}(X) \to \mathrm{QStk}^{\mathrm{St}}(X)$ for each $X \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, which we will also denote by $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$. More informally, this functor is given by the formula $\mathrm{Sp}(\mathcal{C})_\eta = \mathrm{Sp}(\mathcal{C}_\eta)$. It can is a left adjoint to the inclusion $\mathrm{QStk}^{\mathrm{St}}(X) \hookrightarrow \mathrm{QStk}^{\mathrm{PSt}}(X)$ of Remark 10.1.2.2.

Construction 10.1.2.8 (Hearts of Quasi-Coherent Stacks)**.** For every connective $\mathbb{E}_\infty$-ring R, the construction $\mathcal{C} \mapsto \mathcal{C}^\heartsuit$ determines a functor $\mathrm{LinCat}_R^{\mathrm{PSt}} \to \mathrm{LinCat}_R^{\mathrm{Ab}}$. These functors depend functorially on R and can therefore be extended to a functor $\mathrm{QStk}^{\mathrm{PSt}}(X) \to \mathrm{QStk}^{\mathrm{Ab}}(X)$ for each $X \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, which we will also denote by $\mathcal{C} \mapsto \mathcal{C}^\heartsuit$. More informally, this functor is given by the formula $(\mathcal{C}^\heartsuit)_\eta = (\mathcal{C}_\eta)^\heartsuit$.

Remark 10.1.2.9 (Descent for Prestable Quasi-Coherent Stacks)**.** Let $f : X \to Y$ be a map in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which induces an equivalence after sheafification with respect to the étale topology. Then the pullback functor $f^* : \mathrm{QStk}^{\mathrm{PSt}}(Y) \to \mathrm{QStk}^{\mathrm{PSt}}(X)$ is an equivalence of ∞-categories. This follows immediately from Theorem D.4.1.2 (together with the characterization of the functor $\mathrm{QStk}^{\mathrm{PSt}}$ supplied by Remark 10.1.2.4).

Remark 10.1.2.10 (Descent for Stable Quasi-Coherent Stacks)**.** Let $f : X \to Y$ be a map in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which induces an equivalence after sheafification with respect to the étale topology. Then the pullback functor $f^* : \mathrm{QStk}^{\mathrm{St}}(Y) \to \mathrm{QStk}^{\mathrm{St}}(X)$ is an equivalence of ∞-categories. This follows from Remark 10.1.2.9, together with the fact that the stability of a prestable R-linear ∞-category can be tested locally with respect to the étale topology on R (Proposition D.5.1.1).

Remark 10.1.2.11 (Descent for Abelian Quasi-Coherent Stacks)**.** Let $f : X \to Y$ be a map in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which induces an equivalence after sheafification with respect to the fpqc topology. Then the pullback functor $f^* : \mathrm{QStk}^{\mathrm{Ab}}(Y) \to \mathrm{QStk}^{\mathrm{Ab}}(X)$ is an equivalence of ∞-categories. This follows from Corollary D.6.8.4 (together with Remark 10.1.2.4).

10.1.3 Limits and Colimits of Prestable Quasi-Coherent Stacks

Let A be a connective $\mathbb{E}_\infty$-ring. Then the ∞-category $\mathrm{LinCat}_A^{\mathrm{Add}}$ of additive A-linear ∞-categories admits small limits and colimits. Moreover, for every morphism $f : A \to B$, the extension of scalars functor $\mathrm{LinCat}_A^{\mathrm{Add}} \to \mathrm{LinCat}_B^{\mathrm{Add}}$ preserves small limits and colimits. From this, it follows formally that for any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the ∞-category $\mathrm{QStk}^{\mathrm{Add}}(X)$ of additive quasi-coherent stacks on X admits small limits and colimits (which can be computed "levelwise"). However, if we restrict our attention to *prestable* quasi-coherent stacks, then the situation is more delicate. In this case, the existence of limits and colimits requires some additional hypotheses.

Definition 10.1.3.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathrm{QStk}^{\mathrm{PSt}}(X)$. We will say that F is *left exact* if, for every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, the underlying A-linear functor $F_\eta : \mathcal{C}_\eta \to \mathcal{D}_\eta$ is left exact (see Proposition C.3.2.1). We will say that F is *compact* if, for every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, the underlying functor $F_\eta : \mathcal{C}_\eta \to \mathcal{D}_\eta$ is compact (that is, the right adjoint of F_η commutes with filtered colimits).

We let $\mathrm{QStk}^{\mathrm{lex}}(X)$ denote the subcategory of $\mathrm{QStk}^{\mathrm{PSt}}(X)$ whose morphisms are left exact morphisms in $\mathrm{QStk}^{\mathrm{PSt}}(X)$, and we let $\mathrm{QStk}^{c}(X)$ denote the subcategory of $\mathrm{QStk}^{\mathrm{PSt}}(X)$ whose morphisms are compact morphisms in $\mathrm{QStk}^{\mathrm{PSt}}(X)$.

Remark 10.1.3.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism between prestable quasi-coherent stacks on a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. If $\mathcal{C}$ is stable, then F is automatically left exact. Consequently, we can regard the ∞-category $\mathrm{QStk}^{\mathrm{St}}(X)$ as a full subcategory the ∞-category $\mathrm{QStk}^{\mathrm{lex}}(X)$.

Remark 10.1.3.3. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of quasi-coherent stacks on Y. If F is left exact (compact), then so is the induced map $f^*(F) : f^*\mathcal{C} \to f^*\mathcal{D}$. Conversely, if $f : X \to Y$ induces an effective epimorphism after sheafification with respect to the étale topology and $f^*\mathcal{C}$ is left exact (compact), then so is $\mathcal{C}$: this follows from Proposition D.5.2.1 (D.5.2.2).

Proposition 10.1.3.4 (Limits of Prestable Quasi-Coherent Stacks)**.** *Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. Then:*

- (a) *The ∞-category $\mathrm{QStk}^{\mathrm{lex}}(Y)$ admits small limits.*
- (b) *The inclusion functor $\mathrm{QStk}^{\mathrm{lex}}(Y) \hookrightarrow \mathrm{QStk}^{\mathrm{Add}}(Y)$ preserves small limits.*
- (c) *For every map $f : X \to Y$, the pullback functor $f^* : \mathrm{QStk}^{\mathrm{lex}}(Y) \to \mathrm{QStk}^{\mathrm{lex}}(X)$ preserves small limits.*

In particular, for every point $\eta \in Y(R)$, the construction

$$(\mathcal{C} \in \mathrm{QStk}^{\mathrm{lex}}(Y)) \mapsto (\mathcal{C}_\eta \in \mathrm{LinCat}_R^{\mathrm{Add}})$$

preserves small limits.

Proof. In the case where $Y = \operatorname{Spec} A$ is affine, assertions (a) and (b) follow from Remark D.1.6.1. If, in addition, $X = \operatorname{Spec} B$ is affine, then assertion (c) follows from the observation that $\mathrm{Mod}_B^{\mathrm{cn}}$ is dualizable as an object of $\mathrm{LinCat}_A^{\mathrm{Add}}$ (so that the extension of scalars functor $\mathrm{LinCat}_A^{\mathrm{Add}} \to \mathrm{LinCat}_B^{\mathrm{Add}}$ preserves all limits which exist in $\mathrm{LinCat}_A^{\mathrm{Add}}$). The general case reduces to the affine case by formal arguments. $\square$

Variant 10.1.3.5 (Limits of Stable Quasi-Coherent Stacks)**.** Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. Then the collection of stable quasi-coherent stacks on Y is closed under small limits in the ∞-category $\mathrm{QStk}^{\mathrm{lex}}(Y)$. Consequently, we have the following analogue of Proposition 10.1.3.4:

- (a) The ∞-category $\mathrm{QStk}^{\mathrm{St}}(Y)$ admits small limits.
- (b) The inclusion functor $\mathrm{QStk}^{\mathrm{St}}(Y) \hookrightarrow \mathrm{QStk}^{\mathrm{Add}}(Y)$ preserves small limits.

(c) For every map $f : X \to Y$, the pullback functor $f^* : \mathrm{QStk}^{\mathrm{St}}(Y) \to \mathrm{QStk}^{\mathrm{St}}(X)$ preserves small limits.

Variant 10.1.3.6 (Limits of Abelian Quasi-Coherent Stacks)**.** Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathrm{QStk}^{\mathrm{Ab}}(Y)$. We will say that F is *left exact* if, for every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in Y(A)$, the underlying A-linear functor $F_\eta : \mathcal{C}_\eta \to \mathcal{D}_\eta$ is a left exact (and therefore exact) functor of abelian categories. We let $\mathrm{QStk}^{\mathrm{Ab,lex}}(Y)$ denote the subcategory of $\mathrm{QStk}^{\mathrm{Ab}}(Y)$ whose morphisms are left exact functors. Arguing as in the proof of Proposition 10.1.3.4 (using Remark D.1.6.5 in place of Remark D.1.6.1), we deduce that the ∞-category $\mathrm{QStk}^{\mathrm{Ab,lex}}(Y)$ admits small limits which can be computed levelwise (and are therefore preserved by pullback along any map $f : X \to Y$). Moreover, the construction $\mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$ of Construction 10.1.2.8 restricts to a functor $\mathrm{QStk}^{\mathrm{lex}}(X) \to \mathrm{QStk}^{\mathrm{Ab,lex}}(X)$ which preserves small limits (see Remark D.1.6.8).

Proposition 10.1.3.7 (Filtered Colimits of Prestable Quasi-Coherent Stacks)**.** *Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. Then:*

(a) *The ∞-category $\mathrm{QStk}^{\mathrm{PSt}}(Y)$ admits small filtered colimits.*

(b) *The inclusion functor $\mathrm{QStk}^{\mathrm{PSt}}(Y) \hookrightarrow \mathrm{QStk}^{\mathrm{Add}}(Y)$ preserves small filtered colimits.*

(c) *For every map $f : X \to Y$, the pullback functor $f^* : \mathrm{QStk}^{\mathrm{PSt}}(Y) \to \mathrm{QStk}^{\mathrm{PSt}}(X)$ preserves small filtered colimits.*

In particular, for every point $\eta \in Y(R)$, the construction

$$(\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(Y)) \mapsto (\mathcal{C}_\eta \to \mathrm{LinCat}_R^{\mathrm{PSt}})$$

preserves small filtered colimits.

Proof. In the case where $Y = \operatorname{Spec} A$ is affine, assertions (a) and (b) follow from Remark D.1.6.2. If, in addition, $X = \operatorname{Spec} B$ is affine, then assertion (c) follows from the observation that the extension of scalars functor $\mathrm{LinCat}_A^{\mathrm{Add}} \to \mathrm{LinCat}_B^{\mathrm{Add}}$ preserves small colimits. The general case reduces to the affine case by formal arguments. □

Variant 10.1.3.8 (Colimits of Stable Quasi-Coherent Stacks)**.** For any $\mathbb{E}_\infty$-ring A, the ∞-category $\mathrm{LinCat}_A^{\mathrm{St}}$ of *stable* A-linear ∞-categories admits all small colimits, which are preserved by the extension of scalars functor $\mathrm{LinCat}_A^{\mathrm{St}} \to \mathrm{LinCat}_B^{\mathrm{St}}$ associated to any morphism of $\mathbb{E}_\infty$-rings $A \to B$, and by the inclusion functor $\mathrm{LinCat}_A^{\mathrm{St}} \hookrightarrow \mathrm{LinCat}_A^{\mathrm{Add}}$ whenever A is connective. We therefore have the following analogue of Proposition 10.1.3.7:

(a) For any functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the ∞-category $\mathrm{QStk}^{\mathrm{St}}(Y)$ admits small colimits.

(*b*) For any functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the inclusion functors $\mathrm{QStk}^{\mathrm{St}}(Y) \hookrightarrow \mathrm{QStk}^{\mathrm{Add}}(Y)$ preserves small colimits.

(*c*) For every natural transformation $f : X \to Y$ between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the pullback functor $f^* : \mathrm{QStk}^{\mathrm{St}}(Y) \to \mathrm{QStk}^{\mathrm{St}}(X)$ preserves small colimits.

Variant 10.1.3.9 (Filtered Colimits of Abelian Quasi-Coherent Stacks)**.** Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. Then:

(*a*) The ∞-category $\mathrm{QStk}^{\mathrm{Ab}}(Y)$ admits small filtered colimits.

(*b*) The inclusion functor $\mathrm{QStk}^{\mathrm{Ab}}(Y) \hookrightarrow \mathrm{QStk}^{\mathrm{Add}}(Y)$ preserves small filtered colimits.

(*c*) For every map $f : X \to Y$, the pullback functor $f^* : \mathrm{QStk}^{\mathrm{Ab}}(Y) \to \mathrm{QStk}^{\mathrm{Ab}}(X)$ preserves small filtered colimits.

This can be proven in the same way as Proposition 10.1.3.7, using Remark D.1.6.6 in place of Remark D.1.6.2.

Remark 10.1.3.10. Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. Then the ∞-categories $\mathrm{QStk}^{\mathrm{PSt,lex}}(Y)$ and $\mathrm{QStk}^{\mathrm{Ab,lex}}(Y)$ admit small filtered colimits, which are preserved by the inclusion functors

$$\mathrm{QStk}^{\mathrm{PSt,lex}}(Y) \hookrightarrow \mathrm{QStk}^{\mathrm{PSt}}(Y) \qquad \mathrm{QStk}^{\mathrm{Ab,lex}}(Y) \hookrightarrow \mathrm{QStk}^{\mathrm{Ab}}(Y).$$

To prove this, we can immediately reduce to the case where Y is affine, in which case the desired result follows from Remarks D.1.6.2 and **??**.

Remark 10.1.3.11. Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. Then the constructions

$$\mathrm{QStk}^{\mathrm{St}}(Y) \xleftarrow{\mathrm{Sp}} \mathrm{QStk}^{\mathrm{PSt}}(Y) \xrightarrow{\heartsuit} \mathrm{QStk}^{\mathrm{Ab}}(Y)$$

preserve small filtered colimits. To see this, we can reduce to the case where Y is affine, in which case the desired result follows from Remark D.1.6.7.

If we restrict our attention to diagrams involving compact transition maps, then we can obtain a stronger version of Proposition 10.1.3.7:

Proposition 10.1.3.12 (Colimits of Quasi-Coherent Stacks)**.** *Let* $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ *be a functor. Then:*

(*a*) *The* ∞*-category* $\mathrm{QStk}^{c}(Y)$ *admits small colimits.*

(*b*) *The inclusion functor* $\mathrm{QStk}^{c}(Y) \hookrightarrow \mathrm{QStk}^{\mathrm{PSt}}(Y)$ *preserves small colimits.*

(*c*) *For every map* $f : X \to Y$*, the pullback functor* $f^* : \mathrm{QStk}^{c}(Y) \to \mathrm{QStk}^{c}(X)$ *preserves small colimits.*

In particular, for every point $\eta \in Y(R)$, the construction

$$(\mathcal{C} \in \mathrm{QStk}^c(Y)) \mapsto (\mathcal{C}_\eta \to \mathrm{LinCat}_R^{\mathrm{PSt}})$$

preserves small colimits.

Proof. In the case where $Y = \operatorname{Spec} A$ is affine, assertions (a) and (b) follow from Remark D.1.6.3. If, in addition, $X = \operatorname{Spec} B$ is affine, then assertion (c) follows from the observation that $\mathrm{Mod}_B^{\mathrm{cn}}$ is dualizable as an object of $\mathrm{LinCat}_A^{\mathrm{PSt}}$ (so that the extension of scalars functor $\mathrm{LinCat}_A^{\mathrm{PSt}} \to \mathrm{LinCat}_B^{\mathrm{PSt}}$ preserves all colimits which exist in $\mathrm{LinCat}_A^{\mathrm{PSt}}$). The general case reduces to the affine case by formal arguments. □

Warning 10.1.3.13. In the situation of Proposition 10.1.3.12, the inclusion functor $\mathrm{QStk}^c(Y) \hookrightarrow \mathrm{QStk}^{\mathrm{Add}}(Y)$ usually does not preserve small colimits.

Warning 10.1.3.14. If $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is a functor, then the stabilization construction $\mathrm{QStk}^c(Y) \subseteq \mathrm{QStk}^{\mathrm{PSt}}(Y) \xrightarrow{\mathrm{Sp}} \mathrm{QStk}^{\mathrm{St}}(Y)$ generally does not preserve small colimits, even when Y is corepresentable (see Warning D.1.6.9).

10.1.4 Direct Images of Quasi-Coherent Stacks

Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Given a quasi-coherent stack $\mathcal{C}$ on X, one can attempt to define a quasi-coherent stack $f_* \mathcal{C}$ on Y with the following universal property: for every quasi-coherent stack $\mathcal{D}$ on Y, giving a map of quasi-coherent stacks $\mathcal{D} \to f_* \mathcal{C}$ (on Y) is equivalent to giving a map of quasi-coherent stacks $f^* \mathcal{D} \to \mathcal{C}$ (on X). At least heuristically, this construction can be given explicitly by the formula $(f_* \mathcal{C})_\eta = \varprojlim_{\overline{\eta}} \mathcal{C}_{\overline{\eta}}$, where $\eta : \operatorname{Spec} A \to Y$ is an A-valued point of Y and the limit on the right hand side is taken over all commutative diagrams σ :

$$\begin{array}{ccc} \operatorname{Spec} B & \xrightarrow{\overline{\eta}} & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} A & \xrightarrow{\eta} & Y. \end{array}$$

Here we encounter a technical point: the ∞-category $\mathrm{LinCat}_A^{\mathrm{Add}}$ admits all *small* limits, but the collection of all commutative diagrams σ as above need not be small. To circumvent this difficulty, we will make the following additional assumptions:

(i) The morphism $f : X \to Y$ representable, in the sense of Definition 6.3.2.1. That is, for every map $\operatorname{Spec} A \to Y$, the fiber product $X_A = X \times_Y \operatorname{Spec} A$ is (representable by) a spectral Deligne-Mumford stack.

(*ii*) The quasi-coherent stack $\mathcal{C}$ is either prestable or abelian: that is, it belongs to a class of objects which satisfy descent for the étale topology (see Remarks 10.1.2.9 and 10.1.2.11).

We will see below that under these assumptions, we do not need to take the limit of *all* diagrams σ as above: it suffices to consider those diagrams which classify an étale morphism $\operatorname{Spec} B \to X_A$.

Proposition 10.1.4.1. (1) *Let $f : X \to Y$ be a representable morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Then the pullback functors*

$$f^* : \mathrm{QStk}^{\mathrm{PSt}}(Y) \to \mathrm{QStk}^{\mathrm{PSt}}(X) \qquad f^* : \mathrm{QStk}^{\mathrm{Ab}}(Y) \to \mathrm{QStk}^{\mathrm{Ab}}(X)$$

admit right adjoints, which we will denote by f_*.

(2) *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$*, where f is representable. Then the associated diagram*

$$\begin{array}{ccc} \mathrm{QStk}^{\mathrm{PSt}}(Y) & \xrightarrow{f^*} & \mathrm{QStk}^{\mathrm{PSt}}(X) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ \mathrm{QStk}^{\mathrm{PSt}}(Y') & \xrightarrow{f'^*} & \mathrm{QStk}^{\mathrm{PSt}}(X') \end{array}$$

is right adjointable: that is, the canonical natural transformation $g^* f_* \to f'_* g'^*$ is an equivalence.

(2′) *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$*. Then the associated diagram*

$$\begin{array}{ccc} \mathrm{QStk}^{\mathrm{Ab}}(Y) & \xrightarrow{f^*} & \mathrm{QStk}^{\mathrm{Ab}}(X) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ \mathrm{QStk}^{\mathrm{Ab}}(Y') & \xrightarrow{f'^*} & \mathrm{QStk}^{\mathrm{Ab}}(X') \end{array}$$

is right adjointable: that is, the canonical natural transformation $g^* f_* \to f'_* g'^*$ is an equivalence.

Proof of Proposition 10.1.4.1. We will prove the existence of the pushforward $f_* : \mathrm{QStk}^{\mathrm{PSt}}(X) \to \mathrm{QStk}^{\mathrm{PSt}}(Y)$ and verify part (2); the analogous assertions in the setting of abelian quasi-coherent stacks can be proven using the same argument (with some slight changes in notation). We begin by proving (1) in the special case where $Y = \operatorname{Spec} A$ is affine. Let us identify $\mathrm{QStk}^{\mathrm{PSt}}(Y)$ with the ∞-category $\mathrm{LinCat}_A^{\mathrm{PSt}}$ of prestable A-linear ∞-categories. Our assumption that f is representable guarantees that X is (representable by) a spectral Deligne-Mumford stack $(\mathcal{X}, \mathscr{O}_X)$.

Fix an object $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$. For every object $U \in \mathcal{X}$, let X_U denote the functor represented by the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathscr{O}_X|_U)$, let $f_U : X_U \to Y$ be the restriction of f, and let let $\mathcal{C}_U \in \mathrm{QStk}^{\mathrm{PSt}}(X_U)$ denote the restriction of $\mathcal{C}$. We define a functor $\chi_U : (\mathrm{LinCat}_A^{\mathrm{PSt}})^{\mathrm{op}} \to \widehat{\mathcal{S}}$ by the formula $\chi_U(\mathcal{D}) = \mathrm{Map}_{\mathrm{QStk}^{\mathrm{PSt}}(X_U)}(f_U^* \mathcal{D}, \mathcal{C}_U)$. Let us say that a point $\eta \in \chi_U(\mathcal{D})$ is *left exact* if, for every map $V \to U$ in $\mathcal{X}$ where $\mathsf{X}_V \simeq \operatorname{Spec} B$ is affine, the image of η in the mapping space

$$\chi_V(\mathcal{D}) = \mathrm{Map}_{\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}_V)}(f_V^* \mathcal{D}, \mathcal{C}_V) \simeq \mathrm{Map}_{\mathrm{LinCat}_A^{\mathrm{PSt}}}(\mathcal{D}, \mathcal{C}_V)$$

determines a left exact functor from $\mathcal{D}$ to $\mathcal{C}_V$ (here we abuse notation by identifying $\mathcal{C}_V$ with the associated prestable B-linear ∞-category, regarded as a prestable A-linear ∞-category by restriction of scalars). Let $\chi'_U(\mathcal{D})$ denote the summand of $\chi_U(\mathcal{D})$ consisting of the left exact points, so that we can regard the construction $\mathcal{D} \mapsto \chi'_U(\mathcal{D})$ as a functor $\chi'_U : (\mathrm{LinCat}_A^{\mathrm{lex}})^{\mathrm{op}} \to \widehat{\mathcal{S}}$.

Note that the construction $U \mapsto \chi_U$ carries colimits in $\mathcal{X}$ to limits in the ∞-category $\mathrm{Fun}((\mathrm{LinCat}_A^{\mathrm{PSt}})^{\mathrm{op}}, \widehat{\mathcal{S}})$. Using Proposition D.5.2.1, we see that the condition that a point $\eta \in \chi_U(\mathcal{D})$ is left exact can be tested locally on U, so that the functor $U \mapsto \chi'_U$ carries colimits in $\mathcal{X}$ to limits in the ∞-category $\mathrm{Fun}((\mathrm{LinCat}_A^{\mathrm{lex}})^{\mathrm{op}}, \widehat{\mathcal{S}})$ Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the full subcategory spanned by those objects U for which the functor χ'_U is representable by an object of $\mathrm{LinCat}_A^{\mathrm{lex}}$. Since the ∞-categoy $\mathrm{LinCat}_A^{\mathrm{lex}}$ admits small limits (Remark D.1.6.1), it follows that $\mathcal{X}_0$ is closed under small colimits in $\mathcal{X}$. Moreover, if $U \in \mathcal{X}$ is affine so that $\mathsf{X}_U \simeq \operatorname{Spec} B$ for some connective A-algebra B, then we can regard $\mathcal{C}_U$ as a prestable B-linear ∞-category. In this case, the underlying prestable A-linear ∞-category of $\mathcal{C}_U$ represents the functor χ'_U, so that $U \in \mathcal{X}_0$. Since $\mathcal{X}$ is generated under small colimits by its affine objects (see Lemma **??**), it follows that $\mathcal{X}_0 = \mathcal{X}$.

For each object $U \in \mathcal{X}$, let $\Gamma(U; \mathcal{C})$ be a prestable A-linear ∞-category which represents the functor χ'_U. Then we can regard the construction $U \mapsto \Gamma(U; \mathcal{C})$ as a functor from $\mathcal{X}^{\mathrm{op}}$ to the ∞-category $\mathrm{LinCat}_A^{\mathrm{lex}}$ which preserves small limits. By construction, each $\Gamma(U; \mathcal{C})$ is equipped with a map $v_U : f_U^* \Gamma(U; \mathcal{C}) \to \mathcal{C}_U$ having the following universal property: for every prestable A-linear ∞-category $\mathcal{D}$, composition with v_U induces a homotopy equivalence from

$\mathrm{Map}_{\mathrm{LinCat}_A^{\mathrm{lex}}}(\mathcal{D}, \Gamma(U;\mathcal{C}))$ to the summand of $\mathrm{Map}_{\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}_U)}(f_U^* \mathcal{D}, \mathcal{C}_U)$ consisting of the left exact points. Let $\mathcal{X}_1$ be the full subcategory of $\mathcal{X}$ spanned by those objects U having the property that, for every prestable A-linear ∞-category $\mathcal{D}$, composition with v_U also induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{LinCat}_A^{\mathrm{PSt}}}(\mathcal{D}, \Gamma(U;\mathcal{C})) \to \mathrm{Map}_{\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}_U)}(f_U^* \mathcal{D}, \mathcal{C}_U).$$

It follows immediately from the construction of $\Gamma(U;\mathcal{C})$ given above that every affine object of $\mathcal{X}$ is contained in $\mathcal{X}_1$. Because the forgetful functor $\mathrm{LinCat}_A^{\mathrm{lex}} \hookrightarrow \mathrm{LinCat}_A^{\mathrm{PSt}}$ preserves small limits (Remark D.1.6.1), the full subcategory $\mathcal{X}_1 \subseteq \mathcal{X}$ is closed under small colimits. Since $\mathcal{X}$ is generated under small colimits by its affine objects (Lemma **??**), we conclude that $\mathcal{X}_1 = \mathcal{X}$. In particular, the final object $\mathbf{1} \in \mathcal{X}$ is good. Setting $\Gamma(X;\mathcal{C}) = \Gamma(\mathbf{1};\mathcal{C})$, we deduce the existence of a map $v = v_{\mathbf{1}} : f^*\Gamma(X;\mathcal{C}) \to \mathcal{C}$ which induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{LinCat}_A^{\mathrm{PSt}}}(\mathcal{D}, \Gamma(\mathcal{X};\mathcal{C})) \to \mathrm{Map}_{\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})}(f^* \mathcal{D}, \mathcal{C})$$

for every prestable A-linear ∞-category $\mathcal{D}$. Allowing $\mathcal{C}$ to vary, we deduce that the pullback functor $f^* : \mathrm{LinCat}_A^{\mathrm{PSt}} \to \mathrm{QStk}^{\mathrm{PSt}}(X)$ admits a right adjoint, given on objects by the construction $\mathcal{C} \mapsto \Gamma(X;\mathcal{C})$.

We now prove (2) under the assumption that $Y = \mathrm{Spec}\, A$ and $Y' = \mathrm{Spec}\, A'$ are both affine. It follows that X and X' are representable by spectral Deligne-Mumford stacks $(\mathcal{X}, \mathscr{O}_X)$ and $(\mathcal{X}', \mathscr{O}_{X'})$. For each $U \in \mathcal{X}$, let U' denote its inverse image in $\mathcal{X}'$, and let $\Gamma(U';\bullet) : \mathrm{QStk}^{\mathrm{PSt}}(X') \to \mathrm{LinCat}_{A'}^{\mathrm{lex}}$ be defined as above. We will prove that for each $U \in \mathcal{X}$, the canonical map $\alpha_U : A' \otimes_A \Gamma(U;\mathcal{C}) \to \Gamma(U'; g'^* \mathcal{C})$. is an equivalence. When regarded as a functors of U, both the domain and codomain of α_U carry colimits in $\mathcal{X}$ to limits in $\mathrm{LinCat}_{A'}^{\mathrm{lex}}$. It will therefore suffice to prove that α_U is an equivalence when U is affine. We may therefore reduce to the case where $X \simeq \mathrm{Spec}\, B$, so that $X' = \mathrm{Spec}\, B'$ for $B' = A' \otimes_A B$. The desired result now follows from Lemma D.3.5.6, since the canonical map $\mathrm{Mod}_{A'}^{\mathrm{cn}} \otimes_{\mathrm{Mod}_A^{\mathrm{cn}}} \mathrm{Mod}_B^{\mathrm{cn}} \to \mathrm{Mod}_{B'}^{\mathrm{cn}}$ is an equivalence of ∞-categories.

We now treat the general case of (1). Write Y as the colimit of a (not necessarily small) diagram $q : S \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, where each $Y_s = q(s)$ is affine. For each $s \in S$, set $X_s = X \times_Y Y_s$. Every edge $s \to s'$ in S determines a pullback diagram

$$\begin{array}{ccc} X_s & \longrightarrow & X_{s'} \\ \downarrow & & \downarrow \\ Y_s & \longrightarrow & Y_{s'}. \end{array}$$

Using assertion (2) in the affine case, we deduce that the associated diagram of pullback functors

$$\begin{array}{ccc} \mathrm{QStk}^{\mathrm{PSt}}(Y_{s'}) & \longrightarrow & \mathrm{QStk}^{\mathrm{PSt}}(X_{s'}) \\ \downarrow & & \downarrow \\ \mathrm{QStk}^{\mathrm{PSt}}(Y_s) & \longrightarrow & \mathrm{QStk}^{\mathrm{PSt}}(X_s) \end{array}$$

is right adjointable. Since the construction $Z \mapsto \mathrm{QStk}^{\mathrm{PSt}}(Z)$ carries colimits to limits, Corollary HA.4.7.4.18 implies the following:

(*i*) The functor $f^* : \mathrm{QStk}^{\mathrm{PSt}}(Y) \to \mathrm{QStk}^{\mathrm{PSt}}(X)$ admits a right adjoint f_*.

(*ii*) For each $s \in S$, the diagram

$$\begin{array}{ccc} \mathrm{QStk}^{\mathrm{PSt}}(Y) & \xrightarrow{f^*} & \mathrm{QStk}^{\mathrm{PSt}}(X) \\ \downarrow & & \downarrow \\ \mathrm{QStk}^{\mathrm{PSt}}(Y_s) & \longrightarrow & \mathrm{QStk}^{\mathrm{PSt}}(X_s) \end{array}$$

is right adjointable.

This proves (1). Moreover, we can assume without loss of generality that every morphism $\mathrm{Spec}\, A \to Y$ appears as a map $q(s) \to Y$ for some $s \in S$, so that (*ii*) implies that (2) is satisfied whenever Y' is affine. To prove (2) in general, consider a pullback square σ :

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$; we wish to show that the Beck-Chevalley map $\theta : g^* f_* \mathcal{C} \to f'_* g'^* \mathcal{C}$ is an equivalence. Unwinding the definitions, this is equivalent to the assertion that for every map $h : \mathrm{Spec}\, A \to Y'$, the pullback $h^*(\theta)$ is an equivalence in $\mathrm{QStk}^{\mathrm{PSt}}(\mathrm{Spec}\, A)$. Extending σ to a rectangular diagram

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}\, A & \xrightarrow{h} & Y' & \xrightarrow{g} & Y \end{array}$$

where both squares are pullbacks, we see that $h^*(\theta)$ fits into a commutative diagram

$$\begin{array}{ccccc} & & h^* f'_* g'^* \mathcal{C} & & \\ & \nearrow^{h^*(\theta)} & & \searrow^{\theta'} & \\ h^* g^* f_* \mathcal{C} & & \xrightarrow{\theta''} & & f''_* h'^* g'^* \mathcal{C}, \end{array}$$

where θ' and θ'' are equivalences by virtue of the fact that (2) holds in the special case treated above. □

10.1.5 Properties of the Direct Image

Proposition 10.1.4.1 establishes the existence of pushforwards in both the prestable and abelian settings. We now show that these are compatible:

Proposition 10.1.5.1. *Let $f : X \to Y$ be a representable morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Then the diagram*

$$\begin{array}{ccc} \mathrm{QStk}^{\mathrm{PSt}}(Y) & \xrightarrow{f^*} & \mathrm{QStk}^{\mathrm{PSt}}(X) \\ \downarrow \heartsuit & & \downarrow \heartsuit \\ \mathrm{QStk}^{\mathrm{Ab}}(Y) & \xrightarrow{f^*} & \mathrm{QStk}^{\mathrm{Ab}}(X) \end{array}$$

is right adjointable. In other words, for every object $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$, the canonical map $(f_* \mathcal{C})^{\heartsuit} \to f_*(\mathcal{C}^{\heartsuit})$ is an equivalence of abelian quasi-coherent stacks on Y.

Proof. Using assertions (2) and (2′) of Proposition 10.1.4.1, we can reduce to the case where $Y = \operatorname{Spec} A$ is affine. Since f is representable, the functor X is representable by some spectral Deligne-Mumford stack $(\mathcal{X}, \mathscr{O}_X)$. For each $U \in \mathcal{X}$, define $\mathcal{C}_U$ and $\Gamma(U; \mathcal{C}_U)$ as in the proof of Proposition 10.1.4.1, and define $(\mathcal{C}^{\heartsuit})_U$ and $\Gamma(U; (\mathcal{C}^{\heartsuit})_U)$ similarly, so that we have an evident map $\theta_U : \Gamma(U; \mathcal{C}_U)^{\heartsuit} \to \Gamma(U; (\mathcal{C}^{\heartsuit})_U)$. It follows from the definition of $\mathcal{C}^{\heartsuit}$ that the map θ_U is an equivalence whenever U is affine. Since the construction $U \mapsto \theta_U$ carries colimits in $\mathcal{X}$ to limits in $\mathrm{LinCat}_A^{\mathrm{Ab,lex}}$ (see Remark D.1.6.8), it follows that θ_U is an equivalence when U is a final object of $\mathcal{X}$, which is an equivalent formulation of Proposition 10.1.5.1. □

We now specialize Proposition 10.1.4.1 to the stable setting.

Proposition 10.1.5.2. *Let $f : X \to Y$ be a representable morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Then the pushforward functor $f_* : \mathrm{QStk}^{\mathrm{PSt}}(X) \to \mathrm{QStk}^{\mathrm{PSt}}(Y)$ of Proposition 10.1.4.1 carries stable quasi-coherent stacks on X to stable quasi-coherent stacks on Y.*

Proof. Using Proposition 10.1.4.1, we can assume without loss of generality that $Y = \operatorname{Spec} A$ is affine. Since f is representable, the functor X is representable by some spectral Deligne-Mumford stack $(\mathcal{X}, \mathscr{O}_X)$. For each $U \in \mathcal{X}$, define $\mathcal{C}_U$ and $\Gamma(U; \mathcal{C}_U)$ as in the proof of Proposition 10.1.4.1. Let $\mathcal{X}_0$ be the full subcategory of $\mathcal{X}$ spanned by those objects U for which the prestable A-linear ∞-category $\Gamma(U; \mathcal{C}_U)$ is stable. The construction $U \mapsto \Gamma(U; \mathcal{C}_U)$ carries colimits in $\mathcal{X}$ to limits in $\mathrm{LinCat}_A^{\mathrm{PSt}}$, and the full subcategory $\mathrm{LinCat}_A^{\mathrm{St}} \subseteq \mathrm{LinCat}_A^{\mathrm{PSt}}$

is closed under limits (since the inclusion admits a left adjoint). It follows that $\mathcal{X}_0$ is closed under small colimits in $\mathcal{X}$. Our assumption that $\mathcal{C}$ is stable guarantees that $\mathcal{X}_0$ contains all affine objects of $\mathcal{X}$. Using Lemma **??**, we deduce that $\mathcal{X}_0 = \mathcal{X}$. In particular, $\mathcal{X}_0$ contains a final object of $\mathcal{X}$, so that $f_* \mathcal{C}$ is stable. □

Combining Propositions 10.1.4.1 and 10.1.5.2, we obtain the following:

Corollary 10.1.5.3. (1) *Let $f : X \to Y$ be a representable morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Then the pullback functors $f^* : \mathrm{QStk}^{\mathrm{St}}(Y) \to \mathrm{QStk}^{\mathrm{St}}(X)$ admits a right adjoint $f_* : \mathrm{QStk}^{\mathrm{St}}(X) \to \mathrm{QStk}^{\mathrm{St}}(Y)$.*

(2) *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, where f is representable. Then the associated diagram

$$\begin{array}{ccc} \mathrm{QStk}^{\mathrm{St}}(Y) & \xrightarrow{f^*} & \mathrm{QStk}^{\mathrm{St}}(X) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ \mathrm{QStk}^{\mathrm{St}}(Y') & \xrightarrow{f'^*} & \mathrm{QStk}^{\mathrm{St}}(X') \end{array}$$

is right adjointable: that is, the canonical natural transformation $g^* f_* \to f'_* g'^*$ is an equivalence.

We also have the following analogue of Proposition 10.1.5.1:

Proposition 10.1.5.4. *Let $f : X \to Y$ be a representable morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Then the diagram*

$$\begin{array}{ccc} \mathrm{QStk}^{\mathrm{PSt}}(Y) & \xrightarrow{f^*} & \mathrm{QStk}^{\mathrm{PSt}}(X) \\ \downarrow{\scriptstyle \mathrm{Sp}} & & \downarrow{\scriptstyle \mathrm{Sp}} \\ \mathrm{QStk}^{\mathrm{St}}(Y) & \xrightarrow{f^*} & \mathrm{QStk}^{\mathrm{St}}(X) \end{array}$$

is right adjointable. In other words, for every object $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$, the canonical map $\mathrm{Sp}(f_* \mathcal{C}) \to f_*(\mathrm{Sp}(\mathcal{C}))$ is an equivalence of stable quasi-coherent stacks on Y.

Proof. Using Proposition 10.1.4.1, we can reduce to the case where $Y = \operatorname{Spec} A$ is affine. Since f is representable, the functor X is representable by some spectral Deligne-Mumford stack $(\mathcal{X}, \mathscr{O}_X)$. For each $U \in \mathcal{X}$, let $\mathcal{C}_U$ and $\Gamma(U; \mathcal{C}_U)$ be as in the proof of Proposition 10.1.4.1, and define $\operatorname{Sp}(\mathcal{C})_U$ and $\Gamma(U; \operatorname{Sp}(\mathcal{C})_U)$ similarly. Let $\mathcal{X}_0$ denote the full subcategory of $\mathcal{X}$ for which the canonical map $\operatorname{Sp}(\Gamma(U; \mathcal{C}_U)) \to \Gamma(U; \operatorname{Sp}(\mathcal{C})_U)$ is an equivalence. It follows from Remark D.1.6.8 that the subcategory $\mathcal{X}_0$ is closed under colimits in $\mathcal{X}$, and it follows from the definition of $\operatorname{Sp}(\mathcal{C})$ that $\mathcal{X}_0$ contains every affine object of $\mathcal{X}$. Applying Lemma **??**, we deduce that $\mathcal{X}_0 = \mathcal{X}$. In particular, $\mathcal{X}_0$ contains a final object of $\mathcal{X}$, so that the canonical map $\operatorname{Sp}(f_* \mathcal{C}) \to f_*(\operatorname{Sp}(\mathcal{C}))$ is an equivalence. □

10.1.6 Tensor Products of Quasi-Coherent Stacks

Let A be a connective $\mathbb{E}_\infty$-ring. If $\mathcal{C}$ and $\mathcal{D}$ are additive A-linear ∞-categories, then the relative tensor product $\mathcal{C} \otimes_A \mathcal{D}$ of Construction D.2.1.1 inherits the structure of an additive A-linear ∞-category. By means of this construction, we obtain a symmetric monoidal structure on the ∞-category $\operatorname{LinCat}_A^{\mathrm{Add}} \simeq \operatorname{Mod}_{\operatorname{Mod}_A^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ of additive A-linear ∞-categories (see Remark D.2.3.1). This symmetric monoidal structure depends functorially on A: that is, we can regard the construction $A \mapsto \operatorname{LinCat}_A^{\mathrm{PSt}}$ as a functor from $\mathrm{CAlg}^{\mathrm{cn}}$ to the ∞-category $\mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ of symmetric monoidal ∞-categories. Taking a right Kan extension along the Yoneda embedding $\mathrm{CAlg}^{\mathrm{cn}} \hookrightarrow \operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}}$, we obtain a functor

$$\mathrm{QStk}^{\mathrm{Add}} : \operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}}) \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty).$$

It follows from Remark 10.1.1.2 that the composition of $\mathrm{QStk}^{\mathrm{Add}}$ with the forgetful functor $\mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ agrees with Construction 10.1.1.1. We can summarize the situation more informally as follows:

(*a*) For every functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the ∞-category $\mathrm{QStk}^{\mathrm{Add}}(X)$ of additive quasi-coherent stacks on X inherits the structure of a symmetric monoidal ∞-category.

(*b*) For every map $f : X \to Y$ in $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, the pullback functor $f^* : \mathrm{QStk}^{\mathrm{Add}}(Y) \to \mathrm{QStk}^{\mathrm{Add}}(X)$ is symmetric monoidal.

(*c*) When $X = \operatorname{Spec} A$ is a corepresentable functor, the symmetric monoidal structure on $\mathrm{QStk}^{\mathrm{Add}}(X) \simeq \operatorname{LinCat}_A^{\mathrm{Add}}$ is given by the relative tensor product $(\mathcal{C}, \mathcal{D}) \mapsto \mathcal{C} \otimes_A \mathcal{D}$.

Remark 10.1.6.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. It follows from (*a*), (*b*) and (*c*) above that the tensor product on $\mathrm{QStk}^{\mathrm{Add}}(X)$ is computed "pointwise" in the following sense: for every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, we have a canonical equivalence of A-linear ∞-categories $(\mathcal{C} \otimes \mathcal{D})_\eta \simeq \mathcal{C}_\eta \otimes_A \mathcal{D}_\eta$.

Example 10.1.6.2. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. We let $\mathcal{Q}_X^{\mathrm{cn}}$ denote the unit object of the symmetric monoidal ∞-category $\mathrm{QStk}^{\mathrm{Add}}(X)$. The quasi-coherent stack $\mathcal{Q}_X^{\mathrm{cn}}$ can be described more informally as follows: to every point $\eta \in X(A)$, it assigns the prestable A-linear ∞-category $\mathrm{Mod}_A^{\mathrm{cn}}$ (which is the unit object of the symmetric monoidal ∞-category $\mathrm{LinCat}_A^{\mathrm{Add}} = \mathrm{Mod}_{\mathrm{Mod}_A^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$).

We now restrict our attention to tensor products of prestable, stable, and abelian quasi-coherent stacks.

Proposition 10.1.6.3 (Tensor Products of Prestable Quasi-Coherent Stacks)**.** *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. Then the full subcategory $\mathrm{QStk}^{\mathrm{PSt}}(X) \subseteq \mathrm{QStk}^{\mathrm{Add}}(X)$ of prestable quasi-coherent stacks on X contains the unit object $\mathcal{Q}_X^{\mathrm{cn}} \in \mathrm{QStk}^{\mathrm{Add}}(X)$ and is closed under tensor products. Consequently, $\mathrm{QStk}^{\mathrm{PSt}}(X)$ inherits the structure of a symmetric monoidal ∞-category (for which the inclusion $\mathrm{QStk}^{\mathrm{PSt}}(X) \hookrightarrow \mathrm{QStk}^{\mathrm{Add}}(X)$ is symmetric monoidal).*

Proof. Using Remark 10.1.6.1, we can reduce to the case where $X = \operatorname{Spec} A$ is affine, in which case the desired result follows from Proposition D.2.2.1. □

Proposition 10.1.6.4 (Tensor Products of Stable Quasi-Coherent Stacks)**.** *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor, and let $L : \mathrm{QStk}^{\mathrm{Add}}(X) \to \mathrm{QStk}^{\mathrm{St}}(X)$ be a left adjoint to the inclusion (given by $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$). Then the localization functor L is compatible with the symmetric monoidal structure on $\mathrm{QStk}^{\mathrm{Add}}(X)$, in the sense of Definition HA.2.2.1.6. Consequently, the ∞-category $\mathrm{QStk}^{\mathrm{St}}(X)$ inherits a symmetric monoidal structure, which is determined (up to essentially unique equivalence) by the requirement that the functor L is symmetric monoidal.*

Proof. Unwinding the definitions, we must show that for every pair of objects $\mathcal{C}, \mathcal{D} \in \mathrm{QStk}^{\mathrm{Add}}(X)$, the canonical map

$$\mathrm{Sp}(\mathcal{C} \otimes \mathcal{D}) \to \mathrm{Sp}(\mathrm{Sp}(\mathcal{C}) \otimes \mathrm{Sp}(\mathcal{D}))$$

is an equivalence. Without loss of generality, we can assume that $X = \operatorname{Spec} A$ is affine, in which case the desired result follows from the observation that Mod_A is an idempotent object in the symmetric monoidal ∞-category $\mathrm{LinCat}_A^{\mathrm{Add}}$. □

Remark 10.1.6.5. Let $\mathcal{C}$ and $\mathcal{D}$ be additive quasi-coherent stacks on a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. If either $\mathcal{C}$ or $\mathcal{D}$ is stable, then the tensor product $\mathcal{C} \otimes \mathcal{D}$ is stable. It follows that the tensor product on $\mathrm{QStk}^{\mathrm{St}}(X)$ agrees with the tensor product on $\mathrm{QStk}^{\mathrm{Add}}(X)$: in other words, the diagram

$$\begin{array}{ccc} \mathrm{QStk}^{\mathrm{St}}(X) \times \mathrm{QStk}^{\mathrm{St}}(X) & \xrightarrow{\otimes} & \mathrm{QStk}^{\mathrm{St}}(X) \\ \downarrow & & \downarrow \\ \mathrm{QStk}^{\mathrm{Add}}(X) \times \mathrm{QStk}^{\mathrm{Add}}(X) & \xrightarrow{\otimes} & \mathrm{QStk}^{\mathrm{Add}}(X). \end{array}$$

However, the inclusion $\mathrm{QStk}^{\mathrm{St}}(X) \hookrightarrow \mathrm{QStk}^{\mathrm{Add}}(X)$ is not symmetric monoidal, because the unit objects are different: the unit object of $\mathrm{QStk}^{\mathrm{St}}(X)$ is the stable quasi-coherent stack $\mathcal{Q}_X = \mathrm{Sp}(\mathcal{Q}_X^{\mathrm{cn}})$, which can be described more informally as follows: to every point $\eta \in X(A)$, it assigns the stable A-linear ∞-category Mod_A (which is the unit object of the symmetric monoidal ∞-category $\mathrm{LinCat}_A^{\mathrm{St}} = \mathrm{Mod}_{\mathrm{Mod}_A}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$).

Remark 10.1.6.6. For each functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the stabilization construction $\mathrm{QStk}^{\mathrm{PSt}}(X) \to \mathrm{QStk}^{\mathrm{St}}(X)$ is symmetric monoidal functor. Moreover, it is given by the construction $\mathcal{C} \mapsto \mathcal{Q}_X \otimes \mathcal{C}$.

Remark 10.1.6.7. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. Then the ∞-category $\mathrm{QStk}^{\mathrm{St}}(X)$ admits small colimits (see Variant 10.1.3.8) and the tensor product functor $\otimes : \mathrm{QStk}^{\mathrm{St}}(X) \times \mathrm{QStk}^{\mathrm{St}}(X) \to \mathrm{QStk}^{\mathrm{St}}(X)$ preserves small colimits separately in each variable. However, the analogous statement for prestable quasi-coherent stacks is false, even if we restrict our attention to compact morphisms (see Warning **??**).

Variant 10.1.6.8 (Tensor Products of Abelian Quasi-Coherent Stacks)**.** For every connective $\mathbb{E}_\infty$-ring A, the ∞-category $\mathrm{LinCat}_A^{\mathrm{Ab}}$ of abelian A-linear ∞-categories is equipped with a symmetric monoidal structure given by the construction $(\mathcal{C}, \mathcal{D}) \mapsto \mathcal{C} \otimes_A \mathcal{D}$ (see Variant D.2.3.2). Arguing as above, we see that for every functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, we can regard the ∞-category $\mathrm{QStk}^{\mathrm{Ab}}(X)$ as equipped with a symmetric monoidal structure, which depends functorially on X and is given pointwise by the construction $(\mathcal{C} \otimes \mathcal{D})_\eta = \mathcal{C}_\eta \otimes_A \mathcal{D}_\eta)$. In particular, the tensor product on $\mathrm{QStk}^{\mathrm{Ab}}(X)$ agrees with the tensor product on the larger ∞-category $\mathrm{QStk}^{\mathrm{Add}}(X)$. However, the inclusion functor $\mathrm{QStk}^{\mathrm{Ab}}(X) \hookrightarrow \mathrm{QStk}^{\mathrm{Add}}(X)$ is not a symmetric monoidal functor, because it does not preserve unit objects. We will denote the unit object of $\mathrm{QStk}^{\mathrm{Ab}}(X)$ by $\mathcal{Q}_X^{\heartsuit}$: it is given informally by the formula $(\mathcal{Q}_X^{\heartsuit})_\eta = \mathrm{Mod}_A^{\heartsuit}$ for each point $\eta \in X(A)$.

Remark 10.1.6.9. For every functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the functor

$$\mathrm{QStk}^{\mathrm{PSt}}(X) \to \mathrm{QStk}^{\mathrm{Ab}}(X) \qquad \mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$$

of Construction 10.1.2.8 determines a symmetric monoidal functor $\mathrm{QStk}^{\mathrm{PSt}}(X)$ to $\mathrm{QStk}^{\mathrm{Ab}}(X)$, which depends functorially on X. We observe that this construction is given by the formula $\mathcal{C}^{\heartsuit} = \mathcal{Q}_X^{\heartsuit} \otimes \mathcal{C}$, where the tensor product is formed in the ∞-category $\mathrm{QStk}^{\mathrm{Add}}(X)$.

Remark 10.1.6.10. Let $f : X \to Y$ be a representable morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Then the pullback functors

$$f^* : \mathrm{QStk}^{\mathrm{PSt}}(Y) \to \mathrm{QStk}^{\mathrm{PSt}}(X) \qquad f^* : \mathrm{QStk}^{\mathrm{Ab}}(Y) \to \mathrm{QStk}^{\mathrm{Ab}}(X)$$

are symmetric monoidal. It follows that the direct image functors

$$f_* : \mathrm{QStk}^{\mathrm{PSt}}(X) \to \mathrm{QStk}^{\mathrm{PSt}}(Y) \qquad f_* : \mathrm{QStk}^{\mathrm{Ab}}(X) \to \mathrm{QStk}^{\mathrm{Ab}}(Y)$$

of Proposition 10.1.4.1 are lax symmetric monoidal. In particular, they carry commutative algebra objects of $\mathrm{QStk}^{\mathrm{PSt}}(X)$ and $\mathrm{QStk}^{\mathrm{Ab}}(X)$ to commutative algebra objects of $\mathrm{QStk}^{\mathrm{Ab}}(Y)$ and $\mathrm{QStk}^{\mathrm{Ab}}(Y)$, respectively.

10.1.7 Global Sections of Quasi-Coherent Stacks

Let X be a spectral Deligne-Mumford stack and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by X (given by $X(R) = \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X})$). We define a *(additive, stable, prestable, abelian) quasi-coherent stack on* X to be a (additive, stable, prestable, abelian) quasi-coherent stack on the functor X, and we define

$$\mathrm{QStk}^{\mathrm{Add}}(\mathsf{X}) = \mathrm{QStk}^{\mathrm{Add}}(X) \qquad \mathrm{QStk}^{\mathrm{St}}(\mathsf{X}) = \mathrm{QStk}^{\mathrm{St}}(X)$$
$$\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) = \mathrm{QStk}^{\mathrm{PSt}}(X) \qquad \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X}) = \mathrm{QStk}^{\mathrm{PSt}}(X).$$

Construction 10.1.7.1 (Global Sections of Quasi-Coherent Stacks)**.** Let S denote the sphere spectrum, so that we have a canonical equivalences

$$\mathrm{QStk}^{\mathrm{PSt}}(\mathrm{Sp\acute{e}t}\, S) \simeq \mathrm{LinCat}_S^{\mathrm{PSt}} \simeq \mathsf{Groth}_\infty \qquad \mathrm{QStk}^{\mathrm{Ab}}(\mathrm{Sp\acute{e}t}\, S) \simeq \mathrm{LinCat}_S^{\mathrm{Ab}} \simeq \mathsf{Groth}_{\mathrm{ab}}\,.$$

For every spectral Deligne-Mumford stack X, there is an essentially unique morphism $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, S$ which determines a representable morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Applying Proposition 10.1.4.1, we obtain direct image functors

$$\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \xrightarrow{f_*} \mathrm{QStk}^{\mathrm{PSt}}(\mathrm{Sp\acute{e}t}\, S) \simeq \mathsf{Groth}_\infty \qquad \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X}) \xrightarrow{f_*} \mathrm{QStk}^{\mathrm{Ab}}(\mathrm{Sp\acute{e}t}\, S) \simeq \mathsf{Groth}_{\mathrm{ab}}\,.$$

We will refer to either of these functors as the *global sections functor* and denote it by $\mathcal{C} \mapsto \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$.

As the notation suggests, one can view $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ as the ∞-category of quasi-coherent sheaves on X with coefficients in the quasi-coherent stack $\mathcal{C}$.

Example 10.1.7.2. Let X be a spectral Deligne-Mumford stack and let $\mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}}$ be the unit object of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ (see Example 10.1.6.2). Then there is a canonical equivalence of ∞-categories $\mathrm{QCoh}(\mathsf{X}; \mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}}) \simeq \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$.

Variant 10.1.7.3. Let X be a spectral Deligne-Mumford stack and let K be a closed subset of $|\,\mathsf{X}\,|$. For every map $\phi : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, the inverse image of K in $|\,\mathrm{Sp\acute{e}t}\, A| \simeq |\,\mathrm{Spec}\, A|$ is the vanishing locus of some ideal $I_\phi \subseteq \pi_0 A$. Using Remark **??**, we see that the construction

$$(\phi : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}) \mapsto (\mathrm{Mod}_A^{\mathrm{cn}})^{\mathrm{Nil}(I_\phi)}$$

determines a quasi-coherent stack on X. We will denote this quasi-coherent stack by $\mathcal{Q}_K^{\mathrm{cn}}$. By construction, we have a map of quasi-coherent stacks $\mathcal{Q}_K^{\mathrm{cn}} \to \mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}}$, which induces an equivalence

$$\mathrm{QCoh}(\mathsf{X}; \mathcal{Q}_K^{\mathrm{cn}}) \simeq \mathrm{QCoh}_K(\mathsf{X})^{\mathrm{cn}} \subseteq \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} = \Gamma(\mathsf{X}; \mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}}).$$

Remark 10.1.7.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks, and let $\mathcal{C}$ be a prestable or abelian quasi-coherent stack on X. Then we have a canonical equivalence of ∞-categories $\mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \simeq \mathrm{QCoh}(\mathsf{Y}; f_* \mathcal{C})$.

Construction 10.1.7.5. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and let $\mathcal{C}$ be a prestable or abelian quasi-coherent stack on Y. Then the unit map $u_{\mathcal{C}} : \mathcal{C} \to f_* f^* \mathcal{C}$ induces a functor

$$\mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) \to \mathrm{QCoh}(\mathsf{Y}; f_* f^* \mathcal{C}) \simeq \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C}).$$

We will refer to this functor as *pullback along* f and denote it by $f^* : \mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) \to \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C})$. It follows immediately from the definitions that the functor f^* preserves small colimits and therefore admits a right adjoint (see Corollary HTT.5.5.2.9), which we will denote by $f_* : \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C}) \to \mathrm{QCoh}(\mathsf{Y}; \mathcal{C})$.

We now study the functorial behavior of Construction 10.1.7.5. To simplify the exposition, we will restrict our attention to the setting of prestable quasi-coherent stacks. First, we need to introduce some terminology.

Definition 10.1.7.6. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor and let $u : \mathcal{C} \to \mathcal{D}$ be a morphism of prestable quasi-coherent stacks on X. Then:

(1) We will say that u is *left exact* if, for every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, the induced A-linear functor $u_\eta : \mathcal{C}_\eta \to \mathcal{D}_\eta$ is left exact.

(2) We will say that u is *compact* if, for every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, the induced A-linear functor $u_\eta ; \mathcal{C}_\eta \to \mathcal{D}_\eta$ is compact (that is, u_η admits a right adjoint which commutes with filtered colimits).

Remark 10.1.7.7. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, and let $u : \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathrm{QStk}^{\mathrm{PSt}}(Y)$. If u is left exact (compact), then $f^*(u)$ is left exact (compact). The converse holds if f induces an effective epimorphism after sheafification with respect to the étale topology (see Propositions D.5.2.1 and D.5.2.2).

Remark 10.1.7.8. Let A be a connective $\mathbb{E}_\infty$-ring and let $u : \mathcal{C} \to \mathcal{D}$ be an A-linear functor between prestable A-linear ∞-categories. Then u is left exact (compact) if and only if it is left exact (compact) when regarded as a morphism of prestable quasi-coherent stacks on $\operatorname{Spec} A$.

Proposition 10.1.7.9. *Let $f : X \to Y$ be a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which is representable and flat. Then, for every prestable or abelian quasi-coherent stack $\mathcal{C}$ on Y, the unit map $\mathcal{C} \to f_* f^* \mathcal{C}$ is left exact.*

Proof. We will assume that $\mathcal{C}$ is prestable; the proof in the abelian case is the same. By virtue of Proposition 10.1.4.1, we can reduce to the case where $Y = \operatorname{Spec} A$ is affine. Then X is representable by a spectral Deligne-Mumford stack $(\mathcal{X}, \mathscr{O}_X)$ which is flat over A. For each object $U \in \mathcal{X}$, let $\Gamma(U; f^* \mathcal{C})$ be defined as in the proof of Proposition 10.1.4.1. Let us identify $\mathcal{C}$ and $f_* f^* \mathcal{C}$ with prestable A-linear ∞-categories, and let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the full subcategory spanned by those objects U for which the composite map

$$\mathcal{C} \to f_* f^* \mathcal{C} \to \Gamma(U; f^* \mathcal{C})$$

is left exact. We wish to prove that $\mathcal{X}_0$ contains the final object of $\mathcal{X}$. Since the construction $U \mapsto \Gamma(U; f^* \mathcal{C})$ carries colimits in $\mathcal{X}$ to limits in $\mathrm{LinCat}_A^{\mathrm{lex}}$, the ∞-category $\mathcal{X}_0$ is closed under small colimits. It will therefore suffice to show that $\mathcal{X}_0$ contains all affine objects of $\mathcal{X}$. In other words, we are reduced to proving Lemma 10.1.7.9 in the special case where $X \simeq \operatorname{Spec} B$ is also affine. In this case, we wish to show that the extension of scalars functor $\mathcal{C} \to B \otimes_A \mathcal{C}$ is left exact. Equivalently, we wish to show that the construction $C \mapsto B \otimes_A C$ determines a left exact functor from $\mathcal{C}$ to itself. This is clear, since B is flat over A and can therefore be written as a filtered colimit of free A-modules of finite rank (Proposition HA.7.2.2.15). □

Corollary 10.1.7.10. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and let $\mathcal{C}$ be a prestable or abelian quasi-coherent stack on Y. If f is flat, then the induced map $f^* : \mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) \to \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C})$ is left exact.*

Proof. Combine Propositions 10.1.7.9 and 10.3.1.14. □

Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of prestable quasi-coherent stacks on Y. Then we have a commutative diagram of unit maps

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{u_{\mathcal{C}}} & f_* f^* \mathcal{C} \\ \downarrow F & & \downarrow f_* f^* F \\ \mathcal{D} & \xrightarrow{u_{\mathcal{D}}} & f_* f^* \mathcal{D}. \end{array}$$

Passing to global sections, we obtain a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{Y}; \mathcal{D}) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathsf{Y}; \mathcal{D}). \end{array}$$

Proposition 10.1.7.11. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of prestable quasi-coherent stacks on Y. If f is flat and that F is compact, then the diagram of ∞-categories*

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) & \xrightarrow{F} & \mathrm{QCoh}(\mathsf{Y}; \mathcal{D}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C}) & \xrightarrow{F} & \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{D}) \end{array}$$

is right adjointable.

We first verify Proposition 10.1.7.11 in the case where X and Y are affine.

Lemma 10.1.7.12. *Let $\phi : A \to B$ be a flat morphism between connective $\mathbb{E}_\infty$-rings, and let $F : \mathcal{C} \to \mathcal{D}$ be a compact A-linear functor between prestable A-linear ∞-categories. Then the diagram of ∞-categories*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & & \downarrow \\ B \otimes_A \mathcal{C} & \xrightarrow{F} & B \otimes_A \mathcal{D} \end{array}$$

is right adjointable (here the vertical maps are given by extension of scalars).

Proof. Let $G : \mathcal{D} \to \mathcal{C}$ be right adjoint to F. Unwinding the definitions, we wish to show that for every object $D \in \mathcal{D}$, the canonical map $\rho : B \otimes_A G(D) \to G(B \otimes_A D)$ is an equivalence in $\mathcal{C}$. Our assumption that F is compact guarantees that the functor G commutes with filtered colimits. Since B flat over A, we can write B as the colimit of a filtered diagram $\{M_\alpha\}$ where each M_α is a free A-module of finite rank (Proposition HA.7.2.2.15). It follows that ρ can be written as a filtered colimit of maps $\rho_\alpha : M_\alpha \otimes_A G(D) \to G(M_\alpha \otimes_A D)$, each of which is an equivalence because the functor G commutes with finite products. □

Proof of Proposition 10.1.7.11. We first treat the case where the morphism f is étale. Write $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}})$. For each object $U \in \mathcal{Y}$, define $\Gamma(U; \mathcal{C})$ and $\Gamma(U; \mathcal{D})$ as in the proof of Proposition 10.1.4.1. Let us say that a morphism $u : U \to V$ in $\mathcal{Y}$ is *good* if the diagram

$$\begin{array}{ccc} \Gamma(V; \mathcal{C}) & \xrightarrow{F} & \Gamma(V; \mathcal{D}) \\ \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle u^*} \\ \Gamma(U; \mathcal{C}) & \xrightarrow{F} & \Gamma(U; \mathcal{D}) \end{array}$$

is right adjointable. We now proceed in several steps:

(*a*) If $U, V \in \mathcal{Y}$ are affine, then any morphism $u : U \to V$ is good. This follows from Lemma 10.1.7.12.

(b) Suppose we are given a small diagram $\{U_\alpha\}$ in the ∞-category $\mathcal{Y}$ having a colimit U. For each index α, let $u_\alpha : U_\alpha \to U$ denote the tautological map. If each of the transition maps $U_\alpha \to U_\beta$ is good, then each u_α is good. This is a special case of Corollary HA.4.7.4.18 .

(c) Suppose we are given morphisms $\{u_\alpha : U_\alpha \to U\}$ and $v : U \to V$ in the ∞-category $\mathcal{Y}$, where the morphisms u_α are good and the induced map $\coprod U_\alpha \to U$ is an effective epimorphism. Then v is good if and only if each composition $v \circ u_\alpha$ is good.

Let us say that an object $V \in \mathcal{Y}$ is *good* if, for every affine object $U \in \mathcal{Y}$, every morphism $U \to V$ is good.

(d) Every affine object of $\mathcal{Y}$ is good: this is an immediate consequence of (a).

(e) Let $v : U \to V$ be a morphism in $\mathcal{Y}$. If U and V are good, then v is good. To prove this, choose a covering $\{u_\alpha : U_\alpha \to U\}$ where each U_α is affine. Our assumption that U is good now implies that each u_α is good. Our assumption that V is good guarantees that each composition $v \circ u_\alpha$ is good. Applying (c), we deduce that v is good.

(f) Let $\{V_\alpha\}$ be a small diagram in $\mathcal{Y}$ having a colimit V, and suppose that each V_α is good. Then V is good. To prove this, choose an arbitrary map $v : U \to V$ where U is affine; we wish to show that v is good. Choose a covering $\{u_\beta : U_\beta \to U\}$, where each U_β is affine and each composition $v \circ u_\beta$ factors as a composition $U_\beta \xrightarrow{v_\beta} V_\alpha \to V$ for some index α. Our assumption that V_α is good then guarantees that v_β is good. Using (b) and (e), we deduce that the tautological map $V_\alpha \to V$ is good, so that $v \circ u_\beta$ is good. It follows from (a) that each of the maps u_β is good. Applying (c), we deduce that v is good.

(g) Every object of $\mathcal{Y}$ is good: this follows from (d) and (f), together with Proposition 1.4.7.9.

(h) Every morphism in $\mathcal{Y}$ is good: this follows from (e) and (g).

We now prove Proposition 10.1.7.11 for a general flat morphism $f : \mathsf{X} \to \mathsf{Y}$. Choose a mutually surjective collection étale morphisms $u_\alpha : \mathsf{X}_\alpha \to \mathsf{X}$, where each X_α is affine and each of the composite maps $\mathsf{X}_\alpha \xrightarrow{u_\alpha} \mathsf{X} \xrightarrow{f} \mathsf{Y}$ factors through some étale morphism $v_\alpha : \mathsf{Y}_\alpha \to \mathsf{Y}$, where Y_α is affine. For each index α, we have a commutative diagram of ∞-categories

$$\begin{array}{ccc}
\mathrm{QCoh}(\mathsf{Y};\mathcal{C}) & \xrightarrow{F} & \mathrm{QCoh}(\mathsf{Y};\mathcal{D}) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
\mathrm{QCoh}(\mathsf{X}; f^*\mathcal{C}) & \xrightarrow{F} & \mathrm{QCoh}(\mathsf{X}; f^*\mathcal{D}) \\
\downarrow{\scriptstyle u_\alpha^*} & & \downarrow{\scriptstyle u_\alpha^*} \\
\mathrm{QCoh}(\mathsf{X}_\alpha; u_\alpha^* f^*\mathcal{C}) & \xrightarrow{F} & \mathrm{QCoh}(\mathsf{X}_\alpha; u_\alpha^* f^*\mathcal{D}).
\end{array}$$

Applying (h) to the spectral Deligne-Mumford stack X, we see that the lower square in this diagram is right adjointable. Since the morphisms u_α are mutually surjective, the pullback functors $u_\alpha^* : \mathrm{QCoh}(\mathsf{X}; f^*\mathcal{C}) \to \mathrm{QCoh}(\mathsf{X}; u_\alpha^* f^*\mathcal{C})$ are mutually conservative. Consequently, to show that the upper square is right adjointable, it will suffice to show that the outer rectangle is right adjointable. Equivalently, we are reduced to showing that the outer rectangle in the diagram

$$\begin{array}{ccc}
\mathrm{QCoh}(\mathsf{Y};\mathcal{C}) & \xrightarrow{F} & \mathrm{QCoh}(\mathsf{Y};\mathcal{D}) \\
\downarrow{\scriptstyle v_\alpha^*} & & \downarrow{\scriptstyle f^*} \\
\mathrm{QCoh}(\mathsf{Y}_\alpha; v_\alpha^*\mathcal{C}) & \xrightarrow{F} & \mathrm{QCoh}(\mathsf{Y}_\alpha; v_\alpha^*\mathcal{D}) \\
\downarrow & & \downarrow{\scriptstyle v_\alpha^*} \\
\mathrm{QCoh}(\mathsf{X}_\alpha; u_\alpha^* f^*\mathcal{C}) & \xrightarrow{F} & \mathrm{QCoh}(\mathsf{X}_\alpha; u_\alpha^* f^*\mathcal{D}).
\end{array}$$

Applying (h) again, we see that the upper square is right adjointable, and are therefore reduced to verifying the right adjointability of the lower square. In other words, we may replace X by X_α and Y by Y_α, and thereby reduce to the case where $\mathsf{Y} \simeq \operatorname{Spét} A$ and $\mathsf{X} \simeq \operatorname{Spét} B$ are affine. $\square$

Proposition 10.1.7.13. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of prestable quasi-coherent stacks on Y. Suppose that F is left exact and that, for every morphism $\operatorname{Spét} R \to \mathsf{Y}$, the fiber product $\mathsf{X} \times_{\mathsf{Y}} \operatorname{Spét} R$ is a quasi-compact, quasi-separated spectral algebraic space. Then the diagram of ∞-categories*

$$\begin{array}{ccc}
\mathrm{QCoh}(\mathsf{Y};\mathcal{C}) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathsf{X}; f^*\mathcal{C}) \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\
\mathrm{QCoh}(\mathsf{Y};\mathcal{D})) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathsf{X}; f^*\mathcal{D})
\end{array}$$

is right adjointable.

We begin by proving Proposition 10.1.7.13 in the special case where Y is affine.

Lemma 10.1.7.14. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be a quasi-compact, quasi-separated spectral algebraic space equipped with a map $f : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$. Let $\mathcal{C}$ and $\mathcal{D}$ be prestable R-linear ∞-categories and let $f^* \mathcal{C}$ and $f^* \mathcal{D}$ denote the associate prestable quasi-coherent stacks on X. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a left exact R-linear functor. Then the diagram of ∞-categories*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f^*} & \operatorname{QCoh}(\mathsf{X}; f^* \mathcal{C}) \\ \downarrow F & & \downarrow F \\ \mathcal{D} & \xrightarrow{f^*} & \operatorname{QCoh}(\mathsf{X}; f^* \mathcal{D}) \end{array}$$

is right adjointable.

Proof. Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$. For each object $U \in \mathcal{X}$, set $\mathsf{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathsf{X}}|_U)$, let $f_U : \mathsf{X}_U \to \operatorname{Sp\acute{e}t} R$ be the tautological map, and let

$$\Gamma_{\mathcal{C}}(U; \bullet) : \operatorname{QCoh}(\mathsf{X}_U; f_U^* \mathcal{C}) \to \mathcal{C} \qquad \Gamma_{\mathcal{D}}(U; \bullet) : \operatorname{QCoh}(\mathsf{X}_U; f_U^* \mathcal{D}) \to \mathcal{D}$$

denote the pushforward functors associated to the map f_U. Suppose we are given an object $C \in \operatorname{QCoh}(\mathsf{X}; f^* \mathcal{C})$. For each object $U \in \mathcal{X}$, let C_U denote the image of C in $\operatorname{QCoh}(\mathsf{X}_U; f_U^* \mathcal{C})$ and let $F(C_U) \in \operatorname{QCoh}(\mathsf{X}_U; f_U^* \mathcal{D})$ be the image of C_U under the functor determined by F. Then we have a tautological map $\rho_U : F\Gamma(U; C_U) \to \Gamma(U; F(C_U))$, which is an equivalence whenever $U \in \mathcal{X}$ is affine. Since F is left exact, the collection of those objects $U \in \mathcal{X}$ for which ρ_U is an equivalence is closed under finite colimits. Applying Proposition 2.5.3.5 and Theorem 3.4.2.1, we deduce that ρ_U is an equivalence when U is a final object of $\mathcal{X}$, which is equivalent to the statement of Lemma 10.1.7.14. □

Lemma 10.1.7.15. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and let $\mathcal{C}$ be a prestable quasi-coherent stack on Y. Suppose that for every morphism $\operatorname{Sp\acute{e}t} R \to \mathsf{Y}$, the fiber product $\mathsf{X} \times_{\mathsf{Y}} \operatorname{Sp\acute{e}t} R$ is a quasi-compact, quasi-separated spectral algebraic space. Then the unit map $u : \mathcal{C} \to f_* f^* \mathcal{C}$ is a compact morphism of prestable quasi-coherent stacks on Y.*

Proof. The assertion is local on Y. We may therefore assume without loss of generality that $\mathsf{Y} = \operatorname{Sp\acute{e}t} R$ is affine, so that X is a quasi-compact, quasi-separated spectral algebraic space. Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$, and for each $U \in \mathcal{X}$ define $f_U : \mathsf{X}_U \to \operatorname{Sp\acute{e}t} R$ and $\Gamma(U; \bullet) : \operatorname{QCoh}(\mathsf{X}_U; f_U^* \mathcal{C}) \to \mathcal{C}$ as in the proof of Lemma 10.1.7.14. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the full subcategory spanned by those objects U for which the functor $\Gamma(U; \bullet)$ commutes with small filtered colimits. Since $\mathcal{C}$ is a Grothendieck prestable ∞-category, the formation of filtered colimits in $\mathcal{C}$ commutes with finite colimits, so that $\mathcal{X}_0$ is closed under finite colimits in $\mathcal{X}$. Note that if $U \in \mathcal{X}$ is affine, then $\Gamma(U; \bullet)$ is equivalent to the forgetful functor $\operatorname{LMod}_A(\mathcal{C}) \to \mathcal{C}$ for some connective $\mathbb{E}_\infty$-algebra A over R, so that $\Gamma(U; \bullet)$ commutes with all small colimits; in particular, we have $U \in \mathcal{X}_0$. Applying Proposition 2.5.3.5 and Theorem 3.4.2.1, we deduce that $\mathcal{X}_0$ contains the final object of $\mathcal{X}$, so that the unit map $\mathcal{C} \to f_* f^* \mathcal{C}$ is compact as desired. □

Proof of Proposition 10.1.7.13. Set $\mathcal{C}' = f_* f^* \mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y})$ and $\mathcal{D}' = f_* f^* \mathcal{D} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y})$, so that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}' \\ \downarrow & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{D}' \end{array}$$

in $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y})$ where the horizontal morphisms are compact (Lemma 10.1.7.15). Choose a mutually surjective collection of étale morphisms $u_\alpha : \mathsf{Y}_\alpha \to \mathsf{Y}$, where each Y_α is affine, and consider the diagrams

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(\mathsf{Y}; \mathcal{C}') \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{Y}; \mathcal{D}) & \longrightarrow & \mathrm{QCoh}(\mathsf{Y}; \mathcal{D}') \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{Y}_\alpha; u_\alpha^* \mathcal{D}) & \longrightarrow & \mathrm{QCoh}(\mathsf{Y}_\alpha, u_\alpha^* \mathcal{D}'). \end{array}$$

We wish to show that the upper square is right adjointable. Note that each of the lower squares are right adjointable (Proposition 10.1.7.11). Since the pullback functors $u_\alpha^* : \mathrm{QCoh}(\mathsf{Y}; \mathcal{D}) \to \mathrm{QCoh}(\mathsf{Y}_\alpha; u_\alpha^* \mathcal{D})$ are mutually conservative, it will suffice to show that the outer rectangles are right adjointable. Equivalently, we are reduced to proving the right adjointability of the outer rectangle in the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(\mathsf{Y}; \mathcal{C}') \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{Y}_\alpha; u_\alpha^*) & \longrightarrow & \mathrm{QCoh}(\mathsf{Y}_\alpha; u_\alpha^* \mathcal{C}') \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{Y}_\alpha; u_\alpha^* \mathcal{D}) & \longrightarrow & \mathrm{QCoh}(\mathsf{Y}_\alpha, u_\alpha^* \mathcal{D}'). \end{array}$$

Here the upper square is right adjointable by virtue of Proposition 10.1.7.11, so it will suffice to show that the lower square is also right adjointable. In other words, to prove Proposition 10.1.7.13, we can replace Y by Y_α and thereby reduce to the case where Y is affine, in which case the desired result follows from Lemma 10.1.7.14. □

We close this section by establishing a relative version of Theorem 5.6.6.1:

Proposition 10.1.7.16. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a surjective proper morphism of spectral Deligne-Mumford stacks and let $\mathsf{X}_\bullet$ be the Čech nerve of f. For each $n \geqslant 0$, let $f_n : \mathsf{X}_n \to \mathsf{Y}$ denote the associated projection map. Assume that Y is locally Noetherian and that the structure sheaf $\mathscr{O}_{\mathsf{Y}}$ is truncated. Then, for any stable quasi-coherent stack $\mathcal{C}$ on Y, the pullback functors f_n^* induces an equivalence of ∞-categories $\mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) \to \mathrm{Tot}\,\mathrm{QCoh}(\mathsf{X}_n; f_n^* \mathcal{C})$.*

Proof. Let us regard $\mathsf{X}_\bullet$ as an *augmented* simplicial object of SpDM via the convention $\mathsf{X}_{-1} = \mathsf{Y}$. We wish to show that the augmented cosimplicial ∞-category $\mathrm{QCoh}(\mathsf{X}_\bullet; f_n^* \mathcal{C})$ determines a limit diagram $\mathbf{\Delta}_+ \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. Using (the dual of) Corollary HA.4.7.5.3 , it will suffice to verify the following:

(*a*) The adjunction $\mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C})$ is comonadic.

(*b*) The augmented cosimplicial ∞-category $\mathrm{QCoh}(\mathsf{X}_\bullet; f_\bullet^* \mathcal{C})$ satisfies the Beck-Chevalley condition. More precisely, for every morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}_+$, the diagram of pullback functors

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X}_m; f_m^* \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(\mathsf{X}_{m+1}; f_{m+1}^* \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{X}_n; f_n^* \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(\mathsf{X}_{n+1}; f_{n+1}^* \mathcal{C}) \end{array}$$

is right adjointable.

Assertion (*b*) follows from Propositions 10.1.4.1 and 10.1.7.13. To prove (*a*), it will suffice (by virtue of Theorem HA.4.7.3.5) to show that the functor $f^* : \mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) \to \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C})$ is conservative and preserves totalizations of f^*-split cosimplicial objects of $\mathrm{QCoh}(\mathsf{Y}; \mathcal{C})$. Both of these assertions can be tested locally on Y. We may assume without loss of generality that $\mathsf{Y} = \mathrm{Spét}\, R$ is affine. Let us abuse notation by identifying $\mathcal{C}$ with the corresponding R-linear ∞-category.

We proceed as in the proof of Proposition 5.6.6.4. We first show that the functor f^* is conservative. Fix an object $C \in \mathcal{C}$, and suppose that $f^* C \in \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C})$ vanishes. $\mathcal{E} \subseteq \mathrm{Mod}_R$ be the full subcategory spanned by those R-modules M such that $M \otimes_R C \simeq 0$. Let us regard $\mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C})$ as tensored over the ∞-category $\mathrm{QCoh}(\mathsf{X})$ of quasi-coherent sheaves on X (see §10.2). For each $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, we have $\Gamma(\mathsf{X}; \mathscr{F}) \otimes_R C \simeq f_*(\mathscr{F} \otimes f^* C) \simeq 0$, so that $\Gamma(\mathsf{X}; \mathscr{F})$ belongs to $\mathcal{E}$. Since $\mathcal{E}$ is evidently a stable subcategory of Mod_R which is closed under retracts, Proposition 5.6.6.3 guarantees that $R \in \mathcal{E}$, so that $C \simeq R \otimes_R C \simeq 0$.

Now suppose that $D^\bullet$ is an f^*-split cosimplicial object of $\mathcal{C}$. Let $\mathcal{E}' \subseteq \mathrm{Mod}_R$ be the full subcategory spanned by those R-modules M for which the tautological map $f^*(\mathrm{Tot}(M \otimes_R$

$C^\bullet)) \to \mathrm{Tot}(f^*(M \otimes_R C^\bullet))$ is an equivalence in $\mathrm{QCoh}(\mathsf{X}; f^*\mathcal{C})$. Note that for each $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, the cosimplicial object $\Gamma(\mathsf{X}; \mathscr{F}) \otimes_R C^\bullet \simeq \Gamma(\mathsf{X}; \mathscr{F} \otimes f^* C^\bullet)$ is split, so that $\Gamma(\mathsf{X}; \mathscr{F})$ belongs to $\mathcal{E}'$. Because $\mathcal{E}'$ is a stable subcategory of Mod_R which is closed under retracts, Proposition 5.6.6.3 guarantees that $R \in \mathcal{E}'$, so that the canonical map $f^*(\mathrm{Tot}(C^\bullet)) \to \mathrm{Tot}(f^* C^\bullet)$ is an equivalence as desired. □

10.2 Quasi-Coherent Stacks on Spectral Algebraic Spaces

Let X be a spectral Deligne-Mumford stack and let $\mathrm{QCoh}(\mathsf{X}; \bullet) : \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \to \mathsf{Groth}_\infty$ be the global sections functor of Construction 10.1.7.1. It follows from Remark 10.1.6.10 that $\mathrm{QCoh}(\mathsf{X}; \bullet)$ is a lax symmetric monoidal functor. In particular, it carries the unit object $\mathcal{Q}^{\mathrm{cn}}_{\mathsf{X}}$ to a commutative algebra object $\mathrm{QCoh}(\mathsf{X}; \mathcal{Q}^{\mathrm{cn}}_{\mathsf{X}})$ of Groth_∞. Unwinding the definitions, we see that this commutative algebra object can be identified with the ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ of connective quasi-coherent sheaves on X (equipped with its usual symmetric monoidal structure). Consequently, we can promote the global sections functor $\mathrm{QCoh}(\mathsf{X}; \bullet)$ to a functor

$$\begin{aligned} \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) & \simeq \mathrm{Mod}_{\mathcal{Q}^{\mathrm{cn}}_{\mathsf{X}}}(\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})) \\ & \xrightarrow{\mathrm{QCoh}(\mathsf{X};\bullet)} \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X}; \mathcal{Q}^{\mathrm{cn}}_{\mathsf{X}})}(\mathsf{Groth}_\infty) \\ & \simeq \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty). \end{aligned}$$

We will abuse notation by denoting this functor also by $\mathcal{C} \mapsto \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$. We can describe the situation more informally as follows: for every quasi-coherent stack $\mathcal{C}$ on X, the ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ of global sections of $\mathcal{C}$ is a Grothendieck prestable ∞-category which is tensored over ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ of connective quasi-coherent sheaves on X, and the action

$$\otimes : \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \times \mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \to \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$$

preserves small colimits separately in each variable.

Our goal in this section is to show that, under some mild hypotheses on X, a prestable quasi-coherent stack $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ can be recovered from the ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ (as an ∞-category which is tensored over $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$). Our main results can be formulated as follows:

Theorem 10.2.0.1. *Let X be a spectral Deligne-Mumford stack, and suppose that X is quasi-geometric (that is, that X is quasi-compact and the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times \mathsf{X}$ is quasi-affine; see Definition 9.1.4.1). Then the global sections functor*

$$\mathrm{QCoh}(\mathsf{X}; \bullet) : \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \to \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$$

is fully faithful.

Theorem 10.2.0.2. *Let X be a quasi-compact, quasi-separated spectral algebraic space. Then the global sections functor*

$$\mathrm{QCoh}(\mathsf{X}; \bullet) : \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \to \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$$

is an equivalence of ∞-categories.

Corollary 10.2.0.3. *Let $f : X \to Y$ be a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which is representable by quasi-compact, quasi-separated spectral algebraic spaces. Then the direct image functor f_* of Proposition 10.1.4.1 induces an equivalence of ∞-categories*

$$\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \to \mathrm{Mod}_{f_* \mathcal{Q}^{\mathrm{cn}}_{\mathsf{X}}}(\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y})).$$

Proof. Using Proposition 10.1.4.1, we can reduce to the case where Y is affine, in which case the desired result reduces to Theorem 10.2.0.2. □

10.2.1 Base Change Along Quasi-Affine Morphisms

In §C.4, we proved that the ∞-category Groth_∞ of Grothendieck prestable ∞-categories admits a symmetric monoidal structure. For every spectral Deligne-Mumford stack, we can regard the ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ as a commutative algebra object of the ∞-category Groth_∞ and study the associated ∞-category of modules $\mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$. However, since the ∞-category Groth_∞ generally does not admit colimits, the study of modules in Groth_∞ can be somewhat delicate. For example, it is not clear that the ∞-category $\mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$ inherits a symmetric monoidal structure: the standard procedure for constructing a tensor product $\mathcal{M} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{N}$ involves forming the geometric realization of a two-sided bar construction $\mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathcal{M}, \mathcal{N})_\bullet$, which might fail to exist.

One way to address this concern is to embed the ∞-category Groth_∞ of Grothendieck prestable ∞-categories into some larger symmetric monoidal ∞-category $\mathcal{E}$ which does admit small colimits, and for which the tensor product $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ preserves small colimits separately in each variable. In this case, the formation of relative tensor products

$$\mathcal{M} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{N} \simeq |\, \mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathcal{M}, \mathcal{N})_\bullet|$$

is a perfectly well-behaved operation *in the larger ∞-category $\mathcal{E}$*, and one can hope that in good cases the resulting object is actually again a Grothendieck prestable ∞-category.

There are (at least) two natural candidates for the ∞-category $\mathcal{E}$:

(*a*) We can take $\mathcal{E}$ to be the ∞-category the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{Add}} \simeq \mathrm{Mod}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ of additive presentable ∞-categories (see Corollary C.4.1.3), equipped with the symmetric monoidal structure of Corollary C.4.1.4.

(b) We can take $\mathcal{E}$ to be the ∞-category Groth_∞^+ of Remark **??**, whose objects are pairs $(\mathcal{C}, \mathcal{C}_{\geqslant 0})$ where $\mathcal{C}$ is a presentable stable ∞-category and $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$ is a full subcategory which is closed under small colimits and extensions. The ∞-category Groth_∞^+ can be regarded as an enlargement of Groth_∞ (Remark **??**) which admits small limits and colimits (Remark C.3.1.7) and inherits a tensor product (Remark C.4.2.3) which preserves small colimits separately in each variable (Proposition C.4.5.3).

In order to prove Theorems 10.2.0.1 and 10.2.0.2, we will need to study relative tensor products of the form $\mathrm{QCoh}^{\mathrm{cn}}(\mathsf{U}) \times_{\mathrm{QCoh}^{\mathrm{cn}}(\mathsf{X})} \mathcal{C}$, where $f : \mathsf{U} \to \mathsf{X}$ is a quasi-affine morphism of spectral Deligne-Mumford stacks. We now show that this construction is well-behaved *if* we adopt convention (b): that is, if we form the relative tensor product $\mathrm{QCoh}^{\mathrm{cn}}(\mathsf{U}) \times_{\mathrm{QCoh}^{\mathrm{cn}}(\mathsf{X})} \mathcal{N}$ in the ∞-category Groth_∞^+, then it belongs to the essential image of the fully faithful embedding $\mathsf{Groth}_\infty \hookrightarrow \mathsf{Groth}_\infty^+$ (Proposition 10.2.1.3). Beware that this is *a priori* different from the relative tensor product in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{Add}}$ of presentable additive ∞-categories: see Warnings 10.2.5.4 and 10.2.1.5. However, we will see later that there is no difference when X is a quasi-compact algebraic space with affine diagonal (Proposition 10.2.6.7).

Construction 10.2.1.1. Let $f : \mathsf{U} \to \mathsf{X}$ be a quasi-affine morphism of spectral Deligne-Mumford stacks and let $\mathcal{C}$ be a Grothendieck prestable ∞-category equipped with an action of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ (which we view as a commutative algebra object of the ∞-category Groth_∞). Then the stable ∞-category $\mathrm{Sp}(\mathcal{C})$ inherits an action of the ∞-category $\mathrm{QCoh}(\mathsf{X})$ (viewed as a commutative algebra object in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ of presentable stable ∞-categories).

Set $\mathscr{A} = f_* \mathscr{O}_{\mathsf{U}} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X}))$, let $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$ denote the ∞-category of $\mathscr{A}$-module objects of $\mathrm{Sp}(\mathcal{C})$, and let $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ be the smallest full subcategory of $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$ which contains the essential image of the composite functor

$$\rho : \mathcal{C} \xrightarrow{\Sigma^\infty} \mathrm{Sp}(\mathcal{C}) \xrightarrow{\mathscr{A} \otimes} \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$$

and is closed under small colimits and extensions. Using Proposition HA.1.4.4.11 , we deduce that the stable ∞-category $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$ admits an accessible t-structure

$$(\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}, \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0}).$$

Note that an object $X \in \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$ belongs to $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0}$ if and only if the abelian groups

$$\mathrm{Ext}^n_{\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))}(\rho(C), X) \quad \simeq \quad \mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty C, X)$$

vanish for $n < 0$: that is, if and only if the image of X in $\mathrm{Sp}(\mathcal{C})$ belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. Since the forgetful functor $\theta : \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) \to \mathrm{Sp}(\mathcal{C})$ commutes with filtered colimits and the t-structure on $\mathrm{Sp}(\mathcal{C})$ is compatible with filtered colimits, it follows that the t-structure $(\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}, \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0})$ is also compatible with filtered colimits. In particular, $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ is a Grothendieck prestable ∞-category. Moreover, the intersection $\bigcap_{n \geqslant 0} \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\leqslant -n}$ is the inverse image under θ of the intersection $\bigcap_{n \geqslant 0} \mathrm{Sp}(\mathcal{C})_{\leqslant -n}$. Since the t-structure on $\mathrm{Sp}(\mathcal{C})$ is right complete and the functor θ is conservative, it follows that the intersection $\bigcap_{n \geqslant 0} \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\leqslant -n}$ consists only of zero object of $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$. Applying Proposition HA.1.2.1.19 , we deduce that the t-structure $(\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}, \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0})$ is right complete: that is, we can identify $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$ with the stabilization of the Grothendieck prestable ∞-category $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$.

Warning 10.2.1.2. In the situation of Construction 10.2.1.1, the full subcategory

$$\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0} \subseteq \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$$

is the inverse image of $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ under the forgetful functor $\theta : \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) \to \mathrm{Sp}(\mathcal{C})$. However, it is usually *not* true that we can identify $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ with the inverse image of $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ under the functor θ. In general, this holds only when the algebra object $\mathscr{A} \in \mathrm{QCoh}(\mathsf{X})$ is connective: that is, when the morphism $f : \mathsf{U} \to \mathsf{X}$ is assumed to be affine.

Proposition 10.2.1.3. *Let $f : \mathsf{U} \to \mathsf{X}$ be a quasi-affine map and let $\mathcal{C}$ be a Grothendieck prestable ∞-category equipped with an action of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$. Then:*

(*a*) *The pair $(\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})), \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0})$ appearing Construction 10.2.1.1 can be identified with the relative tensor product*

$$(\mathrm{QCoh}(\mathsf{U}), \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}}) \otimes_{(\mathrm{QCoh}(\mathsf{X}), \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}})} (\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{C})_{\geqslant 0})$$

in the ∞-category $\mathsf{Groth}_{\infty}^{+}$.

(*b*) *The Grothendieck prestable ∞-category $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ appearing in Construction 10.2.1.1 can be identified with the geometric of the two-sided bar construction $\mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}}, \mathcal{C})$ in the ∞-category Groth_{∞}; moreover, this geometric realization is preserved by the fully faithful embedding $\mathsf{Groth}_{\infty} \hookrightarrow \mathsf{Groth}_{\infty}^{+}$ of Remark* **??**.

Remark 10.2.1.4 (Relative Tensor Products in $\mathsf{Groth}_{\infty}^{+}$)**.** Let $\mathsf{Groth}_{\infty}^{+}$ be the ∞-category introduced in Remark **??**, equipped with the symmetric monoidal structure described in Remark C.4.2.3. Suppose that $(\mathcal{C}, \mathcal{C}_{\geqslant 0})$ is an algebra object of $\mathsf{Groth}_{\infty}^{+}$, and let $(\mathcal{M}, \mathcal{M}_{\geqslant 0})$ and $(\mathcal{N}, \mathcal{N}_{\geqslant 0})$ be right and left modules over $(\mathcal{C}, \mathcal{C}_{\geqslant 0})$, respectively. More concretely, this means that we have the following data:

(*a*) A presentable stable ∞-category $\mathcal{C}$ equipped with a monoidal structure $m : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which preserves small colimits separately in each variable.

(*b*) A presentable stable ∞-category $\mathcal{M}$ which is right-tensored over $\mathcal{C}$ for which the action $a : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ preserves small colimits separately in each variable.

(*c*) A presentable stable ∞-category $\mathcal{N}$ which is left-tensored over $\mathcal{C}$ for which the action map $a' : \mathcal{C} \times \mathcal{N} \to \mathcal{N}$ preserves small colimits separately in each variable.

(*d*) Full subcategories

$$\mathcal{M}_{\geqslant 0} \subseteq \mathcal{M} \qquad \mathcal{C}_{\geqslant 0} \subseteq \mathcal{C} \qquad \mathcal{N}_{\geqslant 0} \subseteq \mathcal{N}$$

which are closed under colimits and extensions and satisfy

$$\mathbf{1}_{\mathcal{C}} \in \mathcal{C}_{\geqslant 0} \qquad m(\mathcal{C}_{\geqslant 0} \times \mathcal{C}_{\geqslant 0}) \subseteq \mathcal{C}_{\geqslant 0}$$

$$a(\mathcal{M}_{\geqslant 0} \times \mathcal{C}_{\geqslant 0}) \subseteq \mathcal{M}_{\geqslant 0} \qquad a'(\mathcal{C}_{\geqslant 0} \times \mathcal{N}_{\geqslant 0}) \subseteq \mathcal{N}_{\geqslant 0}.$$

Since the ∞-category $\mathsf{Groth}_{\infty}^{+}$ admits small colimits and the tensor product on $\mathsf{Groth}_{\infty}^{+}$ preserves small colimits separately in each variable, the relative tensor product

$$(\mathcal{M}, \mathcal{M}_{\geqslant 0}) \otimes_{(\mathcal{C}, \mathcal{C}_{\geqslant 0})} (\mathcal{N}, \mathcal{N}_{\geqslant 0})$$

is well-defined, given by the geometric realization (as an object of the ∞-category $\mathsf{Groth}_{\infty}^{+}$) of the two-sided bar construction $\mathrm{Bar}_{(\mathcal{C}, \mathcal{C}_{\geqslant 0})}((\mathcal{M}, \mathcal{M}_{\geqslant 0}), (\mathcal{N}, \mathcal{N}_{\geqslant 0}))_{\bullet}$. Unwinding the definitions, we see that this relative tensor product is given by the pair $(\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}, \mathcal{E})$, where the relative tensor product $\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$ is computed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ of presentable stable ∞-categories and $\mathcal{E} \subseteq \mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$ is the smallest full subcategory which is closed under colimits and extensions and contains the essential image of the composite map

$$\mathcal{M}_{\geqslant 0} \times \mathcal{N}_{\geqslant 0} \to \mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}.$$

Proof of Proposition 10.2.1.3. Because f is quasi-affine, the pushforward functor f_* induces an equivalence of ∞-categories $\mathrm{QCoh}(\mathsf{U}) \simeq \mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{X}))$ (Corollary **??**). Applying Theorem HA.4.8.4.6 , we can identify the ∞-category $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$ with the tensor product

$$\mathrm{QCoh}(\mathsf{U}) \otimes_{\mathrm{QCoh}(\mathsf{X})} \mathrm{Sp}(\mathcal{C})$$

in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ of presentable stable ∞-categories. By virtue of Remark 10.2.1.4, it remains only to prove that $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ is the smallest full subcategory of $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$ which is closed under colimits and extensions and contains the essential image of the functor

$$\psi : \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \times \mathcal{C} \to \mathrm{QCoh}(\mathsf{U}) \otimes \mathrm{Sp}(\mathcal{C}) \to \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})).$$

Here the only nontrivial point is to show that the functor ψ factors through $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$. Fix an object $C \in \mathcal{C}$, and let $\mathrm{QCoh}'(\mathsf{U})$ be the full subcategory of $\mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}}$ spanned by those sheaves $\mathscr{F}$ for which $\psi(\mathscr{F}, C)$ belongs to $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$. Then $\mathrm{QCoh}'(\mathsf{U})$ is closed under colimits and extensions and contains the essential image of the functor $f^*|_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}$. Since f is quasi-affine, Corollary 2.5.6.3 guarantees that $\mathrm{QCoh}'(\mathsf{U}) = \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}}$, as desired. $\square$

Warning 10.2.1.5. Proposition 10.2.1.3 does *not* assert that the ∞-category $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ is a geometric realization of the bar construction $\mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}}, \mathcal{C})$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable stable ∞-categories.

Remark 10.2.1.6. In the situation of Proposition 10.2.1.3, the pullback map $f^* : \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \to \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}}$ induces a functor $F : \mathcal{C} \to \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C}$ which admits a right adjoint $G : \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C} \to \mathcal{C}$. We claim that G does not annihilate any nonzero objects of $\mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C}$. To prove this, we let $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$ be as in Construction 10.2.1.1, so that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) & \longrightarrow & \mathrm{Sp}(\mathcal{C}) \\ \downarrow \tau_{\geqslant 0} & & \downarrow \Omega^{\infty} \\ \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C} & \xrightarrow{G} & \mathcal{C}. \end{array}$$

Any object X of the tensor product $\mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C}$ satisfying $GX \simeq 0$ can be identified with an object $\overline{X} \in \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ which is annihilated by the composite functor $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) \to \mathrm{Sp}(\mathcal{C}) \xrightarrow{\Omega^{\infty}} \mathcal{C}$. It follows that we can identify $\overline{X}$ with a left $\mathscr{A}$-module object of the subcategory $\mathrm{Sp}(\mathcal{C})_{\leqslant -1} \subseteq \mathrm{Sp}(\mathcal{C})$, so that $\overline{X}$ belongs to the intersection $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0} \cap \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\leqslant -1}$ and is therefore a zero object.

Example 10.2.1.7 (Pullback Diagrams)**.** Suppose we are given a diagram of spectral Deligne-Mumford stacks σ :

$$\begin{array}{ccc} \mathsf{U}' & \longrightarrow & \mathsf{U} \\ \downarrow f' & & \downarrow f \\ \mathsf{X}' & \longrightarrow & \mathsf{X} \end{array}$$

where the morphism f is quasi-affine. Proposition 10.2.1.3 guarantees the existence of a relative tensor product $\mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X}')^{\mathrm{cn}}$ in the ∞-category Groth_{∞} of Grothendieck prestable ∞-categories, and the diagram σ determines a colimit-preserving functor

$$F : \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X}')^{\mathrm{cn}} \to \mathrm{QCoh}(\mathsf{U}')^{\mathrm{cn}}.$$

If σ is a pullback diagram, then F is an equivalence. To prove this, it suffices to verify conditions (a) and (b) of Remark C.3.1.8:

- (a) The induced functor of stable ∞-categories $\mathrm{QCoh}(\mathsf{U}) \otimes_{\mathrm{QCoh}(\mathsf{X})} \mathrm{QCoh}(\mathsf{X}') \to \mathrm{QCoh}(\mathsf{U}')$ is an equivalence; this is a special case of Corollary 6.3.4.7.

- (b) The right adjoint to F does not annihilate any nonzero objects of $\mathrm{QCoh}(\mathsf{U}')^{\mathrm{cn}}$. By virtue of Remark 10.2.1.6, this is equivalent to the assertion that the pullback functor $f'^* : \mathrm{QCoh}(\mathsf{X}')^{\mathrm{cn}} \to \mathrm{QCoh}(\mathsf{U}')^{\mathrm{cn}}$ admits a right adjoint which does not annihilate any nonzero objects of $\mathrm{QCoh}(\mathsf{U}')^{\mathrm{cn}}$. Note that the right adjoint to f'^* is given by the construction $\mathscr{F} \mapsto \tau_{\geqslant 0} f'_* \mathscr{F}$, which annihilates an object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{U}')^{\mathrm{cn}}$ if and only if the pushforward $f'_* \mathscr{F}$ belongs to $\mathrm{QCoh}(\mathsf{X}')_{\leqslant -1}$. Since f' is quasi-affine, this implies that $\mathscr{F}$ belongs to $\mathrm{QCoh}(\mathsf{U}')_{\leqslant -1}$ (Proposition 2.5.6.2), so that $\mathscr{F} \simeq 0$.

10.2.2 Compatibility with Inverse Limits

We next show that, under mild hypotheses, the relative tensor product of Proposition 10.2.1.3 is compatible with the formation of inverse limits.

Notation 10.2.2.1. Let X be a spectral Deligne-Mumford stack. We let $\mathrm{Mod}^{\mathrm{lex}}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$ denote the fiber product $\mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty) \times_{\mathsf{Groth}_\infty} \mathsf{Groth}^{\mathrm{lex}}_\infty$. More concretely, the objects of the ∞-category $\mathrm{Mod}^{\mathrm{lex}}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$ are Grothendieck prestable ∞-categories equipped with an action of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$, and the morphisms in $\mathrm{Mod}^{\mathrm{lex}}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$ are $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$-linear functors which preserve small colimits and finite limits.

Remark 10.2.2.2. According to Proposition C.3.2.4, the ∞-category $\mathsf{Groth}^{\mathrm{lex}}_\infty$ admits small limits which are preserved by the inclusion functor $\mathsf{Groth}^{\mathrm{lex}}_\infty \hookrightarrow \mathsf{Groth}_\infty$. Combining this observation with Corollary HA.4.2.3.3 , we deduce that $\mathrm{Mod}^{\mathrm{lex}}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$ admits small limits, which are preserved by the inclusion functor $\mathrm{Mod}^{\mathrm{lex}}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty) \hookrightarrow \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$ and also by the forgetful functor $\mathrm{Mod}^{\mathrm{lex}}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty) \to \mathsf{Groth}^{\mathrm{lex}}_\infty \subseteq \mathsf{Groth}_\infty$.

Proposition 10.2.2.3. *Let $q : \mathsf{U} \to \mathsf{X}$ be a quasi-affine morphism between spectral Deligne-Mumford stacks. Then:*

(1) *The tensor product construction $\mathcal{C} \mapsto \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C}$ of Proposition 10.2.1.3 carries left exact morphisms in $\mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$ to left exact morphisms in $\mathrm{Mod}_{\mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$.*

(2) *The associated functor $\mathrm{Mod}^{\mathrm{lex}}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty) \to \mathrm{Mod}^{\mathrm{lex}}_{\mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$ preserves small limits.*

Proof. We first prove (1). Let $f : \mathcal{C} \to \mathcal{D}$ be a left exact morphism between $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$-module objects of Groth_∞; we wish to show that the induced map

$$f_{\mathsf{U}} : \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C} \to \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{D}$$

is also left exact. Let $F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ be the map of stable ∞-categories determined by f, which we regard as a morphism of $\mathrm{QCoh}(\mathsf{X})$-module objects of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. Let $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X}))$ denote the direct image of the structure sheaf of U. Proposition 10.2.1.3 shows that we can identify the induced map $F_{\mathsf{U}} : \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) \to \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{D}))$ with the stabilization of the functor f_{U}. To prove that f_{U} is left exact, it will suffice to show that the functor F_{U} is left t-exact (Proposition C.3.2.1). Invoking the definition of the t-structure on $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{D}))$ (see Construction 10.2.1.1), we wish to prove that the composite functor

$$\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0} \subseteq \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) \xrightarrow{F_{\mathsf{U}}} \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{D})) \rightarrow \mathrm{Sp}(\mathcal{D})$$

factors through the full subcategory $\mathrm{Sp}(\mathcal{D})_{\leqslant 0} \subseteq \mathrm{Sp}(\mathcal{D})$. This follows immediately from the left exactness of F and the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) & \xrightarrow{F_{\mathsf{U}}} & \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{D})) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(\mathcal{C}) & \xrightarrow{F} & \mathrm{Sp}(\mathcal{D}). \end{array}$$

We now prove (2). Note that we have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Mod}^{\mathrm{lex}}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_{\infty}) & \longrightarrow & \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})}(\mathcal{P}\mathrm{r}^{\mathrm{St}}) \\ \downarrow {\scriptstyle \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \bullet} & & \downarrow {\scriptstyle \mathrm{QCoh}(\mathsf{U}) \otimes_{\mathrm{QCoh}(\mathsf{X})} \bullet} \\ \mathsf{Groth}^{\mathrm{lex}}_{\infty} & \longrightarrow & \mathcal{P}\mathrm{r}^{\mathrm{St}} \end{array}$$

where the horizontal maps are given by stabilization. We wish to show that the left vertical map preserves small limits (this is equivalent to assertion (2), since the forgetful functor $\mathrm{Mod}^{\mathrm{lex}}_{\mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}}}(\mathsf{Groth}_{\infty}) \rightarrow \mathsf{Groth}^{\mathrm{lex}}_{\infty}$ is conservative and preserves small limits). Note that the horizontal maps preserve small limits (Corollary C.3.2.5). Moreover, the right vertical map preserves small limits: this follows from the fact that $\mathrm{QCoh}(\mathsf{U})$ can be identified with the module ∞-category $\mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{X}))$ (Corollary **??**) and is therefore dualizable (in fact, self-dual) when viewed as a $\mathrm{QCoh}(\mathsf{X})$-module object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$.

Let $\{\mathcal{C}_{\alpha}\}$ be a small diagram in $\mathrm{Mod}^{\mathrm{lex}}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_{\infty})$ having a limit $\mathcal{C}$; we wish to prove that the canonical map

$$\theta : \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C} \rightarrow \varprojlim_{\alpha} \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C}_{\alpha}$$

is an equivalence of Grothendieck prestable ∞-categories. It follows from the above arguments that θ induces an equivalence after passing to the stabilization of both sides: more concretely, the stabilization of both sides can be identified with the ∞-category $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$. To complete the proof that θ is an equivalence, it suffices to show that the domain and codomain of θ determine the same t-structure on the stable ∞-category $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$. This follows immediately from the construction: either t-structure can be characterized by the observation that an object C of $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$ belongs to $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0}$ if and only if, for each index α, the image of C under the composite functor $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) \rightarrow \mathrm{Sp}(\mathcal{C}) \rightarrow \mathrm{Sp}(\mathcal{C}_{\alpha})$ belongs to the subcategory $\mathrm{Sp}(\mathcal{C}_{\alpha})_{\leqslant 0} \subseteq \mathrm{Sp}(\mathcal{C}_{\alpha})$. □

10.2.3 Excision for $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$-Modules

Let X be a spectral Deligne-Mumford stack. It follows from Remark 10.1.2.9 that the theory of prestable quasi-coherent stacks on X satisfies descent for the étale topology, and therefore also with respect to the Nisnevich topology. In particular, it we are given an excision square

$$\begin{array}{ccc} \mathsf{U}' & \longrightarrow & \mathsf{X}' \\ \downarrow & & \downarrow \\ \mathsf{U} & \longrightarrow & \mathsf{X} \end{array}$$

(see Variant 2.5.2.3), then a prestable quasi-coherent stack on X can be recovered from by its restriction to X' and U, together with an identification of their restrictions to U'. The main step in our proof of Theorem 10.2.0.2 is to show that, under some mild assumptions, the theory of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$-module objects of Groth_∞ has the same property.

Proposition 10.2.3.1. *Let X be a quasi-compact, quasi-separated spectral algebraic space and suppose we are given an excision square*

$$\begin{array}{ccc} \mathsf{U}' & \longrightarrow & \mathsf{X}' \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathsf{U} & \xrightarrow{j} & \mathsf{X} \end{array}$$

where the morphism j is a quasi-compact open immersion and the morphism f is étale and quasi-affine. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category equipped with an action of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$. Then the associated diagram $\overline{\sigma}$:

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{U}')^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C} & \longleftarrow & \mathrm{QCoh}(\mathsf{X}')^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C} \\ \uparrow & & \uparrow \\ \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C} & \longleftarrow & \mathcal{C} \end{array}$$

is a pullback square in the ∞-category $\mathsf{Groth}_\infty^{\mathrm{lex}}$ (hence also in the ∞-category Groth_∞).

The proof of Proposition 10.2.3.1 will require some preliminaries.

Lemma 10.2.3.2. *Let X be a quasi-compact, quasi-separated spectral algebraic space and let $\mathcal{C}$ be a Grothendieck prestable ∞-category equipped with an action of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$, so that the presentable stable ∞-category $\mathrm{Sp}(\mathcal{C})$ inherits an action of $\mathrm{QCoh}(\mathsf{X})$. Let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ be a quasi-coherent sheaf having Tor-amplitude $\leqslant 0$. Then the functor*

$$(\mathscr{F} \otimes \bullet) : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$$

is left t-exact.

Proof. Let C be an object of $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$; we wish to show that $\mathscr{F} \otimes C$ also belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. Using Proposition 9.6.7.1, we can write $\mathscr{F}$ as the colimit of a filtered diagram $\{\mathscr{F}_\alpha\}$, where each $\mathscr{F}_\alpha$ is *perfect* and of Tor-amplitude $\leqslant 0$. The assumption that the prestable ∞-category $\mathcal{C}$ is Grothendieck guarantees that $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ is closed under filtered colimits in $\mathrm{Sp}(\mathcal{C})$; it will therefore suffice to show that each tensor product $\mathscr{F}_\alpha \otimes C$ belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. This is equivalent to the assertion that the groups $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(D, \mathscr{F}_\alpha \otimes C)$ vanish for each $D \in \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ and each $n < 0$. Since $\mathscr{F}_\alpha$ is perfect, it admits a dual $\mathscr{F}_\alpha^\vee$ in the symmetric monoidal ∞-category $\mathrm{QCoh}(\mathsf{X})$. We are therefore reduced to proving that the groups $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(\mathscr{F}_\alpha^\vee \otimes D, C)$ vanish for $n < 0$. Because C is assumed to belong to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$, we are reduced to proving that $\mathscr{F}_\alpha^\vee \otimes D$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. This follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \times \mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \\ \downarrow{\scriptstyle \mathrm{id} \times \Sigma^\infty} & & \downarrow{\scriptstyle \Sigma^\infty} \\ \mathrm{QCoh}(\mathsf{X}) \times \mathrm{Sp}(\mathcal{C}) & \xrightarrow{\otimes} & \mathrm{Sp}(\mathcal{C}), \end{array}$$

since the assumption that $\mathscr{F}_\alpha$ has Tor-amplitude $\leqslant 0$ is equivalent to the statement that $\mathscr{F}_\alpha^\vee$ is connective. □

Lemma 10.2.3.3. *Let $f : \mathsf{U} \to \mathsf{X}$ be a flat quasi-affine morphism between quasi-compact, quasi-separated spectral algebraic spaces and let $\mathcal{C}$ be a Grothendieck prestable ∞-category equipped with an action of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{cn}}$. Then the induced map $\mathcal{C} \to \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C}$ is left exact.*

Proof. Let $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X}))$ denote the direct image under f of the structure sheaf of U. Using the description of $\mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C}$ supplied by Proposition 10.2.1.3 and the criterion of Proposition C.3.2.1, we are reduced to showing that the functor

$$(\mathscr{A} \otimes \bullet) : \mathrm{Sp}(\mathcal{C}) \to \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$$

is left t-exact. Here $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C}))$ is equipped with the t-structure appearing in the proof of Construction 10.2.1.1, so that the forgetful functor $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) \to \mathrm{Sp}(\mathcal{C})$ is conservative and left t-exact. It will therefore suffice to show that the formation of tensor product with $\mathscr{A}$ determines a left t-exact functor from $\mathrm{Sp}(\mathcal{C})$ to itself. By virtue of Lemma 10.2.3.2, we are reduced to proving that the quasi-coherent sheaf $\mathscr{A}$ has Tor-amplitude $\leqslant 0$. This assertion can be tested locally on X, so we may assume without loss of generality that $\mathsf{X} \simeq \mathrm{Spét}\, R$ is affine. In this case, we can identify $\mathscr{A}$ with a (nonconnective) $\mathbb{E}_\infty$-algebra A over R, and the assertion that A has Tor-amplitude $\leqslant 0$ is equivalent to the requirement that the construction $M \mapsto A \otimes_R M$ determines a left t-exact functor from the ∞-category Mod_R to itself. Under the identification $\mathrm{Mod}_R \simeq \mathrm{QCoh}(\mathsf{X})$, we can identify this functor which the composition $f_* f^*$. We conclude by observing that the functors f_* and f^* are both left t-exact: in the first case, this is automatic; in the second it follows from our assumption that f is flat. □

Proof of Proposition 10.2.3.1. It follows from Lemma 10.2.3.3 that each of the functors which appear in the diagram $\overline{\sigma}$:

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{U}')^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C} & \longleftarrow & \mathrm{QCoh}(\mathsf{X}')^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C} \\ \uparrow & & \uparrow \\ \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C} & \longleftarrow & \mathcal{C} \end{array}$$

are left exact.

Consequently, the diagram

$$\mathrm{QCoh}(\mathsf{X}')^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C} \to \mathrm{QCoh}(\mathsf{U}')^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C} \leftarrow \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C}$$

admits a limit $\mathcal{D}$ in the ∞-category $\mathsf{Groth}_{\infty}^{\mathrm{lex}}$, and the diagram $\overline{\sigma}$ determines a left exact, colimit-preserving functor $\mathcal{C} \to \mathcal{D}$ which we can identify (using Propositions C.3.1.1 and C.3.2.1) with a t-exact functor from $\mathrm{Sp}(\mathcal{C})$ to $\mathrm{Sp}(\mathcal{D})$. To complete the proof, it will suffice to show that this functor is an equivalence: that is, that the image of $\underline{\sigma}$ under the stabilization functor $\mathsf{Groth}_{\infty}^{\mathrm{lex}} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$ is a pullback diagram of *stable* ∞-categories.

Let $\mathscr{A} \in \mathrm{QCoh}(\mathsf{X})$ denote the direct image of the structure sheaf of X' and let $\mathscr{B} \in \mathrm{QCoh}(\mathsf{X})$ denote the direct image of the structure sheaf of U. Then $\mathscr{A}$ and $\mathscr{B}$ are commutative algebra objects of $\mathrm{QCoh}(\mathsf{X})$ whose tensor product $\mathscr{A} \otimes \mathscr{B}$ is the direct image of the structure sheaf of U'. The proof of Proposition **??** shows that we can identify the image of $\overline{\sigma}$ under the stabilization functor with the diagram of stable ∞-categories

$$\begin{array}{ccc} \mathrm{LMod}_{\mathscr{A} \otimes \mathscr{B}}(\mathrm{Sp}(\mathcal{C})) & \xleftarrow{\mathscr{B} \otimes} & \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) \\ \uparrow{\scriptstyle \mathscr{A} \otimes} & & \uparrow{\scriptstyle \mathscr{A} \otimes} \\ \mathrm{LMod}_{\mathscr{B}}(\mathrm{Sp}(\mathcal{C})) & \xleftarrow{\mathscr{B} \otimes} & \mathrm{Sp}(\mathcal{C}). \end{array}$$

To prove that this diagram is a pullback square, we wish to show that the canonical map

$$u : \mathrm{Sp}(\mathcal{C}) \to \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) \times_{\mathrm{LMod}_{\mathscr{A} \otimes \mathscr{B}}(\mathrm{Sp}(\mathcal{C}))} \mathrm{LMod}_{\mathscr{B}}(\mathrm{Sp}(\mathcal{C}))$$

is an equivalence of ∞-categories. Let us identify objects of the codomain of u with triples (M, N, β) where M is a left $\mathscr{A}$-module objects of $\mathrm{Sp}(\mathcal{C})$, N is a left $\mathscr{B}$-module object of $\mathrm{Sp}(\mathcal{C})$, and $\beta : \mathscr{B} \otimes M \simeq \mathscr{A} \otimes N$ is an equivalence of left $\mathscr{A} \otimes \mathscr{B}$-module objects of $\mathrm{Sp}(\mathcal{C})$. Then the functor u admits a right adjoint v, given on objects by the formula

$$v(M, N, \beta) = M \times_{\mathscr{B} \otimes M} N.$$

We first show that u is fully faithful by establishing that the unit map claim that the unit map $\mathrm{id}_{\mathrm{Sp}(\mathcal{C})} \to v \circ u$ is an equivalence of functors from $\mathrm{Sp}(\mathcal{C})$ to itself. For this, it suffices to show that for each object $C \in \mathrm{Sp}(\mathcal{C})$, the diagram

$$\begin{array}{ccc} C & \longrightarrow & \mathscr{A} \otimes C \\ \downarrow & & \downarrow \\ \mathscr{B} \otimes C & \longrightarrow & \mathscr{A} \otimes \mathscr{B} \otimes C \end{array}$$

is a pullback diagram in $\mathrm{Sp}(\mathcal{C})$. Since $\mathrm{Sp}(\mathcal{C})$ is stable, it suffices to show that this diagram is a pushout square. Since the action $\mathrm{QCoh}(\mathsf{X})$ on $\mathrm{Sp}(\mathcal{C})$ preserves colimits in each variable, we are reduced to proving that the diagram

$$\begin{array}{ccc} \mathscr{O}_{\mathsf{X}} & \longrightarrow & \mathscr{A} \\ \downarrow & & \downarrow \\ \mathscr{B} & \longrightarrow & \mathscr{A} \otimes \mathscr{B}, \end{array}$$

is a pushout square in the ∞-category $\mathrm{QCoh}(\mathsf{X})$. Using stability again, we are reduced to proving that this diagram is a pullback square. This follows from our assumption that σ is a pushout diagram (in the ∞-category of spectral Deligne-Mumford stacks which are étale over X).

To complete the proof that the functor u is an equivalence, it will suffice to show that the functor v is conservative. Suppose we are given an object

$$(M, N, \beta) \in \mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) \times_{\mathrm{LMod}_{\mathscr{A} \otimes \mathscr{B}}(\mathrm{Sp}(\mathcal{C}))} \mathrm{LMod}_{\mathscr{B}}(\mathrm{Sp}(\mathcal{C}))$$

satisfying $g(M, N, \beta) \simeq 0$. Since j is an open immersion, the commutative algebra object $\mathscr{B} \in \mathrm{QCoh}(\mathsf{X})$ is idempotent. It follows that the canonical map $M \to \mathscr{B} \otimes M$ becomes an equivalence after tensoring with $\mathscr{B}$, so that the projection map $g(M, N, \beta) \to N$ becomes an equivalence after tensoring with $\mathscr{B}$. Since N is a $\mathscr{B}$-module, we deduce that

$$N \simeq \mathscr{B} \otimes N \simeq \mathscr{B} \otimes g(M, N, \beta) \simeq \mathscr{B} \otimes 0 \simeq 0.$$

It follows that $\mathscr{B} \otimes M \simeq \mathscr{A} \otimes N \simeq 0$, so that the projection map $g(M, N, \beta) \to M$ is an equivalence. Since $g(M, N, \beta) \simeq 0$, we deduce that $M \simeq 0$, so that (M, N, β) is a zero object of the ∞-category $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathcal{C})) \times_{\mathrm{LMod}_{\mathscr{A} \otimes \mathscr{B}}(\mathrm{Sp}(\mathcal{C}))} \mathrm{LMod}_{\mathscr{B}}(\mathrm{Sp}(\mathcal{C}))$. □

10.2.4 Proofs of Theorems 10.2.0.1 and 10.2.0.2

Let X be a spectral Deligne-Mumford stack, and suppose that the diagonal $\delta : \mathsf{X} \to \mathsf{X} \times \mathsf{X}$ is quasi-affine. To prove Theorem 10.2.0.1, we must show that every prestable quasi-coherent stack $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathcal{C})$ can be recovered from the ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$: in other words, that the "global sections" of $\mathcal{C}$ determine its "local sections."

Construction 10.2.4.1. Let X be a spectral Deligne-Mumford stack and let $\mathcal{C}$ be a quasi-coherent stack on X. Then we can regard $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ as a module over $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ in the ∞-category Groth_∞. This construction depends functorially on X: if $f : \mathsf{U} \to \mathsf{X}$ is a morphism of spectral Deligne-Mumford stacks, then the unit map

$$\mathrm{QCoh}(\mathsf{X};\mathcal{C}) \to \mathrm{QCoh}(\mathsf{X}; f_* f^* \mathcal{C}) \simeq \mathrm{QCoh}(\mathsf{U}; f^* \mathcal{C})$$

can be regarded as morphism of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$-module objects of Groth_∞. In the special case where f is quasi-affine, Proposition 10.2.1.3 allows us to “extend scalars” to obtain a morphism

$$\theta : \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X};\mathcal{C}) \to \mathrm{QCoh}(\mathsf{U}; f^* \mathcal{C})$$

of $\mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}}$-module objects of Groth_∞.

Proposition 10.2.4.2. *Let $f : \mathsf{U} \to \mathsf{X}$ be a quasi-affine morphism of spectral Deligne-Mumford stacks and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$. Then the map*

$$\theta_{\mathcal{C}} : \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X};\mathcal{C}) \to \mathrm{QCoh}(\mathsf{U}; f^* \mathcal{C})$$

of Construction 10.2.4.1 is an equivalence of Grothendieck prestable ∞-categories.

Proof. Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$. For each object $V \in \mathcal{X}$, set $\mathsf{X}_V = (\mathcal{X}_{/V}, \mathscr{O}_{\mathcal{X}}|_V)$, let $g_V : \mathsf{X}_V \to \mathsf{X}$ be the projection map, and let $\mathcal{C}_V \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ be the quasi-coherent stack given by $g_{V*} g_V^* \mathcal{C}$. Let us say that an object $V \in \mathcal{X}$ is *good* if the map $\theta_{\mathcal{C}_V}$ is an equivalence of Grothendieck prestable ∞-categories. To prove Proposition 10.2.4.2, it will suffice to show that the final object of $\mathcal{X}$ is good. In fact, we will prove something stronger: every object of $\mathcal{X}$ is good.

Note that the construction $V \mapsto \mathcal{C}_V$ determines a functor $\mathcal{X}^{\mathrm{op}} \to \mathrm{QStk}^{\mathrm{lex}}(\mathsf{X})$ which preserves small limits. Combining this observation with Propositions 10.2.2.3 and 10.1.3.4, we deduce that the constructions

$$V \mapsto \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X};\mathcal{C}_V) \qquad V \mapsto \mathrm{QCoh}(\mathsf{U}; f^* \mathcal{C}_V)$$

carry colimits in $\mathcal{X}$ to limits in $\mathsf{Groth}_\infty^{\mathrm{lex}}$. Consequently, the collection of those objects $V \in \mathcal{X}$ for which $\theta_{\mathcal{C}_V}$ is an equivalence is closed under small colimits. Since $\mathcal{X}$ is generated under small colimits by the collection of affine objects of $\mathcal{X}$ (Lemma **??**), we are reduced to proving that every affine object $V \in \mathcal{X}$ is good. Let U_V denote the fiber product $\mathsf{X}_V \times_{\mathsf{X}} \mathsf{U}$, and note that U_V is quasi-affine (since it admits a quasi-affine map to the affine spectral Deligne-Mumford stack X_V). Using Example 10.2.1.7, we can identify the domain of the functor $\theta_{\mathcal{C}_V}$ with the relative tensor product

$$\mathrm{QCoh}(\mathsf{U}_V)^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X}_V)^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X};\mathcal{C}_V) \simeq \mathrm{QCoh}(\mathsf{U}_V)^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X}_V)^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X}_V; g_V^* \mathcal{C}).$$

Using Proposition 10.1.4.1, we can identify the codomain of $\theta_{\mathcal{C}_V}$ with the ∞-category $\mathrm{QCoh}(\mathsf{U}_V; f_V^* g_V^* \mathcal{C})$, where $f_V : \mathsf{U}_V \to \mathsf{X}_V$ is the projection onto the first factor. We may therefore replace X by X_V, the morphism f by f_V, and the quasi-coherent stack $\mathcal{C}$ by its pullback $g_V^* \mathcal{C}$ and thereby reduce to the case where X is affine.

Since U is quasi-affine, we can cover U by finitely many affine open substacks $\mathsf{U}_1, \ldots, \mathsf{U}_n \subseteq \mathsf{U}$. We now proceed by induction on n. If $n = 1$, then U and X are both affine and the desired result follows immediately from the definitions. If $n > 1$, we let V be open substack of U given by the union of $\{\mathsf{U}_i\}_{2 \leqslant i \leqslant n}$. Applying Proposition 10.2.3.1 to the diagram

$$\begin{array}{ccc} \mathsf{V} \times_{\mathsf{U}} \mathsf{U}_1 & \longrightarrow & \mathsf{U}_1 \\ \downarrow & & \downarrow \\ \mathsf{V} & \longrightarrow & \mathsf{U}, \end{array}$$

we deduce that the morphism θ_{U} can be identified with the fiber product of θ_{V} with θ_{U_1} over $\theta_{\mathsf{V} \times_{\mathsf{U}} \mathsf{U}_1}$ (as objects of the ∞-category $\mathrm{Fun}(\Delta^1, \mathsf{Groth}_\infty)$). Our inductive hypothesis guarantees that the morphisms θ_{V}, θ_{U_1}, and $\theta_{\mathsf{V} \times_{\mathsf{U}} \mathsf{U}_1}$ are equivalences, so that θ_{U} is an equivalence as well. □

Construction 10.2.4.3. Let X be a spectral Deligne-Mumford stack and suppose that the diagonal $\delta : \mathsf{X} \to \mathsf{X} \times \mathsf{X}$ is quasi-affine. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category equipped with an action of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$. If A is a connective $\mathbb{E}_\infty$-ring, then any map $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$ is quasi-affine. Define $\mathcal{C}_\eta = \mathrm{Mod}_A^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C}$, where the relative tensor product is formed in the ∞-category Groth_∞ (the existence of this tensor product follows from Proposition **??**). It is not difficult to see that the construction $\eta \mapsto \mathcal{C}_\eta$ determines a quasi-coherent stack on X, which we will denote by $\Psi_{\mathsf{X}}(\mathcal{C})$. The construction $\mathcal{C} \mapsto \Psi_{\mathsf{X}}(\mathcal{C})$ determines a functor

$$\Psi_{\mathsf{X}} : \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty) \to \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}).$$

If $q : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, S$ is the projection map, then the functor Ψ_{X} can be described more informally by the formula $\Psi_{\mathsf{X}}(\mathcal{C}) = q^* \mathcal{C} \otimes_{q^* q_* \mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}}} \mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}}$. From this description, it is not hard to see that Ψ_{X} is a left adjoint to the global sections functor $\mathrm{QCoh}(\mathsf{X}; \bullet) : \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \to \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$.

Proof of Theorem 10.2.0.1. Let X be a quasi-geometric spectral Deligne-Mumford stack and let $\mathcal{C}$ be a prestable quasi-coherent stack on X; we wish to show that the counit map $v : \Psi_{\mathsf{X}}\, \mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \to \mathcal{C}$ is an equivalence in $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$. To prove this, it will suffice to show that for every map of spectral Deligne-Mumford stacks $\eta : \mathsf{U} \to \mathsf{X}$ where $\mathsf{U} \simeq \mathrm{Sp\acute{e}t}\, A$ is affine, the pullback

$$\eta^*(v) : \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \to \mathrm{QCoh}(\mathsf{U}; \eta^* \mathcal{C}) = \mathcal{C}_\eta$$

is an equivalence of prestable A-linear ∞-categories. This is a special case of Proposition 10.2.4.2, since our assumption that X is quasi-geometric guarantees that the map η is quasi-affine. □

Proof of Theorem 10.2.0.2. Let X be a quasi-compact, quasi-separated spectral algebraic space; we wish to show that the global sections functor

$$\mathrm{QCoh}(\mathsf{X}; \bullet) : \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \to \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_{\infty})$$

is an equivalence of ∞-categories. Since X is quasi-geometric, Theorem 10.2.0.1 guarantees that the functor $\mathrm{QCoh}(\mathsf{X}; \bullet)$ is fully faithful. To complete the proof, it will suffice to show that the left adjoint $\Psi_{\mathsf{X}} : \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_{\infty}) \to \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ is conservative.

Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$. For each object $U \in \mathcal{X}$, set $\mathsf{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathsf{X}}|_U)$. Let us say that an object $U \in \mathcal{X}$ is *good* if it satisfies the following conditions:

(a) The spectral Deligne-Mumford stack X_U is a quasi-compact, quasi-separated spectral algebraic space.

(b) The functor $\Psi_{\mathsf{X}_U} : \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X}_U)^{\mathrm{cn}}}(\mathsf{Groth}_{\infty}) \to \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}_U)$ is conservative.

We wish to show that the final object of $\mathcal{X}$ is good. Since X admits a scallop decomposition (Theorem 3.4.2.1), it will suffice to show that the collection of good objects of $\mathcal{X}$ satisfies the hypotheses of Proposition 2.5.3.5. This follows immediately from Proposition 10.2.3.1. □

Remark 10.2.4.4. Let X be a spectral Deligne-Mumford stack which is weakly perfect (Definition 9.4.3.3). If $\mathcal{C}$ is a compactly generated stable ∞-category equipped with an action of $\mathrm{QCoh}(\mathsf{X})$, then $\mathcal{C}$ is dualizable as an object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ and therefore also as an object of $\mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$ (Corollary **??**). so that the construction $\mathcal{D} \mapsto \mathcal{D} \otimes_{\mathrm{QCoh}(\mathsf{X})} \mathcal{C}$ commutes with small limits. In particular, the unit

$$\begin{aligned}
\mathcal{C} &\simeq \mathrm{QCoh}(\mathsf{X}) \otimes_{\mathrm{QCoh}(\mathsf{X})} \mathcal{C} \\
&\simeq (\varprojlim_{\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}} \mathrm{Mod}_A) \otimes_{\mathrm{QCoh}(\mathsf{X})} \mathcal{C} \\
&\to \varprojlim_{\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}} (\mathrm{Mod}_A \otimes_{\mathrm{QCoh}(\mathsf{X})} \mathcal{C}) \\
&\simeq \varprojlim_{\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}} (\mathrm{Mod}_A^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathcal{C}) \\
&\simeq \mathrm{QCoh}(\mathsf{X}; \Psi_{\mathsf{X}} \mathcal{C}).
\end{aligned}$$

is an equivalence, where $\Psi_{\mathsf{X}} : \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_{\infty}) \to \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ is as in Construction 10.2.4.3. It follows that $\mathcal{C}$ belongs to the essential image of the fully faithful embedding

$$\mathrm{QCoh}(\mathsf{X}; \bullet) : \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \to \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_{\infty})$$

of Theorem 10.2.0.1.

10.2.5 Digression: Relative Tensor Products in Groth_∞

Let X be a quasi-compact, quasi-separated spectral algebraic space. For applications, it will be useful to know that the equivalence $\mathrm{QCoh}(\mathsf{X};\bullet) : \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \to \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$ of Theorem 10.2.0.2 is compatible with tensor products. This requires a bit of care: as we discussed in §10.2.1, it is not *a priori* clear that the tensor product of of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$-modules is well-defined. However, it is well-defined in the case of interest to us, by virtue of the following:

Proposition 10.2.5.1. *Let $\mathcal{C}$ be a monoidal ∞-category which satisfies the following conditions:*

- (a) *The underlying ∞-category $\mathcal{C}$ is a compactly generated Grothendieck prestable ∞-category.*
- (b) *The unit object $\mathbf{1} \in \mathcal{C}$ is compact.*
- (c) *The tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable, so that we can view $\mathcal{C}$ as an algebra object of the ∞-category Groth_∞.*
- (d) *For every compact object $C \in \mathcal{C}$, the image $\Sigma^\infty C \in \mathrm{Sp}(\mathcal{C})$ is dualizable.*

Then:

1. *The monoidal structure on $\mathcal{C}$ exhibits $\mathcal{C}$ as an algebra object of the subcategory $\mathsf{Groth}_\infty^{\mathrm{c}} \subseteq \mathsf{Groth}_\infty$ of Definition C.3.4.2. In other words, the multiplication and unit maps*
$$\mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \qquad \mathrm{Sp}^{\mathrm{cn}} \to \mathcal{C}$$
admit right adjoints which commute with filtered colimits.

2. *Let $\mathcal{N}$ be a left $\mathcal{C}$-module object of Groth_∞. Then $\mathcal{N}$ is also a left $\mathcal{C}$-module object of $\mathsf{Groth}_\infty^{\mathrm{c}}$. In other words, the action map $a : \mathcal{C} \otimes \mathcal{N} \to \mathcal{N}$ admits a right adjoint which commutes with filtered colimits.*

3. *Let $\mathcal{M}$ and $\mathcal{N}$ be Grothendieck prestable ∞-categories equipped with right and left actions of the monoidal ∞-category $\mathcal{C}$, and suppose that the action maps*
$$\mathcal{M} \times \mathcal{C} \to \mathcal{M} \qquad \mathcal{C} \times \mathcal{N} \to \mathcal{N}$$
*preserve small colimits separately in each variable. Then there exists a relative tensor product $\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$ in the ∞-category Groth_∞ of Grothendieck prestable ∞-categories. In other words, the two-sided bar construction $\mathrm{Bar}_{\mathcal{C}}(\mathcal{M},\mathcal{N})_\bullet$ of Construction HA.4.4.2.7 admits a geometric realization in the ∞-category Groth_∞. Moreover, this geometric realization is preserved by the (symmetric monoidal) embedding $\mathsf{Groth}_\infty \hookrightarrow \mathsf{Groth}_\infty^{+}$ of Remark **??**.*

Remark 10.2.5.2. Let $\mathcal{C}$ be a monoidal ∞-category satisfying conditions (a) through (d) of Proposition 10.2.5.1. Then an object $C \in \mathcal{C}$ is compact if and only if $\Sigma^\infty(C)$ is a dualizable object of $\mathrm{Sp}(\mathcal{C})$. The "only if" direction follows from (d). Conversely, if $\Sigma^\infty(C)$ is dualizable, then the construction

$$\begin{array}{rcl} D & \mapsto & \mathrm{Map}_{\mathcal{C}}(C, D) \\ & \simeq & \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty C, \Sigma^\infty D) \\ & \simeq & \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty \mathbf{1}, (\Sigma^\infty C)^\vee \otimes \Sigma^\infty D) \\ & \simeq & \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, \Omega^\infty((\Sigma^\infty C)^\vee \otimes \Sigma^\infty D) \end{array}$$

commutes with filtered colimits, since the constructions

$$D \mapsto \Sigma^\infty D \qquad X \mapsto (\Sigma^\infty C)^\vee \otimes X$$

$$Y \mapsto \Omega^\infty Y \qquad Z \mapsto \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, Z)$$

individually commute with filtered colimits; it follows that $C \in \mathcal{C}$ is compact.

Example 10.2.5.3. Let X be a quasi-compact, quasi-separated spectral algebraic space. It follows from Proposition 9.6.1.2 that the ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is compactly generated and that an object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is compact if and only if it is perfect: that is, if and only if it is dualizable as an object in the larger ∞-category $\mathrm{QCoh}(\mathsf{X})$. Consequently, the symmetric monoidal ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ satisfies conditions (a) through (d) of Proposition 10.2.5.1.

Warning 10.2.5.4. Proposition 10.2.5.1 is closely related to Proposition D.2.2.1. If R is a connective $\mathbb{E}_2$-ring, then the monoidal ∞-category $\mathrm{LMod}_R^{\mathrm{cn}}$ satisfies hypotheses (a) through (d) of Proposition 10.2.5.1. Consequently Proposition 10.2.5.1 asserts that if $\mathcal{M}$ and $\mathcal{N}$ are Grothendieck prestable ∞-categories equipped with right and left actions of $\mathrm{LMod}_R^{\mathrm{cn}}$ respectively, then there exists a relative tensor product $\mathcal{N} \otimes_{\mathrm{LMod}_R^{\mathrm{cn}}} \mathcal{M}$ in the ∞-category Groth_∞ of Grothendieck prestable ∞-categories. This special case of assertion (3) of Proposition 10.2.5.1 is a formal consequence of Proposition D.2.2.1 (moreover, the analogous special cases of assertions (1) and (2) appear in our proof of Proposition D.2.2.1). However, the assertion of Proposition D.2.2.1 is stronger: it guarantees not only the existence of the relative tensor product $\mathcal{M} \otimes_{\mathrm{LMod}_R^{\mathrm{cn}}} \mathcal{N}$ in the ∞-category Groth_∞, but that this tensor product *coincides* with the analogous construction in larger ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories. For example, this implies that the tensor product $\mathcal{M} \otimes_{\mathrm{LMod}_R^{\mathrm{cn}}} \mathcal{N}$ is generated under small colimits by the essential image of the multiplication map $\mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes_{\mathrm{LMod}_R^{\mathrm{cn}}} \mathcal{N}$.

In the situation of Proposition 10.2.5.1, we do not know if relative tensor product $\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$ is generated under small colimits by the essential image of the multiplication map $m : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$. However assertion that the geometric realization of the bar construction $\mathrm{Bar}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})_\bullet$ is preserved by the embedding $\mathsf{Groth}_\infty \hookrightarrow \mathsf{Groth}_\infty^+$ gives a slightly weaker statement: the ∞-category $\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$ is generated under small colimits *and extensions* by the essential image of the functor m.

We will later see that this unpleasant phenomenon does not occur in the special case $\mathcal{C} = \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$, where X is a quasi-compact spectral algebraic space with *affine* diagonal; see Proposition 10.2.6.7.

Proof of Proposition 10.2.5.1. We first prove (2). Let $\mathcal{C}$ be a monoidal ∞-category satisfying conditions (a) through (d) and let $\mathcal{N}$ be a Grothendieck prestable ∞-category equipped with a left action of $\mathcal{C}$. Since $\mathcal{C}$ is compactly generated, we can identify $\mathcal{C}$ with $\mathrm{Ind}(\mathcal{C}_0)$, where $\mathcal{C}_0$ is the full subcategory of $\mathcal{C}$ spanned by the compact objects. Using Proposition HA.4.8.1.17 , we can identify the tensor product $\mathcal{C} \otimes \mathcal{N}$ (formed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories) with the ∞-category $\mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{N}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{N})$ of left exact functors from $\mathcal{C}_0^{\mathrm{op}}$ to $\mathcal{N}$. Since $\mathcal{N}$ is a Grothendieck prestable ∞-category, the ∞-category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{N})$ is closed under filtered colimits in the ambient ∞-category $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{N})$: in other words, filtered colimits in $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{N})$ can be computed pointwise. The right adjoint to the action map $\mathcal{C} \otimes \mathcal{N} \to \mathcal{N}$ determines a functor $\lambda : \mathcal{N} \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{N})$. For each compact object $C \in \mathcal{C}$, let $\lambda_C : \mathcal{N} \to \mathcal{N}$ denote the composition of λ with the functor $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{N}) \to \mathcal{N}$ given by evaluation at C. We wish to show that each of the functors λ_C commutes with filtered colimits. Unwinding the definitions, we see that λ_C can be identified with the right adjoint of the functor given by tensoring with the object C. Let us regard the stable ∞-category $\mathrm{Sp}(\mathcal{N})$ as equipped with a left action of $\mathrm{Sp}(\mathcal{C})$. By virtue of assumption (d), the object $\Sigma^\infty C \in \mathrm{Sp}(\mathcal{C})$ admits a dual $(\Sigma^\infty C)^\vee$. Using the prestability of $\mathcal{N}$, we compute

$$\begin{aligned}
\mathrm{Map}_{\mathcal{N}}(N, \lambda_C(N')) &\simeq \mathrm{Map}_{\mathcal{N}}(C \otimes N, N') \\
&\simeq \mathrm{Map}_{\mathrm{Sp}(\mathcal{N})}(\Sigma^\infty C \otimes \Sigma^\infty N, \Sigma^\infty N') \\
&\simeq \mathrm{Map}_{\mathrm{Sp}(\mathcal{N})}(\Sigma^\infty N, (\Sigma^\infty C)^\vee \otimes \Sigma^\infty N') \\
&\simeq \mathrm{Map}_{\mathcal{N}}(N, \Omega^\infty((\Sigma^\infty C)^\vee \otimes \Sigma^\infty N')).
\end{aligned}$$

It follows that we can identify λ_C with the composition of functors

$$\mathcal{N} \xrightarrow{\Sigma^\infty} \mathrm{Sp}(\mathcal{N}) \xrightarrow{(\Sigma^\infty C)^\vee \otimes} \mathrm{Sp}(\mathcal{N}) \xrightarrow{\Omega^\infty} \mathcal{N},$$

each of which commutes with filtered colimits, so that λ_C commutes with filtered colimits as desired. This completes the proof of (2).

We now prove (1). Applying assertion (2) to the left action of $\mathcal{C}$ on itself, we deduce that the multiplication map $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ admits a right adjoint which commutes with filtered colimits. It will therefore suffice to show that if $G : \mathcal{C} \to \mathrm{Sp}^{\mathrm{cn}}$ is the right adjoint to the unit map $\mathrm{Sp}^{\mathrm{cn}} \to \mathcal{C}$, then G commutes with filtered colimits. Since the functor $\Omega^\infty : \mathrm{Sp}^{\mathrm{cn}} \to \mathcal{S}$ is conservative and commutes with filtered colimits, it will suffice to show that the composition

$\mathcal{C} \xrightarrow{G} \mathrm{Sp}^{\mathrm{cn}} \xrightarrow{\Omega^\infty} \mathcal{S}$ commutes with filtered colimits. Unwinding the definitions, we see that this composite functor is corepresented by the unit object $\mathbf{1} \in \mathcal{C}$; the desired result now follows from our assumption that the object $\mathbf{1}$ is compact.

Now suppose that $\mathcal{M}$ and $\mathcal{N}$ are Grothendieck prestable ∞-categories equipped with right and left actions of $\mathcal{C}$. It follows from assertions (1) and (2) that we can regard $\mathcal{M}$ and $\mathcal{N}$ as right and left modules over $\mathcal{C}$ in the symmetric monoidal subcategory $\mathsf{Groth}_\infty^{\mathrm{c}} \subseteq \mathsf{Groth}_\infty$, so that the bar construction $\mathrm{Bar}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})_\bullet$ can be viewed as a simplicial object of $\mathsf{Groth}_\infty^{\mathrm{c}}$. Since the ∞-category $\mathsf{Groth}_\infty^{\mathrm{c}}$ admits small colimits which are preserved by the functors $\mathsf{Groth}_\infty^{\mathrm{c}} \hookrightarrow \mathsf{Groth}_\infty \to \mathsf{Groth}_\infty^{+}$ (Propositions C.3.5.1, Remark C.3.5.2, and Proposition C.3.5.3), it follows that $\mathrm{Bar}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})_\bullet$ admits a geometric realization in Groth_∞ which is preserved by the functor $\mathsf{Groth}_\infty \hookrightarrow \mathsf{Groth}_\infty^{+}$. □

10.2.6 The Projection Formula for Quasi-Coherent Stacks

We now extract some easy consequences of Proposition 10.2.5.1 and Theorem 10.2.0.2.

Proposition 10.2.6.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism between quasi-compact, quasi-separated spectral algebraic spaces. For any quasi-coherent stack $\mathcal{C}$ on Y, the canonical map*

$$\mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) \to \mathrm{QCoh}(\mathsf{Y}; f_* f^* \mathcal{C}) \simeq \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C})$$

extends to an equivalence of ∞-categories

$$\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) \to \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C});$$

here the left hand side is defined as in Proposition 10.2.5.1.

Proof. Unwinding the definitions, we wish to prove the left adjointability of the commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) & \xrightarrow{f_*} & \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y}) \\ \downarrow{\scriptstyle \mathrm{QCoh}(\mathsf{X};\bullet)} & & \downarrow{\scriptstyle \mathrm{QCoh}(\mathsf{Y};\bullet)} \\ \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty) & \longrightarrow & \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}}(\mathsf{Groth}_\infty). \end{array}$$

This is an immediate consequence of Theorem 10.2.0.2, which guarantees that the vertical maps are equivalences (under the assumption that X and Y are quasi-compact, quasi-separated spectral algebraic spaces). □

Example 10.2.6.2. In the situation of Proposition 10.2.6.1, suppose that the morphism f is affine and let $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{Y}))$ denote the direct image direct image of the structure sheaf of X. Then we can identify $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ with the ∞-category $\mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}})$. Combining this observation with Theorem HA.4.8.4.6 and Proposition 10.2.6.1, we obtain an equivalence $\mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C}) \simeq \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{Y}; \mathcal{C}))$ for any quasi-coherent stack $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y})$. Under this identification, the "pullback" functor

$$\mathrm{QCoh}(\mathsf{Y}; \mathcal{C}) \to \mathrm{QCoh}(\mathsf{Y}; f_* f^* \mathcal{C}) \simeq \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C})$$

is given by forming the tensor product with $\mathscr{A}$, which is right adjoint to the forgetful functor $\mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{Y}; \mathcal{C})) \to \mathrm{QCoh}(\mathsf{Y}; \mathcal{C})$.

Corollary 10.2.6.3. *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{f'} & \mathsf{Y}' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \mathsf{X} & \xrightarrow{f} & \mathsf{Y}. \end{array}$$

If X *and* Y *are quasi-compact, quasi-separated spectral algebraic spaces, then the induced map*

$$\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{Y}')^{\mathrm{cn}} \to \mathrm{QCoh}(\mathsf{X}')^{\mathrm{cn}}$$

is an equivalence of Grothendieck prestable ∞*-categories.*

Remark 10.2.6.4. In the situation of Corollary 10.2.6.3, it follows that the induced map

$$\mathrm{QCoh}(\mathsf{X}) \otimes_{\mathrm{QCoh}(\mathsf{Y})} \mathrm{QCoh}(\mathsf{Y}') \to \mathrm{QCoh}(\mathsf{X}')$$

is an equivalence of presentable stable ∞-categories. This consequence is valid more generally for pullback diagrams of weakly perfect stacks; see Corollary 9.4.3.8.

Proof of Corollary 10.2.6.3. Using Propositions 10.2.6.1 and 10.1.4.1, we compute

$$\begin{aligned} \mathrm{QCoh}(\mathsf{X}')^{\mathrm{cn}} &\simeq \mathrm{QCoh}(\mathsf{X}; g'_* \mathscr{Q}^{\mathrm{cn}}_{\mathsf{X}'}) \\ &\simeq \mathrm{QCoh}(\mathsf{X}; g'_* f'^* \mathscr{Q}^{\mathrm{cn}}_{\mathsf{Y}'}) \\ &\xleftarrow{\sim} \mathrm{QCoh}(\mathsf{X}; f^* g_* \mathscr{Q}^{\mathrm{cn}}_{\mathsf{Y}'}) \\ &\xleftarrow{\sim} \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{Y}; g_* \mathscr{Q}^{\mathrm{cn}}_{\mathsf{Y}'}) \\ &\simeq \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{Y}')^{\mathrm{cn}}. \end{aligned}$$

□

Corollary 10.2.6.5. *Let* X *be a quasi-compact, quasi-separated spectral algebraic space. For every pair of quasi-coherent stacks* $\mathcal{C}, \mathcal{D} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$*, the canonical map*

$$\mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X}; \mathcal{D}) \to \mathrm{QCoh}(\mathsf{X}; \mathcal{C} \otimes \mathcal{D})$$

is an equivalence of Grothendieck prestable ∞*-categories.*

Corollary 10.2.6.6 (Projection Formula)**.** *Let $f : X \to Y$ be a morphism in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ *which is representable by quasi-compact, quasi-separated spectral algebraic spaces. For every pair of prestable quasi-coherent stacks $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(Y)$, $\mathcal{D} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$, the canonical map*

$$\theta : \mathcal{C} \otimes (f_* \mathcal{D}) \to f_* f^* (\mathcal{C} \otimes f_* \mathcal{D}) \simeq f_*((f^* \mathcal{C}) \otimes (f^* f_* \mathcal{D})) \to f_*((f^* \mathcal{C}) \otimes \mathcal{D})$$

is an equivalence of quasi-coherent stacks on Y.

Proof. Using Proposition 10.1.4.1, we can reduce to the case where $Y = \mathrm{Spec}\, A$ is affine, so that both X and Y are representable by quasi-compact, quasi-separated spectral algebraic spaces. By virtue of Theorem 10.2.0.2, it will suffice to show that θ induces an equivalence of Grothendieck prestable ∞-categories

$$\rho : \mathrm{QCoh}(Y; \mathcal{C} \otimes f_* \mathcal{D}) \to \mathrm{QCoh}(Y; f_*(f^* \mathcal{C} \otimes \mathcal{D})) \simeq \mathrm{QCoh}(X; f^* \mathcal{C} \otimes \mathcal{D}).$$

Using Corollary 10.2.6.5 and Proposition 10.2.6.1, we can identify ρ with the evident equivalence

$$\begin{array}{c} \mathrm{QCoh}(Y; \mathcal{C}) \otimes_{\mathrm{QCoh}(Y)^{\mathrm{cn}}} \mathrm{QCoh}(X; \mathcal{D}) \\ \big\downarrow \sim \\ (\mathrm{QCoh}(Y; \mathcal{C}) \otimes_{\mathrm{QCoh}(Y)^{\mathrm{cn}}} \mathrm{QCoh}(X)^{\mathrm{cn}}) \otimes_{\mathrm{QCoh}(X)^{\mathrm{cn}}} \mathrm{QCoh}(X; \mathcal{D}). \end{array}$$

□

We close this section by partially addressing the technical point raised in Warning 10.2.5.4. Suppose that X is a spectral Deligne-Mumford stack and that we are given a pair of Grothendieck prestable ∞-categories $\mathcal{M}$ and $\mathcal{N}$ equipped with actions of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$. If X is a quasi-compact, quasi-separated spectral algebraic space, then Proposition 10.2.5.1 guarantees that the two-sided bar construction $\mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathcal{M}, \mathcal{N})_\bullet$ admits a colimit in the ∞-category Groth_∞. It also admits an *a priori* different colimit in the larger ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of all presentable ∞-categories. However, these colimits agree whenever X has affine diagonal:

Proposition 10.2.6.7. *Let X be a quasi-compact spectral algebraic space and suppose that the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times \mathsf{X}$ is affine (this condition is satisfied, for example, if X is separated). Let $\mathcal{M}$ and $\mathcal{N}$ be Grothendieck prestable ∞-categories equipped with actions of* $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$*, and let* $\mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathcal{M}, \mathcal{N})_\bullet$ *be the simplicial object of* Groth_∞ *given by applying the two-sided bar construction of Construction HA.4.4.2.7. Then the inclusion functor* $\mathsf{Groth}_\infty \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$ *preserves the geometric realization of* $\mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathcal{M}, \mathcal{N})_\bullet$*. In other words, the geometric realization* $|\mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathcal{M}, \mathcal{N})_\bullet|$ *in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is again a Grothendieck prestable ∞-category.*

Proof. We proceed as in the proof of Proposition D.2.2.1. By virtue of Lemma HTT.6.5.3.7, we can identify the geometric realization of $\mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathcal{M},\mathcal{N})_\bullet$ with the colimit of the underlying semisimplicial object of Groth_∞. To show that this colimit (formed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$) is again a Grothendieck prestable ∞-category, it will suffice to show that this semisimplicial object satisfies the hypotheses of Remark C.3.5.4: that is, for every injective map $\alpha : [m] \hookrightarrow [n]$, the associated functor $f_\alpha : \mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathcal{M},\mathcal{N})_n \to \mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathcal{M},\mathcal{N})_m$ admits a right adjoint which preserves small colimits. Factoring α as a composition, we may reduce to the case $m = n-1$, so that f_α is one of the face maps of the simplicial object $\mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathcal{M},\mathcal{N})_\bullet$. Unwinding the definitions, we see that f_α is obtained from one of the maps

$$m : \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \otimes \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \to \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$$

$$a : \mathcal{M} \otimes \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \to \mathcal{M} \qquad a' : \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \otimes \mathcal{N} \to \mathcal{N}$$

by tensoring with some auxiliary object of Groth_∞; here the functor a is given by the action of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ on $\mathcal{M}$, the functor m is given by the tensor product of quasi-coherent sheaves, and the functor a' is given by the action of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ on $\mathcal{N}$. By virtue of Remark C.4.4.5, it will suffice to show that the functors a, m, and a' admit right adjoints which preserve small colimits. We will prove this for the functor a; the other cases then follow by symmetry.

Since X is a quasi-compact, quasi-separated spectral algebraic space, Theorem 10.2.0.2 implies that we can write $\mathcal{M}$ as the ∞-category $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ for some quasi-coherent stack $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$. Let $p : \mathsf{X} \times \mathsf{X} \to \mathsf{X}$ denote the projection map onto the first factor. Using Proposition 10.2.6.1, we can identify the tensor product $\mathcal{M} \otimes \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ with the ∞-category $\mathrm{QCoh}(\mathsf{X} \times \mathsf{X}; p^* \mathcal{C})$. Under this identification, the map a corresponds to the functor

$$\mathrm{QCoh}(\mathsf{X} \times \mathsf{X}; p^* \mathcal{C}) \to \mathrm{QCoh}(\mathsf{X} \times \mathsf{X}; \delta_* \delta^* p^* \mathcal{C}) \simeq \mathrm{QCoh}(\mathsf{X} \times \mathsf{X}; \delta_* \mathcal{C}) \simeq \mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \simeq \mathcal{M}$$

given informally by "pullback along δ." Since δ is an affine morphism, the desired result is a consequence of Example 10.2.6.2. □

10.3 Local Properties of Quasi-Coherent Stacks

We now study various local conditions which can be imposed on quasi-coherent stacks.

Definition 10.3.0.1. Let P be a condition on pairs $(A, \mathcal{C})$, where A is a connective $\mathbb{E}_\infty$-ring and $\mathcal{C}$ is a prestable A-linear ∞-category. We will say that P is *local for the étale topology* if the following conditions are satisfied:

(i) Whenever a pair $(A, \mathcal{C})$ satisfies the condition P and $\phi : A \to B$ is an étale morphism, the pair $(B, B \otimes_A \mathcal{C})$ also satisfies condition P.

(ii) Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable A-linear ∞-category. If there exists a finite collection of étale morphisms $\{A \to B_\alpha\}$ for which the induced map $A \to \prod B_\alpha$ is faithfully flat and each pair $(B_\alpha, B_\alpha \otimes_A \mathcal{C})$ satisfies the condition P, then $(A, \mathcal{C})$ satisfies the condition P.

Example 10.3.0.2. The following conditions on a pair $(A, \mathcal{C})$ are local for the étale topology:

(1) The condition that $\mathcal{C}$ is stable (Proposition D.5.1.1).

(2) The condition that $\mathcal{C}$ is separated (Proposition D.5.1.2).

(3) The condition that $\mathcal{C}$ is complete (Proposition D.5.1.3).

(4) The condition that $\mathcal{C}$ is compactly generated (Theorem D.5.3.1).

(5) The condition that $\mathcal{C}$ is anticomplete (Theorems D.5.4.1 and D.5.4.9).

(6) The condition that $\mathcal{C}$ is weakly coherent (Theorem D.5.5.1 and Corollary D.5.5.11).

(7) The condition that $\mathcal{C}$ is locally Noetherian (Propositions D.5.6.1 and D.5.6.4).

(8) The condition that $\mathcal{C}$ is weakly n-complicial (Proposition D.5.7.1).

(9) The condition that $\mathcal{C}$ is anticomplete and n-complicial (Corollary D.5.7.3).

(10) The condition that $\mathcal{C}$ is separated and n-complicial (Corollary D.5.7.3).

We have the following easy analogue of Proposition 2.8.1.7:

Proposition 10.3.0.3. *Let P be a condition on pairs $(A, \mathcal{C})$, where A is connective $\mathbb{E}_\infty$-ring and $\mathcal{C}$ is a prestable A-linear ∞-category. Let X be a spectral Deligne-Mumford stack and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ be a prestable quasi-coherent stack on X. Suppose that P is local for the étale topology (in the sense of Definition 10.3.0.1). Then the following conditions are equivalent:*

(a) *For every étale morphism $\eta : \mathrm{Spét}\, A \to \mathsf{X}$, the pair $(A, \mathcal{C}_\eta)$ satisfies condition P.*

(b) *There exists mutually surjective collection of étale morphisms $\{\eta_\alpha : \mathrm{Spét}\, A_\alpha \to \mathsf{X}\}$ such that each of the pairs $(A_\alpha, \mathcal{C}_{\eta_\alpha})$ has the property P.*

In the situation of Proposition 10.3.0.3, we will generally say that a quasi-coherent stack $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ *has the property P* if the equivalent conditions of Proposition 10.3.0.3 are satisfied. Our goal in this section is to address the following:

Question 10.3.0.4. Let X be a spectral Deligne-Mumford stack, let $\mathcal{C}$ be a prestable quasi-coherent stack on X, and let P be as in Proposition 10.3.0.3. If $\mathcal{C}$ has the property P, then does the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ also have the property P? Conversely, if $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ has the property P, then can we conclude that $\mathcal{C}$ has the property P?

10.3.1 Properties Stable Under Base Change

We begin by considering instances of Question 10.3.0.4 which make sense in somewhat greater generality.

Definition 10.3.1.1. Let P be a condition on pairs $(A, \mathcal{C})$, where A is a connective $\mathbb{E}_\infty$-ring and $\mathcal{C}$ is a prestable A-linear ∞-category. We will say that P is *stable under base change* if, for every morphism $\phi : A \to B$ of connective $\mathbb{E}_\infty$-rings and every prestable A-linear ∞-category $\mathcal{C}$ such that $(A, \mathcal{C})$ satisfies the condition P, the pair $(B, B \otimes_A \mathcal{C})$ also satisfies the condition P.

Example 10.3.1.2. The following conditions on a pair $(A, \mathcal{C})$ are stable under base change (in the sense of Definition 10.3.1.1):

(1) The condition that $\mathcal{C}$ is stable (Proposition D.5.1.1).

(2) The condition that $\mathcal{C}$ is separated (Proposition D.5.1.2).

(3) The condition that $\mathcal{C}$ is complete (Proposition D.5.1.3).

(4) The condition that $\mathcal{C}$ is compactly generated (Lemma D.5.3.3).

Beware that the other conditions mentioned in Example 10.3.0.2 are *not* stable under base change.

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be an arbitrary functor and let $\mathcal{C}$ be a prestable quasi-coherent stack on X. If P is as in Definition 10.3.1.1, we will generally say that $\mathcal{C}$ *satisfies the condition* P if, for every point $\eta \in X(A)$, the pair $(A, \mathcal{C}_\eta)$ satisfies the condition P, where $\mathcal{C}_\eta \in \mathrm{LinCat}_A^{\mathrm{PSt}}$ is as in Remark 10.1.1.3. Note that if P is local for the étale topology (in the sense of Defintiion 10.3.0.1) and X is representable by a spectral Deligne-Mumford stack, then this is equivalent to satisfying either of the conditions appearing in the statement of Proposition 10.3.0.3.

Specializing to the conditions appearing in Example 10.3.1.2, we obtain the following:

Definition 10.3.1.3. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor and let $\mathcal{C}$ be a prestable quasi-coherent stack on X.

(1) We will say that $\mathcal{C}$ is *stable* if, for every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, the A-linear ∞-category $\mathcal{C}_\eta$ is stable (note that this agrees with Definition 10.1.2.1).

(2) We will say that $\mathcal{C}$ is *separated* if, for every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, the A-linear ∞-category $\mathcal{C}_\eta$ is separated.

(3) We will say that $\mathcal{C}$ is *complete* if, for every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, the A-linear ∞-category $\mathcal{C}_\eta$ is complete.

(4) We will say that $\mathcal{C}$ is *compactly generated* if, for every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, the A-linear ∞-category $\mathcal{C}_\eta$ is compactly generated.

Remark 10.3.1.4. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, and let $\mathcal{C}$ be a prestable quasi-coherent stack on Y. If $\mathcal{C}$ is stable (separated, complete, compactly generated), then $f^* \mathcal{C}$ is also stable (separated, complete, compactly generated). The converse holds if f induces an effective epimorphism after sheafifying with respect to the étale topology (since the conditions of being stable, separated, complete, and compactly generated can be tested locally with respect to the étale topology: see Example 10.3.0.2.

Remark 10.3.1.5. Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable A-linear ∞-category. Then $\mathcal{C}$ is stable (separated, complete, compactly generated) when regarded as a Grothendieck prestable ∞-category if and only if it is stable (separated, complete, compactly generated) when regarded as a prestable quasi-coherent stack on $\operatorname{Spec} A$.

Example 10.3.1.6. Let X be a spectral Deligne-Mumford stack, let $K \subseteq |\mathsf{X}|$ be a cocompact closed subset, and let $\mathcal{Q}_K^{\mathrm{cn}}$ be as in Variant 10.1.7.3. Then the prestable quasi-coherent stack $\mathcal{Q}_K \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ is compactly generated (see Proposition **??**).

Proposition 10.3.1.7. *Let $f : X \to Y$ be a representable morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Then:*

(*a*) *If $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$ is stable, then $f_* \mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(Y)$ is stable.*

(*b*) *If $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$ is separated, then $f_* \mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(Y)$ is separated.*

(*c*) *If $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$ is complete, then $f_* \mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(Y)$ is complete.*

Proof. We will give the proof of (c); the proofs of (a) and (b) are similar. Using Proposition 10.1.5.1, we can reduce to the case where $Y = \operatorname{Spec} A$ is affine. Since f is representable, it follows that X is representable by a spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathcal{O}_X)$. For each object $U \in \mathcal{X}$, let $\Gamma(U; \mathcal{C})$ be as in the proof of Proposition 10.1.4.1. The construction $(U \in \mathcal{X}) \mapsto \Gamma(U; \mathcal{C})$ determines a functor $\mathcal{X}^{\mathrm{op}} \to \mathsf{Groth}_\infty^{\mathrm{lex}}$ which preserves small limits. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the full subcategory spanned by those objects U for which the Grothendieck prestable ∞-category $\Gamma(U; \mathcal{C})$ is complete. Since the full subcategory of $\mathsf{Groth}_\infty^{\mathrm{lex}}$ spanned by the complete Grothendieck prestable ∞-categories is closed under small limits, it follows that $\mathcal{X}_0 \subseteq \mathcal{X}$ is closed under small colimits. If $\mathcal{C}$ is complete, then $\mathcal{X}_0$ contains all affine objects of $\mathcal{X}$. Applying Proposition 1.4.7.9, we deduce that $\mathcal{X}_0 = \mathcal{X}$. In particular, $\mathcal{X}_0$ contains the final object of $\mathcal{X}$, which proves that the Grothendieck prestable ∞-category $\mathrm{QCoh}(X; \mathcal{C}) \simeq \mathrm{QCoh}(Y; f_* \mathcal{C})$ is complete. Applying Remark 10.3.1.5, we deduce that $f_* \mathcal{C}$ is complete. □

If X is a quasi-geometric spectral Deligne-Mumford stack, then Theorem 10.2.0.1 guarantees that a prestable quasi-coherent stack $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ is determined by the ∞-category of global sections $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$. In this case, there is a close relationship between local and global properties of $\mathcal{C}$.

Proposition 10.3.1.8. *Let X be a spectral Deligne-Mumford stack and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$. Then:*

(*a*) *If $\mathcal{C}$ is stable, then the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is stable. The converse holds if X is quasi-geometric.*

(*b*) *If $\mathcal{C}$ is separated, then the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is separated. The converse holds if X is quasi-geometric.*

(*c*) *If $\mathcal{C}$ is complete, then the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is complete. The converse holds if X is geometric.*

Remark 10.3.1.9. The analogues of Propositions 10.3.1.7 and 10.3.1.8 for compactly generated prestable quasi-coherent stacks are more subtle; we will discuss the matter in §10.3.2.

Proof of Proposition 10.3.1.8. The first parts of assertions (a) through (c) are special cases of Proposition 10.3.1.7. To prove the converse assertions, let us assume that X is quasi-geometric. In this case, Proposition 10.2.4.2 shows that we can recover the quasi-coherent stack $\mathcal{C}$ from the ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ as follows: for any map $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, we have a canonical equivalence of ∞-categories

$$\mathcal{C}_\eta \simeq \mathrm{Mod}_A^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X}; \mathcal{C}).$$

If $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is stable, then the two-sided bar construction $\mathrm{Bar}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathrm{Mod}_A^{\mathrm{cn}}, \mathrm{QCoh}(\mathsf{X}; \mathcal{C}))_\bullet$ is stable in each degree, so its geometric realization $\mathcal{C}_\eta$ is also stable; this proves (a).

We now prove (b). Let $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$ be as above and let $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X}))$ be the direct of the structure sheaf of $\mathrm{Sp\acute{e}t}\, A$. Since X is quasi-compact, Proposition 2.4.1.5 guarantees that there exists an integer $n \geqslant 0$ such that $\mathscr{A} \in \mathrm{QCoh}(\mathsf{X})_{\geqslant -n}$. Let $\mathcal{E}$ denote the stabilization of the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$, so that $\mathcal{E}$ inherits a t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ and the functor Σ^∞ induces an equivalence $\mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \simeq \mathcal{E}_{\geqslant 0}$. Let us regard the ∞-category $\mathrm{LMod}_{\mathcal{A}}(\mathcal{E})$ as endowed with the t-structure of Construction 10.2.1.1, so that Proposition 10.2.1.3 supplies a t-exact equivalence $\mathcal{C}_\eta \simeq \mathrm{LMod}_{\mathscr{A}}(\mathcal{E})_{\geqslant 0}$. By definition, $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})_{\geqslant 0}$ is the smallest full subcategory of $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$ which is closed under colimits and extensions and contains the tensor product $\mathcal{A} \otimes E$, for each object $E \in \mathcal{E}_{\geqslant 0}$. It follows that the forgetful functor $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E}) \to \mathcal{E}$ carries $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})_{\geqslant k}$ into $\mathcal{E}_{\geqslant k-n}$ for each integer k. If $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is separated, then the intersection $\bigcap_{k \in \mathbf{Z}} \mathcal{E}_{\geqslant k-n}$ contains only zero objects of $\mathcal{E}$. It follows that the intersection $\bigcap_{k \in \mathbf{Z}} \mathrm{LMod}_{\mathscr{A}}(\mathcal{E})_{\geqslant k}$ contains only zero objects of $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$, so that the Grothendieck prestable ∞-category $\mathcal{C}_\eta$ is also separated. This completes the proof of (b).

We now prove (c). Assume that X is geometric and that the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is complete; we wish to show that $\mathcal{C}_\eta$ is also complete, for any morphism $\eta : \mathrm{Spét}\, A \to \mathsf{X}$. The assumption that X is geometric guarantees that the algebra $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X}))$ is connective. In this case, the forgetful functor $\theta : \mathrm{LMod}_{\mathscr{A}}(\mathcal{E}) \to \mathcal{E}$ is t-exact: left t-exactness is automatic, and right t-exactness follows from the observation that the fiber product $\mathcal{E}_{\geqslant 0} \times_{\mathcal{E}} \mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$ is a full subcategory of $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$ which is closed under small colimits, extensions, and contains all objects of the form $\mathscr{A} \otimes E$ where $E \in \mathcal{E}_{\geqslant 0}$. We will complete the proof by showing that the ∞-category $\mathcal{E}_\eta \in \mathrm{LMod}_{\mathscr{A}}(\mathcal{E})_{\geqslant 0}$ satisfies the criterion of Proposition HTT.5.5.6.26 : that is, that a diagram

$$X \to \cdots \to X(3) \to X(2) \to X(1) \to X(0)$$

in $\mathcal{C}_\eta$ is a Postnikov tower if and only if it exhibits each $X(n)$ as an n-truncation of $X(n+1)$ and induces an equivalence $X \to \varprojlim_{n \geqslant 0} X(n)$. Since the forgetful functor θ is t-exact, conservative, and preserves small limits, it suffices to observe that the diagram

$$\theta X \to \cdots \to \theta X(3) \to \theta X(2) \to \theta X(1) \to \theta X(0)$$

in $\mathcal{E}_{\geqslant 0}$ is a Postnikov tower if and only if exhibit each $\theta X(n)$ as the n-truncation of $\theta X(n+1)$ and induces an equivalence $\theta X \to \varprojlim_{n \geqslant 0} \theta X(n)$. This follows from our assumption that $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is complete. □

Notation 10.3.1.10. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. We let $\mathrm{QStk}^{\mathrm{comp}}(X)$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{PSt}}(X)$ spanned by the complete prestable quasi-coherent stacks on X, and we let $\mathrm{QStk}^{\mathrm{sep}}(X)$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{PSt}}(X)$ spanned by the the separated prestable quasi-coherent stacks on X.

Proposition 10.3.1.11. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. Then:*

- (a) *The inclusion functor $\mathrm{QStk}^{\mathrm{comp}}(X) \hookrightarrow \mathrm{QStk}^{\mathrm{PSt}}(X)$ admits a left adjoint which we will denote by $\mathcal{C} \mapsto \widehat{\mathcal{C}}$. Moreover, if $\mathcal{C}$ is a prestable quasi-coherent stack on X, then the canonical map $\mathcal{C} \to \widehat{\mathcal{C}}$ exhibits $\widehat{\mathcal{C}}_\eta$ as the completion of $\mathcal{C}_\eta$ for each point $\eta \in X(A)$ (see §C.3.6).*

- (b) *The inclusion functor $\mathrm{QStk}^{\mathrm{sep}}(X) \hookrightarrow \mathrm{QStk}^{\mathrm{PSt}}(X)$ admits a left adjoint which we will denote by $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{sep}}$. Moreover, if $\mathcal{C}$ is a prestable quasi-coherent stack on X, then the canonical map $\mathcal{C} \to \mathcal{C}^{\mathrm{sep}}$ exhibits $\mathcal{C}^{\mathrm{sep}}_\eta$ as the separated quotient of $\mathcal{C}_\eta$ for each point $\eta \in X(A)$.*

Remark 10.3.1.12. For $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$, we will refer to the quasi-coherent stacks $\widehat{\mathcal{C}}$ and $\mathcal{C}^{\mathrm{sep}}$ of Proposition 10.3.1.11 as the *completion* and *separated quotient* of $\mathcal{C}$, respectively.

Proof. We will give the proof of (a); the proof of (b) is similar. Let $\mathrm{LinCat}^{\mathrm{Add}}$ be as in Definition 10.1.1.1, let $\mathrm{LinCat}^{\mathrm{PSt}}$ be the full subcategory of $\mathrm{LinCat}^{\mathrm{Add}}$ spanned by those pairs $(A, \mathcal{C})$ where A is a connective $\mathbb{E}_\infty$-ring and $\mathcal{C}$ is a prestable A-linear ∞-category, and let $\mathrm{LinCat}^{\mathrm{comp}} \subseteq \mathrm{LinCat}^{\mathrm{PSt}}$ be the full subcategory spanned by those pairs $(A, \mathcal{C})$ where $\mathcal{C}$ is complete. Then the forgetful functor $\theta : \mathrm{LinCat}^{\mathrm{PSt}} \to \mathrm{CAlg}^{\mathrm{cn}}$ is a coCartesian fibration which restricts to a coCartesian fibration $\theta' : \mathrm{LinCat}^{\mathrm{comp}} \to \mathrm{CAlg}^{\mathrm{cn}}$ (moreover, every θ'-coCartesian morphism in $\mathrm{LinCat}^{\mathrm{comp}}$ is a θ-coCartesian morphism in $\mathrm{LinCat}^{\mathrm{PSt}}$: see Proposition D.5.1.3). It follows from Proposition C.4.6.1 that for every connective $\mathbb{E}_\infty$-ring A, the inclusion $\mathrm{LinCat}^{\mathrm{comp}}_A \hookrightarrow \mathrm{LinCat}^{\mathrm{PSt}}_A$ admits a left adjoint, given by the completion construction $\mathcal{C} \mapsto \widehat{\mathcal{C}}$. Moreover, if $f : \mathcal{C} \to \mathcal{D}$ is an A-linear functor which induces an equivalence of completions, then the induced map $f_B : B \otimes_A \mathcal{C} \to B \otimes_A \mathcal{D}$ also induces an equivalence of completions, for any $\mathbb{E}_\infty$-algebra B over A. Applying Proposition HA.7.3.2.11 , we deduce that the inclusion functor $\mathrm{LinCat}^{\mathrm{comp}} \hookrightarrow \mathrm{LinCat}^{\mathrm{PSt}}$ admits a left adjoint L relative to $\mathrm{CAlg}^{\mathrm{cn}}$ (see Definition HA.7.3.2.2). If $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ classifies a left fibration of ∞-categories $\mathcal{E} \to \mathrm{CAlg}^{\mathrm{cn}}$, then we can identify $\mathrm{QStk}^{\mathrm{PSt}}(X)$ and $\mathrm{QStk}^{\mathrm{comp}}(X)$ with the full subcategories of $\mathrm{Fun}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathcal{E}, \mathrm{LinCat}^{\mathrm{PSt}})$ and $\mathrm{Fun}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathcal{E}, \mathrm{LinCat}^{\mathrm{comp}})$ spanned by those functors which carry morphisms in $\mathcal{E}$ to θ-coCartesian morphisms in $\mathrm{LinCat}^{\mathrm{PSt}}$ and θ'-coCartesian morphisms in $\mathrm{LinCat}^{\mathrm{comp}}$, respectively. Using these identifications, we see that pointwise composition with L determines a functor $\mathrm{QStk}^{\mathrm{PSt}}(X) \to \mathrm{QStk}^{\mathrm{comp}}(X)$ having the desired properties. □

Proposition 10.3.1.7 has the following analogue for properties of morphisms between quasi-coherent stacks:

Proposition 10.3.1.13. *Let $f : X \to Y$ be a representable morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ and let $u : \mathcal{C} \to \mathcal{D}$ be a morphism of prestable quasi-coherent stacks on X. Then:*

- (a) *If u is left exact, then the induced map $f_*(u) : f_* \mathcal{C} \to f_* \mathcal{D}$ is left exact.*
- (b) *If u is compact, then the induced map $f_*(u) : f_* \mathcal{C} \to f_* \mathcal{D}$ is compact.*

We now consider the converse:

Proposition 10.3.1.14. *Let X be a spectral Deligne-Mumford stack and let $u : \mathcal{C} \to \mathcal{D}$ be a morphism of prestable quasi-coherent stacks on X. Then:*

- (a) *If u is left exact, then the induced map $\mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \to \mathrm{QCoh}(\mathsf{Y}; \mathcal{D})$ is left exact. The converse holds if X is quasi-geometric.*

(b) *If u is compact, then the induced map* $\mathrm{QCoh}(\mathsf{X};\mathcal{C}) \to \mathrm{QCoh}(\mathsf{Y};\mathcal{D})$ *is compact. The converse holds if* X *is quasi-geometric.*

Proof. The first parts of assertions (a) and (b) follow immediately from Proposition **??**. For the converse, assume that X is quasi-geometric and that u induces a functor $U : \mathrm{QCoh}(\mathsf{X};\mathcal{C}) \to \mathrm{QCoh}(\mathsf{X};\mathcal{D})$ which is left exact (compact); we wish to show that for each map $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, the induced A-linear functor $u_\eta : \mathcal{C}_\eta \to \mathcal{D}_\eta$ is also left exact (compact). As in the proof of Proposition 10.3.1.8, we can identify u_η with the functor

$$\mathrm{Mod}_A^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X};\mathcal{C}) \to \mathrm{Mod}_A^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X};\mathcal{D})$$

determined by U. Assertion (a) is now a special case of Proposition 10.2.2.3. To prove (b), we can replace $\mathcal{C}$ and $\mathcal{D}$ by their stabilizations (see Proposition C.3.4.1). In this case, we can identify u_η with the functor $\mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{X};\mathcal{C})) \to \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{X};\mathcal{D}))$ determined by U, where $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X}))$ is the direct image of the structure sheaf of $\mathrm{Sp\acute{e}t}\, A$. Let G_η denote the right adjoint to u_η, so that G_η fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{X};\mathcal{D})) & \xrightarrow{G_\eta} & \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{X};\mathcal{C})) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{X};\mathcal{D}) & \xrightarrow{G} & \mathrm{QCoh}(\mathsf{X};\mathcal{C}), \end{array}$$

where G is left adjoint to U. Since U is compact, the functor G commutes with filtered colimits. Since the vertical maps are conservative and commute with filtered colimits, it follows that G_η also commutes with filtered colimits. □

Corollary 10.3.1.15. *Let* X *be a quasi-geometric spectral Deligne-Mumford stack and let* $u : \mathcal{C} \to \mathcal{D}$ *be a left morphism of prestable quasi-coherent stacks on* X*. The following conditions are equivalent:*

(i) *The functor u induces an equivalence of completions* $\widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ *(see Proposition 10.3.1.11).*

(ii) *The induced map of Grothendieck prestable ∞-categories* $\mathrm{QCoh}(\mathsf{X};\mathcal{C}) \to \mathrm{QCoh}(\mathsf{X};\mathcal{D})$ *induces an equivalence of completions.*

Proof. The implication (i) $\Rightarrow$ (ii) is immediate. Conversely, suppose that (ii) is satisfied. We wish to show that for every étale morphism $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, the A-linear functor $u_\eta : \mathcal{C}_\eta \to \mathcal{D}_\eta$ induces an equivalence of completions. Equivalently, we must show that u_η induces an equivalence on n-truncated objects for each $n \geqslant 0$. For this, it will suffice to show that the t-exact functor of stable ∞-categories $\mathrm{Sp}(u)_\eta : \mathrm{Sp}(\mathcal{C}_\eta) \to \mathrm{Sp}(\mathcal{D}_\eta)$ induces an equivalence on n-truncated objects. Let $\mathscr{A}$ be as in the proof of Proposition 10.3.1.13, so that we have canonical equivalences

$$\mathrm{Sp}(\mathcal{C}_\eta) \simeq \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{X};\mathrm{Sp}(\mathcal{C}))) \qquad \mathrm{Sp}(\mathcal{D}_\eta) \simeq \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{X};\mathrm{Sp}(\mathcal{D})))$$

which restrict to equivalences

$$\mathrm{Sp}(\mathcal{C}_\eta)_{\leqslant n} \simeq \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))_{\leqslant n}) \qquad \mathrm{Sp}(\mathcal{D}_\eta) \simeq \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{D})_{\leqslant n})).$$

It will therefore suffice to show that the functor u induces an equivalence of ∞-categories $\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))_{\leqslant n} \to \mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{D}))_{\leqslant n}$, which follows immediately from assumption (ii). □

10.3.2 Compact Generation

Let X be a quasi-compact, quasi-separated spectral algebraic space. Proposition 9.6.1.2 asserts that the ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ of connective quasi-coherent sheaves on X is compactly generated. Our goal in this section is to prove a relative version of this result:

Theorem 10.3.2.1. *Let X be a spectral Deligne-Mumford stack and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ be a prestable quasi-coherent stack on X. Then:*

- (a) *If X is quasi-geometric and the ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is compactly generated, then the quasi-coherent stack $\mathcal{C}$ is compactly generated.*
- (b) *If X is a quasi-compact, quasi-separated spectral algebraic space and the quasi-coherent stack $\mathcal{C}$ is compactly generated, then the prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is compactly generated.*

Remark 10.3.2.2. In the special case where $\mathcal{C}$ is the unit object of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$, Theorem 10.3.2.1 reduces to Proposition 9.6.1.2.

Corollary 10.3.2.3. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks. Suppose that f is representable by quasi-compact, quasi-separated spectral algebraic spaces. If $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ is compactly generated, then $f_* \mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y})$ is compactly generated.*

Proof. Using Proposition 10.1.4.1, we can reduce to the case where Y is affine, in which case the desired result follows from Theorem 10.3.2.1. □

As a first step towards a proof of Theorem 10.3.2.1, we establish a criterion for recognizing compact objects in an ∞-category of the form $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$.

Lemma 10.3.2.4. *Let X be a spectral Deligne-Mumford stack, let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ be compactly generated, and let $C \in \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ be an object. The following conditions are equivalent:*

1. *For every map $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, the image of C under the forgetful functor $\mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \to \mathcal{C}_\eta$ is compact.*

(2) *For every étale* $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$*, the image of* C *under the forgetful functor* $\mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \to \mathcal{C}_\eta$ *is compact.*

(3) *There exists a jointly surjective collection of étale maps* $\eta_\alpha : \mathrm{Sp\acute{e}t}\, A_\alpha \to \mathsf{X}$ *such that the image of* C *under each of the forgetful functors* $\mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \to \mathcal{C}_{\eta_\alpha}$ *is compact.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious. We prove that (3) ⇒ (1). Assume that (3) is satisfied for some jointly surjective $\eta_\alpha : \mathrm{Spec}\, A_\alpha \to \mathsf{X}$, and choose any map $\xi : \mathrm{Spec}\, R \to \mathsf{X}$. Then there exists a finite collection of étale maps $\{R \to R_\beta\}$ for which the induced map $R \to \prod_\beta R_\beta$ is faithfully flat, such that each of the induced maps $\mathrm{Spec}\, R_\beta \to \mathrm{Spec}\, R$ fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\, R_\beta & \longrightarrow & \mathrm{Spec}\, A_\alpha \\ \downarrow & & \downarrow \eta_\alpha \\ \mathrm{Spec}\, R & \xrightarrow{\xi} & \mathsf{X} \end{array}$$

for some index α. It follows that the image of C in $\mathrm{Mod}_{R_\beta}(\mathcal{C}_\xi)$ is compact for every index β. Applying Remark D.5.2.3, we deduce that the image of C in $\mathcal{C}_\xi$ is compact. □

Definition 10.3.2.5. Let X be a spectral Deligne-Mumford stack and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ be compactly generated. We will say that an object of $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is *locally compact* if it satisfies the equivalent conditions of Lemma 10.3.2.4.

We now prove a decategorified version of Theorem 10.3.2.1:

Proposition 10.3.2.6. *Let* X *be a spectral Deligne-Mumford stack, let* $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ *be compactly generated, and let* C *be an object of* $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$*. Then:*

(a) *If* X *is quasi-geometric and* C *is a compact object of* $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$*, then* C *is locally compact.*

(b) *If* X *is a quasi-compact, quasi-separated spectral algebraic space and* C *is locally compact, then* C *is a compact object of* $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$*.*

Proof. We first prove (a). Assume that X is quasi-geometric and let $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$; we wish to show that pullback along η determines a functor $f : \mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \to \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A; \eta^* \mathcal{C}) = \mathcal{C}_\eta$ which preserves compact objects. In fact, we claim that the functor f is compact. To prove this, it will suffice to show that the induced map of stable ∞-categories $F : \mathrm{Sp}(\mathrm{QCoh}(\mathsf{X}; \mathcal{C})) \to \mathrm{Sp}(\mathcal{C}_\eta)$ admits a right adjoint G which commutes with small colimits (see Proposition C.3.4.1). This follows from Propositions 10.2.4.2 and 10.2.1.3, which allow us to identify G with the forgetful functor $\mathrm{LMod}_{\mathscr{A}}(\mathrm{Sp}(\mathrm{QCoh}(\mathsf{X}; \mathcal{C}))) \to \mathrm{Sp}(\mathrm{QCoh}(\mathsf{X}; \mathcal{C}))$, where $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X}))$ denotes the direct image of the structure sheaf of $\mathrm{Sp\acute{e}t}\, A$.

We now prove (b). Every object $C \in \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ determines a colimit-preserving functor $\lambda : \mathrm{Sp}^{\mathrm{cn}} \to \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ satisfying $\lambda(S) = C$ (Example C.3.1.2), which we can identify with a map of quasi-coherent stacks $\overline{\lambda} : \mathcal{Q}^{\mathrm{cn}}_{\mathsf{X}} \to \mathcal{C}$. Note that if C is locally compact, then the morphism quasi-coherent stacks $\overline{\lambda}$ is compact. Invoking Proposition 10.3.1.14, we deduce that the induced map of global sections

$$\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \simeq \mathrm{QCoh}(\mathsf{X}; \mathcal{Q}^{\mathrm{cn}}_{\mathsf{X}}) \xrightarrow{\mathrm{QCoh}(\mathsf{X}; \overline{\lambda})} \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$$

is compact. Unwinding the definitions, we see that this functor carries the structure sheaf of X to the object $C \in \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$. We are therefore reduced to proving that the structure sheaf of X is a compact object of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$, which is a special case of Proposition 9.6.1.1. □

The proof of Theorem 10.3.2.1 will require the following more elaborate version of Proposition D.5.3.4:

Lemma 10.3.2.7. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a compactly generated prestable R-linear ∞-category, and let $I \subseteq \pi_0 R$ be a finitely generated ideal. For each object $C \in \mathrm{Sp}(\mathcal{C})$, set $T(C) = C \oplus \Sigma(C)$. Suppose that $C \in \mathrm{Sp}(\mathcal{C})$ is an object satisfying the following condition:*

- $(*)$ *For every element $a \in I$, the localization $C[a^{-1}] = R[a^{-1}] \otimes_R C$ is a compact object of $\mathrm{LMod}_{R[a^{-1}]}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$.*

Then there exists an integer $e \geqslant 0$ and a morphism $\alpha : D \to T^e(C)$ in $\mathrm{Sp}(\mathcal{C})$, where D is a compact object of $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ and α induces an equivalence $D[a^{-1}] \to T^e(C)[a^{-1}]$ for all $a \in I$.

Proof. Since I is finitely generated, we can write $I = (a_1, \ldots, a_n)$ for some nonnegative integer n. We proceed by induction on n. If $n = 0$, then $I = (0)$ and there is nothing to prove. Otherwise, set $J = (a_1, \ldots, a_{n-1})$. Applying the inductive hypothesis, we can choose a compact object $D_0 \in \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ and a map $\beta : D_0 \to T^d(C)$ which induces an equivalence $D_0[a^{-1}] \simeq T^d(C)[a^{-1}]$ for all $a \in J$. Replacing C by $T^d(C)$, we can assume without loss of generality that $d = 0$. Let $C' = \mathrm{cofib}(\beta)$. Then $C'[a_n^{-1}]$ is a compact object of $\mathrm{LMod}_{R[a_n^{-1}]}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$. Arguing as in the proof of Proposition D.5.3.4, we deduce that $T(C'[a_n^{-1}])$ can be written as $E[a_n^{-1}]$ for some compact object $E \in \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Replacing C by $T(C)$, we may assume that $C'[a_n^{-1}] \simeq E[a_n^{-1}]$, where E is a compact object of $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$.

Fix $1 \leqslant i < n$. By construction, we have $C'[a_i^{-1}] \simeq 0$. Since $C'[a_n^{-1}]$ is a compact object of $\mathrm{LMod}_{R[a_n^{-1}]}(\mathrm{Sp}(\mathcal{C}))$, it follows that the map $a_i^m : C'[a_n^{-1}] \to C'[a_n^{-1}]$ is nullhomotopic for $m \gg 0$. Consequently, $T(C'[a_n^{-1}])$ is equivalent to $E'[a_n^{-1}]$, where $E' = \mathrm{cofib}(a_i^m : E \to E)$ is a compact object of $\mathcal{C}_{\geqslant 0}$ satisfying $E[a_i^{-1}] \simeq 0$. Replacing C by $T(C)$ and E by E', we can assume that $E[a_i^{-1}] \simeq 0$. Applying this argument repeatedly, we can arrange that $E[a_i^{-1}] \simeq 0$ for *all* $i \in \{1, 2, \ldots, n-1\}$.

Fix an equivalence $\gamma_0 : E[a_n^{-1}] \simeq C'[a_n^{-1}]$. Since E is a compact object of $\mathrm{Sp}(\mathcal{C})$, we may assume (after multiplying γ_0 by a suitable power of a_n) that γ_0 is induced by a morphism

$\gamma : E \to C'$ in the ∞-category $\mathrm{Sp}(\mathcal{C})$. We now define D to be the pullback $C \times_{C'} E$. The existence of a fiber sequence $D_0 \to D \to E$ shows that D is a compact object of $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. We claim that the projection map $e : D \to C$ induces an equivalence $D[a^{-1}] \to C[a^{-1}]$ whenever $a \in I$. To prove this, we observe that the condition $a \in I$ guarantees that the map $R[a^{-1}] \to \prod_{1 \leqslant i \leqslant n} R[(aa_i)^{-1}]$ is étale and faithfully flat; it will therefore suffice to show that e becomes an equivalence after tensoring with $R[a_i^{-1}]$ for $1 \leqslant i \leqslant n$. For $i < n$, this follows from the fact that both sides vanish; when $i = n$, it follows from our construction. □

Proof of Theorem 10.3.2.1. We first prove (a). Suppose that X is a quasi-geometric spectral Deligne-Mumford stack and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ have the property that $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is compactly generated; we wish to show that the prestable quasi-coherent stack $\mathcal{C}$ is compactly generated. Fix a map $\eta : \mathrm{Spét}\, A \to \mathsf{X}$; we will show that the A-linear ∞-category $\mathcal{C}_\eta$ is compactly generated. Set $\mathcal{E} = \mathrm{Sp}(\mathrm{QCoh}(\mathsf{X}; \mathcal{C}))$, so $\mathcal{E}$ admits a t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ with $\mathcal{E}_{\geqslant 0} \simeq \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$. Let $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X}))$ denote the direct image of the structure sheaf of $\mathrm{Spét}\, A$ and let us regard $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$ as equipped with the t-structure described in Construction 10.2.1.1, so that Theorem 10.2.0.1 supplies an equivalence $\mathcal{C}_\eta \simeq \mathrm{LMod}_{\mathscr{A}}(\mathcal{E})_{\geqslant 0}$. By definition, $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})_{\geqslant 0}$ is the smallest full subcategory of $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$ which is closed under colimits and extensions and contains each tensor product $\mathscr{A} \otimes E$, where $E \in \mathcal{E}_{\geqslant 0}$. Since $\mathcal{E}_{\geqslant 0} \simeq \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is compactly generated, we can also describe $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})_{\geqslant 0}$ as the smallest full subcategory of $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$ which is closed under colimits and extensions and contains $\mathscr{A} \otimes E$ where E is a *compact* object of $\mathcal{E}_{\geqslant 0}$. In this case, $\mathscr{A} \otimes E$ is a compact object of $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$. Applying Proposition C.6.3.1, we deduce that the ∞-category $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})_{\geqslant 0}$ is compactly generated.

We now prove (b). Assume that X is a quasi-compact, quasi-separated spectral algebraic space and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ be compactly generated; we wish to prove that the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is compactly generated. We will show that $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ satisfies the criterion of Corollary C.6.3.3: that is, for every nonzero object $N \in \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$, we will show that there exists a nonzero map $C \to N$ in $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$, where C is compact (by virtue of Proposition 10.3.2.6, this is equivalent to the requirement that C is locally compact).

Since X is a quasi-compact, quasi-separated spectral algebraic space, we can choose a scallop decomposition $\varnothing = \mathsf{U}_0 \to \mathsf{U}_1 \to \cdots \to \mathsf{U}_n \simeq \mathsf{X}$ and excision squares

$$\begin{array}{ccc} \mathsf{V}_i & \longrightarrow & \mathrm{Spét}\, R_i \\ \downarrow & & \downarrow{\scriptstyle \eta_i} \\ \mathsf{U}_{i-1} & \longrightarrow & \mathsf{U}_i; \end{array}$$

see Theorem 3.4.2.1. For $0 \leqslant i \leqslant n$, let $\mathcal{C}_{\mathsf{U}_i}$ denote the restriction of $\mathcal{C}$ to the open substack $\mathsf{U}_i \subseteq \mathsf{X}$ and let N_i denote the image of N in the ∞-category $\mathrm{QCoh}(\mathsf{U}_i; \mathcal{C}_{\mathsf{U}_i})$. Since $N \neq 0$, there exists some smallest integer i such that $N_i \neq 0$. Since $\mathsf{U}_0 = \emptyset$, we have $i > 0$, and the minimality of i guarantees that the image of N_i in $\mathrm{QCoh}(\mathsf{U}_{i-1}; \mathcal{C}_{\mathsf{U}_{i-1}})$ vanishes. It follows that N_i has nonzero image in the prestable R_i-linear ∞-category $\mathcal{C}_{\eta_i}$. Let us denote this image by N'. Our assumption that the quasi-coherent stack $\mathcal{C}$ is compactly generated guarantees that the prestable ∞-category $\mathcal{C}_{\eta_i}$ is compactly generated. Let $I \subseteq \pi_0 R_i$ be a finitely generated ideal which defines the open substack $\mathsf{V}_i \subseteq \mathrm{Spét}\, R_i$, so that the vanishing of N_{i-1} guarantees that N' is I-nilpotent. Proposition 7.1.1.12 guarantees that the collection of I-nilpotent objects of $\mathcal{C}_{\eta_i}$ form a compactly generated prestable ∞-category. It then follows from Corollary C.6.3.3 (or from the proof of Proposition 7.1.1.12) that there exists a nonzero map $\rho : M' \to N'$ in $\mathcal{C}_{\eta_i}$, where M' is compact and I-nilpotent (Proposition 7.1.1.12). Note that the domain and codomain of ρ vanish when restricted to the open substack $\mathsf{V}_i \subseteq \mathrm{Spét}\, R_i$. Using the pullback diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{U}_i; \mathcal{C}_{\mathsf{U}_i}) & \longrightarrow & \mathrm{QCoh}(\mathsf{U}_{i-1}; \mathcal{C}_{\mathsf{U}_{i-1}}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathrm{Spét}\, R_i; \mathcal{C}_{\eta_i}) & \longrightarrow & \mathrm{QCoh}(\mathsf{V}_i; \mathcal{C}_{\mathsf{V}_i}), \end{array}$$

we can lift the map $\rho : M' \to N'$ (in an essentially unique way) to a morphism $\rho_i : M_i \to N_i$ in the ∞-category $M_i \in \mathrm{QCoh}(\mathsf{U}_i; \mathcal{C})$, where M_i has vanishing image in $\mathrm{QCoh}(\mathsf{U}_{i-1}; \mathcal{C}_{\mathsf{U}_i})$. It follows from Lemma 10.3.2.4 that the object M_i is locally compact.

We now prove the following assertion:

($*$) For $i \leqslant j \leqslant n$, there exists a nonzero morphism $\rho_j : M_j \to N_j$ in $\mathrm{QCoh}(\mathsf{U}_j; \mathcal{C}_{\mathsf{U}_j})$, where M_j is locally compact.

The proof proceeds by induction on j, the case $j = i$ having been handled above. When $j = n$, we will obtain a nonzero morphism from a locally compact object of $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ into N, which will complete the proof of Theorem 10.3.2.1.

Let us assume that $i < j$ and that $\rho_{j-1} : M_{j-1} \to N_{j-1}$ has been constructed. We let u denote the composite map $M_{j-1} \oplus \Sigma M_{j-1} \to M_{j-1} \xrightarrow{\rho_{j-1}} N_{j-1}$, and let u_0 be the image of u in $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{V}_j; \mathcal{C}_{\mathsf{V}_j})$. Let N'' denote the image of N in $\mathrm{QCoh}(\mathrm{Spét}\, R_j; \mathcal{C}_{\eta_j})$. Using the pullback diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{U}_j; \mathcal{C}_{\mathsf{U}_j}) & \longrightarrow & \mathrm{QCoh}(\mathsf{U}_{j-1}; \mathcal{C}_{\mathsf{U}_{j-1}}) \\ \downarrow & & \downarrow g^* \\ \mathrm{QCoh}(\mathrm{Spét}\, R_j; \mathcal{C}_{\eta_j}) & \xrightarrow{h^*} & \mathrm{QCoh}(\mathsf{V}_j; \mathcal{C}_{\mathsf{V}_j}), \end{array}$$

we are reduced to proving that u_0 can be lifted to a morphism $v : M'' \to N''$ in $\mathrm{QCoh}(\mathrm{Spét}\, R_j; \mathcal{C}_{\eta_j})$ for some compact object $M'' \in \mathrm{QCoh}(\mathrm{Spét}\, R_j; \mathcal{C})$.

It follows from Lemma 10.3.2.7 that, after replacing M_{j-1} by the direct sum $M_{j-1} \oplus \Sigma M_{j-1}$ finitely many times, we can assume that the object $g^* M_{j-1}$ can be lifted to a compact object

$M'' \in \mathcal{C}_{\eta_j}$. Then we can regard u_0 as a morphism from $h^* M''$ to $h^* N''$, which determines a map $v_0 : M'' \to h_* h^* N''$ in the ∞-category. Let K denote the cofiber of the unit map $N'' \to h^* h^* N''$, let Q denote the cokernel of the composite map $\pi_0 M'' \xrightarrow{v_0} \pi_0(h_* h^* N'') \to \pi_0 K$ (formed in the abelian category $\mathcal{C}_{\eta_j}^{\heartsuit}$), and let K_0 denote the fiber of the map $K \to \pi_0 K \to Q$. By construction, the object $K \in \mathcal{C}_{\eta_j}^{\heartsuit}$ is I-nilpotent (in the sense of Proposition 7.1.1.12). For each $x \in I$, we can identify $Q[x^{-1}]$ with a quotient of $\pi_0 K[x^{-1}]$. It follows that Q is also I-nilpotent, so that the fiber K_0 is I-nilpotent. Using Proposition 7.1.1.12, we can write K_0 as a filtered colimit of I-nilpotent compact objects $K_\alpha \in \mathcal{C}_\alpha$. By construction, the composite map $M'' \to h_* h^* N'' \to K$ factors K_0 and therefore (by virtue of the compactness of M'') through some K_α. Let L denote the fiber of the map $K_\alpha \to K_0$, so that L is also an I-nilpotent object of $\mathcal{C}_{\eta_j}$. Applying Proposition 7.1.1.12 again, we can write L as a filtered colimit $\varinjlim L_\beta$, where each L_β is a compact I-nilpotent object. For each β, let $P_\beta \in \mathcal{C}_{\eta_j}^{\heartsuit}$ be the cokernel of the induced map $\pi_0(M'' \oplus L_\beta) \to \pi_0 K_\alpha$. Then

$$\varinjlim P_\beta \simeq \operatorname{coker}(\pi_0(M'' \oplus L) \to \pi_0 K_\alpha) \simeq \operatorname{coker}(\pi_0(M'') \to \pi_0 K_0) \simeq 0.$$

It follows that the map $K_\alpha \to \pi_0 K_\alpha \to \varinjlim P_\beta$ is nullhomotopic. Using the compactness of K_α, we deduce that there exists an index β for which the composite map $K_\alpha \to \pi_0 K_\alpha \to P_\beta$ is nullhomotopic: that is, the map $\epsilon : M'' \oplus L_\beta \to K_\alpha$ is an epimorphism on π_0. Let F denote the fiber of ϵ, so that the fiber sequence $F \to M'' \oplus L_\beta \to K_\alpha$ remains a fiber sequence after applying the functor $\Sigma^\infty : \mathcal{C}_{\eta_j} \to \operatorname{Sp}(\mathcal{C}_{\eta_j})$. Since M'', L_β, and K_α are compact, the object F is also compact. Moreover, since the objects L_β and K_α are I-nilpotent, the map $F \to M''$ induces an equivalence $h^* F \to h^* M''$. We may therefore replace M'' by F and thereby reduce to the case where the composite map $M'' \xrightarrow{v_0} h_* h^* N'' \to K$ is nullhomotopic, so that v_0 factors through the unit map $h_* h^* N''$ as desired. □

10.3.3 Anticomplete Quasi-Coherent Stacks

We now consider some more subtle variants of Definition 10.3.1.3.

Definition 10.3.3.1. Let X be a spectral Deligne-Mumford stack and let $\mathcal{C}$ be a prestable quasi-coherent stack on X.

(1) We will say that $\mathcal{C}$ is *anticomplete* if, for every étale morphism $\eta : \operatorname{Spét} A \to \mathsf{X}$, the A-linear ∞-category $\mathcal{C}_\eta$ is anticomplete.

(2) We will say that $\mathcal{C}$ is *weakly coherent* if, for every étale morphism $\eta : \operatorname{Spét} A \to \mathsf{X}$, the A-linear ∞-category $\mathcal{C}_\eta$ is weakly coherent.

Warning 10.3.3.2. In the situation of Definition 10.3.3.1, the assumption that $\eta : \operatorname{Spét} A \to \mathsf{X}$ is étale plays an essential role: the condition that a prestable A-linear ∞-category $\mathcal{C}$ is anticomplete (weakly coherent) is not preserved by arbitrary base change. Consequently, Definition 10.3.3.1 makes sense when X is a spectral Deligne-Mumford stack, but not for an arbitrary functor $\mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$.

Remark 10.3.3.3. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and let $\mathcal{C}$ be a prestable quasi-coherent stack on Y. Then:

- (i) Suppose that f is flat and that the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ has finite Tor-amplitude. If $\mathcal{C}$ is anticomplete, then $f^* \mathcal{C}$ is anticomplete: this follows from Theorems D.5.4.1 and D.5.4.9.
- (ii) Suppose that f is locally almost of finite presentation and locally quasi-finite. If $\mathcal{C}$ is weakly coherent, then $f^* \mathcal{C}$ is weakly coherent: this follows from Theorem D.5.5.1 and Corollary D.5.5.11.

In particular, if f is étale and $\mathcal{C}$ is anticomplete (weakly coherent), then $f^* \mathcal{C}$ is also anticomplete (weakly coherent). The converse holds if f is an étale surjection (see Proposition 10.3.0.3).

Remark 10.3.3.4. Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable A-linear ∞-category. Then $\mathcal{C}$ is anticomplete (weakly coherent) when regarded as a Grothendieck prestable ∞-category if and only if it is anticomplete (weakly coherent) when regarded as a prestable quasi-coherent stack on $\mathrm{Sp\acute{e}t}\, A$.

We now establish a global analogue of Lemma D.5.4.5.

Proposition 10.3.3.5. *Let X be a spectral Deligne-Mumford stack, let $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ denote the subcategory of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ whose morphisms are left exact, and let $\mathrm{QStk}^{\mathrm{ch}}(\mathsf{X})$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ whose objects are anticomplete prestable quasi-coherent stacks on X (see Definition 10.3.3.1). Then:*

- (a) *The inclusion functor $\mathrm{QStk}^{\mathrm{ch}}(\mathsf{X}) \hookrightarrow \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ admits a right adjoint, which we will denote by $\mathcal{C} \mapsto \check{\mathcal{C}}$.*
- (b) *Let $f : \mathcal{C} \to \mathcal{D}$ be a left exact morphism of prestable quasi-coherent stacks on X. Then f induces an equivalence $\check{\mathcal{C}} \to \check{\mathcal{D}}$ if and only if, for each étale morphism $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, the induced A-linear functor $f_\eta : \mathcal{C}_\eta \to \mathcal{D}_\eta$ induces an equivalence of completions.*

Proof. Write $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathsf{X}})$. For each object $U \in \mathcal{X}$, set $\mathsf{X}_U = (\mathcal{X}_{/U}, \mathcal{O}_{\mathsf{X}}|_U)$. Let us say that $U \in \mathcal{X}$ is *good* if the conclusions of Proposition 10.3.3.5 hold when X is replaced by X_U. By virtue of Proposition 1.4.7.9, it will suffice to prove the following:

- (i) Every affine object $U \in \mathcal{X}$ is good.

(ii) The collection of good objects of $\mathcal{X}$ is closed under small colimits.

Assertion (i) follows immediately from Lemma D.5.4.5. To prove (ii), suppose we are given a small diagram $\rho : \mathcal{J} \to \mathcal{X}$ having colimit $X \in \mathcal{X}$, where $\rho(J)$ is good for each object $J \in \mathcal{J}$. The construction $(J \in \mathcal{J}) \mapsto \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}_{\rho(J)})^{\mathrm{lex}}$ classifies a coCartesian fibration of ∞-categories $q : \mathcal{E} \to \mathcal{J}^{\mathrm{op}}$. Let us identify objects of $\mathcal{E}$ with pairs $(J, \mathcal{C})$, where $J \in \mathcal{J}$ and $\mathcal{C}$ is a prestable quasi-coherent stack on $\mathsf{X}_{\rho(J)}$. Let $\mathcal{E}^{\mathrm{ch}}$ denote the full subcategory of $\mathcal{E}$ spanned by those pairs $(J, \mathcal{C})$ where $\mathcal{C}$ is anticomplete. It follows from Remark 10.3.1.4 that the restriction $q|_{\mathcal{E}^{\mathrm{ch}}} : \mathcal{E}^{\mathrm{ch}} \to \mathcal{J}^{\mathrm{op}}$ is also a coCartesian fibration, and that the inclusion $\mathcal{E}^{\mathrm{ch}} \hookrightarrow \mathcal{E}$ carries $q|_{\mathcal{E}^{\mathrm{ch}}}$-coCartesian morphisms to q-coCartesian morphisms. Since each $\rho(J)$ is good, each of the inclusion maps $\mathcal{E}^{\mathrm{ch}} \times_{\mathcal{J}^{\mathrm{op}}} \{J\} \hookrightarrow \mathcal{E} \times_{\mathcal{J}^{\mathrm{op}}} \{J\}$ admits a right adjoint. Applying Proposition HA.7.3.2.6 , we deduce that the inclusion $\mathcal{E}^{\mathrm{ch}} \hookrightarrow \mathcal{E}$ admits a right adjoint $G : \mathcal{E} \to \mathcal{E}^{\mathrm{ch}}$ relative to $\mathcal{J}^{\mathrm{op}}$. Using the fact that each $\mathsf{X}_{\rho(J)}$ satisfies condition (b) of Proposition 10.3.3.5, we see that the functor G carries q-coCartesian morphisms of $\mathcal{E}$ to $q|_{\mathcal{E}^{\mathrm{ch}}}$-coCartesian morphisms of $\mathcal{E}^{\mathrm{ch}}$. Using Proposition HTT.3.3.3.1 , we obtain equivalences

$$\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}_X)^{\mathrm{lex}} \simeq \varprojlim_{J \in \mathcal{J}} \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}_{\rho(J)})^{\mathrm{lex}} \simeq \mathrm{Fun}'_{\mathcal{J}^{\mathrm{op}}}(\mathcal{J}^{\mathrm{op}}, \mathcal{E})$$

$$\mathrm{QStk}^{\mathrm{ch}}(\mathsf{X}_X) \simeq \varprojlim_{J \in \mathcal{J}} \mathrm{QStk}^{\mathrm{ch}}(\mathsf{X}_{\rho(J)}) \simeq \mathrm{Fun}'_{\mathcal{J}^{\mathrm{op}}}(\mathcal{J}^{\mathrm{op}}, \mathcal{E}^{\mathrm{ch}}),$$

where $\mathrm{Fun}'_{\mathcal{J}^{\mathrm{op}}}(\mathcal{J}^{\mathrm{op}}, \mathcal{E})$ denotes the full subcategory of $\mathrm{Fun}'_{\mathcal{J}^{\mathrm{op}}}(\mathcal{J}^{\mathrm{op}}, \mathcal{E})$ spanned by those functors which carry every morphism in $\mathcal{J}^{\mathrm{op}}$ to a q-coCartesian morphism in $\mathcal{E}$ and $\mathrm{Fun}'_{\mathcal{J}^{\mathrm{op}}}(\mathcal{J}^{\mathrm{op}}, \mathcal{E}^{\mathrm{ch}})$ is defined similarly. Under these equivalences, pointwise composition with G determines a functor $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}_X)^{\mathrm{lex}} \to \mathrm{QStk}^{\mathrm{ch}}(\mathsf{X}_X)$ which is a right adjoint to the inclusion, so that X_X satisfies condition (a) of Proposition 10.3.3.5. Moreover, the proof yields the following version of (b):

(b') Let $f : \mathcal{C} \to \mathcal{D}$ be a left exact morphism of prestable quasi-coherent stacks on X_X. Then f induces an equivalence $\check{\mathcal{C}} \to \check{\mathcal{D}}$ if and only if, for each étale morphism $\eta : \mathrm{Spét}\, A \to \mathsf{X}_X$ which factors through $\mathsf{X}_{\rho(J)}$ for some $J \in \mathcal{J}$, the induced A-linear functor $f_\eta : \mathcal{C}_\eta \to \mathcal{D}_\eta$ induces an equivalence of completions.

To complete the proof, it suffices to observe that if $f : \mathcal{C} \to \mathcal{D}$ satisfies the criterion of (b'), then the induced functor $f_\eta : \mathcal{C}_\eta \to \mathcal{D}_\eta$ induces an equivalence of completions for *every* étale morphism $\eta : \mathrm{Spét}\, A \to \mathsf{X}_X$, since this can be checked locally with respect to the étale topology on $\mathrm{Spét}\, A$. □

Let X be a spectral Deligne-Mumford stack on X and let $\mathcal{C}$ be a prestable quasi-coherent stack on X. Unwinding the definitions, we see that the quasi-coherent stack $\check{\mathcal{C}}$ is characterized by the following:

(i) There exists a left exact morphism $\lambda : \check{\mathcal{C}} \to \mathcal{C}$ in $\mathrm{QStk}^{\mathrm{PSt}}(\mathcal{C})$.

(ii) The morphism λ induces an equivalence of completions.

(iii) The prestable quasi-coherent stack $\check{\mathcal{C}}$ is anticomplete.

Combining this observation with Remark 10.3.3.3, we obtain the following:

Corollary 10.3.3.6. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Assume that f is flat and that the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ has finite* Tor*-amplitude. Then the diagram of ∞-categories*

$$\begin{array}{ccc} \mathrm{QStk}^{\mathrm{ch}}(\mathsf{X}) & \longrightarrow & \mathrm{QStk}^{\mathrm{PSt,lex}}(\mathsf{X}) \\ \downarrow f^* & & \downarrow f^* \\ \mathrm{QStk}^{\mathrm{ch}}(\mathsf{Y}) & \longrightarrow & \mathrm{QStk}^{\mathrm{PSt,lex}}(\mathsf{Y}) \end{array}$$

is right adjointable. In other words, for every prestable quasi-coherent stack $\mathcal{C}$ on Y, the canonical map $\widetilde{f^* \mathcal{C}} \to f^*(\check{\mathcal{C}})$ is an equivalence in $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$.

We now consider local-to-global principles which are satisfied by the notions introduced in Definition 10.3.3.1.

Proposition 10.3.3.7. *Let X be a quasi-compact, quasi-separated spectral algebraic space and let $\mathcal{C}$ be a prestable quasi-coherent stack on X. The following conditions are equivalent:*

(a) The quasi-coherent stack $\mathcal{C}$ is anticomplete.

(b) The Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is anticomplete.

Proof. Let $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ denote the subcategory of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ whose morphisms are left exact and define $\mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)^{\mathrm{lex}} \subseteq \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$ similarly. Let S be the collection of those morphisms in $\mathrm{QStk}^{\mathrm{PSt,lex}}(\mathsf{X})^{\mathrm{lex}}$ which induce equivalences of completions and define $S' \subseteq \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)^{\mathrm{lex}}$ similarly. It follows from Proposition 10.3.1.14 that the equivalence

$$\mathrm{QCoh}(\mathsf{X}; \bullet) : \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \to \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$$

of Theorem 10.2.0.2 restricts to an equivalence of ∞-categories

$$\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}} \simeq \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)^{\mathrm{lex}},$$

and from Corollary 10.3.1.15 that this equivalence carries S to S'. Using Proposition 10.3.3.5, we see that a prestable quasi-coherent stack $\mathcal{C}$ on X is anticomplete if and only if it is an S-local object of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$. Similarly, since $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is a compactly generated prestable ∞-category whose stabilization is locally rigid (see Proposition 9.6.1.2), Lemma D.5.4.5 implies that $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is anticomplete if and only if it is an S'-local object of $\mathrm{Mod}_{\mathrm{Coh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)^{\mathrm{lex}}$. □

Lemma 10.3.3.8. *Let* X *be a quasi-compact, quasi-separated spectral algebraic space and let* $\mathcal{C}$ *be a prestable quasi-coherent stack on* X*. If* $\mathcal{C}$ *is anticomplete and weakly coherent, then* $\Gamma(\mathsf{X};\mathcal{C})$ *is anticomplete and weakly coherent.*

Proof. Applying Theorem C.6.7.1 (and the quasi-compactness of X), we see that $\mathcal{C}$ is compactly generated, that the loop functor $\Omega : \mathrm{QCoh}(\mathsf{X};\mathcal{C}) \to \mathrm{QCoh}(\mathsf{X};\mathcal{C})$ carries locally compact objects to locally compact objects (Definition 10.3.2.5), and that every locally compact object of $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is truncated. Using Theorem 10.3.2.1 and Proposition 10.3.2.6, we deduce that $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is compactly generated and that an object of $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is compact if and only if it is locally compact. The desired result now follows from Theorem C.6.7.1. □

Proposition 10.3.3.9. *Let* X *be a quasi-compact, quasi-separated spectral algebraic space and let* $\mathcal{C}$ *be a prestable quasi-coherent stack on* X*. If* $\mathcal{C}$ *is weakly coherent, then* $\Gamma(\mathsf{X};\mathcal{C})$ *is weakly coherent.*

Proof. Let $\check{\mathcal{C}} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ be as in Proposition 10.3.3.5. Then the canonical map $\lambda : \check{\mathcal{C}} \to \mathcal{C}$ induces an equivalence of completions. It follows from Corollary C.6.5.5 that $\check{\mathcal{C}}$ is weakly coherent, so that Lemma 10.3.3.8 implies that $\Gamma(\mathsf{X};\check{\mathcal{C}})$ is anticomplete and weakly coherent. Since the induced map $\mathrm{QCoh}(\mathsf{X};\check{\mathcal{C}}) \to \mathrm{QCoh}(\mathsf{X};\mathcal{C})$ also induces an equivalence of completions, Corollary C.6.5.5 guarantees that $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is also weakly coherent. □

10.3.4 Complicial Quasi-Coherent Stacks

Let X be a spectral Deligne-Mumford stack and let $\mathrm{QCoh}(\mathsf{X})$ denote the ∞-category of quasi-coherent sheaves on X. Then $\mathrm{QCoh}(\mathsf{X})$ is equipped with a t-structure, whose heart $\mathrm{QCoh}(\mathsf{X})^{\heartsuit}$ is the abelian category of quasi-coherent sheaves on the underlying ordinary Deligne-Mumford stack of X (see §2.2.6). Using Theorem C.5.4.9, we see that the inclusion $\mathrm{QCoh}(\mathsf{X})^{\heartsuit} \hookrightarrow \mathrm{QCoh}(\mathsf{X})$ admits an essentially unique extension to a colimit-preserving t-exact functor

$$F : \mathcal{D}(\mathrm{QCoh}(\mathsf{X})^{\heartsuit}) \to \mathrm{QCoh}(\mathsf{X}).$$

We now consider the following:

Question 10.3.4.1. Let X be a spectral Deligne-Mumford stack. Under what conditions can we assert that the functor $F : \mathcal{D}(\mathrm{QCoh}(\mathsf{X})^{\heartsuit}) \to \mathrm{QCoh}(\mathsf{X})$ is an equivalence of ∞-categories?

In the special case where X is schematic and 0-truncated, Question 10.3.4.1 was studied by Bökstedt and Neeman. In this case, X can be identified with an ordinary scheme $(X, \mathscr{O}_X)$, and the ∞-category $\mathrm{QCoh}(\mathsf{X})$ can be identified with the full subcategory $\mathcal{D}_{\mathrm{qc}}(\mathrm{Mod}^{\heartsuit}_{\mathscr{O}_X}) \subseteq \mathcal{D}(\mathrm{Mod}^{\heartsuit}_{\mathscr{O}_X})$ spanned by chain complexes of $\mathscr{O}_X$-modules having quasi-coherent cohomology sheaves. Under the assumption that X is quasi-compact and separated, Bökstedt and Neeman show that $\mathcal{D}_{\mathrm{qc}}(\mathrm{Mod}^{\heartsuit}_{\mathscr{O}_X})$ can be recovered as the derived ∞-category of its heart.

Our goal in this section is formulate and prove a relative version of the Bökstedt-Neeman result (we recover a slight variant of their result as Corollary 10.3.4.13 below). We begin by noting that for *any* spectral Deligne-Mumford stack X, the t-structure on $\mathrm{QCoh}(\mathsf{X})$ is right and left complete and compatible with filtered colimits (Proposition 2.2.5.4). Using Remark C.5.4.11, we can reformulate Question 10.3.4.1 as follows:

Question 10.3.4.2. Let X be a spectral Deligne-Mumford stack. Under what conditions can we assert that the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is 0-complicial (see Definition C.5.3.1)?

Question 10.3.4.2 is an instance of the following more general question:

Question 10.3.4.3. Let X be a spectral Deligne-Mumford stack, let $\mathcal{C}$ be a quasi-coherent stack on X, and let $n \geqslant 0$ be an integer. Under what conditions can we assert that the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is n-complicial?

To address Question **??**, we will need to introduce some terminology.

Definition 10.3.4.4. Let X be a spectral Deligne-Mumford stack, let $n \geqslant 0$ be an integer, and let $\mathcal{C}$ be a prestable quasi-coherent stack on X.

(1) We will say that $\mathcal{C}$ is *weakly n-complicial* if, for every étale morphism $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, the A-linear ∞-category $\mathcal{C}_\eta$ is weakly n-complicial.

(2) Assume that $\mathcal{C}$ is either separated or anticomplete. We will say that $\mathcal{C}$ is *n-complicial* if, for every étale morphism $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, the A-linear ∞-category $\mathcal{C}_\eta$ is n-complicial.

Warning 10.3.4.5. We do not know if the condition that a prestable A-linear ∞-category $\mathcal{C}$ is n-complicial can be tested locally with respect to the étale topology on A. However, this is true if we restrict our attention to prestable A-linear ∞-categories which are either anticomplete or separated (Corollary D.5.7.3), which is why these conditions appear in part (2) of Definition 10.3.4.4.

Remark 10.3.4.6. Let $f : \mathsf{X} \to \mathsf{Y}$ be flat morphism of spectral Deligne-Mumford stacks and let $\mathcal{C}$ be a prestable quasi-coherent stack on Y.

(*i*) If $\mathcal{C}$ is weakly n-complicial, then $f^* \mathcal{C}$ is weakly n-complicial.

(*ii*) If $\mathcal{C}$ is separated and n-complicial, then $f^* \mathcal{C}$ is separated and n-complicial.

(*iii*) If the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ has finite Tor-amplitude and $\mathcal{C}$ is anticomplete and n-complicial, then $f^* \mathcal{C}$ is anticomplete and n-complicial.

In particular, if f is étale and $\mathcal{C}$ is weakly n-complicial (separated and n-complicial, anticomplete and n-complicial), then $f^* \mathcal{C}$ is also weakly n-complicial (separated and n-complicial, anticomplete and n-complicial). The converse holds if f is an étale surjection (see Proposition 10.3.0.3).

Remark 10.3.4.7. Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable A-linear ∞-category. Then $\mathcal{C}$ is weakly n-complicial (separated and n-complicial, anticomplete and n-complicial) when regarded as a Grothendieck prestable ∞-category if and only if it is weakly n-complicial (separated and n-complicial, anticomplete and n-complicial) when regarded as a prestable quasi-coherent stack on Spét A.

Example 10.3.4.8. Let X be a spectral Deligne-Mumford stack and let $\mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}}$ be the unit object of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ (see Example 10.1.6.2). The following conditions are equivalent (see Proposition C.5.5.15):

(*i*) The structure sheaf $\mathscr{O}_{\mathsf{X}}$ is n-truncated.

(*ii*) The quasi-coherent stack $\mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}}$ is weakly n-complicial.

(*iii*) The quasi-coherent stack $\mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}}$ is separated and n-complicial.

Proposition 10.3.4.9. *Let X be a spectral Deligne-Mumford stack, let $\mathcal{C}$ be a prestable quasi-coherent stack on X, and let $n \geqslant 0$ be an integer. If X is geometric, then $\mathcal{C}$ is weakly n-complicial if and only if the ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is weakly n-complicial.*

Proof. Using Remark C.5.5.14, we can replace $\mathcal{C}$ by its completion $\widehat{\mathcal{C}}$ and thereby reduce to the case where $\mathcal{C}$ is complete. In this case, Proposition 10.3.4.9 is a special case of Proposition 10.4.6.6, which we will discuss in §10.4. □

Proposition 10.3.4.10. *Let X be a quasi-compact, quasi-separated spectral algebraic space, let $\mathcal{C}$ be a prestable quasi-coherent stack on X, and let $n \geqslant 0$. Assume that the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times \mathsf{X}$ is affine. Then $\mathcal{C}$ is anticomplete and n-complicial if and only if the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C}$ is anticomplete and n-complicial.*

Proof. Combine Propositions C.5.5.16, 10.3.3.7, and 10.3.4.9. □

Proposition 10.3.4.11. *Let X be a geometric spectral Deligne-Mumford stack, let $\mathcal{C}$ be a prestable quasi-coherent stack on X, and let $n \geqslant 0$. Then:*

(1) *If $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is separated and n-complicial, then $\mathcal{C}$ is separated and n-complicial.*

(2) *If X is a quasi-compact, quasi-separated spectral algebraic space and $\mathcal{C}$ is separated and n-complicial, then $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is separated and n-complicial.*

Proof. We first prove (1). Assume that $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is separated and n-complicial. Applying Proposition 10.3.1.8, we deduce that $\mathcal{C}$ is separated. To complete the proof, it will suffice to show that for every étale morphism $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, the prestable A-linear ∞-category $\mathcal{C}_\eta$ is n-complicial. Set $\mathscr{A} = \eta_* \mathscr{O}_{\mathrm{Sp\acute{e}t}\, A} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}})$, so that Proposition 10.2.4.2 supplies an equivalence $\mathcal{C}_\eta \simeq \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{X};\mathcal{C}))$. Let X be an object of $\mathcal{C}_\eta$, which we will identify with an $\mathscr{A}$-module object of $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$. Our assumption that $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is n-complicial guarantees that there exists a morphism $u : Y \to X$ in $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ which is surjective on π_0, where Y is n-truncated. Extending scalars, we can identify u with a morphism $u' : \mathscr{A} \otimes Y \to X$ whose domain is also n-truncated (since $\mathscr{A} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is flat) which is also an epimorphism on π_0. Allowing X to vary, we conclude that $\mathcal{C}_\eta \simeq \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(\mathsf{X};\mathcal{C}))$ is n-complicial, which completes the proof of (1).

We now prove (2). Assume that $\mathcal{C}$ is separated and n-complicial. Let $\lambda : \check{\mathcal{C}} \to \mathcal{C}$ be as in Proposition 10.3.3.5 and $\check{\mathcal{C}}^{\mathrm{sep}}$ denote the separated quotient of $\check{\mathcal{C}}$ (see Proposition 10.3.1.11). Applying Remark C.5.5.14, we deduce that $\check{\mathcal{C}}$ is weakly n-complicial. Since $\check{\mathcal{C}}$ is anticomplete, it follows from Proposition C.5.5.16 that $\check{\mathcal{C}}$ is n-complicial. Applying Proposition C.5.3.3, we see that $\check{\mathcal{C}}^{\mathrm{sep}}$ is n-complicial. Because $\mathcal{C}$ is separated, the map λ factors as a composition $\check{\mathcal{C}} \to \check{\mathcal{C}}^{\mathrm{sep}} \xrightarrow{\mu} \mathcal{C}$. The domain and codomain of μ are both separated and n-complicial. Since μ induces an equivalence of completions, it follows from Proposition C.5.3.9 that μ is an equivalence: that is, the map λ exhibits $\mathcal{C}$ as the separated quotient of $\check{\mathcal{C}}$. Passing to global sections, we obtain an equivalence $\mathrm{QCoh}(\mathsf{X};\mathcal{C}) \simeq \mathrm{QCoh}(\mathsf{X};\check{\mathcal{C}})^{\mathrm{sep}}$. It follows from Proposition 10.3.4.10 that the ∞-category $\mathrm{QCoh}(\mathsf{X};\check{\mathcal{C}})$ is n-complicial, so that $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is also n-complicial by virtue of Proposition C.5.3.3. We conclude the proof by observing that $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is also separated (Proposition 10.3.1.8). □

Specializing to the case $\mathcal{C} = \mathcal{Q}^{\mathrm{cn}}_X$, we obtain the following:

Corollary 10.3.4.12. *Let X be a spectral Deligne-Mumford stack. Then:*

(1) *If X is geometric, then the structure sheaf $\mathscr{O}_{\mathsf{X}}$ is n-truncated if and only if the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is weakly n-complicial.*

(2) *If X is a geometric spectral algebraic space, then the structure sheaf $\mathscr{O}_{\mathsf{X}}$ is n-truncated if and only if the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is n-complicial.*

Proof. Combine Example 10.3.4.8, Proposition 10.3.4.9, and Proposition 10.3.4.11. □

Specializing to the case $n = 0$, we obtain the following:

Corollary 10.3.4.13. *Let X be a spectral Deligne-Mumford stack. Then:*

(1) *If X is 0-truncated and geometric, then the inclusion $\mathrm{QCoh}(\mathsf{X})^{\heartsuit} \hookrightarrow \mathrm{QCoh}(\mathsf{X})$ extends to a t-exact equivalence $\widehat{\mathcal{D}}(\mathrm{QCoh}(\mathsf{X})^{\heartsuit}) \simeq \mathrm{QCoh}(\mathsf{X})$.*

(2) *If* X *is a* 0*-truncated geometric spectral algebraic space, then the inclusion* $\mathrm{QCoh}(\mathsf{X})^{\heartsuit} \hookrightarrow \mathrm{QCoh}(\mathsf{X})$ *extends ot an equivalence* $\mathcal{D}(\mathrm{QCoh}(\mathsf{X})^{\heartsuit}) \simeq \mathrm{QCoh}(\mathsf{X})$.

Remark 10.3.4.14. In the special case where X is schematic and separated, part (2) of Corollary 10.3.4.13 was proven by Bökstedt and Neeman; see [29].

Proof of Corollary 10.3.4.13. Note that for any spectral Deligne-Mumford stack X, the t-structure on $\mathrm{QCoh}(\mathsf{X})$ is both left and right complete (Proposition 2.2.5.4). Assertion (1) now follows from Corollaries 10.3.4.12 and C.5.9.7, while (2) follows from Corollaries 10.3.4.12 and C.5.8.11. □

10.4 Complete Quasi-Coherent Stacks

Let X be a quasi-compact, quasi-separated spectral algebraic space. According to Theorem 10.2.0.2, the global sections functor $\mathcal{C} \mapsto \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ establishes an equivalence between quasi-coherent stacks on X and Grothendieck prestable ∞-categories equipped with an action of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$. Our goal in this section is to establish an analogous result in the case where X is replaced by a geometric stack $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ (Definition 9.3.0.1). More precisely, we will prove (Theorem 10.4.2.3) that for every geometric stack X, the following data are equivalent:

- Prestable quasi-coherent stacks on X which are *complete* (Definition 10.3.1.3).
- Complete Grothendieck prestable ∞-categories $\mathcal{C}$ equipped with an action of $\mathrm{QCoh}(X)^{\mathrm{cn}}$, having the additional property that tensor product with any *flat* object of $\mathrm{QCoh}(X)^{\mathrm{cn}}$ determines a left exact functor from $\mathcal{C}$ to itself.

This equivalence can be regarded as a categorification of the fact that for an affine scheme X, the category of quasi-coherent sheaves on X is equivalent to the category of modules over the commutative ring $\Gamma(X; \mathscr{O}_X)$. However, our result requires a much weaker hypothesis: we need affineness only for the diagonal $X \to X \times X$, rather than for X itself.

10.4.1 The Global Sections Functor

Let S denote the sphere spectrum. For any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, the projection map $q : X \to \operatorname{Spec} S$ induces a pullback functor

$$\mathsf{Groth}_\infty \simeq \mathrm{QStk}^{\mathrm{PSt}}(\operatorname{Spec} S) \xrightarrow{q^*} \mathrm{QStk}^{\mathrm{PSt}}(X).$$

When X is representable by a spectral Deligne-Mumford stack, Construction 10.1.7.1 shows that this pullback functor admits a right adjoint $\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{PSt}}(X) \to \mathsf{Groth}_\infty$. We would like to prove an analogous assertion in the case where X is a quasi-geometric stack.

Here we encounter a technical obstacle: if X is representable by a Deligne-Mumford stack, then we can use an étale surjection $u : \operatorname{Spec} A \to X$ and étale descent for Grothendieck prestable ∞-categories (Theorem D.4.1.2) to reduce to the case where X is affine. If X is assumed only to be a quasi-geometric stack, then we can choose a faithfully flat map $u : \operatorname{Spec} A \to X$, but we generally cannot arrange that u is étale. We do not know if the theory of Grothendieck prestable ∞-categories satisfies descent for the flat topology. Consequently, we will restrict our attention to *complete* Grothendieck prestable ∞-categories (where the requisite descent property follows from Proposition D.6.6.2).

Remark 10.4.1.1. The completeness hypotheses we use in this section can be removed if we are willing to impose some mild additional hypotheses on X, like the requirement that X admits a map $u : \operatorname{Spec} A \to X$ which both is faithfully flat and locally almost of finite presentation (this condition is satisfied, for example, if X is obtained from an Artin stack in classical algebraic geometry): in this case, we can use Theorem D.4.1.6 as a replacement for Theorem D.4.1.2. However, completeness assumptions will be needed again to characterize the essential image of the global sections functor $\operatorname{QCoh}(X; \bullet)$ (Theorem 10.4.2.3).

Notation 10.4.1.2. For each functor $X : \operatorname{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, we let $\operatorname{QStk}^{\mathrm{comp}}(X)$ denote the full subcategory of $\operatorname{QStk}^{\mathrm{PSt}}(X)$ spanned by the complete prestable quasi-coherent stacks on X. It follows from Remark **??** that we can regard the construction $X \mapsto \operatorname{QStk}^{\mathrm{comp}}(X)$ as defining a functor $\operatorname{QStk}^{\mathrm{comp}} : \operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$.

Remark 10.4.1.3. Using Lemma 6.2.1.13, we see that the functor $\operatorname{QStk}^{\mathrm{comp}}$ can be regarded as a right Kan extension of the functor

$$\operatorname{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty} \qquad A \mapsto \operatorname{LinCat}_A^{\mathrm{comp}}$$

along the Yoneda embedding $j : \operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}}$. Since the functor $A \mapsto \operatorname{LinCat}_A^{\mathrm{comp}}$ is a sheaf with respect to the fpqc topology (Theorem D.6.8.1), it follows that the construction $\operatorname{QStk}^{\mathrm{comp}}$ factors through the formation of sheafification for the fpqc topology. In other words, if $f : X \to Y$ is a natural transformation between functors $X, Y : \operatorname{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ which induces an equivalence after sheafification with respect to the fpqc topology, then the pullback functor $f^* \operatorname{QStk}^{\mathrm{comp}}(Y) \to \operatorname{QStk}^{\mathrm{comp}}(X)$ is an equivalence of ∞-categories.

Proposition 10.4.1.4. *Let $X : \operatorname{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a quasi-geometric stack, let S denote the sphere spectrum, and let $q : X \to \operatorname{Spec} S$ be the projection map. Then the pullback functor*

$$q^* : \mathsf{Groth}_{\infty}^{\mathrm{comp}} \simeq \operatorname{QStk}^{\mathrm{comp}}(\operatorname{Sp\acute{e}t} S) \to \operatorname{QStk}^{\mathrm{comp}}(X)$$

admits a right adjoint $\operatorname{QCoh}(X; \bullet) : \operatorname{QStk}^{\mathrm{comp}}(X) \to \mathsf{Groth}_{\infty}^{\mathrm{comp}}$.

Proof. Let X be a geometric stack and let $q : X \to \operatorname{Spec} S$ be the projection map. Fix an object $\mathcal{C} \in \mathrm{QStk}^{\mathrm{comp}}(X)$. We wish to show that the functor

$$F : \mathsf{Groth}_{\infty}^{\mathrm{comp}} \to \widehat{\mathcal{S}} \qquad \mathcal{E} \mapsto \mathrm{Map}_{\mathrm{QStk}^{\mathrm{comp}}(X)}(q^{*}\mathcal{E}, \mathcal{C})$$

is representable by an object of $\mathsf{Groth}_{\infty}^{\mathrm{comp}}$. Choose a faithfully flat map $f_0 : U_0 \to X$ where $U_0 \simeq \operatorname{Spec} A^0$ is affine and let $U_\bullet$ denote the Čech nerve of f_0. For each $n \geqslant 0$, let $f_n : U_n \to X$ be the tautological map and let $F^n : \mathsf{Groth}_{\infty}^{\mathrm{comp}} \to \widehat{\mathcal{S}}$ denote the functor given by

$$\mathcal{E} \mapsto \mathrm{Map}_{\mathrm{QStk}^{\mathrm{comp}}(U_n)}(f_n^{*} q^{*}\mathcal{E}, f_n^{*}\mathcal{C}).$$

Note that each F^n is representable by the complete Grothendieck prestable ∞-category $\mathrm{QCoh}(U_n; f_n^{*}\mathcal{C})$ of Construction 10.1.7.1.

Since the natural map $|U_\bullet| \to X$ is an equivalence after sheafification for the fpqc topology, the ∞-category $\mathrm{QStk}^{\mathrm{comp}}(X)$ can be identified with the totalization of the cosimplicial ∞-category $\mathrm{QStk}^{\mathrm{comp}}(U_\bullet)$ (see Remark 10.4.1.3), so that $F \simeq \varprojlim_{[n] \in \mathbf{\Delta}} F^n$. Consequently, to show that F is representable, it will suffice to show that the cosimplicial object $[n] \mapsto \mathrm{QCoh}(U_n; f_n^{*}\mathcal{C})$ admits a limit in the ∞-category $\mathsf{Groth}_{\infty}^{\mathrm{comp}}$. Using the right cofinality of the inclusion $\mathbf{\Delta}_s \hookrightarrow \mathbf{\Delta}$ (Lemma HTT.6.5.3.7) and the fact that $\mathsf{Groth}_{\infty}^{\mathrm{comp}}$ is closed under limits in Groth_{∞} (Proposition C.3.6.3), it will suffice to show that the underlying cosemisimplicial object of Groth_{∞} admits a limit. Note that if $\alpha : [m] \to [n]$ is a morphism in $\mathbf{\Delta}_{+}$, then the associated map $U_n \to U_m$ is flat (since it is a composition of pullbacks of the flat map $f_0 : U_0 \to X$), so that the associated pullback functor $\mathrm{QCoh}(U_m; f_m^{*}\mathcal{C}) \to \mathrm{QCoh}(U_n; f_n^{*}\mathcal{C})$ is left exact (Corollary 10.1.7.10). It follows that the construction $[n] \mapsto \mathrm{QCoh}(U_n; f_n^{*}\mathcal{C})$ determines a functor from $\mathbf{\Delta}_s$ to the subcategory $\mathsf{Groth}_{\infty}^{\mathrm{lex}} \subseteq \mathsf{Groth}_{\infty}$ of Notation C.3.2.3, so the existence of the limit $\varprojlim \mathrm{QCoh}(U_\bullet; f_\bullet^{*}\mathcal{C})$ follows from Proposition C.3.2.4. □

Example 10.4.1.5. Let X be a geometric stack and $\mathcal{C}$ be a complete quasi-coherent stack on X. The proof of Proposition 10.4.1.4 shows that the ∞-category $\mathrm{QCoh}(X; \mathcal{C})$ can be identified with the totalization $\varprojlim \mathrm{QCoh}(U_\bullet; f_\bullet^{*}\mathcal{C})$, where $f_\bullet : U_\bullet \to X$ is the Čech nerve of any faithfully flat map $f_0 : U_0 = \operatorname{Spec} A \to X$. Taking $\mathcal{C} = \mathcal{Q}_X^{\mathrm{cn}}$, we obtain an equivalence of ∞-categories

$$\mathrm{QCoh}(X; \mathcal{Q}_X^{\mathrm{cn}}) \simeq \varprojlim \mathrm{QCoh}(U_\bullet; \mathcal{Q}_{U_\bullet}^{\mathrm{cn}}) = \varprojlim \mathrm{QCoh}(U_\bullet)^{\mathrm{cn}} \simeq \mathrm{QCoh}(X)^{\mathrm{cn}}.$$

10.4.2 Recovering $\mathcal{C}$ from $\mathrm{QCoh}(X; \mathcal{C})$

Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathrm{LinCat}_A^{\mathrm{comp}}$ denote the ∞-category of complete A-linear prestable ∞-categories. Then the inclusion $\mathrm{LinCat}_A^{\mathrm{comp}} \hookrightarrow \mathrm{LinCat}_A^{\mathrm{PSt}}$ admits a left adjoint L, given by the completion functor of Proposition C.3.6.3. This left adjoint is compatible with the symmetric monoidal structure of Remark D.2.3.1 (in the sense of

Definition HA.2.2.1.6). It follows that $\mathrm{LinCat}_A^{\mathrm{comp}}$ inherits a symmetric monoidal structure, whose tensor product $(\mathcal{C}, \mathcal{D}) \mapsto \mathcal{C} \widehat{\otimes}_A \mathcal{D}$ is the completion of the usual A-linear tensor product $\mathcal{C} \otimes_A \mathcal{D}$ (in the special case where A is the sphere spectrum, this is the completed tensor product of Warning C.4.6.3). We can therefore regard the construction $A \mapsto \mathrm{LinCat}_A^{\mathrm{comp}}$ as a functor from the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-rings to the ∞-category $\mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ of (not necessarily small) symmetric monoidal ∞-categories.

Construction 10.4.2.1. [Completed Tensor Products of Quasi-Coherent Stacks] The construction $X \mapsto \mathrm{QStk}^{\mathrm{comp}}(X)$ of Notation 10.4.1.2 can be regarded as a right Kan extension of the functor $A \mapsto \mathrm{LinCat}_A^{\mathrm{comp}}$ along the Yoneda embedding $\mathrm{CAlg}^{\mathrm{cn}} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{\mathrm{op}}$. Since the forgetful functor $\mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ commutes with limits, it commutes with the formation of right Kan extensions along j. Consequently, for every functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, we can regard $\mathrm{QStk}^{\mathrm{comp}}(X)$ as a symmetric monoidal ∞-category.

Remark 10.4.2.2. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be any functor. The tensor product on $\mathrm{QStk}^{\mathrm{comp}}(X)$ is computed "pointwise" in the following sense: for every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, we have a canonical equivalence of A-linear ∞-categories $(\mathcal{C} \otimes \mathcal{D})_\eta \simeq \mathcal{C}_\eta \widehat{\otimes}_A \mathcal{D}_\eta$ for any pair of complete prestable quasi-coherent stacks $\mathcal{C}, \mathcal{D} \in \mathrm{QStk}^{\mathrm{comp}}(X)$.

Suppose now that X is a quasi-geometric stack and let $q : X \to \operatorname{Spec} S$ be as in Proposition 10.4.1.4. Then the pullback functor $q^* : \mathsf{Groth}_\infty^{\mathrm{comp}} \simeq \mathrm{QStk}^{\mathrm{comp}}(\operatorname{Spec} S) \to \mathrm{QStk}^{\mathrm{comp}}(X)$ is symmetric monoidal. It follows that the global sections functor $\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{comp}}(X) \to \mathsf{Groth}_\infty^{\mathrm{comp}}$ is lax symmetric monoidal. Consequently, we can promote the functor $\mathrm{QCoh}(X; \bullet)$ to a functor

$$\begin{aligned} \mathrm{QStk}^{\mathrm{comp}}(X) &\simeq \mathrm{Mod}_{\mathcal{Q}_X^{\mathrm{cn}}}(\mathrm{QStk}^{\mathrm{comp}}(X)) \\ &\xrightarrow{\mathrm{QCoh}(X;\bullet)} \mathrm{Mod}_{\mathrm{QCoh}(X;\mathcal{Q}_X^{\mathrm{cn}})}(\mathsf{Groth}_\infty^{\mathrm{comp}}) \\ &\simeq \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_\infty^{\mathrm{comp}}). \end{aligned}$$

We will abuse notation by denoting this functor also by $\mathcal{C} \mapsto \mathrm{QCoh}(X; \mathcal{C})$. We can describe the situation more informally as follows: for every complete quasi-coherent stack $\mathcal{C}$ on X, the ∞-category $\mathrm{QCoh}(X; \mathcal{C})$ of global sections of $\mathcal{C}$ is tensored over ∞-category $\mathrm{QCoh}(X)^{\mathrm{cn}}$ of connective quasi-coherent sheaves on X, and the action

$$\mathrm{QCoh}(X)^{\mathrm{cn}} \times \mathrm{QCoh}(X; \mathcal{C}) \to \mathrm{QCoh}(X; \mathcal{C})$$

preserves small colimits separately in each variable. We can now formulate the main result of this section (which we will prove in §10.4.4):

Theorem 10.4.2.3. *Let X be a geometric stack. Then the global sections functor*

$$\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{comp}}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_\infty^{\mathrm{comp}})$$

is fully faithful. Moreover, for an object $\mathcal{E} \in \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_\infty^{\mathrm{comp}})$*, the following conditions are equivalent:*

(*i*) *The* ∞*-category* $\mathcal{E}$ *belongs to the essential image of the functor* $\mathrm{QCoh}(X; \bullet)$.

(*ii*) *For every flat quasi-coherent sheaf* $\mathscr{F} \in \mathrm{QCoh}(X)$*, the construction* $E \mapsto \mathscr{F} \otimes E$ *determines a left exact functor from* $\mathcal{E}$ *to itself.*

(*iii*) *There exists a faithfully flat map* $\eta : \operatorname{Spec} A \to X$ *for which the construction* $E \mapsto \mathscr{F} \otimes E$ *determines a left exact functor from* $\mathcal{E}$ *to itself, where* $\mathcal{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$ *is the cofiber of the unit map* $\mathscr{O}_X \to \eta_* \mathscr{O}_{\operatorname{Spec} A}$.

10.4.3 Digression: Module Objects of Prestable ∞-Categories

Our proof of Theorem 10.4.2.3 will use the following general observation about Grothendieck prestable ∞-categories:

Proposition 10.4.3.1. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be presentable* ∞*-categories and suppose that we are given a functor* $G : \mathcal{D} \to \mathcal{C}$ *which is conservative and preserves small limits and colimits. If* $\mathcal{C}$ *is a Grothendieck prestable* ∞*-category, then so is* $\mathcal{D}$*. Moreover, if* $\mathcal{C}$ *is stable (separated, complete), then so is* $\mathcal{D}$*.*

Example 10.4.3.2. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category which is tensored over a monoidal ∞-category $\mathcal{A}$. Suppose that A is an algebra object of $\mathcal{A}$ and that the functor

$$(C \in \mathcal{C}) \mapsto (A \otimes C \in \mathcal{C})$$

commutes with small colimits. Then the forgetful functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$ is conservative and preserves small limits and colimits (see Corollaries HA.4.2.3.3 and HA.4.2.3.5). It follows from Proposition 10.4.3.1 that $\mathrm{LMod}_A(\mathcal{C})$ is a Grothendieck prestable ∞-category, which is complete (separated, stable) whenever $\mathcal{C}$ is complete (separated, stable).

Remark 10.4.3.3. Proposition 10.4.3.1 can be deduced from Example 10.4.3.2: if $G : \mathcal{D} \to \mathcal{C}$ is a functor between presentable ∞-categories which preserves small limits and colimits, then G admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$. It follows that the composition $T = G \circ F$ can be regarded as an algebra object of the monoidal ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$. If G is conservative then Theorem HA.4.7.3.5 supplies an equivalence $\mathcal{D} \simeq \mathrm{LMod}_T(\mathcal{C})$ under which G corresponds to the forgetful functor $\mathrm{LMod}_T(\mathcal{C}) \to \mathcal{C}$.

Proof of Proposition 10.4.3.1. Assume that $\mathcal{C}$ is a Grothendieck prestable ∞-category. We first prove that $\mathcal{D}$ is pointed. Let $\varnothing$ and $\mathbf{1}$ denote initial and final objects of $\mathcal{D}$, respectively, so that there is an essentially unique map $\alpha : \varnothing \to \mathbf{1}$ in $\mathcal{D}$. We wish to prove that α is an equivalence. Since G preserves small limits and colimits, it preserves initial and final objects, so that $G(\alpha)$ is a morphism from an initial object of $\mathcal{C}$ to a final object of $\mathcal{C}$. Because $\mathcal{C}$ is pointed, it follows that $G(\alpha)$ is an equivalence. Since the functor G is conservative, it follows that α is an equivalence.

We next claim that that the suspension functor $\Sigma_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is fully faithful. Let D be an object of $\mathcal{D}$ and let $u : D \to \Omega_{\mathcal{D}}\Sigma_{\mathcal{D}}D$ be the unit map. Since the functor G commutes with limits and colimits, we can identify $G(u)$ with the unit map $G(D) \to \Omega_{\mathcal{C}}\Sigma_{\mathcal{C}}G(D)$, which is an equivalence by virtue of prestability of $\mathcal{C}$. Since G is conservative, it follows that u is an equivalence.

We now complete the proof that $\mathcal{D}$ is prestable by verifying condition (c) of Definition **??**. Suppose we are given a morphism $f : Y \to \Sigma_{\mathcal{D}}Z$ in the ∞-category $\mathcal{D}$. We can then form a pullback diagram σ :

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma_{\mathcal{D}}Z. \end{array}$$

Since G commutes with finite colimits, we can identify $G(\sigma)$ with a diagram

$$\begin{array}{ccc} G(X) & \longrightarrow & G(Y) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma_{\mathcal{C}}G(Z). \end{array}$$

Since G commutes with finite limits, the diagram $G(\sigma)$ is a pullback square. Using the prestability of $\mathcal{C}$, we conclude that $G(\sigma)$ is also a pushout square. Because G is conservative and commutes with pushouts, it follows that σ is a pushout square in $\mathcal{D}$. This completes the proof that $\mathcal{D}$ is prestable.

We now claim that $\mathcal{D}$ is a *Grothendieck* prestable ∞-category: that is, that filtered colimits in $\mathcal{D}$ are left exact. Let K be a finite simplicial set and let $\varprojlim_K : \mathrm{Fun}(K, \mathcal{D}) \to \mathcal{D}$ be a right adjoint to the diagonal map; we wish to show that $\varprojlim_K$ commutes with filtered colimits. Since the functor G is conservative and commutes with filtered colimits, it will suffice to show that the composite functor $\mathrm{Fun}(K, \mathcal{D}) \xrightarrow{\varprojlim_K} \mathcal{D} \xrightarrow{G} \mathcal{C}$ commutes with filtered colimits. Because G commutes with finite limits, this functor also factors as a composition $\mathrm{Fun}(K, \mathcal{D}) \xrightarrow{G\circ} \mathrm{Fun}(K, \mathcal{C}) \to \mathcal{C}$, which commutes with filtered colimits because G commutes with all colimits and filtered colimits in $\mathcal{C}$ are left exact.

Suppose that $\mathcal{C}$ is stable; we wish to show that $\mathcal{D}$ is also stable. Equivalently, we wish to show that for each object $D \in \mathcal{D}$, the counit map $v : \Sigma_{\mathcal{D}}\Omega_{\mathcal{D}}D \to D$ is an equivalence. Since the functor G commutes with limits and colimits, we can identify $G(v)$ with the counit map $\Sigma_{\mathcal{C}}\Omega_{\mathcal{C}}G(D) \to G(D)$, which is an equivalence by virtue of the stability of $\mathcal{C}$. Because G is conservative, we conclude that v is also an equivalence, as desired.

Suppose now that $\mathcal{C}$ is separated; we wish to prove that $\mathcal{D}$ is also separated. Fix an object $D \in \mathcal{D}$ satisfying $\tau_{\leqslant n} D \simeq 0$ for each $n \geqslant 0$; we wish to prove that $D \simeq 0$. For each integer $n \geqslant 0$, we have a cofiber sequence $D_{>n} \to D \to \tau_{\leqslant n} D$ in the ∞-category $\mathcal{D}$, where $D_{>n}$ belongs to the essential image of the iterated suspension functor $\Sigma^{n+1}_{\mathcal{D}}$. Because the functor G commutes with suspension, the object $GD_{>n}$ belongs to the essential image of $\Sigma^{n+1}_{\mathcal{C}}$. Because G is left exact, the object $G(\tau_{\leqslant n} D) \in \mathcal{C}$ is n-truncated. It follows that the cofiber sequence $GD_{>n} \to GD \to G(\tau_{\leqslant n} D)$ exhibits $G(\tau_{\leqslant n} D)$ as the n-truncation of the object $GD \in \mathcal{C}$. Consequently, if $\tau_{\leqslant n} D$ vanishes in $\mathcal{D}$ for each $n \geqslant 0$, then $\tau_{\leqslant n} GD$ vanishes in $\mathcal{C}$ for each $n \geqslant 0$. The separatedness of $\mathcal{C}$ then implies that $GD \simeq 0$, and the conservativity of G implies that $D \simeq 0$.

Assume now that $\mathcal{C}$ is complete; we wish to prove that $\mathcal{D}$ is also complete. Suppose we are given a tower

$$\cdots \to D(2) \to D(1) \to D(0)$$

be a tower of objects of $\mathcal{D}$ which exhibits each $D(n)$ as the n-truncation of $D(n+1)$, and set $D = \varprojlim D(n)$. We wish to show that each of the maps $D \to D(n)$ induces an equivalence $\tau_{\leqslant n} D \to D(n)$. By virtue of the fact that G is conservative, it will suffice to show that the underlying map $G(\tau_{\leqslant n} D) \to G(D(n))$ is an equivalence in $\mathcal{C}$. The preceding argument shows that the functor G commutes with n-truncation, so it will suffice to show that the natural map $D \to D(n)$ exhibits $G(D(n))$ as an n-truncation of $G(D)$. This follows from the completeness of $\mathcal{C}$, because we can identify $G(D)$ with the limit of the Postnikov tower

$$\cdots \to G(D(2)) \to G(D(1)) \to G(D(0))$$

in the ∞-category $\mathcal{C}$. □

10.4.4 The Proof of Theorem 10.4.2.3

Let X be a geometric stack and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_{\infty})$ be a Grothendieck prestable ∞-category equipped with an action of $\mathrm{QCoh}(X)^{\mathrm{cn}}$. For every connective $\mathbb{E}_{\infty}$-ring A and every point $\eta \in X(A)$, let $\mathcal{C}_{\eta}$ denote the tensor product $\mathrm{Mod}^{\mathrm{cn}}_A \otimes_{\mathrm{QCoh}(X)^{\mathrm{cn}}} \mathcal{C}$, formed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories. Our assumption that X is a geometric stack guarantees that η is an affine morphism, so that we can identify $\mathrm{Mod}^{\mathrm{cn}}_A$ with the ∞-category $\mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(X)^{\mathrm{cn}})$ where $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}})$ denotes the direct image of the structure sheaf of Spét A. Applying Theorem HA.4.8.4.6 , we obtain an equivalence $\mathcal{C}_{\eta} \simeq \mathrm{LMod}_{\mathscr{A}}(\mathcal{C})$. It follows from Example 10.4.3.2 that $\mathcal{C}_{\eta}$ is an (A-linear) Grothendieck prestable ∞-category, which is complete (separated, stable) if $\mathcal{C}$ is complete (separated, stable). In particular, the construction $\mathcal{C} \mapsto \{\mathcal{C}_{\eta}\}_{\eta \in X(A)}$ determines a functor

$$\Phi^{\mathrm{comp}}_X : \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}^{\mathrm{comp}}_{\infty}) \to \mathrm{QStk}^{\mathrm{comp}}(X).$$

It is not difficult to see that the functor Φ_X^{comp} is left adjoint to the global sections functor $\mathrm{QCoh}(X;\bullet) : \mathrm{QStk}^{\mathrm{comp}}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_\infty^{\mathrm{comp}})$.

Proof of Theorem 10.4.2.3. Let X be a geometric stack and choose a faithfully flat map $\eta : U_0 \to X$, where $U_0 \simeq \operatorname{Spec} A$ is affine. We first claim that the global sections functor

$$\mathrm{QCoh}(X;\bullet) : \mathrm{QStk}^{\mathrm{comp}}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_\infty^{\mathrm{comp}})$$

is fully faithful. Let $\mathcal{C}$ be a complete quasi-coherent stack on X; we wish to show that the counit map $v : \Phi_X^{\mathrm{comp}} \mathrm{QCoh}(X;\mathcal{C}) \to \mathcal{C}$ is an equivalence of quasi-coherent stacks on X. Since η is an effective epimorphism of fpqc sheaves, it will suffice to show that $\eta^*(v)$ is an equivalence (see Remark 10.4.1.3). Unwinding the definitions, we wish to prove that the canonical map $\theta : \mathrm{Mod}_A^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(X)^{\mathrm{cn}}} \mathrm{QCoh}(X;\mathcal{C}) \to \mathcal{C}_\eta$ is an equivalence of A-linear prestable ∞-categories. Let $U_\bullet$ denote the Čech nerve of η, and for each $n \geqslant 0$ let $f_n : U_n \to X$ be the canonical map. Then the proof of Proposition 10.4.1.4 supplies a canonical equivalence $\mathrm{QCoh}(X;\mathcal{C}) \simeq \varprojlim \mathrm{QCoh}(U_\bullet; f_\bullet^* \mathcal{C})$.

Set $\mathscr{A} = \eta_* \mathscr{O}_{U_0} \in \mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}})$. Since X is a geometric stack, the map η is affine and we can identify $\mathrm{Mod}_A^{\mathrm{cn}}$ with the ∞-category $\mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(X)^{\mathrm{cn}})$. It follows that $\mathrm{Mod}_A^{\mathrm{cn}}$ is self-dual as an object of the ∞-category $\mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_\infty^{\mathrm{comp}})$, so that the formation of tensor products with $\mathrm{Mod}_A^{\mathrm{cn}}$ commutes with limits. We therefore have canonical equivalences

$$\begin{aligned} \mathrm{Mod}_A^{\mathrm{cn}} \otimes_{\mathrm{QCoh}(X)^{\mathrm{cn}}} \mathrm{QCoh}(X;\mathcal{C}) &\simeq \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(X;\mathcal{C})) \\ &\simeq \varprojlim \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(U_\bullet; f_\bullet^* \mathcal{C})) \\ &\simeq \mathrm{QCoh}(U_{\bullet+1}; f_{\bullet+1}^* \mathcal{C}). \end{aligned}$$

The statement that θ is an equivalence now follows from the observation that the augmented simplicial object $\mathrm{QCoh}(U_{\bullet+1}; f_{\bullet+1}^* \mathcal{C})$ is split.

We next show that for every complete quasi-coherent stack $\mathcal{C} \in \mathrm{QStk}^{\mathrm{comp}}(X)$, the ∞-category $\mathrm{QCoh}(X;\mathcal{C})$ satisfies condition (ii) of Theorem 10.4.2.3: that is, for every flat object $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$, the functor

$$F : \mathrm{QCoh}(X;\mathcal{C}) \to \mathrm{QCoh}(X;\mathcal{C}) \qquad E \mapsto \mathscr{F} \otimes E$$

is left exact. For each $n \geqslant 0$, we can write $U_n \simeq \operatorname{Spec} A^n$ for some connective $\mathbb{E}_\infty$-ring A^n. The proof of Proposition 10.4.1.4 shows that we can identify $\mathrm{QCoh}(X;\mathcal{C})$ with the totalization of the cosemisimplicial object $([n] \in \mathbf{\Delta}_s) \mapsto \mathrm{QCoh}(U_\bullet; f_\bullet^* \mathcal{C})$ of the ∞-category $\mathsf{Groth}_\infty^{\mathrm{lex}}$. It will therefore suffice to show that for each $n \geqslant 0$, tensor product with $f_n^* \mathscr{F} \in \mathrm{QCoh}(U_n)^{\mathrm{cn}}$ induces a left exact functor from $f_n^* \mathcal{C}$ to itself, where we regard $f_n^* \mathcal{C}$ as a prestable A^n-linear ∞-category. Since $\mathscr{F}$ is flat, the pullback $f_n^* \mathscr{F}$ is a flat A^n-module and can therefore be written as a filtered colimit $\varinjlim M_\alpha$, where each M_α is a free A^n-module of finite rank. Because $f_n^* \mathcal{C}$ is a Grothendieck prestable ∞-category, filtered colimits in $f_n^* \mathcal{C}$ are left exact.

It will therefore suffice to show that for each index α, tensor product with M_α determines a left exact functor from $f_n^* \mathcal{C}$ to itself. This is clear: the formation of tensor product with M_α preserves all small limits (since it admits a right adjoint, given by tensor product with the A-linear dual $M_\alpha^\vee$).

The implication $(ii) \Rightarrow (iii)$ of Theorem 10.4.2.3 is clear. We will complete the proof by showing that $(iii) \Rightarrow (i)$. Let $\mathcal{E}$ be a complete Grothendieck prestable ∞-category equipped with an action of $\mathrm{QCoh}(X)^{\mathrm{cn}}$ which satisfies the following condition:

$(*)$ Let $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$ denote the cofiber of the unit map $\mathscr{O}_X \to \mathscr{A}$. Then the construction $E \mapsto \mathscr{F} \otimes E$ determines a left exact functor from $\mathcal{E}$ to itself.

We wish to show that $\mathcal{E}$ belongs to the essential image of the global sections functor

$$\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{comp}}(\mathsf{X}) \to \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_\infty^{\mathrm{comp}}).$$

Equivalently, we show that if $\mathcal{E}$ satisfies condition $(*)$, then the unit map $u : \mathcal{E} \to \mathrm{QCoh}(X; \Phi_X^{\mathrm{comp}} \mathcal{E})$ is an equivalence of ∞-categories.

For each $n \geqslant 0$, let $\mathscr{A}^n \in \mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}})$ denote the direct image of the structure sheaf of U_n, so that $f_n^* \Phi_X^{\mathrm{comp}} \mathcal{E}$ can be identified with the ∞-category $\mathcal{E}^n = \mathrm{LMod}_{\mathscr{A}^n}(\mathcal{E})$. Let us regard $\mathscr{A}^\bullet$ as an augmented cosimplicial object of $\mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}})$ by taking $\mathscr{A}^{-1} = \mathscr{O}_X$, so that $\mathcal{E}^{-1} = \mathcal{E}$. The proof of Proposition 10.4.1.4 then supplies an identification of $\mathrm{QCoh}(X; \Phi_X^{\mathrm{comp}} \mathcal{E})$ with the totalization of the underlying cosemisimplicial object of $\mathcal{E}^\bullet$. Under this identification, the functor u is given by the natural map $\mathcal{E}^{-1} \to \varprojlim_{[n] \in \mathbf{\Delta}_s} \mathcal{E}^n$ is an equivalence in the ∞-category $\mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$. Using Corollary HA.4.2.3.3 and Proposition C.3.2.4, we see that this is equivalent to the assertion that $\mathcal{E}^{-1}$ is a limit of the diagram $\{\mathcal{E}^n\}_{[n] \in \mathbf{\Delta}_s}$ in $\widehat{\mathcal{C}\mathrm{at}}_\infty$. Let $F : \mathcal{E} = \mathcal{E}^{-1} \to \mathcal{E}^0 = \mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$ be the functor given $E \mapsto \mathscr{A} \otimes E$. Since the augmented cosimplicial ∞-category $\mathcal{E}^\bullet$ satisfies the Beck-Chevalley condition of Corollary HA.4.7.5.3 , we are reduced to proving the following concrete assertions:

(a) The functor F is conservative.

(b) The functor $F : \mathcal{E} \to \mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$ preserves limits of F-split cosimplicial objects.

To prove (a), it suffices to show that if $E \in \mathcal{E}$ and $F(E) \simeq 0$, then $E \simeq 0$. Note that condition $(*)$ implies that F is t-exact, so that $F(\pi_n E) \simeq 0$ for each integer $n \geqslant 0$. Condition $(*)$ then supplies a short exact sequence

$$0 \to \pi_n E \to \pi_n F(E) \to \pi_n(\mathscr{F} \otimes E) \to 0$$

in the abelian category $\mathcal{E}^\heartsuit$. It follows that $\pi_n E \simeq 0$ for all n, so that $E \simeq 0$ by virtue of our assumption that $\mathcal{E}$ is complete.

We now prove (b). Assumption $(*)$ guarantees that tensor product with $\mathscr{A}$ determines a left exact functor from $\mathcal{E}$ to itself, which restricts to an exact functor T from the abelian category $\mathcal{E}^{\heartsuit}$ to itself. Similarly, tensor product with $\mathscr{F}$ determines an exact functor T' from $\mathcal{E}^{\heartsuit}$ to itself, and we have an exact sequence of functors

$$0 \to \mathrm{id} \to T \to T' \to 0$$

In particular, the functor T is conservative (as noted in our proof of (a)).

Let $E^\bullet$ be an F-split cosimplicial object of $\mathcal{E}$. Then $\mathscr{A} \otimes E^\bullet$ is a split cosimplicial object of $\mathcal{E}$. For every integer n, we obtain a cosimplicial object $\pi_n E^\bullet$ of the abelian category $\mathcal{E}^{\heartsuit}$, which has an associated cochain complex

$$\pi_n E^0 \xrightarrow{d_n} \pi_n E^1 \to \pi_n E^2 \to \cdots$$

After applying the functor T, this cochain complex becomes a split exact resolution of $\pi_n \varprojlim(\mathscr{A} \otimes E^\bullet)$. Since T is exact and conservative, we conclude that the cochain complex above is acyclic in positive degrees. Let $E = \varprojlim E^\bullet$; since $\mathcal{E}$ is complete, Corollary HA.1.2.4.12 guarantees that the map $E \to E^0$ induces an isomorphism $\pi_n E \simeq \ker(d_n)$ for each $n \geqslant 0$ (in the abelian category $\mathcal{E}^{\heartsuit}$). We claim that the induced map $F(E) \to \varprojlim F(E^\bullet)$ is an equivalence: in other words, that $\mathscr{A} \otimes E \simeq \varprojlim(\mathscr{A} \otimes E^\bullet)$. Because the prestable ∞-category $\mathcal{E}$ is separated, it will suffice to show that the induced map $T(\pi_n E) \simeq \pi_n(\mathscr{A} \otimes E) \to \pi_n \varprojlim(\mathscr{A} \otimes E^\bullet)$ is an equivalence for each integer n. This follows immediately from Corollary HA.1.2.4.12 and the exactness of the functor T. □

10.4.5 Geometric Stacks with the Resolution Property

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a geometric stack. Then Theorem 10.4.2.3 supplies a fully faithful embedding $\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{comp}}(X) \hookrightarrow \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_\infty^{\mathrm{comp}})$. In this section, we consider two special cases in which we can prove that this functor is an equivalence of ∞-categories:

Theorem 10.4.5.1. *Let X be a geometric stack which either has the resolution property (Definition 9.3.3.2) or is locally Noetherian (Definition 9.5.1.1). Then the global sections functor*

$$\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{comp}}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_\infty^{\mathrm{comp}})$$

of §10.4.2 is an equivalence of ∞-categories.

Theorem 10.4.5.1 is an immediate consequence of Theorem 10.4.2.3 together with the following pair of results:

Proposition 10.4.5.2. *Let X be a geometric stack with the resolution property, let $f : \operatorname{Spec} A \to X$ be a faithfully flat morphism, and let $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$ be the cofiber of the unit map $\mathscr{O}_X \to f_* \mathscr{O}_{\operatorname{Spec} A}$. Suppose that $\mathcal{E}$ is a Grothendieck prestable ∞-category which* *is tensored over* $\mathrm{QCoh}(X)^{\mathrm{cn}}$*, and that the action map* $\otimes : \mathrm{QCoh}(X)^{\mathrm{cn}} \times \mathcal{E} \to \mathcal{E}$ *preserves small colimits separately in each variable. Then the construction* $E \mapsto \mathscr{F} \otimes E$ *determines a left exact functor from* $\mathcal{E}$ *to itself.*

Proposition 10.4.5.3. *Let* X *be a locally Noetherian geometric stack, let* $\mathcal{E}$ *be a separated Grothendieck prestable* ∞*-category which is tensored over* $\mathrm{QCoh}(X)^{\mathrm{cn}}$*, and suppose that the action map* $\otimes : \mathrm{QCoh}(X)^{\mathrm{cn}} \times \mathcal{E} \to \mathcal{E}$ *preserves small colimits separately in each variable. Then, for any flat object* $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$*, the construction* $E \mapsto \mathscr{F} \otimes E$ *determines a left exact functor from* $\mathcal{E}$ *to itself.*

Proof of Proposition 10.4.5.2. By virtue of Proposition C.3.2.1, it will suffice to show that for every discrete object $E \in \mathcal{E}$, the tensor product $\mathscr{F} \otimes E$ is also discrete. Note that we can promote E to a module over the commutative algebra object $\pi_0 \mathscr{O}_X \in \mathrm{QCoh}(X)^{\mathrm{cn}}$. Replacing $\mathcal{E}$ by $\mathrm{LMod}_{\pi_0 \mathscr{O}_X}(\mathcal{E})$ and X by its 0-truncation (see §9.1.6), we can reduce to the case where X is 0-truncated.

Let $\mathcal{C} = \mathrm{Vect}(X) \times_{\mathrm{QCoh}(X)} \mathrm{QCoh}(X)_{/ f_* \mathscr{O}_{\mathrm{Spec}\, A}}$ be the ∞-category of vector bundles $\mathscr{G}$ on X equipped with a map $\mathscr{G} \to f_* \mathscr{O}_{\mathrm{Spec}\, A}$. According to Lemma 9.3.4.13, the ∞-category $\mathcal{C}$ is filtered. The proof of Proposition 9.3.4.11 shows that the canonical map $\varinjlim_{\mathscr{G} \in \mathcal{C}} \mathscr{G} \to f_* \mathscr{O}_{\mathrm{Spec}\, A}$ is an equivalence. Let us abuse notation by regarding the structure sheaf $\mathscr{O}_X$ as an object of $\mathcal{C}$ (by equipping it with the unit map $\mathscr{O}_X \to f_* \mathscr{O}_{\mathrm{Spec}\, A}$), so that $\mathcal{C}_{\mathscr{O}_X /}$ is also a filtered ∞-category and the projection map $\mathcal{C}_{\mathscr{O}_X /} \to \mathcal{C}$ is left cofinal. For each object $\mathscr{G} \in \mathcal{C}_{\mathscr{O}_X /}$, let $\mathscr{G}'$ denote the cofiber of the map $e : \mathscr{O}_X \to \mathscr{G}$. Note that $f^*(e)$ admits a left homotopy inverse, so that $f^* \mathscr{G}'$ is a direct summand of $f^* \mathscr{G}$. We can therefore write $\mathscr{F} = \mathrm{cofib}(\mathscr{O}_X \to f_* \mathscr{O}_{\mathrm{Spec}\, A})$ as the colimit of a filtered diagram $\{\mathscr{G}'\}_{\mathscr{G} \in \mathcal{C}_{\mathscr{O}_X /}}$. Consequently, to show that $\mathscr{F} \otimes E$ is a discrete object of $\mathcal{E}$, it will suffice to show that $\mathscr{G}' \otimes E$ is a discrete object of $\mathcal{E}$ for each $\mathscr{G}' \in \mathrm{Vect}(X)$. This is clear: every object $\mathscr{G}' \in \mathrm{Vect}(X)$ is dualizable as an object of $\mathrm{QCoh}(X)^{\mathrm{cn}}$, so the functor $E \mapsto \mathscr{G}' \otimes E$ preserves small limits (since it has a left adjoint, given by $E \mapsto \mathscr{G}'^{\vee} \otimes E$). □

We now prove Proposition **??** using the strategy outlined in §9.5.4. The main step is the following variant of Lemma 9.5.4.5:

Lemma 10.4.5.4. *Let* X *be a* 0*-truncated geometric stack, let* $\mathcal{E}$ *be a complete Grothendieck prestable* ∞*-category which is tensored over* $\mathrm{QCoh}(X)^{\mathrm{cn}}$*, and suppose that the action map* $\otimes : \mathrm{QCoh}(X)^{\mathrm{cn}} \times \mathcal{E} \to \mathcal{E}$ *preserves small colimits separately in each variable. Let* $\alpha : \mathscr{F} \to \mathscr{F}'$ *be a morphism in* $\mathrm{QCoh}(X)^{\mathrm{cn}}$*, where* $\mathscr{F}$ *is almost perfect and* $\mathscr{F}'$ *is flat. Then, for every object* $E \in \mathcal{E}^{\heartsuit}$*, the induced map* $\alpha_E : \mathscr{F} \otimes E \to \mathscr{F}' \otimes E$ *factors through an object of* $\mathcal{E}^{\heartsuit}$*.*

Proof. We proceed as in the proof of Lemma 9.5.4.5. Let $\mathscr{F}^{\vee} = \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{O}_X) \in \mathrm{QCoh}(X)$ be defined as in §9.5.3. Since $\mathscr{F}$ is almost perfect, the structure sheaf $\mathscr{O}_X$ is 0-truncated, and $\mathscr{F}'$ has Tor-amplitude $\leqslant 0$, it follows from Corollary 9.5.3.6 that the canonical map $\rho : \mathscr{F}' \otimes \mathscr{F}^\vee \to \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{F}')$ is an equivalence in $\mathrm{QCoh}(X)$. Let $\mathscr{G} = \tau_{\geqslant 0} \mathscr{F}^\vee$ denote the connective cover of $\mathscr{F}^\vee$. Since $\mathscr{F}'$ is flat, we can identify $\mathscr{F}' \otimes \mathscr{G}$ with the connective cover of $\mathscr{F}' \otimes \mathscr{F}^\vee$. It follows that we can choose a map $c : \mathscr{O}_X \to \mathscr{F}' \otimes \mathscr{G}$ which is determined (uniquely up to homotopy) by the requirement that the composition

$$\mathscr{O}_X \xrightarrow{c} \mathscr{F}' \otimes \mathscr{G} \to \mathscr{F}' \otimes \mathscr{F}^\vee \xrightarrow{\rho} \underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{F}')$$

classifies α. The tautological pairing $\mathscr{F}^\vee \otimes \mathscr{F} \to \mathscr{O}_X$ determines an "evaluation map" $e : \mathscr{G} \otimes \mathscr{F} \to \mathscr{O}_X$. Unwinding the definitions, we see that the composite map

$$\mathscr{F} \xrightarrow{c \otimes \mathrm{id}} \mathscr{F}' \otimes \mathscr{G} \otimes \mathscr{F} \xrightarrow{\mathrm{id} \otimes e} \mathscr{F}'$$

is homotopic to α.

Let E be a discrete object of $\mathcal{E}$. The construction $(E' \in \mathcal{E}) \mapsto \mathrm{Map}_{\mathcal{E}}(\mathscr{G} \otimes E', E)$ carries colimits in $\mathcal{E}$ to limits in $\mathcal{S}$, and is therefore representable by an object of $\mathcal{E}$ which we will denote by $\underline{\mathrm{Map}}(\mathscr{G}, E)$. The evaluation map e induces a morphism $e_E : \mathscr{G} \otimes \mathscr{F} \otimes E \to E$, which is classified by a morphism of R-modules $\beta : \mathscr{F} \otimes E \to \underline{\mathrm{Map}}(\mathscr{G}, E)$. Let γ denote the composite map

$$\underline{\mathrm{Map}}(\mathscr{G}, E) \xrightarrow{c \otimes \mathrm{id}} \mathscr{F}' \otimes \mathscr{G} \otimes \underline{\mathrm{Map}}(\mathscr{G}, E) \to \mathscr{F}' \otimes E.$$

Using the fact that $(\mathrm{id} \otimes e) \circ (c \otimes \mathrm{id})$ is homotopic to α, we deduce that the composition $\gamma \circ \beta$ is homotopic to α_E. We conclude by observing that since E is discrete, the mapping space $\mathrm{Map}_{\mathcal{E}}(\mathscr{G} \otimes E', E)$ is discrete for each $E' \in \mathcal{E}$, so that $\underline{\mathrm{Map}}(\mathscr{G}, E)$ is also discrete. □

Proof of Proposition **??**. Let X be a locally Noetherian geometric stack, let $\mathcal{E}$ be a separated Grothendieck prestable ∞-category which is tensored over $\mathrm{QCoh}(X)^{\mathrm{cn}}$, and suppose that the action map $\otimes : \mathrm{QCoh}(X)^{\mathrm{cn}} \times \mathcal{E} \to \mathcal{E}$ preserves small colimits separately in each variable. Let $\mathscr{F}$ be a flat object of $\mathrm{QCoh}(X)$. We wish to show that the functor $E \mapsto \mathscr{F} \otimes E$ is a left exact functor from $\mathcal{E}$ to itself. To prove this, it will suffice to show that if $E \in \mathcal{E}$ is discrete, then $\mathscr{F} \otimes E$ is also discrete (Proposition C.3.2.1).

Set $\mathscr{A} = \pi_0 \mathscr{O}_X$. Since E is discrete, it admits the structure of a $\mathscr{A}$-module object of $\mathcal{E}$. We may therefore replace $\mathcal{E}$ by $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$ (which is prestable and separated by virtue of Example 10.4.3.2) and thereby reduce to the case where the action of $\mathrm{QCoh}(X)^{\mathrm{cn}}$ on $\mathcal{E}$ factors through an action of $\mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(X)^{\mathrm{cn}}) \simeq \mathrm{QCoh}(X_0)^{\mathrm{cn}}$, where X_0 is the 0-truncation of X (see §9.1.6). Replacing X by X_0 (and $\mathscr{F}$ by its restriction $\mathscr{F}|_{X_0}$), we can reduce to the case where X is 0-truncated.

Since X is 0-truncated and $\mathscr{F} \in \mathrm{QCoh}(X)$ is flat, the sheaf $\mathscr{F}$ is also 0-truncated. Invoking our assumption that X is locally Noetherian, we deduce that $\mathscr{F}$ can be written as a filtered colimit $\varinjlim \mathscr{F}_\alpha$ where each $\mathscr{F}_\alpha \in \mathrm{QCoh}(X)^\heartsuit$ is almost perfect (Proposition 9.5.2.3). It follows that for each $n \geqslant 0$, we have an equivalence $\pi_n(\mathscr{F} \otimes E) \simeq \varinjlim_\alpha \pi_n(\mathscr{F}_\alpha \otimes E)$ in the abelian category $\mathcal{E}^\heartsuit$. Lemma 9.5.4.5 implies that for $n > 0$, the maps $\pi_n(\mathscr{F}_\alpha \otimes E) \to$

$\pi_n(\mathscr{F} \otimes E)$ vanish for each index α. It follows that $\pi_n(\mathscr{F} \otimes E)$ vanishes for $n > 0$. Since $\mathcal{E}$ is separated, this implies that $\mathscr{F} \otimes E$ is discrete as desired. □

We close this section by discussing a variant of Theorem 10.4.5.1.

Theorem 10.4.5.5. *Let X be a geometric stack with the resolution property. Assume either that X is n-truncated for some integer $n \geqslant 0$ or that $\mathscr{O}_X$ is a compact object of $\mathrm{QCoh}(X)$. Then the global sections functor $\mathcal{C} \mapsto \mathrm{QCoh}(X; \mathcal{C})$ determines a fully faithful embedding*

$$\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{comp}}(X) \to \mathrm{Mod}_{\mathrm{Vect}(X)}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$$

whose essential image is spanned by those ∞-categories $\mathcal{E}$ with an action of $\mathrm{Vect}(X)$ which satisfy the following conditions:

(*i*) *The ∞-category $\mathcal{E}$ is a complete Grothendieck prestable ∞-category.*

(*ii*) *If $0 \in \mathrm{Vect}(X)$ is a zero object, then $0 \otimes E$ is a zero object of $\mathcal{E}$ for each $E \in \mathcal{E}$.*

(*iii*) *For every diagram*

$$\begin{array}{ccc} \mathscr{F}' & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{F}'' \end{array}$$

in $\mathrm{Vect}(X)$ which is a cofiber sequence in $\mathrm{QCoh}(X)$ and every object $E \in \mathcal{E}$, the resulting diagram

$$\begin{array}{ccc} \mathscr{F}' \otimes E & \longrightarrow & \mathscr{F} \otimes E \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{F}'' \otimes E \end{array}$$

is a cofiber sequence in $\mathcal{E}$.

Proof. Let $\mathcal{C} = \mathrm{Fun}^{\pi}(\mathrm{Vect}(X)^{\mathrm{op}}, \mathcal{S})$ be the ∞-category of functors $\mathrm{Vect}(X)^{\mathrm{op}} \to \mathcal{S}$ which preserve finite products. Then $\mathcal{C}$ is a complete Grothendieck prestable ∞-category (see Proposition C.1.5.7). Since the tensor product on $\mathrm{Vect}(X)$ preserves finite direct sums in each variable, there is an essentially unique symmetric monoidal structure on $\mathcal{C}$ for which the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable and the Yoneda embedding $j : \mathrm{Vect}(X) \to \mathcal{C}$ is symmetric monoidal. Moreover, composition with j induces a fully faithful embedding $\rho : \mathrm{Mod}_{\mathcal{C}}(\mathsf{Groth}_\infty^{\mathrm{comp}}) \to \mathrm{Mod}_{\mathrm{Vect}(X)}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ whose essential image consists of those complete Grothendieck prestable ∞-categories $\mathcal{E}$ with an action of $\mathrm{Vect}(X)$ for which the action map $a : \mathrm{Vect}(X) \times \mathcal{E} \to \mathcal{E}$ preserves finite coproducts in the first variable and small colimits in the second variable. Note that since every object $\mathscr{F} \in \mathrm{Vect}(X)$ is dualizable, this second condition is automatic: the construction $E \mapsto \mathscr{F} \otimes E$ automatically preserves small colimits, since it is left adjoint to the construction $E \mapsto \mathscr{F}^{\vee} \otimes E$. It follows that the essential image of ρ can be characterized as those objects $\mathcal{E} \in \mathrm{Mod}_{\mathrm{Vect}(X)}(\widehat{\mathcal{C}\mathrm{at}}_{\infty})$ which satisfy conditions (i), (ii), and the following weaker version of (iii'):

(iii') For every diagram

$$\begin{array}{ccc} \mathscr{F}' & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{F}'' \end{array}$$

in $\mathrm{Vect}(X)$ which is a *split* cofiber sequence in $\mathrm{QCoh}(X)$ and every object $E \in \mathcal{E}$, the associated diagram

$$\begin{array}{ccc} \mathscr{F}' \otimes E & \longrightarrow & \mathscr{F} \otimes E \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{F}'' \otimes E \end{array}$$

is also a (split) cofiber sequence in $\mathcal{E}$.

Note that the inclusion functor $\mathrm{Vect}(X) \to \mathrm{QCoh}(X)^{\mathrm{cn}}$ is symmetric monoidal, and therefore admits an essentially unique extension to a symmetric monoidal functor $F : \mathcal{C} \to \mathrm{QCoh}(X)^{\mathrm{cn}}$ which preserves small colimits. To prove Theorem 10.4.5.5, we must show that composition with F induces a fully faithful embedding

$$\theta : \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}}(\mathsf{Groth}_{\infty}^{\mathrm{comp}}) \to \mathrm{Mod}_{\mathcal{C}}(\mathsf{Groth}_{\infty}^{\mathrm{comp}}),$$

whose essential image is spanned by those $\mathcal{C}$-modules $\mathcal{E}$ which satisfy condition (iii).

For each $m \geqslant 0$, let Groth_m denote the ∞-category of Grothendieck abelian $(m+1)$-categories (see Definition C.5.4.1). Then Theorem C.5.4.8 supplies an equivalence of ∞-categories $\mathsf{Groth}_{\infty}^{\mathrm{comp}} \simeq \varprojlim_m \mathsf{Groth}_m$. It follows that we can identify θ with the limit of a tower of functors

$$\theta_m : \mathrm{Mod}_{\mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant m}}(\mathsf{Groth}_m) \to \mathrm{Mod}_{\tau_{\leqslant m}\mathcal{C}}(\mathsf{Groth}_m).$$

Let $e : \mathrm{Vect}(X) \to \tau_{\leqslant m}\mathcal{C}$ be the symmetric monoidal functor given by $e(\mathscr{F}) = \tau_{\leqslant m} j(\mathscr{F})$. To complete the proof, it will suffice to show that each θ_m is a fully faithful embedding, whose essential image is spanned by those Grothendieck abelian $(m+1)$-categories $\mathcal{E}$ equipped with an action of $\tau_{\leqslant m}\mathcal{C}$ for which the induced action of $\mathrm{Vect}(X)$ on $\mathcal{E}$ (via the functor e) satisfies condition (iii).

For every commutative diagram σ :

$$\begin{array}{ccc} \mathscr{F}' & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{F}'' \end{array}$$

in $\mathrm{Vect}(X)$ which is a pushout square in $\mathrm{QCoh}(X)$, let $f_\sigma : \mathrm{cofib}(e(\mathscr{F}') \to e(\mathscr{F})) \to e(\mathscr{F}')$ be the induced map in the ∞-category $\tau_{\leqslant m}\,\mathcal{C}$. Let S be the collection of all morphisms in $\tau_{\leqslant m}\,\mathcal{C}$ having the form f_σ. Unwinding the definitions, we see that an object $\mathcal{E} \in \mathrm{Mod}_{\tau_{\leqslant m}\,\mathcal{C}}(\mathsf{Groth}_m)$ satisfies condition (iii) if and only if the induced map $\tau_{\leqslant m}\,\mathcal{C} \to \mathrm{Fun}(\mathcal{E}, \mathcal{E})$ factors through the localization $S^{-1}(\tau_{\leqslant m}\,\mathcal{C})$. To complete the proof, it will suffice to show that the functor F induces an equivalence $S^{-1}(\tau_{\leqslant m}\,\mathcal{C}) \to \mathrm{QCoh}(X)^{\mathrm{cn}}_{\leqslant m}$, which is a reformulation of Proposition 9.3.7.7. □

10.4.6 Weakly Complicial Quasi-Coherent Stacks

We now generalize some of the results of §**??** to the setting of geometric stacks.

Proposition 10.4.6.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a quasi-geometric stack, let $\mathcal{C}$ be a quasi-coherent stack on X, and let n be a nonnegative integer. The following conditions are equivalent:*

- (a) *For every flat morphism $\eta : \mathrm{Spec}\,A \to X$, the prestable A-linear ∞-category $\mathcal{C}_\eta$ is weakly n-complicial.*
- (b) *There exists a faithfully flat morphism $\eta : \mathrm{Spec}\,A \to X$ such that $\mathcal{C}_\eta$ is weakly n-complicial.*

Proof. The implication $(a) \Rightarrow (b)$ is clear. Suppose that (b) is satisfied: that is, there exists a faithfully flat map $\eta : \mathrm{Spec}\,A \to X$ such that $\mathcal{C}_\eta$ is weakly n-complicial. Let $\eta' : \mathrm{Spec}\,A' \to X$ be any flat map. Choose a faithfully flat map $\mathrm{Spec}\,B \to \mathrm{Spec}\,A \times_X \mathrm{Spec}\,A'$. The flatness of η' guarantees that B is flat over A, so that $B \otimes_A \mathcal{C}_\eta$ is weakly n-complicial (Proposition D.5.7.1). Using the equivalence $B \otimes_A \mathcal{C}_\eta \simeq B \otimes_{A'} \mathcal{C}_{\eta'}$, we conclude that $B \otimes_{A'} \mathcal{C}_{\eta'}$ is weakly n-complicial. Since the morphism η is faithfully flat, B is faithfully flat over A'. Applying Proposition D.5.7.1 again, we deduce that $\mathcal{C}_{\eta'}$ is also weakly n-complicial. □

Definition 10.4.6.2. Let X be a quasi-geometric stack and let $\mathcal{C}$ be a prestable quasi-coherent stack on X. We will say that $\mathcal{C}$ is *weakly n-complicial* if it satisfies the equivalent conditions of Proposition 10.4.6.1.

Remark 10.4.6.3. Let X be a quasi-geometric Deligne-Mumford stack and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$. We then $\mathcal{C}$ is weakly n-complicial in the sense of Definition 10.4.6.2 if and only if it is weakly n-complicial in the sense of Definition 10.3.4.4 (this is an elementary consequence of Proposition 10.4.6.1).

Remark 10.4.6.4. Let $f : X \to Y$ be a morphism of quasi-geometric stacks and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(Y)$. If $\mathcal{C}$ is weakly n-complicial and f is flat, then $f^*\,\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$ is weakly n-complicial. Beware that this need not be true if we drop the assumption that f is flat.

Example 10.4.6.5. Let X be a quasi-geometric stack and let $\mathcal{Q}_X^{\mathrm{cn}}$ be the unit object of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ (see Example 10.1.6.2). Then $\mathcal{Q}_X^{\mathrm{cn}}$ is weakly n-complicial if and only if X is n-truncated, in the sense of Definition 9.1.6.2: this follows from Proposition C.5.5.15.

We now establish a local-to-global principle for Definition 10.4.6.2:

Proposition 10.4.6.6. *Let X be a geometric stack, let $\mathcal{C}$ be a complete prestable quasi-coherent stack on X, and let $n \geqslant 0$ be an integer. Then $\mathcal{C}$ is weakly n-complicial if and only if the Grothendieck prestable ∞-category $\mathrm{QCoh}(X;\mathcal{C})$ is weakly n-complicial.*

Proof. Choose a faithfully flat map $\eta : \operatorname{Spec} A \to X$ and let $\mathcal{C}_\eta$ denote the corresponding A-linear ∞-category. Since η is flat, the pullback functor $\eta^* : \mathrm{Sp}(\mathrm{QCoh}(X;\mathcal{C})) \to \mathrm{Sp}(\mathcal{C}_\eta)$ is t-exact (Corollary 10.1.7.10). Consequently, the right adjoint $\eta_* : \mathrm{Sp}(\mathcal{C}_\eta) \to \mathrm{Sp}(\mathrm{QCoh}(X;\mathcal{C}))$ carries injective objects of $\mathrm{Sp}(\mathcal{C}_\eta)_{\leqslant 0}$ to injective objects of $\mathrm{Sp}(\mathrm{QCoh}(X;\mathcal{C}))_{\leqslant 0}$ (see Definition HA.**??**).

Suppose that $\mathrm{QCoh}(X;\mathcal{C})$ is weakly n-complicial. Using Proposition C.5.7.11, we deduce that every injective object of $\mathrm{Sp}(\mathrm{QCoh}(X;\mathcal{C}))_{\leqslant 0}$ belongs to $\mathrm{Sp}(\mathrm{QCoh}(X;\mathcal{C}))_{\geqslant -n}$. It follows that if $\overline{Q}$ is an injective object of $\mathrm{Sp}(\mathcal{C}_\eta)_{\leqslant 0}$, then $\eta_* \overline{Q} \in \mathrm{Sp}(\mathrm{QCoh}(X;\mathcal{C}))_{\geqslant -n}$. Since η is affine, the pushforward functor η_* is t-exact and conservative. We therefore conclude that every injective object of $\mathrm{Sp}(\mathcal{C}_\eta)_{\leqslant 0}$ belongs to $\mathrm{Sp}(\mathcal{C}_\eta)_{\geqslant -n}$. Applying Proposition C.5.7.11 again, we conclude that $\mathcal{C}_\eta$ is weakly n-complicial, and therefore $\mathcal{C}$ is weakly n-complicial (Proposition 10.4.6.1).

To prove the converse, we will need the following:

$(*)$ Every injective object Q of $\mathrm{Sp}(\mathrm{QCoh}(X;\mathcal{C}))_{\leqslant 0}$ is a direct summand of $\eta_* \overline{Q}$, for some injective object $\overline{Q} \in \mathrm{Sp}(\mathcal{C}_\eta)_{\leqslant 0}$.

To prove $(*)$, we note that the abelian category $\mathcal{C}_\eta^\heartsuit$ has enough injectives, so there exists an injective object $I \in \mathcal{C}_\eta^\heartsuit$ and a monomorphism $\rho : \pi_0 \eta^* Q \to I$. Using Proposition HA.**??** , we can write $I = \pi_0 \overline{Q}$ for some injective object $\overline{Q} \in \mathrm{Sp}(\mathcal{C}_\eta)_{\leqslant 0}$. The injectivity of $\overline{Q}$ implies that the composite map $\pi_0 \eta^* Q \xrightarrow{\rho} I \to \overline{Q}$ can be extended to a map $\eta^* Q \to \overline{Q}$, which we can identify with a map $\rho_0 : Q \to \eta_* \overline{Q}$. By construction, $\eta^* \rho_0$ induces a monomorphism on π_0. Since η is faithfully flat, it follows that ρ_0 induces a monomorphism on π_0. Since $\eta_* \overline{Q}$ is an injective object of $\mathrm{Sp}(\mathcal{C}_\eta)_{\leqslant 0}$, it follows that ρ_0 admits a left homotopy inverse: that is, it exhibits Q as a direct summand of $\eta_* \overline{Q}$.

Now suppose that $\mathcal{C}$ is weakly n-complicial. Then every injective object of $\mathrm{Sp}(\mathcal{C}_\eta)_{\leqslant 0}$ belongs to $\mathrm{Sp}(\mathcal{C}_\eta)_{\geqslant -n}$ (Proposition C.5.7.11). Since the functor η_* is t-exact, assertion $(*)$ implies that every injective object of $\mathrm{Sp}(\mathrm{QCoh}(X;\mathcal{C}))_{\leqslant 0}$ belongs to $\mathrm{Sp}(\mathrm{QCoh}(X;\mathcal{C})_{\geqslant -n}$. Applying Proposition C.5.7.11 again, we conclude that $\mathrm{QCoh}(X;\mathcal{C})$ is weakly n-complicial. □

Specializing to the case $\mathcal{C} = \mathcal{Q}_X^{\mathrm{cn}}$, we obtain the following:

Corollary 10.4.6.7. *Let X be a geometric stack. Then X is n-truncated if and only if the Grothendieck prestable ∞-category $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is weakly n-complicial.*

Proof. Combine Example 10.4.6.5 with Proposition 10.4.6.6. □

Corollary 10.4.6.8. *Let X be a geometric stack. Then the following conditions are equivalent:*

(a) *The structure sheaf $\mathscr{O}_X$ is 0-truncated.*

(b) *The inclusion $\mathrm{QCoh}(X)^{\heartsuit} \hookrightarrow \mathrm{QCoh}(X)$ extends to a t-exact equivalence of ∞-categories $\widehat{\mathcal{D}}(\mathrm{QCoh}(X)^{\heartsuit}) \simeq \mathrm{QCoh}(X)$*

Proof. Combine Corollaries 10.4.6.7 and C.5.9.7. □

Warning 10.4.6.9. In general, the completion which appears in the statement of Corollary 10.4.6.8 is necessary. For example, if G is the classifying stack of the additive group $\mathbf{G}_a$ over a field κ of characteristic $p > 0$, then the derived ∞-category $\mathcal{D}(\mathrm{QCoh}(X)^{\heartsuit})$ is not left complete (see [93]).

10.5 Locally Noetherian Quasi-Coherent Stacks

Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. We say that $\mathcal{C}$ is *locally Noetherian* if the abelian category $\mathcal{C}^{\heartsuit}$ is locally Noetherian and each Noetherian object $C \in \mathcal{C}^{\heartsuit}$ is compact when viewed as an object of $\tau_{\leqslant n}\mathcal{C}$, for each $n \geqslant 0$ (see Proposition C.6.9.8). Our goal in this section is to globalize the theory of locally Noetherian prestable ∞-categories by introducing the notion of a *locally Noetherian quasi-coherent stack*.

10.5.1 On Spectral Deligne-Mumford Stacks

It follows from Propositions D.5.6.1 and D.5.6.4 that the condition that a prestable A-linear ∞-category $\mathcal{C}$ is locally Noetherian can be tested locally with respect to the étale topology on $\mathrm{Spét}\, A$. This motivates the following:

Definition 10.5.1.1. Let X be a spectral Deligne-Mumford stack and let $\mathcal{C}$ be a prestable quasi-coherent stack on X. We will say that $\mathcal{C}$ is *locally Noetherian* if, for every étale morphism $\eta : \mathrm{Spét}\, A \to \mathsf{X}$, the prestable A-linear ∞-category $\mathcal{C}_\eta$ is locally Noetherian.

Remark 10.5.1.2. Let $f : \mathsf{X} \to \mathsf{Y}$ be morphism of spectral Deligne-Mumford stacks and let $\mathcal{C}$ be a prestable quasi-coherent stack on Y. If f is locally almost of finite presentation and $\mathcal{C}$ is locally Noetherian, then $f^* \mathcal{C}$ is locally Noetherian (see Proposition D.5.6.1). Conversely, if

f is a flat covering and $f^* \mathcal{C}$ is locally Noetherian, then $\mathcal{C}$ is locally Noetherian (Proposition D.5.6.4).

Remark 10.5.1.3. Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable A-linear ∞-category. Then $\mathcal{C}$ is locally Noetherian when regarded as a Grothendieck prestable ∞-category if and only if it is locally Noetherian when regarded as a prestable quasi-coherent stack on Spét A.

Example 10.5.1.4. Let X be a spectral Deligne-Mumford stack and let $\mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}}$ be the unit object of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ (see Example 10.1.6.2). Then$\mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}}$ is locally Noetherian if and only X is locally Noetherian (see Example C.6.9.4).

In the setting of Definition 10.5.1.1, we have the following local-to-global principle:

Proposition 10.5.1.5. *Let X be a quasi-geometric spectral Deligne-Mumford stack and let $\mathcal{C}$ be a prestable quasi-coherent stack on X. If $\mathcal{C}$ is locally Noetherian, then the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is locally Noetherian.*

Remark 10.5.1.6. In the statement of Proposition 10.5.1.5, the hypothesis that X is quasi-geometric can be weakened: it is enough to assume that X is ∞-quasi-compact (Definition 2.3.1.1).

Proof of Proposition 10.5.1.5. Let $\widehat{\mathcal{C}}$ be the completion of $\mathcal{C}$, so that $\mathrm{QCoh}(\mathsf{X}; \widehat{\mathcal{C}})$ is the completion of $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$. It follows from Proposition 10.5.2.7 that $\mathrm{QCoh}(\mathsf{X}; \widehat{\mathcal{C}})$ is locally Noetherian. Invoking Remark C.6.9.2, we see that $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is also locally Noetherian. □

Proposition 10.5.1.7. *Let X be a quasi-compact, quasi-separated spectral algebraic space and let $\mathcal{C}$ be a prestable quasi-coherent stack on X. If X is schematic and $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is locally Noetherian, then $\mathcal{C}$ is locally Noetherian.*

Proof. Since X is quasi-compact and schematic, it admits a finite covering by affine open substacks $\mathsf{U}_\alpha \subseteq \mathsf{X}$. By virtue of Remark 10.3.1.4, it will suffice to show that the restriction of $\mathcal{C}$ to each U_α is locally Noetherian. Using Remark 10.5.1.3, we see that this is equivalent to the assertion that each of the Grothendieck prestable ∞-categories $\mathrm{QCoh}(\mathsf{U}_\alpha; \mathcal{C}|_{\mathsf{U}_\alpha})$ is locally Noetherian. Let $j : \mathsf{U}_\alpha \to \mathsf{X}$ be the inclusion map, so that we have a pair of adjoint functors

$$\mathrm{QCoh}(\mathsf{X}; \mathcal{C}) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{QCoh}(\mathsf{U}_\alpha; \mathcal{C}|_{\mathsf{U}_\alpha})$$

which exhibit $\mathrm{QCoh}(\mathsf{U}_\alpha; \mathcal{C}|_{\mathsf{U}_\alpha})$ as a left exact localization of $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$. The desired result now follows from Proposition C.6.9.9. □

10.5.2 On Quasi-Geometric Stacks

We now adapt Definition 10.5.1.1 to the setting of quasi-geometric stacks.

Definition 10.5.2.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a quasi-geometric stack and let $\mathcal{C}$ be a prestable quasi-coherent stack on X. We will say that $\mathcal{C}$ is *locally Noetherian* if there exists a faithfully flat morphism $\eta : \mathrm{Spec}\, A \to X$ for which the prestable A-linear ∞-category $\mathcal{C}_\eta$ is locally Noetherian.

Warning 10.5.2.2. Let X be a quasi-geometric stack and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$ be locally Noetherian. It does *not* follow that for every faithfully flat morphism $\eta : \mathrm{Spec}\, A \to X$, the prestable A-linear ∞-category $\mathcal{C}_\eta$ is locally Noetherian. This stronger condition is never satisfied, except in trivial cases.

Remark 10.5.2.3. Let X be a quasi-geometric spectral Deligne-Mumford stack. We then have two *a priori* different notions of locally Noetherian quasi-coherent stack on X: one given by Definition 10.5.1.1 and one given by Definition 10.5.2.1. However, we will show in a moment that these definitions are equivalent: this is an immediate consequence of Corollary 10.5.2.6.

Example 10.5.2.4. Let X be a spectral Deligne-Mumford stack and let $\mathcal{Q}_X^{\mathrm{cn}}$ be the unit object of $\mathrm{QStk}^{\mathrm{PSt}}(X)$ (see Example 10.1.6.2). Then $\mathcal{Q}_X^{\mathrm{cn}}$ is locally Noetherian if and only X is locally Noetherian, in the sense of Definition 9.5.1.1: this follows immediately from Example C.6.9.4.

In spite of Warning 10.5.2.2, we have the following:

Proposition 10.5.2.5. *Let X be a quasi-geometric stack, let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$ be locally Noetherian, and let $\eta : \mathrm{Spec}\, A \to X$ be a morphism which is locally almost of finite presentation. Then the prestable A-linear ∞-category $\mathcal{C}_\eta$ is locally Noetherian.*

Proof. Since $\mathcal{C}$ is locally Noetherian, we can choose a faithfully flat morphism $\eta' : \mathrm{Spec}\, A' \to X$ such that $\mathcal{C}_{\eta'}$ is locally Noetherian. Choose an étale surjection $\mathrm{Spec}\, B \to \mathrm{Spec}\, A \times_X \mathrm{Spec}\, A'$. Since η is locally almost of finite presentation, the $\mathbb{E}_\infty$-ring B is almost of finite presentation over A'. It follows from Proposition D.5.6.1 that the Grothendieck prestable ∞-category $B \otimes_{A'} \mathcal{C}_{\eta'}$ is locally Noetherian. Using the equivalence $B \otimes_{A'} \mathcal{C}_{\eta'} \simeq B \otimes_A \mathcal{C}_\eta$, we deduce that $B \otimes A\, \mathcal{C}_\eta$ is locally Noetherian. Since η' is faithully flat, the $\mathbb{E}_\infty$-ring B is faithfully flat over A. Applying Proposition D.5.6.4, we deduce that $\mathcal{C}_\eta$ is locally Noetherian. □

Corollary 10.5.2.6. *Let X be a quasi-geometric stack and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$. Suppose that there exists a morphism $\eta : \mathrm{Spec}\, A \to X$ which is faithfully flat and locally almost of finite presentation. Then $\mathcal{C}$ is locally Noetherian if and only if the prestable A-linear ∞-category $\mathcal{C}_\eta$ is locally Noetherian.*

The following result can be regarded as a relative version of Proposition 9.5.2.3:

Proposition 10.5.2.7. *Let X be a quasi-geometric stack and let $\mathcal{C}$ be a complete prestable quasi-coherent stack on X. If $\mathcal{C}$ is locally Noetherian, then the Grothendieck prestable ∞-category $\mathrm{QCoh}(X;\mathcal{C})$ is locally Noetherian.*

Proof. We proceed as in the proof of Proposition D.5.6.4 (note that when X is affine, Proposition 10.5.2.7 reduces to Proposition D.5.6.4). Choose a faithfully flat map $\eta : \operatorname{Spec} A \to X$ for which the prestable A-linear ∞-category $\mathcal{C}_\eta$ is locally Noetherian. We wish to prove that $\mathrm{QCoh}(X;\mathcal{C})$ is locally Noetherian.

We begin by showing that the abelian category $\mathrm{QCoh}(X;\mathcal{C})^{\heartsuit}$ is locally Noetherian. Note that we have a pair of adjoint functors $\mathrm{QCoh}(X;\mathcal{C}) \underset{\eta_*}{\overset{\eta^*}{\rightleftarrows}} \mathcal{C}_\eta$ which are left exact, since the morphism η is flat. Fix an object $C \in \mathrm{QCoh}(X;\mathcal{C})^{\heartsuit}$. Since the Grothendieck abelian category $\mathcal{C}_\eta^{\heartsuit}$ is locally Noetherian, we can write $\eta^* C \in \mathcal{C}_\eta^{\heartsuit}$ as a union of Noetherian subobjects $\{C'_\alpha\}$. For each index α, set $C_\alpha = C \times_{\eta_*\eta^* C} \eta_* C'_\alpha$. Since filtered colimits in $\mathrm{QCoh}(X;\mathcal{C})$ are exact, we have $C = \varinjlim C_\alpha$. We will show that each C_α is a Noetherian object of $\mathrm{QCoh}(X;\mathcal{C})^{\heartsuit}$. Note that the functor η^* induces an injection from the set of isomorphism classes of subobjects of C to the set of isomorphism classes of subobjects of $\eta^* C$. It will therefore suffice to show that $\eta^* C_\alpha$ is a Noetherian subobject of $\eta^* C$ in the abelian category $\mathcal{C}_\eta^{\heartsuit}$. This is clear, since the inclusion $\eta^* C_\alpha \hookrightarrow \eta^* C$ factors through C'_α (which is a Noetherian subobject of $\eta^* C$ by construction).

To complete the proof that $\mathrm{QCoh}(X;\mathcal{C})$ is locally Noetherian, it will suffice to show that if C is a Noetherian object of $\mathrm{QCoh}(X;\mathcal{C})^{\heartsuit}$, then C is compact when viewed as an object of $\tau_{\leqslant n} \mathrm{QCoh}(X;\mathcal{C})$ for each $n \geqslant 0$ (see Proposition C.6.9.8). Fix a filtered diagram $\{D_\alpha\}$ of n-truncated objects of $\mathcal{C}$ having colimit D; we wish to show that the canonical map $\rho : \varinjlim \mathrm{Map}_{\mathrm{QCoh}(X,\mathcal{C})}(C, D_\alpha) \to \mathrm{Map}_{\mathrm{QCoh}(X,\mathcal{C})}(C, D)$ is a homotopy equivalence. We will prove that the homotopy fibers of ρ are m-truncated for every integer $m \geqslant -2$. Note that this is trivial when $m = n$ (since the domain and codomain of ρ are both n-truncated). We will handle the general case using descending induction on m. For each index α, form a cofiber sequence $D_\alpha \to \eta_*\eta^* D_\alpha \to D'_\alpha$, and set $D' = \varinjlim D'_\alpha$. Note that since η is faithfully flat, the objects D'_α are also n-truncated. We have a commutative diagram of fiber sequences

$$\begin{array}{ccccc}
\varinjlim \mathrm{Map}_{\mathrm{QCoh}(X;\mathcal{C})}(C, D_\alpha) & \longrightarrow & \varinjlim \mathrm{Map}_{\mathrm{QCoh}(X;\mathcal{C})}(C, \eta_*\eta^* D_\alpha) & \longrightarrow & \varinjlim \mathrm{Map}_{\mathrm{QCoh}(X;\mathcal{C})}(C, D'_\alpha) \\
\downarrow \rho & & \downarrow \phi & & \downarrow \rho' \\
\mathrm{Map}_{\mathrm{QCoh}(X;\mathcal{C})}(C, D) & \longrightarrow & \mathrm{Map}_{\mathrm{QCoh}(X;\mathcal{C})}(C, \eta_*\eta^* D) & \longrightarrow & \mathrm{Map}_{\mathrm{QCoh}(X;\mathcal{C})}(C, D').
\end{array}$$

It follows from our inductive hypothesis that the map ρ' has $(m+1)$-truncated homotopy fibers. Consequently, to show that the homotopy fibers of ρ are m-truncated, it will suffice to show that ϕ is a homotopy equivalence. Unwinding the definitions, we can identify ϕ with the canonical map $\varinjlim \mathrm{Map}_{\mathcal{C}_\eta}(\eta^* C, \eta^* D_\alpha) \to \mathrm{Map}_{\mathcal{C}_\eta}(\eta^* C, \eta^* D)$. To show that this map is a homotopy equivalence, it will suffice to show that $\eta^* C$ is an almost compact object of $\mathcal{C}_\eta$. Using Proposition C.6.9.3 and Corollary C.6.8.9 (and our assumption that $\mathcal{C}_\eta$ is locally Noetherian), we are reduced to showing that $\eta^* C$ is compact as an object of $\mathcal{C}_\eta^\heartsuit$. This is clear: the functor η_* commutes with filtered colimits, so the pullback functor $\eta^* : \mathrm{QCoh}(X; \mathcal{C})^\heartsuit \to \mathcal{C}_\eta^\heartsuit$ preserves compact objects (and C is a compact object of $\mathrm{QCoh}(X; \mathcal{C})$ by virtue of Proposition C.6.8.7). □

10.5.3 Injective and Locally Injective Objects

Let A be a connective $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be a prestable A-linear ∞-category, and let Q be an object of $\mathrm{Sp}(\mathcal{C})$. If $\mathcal{C}$ is locally Noetherian, then Corollary 7.2.5.19 and Proposition 7.2.5.20 imply that the condition that Q is injective (in the sense of Definition C.5.7.2) can be tested locally with respect to the étale topology on A. Motivated by this observation, we introduce the following variant of Definition C.5.7.2:

Definition 10.5.3.1. Let X be a spectral Deligne-Mumford stack, let $\mathcal{C}$ be a locally Noetherian prestable quasi-coherent stack on X, and let $\mathscr{G}$ be an object of the ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))$. We will say that $\mathscr{G}$ is *locally injective* if, for every étale morphism $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$, the object $\mathscr{G}_\eta \in \mathrm{Sp}(\mathcal{C}_\eta)$ is injective.

Remark 10.5.3.2. Let A be a connective $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be a prestable A-linear ∞-category, and let Q be an object of $\mathrm{Sp}(\mathcal{C})$. Then Q is injective (in the sense of Definition C.5.7.2) if and only if it is locally injective when regarded as an object of $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A; \mathrm{Sp}(\mathcal{C}))$ (in the sense of Definition 10.5.3.1). The “if” direction is obvious, and the converse follows from Corollary 7.2.5.19.

Remark 10.5.3.3. Let $f : \mathsf{X} \to \mathsf{Y}$ be an étale morphism between spectral Deligne-Mumford stacks, let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y})$ be locally Noetherian, and let $\mathscr{G} \in \mathrm{QCoh}(\mathsf{Y}; \mathrm{Sp}(\mathcal{C}))$. If $\mathscr{G}$ is locally injective, then $f^* \mathscr{G} \in \mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(f^* \mathcal{C}))$ is also locally injective. The converse holds if f is an étale surjection (this follows from Proposition 7.2.5.20).

Our goal in this section is to establish the following generalization of Remark 10.5.3.2:

Theorem 10.5.3.4. *Let X be a quasi-compact, quasi-separated spectral algebraic space, let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ be locally Noetherian, and let $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C})) \simeq \mathrm{Sp}(\mathrm{QCoh}(\mathsf{X}; \mathcal{C}))$. Then $\mathscr{G}$ is locally injective (in the sense of Definition 10.5.3.1) if and only if it is injective (in the sense of Definition C.5.7.2).*

The proof of Theorem 10.5.3.4 will require some preliminaries.

Lemma 10.5.3.5. *Let $j : \mathsf{U} \hookrightarrow \mathsf{X}$ be a quasi-compact open immersion between spectral Deligne-Mumford stacks and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ be locally Noetherian. If $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))$ is locally injective, then the direct image $j_* j^* \mathscr{G}$ and the fiber $\mathrm{fib}(\mathscr{G} \to j_* j^*)$ are also locally injective objects of $\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))$.*

Proof. Working locally on X, we can reduce to the case where $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ is affine, in which case the desired result follows from Corollary 7.2.5.13. □

Lemma 10.5.3.6. *Let X be a quasi-compact, quasi-separated spectral algebraic space, let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ be locally Noetherian, and let $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C})) \simeq \mathrm{Sp}(\mathrm{QCoh}(\mathsf{X}; \mathcal{C}))$. If $\mathscr{G}$ is locally injective, then $\mathscr{G}$ is injective.*

Proof. Using Theorem 3.4.2.1, we can choose a scallop decomposition

$$\varnothing = \mathsf{U}_0 \hookrightarrow \mathsf{U}_1 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n = \mathsf{X}.$$

We will prove that $\mathscr{G}|_{\mathsf{U}_i}$ is an injective object of $\mathrm{QCoh}(\mathsf{U}_i; \mathcal{C}|_{\mathsf{U}_i})$ for $0 \leqslant i \leqslant n$. The proof proceeds by induction on i, the case $i = 0$ being trivial. To carry out the inductive step, let us assume that $i > 0$ and that $\mathscr{G}|_{\mathsf{U}_{i-1}}$ is an injective object of $\mathrm{QCoh}(\mathsf{U}_{i-1}; \mathcal{C}|_{\mathsf{U}_{i-1}})$. Choose an excision square σ :

$$\begin{array}{ccc} \mathsf{V}' & \longrightarrow & \mathsf{V} \\ \downarrow & & \downarrow q \\ \mathsf{U}_{i-1} & \xrightarrow{j} & \mathsf{U}_i, \end{array}$$

where V is affine. Form a fiber sequence $\mathscr{G}' \to \mathscr{G}|_{\mathsf{U}_i} \to j_* \mathscr{G}|_{\mathsf{U}_{i-1}}$ in the ∞-category $\mathrm{QCoh}(\mathsf{U}_i; \mathrm{Sp}(\mathcal{C}|_{\mathsf{U}_i}))$. Since the functor

$$j_* : \mathrm{QCoh}(\mathsf{U}_{i-1}; \mathrm{Sp}(\mathcal{C}|_{\mathsf{U}_{i-1}})) \to \mathrm{QCoh}(\mathsf{U}_i; \mathrm{Sp}(\mathcal{C}|_{\mathsf{U}_i}))$$

has a t-exact left adjoint (Corollary 10.1.7.10), and therefore carries injective objects to injective objects. It follows that $j_* \mathscr{G}|_{\mathsf{U}_{i-1}}$ is an injective object of $\mathrm{QCoh}(\mathsf{U}_i; \mathrm{Sp}(\mathcal{C}|_{\mathsf{U}_i}))$. Since the collection of injective objects is closed under extensions, it will suffice to show that $\mathscr{G}'$ is an injective object of $\mathrm{QCoh}(\mathsf{U}_i; \mathrm{Sp}(\mathcal{C}|_{\mathsf{U}_i}))$. Lemma 10.5.3.5 shows that $\mathscr{G}'$ is locally injective. Since V is affine, the object $q^* \mathscr{G}' \in \mathrm{QCoh}(\mathsf{V}; \mathrm{Sp}(\mathcal{C}|_{\mathsf{V}}))$ is injective. Since the functor q^* is t-exact (Corollary 10.1.7.10), the pushforward functor $q_* : \mathrm{QCoh}(\mathsf{V}; \mathrm{Sp}(\mathcal{C}|_{\mathsf{V}})) \to \mathrm{QCoh}(\mathsf{U}_i; \mathrm{Sp}(\mathcal{C}|_{\mathsf{U}_i}))$ carries injective objects to injective objects. It follows that $q_* q^* \mathscr{G}'$ is an injective object of $\mathrm{QCoh}(\mathsf{U}_i; \mathrm{Sp}(\mathcal{C}|_{\mathsf{U}_i}))$. We now complete the proof by observing that the unit map $\mathscr{G}' \to q_* q^* \mathscr{G}'$ is an equivalence (since σ is an excision square and $j^* \mathscr{G}' \simeq 0$), so that $\mathscr{G}'$ is also an injective object of $\mathrm{QCoh}(\mathsf{U}_i; \mathrm{Sp}(\mathcal{C}|_{\mathsf{U}_i}))$. □

Lemma 10.5.3.7. *Let $f : \mathsf{X} \to \mathsf{Y}$ be an étale morphism between quasi-compact, quasi-separated spectral algebraic spaces. Let $\mathcal{C}$ be a locally Noetherian prestable quasi-coherent* *stack on* Y *and let* $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(f^* \mathcal{C}))$ *be locally injective. Then* $f_* \mathscr{G} \in \mathrm{QCoh}(\mathsf{Y}; \mathrm{Sp}(\mathcal{C}))$ *is also locally injective.*

Proof. The assertion is local on Y, so we may assume without loss of generality that Y is affine. Note that $\mathscr{G}$ is an injective object of $\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(f^* \mathcal{C}))$ by virtue of Lemma 10.5.3.6. Since the functor $f_* : \mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(f^* \mathcal{C})) \to \mathrm{QCoh}(\mathsf{Y}; \mathrm{Sp}(\mathcal{C}))$ admits a t-exact left adjoint (Corollary 10.1.7.10), it carries injective objects to injective objects. In particular, the object $f_* \mathscr{G} \in \mathrm{QCoh}(\mathsf{Y}; \mathrm{Sp}(\mathcal{C}))$ is injective, and is therefore locally injective by virtue of Remark 10.5.3.2. □

Proof of Theorem 10.5.3.4. Let X be a quasi-compact, quasi-separated spectral algebraic space, let $\mathcal{C}$ be a locally Noetherian prestable quasi-coherent stack on X, and let $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C})) \simeq \mathrm{Sp}(\mathrm{QCoh}(\mathsf{X}; \mathcal{C}))$. We wish to show that $\mathscr{G}$ is locally injective if and only if it is injective. The "only if" direction follows from Lemma 10.5.3.6. For the converse, suppose that $\mathscr{G}$ is injective. Choose an étale surjection $q : \mathsf{U} \to \mathsf{X}$, where U is affine. Choose a morphism $f : q^* \mathscr{G} \to \mathscr{G}'$ in $\mathrm{QCoh}(\mathsf{U}; \mathrm{Sp}(q^* \mathcal{C}))$ which exhibits $\mathscr{G}'$ as an injective hull of $q^* \mathscr{G}$ (see Example C.5.7.9), so that $\mathrm{cofib}(f)$ belongs to $\mathrm{QCoh}(\mathsf{U}; \mathrm{Sp}(q^* \mathcal{C}))_{\leqslant 0}$. Let f' denote the composite map

$$\mathscr{G} \xrightarrow{u} q_* q^* \mathscr{G} \xrightarrow{q_*(f)} q_* \mathscr{G}' .$$

Since q is surjective, we have $\mathrm{cofib}(u) \in \mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))_{\leqslant 0}$. Since q_* is right t-exact, we also have $\mathrm{cofib}(q_*(f)) \simeq q_* \mathrm{cofib}(f) \in \mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))_{\leqslant 0}$. It follows that $\mathrm{cofib}(f') \in \mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))_{\leqslant 0}$. Since $\mathscr{G}$ is an injective object of $\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))$, it follows that u admits a left homotopy inverse: that is, it exhibits $\mathscr{G}$ as a direct summand of $q_* \mathscr{G}'$. Consequently, to show that $\mathscr{G}$ is locally injective, it will suffice to show that $q_* \mathscr{G}'$ is locally injective. This is an immediate consequence of Lemma 10.5.3.7 (since the object $\mathscr{G}' \in \mathrm{QCoh}(\mathsf{U}; \mathrm{Sp}(f^* \mathcal{C}))$ is locally injective by virtue of Remark 10.5.3.2). □

10.5.4 Spectral Decompositions of Injective Objects

Let A be a connective $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be a prestable A-linear ∞-category, and suppose that $\mathcal{C}$ is locally Noetherian. In §7.2.5, we saw that to every indecomposable injective object $Q \in \mathrm{Sp}(\mathcal{C})$, we can associate a unique prime ideal $\mathfrak{p} \in |\operatorname{Spec} A|$ such that Q is centered at $\mathfrak{p}$ (Proposition 7.2.5.11). Our goal in this section is to prove a global version of Proposition 7.2.5.11, replacing $\operatorname{Sp\acute{e}t} A$ by an arbitrary quasi-compact, quasi-separated spectral algebraic space X (Proposition 10.5.4.6).

Definition 10.5.4.1. Let X be a quasi-compact, quasi-separated spectral algebraic space, let $\mathcal{C}$ be a stable quasi-coherent stack on X, and let x be a point of the underlying topological space $|\mathsf{X}|$. We will say that an object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is *centered at x* if the following condition is satisfied:

(∗) Let $U \subseteq |\mathsf{X}|$ be a quasi-compact open subset corresponding to an open immersion $j : \mathsf{U} \hookrightarrow \mathsf{X}$. If $x \in U$, then the unit map $\mathscr{F} \to j_* j^* \mathscr{F}$ is an equivalence. If $x \notin U$, then the pullback $j^* \mathscr{F}$ is a zero object of $\mathrm{QCoh}(\mathsf{U}; \mathcal{C}|_{\mathsf{U}})$.

Example 10.5.4.2. In the situation of Definition 10.5.4.1, suppose that $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ is affine and that $x \in |\mathsf{X}|$ corresponds to a prime ideal $\mathfrak{p} \subseteq \pi_0 A$. Then $\mathscr{F}$ is centered at x (in the sense of Definition 10.5.4.1) if and only if it is centered at $\mathfrak{p}$ (in the sense of Definition 7.2.5.6).

Remark 10.5.4.3. In the situation of Definition 10.5.4.1, suppose that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is nonzero. Then there is at most one point $x \in |\mathsf{X}|$ such that $\mathscr{F}$ is centered at x. To prove this, we note that if x and y are two distinct points of $|\mathsf{X}|$, then we can choose a quasi-compact open subset $U \subseteq |\mathsf{X}|$ which contains one but not the other. In this case, the assumption that $\mathscr{F}$ is centered at both x and y guarantees that the unit map $u : \mathscr{F} \to j_* j^* \mathscr{F}$ is an equivalence and that the codomain of u vanishes; here $j : \mathsf{U} \hookrightarrow \mathsf{X}$ denotes the open immersion corresponding to U. This contradicts our assumption that $\mathscr{F}$ is nonzero.

Remark 10.5.4.4. Let X be a quasi-compact, quasi-separated spectral algebraic space, let $\mathcal{C}$ be a stable quasi-coherent stack on X, and let x be a point of the underlying topological space $|\mathsf{X}|$. The collection of those objects $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ which are centered at x is a stable subcategory of $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ which is closed under suspensions.

Remark 10.5.4.5. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism between quasi-compact, quasi-separated spectral algebraic spaces, let $\mathcal{C}$ be a stable quasi-coherent stack on Y, and let $x \in |\mathsf{X}|$ be a point having image $y = f(x) \in |\mathsf{Y}|$. If $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}; f^* \mathcal{C})$ is centered at x, then $f_* \mathscr{F} \in \mathrm{QCoh}(\mathsf{Y}; \mathcal{C})$ is centered at y.

Proposition 10.5.4.6. *Let X be a quasi-compact, quasi-separated spectral algebraic space, let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ be locally Noetherian, and let $\mathscr{F}$ be an indecomposable injective object of $\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))$. Then there exists a unique point $x \in |\mathsf{X}|$ such that $\mathscr{F}$ is centered at x.*

Corollary 10.5.4.7. *Let X be a quasi-compact, quasi-separated spectral algebraic space, let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ be locally Noetherian, and let $\mathscr{F}$ be an injective object of $\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))$. Then $\mathscr{F}$ can be written as a coproduct $\bigoplus_{x \in |\mathsf{X}|} \mathscr{F}_x$, where each $\mathscr{F}_x$ is an injective object of $\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))$ which is centered at x.*

Proof. It follows from Proposition 10.5.1.5 that the Grothendieck prestable ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is locally Noetherian. Consequently, every injective object $\mathscr{F} \in \mathrm{Sp}(\mathrm{QCoh}(\mathsf{X}; \mathcal{C})) \simeq \mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))$ can be written as a direct sum of indecomposable injective objects of $\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))$ (Proposition C.6.10.6). We can therefore reduce to the case where $\mathscr{F}$ is indecomposable, in which case the desired result follows from Proposition 10.5.4.6. □

Warning 10.5.4.8. In the situation of Corollary 10.5.4.7, one can show that the summands $\mathscr{F}_x$ are determined by $\mathscr{F}$ up to equivalence. However, the decomposition $\mathscr{F} \simeq \bigoplus_{x \in |\mathsf{X}|} \mathscr{F}_x$ is not unique, since objects of the form $\bigoplus_{x \in |\mathsf{X}|} \mathscr{F}_x$ can admit automorphisms which do not preserve their direct sum decomposition.

Proof of Proposition 10.5.4.6. We proceed as in the proof of Lemma 10.5.3.6. Using Theorem 3.4.2.1, we can choose a scallop decomposition

$$\varnothing = \mathsf{U}_0 \hookrightarrow \mathsf{U}_1 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n = \mathsf{X}.$$

Let m be the smallest integer for which the restriction $\mathscr{F}|_{\mathsf{U}_m}$ is nonzero and let $j : \mathsf{U}_m \hookrightarrow \mathsf{X}$ denote the inclusion map. Form a fiber sequence $\mathscr{F}' \to \mathscr{F} \to j_* j^* \mathscr{F}$ in the ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))$. Using Theorem 10.5.3.4 and Lemma 10.5.3.5, we deduce that $\mathscr{F}'$ and $j_* j^* \mathscr{F}$ are also injective objects of $\mathrm{QCoh}(\mathsf{X}; \mathrm{Sp}(\mathcal{C}))$. It follows that $\mathscr{F}$ splits as a direct sum $\mathscr{F}' \oplus j_* j^* \mathscr{F}$, where the second summand does not vanish (by virtue of our assumption that $\mathscr{F}|_{\mathsf{U}_m} \neq 0$). The indecomposability of $\mathscr{F}$ now guarantees that $\mathscr{F}' \simeq 0$, so that the unit map $u : \mathscr{F} \to j_* j^* \mathscr{F}$ is an equivalence.

Choose an excision square σ :

$$\begin{array}{ccc} \mathsf{V}' & \longrightarrow & \mathsf{V} \\ \downarrow & & \downarrow q \\ \mathsf{U}_{m-1} & \longrightarrow & \mathsf{U}_m, \end{array}$$

where V is affine. The minimality of m implies that $\mathscr{F}|_{\mathsf{U}_{m-1}} \simeq 0$, so that the unit map $u' : \mathscr{F}|_{\mathsf{U}_m} \to q_* \mathscr{F}|_{\mathsf{V}}$ is an equivalence. Theorem 10.5.3.4 guarantees that $\mathscr{F}|_{\mathsf{V}}$ is an injective object of $\mathrm{QCoh}(\mathsf{V}; \mathrm{Sp}(\mathcal{C}|_{\mathsf{V}}))$. Applying Corollary 7.2.5.12, we deduce that $\mathscr{F}|_{\mathsf{V}}$ splits as a coproduct $\bigoplus_{v \in |\mathsf{V}|} \mathscr{G}_v$, where each $\mathscr{G}_v$ is centered at $v \in |\mathsf{V}|$. It follows that $\mathscr{F} \simeq j_* q_* \mathscr{F}|_{\mathsf{V}}$ splits as a direct sum $\bigoplus_{v \in |\mathsf{V}|} \mathscr{F}_v$ where $\mathscr{F}_v = j_* q_* \mathscr{G}_v$. The indecomposability of $\mathscr{F}$ now guarantees that exactly one of these summands does not vanish: that is, we have $\mathscr{F} \simeq j_* q_* \mathscr{G}_v$ for some point $v \in |\mathsf{V}|$. Applying Remark 10.5.4.5, we deduce that $\mathscr{F}$ is centered at x, where x denotes the image of v under the continuous map $|\mathsf{V}| \xrightarrow{j \circ q} |\mathsf{X}|$. □

10.6 Abelian Quasi-Coherent Stacks

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a geometric stack. In §10.4, we proved that a complete prestable quasi-coherent stack $\mathcal{C}$ on X can be recovered from its global sections $\mathrm{QCoh}(X; \mathcal{C})$, regarded as an ∞-category tensored over $\mathrm{QCoh}(X)^{\mathrm{cn}}$ (Theorem 10.4.2.3). Our primary goal in this section is to prove an analogous statement in the setting of abelian quasi-coherent stacks. Our main results can be summarized as follows:

- Let $q : X \to \operatorname{Spec} S$ be the projection map. Then the pullback functor
$$q^* : \mathsf{Groth}_{\mathrm{ab}} \simeq \mathrm{QStk}^{\mathrm{Ab}}(\operatorname{Spec} S) \to \mathrm{QStk}^{\mathrm{Ab}}(X)$$
admits a right adjoint $\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{Ab}}(X) \to \mathsf{Groth}_{\mathrm{ab}}$, which we will refer to as the *global sections functor* (Proposition 10.6.1.1).

- For every abelian quasi-coherent stack $\mathcal{C}$ on X, the Grothendieck abelian category $\mathrm{QCoh}(X; \mathcal{C})$ is equipped with an action of the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$. This action depends functorially on $\mathcal{C}$: that is, we can regard the construction $\mathcal{C} \mapsto \mathrm{QCoh}(X; \mathcal{C})$ as a functor from the ∞-category $\mathrm{QStk}^{\mathrm{Ab}}(X)$ to the ∞-category $\mathrm{Mod}_{\mathrm{QCoh}(X)^{\heartsuit}}(\mathsf{Groth}_{\mathrm{ab}})$.

- The construction $\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{Ab}}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)^{\heartsuit}}(\mathsf{Groth}_{\mathrm{ab}})$ is fully faithful, and its essential image admits a simple description (see Theorem 10.6.2.1).

- The theory of abelian quasi-coherent stacks on X is closely related to the theory of prestable quasi-coherent stacks on X. More precisely, there is a full subcategory $\mathrm{QStk}^{\%}(X) \subseteq \mathrm{QStk}^{\mathrm{PSt}}(X)$ whose objects we will refer to as *complicial* quasi-coherent stacks (Definition **??**), for which the composite functor
$$\mathrm{QStk}^{\%}(X) \hookrightarrow \mathrm{QStk}^{\mathrm{PSt}}(X) \xrightarrow{\heartsuit} \mathrm{QStk}^{\mathrm{Ab}}(X)$$
induces a homotopy equivalence $\mathrm{QStk}^{\%}(X)^{\heartsuit} \to \mathrm{QStk}^{\mathrm{Ab}}(X)^{\heartsuit}$ (Proposition **??**).

Remark 10.6.0.1. According to Remark 10.1.2.6, the ∞-category $\mathrm{QStk}^{\mathrm{Ab}}(X)$ of abelian quasi-coherent stacks on X depends only on the restriction of X to the category of ordinary commutative rings (which we can regard as a full subcategory of the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$). Consequently, the results of this section can be viewed as belonging to classical algebraic geometry, rather than spectral algebraic geometry.

Remark 10.6.0.2. Many of the results we prove in this section for abelian quasi-coherent stacks can be deduced from their counterparts for (complete) prestable quasi-coherent stacks using the equivalence $\mathrm{QStk}^{\%}(X)^{\simeq} \to \mathrm{QStk}^{\mathrm{Ab}}(X)^{\simeq}$ of Proposition **??**. However, this approach is unnecessarily convoluted: it will be easier to instead to give direct proofs by imitating the arguments used in §10.4.

10.6.1 Global Sections of Abelian Quasi-Coherent Stacks

We begin by establishing an analogue of Proposition 10.4.1.4:

Proposition 10.6.1.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a quasi-geometric stack, let S denote the sphere spectrum, and let $q : X \to \operatorname{Spec} S$ be the projection map. Then the pullback functor $q^* : \mathsf{Groth}_{\mathrm{ab}} \simeq \mathrm{QStk}^{\mathrm{Ab}}(\operatorname{Spec} S) \to \mathrm{QStk}^{\mathrm{Ab}}(X)$ admits a right adjoint* $\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{Ab}}(X) \to \mathsf{Groth}_{\mathrm{ab}}$.

Proof. Fix an object $\mathcal{C} \in \mathrm{QStk}^{\mathrm{Ab}}(X)$. We wish to show that the functor

$$F : \mathsf{Groth}_{\mathrm{ab}} \to \widehat{\mathcal{S}} \qquad \mathcal{E} \mapsto \mathrm{Map}_{\mathrm{QStk}^{\mathrm{Ab}}(X)}(q^* \mathcal{E}, \mathcal{C})$$

is representable by an object of $\mathsf{Groth}_{\mathrm{ab}}$.

Choose a faithfully flat map $f_0 : U_0 \to X$ where $U_0 \simeq \operatorname{Spec} A$ is affine, and let $U_\bullet$ denote the Čech nerve of f_0. For each $n \geqslant 0$, let $f_n : U_n \to X$ be the tautological map and let $F^n : \mathsf{Groth}_{\mathrm{ab}} \to \widehat{\mathcal{S}}$ denote the functor given by

$$\mathcal{E} \mapsto \mathrm{Map}_{\mathrm{QStk}^{\mathrm{Ab}}(U_n)}(f_n^* q^* \mathcal{E}, f_n^* \mathcal{C}).$$

Since each U_n is representable by a spectral Deligne-Mumford stack, each of the functors F^n is representable by the Grothendieck abelian category $\mathrm{QCoh}(U_n; f_n^* \mathcal{C})$ of Construction 10.1.7.1.

Since the natural map $|U_\bullet| \to X$ is an equivalence after sheafification for the fpqc topology, the ∞-category $\mathrm{QStk}^{\mathrm{Ab}}(X)$ can be identified with the totalization of the cosimplicial ∞-category $\mathrm{QStk}^{\mathrm{Ab}}(U_\bullet)$ (see Remark 10.1.2.11), so that $F \simeq \varprojlim_{[n] \in \mathbf{\Delta}} F^n$. Consequently, to show that F is representable, it will suffice to show that the cosimplicial object $[n] \mapsto f_n^* \mathcal{C}$ admits a limit in the ∞-category $\mathsf{Groth}_{\mathrm{ab}}$. Using the right cofinality of the inclusion $\mathbf{\Delta}_s \hookrightarrow \mathbf{\Delta}$ (Lemma HTT.6.5.3.7), it will suffice to show that the underlying cosemisimplicial object of $\mathsf{Groth}_{\mathrm{ab}}$ admits a limit. Note that if $\alpha : [m] \to [n]$ is a morphism in $\mathbf{\Delta}_+$, then the associated map $U_n \to U_m$ is flat (since it is a composition of pullbacks of the flat map $f_0 : U_0 \to X$). Using Corollary 10.1.7.10, we see that the natural map $\mathrm{QCoh}(U_m; f_m^* \mathcal{C}) \to \mathrm{QCoh}(U_n; f_n^* \mathcal{C})$ is left exact. It follows that the construction $[n] \mapsto f_n^*$ determines a functor from $\mathbf{\Delta}_s$ to the subcategory $\mathsf{Groth}_{\mathrm{ab}}^{\mathrm{lex}} \subseteq \mathsf{Groth}_{\mathrm{ab}}$ of Definition C.5.4.1, so the existence of the limit $\varprojlim f_\bullet^* \mathcal{C}$ follows from Proposition C.5.4.21. □

Example 10.6.1.2. Let X be a quasi-geometric stack and $\mathcal{C}$ be an abelian quasi-coherent stack on X. The proof of Proposition 10.6.1.1 shows that the ∞-category $\mathrm{QCoh}(X; \mathcal{C})$ can be identified with the totalization $\varprojlim \mathrm{QCoh}(U_\bullet; f_\bullet^* \mathcal{C})$, where $f_\bullet : U_\bullet \to X$ is the Čech nerve of any faithfully flat map $f_0 : U_0 = \operatorname{Spec} A \to X$. Taking $\mathcal{C} = \mathcal{Q}_X^\heartsuit$, we obtain an equivalence of Grothendieck abelian categories

$$\mathrm{QCoh}(X; \mathcal{Q}_X^\heartsuit) \simeq \varprojlim \mathrm{QCoh}(U_\bullet; \mathcal{Q}_{U_\bullet}^\heartsuit) = \varprojlim \mathrm{QCoh}(U_\bullet)^\heartsuit \simeq \mathrm{QCoh}(X)^\heartsuit.$$

10.6.2 Recovering $\mathcal{C}$ from $\mathrm{QCoh}(X; \mathcal{C})$

Let X be a quasi-geometric stack and let $q : X \to \operatorname{Spec} S$ be the projection map. Then the pullback functor

$$q^* : \mathsf{Groth}_{\mathrm{ab}} \simeq \mathrm{QStk}^{\mathrm{Ab}}(\operatorname{Spec} S) \to \mathrm{QStk}^{\mathrm{Ab}}(X)$$

is symmetric monoidal. It follows that the global sections functor $\mathrm{QCoh}(X;\bullet)$ of Proposition 10.6.1.1 is lax symmetric monoidal. We can therefore promote $\mathrm{QCoh}(X;\bullet)$ to a functor

$$\begin{aligned}\mathrm{QStk}^{\mathrm{Ab}}(X) &\simeq \mathrm{Mod}_{\mathcal{Q}_X^{\heartsuit}}(\mathrm{QStk}^{\mathrm{Ab}}(X)) \\ &\xrightarrow{\mathrm{QCoh}(X;\bullet)} \mathrm{Mod}_{\mathrm{QCoh}(X;\mathcal{Q}_X^{\heartsuit})}(\mathsf{Groth}_{\mathrm{ab}}) \\ &\simeq \mathrm{Mod}_{\mathrm{QCoh}(X)^{\heartsuit}}(\mathsf{Groth}_{\mathrm{ab}}).\end{aligned}$$

We will abuse notation by denoting this functor also by $\mathcal{C} \mapsto \mathrm{QCoh}(X;\mathcal{C})$. We can describe the situation more informally as follows: for every abelian quasi-coherent stack $\mathcal{C}$ on X, the Grothendieck abelian category $\mathrm{QCoh}(X;\mathcal{C})$ is tensored over the Grothendieck abelian category $\mathrm{QCoh}(X)^{\heartsuit}$ of discrete quasi-coherent sheaves on X, and the action

$$\mathrm{QCoh}(X)^{\heartsuit} \times \mathrm{QCoh}(X;\mathcal{C}) \to \mathrm{QCoh}(X;\mathcal{C})$$

preserves small colimits separately in each variable.

We can now formulate the main result of this section.

Theorem 10.6.2.1. *Let X be a geometric stack. Then the global sections functor*

$$\mathrm{QCoh}(X;\bullet) : \mathrm{QStk}^{\mathrm{Ab}}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)^{\heartsuit}}(\mathsf{Groth}_{\mathrm{ab}})$$

is a fully faithful embedding, whose essential image is spanned by those $\mathcal{E} \in \mathrm{Mod}_{\mathrm{QCoh}(X)^{\heartsuit}}(\mathsf{Groth}_{\mathrm{ab}})$ which satisfy the following additional conditions:

- $(*)$ *Let $\mathscr{F} \in \mathrm{QCoh}(X)^{\heartsuit}$. If $\mathscr{F}$ can be written as $\pi_0 \mathscr{G}$ where $\mathscr{G} \in \mathrm{QCoh}(X)$ is flat, then the construction $E \mapsto \mathscr{F} \otimes E$ determines a left exact functor from the Grothendieck abelian category $\mathcal{E}$ to itself.*

- $(*')$ *Suppose we are given an exact sequence*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

in the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$. If $\mathscr{F}'' \simeq \pi_0 \mathscr{G}$ where $\mathscr{G} \in \mathrm{QCoh}(X)$ is flat, then for every object $E \in \mathcal{E}$ the sequence

$$0 \to \mathscr{F}' \otimes E \to \mathscr{F} \otimes E \to \mathscr{F}'' \otimes E \to 0$$

is also exact.

10.6.3 Digression: Module Objects of Grothendieck Abelian Categories

Our proof of Theorem 10.6.2.1 will require the following standard fact about Grothendieck abelian categories:

Proposition 10.6.3.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable categories and suppose we are given a functor $G : \mathcal{D} \to \mathcal{C}$ which is conservative and preserves small limits and colimits. If $\mathcal{C}$ is a Grothendieck abelian category, then so is $\mathcal{D}$.*

Example 10.6.3.2. Let $\mathcal{C}$ be a Grothendieck abelian category which is tensored over a monoidal ∞-category $\mathcal{A}$. Suppose that A is an algebra object of $\mathcal{A}$ and that the functor $(C \in \mathcal{C}) \mapsto (A \otimes C \in \mathcal{C})$ commutes with small colimits. Then the forgetful functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$ is conservative and preserves small limits and colimits (Corollaries HA.4.2.3.3 and HA.4.2.3.5). It follows from Proposition 10.6.3.1 that $\mathrm{LMod}_A(\mathcal{C})$ is also Grothendieck abelian category.

Proof of Proposition 10.6.3.1. We proceed as in the proof of Proposition 10.4.3.1. The first step is to show that $\mathcal{D}$ is pointed. Let $\varnothing$ and $\mathbf{1}$ denote initial and final objects of $\mathcal{D}$, respectively, so that there is a unique map $\alpha : \varnothing \to \mathbf{1}$ in $\mathcal{D}$. We wish to prove that α is an isomorphism. Since G preserves small limits and colimits, it preserves initial and final objects, so that $G(\alpha)$ is a morphism from an initial object of $\mathcal{C}$ to a final object of $\mathcal{C}$. Because $\mathcal{C}$ is pointed, it follows that $G(\alpha)$ is an equivalence. Since the functor G is conservative, it follows that α is an equivalence.

We next claim that $\mathcal{D}$ is semiadditive: that is, for every pair of objects $D, D' \in \mathcal{D}$, the map

$$\begin{pmatrix} \mathrm{id}_D & 0 \\ 0 & \mathrm{id}_{D'} \end{pmatrix} : D \amalg D' \to D \times D'$$

is an isomorphism. Since the functor G preserves small limits and colimits, we can identify $G \begin{pmatrix} \mathrm{id}_D & 0 \\ 0 & \mathrm{id}_{D'} \end{pmatrix}$ with the natural map

$$\begin{pmatrix} \mathrm{id}_{G(D)} & 0 \\ 0 & \mathrm{id}_{G(D')} \end{pmatrix} : G(D) \amalg G(D') \to G(D) \times G(D'),$$

which is an isomorphism by virtue of the fact that $\mathcal{C}$ is semiadditive. Since G is conservative, it follows that the map $\begin{pmatrix} \mathrm{id}_D & 0 \\ 0 & \mathrm{id}_{D'} \end{pmatrix}$ is an isomorphism in $\mathcal{D}$.

To complete the proof that $\mathcal{D}$ is additive, it will suffice to show that for every object $D \in \mathcal{D}$, the "shearing" map

$$\begin{pmatrix} \mathrm{id}_D & \mathrm{id}_D \\ 0 & \mathrm{id}_D \end{pmatrix} : D \oplus D \to D \oplus D$$

is an isomorphism (see the proof of Theorem C.4.1.1). This follows from the fact that G is conservative, since the map

$$G\begin{pmatrix} \mathrm{id}_D & \mathrm{id}_D \\ 0 & \mathrm{id}_D \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{G(D)} & \mathrm{id}_{G(D)} \\ 0 & \mathrm{id}_{G(D)} \end{pmatrix} : G(D) \oplus G(D) \to G(D) \oplus G(D)$$

is an isomorphism by virtue of our assumption that the category $\mathcal{C}$ is additive.

We next show that $\mathcal{D}$ is abelian. Let $f : D \to D'$ be a morphism in $\mathcal{D}$; we wish to show that the natural map $\alpha : \mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism in $\mathcal{D}$. Since the functor G preserves limits and colimits, it commutes with the formation of images and coimages. Consequently, we can identify $G(\alpha)$ with the canonical map from coimage of $G(f)$ to the image of $G(f)$. Since $\mathcal{C}$ is an abelian category, it follows that $G(\alpha)$ is an isomorphism. Using the fact that G is conservative, we conclude that α is an isomorphism.

We now complete the proof by showing that the abelian category $\mathcal{D}$ is Grothendieck. Since $\mathcal{D}$ is assumed to be presentable, it will suffice to show that the collection of monomorphisms in $\mathcal{D}$ is closed under filtered colimits. Suppose we are given a filtered diagram $\{f_\alpha : D_\alpha \to D'_\alpha\}$ of monomorphisms in $\mathcal{D}$ having a colimit $f : D \to D'$. Since G commutes with limits, it is left exact. Consequently, each $G(f_\alpha)$ is a monomorphism in $\mathcal{C}$. Using the assumption that $\mathcal{C}$ is a Grothendieck abelian category, we deduce that the colimit $\varinjlim G(f_\alpha)$ is also a monomorphism in $\mathcal{C}$. Because G commutes with colimits, we can identify $\varinjlim G(f_\alpha)$ with $G(f)$. Using the left exactness of G, we deduce that $G(\ker(f)) \simeq \ker(G(f)) \simeq 0$. Since G is conservative, it follows that $\ker(f) \simeq 0$, so that f is a monomorphism. □

10.6.4 The Proof of Theorem 10.6.2.1

Let X be a geometric stack and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{QCoh}(X)^\heartsuit}(\mathsf{Groth}_{\mathrm{ab}})$ be a Grothendieck abelian ∞-category equipped with an action of $\mathrm{QCoh}(X)^\heartsuit$. For every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, let $\mathcal{C}_\eta$ denote the tensor product $\mathrm{Mod}_A^\heartsuit \otimes_{\mathrm{QCoh}(X)^\heartsuit} \mathcal{C}$, formed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories. Our assumption that X is a geometric stack guarantees that η is an affine morphism, so that we can identify $\mathrm{Mod}_A^\heartsuit$ with the ∞-category

$$\mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(X))^\heartsuit \simeq \mathrm{Mod}_{\pi_0 \mathscr{A}}(\mathrm{QCoh}(X)^\heartsuit)$$

where $\mathscr{A} = \eta_* \mathscr{O}_{\mathrm{Spec}\, A} \in \mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}})$. Applying Theorem HA.4.8.4.6 , we obtain an equivalence $\mathcal{C}_\eta \simeq \mathrm{LMod}_{\pi_0 \mathscr{A}}(\mathcal{C})$. It follows from Example 10.6.3.2 that $\mathcal{C}_\eta$ is an (A-linear) Grothendieck abelian category.

The construction $\mathcal{C} \mapsto \{\mathcal{C}_\eta\}_{\eta \in X(A)}$ determines a functor

$$\Phi_X^{\mathrm{Ab}} : \mathrm{Mod}_{\mathrm{QCoh}(X)^\heartsuit}(\mathsf{Groth}_{\mathrm{ab}}) \to \mathrm{QStk}^{\mathrm{Ab}}(X).$$

It is not difficult to see that the functor Φ_X^{Ab} is left adjoint to the global sections functor $\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{Ab}}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)^\heartsuit}(\mathsf{Groth}_{\mathrm{ab}})$ appearing in the statement of Theorem 10.6.2.1.

Proof of Theorem 10.6.2.1. Let X be a geometric stack and choose a faithfully flat map $\eta : U_0 \to X$, where $U_0 \simeq \operatorname{Spec} A$ is affine. We first show that the global sections functor

$$\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{Ab}}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)^{\heartsuit}}(\mathsf{Groth}_{\mathrm{ab}})$$

is fully faithful. Let $\mathcal{C}$ be an abelian quasi-coherent stack on X; we wish to show that the counit map $v : \Phi_X^{\mathrm{Ab}} \mathrm{QCoh}(X; \mathcal{C}) \to \mathcal{C}$ is an equivalence of abelian quasi-coherent stacks on X. Since η is an effective epimorphism of fpqc sheaves, it will suffice to show that $\eta^*(v)$ is an equivalence of A-linear abelian categories (Remark 10.1.2.11).

Unwinding the definitions, we wish to prove that the canonical map

$$\theta : \mathrm{Mod}_A^{\heartsuit} \otimes_{\mathrm{QCoh}(X)^{\heartsuit}} \mathrm{QCoh}(X; \mathcal{C}) \to \mathcal{C}_\eta$$

is an equivalence of A-linear abelian categories. Let $U_\bullet$ denote the Čech nerve of η, so that we can write $U_\bullet \simeq \operatorname{Spec} A^\bullet$ for some cosimplicial $\mathbb{E}_\infty$-ring $A^\bullet$. For each $n \geqslant 0$, let $f_n : U_n \to X$ be the canonical map, and let us abuse notation by identifying each pullback $f_n^* \mathcal{C}$ with the corresponding (A^n-linear) Grothendieck abelian category. Then the proof of Proposition 10.6.1.1 supplies a canonical equivalence $\mathrm{QCoh}(X; \mathcal{C}) \simeq \varprojlim f_\bullet^* \mathcal{C}$. Let $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}})$ denote the direct image under η of the structure sheaf of $\operatorname{Spec} A$. Since X is a geometric stack, the map η is affine and we can identify $\mathrm{Mod}_A^{\heartsuit}$ with the ∞-category $\mathrm{Mod}_{\mathscr{A}}(\mathrm{QCoh}(X))^{\heartsuit} \simeq \mathrm{Mod}_{\pi_0 \mathscr{A}}(\mathrm{QCoh}(X)^{\heartsuit})$. It follows that $\mathrm{Mod}_A^{\heartsuit}$ is self-dual as an object of the ∞-category $\mathrm{Mod}_{\mathrm{QCoh}(X)^{\heartsuit}}(\mathsf{Groth}_{\mathrm{ab}})$, so that the formation of tensor products with $\mathrm{Mod}_A^{\heartsuit}$ commutes with limits. We therefore have canonical equivalences

$$\begin{aligned} \mathrm{Mod}_A^{\heartsuit} \otimes_{\mathrm{QCoh}(X)^{\heartsuit}} \mathrm{QCoh}(X; \mathcal{C}) &\simeq \mathrm{LMod}_{\pi_0 \mathscr{A}}(\mathrm{QCoh}(X; \mathcal{C})) \\ &\simeq \varprojlim \mathrm{LMod}_{\mathscr{A}}(\mathrm{QCoh}(U_\bullet; f_\bullet^* \mathcal{C})) \\ &\simeq \mathrm{QCoh}(U_{\bullet+1}; f_{\bullet+1}^* \mathcal{C}). \end{aligned}$$

The statement that θ is an equivalence now follows from the observation that the augmented simplicial object $\mathrm{QCoh}(U_{\bullet+1}; f_{\bullet+1}^* \mathcal{C})$ is split.

We next show that for every abelian quasi-coherent stack $\mathcal{C} \in \mathrm{QStk}^{\mathrm{Ab}}(X)$, the Grothendieck abelian category $\mathrm{QCoh}(X; \mathcal{C})$ satisfies conditions $(*)$ and $(*')$ of Theorem 10.6.2.1. We first verify $(*)$. Let $\mathscr{F} \in \mathrm{QCoh}(X)^{\heartsuit}$, and suppose that we can write $\mathscr{F} = \pi_0 \mathscr{G}$ where $\mathscr{G} \in \mathrm{QCoh}(X)$ is flat. Let $F : \mathrm{QCoh}(X; \mathcal{C}) \to \mathrm{QCoh}(X; \mathcal{C})$ be the functor given by tensor product with $\mathscr{F}$. We wish to prove that the functor F is exact. Recall that we can identify $\mathrm{QCoh}(X; \mathcal{C})$ with the inverse limit of the cosemisimplicial object $f_\bullet^* \mathcal{C}$ in the ∞-category $\mathsf{Groth}_{\mathrm{ab}}^{\mathrm{lex}}$ (see the proof of Proposition 10.6.1.1); consequently, the functor F is exact if and only if each of the composite functors $\mathrm{QCoh}(X; \mathcal{C}) \xrightarrow{F} \mathrm{QCoh}(X; \mathcal{C}) \xrightarrow{f_n^*} f_n^* \mathcal{C}$ is exact. Rewriting this functor as a composition $\mathrm{QCoh}(X; \mathcal{C}) \xrightarrow{f_n^*} f_n^* \mathcal{C} \xrightarrow{(f_n^* \mathscr{F}) \otimes} f_n^* \mathcal{C}$, we are reduced to proving that tensor product with the object $f_n^* \mathscr{F} \in \mathrm{QCoh}(U_n)^{\heartsuit} \simeq \mathrm{Mod}_{A^n}^{\heartsuit}$ determines an exact functor F_n from the abelian category $f_n^* \mathcal{C}$ to itself. To see this, we observe that we can identify $f_n^* \mathscr{F}$ with $\pi_0(f_n^* \mathscr{G})$. Since $\mathscr{G}$ is flat, Proposition HA.7.2.2.15 implies that we can write f_n^* as a filtered colimit of free A^n-modules of finite rank. Consequently, the functor F_n can be written as a filtered colimit of endofunctors of $f_n^* \mathcal{C}$ having the form $C \mapsto C^k$, and is therefore exact.

We now prove that $\mathrm{QCoh}(X; \mathcal{C})$ satisfies condition $(*')$. Suppose we are given an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ in the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$, where $\mathscr{F}'' \simeq \pi_0 \mathscr{G}$ for some flat object $\mathscr{G} \in \mathrm{QCoh}(X)$. We wish to show that for each object $C \in \mathrm{QCoh}(X; \mathcal{C})$, the sequence

$$0 \to \mathscr{F}' \otimes E \to \mathscr{F} \otimes E \to \mathscr{F}'' \otimes E \to 0$$

is exact. Arguing as above, it will suffice to show that the sequence σ :

$$0 \to f_n^*(\mathscr{F}' \otimes E) \to f_n^*(\mathscr{F} \otimes E) \to f_n^*(\mathscr{F}'' \otimes E) \to 0$$

is exact in the Grothendieck abelian category $f_n^* \mathcal{C}$ for each $n \geqslant 0$. Let E^n denote the image of E in $f_n^* \mathcal{C}$, so that we can rewrite the sequence σ as

$$0 \to f_n^* \mathscr{F}' \otimes_{A^n} E^n \to f_n^* \mathscr{F} \otimes_{A^n} E^n \to f_n^* \mathscr{F}'' \otimes_{A^n} E^n \to 0.$$

Note that we can identify $f_n^* \mathscr{F}''$ with $\pi_0 f_n^* \mathscr{G}$. Since the quasi-coherent sheaf $\mathscr{G}$ is flat, Proposition HA.7.2.2.15 implies that we can write $f_n^* \mathscr{G}$ as a filtered colimit of free A^n-modules of finite rank. Consequently, we can write $f_n^* \mathscr{F}''$ as a filtered colimit $\varinjlim M_\alpha$, where each M''_α is a free module over $\pi_0 A^n$. Set $M_\alpha = f_n^* \mathscr{F} \times_{f_n^* \mathscr{F}''} M''_\alpha$, so that we can write σ as a filtered colimit of complexes σ_α:

$$0 \to f_n^* \mathscr{F}' \otimes_{A^n} E^n \to M_\alpha \otimes_{A^n} E^n \to M''_\alpha \otimes_{A^n} E^n \to 0.$$

Since $f_n^* \mathcal{C}$ is a Grothendieck abelian category, the collection of exact sequences in $f_n^* \mathcal{C}$ is closed under filtered colimits. Consequently, to prove that the sequence σ is exact, it will suffice to prove that each of the sequences σ_α is exact. In fact, we claim that σ_α is split exact. This follows from the observation that σ_α is obtained by tensoring the object $E^n \in f_n^* \mathcal{C}$ with the exact sequence

$$0 \to f_n^* \mathscr{F}' \to M_\alpha \to M''_\alpha \to 0$$

in $\mathrm{Mod}_{A^n}^{\heartsuit}$, which is split by virtue of the fact that M''_α is a free module over the commutative ring $\pi_0 A^n$.

To complete the proof of Theorem 10.6.2.1, we must show that if $\mathcal{E}$ is a Grothendieck abelian category equipped with an action of $\mathrm{QCoh}(X)^{\heartsuit}$ satisfying conditions $(*)$ and $(*')$, then $\mathcal{E}$ belongs to the essential image of the global sections functor

$$\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X}) \to \mathrm{Mod}_{\mathrm{QCoh}(X)^{\heartsuit}}(\mathsf{Groth}_{\mathrm{ab}}).$$

Equivalently, we show that $\mathcal{E}$ satisfies conditions $(*)$ and $(*')$, then the unit map $u : \mathcal{E} \to \mathrm{QCoh}(X; \Phi_X^{\mathrm{Ab}} \mathcal{E})$ is an equivalence of abelian categories.

For each $n \geqslant 0$, let $\mathscr{A}^n \in \mathrm{CAlg}(\mathrm{QCoh}(X))$ denote the direct image of the structure sheaf of U_n, so that $f_n^* \Phi_X^{\mathrm{Ab}} \mathcal{E}$ can be identified with the abelian category $\mathcal{E}^n = \mathrm{LMod}_{\pi_0 \mathscr{A}^n}(\mathcal{E})$. Let us regard $\mathscr{A}^\bullet$ as an augmented cosimplicial object of $\mathrm{CAlg}(\mathrm{QCoh}(X))$ by taking $\mathscr{A}^{-1} = \mathscr{O}_X$, so that $\mathcal{E}^{-1} = \mathcal{E}$. The proof of Proposition 10.6.1.1 then supplies an identification of $\mathrm{QCoh}(X; \Phi_X^{\mathrm{Ab}} \mathcal{E})$ with the totalization of the underlying cosemisimplicial object of $\mathcal{E}^\bullet$. Under this identification, the functor u is given by the natural map $\mathcal{E}^{-1} \to \varprojlim_{[n] \in \mathbf{\Delta}_s} \mathcal{E}^n$ is an equivalence in the ∞-category $\mathrm{Mod}_{\mathrm{QCoh}(X)^\heartsuit}(\mathsf{Groth}_{\mathrm{ab}})$. Using Corollary HA.4.2.3.3 and Proposition C.5.4.21, we see that this map is an equivalence if and only if $\mathcal{E}^{-1}$ is a limit of the diagram $\{\mathcal{E}^n\}_{[n] \in \mathbf{\Delta}_s}$ in $\widehat{\mathcal{C}\mathrm{at}}_\infty$. Let $F : \mathcal{E} = \mathcal{E}^{-1} \to \mathcal{E}^0 = \mathrm{LMod}_{\pi_0 \mathscr{A}}(\mathcal{E})$ be the functor given $E \mapsto (\pi_0 \mathscr{A}) \otimes E$. Since the augmented cosimplicial ∞-category $\mathcal{E}^\bullet$ satisfies the Beck-Chevalley condition of Corollary HA.4.7.5.3 , we are reduced to proving the following concrete assertions:

- (a) The functor F is conservative.
- (b) The functor $F : \mathcal{E} \to \mathrm{LMod}_{\pi_0 \mathscr{A}}(\mathcal{E})$ preserves totalizations of F-split cosimplicial objects.

We first prove (b). Note that $\mathcal{E}$ and $\mathrm{LMod}_{\pi_0 \mathscr{A}}(\mathcal{E})$ are ordinary categories, so that the limit of a cosimplicial object $E^\bullet$ (in either setting) can be described as the equalizer of the pair of coface maps $d^0, d^1 : E^0 \to E^1$. Consequently, to prove that condition (b) is satisfied, it will suffice to show that the functor F preserves finite limits. Since the forgetful functor $G : \mathrm{LMod}_{\pi_0 \mathscr{A}}(\mathcal{E}) \to \mathcal{E}$ is conservative and preserves finite limits, we are reduced to proving that the composite functor $G \circ F$ preserves finite limits. This follows from condition $(*)$, applied to the object $\pi_0 \mathscr{A} \in \mathrm{QCoh}(X)^\heartsuit$.

To prove (a), let $e : \mathscr{O}_X \to \mathscr{A}$ denote the unit map of the algebra $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(X))$. Since the map η is faithfully flat, the cofiber $\mathrm{cofib}(e) \in \mathrm{QCoh}(X)$ is flat. We have an exact sequence

$$0 \to \pi_0 \mathscr{O}_X \to \pi_0 \mathscr{A} \to \pi_0 \mathrm{cofib}(e) \to 0$$

in the abelian category $\mathrm{QCoh}(X)^\heartsuit$. If $\mathcal{E}$ satisfies condition $(*)$, then for every object $E \in \mathcal{E}$ the sequence

$$0 \to E \to (\pi_0 \mathscr{A}) \otimes E \to (\pi_0 \mathrm{cofib}(e)) \otimes E \to 0$$

is also exact. Consequently, if $E \in \mathcal{E}$ is annihilated by the functor F, then $E \simeq 0$. Applying this observation to the kernel and cokernel of a morphism $\alpha : X \to Y$ in $\mathcal{E}$ (and using the exactness of F established above), we deduce that if $F(\alpha)$ is an equivalence, then α is an equivalence. □

10.6.5 The Resolution Property

For geometric stacks with the resolution property (see Definition 9.3.3.2), the statement of Theorem 10.6.2.1 can be simplified:

Theorem 10.6.5.1. *Let X be a geometric stack with the resolution property. Then the global sections functor* $\mathrm{QCoh}(X; \bullet) : \mathrm{QStk}^{\mathrm{Ab}}(X) \to \mathrm{Mod}_{\mathrm{QCoh}(X)^{\heartsuit}}(\mathsf{Groth}_{\mathrm{ab}})$ *is an equivalence of ∞-categories.*

Proof. Without loss of generality, we may assume that X is 0-truncated. By virtue of Theorem **??**, it will suffice to show that for every object $\rho : \mathcal{E} \in \mathrm{Mod}_{\mathrm{QCoh}(X)^{\heartsuit}}(\mathsf{Groth}_{\mathrm{ab}})$, the unit map $u : \mathcal{E} \to \mathrm{QCoh}(X; \Phi_X^{\mathrm{Ab}} \mathcal{E})$ is an equivalence of categories. Choose a faithfully flat map $f : U \to X$, where U is affine, and set $\mathscr{A} = f_* \mathscr{O}_U$. Let $F : \mathcal{E} \to \mathrm{LMod}_{\pi_0 \mathscr{A}}(\mathcal{E})$ be a left adjoint to the forgetful functor. Arguing as in the proof of Theorem **??**, we see that ρ is an equivalence if and only if the following conditions are satisfied:

(a) The functor F is conservative.

(b) The functor $F : \mathcal{E} \to \mathrm{LMod}_{\pi_0 \mathscr{A}}(\mathcal{E})$ preserves totalizations of F-split cosimplicial objects.

We first prove (b). Note that $\mathcal{E}$ and $\mathrm{LMod}_{\pi_0 \mathscr{A}}(\mathcal{E})$ are ordinary categories, so that the limit of a cosimplicial object $E^{\bullet}$ (in either setting) can be described as the equalizer of the pair of coface maps $d^0, d^1 : E^0 \to E^1$. Consequently, to prove that condition (b) is satisfied, it will suffice to show that the functor F preserves finite limits. Since the forgetful functor $G : \mathrm{LMod}_{\pi_0 \mathscr{A}}(\mathcal{E}) \to \mathcal{E}$ is conservative and preserves finite limits, we are reduced to proving that the composite functor $G \circ F$, given by $E \mapsto (\pi_0 \mathscr{A}) \otimes E$, preserves finite limits.

We now proceed as in the proof of Proposition 10.4.5.2. Let $\mathcal{C} = \mathrm{Vect}(X) \times_{\mathrm{QCoh}(X)} \mathrm{QCoh}(X)_{/\mathscr{A}}$ be the ∞-category of vector bundles $\mathscr{F}$ on X equipped with a map $\mathscr{F} \to \mathscr{A}$. According to Lemma 9.3.4.13, the ∞-category $\mathcal{C}$ is filtered. The proof of Proposition 9.3.4.11 shows that the canonical map $\varinjlim_{\mathscr{F} \in \mathcal{C}} \mathscr{F} \to \mathscr{A}$ is an equivalence. It follows that we can write $G \circ F$ as the colimit of a filtered diagram of functors of the form $E \mapsto (\pi_0 \mathscr{F}) \otimes E$ for $\mathscr{F} \in \mathrm{Vect}(X)$. Each of these functors is left exact, since it admits a left adjoint (given by the construction $E \mapsto (\pi_0 \mathscr{F}^{\vee}) \otimes E$).

We now prove (a). Since F is an exact functor, it will suffice to show that any object $E \in \mathcal{E}$ satisfying $F(E) \simeq 0$ must itself be a zero object of $\mathcal{E}$. Let us regard the structure sheaf $\mathscr{O}_X$ as an object of $\mathcal{C}$ (by equipping it with the unit map $e : \mathscr{O}_X \to \mathscr{A}$), so that $\mathcal{C}_{\mathscr{O}_X /}$ is also a filtered ∞-category and the projection map $\mathcal{C}_{\mathscr{O}_X /} \to \mathcal{C}$ is left cofinal. It follows that e can be written as the colimit of a filtered diagram of maps $e_\alpha : \mathscr{O}_X \to \mathscr{F}_\alpha$ in $\mathrm{Vect}(X)$, so that the unit map $u_E : E \to (G \circ F)(E)$ can be computed as a filtered colimit of maps $u_\alpha : E \to (\pi_0 \mathscr{F}_\alpha) \otimes E$. Note that $f^*(e_\alpha)$ admits a left homotopy inverse, so that

$f^* \operatorname{cofib}(e_\alpha)$ is a direct summand of $f^* \mathscr{F}_\alpha$ and therefore belongs to $\operatorname{Vect}(U)$. It follows that the dual map $e_\alpha^\vee : \mathscr{F}_\alpha^\vee \to \mathscr{O}_X$ is an epimorphism on π_0. Unwinding the definitions, we see that the composite map

$$(\pi_0 \mathscr{F}_\alpha^\vee) \otimes \ker(u_\alpha) \xrightarrow{e_\alpha^\vee \otimes \mathrm{id}} \ker(u_\alpha) \to E$$

vanishes. Since $e_\alpha^\vee \otimes \mathrm{id}$ is an epimorphism, it follows that the map $\ker(u_\alpha) \to E$ vanishes, so that u_α is a monomorphism. Because filtered colimits in $\mathcal{E}$ are left exact, we conclude that the unit map $u_E : E \to (G \circ F)(E)$ is also a monomorphism. Since $(G \circ F)(E)$ vanishes by assumption, we conclude that $E \simeq 0$ as desired. □

We also have an abelian analogue of Theorem 10.4.5.5:

Theorem 10.6.5.2. *Let X be a 0-truncated geometric stack with the resolution property. Then the global sections functor $\mathcal{C} \mapsto \operatorname{QCoh}(X; \mathcal{C})$ determines a fully faithful embedding*

$$\operatorname{QCoh}(X; \bullet) : \operatorname{QStk}^{\mathrm{Ab}}(X) \to \operatorname{Mod}_{\operatorname{Vect}(X)}(\widehat{\mathcal{C}\mathrm{at}})$$

whose essential image is spanned by those $\operatorname{Vect}(X)$-modules $\mathcal{E}$ satisfying the following conditions:

(i) The category $\mathcal{E}$ is Grothendieck abelian.

(ii) If $0 \in \operatorname{Vect}(X)$ is a zero object, then $0 \otimes E$ is a zero object of $\mathcal{E}$ for each $E \in \mathcal{E}$.

(iii) For every diagram $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ in $\operatorname{Vect}(X)$ which is exact in the abelian category $\operatorname{QCoh}(X)^\heartsuit$ and every object $E \in \mathcal{E}$, the sequence

$$\mathscr{F}' \otimes E \to \mathscr{F} \otimes E \to \mathscr{F}'' \otimes E \to 0$$

is exact in the abelian category $\mathcal{E}$.

*Proof of Theorem **??**.* Let $\mathcal{C} = \tau_{\leqslant 0} \mathcal{P}_\Sigma(\operatorname{Vect}(X)) = \operatorname{Fun}^\pi(\operatorname{Vect}(X)^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ be the abelian category of functors $\operatorname{Vect}(X)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ which preserve finite products. Let us abuse notation by identifying $\operatorname{Vect}(X)$ with its essential image under the Yoneda embedding $\operatorname{Vect}(X) \hookrightarrow \mathcal{C}$. Since X has the resolution property, the Gabriel-Popescu theorem (Theorem C.2.2.1) implies that the inclusion $\operatorname{Vect}(X) \hookrightarrow \operatorname{QCoh}(X)^\heartsuit$ extends to an exact functor $F : \mathcal{C} \to \operatorname{QCoh}(X)^\heartsuit$ which admits a fully faithful right adjoint G. Using Proposition 9.3.7.7, we see that G induces an equivalence from $\operatorname{QCoh}(X)^\heartsuit$ to the full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ spanned by those functors $\lambda : \operatorname{Vect}(X)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ which preserve finite products and satisfy the following additional condition:

$(\star)$ For every diagram $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ in $\operatorname{Vect}(X)$ which is exact in the abelian category $\operatorname{QCoh}(X)^\heartsuit$, the diagram $0 \to \lambda(\mathscr{F}'') \to \lambda(\mathscr{F}) \to \lambda(\mathscr{F}')$ is exact (in the category of abelian groups).

Since the tensor product on $\mathrm{Vect}(X)$ preserves finite direct sums in each variable, it admits an essentially unique extension to a symmetric monoidal structure on $\mathcal{C}$ for which the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Moreover, composition with the inclusion $\mathrm{Vect}(X) \hookrightarrow \mathcal{C}$ determines a fully faithful functor $\rho : \mathrm{Mod}_{\mathcal{C}}(\mathsf{Groth}_{\mathrm{ab}}) \to \mathrm{Mod}_{\mathrm{Vect}(X)}(\widehat{\mathcal{C}\mathrm{at}})$ whose essential image consists of those objects which satisfy conditions (i), (ii), and the following weaker version of (iii'):

(iii') For every diagram $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ in $\mathrm{Vect}(X)$ which is split exact in the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$ and every object $E \in \mathcal{E}$, the sequence

$$\mathscr{F}' \otimes E \to \mathscr{F} \otimes E \to \mathscr{F}'' \otimes E \to 0$$

is exact in the abelian category $\mathcal{E}$.

Let $\mathrm{Mod}^0_{\mathcal{C}}(\mathsf{Groth}_{\mathrm{ab}})$ denote the full subcategory of $\mathrm{Mod}_{\mathcal{C}}(\mathsf{Groth}_{\mathrm{ab}})$ spanned by those $\mathcal{C}$-modules $\mathcal{E}$ for which $\rho(\mathcal{E})$ satisfies condition (iii). Using Theorem 10.6.5.1, we see that Theorem 10.6.5.2 is equivalent to the statement that F induces an equivalence of ∞-categories $\mathrm{Mod}_{\mathrm{QCoh}(X)^{\heartsuit}}(\mathsf{Groth}_{\mathrm{ab}}) \to \mathrm{Mod}^0_{\mathcal{C}}(\mathsf{Groth}_{\mathrm{ab}})$. We are therefore reduced to showing that a monoidal functor $\mathcal{C} \to \mathrm{LFun}(\mathcal{E}, \mathcal{E})$ factors through F if and only if the induced action of $\mathrm{Vect}(X)$ on $\mathcal{E}$ satisfies condition (iii), which follows immediately from $(\star)$. □

10.6.6 Comparison with Prestable Quasi-Coherent Stacks

We now study the relationship between abelian and prestable quasi-coherent stacks in greater detail. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be an arbitrary functor. Then the construction $\mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$ determines a functor $\mathrm{QStk}^{\mathrm{PSt}}(X) \to \mathrm{QStk}^{\mathrm{Ab}}(X)$ (see Construction 10.1.2.8). This functor carries left exact morphisms to left exact morphisms, and therefore restricts to a functor $\mathrm{QStk}^{\mathrm{PSt}}(X)^{\mathrm{lex}} \to \mathrm{QStk}^{\mathrm{Ab}}(X)^{\mathrm{lex}}$; here $\mathrm{QStk}^{\mathrm{PSt}}(X)^{\mathrm{lex}}$ is the subcategory of $\mathrm{QStk}^{\mathrm{PSt}}(X)^{\mathrm{lex}}$ whose morphisms are left exact and $\mathrm{QStk}^{\mathrm{Ab}}(X)^{\mathrm{lex}}$ is defined similarly. We now prove a relative version of Proposition C.5.5.20:

Theorem 10.6.6.1. *Let X be a spectral Deligne-Mumford stack. Then Construction 10.1.2.8 determines a functor*

$$\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}} \to \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}} \qquad \mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$$

which admits a fully faithful left adjoint $\check{\mathcal{D}}$. The essential image of $\check{\mathcal{D}}$ consists of those prestable quasi-coherent stacks on X which are anticomplete and 0-complicial.

The proof of Theorem 10.6.6.1 will proceed by reduction to the case where X is affine, in which case the desired result is a consequence of the following more general assertion:

Lemma 10.6.6.2. *Let R be a connective $\mathbb{E}_\infty$-ring and let n be a nonnegative integer. Then:*

(a) *Suppose that $\mathcal{A}$ is a Grothendieck abelian $(n+1)$-category (Definition C.5.4.1) equipped with an action of R. Then there exists an R-linear Grothendieck prestable ∞-category $\mathcal{C}$ and an R-linear equivalence $\mathcal{A} \simeq \tau_{\leqslant n}\mathcal{C}$, where $\mathcal{C}$ is anticomplete and n-complicial.*

(b) *Let $\mathcal{C}$ and $\mathcal{D}$ be R-linear Grothendieck prestable ∞-categories, let $\mathrm{LFun}^{\mathrm{lex}}_R(\mathcal{C},\mathcal{D})$ denote the ∞-category of left exact R-linear functors from $\mathcal{C}$ to $\mathcal{D}$, and define the ∞-category $\mathrm{LFun}^{\mathrm{lex}}_R(\tau_{\leqslant n}\mathcal{C},\tau_{\leqslant n}\mathcal{D})$ similarly. If $\mathcal{C}$ is anticomplete and n-complicial, then the restriction map $\mathrm{LFun}^{\mathrm{lex}}_R(\mathcal{C},\mathcal{D}) \to \mathrm{LFun}^{\mathrm{lex}}_R(\tau_{\leqslant n}\mathcal{C},\tau_{\leqslant n}\mathcal{D})$ is an equivalence of ∞-categories.*

Proof. We first prove (a). Let $\mathcal{A}$ be a Grothendieck abelian $(n+1)$-category. Using Proposition C.5.4.5, we deduce that there exists a Grothendieck prestable ∞-category $\mathcal{C}$ which is anticomplete and n-complicial and an equivalence $\mathcal{A} \simeq \tau_{\leqslant n}\mathcal{C}$. It then follows from Proposition C.5.5.19 that the restriction functor $\rho : \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C},\mathcal{C}) \to \mathrm{LFun}^{\mathrm{lex}}(\mathcal{A},\mathcal{A})$ is an equivalence of monoidal ∞-categories. An action of R on $\mathcal{A}$ can be identified with a monoidal functor $\mathrm{Mod}^{\mathrm{ff}}_R \to \mathrm{LFun}^{\mathrm{lex}}(\mathcal{A},\mathcal{A})$ (see Definition D.1.1.1). Composing with a homotopy inverse to ρ, we obtain a compatible action of R on the Grothendieck prestable ∞-category $\mathcal{C}$.

We now prove (b). Let $F : \mathrm{LinCat}^{\mathrm{Add}}_R = \mathrm{Mod}_{\mathrm{Mod}^{\mathrm{cn}}_R}(\mathcal{P}\mathrm{r}^{\mathrm{Add}}) \to \mathcal{P}\mathrm{r}^{\mathrm{Add}}$ be the forgetful functor. Then F admits a right adjoint $G : \mathcal{P}\mathrm{r}^{\mathrm{Add}} \to \mathrm{LinCat}^{\mathrm{Add}}_R$, given by the construction

$$F(\mathcal{C}) = \mathrm{LFun}(\mathrm{Mod}^{\mathrm{cn}}_R, \mathcal{C}) \simeq \mathrm{LMod}_R(\mathcal{C}),$$

where the equivalence is given by Theorem HA.4.8.4.1 . The functor F is conservative (by construction) and preserves small limits (Corollary HA.4.2.3.3). Applying Theorem HA.4.7.3.5 , we see that the adjunction $\mathrm{LinCat}^{\mathrm{Add}}_R \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{P}\mathrm{r}^{\mathrm{Add}}$ is comonadic: that is, it exhibits $\mathrm{LinCat}^{\mathrm{Add}}_R$ as the ∞-category of comodules over the comonad $U = F \circ G : \mathcal{P}\mathrm{r}^{\mathrm{Add}} \to \mathcal{P}\mathrm{r}^{\mathrm{Add}}$. Note that an additive R-linear ∞-category $\mathcal{C}$ is prestable if and only if its image under F is prestable and that an R-linear functor is left exact if and only if its image under F is left exact. Consequently, F and G restrict to a comonadic adjunction

$$\mathrm{LinCat}^{\mathrm{PSt,lex}}_R \underset{G_0}{\overset{F_0}{\rightleftarrows}} \mathsf{Groth}^{\mathrm{lex}}_{\infty}.$$

. In particular, every object $\mathcal{D} \in \mathrm{LinCat}^{\mathrm{PSt,lex}}_R$ can be written as a limit of objects which belong to the essential image of the functor G_0 (see Proposition HA.4.7.3.14). Consequently, to prove (b), we may assume without loss of generality that $\mathcal{D} = G_0(\mathcal{D}_0)$ for some Grothendieck prestable ∞-category $\mathcal{D}_0$. In this case, we are reduced to proving that the restriction map $\mathrm{LFun}^{\mathrm{lex}}(\mathcal{C},\mathcal{D}_0) \to \mathrm{LFun}^{\mathrm{lex}}(\tau_{\leqslant n}\mathcal{C},\tau_{\leqslant n}\mathcal{D}_0)$ is an equivalence of ∞-categories, which follows from Proposition C.5.3.9. □

Proof of Theorem 10.6.6.1. Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$. For each object $U \in \mathcal{X}$, let X_U denote the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathscr{O}_{\mathsf{X}}|_U)$, and let $\mathcal{E}_U$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}_U)^{\mathrm{lex}}$ spanned by those prestable quasi-coherent stacks on X_U which are anticomplete and 0-complicial. It follows from Corollary D.5.7.3 that the construction $U \mapsto \mathcal{E}_U$ is (contravariantly) functorial in U. Moreover, for each $U \in \mathcal{X}$, the construction $\mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$ determines a functor $\theta_U : \mathcal{E}_U \to \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X}_U)^{\mathrm{lex}}$, which also depends functorially on U.

Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the full subcategory spanned by those objects $U \in \mathcal{X}$ for which the functor θ_U is an equivalence of ∞-categories. Since the condition of being anticomplete and 0-complicial can be tested locally with respect to the étale topology (Corollary D.5.7.3), the construction $U \mapsto \theta_U$ carries colimits in $\mathcal{X}$ to limits in the ∞-category $\mathrm{Fun}(\Delta^1, \widehat{\mathcal{C}\mathrm{at}}_\infty)$. It follows that $\mathcal{X}_0$ is closed under small colimits. Moreover, Lemma 10.6.6.2 shows that θ_U is an equivalence whenever U is affine. Applying Proposition 1.4.7.9, we deduce that $\mathcal{X}_0 = \mathcal{X}$. In particular, $\mathcal{X}_0$ contains a final object $\mathbf{1} \in \mathcal{X}$.

Let $\check{\mathcal{D}} : \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}} \to \mathcal{E}_{\mathbf{1}} \subseteq \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ be a homotopy inverse to the equivalence $\theta_{\mathbf{1}}$. To complete the proof, it will suffice to show that the identification $\mathrm{id}_{\mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}}} \simeq \check{\mathcal{D}}(\bullet)^{\heartsuit}$ is the unit of an adjunction. In other words, it will suffice to show that for every abelian quasi-coherent stack $\mathcal{A}$ on X and every prestable quasi-coherent stack $\mathcal{C}$ on X, the canonical map

$$\rho : \mathrm{Map}_{\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}}(\check{\mathcal{D}}(\mathcal{A}), \mathcal{C}) \to \mathrm{Map}_{\mathrm{QStk}^{\mathrm{Ab}}}(\mathsf{X})^{\mathrm{lex}}(\mathcal{A}, \mathcal{C}^{\heartsuit})$$

is a homotopy equivalence. For every object $U \in \mathcal{X}$, we have an analogous map

$$\rho_U : \mathrm{Map}_{\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}_U)^{\mathrm{lex}}}(\check{\mathcal{D}}(\mathcal{A})|_{\mathsf{X}_U}, \mathcal{C}|_{\mathsf{X}_U}) \to \mathrm{Map}_{\mathrm{QStk}^{\mathrm{Ab}}}(\mathsf{X}_U)^{\mathrm{lex}}(\mathcal{A}|_{\mathsf{X}_U}, \mathcal{C}^{\heartsuit}|_{\mathsf{X}_U}).$$

Let $\mathcal{X}_1 \subseteq \mathcal{X}$ be the full subcategory spanned by those objects for which ρ_U is a homotopy equivalence. Since the construction $U \mapsto \rho_U$ carries colimits in $\mathcal{X}$ to limits in the ∞-category $\mathrm{Fun}(\Delta^1, \widehat{\mathcal{S}})$, the full subcategory $\mathcal{X}_1$ is closed under small colimits. Lemma 10.6.6.2 shows that $\mathcal{X}_1$ contains all affine objects of $\mathcal{X}$, so that $\mathcal{X}_1 = \mathcal{X}$ by virtue of Proposition 1.4.7.9. In particular, $\mathcal{X}_1$ contains a final object of $\mathcal{X}$, which shows that ρ is a homotopy equivalence as desired. □

The functor $\mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}} \to \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ can be regarded as a global version of the construction $\mathcal{A} \mapsto \check{\mathcal{D}}(\mathcal{A})$ studied in §C.5.8. We now consider analogous globalizations of the related constructions $\mathcal{A} \mapsto \mathcal{D}(\mathcal{A})$ and $\mathcal{A} \mapsto \widehat{\mathcal{D}}(\mathcal{A})$:

Corollary 10.6.6.3. *Let X be a spectral Deligne-Mumford stack and let $\mathrm{QStk}^{\mathrm{sep}}(\mathsf{X})^{\mathrm{lex}}$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ spanned by the separated prestable quasi-coherent stacks on X. Then Construction 10.1.2.8 determines a functor*

$$\mathrm{QStk}^{\mathrm{sep}}(\mathsf{X})^{\mathrm{lex}} \to \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}} \qquad \mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$$

which admits a fully faithful left adjoint $\mathcal{D}$. The essential image of $\mathcal{D}$ consists of those prestable quasi-coherent stacks on X which are separated and 0-complicial.

Proof. Let $\check{\mathcal{D}} : \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}} \to \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ be as in Theorem 10.6.6.1. Note that the inclusion $\mathrm{QStk}^{\mathrm{sep}}(\mathsf{X})^{\mathrm{lex}} \hookrightarrow \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ admits a left adjoint, given by the construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{sep}}$ of Proposition 10.3.1.11. It follows that the construction

$$(\mathcal{A} \in \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}}) \mapsto (\check{\mathcal{D}}(\mathcal{A})^{\mathrm{sep}} \in \mathrm{QStk}^{\mathrm{sep}}(\mathsf{X})^{\mathrm{lex}})$$

is a left adjoint to the functor $\mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$. To show that this functor is fully faithful, we must show that for every abelian quasi-coherent stack $\mathcal{A} \in \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}}$, the composite map

$$\mathcal{A} \xrightarrow{u} \check{\mathcal{D}}(\mathcal{A})^{\heartsuit} \xrightarrow{v} (\check{\mathcal{D}}(\mathcal{A})^{\mathrm{sep}})^{\heartsuit}$$

is an equivalence of abelian quasi-coherent stacks. Here u is the unit map for the adjunction of Theorem 10.6.6.1, which is an equivalence since the functor $\check{\mathcal{D}}$ is fully faithful. The map u' is also an, equivalence because the canonical map $\mathcal{C} \to \mathcal{C}^{\mathrm{sep}}$ induces an equivalence on hearts for any $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$.

Note that if $\mathcal{A}$ is an abelian quasi-coherent stack on X, then the prestable quasi-coherent stack $\check{\mathcal{D}}(\mathcal{A})$ is 0-complicial (Theorem 10.6.6.1), so the separated quotient $\check{\mathcal{D}}(\mathcal{A})^{\mathrm{sep}}$ is separated and 0-complicial (Proposition C.5.3.3). Conversely, suppose that $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ is separated and 0-complicial. Then the counit map $v : \check{\mathcal{D}}(\mathcal{C}^{\heartsuit})^{\mathrm{sep}} \to \mathcal{C}$ is left exact and induces an equivalence on hearts. Since the domain and codomain of v are separated and 0-complicial, it follows from Proposition C.5.4.5 that v is an equivalence, so that $\mathcal{C}$ belongs to the essential image of the functor $\mathcal{A} \mapsto \check{\mathcal{D}}(\mathcal{A})^{\mathrm{sep}}$. □

Corollary 10.6.6.4. *Let X be a spectral Deligne-Mumford stack and let $\mathrm{QStk}^{\mathrm{comp}}(\mathsf{X})^{\mathrm{lex}}$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ spanned by the complete prestable quasi-coherent stacks on X. Then Construction 10.1.2.8 determines a functor*

$$\mathrm{QStk}^{\mathrm{comp}}(\mathsf{X})^{\mathrm{lex}} \to \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}} \qquad \mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$$

which admits a fully faithful left adjoint $\widehat{\mathcal{D}}$. The essential image of $\widehat{\mathcal{D}}$ consists of those prestable quasi-coherent stacks on X which are complete and weakly 0-complicial.

Proof. Let $\check{\mathcal{D}} : \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}} \to \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ be as in Theorem 10.6.6.1. Note that the inclusion $\mathrm{QStk}^{\mathrm{sep}}(\mathsf{X})^{\mathrm{lex}} \hookrightarrow \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ admits a left adjoint, given by the construction $\mathcal{C} \mapsto \widehat{\mathcal{C}}$ of Proposition 10.3.1.11. Composing these left adjoints, we obtain a functor $\widehat{\mathcal{D}} : \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}} \to \mathrm{QStk}^{\mathrm{comp}}(\mathsf{X})^{\mathrm{lex}}$ which is left adjoint to the formation of hearts. To show that this functor is fully faithful, we must show that for every abelian quasi-coherent stack $\mathcal{A} \in \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}}$, the composite map $\mathcal{A} \xrightarrow{u} \check{\mathcal{D}}(\mathcal{A})^{\heartsuit} \xrightarrow{u'} \widehat{\mathcal{D}}(\mathcal{A})^{\heartsuit}$ is an equivalence of abelian quasi-coherent stacks. Here u is the unit map for the adjunction of Theorem 10.6.6.1, which is an equivalence since the functor $\check{\mathcal{D}}$ is fully faithful, and the map u' is evidently an equivalence as well.

Note that if $\mathcal{A}$ is an abelian quasi-coherent stack on X, then the prestable quasi-coherent stack $\check{\mathcal{D}}(\mathcal{A})$ is weakly 0-complicial (Theorem 10.6.6.1), so its completion $\hat{\mathcal{D}}(\mathcal{A})$ is complete and weakly 0-complicial (Remark C.5.5.14). Conversely, suppose that $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ is complete and weakly 0-complicial. Then the counit map $v : \widehat{\mathcal{D}}(\mathcal{C}^{\heartsuit}) \to \mathcal{C}$ is left exact and induces an equivalence on hearts. Since the domain and codomain of v are complete and weakly 0-complicial, it follows from Proposition C.5.9.3 that v is an equivalence, so that $\mathcal{C}$ belongs to the essential image of the functor $\widehat{\mathcal{D}}$. □

Warning 10.6.6.5. The constructions $\check{\mathcal{D}}, \mathcal{D}, \widehat{\mathcal{D}} : \mathrm{QStk}^{\mathrm{Ab}}(\mathsf{X})^{\mathrm{lex}} \to \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})^{\mathrm{lex}}$ of Theorem 10.6.6.1 and Corollaries 10.6.6.3 and 10.6.6.4 are in general not compatible with base change. If $\mathcal{A}$ is an abelian quasi-coherent stack on X and $f : \mathsf{Y} \to \mathsf{X}$ is a morphism of spectral Deligne-Mumford stacks, then we have canonical maps

$$\check{v} : \check{\mathcal{D}}(f^* \mathcal{A}) \to f^* \check{\mathcal{D}}(\mathcal{A}) \qquad v : \mathcal{D}(f^* \mathcal{A}) \to f^* \mathcal{D}(\mathcal{A}) \qquad \hat{v} : \widehat{\mathcal{D}}(f^* \mathcal{A}) \to f^* \widehat{\mathcal{D}}(\mathcal{A}).$$

in $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y})^{\mathrm{lex}}$ which induce equivalences after passing to hearts. Using Remark 10.3.4.6, we obtain the following:

- (i) The morphism $\check{v}$ is an equivalence if and only if $f^* \check{\mathcal{D}}(\mathcal{A})$ is anticomplete and 0-complicial. This condition is satisfied whenever f is étale (or, more generally, when f is flat and the diagonal map $\mathsf{Y} \to \mathsf{Y} \times_{\mathsf{X}} \mathsf{Y}$ has finite Tor-amplitude).
- (ii) The morphism v is an equivalence if and only if $f^* \mathcal{D}(\mathcal{A})$ is separated and 0-complicial. This condition is satisfied whenever f is flat.
- (iii) The morphism $\hat{v}$ is an equivalence if and only if $f^* \widehat{\mathcal{D}}(\mathcal{A})$ is complete and and weakly 0-complicial. This condition is satisfied whenever f is flat.

Because the constructions of Theorem 10.6.6.1 and Corollaries 10.6.6.3 and 10.6.6.4 are not stable under arbitrary base change, they do not extend to quasi-coherent stacks on an arbitrary functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. However, Corollary **??** has an analogue for quasi-geometric stacks:

Corollary 10.6.6.6. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a quasi-geometric stack and let $\mathrm{QStk}^{\mathrm{comp}}(X)^{\mathrm{lex}}$ be defined as in Corollary 10.6.6.4. Then Construction 10.1.2.8 determines a functor*

$$\mathrm{QStk}^{\mathrm{comp}}(X)^{\mathrm{lex}} \to \mathrm{QStk}^{\mathrm{Ab}}(X)^{\mathrm{lex}} \qquad \mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$$

which admits a fully faithful left adjoint $\widehat{\mathcal{D}}$. The essential image of $\widehat{\mathcal{D}}$ consists of those prestable quasi-coherent stacks on X which are complete and weakly 0-complicial.

Proof. We proceed as in the proof of Theorem 10.6.6.1. For every quasi-geometric stack Y, let $\mathrm{QStk}^{(0)}(Y)^{\mathrm{lex}}$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{comp}}(Y)^{\mathrm{lex}}$ spanned by those prestable quasi-coherent stacks on Y which are complete and weakly 0-complicial. Note that the construction $Y \mapsto \mathrm{QStk}^{(0)}(Y)^{\mathrm{lex}}$ is contravariantly functorial for *flat* maps between quasi-geometric stacks (Remark 10.4.6.4).

We first prove that the construction $\mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$ induces an equivalence of ∞-categories $\theta_X : \mathrm{QStk}^{(0)}(X)^{\mathrm{lex}} \to \mathrm{QStk}^{\mathrm{Ab}}(X)^{\mathrm{lex}}$. Choose a faithfully flat $u : X_0 \to X$, where $X_0 \simeq \operatorname{Spec} A$ is affine, and let $X_\bullet$ denote the Čech nerve of u. Then the construction $[n] \mapsto \theta_{X_n}$ determines a cosemisimplicial object of $\mathrm{Fun}(\Delta^1, \widehat{\mathcal{C}\mathrm{at}_\infty})$. Since each X_n is representable by a (quasi-affine) spectral Deligne-Mumford stack, Corollary 10.6.6.4 guarantees that each of the functors θ_{X_n} is an equivalence of ∞-categories. It follows that $\theta_X \simeq \mathrm{Tot}(\theta_{X_\bullet})$ is also an equivalence of ∞-categories. In particular, θ_X admits a homotopy inverse $\widehat{\mathcal{D}} : \mathrm{QStk}^{\mathrm{Ab}}(X)^{\mathrm{lex}} \to \mathrm{QStk}^{(0)}(X)^{\mathrm{lex}} \subseteq \mathrm{QStk}^{\mathrm{comp}}(X)^{\mathrm{lex}}$. To complete the proof, it will suffice to show that the natural map $\mathrm{id}_{\mathrm{QStk}^{\mathrm{Ab}}(X)^{\mathrm{lex}}} \to \widehat{\mathcal{D}}(\bullet)^{\heartsuit}$ is the unit of an adjunction. In other words, it will suffice to show that if $\mathcal{A}$ is an abelian quasi-coherent stack on X and $\mathcal{C}$ is complete prestable quasi-coherent stack on X, then the canonical map

$$\rho : \mathrm{Map}_{\mathrm{QStk}^{\mathrm{PSt}}(X)^{\mathrm{lex}}}(\widehat{\mathcal{D}}(\mathcal{A}), \mathcal{C}) \to \mathrm{Map}_{\mathrm{QStk}^{\mathrm{Ab}}(X)^{\mathrm{lex}}}(\mathcal{A}, \mathcal{C}^{\heartsuit})$$

is a homotopy equivalence. Note that ρ can be realized as the totalization of a map of cosimplicial spaces

$$\rho^\bullet : \mathrm{Map}_{\mathrm{QStk}^{\mathrm{PSt}}(X_\bullet)^{\mathrm{lex}}}(\widehat{\mathcal{D}}(\mathcal{A})|_{X_\bullet}, \mathcal{C}\,|_{X_\bullet}) \to \mathrm{Map}_{\mathrm{QStk}^{\mathrm{Ab}}(X)^{\mathrm{lex}}}(\mathcal{A}\,|_{X_\bullet}, \mathcal{C}^{\heartsuit}\,|_{X_\bullet}).$$

It will therefore suffice to show that each of the maps ρ^n is a homotopy equivalence, which follows from Corollary 10.6.6.4. □

Chapter 11

Smooth and Proper Linear ∞-Categories

Let X be a Noetherian scheme. A theorem of Gabriel ([74]) implies that X can be recovered, up to canonical isomorphism, from the abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X (this result has been extended to non-Noetherian schemes by Rosenberg). In particular, any geometric condition on X can, at least in principle, be reformulated as a condition on the category $\mathrm{Coh}(X)$.

Example 11.0.0.1. Let X be a quasi-projective variety over a field κ. Then X is projective if and only if, for every pair of coherent sheaves $\mathscr{F}, \mathscr{G} \in \mathrm{Coh}(X)$, the vector space $\mathrm{Hom}_{\mathrm{Coh}(X)}(\mathscr{F}, \mathscr{G})$ has finite dimension over κ. The "only if" direction is a fundamental finiteness theorem of Serre. The converse follows from the observation that if X is not projective, then there exists a closed 1-dimensional subvariety $C \subseteq X$ which is not projective, so that the vector space $\mathrm{Hom}_{\mathrm{Coh}(X)}(\mathscr{O}_X, \mathscr{O}_C)$ is isomorphic to the ring of functions on C (and is therefore infinite-dimensional as a vector space over κ).

By a slight abuse of notation, we can regard every scheme X as a spectral algebraic space and consider the stable ∞-category $\mathrm{QCoh}(X)$ introduced in Chapter 2. Let us assume for simplicity that X is quasi-compact and separated, so that $\mathrm{QCoh}(X)$ is the derived category of usual abelian category $\mathrm{QCoh}(X)^{\heartsuit}$ of quasi-coherent sheaves on X (Corollary 10.3.4.13). It follows from Theorem 9.6.0.1 that we can recover X (up to canonical isomorphism) from the ∞-category $\mathrm{QCoh}(X)$, together with its symmetric monoidal structure. However, the symmetric monoidal structure is now essential:

Example 11.0.0.2. Let X be an abelian variety dimension > 0 over a field κ and let $X^{\vee}$ be the dual abelian variety. Then the Fourier-Mukai transform (see [158]) determines an equivalence of ∞-categories $\alpha : \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X^{\vee})$ which is not induced by an isomorphism of schemes $X \simeq X^{\vee}$ (for example, α is not t-exact).

In general, the existence of an equivalence of stable ∞-categories $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(Y)$ need not imply that X and Y are isomorphic. However, it *does* imply that X and Y have many common geometric features. For example, if X and Y are quasi-projective varieties over a field κ and there exists a κ-linear equivalence $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(Y)$, then X is projective if and only if Y is projective. This is a consequence of the following slight variant of Example 11.0.0.1:

Example 11.0.0.3. Let X be a quasi-projective variety over a field κ. Then X is projective if and only if, for every pair of compact objects $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(X)$, the mapping object $\underline{\mathrm{Map}}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{G}) \in \mathrm{Mod}_\kappa$ is perfect (see Notation D.7.1.1).

Example illustrates one of the basic insights of (homological) *non-commutative geometry*: important geometric features of an algebro-geometric object X can be reformulated as conditions on the stable ∞-category $\mathrm{QCoh}(X)$. Such reformulations allow us to apply geometric ideas in the study of stable ∞-categories which do not arise (directly) from algebraic geometry. Though we will not delve deeply into the theory of non-commutative geometry in this book, we devote this chapter to investigating some of the simplest examples of this paradigm.

We begin in §11.1 by introducing the notion of a *proper R-linear ∞-category*, where R is an arbitrary $\mathbb{E}_\infty$-ring (or every $\mathbb{E}_2$-ring; see Definition 11.1.0.1). The definition is motivated by Example 11.0.0.3: we say that $\mathcal{C}$ is proper if it is compactly generated and, for every pair of compact objects $C, D \in \mathcal{C}$, the R-module $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ is perfect. Our main goal is to establish the following more precise version of Example 11.0.0.3: if R is connective and X is a spectral algebraic space which is quasi-compact, separated, and locally almost of finite presentation over R, then $\mathrm{QCoh}(\mathsf{X})$ is a proper R-linear ∞-category if and only if X is proper and of finite Tor-amplitude over R (Theorem 11.1.4.1).

Another essential notion from classical algebraic geometry is *smoothness*. Recall that a map of schemes $f : X \to Y$ is said to be *smooth* if it is locally of finite presentation and satisfies the "infinitesimal lifting property":

$(*)$ If $\phi : B \to \overline{B}$ is a surjective morphism of commutative rings whose kernel $\ker(\phi)$ is a nilpotent ideal of B, then every lifting problem

$$\begin{array}{ccc} \operatorname{Spec} \overline{B} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} B & \longrightarrow & Y \end{array}$$

admits a solution.

In §11.2, we will study some analogues of condition $(*)$ in the setting of spectral algebraic geometry. Here the situation is more complicated: there are (at least) two useful notions of smooth morphism, depending on whether one requires condition $(*)$ to hold only for homomorphisms ϕ between ordinary commutative rings (and demand that f be flat), or require the analogue of $(*)$ for more general $\mathbb{E}_\infty$-rings (which is generally incompatible with the requirement that f be flat). These notions are equivalent in characteristic zero but otherwise distinct; we will refer to them as *fiber smoothness* and *differential smoothness*, respectively.

Let R be a connective $\mathbb{E}_\infty$-ring and let $q : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ be a morphism of spectral algebraic spaces. In §11.3, we show that (under some mild additional hypotheses) the condition that q is fiber smooth depends only on the structure of $\mathrm{QCoh}(\mathsf{X})$ as an R-linear ∞-category. More precisely, we introduce the notion of a *smooth R-linear ∞-category* (Definition 11.3.1.1) and show that $\mathrm{QCoh}(\mathsf{X})$ is smooth if and only if q is fiber smooth (Theorem 11.3.6.1).

The hypotheses of smoothness and properness for R-linear ∞-categories are very powerful when used in conjunction with one another. In §11.4, we will show that if X is a quasi-compact, quasi-separated spectral algebraic space over R, then $\mathrm{QCoh}(\mathsf{X})$ is smooth and proper if and only if the structure map $q : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ is proper and fiber smooth (Theorem 11.4.2.1). In particular, the requirement that $\mathrm{QCoh}(\mathsf{X})$ is smooth and proper guarantees that X is almost of finite presentation over R.

One way to guarantee that a (compactly generated) R-linear ∞-category $\mathcal{C}$ be smooth and proper is to assume that it is *invertible*: that is, that there exists another R-linear ∞-category $\mathcal{C}^{-1}$ such that $\mathcal{C} \otimes_R \mathcal{C}^{-1} \simeq \mathrm{Mod}_R$. The collection of all equivalences classes of (compactly generated) invertible R-linear ∞-categories can be organized into an abelian group $\mathrm{Br}^\dagger(R)$, which we will refer to as the *extended Brauer group* of R. In §11.5, we will investigate the structure of invertible R-linear ∞-categories and the extended Brauer group $\mathrm{Br}^\dagger(R)$. In particular, we will show that every invertible R-linear ∞-category is locally equivalent to Mod_R with respect to the étale topology (Theorem 11.5.5.1) and that $\mathrm{Br}^\dagger(R)$ depends only on the commutative ring $\pi_0 R$ (Proposition 11.5.5.6).

Remark 11.0.0.4. The notions of smooth and proper R-linear ∞-categories were introduced by Kontsevich (at least when R is an ordinary commutative ring), using the language of differential graded categories. They have subsequently been studied by a number of authors: see, for example, [31], [207], [208], [211], [123], [212], and [163]. See also [2] for a discussion in the setting where R is a structured ring spectrum, which has considerable overlap with the material of this chapter.

Contents

11.1 Properness for Linear ∞-Categories and Quasi-Coherent Stacks

Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category (Definition D.1.4.1). For every pair of objects $C, D \in \mathcal{C}$, we let $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ denote the R-module of morphisms from

C to D (see Example D.7.1.2), which is characterized by the following universal property: for every left R-module M, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{LMod}_R}(M, \underline{\mathrm{Map}}_{\mathcal{C}}(C, D)) \simeq \mathrm{Map}_{\mathcal{C}}(M \otimes_R C, D).$$

Definition 11.1.0.1. Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. We will say that $\mathcal{C}$ is *proper* if the following conditions are satisfied:

(1) The ∞-category $\mathcal{C}$ is compactly generated.

(2) For every pair of compact objects $C, D \in \mathcal{C}$, the left R-module $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ is perfect.

When restricted to R-linear ∞-categories of the form LMod_A for $A \in \mathrm{Alg}_R$, Definition 11.1.0.1 recovers the notion of proper R-algebra studied in §HA.4.6.4 :

Proposition 11.1.0.2. *Let R be an $\mathbb{E}_2$-ring and let A be an $\mathbb{E}_1$-algebra over R. The following conditions are equivalent:*

(1) *The algebra A is perfect when regarded as a left R-module (that is, the object $A \in \mathrm{Alg}_R$ is proper).*

(2) *The stable R-linear ∞-category LMod_A is proper.*

Proof. Let $\mathcal{C} = \mathrm{LMod}_A$. Then we can regard A as a compact object of $\mathcal{C}$, with $\underline{\mathrm{Map}}_{\mathcal{C}}(A, A) \simeq A$. It follows immediately that (2) $\Rightarrow$ (1). Conversely, suppose that (1) is satisfied. We must show that if C and D are compact objects of $\mathcal{C}$, then $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ is a perfect R-module. If we regard D as fixed, then the collection of those objects $C \in \mathcal{C}$ for which $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ is perfect is a stable subcategory of $\mathcal{C}$ which is closed under retracts. Consequently, to prove that $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ is perfect for every compact object $C \in \mathcal{C}$, it will suffice to treat the case $C = A$. Similarly, we may reduce to the case $N = A$, so that $\underline{\mathrm{Map}}_{\mathcal{C}}(M, N) \simeq A$ is perfect provided that condition (1) is satisfied. □

Our goal in this section is to study a relative version of Definition 11.1.0.1. We begin in §11.1.1 by showing that the condition that a stable R-linear ∞-category $\mathcal{C}$ is proper can be tested locally with respect to the étale topology (Proposition 11.1.1.1). Consequently, there is a robust notion of proper quasi-coherent stacks on an arbitrary spectral Deligne-Mumford stack X (Definition 11.1.1.3). Our main objective is to establish the following direct image theorem: if $f : \mathsf{X} \to \mathsf{Y}$ is a morphism which is proper, locally almost of finite presentation, and of finite Tor-amplitude, then the pushforward functor $f_* : \mathrm{QStk}^{\mathrm{St}}(\mathsf{X}) \to \mathrm{QStk}^{\mathrm{St}}(\mathsf{Y})$ carries proper quasi-coherent stacks on X to proper quasi-coherent stacks on Y (Theorem 11.1.4.1).

11.1.1 Proper Quasi-Coherent Stacks

Our next goal is to show collection of proper R-linear ∞-categories is stable under base change and satisfies étale descent.

Proposition 11.1.1.1. *Let R be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. Then:*

(1) *If $\mathcal{C}$ is proper as an R-linear ∞-category and $R \to R'$ is an arbitrary morphism of $\mathbb{E}_\infty$-rings, then the tensor product $R' \otimes_R \mathcal{C} = \mathrm{LMod}_{R'}(\mathcal{C})$ is a proper R'-linear ∞-category.*

(2) *Suppose there exists a finite collection of étale morphisms $\{R \to R_\alpha\}$ such that the induced map $R \to \prod_\alpha R_\alpha$ is faithfully flat. If R is connective and each $R_\alpha \otimes_R \mathcal{C}$ is a proper R_α-linear ∞-category, then $\mathcal{C}$ is a proper R-linear ∞-category.*

Remark 11.1.1.2. Proposition 11.1.1.1 remains true (with an essentially identical proof) in the setting of linear ∞-categories over $\mathbb{E}_2$-rings.

Proof of Proposition 11.1.1.1. We first prove (1). If $\mathcal{C}$ is proper, then it is compactly generated and therefore $\mathrm{LMod}_{R'}(\mathcal{C})$ is compactly generated (Example **??**). We must show that for every pair of compact objects $C, D \in \mathrm{LMod}_{R'}(\mathcal{C})$, the R'-module $\underline{\mathrm{Map}}_{\mathrm{LMod}_{R'}(\mathcal{C})}(C, D)$ is perfect. Let us first regard Y as fixed, and let $\mathcal{X} \subseteq \mathrm{LMod}_{R'}(\mathcal{C})$ be the full subcategory spanned by those compact objects X for which $\underline{\mathrm{Map}}_{\mathrm{LMod}_{R'}(\mathcal{C})}(C, D)$ is perfect. Then $\mathcal{X}$ is an idempotent complete, stable subcategory of $\mathrm{LMod}_{R'}(\mathcal{C})$. To show that it contains every compact object of $\mathrm{LMod}_{R'}(\mathcal{C})$, it will suffice to show that it contains every object of the form $R' \otimes_R C_0$, where C_0 is a compact object of $\mathcal{C}$. Let us now regard C_0 as fixed; we wish to show that $\underline{\mathrm{Map}}_{\mathrm{LMod}_{R'}(\mathcal{C})}(R' \otimes_R C_0, D) \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(C_0, D)$ is a perfect R'-module for every compact object $D \in \mathrm{LMod}_{R'}(\mathcal{C})$. Arguing as above, we may suppose that $D \simeq R' \otimes_R D$ for some compact object $D \in \mathcal{C}$. Using the compactness of C_0, we obtain an equivalence of R'-modules $R' \otimes_R \underline{\mathrm{Map}}_{\mathcal{C}}(C_0, D_0) \to \underline{\mathrm{Map}}_{\mathrm{LMod}_{R'}(\mathcal{C})}(C, D)$. It will therefore suffice to show that $\underline{\mathrm{Map}}_{\mathcal{C}}(C_0, D_0)$ is a perfect R-module, which follows from our assumption that $\mathcal{C}$ is proper.

We now prove (2). Assume that R is connective $\mathbb{E}_\infty$-ring and that each $R_\alpha \otimes_R \mathcal{C}$ is a proper R_α-linear ∞-category, for some étale covering $\{R \to R_\alpha\}$. Using Theorem D.5.3.1, we conclude that $\mathcal{C}$ is a compactly generated ∞-category. Fix compact objects $C, D \in \mathcal{C}$. For every index α, the tensor product

$$R_\alpha \otimes_R \underline{\mathrm{Map}}_{\mathcal{C}}(C, D) \simeq \underline{\mathrm{Map}}_{\mathrm{LMod}_{R_\alpha}(\mathcal{C})}(R_\alpha \otimes C, R_\alpha \otimes D)$$

is a perfect module over R_α. Applying Proposition 2.8.4.2, we deduce that $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ is a perfect R-module. $\square$

Proposition 11.1.1.1 asserts that the condition of properness can be tested locally for the étale topology. This motivates the following:

Definition 11.1.1.3. Let X be a spectral Deligne-Mumford stack and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$ be an stable quasi-coherent stack on X. We will say that $\mathcal{C}$ is *proper* if, for every connective $\mathbb{E}_\infty$-ring R and every morphism $\eta : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$, the stable R-linear ∞-category $\mathcal{C}_\eta$ is proper.

Example 11.1.1.4. Let $\mathsf{X} \simeq \mathrm{Sp\acute{e}t}\, R$ be an affine spectral Deligne-Mumford stack and let $\mathcal{C}$ be a stable quasi-coherent stack on X. Then $\mathcal{C}$ is proper as a quasi-coherent stack on X if and only if it is proper as an R-linear ∞-category: this follows immediately from the first assertion of Proposition 11.1.1.1.

Proposition 11.1.1.1 immediately implies the following:

Proposition 11.1.1.5. *Let X be a spectral Deligne-Mumford stack and let $\mathcal{C}$ be a stable quasi-coherent stack on X. Then:*

(1) *Let $f : \mathsf{Y} \to \mathsf{X}$ be any map of spectral Deligne-Mumford stacks. If $\mathcal{C}$ is proper, then $f^* \mathcal{C} \in \mathrm{QStk}^{\mathrm{St}}(\mathsf{Y})$ is proper.*

(2) *Suppose we are given a collection of étale maps $\{f_\alpha : \mathsf{X}_\alpha \to \mathsf{X}\}$ which induce an étale surjection $\coprod_\alpha \mathsf{X}_\alpha \to \mathsf{X}$. If each pullback $f_\alpha^* \mathcal{C} \in \mathrm{QStk}^{\mathrm{St}}(\mathsf{X}_\alpha)$ is proper, then $\mathcal{C}$ is proper.*

11.1.2 Digression: Perfect Objects with Prescribed Support

Let X be a quasi-separated spectral algebraic space. We say that a closed subset $K \subseteq |\mathsf{X}|$ is *cocompact* if the complementary open set $|\mathsf{X}| - K$ is quasi-compact. According to Proposition 7.1.5.5, the support $\mathrm{Supp}(\mathscr{F}) \subseteq |\mathsf{X}|$ is a cocompact closed set for every perfect object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$. We will need the following converse:

Proposition 11.1.2.1. *Let X be a quasi-compact, quasi-separated spectral algebraic space, and let K be a cocompact closed subset of $|\mathsf{X}|$. Then there exists a perfect object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ such that $K = \mathrm{Supp}(\mathscr{F})$.*

We first treat the case where X is affine.

Lemma 11.1.2.2. *Let A be a connective $\mathbb{E}_\infty$-ring, and let K be a cocompact closed subset of $|\mathrm{Spec}\, A|$. Then there exists a perfect A-module M such that $\mathrm{Supp}(\mathscr{F}) = K$, where $\mathscr{F}$ denotes the image of M under the equivalence of ∞-categories $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A) \simeq \mathrm{Mod}_A$.*

Proof. Since K is cocompact, we can write K as the vanishing locus of an ideal $I \subseteq \pi_0 A$ generated by finitely many elements $x_1, \ldots, x_n \in \pi_0 A$. We can now take M to be the tensor product of the cofibers of the maps $x_i : A \to A$. □

Proof of Proposition 11.1.2.1. Using Theorem 3.4.2.1, we can choose a scallop decomposition

$$\varnothing = \mathsf{U}_0 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n = \mathsf{X}.$$

For $0 \leqslant m \leqslant n$, let K_m denote the inverse image of K in $|\,\mathsf{U}_m\,|$. We will prove by induction on m that there exists a perfect object $\mathscr{F}_m \in \mathrm{QCoh}(\mathsf{U}_m)$ with $\mathrm{Supp}(\mathscr{F}_m) = K_m$. The case $m = 0$ is trivial. To carry out the inductive step, suppose that $\mathscr{F}_m$ has been constructed for $m < n$ and choose an excision square

$$\begin{array}{ccc} \mathsf{V} & \xrightarrow{j} & \mathsf{Y} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ \mathsf{U}_m & \longrightarrow & \mathsf{U}_{m+1} \end{array}$$

where Y is affine and j is a quasi-compact open immersion. Let K' denote the inverse image of K in $|\,\mathsf{Y}\,|$. Using the argument of Theorem **??** (applied to the quasi-coherent stack $\mathcal{Q}_K$), we deduce that there exists an object $\mathscr{G} \in \mathrm{QCoh}_{K'}(\mathsf{Y})$ with $j^* \mathscr{G} \simeq \phi^*(\mathscr{F}_m \oplus \Sigma \mathscr{F}_m)$, so that $\mathscr{G}$ and $\mathscr{F}_m \oplus \Sigma \mathscr{F}_m$ can be glued to obtain a perfect object $\mathscr{F}' \in \mathrm{QCoh}_{K_{m+1}}(\mathsf{U}_{m+1})$ whose support contains K_m. Let $Z = K_{m+1} - K_m$, so that Z is a closed subset of $|\,\mathsf{U}_{m+1}\,|$ which does not intersect $|\,\mathsf{U}_m\,|$. Let Z' denote the inverse image of Z in $|\,\mathsf{Y}\,|$. It follows from descent that the pullback functor ψ^* induces an equivalence of ∞-categories

$$\mathrm{QCoh}_Z(\mathsf{U}_{m+1}) \to \mathrm{QCoh}_{Z'}(\mathsf{Y}).$$

Applying Lemma 11.1.2.2, we deduce the existence of a perfect object $\mathscr{F}'' \in \mathrm{QCoh}_Z(\mathsf{U}_{m+1})$ such that $\mathrm{Supp}(\psi^* \mathscr{F}'') = Z'$. We now complete the proof by setting $\mathscr{F}_{m+1} = \mathscr{F}' \oplus \mathscr{F}''$. □

11.1.3 Properness and Dualizability

Let X be a quasi-compact, quasi-separated spectral algebraic space. Combining Theorem 10.2.0.2 with Proposition D.5.1.1, we deduce that the construction $\mathcal{C} \mapsto \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ induces an equivalence of ∞-categories

$$\mathrm{QCoh}(\mathsf{X}; \bullet) : \mathrm{QStk}^{\mathrm{St}}(\mathsf{X}) \to \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})}(\mathcal{P}\mathrm{r}^{\mathrm{St}});$$

here $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ denotes the ∞-category of presentable stable ∞-categories. Consequently, if $\mathcal{C}$ is a stable quasi-coherent stack on X, then the condition that $\mathcal{C}$ is proper (in the sense of Definition 11.1.1.3) depends only on the ∞-category $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ (as an ∞-category tensored over $\mathrm{QCoh}(\mathsf{X})$). We now make this observation more explicit.

Construction 11.1.3.1. Let X be a quasi-compact, quasi-separated spectral algebraic space, and let $\mathcal{C}$ be a stable quasi-coherent stack on X. Assume that $\mathcal{C}$ is compactly generated.

Then the ∞-category $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is compactly generated (Theorem 10.3.2.1). Since the ∞-category $\mathrm{QCoh}(\mathsf{X})$ is locally rigid (Proposition 9.6.1.2), it follows that $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is dualizable as a module over $\mathrm{QCoh}(\mathsf{X})$ (Proposition D.7.5.1 and Proposition HA.4.6.5.11). Let us denote the its dual by ${}^{\vee}\mathrm{QCoh}(\mathsf{X};\mathcal{C})$, so that we have a duality datum

$$e : \mathrm{QCoh}(\mathsf{X};\mathcal{C}) \otimes_{\mathrm{QCoh}(\mathsf{X})} {}^{\vee}\mathrm{QCoh}(\mathsf{X};\mathcal{C}) \to \mathrm{QCoh}(\mathsf{X}).$$

Moreover, if $\lambda : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Sp}$ denotes the global sections functor, then the composition

$$\mathrm{QCoh}(\mathsf{X};\mathcal{C}) \otimes {}^{\vee}\mathrm{QCoh}(\mathsf{X};\mathcal{C}) \to \mathrm{QCoh}(\mathsf{X};\mathcal{C}) \otimes_{\mathrm{QCoh}(\mathsf{X})} {}^{\vee}\mathrm{QCoh}(\mathsf{X};\mathcal{C}) \xrightarrow{e} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\lambda} \mathrm{Sp}$$

exhibits ${}^{\vee}\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ as a dual of $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ in the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. We may therefore use Proposition D.7.2.3 to identify ${}^{\vee}\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ with the ∞-category $\mathrm{Ind}((\mathrm{QCoh}(\mathsf{X};\mathcal{C})_c^{\mathrm{op}})$, where $\mathrm{QCoh}(\mathsf{X};\mathcal{C})_c$ denotes the full subcategory of $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ spanned by the compact objects.

Unwinding the definitions, we see that e restricts to a functor

$$\underline{\mathrm{Map}}_{\mathcal{C}} : \mathrm{QCoh}(\mathsf{X};\mathcal{C})_c^{\mathrm{op}} \times \mathrm{QCoh}(\mathsf{X};\mathcal{C}) \to \mathrm{QCoh}(\mathsf{X}),$$

$$(\mathscr{F},\mathscr{G}) \mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(\mathscr{F},\mathscr{G}),$$

which is characterized by the universal property

$$\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{E},\underline{\mathrm{Map}}_{\mathcal{C}}(\mathscr{F},\mathscr{G})) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X};\mathcal{C})}(\mathscr{E}\otimes\mathscr{F},\mathscr{G}).$$

Remark 11.1.3.2. Let X be a quasi-compact, quasi-separated spectral algebraic space and let $\mathcal{C}$ be a stable quasi-coherent stack on X. Assume that $\mathcal{C}$ is compactly generated and let ${}^{\vee}\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ be as in Construction 11.1.3.1. Since the functor $\mathrm{QCoh}(\mathsf{X};\bullet) : \mathrm{QStk}^{\mathrm{St}}(\mathsf{X}) \to \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$ is a symmetric monoidal equivalence, we can write ${}^{\vee}\mathrm{QCoh}(\mathsf{X};\mathcal{C}) \simeq \mathrm{QCoh}(\mathsf{X};{}^{\vee}\mathcal{C})$, where ${}^{\vee}\mathcal{C}$ is a dual of $\mathcal{C}$ in the symmetric monoidal ∞-category $\mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$.

Proposition 11.1.3.3. *Let X be a quasi-compact, quasi-separated spectral algebraic space and let $\mathcal{C}$ be a stable quasi-coherent stack on X. Assume that $\mathcal{C}$ is compactly generated, so that $\mathcal{C}$ admits a dual ${}^{\vee}\mathcal{C}$ in $\mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$. The following conditions are equivalent:*

(a) *The stable quasi-coherent stack $\mathcal{C}$ is proper (in the sense of Definition 11.1.1.3).*

(b) *The evaluation map $e : \mathcal{C}\otimes{}^{\vee}\mathcal{C} \to \mathcal{Q}_{\mathsf{X}}$ is compact (in the sense of Definition 10.1.3.1).*

(c) *The evaluation map $e' : \mathrm{QCoh}(\mathsf{X};\mathcal{C}) \otimes_{\mathrm{QCoh}(\mathsf{X})} {}^{\vee}\mathrm{QCoh}(\mathsf{X};\mathcal{C}) \to \mathrm{QCoh}(\mathsf{X})$ is compact (in the sense of Definition C.3.4.2).*

(d) *For every pair of compact objects $\mathscr{F},\mathscr{G} \in \mathrm{QCoh}(\mathsf{X};\mathcal{C})$, the object $\underline{\mathrm{Map}}_{\mathcal{C}}(\mathscr{F},\mathscr{G})$ (see Construction 11.1.3.1) is perfect.*

Proof. The equivalence of (b) and (c) follows from Proposition 10.3.1.14, and the equivalence of (b) and (c) follows from the observation that the ∞-category of compact objects of $\mathrm{QCoh}(\mathsf{X};\mathcal{C})\otimes_{\mathrm{QCoh}(\mathsf{X})}{}^{\vee}\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is generated (under finite colimits and retracts) by objects of the form $\mathscr{F}\otimes\mathscr{G}$, where $\mathscr{F}\in\mathrm{QCoh}(\mathsf{X};\mathcal{C})_c$ and $\mathscr{G}\in\mathrm{QCoh}(\mathsf{X};\mathcal{C})_c^{\mathrm{op}}$ (see Construction 11.1.3.1). To prove the equivalence of (a) and (b), we can work locally on X and thereby reduce to the case where $\mathsf{X}=\mathrm{Sp\acute{e}t}\,R$ is affine. In this case, the desired result reduces to the equivalence $(b)\Leftrightarrow(d)$. □

Remark 11.1.3.4. The equivalence $(a)\Leftrightarrow(b)$ of Proposition 11.1.3.3 is valid for *any* spectral Deligne-Mumford stack X.

11.1.4 Direct Images of Proper Quasi-Coherent Stacks

We now consider the relationship between properness of quasi-coherent stacks and proper morphisms between spectral algebraic spaces.

Theorem 11.1.4.1. *Let $f:\mathsf{X}\to\mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Assume that f is quasi-compact, separated, and locally almost of finite presentation. Then the following conditions are equivalent:*

(1) *The map f is proper and locally of* Tor*-amplitude $\leqslant n$, for some integer n.*

(2) *For every proper object $\mathcal{C}\in\mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$, the pushforward $f_*\,\mathcal{C}$ is also proper.*

(3) *The pushforward $f_*\,\mathcal{Q}_{\mathsf{X}}\in\mathrm{QStk}^{\mathrm{St}}(\mathsf{Y})$ is proper (here $\mathcal{Q}_{\mathsf{X}}$ denotes the unit object of $\mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$).*

In the situation of Theorem 11.1.4.1, the implication $(2)\Rightarrow(3)$ is immediate, and the implication $(1)\Rightarrow(3)$ is a reformulation of Theorem 6.1.3.2. The remaining implications are consequences of the following three results:

Proposition 11.1.4.2. *Let R be a connective $\mathbb{E}_\infty$-ring, let X be a spectral algebraic space which is proper, locally almost of finite presentation, and of finite* Tor*-amplitude over R. If $\mathcal{C}\in\mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$ is a proper quasi-coherent stack on X, then $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is a proper R-linear ∞-category.*

Proposition 11.1.4.3. *Let R be a connective $\mathbb{E}_\infty$-ring, let X be a quasi-compact separated spectral algebraic space which is locally almost of finite presentation over R, and suppose that $\mathrm{QCoh}(\mathsf{X})$ is a proper R-linear ∞-category. Then X is proper over R.*

Proposition 11.1.4.4. *Let R be a connective $\mathbb{E}_\infty$-ring, let X be a quasi-compact, quasi-separated spectral algebraic space which is locally almost of finite presentation over R, and suppose that $\mathrm{QCoh}(\mathsf{X})$ is a proper R-linear ∞-category. Then X has* Tor*-amplitude $\leqslant n$ over R, for some $n\geqslant 0$.*

Proof of Proposition 11.1.4.2. Let X be a spectral algebraic space which is proper, locally almost of finite presentation, and locally of Tor-amplitude $\leqslant n$ over a connective $\mathbb{E}_\infty$-ring R, and let $\mathcal{C}$ be a proper quasi-coherent stack on X. It follows from Theorem 10.3.2.1 that $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is a compactly generated R-linear ∞-category, and Proposition 10.3.2.6 implies that an object of $\mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ is compact if and only if it is locally compact. It will therefore suffice to prove that if $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$ are locally compact, then the R-module $\underline{\mathrm{Map}}_{\mathrm{QCoh}(\mathsf{X};\mathcal{C})}(\mathscr{F}, \mathscr{G}) \in \mathrm{Mod}_R$ is perfect. Unwinding the definitions, we see that $\underline{\mathrm{Map}}_{\mathrm{QCoh}(\mathsf{X};\mathcal{C})}(\mathscr{F}, \mathscr{G})$ can be identified with the R-module of global sections of the quasi-coherent sheaf $\underline{\mathrm{Map}}_{\mathcal{C}}(\mathscr{F}, \mathscr{G}) \in \mathrm{QCoh}(\mathsf{X})$ described in Construction 11.1.3.1. The assumption that $\mathcal{C}$ is proper guarantees that $\underline{\mathrm{Map}}_{\mathcal{C}}(\mathscr{F}, \mathscr{G})$ is a perfect object of $\mathrm{QCoh}(\mathsf{X})$, so that the desired result reduces to Theorem 6.1.3.2. □

The proof of Proposition 11.1.4.3 will require some preliminaries.

Lemma 11.1.4.5. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack, let $\mathscr{A}$ be an algebra object of $\mathrm{QCoh}(\mathsf{X})$, and let $A = \Gamma(\mathcal{X}; \mathscr{A}) \in \mathrm{Alg}$ denote the $\mathbb{E}_1$-algebra of global sections of $\mathscr{A}$. Then $\mathscr{A} \simeq 0$ if and only if $A \simeq 0$.*

Proof. The "only if" direction is obvious. The converse follows from the observation that $\mathscr{A}$ is a sheaf of A-module spectra on $\mathcal{X}$, which automatically vanishes if $A \simeq 0$. □

Lemma 11.1.4.6. *Let R be a connective $\mathbb{E}_\infty$-ring, let X be a spectral algebraic space over R which is quasi-compact and quasi-separated, and assume that the R-linear ∞-category $\mathrm{QCoh}(\mathsf{X})$ is proper. Then the map of topological spaces $\phi : |\mathsf{X}| \to |\operatorname{Spec} R|$ is closed.*

Proof. Let K be a closed subset of $|\mathsf{X}|$; we wish to show that $\phi(K) \subseteq |\operatorname{Spec} R|$ is closed. Let $U = |\mathsf{X}| - K$, and write U as a union of quasi-compact open subsets U_α. For each index α, let U_α be the corresponding open substack of X. Since X is quasi-separated, each of the maps $\mathsf{U}_\alpha \to \mathsf{X}$ is quasi-compact, so that the closed sets $K_\alpha = |\mathsf{X}| - U_\alpha$ are cocompact. Since the fibers of ϕ are quasi-compact topological spaces, we have $\phi(K) = \bigcap_\alpha \phi(K_\alpha)$. It will therefore suffice to show that each $\phi(K_\alpha)$ is a closed subset of $|\operatorname{Spec} R|$. Replacing K by K_α, we may reduce to the case where K is cocompact, so that $K = \mathrm{Supp}(\mathscr{F})$ for some perfect object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ (Proposition 11.1.2.1). Replacing $\mathscr{F}$ by $\mathrm{End}(\mathscr{F})$ if necessary, we may suppose that $\mathscr{F}$ is an algebra object of $\mathrm{QCoh}(\mathsf{X})$ (Corollary 7.1.5.6). Let $A \in \mathrm{Alg}(\mathrm{Mod}_R)$ denote the R-module of global sections of $\mathscr{F}$. Since $\mathrm{QCoh}(\mathsf{X})$ is proper, the algebra $A = \underline{\mathrm{Map}}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{O}, \mathscr{F})$ is a perfect R-module. We will complete the proof by showing that $\phi(K) = \mathrm{Supp}(A)$ (where we abuse notation by identifying A with the corresponding quasi-coherent sheaf on $\operatorname{Spét} R$). Equivalently, we show that for any field κ and any map of $\mathbb{E}_\infty$-rings $R \to \kappa$, the tensor product $\kappa \otimes_R A$ vanishes if and only if the pullback of $\mathscr{F}$ to $\mathsf{X}_0 = \operatorname{Spét} \kappa \times_{\operatorname{Spét} R} \mathsf{X}$ vanishes. Replacing R by κ and X by X_0, we deduce the desired result from Lemma 11.1.4.5. □

Proof of Proposition 11.1.4.3. Let X be a spectral algebraic space which is separated and locally almost finite presentation over a connective $\mathbb{E}_\infty$-ring R, and suppose that $\mathrm{QCoh}(\mathsf{X})$ is a proper R-linear ∞-category. We wish to prove that X is proper over R. Equivalently, we must show that every pullback diagram

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}\, R' & \longrightarrow & \mathrm{Sp\acute{e}t}\, R, \end{array}$$

induces a closed map of topological spaces $|\mathsf{X}'| \to |\mathrm{Spec}\, R'|$. This follows from Lemma 11.1.4.6, since $\mathrm{QCoh}(\mathsf{X}') \simeq \mathrm{Mod}_{R'}(\mathrm{QCoh}(\mathsf{X}))$ is a proper R'-linear ∞-category (Proposition 11.1.1.1). □

We now turn to the proof of Proposition 11.1.4.4.

Lemma 11.1.4.7. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of quasi-compact, quasi-separated spectral algebraic spaces. Suppose that the pushforward functor f_* carries $\mathrm{QCoh}(\mathsf{X})^{\mathrm{perf}}$ into $\mathrm{QCoh}(\mathsf{Y})^{\mathrm{perf}}$, and let $f_+ : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ be as in Proposition 6.4.5.4. Then there exists an integer n such that f_+ carries $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$ into $\mathrm{QCoh}(\mathsf{Y})_{\geqslant -n}$.*

Proof. Choose a perfect object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ satisfying the requirements of Proposition 9.6.3.1. Then $f_+ \mathscr{F}$ is a perfect object of $\mathrm{QCoh}(\mathsf{Y})$. Since Y is quasi-compact, there exists an integer n such that $f_+ \mathscr{F} \in \mathrm{QCoh}(\mathsf{Y})_{\geqslant -n}$. The collection of those objects $\mathscr{G} \in \mathrm{QCoh}(\mathsf{X})$ such that $f_+ \mathscr{G} \in \mathrm{QCoh}(\mathsf{Y})_{\geqslant -n}$ contains $\mathscr{F}$ and is closed under colimits and extensions, and therefore contains $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$. □

Lemma 11.1.4.8. *Let R be a connective $\mathbb{E}_\infty$-ring, let X be a quasi-compact, quasi-separated spectral algebraic space over R, and suppose that $\mathrm{QCoh}(\mathsf{X})$ is a proper R-linear ∞-category. Then there exists an integer c with the following property:*

- $(*)$ *Let $\phi : R \to R'$ be a map of connective $\mathbb{E}_\infty$-rings, let $\mathsf{X}' = \mathrm{Sp\acute{e}t}\, R' \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$, and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}')$ be perfect. If $\mathscr{F}$ has* Tor*-amplitude $\leqslant n$, then the R'-module $\Gamma(\mathsf{X}'; \mathscr{F})$ has* Tor*-amplitude $\leqslant n + c$.*

Proof. According to Lemma **??**, there exists an integer n such that the functor f_+ : $\mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R) \simeq \mathrm{Mod}_R$ carries $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$ into $(\mathrm{Mod}_R)_{\geqslant -c}$. We claim that c has the desired property. For any map of connective $\mathbb{E}_\infty$-rings $R \to R'$, form a pullback diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X}') & \xrightarrow{g'} & \mathrm{QCoh}(\mathsf{X}) \\ \downarrow{\scriptstyle f'} & & \downarrow \\ \mathrm{Sp\acute{e}t}\, R' & \longrightarrow & \mathrm{Sp\acute{e}t}\, R. \end{array}$$

The map g' is affine, so that g'_* is right t-exact. It follows that the composite functor $f_+ \circ g'_*$ carries $\mathrm{QCoh}(\mathsf{X}')_{\geqslant 0}$ into $(\mathrm{Mod}_R)_{\geqslant -c}$. Using Remark 6.4.5.7, we deduce that $g_* f'_+$ carries $\mathrm{QCoh}(\mathsf{X}')_{\geqslant 0}$ into $(\mathrm{Mod}_R)_{\geqslant -c}$, so that f'_+ carries $\mathrm{QCoh}(\mathsf{X}')_{\geqslant 0}$ into $(\mathrm{Mod}_{R'})_{\geqslant -c}$. If $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}')$ is perfect of Tor-amplitude $\leqslant n$, then $\mathscr{F}^\vee \in \mathrm{QCoh}(\mathsf{X}')_{\geqslant -n}$, so that $(f_+ \mathscr{F}^\vee) \simeq (f_* \mathscr{F})^\vee \in (\mathrm{Mod}_{R'})_{\geqslant -n-c}$, and therefore $f_* \mathscr{F}$ has Tor-amplitude $\leqslant n + c$. □

Proof of Proposition 11.1.4.4. Let R be a connective $\mathbb{E}_\infty$-ring, let X be a quasi-compact, quasi-separated spectral algebraic space which is locally almost of finite presentation over R, and assume that $\mathrm{QCoh}(\mathsf{X})$ is proper as an R-linear ∞-category. We wish to prove that X has finite Tor-amplitude over R. Since X is quasi-compact, we can choose an étale surjection $\mathrm{Spét}\, A \to \mathsf{X}$. Our assumption that X is locally almost of finite presentation over R guarantees that $\pi_0 A$ is finitely generated as a commutative algebra over $\pi_0 R$. Choose a finite collection of elements $x_1, \ldots, x_d \in \pi_0 A$ which generate $\pi_0 A$ as an algebra over $\pi_0 R$, and choose an integer c satisfying condition $(*)$ of Lemma 11.1.4.8. We will prove that X has Tor-amplitude $\leqslant c + d$ over R. By virtue of Corollary 6.1.4.8, it will suffice to show that for every field κ and every map $R \to \kappa$, the fiber product $\mathrm{Spét}\, \kappa \times_{\mathrm{Spét}\, R} \mathsf{X}$ is $(c + d)$-truncated. Replacing R by κ, we may suppose that R is a field.

Using Theorem 3.4.2.1, we can choose a scallop decomposition

$$\varnothing = \mathsf{U}_0 \hookrightarrow \mathsf{U}_1 \hookrightarrow \cdots \hookrightarrow \mathsf{U}_n = \mathsf{X}$$

of X. We will prove that each U_i is $(c + d)$-truncated, using induction on i. The case $i = 0$ is trivial. To carry out the inductive step, assume that $1 \leqslant i \leqslant n$ and that U_{i-1} is $(c + d)$-truncated; we wish to show that U_i is $(c + d)$-truncated. Choose an excision square σ :

$$\begin{array}{ccc} \mathsf{V} & \xrightarrow{f} & \mathrm{Spét}\, B \\ \downarrow & & \downarrow u \\ \mathsf{U}_{i-1} & \longrightarrow & \mathsf{U}_i . \end{array}$$

The map f is an open immersion whose image is an open subset $V \subseteq |\mathrm{Spec}\, B|$. Since U_{i-1} and $\mathrm{Spét}\, B$ constitute an étale covering of U_i, it will suffice to show that B is $(c + d)$-truncated. Fix $k > c + d$; we will show that $\pi_k B \simeq 0$. For this, it will suffice to show that the localization $\pi_k B_{\mathfrak{m}} \simeq 0$, for every maximal ideal $\mathfrak{m} \subseteq \pi_0 B$. Without loss of generality, we may assume that $\mathfrak{m} \in |\mathrm{Spec}\, B|$ does not belong to V (otherwise, the desired result follows from our inductive hypothesis).

Our choice of c guarantees that the commutative ring $\pi_0 B$ has Krull dimension at most c, so we can choose a sequence of elements $y_1, \ldots, y_d$ belonging to the maximal ideal of $\pi_0 B_{\mathfrak{m}}$, having the property that $(\pi_0 B_{\mathfrak{m}})/(y_1, \ldots, y_d)$ is a finite-dimensional vector space over κ. For $1 \leqslant i \leqslant d$, let M_i denote the $B_{\mathfrak{m}}$-module given by the cofiber of the map $y_i : B_{\mathfrak{m}} \to B_{\mathfrak{m}}$, and let M denote the tensor product $\bigotimes_{1 \leqslant i \leqslant d} M_i$ (formed in the ∞-category $\mathrm{Mod}_{B_{\mathfrak{m}}}$). Then M is a perfect $B_{\mathfrak{m}}$-module, which has Tor-amplitude $\leqslant d$ over $B_{\mathfrak{m}}$. Moreover, M is supported at the closed point of $|\operatorname{Spec} B_{\mathfrak{m}}|$, so that each homotopy group $\pi_i M$ is finite-dimensional as a vector space over κ. It follows that M is almost perfect as a B-module (Proposition HA.7.2.4.17). Since $B_{\mathfrak{m}}$ is flat over B, M has Tor-amplitude $\leqslant c$ as a B-module (Lemma 6.1.1.6). It follows that M is perfect as a B-module (Proposition HA.7.2.4.23). By construction, the support of M (as a B-module) is contained in the closed subset $\overline{K} = |\operatorname{Spec} B| - V$.

Let K be the closed subset of $|\mathsf{U}_i|$ complementary to the open immersion $\mathsf{U}_{i-1} \hookrightarrow \mathsf{U}_i$. Since σ is an excision square, the adjoint functors

$$\operatorname{QCoh}_K(\mathsf{U}_i) \underset{u_*}{\overset{u^*}{\rightleftarrows}} \operatorname{QCoh}_{\overline{K}}(\operatorname{Sp\acute{e}t} B)$$

are mutually inverse equivalence of ∞-categories. In particular, we can write M as the image of some sheaf $\mathscr{F}_i \in \operatorname{QCoh}_K(\mathsf{U}_i)$ under the composite functor

$$\operatorname{QCoh}_K(\mathsf{U}_i) \xrightarrow{u^*} \operatorname{QCoh}_{\overline{K}}(\operatorname{Sp\acute{e}t} B) \subseteq \operatorname{QCoh}(\operatorname{Sp\acute{e}t} B) \simeq \operatorname{Mod}_B .$$

For $j \geqslant i$, let $\mathscr{F}_j$ denote the direct image of $\mathscr{F}_i$ under the open immersion $\mathsf{U}_i \to \mathsf{U}_j$. We will prove by induction on j that each $\mathscr{F}_j$ is a perfect object of $\operatorname{QCoh}(\mathsf{U}_j)$ of Tor-amplitude $\leqslant d$. In the case $j = i$, we observe that $\operatorname{Sp\acute{e}t} B$ and U_{i-1} comprise an étale cover of U_i. Since $\mathscr{F}|_{\mathsf{U}_i} \simeq 0$, it suffices to show that $u^* \mathscr{F}$ is a perfect object of Tor-amplitude $\leqslant c$, which follows from our construction. To carry out the inductive step, suppose that $i < j \leqslant n$, and that $\mathscr{F}_{j-1}$ is perfect of Tor-amplitude $\leqslant d$. We have an excision square τ :

$$\begin{array}{ccc} \mathsf{V}' & \xrightarrow{g'} & \operatorname{Sp\acute{e}t} C \\ \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} \\ \mathsf{U}_{j-1} & \xrightarrow{g} & \mathsf{U}_j . \end{array}$$

To prove that $\mathscr{F}_j \simeq g_* \mathscr{F}_{j-1}$ is perfect of Tor-amplitude $\leqslant d$, it will suffice to show that $g^* g_* \mathscr{F}_{j-1} \simeq \mathscr{F}_{j-1}$ and $v^* g_* \mathscr{F}_{j-1} \simeq g'_* v'^* \mathscr{F}_{j-1}$ are perfect of Tor-amplitude $\leqslant d$. In the former case, this follows from the inductive hypothesis. To handle the latter case, let $Y \subseteq |\mathsf{V}'|$ be the support of $v'^* \mathscr{F}_{j-1}$, and let $\overline{Y}$ denote the closure of Y in $|\operatorname{Spec} C|$. Then $g'_* v'^* \mathscr{F}_{j-1}$ vanishes on the complement of $\overline{Y}$ and is perfect of Tor-amplitude $\leqslant d$ on V'. We are therefore reduced to proving that V' contains $\overline{Y}$: that is, that Y is closed in $|\operatorname{Spec} C|$. Form a pullback diagram

$$\begin{array}{ccc} \mathsf{Y} & \xrightarrow{h'} & \mathsf{V}' \\ \downarrow{\scriptstyle v''} & & \downarrow{\scriptstyle v'} \\ \operatorname{Sp\acute{e}t} B & \xrightarrow{h} & \mathsf{U}_{j-1} \end{array} .$$

Then

$$v'^* \mathscr{F}_{j-1} \simeq v'^* h_* u^* \mathscr{F}_i \simeq h'_* v''^* u^* \mathscr{F}_i .$$

Since $u^* \mathscr{F}_i$ is supported on the set $\{\mathfrak{m}\} \subseteq |\operatorname{Spec} B|$, $v'^* \mathscr{F}_{j-1}$ is supported on the closure of $v''(h'^{-1}\{\mathfrak{m}\})$. However, v'' is a quasi-compact étale map, so $h^{-1}\{\mathfrak{m}\}$ consists of finitely many points whose residue fields are finite extensions of κ. It follows that Y consists of finitely many points of $|\mathsf{V}'|$, whose residue fields are finite extensions of κ, and therefore Y is a closed subset of $|\operatorname{Spec} C|$. This completes the proof the proof that each $\mathscr{F}_j$ is perfect of Tor-amplitude $\leqslant d$. In particular, $\mathscr{F}_n \in \operatorname{QCoh}(\mathsf{X})$ is perfect of Tor-amplitude $\leqslant d$. It follows that $M \simeq \Gamma(\mathsf{X}; \mathscr{F})$ has Tor-amplitude $\leqslant c + d$ as a module over κ. In particular, we have $\pi_k M \simeq 0$.

For $1 \leqslant a \leqslant d$, let $M(a)$ denote the tensor product $\bigotimes_{1 \leqslant i \leqslant a} M_i$ (formed in the ∞-category $\operatorname{Mod}_{B_{\mathfrak{m}}}$. We then have fiber sequences

$$M(a) \xrightarrow{y_{a+1}} M(a) \to M(a+1),$$

hence short exact sequences

$$\pi_k M(a) \overset{y_{a+1}}{\to} \pi_k M(a) \to \pi_k M(a+1).$$

Since $\pi_k M(d) \simeq 0$, it follows by descending induction on a (using Nakayama's lemma) that $\pi_k M(a) \simeq 0$ for all a. Taking $a = 0$, we deduce that $\pi_k B_{\mathfrak{m}} \simeq 0$, as desired. □

11.1.5 Serre Functors and Relative Dualizing Sheaves

Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which is proper, locally almost of finite presentation, and locally of finite Tor-amplitude. Theorem **??** implies that the direct image $f_* \mathcal{Q}_{\mathsf{X}}$ is a proper quasi-coherent stack on Y. In this section, we show that the relative dualizing sheaf $\omega_{\mathsf{X}/\mathsf{Y}}$ of Definition 6.4.2.4 admits a categorical interpretation: it controls the *Serre functor* of the quasi-coherent stack $f_* \mathcal{Q}_{\mathsf{X}}$ (see Construction 11.1.5.1 below). To simplify the exposition, we will assume throughout this section that the spectral Deligne-Mumford stack Y is affine (though all constructions we consider can be globalized).

Construction 11.1.5.1 (The Serre Functor)**.** Let A be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a proper A-linear ∞-category. In particular, $\mathcal{C}$ is compactly generated and is therefore dualizable as an object of $\operatorname{LinCat}^{\operatorname{St}}_A$. Choose an A-linear functor $e : \mathcal{C} \otimes_A \mathcal{C}^\vee \to \operatorname{Mod}_A$ which exhibits $\mathcal{C}^\vee$ as a dual of $\mathcal{C}$ in the symmetric monoidal ∞-category $\operatorname{LinCat}^{\operatorname{St}}_A$. Let $e^R : \operatorname{Mod}_A \to \mathcal{C} \otimes_A \mathcal{C}^\vee$ be a right adjoint of e. Our assumption that $\mathcal{C}$ is proper guarantees that the functor e preserves compact objects (Proposition 11.1.3.3), so that the functor e^R commutes with filtered colimits (Proposition HTT.5.5.7.2). Applying Remark D.1.5.3, we can regard e^R as an A-linear functor from Mod_A to $\mathcal{C} \otimes_A \mathcal{C}^\vee$, which we can identify with an A-linear functor

$S_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. We will refer to $S_{\mathcal{C}}$ as the *Serre functor of* $\mathcal{C}$; it can be described as the composition

$$\mathcal{C} \simeq \mathrm{Mod}_A \otimes_A \mathcal{C} \xrightarrow{e^R \otimes \mathrm{id}} \mathcal{C} \otimes_A \mathcal{C}^\vee \otimes_A \mathcal{C} \xrightarrow{\mathrm{id} \otimes e} \mathcal{C} \otimes_A \mathrm{Mod}_A \simeq \mathcal{C}.$$

Remark 11.1.5.2. In the situation of Construction 11.1.5.1, the Serre functor $S_{\mathcal{C}}$ has the following property: for every pair of compact objects $C, D \in \mathcal{C}$, there is a canonical equivalence of A-modules $\underline{\mathrm{Map}}_{\mathcal{C}}(C, S_{\mathcal{C}}(D)) \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(D, C)^\vee$. Note that this description *essentially* characterizes the functor $S_{\mathcal{C}}$ (if we ignore issues of A-linearity), by virtue of our assumption that $\mathcal{C}$ is compactly generated.

We now specialize Construction 11.1.5.1 to the geometric setting:

Proposition 11.1.5.3. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral algebraic spaces which is proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude. Assume that* $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, A$*, so that* $\mathrm{QCoh}(\mathsf{X})$ *is a proper A-linear ∞-category (Proposition 11.1.4.2), and let* $S : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X})$ *be its Serre functor (Construction 11.1.5.1). Then S is given by the construction $\mathscr{F} \mapsto \mathscr{F} \otimes \omega_{\mathsf{X}/\mathsf{Y}}$, where $\omega_{\mathsf{X}/\mathsf{Y}}$ denotes the relative dualizing sheaf of f (Definition 6.4.2.4).*

Remark 11.1.5.4. Combining Remark 11.1.5.2 with Proposition 11.1.5.3, we see that for $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(\mathsf{Y})^{\mathrm{perf}}$, we have a canonical equivalence of A-modules

$$\underline{\mathrm{Map}}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \mathscr{G} \otimes \omega_{\mathsf{X}/\mathsf{Y}}) \simeq \underline{\mathrm{Map}}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{G}, \mathscr{F})^\vee.$$

In the special case where $\mathscr{G} = \mathscr{O}_{\mathsf{X}}$, this reduces to Serre-Grothendieck duality (which, in the present setting, is essentially immediate from our definition of $\omega_{\mathsf{X}/\mathsf{Y}}$). The terminology of Construction 11.1.5.1 is motivated by this observation: for a general proper A-linear ∞-category $\mathcal{C}$, the functor $S_{\mathcal{C}}$ controls the "twist" which needs to be added for $\mathcal{C}$ to enjoy some analogue of Serre-Grothendieck duality.

Proof of Proposition 11.1.5.3. Note that the A-linear ∞-category $\mathrm{QCoh}(\mathsf{X})$ is canonically self-dual via the evaluation map $e : \mathrm{QCoh}(\mathsf{X}) \otimes_A \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Mod}_A$ given by the composition

$$\mathrm{QCoh}(\mathsf{X}) \otimes_A \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\otimes} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\Gamma(\mathsf{X}; \bullet)} \mathrm{Mod}_A.$$

For each $n \geqslant 0$, let $\mathsf{X}^{(n)}$ denote the n-fold fiber power of X over Y. In what follows, we will use Corollary 9.4.3.8 to identify $\mathrm{QCoh}(\mathsf{X}^{(n)})$ with the n-fold tensor power of $\mathrm{QCoh}(\mathsf{X})$ in the ∞-category $\mathrm{LinCat}_A^{\mathrm{St}}$. Under this identification, the evaluation functor e corresponds to the composition $\mathrm{QCoh}(\mathsf{X}^{(2)}) \xrightarrow{\delta^*} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{f_*} \mathrm{QCoh}(\mathsf{Y})$, where $\delta : \mathsf{X} \to \mathsf{X}^{(2)}$ denotes the diagonal map. It follows that the right adjoint e^R is given by the composition $\mathrm{QCoh}(\mathsf{Y}) \xrightarrow{f^!}$

$\mathrm{QCoh}(\mathsf{X}) \xrightarrow{\delta_*} \mathrm{QCoh}(\mathsf{X}^{(2)})$. Let $\pi : \mathsf{X}^{(2)} \to \mathsf{X}$ denote the projection onto the second factor, so that we have a commutative diagram of stable A-linear ∞-categories

$$\begin{array}{ccccccccc} & & & & \mathrm{QCoh}(\mathsf{X}) & & & & \\ & & & \nearrow^{\delta^*} & & \searrow^{\delta_*} & & & \\ & & \mathrm{QCoh}(\mathsf{X}^{(2)}) & & & & \mathrm{QCoh}(\mathsf{X}^{(2)}) & & \\ & \nearrow^{\pi^!} & & \searrow^{(\mathrm{id}\times\delta)_*} & & \nearrow^{(\delta\times\mathrm{id})^*} & & \searrow^{\pi_*} & \\ \mathrm{QCoh}(\mathsf{X}) & & & & \mathrm{QCoh}(\mathsf{X}^{(3)}) & & & & \mathrm{QCoh}(\mathsf{X}). \end{array}$$

Unwinding the definitions, we see that the Serre functor $S_{\mathrm{QCoh}(\mathsf{X})} = (\mathrm{id} \otimes e) \circ (e^R \otimes \mathrm{id})$ can be described as the composition of the four lower arrows in this diagram. Using the commutativity of the diagram (and the observation that $\pi_* \circ \delta_* \simeq (\pi \circ \delta)_*$ is equivalent to the identity functor), we see that the Serre functor $S_{\mathrm{QCoh}(\mathsf{X})}$ can be identified with the composition $\delta^* \circ \pi^!$, which is given by $\mathscr{F} \mapsto \mathscr{F} \otimes \omega_{\mathsf{X}/\mathsf{Y}}$ by virtue of Corollary 6.4.2.7 and Remark 6.4.2.6. $\square$

11.2 Smooth Morphisms in Spectral Algebraic Geometry

Let $f : X \to Y$ be a morphism of schemes. Recall that f is said to be *smooth* if it is locally of finite presentation and satisfies the following additional condition:

$(*)$ If $\phi : B \to \overline{B}$ is a surjective morphism of commutative rings whose kernel $\ker(\phi)$ is a nilpotent ideal of B, then every lifting problem

$$\begin{array}{ccc} \operatorname{Spec} \overline{B} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} B & \longrightarrow & Y \end{array}$$

admits a solution.

Suppose that we wish to consider an analogous theory of smoothness in the setting of spectral algebraic geometry. We first ask: in the formulation of condition $(*)$, what is a reasonable analogue of the hypothesis that $\ker(\phi)$ is nilpotent, if we allow ϕ to be a morphism of $\mathbb{E}_\infty$-rings? Motivated by the philosophy that the spectrum of a connective $\mathbb{E}_\infty$-ring B should be regarded as a infinitesimal enlargement the ordinary scheme $\operatorname{Spec} \pi_0 B$, let us say that a map of connective $\mathbb{E}_\infty$-rings $B \to \overline{B}$ is a *nilpotent thickening* if it induces a surjection $\pi_0 B \to \pi_0 \overline{B}$ whose whose kernel is a nilpotent ideal of $\pi_0 B$. We might then demand that the lifting criterion $(*)$ is satisfied whenever $\phi : B \to \overline{B}$ is an infinitesimal thickening of

$\mathbb{E}_\infty$-rings. This leads to the notion of a *differentially smooth* morphism $f : \mathsf{X} \to \mathsf{Y}$ between spectral Deligne-Mumford stacks.

A fundamental feature of classical algebraic geometry is that any morphism of schemes $f : X \to Y$ which is of finite presentation and satisfies condition $(*)$ is automatically flat (see Proposition 11.2.4.1 below). In the setting of spectral algebraic geometry, the analogous statement is false: a differentially smooth morphism of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ is usually not flat. For example, if R is a commutative ring and $R\{x\}$ denotes the free $\mathbb{E}_\infty$-algebra over R on one generator, then the induced map $\operatorname{Spét} R\{x\} \to \operatorname{Spét} R$ is differentially smooth, but is flat only if $\pi_0 R$ is an algebra over the field $\mathbf{Q}$ of rational numbers.

If $f : \mathsf{X} \to \mathsf{Y}$ is a flat morphism of spectral Deligne-Mumford stacks, then it is generally not reasonable to exact that f satisfies the lifting criterion $(*)$ for an arbitrary infinitesimal thickening $B \to \overline{B}$ (unless f is étale or we work in characteristic zero). However, there is a large class of flat morphisms which satisfy $(*)$ in the special case where B and $\overline{B}$ are discrete, which we will refer to as *fiber smooth* morphisms.

Our goal in this section is to introduce the notions of differential smoothness and fiber smoothness and to describe their relationships to one another (and to smoothness in classical algebraic geometry). We will focus our attention primarily on the affine case, postponing global definitions until §11.2.5 (see Definition 11.2.5.5).

11.2.1 Formally Smooth Morphisms of $\mathbb{E}_\infty$-Rings

Our first step is to formulate an infinitesimal lifting criterion in the setting of spectral algebraic geometry.

Definition 11.2.1.1. Let $f : R \to A$ be a map of connective $\mathbb{E}_\infty$-rings. We will say that f is *formally smooth* if the following condition is satisfied: for every morphism $B \to \overline{B}$ in $\mathrm{CAlg}^{\mathrm{cn}}_R$, if the underlying map $\pi_0 B \to \pi_0 \overline{B}$ is a surjection whose kernel is a nilpotent ideal of $\pi_0 B$, the induced map

$$\mathrm{Map}_{\mathrm{CAlg}_R}(A, B) \to \mathrm{Map}_{\mathrm{CAlg}_R}(A, \overline{B})$$

is surjective on connected components.

Formal smoothness can be characterized in terms of the cotangent complex:

Proposition 11.2.1.2. *Let $\phi : R \to A$ be a morphism of connective $\mathbb{E}_\infty$-algebras. The following conditions are equivalent:*

(1) *The morphism ϕ is formally smooth, in the sense of Definition 11.2.1.1.*

(2) *The relative cotangent complex $L_{A/R}$ is a projective A-module (Definition HA.7.2.2.4).*

Proof. Assume first that ϕ is formally smooth; we wish to show that $L_{A/R}$ is projective. By virtue of Proposition HA.7.2.2.6 , it will suffice to show that for every cofiber sequence $N' \to N \to N''$ of connective A-modules, the induced map $\theta : \mathrm{Map}_{\mathrm{Mod}_A}(L_{A/R}, N) \to \mathrm{Map}_{\mathrm{Mod}_A}(L_{A/R}, N'')$ is surjective on connected components: in other words, we wish to show that the homotopy fibers of θ are nonempty. Fix a map $L_{A/k} \to N''$, corresponding to a section of s the projection map $A \oplus N'' \to A$. Invoking the definition of $L_{A/R}$, we see that the homotopy fiber of θ over s can be identified with the homotopy fiber of the map

$$\theta' : \mathrm{Map}_{\mathrm{CAlg}_R}(A, A \oplus N) \to \mathrm{Map}_{\mathrm{CAlg}_R}(A, A \oplus N'').$$

Since the map $A \oplus N \to A \oplus N''$ induces a surjection $\pi_0(A \oplus N) \to \pi_0(A \oplus N'')$ with nilpotent kernel, the homotopy fibers of θ' are nonempty by virtue of (1).

Now suppose that $L_{A/R}$ is projective. We wish to prove that ϕ is formally smooth. Let $B \to \overline{B}$ be a map of connective $\mathbb{E}_\infty$-algebras over R which induces a surjection $\pi_0 B \to \pi_0 \overline{B}$ having nilpotent kernel, and let $\eta \in \mathrm{Map}_{\mathrm{CAlg}_R}(A, \overline{B})$; we wish to show that η can be lifted to a point of $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B)$. We define a tower of B-algebras

$$\ldots \to B(2) \to B(1) \to B(0) = \overline{B}$$

by induction as follows. Assume that $B(i)$ has been constructed, set $M(i) = L_{B(i)/B}$, and let $d : B(i) \to B(i) \oplus M(i)$ be the tautological map. By construction, we have a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & B(i) \\ \downarrow & & \downarrow {\scriptstyle d_0} \\ B(i) & \xrightarrow{d} & B(i) \oplus M(i); \end{array}$$

here d_0 corresponds to the trivial derivation. We now define $B(i+1)$ to be the fiber product $B(i) \times_{B(i) \oplus M(i)} B(i)$.

Let $I \subseteq \pi_0 B$ be the kernel of the surjection $\pi_0 B \to \pi_0 \overline{B}$. We first claim:

($*$) For every integer $n \geqslant 0$, the algebra $B(n)$ is connective. Moreover, the map $\pi_0 B \to \pi_0 B(n)$ is a surjection, whose kernel is the ideal I^{2^n}.

The proof of ($*$) proceeds by induction on n. Assume that ($*$) holds for $B(n)$, and let K denote the fiber of the map $B \to B(n)$. Condition ($*$) guarantees that K is connective, and that the image of the map $\pi_0 K \to \pi_0 B$ is the ideal $J = I^{2^n}$. We have a map of fiber sequences

$$\begin{array}{ccccc} K & \longrightarrow & B & \longrightarrow & B(n) \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-1} M(n) & \longrightarrow & B(n+1) & \longrightarrow & B(n), \end{array}$$

so the fiber K' of the map $B \to B(n+1)$ can be identified with the fiber of the composition

$$K \xrightarrow{\beta} K \otimes_B B(n) \xrightarrow{\alpha} \Sigma^{-1} M(n).$$

To prove $(*)$, it will suffice to show that K' is connective and the image of the map $\pi_0 K' \to \pi_0 B$ is J^2. We have a fiber sequence $\mathrm{fib}(\beta) \to K' \to \mathrm{fib}(\alpha)$. Since K is connective, Theorem HA.7.4.3.1 guarantees that $\mathrm{fib}(\alpha)$ is 1-connective. It follows that the maps $\pi_i \mathrm{fib}(\beta) \to \pi_i K'$ are surjective for $i \leqslant 0$. To complete the proof, it will therefore suffice to show that $\mathrm{fib}(\beta)$ is connective and the map $\pi_0 \mathrm{fib}(\beta) \to \pi_0 B$ has image J^2. This follows from the observation that $\mathrm{fib}(\beta) \simeq K \times_B K$, so that $\pi_0 \mathrm{fib}(\beta) \simeq \mathrm{Tor}_0^{\pi_0 B}(\pi_0 K, \pi_0 K)$. Under this identification, the map $\pi_0 \mathrm{fib}(\beta) \to \pi_0 B$ corresponds to the bilinear multiplication map

$$\pi_0 K \times \pi_0 K \to J \times J \to \pi_0 B,$$

whose image generates the ideal $J^2 \subseteq \pi_0 B$. This completes the proof of $(*)$.

Choose any map of R-algebras $A \to B(n)$. Since $L_{A/R}$ is projective, the mapping space $\mathrm{Map}_{\mathrm{Mod}_A}(L_{A/R}, M(n))$ is connected. It follows that the homotopy fibers of the projection map

$$\mathrm{Map}_{\mathrm{CAlg}_R}(A, B(n) \oplus M(n)) \to \mathrm{Map}_{\mathrm{CAlg}_R}(A, B(n))$$

are connected. Consequently, for any derivation $d : B(n) \to B(n) \oplus M(n)$, the homotopy fibers of the induced map

$$\mathrm{Map}_{\mathrm{CAlg}_R}(A, B(n)) \to \mathrm{Map}_{\mathrm{CAlg}_R}(A, B(n) \oplus M(n))$$

are nonempty. It follows that the map $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B(n+1)) \to \mathrm{Map}_{\mathrm{CAlg}_R}(A, B(n))$ also has nonempty homotopy fibers. Consequently, to prove that a point $\eta \in \mathrm{Map}_{\mathrm{CAlg}_R}(A, \overline{B}) = \mathrm{Map}_{\mathrm{CAlg}_R}(A, B(0))$ can be lifted to $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B)$, we are free to replace $\overline{B}$ by $B(n)$, for any integer $n \geqslant 0$. Since I is nilpotent, condition $(*)$ implies that $\pi_0 B \simeq \pi_0 B(n)$ for $n \gg 0$. We may therefore reduce to the case where $\pi_0 \overline{B} = \pi_0 B$.

We now return to our analysis of the tower

$$\ldots \to B(2) \to B(1) \to B(0)$$

defined above, and prove the following strengthening of $(*)$:

$(*')$ For $n \geqslant 0$, the fiber of the map $B \to B(n)$ is 2^n-connective.

The proof of $(*')$ proceeds by induction on n, the case $n = 0$ being obvious. Assume therefore that $B \to B(n)$ is 2^n-connective, and let K and K' be as in the proof of $(*)$. We wish to prove that K' is 2^{n+1}-connective. As before, we have a fiber sequence

$$\mathrm{fib}(\alpha) \to K' \to \mathrm{fib}(\beta).$$

Here $\mathrm{fib}(\beta) \simeq K \otimes_B K$, and is therefore 2^{n+1}-connective since K is 2^n-connective by the inductive hypothesis. The map α is $(2^{n+1}+1)$-connective by Theorem HA.7.4.3.1 .

As before, each of the maps $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B(n+1)) \to \mathrm{Map}_{\mathrm{CAlg}_R}(A, B(n))$ is surjective on connected components, so we can lift η to a point of $\varprojlim \mathrm{Map}_{\mathrm{CAlg}_R}(A, B(n)) \simeq \mathrm{Map}_{\mathrm{CAlg}_R}(A, \varprojlim B(n))$. To complete, the proof, it suffices to show that the canonical map $B \to \varprojlim B(n)$ is an equivalence. This follows from $(*')$, since the ∞-category $\mathrm{CAlg}_R^{\mathrm{cn}}$ is Postnikov complete (in the sense of Definition A.7.2.1; see Proposition HA.7.1.3.19). □

11.2.2 Differentially Smooth Morphisms of $\mathbb{E}_\infty$-Rings

We now combine Definition 11.2.1.1 with a mild finiteness condition.

Proposition 11.2.2.1. *Let $\phi : R \to A$ be a map of connective $\mathbb{E}_\infty$-rings, and suppose that ϕ exhibits $\pi_0 A$ as a finitely presented commutative algebra over $\pi_0 R$. The following conditions are equivalent:*

(1) *The map ϕ is formally smooth.*

(2) *The relative cotangent complex $L_{A/R}$ is a projective A-module.*

(3) *The relative cotangent complex $L_{A/R}$ is a projective A-module of finite rank.*

(4) *There exist elements $a_1, \ldots, a_n \in \pi_0 A$ which generate the unit ideal of $\pi_0 A$ and a collection of étale maps $R\{x_1, \ldots, x_{m_i}\} \to A[a_i^{-1}]$.*

Here $R\{x_1, \ldots, x_k\} = \mathrm{Sym}_R^*(R^k)$ denotes the free $\mathbb{E}_\infty$-algebra over R on k generators.

Proof. The equivalence (1) ⇔ (2) follows from Proposition 11.2.1.2, the implication (3) ⇒ (2) is obvious, and the implication (2) ⇒ (3) follows from the observation that $\pi_0 L_{A/R}$ is the module of Kähler differentials of $\pi_0 A$ over $\pi_0 R$ (Proposition HA.7.4.3.9) and therefore finitely presented over $\pi_0 A$.

We will complete the proof by showing that (4) ⇔ (3). Assume first that (4) is satisfied. The condition that $L_{A/R}$ is a projective A-module of finite rank is local with respect to the étale topology on A (see Proposition 2.8.4.2); we may therefore assume that $A = R\{x_1, \ldots, x_m\}$ for some integer m, in which case the result is obvious.

Now suppose that (3) is satisfied; we will prove (4). Suppose that the module $\pi_0 L_{A/R}$ is projective and of finite rank over $\pi_0 A$. Then there exist elements $a_1, \ldots, a_n$ generating the unit ideal in $\pi_0 A$ such that each of the modules $(\pi_0 L_{A/R})[a_i^{-1}]$ is a free module of some rank m_i over $(\pi_0 A)[a_i^{-1}]$. Replacing A by $A[a_i^{-1}]$, we may suppose that $\pi_0 L_{A/R}$ is a free module of some rank $m \geqslant 0$. Proposition HA.7.4.3.9 allows us to identify $\pi_0 L_{A/R}$ with the module of Kähler differentials of $\pi_0 A$ over $\pi_0 R$. In particular, $\pi_0 L_{A/R}$ is generated (as an A-module) by finitely many differentials $\{dx_p\}_{1 \leqslant p \leqslant q}$. The identification $\pi_0 L_{A/k} \simeq (\pi_0 A)^m$ allows us to view the differentials $\{dx_q\}_{1 \leqslant p \leqslant q}$ as an m-by-q matrix M. Let $\{b_j\}$ be the collection of all determinants of m-by-m square submatrices appearing in M. Since the elements $\{dx_p\}_{1 \leqslant p \leqslant q}$ generate $(\pi_0 A)^m$, the matrix M has rank m so that the elements b_j generate the unit ideal in A. It therefore suffices to prove that (4) is satisfied by each of the algebras $A[b_j^{-1}]$. We may therefore assume (after discarding some of the elements x_i) that $q = m$ and that $\pi_0 L_{A/R}$ is freely generated by the elements dx_i. The choice of elements $x_1, \ldots, x_m \in \pi_0 A$ determines a map $R\{x_1, \ldots, x_m\} \to A$. The fiber sequence

$$A \otimes_{R\{x_1,\ldots,x_n\}} L_{R\{x_1,\ldots,x_m\}/R} \to L_{A/R} \to L_{A/R\{x_1,\ldots,x_m\}}$$

shows that the relative cotangent complex $L_{A/R\{x_1,\ldots,x_m\}}$ vanishes. It follows from Lemma B.1.3.3 that A is étale over $R\{x_1, \ldots, x_m\}$. □

Definition 11.2.2.2. Let $\phi : R \to A$ be a map of connective $\mathbb{E}_\infty$-rings. We will say that ϕ is *differentially smooth* if it satisfies the equivalent conditions of Proposition 11.2.2.1. That is, ϕ is differentially smooth if it is formally smooth and the commutative ring $\pi_0 A$ is finitely presented as an algebra over $\pi_0 R$.

Remark 11.2.2.3. Let $\phi : R \to A$ be a differentially smooth morphism of connective $\mathbb{E}_\infty$-rings. Then ϕ is locally of finite presentation: that is, ϕ exhibits A as a compact object of CAlg_R. This follows immediately from characterization (4) of Proposition 11.2.2.1.

Conversely, suppose that ϕ is almost of finite presentation. Then ϕ is differentially smooth if and only if $L_{A/R}$ is a flat A-module (Proposition HA.7.2.4.20).

Remark 11.2.2.4. Suppose we are given a pushout diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ A & \longrightarrow & A'. \end{array}$$

If ϕ is differentially smooth, then ϕ' is differentially smooth.

Example 11.2.2.5. Any étale morphism of connective $\mathbb{E}_\infty$-rings is differentially smooth.

Differential smoothness can be tested fiberwise:

Proposition 11.2.2.6. *Let $\phi : R \to A$ be a morphism of connective $\mathbb{E}_\infty$-rings, and suppose that ϕ is almost of finite presentation. Then ϕ is differentially smooth if and only if, for every field κ and every morphism $R \to \kappa$, the tensor product $\kappa \otimes_R A$ is differentially smooth over κ.*

Lemma 11.2.2.7. *Let A be a connective $\mathbb{E}_\infty$-ring, and let M be an A-module which is connective and almost perfect. The following conditions are equivalent:*

(a) *The module M is projective of finite rank.*

(b) *For every field κ and every morphism $A \to \kappa$, the tensor product $\kappa \otimes_A M$ is discrete.*

Proof. It is clear that $(a) \Rightarrow (b)$. Conversely, suppose that (b) is satisfied. To prove that M is projective of finite rank, it will suffice to show that M is flat (Proposition HA.7.2.4.20). That is, we must show that if N is a discrete A-module, then $M \otimes_A N$ is also discrete. For this, it suffices to show that the localization $(M \otimes_A N)_{\mathfrak{p}}$ is discrete, for each prime ideal $\mathfrak{p} \subseteq A$. Replacing A by the localization $A_{\mathfrak{p}}$, we may assume that A is local. Let κ denote the residue field of A, and choose a collection of elements $x_1, \ldots, x_n \in M$ whose images form a basis for $\pi_0(\kappa \otimes_A M)$. These elements determine a map of A-modules $f : A^n \to M$. By construction, f induces a map $\kappa^n \to \kappa \otimes_A M$ which is an isomorphism on π_0, and therefore a homotopy equivalence (since condition (b) implies that $\pi_i(\kappa \otimes_A M) \simeq 0$ for $i \neq 0$). Let $K = \mathrm{fib}(f)$. We claim that $K \simeq 0$, so that f is an equivalence and therefore $M \simeq A^n$ is projective. Assume otherwise. Then, since K is almost perfect, there exists a smallest integer d such that $\pi_d K \neq 0$. In this case, Nakayama's lemma implies that $\pi_d(\kappa \otimes_A K) \simeq \mathrm{Tor}_0^{\pi_0 A}(\kappa, \pi_d K) \neq 0$ and we obtain a contradiction. □

Proof of Proposition 11.2.2.6. Since A is almost of finite presentation over R, the relative cotangent complex $L_{A/R}$ is connective and almost perfect (Theorem HA.7.4.3.18). Consequently, to prove that $L_{A/R}$ is a projective A-module of finite rank, it will suffice to show that $\kappa \otimes_A L_{A/R}$ is discrete, for every field κ and every map $A \to \kappa$. Replacing A by $\kappa \otimes_R A$, we can reduce to the case where R is a field, in which case the desired result follows from our assumption that $\kappa \otimes_R A$ is differentially smooth over κ. □

Corollary 11.2.2.8. *Let $\phi : R \to A$ be a morphism of connective $\mathbb{E}_\infty$-rings. Then ϕ is differentially smooth if and only if the induced map $\phi_0 : \pi_0 R \to (\pi_0 R) \otimes_R A$ is differentially smooth.*

Proof. The "only if" direction is clear. Conversely, suppose that ϕ_0 is differentially smooth. Then ϕ_0 is almost of finite presentation, so that ϕ is almost of finite presentation (Proposition 4.1.3.4). The desired result now follows immediately from Proposition 11.2.2.6. □

Proposition 11.2.2.9. *Let $\phi : A \to B$ and $\psi : B \to C$ be morphisms of connective $\mathbb{E}_\infty$-rings. If ϕ and ψ are formally smooth (differentially smooth), then $\psi \circ \phi$ is also formally smooth (differentially smooth).*

Proof. Suppose first that ϕ and ψ are formally smooth. Using Proposition 11.2.1.2, we deduce that $L_{C/B}$ and $C \otimes_B L_{B/A}$ are projective C-modules. It follows that the fiber sequence

$$C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B}$$

splits, so that $L_{C/A} \simeq L_{C/B} \oplus (C \otimes_B L_{B/A})$ is also a projective C-module, and therefore $\psi \circ \phi$ is formally smooth (Proposition 11.2.1.2). If, in addition, ϕ and ψ are differentially smooth, then ϕ and ψ are almost of finite presentation, so that $\psi \circ \phi$ is also almost finite presentation and therefore differentially smooth. □

11.2.3 Fiber Smooth Morphisms of $\mathbb{E}_\infty$-Rings

From the perspective of classical algebraic geometry, the notion of differential smoothness (Definition 11.2.2.2) suffers from two (closely related) defects:

(*a*) A differentially smooth morphism of $\mathbb{E}_\infty$-rings need not be flat.

(*b*) A morphism of commutative rings which is smooth (in the sense of classical commutative algebra) need not be differentially smooth (when regarded as a morphism of $\mathbb{E}_\infty$-rings).

We now consider a different notion of smoothness, which does not suffer from either of these problems.

Definition 11.2.3.1. Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. We will say that f is *fiber smooth* if the following conditions are satisfied:

(1) The morphism f is flat.

(2) The morphism f is almost of finite presentation.

(3) For every field κ and every morphism $B \to \kappa$, the vector space $\pi_1(\kappa \otimes_B L_{B/A})$ vanishes.

Example 11.2.3.2. Let $f : A \to B$ be an étale morphism between connective $\mathbb{E}_\infty$-rings. Then f is fiber smooth.

Remark 11.2.3.3. Let $f : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings which is faithfully flat and almost of finite presentation. Then f is a universal descent morphism (this follows from Proposition D.3.3.1, since $\pi_0 B$ is countably presented as a module over $\pi_0 A$). In particular, if f is faithfully flat and fiber smooth, then f is a universal descent morphism.

We now establish some basic formal properties of fiber smoothness.

Proposition 11.2.3.4. (1) *Let $f : A \to B$ and $g : B \to C$ be morphisms of connective $\mathbb{E}_\infty$-rings. If f and g are fiber smooth, then $g \circ f$ is fiber smooth.*

(2) *Suppose we are given a pushout diagram of $\mathbb{E}_\infty$-rings*

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \longrightarrow & B'. \end{array}$$

If f is fiber smooth, then f' is fiber smooth.

Proof. Assertion (2) follows immediately from the definitions. We will prove (1). Suppose that $f : A \to B$ and $g : B \to C$ are fiber smooth. It is clear that $g \circ f$ is almost of finite presentation and flat. Using the fiber sequence

$$C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B},$$

we deduce that for every field κ and every map $C \to \kappa$ there is an exact sequence

$$\pi_1(\kappa \otimes_B L_{B/A}) \to \pi_1(\kappa \otimes_C L_{C/A}) \to \pi_1(\kappa \otimes_C L_{C/B}),$$

from which we deduce that $\pi_1(\kappa \otimes_C L_{C/A}) \simeq 0$. □

Remark 11.2.3.5. Let $f : R \to A$ be a map of connective $\mathbb{E}_\infty$-rings, and form a pushout diagram

$$\begin{array}{ccc} R & \longrightarrow & \pi_0 R \\ \downarrow f & & \downarrow f' \\ A & \longrightarrow & \pi_0 R \otimes_R A. \end{array}$$

Then f is fiber smooth if and only if f' is fiber smooth. The "only if" direction follows from Proposition 11.2.3.4, and the converse follows from Remarks 6.1.2.6 and Corollary 4.1.3.5.

We first show that fiber smoothness can be interpreted as a regularity condition on the (geometric) fibers of a morphism.

Proposition 11.2.3.6. *Let $f : R \to A$ be a map of connective $\mathbb{E}_\infty$-rings which is almost of finite presentation. Then the following conditions are equivalent:*

(1) *The morphism f is fiber smooth.*

(2) *For every algebraically closed field κ and every map $R \to \kappa$, the tensor product $\kappa \otimes_R A$ is discrete, and is regular when regarded as a commutative ring.*

Corollary 11.2.3.7. *Every fiber smooth morphism of connective $\mathbb{E}_\infty$-rings is geometrically reduced (see Definition 8.6.2.4).*

Remark 11.2.3.8. Let $f : R \to A$ be a morphism of connective $\mathbb{E}_\infty$-rings which is almost of finite presentation. It follows from Proposition **??** that f is fiber smooth if and only if, for every residue field κ of A, the induced map $\kappa \to \kappa \otimes_A B$ is fiber smooth.

Lemma 11.2.3.9. *Let R be a local Noetherian commutative ring with residue field κ. The following conditions are equivalent:*

(a) *The local ring R is regular. That is, the maximal ideal $\mathfrak{m} \subseteq R$ is generated by a regular sequence $x_1, \ldots, x_n \in \mathfrak{m}$.*

(b) *The vector space* $\pi_2 L_{\kappa/R}$ *vanishes.*

Proof. Choose a sequence of elements $x_1, \ldots, x_n \in \mathfrak{m}$ which form a basis for $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over κ. Let $R[X_1, \ldots, X_n]$ be a polynomial ring over R on n variables, and define $\phi_0, \phi : R[X_1, \ldots, X_n] \to R$ by the formulas

$$\phi(X_i) = x_i \qquad \phi_0(X_i) = 0.$$

Form a pushout diagram

$$\begin{array}{ccc} R[X_1, \ldots, X_n] & \xrightarrow{\phi} & R \\ \downarrow{\scriptstyle \phi_0} & & \downarrow \\ R & \longrightarrow & A \end{array}$$

in the ∞-category CAlg_R, so that we have a canonical $A \to \kappa$ which induces an isomorphism $\pi_0 A \simeq \kappa$.

Let $B = R\{X_1, \ldots, X_n\}$ denote the free $\mathbb{E}_\infty$-algebra over R on n-variables, so that $\pi_0 B \simeq R[X_1, \ldots, X_n]$. Then $L_{B/R} \simeq B^n$, so that

$$\pi_i(\kappa \otimes_B L_{B/R}) \simeq \begin{cases} \kappa^n & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

The truncation map $B \to R[X_1, \ldots, X_n]$ has 2-connective cofiber. Applying Corollary HA.7.4.3.2 , we deduce that $L_{R[X_1,\ldots,X_n]/B}$ is 2-connective. Using the fiber sequence

$$\kappa \otimes_B L_{B/R} \to \kappa \otimes_{R[X_1,\ldots,X_n]} L_{R[X_1,\ldots,X_n]/R} \to \kappa \otimes_{R[X_1,\ldots,X_n]} L_{R[X_1,\ldots,X_n]/B},$$

we conclude that

$$\pi_i(\kappa \otimes_{R[X_1,\ldots,X_n]} L_{R[X_1,\ldots,X_n]/R}) \simeq \begin{cases} \kappa^n & \text{if } i = 0 \\ 0 & \text{if } i = 1. \end{cases}$$

Using the fiber sequence

$$\kappa \otimes_{R[X_1,\ldots,X_n]} L_{R[X_1,\ldots,X_n]/R} \to \kappa \otimes_R L_{R/R} \to \kappa \otimes_R L_{R/R[X_1,\ldots,X_n]},$$

we obtain isomorphisms

$$\begin{aligned} \pi_i(\kappa \otimes_A L_{A/R}) &\simeq \pi_i(\kappa \otimes_R L_{R/R[X_1,\ldots,X_n]}) \\ &\simeq \pi_{i-1}(\kappa \otimes_{R[X_1,\ldots,X_n]} L_{R[X_1,\ldots,X_n]/R}) \\ &\simeq \begin{cases} \kappa^n & \text{if } i = 1 \\ 0 & \text{if } i = 2. \end{cases} \end{aligned}$$

Since the map $A \to \kappa$ has 2-connective cofiber, $L_{\kappa/A}$ is 2-connective. Using the fiber sequence

$$\kappa \otimes_A L_{A/R} \to L_{\kappa/R} \to L_{\kappa/A},$$

we obtain a short exact sequence

$$0 \to \pi_2 L_{\kappa/R} \xrightarrow{\alpha} \pi_2 L_{\kappa/A} \to \kappa^n \xrightarrow{\beta} \pi_1 L_{\kappa/R} \to \pi_1 L_{\kappa/A} = 0.$$

Theorem HA.7.4.3.1 supplies an isomorphism $\pi_1 L_{\kappa/R} \simeq \mathfrak{m}/\mathfrak{m}^2$, so that $\pi_1 L_{\kappa/R}$ is an n-dimensional vector space over κ. Since β is a surjection between vector spaces of the same dimension, it is also injective. It follows that α is an isomorphism.

Suppose now that (a) is satisfied. Then $x_1, \ldots, x_n$ is a regular sequence on R, so that $A \simeq \kappa$. It follows that $\pi_2 L_{\kappa/R} \simeq \pi_2 L_{\kappa/A} \simeq 0$.

We now prove the converse. Suppose that (b) is satisfied, so that $\pi_2 L_{\kappa/R} \simeq 0$. Then we also have $\pi_2 L_{\kappa/A} \simeq 0$. Applying Corollary HA.7.4.3.2 , we deduce that $\pi_1 A \simeq 0$. For $1 \leqslant i \leqslant n$, let M_i denote the cofiber of the map $x_i : R \to R$ in Mod_R, and for $0 \leqslant i \leqslant n$ let $M(i)$ denote the tensor product $\bigotimes_{j \leqslant i} M_i$ in Mod_R. Then $M(n) \simeq A$, so that $\pi_1 M(n) \simeq 0$. To prove that $x_1, \ldots, x_n$ is a regular sequence in R, it suffices to show that each $M(i)$ is discrete. Suppose otherwise, and choose a minimal integer i such that $M(i)$ is not discrete. Using the fiber sequence

$$M(i-1) \xrightarrow{x_i} M(i-1) \to M(i),$$

we deduce that $M(i)$ is 1-truncated and therefore $\pi_1 M(i) \neq 0$. Using the exact sequences

$$\pi_1 M(j) \xrightarrow{x_{j+1}} \pi_1 M(j) \to \pi_1 M(j+1),$$

and the fact that each $\pi_1 M(j)$ is a finitely generated R-module, we deduce that $\pi_1 M(j) \neq 0$ for $j \geqslant i$. It follows that $\pi_1 M(n) = \pi_1 A \neq 0$, a contradiction. □

Proof of Proposition 11.2.3.6. Suppose first that $f : R \to A$ is fiber smooth, and that we are given a morphism $R \to \kappa$, where κ is an algebraically closed field. Since f is flat, A_κ is discrete. By assumption f is almost of finite presentation, so that A_κ is finitely generated as an algebra over κ (and in particular Noetherian). We wish to show that A_κ is regular. Let $\mathfrak{m}$ be a maximal ideal of A_κ. It follows from Hilbert's Nullstellensatz that the composite map $\kappa \to A_\kappa \to A_\kappa/\mathfrak{m}$ is an isomorphism. The fiber sequence

$$\kappa \otimes_{A_\kappa} L_{A_\kappa/\kappa} \to L_{\kappa/\kappa} \to L_{\kappa/A_\kappa}$$

therefore yields an isomorphism

$$\pi_1(\kappa \otimes_A L_{A/R}) \simeq \pi_1(\kappa \otimes_{A_\kappa} L_{A_\kappa/\kappa}) \simeq \pi_2 L_{\kappa/A_\kappa}.$$

Since f is fiber smooth, we conclude that $\pi_2 L_{\kappa/A_\kappa} \simeq 0$, so that A_κ is regular at $\mathfrak{m}$ by Lemma 11.2.3.9.

Conversely, suppose that f satisfies condition (2); we wish to show that f is fiber smooth. The flatness of f follows from Proposition **??**. To complete the proof, it will suffice to show that for every field κ and every map $A \rightarrow \kappa$, the vector space $\pi_1(\kappa \otimes_A L_{A/R})$ vanishes. We may assume without loss of generality that κ is algebraically closed, define A_κ as above, and let $\mathfrak{m}$ denote the maximal ideal of A_κ given by the kernel of the canonical map $A_\kappa \rightarrow \kappa$. Then we have a canonical isomorphism

$$\pi_1(\kappa \otimes_A L_{A/R}) \simeq \pi_1(\kappa \otimes_{A_\kappa} L_{A_\kappa/\kappa}) \simeq \pi_2 L_{\kappa/A_\kappa}.$$

Since A_κ is a regular Noetherian ring, $\pi_2 L_{\kappa/A_\kappa}$ vanishes by Lemma 11.2.3.9. □

11.2.4 Smoothness in Commutative Algebra

If we restrict our attention to ordinary commutative rings, then fiber smoothness reduces to the classical notion of smooth morphism in commutative algebra. This admits several alternative characterizations:

Proposition 11.2.4.1. *Let $f : R \rightarrow A$ be a morphism of commutative rings which exhibits A as a finitely presented algebra over R. The following conditions are equivalent:*

(1) *When regarded as a map of connective $\mathbb{E}_\infty$-rings, f is fiber smooth.*

(2) *For every field κ and every map $A \rightarrow \kappa$, the vector space $\pi_1(\kappa \otimes_A L_{A/R})$ vanishes.*

(3) *The A-module $\pi_0 L_{A/R}$ is projective and $\pi_1 L_{A/R}$ vanishes.*

(4) *For every commutative R-algebra B and every nilpotent ideal $I \subseteq B$, the canonical map $\mathrm{Hom}_R(A, B) \rightarrow \mathrm{Hom}_R(A, B/I)$ is surjective (here the* Hom*-sets are computed in the category of commutative R-algebras).*

(5) *There exists a finite collection of elements $a_\alpha \in A$ which generate the unit ideal, such that each localization $A[a_\alpha^{-1}]$ admits an étale map $R[x_1, \ldots, x_{n_\alpha}] \rightarrow A[a_\alpha^{-1}]$.*

Proof. The implication (1) ⇒ (2) is obvious. We next show that (2) ⇒ (3). Since A is finitely presented as a commutative algebra over R, the map $f : R \rightarrow A$ is of finite generation to order 1 (Remark 4.1.1.9), so that $L_{A/R}$ is perfect to order 1. It follows that $P = \pi_0 L_{A/R}$ is a finitely presented A-module. We wish to show that P is projective. This assertion is local on $|\operatorname{Spec} A|$ with respect to the Zariski topology. It will therefore suffice to show that for each maximal ideal $\mathfrak{m} \subseteq A$, there exists an element $a \in A - \mathfrak{m}$ such that $P[a^{-1}]$ is projective as a module over $M[a^{-1}]$. Choose elements $x_1, \ldots, x_n \in P$ which form a basis for the vector space $P/\mathfrak{m}P$ over $A/\mathfrak{m}$. The elements x_i determine an A-module homomorphism $\theta : A^n \rightarrow P$. Using Nakayama's lemma, we see that this map is surjective after localization at $\mathfrak{m}$. Replacing A by a localization if necessary, we may suppose that θ is surjective. Let $\kappa = A/\mathfrak{p}$. Using the exact sequence

$$\pi_1(\kappa \otimes_A L_{A/R}) \to \pi_1(\kappa \otimes_A P) \to \pi_0(\kappa \otimes_A \tau_{\geqslant 1} L_{A/R})$$

and (2), we deduce that $\pi_1(\kappa \otimes_A P) \simeq 0$, so that we have an exact sequence

$$0 \to \pi_0(\kappa \otimes_A \ker(\theta)) \to \pi_0(\kappa \otimes_A A^n) \to \pi_0(\kappa \otimes_A P) \to 0.$$

It follows that $\pi_0(\kappa \otimes_A \ker(\theta)) \simeq 0$. Since P is finitely presented, $\ker(\theta)$ is a finitely generated A-module. Using Nakayama's lemma, we deduce that $\ker(\theta)_{\mathfrak{m}} \simeq 0$ and therefore, after replacing A by a localization, we may suppose that $\ker(\theta) = 0$: that is, that θ is an isomorphism. This completes the proof that P is projective. It follows that $L_{A/R}$ splits as a direct sum $P \oplus \tau_{\geqslant 1} L_{A/R}$. In particular, $\tau_{\geqslant 1} L_{A/R}$ is perfect to order 1, so that $Q = \pi_1 L_{A/R}$ is a finitely generated A-module. For every maximal ideal $\mathfrak{m}$ of A, the quotient $Q/\mathfrak{m}Q \simeq \pi_1(A/\mathfrak{m} \otimes_A \tau_{\geqslant 1} L_{A/R})$ is a direct summand of $\pi_1(A/\mathfrak{m} \otimes_A L_{A/R})$, and therefore vanishes. It follows from Nakayama's lemma that $Q \simeq 0$. This completes the proof that $(2) \Rightarrow (3)$.

We next prove that $(3) \Rightarrow (4)$. Assume that (3) is satisfied, let B be a commutative ring with a map $R \to B$, and let $I \subseteq B$ be an ideal with $I^k = 0$; we wish to show that the map $\mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, B/I)$ is surjective. Proceeding by induction on k, we can reduce to the case where $I^2 = 0$. Fix an R-algebra map $\phi_0 : A \to B/I$; we wish to show that ϕ_0 can be lifted to an R-algebra map $\phi : A \to B$. Replacing B by the pullback $A \times_{B/I} B$, we can assume that ϕ_0 is an isomorphism, so that B is a square-zero extension of A by some (discrete) B-module M, classified by a map $\eta : L_{A/R} \to \Sigma M$. Using (3), we deduce that $L_{A/R}$ splits as a direct sum $P \oplus \tau_{\geqslant 2} L_{A/R}$. Since $\mathrm{Map}_{\mathrm{Mod}_A}(\tau_{\geqslant 2} L_{A/R}, \Sigma M)$ is contractible and $\mathrm{Map}_{\mathrm{Mod}_A}(P, \Sigma M)$ is connected, we conclude that η is nullhomotopic, so that the square-zero extension $B \to A$ admits a section.

We next prove that $(4) \Rightarrow (2)$. Suppose that (4) is satisfied; we wish to show that for every field κ and every map $A \to \kappa$, we have $\pi_1(\kappa \otimes_A L_{A/R}) \simeq 0$. Suppose otherwise: then there exists a nonzero map of κ-modules $\kappa \otimes_A L_{A/R} \to \Sigma\kappa$, which classifies a nonzero map of A-modules $L_{A/R} \to \Sigma\kappa$. This map classifies a square-zero extension $\overline{A}$ of A by κ (in the category of R-modules). It follows that the reduction map $\mathrm{Hom}_R(A, \overline{A}) \to \mathrm{Hom}_R(A, A)$ is not surjective, contradicting assumption (4).

We next show that $(5) \Rightarrow (1)$. Since the assertion that A is fiber smooth over R is local with respect to the Zariski topology (Proposition 11.2.5.4), we may assume that there exists an étale map $\psi : R[x_1, \ldots, x_n] \to A$, for some integer n. The map ψ is fiber smooth (Example 11.2.3.2). Using Proposition 11.2.3.4, we are reduced to proving that $R[x_1, \ldots, x_n]$ is fiber smooth over R. Using Proposition 11.2.3.4 again, we can reduce to the case where R is Noetherian. In this case, Proposition HA.7.2.4.31 implies that $R[x_1, \ldots, x_n]$ is almost of finite presentation as an object of CAlg_R. Using Remark 11.2.3.8, we may further reduce to the case where R is a field κ, and using Proposition 11.2.5.3 we may suppose that κ is algebraically closed. It is clear that $\kappa[x_1, \ldots, x_n]$ is flat over κ. Moreover, the Nullstellensatz guarantees that every maximal ideal $\kappa[x_1, \ldots, x_n]$ is generated by a regular sequence of the form $x_1 - c_1, \ldots, x_n - c_n$, for some scalars $c_i \in \kappa$.

We now complete the proof by showing that (3) ⇒ (5). Suppose that (3) is satisfied; in particular, this implies that $\pi_0 L_{A/R}$ is a projective A-module of finite rank. Note that $\pi_0 L_{A/R}$ can be identified with the module of Kähler differentials $\Omega_{A/R}$; in particular, it is generated by the set of elements $\{da\}_{a \in A}$ (Proposition HA.7.4.3.9). Assertion (5) is local with respect to the Zariski topology on $|\operatorname{Spec} A|$. We may therefore assume without loss of generality that there exist $y_1, \ldots, y_n \in A$ such that the elements dy_i freely generate $\pi_0 L_{A/R}$ as a module over A. Let $A_0 = R[Y_1, \ldots, Y_n]$, so that there is a unique R-algebra homomorphism $\xi : A_0 \to A$ satisfying $\xi(Y_i) = y_i$. We will complete the proof by showing that ξ is étale.

We have a fiber sequence $A \otimes_{A_0} L_{A_0/R} \xrightarrow{\alpha} L_{A/R} \to L_{A/A_0}$. By construction, the map α induces an isomorphism on π_0. Since $\pi_1 L_{A/R} \simeq 0$, we conclude that $\pi_0 L_{A/A_0} \simeq \pi_1 L_{A/A_0} \simeq 0$. Using Lemma B.1.3.2, we deduce that ξ factors as a composition $A_0 \xrightarrow{\xi'} A_1 \xrightarrow{\xi''} A$, where ξ' is étale and $\mathrm{fib}(\xi'')$ is 2-connective. Since A_0 is discrete and ξ' is étale, we conclude that A_1 is discrete. Then ξ'' is a map between discrete $\mathbb{E}_\infty$-rings which induces an isomorphism $\pi_0 A' \to \pi_0 A$, and is therefore an equivalence. It follows that $\xi = \xi'' \circ \xi'$ is étale, as desired. □

Corollary 11.2.4.2. *Let $\phi : R \to A$ be a morphism of connective $\mathbb{E}_\infty$-rings. Suppose that ϕ is flat and that ϕ exhibits $\pi_0 A$ as a finitely presented commutative algebra over $\pi_0 R$. Then the following conditions are equivalent:*

(1) *The morphism ϕ is fiber smooth.*

(2) *For every surjective map of commutative rings $B \to \overline{B}$ with nilpotent kernel, every lifting problem*

$$\begin{array}{ccc} R & \longrightarrow & B \\ \downarrow & \nearrow & \downarrow \\ A & \longrightarrow & \overline{B} \end{array}$$

admits a solution.

(3) *For every discrete A-module M, the abelian group $\pi_1(M \otimes_A L_{A/R})$ vanishes.*

(4) *For every field κ and every morphism $A \to \kappa$, the vector space $\pi_1(\kappa \otimes_A L_{A/R})$ vanishes.*

Proof. Using Remark 11.2.3.5, we can replace R by $\pi_0 R$ and A by $A \otimes_R \pi_0 R$, and thereby reduce to the case where R is discrete. Since A is flat over R, it is also discrete. The equivalence of (1), (2), and (4) follows from Proposition 11.2.4.1, and the implication (3) ⇒ (4) is immediate. We complete the proof by showing that (1) ⇒ (3). If (1) is satisfied, then $\pi_1 L_{A/R} \simeq 0$ and $\pi_0 L_{A/R}$ is a flat A-module. It follows that the abelian groups $\mathrm{Tor}_0^A(M, \pi_1 L_{A/R})$ and $\mathrm{Tor}_1^A(M, \pi_0 L_{A/R})$ vanish. Using the exact sequence

$$\mathrm{Tor}_0^A(M, \pi_1 L_{A/R}) \to \pi_1(M \otimes_A L_{A/R}) \to \mathrm{Tor}_1^A(M, \pi_0 L_{A/R}),$$

we see that $\pi_1(M \otimes_A L_{A/R})$ vanishes as well. □

We now consider the relationship between differential smoothness and fiber smoothness.

Proposition 11.2.4.3. *Let $\phi : R \to A$ be a morphism of connective $\mathbb{E}_\infty$-rings. If ϕ is differentially smooth, then the induced map of commutative rings $\pi_0 R \to \pi_0 A$ is fiber smooth.*

Proof. This follows immediately from the criteria for smoothness provided by part (4) of Proposition 11.2.2.1 and part (5) of Proposition 11.2.4.1. □

In characteristic zero, fiber smoothness is equivalent to differential smoothness:

Proposition 11.2.4.4. *Let $\phi : R \to A$ be a morphism of connective $\mathbb{E}_\infty$-rings, and suppose that $\pi_0 R$ is an algebra over the field $\mathbf{Q}$ of rational numbers. Then ϕ is fiber smooth if and if ϕ is differentially smooth.*

Proof. Using Corollary 11.2.2.8 and Remark 11.2.3.5, we can reduce to the case where R is discrete. By virtue of the criteria provided by part (4) of Proposition 11.2.2.1 and part (5) of Proposition 11.2.4.1, it will suffice to show that for every integer $n \geqslant 0$, the canonical map $\theta : R\{x_1, \ldots, x_n\} \to R[x_1, \ldots, x_n]$ is an equivalence in CAlg_R. This follows from Proposition HA.7.1.4.20 . □

11.2.5 Smooth Morphisms of Spectral Deligne-Mumford Stacks

Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. Then the condition that ϕ is differentially smooth (fiber smooth) is local with respect to the étale topology on Spét B, and local with respect to the flat topology on Spét A:

Proposition 11.2.5.1. *Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings, and suppose we are given a finite collection of étale B-algebras B_α such that the induced map $B \to \prod_\alpha B_\alpha$ is faithfully flat. Then ϕ is differentially smooth if and only if each of the induced maps $\phi_\alpha : A \to B_\alpha$ is differentially smooth.*

Proof. If ϕ is differentially smooth, then it follows immediately from Proposition 11.2.2.9 and Example 11.2.2.5 that each ϕ_α is differentially smooth. Conversely, suppose that each ϕ_α is differentially smooth. Using Proposition 4.1.4.1 we deduce that ϕ is almost of finite presentation. Consequently, we are reduced to proving that the relative cotangent complex $L_{B/A}$ is flat over B (Remark 11.2.2.3). By virtue of Proposition 2.8.4.2, it will suffice to show that each tensor product $B_\alpha \otimes_B L_{B/A}$ is flat over B_α. This is clear, since $B_\alpha \otimes_B L_{B/A} \simeq L_{B_\alpha/A}$, and B_α is assumed to be differentially smooth over A. □

Proposition 11.2.5.2. *Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings, and suppose we are given a finite collection of A-algebras B_α such that the induced map $A \to \prod_\alpha A_\alpha$ is faithfully flat. If each of the induced maps $\phi_\alpha : A_\alpha \to A_\alpha \otimes_A B$ is differentially smooth, then ϕ is differentially smooth.*

Proof. It follows from Proposition 4.1.4.3 that ϕ is locally almost of finite presentation. We wish to prove that $L_{B/A}$ is flat over B (Remark 11.2.2.3). It now suffices to note that the flatness of $L_{B/A}$ can be tested locally with respect to the flat topology (Proposition 2.8.4.2). □

Proposition 11.2.5.3. *Let $\phi : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Suppose that there exists a finite collection of A-algebras A_α, such that $\prod_\alpha A_\alpha$ is faithfully flat over A. If each tensor product $A_\alpha \otimes_A B$ is fiber smooth over B, then B is fiber smooth over A.*

Proof. It follows from Proposition 4.2.1.5 that B is almost of finite presentation over A, and from Proposition 2.8.4.2 that B is flat over A. To complete the proof that B is fiber smooth over A, it will suffice to show that for every field κ and every morphism $B \to \kappa$, the vector space $\pi_1(\kappa \otimes_B L_{B/A})$ vanishes. Enlarging κ if necessary, we may assume that the composite map $A \to B \to \kappa$ factors through some A_α. Let $B_\alpha = A_\alpha \otimes_A B$. Then we have a canonical isomorphism $\pi_1(\kappa \otimes_B L_{B/A}) \simeq \pi_1(\kappa \otimes_{B_\alpha} L_{B_\alpha/A_\alpha})$, so the desired result follows from our assumption that B_α is fiber smooth over A_α. □

Proposition 11.2.5.4. *Let $\phi : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Suppose that there exists a finite collection of flat morphisms $B \to B_\alpha$ which are almost of finite presentation, such that $\prod_\alpha B_\alpha$ is faithfully flat over B and each B_α is fiber smooth over A. Then B is fiber smooth over A.*

Proof. Set $B' = \prod_\alpha B_\alpha$. Then B' is fiber smooth over A, and the map $B \to B'$ is almost of finite presentation and faithfully flat. Applying Corollary 6.1.6.5, we deduce that B is almost of finite presentation over A. We wish to prove that B is fiber smooth over A. Using Remark 11.2.3.8, we can reduce to the case where $A = \kappa$ is a field. Using Proposition 11.2.5.3, we can reduce to the case where κ is algebraically closed. Since B' is fiber smooth over κ, it is discrete. The faithful flatness of B' over B guarantees that B is also discrete. Note that B is finitely generated as an algebra over κ; we wish to show that B is regular. Let $\mathfrak{m}$ be a maximal ideal of B; we will show that $B/\mathfrak{m}$ is perfect as a B-module (see Lemma 11.3.3.3). Since B is Noetherian and $B/\mathfrak{m}$ is a finitely generated B-module, it is almost perfect (Proposition HA.7.2.4.17). We are therefore reduced to proving that $B/\mathfrak{m}$ has finite Tor-amplitude as a B-module (Proposition HA.7.2.4.23).

Since B' is faithfully flat over B, the map $|\operatorname{Spec} B'| \to |\operatorname{Spec} B|$ is surjective. We may therefore choose a maximal ideal $\mathfrak{m}'$ of B' lying over $\mathfrak{m}$. Since κ is algebraically closed, both $B'/\mathfrak{m}'$ and $B/\mathfrak{m}$ are isomorphic to κ, so that the natural map $B/\mathfrak{m} \to B'/\mathfrak{m}'$ is an isomorphism. Since B' is flat over B, we are reduced to proving that $B'/\mathfrak{m}'$ has finite Tor-amplitude as a B'-module (Lemma 6.1.1.6). This follows from Lemma 11.3.3.3, since B' is a regular Noetherian ring. □

Motivated by the preceding results, we introduce global versions of Definitions 11.2.2.2 and 11.2.3.1:

Definition 11.2.5.5. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. We will say that f is *differentially smooth* (*fiber smooth*) if, for every commutative diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} B & \longrightarrow & \mathsf{X} \\ \downarrow & & \downarrow f \\ \operatorname{Sp\acute{e}t} A & \longrightarrow & \mathsf{Y} \end{array}$$

where the horizontal maps are étale, the underlying map of connective $\mathbb{E}_\infty$-rings $A \to B$ is differentially smooth (fiber smooth).

Remark 11.2.5.6. Differential smoothness and fiber smoothness for maps of spectral Deligne-Mumford stacks can also be defined in terms of infinitesimal lifting properties. We defer a proof until §17 (see Proposition 17.3.9.4), where we will develop the global theory of the cotangent complex.

Example 11.2.5.7. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathbf{P}^n_R$ be projective space of dimension n over R (see Construction 5.4.1.3). Then the projection map $\mathbf{P}^n_R \to \operatorname{Sp\acute{e}t} R$ is fiber smooth.

Remark 11.2.5.8. Let $f : \mathsf{X} \to \mathsf{Y}$ be a fiber smooth morphism of spectral Deligne-Mumford stacks. Then f is Gorenstein, in the sense of Definition 6.6.6.1. To prove this, we first observe that f is flat and locally almost of finite presentation, so we may assume without loss of generality that $\mathsf{Y} = \operatorname{Sp\acute{e}t} \kappa$ for some field κ. The assertion is local on X, so we may assume that $\mathsf{X} = \operatorname{Sp\acute{e}t} A$ for some finitely generated (discrete) κ-algebra A. The fiber smoothness of A over κ guarantees that A is a regular Noetherian ring (Proposition 11.2.3.6), and therefore Gorenstein (Example **??**).

We now summarize some formal properties of Definition 11.2.5.5:

Proposition 11.2.5.9. (1) *The condition that a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks be differentially smooth (fiber smooth) is local on the target with respect to the flat topology, and local on the source with respect to the étale topology.*

(2) *The collection of differentially smooth (fiber smooth) morphisms is closed under composition.*

(3) *Suppose are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}. \end{array}$$

If f is differentially smooth (fiber smooth), so is f'.

Proof. Assertion (1) follows from Propositions 11.2.5.3, 11.2.5.4, 11.2.5.1, and 11.2.5.2 (see Remarks 2.8.1.3 and 6.3.1.3). Assertion (2) follows from Propositions 11.2.2.9 and 11.2.3.4, and assertion (3) follows from Proposition 11.2.3.4 and Remark 11.2.2.4. □

Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Since the property of being fiber smooth is local on the source with respect to the étale topology, there exists a largest open substack $\mathsf{U} \subseteq \mathsf{X}$ for which the restriction $f|_{\mathsf{U}} : \mathsf{U} \to \mathsf{Y}$ is fiber smooth. We will refer to $\mathsf{U} \subseteq \mathsf{X}$ as the *smooth locus* of the morphism f. Note that if we are given any pullback diagram

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y}, \end{array}$$

then we can identify $\mathsf{U} \times_{\mathsf{X}} \mathsf{X}'$ with an open substack of X', and the restriction of f' to this open substack is also fiber smooth (Proposition 11.2.5.9). It follows that $\mathsf{U} \times_{\mathsf{X}} \mathsf{X}'$ is contained in the smooth locus of f'. In good cases, we have equality:

Proposition 11.2.5.10 (Universality of the Smooth Locus)**.** *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}' & \xrightarrow{g} & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y} \end{array}$$

where f and f' are flat and locally almost of finite presentation. If $\mathsf{U} \subseteq \mathsf{X}$ is the smooth locus of f, then $\mathsf{U} \times_{\mathsf{X}} \mathsf{X}'$ is the smooth locus of f'.

Proof. Without loss of generality, we may assume that $\mathsf{Y} \simeq \operatorname{Sp\acute{e}t} A$ and $\mathsf{Y}' \simeq \operatorname{Sp\acute{e}t} A'$ are affine. Similarly, we may assume that $\mathsf{X} \simeq \operatorname{Sp\acute{e}t} B$ is affine, so that $\mathsf{X}' \simeq \operatorname{Sp\acute{e}t} B'$ for $B' = A' \otimes_A B$. Let $x' \in |\operatorname{Spec} B'|$ be a point which belongs to the smooth locus of f'; we wish to show that $x = g(x') \in |\operatorname{Spec} B|$ belongs to the smooth locus of f. Let $\kappa(x')$ denote the residue field of B' at x' and let $\kappa(x)$ denote the residue field of B at x. Then $\pi_0(\kappa(x') \otimes_B L_{B/A})$ is generated as a vector space over $\kappa(x')$ by the image of $\pi_0 L_{B/A}$. We can therefore choose a collection of elements $u_1, \ldots, u_d \in \pi_0 L_{B/A}$ whose images form a basis for $\pi_0(\kappa(x') \otimes_B L_{B/A})$. Let us identify the set $\{u_i\}_{1 \leqslant i \leqslant d}$ with a morphism of B-modules $u : B^d \to L_{B/A}$. Our assumption that x' belongs to the smooth locus of f' guarantees that $\kappa(x') \otimes_B \operatorname{cofib}(u)$ is 2-connective, so that $\kappa(x) \otimes_B \operatorname{cofib}(u)$ is 2-connective. The cofiber $\operatorname{cofib}(u)$ is almost perfect, so Proposition 2.7.4.1 guarantees the existence of an element $b \in \pi_0 B$ which does not vanish at x such that $\operatorname{cofib}(u)[b^{-1}]$ is 2-connective. Then $f|_{\operatorname{Sp\acute{e}t} B[b^{-1}]} : \operatorname{Sp\acute{e}t} B[b^{-1}] \to \operatorname{Sp\acute{e}t} A$ is smooth, so that x is contained in the smooth locus of f as desired. □

11.3 Smoothness for Linear ∞-Categories

Let R be a connective $\mathbb{E}_\infty$-ring, and let X be a spectral algebraic space over R. We will say that X is *fiber smooth* over R if the map $\mathsf{X} \to \operatorname{Sp\acute{e}t} R$ is fiber smooth, in the sense of Definition 11.2.5.5. Our goal in this section is to show that (under some mild hypotheses) the hypothesis that X is fiber smooth over R depends only on the R-linear ∞-category $\operatorname{QCoh}(\mathsf{X})$. For this end, we will introduce the notion of a *smooth* R-linear ∞-category (Definition 11.3.1.1). Our main result asserts that if X is quasi-compact, quasi-separated, locally almost of finite presentation, and of finite Tor-amplitude over R, then $\operatorname{QCoh}(\mathsf{X})$ is smooth if and only if X is separated and fiber smooth over R (Theorem 11.3.6.1).

11.3.1 Definition of Smoothness

We begin with some general remarks.

Definition 11.3.1.1. Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. We will say that $\mathcal{C}$ is *smooth* if the following conditions are satisfied:

(1) The ∞-category $\mathcal{C}$ is compactly generated and is therefore left dualizable as a left LMod_R-module object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ (Corollary D.7.7.4). Let us denote its left dual by ${}^\vee\mathcal{C}$.

(2) Let $c : \mathrm{Sp} \to {}^\vee\mathcal{C} \otimes_R \mathcal{C}$ be a morphism in $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ which exhibits ${}^\vee\mathcal{C}$ as a left dual of $\mathcal{C}$. Then the functor c preserves compact objects.

Warning 11.3.1.2. The condition that a stable R-linear ∞-category $\mathcal{C}$ be smooth depends not only on the underlying stable ∞-category, but also on the action of Mod_R on $\mathcal{C}$. We will sometimes emphasize this dependence by saying that $\mathcal{C}$ is *smooth over* R if it satisfies the requirements of Definition 11.3.1.1.

Remark 11.3.1.3. In the situation of Definition 11.3.1.1, condition (2) is equivalent to the *a priori* weaker hypothesis that the functor $c : \mathrm{Sp} \to {}^{\vee}\mathcal{C} \otimes_R \mathcal{C}$ carries the sphere spectrum S to a compact object of ${}^{\vee}\mathcal{C} \otimes_R \mathcal{C}$.

Remark 11.3.1.4. Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be an R-linear ∞-category, and let $R \to R'$ be a morphism of $\mathbb{E}_2$-rings. Then $R' \otimes_R \mathcal{C}$ is a compactly generated R'-linear ∞-category (Corollary **??**), whose left dual can be identified with the ∞-category

$$ {}^{\vee}\mathcal{C} \otimes_R R' = {}^{\vee}\mathcal{C} \otimes_{\mathrm{LMod}_R} \mathrm{LMod}_{R'} . $$

Moreover, the canonical map

$$ {}^{\vee}\mathcal{C} \otimes_R \mathcal{C} \to {}^{\vee}\mathcal{C} \otimes_R \mathrm{LMod}_{R'} \otimes_R \mathcal{C} \simeq ({}^{\vee}\mathcal{C} \otimes_R R') \otimes_{R'} (R' \otimes_R \mathcal{C}) $$

preserves compact objects. It follows that if $\mathcal{C}$ is smooth over R, then $R' \otimes_R \mathcal{C}$ is smooth over R'.

11.3.2 Comparison with Smooth Algebras

Let R be an $\mathbb{E}_2$-ring and let A be an $\mathbb{E}_1$-algebra over R. Then we can regard RMod_A is a stable R-linear ∞-category. We now show that the condition that RMod_A is smooth over R admits a more concrete formulation in terms of the structure of A as an R-algebra (Proposition 11.3.2.3).

Definition 11.3.2.1. Let R be an $\mathbb{E}_2$-ring and let A be an $\mathbb{E}_1$-algebra over R. We will say that A is *smooth over* R if it is compact when viewed as an object of the stable ∞-category ${}_A\mathrm{BMod}_A(\mathrm{LMod}_R)$ of A-A bimodule objects of LMod_R.

Remark 11.3.2.2. Suppose that R is an $\mathbb{E}_\infty$-ring, let A be an $\mathbb{E}_1$-algebra over R, and let A^{rev} denote the opposite algebra (Remark HA.4.1.1.7). Then we have an equivalence of ∞-categories ${}_A\mathrm{BMod}_A(\mathrm{Mod}_R) \simeq \mathrm{LMod}_{A \otimes_R A^{\mathrm{rev}}}$, so that the algebra A is compact when viewed as an object of ${}_A\mathrm{BMod}_A(\mathrm{Mod}_R)$ if and only if it is perfect when regarded as left module over $A \otimes_R A^{\mathrm{rev}}$. In other words, the algebra A is smooth in the sense of Definition 11.3.2.1 if and only if it is a smooth algebra object of Mod_R, in the sense of Definition HA.4.6.4.13 .

Proposition 11.3.2.3. *Let R be an $\mathbb{E}_2$-ring and let A be an $\mathbb{E}_1$-algebra over R. The following conditions are equivalent:*

(a) *The R-linear ∞-category RMod_A is smooth over R (in the sense of Definition 11.3.1.1).*

(b) *The algebra A is smooth over R (in the sense of Definition 11.3.2.1). That is, A is a compact object of the ∞-category ${}_A\mathrm{BMod}_A(\mathrm{LMod}_R)$.*

Proof. Using Theorem HA.4.8.4.6 , we can identify the tensor product $\mathrm{LMod}_A \otimes_R \mathrm{RMod}_A$ with the ∞-category ${}_A\mathrm{BMod}_A(\mathrm{LMod}_R)$. Under this identification, the coevaluation functor

$$c : \mathrm{Sp} \to \mathrm{LMod}_A \otimes_R \mathrm{RMod}_A$$

carries the sphere spectrum $S \in \mathrm{Sp}$ to the spectrum A, regarded as a bimodule over itself. □

Proposition 11.3.2.3 asserts in particular that if $A \in \mathrm{Alg}_R$ is smooth, then the ∞-category RMod_A is smooth over R. In fact, all smooth R-linear ∞-categories arise in this way:

Proposition 11.3.2.4. *Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is smooth over R.*

(2) *There exists an R-linear equivalence $\mathcal{C} \simeq \mathrm{RMod}_A$, where $A \in \mathrm{Alg}_R$ is smooth.*

Proof. The implication (2) ⇒ (1) follows from Proposition 11.3.2.3. We now prove the converse. Let $\mathcal{C}$ be a smooth R-linear ∞-category. We will show that there exists a single compact object $C \in \mathcal{C}$ which generates $\mathcal{C}$. It will then follow from Corollary D.7.6.4 that $\mathcal{C} \simeq \mathrm{RMod}_A$, where A is the endomorphism algebra of C. Proposition 11.3.2.3 then implies that $A \in \mathrm{Alg}_R$ is smooth, so that (2) is satisfied.

We now prove the existence of C. By assumption, the ∞-category $\mathcal{C}$ is compactly generated. Choose a set $\{C_t\}_{t \in T}$ of compact generators for $\mathcal{C}$. For every finite subset $T' \subseteq T$, let $\mathcal{C}_{T'}$ denote the smallest stable subcategory of $\mathcal{C}$ which contains the objects of T' and is closed under small colimits. Fix an object $X \in \mathcal{C}_{T'}$, and let $\mathcal{D}_X$ be the full subcategory of Mod_R spanned by those left R-modules M for which $M \otimes C$ belongs to $\mathcal{C}_{T'}$. Since $\mathcal{C}_{T'}$ is stable and closed under small colimits in $\mathcal{C}$, $\mathcal{D}_C$ is stable and closed under small colimits in Mod_R. Since $R \in \mathcal{D}_{T'}$, we conclude that $\mathcal{D}_{T'} = \mathrm{LMod}_R$: that is, the action of LMod_R on $\mathcal{C}$ carries $\mathcal{C}_{T'}$ into itself. We may therefore regard each $\mathcal{C}_{T'}$ itself as an R-linear ∞-category.

We next claim that the canonical map $\theta : \varinjlim_{T' \subseteq S} \mathcal{C}_{T'} \to \mathcal{C}$ is an equivalence in $\mathrm{LinCat}_R^{\mathrm{St}}$; here the colimit is taken over the collection of all finite subsets $T' \subseteq T$. Since the forgetful functor $\mathrm{LinCat}_R^{\mathrm{St}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ preserves small colimits, we are reduced to proving that $\mathcal{C}$ is a colimit of the diagram $\{\mathcal{C}_{T'}\}_{T' \subseteq T}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. For each finite subset $T' \subseteq T$, let $\mathcal{C}^0_{T'}$ denote the smallest stable subcategory of $\mathcal{C}$ containing T'. Similarly, let $\mathcal{C}^0$ be the smallest stable subcategory of $\mathcal{C}$ containing T. Let $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ denote the subcategory of $\mathcal{C}\mathrm{at}_\infty$ whose objects are (small) stable ∞-categories and whose morphisms are exact functors. Since each $\mathcal{C}^0_{T'}$ consists of compact objects of $\mathcal{C}$, the inclusion $\mathcal{C}^0_{T'} \hookrightarrow \mathcal{C}$ extends to a fully faithful embedding $\mathrm{Ind}(\mathcal{C}^0_{T'}) \to \mathcal{C}$ which preserves small colimits. The essential image of this embedding is the full subcategory of $\mathcal{C}$ generated by $\mathcal{C}^0_{T'}$ under small colimits, and therefore coincides with $\mathcal{C}_{T'}$. Similarly, we can identify $\mathcal{C}$ with $\mathrm{Ind}(\mathcal{C}^0_{T'})$. We may therefore identify θ with the canonical map

$$\varinjlim_{T' \subseteq T} \mathrm{Ind}(\mathcal{C}^0_{T'}) \to \mathrm{Ind}(\mathcal{C}^0).$$

Since the functor $\mathrm{Ind} : \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ preserves small colimits, we are reduced to showing that $\mathcal{C}^0$ is the colimit of the diagram $\{\mathcal{C}^0_{T'}\}_{T' \subseteq T}$, which follows immediately from the definitions (note that the forgetful functor $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}} \to \mathcal{C}\mathrm{at}_\infty$ preserves filtered colimits).

Note that for every finite subset $T' \subseteq T$, the inclusion $\iota : \mathcal{C}_{T'} \hookrightarrow \mathcal{C}$ preserves compact objects, and therefore admits a right adjoint G which preserves filtered colimits. Using Remark D.1.5.3, we can regard G as an R-linear functor, and the unit map $u : \mathrm{id}_{\mathcal{C}_{T'}} \to G \circ \iota$ as an R-linear natural transformation. Since ι is fully faithful, u is an equivalence. Let $c : \mathrm{Sp} \to {}^\vee\mathcal{C} \otimes_R \mathcal{C}$ be the duality datum appearing in Definition 11.3.1.1. Then we can identify the codomain of c with the colimit

$$\varinjlim_{T' \subseteq T} {}^\vee\mathcal{C} \otimes_R \mathcal{C}_{T'}.$$

Let $S \in \mathrm{Sp}$ be the sphere specturm. Since $\mathcal{C}$ is smooth, $c(S)$ is a compact object of the tensor product ${}^\vee\mathcal{C} \otimes_R \mathcal{C}$ and can therefore be lifted to a compact object of ${}^\vee\mathcal{C} \otimes_R \mathcal{C}_{T'}$ for some finite subset $T' \subseteq T$ (Lemma HA.7.3.5.11). This object determines a map $\overline{c} : \mathrm{Sp} \to {}^\vee\mathcal{C} \otimes_R \mathcal{C}_{T''}$ which classifies an R-linear functor $\mathcal{C} \to \mathcal{C}_{T'}$ which is right homotopy inverse to the identity. It follows that $\mathcal{C}_{T'} = \mathcal{C}$. We now complete the proof by observing that the object $C = \bigoplus_{t \in T'} C_t$ is a compact generator of $\mathcal{C}$. □

11.3.3 Relationship with Fiber Smoothness

Let $f : R \to A$ be a morphism of $\mathbb{E}_\infty$-rings. Then, by neglect of structure, we can regard A as an $\mathbb{E}_1$-algebra over R. We now show that the smoothness of A as an $\mathbb{E}_1$-algebra is closely related to the fiber smoothness of the morphism f:

Proposition 11.3.3.1. *Let $f : R \to A$ be a morphism of connective $\mathbb{E}_\infty$-rings. Then f is fiber smooth if and only if it satisfies the following conditions:*

(1) *The morphism f has finite* Tor*-amplitude.*

(2) *The morphism f is almost of finite presentation.*

(3) *The algebra A is smooth as an $\mathbb{E}_1$-algebra over R (Definition 11.3.2.1). That is, A is perfect when regarded as a module over $A \otimes_R A$.*

Warning 11.3.3.2. Hypothesis (1) of Proposition 11.3.3.1 is necessary. For example, let $\mathbf{Q}$ denote the field of rational numbers (regarded as a discrete $\mathbb{E}_\infty$-ring) and let $A = \mathrm{Sym}^*_{\mathbf{Q}}(\Sigma^2\, \mathbf{Q})$ denote the free $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$ on one generator in homological degree 2. Then A is smooth over $\mathbf{Q}$ in the sense of Definition 11.3.2.1, but is not fiber smooth over $\mathbf{Q}$ (since it is not flat over $\mathbf{Q}$).

Lemma 11.3.3.3. *[Serre's Criterion] Let R be a local Noetherian $\mathbb{E}_\infty$-ring with residue field κ. Assume that R is n-truncated for some integer n and that κ is perfect when regarded as an R-module. Then $\pi_0 R$ is a regular local ring and $\pi_m R \simeq 0$ for $m > 0$.*

Proof. If $R \simeq 0$ there is nothing to prove. Otherwise, our assumption that R is truncated guarantees the existence of a largest integer n such that $\pi_n R \neq 0$. We will prove that R is discrete by showing that $n = 0$. The regularity of $\pi_0 R$ will then follow from Serre's criterion (see, for example, [148]).

If κ is a perfect R-module, then it has Tor-amplitude $\leqslant d$ for some integer d. It follows that for every discrete R-module M, we have $\pi_{d+1}(\kappa \otimes_R M) \simeq 0$. Using Corollary 6.1.4.7, we deduce that if M is finitely generated as a module over $\pi_0 R$, then M also has Tor-amplitude $\leqslant d$.

Assume for a contradiction that $n > 0$. Since R is Noetherian, $\pi_n R$ is a finitely generated module over $\pi_0 R$. Let $\mathfrak{p} \subseteq \pi_0 R$ be a minimal prime ideal belonging to the support of $\pi_0 R$, so that $\pi_n R_{\mathfrak{p}} \neq 0$. Note that the quotient $M = \pi_0 R/\mathfrak{p}$ has Tor-amplitude $\leqslant d$ as an R-module. Then $M_{\mathfrak{p}}$ can be identified with the fraction field of $R_{\mathfrak{p}}$, and therefore has Tor-amplitude $\leqslant d$ as an $R_{\mathfrak{p}}$-module. We may therefore replace R by $R_{\mathfrak{p}}$, and thereby reduce to the case where $\pi_n R$ is supported at the maximal ideal $\mathfrak{m} \subseteq R$ (and is therefore an R-module of finite length). In particular, we may assume that there exists a nonzero element $x \in \pi_n R$ which is annihilated by $\mathfrak{m}$.

To obtain a contradiction, it will suffice to prove the following:

$(*)$ Let N be a nonzero perfect R-module which is connective and of Tor-amplitude $\geqslant k$. Then $\pi_{n+k} N \neq 0$.

We prove $(*)$ using induction on k. If $k = 0$, then N is a projective R-module, so that $\pi_n N \simeq (\pi_n R) \otimes_{\pi_0 R} (\pi_0 N) \neq 0$. Assume that $k > 0$, and choose a finite collection of elements $x_1, \ldots, x_m \in \pi_0 N$ which generate $\pi_0(\kappa \otimes_R N)$ as a vector space over κ. These elements determine a fiber sequence of R-modules $N' \xrightarrow{\phi} R^m \to N$. It follows from Nakayama's lemma that the map $\pi_0 R^m \to \pi_0 N$ is surjective. Since N has Tor-amplitude $\geqslant k > 0$, the R-module N' is nonzero and has Tor-amplitude $> k-1$. We have a short exact sequence of abelian groups $\pi_{n+k} N \to \pi_{n+k-1} N' \xrightarrow{\psi} \pi_{n+k-1} R^m$. It will therefore suffice to show that the map ψ is not injective. If $k > 1$ this is clear, since $\pi_{n+k-1} N' \neq 0$ by the inductive hypothesis, and $\pi_{n+k-1} R \simeq 0$. In the case $k = 1$, N' is a projective R-module. Since R is local, we have $R^a \simeq N'$ for some integer $a > 0$. Then ϕ is given (up to homotopy) by a matrix of coefficients $\{\phi_{i,j} \in \pi_0 R\}_{1 \leqslant i \leqslant a, 1 \leqslant j \leqslant m}$. By construction, the map $(\pi_0 R^m)/\mathfrak{m}(\pi_0 R^m) \to N/\mathfrak{m}N$ is injective, so that the image of $\pi_0 N'$ under ϕ is contained in $\mathfrak{m}^m \subseteq (\pi_0 R)^m$. It follows that each of the coefficients $\phi_{i,j}$ belongs to $\mathfrak{m}$, and therefore ψ annihilates the nonzero element $(x, x, \ldots, x) \in \pi_n R^a \simeq \pi_n N'$. □

Proof of Proposition 11.3.3.1. Suppose first that $f : R \to A$ is fiber smooth. Then f is almost of finite presentation and of finite Tor-amplitude (in fact, it is flat). We must show that A is smooth in the sense of Definition 11.3.2.1: that is, that it is perfect when regarded as a module over $A \otimes_R A$. We first note that since A is almost of finite presentation over R, the tensor product $A \otimes_R A$ is also almost of finite presentation over R, and therefore A is almost of finite presentation over $A \otimes_R A$. Using Corollary 5.2.2.2, we deduce that A is almost perfect as a module over $A \otimes_R A$. To prove that A is perfect over $A \otimes_R A$, it will suffice to show that A has finite Tor-amplitude over $A \otimes_R A$ (Proposition HA.7.2.4.23). Choose a finite collection of elements $x_1, \ldots, x_d \in \pi_0 A$ which generate $\pi_0 A$ as a commutative algebra over $\pi_0 R$. We will show that A has Tor-amplitude $\leqslant d$ over $A \otimes_R A$. Using Proposition **??**, we are reduced to proving that for every algebraically closed field κ and every morphism $\theta : A \otimes_R A \to \kappa$, the tensor product $B = A \otimes_{A \otimes_R A} \kappa$ is d-truncated. After replacing R by κ, we can identify θ with a pair of morphisms $\phi_0, \phi_1 : A \to \kappa$ in CAlg_κ, and B with the pushout $\kappa \otimes_A \kappa$. To verify that B is d-truncated, it will suffice to show that κ has Tor-amplitude $\leqslant d$ when regarded as an A-module (via either ϕ_0). Let $\mathfrak{m}$ denote the maximal ideal of A given by the kernel of ϕ_0. Since f is fiber smooth, the ring $A_\mathfrak{m}$ is regular, so that κ has Tor-amplitude n as a module over $A_\mathfrak{m}$, where n is the Krull dimension of A_κ. The inequality $n \leqslant d$ follows from the fact that A is generated by $\leqslant d$ elements as a commutative algebra over κ.

Now suppose that $f : R \to A$ is a morphism of $\mathbb{E}_\infty$-rings which satisfies conditions (1), (2), and (3); we wish to show that f is fiber smooth. Using Propositions **??** and 11.2.3.6, we are reduced to proving that for every algebraically closed field κ and every morphism $R \to \kappa$, the tensor product $\kappa \otimes_R A$ is discrete and regular. Replacing R by κ, we may suppose that A is almost of finite presentation over κ (and therefore Noetherian) and d-truncated for some integer d. To prove that A is discrete and regular, it suffices to show that the localization $A_\mathfrak{m}$ is discrete and regular, for every maximal ideal $\mathfrak{m} \subseteq \pi_0 A$. Since κ is algebraically closed, the residue field $\pi_0 A/\mathfrak{m}$ is isomorphic to κ and is therefore dualizable when regarded as a κ-module. Since A is smooth as an object of Alg_κ, it follows that $(\pi_0 A)/\mathfrak{m}$ is also dualizable as an A-module (Proposition HA.4.6.4.12). The desired result now follows from Lemma 11.3.3.3. □

11.3.4 Smooth Quasi-Coherent Stacks

The condition of smoothness can be tested locally with respect to the étale topology:

Proposition 11.3.4.1. *Let R be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. Suppose that there exists a finite collection of étale maps $R \to R_\alpha$ such that $\prod_\alpha R_\alpha$ is faithfully flat over R, and each of the ∞-categories $R_\alpha \otimes_R \mathcal{C}$ is smooth over R_α. Then $\mathcal{C}$ is smooth over R.*

Proof. It follows from Proposition **??** that the ∞-category $\mathcal{C}$ is compactly generated. Let $c : \mathrm{Sp} \to {}^\vee\mathcal{C} \otimes_R \mathcal{C}$ be as in Definition 11.3.1.1; we wish to show that $c(S)$ is a compact object of the R-linear ∞-category $\mathcal{D} = {}^\vee\mathcal{C} \otimes_R \mathcal{C}$. Since each $\mathrm{LMod}_{R_\alpha}(\mathcal{C})$ is smooth over R_α, we conclude that the image of $c(S)$ is compact in each of the ∞-categories $\mathrm{LMod}_{R_\alpha}(\mathcal{D})$. Applying Proposition **??**, we conclude that $c(S)$ itself is compact. □

Corollary 11.3.4.2. *For every $\mathbb{E}_\infty$-ring R, let $\mathrm{LinCat}_R^{\mathrm{sm}}$ denote the full subcategory of $\mathrm{LinCat}_R^{\mathrm{St}}$ spanned by the smooth R-linear ∞-categories. Then the functor $R \mapsto \mathrm{LinCat}_R^{\mathrm{sm}}$ is a sheaf with respect to the étale topology on CAlg_R.*

Proof. Combine Theorem **??** with Proposition 11.3.4.1. □

We now introduce a global version of Definition 11.3.1.1.

Definition 11.3.4.3. Let X be a spectral Deligne-Mumford stack and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$ be a quasi-coherent stack on X. We will say that $\mathcal{C}$ is *smooth* if it is stable and, for every map $\eta : \mathrm{Spét}\, R \to \mathsf{X}$, the pullback $\eta^* \mathcal{C} \in \mathrm{LinCat}_R$ is a smooth R-linear ∞-category.

Remark 11.3.4.4. Let $\mathsf{X} \simeq \mathrm{Spét}\, R$ be an affine spectral Deligne-Mumford stack and let $\mathcal{C}$ be a quasi-coherent stack on X. Then $\mathcal{C}$ is smooth (as a quasi-coherent stack on X) if and only if the corresponding stable R-linear ∞-category is stable and smooth over R, in the sense of Definition 11.3.1.1: this follows from Remark 11.3.1.4.

Using Proposition 11.3.4.1, we deduce the following:

Proposition 11.3.4.5. *Let X be a spectral Deligne-Mumford stack and let $\mathcal{C}$ be a quasi-coherent stack on X. Then:*

(1) *Let $f : \mathsf{Y} \to \mathsf{X}$ be any map of spectral Deligne-Mumford stacks. If $\mathcal{C}$ is smooth, then $f^* \mathcal{C} \in \mathrm{QStk}(\mathsf{Y})$ is smooth.*

(2) *Suppose we are given a collection of étale maps $\{f_\alpha : \mathsf{X}_\alpha \to \mathsf{X}\}$ which induce an étale surjection $\coprod_\alpha \mathsf{X}_\alpha \to \mathsf{X}$. If each pullback $f_\alpha^* \mathcal{C} \in \mathrm{QStk}(\mathsf{X}_\alpha)$ is smooth, then $\mathcal{C}$ is smooth.*

Remark 11.3.4.6. Let X be a spectral Deligne-Mumford stack and let $\mathcal{C} \in \mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$ be a stable quasi-coherent stack on X. Assume that $\mathcal{C}$ is compactly generated, so that it is dualizable as an object of the symmetric monoidal ∞-category $\mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$ (see Corollary D.7.7.4). Let $c : \mathcal{Q}_{\mathsf{X}} \to \mathcal{C} \otimes \mathcal{C}^{\vee}$ be a duality datum in the symmetric monoidal ∞-category. $\mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$. Then $\mathcal{C}$ is smooth if and only if c is a compact morphism of quasi-coherent stacks, in the sense of Definition 10.1.3.1.

Combining Remark 11.3.4.6, Theorem 10.2.0.2, and Proposition 10.3.1.14, we obtain the following:

Proposition 11.3.4.7. *Let X be a quasi-compact, quasi-separated spectral algebraic space and let $\mathcal{C}$ be a stable quasi-coherent stack on X, so that $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is a dualizable $\mathrm{QCoh}(\mathsf{X})$-module of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ (see Construction 11.1.3.1). Then $\mathcal{C}$ is smooth if and only if the coevaluation functor $c : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X};\mathcal{C}) \otimes_{\mathrm{QCoh}(\mathsf{X})} \mathrm{QCoh}(\mathsf{X};\mathcal{C})^{\vee}$ preserves compact objects.*

Remark 11.3.4.8. In the situation of Proposition 11.3.4.7, it suffices to show that the functor c carries the structure sheaf $\mathscr{O}_{\mathsf{X}}$ to a compact object of the ∞-category $\mathrm{QCoh}(\mathsf{X};\mathcal{C}) \otimes_{\mathrm{QCoh}(\mathsf{X})} \mathrm{QCoh}(\mathsf{X};\mathcal{C})^{\vee} \simeq \mathrm{QCoh}(\mathsf{X};\mathcal{C} \otimes \mathcal{C}^{\vee})$.

11.3.5 Global Sections of Smooth Quasi-Coherent Stacks

We now establish a local-to-global principle for smooth quasi-coherent stacks.

Proposition 11.3.5.1. *Let R be a connective $\mathbb{E}_{\infty}$-ring, and let X be a spectral algebraic space over R which is quasi-compact, separated, and fiber smooth over R. Let $\mathcal{C}$ be a smooth quasi-coherent stack on X. Then $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is a smooth R-linear ∞-category.*

The proof of Proposition 11.3.5.1 will require some preliminaries.

Lemma 11.3.5.2. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a fiber smooth morphism of spectral Deligne-Mumford stacks. Then the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ is locally of finite* Tor*-amplitude.*

Proof. The assertion is local on X and Y; we may therefore assume without loss of generality that $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, R$ and $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ are affine. Using Remark 6.1.2.6, we can reduce to the case where R is discrete. Using Proposition 11.2.4.1 (and the fact that the assertion is local on X with respect to the étale topology) we can further reduce to the case where $A = R[x_1, \ldots, x_n]$ is a polynomial ring over R. In this case, we observe that A has Tor-amplitude $\leqslant n$ as a module over $A \otimes_R A$. □

Lemma 11.3.5.3. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks which is separated and fiber smooth, and let $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ be the diagonal map. Then the pushforward functor δ_* carries perfect objects of $\mathrm{QCoh}(\mathsf{X})$ to perfect objects of $\mathrm{QCoh}(\mathsf{X} \times_{\mathsf{Y}} \mathsf{X})$.*

Proof. Since f is separated, δ is a closed immersion, and therefore a proper map. Since f is fiber smooth, it is locally almost of finite presentation, so that δ is also locally almost of finite presentation. Lemma 11.3.5.2 implies that δ is locally of finite Tor-amplitude, so the desired result follows from Theorem 6.1.3.2. □

Proof of Proposition 11.3.5.1. Let $\Gamma : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Mod}_R$ be the global sections functor. We first claim that Γ exhibits $\mathrm{QCoh}(\mathsf{X})$ as a Frobenius algebra object of LinCat_R: that is, that the canonical map

$$e : \mathrm{QCoh}(\mathsf{X}) \otimes_R \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\otimes} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\Gamma} \mathrm{Mod}_R$$

is a duality datum in LinCat_R. Since $\mathrm{QCoh}(\mathsf{X})$ is compactly generated, it is dualizable as an R-linear ∞-category (Theorem D.7.0.7). Let us denote its dual by $\mathrm{QCoh}(\mathsf{X})^\vee$, so that we have a duality datum $\overline{e} : \mathrm{QCoh}(\mathsf{X})^\vee \otimes_R \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Mod}_R$. The universal property of $\overline{e}$ implies that e factors as a composition

$$\mathrm{QCoh}(\mathsf{X}) \otimes_R \mathrm{QCoh}(\mathsf{X}) \xrightarrow{u \otimes \mathrm{id}} \mathrm{QCoh}(\mathsf{X})^\vee \otimes_R \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\overline{e}} \mathrm{Mod}_R .$$

We wish to prove that $u : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X})^\vee$ is an equivalence. According to Remark D.7.7.6, the composite map

$$\mathrm{QCoh}(\mathsf{X})^\vee \otimes \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X})^\vee \otimes_R \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\overline{e}} \mathrm{Mod}_R \to \mathrm{Sp}$$

is a duality datum in $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. It will therefore suffice to show that

$$\mathrm{QCoh}(\mathsf{X}) \otimes \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X}) \otimes_R \mathrm{QCoh}(\mathsf{X}) \xrightarrow{e} \mathrm{Mod}_R \to \mathrm{Sp}$$

is also a duality datum in $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. This follows from Proposition D.7.5.1, since the ∞-category $\mathrm{QCoh}(\mathsf{X})$ is locally rigid (Corollary **??**).

Set $\mathcal{D} = \mathrm{QCoh}(\mathsf{X}; \mathcal{C})$. Arguing as in Construction 11.1.3.1, we see that $\mathcal{D}$ is dualizable as a module over $\mathrm{QCoh}(\mathsf{X})$. Let $\beta : \mathcal{D}^\vee \otimes_{\mathrm{QCoh}(\mathsf{X})} \mathcal{D} \to \mathrm{QCoh}(\mathsf{X})$ be a duality datum. Applying Corollary HA.4.6.5.14 , we deduce that the composite map

$$\mathcal{D}^\vee \otimes_R \mathcal{D} \xrightarrow{\alpha} \mathcal{D}^\vee \otimes_{\mathrm{QCoh}(\mathsf{X})} \mathcal{D} \xrightarrow{\beta} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\Gamma} \mathrm{Mod}_R$$

is a duality datum in LinCat_R.

We wish to prove that $\mathcal{D}$ is a smooth R-linear ∞-category. Equivalently, we wish to show that the composition $\gamma \circ \beta \circ \alpha$ is dual to an R-linear functor which preserves compact objects. To prove this, we will show that α, β, and Γ are each R-linear duals of functors which preserve compact objects:

(α) Let $\mathcal{E} = \mathrm{QCoh}(\mathsf{X}) \otimes_R \mathrm{QCoh}(\mathsf{X})$. According to Corollary ??, the canonical map

$$\mathcal{E} \to \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X})$$

is an equivalence of symmetric monoidal ∞-categories. In particular, $\mathcal{E}$ is locally rigid. Note that α is obtained from a map $\alpha_0 : \mathcal{E} \to \mathrm{QCoh}(\mathsf{X})$ of $\mathcal{E}$-modules by tensoring over $\mathcal{E}$ with $\mathcal{D}^\vee \otimes_R \mathcal{D}$. Since $\mathcal{E}$ is locally rigid, α_0 is a morphism between dualizable $\mathcal{E}$-modules, and therefore admits a $\mathcal{E}$-linear dual $\alpha_0^\vee$. It follows that the dual of α is given by tensoring $\alpha_0^\vee$ over $\mathcal{E}$ with $\mathcal{D}^\vee \otimes_R \mathcal{D}$. Consequently, to show that the dual of α preserves compact objects, it will suffice to show that $\alpha_0^\vee$ preserves compact objects.

Since X is separated, the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ is a closed immersion, and in particular an affine map. Let $\mathscr{A} \in \mathrm{CAlg}(\mathcal{E})$ denote the direct image of the structure sheaf of X, so that $\mathrm{QCoh}(\mathsf{X}) \simeq \mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$. Then α_0 can be identified with the free module functor from $\mathcal{E}$ to $\mathrm{LMod}_{\mathscr{A}}(\mathcal{E})$. It follows that the dual $\alpha_0^\vee$ can be identified with the forgetful functor $\mathrm{RMod}_{\mathscr{A}}(\mathcal{E}) \to \mathcal{E}$: that is, with the pushforward functor δ_*. Since X is fiber smooth over R, this functor preserves compact objects by Lemma 11.3.5.3.

(β) The stable ∞-category $\mathcal{D}^\vee$ is tensored over $\mathrm{QCoh}(\mathsf{X})$ and can therefore be written as $\mathrm{QCoh}(\mathsf{X}; \mathcal{C}^\vee)$ for some stable quasi-coherent stack $\mathcal{C}^\vee \in \mathrm{QStk}(\mathsf{X})$ (Theorem 10.2.0.2). Since the equivalence of Theorem 10.2.0.2 is symmetric monoidal, the quasi-coherent stack $\mathcal{C}^\vee$ is a dual of $\mathcal{C}$ in $\mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$; in particular, $\mathcal{C}^\vee$ is smooth. Let $c : \mathcal{Q}_{\mathsf{X}} \to \mathcal{C} \otimes \mathcal{C}^\vee$ be the associated coevaluation map, so that β is dual to the map

$$\mathrm{QCoh}(\mathsf{X}) \simeq \mathrm{QCoh}(\mathsf{X}; \mathcal{Q}_{\mathsf{X}}) \to \mathrm{QCoh}(\mathsf{X}; \mathcal{C} \otimes \mathcal{C}^\vee)$$

determined by c. Since $\mathcal{C}$ is smooth, this functor carries perfect objects of $\mathrm{QCoh}(\mathsf{X})$ to locally compact objects of $\mathrm{QCoh}(\mathsf{X}; \mathcal{C} \otimes \mathcal{C}^\vee)$. It follows from Proposition 10.3.2.6 that $\beta^\vee$ preserves compact objects.

(Γ) Since the global sections functor $\Gamma : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Mod}_R$ exhibits $\mathrm{QCoh}(\mathsf{X})$ as a Frobenius algebra object of LinCat_R, it follows immediately from the definitions that the dual of Γ can be identified with the unit map $\mathrm{Mod}_R \to \mathrm{QCoh}(\mathsf{X})$. Since perfect objects of $\mathrm{QCoh}(\mathsf{X})$ are compact, the dual of Γ preserves compact objects.

□

Remark 11.3.5.4. Let R be a connective $\mathbb{E}_\infty$-ring and suppose that X is a quasi-compact, quasi-separated spectral algebraic space over R. The proof of Proposition 11.3.5.1 shows that the global sections functor $\Gamma : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Mod}_R$ exhibits X as a Frobenius algebra object of LinCat_R. That is, the composite map

$$\mathrm{QCoh}(\mathsf{X}) \otimes_R \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\otimes} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\Gamma} \mathrm{Mod}_R$$

is a duality datum in LinCat_R. Using Proposition 2.5.4.5, we see that a compatible coevaluation map is given by the composition

$$\mathrm{Mod}_R \to \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\delta_*} \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}) \simeq \mathrm{QCoh}(\mathsf{X}) \otimes_R \mathrm{QCoh}(\mathsf{X})$$

(where $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ denotes the diagonal map).

Suppose that $f : \mathsf{X} \to \mathsf{Y}$ is a map of quasi-compact, quasi-separated spectral algebraic spaces over R, so that $f^* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ is an R-linear functor. Using the duality data described above, we can identify $\mathrm{QCoh}(\mathsf{X})$ and $\mathrm{QCoh}(\mathsf{Y})$ with their own duals in LinCat_R. Then the dual of f^* is an R-linear functor from $\mathrm{QCoh}(\mathsf{X})$ to $\mathrm{QCoh}(\mathsf{Y})$. Repeatedly using Corollary **??**, we can identify this dual with the composition

$$\begin{aligned} \mathrm{QCoh}(\mathsf{X}) &\xrightarrow{\pi^*} \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}) \\ &\xrightarrow{\delta_*} \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}) \\ &\xrightarrow{f'^*} \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}) \\ &\xrightarrow{\delta'^*} \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}) \\ &\xrightarrow{\pi'_*} \mathrm{QCoh}(\mathsf{Y}) \end{aligned}$$

where π and π' denote the projection maps from $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}$ onto the first and second factor, δ and δ' are given by the diagonal embeddings, and f' is the product of f with the identity maps on X and Y. We have a pullback diagram of spectral algebraic spaces

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{\mathrm{id} \times f} & \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \\ \downarrow{\scriptstyle \mathrm{id} \times f} & & \downarrow{\scriptstyle f' \circ \delta'} \\ \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} & \xrightarrow{\delta} & \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \end{array}$$

Using Proposition 2.5.4.5, we can identify the dual of f^* with the functor given by composing pullback along the composition $\mathsf{X} \xrightarrow{\mathrm{id} \times f} \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \xrightarrow{\pi} \mathsf{X}$ with pushforward along the composition $\mathsf{X} \xrightarrow{\mathrm{id} \times f} \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \xrightarrow{\pi'} \mathsf{Y}$. The first composition is given by the identity map from X to itself, and the second agrees with f. It follows that the dual of f^* is given by f_*, as an R-linear functor from $\mathrm{QCoh}(\mathsf{X})$ to $\mathrm{QCoh}(\mathsf{Y})$.

11.3.6 Direct Images of Smooth Quasi-Coherent Stacks

We can now formulate the main result of this section:

Theorem 11.3.6.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Assume that f is a quasi-compact, quasi-separated, locally almost of finite presentation, locally of finite* Tor*-amplitude, and a relative spectral algebraic space. Then the following conditions are equivalent:*

(1) *The morphism f is separated and fiber smooth.*

(2) *For every smooth object $\mathcal{C} \in \mathrm{QStk}(\mathsf{X})$, the direct image $f_* \mathcal{C} \in \mathrm{QStk}(\mathsf{Y})$ is smooth.*

(3) *The quasi-coherent stack $f_* \mathcal{Q}_{\mathsf{X}} \in \mathrm{QStk}^{\mathrm{St}}(\mathsf{Y})$ is smooth.*

(4) *Let $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ be the diagonal map. Then the direct image $\delta_* \mathscr{O}_{\mathsf{X}}$ is perfect.*

The main content of Theorem 11.3.6.1 is contained in Proposition 11.3.5.1, which we proved in §11.3.5. We will also need the following:

Lemma 11.3.6.2. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a quasi-affine morphism of spectral Deligne-Mumford stacks. Suppose that $f_* \mathscr{O}_{\mathsf{X}}$ is perfect. Then f is affine. Suppose, in addition, that f satisfies the following condition:*

($*$) *For every discrete $\mathbb{E}_\infty$-ring B, the map*

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, B, \mathsf{X}) \to \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, B, \mathsf{Y})$$

has (-1)-truncated homotopy fibers.

Then f is a closed immersion.

Proof. The assertion is local on Y. We may therefore assume without loss of generality that $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, R$ is affine. Let us identify $f_* \mathscr{O}_{\mathsf{X}}$ with an object $A \in \mathrm{CAlg}_R$. Assume first that A is perfect; we wish to show that X is affine. According to Proposition **??**, the canonical map $\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, B, \mathsf{X}) \to \mathrm{Map}_{\mathrm{CAlg}}(A, B)$ is a homotopy equivalence for every connective $\mathbb{E}_\infty$-ring B. It follows that if A is connective, then $\mathsf{X} \simeq \mathrm{Sp\acute{e}t}\, A$ is affine.

Suppose, for a contradiction, that A is not connective. Then there exists an integer $n < 0$ such that $\pi_n A \neq 0$. Since A is perfect, there exists a smallest such integer n. Then $\pi_n A$ is a finitely generated module over $\pi_0 R$. Using Nakayama's lemma, we deduce that there exists a maximal ideal $\mathfrak{m} \subseteq \pi_0 R$ such that $\mathrm{Tor}_0^{\pi_0 R}(\pi_n A, \kappa) \neq 0$, where κ denotes the residue field $(\pi_0 R)/\mathfrak{m}$. Then $\pi_n(A \otimes_R \kappa) \neq 0$. Replacing R by κ, we can reduce to the case where R is a field. In this case, since A is perfect over κ, $\pi_0 A$ is a finite-dimensional vector space over κ. It follows that $|\operatorname{Spec} A|$ is a finite set equipped with the discrete topology. In particular, any open subset of $|\operatorname{Spec} A|$ is also closed. Using Proposition 2.4.1.3, we deduce that X is an open substack of $\mathrm{Sp\acute{e}t}\, A$, and therefore affine. It follows that $A = \Gamma(\mathcal{X}; \mathscr{O}_{\mathsf{X}})$ is connective, contradicting our assumption that $\pi_n A \neq 0$.

Now suppose that ($*$) is satisfied; we wish to show that f is a closed immersion. Since A is perfect and connective as an R-module, $\pi_0 A$ is a finitely generated as a module over $\pi_0 R$. We wish to prove that the unit map $\pi_0 R \to \pi_0 A$ is surjective. Using Nakayama's lemma, we are reduced to proving that the unit map $\kappa \to \mathrm{Tor}_0^{\pi_0 R}(\pi_0 A, \kappa)$ is surjective, for every residue field κ of R. Replacing R by κ, we may suppose that $R = \kappa$. Let $B = \pi_0 A \otimes_\kappa \pi_0 A$. Then

B is a discrete $\mathbb{E}_\infty$-ring, equipped with two R-algebra maps $\phi, \phi' : A \to B$. It follows from $(*)$ that ϕ and ϕ' are homotopic. In particular, for each $x \in \pi_0 A$, we have $x \otimes 1 = 1 \otimes x$ in $\pi_0 A \otimes_\kappa \pi_0 A$, so that x is a scalar multiple of the identity. □

Proof of Theorem 11.3.6.1. The implication (1) ⇒ (2) follows from Proposition 11.3.5.1, and the implication (2) ⇒ (3) is obvious. Assume next that (3) is satisfied; we will prove (4). The assertion is local on Y; we may therefore suppose without loss of generality that $\mathsf{Y} = \operatorname{Spét} R$ is affine. The proof of Proposition 11.3.5.1 shows that the composition

$$\mathrm{QCoh}(\mathsf{X}) \otimes_{\mathrm{Mod}_R} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\otimes} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{f_*} \mathrm{Mod}_R$$

is a duality datum in LinCat_R. Using Remark 11.3.5.4 and Corollary **??**, we can identify the dual of this map with the composition

$$\mathrm{Mod}_R \xrightarrow{f^*} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\delta_*} \mathrm{QCoh}(\mathsf{X} \times_{\operatorname{Spét} R} \mathsf{X}).$$

If condition (3) is satisfied, then this functor preserves compact objects. In particular $\delta_* \mathscr{O}_{\mathsf{X}} \simeq \delta_*(f^* R)$ is perfect, so that condition (4) is satisfied.

We now prove that (4) ⇒ (1). Assume that $\delta_* \mathscr{O}_{\mathsf{X}}$ is perfect. It follows from Theorem 3.3.0.2 that δ is quasi-affine, so that Lemma 11.3.6.2 implies that δ is is a closed immersion. This proves that X is separated.

We now prove that f is fiber smooth. The assertion is local on Y; we may therefore assume that $\mathsf{Y} = \operatorname{Spét} R$ is affine. Choose an étale map $u : \operatorname{Spét} A \to \mathsf{X}$; we will prove that A is fiber smooth over R. Form a pullback diagram

$$\begin{array}{ccc} \operatorname{Spét} A \times_{\mathsf{X}} \operatorname{Spét} A & \xrightarrow{\delta'} & \operatorname{Spét} A \times_{\operatorname{Spét} R} \operatorname{Spét} A \\ \downarrow & & \downarrow \\ \mathsf{X} & \longrightarrow & \mathsf{X} \times_{\operatorname{Spét} R} \mathsf{X}. \end{array}$$

Let $\mathscr{O}$ denote the structure sheaf of $\operatorname{Spét} A \times_{\mathsf{X}} \operatorname{Spét} A$. It follows from (3) that $\delta'_* \mathscr{O}$ is perfect when regarded as a module over $A \otimes_R A$. Since u is separated and étale, the diagonal map $\operatorname{Spét} A \to \operatorname{Spét} A \times_{\mathsf{X}} \operatorname{Spét} A$ is a clopen immersion. It follows that $\delta'_* \mathscr{O}$ contains A as a direct summand, so that A is perfect when regarded as a module over $A \otimes_R A$. That is, A is a smooth R-algebra in the sense of Definition HA.4.6.4.13. Since X is assumed to be locally almost of finite presentation and of finite Tor-amplitude over R, Proposition 11.3.3.1 implies that A is fiber smooth over R. □

11.4 Smooth and Proper Linear ∞-Categories

Let Y be a spectral Deligne-Mumford stack. In §11.3 and §11.1 we introduced the notions of smooth and proper quasi-coherent stacks $\mathcal{C} \in \mathrm{QStk}^{\mathrm{St}}(\mathsf{Y})$. In this section, we will study quasi-coherent stacks which possess both of these properties.

Notation 11.4.0.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and let $\mathrm{QStk}^{\mathrm{St}}(X)$ be the ∞-category of stable quasi-coherent stacks on X. We let $\mathrm{QStk}^{\mathrm{cg}}(X)$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{St}}(X)$ whose objects are compactly generated stable quasi-coherent stacks (Definition 10.3.1.3) and whose morphisms are compact morphisms of quasi-coherent stacks (Definition 10.1.3.1). It follows from Lemma D.5.3.3 that the subcategory $\mathrm{QStk}^{\mathrm{cg}}(X) \subseteq \mathrm{QStk}^{\mathrm{St}}(X)$ is closed under the tensor product of Proposition 10.1.6.4, and therefore inherits the structure of a symmetric monoidal ∞-category.

Proposition 11.4.0.2. *Let X be a spectral Deligne-Mumford stack. Then a stable quasi-coherent stack $\mathcal{C} \in \mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$ is smooth and proper if and only if it is a dualizable object of the symmetric monoidal ∞-category $\mathrm{QStk}^{\mathrm{cg}}(\mathsf{X})$.*

Proof. It follows from Theorem D.7.0.7 that $\mathcal{C}$ is dualizable as an object of the larger symmetric monoidal ∞-category $\mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$: that is, there exists a stable quasi-coherent stack $\mathcal{C}^\vee$ on X and compatible maps

$$e : \mathcal{C}^\vee \otimes \mathcal{C} \to \mathcal{Q}_{\mathsf{X}} \qquad c : \mathcal{Q}_{\mathsf{X}} \to \mathcal{C} \otimes \mathcal{C}^\vee$$

which exhibit $\mathcal{C}^\vee$ as a dual of $\mathcal{C}$. Note that $\mathcal{C}^\vee$ is compactly generated (Remark D.7.7.6). It follows from Proposition 11.1.3.3 and Remark 11.3.4.6 that the maps c and e are compact if and only if $\mathcal{C}$ is smooth and proper. □

Our main goal is to give a proof of the following result:

Theorem 11.4.0.3. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Assume that f is quasi-compact, quasi-separated, and that $\mathrm{Sp\acute{e}t}\, R \times_{\mathsf{Y}} \mathsf{X}$ is a spectral algebraic space for every map $\mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$. The following conditions are equivalent:*

(1) *The morphism f is proper and fiber smooth.*

(2) *The functor $f_* : \mathrm{QStk}^{\mathrm{St}}(\mathsf{X}) \to \mathrm{QStk}^{\mathrm{St}}(\mathsf{Y})$ carries smooth objects of $\mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$ to smooth objects of $\mathrm{QStk}^{\mathrm{St}}(\mathsf{Y})$ and proper objects of $\mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$ to proper objects of $\mathrm{QStk}^{\mathrm{St}}(\mathsf{Y})$.*

(3) *The functor $f_* : \mathrm{QStk}^{\mathrm{cg}}(\mathsf{X}) \to \mathrm{QStk}^{\mathrm{cg}}(\mathsf{Y})$ carries dualizable objects of $\mathrm{QStk}^{\mathrm{cg}}(\mathsf{X})$ to dualizable objects of $\mathrm{QStk}^{\mathrm{cg}}(\mathsf{Y})$, where $\mathrm{QStk}^{\mathrm{cg}}(\mathsf{X})$ and $\mathrm{QStk}^{\mathrm{cg}}(\mathsf{Y})$ are defined as in Notation 11.4.0.1.*

(4) *The quasi-coherent stack $f_* \mathcal{Q}_{\mathsf{X}} \in \mathrm{QStk}^{\mathrm{St}}(\mathsf{Y})$ is smooth and proper.*

The implication (1) ⇒ (2) of Theorem 11.4.0.3 follows from Propositions 11.3.5.1 and 11.1.4.2, and the implications (2) ⇒ (3) ⇒ (4) follow from Proposition 11.4.0.2. It will therefore suffice to show that (4) ⇒ (1). This is a consequence of Theorem 11.4.2.1, which we will prove in §11.4.2.

11.4.1 Digression: Compactness Conditions on Associative Algebras

We now collect some auxiliary results (Propositions 11.4.1.1 and 11.4.1.3) which will be needed to complete our proof of Theorem 11.4.0.3.

Proposition 11.4.1.1. *Let R be a connective $\mathbb{E}_\infty$-ring, and let $A \in \mathrm{CAlg}^{\mathrm{cn}}_R$. Suppose that A is almost compact when viewed as an object of $\mathrm{Alg}^{\mathrm{cn}}_R$: that is, for every integer $n \geqslant 0$, the truncation $\tau_{\leqslant n} A$ is a compact object of $\tau_{\leqslant n} \mathrm{Alg}^{\mathrm{cn}}_R$. Then A is almost of finite presentation over R.*

Lemma 11.4.1.2. *Let R be a connective $\mathbb{E}_\infty$-ring, and suppose that $A, B \in \mathrm{Alg}^{\mathrm{cn}}_R$ are almost compact. Then the tensor product $A \otimes_R B \in \mathrm{Alg}^{\mathrm{cn}}_R$ is almost compact.*

Proof. Write A as a filtered colimit $\varinjlim A_\alpha$ of compact objects of $\mathrm{Alg}^{\mathrm{cn}}_R$, and similarly write B as a filtered colimit $\varinjlim B_\beta$. We wish to show that, for each $n \geqslant 0$, the tensor product $\tau_{\leqslant n}(A \otimes_R B)$ is a compact object of $\tau_{\leqslant n} \mathrm{Alg}^{\mathrm{cn}}_R$. Since $\tau_{\leqslant n} A$ is a compact object of $\tau_{\leqslant n} \mathrm{Alg}^{\mathrm{cn}}_R$, the equivalence $\tau_{\leqslant n} A \simeq \varinjlim_\alpha (\tau_{\leqslant n} A_\alpha)$ guarantees that $\tau_{\leqslant n} A$ is a retract of $\tau_{\leqslant n} A_\alpha$, for some index α. Similarly, $\tau_{\leqslant n} B$ is a retract of some $\tau_{\leqslant n} B_\beta$. Then $\tau_{\leqslant n}(A \otimes_R B) \simeq \tau_{\leqslant n}(\tau_{\leqslant n} A \otimes_R \tau_{\leqslant n} B)$ is a retract of $\tau_{\leqslant n}(A_\alpha \otimes_R B_\alpha) \simeq \tau_{\leqslant n}(\tau_{\leqslant n} A_\alpha \otimes_R \tau_{\leqslant n} B_\beta)$. We may therefore replace A by A_α and B by B_β, and thereby reduce to the case where A and B are compact. In this case, the desired result follows from Corollary HA.5.3.1.17 . □

Proof of Proposition 11.4.1.1. Let $\mathcal{C} = \tau_{\leqslant n} \mathrm{Alg}^{\mathrm{cn}}_R$. We will regard $\mathcal{C}$ as a symmetric monoidal ∞-category, so that we have equivalences $\mathrm{CAlg}(\mathcal{C}) \simeq \tau_{\leqslant n} \mathrm{CAlg}(\mathrm{Alg}^{\mathrm{cn}}_R) \simeq \tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}_R$. It will therefore suffice to show that $\tau_{\leqslant n} A$ is compact when regarded as a commutative algebra object of $\mathcal{C}$. Since A is almost compact as an object of $\mathrm{Alg}^{\mathrm{cn}}_R$, Lemma 11.4.1.2 implies that each $\tau_{\leqslant n} A^{\otimes m}$ is compact as an object of $\mathcal{C}$. The desired result now follows from Lemma 5.2.2.6. □

Proposition 11.4.1.3. *Let X be a quasi-compact, quasi-separated spectral algebraic space over a connective $\mathbb{E}_\infty$-ring R, let $u : \mathsf{U} \to \mathsf{X}$ be a map which is affine and étale. For each $B \in \mathrm{Alg}_R$, let $T(B)$ denote the full subcategory of $\mathrm{LMod}_B(\mathrm{QCoh}(\mathsf{U}))$ spanned by those objects $\mathscr{F}$ such that $u_* \mathscr{F}$ is compact as an object of $\mathrm{LMod}_B(\mathrm{QCoh}(\mathsf{X}))$. Then the construction $B \mapsto T(B)^{\simeq}$ determines a functor $\mathrm{Alg}_R \to \mathcal{S}$ which commutes with filtered colimits.*

Lemma 11.4.1.4. *Let $\phi : R \to A$ be an étale morphism of $\mathbb{E}_\infty$-rings. Then A is compact when viewed as an object of Alg_R.*

Proof. Let $L^{(1)}_A$ denote the cotangent complex of A regarded as an associative algebra object of Alg_R, so that $L^{(1)}_A$ is an object of $\mathrm{Sp}((\mathrm{Alg}_R)_{/A}) \simeq {}_A\mathrm{BMod}_A(R)$. Theorem HA.7.3.5.1 supplies a fiber sequence $L^{(1)}_A \to A \otimes_R A \to A$. Since A is étale over R, this sequence splits, so that $L^{(1)}_A$ can be regarded as a direct summand of $A \otimes_R A$.

For every object $B \in \mathrm{Alg}_R^{\mathrm{cn}}$ and every integer $n > 0$, we can regard $\tau_{\leqslant n} B$ as a square-zero extension of $\tau_{\leqslant n-1} B$ by $\Sigma^n(\pi_n B)$ (Theorem HA.7.4.1.23), so that we have a fiber sequence of spaces

$$\mathrm{Map}_{\mathrm{Alg}_R}(A, \tau_{\leqslant n} B) \xrightarrow{\phi} \mathrm{Map}_{\mathrm{Alg}_R}(A, \tau_{\leqslant n-1} B) \to \mathrm{Map}_{{}_A\mathrm{BMod}_A(\mathrm{Mod}_R)}(L_A^{(1)}, \Sigma^{n+1}\pi_n B).$$

Since $L_A^{(1)}$ is a direct summand of $A \otimes_R A$, the space $\mathrm{Map}_{{}_A\mathrm{BMod}_A(\mathrm{Mod}_R)}(L_A^{(1)}, \Sigma^{n+1}\pi_n B)$ is a direct factor of an Eilenberg-MacLane space $K(\pi_n B, n+1)$. It follows that the map ϕ has n-connective homotopy fibers. Since $\mathrm{Map}_{\mathrm{Alg}_R}(A, B)$ can be identified with the homotopy limit of the tower

$$\cdots \to \mathrm{Map}_{\mathrm{Alg}_R}(A, \tau_{\leqslant 2} B) \to \mathrm{Map}_{\mathrm{Alg}_R}(A, \tau_{\leqslant 1} B) \to \mathrm{Map}_{\mathrm{Alg}_R}(A, \tau_{\leqslant 0} B),$$

we conclude that each of the maps $\mathrm{Map}_{\mathrm{Alg}_R}(A, B) \to \mathrm{Map}_{\mathrm{Alg}_R}(A, \tau_{\leqslant n} B)$ has $(n+1)$-connective homotopy fibers, and therefore induces a homotopy equivalence

$$\tau_{\leqslant n} \mathrm{Map}_{\mathrm{Alg}_R}(A, B) \to \tau_{\leqslant n} \mathrm{Map}_{\mathrm{Alg}_R}(A, \tau_{\leqslant n} B).$$

Now suppose that we are given a filtered diagram $\{B_\alpha\}$ in Alg_R having colimit B; we wish to show that the induced map

$$\theta : \varinjlim \mathrm{Map}_{\mathrm{Alg}_R}(A, B_\alpha) \to \mathrm{Map}_{\mathrm{Alg}_R}(A, B)$$

is a homotopy equivalence. Without loss of generality, we may assume that each B_α is connective. It will suffice to show that for each $m \geqslant 0$, the map

$$\varinjlim \tau_{\leqslant m} \mathrm{Map}_{\mathrm{Alg}_R}(A, B_\alpha) \to \tau_{\leqslant m} \mathrm{Map}_{\mathrm{Alg}_R}(A, B)$$

is a homotopy equivalence. We may therefore replace each B_α by $\tau_{\leqslant m} B_\alpha$, and thereby reduce to the case where there exists an integer $n = 0$ such that each B_α is n-truncated.

We now proceed by induction on n. In the case $n = 0$, each B_α is discrete. The desired result is therefore equivalent to the fact that $\pi_0 A$ is finitely presented when regarded as an associative algebra over $\pi_0 R$. Let us therefore assume that $n > 0$. We have a diagram of fiber sequences

$$\begin{array}{ccc}
\varinjlim \mathrm{Map}_{\mathrm{Alg}_R}(A, B_\alpha) & \xrightarrow{\theta} & \mathrm{Map}_{\mathrm{Alg}_R}(A, B) \\
\downarrow & & \downarrow \\
\varinjlim \mathrm{Map}_{\mathrm{Alg}_R}(A, \tau_{\leqslant n-1} B_\alpha) & \xrightarrow{\theta'} & \mathrm{Map}_{\mathrm{Alg}_R}(A, \tau_{\leqslant n-1} B) \\
\downarrow & & \downarrow \\
\varinjlim \mathrm{Map}_{{}_A\mathrm{BMod}_A(\mathrm{Mod}_R)}(L_A^{(1)}, \Sigma^{n+1}\pi_n B_\alpha) & \xrightarrow{\theta''} & \mathrm{Map}_{{}_A\mathrm{BMod}_A(\mathrm{Mod}_R)}(L_A^{(1)}, \Sigma^{n+1}\pi_n B).
\end{array}$$

The inductive hypothesis implies that θ' is a homotopy equivalence. Consequently, to show that θ is a homotopy equivalence, it suffices to show that θ'' is a homotopy equivalence. This follows from the observation that $L^{(1)}_A$ is a direct summand of $A \otimes_R A$, and therefore a compact object of ${}_A\mathrm{BMod}_A(\mathrm{Mod}_R)$. □

Proof of Proposition 11.4.1.3. Let $u : \mathsf{U} \to \mathsf{X}$ be an affine étale morphism between quasi-compact, quasi-separated spectral algebraic spaces over R. Write $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$. For each $U \in \mathcal{X}$, let $\mathsf{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathsf{X}}|_U)$, let U_U denote the fiber product $\mathsf{X}_U \times_{\mathsf{X}} \mathsf{U}$. Let $T_U : \mathrm{Alg}_R \to \mathcal{C}\mathrm{at}_\infty$ be the functor which assigns to each $B \in \mathrm{Alg}_R$ the full subcategory of $\mathrm{LMod}_B(\mathrm{QCoh}(\mathsf{U}_U))$ spanned by those objects whose direct image in $\mathrm{LMod}_B(\mathsf{X}_U)$ is locally compact (when viewed as section over the quasi-coherent stack given by $V \mapsto \mathrm{LMod}_B(\mathrm{QCoh}(\mathsf{X}_V))$). The construction $U \mapsto T_U(B)^{\simeq}$ carries colimits in $\mathcal{X}$ to limits of ∞-categories. Let $\mathcal{X}_0$ denote the full subcategory of $\mathcal{X}$ spanned by those objects U for which the functor $B \mapsto T_U(B)^{\simeq}$ preserves filtered colimits. Then $\mathcal{X}_0$ is closed under finite colimits in $\mathcal{X}$. We wish to show that $\mathcal{X}_0$ contains the final object of $\mathcal{X}$. Note that X admits a scallop decomposition (Theorem 3.4.2.1). Using Corollary 2.5.3.6, we are reduced to proving that $\mathcal{X}_0$ contains all affine objects $U \in \mathcal{X}$. Replacing X by X_U, we may reduce to the case where X is affine. Replacing R by the $\mathbb{E}_\infty$-ring of global sections of the structure sheaf of X if necessary, we may suppose that $\mathsf{X} = \mathrm{Sp\acute{e}t}\, R$. In this case, we can write $\mathsf{U} = \mathrm{Sp\acute{e}t}\, A$, where A is étale over R.

Suppose we are given a diagram $\{B_\alpha\}$ in Alg_R which is indexed by a filtered partially ordered sets and having colimit B. We have a commutative diagram of spaces

$$\begin{array}{ccc} \varinjlim T(B_\alpha)^{\simeq} & \longrightarrow & T(B)^{\simeq} \\ \downarrow & & \downarrow \\ \varinjlim (\mathrm{LMod}^{\mathrm{perf}}_{B_\alpha})^{\simeq} & \longrightarrow & (\mathrm{LMod}^{\mathrm{perf}}_{B})^{\simeq}, \end{array}$$

where the bottom horizontal map is a homotopy equivalence by virtue of Proposition HA.4.6.3.11 . To show that the top horizontal map is a homotopy equivalence, it will suffice to show that it induces a homotopy equivalence after passing to the homotopy fibers of the vertical maps over any point $\eta \in \varinjlim (\mathrm{LMod}^{\mathrm{perf}}_{B_\alpha})^{\simeq}$, represented by an index α and a perfect left B_α-module M_α. For each $\beta \geqslant \alpha$, let $M_\beta = A_\beta \otimes_{A_\alpha} M_\alpha$, and let $M = B \otimes_{B_\alpha} M_\alpha$. Unwinding the definitions, we are reduced to proving that the map

$$\theta : \varinjlim_{\beta \geqslant \alpha} {}_{B_\beta}\mathrm{BMod}_A(\mathrm{Mod}_R) \times_{\mathrm{LMod}_{B_\beta}} \{M_\beta\} \to {}_B\mathrm{BMod}_A(\mathrm{Mod}_R) \times_{\mathrm{LMod}_B} \{M\}$$

is a homotopy equivalence.

For each $\beta \geqslant \alpha$, let $E_\beta \in \mathrm{Alg}_R$ denote the endomorphism algebra of M_β (where we regard M_β as an object of the R-linear ∞-category LMod_{B_β}, and let $E \in \mathrm{Alg}_R$ denote the endomorphism algebra of M. Using Corollary HA.4.7.1.40 , we can identify θ with the canonical map

$$\varinjlim_{\beta \geqslant \alpha} \mathrm{Map}_{\mathrm{Alg}_R}(A, E_\beta) \to \mathrm{Map}_{\mathrm{Alg}_R}(A, E).$$

It follows from Proposition HA.4.6.3.11 that the natural map $\varinjlim_{\beta \geqslant \alpha} E_\beta \to E$ is an equivalence in Alg_R. We now conclude by observing that A is compact as an object of Alg_R (Lemma 11.4.1.4). □

11.4.2 The Proof of Theorem 11.4.0.3

We now complete the proof of Theorem 11.4.0.3 by establishing the following:

Theorem 11.4.2.1. *Let $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ be a morphism of quasi-compact, quasi-separated spectral algebraic spaces. If* $\mathrm{QCoh}(\mathsf{X})$ *is smooth and proper as an R-linear ∞-category, then the morphism f is proper and fiber smooth.*

Proof. Since $\mathrm{QCoh}(\mathsf{X})$ is a smooth R-linear ∞-category, the proof of Theorem 11.3.6.1 shows that pushforward along the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ carries the structure sheaf of X to a perfect quasi-coherent sheaf on $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$. Lemma 11.3.6.2 implies that δ is a closed immersion, so that X is separated. Suppose, for the moment, that f is locally almost of finite presentation. Since $\mathrm{QCoh}(\mathsf{X})$ is a proper R-linear ∞-category, Theorem 11.1.4.1 guarantees that f is proper and has Tor-amplitude $\leqslant n$, for some integer n. Applying Theorem 11.3.6.1, we deduce that f is fiber smooth, thereby completing the proof of Theorem 11.4.2.1.

It remains to prove that f is locally almost of finite presentation. Choose an étale map $u : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$. We wish to prove that A is locally almost of finite presentation over R. By virtue of Proposition 11.4.1.1, it will suffice to show that A is compact when regarded as an object of Alg_R.

For every morphism $\phi : A \to B$ in Alg_R, we can regard B as an object of the ∞-category ${}_B\mathrm{BMod}_A(\mathrm{Mod}_R) \simeq \mathrm{LMod}_B(\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A))$. Let $\mathscr{F}_\phi \in \mathrm{LMod}_B(\mathrm{QCoh}(\mathsf{X}))$ denote the image of this object under the pushforward functor u_*. We claim that $\mathscr{F}_\phi$ is a compact object of $\mathrm{LMod}_B(\mathrm{QCoh}(\mathsf{X}))$. To prove this, it will suffice (by Theorem **??**) to show that for every étale map $v : \mathrm{Sp\acute{e}t}\, A' \to \mathsf{X}$, the pullback $v^* \mathscr{F}_\phi$ is a compact object of $\mathrm{LMod}_B(\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A')) \simeq {}_B\mathrm{BMod}_{A'}(\mathrm{Mod}_R)$. Since X is separated, we can write $\mathrm{Sp\acute{e}t}\, A' \times_{\mathsf{X}} \mathrm{Sp\acute{e}t}\, A$ as $\mathrm{Sp\acute{e}t}\, A''$, for some $A'' \in \mathrm{Mod}_R$. Unwinding the definitions, we can identify $v^* \mathscr{F}_\phi$ with the tensor product $B \otimes_A A''$. Since δ_* carries the structure sheaf of X to a perfect object of $\mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X})$, the $\mathbb{E}_\infty$-ring A'' is perfect when regarded as a module over $A \otimes_R A'$, so that $B \otimes_A A''$ is perfect when regarded as a module over $B \otimes_R A'$.

For every algebra object $B \in \mathrm{Alg}_R$, let $\mathcal{C}_B$ denote the full subcategory of

$${}_B\mathrm{BMod}_A(\mathrm{Mod}_R) \simeq \mathrm{LMod}_B(\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A))$$

spanned by those objects have compact image in $\mathrm{LMod}_B(\mathrm{QCoh}(\mathsf{X}))$ under the pushforward functor u_*. Since $\mathrm{QCoh}(\mathsf{X})$ is proper, the global sections functor $\Gamma : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Mod}_R$ preserves compact objects, so that the induced functor $\Gamma_B : \mathrm{LMod}_B(\mathrm{QCoh}(\mathsf{X})) \simeq \mathrm{LMod}_B \otimes_{\mathrm{Mod}_R} \mathrm{QCoh}(\mathsf{X}) \to \mathrm{LMod}_B$ also preserves compact objects. It follows that the composite functor

$$_B\mathrm{BMod}_A(\mathrm{Mod}_R) \xrightarrow{u_*} \mathrm{LMod}_B(\mathrm{QCoh}(\mathsf{X})) \xrightarrow{\Gamma_B} \mathrm{LMod}_B$$

carries $\mathcal{C}_B$ into $\mathrm{LMod}_B^{\mathrm{perf}}$. Applying Corollary HA.4.8.5.6 , we see that the construction

$$(\phi : A \to B) \mapsto \mathscr{F}_\phi$$

induces a homotopy equivalence of Kan complexes

$$\mathrm{Map}_{\mathrm{Alg}_R}(A, B) \to \mathcal{C}_B^{\simeq} \times_{\mathrm{LMod}_B^{\mathrm{perf}}} \{B\}.$$

According to Proposition HA.4.6.3.11 , the construction $B \mapsto \mathrm{LMod}_B^{\mathrm{perf}}$ commutes with filtered colimits. To prove that A is compact as an object of Alg_R, it will suffice to show that the construction $B \mapsto \mathcal{C}_B^{\simeq}$ also preserves filtered colimits, which follows from Proposition 11.4.1.3. □

11.4.3 Variant: Dualizability and Affineness

Let R be an $\mathbb{E}_\infty$-ring and let X be a quasi-compact, quasi-separated spectral algebraic space over R. According to Theorem 11.4.2.1 Proposition 11.4.0.2, the spectral algebraic space X is proper and fiber smooth over R if and only if $\mathrm{QCoh}(\mathsf{X})$ is dualizable as an object of the ∞-category of compactly generated stable R-linear ∞-categories (whose morphisms are functors which preserve small colimits and compact objects). We now consider a variant of this result, where we focus our attention on the *prestable* R-linear ∞-category $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$:

Theorem 11.4.3.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Assume that f is quasi-compact, quasi-separated, and that* $\mathrm{Sp\acute{e}t}\, R \times_{\mathsf{Y}} \mathsf{X}$ *is a spectral algebraic space for every map* $\mathrm{Sp\acute{e}t}\, R \to \mathsf{Y}$*. The following conditions are equivalent:*

(1) *The morphism f is affine.*

(2) *The pushforward functor $f_* : \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \to \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y})$ (see Corollary 10.3.2.3) carries dualizable objects of* $\mathrm{QStk}(\mathsf{X})$ *to dualizable objects of* $\mathrm{QStk}(\mathsf{Y})$*.*

(3) *The quasi-coherent stack $f_* \mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y})$ is a dualizable object of* $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y})$*.*

We will deduce Theorem 11.4.3.2 from Serre's affineness criterion (Proposition 9.6.5.1) together with following observation:

Remark 11.4.3.2. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable R-linear ∞-category. Then the forgetful functor $\mathrm{Mod}_R^{\mathrm{cn}} \to \mathrm{Sp}^{\mathrm{cn}}$ exhibits $\mathrm{Mod}_R^{\mathrm{cn}}$ as a symmetric Frobenius algebra object of the symmetric monoidal ∞-category $\mathrm{Mod}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ (see Remark HA.4.6.5.7). It follows that $\mathcal{C}$ is dualizable as an object $\mathrm{Mod}_{\mathrm{Mod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ if and only if it is dualizable as an object of $\mathrm{Mod}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. Moreover, if these conditions are satisfied, then the dual of $\mathcal{C}$ as an object of $\mathrm{Mod}_{\mathrm{Mod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ is equivalent to the dual of $\mathcal{C}$ as an object of $\mathrm{Mod}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. It follows that $\mathcal{C}$ is dualizable in $\mathrm{LinCat}_R^{\mathrm{PSt}}$ if and only if it is dualizable in Groth_∞.

Proof of Theorem 11.4.3.1. We first show that (1) implies (2). Assume that $f : \mathsf{X} \to \mathsf{Y}$ is affine and that $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ is dualizable; we wish to show that $f_* \mathcal{C}$ is dualizable in $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{Y})$. To prove this, we can work locally on Y and thereby reduce to the case where $\mathsf{Y} \simeq \mathrm{Spét}\, A$ is affine. In this case, our assumption that f is affine guarantees that $\mathsf{X} \simeq \mathrm{Spét}\, B$ is also affine. In this case, $\mathcal{C}$ with a B-linear prestable ∞-category. By assumption, $\mathcal{C}$ admits a dual in the ∞-category $\mathrm{LinCat}_B^{\mathrm{PSt}}$ and we wish to show that the image of $\mathcal{C}$ in $\mathrm{LinCat}_A^{\mathrm{PSt}}$ is also dualizable. This follows from Remark 11.4.3.2: both conditions are equivalent to the requirement that $\mathcal{C}$ is dualizable as an object of Groth_∞.

The implication (2) ⇒ (3) is tautological. To prove that (3) ⇒ (1), we can again reduce to the case where $\mathsf{Y} \simeq \mathrm{Spét}\, R$ is affine. We will prove that X is also affine by verifying that the global sections functor $\Gamma(\mathsf{X}; \bullet) : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Sp}$ is t-exact (see Proposition 9.6.5.1).

Condition (3) guarantees that $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ admits a dual in the ∞-category $\mathrm{LinCat}_R^{\mathrm{PSt}}$. Applying Remark 11.4.3.2, we deduce that $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ is dualizable as an object of Groth_∞. Then we can choose a Grothendieck prestable ∞-category $\mathcal{C}$ and (compatible) functors

$$e : \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \otimes \mathcal{C} \to \mathrm{Sp}^{\mathrm{cn}} \qquad c : \mathrm{Sp}^{\mathrm{cn}} \to \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \otimes \mathcal{C}$$

which exhibit $\mathcal{C}$ as a dual of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ in the ∞-category Groth_∞. Since the stabilization functor $\mathrm{Sp}(\bullet) : \mathsf{Groth}_\infty \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$ is symmetric monoidal, the induced maps

$$\mathrm{Sp}(e) : \mathrm{QCoh}(\mathsf{X}) \otimes \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp} \qquad \mathrm{Sp}(c) : \mathrm{Sp} \to \mathrm{QCoh}(\mathsf{X}) \otimes \mathrm{Sp}(\mathcal{C})$$

exhibit $\mathrm{Sp}(\mathcal{C})$ as the dual of $\mathrm{QCoh}(\mathsf{X})$ as a presentable stable ∞-category. Since X is a quasi-compact, quasi-separated spectral algebraic space, the monoidal stable ∞-category $\mathrm{QCoh}(\mathsf{X})$ is locally rigid. Applying Proposition **??**, we deduce that the composite functor

$$e' : \mathrm{QCoh}(\mathsf{X}) \otimes \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\otimes} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\Gamma(\mathsf{X};\bullet)} \to \mathrm{Sp}$$

exhibits $\mathrm{QCoh}(\mathsf{X})$ as a dual of itself in $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. We can therefore choose an equivalence $\alpha : \mathrm{Sp}(\mathcal{C}) \simeq \mathrm{QCoh}(\mathsf{X})$ for which the functor $\mathrm{Sp}(c)$ is given by

$$\mathrm{QCoh}(\mathsf{X}) \otimes \mathrm{Sp}(\mathcal{C}) \xrightarrow{\alpha} \mathrm{QCoh}(\mathsf{X}) \otimes \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\otimes} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\Gamma(\mathsf{X};\bullet)} \mathrm{Sp}\,.$$

Let us identify $\mathcal{C}$ with a full subcategory of $\mathrm{QCoh}(\mathsf{X})$ via the fully faithful embedding $\mathcal{C} \xrightarrow{\Sigma^\infty} \mathrm{Sp}(\mathcal{C}) \xrightarrow{\alpha} \mathrm{QCoh}(\mathsf{X})$. Using the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \otimes \mathcal{C} & \xrightarrow{e} & \mathrm{Sp}^{\mathrm{cn}} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{X}) \otimes \mathrm{QCoh}(\mathsf{X}) & \xrightarrow{e'} & \mathrm{Sp}, \end{array}$$

we deduce that the spectrum $\Gamma(\mathsf{X}; \mathscr{F} \otimes \mathscr{G})$ is connective whenever $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ and $\mathscr{G} \in \mathcal{C}$. Consequently, to prove that $\Gamma(\mathsf{X}; \bullet)$ is right t-exact, it will suffice to show that the structure sheaf $\mathscr{O}_{\mathsf{X}}$ of X is contained in $\mathcal{C}$. Since $\Gamma(\mathsf{X}; \bullet)$ is automatically left t-exact, this will complete the proof of (c).

Note that the action of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ on itself induces an action of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ on its dual $\mathcal{C}$. Using the equivalence $\mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X}) \simeq \mathrm{Mod}_{\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}}(\mathsf{Groth}_\infty)$ of Theorem 10.2.0.2, we can write $\mathcal{C} = \mathrm{QCoh}(\mathsf{X}; \mathcal{E})$ for some prestable quasi-coherent stack $\mathcal{E} \in \mathrm{QStk}^{\mathrm{PSt}}(\mathsf{X})$ whose stabilization is $\mathcal{Q}_{\mathsf{X}}$. To show that $\mathscr{O}_{\mathsf{X}}$ belongs to $\mathcal{C}$, it will suffice to show that the unit map $\mathcal{Q}_{\mathsf{X}}^{\mathrm{cn}} \to \mathcal{Q}_{\mathsf{X}}$ factors through $\mathcal{E}$. This assertion is local on X: it amounts to the assertion that, for every étale map $\eta : \mathsf{U} \to \mathsf{X}$ where $\mathsf{U} = (\mathcal{U}, \mathscr{O}_{\mathsf{U}})$ is affine, the structure sheaf $\mathscr{O}_{\mathsf{U}}$ belongs to the essential image of the inclusion $\mathcal{E}_\eta \subseteq (\mathcal{Q}_{\mathsf{X}})_\eta \simeq \mathrm{QCoh}(\mathsf{U})$. Let $q_{\mathsf{U}} : \mathsf{U} \times \mathsf{U} \to \mathsf{U}$ denote the projection onto the first factor. Since $\mathsf{U} \simeq \mathrm{Spét}\, A$ is affine, the direct image functor $q_{\mathsf{U}*} : \mathrm{QCoh}(\mathsf{U} \times \mathsf{U}) \to \mathrm{QCoh}(\mathsf{U})$ carries the full subcategory

$$\mathrm{LMod}_A(\mathcal{E}_\eta) \simeq \mathcal{E}_\eta \otimes \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \subseteq \mathrm{QCoh}(\mathsf{U}) \otimes \mathrm{QCoh}(\mathsf{U}) \simeq \mathrm{QCoh}(\mathsf{U} \times \mathsf{U})$$

into $\mathcal{E}_\eta \subseteq \mathrm{QCoh}(\mathsf{U})$.

Write $\mathsf{U} \times_{\mathsf{X}} \mathsf{U} = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}})$ so that the diagonal map $\delta : \mathsf{U} \to \mathsf{U} \times \mathsf{U}$ factors as a composition $\mathsf{U} \xrightarrow{\delta'} (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}}) \xrightarrow{\delta''} \mathsf{U} \times \mathsf{U}$. Since η is étale and separated, the diagonal map δ' is a clopen immersion so that $\delta'_* \mathscr{O}_{\mathsf{U}}$ is a direct summand of $\mathscr{O}_{\mathsf{Y}}$. It follows that $\mathscr{O}_{\mathsf{Y}} \simeq q_{\mathsf{U}*} \delta_* \mathscr{O}_{\mathsf{U}} \simeq q_{\mathsf{U}*} \delta''_* \delta'_* \mathscr{O}_{\mathsf{U}}$ is a direct summand of $q_{\mathsf{U}*} \delta'_* \mathscr{O}_{\mathsf{Y}}$. We are therefore reduced to proving that $\delta'_* \mathscr{O}_{\mathsf{Y}} \in \mathrm{QCoh}(\mathsf{U} \times \mathsf{U})$ belongs to the essential image of the inclusion $\mathcal{E}_\eta \otimes \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \hookrightarrow \mathrm{QCoh}(\mathsf{U} \times \mathsf{U})$. Note that we can identify $\delta'_* \mathscr{O}_{\mathsf{Y}}$ with the pullback $(\eta \times \eta)^* \epsilon_* \mathscr{O}_{\mathsf{X}}$, where $\epsilon : \mathsf{X} \to \mathsf{X} \times \mathsf{X}$ is the diagonal map of X. Using the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{C} \otimes \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} & \longrightarrow & \mathcal{E}_\eta \otimes \mathrm{QCoh}(\mathsf{U})^{\mathrm{cn}} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{X} \times \mathsf{X}) & \xrightarrow{(\eta \times \eta)^*} & \mathrm{QCoh}(\mathsf{U} \times \mathsf{U}), \end{array}$$

we see that it will suffice to show that $\epsilon_* \mathscr{O}_{\mathsf{X}}$ belongs to the essential image of the inclusion $\mathcal{C} \otimes \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \hookrightarrow \mathrm{QCoh}(\mathsf{X} \times \mathsf{X})$. The evaluation map $e' : \mathrm{QCoh}(\mathsf{X}) \otimes \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Sp}$ admits a compatible coevaluation $c' : \mathrm{Sp} \to \mathrm{QCoh}(\mathsf{X}) \otimes \mathrm{QCoh}(\mathsf{X}) \simeq \mathrm{QCoh}(\mathsf{X} \times \mathsf{X})$, which carries the sphere spectrum S to the sheaf $\epsilon_* \mathscr{O}_{\mathsf{X}}$. The desired result now follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Sp}^{\mathrm{cn}} & \xrightarrow{c} & \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \otimes \mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{Sp} & \xrightarrow{c'} & \mathrm{QCoh}(\mathsf{X}) \otimes \mathrm{QCoh}(\mathsf{X}). \end{array}$$

□

11.4.4 Deforming Objects of R-Linear ∞-Categories

We conclude this section with some convergence results concerning the deformation theory of objects of smooth and proper R-linear ∞-categories. In what follows, if $\mathcal{C}$ is a compactly generated stable R-linear ∞-category and A is an $\mathbb{E}_1$-algebra over R, we let $\mathrm{LMod}_A(\mathcal{C})^c$ denote the full subcategory of $\mathrm{LMod}_A(\mathcal{C}) \simeq \mathrm{LMod}_A \otimes_{\mathrm{Mod}_R} \mathcal{C}$ spanned by the compact objects. Our goal is to prove the following:

Theorem 11.4.4.1. *Let R be the limit of a tower of $\mathbb{E}_\infty$-rings*

$$\cdots \to R_2 \to R_1 \to R_0$$

satisfying the following conditions:

(1) *Each R_n is connective.*

(2) *Each of the maps $R_n \to R_{n-1}$ induces a surjection of commutative rings $\pi_0 R_n \to \pi_0 R_{n-1}$, whose kernel is a nilpotent ideal in $\pi_0 R_n$.*

(3) *The canonical map $\pi_0 R \to \lim^0\{\pi_0 R_n\}$ is an isomorphism. Equivalently, the abelian group $\lim^1\{\pi_1 R_n\}$ vanishes.*

Let $\mathcal{C}$ be a compactly generated stable R-linear ∞-category. If $\mathcal{C}$ is proper, then the canonical map $\theta : \mathcal{C}^c \to \varprojlim \mathrm{LMod}_{R_n}(\mathcal{C})^c$ is fully faithful. If $\mathcal{C}$ is smooth and proper, then θ is an equivalence of ∞-categories.

Example 11.4.4.2. Let R be an arbitrary connective $\mathbb{E}_\infty$-ring. Then the Postnikov tower

$$\cdots \to \tau_{\leqslant 2} R \to \tau_{\leqslant 1} R \to \tau_{\leqslant 0} R$$

satisfies the hypotheses of Theorem 11.4.4.1. It follows that for every smooth and proper R-linear ∞-category $\mathcal{C}$, the canonical map $\mathcal{C}^c \to \varprojlim \mathrm{LMod}_{\tau_{\leqslant n} R}(\mathcal{C})^c$ is an equivalence of ∞-categories.

Example 11.4.4.3. Let R be a Noetherian commutative ring which is complete with respect to an ideal $I \subseteq R$. Then the tower

$$\cdots \to R/I^3 \to R/I^2 \to R/I$$

satisfies the hypotheses of Theorem 11.4.4.1. It follows that for every smooth and proper R-linear ∞-category $\mathcal{C}$, the canonical map $\mathcal{C}^c \to \varprojlim \mathrm{LMod}_{R/I^n}(\mathcal{C})^c$ is an equivalence of ∞-categories.

Example 11.4.4.4. Let R be a commutative ring, let p be a prime number which vanishes in R, and let $W(R)$ denote the ring of (p-typical) Witt vectors of R. Then $W(R)$ is the limit of a tower of commutative rings

$$\cdots \to W_3(R) \to W_2(R) \to W_1(R) \simeq R$$

which satisfies the hypotheses of Theorem 11.4.4.1. It follows that if $\mathcal{C}$ is a smooth and proper $W(R)$-linear ∞-category, then the canonical map $\mathcal{C}^c \to \varprojlim \mathrm{LMod}_{W_n(R)}(\mathcal{C})^c$ is an equivalence of ∞-categories.

We begin by proving Theorem 11.4.4.1 in the special case $\mathcal{C} = \mathrm{Mod}_R$.

Lemma 11.4.4.5. *Let R be an associative ring, let $I \subseteq R$ be a nilpotent two-sided ideal, and let e be an endomorphism of the left R-module $(R/I)^k$ such that $e = e^2$. Then e can be lifted to an endomorphism $\overline{e}$ of R^k satisfying $\overline{e}^2 = \overline{e}$.*

Proof. Without loss of generality, we may suppose that $I^2 = 0$. Let e' be an arbitrary endomorphism of R^k lifting e, so that $e'^2 = e' + q$, where q is an endomorphism of R^k whose image belongs to I^k. Then $e'^2 + e'q = e'^3 = e'^2 + qe'$, so that $qe' = e'q$. Using this, a simple calculation shows that $\overline{e} = e' + q - 2e'q$ has the desired property. □

Lemma 11.4.4.6. *Let R be the limit of a tower of $\mathbb{E}_1$-rings*

$$\cdots \to R_2 \to R_1 \to R_0$$

satisfying the following conditions:

(1) *Each R_n is connective.*

(2) *Each of the maps $R_n \to R_{n-1}$ induces a surjection of associative rings $\pi_0 R_n \to \pi_0 R_{n-1}$, whose kernel is a nilpotent ideal in $\pi_0 R_n$.*

Then the canonical map $F : \mathrm{LMod}_R^{\mathrm{perf}} \to \varprojlim \mathrm{LMod}_{R_n}^{\mathrm{perf}}$ is an equivalence of ∞-categories.

Proof. Let $\mathcal{C}$ denote the inverse limit $\varprojlim \mathrm{LMod}^{\mathrm{perf}}_{R_n}$. We will identify objects of $\mathcal{C}$ with inverse systems $\vec{M} = \{M_n \in \mathrm{LMod}^{\mathrm{perf}}_{R_n}\}_{n \geqslant 0}$ satisfying $R_{n-1} \otimes_{R_n} M_n \simeq M_{n-1}$ for $n > 0$. Let us say that such an inverse system $\vec{M}$ is *connective* if M_0 is connective.

We define a functor $G : \mathcal{C} \to \mathrm{Mod}_R$ by the formula $G(\vec{M}) = \varprojlim M_n$. Note that G is a right adjoint to F, in the sense that we have homotopy equivalences

$$\mathrm{Map}_{\mathrm{LMod}_R}(N, G(\vec{M})) \simeq \mathrm{Map}_{\mathcal{C}}(F(N), \vec{M})$$

depending functorially on $\vec{M} \in \mathcal{C}$, $N \in \mathrm{LMod}^{\mathrm{perf}}_R$. Moreover, if $N \in \mathrm{LMod}_R$ is perfect, then the unit map $N \to (G \circ F)(N)$ can be identified with the tensor product of N with the natural map $R \to \varprojlim R_n$, and is therefore an equivalence. It follows that the functor F is fully faithful.

We next claim that if $\vec{M} \in \mathcal{C}$ is connective, then each M_n is a connective module over R_n. Suppose otherwise: then there exists some smallest integer $m < 0$ such that $\pi_m M_n \neq 0$. Then we have an isomorphism $\mathrm{Tor}^{\pi_0 R_n}_0(\pi_0 R_0, \pi_m M_n) \simeq \pi_m M_0 \simeq 0$. Since the map $\pi_0 R_n \to \pi_0 R_0$ is a surjection with nilpotent kernel, we obtain a contradiction.

It follows that if $\vec{M} \in \mathcal{C}$ is connective, then the inverse system

$$\cdots \to \pi_0 M_2 \to \pi_0 M_1 \to \pi_0 M_0$$

consists of surjective maps. It follows that for any projective left R-module P, the maps

$$\cdots \to \mathrm{Map}_{\mathrm{LMod}_R}(P, M_2) \to \mathrm{Map}_{\mathrm{LMod}_R}(P, M_1) \to \mathrm{Map}_{\mathrm{LMod}_R}(P, M_0)$$

are surjective on connected components, so that any map of R-modules $P \to M_0$ can be lifted to a map $F(P) \to \vec{M}$ in the ∞-category $\mathcal{C}$.

We now prove that the functor F is essentially surjective. Fix an object $\vec{M} \in \mathcal{C}$; we wish to show that $\vec{M}$ belongs to the essential image of F. Replacing $\vec{M}$ by a suspension if necessary, we may suppose that $\vec{M}$ is connective. Since M_0 is perfect as a left module over R_0, it has Tor-amplitude $\leqslant k$ for some integer $k \geqslant 0$. We proceed by induction on k. Suppose first that $k > 0$. Since M_0 is perfect and connective, $\pi_0 M_0$ is finitely generated as a module over $\pi_0 R_0$. We may therefore choose a map of left R-modules $R^k \to M_0$ which induces a surjection on π_0. Using the preceding arguments, we can lift this to a map $\alpha : F(R^k) \to \vec{M}$ in the ∞-category $\mathcal{C}$. Consequently, to show that $\vec{M}$ belongs to the essential of F, it will suffice to show that $\mathrm{fib}(\alpha)$ belongs to the essential image of F. This follows from the inductive hypothesis, since $\mathrm{fib}(\alpha)$ is connective and has Tor-amplitude $\leqslant k - 1$.

It remains to treat the case where $k = 0$: that is, the case where M_0 is a finitely generated projective module over R_0. In particular, we can write $\pi_0 M_0$ as the image of an idempotent endomorphism e_0 of $(\pi_0 R_0)^k$, for some integer k. Applying Lemma 11.4.4.5 repeatedly, we lift e_0 to idempotent endomorphisms e_n of $(\pi_0 R_n)^k$, for each integer $n \geqslant 0$. The inverse system $\{e_n\}_{n \geqslant 0}$ determines an idempotent endomorphism of A^k, where $A = \varprojlim \pi_0 R_n$. Note that the map $\pi_0 R \to A$ is a surjection, whose kernel I can be identified with $\lim^1\{\pi_1 R_n\}$. The action of $\pi_0 R$ on I factors through A, so that I is nilpotent. Using Lemma 11.4.4.5 again, we can lift $\{e_n\}_{n \geqslant 0}$ to an idempotent endomorphism e of $(\pi_0 R)^k$, whose image P_0 is a projective left module of finite rank over $\pi_0 R$. Applying Corollary HA.7.2.2.19 , we can lift P_0 to a projective left R-module P of finite rank. By construction, we have a a map of R-modules $P \to M_0$, which we can lift to a map $\beta : F(P) \to \vec{M}$. Replacing $\vec{M}$ by $\mathrm{cofib}(\beta)$, we can reduce to the case where $M_0 \simeq 0$. Then $\Sigma^k \vec{M}$ is a connective object of $\mathcal{C}$ for each integer k. It follows that $\Sigma^k M_n$ is a connective left R_n-module for every integer k, so that each M_n is zero and $\vec{M} \simeq 0$ belongs to the essential image of F, as desired. □

Proof of Theorem 11.4.4.1. We first show that canonical map $\theta : \mathcal{C}^c \to \varprojlim_n \mathrm{LMod}_{R_n}(\mathcal{C})^c$ is fully faithful. Let $X, Y \in \mathcal{C}$ be compact objects. Unwinding the definitions, we wish to show that θ induces a homotopy equivalence

$$\Omega^\infty \underline{\mathrm{Map}}_{\mathcal{C}}(X, Y) \to \varprojlim \Omega^\infty \underline{\mathrm{Map}}_{\mathrm{LMod}_{R_n}(\mathcal{C})}(R_n \otimes X, R_n \otimes Y).$$

In fact, we claim that the map

$$\phi : \underline{\mathrm{Map}}_{\mathcal{C}}(X, Y) \to \varprojlim \underline{\mathrm{Map}}_{\mathrm{LMod}_{R_n}(\mathcal{C})}(R_n \otimes X, R_n \otimes Y) \simeq \varprojlim \underline{\mathrm{Map}}_{\mathcal{C}}(X, R_n \otimes Y)$$

is an equivalence of R-modules. Since $X \in \mathcal{C}$ is compact, we can identify ϕ with the canonical map $R \otimes_R \underline{\mathrm{Map}}_{\mathcal{C}}(X, Y) \to \varprojlim(R_n \otimes_R \underline{\mathrm{Map}}_{\mathcal{C}}(X, Y))$. Since $\mathcal{C}$ is proper, $\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y)$ is a perfect R-module, so tensor product with $\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y)$ commutes with limits. We are therefore reduced to proving that the canonical map $R \to \varprojlim R_n$ is an equivalence, which is true by hypothesis.

We now prove that θ is essentially surjective. Since $\mathcal{C}$ is a smooth R-linear ∞-category, we can write $\mathcal{C} \simeq \mathrm{RMod}_A$ for some smooth $\mathbb{E}_1$-algebra A over R (Proposition 11.3.2.4). It follows from Proposition 11.1.0.2 that A is also a proper $\mathbb{E}_1$-algebra over R, so that an object of $\mathcal{C} \simeq \mathrm{RMod}_A$ is compact if and only if its image in Mod_R is compact. We therefore have a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^c & \xrightarrow{\theta} & \varprojlim \mathrm{LMod}_{\tau_{\leqslant n} R}(\mathcal{C})^c \\ \downarrow & & \downarrow \\ \mathrm{Mod}_R^{\mathrm{perf}} & \xrightarrow{\theta'} & \varprojlim_n \mathrm{Mod}^{\mathrm{perf}}_{\tau_{\leqslant n} R} . \end{array}$$

The map θ' is an equivalence of ∞-categories by Lemma 11.4.4.6. Consequently, to prove that θ is essentially surjective, it will suffice to show that θ induces an essentially surjective map

$$\phi : \mathcal{C}^c \times_{\mathrm{Mod}_R^{\mathrm{perf}}} \{M\} \to (\varprojlim \mathrm{LMod}_{R_n}(\mathcal{C})^c) \times_{\varprojlim_n \mathrm{Mod}_{R_n}^{\mathrm{perf}}} \{M\}$$

for every perfect R-module M. Using Theorem HA.4.7.1.34, we can identify ϕ with the canonical map

$$\begin{aligned} \mathrm{Map}_{\mathrm{Alg}_R}(A^{\mathrm{rev}}, \mathrm{End}(M)) &\rightarrow \varprojlim \mathrm{Map}_{\mathrm{Alg}_{R_n}}(R_n \otimes_R A^{\mathrm{rev}}, \mathrm{End}(R_n \otimes_R M)) \\ &\simeq \varprojlim \mathrm{Map}_{\mathrm{Alg}_R}(A^{\mathrm{rev}}, R_n \otimes \mathrm{End}(M)). \end{aligned}$$

To prove that this map is a homotopy equivalence, it suffices to observe that the canonical map $\mathrm{End}(M) \to \varprojlim(R_n \otimes_R \mathrm{End}(M))$ is an equivalence. Since $\mathrm{End}(M)$ is a perfect R-module, this again follows from our assumption that the map $R \to \varprojlim R_n$ is an equivalence. □

11.5 Brauer Groups in Spectral Algebraic Geometry

For every associative ring A, we let $\mathfrak{Z}(A)$ denote the center of A (that is, the subalgebra consisting of those elements $a \in A$ such that $ab = ba$ for all $b \in A$). Recall that a *division algebra* is a nonzero associative ring A such that every nonzero element $a \in A$ is invertible.

Let κ be a field. A *central division algebra* over κ is a division algebra A equipped with an isomorphism $\iota : \kappa \simeq \mathfrak{Z}(A)$ which exhibits A as a finite-dimensional vector space over κ. Let $\mathrm{Br}(\kappa)$ denote the set of isomorphism classes of central division algebras over κ. We refer to $\mathrm{Br}(\kappa)$ as the *Brauer group* of the field κ. It is naturally endowed with the structure of an abelian group, whose multiplication is characterized by the following requirement:

(∗) Let A, B, and C be central division algebras over κ and denote their isomorphism classes by $[A], [B], [C] \in \mathrm{Br}(\kappa)$. Then $[A][B] = [C]$ if and only if there is a κ-linear algebra isomorphism $A \otimes_\kappa B \simeq M_n(C)$; here $M_n(C)$ denotes the algebra of n-by-n matrices over C.

The theory of the Brauer group was extended to the setting of arbitrary commutative rings by Auslander and Goldman ([9]), following earlier work of Azumaya in the case of local commutative rings ([10]). It was generalized further by Grothendieck, who associated a Brauer group to an arbitrary scheme X ([91]). In fact, there are at least two natural candidates for such an extension:

- The *Brauer-Grothendieck group* $\mathrm{Br}_{\mathsf{Groth}}(X)$, whose elements are isomorphism classes of quasi-coherent sheaves of Azumaya algebras on X.

- The *cohomological Brauer group* $\mathrm{Br}_{\mathrm{coh}}(X)$, given by the étale cohomology group $\mathrm{H}^2_{\text{ét}}(X; \mathbf{G}_m)$.

These groups are related as follows:

(a) For any scheme X, there is a canonical monomorphism $\rho : \mathrm{Br}_{\mathsf{Groth}}(X) \hookrightarrow \mathrm{Br}_{\mathrm{coh}}(X)$.

(*b*) If X is a quasi-compact separated scheme equipped with an ample line bundle $\mathscr{L}$, then a theorem of Gabber asserts that the image of ρ is the torsion subgroup of $\mathrm{Br}_{\mathrm{coh}}(X)$ (see [47]).

(*c*) If X is a regular Noetherian scheme, then the cohomological Brauer group $\mathrm{Br}_{\mathrm{coh}}(X)$ is a torsion group.

In general, the map ρ need not be an isomorphism: that is, a general element $\alpha \in \mathrm{H}^2_{\text{ét}}(X; \mathbf{G}_m)$ need not be representable by an Azumaya algebra on X. To address this point, Toën introduced a mild enlargement of the cohomological Brauer group $\mathrm{Br}_{\mathrm{coh}}(X)$, which we will refer to here as the *extended Brauer group* and denote by $\mathrm{Br}^{\dagger}(X)$. This group admits three different descriptions:

(*i*) For any scheme X, the extended Brauer group $\mathrm{Br}^{\dagger}(X)$ can be defined as the group of equivalence classes of stable quasi-coherent stacks on X (in the sense of Chapter 10) which are invertible and compactly generated.

(*ii*) For any scheme X, the extended Brauer group $\mathrm{Br}^{\dagger}(X)$ admits a cohomological interpretation: it is canonically isomorphic to the Cartesian product $\mathrm{H}^2_{\text{ét}}(X; \mathbf{G}_m) \times \mathrm{H}^1_{\text{ét}}(X; \mathbf{Z})$. In particular, if the group $\mathrm{H}^1_{\text{ét}}(X; \mathbf{Z})$ vanishes (for example, if X is a normal Noetherian scheme), then the extended Brauer group $\mathrm{Br}^{\dagger}(X)$ agrees with the cohomological Brauer group $\mathrm{Br}_{\mathrm{coh}}(X)$.

(*iii*) If X is quasi-compact and quasi-separated, then $\mathrm{Br}^{\dagger}(X)$ can be defined as the group of equivalence classes of *derived Azumaya algebras* on X (see Definition 11.5.3.1).

In [211], Toën proved the equivalence of (*i*), (*ii*), and (*iii*) in the setting of derived algebraic geometry (based on simplicial commutative rings). His results were generalized to the setting of spectral algebraic geometry (based on $\mathbb{E}_\infty$-rings) by Antieau-Gepner (see [2]). In this section, we give an overview of the work of Antieau-Gepner and discuss an analogue of the equivalence of (*i*) and (*ii*) for the (non-extended) cohomological Brauer group (see Theorem 11.5.7.11 and the accompanying discussion).

11.5.1 The Brauer Group of a Field

We begin by reviewing the classical theory of the Brauer group in the case of a field.

Definition 11.5.1.1. Let κ be a field and let $A \in \mathrm{Alg}^{\heartsuit}_{\kappa}$ be an associative algebra over κ. We will say that A is *central simple* if it satisfies the following conditions:

(*i*) The structural map $\kappa \to \mathfrak{Z}(A)$ is an isomorphism (that is, κ is the center of A).

(*ii*) The algebra A is finite-dimensional as a vector space over κ.

(*iii*) The algebra A is simple: that is, any nonzero two-sided ideal $I \subseteq A$ is equal to A.

Definition 11.5.1.1 admits several convenient reformulations:

Proposition 11.5.1.2. *Let κ be a field and let A be an associative algebra over κ. The following conditions are equivalent:*

(1) *The algebra A is central simple over κ.*

(2) *There exists an isomorphism $A \simeq M_k(D)$, where D is a central division algebra over κ and $M_k(D)$ denotes the algebra of k-by-k matrices over D.*

Proof. We first show that (1) implies (2). Assume that A is central simple over κ. Then A is nonzero and is finite-dimensional as a vector space over κ. Let $I \subseteq A$ be a nonzero right ideal whose dimension over κ is minimal. Choose a collection of elements $\{a_1, \ldots, a_k\}$ which is maximal among those for which each $a_i I$ is nonzero and the collection of right ideals $\{a_i I\}_{1 \leqslant i \leqslant k}$ is linearly independent over κ. Set $J = a_1 I + \cdots + a_k I$. Then J is a nonzero right ideal of A. For each element $b \in A$, the maximality of $\{a_1, \ldots, a_k\}$ implies that either $bI = 0$ or $bI \cap J \neq 0$. In the latter case, we have $\dim_\kappa(bI \cap J) \leqslant \dim_\kappa(bI) \leqslant \dim_\kappa(I)$. The minimality of $\dim_\kappa(I)$ then guarantees that equality must hold, so that $bI \subseteq J$. Allowing b to vary, we deduce that J is also a left ideal of A. Our assumption that A is simple then guarantees that $J = A$.

Note that for $1 \leqslant i \leqslant k$, right multiplication by a_i determines a surjective A-module homomorphism $\alpha_i : I \to a_i I$. We then have $0 < \dim_\kappa(a_i I) \leqslant \dim_\kappa(I)$, so the minimality of I guarantees that α_i is an isomorphism. Consequently, the homomorphisms $\{\alpha_i\}_{1 \leqslant i \leqslant k}$ induce a right A-module isomorphism $\alpha : I^k \to A$. We can therefore identify A with the endomorphism ring of I^k as a right A-module, which is given by $M_k(D)$ where D is the endomorphism ring of I as a right A-module.

Note that if $f : I \to I$ is a nonzero element of D, then $\mathrm{im}(f)$ is a nonzero right ideal of A which is contained in I. It follows from the minimality of $\dim_\kappa(I)$ that $\mathrm{im}(f) = I$: that is, the map f is surjective. Since I is finite-dimensional as a vector space over κ, it follows that f is bijective: that is, it is an invertible element of D. This proves that D is a division algebra, and the isomorphism $\mathfrak{Z}(A) \simeq \mathfrak{Z}(M_k(D)) \simeq \mathfrak{Z}(D)$ shows that D is central over κ. This completes the proof of the implication (1) $\Rightarrow$ (2).

We now show that (2) $\Rightarrow$ (1). Suppose that we are given an isomorphism $A \simeq M_k(D)$, where D is a central division algebra over κ. It is then clear that $\mathfrak{Z}(A) \simeq \mathfrak{Z}(M_k(D)) \simeq \mathfrak{Z}(D) \simeq \kappa$ and that A is finite-dimensional as a vector space over κ. To complete the proof, it will suffice to show that A is central. Let us identify A with the endomorphism ring $\mathrm{End}_D(V)$, where V is a right D-module with basis $\{v_1, \ldots, v_k\}$. Let $I \subseteq A$ be a nonzero two-sided ideal; we wish to show that $I = A$. Choose elements $f \in I$ and $x \in V$ such that

$f(x) \neq 0$. For $w \in V$ and $1 \leqslant i \leqslant k$, we can choose elements $g_w, h_i \in A$ such that

$$g_w(f(x)) = w \qquad h_i(v_j) = \begin{cases} x & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}.$$

Then for any $f' \in A$, we have $f' = \sum_{1 \leqslant i \leqslant k} g_{f'(v_i)} \circ f \circ h_i \in I$. □

Remark 11.5.1.3. In the situation of Proposition 11.5.1.2, the division algebra D is well-defined up to isomorphism: it can be recovered as the endomorphism ring of any simple right A-module.

Proposition 11.5.1.4. *Let κ be a field and let A and B be central simple algebras over κ. Then $A \otimes_\kappa B$ is also a central simple algebra over κ.*

Proof. Let x be an element of the center $\mathfrak{Z}(A \otimes_\kappa B)$. Choose a representation $x = a_1 \otimes b_1 + \cdots + a_n \otimes b_n$ where n is minimal, so that the set $\{b_1, \ldots, b_n\}$ is linearly independent over κ. For each $a \in A$, we have

$$0 = ax - xa = \sum (aa_i - a_i a) \otimes b_i.$$

The linear independence of the set $\{b_1, \ldots, b_n\}$ then implies that we must have $aa_i = a_i a$. Allowing a to vary, we deduce that each a_i belongs to the center $\mathfrak{Z}(A) \simeq \kappa$. It follows that x belongs to $\mathfrak{Z}(A) \otimes_\kappa B \simeq B$. Since $\mathfrak{Z}(B) \simeq \kappa$, we deduce that x is a scalar.

It is clear that $A \otimes_\kappa B$ is finite-dimensional as a vector space over κ. To complete the proof, it will suffice to show that every nonzero two-sided ideal $I \subseteq A \otimes_\kappa B$ coincides with $A \otimes_\kappa B$. To prove this, choose a nonzero element $x = a_1 \otimes b_1 + \cdots + a_n \otimes b_n \in I$ where n is chosen as small as possible. Then $a_1 \neq 0$. Our assumption that A is simple guarantees that we can write $1 = \sum a'_j a_1 a''_j$ for some elements $a'_j, a''_j \in A$. Replacing x by $\sum a'_j x a''_j$, we can assume that $a_1 = 1$. For each $a \in A$, we then have

$$ax - xa = (aa_2 - a_2 a) \otimes b_2 + \cdots + (aa_n - a_n a) \otimes b_n \in I.$$

It follows from the minimality of n (and the linear independence of $\{b_2, \ldots, b_n\}$) that we have $aa_i - a_i a = 0$ for each i. Using our assumption that A is central over κ, we conclude that each a_i belongs to κ. The linear independence of $\{a_1, \ldots, a_n\}$ then implies that $n = 1$: that is, the element x has the form $1 \otimes b$ for some nonzero element $b \in B$. The simplicity of B then gives $B = BbB$, so that $I = A \otimes_\kappa B$. □

Proposition 11.5.1.5. *Let κ be a field and let A be an associative algebra over κ which is nonzero and finite-dimensional as a vector space over κ. The following conditions are equivalent:*

(1) *The algebra A is central simple over κ.*

(2) *The left and right actions of A on itself induce an isomorphism $\rho : A \otimes_\kappa A^{\mathrm{rev}} \to \mathrm{End}_\kappa(A)$; here A^{rev} denotes the opposite algebra of A.*

Proof. Suppose first that (1) is satisfied. It follows from Proposition 11.5.1.4 that $A \otimes_\kappa A^{\mathrm{rev}}$ is a central simple algebra over κ. Consequently, the two-sided ideal $\ker(\rho) \subseteq A \otimes_\kappa A^{\mathrm{rev}}$ must be zero, so that ρ is injective. Since the domain and codomain of ρ are vector spaces of the same (finite) dimension over κ, the map ρ is an isomorphism.

Now suppose that (2) is satisfied. An element $x \in A$ belongs to the center $\mathfrak{Z}(A)$ if and only if $\rho(x \otimes 1 - 1 \otimes x) = 0$. Since ρ is injective, we then conclude that $x \otimes 1 - 1 \otimes x$ vanishes in $A \otimes_\kappa A^{\mathrm{rev}}$, so that x belongs to κ. To complete the proof, it will suffice to show that A is simple. Let $I \subseteq A$ be a two-sided ideal. Then $\mathrm{im}(\rho)$ is contained in the subspace of $\mathrm{End}_\kappa(A)$ consisting of those maps $f : A \to A$ satisfying $f(I) = I$. Since ρ is surjective, it follows that I is stable under every endomorphism of A as a κ-vector space, so that either $I = 0$ or $I = A$. □

Proposition 11.5.1.6. *Let κ be a field and let A and B be central simple algebras over κ. The following conditions are equivalent:*

(1) *There is a central division algebra D over κ and isomorphisms $A \simeq M_m(D)$ and $B \simeq M_n(D)$ for $m, n \geqslant 1$.*

(2) *The tensor product $A \otimes_\kappa B^{\mathrm{rev}}$ is isomorphic to a matrix algebra over κ.*

Proof. To show that $(1) = (2)$, we may assume without loss of generality that $m = n$, in which case the desired result follows immediately from Proposition 11.5.1.5. For the converse, we may assume that $A = M_m(D)$ and $B = M_n(D')$ for some central division algebras $D, D' \in \mathrm{Alg}_\kappa^\heartsuit$ (Proposition 11.5.1.2). Proposition 11.5.1.4 guarantees that $D \otimes_\kappa D'^{\mathrm{rev}}$ is central simple over κ, and is therefore isomorphic to $M_d(D'')$ for some central division algebra D'' over κ. We then have $A \otimes_\kappa B^{\mathrm{rev}} \simeq M_{dmn}(D'')$, so condition (2) guarantees that $D'' \simeq \kappa$. We then have an isomorphism $\rho : D \otimes_\kappa D'^{\mathrm{rev}} \simeq \mathrm{End}_\kappa(V)$ where V is a vector space of dimension d over κ. Let us regard V as a D-D' bimodule via the map ρ. We then have $\dim_D(V) \dim_\kappa(D) = \dim_\kappa(V) = \dim_{D'}(V) \dim_\kappa(D')$. Since ρ is an isomorphism, we also have $\dim_\kappa(D) \dim_\kappa(D') = \dim_\kappa(\mathrm{End}_\kappa(V)) = \dim_\kappa(V)^2$. It follows that $\dim_D(V) = \dim_{D'}(V) = 1$, so that $\dim_\kappa(D) = \dim_{\kappa'}(D') = d$. Moreover, the equality $\dim_{D'}(V) = 1$ ensures that the endomorphism algebra of V as a right D'-module is isomorphic to D', so that ρ determines a κ-algebra homomorphism $\beta : D \to D'$. This map is automatically injective (since D is a division algebra), and is therefore an isomorphism (since the domain and codomain of β have the same dimension over κ). □

Definition 11.5.1.7. Let κ be a field and let $A, B \in \mathrm{Alg}_\kappa^\heartsuit$ be central simple algebras over κ. We will say that A and B are *Morita equivalent* if the equivalent conditions of Proposition

11.5.1.6 are satisfied. We let $\mathrm{Br}(\kappa)$ denote the set of Morita equivalence classes of central simple algebras over κ. We will refer to $\mathrm{Br}(\kappa)$ as the *Brauer group* of κ.

Remark 11.5.1.8. Let κ be a field. It follows immediately from characterization (1) of Proposition 11.5.1.6 (together with Remark 11.5.1.3) that Morita equivalence defines an equivalence relation on the collection of all (isomorphism classes of) central simple algebras over κ. Moreover, we can also identify $\mathrm{Br}(\kappa)$ with the set of all isomorphism classes of central division algebras over κ.

Notation 11.5.1.9. Let κ be a field and let A be a central simple algebra over κ. We let $[A]$ denote the Morita equivalence class of A, which we regard as an element of the Brauer group $\mathrm{Br}(\kappa)$.

Proposition 11.5.1.10. *Let κ be a field. Then $\mathrm{Br}(\kappa)$ has the structure of an abelian group, with multiplication given by $[A][B] = [A \otimes_\kappa B]$.*

Proof. It follows immediately from the definition that the construction $[A], [B] \mapsto [A \otimes_\kappa B]$ determines a multiplication $\mathrm{Br}(\kappa) \times \mathrm{Br}(\kappa) \to \mathrm{Br}(\kappa)$ which is commutative and associative. Moreover, it has a unit given by the Morita equivalence class $[\kappa]$. For every central simple algebra A over κ, Proposition 11.5.1.5 gives $[A][A^{\mathrm{rev}}] = [A \otimes_\kappa A^{\mathrm{rev}}] = [\mathrm{End}_\kappa(A)] = [\kappa]$, so that $[A^{\mathrm{rev}}]$ is an inverse of $[A]$ in $\mathrm{Br}(\kappa)$. □

Remark 11.5.1.11 (Functoriality)**.** Let $f : \kappa \to \kappa'$ be a homomorphism of fields. It follows from Proposition 11.5.1.5 that if A is an associative algebra over κ, then A is central simple over κ if and only if $A' = \kappa' \otimes_\kappa A$ is central simple over κ'. Consequently, the construction $A \mapsto \kappa' \otimes_\kappa A$ determines an abelian group homomorphism $\mathrm{Br}(\kappa) \to \mathrm{Br}(\kappa')$.

Proposition 11.5.1.12. *Let κ be a separably closed field. Then the Brauer group $\mathrm{Br}(\kappa)$ is trivial.*

Proof. Let D be a central division algebra over κ. We wish to show that $D = \kappa$. Assume otherwise: then there exists an element $x \in D$ which does not belong to κ. Let D_0 be the κ-subalgebra of D generated by x. Then D_0 is commutative, and is therefore a finite algebraic extension field of κ. Since κ is separably closed, the extension $\kappa \hookrightarrow D_0$ is purely inseparable. It follows that κ has characteristic $p > 0$ and that $x^{p^n} \in \kappa$ for $n \gg 0$. Choose n as small as possible. Replacing x by $x^{p^{n-1}}$, we can assume that $x^p \in \kappa$.

Let $f : D \to D$ be the function given by $f(y) = xy - yx$. Then $f^k(y) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} x^{k-i} y x^i$. Since κ has characteristic p and $x^p \in \kappa$, we have $f^p(y) = x^p y - y x^p = 0$ for each $y \in D$. However, the map f is nonzero (since $x \notin \kappa = \mathfrak{Z}(D)$). We can therefore choose an element $y \in D$ such that $f(y) \neq 0$ but $f^2(y) = 0$. Set $u = y f(y)^{-1} x$. Since $f(y)^{-1}$ commutes with x, we obtain $x u x^{-1} = x y f(y)^{-1} = (yx + f(y)) f(y)^{-1} = y f(y)^{-1} x + 1 = u + 1$.

Let $D_1 \subseteq D$ be the κ-subalgebra generated by u. Since D_1 is commutative, it is also a purely inseparable algebraic extension of κ, so that $u^{p^m} \in \kappa$ for $m \gg 0$. In particular, u^{p^m} commutes with x for $m \gg 0$, so that we obtain the contradiction

$$u^{p^m} = xu^{p^m}x^{-1} = (xux^{-1})^{p^m} = (u+1)^{p^m} = u^{p^m} + 1.$$

□

We conclude this section by recording another standard fact about associative algebras which will be needed in §11.5.3.

Proposition 11.5.1.13. *Let κ be a field and let $A \in \mathrm{Alg}_\kappa^\heartsuit$ be an associative algebra over κ which is finite-dimensional as a vector space over κ. Then there are only finitely many isomorphism classes of simple left A-modules.*

Proof. Let $M_1, M_2, \ldots, M_n \in \mathrm{Mod}_A^\heartsuit$ be a collection of simple left A-modules which are pairwise nonisomorphic. Choose a nonzero element $x_i \in M_i$ for $1 \leqslant i \leqslant n$, and set $x = \{x_i\}_{1 \leqslant i \leqslant n} \in \bigoplus_{1 \leqslant i \leqslant n} M_i$. Let $N \subseteq \bigoplus_{1 \leqslant i \leqslant n} M_i$ be the cyclic submodule generated by x. We claim that $N = \bigoplus_{1 \leqslant i \leqslant n} M_i$. The proof proceeds by induction on n, the case $n = 0$ being trivial. Assume that $n > 0$, and consider the composite map $\phi : N \hookrightarrow \bigoplus_{1 \leqslant i \leqslant n} M_i \to \bigoplus_{1 \leqslant i < n} M_i$. We have a diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \ker(\phi) & \to & N & \to & \mathrm{im}(\phi) & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \to & M_n & \to & \bigoplus_{1 \leqslant i \leqslant n} M_i & \to & \bigoplus_{1 \leqslant i < n} M_i & \to & 0 \end{array}$$

where the vertical maps are injective. Our inductive hypothesis guarantees that γ is surjective. Consequently, to show that β is an isomorphism, it will suffice to show that α is surjective. Assume otherwise. Then, since M_n is simple, the map α vanishes, so that $\ker(\phi) \simeq 0$. It follows that ϕ is an isomorphism, so that the construction $\bigoplus_{1 \leqslant i < n} M_i \xrightarrow{\phi^{-1}} N \to M_n$ determines a nonzero A-module homomorphism from $\bigoplus_{1 \leqslant i < n} M_i$ to M_n. It follows that there exists a nonzero map $\alpha : M_i \to M_n$ for some $i < n$. Since M_i and M_n are simple, the map α is an isomorphism, contradicting our assumption that the modules $\{M_i\}$ are pairwise nonisomorphic.

It follows from the above argument that if $M_1, M_2, \ldots, M_n$ are pairwise nonisomorphic simple left A-modules, then we have $\sum \dim_\kappa(M_i) \leqslant \dim_\kappa(A)$. In particular, there are at most $\dim_\kappa(A)$ isomorphism classes of simple left A-modules. □

11.5.2 The Extended Brauer Group

We now introduce our main objects of interest in this section.

Definition 11.5.2.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and let $\mathrm{QStk}^{\mathrm{cg}}(X)$ be the ∞-category of compactly generated stable quasi-coherent stacks on X (Notation 11.4.0.1). We let $\mathscr{B}\mathrm{r}^{\dagger}(X)$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{cg}}(X)^{\simeq}$ spanned by the invertible objects of $\mathrm{QStk}^{\mathrm{cg}}(X)$. We will refer to $\mathscr{B}\mathrm{r}^{\dagger}(X)$ as the *extended Brauer space* of X. We let $\mathrm{Br}^{\dagger}(X)$ denote the set $\pi_0 \mathscr{B}\mathrm{r}^{\dagger}(X)$. We will refer to $\mathrm{Br}^{\dagger}(X)$ as the *extended Brauer group* of X.

If X is a spectral Deligne-Mumford stack. we let $\mathrm{Br}^{\dagger}(\mathsf{X})$ and $\mathscr{B}\mathrm{r}^{\dagger}(\mathsf{X})$ denote the extended Brauer group and extended Brauer space of the functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ represented by X. If R is a connective $\mathbb{E}_\infty$-ring, we let $\mathrm{Br}^{\dagger}(R)$ and $\mathscr{B}\mathrm{r}^{\dagger}(R)$ denote extended Brauer group and extended Brauer space of the functor $\mathrm{Spec}\, R : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ corepresented by R.

Remark 11.5.2.2. We will see below that if κ is a field, then the extended Brauer group $\mathrm{Br}^{\dagger}(\kappa)$ of Definition 11.5.2.1 coincides with the classical Brauer group $\mathrm{Br}(\kappa)$ of Definition 11.5.1.7 (see Theorem 11.5.3.18).

Warning 11.5.2.3. In the situation of Definition 11.5.2.1, the ∞-category $\mathrm{QStk}^{\mathrm{cg}}(X)$ need not be small (and the ∞-category $\mathrm{QStk}^{\mathrm{St}}(X)$ need not even be locally small). Consequently, it is not obvious from the definition that the extended Brauer space $\mathscr{B}\mathrm{r}^{\dagger}(X)$ is essentially small, or that the extended Brauer group $\mathrm{Br}^{\dagger}(X)$ is small. We will later see that these invariants are small under some mild assumptions on X (see Remark 11.5.3.13 and Construction 11.5.5.3).

Example 11.5.2.4 (The Dualizing Gerbe)**.** Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a Noetherian spectral algebraic space. Let us say that an object $U \in \mathcal{X}$ is *good* if U is affine and the spectral algebraic space $\mathsf{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathsf{X}}|_U)$ admits a dualizing sheaf (Definition 6.6.1.1). In this case, we let $\mathcal{C}(U)$ denote the full subcategory of $\mathrm{QCoh}(\mathsf{X}_U)$ spanned by those objects which are coherent, truncated, and of finite injective dimension. If ω_U is a dualizing sheaf for X_U, then Corollary 6.6.1.11 implies that the construction $\mathscr{F} \mapsto \mathscr{F} \otimes \omega_U$ induces an equivalence $\mathrm{QCoh}(\mathsf{X}_U)^{\mathrm{perf}} \simeq \mathcal{C}(U)$, which extends to a $\mathscr{O}_{\mathsf{X}}(U)$-linear equivalence of ∞-category $\mathrm{Mod}_{\mathscr{O}_{\mathsf{X}}(U)} \simeq \mathrm{Ind}(\mathcal{C}(U))$.

Suppose that $\mathcal{X}$ is covered by good objects: that is, that X admits a dualizing sheaf *locally*. In this case, the construction $U \mapsto \mathrm{Ind}(\mathcal{C}(U))$ extends to a quasi-coherent stack on X which is compactly generated and invertible, and can therefore be identified with an object of $\mathscr{B}\mathrm{r}^{\dagger}(\mathsf{X})$. We will refer to this object as the *dualizing gerbe* of X. The associated element of $\mathrm{Br}^{\dagger}(\mathsf{X})$ vanishes if and only if X admits a dualizing sheaf ω_{X}.

Remark 11.5.2.5. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a compactly generated stable R-linear ∞-category. Then $\mathcal{C}$ is a dualizable object of $\mathrm{LinCat}_R^{\mathrm{St}}$ (Theorem D.7.0.7). Let us denote its dual by $\mathcal{C}^{\vee}$, so that we have evaluation and coevaluation maps

$$e : \mathcal{C}^{\vee} \otimes_R \mathcal{C} \to \mathrm{Mod}_R \qquad c : \mathrm{Mod}_R \to \mathcal{C} \otimes_R \mathcal{C}^{\vee}.$$

Then the following conditions are equivalent:

(a) The functor e is an equivalence of R-linear ∞-categories.

(b) The functor c is an equivalence of R-linear ∞-categories.

(c) As an object of $\mathrm{LinCat}_R^{\mathrm{St}}$, $\mathcal{C}$ is invertible.

Moreover, if these conditions are satisfied, then the inverse $\mathcal{C}^{-1}$ of $\mathcal{C}$ in $\mathrm{LinCat}_R^{\mathrm{St}}$ is also compactly generated (see Remark D.7.7.6), so that $\mathcal{C} \in \mathscr{B}\mathrm{r}^{\dagger}(R) \subseteq \mathrm{LinCat}_R$. In this case, the functors e and c preserve compact objects, so that $\mathcal{C}$ is both smooth and proper as an R-linear ∞-category.

Remark 11.5.2.6. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Since every invertible morphism in $\mathrm{QStk}^{\mathrm{St}}(X)$ is compact, we can identify $\mathscr{B}\mathrm{r}^{\dagger}(X)$ with the full subcategory of $\mathrm{QStk}^{\mathrm{St}}(X)^{\simeq}$ spanned by those stable quasi-coherent stacks $\mathcal{C}$ on X which are compactly generated and invertible as objects of $\mathrm{QStk}^{\mathrm{St}}(X)$ (the inverse of $\mathcal{C}$ is then automatically compactly generated as well; see Remark 11.5.2.5). We do not know if there exist invertible objects of $\mathrm{QStk}^{\mathrm{St}}(X)$ which are not compactly generated.

Remark 11.5.2.7. For every functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, the symmetric monoidal structure on $\mathrm{QStk}(X)$ restricts to a symmetric monoidal structure on $\mathscr{B}\mathrm{r}^{\dagger}(X)$: that is, we may regard $\mathscr{B}\mathrm{r}^{\dagger}(X)$ as a commutative monoid object of the ∞-category $\widehat{\mathcal{S}}$ of (not necessarily small) spaces. By construction, this commutative monoid is grouplike. Neglecting issues of size, we can regard $\mathscr{B}\mathrm{r}^{\dagger}(X)$ as the zeroth space of a connective spectrum (see §HA.5.2.6), so that $\mathrm{Br}^{\dagger}(X) = \pi_0 \mathscr{B}\mathrm{r}^{\dagger}(X)$ has the structure of an abelian group.

Remark 11.5.2.8 (Functoriality)**.** Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Then the pullback functor $f^* : \mathrm{QStk}^{\mathrm{St}}(Y) \to \mathrm{QStk}^{\mathrm{St}}(X)$ restricts to a map $\mathscr{B}\mathrm{r}^{\dagger}(Y) \to \mathscr{B}\mathrm{r}^{\dagger}(X)$, and therefore induces a homomorphism of extended Brauer groups $\mathrm{Br}^{\dagger}(Y) \to \mathrm{Br}^{\dagger}(X)$.

Remark 11.5.2.9. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. The condition that an object $\mathcal{C} \in \mathrm{QStk}^{\mathrm{St}}(X)$ belong to $\mathscr{B}\mathrm{r}^{\dagger}(X)$ can be tested pointwise: that is, $\mathcal{C} \in \mathscr{B}\mathrm{r}^{\dagger}(X)$ if and only if, for every connective $\mathbb{E}_\infty$-ring R and every point $\eta \in X(R)$, we have $\eta^* \mathcal{C} \in \mathscr{B}\mathrm{r}^{\dagger}(R) \subseteq \mathrm{LinCat}_R^{\mathrm{St}}$. The "only if" direction is obvious, and the "if" direction follows from Remark 11.5.2.5.

Remark 11.5.2.10. Using Theorem **??**, we see that the functor $R \mapsto \mathscr{B}\mathrm{r}^{\dagger}(R)$ is a sheaf with respect to the étale topology. Moreover, the functor $\mathscr{B}\mathrm{r}^{\dagger} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})^{\mathrm{op}} \to \widehat{\mathcal{S}}$ is a right Kan extension of the functor $R \mapsto \mathscr{B}\mathrm{r}^{\dagger}(R)$ along the Yoneda embedding $\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})^{\mathrm{op}}$. It follows that if $\alpha : X \to Y$ is a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ which induces an equivalence after sheafification with respect to the étale topology, then the pullback map $\alpha^* : \mathscr{B}\mathrm{r}^{\dagger}(Y) \to \mathscr{B}\mathrm{r}^{\dagger}(X)$ is a homotopy equivalence.

Remark 11.5.2.11. Let R be an $\mathbb{E}_\infty$-ring. Every R-linear functor F from Mod_R to itself is given by tensor product by an object $M \in \mathrm{Mod}_R$, and F is an equivalence of ∞-categories if and only if M is an invertible object of Mod_R. We therefore have a homotopy equivalence

$$\Omega\,\mathscr{B}\mathrm{r}^\dagger(R) \simeq \mathscr{P}\mathrm{ic}^\dagger(R).$$

This homotopy equivalence depends functorially on R, and so globalizes to give an equivalence $\Omega\,\mathscr{B}\mathrm{r}^\dagger(X) \simeq \mathscr{P}\mathrm{ic}^\dagger(X)$ for any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$; here $\mathscr{P}\mathrm{ic}^\dagger(X)$ denotes the full subcategory of $\mathrm{QCoh}(X)^{\simeq}$ spanned by the invertible objects.

Remark 11.5.2.12. The extended Brauer group $\mathrm{Br}^\dagger(R)$ of Definition 11.5.2.1 makes sense for $\mathbb{E}_\infty$-rings R which are not connective. However, we will restrict our attention to the connective setting in what follows.

11.5.3 Azumaya Algebras

Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathrm{Br}^\dagger(R)$ be the extended Brauer group of R (Definition 11.5.2.1). By definition, the elements of $\mathrm{Br}^\dagger(R)$ are equivalence classes of invertible compactly generated stable R-linear ∞-categories. Our next goal is to make this definition more explicit by describing concrete representatives for extended Brauer classes. Recall that when R is a field, the classical Brauer group $\mathrm{Br}(R)$ can be described as the set of Morita equivalence classes of central simple algebras over R (Definition 11.5.1.7). The theory of central simple algebras was generalized by Azumaya to the setting of local commutative rings ([10]). We consider here a further generalization of Azumaya's theory to the setting of ring spectra, introduced by Toën in the setting of simplicial commutative rings and Antieau-Gepner in the setting of $\mathbb{E}_\infty$-rings:

Definition 11.5.3.1. Let R be a connective $\mathbb{E}_\infty$-ring and let A be an $\mathbb{E}_1$-algebra over R. We will say that A is an *Azumaya algebra over* R if it satisfies the following conditions:

(1) The algebra A is a compact generator of Mod_R.

(2) The left and right actions of A on itself induce an equivalence $A \otimes_R A^{\mathrm{rev}} \to \mathrm{End}_R(A)$.

Warning 11.5.3.2. In the case where R is an ordinary commutative ring, Definition 11.5.3.1 does not reduce to the usual notion of Azumaya algebra in commutative algebra because we do not require A to be a projective R-module (see Proposition 11.5.3.3 below). For this reason, Toën refers to an algebra $A \in \mathrm{Alg}_R$ satisfying the requirements of Definition 11.5.3.1 as a *derived Azumaya algebra*. Other variants have appeared in the literature; see [133] and [11].

Proposition 11.5.3.3. *Let R be an $\mathbb{E}_\infty$-ring and let A be an Azumaya algebra over R. The following conditions are equivalent:*

(a) *The algebra A is connective.*

(b) *The algebra A is a locally free of finite rank as an R-module.*

Proof. The implication $(b) \Rightarrow (a)$ is obvious. For the converse, suppose that (a) is satisfied. Then A is a connective perfect R-module. To show that A is locally free of finite rank, it will suffice to show that it has Tor-amplitude $\leqslant 0$ (Remark 2.9.1.2). Equivalently, we must show that the dual $A^\vee$ is connective. Using Corollary 2.7.4.3, we can reduce to the case where $R = \kappa$ is a field. In this case, our assumption that A is an Azumaya algebra supplies an isomorphism of graded vector spaces

$$(\pi_* A) \otimes_\kappa (\pi_* A^{\mathrm{rev}}) \simeq \pi_*(A \otimes_\kappa A^{\mathrm{rev}}) \simeq \pi_* \mathrm{End}_\kappa(A) \simeq (\pi_* A) \otimes (\pi_* A^\vee).$$

Since the left hand side is concentrated in nonnegative degrees, the right hand side must also be concentrated in nonnegative degrees, from which it follows that $A^\vee$ is connective. □

The next result highlights the relevance of Definition 11.5.3.1 to the theory of Brauer groups:

Proposition 11.5.3.4. *Let R be a connective $\mathbb{E}_\infty$-ring and let $A \in \mathrm{Alg}_R$. Then A is an Azumaya algebra over R if and only if the stable R-linear ∞-category RMod_A is an invertible object of $\mathrm{LinCat}_R^{\mathrm{St}}$.*

The proof of Proposition 11.5.3.4 will require a few preliminaries.

Lemma 11.5.3.5. *Let R be an $\mathbb{E}_\infty$-ring and let M be a perfect R-module. The following conditions are equivalent:*

(1) *The object M is a compact generator of Mod_R.*

(2) *Let $\mathcal{C}$ be the smallest stable subcategory of Mod_R which contains M and is idempotent complete. Then $\mathcal{C} = \mathrm{Mod}_R^{\mathrm{perf}}$.*

(3) *The R-linear dual $M^\vee$ is a compact generator of Mod_R.*

Proof. Let $\mathcal{C}$ be as in (2). Since M is perfect, $\mathcal{C} \subseteq \mathrm{Mod}_R^{\mathrm{perf}}$, so the inclusion $\mathcal{C} \hookrightarrow \mathrm{Mod}_R^{\mathrm{perf}}$ extends to a fully faithful embedding $\theta : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{perf}}) \simeq \mathrm{Mod}_R$. Condition (1) is equivalent to the requirement that θ is an equivalence of ∞-categories. Since $\mathcal{C}$ is idempotent complete, this is equivalent to condition (2).

Let $\mathcal{C}^\vee = \{N^\vee : N \in \mathcal{C}\}$. Then $\mathcal{C} = \mathrm{Mod}_R^{\mathrm{perf}}$ if and only if $\mathcal{C}^\vee = \mathrm{Mod}_R^{\mathrm{perf}}$. Since $\mathcal{C}^\vee$ is the smallest stable subcategory of Mod_R which contains $M^\vee$ and is idempotent complete, we deduce that $(2) \Leftrightarrow (3)$. □

Lemma 11.5.3.6. *Let R be an $\mathbb{E}_\infty$-ring, let $A \in \mathrm{Alg}_R$, and let $X \in \mathrm{RMod}_A$. Let $G : \mathrm{Mod}_R \to \mathrm{RMod}_A$ be the R-linear functor given by $G(M) = M \otimes_R X$. The following conditions are equivalent:*

(1) *The functor G is an equivalence of ∞-categories.*

(2) *The image of X under the forgetful functor $\mathrm{RMod}_A \to \mathrm{Mod}_R$ is a compact generator of Mod_R. Moreover, the right action of A on X induces an equivalence $A^{\mathrm{rev}} \to \mathrm{End}_R(X)$ of $\mathbb{E}_1$-algebras over R.*

Proof. Suppose first that (1) is satisfied, and let $F : \mathrm{RMod}_A \to \mathrm{Mod}_R$ be a homotopy inverse of G (so that F is also an R-linear functor). Let $Y = F(A) \in \mathrm{Mod}_R$. According to Theorem HA.4.8.4.1 , we can regard Y as a left A-module object of Mod_R, and the functor F is given by the formula $F(M) = M \otimes_A Y$. Since A is a compact generator of RMod_A and F is an equivalence of ∞-categories, the object Y is a compact generator of Mod_R. In particular, Y is dualizable as an R-module. Let us regard its R-linear dual $Y^\vee$ as a left A-module. Then the construction $P \mapsto P \otimes_R Y^\vee$ determines an R-linear functor $\mathrm{Mod}_R \to \mathrm{RMod}_A$ which is left adjoint to G, and therefore equivalent to F. In particular, we obtain an equivalence of right A-modules $X = F(R) \simeq R \otimes_R Y^\vee \simeq Y^\vee$. Since Y is a compact generator of Mod_R, Lemma 11.5.3.5 implies that $X \simeq Y^\vee$ is also a compact generator of Mod_R. Moreover, the assertion that the right action of A on X induces an equivalence $A^{\mathrm{rev}} \to \mathrm{End}_R(X)$ is equivalent to the assertion that the left action of A on Y induces an equivalence $A \to \mathrm{End}_R(Y)$. Since F is an equivalence of ∞-categories, this follows from the observation that A is a classifying object for endomorphisms of itself, regarded as an object of RMod_A.

We now show that (2) $\Rightarrow$ (1). Suppose that X is a compact generator of Mod_R, and let $A = \mathrm{End}_R(X)^{\mathrm{op}}$, so that we can regard X as a right A-module. We wish to show that the functor $M \mapsto M \otimes_R X$ induces an equivalence $G : \mathrm{Mod}_R \to \mathrm{RMod}_A$. Let Y denote the R-linear dual of X, so that we can regard Y as a left A-module, and let $F : \mathrm{RMod}_A \to \mathrm{Mod}_R$ be given by $F(N) = N \otimes_A Y$. Then F is left adjoint to G; it will therefore suffice to show that F is an equivalence of ∞-categories. We first claim that F is fully faithful. Let P and Q be right A-modules; we wish to show that the canonical map

$$\mu_{P,Q} : \mathrm{Map}_{\mathrm{RMod}_A}(P, Q) \to \mathrm{Map}_{\mathrm{Mod}_R}(P \otimes_A Y, Q \otimes_A Y)$$

is a homotopy equivalence. If we fix P, the collection of those objects $P \in \mathrm{RMod}_A$ for which $\mu_{P,Q}$ is a homotopy equivalence is closed under small colimits. We may therefore reduce to the case where $P = \Sigma^d A$ for some integer d. In particular, P is a compact of RMod_A. Since $P \otimes_A M \simeq \Sigma^d Y$ is a compact object of Mod_R, we conclude that the collection of those Q for which $\mu_{P,Q}$ is an equivalence is closed under filtered colimits in Q. We may therefore reduce to the case where Q is perfect. Since the collection of those Q for which $\mu_{P,Q}$ is an equivalence is closed under finite limits, we may further reduce to the case where $Q = \Sigma^{d'} A$. In this case, we must show that the canonical map

$$\mathrm{Map}_{\mathrm{RMod}_A}(\Sigma^d A, \Sigma^{d'} A) \to \mathrm{Map}_{\mathrm{Mod}_R}(\Sigma^d Y, \Sigma^{d'}_Y)$$

is a homotopy equivalence, which follows immediately from our assumption that the map $A \simeq \mathrm{End}_R(X)^{\mathrm{rev}} \simeq \mathrm{End}_R(Y)$ is an equivalence.

It remains to show that the functor F is essentially surjective. Because F is fully faithful, the essential image of F is a full subcategory of Mod_R which is closed under small colimits. Since $F(A) \simeq Y = X^\vee$ is a compact generator of Mod_R (Lemma 11.5.3.5), we conclude that $F(\mathrm{RMod}_A) = \mathrm{Mod}_R$. □

Proof of Proposition 11.5.3.4. Let A be an arbitrary $\mathbb{E}_1$-algebra over R. According to Remark HA.4.8.4.8, the R-linear ∞-category RMod_A is a dualizable object of $\mathrm{LinCat}^{\mathrm{St}}_R$, whose dual is given by LMod_A. More precisely, we can identify the tensor product $\mathrm{LMod}_A \otimes_R \mathrm{RMod}_A$ with the ∞-category ${}_A\mathrm{BMod}_A(\mathrm{Mod}_R)$ of A-A bimodule objects of Mod_R. The unit object $A \in {}_A\mathrm{BMod}_A(\mathrm{Mod}_R)$ determines an R-linear functor

$$c : \mathrm{Mod}_R \to \mathrm{LMod}_A \otimes_R \mathrm{RMod}_A,$$

which exhibits LMod_A as a dual of RMod_A in $\mathrm{LinCat}^{\mathrm{St}}_R$. Consequently, RMod_A is an invertible object of $\mathrm{LinCat}^{\mathrm{St}}_R$ if and only if the functor c is an equivalence. Proposition 11.5.3.4 now follows by applying Lemma 11.5.3.6 to the ∞-category

$$\mathrm{LMod}_A \otimes \mathrm{RMod}_A \simeq {}_A\mathrm{BMod}_A(\mathrm{Mod}_R) \simeq \mathrm{RMod}_{A \otimes_R A^{\mathrm{rev}}}.$$

□

We now use Proposition 11.5.3.4 to give a concrete description of the extended Brauer group $\mathrm{Br}^\dagger(\mathsf{X})$, where X is a quasi-compact, quasi-separated spectral algebraic space. First, we need a global version of Definition 11.5.3.1.

Definition 11.5.3.7. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and let $\mathscr{A}$ be an associative algebra object of the ∞-category $\mathrm{QCoh}(X)$. We will say that $\mathscr{A}$ is an *Azumaya algebra* if, for every connective $\mathbb{E}_\infty$-ring R and every point $\eta \in X(R)$, the object $\mathscr{A}_\eta \in \mathrm{Alg}_R$ is an Azumaya algebra over R.

Example 11.5.3.8. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathscr{A} \in \mathrm{Alg}(\mathrm{QCoh}(\mathrm{Spec}\, R))$. Then $\mathscr{A}$ is an Azumaya algebra (in the sense of Definition 11.5.3.7) if and only if its image under the equivalence $\mathrm{Alg}(\mathrm{QCoh}(\mathrm{Spec}\, R)) \simeq \mathrm{Alg}_R$ is an Azumaya algebra over R (in the sense of Definition 11.5.3.1).

Construction 11.5.3.9 (The Extended Brauer Class of an Azumaya Algebra)**.** Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and let $\mathscr{A} \in \mathrm{Alg}(\mathrm{QCoh}(X))$ be an Azumaya algebra. Then the construction

$$(\eta \in X(R)) \mapsto (\mathrm{RMod}_{\mathscr{A}_\eta} \in \mathrm{LinCat}_R^{\mathrm{St}})$$

determines a compactly generated stable quasi-coherent stack on X. It follows from Proposition 11.5.3.4 that this quasi-coherent stack is an invertible object of $\mathrm{LinCat}_R^{\mathrm{St}}$, and can therefore be identified with a point of the extended Brauer space $\mathscr{B}\mathrm{r}^\dagger(X)$. We let $[\mathscr{A}] \in \mathrm{Br}^\dagger(X)$ denote the equivalence class of this quasi-coherent stack. We will refer to $[\mathscr{A}]$ as the *extended Brauer class of* $\mathscr{A}$.

In good cases, every element of $\mathrm{Br}^\dagger(X)$ can be represented by an Azumaya algebra:

Proposition 11.5.3.10. *Let* X *be a spectral algebraic space which is quasi-compact and quasi-separated. Then every element of* $\mathrm{Br}^\dagger(\mathsf{X})$ *has the form* $[\mathscr{A}]$*, for some Azumaya algebra* $\mathscr{A} \in \mathrm{Alg}(\mathrm{QCoh}(\mathsf{X}))$*.*

Proof. Let u be an element of $\mathrm{Br}^\dagger(\mathsf{X})$. Choose a compactly generated invertible object $\mathcal{C} \in \mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$ which represents u. It follows from Theorem 10.3.2.1 that the ∞-category $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ is compactly generated. Choose a set of compact generators $\{C_i\}_{i\in I}$ for $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$. Since X is quasi-compact, there exists an étale surjection $\eta : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$. For each $i \in I$, let $\eta^* C_i$ denote the image of C_i in the stable R-linear ∞-category $\mathcal{C}_\eta$. It follows from Remark 11.5.2.5 that $\mathcal{C}_\eta$ is smooth over R. It follows from the proof of Proposition 11.3.2.4 that we can choose a finite subset $I_0 \subseteq I$ such that the objects $\{\eta^* C_i\}_{i\in I_0}$ generate $\mathcal{C}_\eta$. Set $C = \bigoplus_{i\in I_0} C_i$.

Let us regard the ∞-category $\mathrm{QCoh}(\mathsf{X};\mathcal{C})$ as tensored over $\mathrm{QCoh}(\mathsf{X})$, and let $\mathscr{A} \in \mathrm{Alg}(\mathrm{QCoh}(\mathsf{X}))$ be the endomorphism algebra of C. Let $\mathcal{C}' \in \mathrm{QStk}^{\mathrm{St}}(\mathsf{X})$ be the stable quasi-coherent stack on X given by the construction $(\eta' : \mathrm{Sp\acute{e}t}\, R' \to \mathsf{X}) \mapsto (\mathrm{RMod}_{\eta'^* \mathscr{A}} \in \mathrm{LinCat}_{R'}^{\mathrm{St}})$. The operation $\bullet \otimes_{\mathscr{A}} C$ determines a morphism of quasi-coherent stacks $F : \mathcal{C}' \to \mathcal{C}$. Since $\eta^* C$ is a compact generator of $\mathcal{C}_\eta$, the induced map functor $F_\eta : \mathcal{C}'_\eta \to \mathcal{C}_\eta$ is an equivalence of R-linear ∞-categories. In particular, the R-linear ∞-category $\mathrm{RMod}_{\eta^* \mathscr{A}} \simeq \mathcal{C}'_\eta \simeq \mathcal{C}_\eta$ is an invertible object of $\mathrm{LinCat}_R^{\mathrm{St}}$, so that Proposition 11.5.3.4 implies that $\eta^* \mathscr{A} \in \mathrm{Alg}_R$ is an Azumaya algebra over R. Since the map η is an étale surjection, it follows that $\mathscr{A}$ is an Azumaya algebra on X and that F is an equivalence, so that $u = [\mathscr{A}]$. □

In the situation of Proposition 11.5.3.10, the Azumaya algebra $\mathscr{A}$ is not determined by its Brauer class $[\mathscr{A}]$. However, the failure of uniqueness is addressed by the following:

Proposition 11.5.3.11. *Let* $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ *be a functor and let* $\mathscr{A} \in \mathrm{Alg}(\mathrm{QCoh}(X))$*. The following conditions are equivalent:*

(1) *The object* $\mathscr{A}$ *is an Azumaya algebra and the extended Brauer class* $[\mathscr{A}] \in \mathrm{Br}^\dagger(X)$ *vanishes.*

(2) *There exists an equivalence* $\mathscr{A} \simeq \operatorname{End}(\mathscr{F}) = \mathscr{F} \otimes \mathscr{F}^\vee$, *where* $\mathscr{F} \in \operatorname{QCoh}(X)$ *has the property that for each point* $\eta \in X(R)$, *the object* $\eta^* \mathscr{F}$ *is a compact generator of* Mod_R.

Proof. Let $\mathcal{C}$ denote the stable quasi-coherent stack on X given by the construction $(\eta \in X(R)) \mapsto (\operatorname{RMod}_{\eta^* \mathscr{A}} \in \operatorname{LinCat}_R^{\operatorname{St}})$. Using Proposition 11.5.3.4, we see that condition (1) is equivalent to the following:

(1′) There exists a map $\rho : \mathcal{Q}_X^{\operatorname{St}} \to \mathcal{C}$ which is an equivalence of quasi-coherent stacks.

Note that giving a map of quasi-coherent stacks $\rho : \mathcal{Q}_X^{\operatorname{St}} \to \mathcal{C}$ is equivalent to giving an object $\mathscr{G} \in \operatorname{QCoh}(X; \mathcal{C}) \simeq \operatorname{RMod}_{\mathscr{A}}(\operatorname{QCoh}(X))$. Using Lemma **??**, we see that the map ρ is an equivalence if and only if $\mathscr{G}$ satisfies the following additional pair of conditions:

(i) For each point $\eta \in X(R)$, the R-module $\eta^* \mathscr{G}$ is a compact generator of Mod_R.

(ii) The right action of $\mathscr{A}$ on $\mathscr{G}$ induces an equivalence $\mathscr{A} \to \operatorname{End}(\mathscr{G})^{\operatorname{rev}} \simeq \operatorname{End}(\mathscr{G}^\vee)$.

Setting $\mathscr{F} = \mathscr{G}^\vee$ (and applying Lemma 11.5.3.5), we see that conditions (1) and (2) are equivalent. □

Remark 11.5.3.12. Let $X : \operatorname{CAlg}^{\operatorname{cn}} \to \mathcal{S}$ be a functor. We can summarize Construction **??**, Proposition **??**, and Proposition 11.5.3.11 as follows:

(a) Every Azumaya algebra $\mathscr{A}$ on X determines an element $[\mathscr{A}] \in \operatorname{Br}^\dagger(X)$. If X is (representable by) a quasi-compact, quasi-separated spectral algebraic space, then every element of $\operatorname{Br}^\dagger(X)$ is obtained in this way.

(b) The collection of elements of $\operatorname{Br}^\dagger(X)$ having the form $[\mathscr{A}]$ comprise a subgroup of $\operatorname{Br}^\dagger(X)$. More explicitly, the multiplication on $\operatorname{Br}^\dagger(X)$ is given by $[\mathscr{A}][\mathscr{B}] = [\mathscr{A} \otimes \mathscr{B}]$, the identity element of $\operatorname{Br}^\dagger(X)$ is given by $[\mathscr{O}_X]$, and the inverse of $[\mathscr{A}]$ is given by $[\mathscr{A}^{\operatorname{rev}}]$ (the last assertion follows from Proposition 11.5.3.11 together with the equivalence $\mathscr{A} \otimes \mathscr{A}^{\operatorname{rev}} \simeq \operatorname{End}(\mathscr{A})$ supplied by our assumption that $\mathscr{A}$ is an Azumaya algebra).

(c) Given Azumaya algebras $\mathscr{A}, \mathscr{B} \in \operatorname{Alg}(\operatorname{QCoh}(X))$, we have $[\mathscr{A}] = [\mathscr{B}]$ in $\operatorname{Br}^\dagger(X)$ if and only if there is an equivalence $\mathscr{A} \otimes \mathscr{B}^{\operatorname{rev}} \simeq \operatorname{End}(\mathscr{F})$ for some $\mathscr{F} \in \operatorname{QCoh}(X)^{\operatorname{perf}}$ having the property that $\eta^* \mathscr{F} \in \operatorname{Mod}_R$ is a compact generator for each $\eta \in X(R)$ (in this case, we can regard $\mathscr{F}$ as a $\mathscr{A}$-$\mathscr{B}$ bimodule object of $\operatorname{QCoh}(X)$).

Remark 11.5.3.13. Let X be a spectral algebraic space which is quasi-compact and quasi-separated. It follows from Proposition 11.5.3.10 that the Brauer group $\operatorname{Br}^\dagger(\mathsf{X})$ is small. Combining this observation with Remarks **??** and 2.9.5.2, we deduce that the extended Brauer space $\mathscr{B}\mathrm{r}^\dagger(R)$ is essentially small. In fact, these assertions are valid for *any* spectral Deligne-Mumford stack (see Construction 11.5.5.3).

We now use Proposition 11.5.3.11 to relate the extended Brauer group of Definition 11.5.2.1 to the classical Brauer group of Definition 11.5.1.7. The key observation is the following:

Proposition 11.5.3.14. *Let κ be a field and let $A \in \mathrm{Alg}_\kappa$. Then:*

- (a) *The algebra A is an Azumaya algebra over κ (in the sense of Definition 11.5.3.1) if and only if $B = \pi_* A$ is a central simple algebra over κ (in the sense of Definition 11.5.1.1).*
- (b) *The algebra A is a connective Azumaya algebra over κ (in the sense of Definition 11.5.3.1) if and only A is discrete and is a central simple algebra over κ (in the sense of Definition 11.5.1.1).*

Warning 11.5.3.15. Assertion (b) of Proposition 11.5.3.14 follows immediately from Propositions 11.5.3.3 and 11.5.1.5. However, the proof of (a) is a bit more subtle. Let A be an $\mathbb{E}_1$-algebra over a field κ which is perfect as a κ-module. Then we can regard $B = \pi_* A$ as a finite-dimensional associative algebra over κ. Using the left and right actions of A and B on themselves, we obtain maps

$$\rho_A : A \otimes_\kappa A^{\mathrm{rev}} \to \mathrm{End}_\kappa(A) \qquad \rho_B : B \otimes_\kappa B^{\mathrm{rev}} \to \mathrm{End}_\kappa(B).$$

On homotopy groups, ρ_A induces a map $\overline{\rho}_A : (\pi_* A) \otimes_\kappa (\pi_* A^{\mathrm{rev}}) \to \mathrm{End}_\kappa(\pi_* A)$ which does *not* agree with the map ρ_B. Given elements $x \in \pi_a A$, $y \in \pi_b A^{\mathrm{rev}}$, $z \in \pi_c A$, we have

$$\overline{\rho}_A(x \otimes y)(z) = (-1)^{bc} xzy \qquad \rho_B(x \otimes y)(z) = xzy.$$

In order to address the difficulties described in Warning 11.5.3.15, we will need a few auxiliary results.

Lemma 11.5.3.16. *Let κ be a field, let $B = \oplus_{n \in \mathbf{Z}} B_n$ be a graded associative algebra over κ which is finite-dimensional as a vector space over κ, and let $M \in \mathrm{LMod}_B^{\heartsuit}$ be a left B-module. Assume that M is absolutely simple (that is, the tensor product $\overline{\kappa} \otimes_\kappa M$ is a simple module over $\overline{\kappa} \otimes_\kappa B$, where $\overline{\kappa}$ is an algebraic closure of κ). Then M admits the structure of a graded left module over B (that is, there exists a decomposition $M = \bigoplus_{n \in \mathbf{Z}} M_n$ satisfying $bx \in M_{m+n}$ for $b \in B_m$ and $x \in M_n$).*

Proof. For every commutative κ-algebra R, let $G(R)$ denote the subgroup of $R^\times \times \mathrm{Aut}_R(R \otimes_\kappa M)$ consisting of those pairs (λ, f) which satisfy the identity $f(bx) = \lambda^n b f(x)$ for all $b \in B_n$ and $x \in M$. The construction $R \mapsto G(R)$ determines an affine group scheme over κ, which we can regard as a closed subgroup of the product $\mathbf{G}_m \times \mathrm{GL}_d$ for $d = \dim_\kappa(M)$. Let $\pi : G \to \mathbf{G}_m$ be the projection map onto the first factor (given on R-valued points by $\pi(\lambda, f) = \lambda$).

Set $\overline{B} = \overline{\kappa} \otimes_{\kappa} B$ and $\overline{M} = \overline{\kappa} \otimes_{\kappa} M$. For each $\lambda \in \overline{\kappa}^{\times}$, we define a left $\overline{B}$-module $\overline{M}_{\lambda}$ by the following requirement: there exists an isomorphism of $\overline{\kappa}$-vector spaces $\alpha : \overline{M} \to \overline{M}_{\lambda}$, and the actions of $\overline{B}$ on $\overline{M}$ and $\overline{M}_{\lambda}$ are related by the formula $\alpha(bx) = \lambda^n b\alpha(x)$ for $b \in B_n$. Note that $\lambda \in \overline{\kappa}^{\times}$ belongs to the image of the map $\pi : G(\overline{\kappa}) \to \mathbf{G}_m(\overline{\kappa})$ if and only if $\overline{M}$ and $\overline{M}_{\lambda}$ are isomorphic. Since there are only finitely many isomorphism classes of simple $\overline{B}$-modules (Proposition 11.5.1.13), the image of the map $G(\overline{\kappa}) \to \overline{\kappa}^{\times}$ is a finite-index subgroup of $\overline{\kappa}^{\times}$, and is therefore dense in the Zariski topology.

Our assumption that M is absolutely simple guarantees that the kernel $\ker(\pi)$ is isomorphic to the multiplicative group $\mathbf{G}_m$. We therefore have an exact sequence $0 \to \mathbf{G}_m \to G \xrightarrow{\pi} \mathbf{G}_m \to 0$ in the category of affine group schemes over κ. It follows that π admits a section, which determines a grading of M which is compatible with the action of B. □

Lemma 11.5.3.17. *Let κ be a field and let $A \in \mathrm{Alg}_{\kappa}$. Suppose that there exists a finite-dimensional graded vector space V over κ and an isomorphism of graded rings $\pi_* A \simeq \mathrm{End}_{\kappa}(V)$. Then there is a perfect κ-module $\overline{V}$ and an equivalence $A \simeq \mathrm{End}_{\kappa}(\overline{V})$ of $\mathbb{E}_1$-algebras over κ.*

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis for V as a κ-vector space, where each v_i is a homogeneous element of degree d_i. For $1 \leqslant i, j \leqslant n$, let $e_{i,j} \in \mathrm{End}_{\kappa}(V)$ be the endomorphism given by

$$e_{i,j}(v_l) = \begin{cases} v_j & \text{if } i = l \\ 0 & \text{otherwise,} \end{cases}$$

so that $e_{i,j}$ is homogeneous of degree $d_j - d_i$. Let us abuse notation by identifying each $e_{i,j}$ with its image under the isomorphism $\mathrm{End}_{\kappa}(V) \simeq \pi_* A$.

For $1 \leqslant i \leqslant n$, left multiplication by the element $e_{i,i} \in \pi_0 A$ determines an idempotent map from A to itself in the ∞-category RMod_A. We let $e_{i,i}A$ denote the corresponding direct factor of A, so that we have an equivalence $A \simeq \bigoplus e_{i,i}A$ in RMod_A. Note that each $e_{i,i}A$ is a summand of A, and therefore a compact object of RMod_A. Moreover, the elements $e_{i,j} \in \pi_{d_j - d_i} A$ induce equivalences $e_{j,j}A \simeq \Sigma^{d_i - d_j} e_{i,i}A$. It follows that each $e_{i,i}A$ is a compact generator of RMod_A. Note that the unit map $\kappa \to \pi_* \mathrm{End}_{\mathrm{RMod}_A}(e_{i,i}A) \simeq e_{i,i}(\pi_* A)e_{i,i}$ is an isomorphism, so that the construction $\kappa \mapsto e_{i,i}A$ extends to an equivalence of κ-linear stable ∞-categories $\phi : \mathrm{Mod}_{\kappa} \to \mathrm{RMod}_A$. Then $\phi^{-1}(A)$ is a compact object of Mod_{κ}, and we have an equivalence of $\mathbb{E}_1$-algebras $A \simeq \mathrm{End}_{\mathrm{RMod}_A}(A) \simeq \mathrm{End}_{\mathrm{Mod}_{\kappa}}(\phi^{-1}A)$. □

Proof of Proposition 11.5.3.14. Since assertion (b) follows from Propositions 11.5.3.3 and 11.5.1.5, it will suffice to prove (a) (alternatively, we note that (b) is an easy consequence of (a)). Let $A \in \mathrm{Alg}_{\kappa}$ and let $B = \pi_* A$, which we regard as an associative algebra over κ. We wish to show that A is an Azumaya algebra if and only if B is central simple. Without loss of generality, we may assume that κ is algebraically closed (see Remark 11.5.1.11).

Assume first that A is an Azumaya algebra. Then B is finite-dimensional as a vector space over κ. Let M be a simple left B-module. Using Lemma 11.5.3.16, we can choose a grading on M which is compatible with the grading on B. In this case, the left action of B on M determines a map of graded rings $f : B \to \mathrm{End}_\kappa(M)$. Since κ is algebraically closed and M is a simple B-module, the morphism f is surjective. We will complete the proof by showing that that the graded two-sided ideal $I = \ker(f)$ vanishes. Let $\overline{\rho}_A : B \otimes_\kappa B^{\mathrm{rev}} \to \mathrm{End}_\kappa(B)$ be the map appearing in Warning 11.5.3.15. Note that $\overline{\rho}_A$ is an algebra homomorphism, if we regard $B \otimes_\kappa B^{\mathrm{rev}}$ as equipped with the multiplication law given by

$$(x \otimes y)(x' \otimes y') = (-1)^{ij+ik}(xx' \otimes y'y),$$

where y, x', and y' are homogeneous elements of degree i, j, and k respectively. With respect to this multiplication, the tensor product $I \otimes_\kappa B^{\mathrm{rev}}$ is a two-sided ideal in $B \otimes_\kappa B^{\mathrm{rev}}$. Our assumption that A is an Azumaya algebra guarantees that $\overline{\rho}_A$ is an isomorphism, so that $B \otimes_\kappa B^{\mathrm{rev}}$ is central simple over κ (Proposition 11.5.1.2). Since $I \neq B$, it follows that $I = 0$ as desired.

Now suppose that B is central simple. Using Proposition 11.5.1.12 (and our assumption that κ is algebraically closed), we deduce that there exists an isomorphism $\alpha : B \simeq \mathrm{End}_\kappa(V)$, where V is a finite-dimensional vector space over κ. Applying Lemma 11.5.3.17, we conclude that $A \simeq \mathrm{End}_\kappa(\overline{V})$ for some perfect object $\overline{V} \in \mathrm{Mod}_\kappa$, so that A is an Azumaya algebra by virtue of Proposition 11.5.3.11. □

Theorem 11.5.3.18. *Let κ be a field. Then there is a canonical isomorphism from the Brauer group $\mathrm{Br}(\kappa)$ (Definition **??**) to the extended Brauer group $\mathrm{Br}^\dagger(\kappa)$ (Definition 11.5.2.1), which carries the equivalence class $[A] \in \mathrm{Br}(\kappa)$ of a central simple algebra A over κ to class $[A] = \mathrm{RMod}_A \in \mathrm{Br}^\dagger(\kappa)$ of Construction 11.5.3.9.*

Proof. Let A and B be central simple algebras over κ. Using Proposition 11.5.3.11, we see that the identity $[A] = [B]$ holds in $\mathrm{Br}^\dagger(\kappa)$ if and only if the tensor product $A \otimes_\kappa B$ can be identified with the endomorphism algebra $\mathrm{End}_\kappa(V)$ for some nonzero object $V \in \mathrm{Mod}_\kappa^{\mathrm{perf}}$. Note that in this case V must be discrete, so the equality $[A] = [B]$ holds in $\mathrm{Br}^\dagger(\kappa)$ if and only if it holds in $\mathrm{Br}(\kappa)$. We therefore obtain a well-defined monomorphism $\phi : \mathrm{Br}(\kappa) \to \mathrm{Br}^\dagger(\kappa)$.

We now complete the proof by showing that ϕ is surjective. According to Proposition 11.5.3.10, every element of $\mathrm{Br}^\dagger(\kappa)$ has the form $[A]$, where $A \in \mathrm{Alg}_\kappa$ is an Azumaya algebra over κ. Set $B = \pi_* A$, which we regard as a (discrete) associative algebra over κ. Proposition 11.5.3.14 implies that B is central simple. We then have canonical isomorphisms

$$\pi_*(A \otimes_\kappa B^{\mathrm{rev}}) \simeq (\pi_* A) \otimes_\kappa B^{\mathrm{rev}} \simeq B \otimes_\kappa B^{\mathrm{rev}} \simeq \mathrm{End}_\kappa(B).$$

Using Lemma 11.5.3.17, we deduce that $[A] = [B]$ in $\mathrm{Br}^\dagger(\kappa)$. □

Warning 11.5.3.19. Let κ be a field and let $\phi : \mathrm{Br}(\kappa) \to \mathrm{Br}^{\dagger}(\kappa)$ be the isomorphism of Theorem 11.5.3.18. The proof of Theorem 11.5.3.18 shows that the inverse isomorphism $\phi^{-1} : \mathrm{Br}^{\dagger}(\kappa) \to \mathrm{Br}(\kappa)$ is given by the construction $[A] \mapsto [\pi_* A]$. In particular, if A and B are Azumaya algebras over κ, then the κ-algebras $\pi_*(A \otimes_{\kappa} B)$ and $(\pi_* A) \otimes_{\kappa} (\pi_* B)$ are Morita equivalent (and therefore isomorphic, since they have the same dimension over κ). This is not *a priori* obvious: beware that the canonical identification $\pi_*(A \otimes_{\kappa} B) \simeq (\pi_* A) \otimes_{\kappa} (\pi_* B)$ is usually *not* an isomorphism of algebras.

11.5.4 Extended Brauer Groups of Direct and Inverse Limits

By virtue of Remark 11.5.2.8, we can regard the construction $R \mapsto \mathrm{Br}^{\dagger}(R)$ as a functor from the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_{\infty}$-rings to the ordinary category of abelian groups. We now study the compatibility of this functor with direct and inverse limits.

Proposition 11.5.4.1. *The construction $R \mapsto \mathrm{Br}^{\dagger}(R)$ commutes with filtered colimits.*

The proof of Proposition 11.5.4.1 will require some preliminaries.

Remark 11.5.4.2. Let R be a connective $\mathbb{E}_{\infty}$-ring and let $A \in \mathrm{Alg}_R$ be an Azumaya algebra. Then A is smooth and proper over R (Remark 11.5.2.5), and is therefore a compact object of Alg_R (Proposition HA.**??**).

Lemma 11.5.4.3. *For each $\mathbb{E}_2$-ring R, let Alg^c_R denote the full subcategory of Alg_R spanned by the compact objects. Then the construction $R \mapsto \mathrm{Alg}^c_R$ commutes with filtered colimits.*

Proof. Let $\{R_{\alpha}\}$ be a filtered diagram of $\mathbb{E}_2$-rings with colimit R. We wish to show that the canonical map $\theta : \varinjlim \mathrm{Alg}^c_{R_{\alpha}} \to \mathrm{Alg}^c_R$ is an equivalence of ∞-categories. It follows from Lemma HA.**??** that the domain and codomain of θ are both idempotent complete. Using Lemma HA.5.3.2.9 , we are reduced to proving that θ exhibits $\mathrm{Ind}(\mathrm{Alg}^c_R)$ as a colimit of the diagram $\{\mathrm{Ind}(\mathrm{Alg}^c_{R_{\alpha}})\}$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. In other words, we must show that Alg_R is a colimit of the diagram $\{\mathrm{Alg}_{R_{\alpha}}\}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Using Theorem HTT.5.5.3.18 , we are reduced to showing that the forgetful functors $\mathrm{Alg}_R \to \mathrm{Alg}_{R_{\alpha}}$ exhibit Alg_R as a limit $\varprojlim \mathrm{Alg}_{R_{\alpha}}$ in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$. For this, it will suffice to prove that the canonical map $\mathrm{LMod}_R \to \varprojlim \mathrm{LMod}_{R_{\alpha}}$ is an equivalence in $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$, which follows from Proposition HA.**??** (and Theorem HTT.5.5.3.18). □

Lemma 11.5.4.4. *Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a compactly generated R-linear ∞-category, and let $\chi : \mathrm{Alg}_R \to \mathcal{C}\mathrm{at}_{\infty}$ be the functor given by $\chi(A) = \mathrm{LMod}_A(\mathcal{C})^c$, where $\mathrm{LMod}_A(\mathcal{C})^c$ denotes the full subcategory of $\mathrm{LMod}_A(\mathcal{C})$ spanned by the compact objects. Then χ preserves filtered colimits.*

Proof. Let $\{A_\alpha\}$ be a filtered diagram in Alg_R with colimit A. We wish to show that the canonical map $\theta : \varinjlim \mathrm{LMod}_{A_\alpha}(\mathcal{C})^c \to \mathrm{LMod}_A(\mathcal{C})^c$ is an equivalence of ∞-categories. It follows from Lemma HA.**??** that the domain and codomain of θ are both idempotent complete. Using Lemma HA.5.3.2.9 , we are reduced to proving that the $\mathrm{LMod}_A(\mathcal{C})$ is a colimit of the diagram $\{\mathrm{LMod}_{A_\alpha}(\mathcal{C})\}$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Using Theorem HA.4.8.4.6 , we are reduced to proving that $\mathrm{LMod}_A \otimes \mathcal{C}$ is a colimit of the diagram $\{\mathrm{LMod}_{A_\alpha} \otimes_R \mathcal{C}\}$ in the ∞-category LinCat_R, which follows from Proposition HA.**??** (since the tensor product in LinCat_R preserves small colimits separately in each variable). □

Proof of Proposition 11.5.4.1. Let $\{R_\alpha\}$ be a diagram of $\mathbb{E}_\infty$-rings indexed by a filtered partially ordered set, having colimit $R = \varinjlim R_\alpha$. We wish to show that the canonical map $\theta : \varinjlim \mathrm{Br}^\dagger(R_\alpha) \to \mathrm{Br}^\dagger(R)$ is an isomorphism. We first show that θ is injective. Fix an index α and an Azumaya algebra A_α over R_α, and let $A = R \otimes_{R_\alpha} A_\alpha$. Suppose that $[A]$ is trivial in $\mathrm{Br}(R)$. Then we can write $A = \mathrm{End}_R(X)$, where X is a compact generator of Mod_R. Enlarging α if necessary, we can use Lemma 11.5.4.4 to write $X = R \otimes_{R_\alpha} X_\alpha$ for some perfect R_α-module X_α. Let $\mathcal{C} \subseteq \mathrm{Mod}_{R_\alpha}$ be the smallest stable subcategory of Mod_{R_α} which contains X_α and is closed under small colimits. For each $\beta \geqslant \alpha$, we can identify $\mathrm{Mod}_{R_\beta} \otimes_{\mathrm{Mod}_{R_\alpha}} \mathcal{C}$ with the smallest stable subcategory of Mod_{R_β} which contains $X_\beta = R_\beta \otimes_{R_\alpha} X_\alpha$ and is closed under small colimits. Since X is a compact generator of Mod_R, we conclude that $\varinjlim_{\beta \geqslant \alpha} \mathrm{LMod}_{R_\beta}(\mathcal{C}) \simeq \mathrm{Mod}_R$. Using Lemma 11.5.4.4, we conclude that there exists an index $\beta \geqslant \alpha$ and a compact object $Y_\beta \in \mathcal{C}_\beta$ such that $R \otimes_{R_\beta} Y_\beta \simeq R$. Using Lemma 11.5.4.4 again, we may suppose (after enlarging β) that $Y_\beta \simeq R$, so that $\mathrm{LMod}_{R_\beta}(\mathcal{C}) = \mathrm{Mod}_{R_\beta}$. Replacing α by β, we may reduce to the case where X_α is a compact generator of Mod_{R_α}, so that $B_\alpha = \mathrm{End}_{R_\alpha}(X_\alpha)$ is an Azumaya algebra. Remark 11.5.4.2 implies that A_α and B_α are compact objects of Alg_{R_α}. Since the images of A_α and B_α in Alg_R are equivalent, Lemma 11.5.4.3 implies that there exists $\gamma \geqslant \alpha$ such that $R_\gamma \otimes_{R_\alpha} A_\alpha$ and $R_\gamma \otimes_{R_\alpha} B_\alpha$ are equivalent in Alg_{R_γ}. It follows that the image of $[A_\alpha]$ vanishes in $\mathrm{Br}^\dagger(R_\gamma)$, and therefore represents the trivial element in the domain of θ.

We now prove that θ is surjective. According to Proposition 11.5.3.10, every element of $\mathrm{Br}^\dagger(R)$ can be written as $[A]$, where A is an Azumaya algebra over R. Then A is a compact object of Alg_R (Remark 11.5.4.2), so we can write $A = R \otimes_{R_\alpha} A_\alpha$ for some index α and some compact object $A_\alpha \in \mathrm{Alg}_{R_\alpha}$. Using the preceding argument, we may assume (after enlarging α if necessary) that A_α is a compact generator of Mod_{R_α}. The left and right actions of A_α on itself induce a map $u : A_\alpha \otimes_{R_\alpha} A_\alpha^{\mathrm{op}} \to \mathrm{End}_{R_\alpha}(A_\alpha)$. We can regard u as a morphism between perfect R_α-modules. Since A is Azumaya, the image of u in Mod_R is an equivalence. Using Lemma 11.5.4.4, we deduce that there exists an index $\beta \geqslant \alpha$ such that the image of u in Mod_{R_β} is an equivalence. Then $A_\beta = R_\beta \otimes_{R_\alpha} A_\alpha$ is an Azumaya algebra over R_β, so that $[A]$ belongs to the image of the map $\mathrm{Br}^\dagger(R_\beta) \to \mathrm{Br}(R)$. □

We now study the compatibility of the functor $R \mapsto \mathrm{Br}^{\dagger}(R)$ with (certain) inverse limits.

Proposition 11.5.4.5. *Let R be a Noetherian commutative ring which is complete with respect to an ideal I. Then the map $\mathrm{Br}^{\dagger}(R) \to \mathrm{Br}^{\dagger}(R/I)$ is injective.*

Proposition 11.5.4.6. *Let R be a connective $\mathbb{E}_{\infty}$-ring. Then the map $\mathrm{Br}^{\dagger}(R) \to \mathrm{Br}^{\dagger}(\pi_0 R)$ is injective.*

Remark 11.5.4.7. We will show later that the canonical map $\mathrm{Br}^{\dagger}(R) \to \mathrm{Br}^{\dagger}(\pi_0 R)$ is an isomorphism: see Proposition 11.5.5.6.

To prove Propositions 11.5.4.5 and 11.5.4.6, we will need a variant of Proposition 11.5.3.11 which can be used to verify the vanishing of an element $\eta \in \mathrm{Br}^{\dagger}(R)$.

Definition 11.5.4.8. Let R be a connective $\mathbb{E}_{\infty}$-ring, and let $\mathcal{C} \in \mathscr{B}\mathrm{r}^{\dagger}(R)$ be an invertible R-linear ∞-category. We will say that a compact object $X \in \mathcal{C}$ is a *neutralization* of $\mathcal{C}$ if the unit map $R \to \underline{\mathrm{Map}}_{\mathcal{C}}(X, X)$ is an equivalence.

Proposition 11.5.4.9. *Let R be a connective $\mathbb{E}_{\infty}$-ring, let $\mathcal{C} \in \mathscr{B}\mathrm{r}^{\dagger}(R)$, and let $X \in \mathcal{C}$ be a compact object, so that X classifies an R-linear functor $F : \mathrm{Mod}_R \to \mathcal{C}$. The following conditions are equivalent:*

(1) *The map F is an equivalence of ∞-categories.*

(2) *The object X is a neutralization of $\mathcal{C}$.*

Proof. The implication (1) ⇒ (2) is clear. Conversely, suppose that (2) is satisfied. Since F preserves small colimits and compact objects, it admits an R-linear right adjoint G (Remark D.1.5.3). Since X is a neutralization of $\mathcal{C}$, the functor F is fully faithful, so that the unit map $u : \mathrm{id} \to G \circ F$ is an equivalence. Let $F' : \mathcal{C}^{-1} \to \mathrm{Mod}_R$ be the functor obtained by tensoring F over Mod_R with $\mathcal{C}^{-1}$. It follows that F' admits an R-linear right adjoint G', and that the unit map $u' : \mathrm{id} \to G' \circ F'$ is fully faithful. In particular, the functor F' is fully faithful and preserves compact objects. Let Y be a compact generator of $\mathcal{C}^{-1}$, and let $M = F'(Y) \in \mathrm{Mod}_R^{\mathrm{perf}}$. Since F' is fully faithful, the canonical map $\underline{\mathrm{Map}}_{\mathcal{C}^{-1}}(Y, Y) \to \mathrm{End}(M)$ is an equivalence in Alg_R. It follows from Proposition 11.5.3.4 that $\mathrm{End}(M)$ is an Azumaya algebra over R. In particular, $\mathrm{End}(M)$ is a compact generator of Mod_R. Since $\mathrm{End}(M) \simeq M^{\vee} \otimes_R M \simeq F'(M^{\vee} \otimes Y)$ belongs to the essential image of F', we deduce that F' is an equivalence of ∞-categories. We conclude that the functor F is also an equivalence. □

Corollary 11.5.4.10. *Let R be a connective $\mathbb{E}_{\infty}$-ring and let $\mathcal{C} \in \mathscr{B}\mathrm{r}^{\dagger}(R)$. Then $[\mathcal{C}] = 0$ in $\mathrm{Br}^{\dagger}(R)$ if and only if there exists a neutralization of $\mathcal{C}$.*

The proofs of Propositions 11.5.4.6 and 11.5.4.5 will also require a bit of deformation theory.

Lemma 11.5.4.11. *Suppose we are given a pullback diagram of connective $\mathbb{E}_\infty$-rings σ :*

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01}, \end{array}$$

where the underlying maps of commutative rings $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective. Let $\mathcal{C}$ be an R-linear ∞-category which is smooth and proper. Then the canonical map

$$\theta : \mathcal{C}^c \to \mathrm{LMod}_{R_0}(\mathcal{C})^c \times_{\mathrm{LMod}_{R_{01}}(\mathcal{C})} \mathrm{LMod}_{R_1}(\mathcal{C})^c$$

is an equivalence of ∞-categories.

Proof. We proceed as in the proof of Theorem 11.4.4.1. Since $\mathcal{C}$ is a smooth R-linear ∞-category, we can write $\mathcal{C} \simeq \mathrm{RMod}_A$ for some smooth $\mathbb{E}_1$-algebra A over R (Proposition 11.3.2.4). It follows from Proposition 11.1.0.2 that A is also a proper $\mathbb{E}_1$-algebra over R, so that an object of $\mathcal{C} \simeq \mathrm{RMod}_A$ is compact if and only if its image in Mod_R is compact. Let $A_0 = R_0 \otimes_R A$, and define A_1 and A_{01} similarly. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{RMod}_A^{\mathrm{perf}} & \xrightarrow{\theta} & \mathrm{RMod}_{A_0}^{\mathrm{perf}} \times_{\mathrm{RMod}_{A_{01}}^{\mathrm{perf}}} \mathrm{RMod}_{A_1}^{\mathrm{perf}} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_R^{\mathrm{perf}} & \xrightarrow{\theta'} & \mathrm{Mod}_{R_0}^{\mathrm{perf}} \times_{\mathrm{Mod}_{R_{01}}^{\mathrm{perf}}} \mathrm{Mod}_{R_1}^{\mathrm{perf}} . \end{array}$$

Proposition 16.2.1.1 implies that θ is fully faithful. Moreover, Theorem 16.2.0.2 and Proposition 16.2.3.1 guarantee θ' is an equivalence of ∞-categories. Consequently, to prove that θ is an equivalence, it will suffice to show that θ induces an essentially surjective map

$$\phi : \mathrm{RMod}_A^{\mathrm{perf}} \times_{\mathrm{Mod}_R^{\mathrm{perf}}} \{M\} \to (\mathrm{RMod}_{A_0}^{\mathrm{perf}} \times_{\mathrm{RMod}_{A_{01}}^{\mathrm{perf}}} \mathrm{RMod}_{A_1}^{\mathrm{perf}}) \times_{\mathrm{Mod}_R^{\mathrm{perf}}} \{M\}$$

for every perfect R-module M. Let $M_0 = R_0 \otimes_R M \in \mathrm{Mod}_{R_0}^{\mathrm{perf}}$, and define M_1 and M_{01}

similarly. Using Theorem HA.4.7.1.34 , we can identify ϕ with the canonical map

$$\begin{array}{c}
\mathrm{Map}_{\mathrm{Alg}_R}(A^{\mathrm{rev}}, \mathrm{End}(M)) \\
\downarrow \\
\mathrm{Map}_{\mathrm{Alg}_{R_0}}(A_0^{\mathrm{rev}}, \mathrm{End}(M_0)) \times_{\mathrm{Map}_{\mathrm{Alg}_{R_{01}}}(A_{01}^{\mathrm{rev}}, \mathrm{End}(M_{01}))} \mathrm{Map}_{\mathrm{Alg}_{R_1}}(A_1^{\mathrm{rev}}, \mathrm{End}(M_1)) \\
\downarrow \\
\mathrm{Map}_{\mathrm{Alg}_R}(A^{\mathrm{rev}}, \mathrm{End}(M_0)) \times_{\mathrm{Map}_{\mathrm{Alg}_R}(A^{\mathrm{rev}}, \mathrm{End}(M_{01}))} \mathrm{Map}_{\mathrm{Alg}_R}(A^{\mathrm{rev}}, \mathrm{End}(M_1)) \\
\downarrow \\
\mathrm{Map}_{\mathrm{Alg}_R}(A^{\mathrm{rev}}, \mathrm{End}(M_0) \times_{\mathrm{End}(M_{01})} \mathrm{End}(M_1)) \\
\downarrow \\
\mathrm{Map}_{\mathrm{Alg}_R}(A^{\mathrm{rev}}, (R_0 \times_{R_{01}} R_1) \otimes_R M).
\end{array}$$

Since σ is a pullback diagram, this map is an equivalence. □

Lemma 11.5.4.12. *Let R be a connective $\mathbb{E}_\infty$-ring, let $\overline{R}$ be a square-zero extension of R by a connective R-module M, and let $\mathcal{C} \in \mathscr{B}\mathrm{r}^\dagger(\overline{R})$. Then any neutralization of $\mathrm{LMod}_R(\mathcal{C})$ can be lifted to a neutralization of $\mathcal{C}$.*

Proof. Since $\overline{R}$ is a square-zero extension of R by M, we have a pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc}
\overline{R} & \longrightarrow & R \\
\downarrow & & \downarrow \\
R & \longrightarrow & R \oplus \Sigma M.
\end{array}$$

Applying Lemma 11.5.4.11, we obtain a pullback diagram of ∞-categories

$$\begin{array}{ccc}
\mathcal{C}^c & \longrightarrow & \mathrm{LMod}_R(\mathcal{C})^c \\
\downarrow & & \downarrow \beta \\
\mathrm{LMod}_R(\mathcal{C})^c & \xrightarrow{\alpha} & \mathrm{LMod}_{R \oplus \Sigma M}(\mathcal{C})^c.
\end{array}$$

Let $X \in \mathrm{LMod}_R(\mathcal{C})^c$ be a neutralization. We first claim that X can be lifted to a compact object $\overline{X} \in \mathcal{C}$. For this, it suffices to show that $\alpha(X)$ lies in the essential image of β. Using Corollary 11.5.4.10, we can choose an R-linear equivalence $\mathrm{LMod}_R(\mathcal{C}) \simeq \mathrm{Mod}_R$, which induces an equivalence $\mathrm{LMod}_{R \oplus \Sigma M}(\mathcal{C}) \simeq \mathrm{Mod}_{R \oplus \Sigma M}$. Under the latter equivalence, we can identify $\alpha(X)$ with an invertible module L over $R \oplus \Sigma M$. Proposition 2.9.6.2 guarantees that the base change map $\mathrm{Pic}^\dagger(R) \to \mathrm{Pic}^\dagger(R \oplus \Sigma M)$ is an isomorphism of abelian groups, so that L belongs to the essential image of α.

It remains to prove that $\overline{X}$ is a neutralization of $\mathcal{C}$. Let $u : \overline{R} \to \underline{\mathrm{Map}}_{\mathcal{C}}(\overline{X}, \overline{X})$ be the unit map; we wish to show that u is an equivalence. Suppose otherwise. Then $\mathrm{cofib}(u)$

is a nonzero perfect module over $\overline{R}$. It follows that there is a smallest integer m such that $\pi_m \operatorname{cofib}(u) \neq 0$. Using Nakayama's lemma, we deduce that $\pi_m(R \otimes_{\overline{R}} \operatorname{cofib}(u)) \neq 0$, contradicting our assumption that X is a neutralization of $\mathrm{LMod}_R(\mathcal{C})$. □

Proof of Proposition 11.5.4.5. Let $\mathcal{C} \in \mathscr{B}\mathrm{r}^{\dagger}(R)$, and suppose that $\mathrm{LMod}_{R/I}(\mathcal{C})$ represents the zero element of $\mathrm{Br}^{\dagger}(R/I)$. Corollary 11.5.4.10 implies that there exists a neutralization X_1 of $\mathrm{LMod}_{R/I}(\mathcal{C})$. Applying Lemma 11.5.4.12 repeatedly, we can choose a compatible family of neutralizations $X_n \in \mathrm{LMod}_{R/I^n}(\mathcal{C})$. It follows from Theorem 11.4.4.1 that we can write $X_n \simeq R/I^n \otimes X$ for some compact object $X \in \mathcal{C}$. We will complete the proof by showing that X is a neutralization of $\mathcal{C}$. Let $u : R \to \underline{\mathrm{Map}}_{\mathcal{C}}(X, X)$ be the unit map; we wish to prove that $\operatorname{cofib}(u) \simeq 0$. Suppose otherwise. Since $\operatorname{cofib}(u)$ is a perfect R-module, there exists some smallest integer m such that $\pi_m \operatorname{cofib}(u) \neq 0$. Then $\pi_m \operatorname{cofib}(u)$ is a finitely generated module over R, and $\mathrm{Tor}_0^R(R/I, \pi_m \operatorname{cofib}(u)) \simeq \pi_m(R/I \otimes_R \operatorname{cofib}(u)) \simeq 0$. Since R is I-adically complete, it follows that $\pi_m \operatorname{cofib}(u) \simeq 0$, contrary to our choice of m. □

Proof of Proposition 11.5.4.6. Let $\mathcal{C} \in \mathscr{B}\mathrm{r}^{\dagger}(R)$, and suppose that $\mathrm{LMod}_{\pi_0 R}(\mathcal{C})$ represents the zero element of $\mathrm{Br}^{\dagger}(\pi_0 R)$. Corollary 11.5.4.10 implies that there exists a neutralization X_0 of $\mathrm{LMod}_{\pi_0 R}(\mathcal{C})$. Applying Lemma 11.5.4.12 repeatedly, we can choose a compatible family of neutralizations $X_n \in \mathrm{LMod}_{\tau_{\leqslant n} R}(\mathcal{C})$. It follows from Theorem 11.4.4.1 that we can write $X_n \simeq (\tau_{\leqslant n} R) \otimes X$ for some compact object $X \in \mathcal{C}$. We will complete the proof by showing that X is a neutralization of $\mathcal{C}$. Let $u : R \to \underline{\mathrm{Map}}_{\mathcal{C}}(X, X)$ be the unit map; we wish to prove that $\operatorname{cofib}(u) \simeq 0$. Suppose otherwise. Since $\operatorname{cofib}(u)$ is a perfect R-module, there exists some smallest integer m such that $\pi_m \operatorname{cofib}(u) \neq 0$. Then

$$0 \neq \pi_m \operatorname{cofib}(u) \simeq \pi_m(\pi_0 R \otimes_R \operatorname{cofib}(u)) \simeq 0$$

and we obtain a contradiction. □

11.5.5 Cohomological Interpretation of the Extended Brauer Group

We can now formulate the main result of this section:

Theorem 11.5.5.1. *Let X be a spectral Deligne-Mumford stack, and let $u \in \mathrm{Br}^{\dagger}(\mathsf{X})$. Then there exists an étale surjection $f : \mathsf{U} \to \mathsf{X}$ such that $f^* u = 0$ in $\mathrm{Br}^{\dagger}(\mathsf{U})$.*

Remark 11.5.5.2. Theorem 11.5.5.1 was proven by Toën in the setting of simplicial commutative rings ([211]) and by Antieau-Gepner in general ([2]). The proof we present here is slightly different: it avoids theory of higher algebraic stacks in the setting of spectral algebraic geometry, but relies on nontrivial input from commutative algebra (namely, Popescu's smoothing theorem; see Theorem **??**).

Proof of Theorem 11.5.5.1. Let X be a spectral Deligne-Mumford stack and let $u \in \mathrm{Br}^{\dagger}(\mathsf{X})$; we wish to show that u vanishes after passing to an étale cover of X. Without loss of generality that $\mathsf{X} = \mathrm{Spét}\, R$ for some connective $\mathbb{E}_{\infty}$-ring R. Using Proposition 11.5.4.6 (and Theorem HA.7.5.0.6), we can replace R by $\pi_0 R$ and thereby reduce to the case where R is discrete. Write R as a union of its finitely generated subalgebras R_{α}. Using Proposition 11.5.4.1, we may assume that u is the image of some class $u_{\alpha} \in \mathrm{Br}(R_{\alpha})$. Replacing R by R_{α}, we may suppose that R is finitely generated commutative ring.

Let $\mathfrak{m} \subseteq R$ be a maximal ideal. We will show that there exists an étale map $R \to R(\mathfrak{m})$ such that the image of u in $\mathrm{Br}^{\dagger}(R(\mathfrak{m}))$ vanishes, and $\mathfrak{m}$ lies in the image of the induced map $\theta_{\mathfrak{m}} : |\operatorname{Spec} R(\mathfrak{m})| \to |\operatorname{Spec} R|$. Assuming this, the union of the images of the maps $\theta_{\mathfrak{m}}$ is an open subset of $|\operatorname{Spec} R|$ which contains all maximal ideals, and is therefore the whole of $|\operatorname{Spec} R|$. Since $|\operatorname{Spec} R|$ is quasi-compact, we can choose a finite collection of maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ such that $|\operatorname{Spec} R| = \bigcup_{1 \leqslant i \leqslant n} \theta_{\mathfrak{m}_i}(|\operatorname{Spec} R(\mathfrak{m}_i)|)$. We may then complete the proof by taking $R' = \prod_{1 \leqslant i \leqslant n} R(\mathfrak{m}_i)$.

Let $\kappa = R/\mathfrak{m}$ and let $\overline{\kappa}$ be a separable closure of κ. Using Theorem 11.5.3.18 and Proposition 11.5.1.12, we deduce that the image of u in $\mathrm{Br}^{\dagger}(\overline{\kappa})$ vanishes. It follows from Proposition 11.5.4.1 that there exists a separable extension κ' of κ having degree $d < \infty$, such that the image of u in $\mathrm{Br}^{\dagger}(\kappa')$ vanishes. Using the primitive element theorem, we can write $\kappa' = \kappa[x]/(f(x))$, where f is a separable monic polynomial of degree d. Choose a monic polynomial $\overline{f} \in R[x]$ of degree d lifting f. Replacing R by a localization if necessary, we may suppose that $R[x]/(f(x))$ is an étale R-algebra. Replacing R by $R[x]/(f(x))$, we may reduce to the case where the image of u in $\mathrm{Br}^{\dagger}(\kappa)$ vanishes.

Let $R_{\mathfrak{m}}$ denote the localization of R at $\mathfrak{m}$, and let $R^{\wedge}_{\mathfrak{m}}$ denote the completion of $R_{\mathfrak{m}}$ with respect to its maximal ideal. It follows from Proposition 11.5.4.5 that the image of u in $\mathrm{Br}^{\dagger}(R^{\wedge}_{\mathfrak{m}})$ vanishes. Since R is a finitely generated commutative ring, it is a Grothendieck ring (Theorem **??**). It follows that the map $R \to R^{\wedge}_{\mathfrak{m}}$ is geometrically regular, so that we can write $R^{\wedge}_{\mathfrak{m}} = \varinjlim A_{\beta}$, where each A_{β} is fiber smooth over R (Theorem **??**) Using Proposition 11.5.4.1, we deduce that there exists an index β such that the image of u in $\mathrm{Br}^{\dagger}(A_{\beta})$ vanishes. Replacing R by a localization if necessary, we may suppose that the map of affine schemes $\operatorname{Spec} A_{\beta} \to \operatorname{Spec} R$ is smooth and faithfully flat, and therefore admits an étale section. That is, we can find a map $A_{\beta} \to R'$, where R' is étale and faithfully flat over R. By construction the image of u vanishes in $\mathrm{Br}^{\dagger}(R')$. □

Let us now describe some consequences of Theorem 11.5.5.1.

Construction 11.5.5.3 (The Extended Brauer Sheaf)**.** Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack. For each object $U \in \mathcal{X}$, let X_U denote the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}_{\mathsf{X}}|_U)$. The construction $U \mapsto \mathscr{B}\mathrm{r}(\mathsf{X}_U)$ determines a functor $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{S}}$ which preserves small limits. Note that if U is affine, then Remark 11.5.3.13 implies that $\mathscr{B}\mathrm{r}(\mathsf{X}_U)$

is essentially small. It follows that $\mathscr{B}\mathrm{r}(\mathsf{X}_U)$ is essentially small for each $U \in \mathcal{X}$, so that that the functor $U \mapsto \mathscr{B}\mathrm{r}^{\dagger}(\mathsf{X}_U)$ is representable by an object $\underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}} \in \mathcal{X}$. We will refer to $\underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}}$ as the *extended Brauer sheaf* of X.

Note that $\underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}}$ is a group-like commutative monoid of $\mathcal{X}$. In particular, we can view each homotopy group $\pi_n \underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}}$ as an abelian group object in the topos $\mathcal{X}^{\heartsuit}$ of discrete objects of $\mathcal{X}$. Using Theorem 11.5.5.1, Remark 11.5.2.11, and Remark 2.9.5.8, we obtain isomorphisms

$$\pi_n \underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}} = \begin{cases} 0 & \text{if } n = 0 \\ \underline{\mathbf{Z}} & \text{if } n = 1 \\ (\pi_0 \mathscr{O}_{\mathsf{X}})^{\times} & \text{if } n = 2 \\ \pi_{n-2} \mathscr{O}_{\mathsf{X}} & \text{if } n > 2, \end{cases}$$

where $\underline{\mathbf{Z}}$ denotes the constant sheaf associated to the abelian group $\mathbf{Z}$.

Remark 11.5.5.4. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack. Then the invertible sheaf $\Sigma\,\mathscr{O}_{\mathsf{X}}$ determines a global section of the sheaf $\Omega \underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}}$, which we can identify with a map $\alpha : K(\underline{\mathbf{Z}}, 1) \to \underline{\mathscr{B}\mathrm{r}}^{\dagger}_{X}$ of pointed objects of $\mathcal{X}$; here $K(\underline{\mathbf{Z}}, 1)$ denotes the constant sheaf on X whose value is the space $K(\mathbf{Z}, 1) \simeq S^1$. Note that α can be identified with a section of the truncation map $\underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}} \to \tau_{\leqslant 1} \underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}} \simeq K(\underline{\mathbf{Z}}, 1)$. Using the monoid structure on $\underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}}$, we see that α determines an equivalence

$$\tau_{\geqslant 2} \underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}} \times K(\underline{Z}, 1) \to \underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}}.$$

Beware that this equivalence is *not* compatible with the monoid structure on $\underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}}$, because the map α is not multiplicative.

Example 11.5.5.5. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack, and suppose that the structure sheaf $\mathscr{O}_{\mathsf{X}}$ is 0-truncated. Then we can identify $\mathscr{O}_{\mathsf{X}}$ with a commutative ring object of the topos of discrete objects of $\mathcal{X}$; let $\mathscr{O}_{\mathsf{X}}^{\times}$ denote its group of units. Combining the analysis of Construction 11.5.5.3 with Remark 11.5.5.4, we obtain an equivalence $\underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}} \simeq K(\mathscr{O}_{\mathsf{X}}^{\times}, 2) \times K(\underline{\mathbf{Z}}, 1)$ in the ∞-topos $\mathcal{X}$. Passing to global sections, we obtain a bijection $\mathrm{Br}^{\dagger}(\mathsf{X}) \simeq \mathrm{H}^2(\mathcal{X}; \mathscr{O}_{\mathsf{X}}^{\times}) \times \mathrm{H}^1(\mathcal{X}; \underline{\mathbf{Z}})$. It is not hard to see that this bijection is an isomorphism of abelian groups. In particular, if $\mathrm{H}^1(\mathcal{X}; \underline{\mathbf{Z}})$ vanishes (for example, if X is a quasi-compact, quasi-separated spectral algebraic space which is normal and locally Noetherian; see Theorem **??**), then we obtain an isomorphism $\mathrm{Br}^{\dagger}(\mathsf{X}) \simeq \mathrm{H}^2(\mathcal{X}; \mathscr{O}_{\mathsf{X}}^{\times})$.

Using Theorem 11.5.5.1, we can prove the following refinement of Proposition 11.5.4.6:

Proposition 11.5.5.6. *Let R be a connective $\mathbb{E}_{\infty}$-ring. Then the canonical map $\mathrm{Br}^{\dagger}(R) \to \mathrm{Br}^{\dagger}(\pi_0 R)$ is an isomorphism.*

Proof. Let $\mathcal{X} = \mathcal{S}\mathrm{hv}_R^{\text{ét}}$. For each integer n, let $\mathscr{F}_n$ denote the Brauer sheaf of $\operatorname{Sp\acute{e}t} \tau_{\leqslant n} R$, and regard $\mathscr{F}_n$ as a grouplike commutative monoid object of $\mathcal{X}$. Let $\mathscr{F}$ denote the Brauer sheaf of $\operatorname{Sp\acute{e}t} R$, and let $\mathscr{F}_\infty = \varprojlim \mathscr{F}_n$, so that we have maps

$$\mathscr{F} \xrightarrow{\alpha} \mathscr{F}_\infty \xrightarrow{\beta} \mathscr{F}_0 .$$

We will prove the following:

(a) For every étale R-algebra R', the map β induces an isomorphism $\pi_0(\mathscr{F}_\infty(R')) \to \pi_0(\mathscr{F}_0(R'))$.

(b) The map α is an equivalence.

Assuming (a) and (b), we deduce that the composition of α and β induces an isomorphism

$$\mathrm{Br}^\dagger(R) \simeq \pi_0 \mathscr{F}(R) \xrightarrow{\alpha} \pi_0 \mathscr{F}_\infty(R) \xrightarrow{\beta} \pi_0 \mathscr{F}_0(R) \simeq \mathrm{Br}^\dagger(\pi_0 R).$$

We first prove (a). Note that we have an exact sequence

$$\lim{}^1\{\pi_1 \mathscr{F}_n(R')\} \to \pi_0 \mathscr{F}_\infty(R') \to \lim{}^0\{\pi_0 \mathscr{F}_n(R')\} \to 0$$

where the left hand side vanishes by virtue of Proposition 2.9.6.2. Consequently, to prove (a), it will suffice to show that each of the maps $\theta : \pi_0 \mathscr{F}_n(R') \to \pi_0 \mathscr{F}_{n-1}(R')$ is an isomorphism. Enlarging R if necessary, we may suppose that $R' = R$. Let $\mathscr{O}$ denote the structure sheaf of $\operatorname{Sp\acute{e}t} R$. Using the description of the homotopy sheaves of $\mathscr{F}_n$ and $\mathscr{F}_{n-1}$ supplied by Construction 11.5.5.3, we deduce the existence of a fiber sequence

$$\mathscr{F}_n \to \mathscr{F}_{n-1} \to K(\pi_n \mathscr{O}, n+3)$$

in the ∞-topos $\mathcal{X}$, so that we have a short exact sequence

$$\mathrm{H}^{n+2}(\mathcal{X}; \pi_n \mathscr{O}) \to \pi_0 \mathscr{F}_n(R) \xrightarrow{\theta} \pi_0 \mathscr{F}_{n-1}(R) \to \mathrm{H}^{n+3}(\mathcal{X}; \pi_n \mathscr{O}).$$

The desired result now follows from the vanishing of $\mathrm{H}^i(\mathcal{X}; \pi_n \mathscr{O}) \simeq 0$ for $i > 0$.

We now prove (b). Using (a), we deduce that $\pi_0 \mathscr{F}_\infty \simeq \pi_0 \mathscr{F}_0$. The map α induces an isomorphism $\pi_0 \mathscr{F} \to \pi_0 \mathscr{F}_\infty$ (both sides vanish by Theorem 11.5.5.1). Consequently, to prove that α is an equivalence, it will suffice to show that α induces an equivalence of loop objects $\Omega \mathscr{F} \to \Omega \mathscr{F}_\infty \simeq \varprojlim \Omega \mathscr{F}_n$. This follows immediately from Remark 11.5.2.11 and Corollary 2.9.6.3. □

11.5.6 Digression: Connectivity of Compact Objects

Let R be a connective $\mathbb{E}_\infty$-ring and let M be an R-module spectrum. If M is connective, then the fiber $\kappa \otimes_R M \in \mathrm{Mod}_\kappa$ is connective for every residue field κ of R. If M is a perfect R-module, then the converse holds: if M is not connective, then there exists some largest integer $n > 0$ such that $\pi_{-n} M \neq 0$. Then $\pi_{-n} M$ is a finitely generated module over $\pi_0 R$, so Nakayama's lemma implies that there exists a residue field κ of R for which $0 \neq \mathrm{Tor}_0^{\pi_0 R}(\kappa, \pi_{-n} M) \simeq \pi_{-n}(\kappa \otimes_R M)$. We close this section by establishing a "relative version" of this observation, which will be useful in §11.5.7:

Theorem 11.5.6.1. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a compactly generated prestable R-linear ∞-category, and let $C \in \mathrm{Sp}(\mathcal{C})$ be a compact object. Assume that the commutative ring $\pi_0 R$ is Noetherian. Then the following conditions are equivalent:*

(1) *The object C belongs to $\mathrm{Sp}(\mathcal{C})_{\geq 0}$.*

(2) *For every residue field κ of $\pi_0 R$, the object $\kappa \otimes_R C$ belongs to $(\kappa \otimes_R \mathrm{Sp}(\mathcal{C}))_{\geq 0}$.*

The proof of Theorem 11.5.6.1 will require some preliminaries. We first show that (under very mild assumptions) the question can be reduced to the case where R is discrete:

Proposition 11.5.6.2. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be an R-linear prestable ∞-category, and set $\mathcal{C}' = (\pi_0 R) \otimes_R \mathcal{C}$. Let C be an object of $\mathrm{Sp}(\mathcal{C})$ which belongs to $\mathrm{Sp}(\mathcal{C})_{\geq -n}$ for some integer n. Then C belongs to $\mathrm{Sp}(\mathcal{C})_{\geq 0}$ if and only if the image of C in $\mathrm{Sp}(\mathcal{C}')$ belongs to $\mathrm{Sp}(\mathcal{C}')_{\geq 0}$.*

Remark 11.5.6.3. In the situation of Proposition 11.5.6.2, the assumption that C belongs to $\mathrm{Sp}(\mathcal{C})_{\geq -n}$ for some integer n is automatically satisfied if C is a compact object of $\mathrm{Sp}(\mathcal{C})$: the right completeness of $\mathrm{Sp}(\mathcal{C})$ guarantees that C can be realized as the colimit of the sequence of truncations

$$\cdots \rightarrow \tau_{\geq 0} C \rightarrow \tau_{\geq -1} C \rightarrow \tau_{\geq -2} C \rightarrow \cdots,$$

so that the compactness of C implies that C is a retract of $\tau_{\geq -n} C$ for some n.

Proof of Proposition 11.5.6.2. The "only if" direction is obvious. To prove the converse, we show that $C \in \mathrm{Sp}(\mathcal{C})_{\geq -m}$ for every nonnegative integer m using descending induction on m. To carry out the inductive step, we note if $C \in \mathrm{Sp}(\mathcal{C})_{\geq -m}$, then $(\tau_{\geq 1} R) \otimes_R C$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geq 1-m}$, so that we have an exact sequence (in the abelian category $\mathcal{C}^{\heartsuit}$)

$$0 = \pi_{-m}((\tau_{\geq 1} R) \otimes_R C) \rightarrow \pi_{-m} C \rightarrow \pi_{-m}((\pi_0 R) \otimes_R C).$$

Our hypothesis guarantees that $(\pi_0 R) \otimes_R C$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geq 0}$, so that $\pi_{-m}((\pi_0 R) \otimes_R C)$ vanishes for $m > 0$. it follows that $\pi_{-m} C \simeq 0$ and therefore $C \in \mathcal{C}_{\geq 1-m}$. □

Proposition 11.5.6.4. *Let R be a commutative ring (which we regard as a discrete $\mathbb{E}_2$-ring), let $\mathcal{C}$ be a compactly generated prestable R-linear ∞-category. Let $C \in \mathrm{Sp}(\mathcal{C})$ be a compact object and let x be an element of R. Then C belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ if and only if both of the objects $R[x^{-1}] \otimes_R C$ and $R/(x) \otimes_R C$ belong to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$.*

Proof. The "only if" direction is clear. Conversely, suppose that both $R[x^{-1}] \otimes_R C$ and $R/(x) \otimes_R C$ belong to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. For every object $D \in \mathrm{Sp}(\mathcal{C})$, set $D[x^{-1}] = R[x^{-1}] \otimes_R D \in \mathrm{LMod}_{R[x^{-1}]}(\mathrm{Sp}(\mathcal{C}))$. Note that the object C belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ if and only if the direct sum $C \oplus \Sigma C$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Using our assumption that $C[x^{-1}]$ is a compact object of $\mathrm{LMod}_{R[x^{-1}]}(\mathcal{C})$, we see that (after replacing C by $C \oplus \Sigma C$ if necessary) we may assume that there exists a compact object $D \in \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ with $\alpha : D[x^{-1}] \simeq C[x^{-1}]$ (Proposition D.5.3.4). Arguing as in Proposition D.5.3.4, we see that after multipliying the equivalence α by a suitable power of x, we can arrange that α arises from a morphism $\alpha_0 : D \to C$ in $\mathrm{Sp}(\mathcal{C})$. Since the full subcategory $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ is closed under extensions, it will suffice to prove that $\mathrm{cofib}(\alpha_0)$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Replacing C by $\mathrm{cofib}(\alpha_0)$, we can reduce to the case where $C[x^{-1}] \simeq 0$.

The assumption that $C[x^{-1}] \simeq 0$ implies that the canonical map $C \to C[x^{-1}]$ is nullhomotopic. Since C is a compact object of $\mathrm{Sp}(\mathcal{C})$, it follows that the map $x^n : C \to C$ is nullhomotopic for $n \gg 0$. Let $M \in \mathrm{Mod}_R$ denote the cofiber of the map $x^n : R \to R$. Then

$$M \otimes_R C \simeq \mathrm{cofib}(x^n : C \to C) \simeq C \oplus \Sigma(C).$$

It will therefore suffice to show that $M \otimes_R C$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Note that M can be written as a successive extension of finitely many copies of $\mathrm{cofib}(x : R \to R)$. We may therefore reduce to the case $n = 1$. Note that $M \simeq R \otimes_{R[X]} R$ has the structure of a connective $\mathbb{E}_\infty$-algebra over R, with $\pi_0 M \simeq R/(x)$. Our assumption that $R/(x) \otimes_R C$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ implies that $M \otimes_R C$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ by virtue of Proposition 11.5.6.2. □

Proof of Theorem 11.5.6.1. Using Proposition 11.5.6.2 (and Remark 11.5.6.3), we can reduce to the case where R is discrete. Suppose that $C \notin \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Since R is Noetherian, there exists an ideal which is maximal among those ideals I for which $R/I \otimes_R C$ does not belong to $(R/I \otimes_R \mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$. Replacing R by R/I, we may assume that $I = (0)$.

If $R \simeq 0$, there is nothing to prove. Otherwise, we can choose an associated prime ideal $\mathfrak{p}$ of R which appears as the annihilator of a nonzero element $x \in R$. In this case, we have a short exact sequence of discrete R-modules $0 \to R/\mathfrak{p} \to R \to R/(x) \to 0$. It follows from our inductive hypothesis that $R/(x) \otimes_R C$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Since $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ is closed under extensions, we are reduced to proving that $R/\mathfrak{p} \otimes_R C$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. This follows from our inductive hypothesis unless $\mathfrak{p} = (0)$. We may therefore assume without loss of generality that R is an integral domain.

Let κ denote the fraction field of R, so that $\kappa \otimes_R C$ is belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Let $\mathcal{E} \subseteq (\kappa \otimes_R \mathcal{C})$ denote the full subcategory spanned by objects of the form $\kappa \otimes_R D$, where D is a compact object of $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Note that if D is a compact object of $\mathrm{Sp}(\mathcal{C})$ and $D' \in \mathrm{Sp}(\mathcal{C})$ is arbitrary, then the mapping space

$$\mathrm{Map}_{\kappa \otimes_R \mathrm{Sp}(\mathcal{C})}(\kappa \otimes_R D, \kappa \otimes_R D') \simeq \mathrm{Map}_{\mathcal{C}}(D, \kappa \otimes_R D')$$

can be identified with a filtered colimit of copies of the mapping space $\mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(D, D')$, indexed by the nonzero elements of R. Arguing as in the proof of Proposition D.5.3.4, we see that $\mathcal{E}$ is closed under the formation of cofibers and extensions, so that every compact object of $\kappa \otimes_R \mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ can be written as a retract of an object of $\mathcal{E}$. In particular, we can write $(\kappa \otimes_R C) \oplus E \simeq \kappa \otimes_R D$ for some compact object $D \in \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Let f be the endomorphism of $\kappa \otimes_R D$ which is nullhomotopic on $\kappa \otimes_R C$ and an equivalence on E. Since D is compact, we may assume that f is induced by an endomorphism $f_0 : D \to D$ in the ∞-category $\mathrm{Sp}(\mathcal{C})$. Then $\kappa \otimes_R \mathrm{cofib}(f_0) \simeq \kappa \otimes_R (C \oplus \Sigma(C))$.

Replacing C by $C \oplus \Sigma(C)$, we may reduce to the case where there exists a compact object $C' \in \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ and an equivalence $g : \kappa \otimes_R C' \simeq \kappa \otimes_R C$. Since C' is compact, we can assume (after multiplying g by a nonzero element of R) that g is induced by a map $g_0 : C' \to C$. Form a fiber sequence $C' \xrightarrow{g_0} C \to C''$. Then C belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ if and only if C'' belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Replacing C by C'', we may assume that $\kappa \otimes_R C \simeq 0$. It follows that the natural map $C \to \kappa \otimes_R C$ is nullhomotopic. Since C is compact, it follows that there exists a nonzero element $x \in R$ such that the map $x : C \to C$ is nullhomotopic. In this case, C can be identified with a direct summand of $R/(x) \otimes_R C \simeq C \oplus \Sigma(C)$, which belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ by virtue of our inductive hypothesis. □

11.5.7 The Brauer Group

Let X be a spectral Deligne-Mumford stack. In §11.5.2 we introduced the *extended Brauer space* $\mathscr{B}\mathrm{r}^{\dagger}(\mathsf{X})$, an infinite loop space whose first homotopy group $\pi_1 \mathscr{B}\mathrm{r}^{\dagger}(\mathsf{X})$ can be identified with the extended Picard group $\mathrm{Pic}^{\dagger}(\mathsf{X})$ of invertible objects of $\mathrm{QCoh}(\mathsf{X})$. In this section, we will discuss a variant of $\mathscr{B}\mathrm{r}^{\dagger}(\mathsf{X})$ which bears an analogous relation to the usual Picard group $\mathrm{Pic}(\mathsf{X})$ of line bundles on X.

Definition 11.5.7.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and let $\mathrm{QStk}^{\mathrm{PSt}}(X)$ denote the ∞-category of prestable quasi-coherent stacks on X. We let $\mathscr{B}\mathrm{r}(X)$ denote the full subcategory of $\mathrm{QStk}^{\mathrm{PSt}}(X)^{\simeq}$ spanned by the invertible quasi-coherent stacks $\mathcal{C} \in \mathrm{QStk}^{\mathrm{PSt}}(X)$ for which $\mathcal{C}$ and $\mathcal{C}^{-1}$ are compactly generated. We will refer to $\mathscr{B}\mathrm{r}(X)$ as the *Brauer space* of X. We let $\mathrm{Br}(X)$ denote the set $\pi_0 \mathscr{B}\mathrm{r}(X)$, which we will refer to as the *Brauer group* of X.

If X is a spectral Deligne-Mumford stack representing a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, we will denote the Brauer space $\mathscr{B}\mathrm{r}(X)$ and the Brauer group $\mathrm{Br}(X)$ by $\mathscr{B}\mathrm{r}(\mathsf{X})$ and $\mathrm{Br}(\mathsf{X})$, respectively. If R is a connective $\mathbb{E}_\infty$-ring, we will denote $\mathscr{B}\mathrm{r}(\mathrm{Spét}\, R)$ and $\mathrm{Br}(\mathrm{Spét}\, R)$ by $\mathscr{B}\mathrm{r}(R)$ and $\mathrm{Br}(R)$, respectively.

Remark 11.5.7.2. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Then the Brauer space $\mathscr{B}\mathrm{r}(X)$ admits the structure of a symmetric monoidal ∞-category in which every object and every morphism is invertible. In other words, it is a grouplike $\mathbb{E}_\infty$-space. In particular, the set $\mathrm{Br}(X) = \pi_0 \mathscr{B}\mathrm{r}(X)$ has the structure of an abelian group.

Remark 11.5.7.3. Let R be a connective $\mathbb{E}_\infty$-ring. Then the stabilization construction $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ determines a symmetric monoidal functor $\mathrm{LinCat}_R^{\mathrm{PSt}} \to \mathrm{LinCat}_R^{\mathrm{St}}$ which preserves compactly generated objects, and therefore induces a map of $\mathbb{E}_\infty$-spaces $\theta : \mathscr{B}\mathrm{r}(R) \to \mathscr{B}\mathrm{r}^\dagger(R)$.

Let $\mathcal{C}$ be a stable R-linear ∞-category which is compactly generated and invertible. Using Proposition C.3.1.1, we see that the homotopy fiber $\theta^{-1}\{\mathcal{C}\}$ is discrete: more precisely, it can be identified with the collection of all t-structures $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ on $\mathcal{C}$ satisfying the following conditions:

- (i) The t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ is right complete and compatible with filtered colimits.
- (ii) The Grothendieck prestable ∞-category $\mathcal{C}_{\geqslant 0}$ is an invertible object of $\mathrm{LinCat}_R^{\mathrm{PSt}}$.
- (iii) The Grothendieck prestable ∞-category $\mathcal{C}_{\geqslant 0}$ and its inverse $\mathcal{C}_{\geqslant 0}^{-1}$ are compactly generated.

Note that condition (iii) implies that $\mathcal{C}_{\geqslant 0}$ is determined by its intersection with the ∞-category of compact objects of $\mathcal{C}$. In particular, we deduce that the homotopy fibers of θ are essentially small. Since $\mathscr{B}\mathrm{r}^\dagger(R)$ is essentially small (Remark 11.5.3.13), it follows that the Brauer space $\mathscr{B}\mathrm{r}(R)$ is essentially small.

Remark 11.5.7.4. Let X be a spectral Deligne-Mumford stack. Then the canonical equivalence $\Omega \mathscr{B}\mathrm{r}^\dagger(\mathsf{X}) \simeq \mathscr{P}\mathrm{ic}^\dagger(\mathsf{X})$ induces a map $\Omega \mathscr{B}\mathrm{r}(\mathsf{X}) \to \mathscr{P}\mathrm{ic}^\dagger(\mathsf{X})$ which exhibits $\Omega \mathscr{B}\mathrm{r}(\mathsf{X})$ as a summand of $\mathscr{P}\mathrm{ic}^\dagger(\mathsf{X})$. Unwinding the definitions, we see that this summand consists of those invertible quasi-coherent sheaves $\mathscr{L} \in \mathrm{QCoh}(\mathsf{X})$ having the property that for every map $\eta : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$, tensor product with $\eta^* \mathscr{L}$ induces a t-exact equivalence from the ∞-category $\mathrm{Mod}_R^{\mathrm{cn}}$ to itself. We therefore obtain a homotopy equivalence $\Omega \mathscr{B}\mathrm{r}(\mathsf{X}) \simeq \mathscr{P}\mathrm{ic}(\mathsf{X})$, where $\mathscr{P}\mathrm{ic}(\mathsf{X})$ is the Picard space of X (see Definition 2.9.4.1). In particular, we can identify $\pi_1 \mathscr{B}\mathrm{r}(\mathsf{X})$ with the Picard group $\mathrm{Pic}(\mathsf{X})$.

In §11.5.3, we studied the relationship between the extended Brauer group $\mathrm{Br}^\dagger(X)$ and the theory of Azumaya algebras on X. There is an analogous (but somewhat weaker) connection between $\mathrm{Br}(X)$ and connective Azumaya algebras. We begin with an analogue of Proposition 11.5.3.4.

Lemma 11.5.7.5. *Let R be a connective $\mathbb{E}_\infty$-ring, let $A \in \mathrm{Alg}_R^{\mathrm{cn}}$, and let M be a connective right A-module. Let $G : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathrm{RMod}_A^{\mathrm{cn}}$ be the R-linear functor given by $G(N) = N \otimes_R M$. The following conditions are equivalent:*

(1) *The functor G is an equivalence of ∞-categories.*

(2) *As an R-module, M is locally free of rank > 0 and the action of A on M induces an equivalence $A^{\mathrm{rev}} \to \mathrm{End}_R(M)$.*

Proof. Suppose first that (2) is satisfied. Then M is a compact generator of the stable ∞-category Mod_R. Applying Lemma **??**, we deduce that the construction $N \mapsto N \otimes_R M$ induces an equivalence of stable ∞-categories $\mathrm{Mod}_R \simeq \mathrm{RMod}_A$. Since an R-module N is connective if and only if $N \otimes_R M$ is connective, it follows that G is an equivalence of ∞-categories.

Now suppose that (1) is satisfied. Then the induced map $\mathrm{Sp}(G) : \mathrm{Mod}_R \to \mathrm{RMod}_A$ is an equivalence of stable ∞-categories. Applying Lemma **??**, we deduce that M is a perfect R-module and that the canonical map $A^{\mathrm{rev}} \to \mathrm{End}_R(M)$ is an equivalence. We will complete the proof by showing that the R-module M is locally free of rank > 0. By virtue of Proposition 2.9.3.2, it will suffice to show that $\kappa \otimes_R M$ is locally free of rank > 0, for every residue field κ of R. We may therefore replace R by κ and thereby reduce to the case where $R = \kappa$ is a field.

Note that R does not belong to the essential image of the suspension functor $\Sigma : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathrm{Mod}_R^{\mathrm{cn}}$. Using assumption (1), we deduce that $M = G(R)$ does not belong to the essential image of the suspension functor $\Sigma : \mathrm{RMod}_A^{\mathrm{cn}} \to \mathrm{RMod}_A^{\mathrm{cn}}$. It follows that $\pi_0 M \neq 0$. If M is not locally free as an R-module, then we must have $\pi_n M \neq 0$ for some $n > 0$. It then follows that $\pi_{-n}(M^\vee) \simeq (\pi_n M)^\vee \neq 0$ and therefore $\pi_{-n}(M \otimes_\kappa M^\vee) \neq 0$. This is a contradiction, since the tensor product $M \otimes_\kappa M^\vee \simeq A^{\mathrm{rev}}$ is connective by assumption. □

Proposition 11.5.7.6. *Let R be a connective $\mathbb{E}_\infty$-ring and let $A \in \mathrm{Alg}_R^{\mathrm{cn}}$. Then A is an Azumaya algebra if and only if $\mathrm{RMod}_A^{\mathrm{cn}}$ is an invertible object of $\mathrm{LinCat}_R^{\mathrm{PSt}}$.*

Proof. If $\mathrm{RMod}_A^{\mathrm{cn}}$ is an invertible object of $\mathrm{LinCat}_R^{\mathrm{PSt}}$, then $\mathrm{Sp}(\mathrm{RMod}_A^{\mathrm{cn}}) \simeq \mathrm{RMod}_A$ is an invertible object of $\mathrm{LinCat}_R^{\mathrm{St}}$, so that A is an Azumaya algebra by virtue of Proposition 11.5.3.4. Conversely, if suppose that A is an Azumaya algebra. Then we have an equivalence $\mathrm{RMod}_A^{\mathrm{cn}} \simeq \mathrm{RMod}_{A^{\mathrm{rev}}}^{\mathrm{cn}} \simeq \mathrm{RMod}_{A \otimes_R A^{\mathrm{rev}}}^{\mathrm{cn}} \simeq \mathrm{RMod}_{\mathrm{End}_R(A)}^{\mathrm{cn}}$. Since A is a locally free R-module of rank > 0 (Proposition 11.5.3.3), Lemma 11.5.7.5 supplies an R-linear equivalence $\mathrm{RMod}_{\mathrm{End}_R(A)}^{\mathrm{cn}} \simeq \mathrm{Mod}_R^{\mathrm{cn}}$. □

We now introduce a connective analogue of Construction 11.5.7.7:

Construction 11.5.7.7 (The Brauer Class of a Connective Azumaya Algebra)**.** Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and let $\mathscr{A} \in \mathrm{Alg}(\mathrm{QCoh}(X)^{\mathrm{cn}})$ be a connective Azumaya algebra. Then the construction

$$(\eta \in X(R)) \mapsto (\mathrm{RMod}_{\mathscr{A}_\eta}^{\mathrm{cn}} \in \mathrm{LinCat}_R^{\mathrm{PSt}})$$

determines a compactly generated prestable quasi-coherent stack on X. It follows from Proposition 11.5.7.6 that this quasi-coherent stack is an invertible object of $\mathrm{LinCat}_R^{\mathrm{PSt}}$ (whose inverse is also compactly generated), and can therefore be identified with a point of the Brauer space $\mathscr{B}\mathrm{r}(X)$. We let $[\mathscr{A}] \in \mathrm{Br}(X)$ denote the equivalence class of this quasi-coherent stack. We will refer to $[\mathscr{A}]$ as the *Brauer class of $\mathscr{A}$*.

Warning 11.5.7.8. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and let $\mathscr{A}$ be a connective Azumaya algebra on X. We have now assigned two different meanings to the symbol $[\mathscr{A}]$: it can be regarded as an element of the extended Brauer group $\mathrm{Br}^{\dagger}(X)$ (Construction 11.5.3.9) or as an element of the Brauer group $\mathrm{Br}(X)$ (Construction **??**). In practice, there is little danger of ambiguity: the canonical map $\mathrm{Br}(X) \to \mathrm{Br}^{\dagger}(X)$ (see Remark 11.5.7.3) carries the Brauer class of $\mathscr{A}$ to the extended Brauer class of $\mathscr{A}$.

Proposition 11.5.7.9. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and let $\mathscr{A} \in \mathrm{Alg}(\mathrm{QCoh}(X))$. The following conditions are equivalent:*

(1) *The object $\mathscr{A}$ is a connective Azumaya algebra and the Brauer class $[\mathscr{A}] \in \mathrm{Br}(X)$ vanishes.*

(2) *There exists an equivalence $\mathscr{A} \simeq \mathrm{End}(\mathscr{E}) = \mathscr{E} \otimes \mathscr{E}^{\vee}$, where $\mathscr{E} \in \mathrm{Vect}(X)$ is a vector bundle on X of rank > 0.*

Proof. We proceed as in the proof of Proposition 11.5.3.11. Let $\mathcal{C}$ denote the prestable quasi-coherent stack on X given by the construction $(\eta \in X(R)) \mapsto (\mathrm{RMod}^{\mathrm{cn}}_{\eta^* \mathscr{A}} \in \mathrm{LinCat}_R^{\mathrm{PSt}})$. Using Proposition 11.5.7.6, we see that condition (1) is equivalent to the following:

(1′) There exists a map $\rho : \mathcal{Q}_X^{\mathrm{PSt}} \to \mathcal{C}$ which is an equivalence of prestable quasi-coherent stacks on X.

Note that giving a map of quasi-coherent stacks $\rho : \mathcal{Q}_X^{\mathrm{PSt}} \to \mathcal{C}$ is equivalent to giving an object $\mathscr{G} \in \mathrm{QCoh}(X; \mathcal{C}) \simeq \mathrm{RMod}^{\mathrm{cn}}_{\mathscr{A}}(\mathrm{QCoh}(X))$. Using Lemma 11.5.7.5, we see that the map ρ is an equivalence if and only if $\mathscr{G}$ is a vector bundle of rank > 0 on X and ρ induces an equivalence $\mathscr{A} \simeq \mathrm{End}(\mathscr{G})^{\mathrm{rev}} \simeq \mathrm{End}(\mathscr{G}^{\vee})$. Setting $\mathscr{E} = \mathscr{G}^{\vee}$, we see that conditions (1) and (2) are equivalent. □

Remark 11.5.7.10. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Using Construction 11.5.3.9 and Proposition 11.5.7.9, we arrive at the following analogue of Remark **??**:

(a) Every connective Azumaya algebra $\mathscr{A}$ on X determines an element $[\mathscr{A}] \in \mathrm{Br}(X)$.

(b) The collection of elements of $\mathrm{Br}(X)$ having the form $[\mathscr{A}]$ comprise a subgroup of $\mathrm{Br}(X)$. More explicitly, the multiplication on $\mathrm{Br}(X)$ is given by $[\mathscr{A}][\mathscr{B}] = [\mathscr{A} \otimes \mathscr{B}]$, the identity element of $\mathrm{Br}^{\dagger}(X)$ is given by $[\mathscr{O}_X]$, and the inverse of $[\mathscr{A}]$ is given by $[\mathscr{A}^{\mathrm{rev}}]$.

(c) Given connective Azumaya algebras $\mathscr{A}, \mathscr{B} \in \mathrm{Alg}(\mathrm{QCoh}(X))$, we have $[\mathscr{A}] = [\mathscr{B}]$ in $\mathrm{Br}(X)$ if and only if there is an equivalence $\mathscr{A} \otimes \mathscr{B}^{\mathrm{rev}} \simeq \mathrm{End}(\mathscr{E})$ for some vector bundle $\mathscr{E} \in \mathrm{Vect}(X)$ of positive rank (in this case, we can regard $\mathscr{E}$ as a $\mathscr{A}$-$\mathscr{B}$ bimodule object of $\mathrm{Vect}(X)$).

In contrast with Remark **??**, it is generally not true that every element of $\mathrm{Br}(X)$ has the form $[\mathscr{A}]$ for some connective Azumaya algebra $\mathscr{A} \in \mathrm{Alg}(\mathrm{QCoh}(X))$, even if we assume that X is affine. For example, if X is an ordinary scheme, then the subgroup described in (b) coincides with the Brauer-Grothendieck group of X, while the group $\mathrm{Br}(X)$ agrees with the cohomological Brauer group $\mathrm{H}^2_{\text{ét}}(X; \mathbf{G}_m)$ (see Example 11.5.7.15).

Our main result is the following analogue of Theorem 11.5.5.1:

Theorem 11.5.7.11. *Let X be a spectral Deligne-Mumford stack, and let $u \in \mathrm{Br}(\mathsf{X})$. Then there exists an étale surjection $f : \mathsf{U} \to \mathsf{X}$ such that $f^* u = 0$ in $\mathrm{Br}(\mathsf{U})$.*

Before giving the proof of Theorem 11.5.7.11, let us describe some of its consequences.

Corollary 11.5.7.12. *Let X be a spectral Deligne-Mumford stack and let $\mathscr{A} \in \mathrm{Alg}(\mathrm{QCoh}(\mathsf{X}))$ be a connective Azumaya algebra on X. Then there exists an étale surjection $f : \mathsf{U} \to \mathsf{X}$ and an equivalence $f^* \mathscr{A} \simeq \mathrm{End}(\mathscr{E})$, where $\mathscr{E}$ is a vector bundle on U of rank > 0.*

Proof. Combine Theorem 11.5.7.11 with Proposition 11.5.7.9. □

Construction 11.5.7.13 (The Brauer Sheaf)**.** Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack. For each object $U \in \mathcal{X}$, let X_U denote the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathscr{O}_{\mathsf{X}}|_U)$. The construction $U \mapsto \mathscr{B}\mathrm{r}(\mathsf{X}_U)$ determines a functor $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{S}}$ which preserves small limits. Remark 11.5.7.3 implies that $\mathscr{B}\mathrm{r}(\mathsf{X}_U)$ is essentially small whenever U is affine, so that $\mathscr{B}\mathrm{r}(\mathsf{X}_U)$ is essentially small for all objects $U \in \mathcal{X}$. It follows that the construction $U \mapsto \mathscr{B}\mathrm{r}(\mathsf{X}_U)$ is representable by an object $\underline{\mathscr{B}\mathrm{r}}_{\mathsf{X}} \in \mathcal{X}$. We will refer to $\underline{\mathscr{B}\mathrm{r}}_{\mathsf{X}}$ as the *Brauer sheaf* of X.

Note that $\underline{\mathscr{B}\mathrm{r}}_{\mathsf{X}}$ is a grouplike commutative monoid of $\mathcal{X}$. In particular, we can each homotopy group $\pi_n \underline{\mathscr{B}\mathrm{r}}_{\mathsf{X}}$ as an abelian group object in the topos $\mathcal{X}^{\heartsuit}$. Using Theorem 11.5.7.11 and Remark 11.5.7.4, we obtain isomorphisms

$$\pi_n \underline{\mathscr{B}\mathrm{r}}_{\mathsf{X}} = \begin{cases} 0 & \text{if } n \leqslant 1 \\ (\pi_0 \mathscr{O}_{\mathsf{X}})^{\times} & \text{if } n = 2 \\ \pi_{n-2} \mathscr{O}_{\mathsf{X}} & \text{if } n > 2. \end{cases}$$

Remark 11.5.7.14. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack. The forgetful functor $\mathscr{B}\mathrm{r} \to \mathscr{B}\mathrm{r}^{\dagger}$ determines a map of Brauer sheaves $\underline{\mathscr{B}\mathrm{r}}_{\mathsf{X}} \to \underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}}$. Combining the analyses of Construction 11.5.5.3 and **??**, we see that this map exhibits $\underline{\mathscr{B}\mathrm{r}}_{\mathsf{X}}$ as the 2-connective cover of $\underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}}$. In particular, we have a canonical fiber sequence $\underline{\mathscr{B}\mathrm{r}}_{\mathsf{X}} \to \underline{\mathscr{B}\mathrm{r}}^{\dagger}_{\mathsf{X}} \to K(\underline{\mathbf{Z}}, 1)$ of grouplike commutative monoid objects of $\mathcal{X}$ (which admits a non-multiplicative splitting; see Remark 11.5.5.4). Passing to global sections, we obtain a (split) short exact sequence of abelian groups $0 \to \mathrm{Br}(\mathsf{X}) \to \mathrm{Br}^{\dagger}(\mathsf{X}) \to \mathrm{H}^1(\mathcal{X}; \underline{\mathbf{Z}}) \to 0$.

Example 11.5.7.15. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack, and suppose that $\mathscr{O}_{\mathsf{X}}$ is 0-truncated. Then we can identify $\mathscr{O}_{\mathsf{X}}$ with a commutative ring object of the topos of discrete objects of $\mathcal{X}$; let $\mathscr{O}^{\times}_{\mathsf{X}}$ denote its group of units. For each $U \in \mathcal{X}$, $\mathscr{P}\mathrm{ic}(\mathsf{X}_U)$ is equivalent to the nerve of a groupoid, so that the extended Brauer space $\mathscr{B}\mathrm{r}(\mathsf{X}_U)$ is 2-truncated. It follows that the Brauer sheaf $\underline{\mathscr{B}\mathrm{r}}_{\mathsf{X}}$ is 2-truncated, so that the analysis of Construction 11.5.7.13 supplies an equivalence $\underline{\mathscr{B}\mathrm{r}}_{\mathsf{X}} \simeq K(\mathscr{O}^{\times}_{\mathsf{X}}, 2)$. Passing to global sections, we obtain an isomorphism $\mathrm{Br}(\mathsf{X}) \simeq \mathrm{H}^2(\mathcal{X}; \mathscr{O}^{\times}_{\mathsf{X}})$. In other words, the Brauer group $\mathrm{Br}(\mathsf{X})$ of Definition 11.5.7.1 agrees with the classical *cohomological Brauer group* of X.

Corollary 11.5.7.16. *Let R be a connective $\mathbb{E}_\infty$-ring. Then the canonical map $\mathrm{Br}(R) \to \mathrm{Br}(\pi_0 R)$ is an isomorphism of abelian groups.*

Proof. Combine Proposition 11.5.5.6 with Remark 11.5.7.14. □

We now turn to the proof of Theorem 11.5.7.11. We will need a few preliminary results.

Lemma 11.5.7.17. *For every $\mathbb{E}_\infty$-ring R, let $\mathrm{Alg}^{\mathrm{c}}_R$ denote the full subcategory of Alg_R spanned by the compact objects. Then the construction $R \mapsto \mathrm{Alg}^{\mathrm{c}}_R$ commutes with filtered colimits (when regarded as a functor from CAlg to $\mathcal{C}\mathrm{at}_\infty$).*

Proof. The construction $R \mapsto \mathrm{Alg}_R$ carries each $\mathbb{E}_\infty$-ring R to a compactly generated ∞-category and each morphism $R \to R'$ to an extension-of-scalars functor $\mathrm{Alg}_R \to \mathrm{Alg}_{R'}$ which preserves compact objects. By virtue of Lemma HA.7.3.5.11 , it will suffice to show that the construction $R \mapsto \mathrm{Alg}_R$ determines a functor $\rho : \mathrm{CAlg} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ which commutes with filtered colimits. Using Corollary **??**, we can identify ρ with a functor $\rho' : \mathrm{CAlg}^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{R}}$, which carries an $\mathbb{E}_\infty$-ring R to the ∞-category $\mathrm{Alg}(\mathrm{Mod}_R)$ and a morphism $R \to R'$ to the restriction-of-scalars functor $\mathrm{Alg}(\mathrm{Mod}_{R'}) \to \mathrm{Alg}(\mathrm{Mod}_R)$. To show that ρ' commutes with filtered limits, it will suffice to show that the construction $R \mapsto \mathrm{Mod}_R$ carries filtered colimits in CAlg to filtered limits in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$, which follows from Proposition 4.5.1.2. □

Lemma 11.5.7.18. *Let R be a connective $\mathbb{E}_\infty$-ring and let M be a connective perfect R-module. Suppose that:*

(a) *For every residue field κ of R, the vector space $\pi_0(\kappa \otimes_R M)$ is nonzero.*

(b) *The R-module M generates Mod_R as a presentable stable ∞-category: in other words, if $\mathcal{C} \subseteq \mathrm{Mod}_R$ is a full stable subcategory which is closed under small colimits, then $\mathcal{C} = \mathrm{Mod}_R$.*

Then M generates $(\mathrm{Mod}_R)_{\geqslant 0}$ under small colimits and extensions.

Proof. Write R as a colimit $\varinjlim_{\alpha \in A} R_\alpha$ indexed by a filtered partially ordered set A, where each R_α is a compact object of $\mathrm{CAlg}^{\mathrm{cn}}$ (and, in particular, a Noetherian $\mathbb{E}_\infty$-ring). Using Lemma 11.5.7.17, we can choose an index α, a connective perfect R_α-module M_α, and an equivalence $M \simeq R \otimes_{R_\alpha} M_\alpha$. Then $\pi_0 M_\alpha$ is a finitely generated module over $\pi_0 R_\alpha$. Let $I \subseteq \pi_0 R_\alpha$ denote its annihilator ideal. Since R_α is Noetherian, the ideal I is finitely generated. Using (a), we see that the image of I in $\pi_0 R$ is nilpotent. Enlarging α if necessary, we may assume that I is nilpotent.

For each $\beta \geqslant \alpha$, let $M_\beta = R_\beta \otimes_{R_\alpha} M_\alpha$, and let $\mathcal{E}_\beta$ denote the smallest stable subcategory of Mod_{R_β} which contains M_β. Then $\mathcal{E} = \varinjlim \mathcal{E}_\beta$ is the smallest stable subcategory of Mod_R which contains M. Since M is perfect, the inclusion $\mathcal{E} \hookrightarrow \mathrm{Mod}_R$ extends to a fully faithful embedding $\mathrm{Ind}(\mathcal{E}) \to \mathrm{Mod}_R$, and condition (b) implies that this functor is an equivalence of ∞-categories. In particular, since $R \in \mathrm{Mod}_R$ is compact, the $\mathbb{E}_\infty$-ring R itself can be written as a retract of an object of $\mathcal{E}$. It follows that there exists $\beta \geqslant \alpha$ for which R_β can be written as a retract of some object of $\mathcal{E}_\beta$. Consequently, the module $M_\beta \in \mathrm{Mod}_{R_\beta}$ satisfies conditions (a) and (b). We may therefore replace R by R_β and thereby reduce to the case where R is Noetherian.

Let $\mathcal{C}_{\geqslant 0} \subseteq \mathrm{Mod}_R$ be the full subcategory generated by M under small colimits and extensions. We wish to prove that $\mathcal{C}_{\geqslant 0} = (\mathrm{Mod}_R)_{\geqslant 0}$. Since M is connective, we clearly have $\mathcal{C}_{\geqslant 0} \subseteq (\mathrm{Mod}_R)_{\geqslant 0}$. To prove the reverse inclusion, it will suffice to show that $\mathcal{C}_{\geqslant 0}$ contains R. It follows from Proposition C.6.3.1 that $\mathcal{C}_{\geqslant 0}$ is a compactly generated prestable ∞-category. Moreover, the action of $\mathrm{Mod}_R^{\mathrm{cn}}$ on itself restricts to an action of $\mathrm{Mod}_R^{\mathrm{cn}}$ on $\mathcal{C}_{\geqslant 0}$. Applying Theorem 11.5.6.1, we are reduced to proving that $\mathcal{C}_{\geqslant 0}$ contains each residue field κ of R. This is clear, since condition (a) guarantees that κ is a retract of the tensor product $\kappa \otimes_R M$. □

Proof of Theorem 11.5.7.11. Without loss of generality, we may assume that $\mathsf{X} = \mathrm{Spét}\, R$ is affine, so that $u \in \mathrm{Br}(\mathsf{X})$ classifies a compactly generated prestable R-linear ∞-category $\mathcal{C}_{\geqslant 0} \in \mathrm{LinCat}_R^{\mathrm{PSt}}$ which admits a compactly generated inverse $\mathcal{D}_{\geqslant 0} \in \mathrm{LinCat}_R^{\mathrm{PSt}}$. Using Theorem 11.5.5.1, we can further assume that $\mathrm{Sp}(\mathcal{C}_{\geqslant 0}) \simeq \mathrm{Mod}_R$. Without loss of generality, we can identify $\mathcal{C}_{\geqslant 0}$ with its essential image under the full faithful embedding

$$\Sigma^\infty : \mathcal{C}_{\geqslant 0} \simeq \mathrm{Sp}(\mathcal{C}_{\geqslant 0}) \simeq \mathrm{Mod}_R$$

and thereby reduce to the case where $\mathcal{C}_{\geqslant 0}$ is the connective part of a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ on Mod_R; similarly, we may identify $\mathcal{D}_{\geqslant 0}$ with the connective part of a t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ on Mod_R. Because $\mathcal{C}_{\geqslant 0}$ and $\mathcal{D}_{\geqslant 0}$ are compactly generated, we can choose compact objects $\{C_\alpha\}_{\alpha \in A}$ and $\{D_\beta\}_{\beta \in B}$ of Mod_R which generate $\mathcal{C}_{\geqslant 0}$ and $\mathcal{D}_{\geqslant 0}$ under small colimits. Then the collection of tensor products $C_\alpha \otimes D_\beta$ generates $\mathcal{C}_{\geqslant 0} \otimes \mathcal{D}_{\geqslant 0} \simeq \mathrm{Mod}_R^{\mathrm{cn}}$ under small colimits. For every pair of subsets $A_0 \subseteq A$, $B_0 \subseteq B$, let $\mathcal{E}_{A_0, B_0}$ be the smallest full subcategory of Mod_R which contains each tensor product $\{C_\alpha \otimes_R D_\beta\}_{\alpha \in A_0, \beta \in B_0}$ and is closed under finite colimits and extensions. Using Proposition C.6.3.1, we deduce that the inclusion $\mathcal{E}_{A,B} \hookrightarrow \mathrm{Mod}_R$ extends to an equivalence of ∞-categories $\mathrm{Ind}(\mathcal{E}_{A,B}) \simeq \mathrm{Mod}_R^{\mathrm{cn}}$. It follows that the module R appears as a direct summand of some object $E \in \mathcal{E}_{A,B}$. Note that $\mathcal{E}_{A,B}$ can be written as a union $\bigcup_{A_0} \mathcal{E}_{A_0,B}$, where A_0 ranges over all finite subsets of A. We can therefore choose a finite subset $A_0 \subseteq A$ such that $E \in \mathcal{E}_{A_0,B}$. Let $\mathcal{C}'_{\geqslant 0}$ denote the full subcategory of Mod_R generated by the objects $\{C_\alpha\}_{\alpha \in A_0}$ under small colimits and extensions. Enlarging A_0 if necessary, we may assume that $\mathcal{C}'_{\geqslant 0}$ contains $\Sigma^n R$ for $n \gg 0$. Applying Proposition C.6.3.1, we deduce that $\mathcal{C}'_{\geqslant 0}$ is a compactly generated prestable ∞-category with stabilization Mod_R. It follows that the tensor product $\mathcal{C}'_{\geqslant 0} \otimes_R \mathcal{D}_{\geqslant 0}$ is also a Grothendieck prestable ∞-category (Theorem C.4.2.1), so it embeds fully faithfully into its stabilization

$$\mathrm{Sp}(\mathcal{C}'_{\geqslant 0} \otimes_R \mathcal{D}_{\geqslant 0}) \simeq \mathrm{Sp}(\mathcal{C}'_{\geqslant 0}) \otimes_R \mathrm{Sp}(\mathcal{D}_{\geqslant 0}) \simeq \mathrm{Mod}_R .$$

By construction, the essential image of this embedding contains R, so that the natural map

$$\mathcal{C}'_{\geqslant 0} \otimes_R \mathcal{D}_{\geqslant 0} \to \mathcal{C}_{\geqslant 0} \otimes_R \mathcal{D}_{\geqslant 0}$$

is an equivalence of R-linear Grothenieck prestable ∞-categories. Using the invertibility of $\mathcal{D}_{\geqslant 0}$ as an object of $\mathrm{LinCat}_R^{\mathrm{PSt}}$, we deduce that the inclusion $\mathcal{C}'_{\geqslant 0} \hookrightarrow \mathcal{C}_{\geqslant 0}$ is an equivalence: that is, the ∞-category $\mathcal{C}_{\geqslant 0}$ is generated under colimits and extensions by a single perfect R-module $C = \bigoplus_{\alpha \in A_0} C_\alpha$. Similarly, the ∞-category $\mathcal{D}_{\geqslant 0}$ is generated under colimits and extensions by a perfect R-module $D \in \mathcal{D}_{\geqslant 0}$.

Let $\mathfrak{p}$ be a prime ideal of the commutative ring $\pi_0 R$ and let $\kappa(\mathfrak{p})$ denote the residue field of R at $\mathfrak{p}$. Since C generates Mod_R under shifts and colimits, the tensor product $\kappa(\mathfrak{p}) \otimes_R C$ is a nonzero perfect $\kappa(\mathfrak{p})$-module. We let $d_C(\mathfrak{p})$ denote the smallest integer n such that $\pi_n(\kappa(x) \otimes_R C) \neq 0$. We can then regard d_C as a $\mathbf{Z}$-valued function on $|\operatorname{Spec} R|$. We claim that d_C is lower semicontinuous. To prove that, fix a point $\mathfrak{p} \in |\operatorname{Spec} R|$ such that $d_C(\mathfrak{p}) = n$. We claim that there exists an element $a \in (\pi_0 R) - \mathfrak{p}$ such that $C[a^{-1}]$ is n-connective (so that $d_C(\mathfrak{q}) \geqslant n$ for all $\mathfrak{q} \in |\operatorname{Spec} R|$ which do not contain A). To prove this, we argue by induction on $m \leqslant n$ that we can choose $a_m \in (\pi_0 R) - \mathfrak{p}$ for which $C[a_m^{-1}]$ is m-connective. For $m \ll 0$ we can take $a_m = 1$ (since C is perfect). In the general case, we can use the inductive hypothesis to reduce to the case where C is $(m-1)$-connective. In this case, $\pi_{m-1} C$ is a finitely generated $(\pi_0 R)$-module M satisfying $\mathrm{Tor}_0^{\pi_0 R}(\kappa(\mathfrak{p}), M) \simeq 0$, so that $M[a^{-1}] \simeq 0$ for some $a \notin \mathfrak{p}$ by virtue of Nakayama's lemma.

Define functions $d_D, d_{C \otimes D} : |\operatorname{Spec} R| \to \mathbf{Z}$ similarly, so that we have

$$d_C + d_D = d_{C \otimes D}.$$

Since $C \otimes D$ is a connective R-module which generates $(\mathrm{Mod}_R)_{\geqslant 0}$ under colimits and extensions, the function $d_{C \otimes D}$ vanishes, so that $d_C(\mathfrak{p}) = -d_D(\mathfrak{p})$. It follows from the above argument that d_D is lower semi-continuous, so that d_C is upper semi-continuous and therefore continuous. Passing to a étale cover of $\operatorname{Spec} R$, we can assume that d_C is the constant function taking some value n. It now follows from Lemma 11.5.7.18 that $\mathcal{C}_{\geqslant 0} = (\mathrm{Mod}_R)_{\geqslant n}$, which implies that $u = 0 \in \mathrm{Br}(R)$. □

Part IV

Formal Moduli Problems

The following thesis plays a central role in deformation theory:

(∗) If X is a moduli space defined over a field κ of characteristic zero, then a formal neighborhood of any point $x \in X$ can be described by a differential graded Lie algebra over κ.

This idea has been developed in unpublished work of Deligne, Drinfeld, and Feigin, and has powerfully influenced subsequent contributions of Hinich, Kontsevich-Soibelman, Manetti, Pridham, and many others. Our goal in Part IV is to give a precise formulation (and proof) of (∗), together with some of its generalizations.

The first step in formulating (∗) is to decide exactly what we mean by a moduli space. For simplicity, let us work for now over the field $\mathbf{C}$ of complex numbers. We will adopt Grothendieck's "functor of points" philosophy, and identify an algebro-geometric object X (for example, a scheme) with the functor

$$R \mapsto X(R) = \mathrm{Hom}(\mathrm{Spec}\, R, X).$$

This suggests a very general definition:

Definition 11.5.0.1. Let $\mathrm{CAlg}^{\heartsuit}_{\mathbf{C}}$ denote the category of commutative $\mathbf{C}$-algebras and let $\mathcal{S}\mathrm{et}$ denote the category of sets. A *classical moduli problem* is a functor $X : \mathrm{CAlg}^{\heartsuit}_{\mathbf{C}} \to \mathcal{S}\mathrm{et}$.

Unfortunately, Definition 11.5.0.1 is not adequate four our needs. First of all, Definition 11.5.0.1 requires that the functor X take values in the category of sets. In many applications, one would like to consider functors X which assign to each commutative ring R some collection of geometric objects parametrized by the affine scheme $\mathrm{Spec}\, R$. In such cases, it is important to retain information about *automorphisms* of geometric objects.

Example 11.5.0.2. For every commutative $\mathbf{C}$-algebra R, let $X(R)$ denote the category of elliptic curves $E \to \mathrm{Spec}\, R$ (morphisms in the category $X(R)$ are given by isomorphisms of elliptic curves). Then F determines a functor from $\mathrm{CAlg}^{\heartsuit}_{\mathbf{C}}$ to Gpd, where Gpd denotes the 2-category of *groupoids*. In this case, X determines an underlying set-valued functor, which assigns to each commutative ring R the set $\pi_0 X(R)$ of isomorphism classes of elliptic curves over R. However, the groupoid-valued functor $X : \mathrm{CAlg}^{\heartsuit}_{\mathbf{C}} \to \mathrm{Gpd}$ is much better behaved than the set-valued functor $\pi_0 X : \mathrm{CAlg}^{\heartsuit}_{\mathbf{C}} \to \mathcal{S}\mathrm{et}$. For example, the functor X satisfies descent (with respect to the flat topology on the category of commutative rings), while the functor $\pi_0 X$ does not: two elliptic curves which are locally isomorphic need not be globally isomorphic.

Because the functor X of Example 11.5.0.2 is not $\mathcal{S}$et-valued, it cannot be represented by a scheme. However, it is nevertheless a reasonable geometric object: it is representable by a Deligne-Mumford stack. To accommodate Example 11.5.0.2, we would like to adjust Definition 11.5.0.1 to allow groupoid-valued functors.

Variant 11.5.0.3. Let $\mathcal{C}$ be an ∞-category. A *$\mathcal{C}$-valued classical moduli problem* is a functor of ∞-categories $X : \mathrm{CAlg}^{\heartsuit}_{\mathbf{C}} \to \mathcal{C}$.

Remark 11.5.0.4. We recover Definition 11.5.0.1 as a special case of Variant 11.5.0.3, by taking the ∞-category $\mathcal{C}$ to be the ordinary category of sets. In practice, we will be most interested in the special case where $\mathcal{C}$ is the ∞-category $\mathcal{S}$ of spaces.

The next step in formulating $(*)$ is to decide what we mean by a formal neighborhood of a point x in a moduli space X. Suppose, for example, that $X = \operatorname{Spec} A$ is an affine algebraic variety over the field $\mathbf{C}$ of complex numbers. Then a closed point $x \in X$ is determined by a $\mathbf{C}$-algebra homomorphism $\phi : A \to \mathbf{C}$, which is determined a choice of maximal ideal $\mathfrak{m} = \ker(\phi) \subseteq A$. One can define the *formal completion* of X at the point x to be the functor $X^{\wedge} : \mathrm{CAlg}^{\heartsuit}_{\mathbf{C}} \to \mathcal{S}$et given by the formula

$$X^{\wedge}(R) = \{f \in X(R) : f(\operatorname{Spec} R) \subseteq \{x\} \subseteq \operatorname{Spec} A\}.$$

In other words, $X^{\wedge}(R)$ is the collection of commutative ring homomorphisms $\phi : A \to R$ having the property that ϕ carries each element of $\mathfrak{m}$ to a nilpotent element of R. Since $\mathfrak{m}$ is finitely generated, this is equivalent to the condition that ϕ annihilates $\mathfrak{m}^n$ for some integer $n \gg 0$, so that the image of ϕ is a quotient of A by some $\mathfrak{m}$-primary ideal.

Definition 11.5.0.5. Let R be a commutative algebra over the field $\mathbf{C}$ of complex numbers. We will say that R is a *local Artinian* if it is finite dimensional as a $\mathbf{C}$-vector space and has a unique maximal ideal $\mathfrak{m}_R$. The collection of local Artinian $\mathbf{C}$-algebras forms a category, which we will denote by $\mathrm{CAlg}^{\heartsuit\,\mathrm{art}}_{\mathbf{C}}$.

The above analysis shows that if X is an affine algebraic variety over $\mathbf{C}$ containing a point x, then the formal completion $X^{\wedge}$ can be recovered from its values on local Artinian $\mathbf{C}$-algebras. This motivates the following definition:

Definition 11.5.0.6. Let $\mathcal{C}$ be an ∞-category. A *$\mathcal{C}$-valued classical formal moduli problem* is a functor $\mathrm{CAlg}^{\heartsuit\,\mathrm{art}}_{\mathbf{C}} \to \mathcal{C}$.

If X is a set-valued classical moduli problem and we are given a point $\eta \in X(\mathbf{C})$, we can define a set-valued classical formal moduli problem $X^{\wedge}$ by the formula

$$X^{\wedge}(R) = X(R) \times_{X(R/\mathfrak{m}_R)} \{\eta\}.$$

We will refer to $X^\wedge$ as the *completion of X at the point η*. If X is groupoid-valued, the same formula determines a groupoid-valued classical formal moduli problem $X^\wedge$ (in this case, we must interpret $X(R) \times_{X(R/\mathfrak{m}_R)} \{\eta\}$ as the *homotopy fiber product* of the relevant groupoids).

Example 11.5.0.7. For every commutative $\mathbf{C}$-algebra R, let $X(R)$ denote the groupoid whose objects are smooth proper R-schemes and whose morphisms are isomorphisms of R-schemes. Suppose we are given a point $\eta \in X(\mathbf{C})$, corresponding to smooth and proper algebraic variety Z over $\mathbf{C}$. The formal completion $X^\wedge$ assigns to every local Artinian $\mathbf{C}$-algebra R the groupoid $X^\wedge(R)$ of *deformations over Z* over R: that is, smooth proper morphisms $f : \overline{Z} \to \operatorname{Spec} R$ which fit into a pullback diagram

$$\begin{array}{ccc} Z & \longrightarrow & \overline{Z} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbf{C} & \longrightarrow & \operatorname{Spec} R. \end{array}$$

Example 11.5.0.7 is a typical example of the kind of formal moduli problem we would like to study (see §**??**). Let us recall some well-known features of the functor $X^\wedge$ (see Propositions 0.1.3.1 and 0.1.3.4):

(*a*) The functor $X^\wedge$ carries the ring $\mathbf{C}[\epsilon]/(\epsilon^2)$ to the groupoid of first-order deformations of the variety Z. Every first order deformation of Z has an automorphism group which is canonically isomorphic to $\mathrm{H}^0(Z; T_Z)$, where T_Z denotes the tangent bundle of Z.

(*b*) The collection of isomorphism classes of first order deformations of Z can be canonically identified with the cohomology group $\mathrm{H}^1(Z; T_Z)$.

(*c*) To every first order deformation $\overline{Z}$ of Z, we can assign an *obstruction class* $\rho(\overline{Z}) \in \mathrm{H}^2(Z; T_Z)$ which vanishes if and only if $\overline{Z}$ can be extended to a second-order deformation of Z.

As we mentioned in §0.1.3, assertion (*c*) has a natural interpretation in the language of spectral algebraic geometry. More precisely, the spectral language allows us to enlarge the category on which functor X of Example 11.5.0.7 is defined. For every connective $\mathbb{E}_\infty$-algebra R over $\mathbf{C}$, we can define $X(R)$ to be the underlying ∞-groupoid of the ∞-category of spectral schemes which are proper and differentially smooth over R. If R is equipped with an augmentation $\epsilon : R \to \mathbf{C}$, we let $X^\wedge(R)$ denote the fiber product $X(R) \times_{X(\mathbf{C})} \{\eta\}$, which we can think of as a classifying space for *deformations of Z* over $\operatorname{Spec} R$. In the special case where R is a (discrete) local Artinian $\mathbf{C}$-algebra, this agrees with the groupoid described in Example 11.5.0.7. However, we can obtain more information by evaluating the functor $X^\wedge$ on $\mathbb{E}_\infty$-algebras over $\mathbf{C}$ which are not discrete. For example, let $\mathbf{C}[\delta]$ denote the trivial square-zero extension $\mathbf{C} \oplus \Sigma\, \mathbf{C}$. According to As we will see later (Remark 19.4.3.5), there is a canonical isomorphism $\mathrm{H}^2(Z; T_Z) \simeq \pi_0 X^\wedge(\mathbf{C}[\delta])$ (as we promised in Proposition 0.1.3.5). The ordinary commutative ring $\mathbf{C}[\epsilon]/(\epsilon^3)$ is a square-zero extension of $\mathbf{C}[\epsilon]/(\epsilon^2)$ by the ideal $\mathbf{C}\,\epsilon^2$, and therefore fits into a pullback diagram of $\mathbb{E}_\infty$-algebras σ :

$$\begin{array}{ccc} \mathbf{C}[\epsilon]/(\epsilon^3) & \longrightarrow & \mathbf{C}[\epsilon]/(\epsilon^2) \\ \downarrow & & \downarrow \\ \mathbf{C} & \longrightarrow & \mathbf{C}[\delta]. \end{array}$$

In §16.3, we will show that this pullback square determines a pullback diagram of spaces

$$\begin{array}{ccc} X(\mathbf{C}[\epsilon]/(\epsilon^3)) & \longrightarrow & X(\mathbf{C}[\epsilon]/(\epsilon^2)) \\ \downarrow & & \downarrow \\ X(\mathbf{C}) & \longrightarrow & X(\mathbf{C}[\delta]), \end{array}$$

and therefore a fiber sequence

$$X^\wedge(\mathbf{C}[\epsilon]/(\epsilon^3)) \to X^\wedge(\mathbf{C}[\epsilon]/(\epsilon^2)) \xrightarrow{\rho} X^\wedge(\mathbf{C}[\delta]).$$

Passing to connected components, we obtain the obstruction class map $\rho : \mathrm{H}^1(Z; T_Z) \to \mathrm{H}^2(Z; T_Z)$ which vanishes on precisely those elements of $\mathrm{H}^1(Z; T_Z)$ classifying first-order deformations of Z which can be extended to second-order deformations.

The preceding analysis of Example 11.5.0.7 cannot be carried out for an arbitrary classical formal moduli problem (in the sense of Definition 11.5.0.6): it depends crucially on the fact that we can extend the domain of the functor $X^\wedge$ to $\mathbb{E}_\infty$-rings which are not discrete (such as $\mathbf{C}[\delta] = \mathbf{C} \oplus \Sigma\,\mathbf{C}$). Moreover, this extended functor needs to be reasonably well-behaved: for example, it should carry the diagram σ to a pullback diagram of spaces. This motivates another variant of Definition 11.5.0.1:

Definition 11.5.0.8. Let R be an $\mathbb{E}_\infty$-algebra over $\mathbf{C}$. We will say that R is *Artinian* if it is connective, $\pi_* R$ is a finite-dimensional vector space over $\mathbf{C}$, and $\pi_0 R$ is a local ring. Let $\mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}}$ denote the ∞-category whose objects are $\mathbb{E}_\infty$-algebras R over $\mathbf{C}$

A *formal moduli problem over* $\mathbf{C}$ is a functor $X : \mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}} \to \mathcal{S}$ which satisfies the following pair of conditions:

(1) The space $X(\mathbf{C})$ is contractible.

(2) For every pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

in $\mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}}$ for which the underlying ring homomorphisms $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, the diagram

$$\begin{array}{ccc} X(R) & \longrightarrow & X(R_0) \\ \downarrow & & \downarrow \\ X(R_1) & \longrightarrow & X(R_{01}) \end{array}$$

is a pullback square.

Warning 11.5.0.9. The terminology of Definition 11.5.0.8 is potentially misleading: it might be more appropriate to refer to an object $R \in \mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}}$ as a *local Artinian* $\mathbb{E}_\infty$-algebra over $\mathbf{C}$. However, we will abuse terminology throughout Part IV by referring to such an $\mathbb{E}_\infty$-algebra simply as *Artinian* (since we will have no cause to consider Artinian algebras which are not local). See Warning 12.1.2.6.

Remark 11.5.0.10. Let $\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{C}}$ denote the ∞-category of connective $\mathbb{E}_\infty$-algebras over the field $\mathbf{C}$ of complex numbers, and let $X : \mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{C}} \to \mathcal{S}$ be a functor. Given a point $x \in X(\mathbf{C})$, we define the *completion of* X *at the point* x to be the functor $X^\wedge : \mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}} \to \mathcal{S}$ given by the formula $X^\wedge(R) = X(R) \times_{X(\mathbf{C})} \{x\}$. The functor $X^\wedge$ automatically satisfies condition (1) of Definition 11.5.0.8. Condition (2) is not automatic, but holds for a wide variety of functors; for example, it is satisfied whenever X is representable by a spectral Deligne-Mumford stack (see Corollary **??**).

Remark 11.5.0.11. Let $X : \mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}} \to \mathcal{S}$ be a formal moduli problem. Then X determines a functor between ordinary categories $X_0 : \mathrm{hCAlg}^{\mathrm{art}}_{\mathbf{C}} \to \mathcal{S}\mathrm{et}$, where $\mathrm{hCAlg}^{\mathrm{art}}_{\mathbf{C}}$ denotes the homotopy category of $\mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}}$, given by the formula $X_0(A) = \pi_0 X(A)$. It follows from condition (2) of Definition 11.5.0.8 that if we are given maps of Artinian $\mathbb{E}_\infty$-algebras $R_0 \to R_{01} \leftarrow R_1$ which induce surjections $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$, then the induced map

$$X_0(R_0 \times_{R_{01}} R_1) \to X_0(R_0) \times_{X_0(R_{01})} X_0(R_1)$$

is a surjection of sets. There is a substantial literature on set-valued moduli functors of this type; see, for example, [146] and [122].

Warning 11.5.0.12. If X is a formal moduli problem over $\mathbf{C}$, then X determines a classical formal moduli problem (with values in the ∞-category $\mathcal{S}$) simply by restricting the functor X to the subcategory of $\mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}}$ consisting of ordinary local Artinian $\mathbf{C}$-algebras (which are precisely the discrete objects of $\mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}}$).

If $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ is a spectral Deligne-Mumford stack over $\mathbf{C}$ equipped with a point $\eta : \mathrm{Spec}^{\text{ét}}\,\mathbf{C} \to \mathsf{X}$ and X is defined as in Remark 11.5.0.10, then the restriction $X_0 = X|_{\mathrm{CAlg}^{\heartsuit\,\mathrm{art}}_{\mathbf{C}}}$ depends only on the pair $(\mathcal{X}, \pi_0\,\mathcal{O})$. In particular, the functor X cannot be recovered from X_0.

In general, if we are given a classical formal moduli problem $X_0 : \mathrm{CAlg}_{\mathbf{C}}^{\heartsuit\,\mathrm{art}} \to \mathcal{S}$, there may or may not exist a formal moduli problem X such that $X_0 = X|_{\mathrm{CAlg}_{\mathbf{C}}^{\heartsuit\,\mathrm{art}}}$. Moreover, if X exists, then it need not be unique. Nevertheless, classical formal moduli problems X_0 which arise naturally are often equipped with a natural extension $X : \mathrm{CAlg}_{\mathbf{C}}^{\mathrm{art}} \to \mathcal{S}$ (as in our discussion of Example 11.5.0.7 above), which can be quite useful in the study of X_0.

We are now ready to articulate a precise version of $(*)$:

Theorem 11.5.0.13. *Let* $\mathrm{Moduli}_{\mathbf{C}}$ *denote the full subcategory of* $\mathrm{Fun}(\mathrm{CAlg}_{\mathbf{C}}^{\mathrm{art}}, \mathcal{S})$ *spanned by the formal moduli problems, and let* $\mathrm{Lie}_{\mathbf{C}}^{\mathrm{dg}}$ *denote the category of differential graded Lie algebras over* $\mathbf{C}$ *(see §13.1). Then there is a functor* $\theta : \mathrm{Lie}_{\mathbf{C}}^{\mathrm{dg}} \to \mathrm{Moduli}$ *with the following universal property: for any* ∞*-category* $\mathcal{C}$*, composition with* θ *induces a fully faithful embedding* $\mathrm{Fun}(\mathrm{Moduli}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Lie}_{\mathbf{C}}^{\mathrm{dg}}, \mathcal{C})$*, whose essential image is the collection of all functors* $F : \mathrm{Lie}_{\mathbf{C}}^{\mathrm{dg}} \to \mathcal{C}$ *which carry quasi-isomorphisms of differential graded Lie algebras to equivalences in* $\mathcal{C}$*.*

Remark 11.5.0.14. An equivalent version of Theorem 11.5.0.13 has been established by Pridham; we refer the reader to [165] for details.

Remark 11.5.0.15. Let W be the collection of all quasi-isomorphisms in the category $\mathrm{Lie}_{\mathbf{C}}^{\mathrm{dg}}$, and let $\mathrm{Lie}_{\mathbf{C}}^{\mathrm{dg}}[W^{-1}]$ denote the ∞-category obtained from $\mathrm{Lie}_{\mathbf{C}}^{\mathrm{dg}}$ by formally inverting the morphisms in W. Theorem 11.5.0.13 asserts that there is an equivalence of ∞-categories $\mathrm{Lie}_{\mathbf{C}}^{\mathrm{dg}}[W^{-1}] \simeq \mathrm{Moduli}$. In particular, every differential graded Lie algebra over $\mathbf{C}$ determines a formal moduli problem, and two differential graded Lie algebras $\mathfrak{g}_*$ and $\mathfrak{g}'_*$ determine equivalent formal moduli problems if and only if they can be joined by a chain of quasi-isomorphisms.

Theorem 11.5.0.13 articulates a sense in which the theories of commutative algebras and Lie algebras are closely related. In concrete terms, this relationship is controlled by the *Chevalley-Eilenberg* functor, which associates to a differential graded Lie algebra $\mathfrak{g}_*$ a cochain complex of vector spaces $C^*(\mathfrak{g}_*)$. The cohomology of this cochain complex is the *Lie algebra cohomology* of the Lie algebra $\mathfrak{g}_*$, and is endowed with a commutative multiplication. In fact, this multiplication is defined at the level of cochains: the construction $\mathfrak{g}_* \mapsto C^*(\mathfrak{g}_*)$ determines a functor C^* from the (opposite of) the category $\mathrm{Lie}_{\mathbf{C}}^{\mathrm{dg}}$ of differential graded Lie algebras over $\mathbf{C}$ to the category $\mathrm{CAlg}_{\mathbf{C}}^{\mathrm{dg}}$ of commutative differential graded algebras over $\mathbf{C}$. This functor carries quasi-isomorphisms to quasi-isomorphisms, and therefore induces a functor of ∞-categories

$$C^* : \mathrm{Lie}_{\mathbf{C}}^{\mathrm{dg}}[W^{-1}]^{\mathrm{op}} \to \mathrm{CAlg}_{\mathbf{C}}^{\mathrm{dg}}[W'^{-1}];$$

here W is the collection of quasi-isomorphisms in $\mathrm{Lie}_{\mathbf{C}}^{\mathrm{dg}}$ (as in Remark 11.5.0.15) and W' is the collection of quasi-isomorphisms in $\mathrm{CAlg}_{\mathbf{C}}^{\mathrm{dg}}$ (here the ∞-category $\mathrm{CAlg}_{\mathbf{C}}^{\mathrm{dg}}[W'^{-1}]$

can be identified $\mathrm{CAlg}_{\mathbf{C}}$ of $\mathbb{E}_\infty$-algebras over $\mathbf{C}$: see Proposition HA.7.1.4.11). Every differential graded Lie algebra $\mathfrak{g}_*$ admits a canonical map $\mathfrak{g}_* \to 0$, so that its Chevalley-Eilenberg complex is equipped with an augmentation $C^*(\mathfrak{g}_*) \to C^*(0) \simeq \mathbf{C}$. We may therefore refine C^* to a functor $\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}[W^{-1}]^{\mathrm{op}} \to \mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}}$ taking values in the ∞-category $\mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}}$ of *augmented* $\mathbb{E}_\infty$-algebras over $\mathbf{C}$. We will see that this functor admits a left adjoint $\mathfrak{D} : \mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}} \to \mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}[W^{-1}]^{\mathrm{op}}$ (Theorem 13.3.0.1). The functor $\theta : \mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}} \to \mathrm{Moduli}$ appearing in the statement of Theorem 11.5.0.13 can then be defined by the formula

$$\theta(\mathfrak{g}_*)(R) = \mathrm{Map}_{\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}[W^{-1}]}(\mathfrak{D}(R), \mathfrak{g}_*).$$

In more abstract terms, the relationship between commutative algebras and Lie algebras suggested by Theorem 11.5.0.13 is an avatar of *Koszul duality*. More specifically, Theorem 11.5.0.13 reflects the fact that the commutative operad is Koszul dual to the Lie operad (see [114]). This indicates that should be many other versions of Theorem 11.5.0.13, where we replace commutative and Lie algebras by algebras over some other pair of Koszul dual operads. For example, the Koszul self-duality of the $\mathbb{E}_n$-operads (see [68]) suggests an analogue of Theorem 11.5.0.13 in the setting of “noncommutative” derived algebraic geometry, which we will also prove here (see Theorems 14.0.0.5 and 15.0.0.9).

Let us now outline the contents of Part IV. In Chapter 12, we will introduce the general notion of a *deformation theory*: a functor of ∞-categories $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ satisfying a suitable list of axioms (see Definitions 12.3.1.1 and 12.3.3.2). We will then prove an abstract version of Theorem 11.5.0.13: every deformation theory $\mathfrak{D}$ determines an equivalence $\mathbf{B} \simeq \mathrm{Moduli}^{\mathbf{A}}$, where $\mathrm{Moduli}^{\mathbf{A}}$ is a suitably defined ∞-category of formal moduli problems (Theorem 12.3.3.5). This result is not very difficult in itself: it can be regarded as a distillation of the purely formal ingredients needed for the proof of results like Theorem 11.5.0.13. In practice, the hard part is to construct the functor $\mathfrak{D}$ and to prove that it satisfies the axioms of Definitions 12.3.1.1 and 12.3.3.2. We will give a detailed treatment of three special cases:

(a) In Chapter 13, we treat the case where $\mathbf{A}$ is the ∞-category $\mathrm{CAlg}^{\mathrm{aug}}_{\kappa}$ of augmented $\mathbb{E}_\infty$-algebras over a field κ of characteristic zero, and use Theorem 12.3.3.5 to prove a version of Theorem 11.5.0.13 (Theorem 13.0.0.2).

(b) In Chapter 14, we treat the case where $\mathbf{A}$ is the ∞-category $\mathrm{Alg}^{\mathrm{aug}}_{\kappa}$ of augmented $\mathbb{E}_1$-algebras over a field κ (of arbitrary characteristic), and use Theorem 12.3.3.5 to prove a noncommutative analogue of Theorem 11.5.0.13 (Theorem 14.0.0.5).

(c) In Chapter 15, we treat the case where $\mathbf{A}$ is the ∞-category $\mathrm{Alg}^{(n),\mathrm{aug}}_{\kappa}$ of augmented $\mathbb{E}_n$-algebras over a field κ (again of arbitrary characteristic), and use Theorem 12.3.3.5 to prove a more general noncommutative analogue of Theorem 11.5.0.13 (Theorem 15.0.0.9).

In each case, the relevant deformation functor $\mathfrak{D}$ is given by some version of Koszul duality, and our main result gives an algebraic model for the ∞-category of formal moduli problems $\mathrm{Moduli}^{\mathbf{A}}$. In Chapter 16, we will use these results to study some concrete examples of formal moduli problems which arise naturally in deformation theory.

Remark 11.5.0.16. The notion that differential graded Lie algebras should play an important role in the description of moduli spaces goes back to Quillen's work on rational homotopy theory ([220]), and was developed further in unpublished work of Deligne, Drinfeld, and Feigin. Many other mathematicians have subsequently taken up these ideas: see, for example, the book of Kontsevich and Soibelman ([122]).

Chapter 12

Deformation Theories: Axiomatic Approach

Our in Part IV is to prove several variants of Theorem 11.5.0.13, which supply algebraic descriptions of various ∞-categories of formal moduli problems. Here is a basic prototype for the kind of result we are after:

$(*)$ Let $\mathbf{A}$ be an ∞-category of algebraic objects, let $\mathbf{A}^{\mathrm{art}} \subseteq \mathbf{A}$ be a full subcategory of "Artinian" objects, and let $\mathrm{Moduli}^{\mathbf{A}} \subseteq \mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ be the ∞-category of functors $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ which satisfy a suitable gluing condition (as in Definition 11.5.0.8). Then there is an equivalence of ∞-categories $\mathrm{Moduli}^{\mathbf{A}} \simeq \mathbf{B}$, where $\mathbf{B}$ is some other ∞-category of algebraic objects.

Our goal in this section is to flesh out assertion $(*)$. We begin in §12.1 by introducing the notion of a *deformation context* (Definition 12.1.1.1). A deformation context is a presentable ∞-category $\mathbf{A}$ equipped with some additional data (namely, a collection of spectrum objects $E_\alpha \in \mathrm{Sp}(\mathbf{A})$). Using this data, we define a full subcategory $\mathbf{A}^{\mathrm{art}} \subseteq \mathbf{A}$ of *Artinian* objects of $\mathbf{A}$ (Definition 12.1.2.4) and a full subcategory $\mathrm{Moduli}^{\mathbf{A}} \subseteq \mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ of *formal moduli problems* (Definition 12.1.3.1). Our definition is quite general, but nevertheless sufficient to guarantee the existence of a reasonable *differential* theory of formal moduli problems. In §12.2 we will explain how to associate to every formal moduli problem X a collection of spectra $X(E_\alpha)$, which we call the *tangent complex*(es) of X. The construction is functorial: every map between formal moduli problems $u : X \to Y$ can be differentiated to obtain maps of spectra $X(E_\alpha) \to Y(E_\alpha)$. Moreover, if each of these maps is a homotopy equivalence, then u is an equivalence (Proposition 12.2.2.6).

In §12.3, we will formulate a general version of $(*)$. For this, we will introduce the notion of a *deformation theory*. A deformation theory is a functor $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ satisfying a collection of axioms (see Definitions 12.3.1.1 and 12.3.3.2). Our main result (Theorem 12.3.3.5) can then be stated as follows: if $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ is a deformation theory, then $\mathfrak{D}$ determines an equivalence of ∞-categories $\mathbf{B} \simeq \mathrm{Moduli}^{\mathbf{A}}$. The proof of this result will be given in §12.5, using an ∞-categorical variant of Quillen's small object argument which we review in §12.4.

Our work in this section should be regarded as providing a sort of formal outline for proving results like Theorem 11.5.0.13. For practical purposes, the main difficulty is not in proving Theorem 12.3.3.5 but in verifying its hypotheses: that is, in constructing a functor $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ which satisfies the axioms listed in Definitions 12.3.1.1 and 12.3.3.2. The later chapters of Part IV are devoted to carrying this out in special cases (we will treat the case of commutative algebras in Chapter 13, associative algebras in Chapter 14, and $\mathbb{E}_n$-algebras in Chapter 15).

Contents

12.1 Formal Moduli Problems

In this section, we introduce a general axiomatic paradigm for the study of deformation theory. Let us begin by outlining the basic idea. We are ultimately interesting in studying some class of algebro-geometric objects (such as schemes, or algebraic stacks, or their spectral analogues). Using the functor of points philosophy, we will view these geometric objects with functors $X : \mathbf{A} \to \mathcal{S}$, where $\mathbf{A}$ denotes some ∞-category of "test objects". The main example of interest (which we will study in detail in Chapter 13) is the case where $\mathbf{A}$ to be the ∞-category $\mathrm{CAlg}^{\mathrm{aug}}_{\kappa} = (\mathrm{CAlg}_{\kappa})_{/\kappa}$ of augmented $\mathbb{E}_{\infty}$-algebras over a field κ of characteristic zero. In any case, we will always assume that $\mathbf{A}$ contains a final object $*$; we can then define a *point* of a functor $X : \mathbf{A} \to \mathcal{S}$ to be a point of the space $X(*)$. Suppose that we wish to study a *formal neighborhood* of X around some chosen point $x \in X(*)$. This formal neighborhood should encode information about the homotopy fiber products $X(A) \times_{X(*)} \{x\}$ for every object $A \in \mathbf{A}$ which is sufficiently "close" to the final object $*$. In order to make this idea precise, we need to introduce some terminology.

12.1.1 Deformation Contexts

Let $\mathbf{A}$ be a presentable ∞-category. We let $\mathrm{Sp}(\mathbf{A})$ denote the ∞-category of spectrum objects of $\mathbf{A}$: that is, the full subcategory of $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_{*}, \mathbf{A})$ spanned by the reduced and excisive functors (see Definition HA.1.4.2.8). For each integer n, we we let $\Omega^{\infty-n} : \mathrm{Sp}(\mathbf{A}) \to \mathbf{A}$ denote the forgetful functor, given concretely by the formula $\Omega^{\infty-n}(X) = \Omega^{m} X(S^{n+m})$ for any $m \geqslant -n$. We recall that $\mathrm{Sp}(\mathbf{A})$ can also be described as the homotopy limit of the tower of ∞-categories

$$\cdots \to \mathbf{A}_{*} \xrightarrow{\Omega} \mathbf{A}_{*} \xrightarrow{\Omega} \mathbf{A}_{*} \to \cdots ,$$

where $\mathbf{A}_{*}$ denotes the ∞-category of pointed objects of $\mathbf{A}$.

Definition 12.1.1.1. A *deformation context* is a pair $(\mathbf{A}, \{E_{\alpha}\}_{\alpha \in T})$, where $\mathbf{A}$ is a presentable ∞-category and $\{E_{\alpha}\}_{\alpha \in T}$ is a set of objects of the ∞-category $\mathrm{Sp}(\mathbf{A})$.

Example 12.1.1.2. Let κ be an $\mathbb{E}_{\infty}$-ring, and let $\mathbf{A} = \mathrm{CAlg}^{\mathrm{aug}}_{\kappa} = (\mathrm{CAlg}_{\kappa})_{/\kappa}$ denote the ∞-category of augmented $\mathbb{E}_{\infty}$-algebras over κ. Using Theorem HA.7.3.4.13 , we can identify $\mathrm{Sp}(\mathbf{A})$ with the ∞-category Mod_{κ} of modules over κ. Let $E \in \mathrm{Sp}(\mathbf{A})$ be the object which corresponds to $\kappa \in \mathrm{Mod}_{\kappa}$ under this identification, so that for every integer n we can identify $\Omega^{\infty-n} E$ with the trivial square-zero extension $\kappa \oplus \Sigma^{n}(\kappa)$ of κ. Then the pair $(\mathrm{CAlg}^{\mathrm{aug}}_{\kappa}, \{E\})$ is a deformation context.

12.1.2 Artinian Objects

Let κ be a field and let $A \in \mathrm{CAlg}^{\mathrm{aug}}_{\kappa}$ be a local augmented $\mathbb{E}_{\infty}$-algebra over κ. We will say that A is said to be *Artinian* if A is connective and $\pi_{*} A$ is a finite-dimensional vector space over κ. Our next goal is to show that the class of (local) Artinian objects of $\mathrm{CAlg}_\kappa^{\mathrm{aug}}$ is determined by the deformation context of Example 12.1.1.2 (see Proposition 12.1.2.9).

Definition 12.1.2.1. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. We will say that a morphism $\phi : A' \to A$ in $\mathbf{A}$ is *elementary* if there exists an index $\alpha \in T$, an integer $n > 0$, and a pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & * \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_0} \\ A & \longrightarrow & \Omega^{\infty - n} E_\alpha. \end{array}$$

Here ϕ_0 corresponds to the image of E_α under the forgetful functor $\mathrm{Sp}(\mathbf{A}) \xrightarrow{\Omega_*^{\infty-n}} \mathbf{A}_*$.

Example 12.1.2.2. Let κ be a field and let $(\mathbf{A}, \{E\})$ be the deformation context described in Example 12.1.1.2. Suppose that $\phi : A' \to A$ is a map between connective objects of $\mathbf{A} = \mathrm{CAlg}_\kappa^{\mathrm{aug}}$. Using Theorem HA.7.4.1.26 , we deduce that ϕ is elementary if and only if the following conditions are satisfied:

(a) There exists an integer $n \geqslant 0$ and an equivalence $\mathrm{fib}(\phi) \simeq \Sigma^n(\kappa)$ in the ∞-category $\mathrm{Mod}_{A'}$ (here we regard κ as an object of $\mathrm{Mod}_{A'}$ via the augmentation map $A' \to \kappa$).

(b) If $n = 0$, then the multiplication map $\pi_0 \,\mathrm{fib}(\phi) \otimes \pi_0 \,\mathrm{fib}(\phi) \to \pi_0 \,\mathrm{fib}(\phi)$ vanishes.

If (a) is satisfied for $n = 0$, then we can choose a generator $\overline{x}$ for $\pi_0 \,\mathrm{fib}(\phi)$ having image $x \in \pi_0 A'$. Condition (b') is automatic if $x = 0$. If $a \neq 0$, then the map $\pi_0 \,\mathrm{fib}(\phi) \to \pi_0 A'$ is injective, so condition (b) is equivalent to the requirement that $x^2 = 0$ in $\pi_0 A'$.

Remark 12.1.2.3. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and suppose we are given an object $A \in \mathbf{A}$. Every elementary map $A' \to A$ in $\mathbf{A}$ is given by the fiber of a map $A \to \Omega^{\infty-n} E_\alpha$ for some $n > 0$ and some $\alpha \in T$. It follows that the collection of equivalence classes of elementary maps $A' \to A$ is bounded in cardinality.

Definition 12.1.2.4. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. We will say that a morphism $\phi : A' \to A$ in $\mathbf{A}$ is *small* if it can be written as a composition of finitely many elementary morphisms $A' \simeq A_0 \to A_1 \to \cdots \to A_n \simeq A$. We will say that an object $A \in \mathbf{A}$ is *Artinian* if the map $A \to *$ (which is uniquely determined up to homotopy) is small. We let $\mathbf{A}^{\mathrm{art}}$ denote the full subcategory of $\mathbf{A}$ spanned by the Artinian objects.

Warning 12.1.2.5. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. The collection of Artinian objects of $\mathbf{A}$ depends not only on the underlying ∞-category $\mathbf{A}$, but also on the collection of spectrum objects $\{E_\alpha\}_{\alpha \in T}$.

Warning 12.1.2.6. The terminology of Definition 12.1.2.4 is motivated by the observation that, if κ is a field, then the class of Artinian objects of $\mathrm{CAlg}^{\mathrm{aug}}_{\kappa}$ (in the sense of Definition 12.1.2.4) can be regarded as a "derived" version of the theory of augmented local Artinian κ-algebras (in the sense of commutative algebra): see Proposition 12.1.2.9 below. However, our terminology is potentially confusing for two reasons:

(a) If A is an augmented commutative algebra over a field κ which is Artinian in the sense of classical commutative algebra, then A will not be Artinian in the the sense of Definition 12.1.2.4 unless it is local. For example, the product $A = \kappa \times \kappa$ is not Artinian in the sense of Definition 12.1.2.4. (To avoid this confusion, it might be more appropriate to refer to the objects of $\mathbf{A}^{\mathrm{art}}$ as the *local Artinian* objects of $\mathbf{A}$; however, we would prefer to avoid unnecessarily convoluted language: see Warning 11.5.0.9)

(b) If κ is a commutative ring which is not a field, then we can still regard $\mathbf{A} = \mathrm{CAlg}^{\mathrm{aug}}_{\kappa}$ as a deformation context (as in Example 12.1.1.2), but the condition that an augmented commutative κ-algebra A belongs to $\mathbf{A}^{\mathrm{art}}$ is generally unrelated to the condition that A is Artinian (for example, the commutative ring κ itself is always Artinian in the sense of Definition 12.1.2.4). In practice, this will not concern us: we will only consider the deformation context of Example 12.1.1.2 in the case where κ is a field (usually of characteristic zero).

Example 12.1.2.7. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. For every integer $n \geqslant 0$ and every index $\alpha \in T$, we have a pullback diagram

$$\begin{array}{ccc} \Omega^{\infty-n} E_\alpha & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Omega^{\infty-n-1} E_\alpha. \end{array}$$

It follows that the left vertical map is elementary. In particular, $\Omega^{\infty-n} E_\alpha$ is an Artinian object of $\mathbf{A}$.

Remark 12.1.2.8. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. It follows from Remark 12.1.2.3 that the subcategory $\mathbf{A}^{\mathrm{art}} \subseteq \mathbf{A}$ is essentially small.

Proposition 12.1.2.9. *Let κ be a field and let $(\mathbf{A}, \{E\})$ be the deformation context of Example 12.1.1.2. Then an object $A \in \mathbf{A} = \mathrm{CAlg}^{\mathrm{aug}}_{\kappa}$ is Artinian (in the sense of Definition 12.1.2.4) if and only if the following conditions are satisfied:*

(1) *The homotopy groups $\pi_n A$ vanish for $n < 0$ and $n \gg 0$.*

(2) *Each homotopy group $\pi_n A$ is finite-dimensional as a vector space over κ.*

(3) *The commutative ring* $\pi_0 A$ *is local.*

Proof. Suppose first that A is Artinian, so that there there exists a finite sequence of maps

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq \kappa$$

where each A_i is a square-zero extension of A_{i+1} by $\Sigma^{m_i}(\kappa)$, for some $n_i \geqslant 0$. We prove that each A_i satisfies conditions (1), (2), and (3) using descending induction on i. The case $i = n$ is obvious, so let us assume that $i < n$ and that A_{i+1} is known to satisfy conditions (1), (2), and (3). We have a fiber sequence of κ-module spectra

$$\Sigma^{m_i}(\kappa) \to A_i \to A_{i+1}$$

which immediately implies that A_i satisfies (1) and (2). The map $\phi : \pi_0 A_i \to \pi_0 A_{i+1}$ is surjective and $\ker(\phi)^2 = 0$, from which it follows immediately that $\pi_0 A_i$ is local.

Now suppose that A satisfies conditions (1), (2), and (3). We will prove that A is Artinian by induction on the dimension of the κ-vector space $\pi_* A$. Let n be the largest integer for which $\pi_n A$ does not vanish. We first treat the case $n = 0$. We will abuse notation by identifying A with the underlying commutative ring $\pi_0 A$. Condition (3) asserts that A is a local ring; let $\mathfrak{m}$ denote its maximal ideal. Since A is a finite dimensional algebra over κ, we have $\mathfrak{m}^{i+1} \simeq 0$ for $i \gg 0$. Choose i as small as possible. If $i = 0$, then $\mathfrak{m} \simeq 0$ and $A \simeq \kappa$, in which case there is nothing to prove. Otherwise, we can choose a nonzero element $x \in \mathfrak{m}^i \subseteq \mathfrak{m}$. Let A' denote the quotient ring $A/(x)$. It follows from Example 12.1.2.2 that the quotient map $A \to A'$ is elementary. Since A' is Artinian by the inductive hypothesis, we conclude that A is Artinian.

Now suppose that $n > 0$ and let $M = \pi_n A$, so that M has the structure of a module over the ring $\pi_0 A$. Let $\mathfrak{m} \subseteq \pi_0 A$ be as above, and let i be the least integer such that $\mathfrak{m}^{i+1} M \simeq 0$. Let $x \in \mathfrak{m}^i M$ and let M' be the quotient of M by x, so that we have an exact sequence

$$0 \to \kappa \xrightarrow{x} M \to M' \to 0$$

of modules over $\pi_0 A$. We will abuse notation by viewing this sequence as a fiber sequence of A''-modules, where $A'' = \tau_{\leqslant n-1} A$. It follows from Theorem HA.7.4.1.26 that there is a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ A'' & \longrightarrow & \kappa \oplus \Sigma^{n+1}(M). \end{array}$$

Set $A' = A'' \times_{\kappa \oplus \Sigma^{n+1}(M')} \kappa$. Then $A \simeq A' \times_{\kappa \oplus \Sigma^{n+1}(\kappa)} \kappa$ so we have an elementary map $A \to A'$. Using the inductive hypothesis we deduce that A' is Artinian, so that A is also Artinian. □

Remark 12.1.2.10. Let κ be a field and suppose that $A \in \mathrm{CAlg}_{\kappa}^{\mathrm{cn}}$ has the property that $\pi_* A$ is a finite-dimensional vector space over κ and that $\pi_0 A$ is a local ring with residue field κ. Then the mapping space $\mathrm{Map}_{\mathrm{CAlg}_{\kappa}}(A, \kappa)$ is contractible. In particular, A can be promoted (in an essentially unique way) to an Artinian object of $\mathbf{A} = \mathrm{CAlg}_{\kappa}^{\mathrm{aug}}$. Moreover, the forgetful functor $\mathrm{CAlg}_{\kappa}^{\mathrm{aug}} \to \mathrm{CAlg}_{\kappa}$ is fully faithful when restricted to the full subcategory $\mathbf{A}^{\mathrm{art}} \subseteq \mathbf{A}$. We will denote the essential image of this restriction by $\mathrm{CAlg}_{\kappa}^{\mathrm{art}}$. We refer to $\mathrm{CAlg}_{\kappa}^{\mathrm{art}}$ as the *∞-category of Artinian $\mathbb{E}_{\infty}$-algebras over κ*. In the case where $\kappa = \mathbf{C}$, this recovers the definition of $\mathrm{CAlg}_{\mathbf{C}}^{\mathrm{art}}$ appearing in Definition 11.5.0.8.

Remark 12.1.2.11. Let $(\mathbf{A}, \{E_{\alpha}\}_{\alpha \in T})$ be a deformation context. Then the collection of small morphisms in $\mathbf{A}$ is closed under composition. Consequently, if $\phi : A' \to A$ is small and A is Artinian, then A' is also Artinian. In particular, if there exists a pullback diagram

$$\begin{array}{ccc} B' & \longrightarrow & A' \\ \downarrow & & \downarrow \phi \\ B & \longrightarrow & A \end{array}$$

where B is Artinian and ϕ is small, then B' is also Artinian.

12.1.3 Formal Moduli Problems

We are now ready to introduce the main objects of study in Part IV.

Definition 12.1.3.1. Let $(\mathbf{A}, \{E_{\alpha}\}_{\alpha \in T})$ be a deformation context. A *formal moduli problem* is a functor $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ satisfying the following pair of conditions:

(a) The space $X(*)$ is contractible (here $*$ denotes a final object of $\mathbf{A}$).

(b) Let σ :

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \phi \\ A & \longrightarrow & B \end{array}$$

be a diagram in $\mathbf{A}^{\mathrm{art}}$. If σ is a pullback diagram and ϕ is small, then $X(\sigma)$ is a pullback diagram in $\mathcal{S}$.

We let $\mathrm{Moduli}^{\mathbf{A}}$ denote the full subcategory of $\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ spanned by the formal moduli problems. We will refer to $\mathrm{Moduli}^{\mathbf{A}}$ as the *∞-category of formal moduli problems*.

Condition (b) of Definition 12.1.3.1 has a number of equivalent formulations:

Proposition 12.1.3.2. *Let $(\mathbf{A}, \{E_{\alpha}\}_{\alpha \in T})$ be a deformation context, and let $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

(1) *Let* σ :

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \phi \\ A & \longrightarrow & B \end{array}$$

be a diagram in $\mathbf{A}^{\mathrm{art}}$*. If* σ *is a pullback diagram and* ϕ *is small, then* $X(\sigma)$ *is a pullback diagram in* $\mathcal{S}$*.*

(2) *Let* σ *be as in* (1)*. If* σ *is a pullback diagram and* ϕ *is elementary, then* $X(\sigma)$ *is a pullback diagram in* $\mathcal{S}$*.*

(3) *Let* σ *be as in* (1)*. If* σ *is a pullback diagram and* ϕ *is the base point morphism* $* \to \Omega^{\infty-n}E_\alpha$ *for some* $\alpha \in T$ *and* $n > 0$*, then* $X(\sigma)$ *is a pullback diagram in* $\mathcal{S}$*.*

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear. The reverse implications follow from Lemma HTT.4.4.2.1 . □

Example 12.1.3.3. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $A \in \mathbf{A}$ be an object. Let $\mathrm{Spf}(A) : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ be the functor corepresented by A, given on Artinian objects of $\mathbf{A}$ by the formula $\mathrm{Spf}(A)(B) = \mathrm{Map}_{\mathbf{A}}(A, B)$. Then $\mathrm{Spf}(A)$ is a formal moduli problem. We will refer to $\mathrm{Spf}(A)$ as the *formal spectrum* of A. Moreover, the construction $A \mapsto \mathrm{Spf}(A)$ determines a functor $\mathrm{Spf} : \mathbf{A}^{\mathrm{op}} \to \mathrm{Moduli}^{\mathbf{A}}$.

Warning 12.1.3.4. The notion of formal spectrum Spf A of Example 12.1.3.3 is different from the notion of formal spectrum studied in Chapter 8. However, in many situations where both notions are defined, they are interchangeable: see §**??**.

Remark 12.1.3.5. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. The ∞-category $\mathbf{A}^{\mathrm{art}} \subseteq \mathbf{A}$ is essentially small. It follows from Lemmas HTT.5.5.4.19 and HTT.5.5.4.18 that the ∞-category $\mathrm{Moduli}^{\mathbf{A}}$ is an accessible localization of the ∞-category $\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$. In particular, the ∞-category $\mathrm{Moduli}^{\mathbf{A}}$ is presentable.

Remark 12.1.3.6. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ be a functor which satisfies the equivalent conditions of Proposition 12.1.3.2. For every point $\eta \in X(*)$, define a functor $X_\eta : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ by the formula $X_\eta(A) = X(A) \times_{X(*)} \{\eta\}$. Then X_η is a formal moduli problem. We may therefore identify X as a family of formal moduli problems parametrized by the space $X(*)$. Consequently, condition (a) of Definition 12.1.3.1 should be regarded as a harmless simplifying assumption.

In the special case where $\mathbf{A} = \mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}}$, Definition 12.1.3.1 agrees with Definition 11.5.0.8. This is an immediate consequence of the following result:

Proposition 12.1.3.7. *Let κ be a field and let $X : \mathrm{CAlg}^{\mathrm{art}}_{\kappa} \to \mathcal{S}$ be a functor. Then conditions* (1)*,* (2)*, and* (3) *of Proposition 12.1.3.2 are equivalent to the following:*

($*$) *For every pullback diagram*

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

in $\mathrm{CAlg}^{\mathrm{art}}_{\kappa}$ for which the underlying maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, the diagram

$$\begin{array}{ccc} X(R) & \longrightarrow & X(R_0) \\ \downarrow & & \downarrow \\ X(R_1) & \longrightarrow & X(R_{01}) \end{array}$$

is a pullback square.

The proof of Proposition 12.1.3.7 will require the following elaboration on Proposition 12.1.2.9:

Lemma 12.1.3.8. *Let κ be a field and let $f : A \to B$ be a morphism in $\mathrm{CAlg}^{\mathrm{art}}_{\kappa}$. Then f is small (when regarded as a morphism in $\mathrm{CAlg}^{\mathrm{aug}}_{\kappa}$) if and only if it induces a surjection of commutative rings $\pi_0 A \to \pi_0 B$.*

Proof. Let K be the fiber of f, regarded as an A-module. If $\pi_0 A \to \pi_0 B$ is surjective, then K is connective. We will prove that f is small by induction on the dimension of the graded vector space $\pi_* K$. If this dimension is zero, then $K \simeq 0$ and f is an equivalence. Assume therefore that $\pi_* K \neq 0$, and let n be the smallest integer such that $\pi_n K \neq 0$. Let $\mathfrak{m}$ denote the maximal ideal of $\pi_0 A$. Then $\mathfrak{m}$ is nilpotent, so $\mathfrak{m}(\pi_n K) \neq \pi_n K$ and we can choose a map of $\pi_0 A$-modules $\phi : \pi_n K \to \kappa$. According to Theorem HA.7.4.3.1 , we have $(2n+1)$-connective map $K \otimes_A B \to \Sigma^{-1}(L_{B/A})$. In particular, we have an isomorphism $\pi_{n+1} L_{B/A} \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_n K)$ so that ϕ determines a map $L_{B/A} \to \Sigma^{n+1}(\kappa)$. We can interpret this map as a derivation $B \to B \oplus \Sigma^{n+1}(\kappa)$; let $B' = B \times_{B \oplus \Sigma^{n+1}(\kappa)} \kappa$. Then f factors as a composition

$$A \xrightarrow{f'} B' \xrightarrow{f''} B.$$

Since the map f'' is elementary, it will suffice to show that f' is small, which follows from the inductive hypothesis. □

Proof of Proposition 12.1.3.7. The implication $(*) \Rightarrow (3)$ is obvious, and the implication $(1) \Rightarrow (*)$ follows from Lemma 12.1.3.8. □

Remark 12.1.3.9. The proof of Proposition 12.1.3.7 shows that condition $(*)$ is equivalent to the *a priori* stronger condition that the diagram

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

is a pullback square whenever *one* of the maps $\pi_0 R_0 \to \pi_0 R_{01}$ or $\pi_0 R_1 \to \pi_0 R_{01}$ is surjective.

12.2 The Tangent Complex

Let X be an algebraic variety over the field $\mathbf{C}$ of complex numbers, and let $x : \operatorname{Spec} \mathbf{C} \to X$ be a point of X. A *tangent vector* to X at the point x is a dotted arrow rendering the diagram

$$\begin{array}{ccc} \operatorname{Spec} \mathbf{C} & \xrightarrow{x} & X \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} \mathbf{C}[\epsilon]/(\epsilon^2) & \longrightarrow & \operatorname{Spec} \mathbf{C} \end{array}$$

commutative. The collection of tangent vectors to X at x comprise a vector space $T_{X,x}$, which we call the *Zariski tangent space* of X at x. If $\mathcal{O}_{X,x}$ denotes the local ring of X at the point x and $\mathfrak{m} \subseteq \mathcal{O}_{X,x}$ its maximal ideal, then there is a canonical isomorphism of vector spaces $T_{X,\eta} \simeq (\mathfrak{m}/\mathfrak{m}^2)^\vee$.

The tangent space $T_{X,x}$ is among the most basic and useful invariants one can use to study the local structure of an algebraic variety X near a point x. Our goal in this section is to generalize the construction of $T_{X,x}$ to the setting of an arbitrary formal moduli problem, in the sense of Definition 12.1.3.1. Let us identify X with its functor of points, given by $X(A) = \operatorname{Hom}(\operatorname{Spec} A, X)$ for every $\mathbf{C}$-algebra A (here the Hom is computed in the category of schemes over $\mathbf{C}$). Then $T_{X,x}$ can be described as the fiber of the map $X(\mathbf{C}[\epsilon]/(\epsilon^2)) \to X(\mathbf{C})$ over the point $x \in X(\mathbf{C})$. Note that the commutative ring $\mathbf{C}[\epsilon]/(\epsilon^2)$ is given by $\Omega^\infty E$, where E is the spectrum object of $\mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}}$ appearing in Example 12.1.1.2. This suggests a possible generalization:

Definition 12.2.0.1. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $Y : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ be a formal moduli problem. For each $\alpha \in T$, the *tangent space of Y at α* is the space $Y(\Omega^\infty E_\alpha)$.

12.2.1 Delooping the Tangent Space

There is a somewhat unfortunate aspect to the terminology of Definition 12.2.0.1. By definition, a formal moduli problem Y is a $\mathcal{S}$-valued functor, so the evaluation of X on any object $A \in \mathbf{A}^{\mathrm{art}}$ might be called a “space”. The term “tangent space” in algebraic geometry has a different meaning: if X is a complex algebraic variety with a base point x, then we refer to $T_{X,x}$ as the tangent *space* of X not because it is equipped with a topology, but because it has the structure of a vector space over $\mathbf{C}$. In particular, $T_{X,x}$ is equipped with an addition which is commutative and associative. This phenomenon is quite general: for any formal moduli problem $Y : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$, each tangent space $Y(\Omega^{\infty} E_\alpha)$ of Y is an infinite loop space, and therefore equipped with a composition law which is commutative and associative up to coherent homotopy.

Recall that if $\mathcal{C}$ is an ∞-category which admits finite limits, then the ∞-category $\mathrm{Sp}(\mathcal{C})$ of *spectrum objects of* $\mathcal{C}$ is defined to be the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ spanned by those functors which are reduced and excisive (Definition HA.1.4.2.8).

Proposition 12.2.1.1. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. For each $\alpha \in T$, we identify $E_\alpha \in \mathrm{Sp}(\mathbf{A})$ with the corresponding functor $\mathcal{S}_*^{\mathrm{fin}} \to \mathbf{A}$. Then:*

(1) *For every map $f : K \to K'$ of pointed finite spaces which induces a surjection $\pi_0 K \to \pi_0 K'$, the induced map $E_\alpha(K) \to E_\alpha(K')$ is a small morphism in $\mathbf{A}$.*

(2) *For every pointed finite space K, the object $E_\alpha(K) \in \mathbf{A}$ is Artinian.*

Proof. We will prove (1); assertion (2) follows by applying (1) to the constant map $K \to *$. Note that f is equivalent to a composition of maps

$$K = K_0 \to K_1 \to \cdots \to K_n = K',$$

where each K_i is obtained from K_{i-1} by attaching a single cell of dimension n_i. Since $\pi_0 K$ surjects onto $\pi_0 K'$, we may assume that each n_i is positive. It follows that we have pushout diagrams of finite pointed spaces

$$\begin{array}{ccc} K_{i-1} & \longrightarrow & K_i \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^{n_i}. \end{array}$$

Since E_α is excisive, we obtain a pullback square

$$\begin{array}{ccc} E_\alpha(K_{i-1}) & \longrightarrow & E_\alpha(K_i) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Omega^{n_i} E_\alpha, \end{array}$$

so that each of the maps $E_\alpha(K_{i-1}) \to E_\alpha(K_i)$ is elementary. □

It follows from Proposition 12.2.1.1 that if $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ is a deformation context, then each E_α can be regarded as a functor from $\mathcal{S}_*^{\mathrm{fin}}$ to the full subcategory $\mathbf{A}^{\mathrm{art}} \subseteq \mathbf{A}$ spanned by the Artinian objects. It therefore makes sense to compose E_α with any formal moduli problem.

Proposition 12.2.1.2. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $Y : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ be a formal moduli problem. For every $\alpha \in T$, the composite functor*

$$\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{E_\alpha} \mathbf{A}^{\mathrm{art}} \xrightarrow{Y} \mathcal{S}$$

is reduced and excisive.

Proof. It is obvious that $Y \circ E_\alpha$ carries initial objects of $\mathcal{S}_*^{\mathrm{fin}}$ to contractible spaces. Suppose we are given a pushout diagram

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \downarrow & & \downarrow \\ L & \longrightarrow & L' \end{array}$$

of pointed finite spaces; we wish to show that the diagram σ :

$$\begin{array}{ccc} Y(E_\alpha(K)) & \longrightarrow & Y(E_\alpha(K')) \\ \downarrow & & \downarrow \\ Y(E_\alpha(L)) & \longrightarrow & Y(E_\alpha(L')) \end{array}$$

is a pullback square in $\mathcal{S}$. Let K'_+ denote the union of those connected components of K' which meet the image of the map $K \to K'$. There is a retraction of K' onto K'_+, which carries the other connected components of K' to the base point. Define L'_+ and the retraction $L' \to L'_+$ similarly. We have a commutative diagram of pointed finite spaces

$$\begin{array}{ccccc} K & \longrightarrow & K' & \longrightarrow & K'_+ \\ \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & L' & \longrightarrow & L'_+ \end{array}$$

where each square is a pushout, hence a diagram of spaces

$$\begin{array}{ccccc} Y(E_\alpha(K)) & \longrightarrow & Y(E_\alpha(K')) & \longrightarrow & Y(E_\alpha(K'_+)) \\ \downarrow & & \downarrow & & \downarrow \\ Y(E_\alpha(L)) & \longrightarrow & Y(E_\alpha(L')) & \longrightarrow & Y(E_\alpha(L'_+)). \end{array}$$

To prove that the left square is a pullback diagrams, it will suffice to show that the right square and the outer rectangle are pullback diagrams. We may therefore reduce to the case where the map $\pi_0 K \to \pi_0 K'$ is surjective. Then the map $\pi_0 L \to \pi_0 L'$ is surjective, so that $E_\alpha(L) \to E_\alpha(L')$ is a small morphism in $\mathbf{A}^{\mathrm{art}}$ (Proposition 12.2.1.1). Since E_α is excisive, the diagram

$$\begin{array}{ccc} E_\alpha(K) & \longrightarrow & E_\alpha(K') \\ \downarrow & & \downarrow \\ E_\alpha(L) & \longrightarrow & E_\alpha(L') \end{array}$$

is a pullback square in $\mathbf{A}$. Using the assumption that Y is a formal moduli problem, we deduce that σ is a pullback square of spaces. □

12.2.2 The Tangent Complex

Proposition 12.2.1.2 motivates the following:

Definition 12.2.2.1. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $Y : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ be a formal moduli problem. For each $\alpha \in T$, we let $Y(E_\alpha)$ denote the composite functor

$$\mathcal{S}^{\mathrm{fin}}_* \xrightarrow{E_\alpha} \mathbf{A}^{\mathrm{art}} \xrightarrow{Y} \mathcal{S}.$$

We will view $Y(E_\alpha)$ as an object in the ∞-category $\mathrm{Sp} = \mathrm{Sp}(\mathcal{S})$ of spectra, and refer to $Y(E_\alpha)$ as the *tangent complex to* Y at α.

Remark 12.2.2.2. In the situation of Definition 12.2.2.1, suppose that T has a single element, so that $\{E_\alpha\}_{\alpha \in T} = \{E\}$ for some $E \in \mathrm{Sp}(\mathbf{A})$ (this condition is satisfied in all of the main examples we will study in this paper). In this case, we will omit mention of the index α and simply refer to $Y(E)$ as the *tangent complex* to the formal moduli problem Y.

Remark 12.2.2.3. Let $Y : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ be as in Definition 12.2.2.1. For every index α, we can identify the tangent space $Y(\Omega^\infty E_\alpha)$ at α with the 0th space of the tangent complex $Y(E_\alpha)$. More generally, there are canonical homotopy equivalences $Y(\Omega^{\infty-n} E_\alpha) \simeq \Omega^{\infty-n} Y(E_\alpha)$ for $n \geqslant 0$.

Example 12.2.2.4. Let $X = \mathrm{Spec}\, R$ be an affine algebraic variety over the field $\mathbf{C}$ of complex numbers, and suppose we are given a point x of X (corresponding to an augmentation $\epsilon : R \to \mathbf{C}$ of the $\mathbf{C}$-algebra R). Then X determines a formal moduli problem $X^\wedge = \mathrm{Spf}\, R : \mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}} \to \mathcal{S}$, given by the formula $X^\wedge(A) = \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}}}(R, A)$ (here we work in the deformation context $(\mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}}, \{E\})$ of Example 12.1.1.2). Unwinding the definitions, we see that the tangent complex $X^\wedge(E)$ can be identified with the spectrum $\mathrm{Mor}_{\mathrm{Mod}_R}(L_{R/\mathbf{C}}, \mathbf{C})$ classifying maps from the cotangent complex $L_{R/\mathbf{C}}$ into $\mathbf{C}$ (regarded as an R-module via the augmentation ϵ). In particular, the homotopy groups of $X^\wedge(E)$ are given by

$$\pi_i X^\wedge(E) \simeq (\pi_{-i}(\mathbf{C} \otimes_R L_{R/\mathbf{C}}))^\vee.$$

It follows that $\pi_i X^\wedge(E)$ vanishes for $i > 0$, and that $\pi_0 X^\wedge(E)$ is isomorphic to the Zariski tangent space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$ of X at the point x. If X is smooth at the point x, then the negative homotopy groups of $\pi_i X^\wedge(E)$ vanish. In general, the homotopy groups $\pi_i X^\wedge(E)$ encode information about the nature of the singularity of X at the point x. One of our goals in Part IV is to articulate a sense in which the tangent complex $X^\wedge(E)$ encodes *complete* information about the local structure of X near the point x.

Warning 12.2.2.5. The terminology of Definition 12.2.2.1 is potentially misleading. For a general deformation context $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ and formal moduli problem $Y : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$, the tangent complexes $Y(E_\alpha)$ are spectra. If κ is a field and $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T}) = (\mathrm{CAlg}^{\mathrm{aug}}_\kappa, \{E\})$ is the deformation context of Example 12.1.1.2, one can show that the tangent complex $Y(E)$ admits the structure of a κ-module spectrum, and can therefore be identified with a chain complex of vector spaces over κ. This observation motivates our use of the term "tangent complex." In the general case, it might be more appropriate to refer to $Y(E_\alpha)$ as a "tangent spectrum" to the formal moduli problem Y.

The tangent complex of a formal moduli problem Y is a powerful invariant of Y. We close this section with a simple illustration:

Proposition 12.2.2.6. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $u : X \to Y$ be a map of formal moduli problems. Suppose that u induces an equivalence of tangent complexes $X(E_\alpha) \to Y(E_\alpha)$ for each $\alpha \in T$. Then u is an equivalence.*

Proof. Consider an arbitrary object $A \in \mathbf{A}^{\mathrm{art}}$, so that there exists a sequence of elementary morphisms

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq *$$

in $\mathbf{A}$. We prove that the map $u(A_i) : X(A_i) \to Y(A_i)$ is a homotopy equivalence using descending induction on i, the case $i = n$ being trivial. Assume therefore that $i < n$ and that $u(A_{i+1})$ is a homotopy equivalence. Since $A_i \to A_{i+1}$ is elementary, we have a fiber sequence of maps

$$u(A_i) \to u(A_{i+1}) \to u(\Omega^{\infty - n} E_\alpha)$$

for some $n > 0$ and $\alpha \in T$. To prove that $u(A_i)$ is a homotopy equivalence, it suffices to show that $u(\Omega^{\infty - n} E_\alpha)$ is a homotopy equivalence, which follows immediately from our assumption that u induces an equivalence $X(E_\alpha) \to Y(E_\alpha)$. □

12.3 Deformation Theories

Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. Our main goal in Part IV is to show that, in many cases of interest, the ∞-category $\mathrm{Moduli}^{\mathbf{A}} \subseteq \mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ admits a concrete algebraic description. To prove this, we will need to be able to address the following question:

(Q) Given an ∞-category $\mathbf{B}$, how can we recognize the existence of an equivalence $\mathrm{Moduli}^{\mathbf{A}} \simeq \mathbf{B}$?

12.3.1 Weak Deformation Theories

We take our first cue from Example 12.1.3.3. If $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T}$ is a deformation context, then to every object $A \in \mathbf{A}$ we can associate a formal moduli problem $\operatorname{Spf} A \in \mathrm{Moduli}^{\mathbf{A}}$ using the formula $(\operatorname{Spf} A)(R) = \mathrm{Map}_{\mathbf{A}}(A, R)$. Composing this construction with a prospective equivalence $\mathrm{Moduli}^{\mathbf{A}} \simeq \mathbf{B}$, we obtain a functor $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$. We begin by axiomatizing the expected properties of such a functor:

Definition 12.3.1.1. A *weak deformation theory* for a deformation context $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ is a functor $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ satisfying the following axioms:

($D1$) The ∞-category $\mathbf{B}$ is presentable.

($D2$) The functor $\mathfrak{D}$ admits a left adjoint $\mathfrak{D}' : \mathbf{B} \to \mathbf{A}^{\mathrm{op}}$.

($D3$) There exists a full subcategory $\mathbf{B}_0 \subseteq \mathbf{B}$ satisfying the following conditions:

- (a) For every object $K \in \mathbf{B}_0$, the unit map $K \to \mathfrak{D}\mathfrak{D}'K$ is an equivalence.
- (b) The full subcategory $\mathbf{B}_0$ contains the initial object $\varnothing \in \mathbf{B}$. It then follows from ($a$) that $\varnothing \simeq \mathfrak{D}\mathfrak{D}'\varnothing \simeq \mathfrak{D}(*)$, where $*$ denotes the final object of $\mathbf{A}$.
- (c) For every index $\alpha \in T$ and every $n \geqslant 1$, there exists an object $K_{\alpha,n} \in \mathbf{B}_0$ and an equivalence $\Omega^{\infty-n} E_\alpha \simeq \mathfrak{D}' K_{\alpha,n}$. It follows that the base point of $\Omega^{\infty-n} E_\alpha$ determines a map

$$v_{\alpha,n} : K_{\alpha,n} \simeq \mathfrak{D}\mathfrak{D}' K_{\alpha,n} \simeq \mathfrak{D}(\Omega^{\infty-n} E_\alpha) \to \mathfrak{D}(*) \simeq \varnothing.$$

- (d) For every pushout diagram

$$\begin{array}{ccc} K_{\alpha,n} & \longrightarrow & K \\ \big\downarrow{\scriptstyle v_{\alpha,n}} & & \big\downarrow \\ \varnothing & \longrightarrow & K' \end{array}$$

where $\alpha \in T$ and $n > 0$, if K belongs to to $\mathbf{B}_0$ then K' also belongs to $\mathbf{B}_0$.

Definition 12.3.1.1 might seem a bit complicated at a first glance. We can summarize axioms ($D2$) and ($D3$) informally by saying that the functor $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ is not far from being an equivalence. Axiom ($D2$) requires that there exists an adjoint $\mathfrak{D}'$ to $\mathfrak{D}$, and axiom ($D3$) requires that $\mathfrak{D}'$ behave as a homotopy inverse to $\mathfrak{D}$, at least on a subcategory $\mathbf{B}_0 \subseteq \mathbf{B}$ with good closure properties.

Example 12.3.1.2. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. The functor $\mathrm{Spf} : \mathbf{A}^{\mathrm{op}} \to \mathrm{Moduli}^{\mathbf{A}}$ of Example 12.1.3.3 satisfies conditions $(D1)$, $(D2)$ and $(D3)$ of Definition 12.3.1.1, and therefore defines a weak deformation theory (we can define the full subcategory $\mathrm{Moduli}^{\mathbf{A}}_0 \subseteq \mathrm{Moduli}^{\mathbf{A}}$ whose existence is required by $(D3)$ to be spanned by objects of the form $\mathrm{Spf}(A)$, where $A \in \mathbf{A}^{\mathrm{art}}$).

Example 12.3.1.3. Let κ be a field of characteristic zero and let $(\mathrm{CAlg}^{\mathrm{aug}}_\kappa, \{E\})$ be the deformation context described in Example 12.1.1.2. In Chapter 13, we will construct a weak deformation theory $\mathfrak{D} : (\mathrm{CAlg}^{\mathrm{aug}}_\kappa)^{\mathrm{op}} \to \mathrm{Lie}_\kappa$, where Lie_κ denotes the ∞-category of differential graded Lie algebras over κ (Definition 13.1.4.1). Here the adjoint functor $\mathfrak{D}' : \mathrm{Lie}_\kappa \to (\mathrm{CAlg}^{\mathrm{aug}}_\kappa)^{\mathrm{op}}$ assigns to each differential graded Lie algebra $\mathfrak{g}_*$ its cohomology Chevalley-Eilenberg complex $C^*(\mathfrak{g}_*)$ (see Construction 13.2.5.1). In fact, the functor $\mathfrak{D} : (\mathrm{CAlg}^{\mathrm{aug}}_\kappa)^{\mathrm{op}} \to \mathrm{Lie}_\kappa$ is even a *deformation theory*: it satisfies condition $(D4)$ appearing in Definition 12.3.3.2 below.

Remark 12.3.1.4. By virtue of Corollary HTT.5.5.2.9 and Remark HTT.5.5.2.10 , condition $(D2)$ of Definition 12.3.1.1 is equivalent to the requirement that the functor $\mathfrak{D}$ preserves small limits.

Remark 12.3.1.5. In the situation of Definition 12.3.1.1, the objects $K_{\alpha,n} \in \mathbf{B}_0$ are determined up to canonical equivalence: it follows from (a) that they are given by $K_{\alpha,n} \simeq \mathfrak{D}\mathfrak{D}' K_{\alpha,n} \simeq \mathfrak{D}(\Omega^{\infty-n} E_\alpha)$. In particular, Definition 12.3.1.1 requires that the objects $\Omega^{\infty-n} E_\alpha$ belong to $\mathbf{B}_0$.

12.3.2 Formal Moduli Problems from Weak Deformation Theories

Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$. According to Example 12.3.1.2, we can always regard the formal spectrum construction $\mathrm{Spf} : \mathbf{A}^{\mathrm{op}} \to \mathrm{Moduli}^{\mathbf{A}}$ as a weak deformation theory. Our next goal is to show that the functor Spf is actually *universal* with respect to this property:

Proposition 12.3.2.1. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, let $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ a weak deformation theory, and let $j : \mathbf{B} \to \mathrm{Fun}(\mathbf{B}^{\mathrm{op}}, \mathcal{S})$ denote the Yoneda embedding. Then:*

(a) *For every object $B \in \mathbf{B}$, the composition*

$$\mathbf{A}^{\mathrm{art}} \subseteq \mathbf{A} \xrightarrow{\mathfrak{D}} \mathbf{B}^{\mathrm{op}} \xrightarrow{j(B)} \mathcal{S}$$

is a formal moduli problem.

(b) *The construction $(B \in \mathbf{B}) \mapsto (j(B) \circ \mathfrak{D})|_{\mathbf{A}^{\mathrm{art}}}$ determines a functor $\Psi : \mathbf{B} \to \mathrm{Moduli}^{\mathbf{A}}$.*

(*c*) *The diagram*

$$\begin{array}{ccc} \mathbf{A}^{\mathrm{op}} & \xrightarrow{\mathrm{Spf}} & \mathrm{Moduli}^{\mathbf{A}} \\ & \searrow^{\mathfrak{D}} \quad \nearrow_{\Psi} & \\ & \mathbf{B} & \end{array}$$

commutes up to (canonical) homotopy.

Proposition 12.3.2.1 is an easy consequence of the following result, which summarizes some of the formal properties of Definition 12.3.1.1:

Proposition 12.3.2.2. *Let* $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ *be a deformation context and* $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ *a weak deformation theory. Let* $\mathbf{B}_0 \subseteq \mathbf{B}$ *be a full subcategory which is stable under equivalence and satisfies condition* (3) *of Definition 12.3.1.1. Then:*

(1) *The functor* $\mathfrak{D}$ *carries final objects of* $\mathbf{A}$ *to initial objects of* $\mathbf{B}$.

(2) *Let* $A \in \mathbf{A}$ *be an object having the form* $\mathfrak{D}'(K)$, *where* $K \in \mathbf{B}_0$. *Then the unit map* $A \to \mathfrak{D}'\mathfrak{D}(A)$ *is an equivalence in* $\mathbf{A}$.

(3) *If* $A \in \mathbf{A}$ *is Artinian, then* $\mathfrak{D}(A) \in \mathbf{B}_0$ *and the unit map* $A \to \mathfrak{D}'\mathfrak{D}A$ *is an equivalence in* $\mathbf{A}$.

(4) *Suppose we are given a pullback diagram* σ :

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ A & \longrightarrow & B \end{array}$$

in $\mathbf{A}$, *where* A *and* B *are Artinian and the morphism* ϕ *is small. Then* $\mathfrak{D}(\sigma)$ *is a pushout diagram in* $\mathbf{B}$.

Proof of Proposition 12.3.2.1. Part (*a*) is an immediate consequence of assertions (1) and (4) of Proposition 12.3.2.2, and (*b*) is a restatement of (*a*). To prove (*c*), we note that for $A \in \mathbf{A}$ and $A' \in \mathbf{A}^{\mathrm{art}}$, we have canonical homotopy equivalences

$$\begin{array}{rcl} \Psi(\mathbf{D}(A))(A') & = & \mathrm{Map}_{\mathbf{B}}(\mathfrak{D}(A'), \mathfrak{D}(A)) \\ & \simeq & \mathrm{Map}_{\mathbf{A}}(A, \mathfrak{D}'\mathfrak{D}(A')) \\ & \xleftarrow{\sim} & \mathrm{Map}_{\mathbf{A}}(A, A') \\ & = & (\mathrm{Spf}\, A)(A') \end{array}$$

where the displayed arrow is a homotopy equivalence by virtue of assertion (3) of Proposition 12.3.2.2. $\square$

Proof of Proposition 12.3.2.2. Let $\varnothing$ denote an initial object of $\mathbf{B}$. Then $\varnothing \in \mathbf{B}_0$ so that the adjunction map $\varnothing \to \mathfrak{D}\mathfrak{D}'\varnothing$ is an equivalence. Since $\mathfrak{D}' : \mathbf{B} \to \mathbf{A}^{\mathrm{op}}$ is left adjoint to $\mathfrak{D}$, it carries $\varnothing$ to a final object $* \in \mathbf{A}$. This proves (1). To prove (2), suppose that $A = \mathfrak{D}'(K)$ for $K \in \mathbf{B}_0$. Then the unit map $u : A \to \mathfrak{D}'\mathfrak{D}A$ has a left homotopy inverse, given by applying $\mathfrak{D}'$ to the the map $v : K \to \mathfrak{D}\mathfrak{D}'K$ in $\mathbf{B}$. Since v is an equivalence (part (a) of Definition 12.3.1.1), we conclude that u is an equivalence.

We now prove (3). Let $A \in \mathbf{A}$ be Artinian, so that there exists a sequence of elementary morphisms

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq *.$$

We will prove that $\mathfrak{D}A_i \in \mathbf{B}_0$ using descending induction on i. If $i = n$, the desired result follows from (1). Assume therefore that $i < n$, so that the inductive hypothesis guarantees that $\mathfrak{D}(A_{i+1}) \in \mathbf{B}_0$. Choose a pullback diagram σ :

$$\begin{array}{ccc} A_i & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ A_{i+1} & \xrightarrow{\psi} & \Omega^{\infty-n}E_\alpha \end{array}$$

where $n > 0$, $\alpha \in T$, and ϕ is the base point of $\Omega^{\infty-n}E_\alpha$. Form a pushout diagram τ :

$$\begin{array}{ccc} \mathfrak{D}(\Omega^{\infty-n}E_\alpha) & \xrightarrow{\mathfrak{D}\psi} & \mathfrak{D}A_{i+1} \\ \downarrow{\scriptstyle \mathfrak{D}\phi} & & \downarrow \\ \mathfrak{D}(*) & \longrightarrow & X \end{array}$$

in $\mathbf{B}$. There is an evident transformation $\xi : \sigma \to \mathfrak{D}'(\tau)$ of diagrams in $\mathbf{A}$. Since both σ and $\mathfrak{D}'(\tau)$ are pullback diagrams and the objects A_{i+1}, $\Omega^{\infty-n}E_\alpha$, and $*$ belong to the essential image of $\mathfrak{D}'|_{\mathbf{B}_0}$, it follows from assertion (2) that ξ is an equivalence, so that $A_i \simeq \mathfrak{D}'(X)$. Assumption (d) of Definition 12.3.1.1 guarantees that $X \in \mathbf{B}_0$, so that A_i lies in the essential image of $\mathfrak{D}'|_{\mathbf{B}_0}$.

We now prove (4). The class of morphisms ϕ for which the conclusion holds (for an arbitrary morphism $A \to B$ between Artinian objects of $\mathbf{A}$) is evidently stable under composition. We may therefore reduce to the case where ϕ is elementary, and further to the case where ϕ is the base point map $* \to \Omega^{\infty-n}E_\alpha$ for some $\alpha \in T$ and some $n > 0$. Arguing as above, we deduce that the pullback diagram σ :

$$\begin{array}{ccc} A' & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ A & \longrightarrow & \Omega^{\infty-n}E_\alpha \end{array}$$

is equivalent to $\mathfrak{D}'(\tau)$, where τ is a diagram in $\mathbf{B}_0$ which is a pushout square in $\mathbf{B}$. Then $\mathfrak{D}(\sigma) \simeq \mathfrak{D}\mathfrak{D}'(\tau) \simeq \tau$ is a pushout diagram, by virtue of condition (a) of Definition 12.3.1.1. □

12.3.3 Deformation Theories

Our next goal is to formulate an additional hypothesis on a weak deformation theory $\mathbf{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ which is sufficient (but not necessary: see Warning 12.3.3.3) to guarantee that the functor $\Psi : \mathbf{B} \to \mathrm{Moduli}^{\mathbf{A}}$ is an equivalence of ∞-categories. First, we note the following immediate consequence of Propositions 12.3.2.1 and 12.2.1.2:

Proposition 12.3.3.1. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ a weak deformation theory. For each $\alpha \in T$ and each $K \in \mathbf{B}$, the composite map*

$$\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{E_\alpha} \mathbf{A} \xrightarrow{\mathfrak{D}} \mathbf{B}^{\mathrm{op}} \xrightarrow{j(K)} \mathcal{S}$$

is reduced and excisive, and can therefore be identified with a spectrum which we will denote by $e_\alpha(K)$. This construction determines a functor $e_\alpha : \mathbf{B} \to \mathrm{Sp} = \mathrm{Sp}(\mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$.

Definition 12.3.3.2. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. A *deformation theory* for $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ is a weak deformation theory $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ which satisfies the following additional condition:

$(D4)$ For each $\alpha \in T$, let $e_\alpha : \mathbf{B} \to \mathrm{Sp}$ be the functor described in Proposition 12.3.3.1. Then e_α preserves small sifted colimits. Moreover, a morphism f in $\mathbf{B}$ is an equivalence if and only if each $e_\alpha(f)$ is an equivalence of spectra.

Warning 12.3.3.3. Let $(\mathbf{A}, \{X_\alpha\}_{\alpha \in T})$ be a deformation context, and let $\mathrm{Spf} : \mathbf{A}^{\mathrm{op}} \to \mathrm{Moduli}^{\mathbf{A}}$ be given by the Yoneda embedding (see Example 12.3.1.2). The resulting functors $e_\alpha : \mathrm{Moduli}^{\mathbf{A}} \to \mathrm{Sp}$ are then given by evaluation on the spectrum objects $E_\alpha \in \mathrm{Sp}(\mathbf{A})$, and are therefore jointly conservative (Proposition 12.2.2.6) and preserve filtered colimits. However, it is not clear that Spf is a deformation theory, since the tangent complex constructions $X \mapsto X(E_\alpha)$ do not obviously commute with sifted colimits.

Remark 12.3.3.4. Let $(\mathbf{A}, \{E\})$ be a deformation context, let $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ be a deformation theory for $\mathbf{A}$, and let $e : \mathbf{B} \to \mathrm{Sp}$ be as in Proposition 12.3.3.1. The functor e preserves small limits, and condition $(D4)$ of Definition 12.3.3.2 implies that e preserves sifted colimits. It follows that e admits a left adjoint $F : \mathrm{Sp} \to \mathbf{B}$. The composite functor $e \circ F$ has the structure of a monad U on Sp. Since e is conservative and commutes with sifted colimits, Theorem HA.4.7.3.5 gives us an equivalence of ∞-categories $\mathbf{B} \simeq \mathrm{LMod}_U(\mathrm{Sp})$ with the ∞-category of algebras over the monad U. In other words, we can think of $\mathbf{B}$ as an

∞-category whose objects are spectra which are equipped some additional structure (namely, a left action of the monad U).

More generally, if $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ is a deformation context equipped with a deformation theory $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$, the same argument supplies an equivalence $\mathbf{B} \simeq \mathrm{LMod}_U(\mathrm{Sp}^T)$: that is, we can think of objects of $\mathbf{B}$ as determines by a collection of spectra (indexed by T), together with some additional structure.

We can now formulate the main result of this section:

Theorem 12.3.3.5. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ be a deformation theory. Then the functor $\Psi : \mathbf{B} \to \mathrm{Moduli}^{\mathbf{A}}$ of Proposition 12.3.2.1 is an equivalence of ∞-categories.*

The proof of Theorem 12.3.3.5 will be given in §12.5.

Remark 12.3.3.6. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ a deformation theory. Using the commutativity of the diagram

$$\begin{array}{ccc} \mathbf{A}^{\mathrm{op}} & \xrightarrow{\mathrm{Spf}} & \mathrm{Moduli}^{\mathbf{A}} \\ & {\scriptstyle \mathfrak{D}} \searrow \quad \nearrow {\scriptstyle \Psi} & \\ & \mathbf{B} & \end{array}$$

(see Proposition 12.3.2.1) and Theorem 12.3.3.5, we see that $\mathfrak{D}$ can be identified (in an essentially unique way) with the formal spectrum functor $\mathrm{Spf} : \mathbf{A}^{\mathrm{op}} \to \mathrm{Moduli}^{\mathbf{A}}$. In other words, if the deformation context $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T}$ admits a deformation theory, then that deformation theory is essentially unique. The existence is not automatic (see Warning 12.3.3.3): it is equivalent to the requirement that each of the tangent complex functors $\mathrm{Moduli}^{\mathbf{A}} \to \mathrm{Sp}$ commutes with sifted colimits.

Remark 12.3.3.7. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. Then the ∞-category $\mathbf{A}^{\mathrm{art}}$ admits finite products. Let $\mathcal{P}_\Sigma(\mathbf{A}^{\mathrm{art,op}})$ be the full subcategory of $\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ spanned by those functors which preserve finite products. Using Proposition HTT.5.5.8.15 , we deduce that for each index $\alpha \in T$, there is an essentially unique functor $F_\alpha : \mathcal{P}_\Sigma(\mathbf{A}^{\mathrm{art,op}}) \to \mathrm{Sp}$ which preserves sifted colimits, having the property that the composite functor

$$\mathbf{A}^{\mathrm{art,op}} \xhookrightarrow{j} \mathcal{P}_\Sigma(\mathbf{A}^{\mathrm{art,op}}) \xrightarrow{F_\alpha} \mathrm{Sp}$$

is given by $A \mapsto (\mathrm{Spf}(A))(E_\alpha)$; here j denotes the Yoneda embedding. Note that the ∞-category $\mathcal{P}_\Sigma(\mathbf{A}^{\mathrm{art,op}})$ contains $\mathrm{Moduli}^{\mathbf{A}}$, and that each of the functors F_α is a left Kan extension of its restriction to the full subcategory of $\mathrm{Moduli}^{\mathbf{A}}$ spanned by those formal moduli problems of the form $\mathrm{Spf}(A)$, where $A \in \mathbf{A}^{\mathrm{art}}$. In particular, for every formal moduli problem X, we have a canonical map $\theta_X : F_\alpha(X) \to X(E_\alpha)$. We claim that each of these maps is an equivalence. To prove this, choose a smooth hypercovering $X_\bullet$ of X, where each X_n is prorepresentable. Then θ_X is a colimit of the maps θ_{X_n}, each of which is a filtered colimit of a collection of maps of the form $\theta_{\mathrm{Spf}(A)}$, hence an equivalence.

Let $L : \mathcal{P}_\Sigma(\mathbf{A}^{\mathrm{art,op}}) \to \mathrm{Moduli}^{\mathbf{A}}$ denote a left adjoint to the inclusion. We claim that the following conditions are equivalent:

(1) For each $\alpha \in T$, the functor F_α carries L-equivalences to equivalences in the ∞-category Sp (in other words, each of the functors F_α factors through L, up to homotopy).

(2) For each $\alpha \in T$, the construction $X \mapsto X(E_\alpha)$ determines a functor $\mathrm{Moduli}^{\mathbf{A}} \to \mathrm{Sp}$ which commutes with sifted colimits.

(3) The deformation context $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ admits a deformation theory $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$.

The equivalence (2) $\Leftrightarrow$ (3) follows from Remark 12.3.3.6. Note that (2) is equivalent to the requirement that each of the constructions $X \mapsto (LX)(E_\alpha)$ determines a functor $F'_\alpha : \mathcal{P}_\Sigma(\mathbf{A}^{\mathrm{art,op}}) \to \mathrm{Sp}$ which commutes with sifted colimits. Note that F_α and F'_α agree on objects of the form $\mathrm{Spf}(A)$, so there is a canonical natural transformation $\mu_\alpha : F_\alpha \to F'_\alpha$. Conditions (1) and (2) are both equivalent to the requirement that each μ_α is an equivalence.

12.4 Digression: The Small Object Argument

Let $\mathcal{C}$ be a category containing a collection of morphisms $\{f_\alpha : C_\alpha \to D_\alpha\}$, and let $g : X \to Z$ be another morphism in $\mathcal{C}$. Under some mild hypotheses, Quillen's *small object argument* can be used to produce a factorization $X \xrightarrow{g'} Y \xrightarrow{g''} Z$ where g' is "built from" the morphisms f_α, and g'' has the right lifting property with respect to the morphisms f_α (see §HTT.A.1.2 for a more detailed discussion). The small object argument was used by Grothendieck to prove that every Grothendieck abelian category has enough injective objects (see [87] or Corollary HA.1.3.5.7). It is now a basic tool in the theory of model categories. Our goal in this section is to formulate an ∞-categorical version of the small object argument (Proposition 12.4.2.1).

12.4.1 Lifting Properties and Saturation

We begin by introducing some terminology.

Definition 12.4.1.1. Let $\mathcal{C}$ be an ∞-category. Let $f : C \to D$ and $g : X \to Y$ be morphisms in $\mathcal{C}$. We will say that g has the *right lifting property* with respect to f if every commutative

diagram

$$\begin{array}{ccc} C & \longrightarrow & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ D & \longrightarrow & Y \end{array}$$

can be extended to a 3-simplex of $\mathcal{C}$, as depicted by the diagram

$$\begin{array}{ccc} C & \longrightarrow & X \\ \downarrow{\scriptstyle f} & \nearrow & \downarrow{\scriptstyle g} \\ D & \longrightarrow & Y. \end{array}$$

In this case, we will also say that f has the *left lifting property* with respect to g.

More generally, if S is any set of morphisms in $\mathcal{C}$, we will say that a morphism g has the *right lifting property* with respect to S if it has the right lifting property with respect to every morphism in S, and that a morphism f has the *left lifting property* with respect to S if f has the left lifting property with respect to every morphism in S.

Definition 12.4.1.2. Let $\mathcal{C}$ be an ∞-category and let S be a collection of morphisms in $\mathcal{C}$. We will say that a morphism f in $\mathcal{C}$ is a *transfinite pushout of morphisms in* S if there exists an ordinal α and a diagram $F : [\alpha] \to \mathcal{C}$ (here $[\alpha]$ denotes the linearly ordered set of ordinals $\{\beta : \beta \leqslant \alpha\}$) with the following properties:

(1) For every nonzero limit ordinal $\lambda \leqslant \alpha$, the restriction $F|_{[\lambda]}$ is a colimit diagram in $\mathcal{C}$.

(2) For every ordinal $\beta < \alpha$, the morphism $F(\beta) \to F(\beta + 1)$ is a pushout of a morphism in S.

(3) The morphism $F(0) \to F(\alpha)$ coincides with f.

Remark 12.4.1.3. Let $\mathcal{C}$ be an ∞-category, and let S and T be collections of morphisms in $\mathcal{C}$. Suppose that every morphism belonging to T is a transfinite pushout of morphisms in S. If f is a transfinite pushout of morphisms in T, then f is a transfinite pushout of morphisms in S.

Definition 12.4.1.4. Let $\mathcal{C}$ be an ∞-category and let S be a collection of morphisms in $\mathcal{C}$. We will say that S is *weakly saturated* if it has the following properties:

(1) If f is a morphism in $\mathcal{C}$ which is a transfinite pushout of morphisms in S, then $f \in S$.

(2) The set S is closed under retracts. In other words, if we are given a commutative diagram

$$\begin{array}{ccccc} C & \longrightarrow & C' & \longrightarrow & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ D & \longrightarrow & D' & \longrightarrow & D \end{array}$$

in which both horizontal compositions are the identity and f' belongs to S, then so does f.

Remark 12.4.1.5. If $\mathcal{C}$ is the nerve of an ordinary category (which admits small colimits), then Definition 12.4.1.4 reduces to Definition HTT.A.1.2.2 .

Remark 12.4.1.6. Let S be a weakly saturated collection of morphisms in an ∞-category $\mathcal{C}$. Any identity map in $\mathcal{C}$ can be written as a transfinite composition of morphisms in S (take $\alpha = 0$ in Definition 12.4.1.2). Condition (2) of Definition 12.4.1.4 guarantees that the class of morphisms is stable under equivalence; it follows that every equivalence in $\mathcal{C}$ belongs to S. Condition (1) of Definition 12.4.1.4 also implies that S is closed under composition (take $\alpha = 2$ in Definition 12.4.1.2).

12.4.2 The Small Object Argument for ∞-Categories

We can now formulate our main result:

Proposition 12.4.2.1 (Small Object Argument)**.** *Let $\mathcal{C}$ be a presentable ∞-category and let S be a small collection of morphisms in $\mathcal{C}$. Then every morphism $f : X \to Z$ admits a factorizaton $X \xrightarrow{f'} Y \xrightarrow{f''} Z$ where f' is a transfinite pushout of morphisms in S and f'' has the right lifting property with respect to S.*

Warning 12.4.2.2. In contrast with the ordinary categorical setting (see Proposition HTT.A.1.2.5), the factorization $X \xrightarrow{f'} Y \xrightarrow{f''} Z$ of Proposition 12.4.2.1 cannot generally be chosen to depend functorially on f.

Proof of Proposition 12.4.2.1. Let $S = \{g_i : C_i \to D_i\}_{i \in I}$. Choose a regular cardinal κ such that each of the objects C_i is κ-compact. We construct a diagram $F : [\kappa] \to \mathcal{C}_{/Z}$ as the union of maps $\{F_\alpha : [\alpha] \to \mathcal{C}_{/Z}\}_{\alpha \leqslant \kappa}$; here $[\alpha]$ denotes the linearly ordered set of ordinals $\{\beta : \beta \leqslant \alpha\}$. The construction proceeds by induction: we let F_0 be the morphism $f : X \to Z$, and for a nonzero limit ordinal $\lambda \leqslant \kappa$ we let F_λ be a colimit of the diagram obtained by amalgamating themaps $\{F_\alpha\}_{\alpha < \lambda}$. Assume that $\alpha < \kappa$ and that F_α been constructed. Then $F_\alpha(\alpha)$ corresponds to a map $X' \to Z$. Let $T(\alpha)$ be a set of representantives for all equivalence classes of diagrams equivalence classes of diagrams σ_t :

$$\begin{array}{ccc} C_t & \longrightarrow & X' \\ \big\downarrow{\scriptstyle g_t} & & \big\downarrow \\ D_t & \longrightarrow & Z, \end{array}$$

where g_t is a morphism belonging to S. Choose a pushout diagram

$$\begin{array}{ccc} \amalg_{t \in T(\alpha)} C_t & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \amalg_{t \in T(\alpha)} D_t & \longrightarrow & X'' \end{array}$$

in $\mathcal{C}_{/Z}$. We regard X'' as an object of $\mathcal{C}_{X'//Z}$. Since the map $(\mathcal{C}_{/Z})_{F_\alpha/} \to \mathcal{C}_{X'//Z}$ is a trivial Kan fibration, we can lift X' to an object of $(\mathcal{C}_{/Z})_{F_\alpha/}$, which determines the desired map $F_{\alpha+1}$.

For each $\alpha \leqslant \kappa$, let $f_\alpha : Y_\alpha \to Z$ be the image $F(\alpha) \in \mathcal{C}_{/Z}$. Let $Y = Y_\kappa$ and $f'' = f_\kappa$. We claim that f'' has the right lifting property with respect to every morphism in S. In other words, we wish to show that for each $i \in I$ and every map $D_i \to Z$, the induced map

$$\mathrm{Map}_{\mathcal{C}_{/Z}}(D_i, Y) \to \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y)$$

is surjective on connected components. Choose a point $\eta \in \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y)$. Since C_i is κ-compact, the space $\mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y)$ can be realized as the filtered colimit of mapping spaces $\varinjlim_\alpha \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y_\alpha)$, so we may assume that η is the image of a point $\eta_\alpha \in \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y_\alpha)$ for some $\alpha < \kappa$. The point η_α determines a commutative diagram

$$\begin{array}{ccc} C_i & \longrightarrow & Y_\alpha \\ \downarrow{\scriptstyle g_i} & & \downarrow \\ D_i & \longrightarrow & Z \end{array}$$

which is equivalent to σ_t for some $t \in T(\alpha)$. It follows that the image of η_α in $\mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y_{\alpha+1})$ extends to D_i, so that η lies in the image of the map $\mathrm{Map}_{\mathcal{C}_{/Z}}(D_i, Y_{\alpha+1}) \to \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y)$.

The morphism $F(0) \to F(\kappa)$ in $\mathcal{C}_{/Z}$ induces a morphism $f' : X \to Y$ in $\mathcal{C}$; we will complete the proof by showing that f' is a transfinite pushout of morphisms in S. Using Remark 12.4.1.3, we are reduced to showing that for each $\alpha < \kappa$, the map $Y_\alpha \to Y_{\alpha+1}$ is a transfinite pushout of morphisms in S. To prove this, choose a well-ordering of $T(\alpha)$ having order type β. For $\gamma < \beta$, let t_γ denote the corresponding element of $T(\alpha)$. We define a functor $G : [\beta] \to \mathcal{C}$ so that, for each $\beta' \leqslant \beta$, we have a pushout diagram

$$\begin{array}{ccc} \amalg_{\gamma < \beta'} C_{t_\gamma} & \longrightarrow & Y_\alpha \\ \downarrow & & \downarrow \\ \amalg_{\gamma < \beta'} D_{t_\gamma} & \longrightarrow & G(\beta'). \end{array}$$

It is easy to see that G satisfies the conditions of Definition 12.4.1.2 and therefore exhibits $Y_\alpha \to Y_{\alpha+1}$ as a transfinite pushout of morphisms in S. □

12.4.3 Applications of the Small Object Argument

To apply Proposition 12.4.2.1, the following observation is often useful:

Proposition 12.4.3.1. *Let $\mathcal{C}$ be an ∞-category and let T be a collection of morphisms in $\mathcal{C}$. Let S denote the collection of all morphisms in $\mathcal{C}$ which have the left lifting property with respect to T. Then S is weakly saturated.*

Proof. Since the intersection of a collection of weakly saturated collections is weakly saturated, it will suffice to treat the case where T consists of a single morphism $g : X \to Y$. Note that a morphism $f : C \to D$ has the left lifting property with respect to g if and only if, for every lifting of Y to $\mathcal{C}_{f/}$, the induced map $\theta_f : \mathcal{C}_{f//Y} \to \mathcal{C}_{C//Y}$ is surjective on objects which lie over $g \in \mathcal{C}_{/Y}$. Since θ_f is a left fibration, it is a categorical fibration; it therefore suffices to show that object of $\mathcal{C}_{C//Y}$ which lies over g is in the essential image of θ_f. We begin by showing that S is stable under pushouts. Suppose we are given a pushout diagram σ :

$$\begin{array}{ccc} C' & \xrightarrow{f'} & D' \\ \downarrow{\scriptstyle \lambda} & & \downarrow \\ C & \xrightarrow{f} & D \end{array}$$

in $\mathcal{C}$, where $f' \in S$. We wish to prove that $f \in S$. Consider a lifting of Y to $\mathcal{C}_{f/}$ which we can lift further to $\mathcal{C}_{\sigma/}$. The map θ_f is equivalent to the left fibration $\mathcal{C}_{\sigma//Y} \to \mathcal{C}_{\lambda/Y}$. Since σ is a pushout diagram, this is equivalent to the map $\theta : \mathcal{C}_{f'//Y} \times_{\mathcal{C}_{C'//Y}} \mathcal{C}_{\lambda//Y} \to \mathcal{C}_{\lambda//Y}$. It will therefore suffice to show that every lifting of g to $\mathcal{C}_{\lambda//Y}$ lies in the essential image of θ, which follows from our assumption that every lifting of g to $\mathcal{C}_{C'//Y}$ lies in the essential image of $\theta_{f'}$.

We now verify condition (1) of Definition 12.4.1.4. Fix an ordinary α and a diagram $F : [\alpha] \to \mathcal{C}$ satisfying the hypotheses of Definition 12.4.1.2, and let $f : F(0) \to F(\alpha)$ be the induced map. Choose a lifting of Y to $\mathcal{C}_{f/}$ which we can lift further to $\mathcal{C}_{F/}$. Then θ_f is equivalent to the map $\theta : \mathcal{C}_{F//Y} \to \mathcal{C}_{F(0)//Y}$. It will therefore suffice to show that every lift of g to an object of $\overline{X} \in \mathcal{C}_{F(0)//Y}$ lies in the image of θ. For each $\beta < \alpha$, we let $F_\beta = F|_{[\beta]}$; we will construct a compatible sequence of objects $\overline{X}_\beta \in \mathcal{C}_{F_\beta//Y}$ by induction on β. If $\beta = 0$, we take $\overline{X}_\beta = \overline{X}$. If β is a nonzero limit ordinal, then our assumption that F_β is a colimit diagram guarantees that the map $\mathcal{C}_{F_\beta//Y} \to \varprojlim_{\gamma<\beta} \mathcal{C}_{F_\gamma//Y}$ is a trivial Kan fibration so that $\overline{X}_\beta$ can be defined. It remains to treat the case of a successor ordinal: let $\beta < \alpha$ and assume that $\overline{X}_\beta$ has been defined; we wish to show that the vertex $\overline{X}_\beta$ lies in the image of the map $\theta_\beta : \mathcal{C}_{F_{\beta+1}//Y} \to \mathcal{C}_{F_\beta//Y}$. Let $u : F(\beta) \to F(\beta+1)$ be the morphism determined by F, so that θ_β is equivalent to the map θ_u. Since the image of $\overline{X}_\beta$ in $\mathcal{C}_{F(\beta)//Y}$ lies over g, the existence of the desired lifting follows from our assumption that $u \in S$.

We now verify (2). Consider a diagram $\sigma : \Delta^2 \times \Delta^1 \to \mathcal{C}$, given by

$$\begin{array}{ccccc} C & \xrightarrow{\lambda} & C' & \longrightarrow & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ D & \longrightarrow & D' & \longrightarrow & D. \end{array}$$

Assume that $f' \in S$; we wish to prove $f \in S$. Choose a lifting of Y to $\mathcal{C}_{f/}$, and lift Y further to $\mathcal{C}_{\sigma/}$ (here we identify f with $\sigma|_{\{2\} \times \Delta^1}$. Let $\overline{X}$ be a lifting of g to $\mathcal{C}_{C//Y}$; we wish to show that $\overline{X}$ lies in the image of θ_f. We can lift $\overline{X}$ further to an object $\widetilde{X} \in \mathcal{C}_{\sigma_0//Y}$, where $\sigma_0 = \sigma|_{\Delta^2 \times \{0\}}$. Let $\sigma' = \sigma|_{\Delta^1 \times \Delta^1}$. The forgetful functor $\theta : \mathcal{C}_{\sigma'//Y} \to \mathcal{C}_{\lambda//Y}$ is equivalent to $\theta_{f'}$, so that the image of $\widetilde{X}$ in $\mathcal{C}_{\lambda//Y}$ lies in the image of θ. It follows immediately that $\overline{X}$ lies in the image of θ_f. □

Corollary 12.4.3.2. *Let $\mathcal{C}$ be a presentable ∞-category, let S be a small collection of morphisms of $\mathcal{C}$, let T be the collection of all morphisms in $\mathcal{C}$ which have the right lifting property with respect to every morphism in S, and let $S^\vee$ be the collection of all morphisms in $\mathcal{C}$ which have the left lifting property with respect to every morphism in T. Then $S^\vee$ is the smallest weakly saturated collection of morphisms which contains S.*

Proof. Proposition 12.4.3.1 implies that $S^\vee$ is weakly saturated, and it is obvious that $S^\vee$ contains S. Suppose that $\overline{S}$ is any weakly saturated collection of morphisms which contains S; we will show that $S^\vee \subseteq \overline{S}$. Let $f : X \to Z$ be a morphism in $S^\vee$, and choose a factorization

$$X \xrightarrow{f'} Y \xrightarrow{f''} Z$$

as in Proposition 12.4.2.1, so that $f' \in \overline{S}$ and $f'' \in T$. Since $f \in S^\vee$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\ Z & \xrightarrow{\mathrm{id}} & Z \end{array}$$

can be extended to a 3-simplex

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ \downarrow{\scriptstyle f} & \nearrow{\scriptstyle g} & \downarrow{\scriptstyle f''} \\ Z & \xrightarrow{\mathrm{id}} & Z. \end{array}$$

We therefore obtain a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{id}} & X & \longrightarrow & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y & \xrightarrow{f''} & Z \end{array}$$

which shows that f is a retract of f' and therefore belongs to $\overline{S}$ as desired. $\square$

Recall that if $\mathcal{C}$ is an ∞-category which admits finite limits and colimits, then every simplicial object $X_\bullet$ of $\mathcal{C}$ determines *latching* and *matching* objects $L_n(X_\bullet), M_n(X_\bullet)$ for $n \geqslant 0$ (see Remark HTT.A.2.9.16). The following result will play an important role in our proof of Theorem 12.3.3.5:

Corollary 12.4.3.3. *Let $\mathcal{C}$ be a presentable ∞-category and let S be a small collection of morphisms in $\mathcal{C}$. Let Y be any object of $\mathcal{C}$, and let $\phi : \mathcal{C}_{/Y} \to \mathcal{C}$ be the forgetful functor. Then there exists a simplicial object $X_\bullet$ of $\mathcal{C}_{/Y}$ with the following properties:*

(1) *For each $n \geqslant 0$, let $u_n : L_n(X_\bullet) \to X_n$ be the canonical map. Then $\phi(u_n)$ is a transfinite pushout of morphisms in S.*

(2) *For each $n \geqslant 0$, let $v_n : X_n \to M_n(X_\bullet)$ be the canonical map in $\mathcal{C}_{/Y}$. Then $\phi(v_n)$ has the right lifting property with respect to every morphism in S.*

Proof. We construct $X_\bullet$ as the union of a compatible family of diagrams $X_\bullet^{(n)} : \mathbf{\Delta}^{\mathrm{op}}_{\leqslant n} \to \mathcal{C}_{/Y}$, which we construct by induction on n. The case $n = -1$ is trivial (since $\mathbf{\Delta}_{\leqslant -1}$ is empty). Assume that $n \geqslant 0$ and that $X_\bullet^{(n-1)}$ has been constructed, so that the matching and latching objects $L_n(X)$, $M_n(X)$ are defined and we have a map $t : L_n(X) \to M_n(X)$. Using Proposition HTT.A.2.9.14 , we see that it suffices to construct a commutative diagram

$$\begin{array}{ccccc} & & X_n & & \\ & \overset{u_n}{\nearrow} & & \overset{v_n}{\searrow} & \\ L_n(X_\bullet) & & \xrightarrow{\quad t \quad} & & M_n(X_\bullet) \end{array}$$

in $\mathcal{C}_{/Y}$. Since the map $\mathcal{C}_{/Y} \to \mathcal{C}$ is a right fibration, this is equivalent to the problem of producing a commutative diagram

$$\begin{array}{ccccc} & & K_n & & \\ & \overset{u}{\nearrow} & & \searrow & \\ \phi(L_n(X_\bullet)) & & \xrightarrow{\quad \phi(t) \quad} & & \phi(M_n(X_\bullet)) \end{array}$$

in the ∞-category $\mathcal{C}$. Proposition 12.4.2.1 guarantees that we are able to make these choices in such a way that (1) and (2) are satisfied. $\square$

Remark 12.4.3.4. In the situation of Corollary 12.4.3.3, let $\varnothing$ denote the initial object of $\mathcal{C}$. Then for each $n \geqslant 0$, the canonical map $w : \varnothing \to \phi(X_n)$ is a transfinite pushout of morphisms in S. To prove this, we let P denote the full subcategory of $\mathbf{\Delta}_{[n]/}$ spanned by the surjective maps $[n] \to [m]$; we will regard P as a partially ordered set. For each upward-closed subset $P_0 \subseteq P$, we let $Z(P_0)$ denote a colimit of the induced diagram

$$P_0^{\mathrm{op}} \longrightarrow \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}_{/Y} \xrightarrow{\phi} \mathcal{C}\,.$$

Then $Z(\emptyset) \simeq \emptyset$ and $Z(P) \simeq \phi(X_n)$. It will therefore suffice to show that if $P_1 \subseteq P$ is obtained from P_0 by adjoining a new element given by $\alpha : [n] \to [m]$, then the induced map $\theta : Z(P_0) \to Z(P_1)$ is a transfinite pushout of morphisms in S. This follows from assertion (1) of Corollary 12.4.3.3, since θ is a pushout of the map $\phi(u_n) : \phi(L_m(X_\bullet)) \to \phi(X_m)$.

12.5 Proof of the Main Theorem

Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context which admits a deformation theory $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$. Our goal in this section is to prove Theorem 12.3.3.5, which asserts that the construction $K \mapsto \mathrm{Map}_{\mathbf{B}}(\mathfrak{D}(\bullet), K)$ induces an equivalence of ∞-categories $\Psi : \mathbf{B} \to \mathrm{Moduli}^{\mathbf{A}}$. The key step is to prove that every formal moduli problem X admits a "smooth atlas" (Proposition 12.5.3.3).

12.5.1 Digression: Atlases in Algebraic Geometry

We have seen that if X is an algebraic variety over the field $\mathbf{C}$ of complex numbers and $x : \mathrm{Spec}\,\mathbf{C} \to X$ is a point, then X determines a formal moduli problem $X^\wedge : \mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}} \to \mathcal{S}$ (Example 12.2.2.4). However, Definition 12.1.3.1 is far more inclusive than this. For example, we can also obtain formal moduli problems by extracting the formal completions of algebraic stacks.

Example 12.5.1.1. Let $n \geqslant 0$ be an integer, and let A be a connective $\mathbb{E}_\infty$-ring. We say that an A-module M is *projective of rank* n if $\pi_0 M$ is a projective module over $\pi_0 A$ of rank n, and M is flat over A. Let $X(A)$ denote the subcategory of Mod_A whose objects are modules which are locally free of rank n, and whose morphisms are equivalences of modules. It is not difficult to see that the ∞-category $X(A)$ is an essentially small Kan complex. Consequently, the construction $A \mapsto X(A)$ determines a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$.

Let η denote the point of $X(\mathbf{C})$ corresponding to the complex vector space $\mathbf{C}^n$. We define the *formal completion of* X *at* η to be the functor $X^\wedge : \mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}} \to \mathcal{S}$ given by $X^\wedge(R) = X(R) \times_{X(\mathbf{C})} \{\eta\}$. More informally, $X^\wedge(R)$ is a classifying space for projective R-modules M of rank n equipped with a trivialization $\mathbf{C} \otimes_R M \simeq \mathbf{C}^n$. Then $X^\wedge$ is a formal moduli problem (we will prove a more general statement to this effect in §16.5).

If A is a local commutative ring, then every projective A-module of rank of n is isomorphic to A^n. If X is the functor of Example 12.5.1.1, we deduce that $X(A)$ can be identified with the classifying space for the group $\mathrm{GL}_n(A)$ of automorphisms of A^n as an A-module.

For this reason, the functor X is often denoted by BGL_n. It can be described as the geometric realization (in the ∞-category of functors $F : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which are sheaves with respect to the Zariski topology) of a simplicial functor $F_\bullet$, given by the formula $F_m(A) = \mathrm{GL}_n(A)^m$, where $\mathrm{GL}_n(A)$ denotes the subspace of $\mathrm{Map}_{\mathrm{Mod}_A}(A^n, A^n)$ spanned by the invertible morphisms. Similarly, the formal completion $X^\wedge$ can be described as the geometric realization of a simplicial functor $F^\wedge_\bullet$, given by $F^\wedge_m(R) = \mathrm{fib}(\mathrm{GL}_n(R) \to \mathrm{GL}_n(\mathbf{C}))$ for $R \in \mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}}$ (after passing to formal completions, there is no need to sheafify with respect to the Zariski topology: if $R \in \mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}}$, then $\pi_0 R$ is a local Artin ring, so that every projective R-module of rank n is automatically free).

Remark 12.5.1.2. Example 12.5.1.1 can be generalized. Suppose that X is an arbitrary Artin stack over $\mathbf{C}$. Then X can be presented by an atlas, which is a (smooth) groupoid object

$$\cdots \rightrightarrows U_1 \rightrightarrows U_0.$$

in the category of $\mathbf{C}$-schemes. Let $\eta_0 : \mathrm{Spec}\,\mathbf{C} \to U_0$ be any point, so that η_0 determines points $\eta_n : \mathrm{Spec}\,\mathbf{C} \to U_n$ for every integer n. We can then define formal moduli problems $U^\wedge_n : \mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}} \to \mathcal{S}$ by formally completing each U_n at the point η_n. This gives a simplicial object $U^\wedge_\bullet$ in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}}, \mathcal{S})$. The geometric realization $|U^\wedge_\bullet| \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}}, \mathcal{S})$ is also a formal moduli problem which we will denote by $X^\wedge$. One can show that it is canonically independent (up to equivalence) of the atlas $U_\bullet$ chosen.

Our first goal in this section is to formulate a converse to Remark 12.5.1.2. Roughly speaking, we would like to assert that every formal moduli problem $Y : \mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}} \to \mathcal{S}$ admits a description that resembles the formal completion of an algebraic stack. However, the precise context of Remark 12.5.1.2 is too restrictive in several respects:

- (a) We can associate formal completions not only to algebraic stacks, but also to *higher* algebraic stacks. Consequently, rather than trying to realize Y as the geometric realization of a groupoid object $Y_\bullet$ of $\mathrm{Moduli} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}}, \mathcal{S})$, we will allow more general simplicial objects $Y_\bullet$ of Moduli.

- (b) We would like to exhibit Y as the geometric realization of a simplicial object $Y_\bullet$ where each Y_m resembles the formal completion of a $\mathbf{C}$-scheme near some point (which, without loss of generality, we may take to be an affine scheme of the form $\mathrm{Spét}\,R$). Since the construction of the formal completion makes sense not only for schemes but also for spectral Deligne-Mumford stacks (Remark 11.5.0.10), we should allow the possibility that R is a nondiscrete $\mathbb{E}_\infty$-algebra over $\mathbf{C}$.

- (c) If R is an augmented $\mathbb{E}_\infty$-algebra over $\mathbf{C}$, then A determines a formal moduli problem $\mathrm{Spf}\,R : \mathrm{CAlg}^{\mathrm{art}}_{\mathbf{C}} \to \mathcal{S}$ given by the formula $A \mapsto \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}}}(R, A)$. This functor is perhaps better understood (at least in the case where R is Noetherian) as the formal spectrum of $R^{\wedge}$, where $R^{\wedge}$ denotes the completion of R along the augmentation ideal in $\pi_0 R$. To incorporate a wider class of examples, we should allow arbitrary (possibly infinite-dimensional) affine formal schemes, not only those which arise as the formal completions of actual schemes.

12.5.2 Smooth Morphisms

We now translate some of the ideas of Remark 12.5.1.2 to the setting of an arbitrary deformation context.

Proposition 12.5.2.1. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $u : X \to Y$ be a map of formal moduli problems $X, Y : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$. The following conditions are equivalent:*

(1) *For every small map $\phi : A \to B$ in $\mathbf{A}^{\mathrm{art}}$, u has the right lifting property with respect to $\mathrm{Spf}(\phi) : \mathrm{Spf}(B) \to \mathrm{Spf}(A)$.*

(2) *For every small map $\phi : A \to B$ in $\mathbf{A}^{\mathrm{art}}$, the induced map $X(A) \to X(B) \times_{Y(B)} Y(A)$ is surjective on connected components.*

(3) *For every elementary map $\phi : A \to B$ in $\mathbf{A}^{\mathrm{art}}$, the induced map $X(A) \to X(B) \times_{Y(B)} Y(A)$ is surjective on connected components.*

(4) *For every $\alpha \in T$ and every $n > 0$, the homotopy fiber of the map $X(\Omega^{\infty-n} E_\alpha) \to Y(\Omega^{\infty-n} E_\alpha)$ (taken over the point determined by the base point of $\Omega^{\infty-n} E_\alpha$) is connected.*

(5) *For every $\alpha \in T$, the map of spectra $X(E_\alpha) \to Y(E_\alpha)$ is connective (that is, it has a connective homotopy fiber).*

Proof. The equivalence (1) $\Leftrightarrow$ (2) is tautological, and the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are evident. Let S be the collection of all small morphisms $A \to B$ in $\mathbf{A}^{\mathrm{art}}$ for which the map $X(A) \to X(B) \times_{Y(B)} Y(A)$ is surjective on connected components. The implication (3) $\Rightarrow$ (2) follows from the observation that S is closed under composition, and the implication (4) $\Rightarrow$ (3) from the observation that S is stable under the formation of pullbacks. The equivalence (4) $\Leftrightarrow$ (5) follows from fact that a map of $M \to M'$ of spectra is connective if and only if the induced map $\Omega^{\infty-n} M \to \Omega^{\infty-n} M'$ has connected homotopy fibers for each $n > 0$. $\square$

Definition 12.5.2.2. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $u : X \to Y$ be a map of formal moduli problems. We will say that u is *smooth* if it satisfies the equivalent conditions of Proposition 12.5.2.1. We will say that a formal moduli problem X is smooth if the map $X \to *$ is smooth, where $*$ denotes the final object of $\mathrm{Moduli}^{\mathbf{A}}$ (that is, the constant functor $\mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ taking the value $* \in \mathcal{S}$).

Remark 12.5.2.3. We can regard condition (5) of Proposition 12.5.2.1 as providing a differential criterion for smoothness: a map of formal moduli problems $X \to Y$ is smooth if and only if it induces a connective map of tangent complexes $X(E_\alpha) \to Y(E_\alpha)$. This should be regarded as an analogue of the condition that a map of smooth algebraic varieties $f : X \to Y$ induce a surjective map of tangent sheaves $T_X \to f^* T_Y$.

12.5.3 Existence of Smooth Hypercoverings

Our next goal is to show that in the setting of an arbitrary deformation context, every formal moduli problem admits a smooth hypercovering by "affine" objects.

Definition 12.5.3.1. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. We let $\mathrm{Pro}(\mathbf{A}^{\mathrm{art}})$ denote the ∞-category of pro-objects of $\mathbf{A}^{\mathrm{art}}$: that is, the smallest full subcategory of $\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})^{\mathrm{op}}$ which contains all corepresentable functors and is closed under filtered colimits. We will say that a functor $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ is *prorepresentable* if it belongs to the full subcategory $\mathrm{Pro}(\mathbf{A}^{\mathrm{art}}) \subseteq \mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})^{\mathrm{op}}$.

Remark 12.5.3.2. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. Since filtered colimits in $\mathcal{S}$ are left exact (Example HTT.7.3.4.4), the full subcategory $\mathrm{Moduli}^{\mathbf{A}}$ is stable under filtered colimits in $\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$. Since every corepresentable functor is a formal moduli problem (Example 12.1.3.3), we conclude that $\mathrm{Pro}(\mathbf{A}^{\mathrm{art}})^{\mathrm{op}}$ is contained in $\mathrm{Moduli}^{\mathbf{A}}$ (as a full subcategory of $\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$). That is, every prorepresentable functor $\mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ is a formal moduli problem.

Proposition 12.5.3.3. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ be a formal moduli problem. Then there exists a simplicial object $X_\bullet$ in $\mathrm{Moduli}^{\mathbf{A}}_{/X}$ with the following properties:*

(1) *Each X_n is prorepresentable.*

(2) *For each $n \geqslant 0$, let $M_n(X_\bullet)$ denote the nth matching object of the simplicial object $X_\bullet$ (computed in the ∞-category $\mathrm{Moduli}^{\mathbf{A}}_{/X}$). Then the canonical map $X_n \to M_n(X_\bullet)$ is smooth.*

In particular, X is equivalent to the geometric realization $|X_\bullet|$ in $\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$.

The proof of Proposition 12.5.3.3 will use the following simple observation:

Lemma 12.5.3.4. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let S be the collection of all morphisms in the ∞-category $\mathrm{Moduli}^{\mathbf{A}}$ of the form $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$, where the* *underlying map $A \to B$ is a small morphism in $\mathbf{A}^{\mathrm{art}}$. Let $f : X \to Y$ be a morphism in $\mathrm{Moduli}^{\mathbf{A}}$, and suppose that f is a transfinite pushout of morphisms in S. If X is prorepresentable, then Y is prorepresentable.*

Proof. Since the collection of prorepresentable objects of $\mathrm{Moduli}^{\mathbf{A}}$ is closed under filtered colimits, it will suffice to prove the following:

(∗) If $\phi : A \to B$ is a small morphism in $\mathrm{Moduli}^{\mathbf{A}}$ and we are given a pushout diagram

$$\begin{array}{ccc} \mathrm{Spf}(B) & \longrightarrow & X \\ \Big\downarrow{\scriptstyle \mathrm{Spf}(\phi)} & & \Big\downarrow{\scriptstyle f} \\ \mathrm{Spf}(A) & \longrightarrow & Y \end{array}$$

where X is prorepresentable, then Y is also prorepresentable.

To prove (∗), we note that X can be regarded as an object of $\mathrm{Pro}(\mathbf{A}^{\mathrm{art}})^{\mathrm{op}}_{\mathrm{Spf}(B)/} \simeq \mathrm{Ind}((\mathbf{A}^{\mathrm{art}}_{/B})^{\mathrm{op}})$. In other words, we have $X \simeq \varinjlim_\beta \mathrm{Spf}(B_\beta)$ for some filtered diagram $\{B_\beta\}$ in $\mathbf{A}^{\mathrm{art}}_{/B}$. Then

$$Y \simeq \varinjlim (\mathrm{Spf}(B_\beta) \amalg_{\mathrm{Spf}(B)} \mathrm{Spf}(A)).$$

For any formal moduli problem Z, we have

$$\mathrm{Map}_{\mathrm{Moduli}^{\mathbf{A}}}(\mathrm{Spf}(B_\beta) \amalg_{\mathrm{Spf}(B)} \mathrm{Spf}(A), Z) \simeq Z(B_\beta) \times_{Z(B)} Z(A) \simeq Z(B_\beta \times_B A)$$

(since the map $\phi : A \to B$ is small), so that $\mathrm{Spf}(B_\beta) \amalg_{\mathrm{Spf}(B)} \mathrm{Spf}(A) \simeq \mathrm{Spf}(B_\beta \times_B A)$ is corepresented by an object $B_\beta \times_B A$. It follows that Y is prorepresentable, as desired. □

Proof of Proposition 12.5.3.3. Let X be an arbitrary formal moduli problem. Applying Proposition **??**, we can choose a simplicial object $X_\bullet$ of $\mathrm{Moduli}^{\mathbf{A}}_{/X}$ such that each of the maps $X_n \to M_n(X_\bullet)$ is smooth, and each of the maps $L_n(X_\bullet) \to X_n$ is a transfinite pushout of morphisms of the form $\mathrm{Spf}\, B \to \mathrm{Spf}\, A$, where $A \to B$ is an elementary morphism in $\mathbf{A}^{\mathrm{art}}$. Using Remark 12.4.3.4 and Lemma 12.5.3.4, we conclude that each X_n is prorepresentable. This proves (1) and (2). To prove that $X \simeq |X_\bullet|$ in $\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$, it suffices to observe that condition (2) implies that $X_\bullet(A)$ is a hypercovering of $X(A)$ for every $A \in \mathbf{A}^{\mathrm{art}}$. □

12.5.4 The Proof of Theorem 12.3.3.5

Our proof of Theorem 12.3.3.5 will make use of the following:

Lemma 12.5.4.1. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ be a deformation theory. For every Artinian object $A \in \mathbf{A}^{\mathrm{art}}$, $\mathfrak{D}(A)$ is a compact object of the ∞-category $\mathbf{B}$.*

Proof. Since A is Artinian, there exists a sequence of elementary morphisms

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq *$$

in $\mathbf{A}$. We will prove that $\mathfrak{D}(A_i)$ is a compact object of $\mathbf{B}$ by descending induction on i. When $i = n$, the desired result follows from the observation that $\mathfrak{D}$ carries final objects of $\mathbf{A}$ to initial objects of $\mathbf{B}$ (Proposition 12.3.2.2). Assume therefore that $i < n$ and that $\mathfrak{D}(A_{i+1}) \in \mathbf{B}$ is compact. Since the map $A_i \to A_{i+1}$ is elementary, we have a pullback diagram σ :

$$\begin{array}{ccc} A_i & \longrightarrow & * \\ \downarrow & & \downarrow \\ A_{i+1} & \longrightarrow & \Omega^{\infty-n} E_\alpha \end{array}$$

for some $\alpha \in T$ and some $n > 0$. It follows from Proposition 12.3.2.2 that $\mathfrak{D}(\sigma)$ is a pushout square in $\mathbf{B}$. Consequently, to show that $\mathfrak{D}(A_i)$ is a compact object of $\mathbf{B}$, it will suffice to show that $\mathfrak{D}(A_{i+1})$, $\mathfrak{D}(*)$, and $\mathfrak{D}(\Omega^{\infty-n} E_\alpha)$ are compact objects of $\mathbf{B}$. In the first two cases, this follows from the inductive hypothesis. For the third, we note that the functor corepresented by $\mathfrak{D}(\Omega^{\infty-n} E_\alpha)$ is given by the composition $\mathbf{B} \xrightarrow{e_\alpha} \mathrm{Sp} \xrightarrow{\Omega^{\infty-n}} \mathcal{S}$, where e_α is the functor described in Proposition 12.3.3.1. Our assumption that $\mathfrak{D}$ is a deformation theory guarantees that e_α commutes with sifted colimits. The functor $\Omega^{\infty-n} : \mathrm{Sp} \to \mathcal{S}$ commutes with filtered colimits, so the composite functor $\mathbf{B} \to \mathcal{S}$ commutes with filtered colimits which implies that $\Omega^{\infty-n} E_\alpha$ is a compact object of $\mathbf{B}$. $\square$

Proof of Theorem 12.3.3.5. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, let $\mathfrak{D} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ be a deformation theory, and let $\Psi : \mathbf{B} \to \mathrm{Moduli}^{\mathbf{A}} \subseteq \mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ denote the functor given by the formula $\Psi(K)(A) = \mathrm{Map}_{\mathbf{B}}(\mathfrak{D}(A), K)$. We wish to prove that Ψ is an equivalence of ∞-categories. It is clear that Ψ preserves small limits. It follows from Lemma 12.5.4.1 that Ψ preserves filtered colimits and is therefore accessible. Using Corollary HTT.5.5.2.9 we conclude that Ψ admits a left adjoint Φ. To prove that Ψ is an equivalence, it will suffice to show:

- (a) The functor Ψ is conservative.
- (b) The unit transformation $u : \mathrm{id}_{\mathrm{Moduli}} \to \Psi \circ \Phi$ is an equivalence.

We begin with the proof of (a). Suppose we are given a morphism $f : K \to K'$ in $\mathbf{B}$ have that $\Psi(f)$ is an equivalence. In particular, for each $\alpha \in T$ and each $n \geqslant 0$, we have homotopy equivalences

$$\begin{aligned} \mathrm{Map}_{\mathbf{B}}(\mathfrak{D}(\Omega^{\infty-n} E_\alpha), K) &\simeq \Psi(K)(\mathfrak{D}\Omega^{\infty-n} E_\alpha) \\ &\to \Psi(K')(\mathfrak{D}\Omega^{\infty-n} E_\alpha) \\ &\simeq \mathrm{Map}_{\mathbf{B}}(\mathfrak{D}(\Omega^{\infty-n} E_\alpha), K'). \end{aligned}$$

It follows that $e_\alpha(K) \simeq e_\alpha(K')$, where $e_\alpha : \mathbf{B} \to \mathrm{Sp}$ is the functor described in Proposition 12.3.3.1. Since the functors e_α are jointly conservative (Definition 12.3.3.2), we conclude that f is an equivalence.

We now prove (b). Let $X \in \mathrm{Moduli}^{\mathbf{A}}$ be a formal moduli problem; we wish to show that u induces an equivalence $X \to (\Psi \circ \Phi)(X)$. According to Proposition 12.2.2.6, it suffices to show that for each $\alpha \in T$, the induced map

$$\theta : X(E_\alpha) \to (\Psi \circ \Phi)(X)(E_\alpha) \simeq e_\alpha(\Phi X)$$

is an equivalence of spectra. To prove this, choose a simplicial object $X_\bullet$ of $\mathrm{Moduli}^{\mathbf{A}}_{/X}$ satisfying the requirements of Proposition 12.5.3.3. For every object $A \in \mathbf{A}^{\mathrm{art}}$, the simplicial space $X_\bullet(A)$ is a hypercovering of $X(A)$ so that the induced map $|X_\bullet(A)| \to X(A)$ is a homotopy equivalence. It follows that X is a colimit of the diagram $X_\bullet$ in the ∞-category $\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ and therefore also in the ∞-category $\mathrm{Moduli}^{\mathbf{A}}$. Similarly, $X(E_\alpha)$ is equivalent to the geometric realization $|X_\bullet(E_\alpha)|$ in the ∞-category $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{S})$ and therefore also in the ∞-category of spectra. Since Φ preserves small colimits and e_α preserves sifted colimits, we have

$$e_\alpha(\Phi(X)) \simeq e_\alpha(\Phi|X_\bullet|) \simeq |e_\alpha(\Phi X_\bullet)|.$$

It follows that θ is a geometric realization of a simplicial morphism $\theta_\bullet : X_\bullet(E_\alpha) \to e_\alpha(\Phi X_\bullet)$. It will therefore suffice to prove that each θ_n is an equivalence, which is equivalent to the requirement that u induces an equivalence $X_n \to (\Psi \circ \Phi)(X_n)$. In other words, we may replace X by X_n, and thereby reduce to the case where X is prorepresentable. Since the functors Φ and Ψ both commute with filtered colimits, we may further reduce to the case where $X = \mathrm{Spf}(A)$ for some $A \in \mathbf{A}^{\mathrm{art}}$. Since $\Phi(\mathrm{Spf}(A)) = \mathfrak{D}(A)$, it suffices to show that for each $B \in \mathbf{A}^{\mathrm{art}}$, the map

$$\mathrm{Map}_{\mathbf{A}}(A, B) \to \mathrm{Map}_{\mathbf{B}}(\mathfrak{D}(B), \mathfrak{D}(A)) \simeq \mathrm{Map}_{\mathbf{A}}(A, \mathfrak{D}'\mathfrak{D}(B))$$

is a homotopy equivalence, which follows immediately from Proposition 12.3.2.2. □

Chapter 13

Moduli Problems for Commutative Algebras

Let κ be a field of characteristic zero. We will say that an augmented $\mathbb{E}_\infty$-algebra $A \in \mathrm{CAlg}_\kappa^{\mathrm{aug}}$ is *Artinian* if it satisfies the conditions of Proposition 12.1.2.9: that is, if A is connective, $\pi_* A$ is a finite dimensional vector space over κ, and $\pi_0 A$ is a local ring (note that this terminology is slightly misleading, since it includes the hypothesis that A is local: see Warning 12.1.2.6). We let $\mathrm{CAlg}_\kappa^{\mathrm{art}}$ denote the full subcategory of $\mathrm{CAlg}_\kappa^{\mathrm{aug}}$ spanned by the Artinian $\mathbb{E}_\infty$-algebra over κ (which we can also identify with a full subcategory of the ∞-category CAlg_κ: see Remark 12.1.2.10).

Recall that a functor $X : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ is called a *formal moduli problem* if it satisfies the following pair of conditions (see Proposition 12.1.3.7):

(*a*) The space $X(\kappa)$ is contractible.

(*b*) For every pullback square

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

in $\mathrm{CAlg}_\kappa^{\mathrm{art}}$, if both the maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, then the diagram of spaces

$$\begin{array}{ccc} X(R) & \longrightarrow & X(R_0) \\ \downarrow & & \downarrow \\ X(R_1) & \longrightarrow & X(R_{01}) \end{array}$$

is also a pullback square.

In this section, we will study the full subcategory $\mathrm{Moduli}_\kappa \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_\kappa, \mathcal{S})$ spanned by the formal moduli problems.

We begin by applying the general formalism of §12.2. To every formal moduli problem $X \in \mathrm{Moduli}_\kappa$, we can associate a spectrum $T_X \in \mathrm{Sp}$, which is given informally by the formula $\Omega^{\infty-n} T_X = X(\kappa \oplus \Sigma^n(\kappa))$ for $n \geqslant 0$. In particular, we can identify the 0th space $\Omega^\infty T_X$ with $X(\kappa[\epsilon]/(\epsilon^2))$, an analogue of the classical Zariski tangent space. We refer to T_X as the *tangent complex* of the formal moduli problem X.

The construction $X \mapsto T_X$ commutes with finite limits. In particular, we have a homotopy equivalence of spectra $\Sigma^{-1} T_X \simeq T_{\Omega X}$, where ΩX denotes the formal moduli problem given by the formula $(\Omega X)(R) = \Omega X(R)$ (note that a choice of point η in the contractible space $X(\kappa)$ determines a base point of each $X(R)$, so the loop space $\Omega X(R)$ is well-defined). The formal moduli problem ΩX is equipped with additional structure: it can be regarded as a group object of Moduli_κ. It is therefore natural to expect that the tangent complex $T_{\Omega X}$ should behave somewhat like the tangent space to an algebraic group. We can formulate this expectation more precisely as follows:

(∗) Let $X \in \mathrm{Moduli}_\kappa$ be a formal moduli problem. Then the shifted tangent complex $\Sigma^{-1} T_X \simeq T_{\Omega X}$ can be identified with the underlying spectrum of a differential graded Lie algebra over κ.

Example 13.0.0.1. Suppose that A is a commutative κ-algebra equipped with an augmentation $\epsilon : A \to \kappa$. Then R defines a formal moduli problem X over κ, which carries a Artinian $\mathbb{E}_\infty$-algebra R over κ to the mapping space $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}_\kappa}(A, R)$. When κ is of characteristic zero, the tangent complex T_X can be identified with the complex of Andre-Quillen cochains taking values in κ. In this case, the existence of a natural differential graded Lie algebra structure on $\Sigma^{-1} T_X$ is proven in [183].

Assertion (∗) has a converse: every differential graded Lie algebra $\mathfrak{g}_*$ arises (up to quasi-isomorphism) as the shifted tangent complex $\Sigma^{-1} T_X$ of some $X \in \mathrm{Moduli}_\kappa$. Moreover, the formal moduli problem X is determined by $\mathfrak{g}_*$ up to equivalence. More precisely, we have the following stronger version of Theorem 11.5.0.13:

Theorem 13.0.0.2. *Let κ be a field of characteristic zero and let Lie_κ denote the ∞-category of differential graded Lie algebras over κ (see Definition 13.1.4.1). Then there is an equivalence of ∞-categories $\Psi : \mathrm{Lie}_\kappa \to \mathrm{Moduli}_\kappa$. Moreover, the functor $\mathfrak{g}_* \mapsto \Sigma^{-1} T_{\Psi(\mathfrak{g}_*)}$ is equivalent to the forgetful functor $\mathrm{Lie}_\kappa \to \mathrm{Sp}$ (which carries a differential graded Lie algebra $\mathfrak{g}_*$ to the generalized Eilenberg-MacLane spectrum determined by its underlying chain complex).*

Our main goal in this section is to prove Theorem 13.0.0.2. The first step is to construct the functor $\Psi : \mathrm{Lie}_\kappa \to \mathrm{Moduli}_\kappa$. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ, and let $R \in \mathrm{CAlg}_\kappa^{\mathrm{art}}$. Since κ has characteristic zero, we can identify R with an (augmented) commutative differential graded algebra over κ; let us denote its augmentation ideal by $\mathfrak{m}_R$. The tensor product $\mathfrak{m}_R \otimes_\kappa \mathfrak{g}_*$ then inherits the structure of a differential graded Lie algebra over κ. Heuristically, $\Psi(\mathfrak{g}_*)(R)$ can be described as the space of *Maurer-Cartan elements* of the differential graded Lie algebra $\mathfrak{m}_R \otimes_\kappa \mathfrak{g}_*$: that is, the space of solutions to the Maurer-Cartan equation $dx = [x, x]$. There does not seem to be a homotopy-invariant definition for the space $\mathrm{MC}(\mathfrak{g}_*)$ of Maurer-Cartan elements of an arbitrary differential graded Lie algebra over κ: the well-definedness of $\mathrm{MC}(\mathfrak{m}_R \otimes_\kappa \mathfrak{g}_*)$ relies on the nilpotence properties of the tensor product $\mathfrak{m}_R \otimes_\kappa \mathfrak{g}_*$ (which result from our assumption that R is Artinian). Nevertheless, there is a well-defined *bifunctor* $\mathrm{MC} : \mathrm{CAlg}_\kappa^{\mathrm{aug}} \times \mathrm{Lie}_\kappa \to \mathcal{S}$ which is given heuristically by $(R, \mathfrak{g}_*) \mapsto \mathrm{MC}(\mathfrak{m}_R \otimes_\kappa \mathfrak{g}_*)$. This functor can be defined rigorously by the formula $\mathrm{MC}(R, \mathfrak{g}_*) = \mathrm{Map}_{\mathrm{Lie}_\kappa}(\mathfrak{D}(R), \mathfrak{g}_*)$, where $\mathfrak{D} : (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Lie}_\kappa$ is the *Koszul duality* functor that we will describe in §13.3. Roughly speaking, the Koszul dual of an augmented $\mathbb{E}_\infty$-algebra R is a differential graded Lie algebra $\mathfrak{D}(R) \in \mathrm{Lie}_\kappa$ which *corepresents* the functor $\mathfrak{g}_* \mapsto \mathrm{MC}(\mathfrak{m}_R \otimes_\kappa \mathfrak{g}_*)$. However, it will be more convenient for us to describe $\mathfrak{D}(R)$ instead by the functor that *represents*. We will define $\mathfrak{D}$ as the right adjoint to the functor $C^* : \mathrm{Lie}_\kappa \to (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}}$, which assigns to each differential graded Lie algebra $\mathfrak{g}_*$ the commutative differential graded algebra $C^*(\mathfrak{g}_*)$ of Lie algebra cochains on $\mathfrak{g}_*$ (see Construction 13.2.5.1).

Remark 13.0.0.3. For our purposes, the Maurer-Cartan equation $dx = [x, x]$ (and the associated space $\mathrm{MC}(\mathfrak{g}_*)$ of Maurer-Cartan elements of a differential graded Lie algebra $\mathfrak{g}_*$) are useful heuristics for understanding the functor Ψ appearing in Theorem 13.0.0.2. However, they will play no further role in our discussion. For a construction of the functor Ψ which makes direct use of the Maurer-Cartan equation, we refer the reader to the work of Hinich (see [95]). We also refer the reader to the work of Goldman and Millson ([82] and [83]).

Let us now outline the contents of this section. We begin in §13.1 with a brief review of the theory of differential graded Lie algebras and a definition of the ∞-category Lie_κ. In §13.2, we will review the homology and cohomology of (differential graded) Lie algebras, which are computed by the Chevalley-Eilenberg constructions $\mathfrak{g}_* \mapsto C_*(\mathfrak{g}_*)$ and $\mathfrak{g}_* \mapsto C^*(\mathfrak{g}_*)$. The functor C^* carries quasi-isomorphisms of differential graded Lie algebras to quasi-isomorphisms between (augmented) commutative differential graded algebras, and therefore descends to a functor from the ∞-category Lie_κ to the ∞-category $(\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}}$. We will show that this functor admits a right adjoint $\mathfrak{D}$ and study its properties. The main point is to show that $\mathfrak{D}$ is a deformation theory (in the sense of Definition 12.3.3.2) on the deformation context $(\mathrm{CAlg}_\kappa^{\mathrm{aug}}, \{E\})$ of Example 12.1.1.2. We will use this fact in §13.3 to deduce Theorem 13.0.0.2 from Theorem 12.3.3.5.

To every formal modui problem $X : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathcal{S}$, we can introduce an ∞-category

QCoh(X) of *quasi-coherent sheaves* on X. It follows from Theorem 13.0.0.2 that X is completely determined by a differential graded Lie algebra $\mathfrak{g}_*$ (which is well-defined up to quasi-isomorphism). In §13.4, we will show that QCoh(X) can be identified with a full subcategory of the ∞-category of (differential graded) representations of $\mathfrak{g}_*$ (Theorem 13.4.0.1).

Contents

13.1 Differential Graded Lie Algebras

Let κ be a field of characteristic zero. Theorem 13.0.0.2 asserts that the ∞-category Moduli_{κ} of formal moduli problems over κ is equivalent to the ∞-category Lie_{κ} of differential graded Lie algebras over κ. Our goal in this section is to explain the definition of Lie_κ and establish some of its basic properties. Along the way, we will introduce some of the notation and constructions which will play a role in our proof of Theorem 13.0.0.2.

13.1.1 Chain Complexes of Vector Spaces

We begin by reviewing some terminology.

Notation 13.1.1.1. Let κ be a field. We let $\mathrm{Vect}_\kappa^{\mathrm{dg}}$ denote the category of differential graded vector spaces over κ: that is, the category whose objects are chain complexes

$$\cdots \to V_1 \to V_0 \to V_{-1} \to \cdots$$

and whose morphisms are maps of chain complexes. We will regard $\mathrm{Vect}_\kappa^{\mathrm{dg}}$ as a symmetric monoidal category with respect to the tensor product of chain complexes described by the formula

$$(V \otimes W)_p = \bigoplus_{p=p'+p''} V_{p'} \otimes_\kappa W_{p''},$$

and the symmetry isomorphism $V \otimes W \simeq W \otimes V$ is the sum of the isomorphisms $V_{p'} \otimes_\kappa W_{p''} \simeq W_{p''} \otimes_\kappa V_{p'}$, multiplied by the factor $(-1)^{p'p''}$.

Recall that the category $\mathrm{Vect}_\kappa^{\mathrm{dg}}$ admits a model structure, where:

(C) A map of chain complexes $f : V_* \to W_*$ is a cofibration it is degreewise monic: that is, each of the induced maps $V_n \to W_n$ is injective.

(F) A map of chain complexes $f : V_* \to W_*$ is a fibration it is degreewise epic: that is, each of the induced maps $V_n \to W_n$ is surjective.

(W) A map of chain complexes $f : V_* \to W_*$ is a weak equivalence if it is a *quasi-isomorphism*: that is, if it induces an isomorphism of homology groups $\mathrm{H}_n(V) \to \mathrm{H}_n(W)$ for every integer n.

Moreover, the underlying ∞-category of $\mathrm{Vect}_\kappa^{\mathrm{dg}}$ can be identified with the ∞-category Mod_κ of κ-module spectra (see Remark HA.7.1.1.16).

Notation 13.1.1.2. Let V be a graded vector space over κ. We let $V^\vee$ denote the graded dual of V, given by $(V^\vee)_p = \mathrm{Hom}_\kappa(V_{-p}, \kappa)$. For each integer n, we let $V[n]$ denote the same vector space with grading shifted by n, so that $V[n]_p = V_{p-n}$.

Definition 13.1.1.3. We let $\mathrm{Alg}_\kappa^{\mathrm{dg}}$ denote the category of associative algebra objects of $\mathrm{Vect}_\kappa^{\mathrm{dg}}$, and $\mathrm{CAlg}_\kappa^{\mathrm{dg}}$ the category of commutative algebra objects of $\mathrm{Vect}_\kappa^{\mathrm{dg}}$. We will refer to objects of $\mathrm{Alg}_\kappa^{\mathrm{dg}}$ as *differential graded algebras* over κ and objects of $\mathrm{CAlg}_\kappa^{\mathrm{dg}}$ as *commutative differential graded algebras over* κ. According to Propositions HA.4.1.8.3 and HA.4.5.4.6 ,

$\mathrm{Alg}_\kappa^{\mathrm{dg}}$ and $\mathrm{CAlg}_\kappa^{\mathrm{dg}}$ admit combinatorial model structures, where a map $f : A_* \to B_*$ of (commutative) differential graded algebras is a weak equivalence or fibration if the underlying map of chain complexes is a weak equivalence or fibration.

Remark 13.1.1.4. In more concrete terms, a differential graded algebra A is a chain complex (A_*, d) together with a unit $1 \in A_0$ and a collection of κ-bilinear multiplication maps $A_p \times A_q \to A_{p+q}$ satisfying

$$1x = x1 = x \qquad x(yz) = (xy)z \qquad d(xy) = dxy + (-1)^p xdy$$

for $x \in A_p$, $y \in A_q$, and $z \in A_r$. The differential graded algebra A is commutative if $xy = (-1)^{pq} yx$ for $x \in A_p$, $y \in A_q$.

13.1.2 The Ordinary Category of Differential Graded Lie Algebras

We now introduce our main objects of interest.

Definition 13.1.2.1. A *differential graded Lie algebra* over κ is a chain complex $(\mathfrak{g}_*, d)$ of κ-vector spaces equipped with a Lie bracket $[,] : \mathfrak{g}_p \otimes_\kappa \mathfrak{g}_q \to \mathfrak{g}_{p+q}$ satisfying the following conditions:

(1) For $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$, we have $[x, y] + (-1)^{pq}[y, x] = 0$.

(2) For $x \in \mathfrak{g}_p$, $y \in \mathfrak{g}_q$, and $z \in \mathfrak{g}_r$, we have

$$(-1)^{pr}[x, [y, z]] + (-1)^{pq}[y, [z, x]] + (-1)^{qr}[z, [x, y]] = 0.$$

(3) The differential d is a derivation with respect to the Lie bracket. That is, for $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$, we have $d[x, y] = [dx, y] + (-1)^p[x, dy]$.

Given a pair of differential graded Lie algebras $(\mathfrak{g}_*, d)$ and $(\mathfrak{g}'_*, d')$, a *morphism of differential graded Lie algebras* from $(\mathfrak{g}_*, d)$ to $(\mathfrak{g}'_*, d')$ is a map of chain complexes $F : (\mathfrak{g}_*, d) \to (\mathfrak{g}'_*, d')$ such that $F([x, y]) = [F(x), F(y)]$ for $x \in \mathfrak{g}_p$, $y \in \mathfrak{g}_q$. With this notion of morphism, we can regard the collection of differential graded Lie algebras over κ as a category which we will denote by $\mathrm{Lie}_\kappa^{\mathrm{dg}}$.

Example 13.1.2.2. Let A_* be a (possibly nonunital) differential graded algebra over κ. Then A_* has the structure of a differential graded Lie algebra, where the Lie bracket $[,] : A_p \otimes_\kappa A_q \to A_{p+q}$ is given by the graded commutator $[x, y] = xy - (-1)^{pq} yx$.

Remark 13.1.2.3. The construction of Example 13.1.2.2 determines a forgetful functor $\mathrm{Alg}_\kappa^{\mathrm{dg}} \to \mathrm{Lie}_\kappa^{\mathrm{dg}}$. This functor admits a left adjoint $U : \mathrm{Lie}_\kappa^{\mathrm{dg}} \to \mathrm{Alg}_\kappa^{\mathrm{dg}}$, which assigns to every differential graded Lie algebra $\mathfrak{g}_*$ its *universal enveloping algebra* $U(\mathfrak{g}_*)$. The universal enveloping algebra $U(\mathfrak{g}_*)$ can be described as the quotient of the tensor algebra $\bigoplus_{n \geqslant 0} \mathfrak{g}_*^{\otimes n}$ by the two-sided ideal generated by all expressions of the form $(x \otimes y) - (-1)^{pq}(y \otimes x) - [x, y]$, where $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$. The collection of such expressions is stable under the differential on $\bigoplus_{n \geqslant 0} \mathfrak{g}_*^{\otimes n}$, so that $U(\mathfrak{g}_*)$ inherits the structure of a differential graded algebra.

The universal enveloping algebra $U(\mathfrak{g}_*)$ admits a natural filtration

$$U(\mathfrak{g}_*)^{\leqslant 0} \subseteq U(\mathfrak{g}_*)^{\leqslant 1} \subseteq \cdots,$$

where $U(\mathfrak{g}_*)^{\leqslant n}$ is the image of $\bigoplus_{0 \leqslant i \leqslant n} \mathfrak{g}_*^{\otimes i}$ in $U(\mathfrak{g}_*)$. The associated graded algebra of $U(\mathfrak{g}_*)$ is commutative (in the graded sense), so that the canonical map $\mathfrak{g}_* \to U(\mathfrak{g}_*)^{\leqslant 1}$ induces a map of differential graded algebras $\theta : \mathrm{Sym}_\kappa^* \mathfrak{g}_* \to \mathrm{gr}\, U(\mathfrak{g}_*)$. According to the Poincare-Birkhoff-Witt theorem, the map θ is an isomorphism (see Theorem 2.3 of [220] for a proof in the setting of differential graded Lie algebras).

Remark 13.1.2.4. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ. For each integer n, we let $\psi : \mathfrak{g}_*^{\otimes n} \to U(\mathfrak{g}_*)$ denote the multiplication map. For every permutation σ of the set $\{1, 2, \ldots, n\}$, let ϕ_σ denote the induced automorphism of $\mathfrak{g}_*^{\otimes n}$. The map $\frac{1}{n!} \sum_\sigma \psi \circ \phi_\sigma$ is invariant under precomposition with each of the maps ϕ_σ, and therefore factors as a composition $\mathfrak{g}_*^{\otimes n} \to \mathrm{Sym}_\kappa^n(\mathfrak{g}_*) \xrightarrow{\Psi_n} U(\mathfrak{g}_*)^{\leqslant n} \subseteq U(\mathfrak{g}_*)$. We observe that the composite map $\mathrm{Sym}_\kappa^n(\mathfrak{g}_*) \xrightarrow{\Psi_n} U(\mathfrak{g}_*)^{\leqslant n} \to \mathrm{gr}^n U(\mathfrak{g}_*)$ coincides with the isomorphism of Remark 13.1.2.3. It follows that the direct sum of the maps Ψ_n determines an isomorphism of chain complexes $\overline{\theta} : \mathrm{Sym}_\kappa^*(\mathfrak{g}_*) \to U(\mathfrak{g}_*)$.

13.1.3 The Model Structure on $\mathrm{Lie}_\kappa^{\mathrm{dg}}$

We now show that the category of differential graded Lie algebras over a field κ of characteristic zero admits a model structure.

Definition 13.1.3.1. Let $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ be a map of differential graded Lie algebras over κ. We will say that f is a *quasi-isomorphism* if the underlying map of chain complexes is a quasi-isomorphism: that is, if F induces an isomorphism on homology.

Proposition 13.1.3.2. *Let κ be a field of characteristic zero. Then the category $\mathrm{Lie}_\kappa^{\mathrm{dg}}$ of differential graded Lie algebras over κ has the structure of a left proper combinatorial model category, where:*

- (W) *A map of differential graded Lie algebras $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ is a weak equivalence if and only if it is a quasi-isomorphism (Definition 13.1.3.1).*

- (F) *A map of differential graded Lie algebras $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ is a fibration if and only if it is a fibration of chain complexes: that is, if and only if each of the induced maps $\mathfrak{g}_p \to \mathfrak{g}'_p$ is a surjective map of vector spaces over κ.*

(C) *A map of differential graded Lie algebras $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ is a cofibration if and only if it has the left lifting property with respect to every map of differential graded Lie algebras which is simultaneously a fibration and a weak equivalence.*

Remark 13.1.3.3. In the situation of Proposition 13.1.3.2, the forgetful functor $\mathrm{Alg}_\kappa^{\mathrm{dg}} \to \mathrm{Lie}_\kappa^{\mathrm{dg}}$ of Example 13.1.2.2 preserves fibrations and weak equivalences, and is therefore a right Quillen functor. It follows that the universal enveloping algebra functor $U : \mathrm{Lie}_\kappa^{\mathrm{dg}} \to \mathrm{Alg}_\kappa^{\mathrm{dg}}$ is a left Quillen functor.

Lemma 13.1.3.4. *Let $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ be a map of differential graded Lie algebras over κ. The following conditions are equivalent:*

(1) *The map f is a quasi-isomorphism.*

(2) *The induced map $U(\mathfrak{g}_*) \to U(\mathfrak{g}'_*)$ is a quasi-isomorphism of differential graded algebras.*

Proof. We note that if $g : V_* \to W_*$ is any map of chain complexes of κ-vector spaces, then g is a quasi-isomorphism if and only if g induces a quasi-isomorphism $\mathrm{Sym}_\kappa^*(V_*) \to \mathrm{Sym}_\kappa^*(W_*)$. The desired assertion now follows immediately from Remark 13.1.2.4. □

Proof of Proposition 13.1.3.2. The forgetful functor $\mathrm{Lie}_\kappa^{\mathrm{dg}} \to \mathrm{Vect}_\kappa^{\mathrm{dg}}$ has a left adjoint (the free Lie algebra functor), which we will denote by $\mathrm{Free} : \mathrm{Vect}_\kappa^{\mathrm{dg}} \to \mathrm{Lie}_\kappa^{\mathrm{dg}}$. For every integer n, let $E(n)_*$ denote the acyclic chain complex

$$\cdots \to 0 \to 0 \to \kappa \simeq \kappa \to 0 \to 0 \to \cdots$$

which is nontrivial only in degrees n and $(n-1)$, and let $\partial\, E(n)_*$ be the subcomplex of $E(n)_*$ which is nontrivial only in degree $(n-1)$. Let C_0 be the collection of morphisms in $\mathrm{Lie}_k^{\mathrm{dg}}$ of the form $\mathrm{Free}(\partial\, E(n)_*) \to \mathrm{Free}(E(n)_*)$, and let W be the collection of all quasi-isomorphisms in $\mathrm{Lie}_\kappa^{\mathrm{dg}}$. We claim that the collection of morphisms C_0 and W satisfy the hypotheses of Proposition HTT.A.2.6.15:

(1) The collection W of quasi-isomorphisms is perfect, in the sense of Definition HTT.A.2.6.12 . This follows immediately from Corollary HTT.A.2.6.14 , applied to the forgetful functor $\mathrm{Lie}_\kappa^{\mathrm{dg}} \to \mathrm{Vect}_\kappa^{\mathrm{dg}}$.

(2) The collection of weak equivalences is stable under pushouts of morphisms in C_0. In other words, if $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ is a quasi-isomorphism of differential graded Lie algebras over κ and $x \in \mathfrak{g}_{n-1}$ is a cycle classifying a map $\mathrm{Free}(\partial\, E(n)_*) \to \mathfrak{g}_*$, we must show that the induced map

$$\mathfrak{g}_* \amalg_{\mathrm{Free}(\partial\, E(n)_*)} \mathrm{Free}(E(n)_*) \to \mathfrak{g}'_* \amalg_{\mathrm{Free}(\partial\, E(n)_*)} \mathrm{Free}(E(n)_*)$$

is also a quasi-isomorphism of differential graded Lie algebras. Let $A_* = U(\mathfrak{g}_*)$, let $A'_* = U(\mathfrak{g}'_*)$, and let $F : A_* \to A'_*$ be the map induced by f. We will abuse notation and identify x with its image in A_{n-1}. Using Lemma 13.1.3.4, we see that F is a quasi-isomorphism, and we are reduced to showing that F induces a quasi-isomorphism $B_* \to B'_*$, where B_* is the differential graded algebra obtained from A_* by adjoining a class y in degree n with $dy = x$, and B'_* is defined similarly. To prove this, we note that B_* admits an exhaustive filtration

$$A_* \simeq B_*^{\leqslant 0} \subseteq B_*^{\leqslant 1} \subseteq B_*^{\leqslant 2} \subseteq \cdots$$

where $B^{\leqslant m}$ is the subspace of B spanned by all expressions of the form $a_0 y a_1 y \cdots y a_k$, where $k \leqslant m$ and each a_i belongs to the image of A_* in B_*. Similarly, we have a filtration

$$A'_* \simeq B_*'^{\leqslant 0} \subseteq B_*'^{\leqslant 1} \subseteq B_*'^{\leqslant 2} \subseteq \cdots$$

of B'_*. Since the collection of quasi-isomorphisms is stable under filtered colimits, it will suffice to show that for each $m \geqslant 0$, the map of chain complexes $B_*^{\leqslant m} \to B_*'^{\leqslant m}$ is a quasi-isomorphism. The proof proceeds by induction on m, the case $m = 0$ being true by assumption. If $m > 0$, we have a diagram of short exact sequences of chain complexes

$$\begin{array}{ccccccccc} 0 & \to & B_*^{\leqslant m-1} & \to & B_*^{\leqslant m} & \to & B_*^{\leqslant n}/B_*^{\leqslant m-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} & & \\ 0 & \to & B_*'^{\leqslant m-1} & \to & B_*'^{\leqslant m} & \to & B_*'^{\leqslant m}/B_*'^{\leqslant m-1} & \to & 0. \end{array}$$

The inductive hypothesis implies that the left vertical map is a quasi-isomorphism. To complete the inductive step, it will suffice to show that ϕ is a quasi-isomorphism. For this, we observe that the construction $a_0 \otimes \cdots \otimes a_n \mapsto a_0 y a_1 y \cdots y a_m$ determines an isomorphism of chain complexes $A_*^{\otimes m+1} \to B_*^{\leqslant m}/B_*^{\leqslant m-1}$, and similarly we have an isomorphism $A_*'^{\otimes m+1} \to B_*'^{\leqslant m}/B_*'^{\leqslant m-1}$. Under these isomorphisms, ϕ corresponds to the map $A_*^{\otimes m+1} \to A_*'^{\otimes m+1}$ given by the $(m+1)$st tensor power of F, which is a quasi-isomorphism by assumption.

(3) Let $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ be a map of differential graded Lie algebras which has the right lifting property with respect to every morphism in C_0. We claim that f is a quasi-isomorphism. To prove this, we must show that f induces an isomorphism $\theta_n : \mathrm{H}_n(\mathfrak{g}_*) \to \mathrm{H}_n(\mathfrak{g}'_*)$ for every integer n (here $\mathrm{H}_n(\mathfrak{h}_*)$ denotes the homology of the underlying chain complex of $\mathfrak{h}_*$). We first show that θ_n is surjective. Choose a class $\eta \in \mathrm{H}_n(\mathfrak{g}'_*)$, represented by a cycle $x \in \mathfrak{g}'_n$. Then x determines a map $u : \mathrm{Free}(E(n)_*) \to \mathfrak{g}'_*$ which vanishes on $\mathrm{Free}(\partial\, E(n)_*)$. It follows that $u = f \circ v$, where $v : \mathrm{Free}(E(n)_*) \to \mathfrak{g}_*$ is a map of

differential graded Lie algebras which vanishes on $\mathrm{Free}(\partial\, E(n-1)_*)$. The map v is determined by a cycle $\overline{x} \in \mathfrak{g}_n$ which represents a homology class lifting η.

We now prove that θ_n is injective. Let $\eta \in \mathrm{H}_n(\mathfrak{g}_*)$ be a class whose image in $\mathrm{H}_n(\mathfrak{g}'_*)$ vanishes. Then η is represented by a cycle $x \in \mathfrak{g}_n$ such that $f(x) = dy$, for some $y \in \mathfrak{g}'_{n+1}$. Then y determines a map of differential graded Lie algebras $u : \mathrm{Free}(E(n+1)_*) \to \mathfrak{g}'$ such that $u|_{\mathrm{Free}(\partial\, E(n+1)_*)}$ lifts to $\mathfrak{g}_*$. It follows that $u = f \circ v$, for some map of differential graded Lie algebras $\mathrm{Free}(E(n+1)_*) \to \mathfrak{g}_*$ such that $v|_{\mathrm{Free}(\partial\, E(n+1)_*)}$ classifies x. It follows that x is a boundary, so that $\eta = 0$.

It follows from Proposition HTT.A.2.6.15 that $\mathrm{Lie}^{\mathrm{dg}}_{\kappa}$ admits a left proper combinatorial model structure having W as the class of weak equivalences and C_0 as a class of generating cofibrations. To complete the proof, it will suffice to show that a morphism $u : \mathfrak{g}_* \to \mathfrak{g}'_*$ in $\mathrm{Lie}^{\mathrm{dg}}_{\kappa}$ is a fibration if and only if it is degreewise surjective. Suppose first that u is a fibration. For each integer n, let $i_n : 0 \to \mathrm{Free}(E(n)_*)$ be the evident map of differential graded Lie algebras. Then i_n factors as a composition

$$0 \to 0 \amalg_{\mathrm{Free}(\partial\, E(n-1)_*)} \mathrm{Free}(E(n-1)_*) \simeq \mathrm{Free}(\partial\, E(n)_*) \to \mathrm{Free}(E(n)_*),$$

and is therefore a cofibration. The unit map $\kappa \simeq U(0) \to U(\mathrm{Free}(E(n)_*)) \simeq \bigoplus_{m \geqslant 0} E(n)_*^{\otimes m}$ is a quasi-isomorphism (since $E(n)$ is acyclic and therefore each $E(n)_*^{\otimes m}$ is acyclic for $m > 0$). It follows that i_n is a trivial cofibration, so that u has the right lifting property with respect to i_n. Unwinding the definitions, we conclude that the map $\mathfrak{g}_n \to \mathfrak{g}'_n$ is surjective.

Now suppose that u is degreewise surjective; we wish to show that u is a fibration. Let S be the collection of all trivial cofibrations in $\mathrm{Lie}^{\mathrm{dg}}_{\kappa}$ which have the left lifting property with respect to u. Let $f : \mathfrak{h}_* \to \mathfrak{h}''_*$ be a trivial cofibration in $\mathrm{Lie}^{\mathrm{dg}}_{k}$; we will prove that $f \in S$. Note that f contains each of the trivial cofibrations $i_n : 0 \to \mathrm{Free}(E(n)_*)$ above. Using the small object argument, we can factor f as a composition $\mathfrak{h}_* \xrightarrow{f'} \mathfrak{h}'_* \xrightarrow{f''} \mathfrak{h}''_*$ where $f' \in S$ and f'' has the right lifting property with respect to each of the morphisms i_n: that is, f'' is degreewise surjective. Since f and f' are quasi-isomorphisms, we conclude that f'' is a quasi-isomorphism. It follows that f'' is a trivial fibration in the category of chain complexes and therefore a trivial fibration in the category $\mathrm{Lie}^{\mathrm{dg}}_{\kappa}$. Because f is a cofibration, the lifting problem

$$\begin{array}{ccc} \mathfrak{h}_* & \xrightarrow{f'} & \mathfrak{h}'_* \\ \downarrow{\scriptstyle f} & \nearrow & \downarrow{\scriptstyle f''} \\ \mathfrak{g}''_* & \xrightarrow{\mathrm{id}} & \mathfrak{g}''_* \end{array}$$

admits a solution. We conclude that f is a retract of f', and therefore also belongs to S. $\square$

13.1.4 The ∞-Category of Differential Graded Lie Algebras

For our purposes, it will be convenient to view the collection of differential graded Lie algebras (over a field κ of characteristic zero) not as an ordinary category or as a model category, but as an ∞-category.

Definition 13.1.4.1. Let κ be a field of characteristic zero. We let Lie_κ denote the underlying ∞-category of the model category $\mathrm{Lie}_k^{\mathrm{dg}}$. More precisely, Lie_κ denotes an ∞-category equipped with a functor $u : \mathrm{Lie}_\kappa^{\mathrm{dg}} \to \mathrm{Lie}_\kappa$ having the following universal property: for every ∞-category $\mathcal{C}$, composition with u induces an equivalence from $\mathrm{Fun}(\mathrm{Lie}_\kappa, \mathcal{C})$ to the full subcategory of $\mathrm{Fun}(\mathrm{Lie}_\kappa^{\mathrm{dg}}, \mathcal{C})$ spanned by those functors $F : \mathrm{Lie}_\kappa^{\mathrm{dg}} \to \mathcal{C}$ which carry quasi-isomorphisms in $\mathrm{Lie}_\kappa^{\mathrm{dg}}$ to equivalences in $\mathcal{C}$ (see Definition HA.1.3.4.15 and Remark HA.1.3.4.16). We will refer to Lie_κ as the *∞-category of differential graded Lie algebras over κ.*

Remark 13.1.4.2. Using Proposition HA.7.1.1.15 , we conclude that the underlying ∞-category of the model category $\mathrm{Vect}_\kappa^{\mathrm{dg}}$ can be identified with the ∞-category $\mathrm{Mod}_\kappa = \mathrm{Mod}_\kappa(\mathrm{Sp})$ of κ-module spectra. The forgetful functor $\mathrm{Lie}_\kappa^{\mathrm{dg}} \to \mathrm{Vect}_\kappa^{\mathrm{dg}}$ preserves quasi-isomorphisms, and therefore induces a forgetful functor $\mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa$.

Proposition 13.1.4.3. *Let $\mathcal{J}$ be a small category which is sifted (when viewed as an ∞-category). The forgetful functor $G : \mathrm{Lie}_\kappa^{\mathrm{dg}} \to \mathrm{Vect}_\kappa^{\mathrm{dg}}$ preserves $\mathcal{J}$-indexed homotopy colimits.*

Proof. Let $G' : \mathrm{Alg}_\kappa^{\mathrm{dg}} \to \mathrm{Vect}_\kappa^{\mathrm{dg}}$ be the forgetful functor. It follows from Remark 13.1.2.4 that the functor G is a retract of $G' \circ U$. It will therefore suffice to show that $G' \circ U$ preserves $\mathcal{J}$-indexed homotopy colimits. The functor U is a left Quillen functor (Remark 13.1.3.3) and therefore preserves all homotopy colimits. We are therefore reduced to showing that G' preserves $\mathcal{J}$-indexed homotopy colimits, which is a special case of Lemma HA.4.1.8.13 . □

Proposition 13.1.4.4. *The ∞-category Lie_κ is presentable, and the forgetful functor $\theta : \mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa$ of Remark 13.1.4.2 preserves small sifted colimits.*

Proof. The first assertion follows from Proposition HA.1.3.4.22 . Using Propositions HA.1.3.4.24 , HA.1.3.4.25 , and 13.1.4.3, we conclude that θ preserves colimits indexed by small categories $\mathcal{J}$ which are sifted (when viewed as ∞-categories). Since any filtered ∞-category $\mathcal{I}$ admits a left cofinal map $A \to \mathcal{I}$ where A is a filtered partially ordered set (Proposition HTT.5.3.1.18), we conclude that θ preserves small filtered colimits. Since θ also preserves geometric realizations of simplicial objects, it preserves all small sifted colimits (Corollary HTT.5.5.8.17). □

Remark 13.1.4.5. The forgetful functor $\theta : \mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa$ is monadic: that is, θ admits a left adjoint $\mathrm{Free} : \mathrm{Mod}_\kappa \to \mathrm{Lie}_\kappa$, and induces an equivalence of Lie_κ with $\mathrm{LMod}_T(\mathrm{Mod}_\kappa)$, where T is the monad on Mod_κ given by the composition $\theta \circ \mathrm{Free}$. This follows from Theorem HA.4.7.0.3 and Proposition 13.1.4.4.

13.2 Homology and Cohomology of Lie Algebras

Throughout this section, we fix a field κ, which we will assume to be of characteristic zero unless otherwise specified. Let $\mathfrak{g}$ be a Lie algebra over κ and let $U(\mathfrak{g})$ denote its universal enveloping algebra. We can regard κ as a (left or right) module over $U(\mathfrak{g})$, with each element of $\mathfrak{g}$ acting trivially on κ. The *homology and cohomology* groups of $\mathfrak{g}$ are defined by

$$\mathrm{H}_n(\mathfrak{g}) = \mathrm{Tor}_n^{U(\mathfrak{g})}(\kappa, \kappa) \qquad \mathrm{H}^n(\mathfrak{g}) = \mathrm{Ext}^n_{U(\mathfrak{g})}(\kappa, \kappa).$$

These groups can be described more explicitly as the homology groups of chain complexes $C_*(\mathfrak{g})$ and $C^*(\mathfrak{g})$, called the (homological and cohomological) *Chevalley-Eilenberg complexes* of $\mathfrak{g}$. In this section, we will review the definition of these chain complexes (in the more general setting of differential graded algebras) and establish some of their basic properties. These constructions will play an important role in the construction of the deformation theory $\mathfrak{D} : (\mathrm{CAlg}^{\mathrm{aug}}_\kappa)^{\mathrm{op}} \to \mathrm{Lie}_\kappa$ required for the proof of Theorem 13.0.0.2.

13.2.1 The Homological Chevalley-Eilenberg Complex

Suppose that $\mathfrak{g}$ is a Lie algebra over κ. To obtain a concrete description of the homology and cohomology groups $\mathrm{H}_*(\mathfrak{g})$ and $\mathrm{H}^*(\mathfrak{g})$, it is convenient to choose explicit resolution of the ground field κ as a (left) module over the universal enveloping algebra $U(\mathfrak{g})$. We can obtain such a resolution by taking the universal enveloping algebra of an acyclic differential graded Lie algebra which contains $\mathfrak{g}$.

Construction 13.2.1.1. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ. We define another differential graded Lie algebra $\mathrm{Cn}(\mathfrak{g})_*$ as follows:

(1) For each $n \in \mathbf{Z}$, the vector space $\mathrm{Cn}(\mathfrak{g})_*$ is given by $\mathfrak{g}_n \oplus \mathfrak{g}_{n-1}$. We will denote the elements of $\mathrm{Cn}(\mathfrak{g})_n$ by $x + \epsilon y$, where $x \in \mathfrak{g}_n$ and $y \in \mathfrak{g}_{n-1}$.

(2) The differential on $\mathrm{Cn}(\mathfrak{g})_*$ is given by the formula $d(x + \epsilon y) = dx + y - \epsilon dy$.

(3) The Lie bracket on $\mathrm{Cn}(\mathfrak{g})_*$ is given by $[x+\epsilon y, x'+\epsilon y'] = [x, x']+\epsilon([y, x']+(-1)^p[x, y'])$, where $x \in \mathfrak{g}_p$.

We will refer to $\mathrm{Cn}(\mathfrak{g})_*$ as the *cone* on $\mathfrak{g}_*$.

Remark 13.2.1.2. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ. Then the underlying chain complex $\mathrm{Cn}(\mathfrak{g})_*$ can be identified with the mapping cone for the identity $\mathrm{id} : \mathfrak{g}_* \to \mathfrak{g}_*$. It follows that $\mathrm{Cn}(\mathfrak{g})_*$ is a contractible chain complex. In particular, the map $0 \to \mathrm{Cn}(\mathfrak{g})_*$ is a quasi-isomorphism of differential graded Lie algebras.

Construction 13.2.1.3. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ. The zero map $\mathfrak{g}_* \to 0$ induces a morphism of differential graded algebras $U(\mathfrak{g}_*) \to U(0) \simeq \kappa$. There is an evident map of differential graded Lie algebras $\mathfrak{g}_* \to \mathrm{Cn}(\mathfrak{g})_*$. We let $C_*(\mathfrak{g}_*)$ denote the chain complex given by the tensor product $U(\mathrm{Cn}(\mathfrak{g})_*) \otimes_{U(\mathfrak{g}_*)} \kappa$. We will refer to $C_*(\mathfrak{g}_*)$ as the *homological Chevalley-Eilenberg complex of* $\mathfrak{g}_*$.

Remark 13.2.1.4. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field κ, and regard the shifted chain complex $\mathfrak{g}_*[1]$ as a graded Lie algebra with a vanishing Lie bracket. There is an evident map of graded Lie algebras (without differential) $\mathfrak{g}_*[1] \to \mathrm{Cn}(\mathfrak{g})_*$. This map induces a map of graded vector spaces $\mathrm{Sym}^*_\kappa(\mathfrak{g}_*[1]) \simeq U(\mathfrak{g}_*[1]) \to U(\mathrm{Cn}(\mathfrak{g})_*)$. Using the Poincare-Birkhoff-Witt theorem, we obtain an isomorphism of graded right $U(\mathfrak{g}_*)$-modules

$$U(\mathrm{Cn}(\mathfrak{g})_*) \simeq \mathrm{Sym}^*_\kappa(\mathfrak{g}_*[1]) \otimes_\kappa U(\mathfrak{g}_*),$$

hence an isomorphism of graded vector spaces $\phi : \mathrm{Sym}^*_\kappa(\mathfrak{g}_*[1]) \to C_*(\mathfrak{g}_*)$. We will often identify $C_*(\mathfrak{g}_*)$ with the symmetric algebra $\mathrm{Sym}^*_\kappa(\mathfrak{g}_*[1])$ using the isomorphism ϕ. Note that ϕ is not an isomorphism of differential graded vector spaces. Unwinding the definitions, we see that the differential on $C_*(\mathfrak{g}_*)$ is given by the formula

$$\begin{aligned} D(x_1 \dots x_n) &= \sum_{1 \leqslant i \leqslant n} (-1)^{p_1 + \cdots + p_{i-1}} x_1 \dots x_{i-1} dx_i x_{i+1} \dots x_n + \\ & \sum_{1 \leqslant i < j \leqslant n} (-1)^{p_i(p_{i+1} + \cdots + p_{j-1})} x_1 \dots x_{i-1} x_{i+1} \dots x_{j-1} [x_i, x_j] x_{j+1} \dots x_n. \end{aligned}$$

Remark 13.2.1.5. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ. The filtration of $\mathrm{Sym}^*_\kappa(\mathfrak{g}_*)$ by the subspaces $\mathrm{Sym}^{\leqslant n}_\kappa(\mathfrak{g}_*) \simeq \bigoplus_{i \leqslant n} \mathrm{Sym}^i_\kappa(\mathfrak{g}_*)$ determines a filtration

$$\kappa \simeq C^{\leqslant 0}_*(\mathfrak{g}_*) \hookrightarrow C^{\leqslant 1}_*(\mathfrak{g}_*) \hookrightarrow C^{\leqslant 2}_*(\mathfrak{g}_*) \hookrightarrow \cdots$$

Using the formula for the differential on $C_*(\mathfrak{g}_*)$ given in Remark 13.2.1.4, we deduce the existence of canonical isomorphisms $C^{\leqslant n}_*(\mathfrak{g}_*)/C^{\leqslant n-1}_*(\mathfrak{g}_*) \simeq \mathrm{Sym}^n_\kappa(\mathfrak{g}_*)$ in the category of differential graded vector spaces over κ.

13.2.2 Lie Algebra Homology

If $\mathfrak{g}_*$ is a differential graded Lie algebra, we will refer to the homology groups of the chain complex $C_*(\mathfrak{g}_*)$ as the *Lie algebra homology groups* of $\mathfrak{g}_*$. This is a quasi-isomorphism invariant:

Proposition 13.2.2.1. *Let $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ be a quasi-isomorphism between differential graded Lie algebras over κ. Then the induced map $C_*(\mathfrak{g}_*) \to C_*(\mathfrak{g}'_*)$ is a quasi-isomorphism of chain complexes.*

Proof. Since the collection of quasi-isomorphisms is closed under filtered colimits, it will suffice to show that the induced map $\theta_n : C_*^{\leqslant n}(\mathfrak{g}_*) \to C_*^{\leqslant n}(\mathfrak{g}'_*)$ is a quasi-isomorphism for each $n \geqslant 0$. We proceed by induction on n. When $n = 0$, the map θ is an isomorphism and there is nothing to prove. Assume therefore that $n > 0$, so that we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*^{\leqslant n-1}(\mathfrak{g}_*) & \longrightarrow & C_*^{\leqslant n}(\mathfrak{g}_*) & \longrightarrow & \mathrm{Sym}^n_\kappa(\mathfrak{g}_*[1]) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_{n-1}} & & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & C_*^{\leqslant n-1}(\mathfrak{g}'_*) & \longrightarrow & C_*^{\leqslant n}(\mathfrak{g}'_*) & \longrightarrow & \mathrm{Sym}^n_\kappa(\mathfrak{g}'_*[1]) & \longrightarrow & 0 \end{array}$$

Using the inductive hypothesis, we are reduced to showing that the map ϕ is a quasi-isomorphism. Since κ is a field of characteristic zero, the map ϕ is a retract of the map $\mathfrak{g}_*^{\otimes n}[n] \to \mathfrak{g}_*'^{\otimes n}[n]$, which is a quasi-isomorphism by virtue of our assumption that f is a quasi-isomorphism. □

Remark 13.2.2.2. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ. Then $U(\mathrm{Cn}(\mathfrak{g})_*)$ can be regarded as a cofibrant replacement for κ in the model category of differential graded right modules over $U(\mathfrak{g}_*)$. The tensor product functor $M_* \mapsto M_* \otimes_{U(\mathfrak{g}_*)} \kappa$ is a left Quillen functor. It follows that $C_*(\mathfrak{g}_*)$ is an explicit model for the left derived tensor product $\kappa \otimes^L_{U(\mathfrak{g}_*)} \kappa$. Equivalently, the image of $C_*(\mathfrak{g}_*)$ in Mod_κ can be identified with the ∞-categorical relative tensor product $\kappa \otimes_A \kappa$, where $A \in \mathrm{Alg}_\kappa$ is the image of $U(\mathfrak{g}_*)$ under the functor $\mathrm{Alg}^{\mathrm{dg}}_\kappa \to \mathrm{Alg}_\kappa$.

13.2.3 The Case of a Free Algebra

If $\mathfrak{g}_*$ is a free differential graded Lie algebra, then the homology of $\mathfrak{g}_*$ is easy to describe.

Proposition 13.2.3.1. *Let V_* be a differential graded vector spaces over κ and let $\mathfrak{g}_*$ be free differential graded Lie algebra generated by V_*. Then the inclusion of chain complexes*

$$\xi : \kappa \oplus V_*[1] \hookrightarrow \kappa \oplus \mathfrak{g}_*[1] \simeq C_*^{\leqslant 1}(\mathfrak{g}_*) \hookrightarrow C_*(\mathfrak{g}_*)$$

is a quasi-isomorphism.

To prove Proposition 13.2.3.1, we will need some general observations about differential graded algebras and their modules.

Lemma 13.2.3.2. *Let A_* be a differential graded algebra over a field κ (not necessarily assumed to be of characteristic zero), and let $f : M_* \to N_*$ be a map of differential graded right modules over A_*. Assume that:*

(1) *The differential graded module M_* can be written as a union of submodules*

$$0 = M(0)_* \subseteq M(1)_* \subseteq M(2)_* \subseteq \cdots$$

where each successive quotient $M(n)_*/M(n-1)_*$ is isomorphic (as a differential graded A_*-module) to a free differential graded module of the form $\bigoplus_\alpha A_*[e_\alpha]$.

(2) *The chain complex N_* is acyclic.*

Then the map f is nullhomotopic. That is, there exists a map of graded A_*-modules $h : M_* \to N_{*+1}$ satisfying $dh + hd = f$.

Proof. We construct a compatible family of nullhomotopies $h(n) : M(n)_* \to N_{*+1}$ for the maps $f(n) = f|_{M(n)_*}$. When $n = 0$, such a nullhomotopy exists and is unique (since $M(0)_* \simeq 0$). Assume therefore that $n > 0$ and that $h(n-1)$ has been constructed. Condition (1) guarantees that $M(n)_*/M(n-1)_*$ is freely generated (as an A_*-module) by generators $\overline{x}_\alpha \in (M(n)/M(n-1))_{e_\alpha}$. Choose $x_\alpha \in M(n)_{e_\alpha}$ representing $\overline{x}_\alpha$. We compute

$$d(f(x_\alpha) - h(n-1)dx_\alpha) = f(dx_\alpha) - d(h(n-1)dx_\alpha) = h(n-1)d^2x_\alpha = 0.$$

Since N_* is acyclic, we can choose $y_\alpha \in N_{e_\alpha+1}$ with $dy_\alpha = f(x_\alpha) - h(n-1)dx_\alpha$. We now define $h(n)$ to be the unique map of graded A_*-modules from $M(n)_*$ to N_{*+1} which extends $h(n-1)$ and carries x_α to y_α; it is easy to see that $h(n)$ has the desired properties. □

Lemma 13.2.3.3. *Let A_* be a differential graded algebra over a field κ (not necessarily assumed to be of chracteristic zero), and let M_* be a chain complex of differential graded right modules over A_*. Assume that M_* is acyclic and satisfies condition* (1) *of Lemma 13.2.3.2. Then, for any differential graded left A_*-module N_*, the tensor product $M_* \otimes_{A_*} N_*$ is acyclic.*

Proof. It follows from Lemma 13.2.3.2 that identity map $\mathrm{id} : M_* \to M_*$ is chain homotopic to zero: that is, there exists a map $h : M_* \to M_{*+1}$ such that $dh + hd = \mathrm{id}$. Then h determines a contracting homotopy for $M_* \otimes_{A_*} N_*$, so that $M_* \otimes_{A_*} N_*$ is also acyclic. □

Proof of Proposition 13.2.3.1. Note that the universal enveloping algebra $U(\mathfrak{g}_*)$ can be identified with the tensor algebra $T(V_*) \simeq \bigoplus_{n \geqslant 0} V_*^{\otimes n}$. Let $M_* \subseteq U(\mathrm{Cn}(\mathfrak{g})_*)$ be the right $T(V_*)$-submodule generated by $\kappa \oplus V_*[1]$. Unwinding the definitions, we see that M_* is isomorphic (as a chain complex) to the direct sum $\kappa \oplus M'_*$, where M'_* is isomorphic to mapping cone of the identity map from $\bigoplus_{n \geqslant 1} V_*^{\otimes n}$ to itself. It follows that the inclusion $\kappa \hookrightarrow M_*$ is a quasi-isomorphism. The composite inclusion $\kappa \hookrightarrow M_* \to U(\mathrm{Cn}(\mathfrak{g})_*)$ is given by applying the universal enveloping algebra functor U to the inclusion of differential graded Lie algebras $0 \to \mathrm{Cn}(\mathfrak{g})_*$, and is therefore a quasi-isomorphism by Remark 13.2.1.2 and

Lemma 13.1.3.4. It follows that the inclusion $M_* \subseteq U(\mathrm{Cn}(\mathfrak{g})_*)$ is a quasi-isomorphism, so that the quotient $Q_* = U(\mathrm{Cn}(\mathfrak{g})_*)/M_*$ is acyclic. It is not difficult to see that Q_* satisfies hypothesis (1) of Lemma 13.2.3.2; in particular, Q_* is free as a graded A_*-module. It follows that we have an exact sequence of chain complexes

$$0 \to M_* \otimes_{T(V_*)} \kappa \xrightarrow{\theta} U(\mathrm{Cn}(\mathfrak{g})_*) \otimes_{T(V_*)} \kappa \to Q_* \otimes_{T(V_*)} \kappa \to 0.$$

Lemma 13.2.3.3 guarantees that $Q_* \otimes_{T(V_*)} \kappa$ is acyclic, so that θ determines a quasi-isomorphism $\kappa \oplus V_*[1] \to C_*(\mathfrak{g}_*)$. □

13.2.4 Compatibility with Colimits

It follows from Proposition 13.2.2.1 that the Chevalley-Eilenberg construction $C_* : \mathrm{Lie}_\kappa^{\mathrm{dg}} \to \mathrm{Vect}_\kappa^{\mathrm{dg}}$ induces a functor of ∞-categories $\mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa$, which we will also denote by C_*. Note that C_* carries the initial object $0 \in \mathrm{Lie}_\kappa$ to $C_*(0) \simeq \kappa$, and therefore induces a functor $\mathrm{Lie}_\kappa \to (\mathrm{Mod}_\kappa)_{\kappa/}$. We will abuse notation by denoting this functor also by C_*.

Proposition 13.2.4.1. *The functor of ∞-categories $C_* : \mathrm{Lie}_\kappa \to (\mathrm{Mod}_\kappa)_{\kappa/}$ preserves small colimits.*

Proof. By virtue of Corollary HTT.4.2.3.11 and Lemma HA.1.3.3.10 , it will suffice to show that C_* preserves finite coproducts and small sifted colimits. We begin by showing that C_* preseres small sifted colimits. Using Lemma HTT.4.4.2.8 and Proposition HTT.4.3.1.5 , we are reduced to showing that the composite functor $\mathrm{Lie}_\kappa \to (\mathrm{Mod}_\kappa)_{\kappa/} \to \mathrm{Mod}_\kappa$ preserves small sifted colimits. The proof of Proposition 13.2.2.1 shows that for each $n \geqslant 0$, the functor $C_*^{\leqslant n}$ preserves quasi-isomorphisms and therefore induces a functor of ∞-categories $\mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa$. Since the collection of quasi-isomorphisms in $\mathrm{Vect}_\kappa^{\mathrm{dg}}$ is closed under filtered colimits, every colimit diagram in $\mathrm{Vect}_\kappa^{\mathrm{dg}}$ indexed by a filtered category determines a homotopy colimit diagram in $\mathrm{Vect}_\kappa^{\mathrm{dg}}$ and therefore a colimit diagram in Mod_κ (Proposition HA.1.3.4.24). It follows that the functor $C_* : \mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa$ is a colimit of the functors $C_*^{\leqslant n} : \mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa$. Using Proposition HTT.5.5.2.3 , we are reduced to proving that each of the functors $C_*^{\leqslant n}$ preserves small sifted colimits. We proceed by induction on n, the case $n < 0$ being trivial. Since the field κ has characteristic zero, the construction $V_* \mapsto \mathrm{Sym}_\kappa^n V_*$ preserves quasi-isomorphisms and therefore induces a functor $\mathrm{Sym}_\kappa^n : \mathrm{Mod}_\kappa \to \mathrm{Mod}_\kappa$. Let $\theta : \mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa$ be the forgetful functor. Using Remark 13.2.1.5 and Corollary HA.1.3.2.16 , we obtain a fiber sequence of functors

$$C_*^{\leqslant n-1} \to C_*^{\leqslant n} \to \mathrm{Sym}_\kappa^n \circ \theta[1]$$

from Lie_κ to Mod_κ. Since $C_*^{\leqslant n-1}$ preserves sifted colimits by the inductive hypothesis and $\theta[1]$ preserves sifted colimits by Proposition 13.1.4.4, it will suffice to show that the functor Sym_κ^n preserves sifted colimits. Since the characteristic of κ is zero, the functor Sym_κ^n is a retract of the functor $V_* \mapsto V_*^{\otimes n}$, which evidently preserves sifted colimits.

We now prove that $C_* : \mathrm{Lie}_\kappa \to (\mathrm{Mod}_\kappa)_{\kappa/}$ preserves finite coproducts. Since C_* preserves initial objects by construction, it will suffice to show that C_* preserves pairwise coproducts. That is, we must show that for every pair of differential graded Lie algebras $\mathfrak{g}_*$ and $\mathfrak{g}'_*$ having a coproduct $\mathfrak{g}''_*$ in Lie_κ, the diagram σ :

$$\begin{array}{ccc} \kappa & \longrightarrow & C_*(\mathfrak{g}_*) \\ \downarrow & & \downarrow \\ C_*(\mathfrak{g}'_*) & \longrightarrow & C_*(\mathfrak{g}''_*) \end{array}$$

is a pushout square in Mod_κ.

Let $\mathrm{Free} : \mathrm{Mod}_\kappa \to \mathrm{Lie}_\kappa$ be a left adjoint to the forgetful functor. Using Proposition HA.4.7.3.14 and Proposition 13.1.4.4, we deduce that $\mathfrak{g}_*$ can be obtained as the geometric realization of a simplicial object $(\mathfrak{g}_*)_\bullet$ of Lie_κ, where each $(\mathfrak{g}_*)_n$ lies in the essential image of Free. Similarly, we can write $\mathfrak{g}'_*$ as the geometric realization of a simplicial object $(\mathfrak{g}'_*)_\bullet$. Then $(\mathfrak{g}''_*)_\bullet$ is the geometric realization of a simplicial object $(\mathfrak{g}''_*)_\bullet$ of Lie_κ, given by $[n] \mapsto (\mathfrak{g}_*)_n \amalg (\mathfrak{g}'_*)_n$. Since the functor C_* commutes with geometric realization of simplicial objects, it will suffice to show that the diagram

$$\begin{array}{ccc} \kappa & \longrightarrow & C_*((\mathfrak{g}_*)_n) \\ \downarrow & & \downarrow \\ C_*((\mathfrak{g}'_*)_n) & \longrightarrow & C_*((\mathfrak{g}''_*)_n) \end{array}$$

is a pushout square in Mod_κ, for each $n \geqslant 0$. We may therefore reduce to the case where $\mathfrak{g}_* \simeq \mathrm{Free}(V_*)$, $\mathfrak{g}'_* \simeq \mathrm{Free}(V'_*)$ for some objects $V_*, V'_* \in \mathrm{Mod}_\kappa$. Then $\mathfrak{g}''_* \simeq \mathrm{Free}(V_* \oplus V'_*)$. Using Proposition 13.2.3.1, we can identify σ with the diagram

$$\begin{array}{ccc} \kappa & \longrightarrow & \kappa \oplus V_*[1] \\ \downarrow & & \downarrow \\ \kappa \oplus V'_*[1] & \longrightarrow & \kappa \oplus V_*[1] \oplus V'_*[1], \end{array}$$

which is evidently a pushout square in the ∞-category Mod_κ. □

13.2.5 The Cohomological Chevalley-Eilenberg Complex

We now turn our attention to the cohomology of (differential graded) Lie algebras.

Construction 13.2.5.1. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ. We let $C^*(\mathfrak{g}_*)$ denote the linear dual of the chain complex dual to $C_*(\mathfrak{g}_*)$. We will refer to $C^*(\mathfrak{g}_*)$ as the *cohomological Chevalley-Eilenberg complex* of $\mathfrak{g}_*$. We will identify elements $\lambda \in C^n(\mathfrak{g}_*)$ with the dual space of the degree n part of the graded vector space $\mathrm{Sym}^*_\kappa(\mathfrak{g}_*[1])$.

There is a natural multiplication on $C^*(\mathfrak{g}_*)$, which carries $\lambda \in C^p(\mathfrak{g}_*)$ and $\mu \in C^q(\mathfrak{g}_*)$ to the element $\lambda\mu \in C^{p+q}(\mathfrak{g}_*)$ characterized by the formula

$$(\lambda\mu)(x_1 \ldots x_n) = \sum_{S,S'} \epsilon(S,S')\lambda(x_{i_1} \ldots x_{i_m})\mu(x_{j_1} \ldots x_{j_{n-m}}).$$

Here $x_i \in \mathfrak{g}_{r_i}$ denotes a sequence of homogeneous elements of $\mathfrak{g}_*$, the sum is taken over all disjoint sets $S = \{i_1 < \ldots < i_m\}$ and $S' = \{j_1 < \ldots < j_{n-m}\}$ range with $S \cup S' = \{1, \ldots, n\}$ and $r_{i_1} + \cdots + r_{i_m} = p$, and $\epsilon(S,S') = \prod_{i \in S', j \in S, i<j} (-1)^{r_i r_j}$. With this multiplication, $C^*(\mathfrak{g}_*)$ has the structure of a commutative differential graded algebra over κ.

Remark 13.2.5.2. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ. Unwinding the definitions, we can identify $C^*(\mathfrak{g}_*)$ with the chain complex of right $U(\mathfrak{g}_*)$-linear maps from $U(\mathrm{Cn}(\mathfrak{g})_*)$ into κ. Arguing as in Remark 13.2.2.2, we see that $C^*(\mathfrak{g}_*)$ is a model for the right derived mapping complex of right $U(\mathfrak{g}_*)$-module maps from κ to itself.

Remark 13.2.5.3. Let κ be a field of characteristic zero, let V_* be a chain complex of vector spaces over κ, and let $\mathfrak{g}_*$ be the free differential graded Lie algebra generated by V_*. The quasi-isomorphism $\kappa \oplus V_*[1] \to C_*(\mathfrak{g}_*)$ of Proposition 13.2.3.1 induces a quasi-isomorphism of chain complexes $C^*(\mathfrak{g}_*) \to \kappa \oplus V_*^\vee[-1]$, where $V_*^\vee$ denote the dual of the chain complex V_*. In fact, this map is a quasi-isomorphism of commutative differential graded algebras (where we regard $\kappa \oplus V_*^\vee[-1]$ as a trivial square-zero extension of κ).

Notation 13.2.5.4. If the field κ has characteristic zero, Proposition 13.2.2.1 implies that the construction $\mathfrak{g}_* \mapsto C^*(\mathfrak{g}_*)$ carries quasi-isomorphisms of differential graded Lie algebras to quasi-isomorphisms of commutative differential graded algebras. Consequently, we obtain a functor between ∞-categories $\mathrm{Lie}_\kappa \to \mathrm{CAlg}_\kappa^{\mathrm{op}}$, which we will also denote by C^*.

Note that the functor C^* carries the initial object $0 \in \mathrm{Lie}_\kappa$ to the final object $\kappa \in \mathrm{CAlg}_\kappa^{\mathrm{op}}$. We therefore obtain a functor $\mathrm{Lie}_\kappa \to (\mathrm{CAlg}_\kappa^{\mathrm{op}})_{\kappa/} \simeq (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}}$, where $\mathrm{CAlg}_\kappa^{\mathrm{aug}} = (\mathrm{CAlg}_\kappa)_{/\kappa}$ denotes the ∞-category of augmented $\mathbb{E}_\infty$-algebras over κ. We will abuse notation by denoting this functor also by C^*.

Proposition 13.2.5.5. *The functor* $C^* : \mathrm{Lie}_\kappa \to (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}}$ *preserves small colimits.*

Proof. Using Corollary HA.3.2.2.5 , we are reduced to proving that the composite functor

$$\mathrm{Lie}_\kappa \xrightarrow{C^*} (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \longrightarrow (\mathrm{Mod}_\kappa^{\mathrm{op}})_{\kappa/}$$

preserves small colimits. We note that this composition can be identified with the functor

$$\mathrm{Lie}_\kappa \xrightarrow{C_*} (\mathrm{Mod}_\kappa)_{\kappa/} \xrightarrow{D} (\mathrm{Mod}_\kappa^{\mathrm{op}})_{\kappa/},$$

where D is induced by the κ-linear duality functor $V_* \mapsto V_*^\vee$ from $\mathrm{Vect}_\kappa^{\mathrm{dg}}$ to itself. According to Proposition 13.2.4.1, it will suffice to show that D preserves small colimits. Using

Propositions HA.1.3.4.23 , HA.1.3.4.24 , and HA.1.3.4.25 , we are reduced to the problem of showing that the functor $V_* \mapsto V_*^\vee$ carries homotopy colimits in $\mathrm{Vect}_\kappa^{\mathrm{dg}}$ to homotopy limits in $\mathrm{Vect}_\kappa^{\mathrm{dg}}$, which is clear. □

13.3 Koszul Duality

Let κ be a field of characteristic zero and let $C^* : \mathrm{Lie}_\kappa \to (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}}$ be the functor constructed in Notation 13.2.5.4. Proposition 13.2.5.5 implies that C^* preserves small colimits. Since the ∞-category Lie_κ is presentable (Proposition 13.1.4.4), Corollary HTT.5.5.2.9 (and Remark HTT.5.5.2.10) imply that C^* admits a right adjoint $\mathfrak{D} : (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Lie}_\kappa$. We will refer to the functor $\mathfrak{D}$ as *Koszul duality.* The main goal of this section is to prove the following result:

Theorem 13.3.0.1. *Let κ be a field of characteristic zero and let $(\mathrm{CAlg}_\kappa^{\mathrm{aug}}, \{E\})$ be the deformation context of Example 12.1.1.2. Then the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Lie}_\kappa$ is a deformation theory (see Definition 12.3.3.2).*

We will then deduce Theorem 13.0.0.2 by combining Theorems 13.3.0.1 and 12.3.3.5.

Remark 13.3.0.2. Let us temporarily distinguish in notation between the functor of ∞-categories $C^* : \mathrm{Lie}_\kappa \to (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}}$ and the functor of ordinary categories $\mathrm{Lie}_\kappa^{\mathrm{dg}} \to (\mathrm{CAlg}_\kappa^{\mathrm{dg}})^{\mathrm{op}}_{/\kappa}$ of Construction **??**, denoting the latter by C^*_{dg}. It follows from Proposition 13.2.5.5 and the adjoint functor theorem that the functor C^* admits a right adjoint, the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Lie}_\kappa$. Consequently, the functor C^*_{dg} descends to a functor from the homotopy category of $\mathrm{Lie}_\kappa^{\mathrm{dg}}$ to the homotopy category of $(\mathrm{CAlg}_\kappa^{\mathrm{dg}})^{\mathrm{op}}_{/\kappa}$ which admits a right adjoint. However, the functor C^*_{dg} itself does not admit a right adjoint; in particular, it is not a left Quillen functor. Consequently, it is not so easy to give a concrete description of the functor $\mathfrak{D}$ using the formalism of differential graded Lie algebras. To obtain a more explicit construction of $\mathfrak{D}$, it is convenient to work in the setting of L_∞-algebras. Since we will not need this construction, we do not describe it here.

Remark 13.3.0.3. We will often abuse notation by identifying the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Lie}_\kappa$ with the induced functor between opposite ∞-categories $\mathrm{CAlg}_\kappa^{\mathrm{aug}} \to \mathrm{Lie}_\kappa^{\mathrm{op}}$.

13.3.1 The Double Dual

To prove Theorem 13.3.0.1, we must show that the adjunction $\mathrm{Lie}_\kappa \underset{\mathfrak{D}}{\overset{C^*}{\rightleftarrows}} (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}}$ is not too far from being an equivalence of ∞-categories. More precisely, we have the following result:

Proposition 13.3.1.1. *Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field κ of characteristic zero. Assume that:*

(a) *For every integer n, the vector space $\mathfrak{g}_n$ is finite dimensional.*

(b) *The vector space $\mathfrak{g}_n$ is trivial for $n \geqslant 0$.*

Then the unit map $u : \mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g})$ is an equivalence in Lie_κ.

The proof of Proposition 13.3.1.1 will require some preliminaries.

Notation 13.3.1.2. Let $F : (\mathrm{Vect}_\kappa^{\mathrm{dg}})^{\mathrm{op}} \to \mathrm{Vect}_\kappa^{\mathrm{dg}}$ be the functor between ordinary categories which carries each chain complex (V_*, d) to the dual chain complex $(V_*^\vee, d^\vee)$, where $V_n^\vee = \mathrm{Hom}_\kappa(V_{-n}, \kappa)$ and the differential $d^\vee$ is characterized by the formula $d^\vee(\lambda)(v) + (-1)^n \lambda(dv) = 0$ for $\lambda \in V_n^\vee$. The construction $V_* \mapsto V_*^\vee$ preserves quasi-isomorphisms and therefore induces a functor $\mathrm{Mod}_\kappa^{\mathrm{op}} \to \mathrm{Mod}_\kappa$, which we will denote by $V \mapsto V^\vee$. We will refer to this functor as *κ-linear duality.*

Remark 13.3.1.3. For every pair of κ-module spectra $V, W \in \mathrm{Mod}_\kappa$, we have canonical homotopy equivalences

$$\mathrm{Map}_{\mathrm{Mod}_\kappa}(V, W^\vee) \simeq \mathrm{Map}_{\mathrm{Mod}_\kappa}(V \otimes W, \kappa) \simeq \mathrm{Map}_{\mathrm{Mod}_\kappa}(W, V^\vee).$$

It follows that κ-linear duality, when regarded as a functor $\mathrm{Mod}_\kappa \to \mathrm{Mod}_\kappa^{\mathrm{op}}$, is canonically equivalent to the left adjoint of the κ-linear duality functor $\mathrm{Mod}_\kappa^{\mathrm{op}} \to \mathrm{Mod}_\kappa$.

Let us now study the composite functor $(\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \xrightarrow{\mathfrak{D}} \mathrm{Lie}_\kappa \xrightarrow{\theta} \mathrm{Mod}_\kappa$, where θ denotes the forgetful functor. This composition admits a left adjoint $\mathrm{Mod}_\kappa \xrightarrow{\mathrm{Free}} \mathrm{Lie}_\kappa \xrightarrow{C^*} (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}}$, which is in turn induced by the map of ordinary categories $\mathrm{Vect}_\kappa^{\mathrm{dg}} \to \mathrm{CAlg}_\kappa^{\mathrm{dg}}$ given by $V_* \mapsto C^*(\mathrm{Free}(V_*))$. Remark 13.2.5.3 supplies a (functorial) quasi-isomorphism of commutative differential graded algebras $C^*(\mathrm{Free}(V_*)) \to \kappa \oplus V_*^\vee[-1]$. It follows that the underlying functor of ∞-categories $\mathrm{Mod}_\kappa \to (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}}$ is given by composing the κ-linear duality functor $\mathrm{Mod}_\kappa \to \mathrm{Mod}_\kappa^{\mathrm{op}}$ with the functor $\mathrm{Mod}_\kappa^{\mathrm{op}} \to (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}}$ given by the formation of square-zero extensions $M \mapsto \kappa \oplus \Sigma^{-1} M$. Both of these functors admit left adjoints: in the first case, the left adjoint is given by κ-linear duality (Remark 13.3.1.3), and in the second it is given by the formation of the relative cotangent complex $A \mapsto \Sigma^{-1}(L_{A/\kappa} \otimes_A \kappa) \simeq L_{\kappa/A}$ respectively. We have proven:

Proposition 13.3.1.4. *Let κ be a field of characteristic zero and let $\theta : \mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa$ be the forgetful functor. Then the composite functor $(\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \xrightarrow{\mathfrak{D}} \mathrm{Lie}_\kappa \xrightarrow{\theta} \mathrm{Mod}_\kappa$ is given on objects by $A \mapsto L_{\kappa/A}^\vee$.*

To prove Proposition 13.3.1.1, we need to analyze the unit map $\mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g}_*)$ associated to a differential graded Lie algebra $\mathfrak{g}_*$. We begin with a few preliminary remarks regarding explicit models for the cotangent fiber of a commutative differential graded algebra.

Definition 13.3.1.5. Let A_* be a commutative differential graded algebra over κ equipped with an augmentation $u : A_* \to \kappa$. The kernel of u is an ideal $\mathfrak{m}_A \subseteq A_*$. We let $\mathrm{Indec}(A)_*$ denote the quotient $\mathfrak{m}_A/\mathfrak{m}_A^2$, which we regard as a complex of κ-vector spaces. We will refer to $\mathrm{Indec}(A)_*$ as the *chain complex of indecomposables in* A_*.

Remark 13.3.1.6. The construction $V_* \mapsto \kappa \oplus V_*$ determines a right Quillen functor from $\mathrm{Vect}_\kappa^{\mathrm{dg}}$ to $(\mathrm{CAlg}_\kappa^{\mathrm{dg}})_{/\kappa}$, whose left adjoint is given by $A_* \mapsto \mathrm{Indec}(A)_*$. It follows that the functor $\mathrm{Indec}(A)_*$ preserves weak equivalences between cofibrant objects of $(\mathrm{CAlg}_\kappa^{\mathrm{dg}})_{/\kappa}$, and induces a functor of ∞-categories $\mathrm{CAlg}_\kappa^{\mathrm{aug}} \to \mathrm{Mod}_\kappa$. This functor is evidently left adjoint to the formation of trivial square-zero extensions, and is therefore given by $A \mapsto L_{A/\kappa} \otimes_A \kappa \simeq \Sigma^{-1} L_{\kappa/A}$. It follows that for every cofibrant augmented commutative differential graded algebra A_*, the canonical map $A_* \to \kappa \oplus \mathrm{Indec}(A)_*$ induces an equivalence $\Sigma^{-1} L_{\kappa/A_*} \simeq L_{A_*/\kappa} \otimes_{A_*} \kappa \to \mathrm{Indec}(A)_*$ in Mod_κ (here we abuse notation by identifying A_* with its image in the ∞-category $\mathrm{CAlg}_\kappa^{\mathrm{aug}}$).

Proof of Proposition 13.3.1.1. Let $\mathfrak{g}_*$ be a differential graded Lie algebra which is concentrated in negative degrees and finite-dimensional in each degree. We wish to show that the unit map $\mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g})$ is an equivalence in the ∞-category Lie_κ. Since the forgetful functor $\mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa$ is conservative, it will suffice to show that u induces an equivalence $\mathfrak{g}_* \to L^\vee_{\kappa/C^*(\mathfrak{g}_*)}$ in Mod_κ (see Proposition 13.3.1.4). This map admits a predual $u^\vee : L_{C^*(\mathfrak{g}_*)/\kappa} \otimes_{C^*(\mathfrak{g})} \kappa \to \Sigma^{-1} \mathfrak{g}_*^\vee$. We will prove that $u^\vee$ is an equivalence.

Consider the isomorphism of graded vector spaces $C^*(\mathfrak{g}_*) \simeq \prod_{n \geqslant 0} (\mathrm{Sym}^n_\kappa \mathfrak{g}_*[1])^\vee$. Choose a basis $\{y_1, \ldots, y_p\}$ for the vector space $\mathfrak{g}_{-1}$, and let $\{x_1, \ldots, x_p\}$ be the dual basis for $\mathfrak{g}_1^\vee$, so that $C^0(\mathfrak{g}_*)$ can be identified with the power series ring $\kappa[[x_1, \ldots, x_p]]$. Let $A_* = \bigoplus_{n \geqslant 0} (\mathrm{Sym}^n_\kappa(\mathfrak{g}_*[1]))^\vee$, and regard A_* as a graded subalgebra of $C^*(\mathfrak{g}_*)$. It is easy to see that A_* is a differential graded subalgebra of $C^*(\mathfrak{g}_*)$, and that A_0 contains the polynomial ring $\kappa[x_1, \ldots, x_p]$. Using (a) and (b), we deduce that A_* is a graded polynomial ring generated by $g_*^\vee[-1]$, and that the natural map

$$A_* \otimes_{\kappa[x_1, \ldots, x_p]} \kappa[[x_1, \ldots, x_p]] \to C^*(\mathfrak{g}_*)$$

is an isomorphism of commutative differential graded algebras. Since $\kappa[[x_1, \ldots, x_p]]$ is flat over $\kappa[x_1, \ldots, x_p]$, it follows that for each $n \in \mathbf{Z}$ we have an isomorphism in homology

$$\mathrm{H}_n(A_*) \otimes_{\kappa[x_1, \ldots, x_p]} \kappa[[x_1, \ldots, x_p]] \to \mathrm{H}_n(C^*(\mathfrak{g}_*)),$$

so that the diagram

$$\begin{array}{ccc} \kappa[x_1, \ldots, x_p] & \longrightarrow & \kappa[[x_1, \ldots, x_p]] \\ \downarrow & & \downarrow \\ A_* & \longrightarrow & C^*(\mathfrak{g}_*) \end{array}$$

is a pushout square in the ∞-category CAlg_κ. We therefore obtain equivalences

$$L_{C^*(\mathfrak{g}_*)/A_*} \otimes_{C^*(\mathfrak{g}_*)} \kappa \simeq L_{\kappa[[x_1,\ldots,x_p]]/\kappa[x_1,\ldots,x_p]} \otimes_{\kappa[[x_1,\ldots,x_p]]} \kappa \simeq L_{R/\kappa} \simeq 0.$$

where R denotes the tensor product $\kappa[[x_1, \ldots, x_p]] \otimes_{\kappa[x_1,\ldots,x_p]} \kappa \simeq \kappa$. It follows that u can be identified with the map $L_{A_*/\kappa} \otimes_{A_*} \kappa \to \Sigma^{-1}\mathfrak{g}_*^\vee$ which classifies the morphism $A_* \to \kappa \oplus \mathfrak{g}_*^\vee[-1] \simeq \kappa \oplus \mathrm{Indec}(A)_*$. Since A_* is a cofibrant differential graded algebra, Remark 13.3.1.6 implies that u is an equivalence in Mod_κ. □

13.3.2 The Main Theorem

The first step in our proof of Theorem 13.3.0.1 is to show that the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Lie}_\kappa$ is a weak deformation theory: that is, it satisfies axioms $(D1)$, $(D2)$, and $(D3)$ of Definition 12.3.1.1. Axioms $(D1)$ and $(D2)$ are easy: we have already seen that Lie_κ is presentable (Proposition 13.1.4.4), and the functor $\mathfrak{D}$ admits a left adjoint by construction. Axiom $(D3)$ is a consequence of the following more precise assertion:

Proposition 13.3.2.1. *Let κ be a field of characteristic zero and let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ. We will say that $\mathfrak{g}_*$ is good if it is cofibrant (with respect to the model structure on $\mathrm{Lie}_\kappa^{\mathrm{dg}}$ described in Proposition 13.1.3.2) and there exists a graded vector subspace $V_* \subseteq \mathfrak{g}_*$ satisfying the following conditions:*

(i) For every integer n, the vector space V_n is finite dimensional.

(ii) For every nonnegative integer n, the vector space V_n is trivial.

(iii) The graded vector space V_* freely generates $\mathfrak{g}_*$ as a graded Lie algebra.

Let $\mathcal{C}$ be the full subcategory of Lie_κ spanned by those objects which can be represented by good objects of $\mathrm{Lie}_\kappa^{\mathrm{dg}}$. Then $\mathcal{C}$ satisfies conditions (a), (b), (c), and (d) of Definition 12.3.1.1.

Proof. We verify each condition in turn:

(a) Let $\mathfrak{g}_* \in \mathcal{C}$; we wish to prove that the unit map $\mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g})$ is an equivalence in Lie_κ. We may assume without loss of generality that $\mathfrak{g}_*$ is good, so there is a graded subspace $V_* \subseteq \mathfrak{g}_*$ satisfying conditions (i), (ii), and (iii). As a graded vector space, $\mathfrak{g}_*$ is isomorphic to a direct summand of the augmentation ideal in $U(\mathfrak{g}) \simeq \bigoplus_{n \geqslant 0} V_*^{\otimes n}$. It follows that each $\mathfrak{g}_n$ is finite dimensional, and that $\mathfrak{g}_n \simeq 0$ for $n \geqslant 0$. The desired result now follows from Proposition 13.3.1.1.

(b) The initial object $0 \in \mathrm{Lie}_\kappa$ obviously belongs to $\mathcal{C}$.

(c) We must show that for each $n \geqslant 0$, the square-zero algebra $\kappa \oplus \Sigma^n(\kappa) \in \mathrm{CAlg}_\kappa^{\mathrm{aug}}$ is equivalent to $C^*(\mathfrak{g})$ for some object $\mathfrak{g}_* \in \mathcal{C}$. In fact, we can take $\mathfrak{g}_*$ to be the differential graded Lie algebra freely generated by the complex $\kappa[-n-1]$ (see Remark 13.2.5.3).

(d) Suppose that $n \leqslant -2$ and that we are given a pushout diagram

$$\begin{array}{ccc} \mathrm{Free}(\kappa[n]) & \xrightarrow{\alpha} & \mathfrak{g}_* \\ \downarrow{\scriptstyle v} & & \downarrow \\ 0 & \longrightarrow & \mathfrak{g}'_* \end{array}$$

in the ∞-category Lie_κ. Here $\mathrm{Free} : \mathrm{Mod}_\kappa \to \mathrm{Lie}_\kappa$ denotes the left adjoint to the forgetful functor. We wish to show that if $\mathfrak{g}_* \in \mathcal{C}$, then $\mathfrak{g}'_* \in \mathcal{C}$. We may assume without loss of generality that $\mathfrak{g}_*$ is good. Since $\mathrm{Free}(\kappa[n])$ is a cofibrant object of $\mathrm{Lie}_\kappa^{\mathrm{dg}}$ and $\mathfrak{g}_*$ is fibrant, we can assume that α is given by a morphism $\mathrm{Free}(\kappa[n]) \to \mathfrak{g}_*$ in the category $\mathrm{Lie}_\kappa^{\mathrm{dg}}$ (determined by a cycle $x \in \mathfrak{g}_n$). The morphism v in $\mathrm{Lie}\,\kappa$ is represented by the cofibration of differential graded Lie algebras $j : \mathrm{Free}(\partial\, E(n+1)_*) \hookrightarrow \mathrm{Free}(E(n+1)_*)$ (see the proof of Proposition 13.1.3.2). Form a pushout diagram σ :

$$\begin{array}{ccc} \mathrm{Free}(\partial\, E(n)_*) & \longrightarrow & \mathfrak{g}_* \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ \mathrm{Free}(E(n+1)_*) & \longrightarrow & \mathfrak{h}_*. \end{array}$$

Since j is a cofibration and $\mathfrak{g}_*$ is cofibrant, σ is a homotopy pushout diagram in $\mathrm{Lie}_\kappa^{\mathrm{dg}}$, so that $\mathfrak{h}_*$ and $\mathfrak{g}'_*$ are equivalent in Lie_κ (Proposition HA.1.3.4.24). It will therefore suffice to show that the object $\mathfrak{h}_* \in \mathrm{Lie}_\kappa^{\mathrm{dg}}$ is good.

The differential graded Lie algebra $\mathfrak{h}_*$ is cofibrant by construction. Let $V_* \subseteq \mathfrak{g}$ be a subspace satisfying conditions (i), (ii), and (iii), and let $y \in \mathfrak{h}_{n+1}$ be the image of a generator of $E(n+1)_{n+1}$. Let V'_* be the graded subspace of $\mathfrak{h}_*$ generated by V_* and y. It is trivial to verify that V'_* satisfies conditions (i), (ii), and (iii).

□

Proof of Theorem 13.3.0.1. Proposition 13.3.2.1 shows that the functor $\mathfrak{D} : (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Lie}_\kappa$ is a weak deformation theory. We will show that it satisfies axiom (D4) of Definition 12.3.3.2. Let $E \in \mathrm{Sp}(\mathrm{CAlg}_\kappa^{\mathrm{aug}})$ be the spectrum object of Example 12.1.1.2, so that $\Omega^{\infty-n}E \simeq \kappa \oplus \Sigma^n(\kappa)$. The proof of Proposition 13.3.2.1 shows that $\mathfrak{D}(E)$ is given by the infinite loop object $\{\mathrm{Free}(\kappa[-n-1])\}_{n\geqslant 0}$ in $\mathrm{Lie}_\kappa^{\mathrm{op}}$; here $\mathrm{Free} : \mathrm{Mod}_\kappa \to \mathrm{Lie}_\kappa$ denotes a left adjoint to the forgetful functor $\theta : \mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa$. It follows that the functor $e : \mathrm{Lie}_\kappa \to \mathrm{Sp}$

appearing in Definition 12.3.3.2 is given by $\Sigma(F \circ \theta)$, where $F : \mathrm{Mod}_\kappa = \mathrm{Mod}_\kappa(\mathrm{Sp}) \to \mathrm{Sp}$ and $\theta : \mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa$ are the forgetful functors. Since F is conservative and commutes with all colimits, it will suffice to observe that θ is conservative (which is obvious) and preserves sifted colimits (Proposition 13.1.4.4). □

Proof of Theorem 13.0.0.2. Let κ be a field of characteristic zero, and let $\Psi : \mathrm{Lie}_\kappa \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_\kappa, \mathcal{S})$ denote the functor given on objects by the formula

$$\Psi(\mathfrak{g}_*)(R) = \mathrm{Map}_{\mathrm{Lie}_\kappa}(\mathfrak{D}(R), \mathfrak{g}_*).$$

Combining Theorems 13.3.0.1 and 12.3.3.5, we deduce that Ψ is a fully faithful embedding whose essential image is the full subcategory $\mathrm{Moduli}_\kappa \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_\kappa, \mathcal{S})$ spanned by the formal moduli problems. Let $X \mapsto T_X$ denote tangent complex functor $\mathrm{Moduli}_\kappa \to \mathrm{Sp}$, given by evaluation on the spectrum object $E \in \mathrm{Sp}(\mathrm{CAlg}^{\mathrm{art}}_\kappa)$ appearing in Example 12.1.1.2. Then the functor $\mathfrak{g}_* \mapsto \Sigma^{-1} T_{\Psi(\mathfrak{g}_*)}$ coincides with the functor $\Sigma^{-1} e$, where $e : \mathrm{Lie}_\kappa \to \mathrm{Sp}$ is the functor appearing in Definition 12.3.3.2. The proof of Theorem 13.3.0.1 supplies an equivalence of $\Sigma^{-1} e$ with the forgetful functor $\mathrm{Lie}_\kappa \to \mathrm{Mod}_\kappa = \mathrm{Mod}_\kappa(\mathrm{Sp}) \to \mathrm{Sp}$. □

13.3.3 Classification of Prorepresentable Formal Moduli Problems

We close this section with an application of Theorem 13.3.0.1.

Proposition 13.3.3.1. *Let κ be a field of characteristic zero and let $X : \mathrm{CAlg}^{\mathrm{art}}_\kappa \to \mathcal{S}$ be a formal moduli problem. Then following conditions are equivalent:*

(1) *The formal moduli problem X is prorepresentable (see Definition 12.5.3.1).*

(2) *Let T_X denote the tangent complex of X. Then $\pi_i T_X \simeq 0$ for $i > 0$.*

Proof. Suppose first that X is prorepresentable; we wish to show that the homotopy groups $\pi_i T_X$ vanish for $i > 0$. The construction $X \mapsto \pi_i T_X$ commutes with filtered colimits. It will therefore suffice to show that $\pi_i T_X \simeq 0$ when $X = \mathrm{Spf}\, A$ is the the functor corepresented by an object $A \in \mathrm{CAlg}^{\mathrm{art}}_\kappa$. This is clear: the homotopy group $\pi_i T_X \simeq \pi_i \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}_\kappa}(A, \kappa[\epsilon]/(\epsilon^2))$ vanishes because A is connective and $\kappa[\epsilon]/(\epsilon^2)$ is discrete.

We now prove the converse. Let X be a formal moduli problem such that $\pi_i T_X \simeq 0$ for $i > 0$; we wish to prove that X is prorepresentable. Let $\Psi : \mathrm{Lie}_\kappa \to \mathrm{Moduli}_\kappa$ be the equivalence of ∞-categories of Theorem 13.0.0.2. Then we can assume that $X = \Psi(\mathfrak{g}_*)$ for some differential graded Lie algebra $\mathfrak{g}_*$ satisfying $\mathrm{H}_i(\mathfrak{g}_*) \simeq 0$ for $i \geqslant 0$ (here we let $\mathrm{H}_i(\mathfrak{g}_*)$ denote the ith homology group of the underlying chain complex of $\mathfrak{g}_*$, rather than the Lie algebra homology of $\mathfrak{g}_*$ computed by the Chevalley-Eilenberg complex $C_*(\mathfrak{g}_*)$ of §13.2).

We will construct a sequence of differential graded Lie algebras

$$0 = \mathfrak{g}(0)_* \to \mathfrak{g}(1)_* \to \mathfrak{g}(2)_* \to \cdots$$

equipped with maps $\phi(i) : \mathfrak{g}(i)_* \to \mathfrak{g}_*$. For every integer n, choose a graded subspace $V_n \subseteq \mathfrak{g}_n$ consisting of cycles which maps isomorphically onto the homology $\mathrm{H}_n(\mathfrak{g}_*)$. Then we can regard V_* as a differential graded vector space with trivial differential. Let $\mathfrak{g}(1)_*$ denote the free differential graded Lie algebra generated by V_*, and $\phi(1) : \mathfrak{g}(1)_* \to \mathfrak{g}_*$ the canonical map. Assume now that $i \geqslant 1$ and that we have constructed a map $\phi(i) : \mathfrak{g}(i)_* \to \mathfrak{g}_*$ extending $\phi(1)$. Then $\phi(i)$ induces a surjection $\theta : \mathrm{H}_n(\mathfrak{g}(i)_*) \to \mathrm{H}_*(\mathfrak{g}_*)$. Choose a collection of cycles $x_\alpha \in \mathfrak{g}(i)_{n_\alpha}$ whose images form a basis for $\ker(\theta)$. Then we can write $\phi(i)(x_\alpha) = dy_\alpha$ for some $y_\alpha \in \mathfrak{g}_{n_\alpha+1}$. Let $\mathfrak{g}(i+1)_*$ be the differential graded Lie algebra obtained from $\mathfrak{g}(i)_*$ by freely adjoining elements Y_α (in degrees $n_\alpha + 1$) satisfying $dY_\alpha = x_\alpha$. We let $\phi(i+1) : \mathfrak{g}(i+1)_* \to \mathfrak{g}_*$ denote the unique extension of $\phi(i)$ satisfying $\phi(i+1)(Y_\alpha) = y_\alpha$.

We will establish the following assertion for each integer $i \geqslant 1$:

$(*_i)$ The inclusion $V_{-1} \hookrightarrow \mathfrak{g}(i)_{-1}$ induces an isomorphism $V_{-1} \to \mathrm{H}_{-1}(\mathfrak{g}(i)_*)$, and the groups $\mathfrak{g}(i)_n$ vanish for $n \geqslant 0$.

Assertion $(*_i)$ is easy when $i = 1$. Let us assume that $(*_i)$ holds, and let θ be defined as above. Then θ is an isomorphism in degrees $\geqslant -1$, so that $\mathfrak{g}(i+1)_*$ is obtained from $\mathfrak{g}(i)_*$ by freely adjoining generators Y_α in degrees $\leqslant -1$. It follows immediately that $\mathfrak{g}(i+1)_n \simeq 0$ for $n \geqslant 0$. Moreover, we can write $\mathfrak{g}(i+1)_{-1} \simeq \mathfrak{g}(i)_{-1} \oplus W$, where W is the subspace spanned by elements of the form Y_α where $n_\alpha = -2$. By construction, the differential on $\mathfrak{g}(i+1)_*$ induces a monomorphism from W to the quotient $\mathfrak{g}(i)_{-2}/d\mathfrak{g}(i)_{-1} \subseteq \mathfrak{g}(i+1)_{-2}/d\mathfrak{g}(i)_{-1}$, so that the Lie algebras $\mathfrak{g}(i+1)_*$ and $\mathfrak{g}(i)_*$ have the same homology in degree -1.

Let $\mathfrak{g}'_*$ denote the colimit of the sequence $\{\mathfrak{g}(i)_*\}_{i \geqslant 0}$. The evident map $\mathfrak{g}'_* \to \mathfrak{g}_*$ is surjective on homology (since the map $\mathfrak{g}(1)_* \to \mathfrak{g}_*$ is surjective on homology). If $\eta \in \ker(\mathrm{H}_*(\mathfrak{g}'_*) \to \mathrm{H}_*(\mathfrak{g}_*)$, then η is represented by a class $\overline{\eta} \in \ker(\mathrm{H}_*(\mathfrak{g}(i)_*) \to \mathrm{H}_*(\mathfrak{g}_*))$ for $i \gg 0$. By construction, the image of $\overline{\eta}$ vanishes in $\mathrm{H}_*(\mathfrak{g}(i+1)_*)$, so that $\eta = 0$. It follows that the map $\mathfrak{g}'_* \to \mathfrak{g}_*$ is a quasi-isomorphism. Since the collection of quasi-isomorphisms in $\mathrm{Lie}_\kappa^{\mathrm{dg}}$ is closed under filtered colimits, we conclude that $\mathfrak{g}_*$ is a homotopy colimit of the sequence $\{\mathfrak{g}(i)_*\}_{i \geqslant 0}$ in the model category $\mathrm{Lie}_\kappa^{\mathrm{dg}}$, and therefore a colimit of $\{\mathfrak{g}(i)_*\}_{i \geqslant 0}$ in the ∞-category Lie_κ. Setting $X(i) = \Psi(\mathfrak{g}(i)_*) \in \mathrm{Moduli}_\kappa$, we deduce that $X \simeq \varinjlim X(i)$. To prove that X is prorepresentable, it will suffice to show that each $X(i)$ is prorepresentable.

We now proceed by induction on i, the case $i = 0$ being trivial. To carry out the inductive step, we note that each of the Lie algebras $\mathfrak{g}(i+1)_*$ is obtained from $\mathfrak{g}(i)_*$ by freely adjoining a set of generators $\{Y_\alpha\}_{\alpha \in A}$ of degrees $n_\alpha + 1 \leqslant -1$, satisfying $dY_\alpha = x_\alpha \in \mathfrak{g}(i)_{n_\alpha}$ (this is obvious when $i = 0$, and follows from $(*_i)$ when $i > 0$). Choose a well-ordering of the set A. For each $\alpha \in A$, we let $\mathfrak{g}_*^{<\alpha}$ denote the Lie subalgebra of $\mathfrak{g}(i+1)_*$ generated by $\mathfrak{g}(i)_*$ and the elements Y_β for $\beta < \alpha$, and let $\mathfrak{g}_*^{\leqslant\alpha}$ be defined similarly. Set $X^{<\alpha} = \Psi(\mathfrak{g}_*^{<\alpha})$ and $X^{\leqslant\alpha} = \Psi(\mathfrak{g}_*^{\leqslant\alpha})$. For each $\alpha \in A$, we have a homotopy pushout diagram of differential graded

Lie algebras

$$\begin{array}{ccc} \mathrm{Free}(\partial E(n_\alpha+1)_*) & \longrightarrow & \mathrm{Free}(E(n_\alpha+1)_*) \\ \downarrow & & \downarrow \\ \mathfrak{g}_*^{<\alpha} & \longrightarrow & \mathfrak{g}_*^{\leqslant\alpha}, \end{array}$$

hence a pushout diagram of formal moduli problems

$$\begin{array}{ccc} \mathrm{Spf}(\kappa \oplus \Sigma^{n_\alpha+1}\kappa) & \longrightarrow & \mathrm{Spf}(\kappa) \\ \downarrow & & \downarrow \\ X^{<\alpha} & \longrightarrow & X^{\leqslant\alpha}. \end{array}$$

It follows that the map $X(i) \to X(i+1)$ satisfies the criterion of Lemma 12.5.3.4. Since $X(i)$ is prorepresentable, we conclude that $X(i+1)$ is prorepresentable. □

13.4 Quasi-Coherent Sheaves

Let κ be a field and let $X : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ be a formal moduli problem over κ. Following the ideas introduced in §6.2, we can define a symmetric monoidal ∞-category $\mathrm{QCoh}(X)$ of *quasi-coherent sheaves* on X. Roughly speaking, a quasi-coherent sheaf $\mathscr{F}$ on X is a rule which assigns to each point $\eta \in X(R)$ an R-module $\eta^* \mathscr{F} \in \mathrm{Mod}_R$, which is functorial in the following sense: if $\phi : R \to R'$ is a morphism in $\mathrm{CAlg}_\kappa^{\mathrm{art}}$ and η' denotes the image of η in $X(R')$, then there is an equivalence $\eta'^* \mathscr{F} \simeq R' \otimes_R \eta^* \mathscr{F}$ in the ∞-category $\mathrm{Mod}_{R'}$.

If the field κ has characteristic zero, Theorem 13.0.0.2 provides an equivalence of ∞-categories $\Psi : \mathrm{Lie}_\kappa \to \mathrm{Moduli}_\kappa$. In particular, every formal moduli problem X is equivalent to $\Psi(\mathfrak{g}_*)$, for some differential graded Lie algebra $\mathfrak{g}_*$ which is well-defined up to quasi-isomorphism. In this section, we will explore the relationship between $\mathfrak{g}_*$ and the ∞-category $\mathrm{QCoh}(X)$. Our main result is the following:

Theorem 13.4.0.1. *Let κ be a field of characteristic zero, let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ, and let $X = \Psi(\mathfrak{g}_*)$ be the associated formal moduli problem. Then there is a fully faithful symmetric monoidal embedding $\mathrm{QCoh}(X) \hookrightarrow \mathrm{Rep}_{\mathfrak{g}_*}$, where $\mathrm{Rep}_{\mathfrak{g}_*}$ denotes the ∞-category of representations of $\mathfrak{g}_*$ (see Notation 13.4.1.4).*

Remark 13.4.0.2. It follows from Theorem 13.4.0.1 that the ∞-category $\mathrm{Rep}_{\mathfrak{g}_*}$ can be regarded as a (symmetric monoidal) *enlargement* of the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on the formal moduli problem determined by $\mathfrak{g}_*$. This enlargement can be described geometrically as the ∞-category of Ind-coherent sheaves on X. We refer the reader to §14.5 for a discussion of Ind-coherent sheaves in the noncommutative setting, and to §14.6 for a noncommutative analogue of Theorem 13.4.0.1.

13.4.1 Representations of Lie Algebras

We begin by reviewing some definitions.

Definition 13.4.1.1. Let κ be a field of characteristic zero and let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ. A *representation* of $\mathfrak{g}_*$ is a differential graded vector space V_* equipped with a map $\mathfrak{g}_* \otimes_\kappa V_* \to V_*$ satisfying the identity $[x,y]v = x(yv) + (-1)^{pq} y(xv)$ for all $x \in \mathfrak{g}_p$, $y \in \mathfrak{g}_q$, and $v \in V_r$. The representations of $\mathfrak{g}_*$ form a category which we will denote by $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$.

Example 13.4.1.2. For every differential graded vector space V_*, the zero map $\mathfrak{g}_* \otimes_\kappa V_* \to V_*$ exhibits V_* as a representation of $\mathfrak{g}_*$. In particular, taking $V_* = \kappa$ (regarded as a graded vector space concentrated in degree zero), we obtain a representation of $\mathfrak{g}_*$ on κ which we call the *trivial representation*.

Note that a representation of a differential graded Lie algebra $\mathfrak{g}_*$ is the same data as a (left) module over the universal enveloping algebra $U(\mathfrak{g}_*)$. Using Proposition HA.4.3.3.15 , we deduce the following:

Proposition 13.4.1.3. *Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field κ. Then the category $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$ of representations of $\mathfrak{g}_*$ admits a combinatorial model structure, where:*

(W) *A map $f : V_* \to W_*$ of representations of $\mathfrak{g}_*$ is a weak equivalence if and only if it induces an isomorphism on homology.*

(F) *A map $f : V_* \to W_*$ of representations of $\mathfrak{g}_*$ is a fibration if and only if it is degreewise surjective.*

Notation 13.4.1.4. If $\mathfrak{g}_*$ is a differential graded Lie algebra over a field κ, we let $W_{\mathfrak{g}_*}$ denote the collection of all weak equivalences in $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$, and we let $\mathrm{Rep}_{\mathfrak{g}_*} = \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}[W^{-1}_{\mathfrak{g}_*}]$ denote the ∞-category obtained from $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$ by formally inverting all quasi-isomorphisms: that is, the underlying ∞-category of the model category described in Proposition 13.4.1.3.

It follows from Theorem HA.4.3.3.17 that we can identify $\mathrm{Rep}_{\mathfrak{g}_*}$ with the ∞-category $\mathrm{LMod}_{U(\mathfrak{g}_*)}$ of left modules over the universal enveloping algebra $U(\mathfrak{g}_*)$ (which we regard as an $\mathbb{E}_1$-ring). In particular, $\mathrm{Rep}_{\mathfrak{g}_*}$ is a stable ∞-category.

13.4.2 Cohomology with Coefficients

Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field κ and let V_* be a representation of $\mathfrak{g}_*$. We let $C^*(\mathfrak{g}_*; V_*)$ denote the differential graded vector space of $U(\mathfrak{g}_*)$-module maps from $U(\mathrm{Cn}(\mathfrak{g})_*)$ into V_*. We will refer to $C^*(\mathfrak{g}_*; V_*)$ as the *cohomological Chevalley-Eilenberg complex of $\mathfrak{g}_*$ with coefficients in V_*.*

Remark 13.4.2.1. Unwinding the definitions, we see that the graded pieces $C^n(\mathfrak{g}_*; V_*)$ can be identified with the set of graded vector space maps $\mathrm{Sym}^*_\kappa(\mathfrak{g}_*[1]) \to V_*[-n]$.

We note that $C^*(\mathfrak{g}_*; V_*)$ has the structure of a module over the differential graded algebra $C^*(\mathfrak{g}_*)$. The action is given by κ-bilinear maps

$$C^p(\mathfrak{g}_*) \times C^q(\mathfrak{g}_*; V_*) \to C^{p+q}(\mathfrak{g}_*; V_*),$$

which carries a class $\lambda \in C^p(\mathfrak{g}_*)$ and $\mu \in C^q(\mathfrak{g}_*; V_*)$ to the element $\lambda\mu \in C^{p+q}(\mathfrak{g}_*; V_*)$ given by the formula

$$(\lambda\mu)(x_1 \ldots x_n) = \sum_{S,S'} \epsilon(S, S')\lambda(x_{i_1} \ldots x_{i_m})\mu(x_{j_1} \ldots x_{j_{n-m}}),$$

as in Construction 13.2.5.1.

Remark 13.4.2.2. It follows from general nonsense that the (differential graded) endomorphism ring of the functor $C^*(\mathfrak{g}_*; \bullet) : \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*} \to \mathrm{Mod}^{\mathrm{dg}}_\kappa$ is isomorphic to the (differential graded) endomorphism ring of $U(\mathrm{Cn}(\mathfrak{g})_*)$ (regarded as a representation of $\mathfrak{g}_*$). In particular, the action of $C^*(\mathfrak{g}_*)$ on $C^*(\mathfrak{g}_*; \bullet)$ arises from an action of $C^*(\mathfrak{g}_*)$ on $U(\mathrm{Cn}(\mathfrak{g})_*)$, which commutes with the left action of $U(\mathfrak{g}_*)$. For an alternative description of this action, we refer the reader to the proof of Proposition 14.3.2.2.

Proposition 13.4.2.3. *Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field κ of characteristic zero. Then the functor $V_* \mapsto C^*(\mathfrak{g}_*; V_*)$ preserves quasi-isomorphisms.*

Proof. For each $n \geqslant 0$ and each $V_* \in \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$, we let $F_n(V_*)$ denote the quotient of $C^*(\mathfrak{g}_*; V_*)$ given by maps from $\mathrm{Sym}^{\leqslant n}_\kappa(\mathfrak{g}_*[1])$ into V_*. Then $C^*(\mathfrak{g}_*; V_*)$ is given by the inverse limit of a tower of fibrations

$$\cdots \to F_2(V_*) \to F_1(V_*) \to F_0(V_*).$$

It will therefore suffice to show that each of the functors F_n preserves quasi-isomorphisms. We proceed by induction on n. If $n = 0$, then F_n is the identity functor and the result is obvious. Assume therefore that $n > 0$. Let $K : \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*} \to \mathrm{Mod}^{\mathrm{dg}}_\kappa$ be the functor given by the kernel of the surjection $F_n \to F_{n-1}$, so that we have a short exact sequence of functors

$$0 \to K \to F_n \to F_{n-1} \to 0.$$

It will therefore suffice to show that the functor K preserves quasi-isomorphisms. Unwinding the definitions, we see that K carries a representation V_* to the chain complex of Σ_n-equivariant maps from $(\mathfrak{g}_*[1])^{\otimes n}$ into V_*, regarded as objects of $\mathrm{Mod}^{\mathrm{dg}}_\kappa$. Since κ has characteristic zero, the functor K is a direct summand of the functor $K' : \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*} \to \mathrm{Mod}^{\mathrm{dg}}_\kappa$, which carries V_* to the chain complex of maps from $(\mathfrak{g}_*[1])^{\otimes n}$ into V_*. This functor evidently preserves quasi-isomorphisms. □

Remark 13.4.2.4. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field κ of characteristic zero, and let $\mathrm{Mod}^{\mathrm{dg}}_{C^*(\mathfrak{g}_*)}$ denote the category of differential graded modules over $C^*(\mathfrak{g}_*)$. The functor $C^*(\mathfrak{g}_*; \bullet) : \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*} \to \mathrm{Mod}^{\mathrm{dg}}_{C^*(\mathfrak{g}_*)}$ preserves weak equivalences and fibrations. Moreover, it has a left adjoint F, given by $M_* \mapsto U(\mathrm{Cn}(\mathfrak{g})_*) \otimes_{C^*(\mathfrak{g}_*)} M_*$ (see Remark 13.4.2.2). It follows that $C^*(\mathfrak{g}_*; \bullet)$ is a right Quillen functor, which induces a map between the underlying ∞-categories $\mathrm{Rep}_{\mathfrak{g}_*} \to \mathrm{Mod}_{C^*(\mathfrak{g}_*)}$. We will generally abuse notation by denoting this functor also by $C^*(\mathfrak{g}_*; \bullet)$. It admits a left adjoint $f : \mathrm{Mod}_{C^*(\mathfrak{g}_*)} \to \mathrm{Rep}_{\mathfrak{g}_*}$ (given by the left derived functor of F).

13.4.3 Koszul Duality with Coefficients

In good cases, there is a close relationship between the representations of a differential graded Lie algebra $\mathfrak{g}_*$ and the modules over its Chevalley-Eilenberg complex $C^*(\mathfrak{g}_*)$.

Proposition 13.4.3.1. *Let κ be a field of characteristic zero and let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ. Assume that the underlying graded Lie algebra is freely generated by a finite sequence of homogeneous elements $x_1, \ldots, x_n$ such that each dx_i belongs to the Lie subalgebra of $\mathfrak{g}_*$ generated by $x_1, \ldots, x_{i-1}$. Let $f : \mathrm{Mod}_{C^*(\mathfrak{g}_*)} \to \mathrm{Rep}_{\mathfrak{g}_*}$ denote the left adjoint to the functor $C^*(\mathfrak{g}_*; \bullet)$ (see Remark 13.4.2.4). Then f is a fully faithful embedding.*

Lemma 13.4.3.2. *Let κ be a field and let A_* be an augmented differential graded algebra over κ; we will abuse notation by identifying A_* with its image in Mod_κ. Assume that A_* is freely generated (as a graded algebra) by a finite sequence of homogeneous elements $x_1, \ldots, x_n$, such that each dx_i lies in the subalgebra generated by $x_1, \ldots, x_{i-1}$. Then the field κ is a compact object of the stable ∞-category LMod_{A_*}.*

Proof. Adding scalars to the elements x_i if necessary, we may assume that the augmentation $A_* \to \kappa$ annihilates each x_i. For $0 \leqslant i \leqslant n$, let $M(i)_*$ denote the quotient of A_* by the left ideal generated by the elements $x_1, \ldots, x_i$. We will prove that each $M(i)_*$ is perfect as a left A_*-module; taking $i = n$, this will imply the desired result. The proof proceeds by induction on i. If $i = 0$, then $M(i)_* \simeq A_*$ and the result is obvious. If $i > 0$, then the image of x_i in $M(i-1)_*$ is a cycle. It follows that right multiplication by x_i induces a map of left A_*-modules $A_* \to M(i-1)_*$, fitting into an exact sequence

$$0 \to A_* \xrightarrow{x_i} M(i-1)_* \to M(i)_* \to 0.$$

Since $M(i-1)_*$ is perfect by the inductive hypothesis, we deduce that $M(i)_*$ is perfect. □

Proof of Proposition 13.4.3.1. We first show that f is fully faithful when restricted to the full subcategory $\mathrm{Mod}^{\mathrm{perf}}_{C^*(\mathfrak{g}_*)} \subseteq \mathrm{Mod}_{C^*(\mathfrak{g}_*)}$ spanned by the perfect $C^*(\mathfrak{g}_*)$-modules. Let M and N be perfect $C^*(\mathfrak{g}_*)$-modules. We wish to show that f induces an isomorphism

$$\theta : \mathrm{Ext}^*_{C^*(\mathfrak{g}_*)}(M, N) \to \mathrm{Ext}^*_{U(\mathfrak{g}_*)}(fM, fN).$$

Regard M as fixed. The collection of those modules N for which θ is an isomorphism is closed under retracts, shifts, and extensions. To prove that θ is an isomorphism for each $N \in \mathrm{Mod}^{\mathrm{perf}}_{C^*(\mathfrak{g}_*)}$, it will suffice to prove that θ is an isomorphism for $N = C^*(\mathfrak{g}_*)$. By the same reasoning, we can reduce to the case where $M = C^*(\mathfrak{g}_*)$. Then $fM \simeq U(\mathrm{Cn}(\mathfrak{g})_*)$ and $fN \simeq U(\mathrm{Cn}(\mathfrak{g}_*) \simeq \kappa$, so that $\mathrm{Ext}^*_{U(\mathfrak{g}_*)}(fM, fN) \simeq \mathrm{Ext}^*_{U(\mathfrak{g}_*)}(U(\mathrm{Cn}(\mathfrak{g})_*), \kappa)$ is canonically isomorphic to the Lie algebra cohomology of $\mathfrak{g}_*$. Under this isomorphism, θ corresponds to the identity map.

We now prove that f is fully faithful in general. Since $\mathrm{Mod}_{C^*(\mathfrak{g}_*)} \simeq \mathrm{Ind}(\mathrm{Mod}^{\mathrm{perf}}_{C^*(\mathfrak{g}_*)})$ and the functor f preserves filtered colimits, it will suffice to show that f carries objects $\mathrm{Mod}^{\mathrm{perf}}_{C^*(\mathfrak{g}_*)}$ to perfect $U(\mathfrak{g}_*)$-modules. The collection of those $M \in \mathrm{Mod}_{C^*(\mathfrak{g}_*)}$ for which fM is perfect is closed under extensions, shifts, and retracts. It will therefore suffice to show that $fC^*(\mathfrak{g}_*) \simeq \kappa$ is perfect as a $U(\mathfrak{g}_*)$-module, which follows from Lemma 13.4.3.2. □

Notation 13.4.3.3. Let $\mathfrak{g}_*$ be a differential graded Lie algebra and let V_* be a representation of $\mathfrak{g}_*$. We will say that V_* is *connective* if its image in Mod_κ is connective: that is, if the homology groups of the chain complex V_* are concentrated in non-negative degrees. We let $\mathrm{Rep}^{\mathrm{cn}}_{\mathfrak{g}_*}$ denote the full subcategory of $\mathrm{Rep}_{\mathfrak{g}_*}$ spanned by the connective $\mathfrak{g}_*$-modules.

Proposition 13.4.3.4. *Let $\mathfrak{g}_*$ be as in the statement of Proposition 13.4.3.1, and assume that each of the generators x_i of $\mathfrak{g}_*$ has negative homological degree. Then the fully faithful embedding $f : \mathrm{Mod}_{C^*(\mathfrak{g}_*)} \to \mathrm{Rep}_{\mathfrak{g}_*}$ induces an equivalence of ∞-categories $\mathrm{Mod}^{\mathrm{cn}}_{C^*(\mathfrak{g}_*)} \to \mathrm{Rep}^{\mathrm{cn}}_{\mathfrak{g}_*}$.*

Lemma 13.4.3.5. *Let κ be a field and let A be a coconnective $\mathbb{E}_1$-algebra over κ (see Definition 14.1.3.1), equipped with an augmentation $\epsilon : A \to \kappa$. Let $\mathcal{C} \subseteq \mathrm{LMod}_A$ be a full subcategory which contains κ (regarded as a left A-module via the augmentation ϵ) and is closed under colimits and extensions. Then $\mathcal{C}$ contains every left A-module whose underlying spectrum is connective.*

Proof. Let M be a left A-module whose underlying spectrum is connective. We will construct a sequence of objects $0 = M(0) \to M(1) \to M(2) \to \cdots$ in $\mathcal{C}$ and a compatible family of maps $\theta(i) : M(i) \to M$ with the following property:

- $(*)$ The groups $\pi_j M(i)$ vanish unless $0 \leqslant j < i$, and the maps $\pi_j M(j) \to \pi_j M$ are isomorphisms for $0 \leqslant j < i$.

Assume that $i \geqslant 0$ and that we have already constructed a map $\theta(i)$ satisfying $(*)$. Let $M' = \mathrm{fib}(\theta(i))$, so that $\pi_j M' \simeq 0$ for $j < i-1$. Using Proposition 14.1.4.1, we can construct a map of left A-modules $N \to M'$ which induces an isomorphism $\pi_{i-1}N \to \pi_{i-1}M'$, with $\pi_j N \simeq 0$ for $j \neq i-1$. Let $M(i+1)$ denote the cofiber of the composite map $N \to M' \to M(i)$. There is an evident map $\theta(i+1) : M(i+1) \to M$ satisfying $(*)$. We will complete the proof by showing that $M(i+1) \in \mathcal{C}$. We have a fiber sequence $M(i) \to M(i+1) \to \Sigma N$. Lemma **??** implies that ΣN is equivalent to a direct sum of copies of $\Sigma^i(\kappa)$. Since $\mathcal{C}$ contains κ and is closed under colimits, we conclude that $\Sigma N \in \mathcal{C}$. The module $M(i)$ belong to $\mathcal{C}$ by the inductive hypothesis. Since $\mathcal{C}$ is closed under extensions, we deduce that $M(i+1) \in \mathcal{C}$. □

Proof of Proposition 13.4.3.4. Since $C^*(\mathfrak{g}_*)$ is connective, we can characterize as the smallest full subcategory of $\mathrm{Mod}^{\mathrm{cn}}_{C^*(\mathfrak{g}_*)}$ which contains $C^*(\mathfrak{g}_*)$ and is closed under colimits and extensions. It follows that f induces an equivalence from $\mathrm{Mod}^{\mathrm{cn}}_{C^*(\mathfrak{g}_*)}$ to the smallest full subcategory of $\mathrm{Rep}^{\mathrm{cn}}_{\mathfrak{g}_*}$ which contains $fC^*(\mathfrak{g}_*) \simeq \kappa$ and is closed under colimits and extensions. It is clear that this full subcategory is contained in $\mathrm{Rep}^{\mathrm{cn}}_{\mathfrak{g}_*}$, and the reverse inclusion follows from Lemma 13.4.3.5. □

13.4.4 Tensor Products of Lie Algebra Representations

If $\mathfrak{g}_*$ is a differential graded Lie algebra over a field κ, then the category $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$ of representations of $\mathfrak{g}_*$ has a natural symmetric monoidal structure: if V_* and W_* are representations of $\mathfrak{g}_*$, then the tensor product $V_* \otimes_\kappa W_*$ can also be regarded as a representation of $\mathfrak{g}_*$, with action given by the formula

$$x(v \otimes w) = (xv) \otimes w + (-1)^{pq} v \otimes (xw)$$

for homogeneous elements $x \in \mathfrak{g}_p$, $v \in V_q$, and $w \in W_r$. For fixed $V_* \in \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$, the construction $W_* \mapsto V_* \otimes_\kappa W_*$ preserves quasi-isomorphisms. It follows from Proposition HA.4.1.7.4 that the underlying ∞-category $\mathrm{Rep}_{\mathfrak{g}_*} = \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}[W^{-1}_{\mathfrak{g}_*}]$ inherits a symmetric monoidal structure.

Remark 13.4.4.1. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field κ. Then the diagram

$$\begin{array}{ccc} \mathrm{Rep}_{\mathfrak{g}_*} \times \mathrm{Rep}_{\mathfrak{g}_*} & \xrightarrow{\otimes} & \mathrm{Rep}_{\mathfrak{g}_*} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_\kappa \times \mathrm{Mod}_\kappa & \xrightarrow{\otimes} & \mathrm{Mod}_\kappa \end{array}$$

commutes up to equivalence. It follows that the tensor product functor $\otimes : \mathrm{Rep}_{\mathfrak{g}_*} \times \mathrm{Rep}_{\mathfrak{g}_*} \to \mathrm{Rep}_{\mathfrak{g}_*}$ preserves small colimits separately in each variable.

To discuss the functorial dependence of the symmetric monoidal structure of Remark 13.4.4.1 on $\mathfrak{g}_*$, it is convenient to introduce the following more elaborate construction:

Construction 13.4.4.2. Let κ be a field. We define a category $\mathrm{Rep}^{\otimes}_{\mathrm{dg}}$ as follows:

(1) An object of $\mathrm{Rep}^{\otimes}_{\mathrm{dg}}$ is a tuple $(\mathfrak{g}_*, V^1_*, \ldots, V^n_*)$, where $\mathfrak{g}_*$ is a differential graded Lie algebra over κ and each V^i_* is a representation of $\mathfrak{g}_*$.

(2) Given a pair of objects $(\mathfrak{g}_*, V^1_*, \ldots, V^m_*), (\mathfrak{h}_*, W^1_*, \ldots, W^n_*) \in \mathrm{Rep}^{\otimes}_{\mathrm{dg}}$, a morphism

$$(\mathfrak{g}_*, V^1_*, \ldots, V^m_*) \to (\mathfrak{h}_*, W^1_*, \ldots, W^n_*)$$

is given by a map $\alpha : \langle m \rangle \to \langle n \rangle$ of pointed finite sets, a morphism $\phi : \mathfrak{h}_* \to \mathfrak{g}_*$ of differential graded Lie algebras, and, for each $1 \leqslant j \leqslant n$, a map $\bigotimes_{\alpha(i)=j} V^i_* \to W^j_*$ of representations of $\mathfrak{h}_*$ (here we regard each V^i_* as a representation of $\mathfrak{h}_*$ via the morphism ϕ).

The category $\mathrm{Rep}^{\otimes}_{\mathrm{dg}}$ is equipped with an evident forgetful functor $\mathrm{Rep}^{\otimes}_{\mathrm{dg}} \to (\mathrm{Lie}^{\mathrm{dg}}_{\kappa})^{\mathrm{op}} \times \mathcal{F}\mathrm{in}_*$, which induces a coCartesian fibration $\mathrm{Rep}^{\otimes}_{\mathrm{dg}} \to (\mathrm{Lie}^{\mathrm{dg}}_{\kappa})^{\mathrm{op}} \times \mathcal{F}\mathrm{in}_*$.

For our applications of Construction 13.4.4.2, we will need the following general result:

Proposition 13.4.4.3. *Let $p : \mathcal{C} \to \mathcal{D}$ be a coCartesian fibration of ∞-categories. Suppose that we are given, for each $D \in \mathcal{D}$, a collection of morphisms W_D in the fiber $\mathcal{C}_D$. Suppose further that for each morphism $D \to D'$ in $\mathcal{D}$, the induced functor $\mathcal{C}_D \to \mathcal{C}_{D'}$ carries W_D into $W_{D'}$. Let $W = \bigcup_{D \in \mathcal{D}} W_D$. Since p carries each morphism of W to an equivalence in $\mathcal{D}$, it factors as a composition $\mathcal{C} \xrightarrow{\theta} \mathcal{C}[W^{-1}] \xrightarrow{q} \mathcal{D}$. Replacing $\mathcal{C}[W^{-1}]$ by an equivalent ∞-category if necessary, we may assume that q is a categorical fibration. Then:*

(1) *The map q is a coCartesian fibration.*

(2) *The functor θ carries p-coCartesian morphisms in $\mathcal{C}$ to q-coCartesian morphisms in $\mathcal{C}[W^{-1}]$.*

(3) *For each $D \in \mathcal{D}$, the map θ induces an equivalence $\mathcal{C}[W_D^{-1}] \to (\mathcal{C}[W^{-1}])_D$.*

Corollary 13.4.4.4. *Let κ be a field and let W be the collection of all morphisms in $\mathrm{Rep}^{\otimes}_{\mathrm{dg}}$ of the form $\alpha : (\mathfrak{g}_*, V^1_*, \ldots, V^n_*) \to (\mathfrak{g}_*, V'^1_*, \ldots, V'^n_*)$ where the image of α in both $\mathrm{Lie}^{\mathrm{dg}}_{\kappa}$ and $\mathcal{F}\mathrm{in}_*$ is an identity map, and α induces a quasi-isomorphism $V^i_* \to V'^i_*$ for $1 \leqslant i \leqslant n$. Then we have a coCartesian fibration $\mathrm{Rep}^{\otimes}_{\mathrm{dg}}[W^{-1}] \to (\mathrm{Lie}^{\mathrm{dg}}_{\kappa})^{\mathrm{op}} \times \mathcal{F}\mathrm{in}_*$. For every differential graded Lie algebra $\mathfrak{g}_*$ over κ, we can identify the fiber $\mathrm{Rep}^{\otimes}_{\mathrm{dg}}[W^{-1}]_{\mathfrak{g}_*, \langle 1 \rangle}$ with the ∞-category $\mathrm{Rep}^{\otimes}_{\mathfrak{g}_*}$.*

Proof of Proposition 13.4.4.3. Let $\chi : \mathcal{D} \to \mathcal{C}\mathrm{at}_{\infty}$ classify the Cartesian fibration p. For each $D \in \mathcal{D}$, we have a canonical equivalence $\chi(D) \simeq \mathcal{C}_D$; let W'_D denote the collection of morphisms in $\chi(D)$ whose in $\mathcal{C}_D$ are equivalent to morphisms belonging to W_D. Then the construction $D \mapsto (\chi(D), W)$ determines a functor $\chi_W : \mathcal{D} \to \mathcal{W}\mathcal{C}\mathrm{at}_{\infty}$, where $\mathcal{W}\mathcal{C}\mathrm{at}_{\infty}$ is defined as in Construction HA.4.1.7.1 . Composing with the left adjoint to the inclusion $\mathcal{C}\mathrm{at}_{\infty} \to \mathcal{W}\mathcal{C}\mathrm{at}_{\infty}$, we obtain a new functor $\chi' : \mathcal{D} \to \mathcal{C}\mathrm{at}_{\infty}$, given on objects by $\chi'(D) = \chi(D)[W'^{-1}_D] \simeq \mathcal{C}_D[W_D^{-1}]$. The functor χ' classifies a coCartesian fibration $p' : \mathcal{C}' \to \mathcal{D}$. We have an evident natural transformation $\chi \to \chi'$, which determines a functor $\phi \in \mathrm{Fun}_{\mathcal{D}}(\mathcal{C}, \mathcal{C}')$

which carries p-coCartesian morphisms to p'-coCartesian morphisms. To complete the proof, it will suffice to show that ϕ induces an equivalence $\mathcal{C}[W^{-1}] \to \mathcal{C}'$. Equivalently, we must show that for any ∞-category $\mathcal{E}$, composition with ϕ induces a fully faithful embedding $v : \mathrm{Fun}(\mathcal{C}', \mathcal{E})^{\simeq} \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}$ whose essential image consists of those functors $F : \mathcal{C} \to \mathcal{E}$ which carry each morphism of W to an equivalence in $\mathcal{E}$.

Evaluation at the vertex $0 \in \Delta^1$ induces a Cartesian fibration $\mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{D}$. We define a new simplicial set $\mathcal{E}'$ with a map $r : \mathcal{E}' \to \mathcal{D}$ so that the following universal property is satisfied: for every map of simplicial sets $K \to \mathcal{D}$, we have a canonical bijection

$$\mathrm{Hom}_{(\mathcal{S}\mathrm{et}_{\Delta})_{/\mathcal{D}}}(K, \mathcal{E}') = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta}}(K \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D}), \mathcal{E}).$$

Using Corollary HTT.3.2.2.12 , we deduce that the map $r : \mathcal{E}' \to \mathcal{D}$ is a coCartesian fibration. The diagonal inclusion $\mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{D})$ induces a map $K \to K \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$ for every map $K \to \mathcal{D}$. Composition with these maps gives a functor $u : \mathcal{E}' \to \mathcal{E}$. We claim:

($*$) Let $\mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}, \mathcal{E}')$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{D}}(\mathcal{C}, \mathcal{E}')$ spanned by those functors which carry p-coCartesian morphisms to r-coCartesian morphisms. Then composition with u induces a trivial Kan fibration $\mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}, \mathcal{E}') \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$

To prove ($*$), we note that $\mathrm{Fun}_{\mathcal{D}}(\mathcal{C}, \mathcal{E}')$ can be identified with the ∞-category

$$\mathrm{Fun}(\mathcal{C} \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D}), \mathcal{E}).$$

Under this identification, $\mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}, \mathcal{E}')$ corresponds to the full subcategory spanned by those functors F which are right Kan extensions of their restrictions to $\mathcal{C} \hookrightarrow \mathcal{C} \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$. Assertion ($*$) now follows from Proposition HTT.4.3.2.15 . A similar argument gives:

($*$) Let $\mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}', \mathcal{E}')$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{D}}(\mathcal{C}', \mathcal{E}')$ spanned by those functors which carry p'-coCartesian morphisms to r-coCartesian morphisms. Then composition with u induces a trivial Kan fibration $\mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}', \mathcal{E}') \to \mathrm{Fun}(\mathcal{C}', \mathcal{E})$.

It follows that we can identify v with the map $\mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}', \mathcal{E}')^{\simeq} \to \mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}, \mathcal{E}')^{\simeq}$. Let $\nu : \mathcal{D} \to \mathcal{C}\mathrm{at}_{\infty}$ classify the coCartesian fibration r, so that v is given by the map $\mathrm{Map}_{\mathrm{Fun}(\mathcal{D},\mathcal{C}\mathrm{at}_{\infty})}(\chi', \nu) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{D},\mathcal{C}\mathrm{at}_{\infty})}(\chi, \nu)$ given by composition with the natural transformation α. The desired result now follows from the construction of the natural transformation χ'. $\square$

13.4.5 Tensor Products and Cohomology

Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field κ. Then the Chevalley-Eilenberg construction $V_* \mapsto C^*(\mathfrak{g}_*; V_*)$ is a lax symmetric monoidal functor: for every pair of representations $V_*, W_* \in \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$, there is a canonical map

$$C^*(\mathfrak{g}_*; V_*) \otimes_{\kappa} C^*(\mathfrak{g}_*; W_*) \to C^*(\mathfrak{g}_*; V_* \otimes_{\kappa} W_*),$$

which classifies bilinear maps $C^p(\mathfrak{g}_*; V_*) \times C^q(\mathfrak{g}_*; W_*) \to C^{p+q}(\mathfrak{g}_*; V_* \otimes_\kappa W_*)$ carrying classes $\lambda \in C^p(\mathfrak{g}_*; V_*)$ and $\mu \in C^q(\mathfrak{g}_*; W_*)$ to the product $\lambda\mu \in C^{p+q}(\mathfrak{g}_*; V_* \otimes_\kappa W_*)$ given by

$$(\lambda\mu)(x_1 \ldots x_n) = \sum_{S,S'} \epsilon(S, S')\lambda(x_{i_1} \ldots x_{i_m}) \otimes \mu(x_{j_1} \ldots x_{j_{n-m}}).$$

Remark 13.4.5.1. Taking V_* and W_* to be the trivial representation of $\mathfrak{g}_*$, we recover the multiplication on $C^*(\mathfrak{g}_*)$ described in Construction 13.2.5.1. Taking V_* to be the trivial representation, we recover the action of $C^*(\mathfrak{g}_*)$ on $C^*(\mathfrak{g}_*; W_*)$ described in Remark 13.4.2.1. It follows from general nonsense that the multiplication maps

$$C^*(\mathfrak{g}_*; V_*) \otimes_\kappa C^*(\mathfrak{g}_*; W_*) \to C^*(\mathfrak{g}_*; V_* \otimes_k W_*)$$

are $C^*(\mathfrak{g}_*)$-bilinear, and therefore descend to give maps

$$C^*(\mathfrak{g}_*; V_*) \otimes_{C^*(\mathfrak{g}_*)} C^*(\mathfrak{g}_*; W_*) \to C^*(\mathfrak{g}_*; V_* \otimes_\kappa W_*).$$

Notation 13.4.5.2. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We let $\mathrm{Mod}(\mathcal{C})^{\otimes} = \mathrm{Mod}^{\mathrm{Comm}}(\mathcal{C})^{\otimes}$ be as in Definition HA.3.3.3.8: more informally, the objects of $\mathrm{Mod}(\mathcal{C})^{\otimes}$ are given by tuples $(A, M_1, \ldots, M_n)$ where $A \in \mathrm{CAlg}(\mathcal{C})$ and each M_i is a module over A. Note that if $\mathcal{C}$ is (equivalent to) an ordinary symmetric monoidal category, then $\mathrm{Mod}(\mathcal{C})^{\otimes}$ is also (equivalent to) an ordinary category.

The lax symmetric monoidal structure on the functor $C^*(\mathfrak{g}_*; \bullet)$, and its dependence on $\mathfrak{g}_*$, are encoded by a map of categories $\mathrm{Rep}^{\otimes}_{\mathrm{dg}} \to \mathrm{Mod}(\mathrm{Mod}^{\mathrm{dg}}_\kappa)^{\otimes}$, given on objects by $(\mathfrak{g}_*, V^1_*, \ldots, V^n_*) \mapsto (C^*(\mathfrak{g}_*), C^*(\mathfrak{g}_*; V^1_*), \ldots, C^*(\mathfrak{g}_*; V^n_*))$. Composing this with the map of symmetric monoidal functor $\mathrm{Mod}^{\mathrm{dg}}_\kappa \to \mathrm{Mod}_\kappa$, we obtain a map $\mathrm{Rep}^{\otimes}_{\mathrm{dg}} \to \mathrm{Mod}(\mathrm{Mod}_\kappa)^{\otimes}$. If the field κ has characteristic zero, then Proposition 13.4.2.3 implies that this functor carries morphisms of W (where W is defined as in Corollary 13.4.4.4) to equivalences in $\mathrm{Mod}(\mathrm{Mod}_\kappa)^{\otimes}$, and therefore induces a lax symmetric monoidal functor

$$G : \mathrm{Rep}^{\otimes}_{\mathrm{dg}}[W^{-1}] \to (\mathrm{Lie}^{\mathrm{dg}}_\kappa)^{\mathrm{op}} \times_{\mathrm{CAlg}_\kappa} \mathrm{Mod}(\mathrm{Mod}_\kappa)^{\otimes}.$$

Proposition 13.4.5.3. *Let κ be a field of characteristic zero, and consider the commutative diagram*

$$\begin{array}{ccc} \mathrm{Rep}^{\otimes}_{\mathrm{dg}}[W^{-1}] & \xrightarrow{G} & (\mathrm{Lie}^{\mathrm{dg}}_\kappa)^{\mathrm{op}} \times_{\mathrm{CAlg}_\kappa} \mathrm{Mod}(\mathrm{Mod}_\kappa)^{\otimes} \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & (\mathrm{Lie}^{\mathrm{dg}}_\kappa)^{\mathrm{op}} \times \mathcal{F}\mathrm{in}_* . & \end{array}$$

Then:

(1) *The functor G admits a left adjoint F relative to $(\mathrm{Lie}_\kappa^{\mathrm{dg}})^{\mathrm{op}} \times \mathcal{F}\mathrm{in}_*$ (see Definition HA.7.3.2.2).*

(2) *The functor F carries q-coCartesian morphisms to p-coCartesian morphisms.*

Remark 13.4.5.4. We can summarize Proposition 13.4.5.3 more informally as follows. For every differential graded Lie algebra $\mathfrak{g}_*$ over κ, the construction $V_* \mapsto C^*(\mathfrak{g}_*; V_*)$ determines a lax symmetric monoidal functor from $\mathrm{Rep}_{\mathfrak{g}_*}$ to $\mathrm{Mod}_{C^*(\mathfrak{g}_*)}$. This functor admits a symmetric monoidal left adjoint $f : \mathrm{Mod}_{C^*(\mathfrak{g}_*)} \to \mathrm{Rep}_{\mathfrak{g}_*}$. Moreover, the functor f depends functorially on the differential graded Lie algebra $\mathfrak{g}_*$.

Proof of Proposition 13.4.5.3. We will prove the existence of F; since F admits a right adjoint relative to $(\mathrm{Lie}_\kappa^{\mathrm{dg}})^{\mathrm{op}} \times \mathcal{F}\mathrm{in}_*$, it will follow automatically that F carries q-coCartesian morphisms to p-coCartesian morphisms (see Proposition HA.7.3.2.6). To prove the existence of F, we will check that G satisfies the criterion of Proposition HA.7.3.2.11 . For each differential graded Lie algebra $\mathfrak{g}_*$ and each $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$, the induced functor

$$G_{\mathfrak{g}_*, \langle n \rangle} : \mathrm{Rep}_{\mathrm{dg}}^{\otimes}[W^{-1}]_{\mathfrak{g}_*, \langle n \rangle} \to (\mathrm{Mod}_{C^*(\mathfrak{g}_*)})^{\otimes}_{\langle n \rangle}$$

is equivalent to a product of n copies of the functor $C^*(\mathfrak{g}_*; \bullet) : \mathrm{Rep}_{\mathfrak{g}_*} \to \mathrm{Mod}_{C^*(\mathfrak{g}_*)}$, and therefore admits a left adjoint $f_{\mathfrak{g}_*}$ by Remark 13.4.2.4. Unwinding the definitions, we are reduced to proving that for every finite sequence of $C^*(\mathfrak{g}_*)$-modules $M_1, \ldots, M_n$, and every map of differential graded Lie algebras $\mathfrak{h}_* \to \mathfrak{g}_*$, the canonical map

$$f_{\mathfrak{h}_*}(C^*(\mathfrak{h}_*) \otimes_{C^*(\mathfrak{g}_*)} M_1 \otimes_{C^*(\mathfrak{g}_*)} \cdots \otimes_{C^*(\mathfrak{g}_*)} M_n) \to f_{\mathfrak{g}_*}(M_1) \otimes_\kappa \cdots \otimes_\kappa f_{\mathfrak{g}_*}(M_n)$$

is an equivalence. We observe that both sides are compatible with colimits in each M_i (see Remark 13.4.4.1). Since $\mathrm{Mod}_{C^*(\mathfrak{g}_*)}$ is generated under small colimits by the modules $\Sigma^k C^*(\mathfrak{g}_*)$ for $k \in \mathbf{Z}$, we can reduce to the case where $M_i = C^*(\mathfrak{g}_*)$ for $1 \leqslant i \leqslant n$. In this case, the result is obvious. □

13.4.6 Quasi-Coherent Sheaves on a Formal Moduli Problem

We now define the stable ∞-category $\mathrm{QCoh}(X)$ associated to a formal moduli problem $X : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathcal{S}$.

Construction 13.4.6.1. Let κ be a field. The coCartesian fibration $\mathrm{Mod}(\mathrm{Mod}_\kappa)^{\otimes} \to \mathrm{CAlg}_\kappa \times \mathcal{F}\mathrm{in}_*$ is classified by a map $\chi : \mathrm{CAlg}_\kappa \to \mathrm{Mon}_{\mathrm{Comm}}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \simeq \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$, which carries an $\mathbb{E}_\infty$-algebra A over κ to Mod_A, regarded as a symmetric monoidal ∞-category. Let χ^{art} denote the restriction of χ to the full subcategory $\mathrm{CAlg}_\kappa^{\mathrm{art}} \subseteq \mathrm{CAlg}_\kappa$ spanned by the Artinian $\mathbb{E}_\infty$-algebras over κ. Applying Theorem HTT.5.1.5.6 , we deduce that χ^{art} admits an essentially unique factorization as a composition

$$\mathrm{CAlg}_\kappa^{\mathrm{art}} \xrightarrow{j} \mathrm{Fun}(\mathrm{CAlg}_\kappa^{\mathrm{art}}, \mathcal{S})^{\mathrm{op}} \xrightarrow{\mathrm{QCoh}} \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty),$$

where the functor QCoh preserves small limits. For every functor $X : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathcal{S}$, we will regard $\mathrm{QCoh}(X) \in \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ as a symmetric monoidal ∞-category, which we call the *∞-category of quasi-coherent sheaves on X.*

Remark 13.4.6.2. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}} \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the subcategory whose objects are presentable ∞-categories and whose morphisms are colimit preserving functors, and regard $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ as a symmetric monoidal ∞-category as explained in §HA.4.8.1 . Note that the functor χ of Construction 13.4.6.1 factors through $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \subseteq \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$. Since this inclusion preserves small limits, we deduce that the functor

$$\mathrm{QCoh} : \mathrm{Fun}(\mathrm{CAlg}_\kappa^{\mathrm{art}}, \mathcal{S})^{\mathrm{op}} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$$

also factors through $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. In other words:

- (a) For every functor $X : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathcal{S}$, the ∞-category $\mathrm{QCoh}(X)$ is presentable.
- (b) For every functor $X : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathcal{S}$, the tensor product $\otimes : \mathrm{QCoh}(X) \times \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ preserves small colimits separately in each variable.
- (c) For every natural transformation $f : X \to Y$ of functors $X, Y : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathcal{S}$, the pullback functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ preserves small colimits.

Remark 13.4.6.3. Let κ be a field and let $X : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ be a functor which classifies a right fibration $\mathcal{X} \to \mathrm{CAlg}_\kappa^{\mathrm{art}}$. Then $\mathrm{QCoh}(X)$ can be identified with the ∞-category coCartesian sections of the coCartesian fibration $\mathcal{X} \times_{\mathrm{CAlg}_\kappa} \mathrm{Mod}(\mathrm{Mod}_\kappa) \to \mathcal{X}$. More informally, an object $\mathscr{F} \in \mathrm{QCoh}(X)$ is a rule which assigns to every point $\eta \in X(A)$ an A-module $\mathscr{F}_\eta$, and to every morphism $f : A \to A'$ carrying η to $\eta' \in X(A')$ an equivalence $\mathscr{F}_{\eta'} \simeq A' \otimes_A \mathscr{F}_\eta$.

We now consider a variant of Construction 13.4.6.1:

Construction 13.4.6.4. Let κ be a field of characteristic zero. The coCartesian fibration $\mathrm{Rep}_{\mathrm{dg}}^{\otimes}[W^{-1}] \to (\mathrm{Lie}_\kappa^{\mathrm{dg}})^{\mathrm{op}} \times \mathcal{F}\mathrm{in}_*$ of Corollary 13.4.4.4 classifies a functor $\overline{\chi}_0 : (\mathrm{Lie}_\kappa^{\mathrm{dg}})^{\mathrm{op}} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$, given on objects by $\mathfrak{g}_* \mapsto \mathrm{Rep}_{\mathfrak{g}_*}$. If $\phi : \mathfrak{h}_* \to \mathfrak{g}_*$ is a quasi-isomorphism of differential graded Lie algebras, then the induced map $U(\mathfrak{h}_*) \to U(\mathfrak{g}_*)$ is an equivalence in Alg_κ, so that the forgetful functor $\mathrm{Rep}_{\mathfrak{g}_*} \to \mathrm{Rep}_{\mathfrak{h}_*}$ is an equivalence of ∞-categories. It follows that $\overline{\chi}_0$ induces a functor $\overline{\chi} : \mathrm{Lie}_\kappa^{\mathrm{op}} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$.

We let $\chi_!^{\mathrm{art}} : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ denote the composition of $\overline{\chi}$ with the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_\kappa^{\mathrm{art}})^{\mathrm{op}} \to \mathrm{Lie}_\kappa$ studied in §13.3. Applying Theorem HTT.5.1.5.6 , we deduce that $\chi_!^{\mathrm{art}}$ admits an essentially unique factorization as a composition

$$\mathrm{CAlg}_\kappa^{\mathrm{art}} \xrightarrow{j} \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{art}}, \mathcal{S})^{\mathrm{op}} \xrightarrow{\mathrm{QCoh}^!} \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty),$$

where j denotes the Yoneda embedding and the functor $\mathrm{QCoh}^!$ preserves small limits.

Remark 13.4.6.5. If $f : X \to Y$ is a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{art}}_{\kappa} \to \mathcal{S}$, we will denote the induced functor $\mathrm{QCoh}^!(Y) \to \mathrm{QCoh}^!(X)$ by $f^!$.

The functor $\chi^{\mathrm{art}}_!$ appearing in Construction 13.4.6.4 factors through the subcategory $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \subseteq \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_{\infty})$. As in Remark 13.4.6.2, we deduce that the functor $\mathrm{QCoh}^!$ factors through $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. That is, each of the ∞-categories $\mathrm{QCoh}^!(X)$ is presentable, each of the functors $f^! : \mathrm{QCoh}^!(Y) \to \mathrm{QCoh}^!(X)$ preserves small colimits, and the tensor product functors $\mathrm{QCoh}^!(X) \times \mathrm{QCoh}^!(X) \to \mathrm{QCoh}^!(X)$ preserve small colimits separately in each variable.

Remark 13.4.6.6. For $A \in \mathrm{Alg}^{\mathrm{art}}_{\kappa}$, the biduality map $A \to C^*(\mathfrak{D}(A))$ is an equivalence. It follows that the functor χ^{art} of Construction 13.4.6.1 is given by the composition

$$\mathrm{CAlg}^{\mathrm{art}}_{\kappa} \xrightarrow{\mathfrak{D}} \mathrm{Lie}^{\mathrm{op}}_{\kappa} \xrightarrow{C^*} \mathrm{CAlg}^{\mathrm{aug}}_{\kappa} \to \mathrm{CAlg}_{\kappa} \xrightarrow{\chi} \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_{\infty}).$$

The functor F of Proposition 13.4.5.3 induces a natural transformation $\chi^{\mathrm{art}} \to \chi^{\mathrm{art}}_!$, and therefore a natural transformation $\mathrm{QCoh} \to \mathrm{QCoh}^!$ of functors $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_{\kappa}, \mathcal{S})^{\mathrm{op}} \to \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

Let $A \in \mathrm{CAlg}^{\mathrm{art}}_{\kappa}$ and let $\mathfrak{g}_* = \mathfrak{D}(A)$ be its Koszul dual. Since A is Artinian, there exists a sequence of maps $A = A_n \to A_{n-1} \to \cdots \to A_0 = \kappa$, where each A_i is a square-zero extension of A_{i-1} by $\Sigma^{n_i}(\kappa)$ for some $n_i \geqslant 0$. We therefore have a sequence of differential graded Lie algebras

$$0 = \mathfrak{D}(A_0) \to \mathfrak{D}(A_1) \to \cdots \to \mathfrak{D}(A_n) \simeq \mathfrak{g}_*,$$

where each $\mathfrak{D}(A_i)$ is obtained from $\mathfrak{D}(A_{i-1})$ by adjoining a cell in dimension $-n_i - 1$. It follows that, up to quasi-isomorphism, $\mathfrak{g}_*$ satisfies the hypotheses of Proposition 13.4.3.1 and Proposition 13.4.3.4. We conclude that the natural transformation $\chi^{\mathrm{art}} \to \chi^{\mathrm{art}}_!$ induces a (symmetric monoidal) fully faithful embedding

$$\chi^{\mathrm{art}}(A) \simeq \mathrm{Mod}_A \simeq \mathrm{Mod}_{C^*(\mathfrak{g}_*)} \to \mathrm{Rep}_{\mathfrak{g}_*} \simeq \chi^{\mathrm{art}}_!(A),$$

which restricts to an equivalence of ∞-categories $\mathrm{Mod}^{\mathrm{cn}}_A \to \mathrm{Rep}^{\mathrm{cn}}_{\mathfrak{g}_*}$. It follows that the natural transformation $\mathrm{QCoh} \to \mathrm{QCoh}^!$ determines a (symmetric monoidal) fully faithful embedding $\mathrm{QCoh}(X) \to \mathrm{QCoh}^!(X)$ for each $X \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_{\kappa}, \mathcal{S})$.

13.4.7 The Proof of Theorem 13.4.0.1

Let κ be a field of characteristic zero and let $X : \mathrm{CAlg}^{\mathrm{art}}_{\kappa} \to \mathcal{S}$ be a formal moduli problem, given by $\Psi(\mathfrak{g}_*)$ for some differential graded Lie algebra $\mathfrak{g}_*$. Remark 13.4.6.6 supplies a symmetric monoidal fully faithful embedding $\mathrm{QCoh}(X) \hookrightarrow \mathrm{QCoh}^!(X)$. To prove Theorem 13.4.0.1, it will suffice to establish the following:

Proposition 13.4.7.1. *Let κ be a field of characteristic zero, let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ and let $X = \Psi(\mathfrak{g}_*)$ be the formal moduli problem given by $X(R) = \mathrm{Map}_{\mathrm{Lie}_\kappa}(\mathfrak{D}(R), \mathfrak{g}_*)$. Then there is a canonical equivalence of symmetric monoidal ∞-categories $\mathrm{QCoh}^!(X) \simeq \mathrm{Rep}_{\mathfrak{g}_*}$.*

Lemma 13.4.7.2. *Let κ be a field and let $\nu : \mathrm{Alg}_\kappa^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classify the Cartesian fibration $\mathrm{LMod}(\mathrm{Mod}_\kappa) \to \mathrm{Alg}_\kappa$ (so that ν is given by the formula $\nu(A) = \mathrm{LMod}_A$). Then ν preserves K-indexed limits for every weakly contractible simplicial set K.*

Proof. Let $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ whose objects are presentable ∞-categories and whose morphisms are functors which admit left adjoints, and define $\mathcal{P}\mathrm{r}^{\mathrm{L}} \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$ similarly. Note that the functor ν factors through $\mathcal{P}\mathrm{r}^{\mathrm{R}}$, and that the inclusion $\mathcal{P}\mathrm{r}^{\mathrm{R}} \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$ preserves small limits (Theorem HTT.5.5.3.18). It will therefore suffice to show that if K is weakly contractible, then ν carries K-indexed limits in $\mathrm{Alg}_\kappa^{\mathrm{op}}$ to K-indexed limits in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$. Using the equivalence $\mathcal{P}\mathrm{r}^{\mathrm{L}} \simeq (\mathcal{P}\mathrm{r}^{\mathrm{R}})^{\mathrm{op}}$ of Corollary HTT.5.5.3.4 , we can identify ν with a functor $\mu : \mathrm{Alg}_\kappa \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ (the functor μ classifies the coCartesian fibration $\mathrm{LMod}(\mathrm{Mod}_\kappa) \to \mathrm{Alg}_\kappa$). Theorem HA.4.8.5.11 implies that the functor $\mathrm{Alg}_\kappa \simeq (\mathrm{Alg}_\kappa)_{\kappa/} \to (\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Mod}_\kappa /}$ admits a right adjoint, and therefore preserves all small colimits. It therefore suffices to verify that the forgetful functor $(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Mod}_\kappa /} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ preserves K-indexed colimits, which follows from Proposition HTT.4.4.2.9 . □

Lemma 13.4.7.3. *Let κ be a field of characteristic zero, and let $\overline{\chi} : \mathrm{Lie}_\kappa^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Construction 13.4.6.4. Then $\overline{\chi}$ preserves K-indexed limits for every weakly contractible simplicial set K.*

Proof. The functor $\overline{\chi}$ factors as a composition $\mathrm{Lie}_\kappa^{\mathrm{op}} \xrightarrow{U} \mathrm{Alg}_\kappa^{\mathrm{op}} \xrightarrow{\nu} \widehat{\mathcal{C}\mathrm{at}}_\infty$, where ν preserves K-indexed limits by Lemma 13.4.7.2, and U preserves all small limits (since it is right adjoint to the forgetful functor $\mathrm{Alg}_\kappa^{\mathrm{op}} \to \mathrm{Lie}_\kappa^{\mathrm{op}}$). □

Proof of Proposition 13.4.7.1. Let $\Psi : \mathrm{Lie}_\kappa \to \mathrm{Moduli}_\kappa$ be the equivalence of ∞-categories appearing in Theorem 13.0.0.2, and let Ψ^{-1} denote a homotopy inverse to Ψ. Let $L : \mathrm{Fun}(\mathrm{CAlg}_\kappa^{\mathrm{art}}, \mathcal{S}) \to \mathrm{Moduli}_\kappa$ denote a left adjoint to the inclusion functor $\mathrm{Moduli}_\kappa \subseteq \mathrm{Fun}(\mathrm{CAlg}_\kappa^{\mathrm{art}}, \mathcal{S})$ (see Remark 12.1.3.5), and let $\widehat{\mathfrak{D}} : \mathrm{Fun}(\mathrm{CAlg}_\kappa^{\mathrm{art}}, \mathcal{S}) \to \mathrm{Lie}_\kappa$ be the composition $\Psi^{-1} \circ L$. The functor $\widehat{\mathfrak{D}}$ preserves small colimits, and the composition of $\widehat{\mathfrak{D}}$ with the Yoneda embedding $(\mathrm{CAlg}_\kappa^{\mathrm{art}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}_\kappa^{\mathrm{art}}, \mathcal{S})$ can be identified with the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_\kappa^{\mathrm{art}})^{\mathrm{op}} \to \mathrm{Lie}_\kappa$. Let $\overline{\chi} : \mathrm{Lie}_\kappa^{\mathrm{op}} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ be as in Construction 13.4.6.4 (given on objects by $\overline{\chi}(\mathfrak{g}_*) = \mathrm{Rep}_{\mathfrak{g}_*}$), and let $F : \mathrm{Fun}(\mathrm{CAlg}_\kappa^{\mathrm{art}}, \mathcal{S})^{\mathrm{op}} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ denote the composite functor

$$\mathrm{Fun}(\mathrm{CAlg}_\kappa^{\mathrm{art}}, \mathcal{S})^{\mathrm{op}} \xrightarrow{\widehat{\mathfrak{D}}} \mathrm{Lie}_\kappa^{\mathrm{op}} \xrightarrow{\overline{\chi}} \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty).$$

Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_{\kappa}, \mathcal{S})$ spanned by the corepresentable functors. By construction, the functors F and $\mathrm{QCoh}^{!}$ agree on the ∞-category $\mathcal{C}$, and by construction $\mathrm{QCoh}^{!}$ is a right Kan extension of its restriction to $\mathcal{C}$. We therefore have a canonical natural transformation $\alpha : F \to \mathrm{QCoh}^{!}$. We will prove the following:

$(*)$ If $X : \mathrm{CAlg}^{\mathrm{art}}_{\kappa} \to \mathcal{S}$ is a formal moduli problem, then α induces an equivalence of ∞-categories $F(X) \to \mathrm{QCoh}^{!}(X))$.

Taking $X = \Psi(\mathfrak{g}_*)$ for $A \in \mathrm{Alg}^{\mathrm{aug}}_{\kappa}$, we see that $(*)$ guarantees an equivalence of symmetric monoidal ∞-categories $\mathrm{Rep}_{\mathfrak{g}_*} \simeq F(X) \to \mathrm{QCoh}^{!}(X)$.

It remains to prove $(*)$. Let $\mathcal{E} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_{\kappa}, \mathcal{S})$ be the full subcategory spanned by those functors X for which α induces an equivalence of ∞-categories $F(X) \to \mathrm{QCoh}^{!}(X)$. The localization functor $L : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_{\kappa}, \mathcal{S}) \to \mathrm{Moduli}_{\kappa}$, the equivalence $\Psi^{-1} : \mathrm{Moduli}_{\kappa} \to \mathrm{Lie}_{\kappa}$ both preserve small colimits. It follows from Lemma 13.4.7.3 that the functor $\overline{\chi} : \mathrm{Lie}^{\mathrm{op}}_{\kappa} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ preserves sifted limits, so that F preserves sifted limits. Since the functor $\mathrm{QCoh}^{!}$ preserves small limits, the ∞-category $\mathcal{E}$ is closed under sifted colimits in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_{\kappa}, \mathcal{S})$. Since $\mathcal{E}$ contains all corepresentable functors and is closed under filtered colimits, it contains it contains all prorepresentable formal moduli problems (see Definition 12.5.3.1). Proposition 12.5.3.3 implies that every formal moduli problem X can be obtained as the geometric realization of a simplicial object $X_{\bullet}$ of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_{\kappa}, \mathcal{S})$, where each X_n is prorepresentable. Since $\mathcal{E}$ is closed under geometric realizations in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{art}}_{\kappa}, \mathcal{S})$, we conclude that $X \in \mathcal{E}$ as desired. □

13.4.8 Connectivity Conditions

Let $X : \mathrm{CAlg}^{\mathrm{art}}_{\kappa} \to \mathcal{S}$ be a formal moduli problem and let $\mathscr{F} \in \mathrm{QCoh}(X)$ be a quasi-coherent sheaf on X, so that $\mathscr{F}$ determines an A-module $\mathscr{F}_{\eta}$ for every $\eta \in X(A)$ (see Remark 13.4.6.3). We will say that $\mathscr{F}$ is *connective* if each $\mathscr{F}_{\eta} \in \mathrm{Mod}_A$ is connective. We let $\mathrm{QCoh}(X)^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{QCoh}(X)$ spanned by the connective objects. It is easy to see that $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is a presentable ∞-category which is closed under colimits and extensions in $\mathrm{QCoh}(X)$, and therefore determines an accessible t-structure on $\mathrm{QCoh}(X)$ (see Proposition HA.1.4.4.11).

Proposition 13.4.8.1. *Let κ be a field of characteristic zero, let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ and let $X = \Psi(\mathfrak{g}_*)$ be the associated formal moduli problem. Then the fully faithful embedding $\theta : \mathrm{QCoh}(X) \to \mathrm{Rep}_{\mathfrak{g}_*}$ induces an equivalence of ∞-categories* $\mathrm{QCoh}(X)^{\mathrm{cn}} \to \mathrm{Rep}^{\mathrm{cn}}_{\mathfrak{g}_*}$.

Proof. If $\phi : \mathfrak{h}_* \to \mathfrak{g}_*$ is a map of differential graded Lie algebras over κ inducing a map of formal moduli problems $Y \to X$, then the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \longrightarrow & \mathrm{Rep}_{\mathfrak{g}_*} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(Y) & \longrightarrow & \mathrm{Rep}_{\mathfrak{g}_*} \end{array}$$

commutes up to canonical homotopy. Taking $\mathfrak{h}_* = 0$, we deduce that the composite functor $\mathrm{QCoh}(X) \to \mathrm{Rep}_{\mathfrak{g}_*} \to \mathrm{Mod}_\kappa$ is given by evaluation at the base point $\eta_0 \in X(\kappa)$. In particular, we deduce that θ carries $\mathrm{QCoh}(X)^{\mathrm{cn}}$ into $\mathrm{Rep}^{\mathrm{cn}}_{\mathfrak{g}_*} = \mathrm{Rep}_{\mathfrak{g}_*} \times_{\mathrm{Mod}_\kappa} \mathrm{Mod}^{\mathrm{cn}}_\kappa$. To complete the proof, it will suffice to show that if $V \in \mathrm{Mod}^{\mathrm{cn}}_{\mathfrak{g}_*}$, then V_* belongs to the essential image of θ. To prove this, it suffices to show that for every point $\eta \in X(A)$ classified by a map of differential graded Lie algebras $\mathfrak{D}(A) \to \mathfrak{g}_*$, the image of V in $\mathrm{Rep}_{\mathfrak{D}(A)}$ belongs to the essential image of the functor $\mathrm{QCoh}(\mathrm{Spf}\, A) \to \mathrm{Rep}_{\mathfrak{D}(A)}$. Since V is connective, this follows from Proposition 13.4.3.4 (note that $\mathfrak{D}(A)$ satisfies the hypotheses of Propositions 13.4.3.1 and 13.4.3.4; see Remark 13.4.6.6. $\square$

Chapter 14

Moduli Problems for Associative Algebras

Let A be a connective $\mathbb{E}_\infty$-ring. Recall that an A-module spectrum M is *projective of rank n* if the following conditions are satisfied:

(1) The group $\pi_0 M$ is a projective $\pi_0 A$-module of rank n.

(2) For every integer n, the canonical map $\mathrm{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 M) \to \pi_n M$ is an isomorphism (that is, M is flat over A).

Let $X(A)$ denote the subcategory of Mod_A whose objects are A-modules which are projective of rank n, and whose morphisms are equivalences. Then $X(A)$ is an essentially small Kan complex. The construction $A \mapsto X(A)$ determines a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$.

Fix a field κ and a point $\eta \in X(\kappa)$, corresponding to a vector space V of dimension n over κ. The formal completion of X (at the point η) is the functor $X^\wedge : \mathrm{CAlg}^{\mathrm{art}}_\kappa \to \mathcal{S}$ given by $X^\wedge(A) = X(A) \times_{X(\kappa)} \{\eta\}$. More informally, $X^\wedge(A)$ is a classifying space for pairs (M, α), where M is a projective A-module of rank n and $\alpha : \kappa \otimes_A M \simeq V$ is an isomorphism of κ-vector spaces. It is not difficult to see that the functor $X^\wedge : \mathrm{CAlg}^{\mathrm{art}}_\kappa \to \mathcal{S}$ is a formal moduli problem (we will give a proof of a stronger assertion in §16.5).

Assume now that κ is a field of characteristic zero. According to Theorem 13.0.0.2, the functor $X^\wedge : \mathrm{CAlg}^{\mathrm{art}}_\kappa \to \mathcal{S}$ is determined (up to equivalence) by a differential graded Lie algebra $\mathfrak{g}_*$. Let $T_{X^\wedge}$ denote the tangent complex of $X^\wedge$, so that $\Sigma^{-1} T_{X^\wedge}$ can be identified with the spectrum underlying the chain complex of vector spaces $\mathfrak{g}_*$. Then $\Omega^\infty T_{X^\wedge} \simeq X^\wedge(\kappa[\epsilon]/(\epsilon^2))$ is a classifying space for the groupoid of order deformations of the vector space V: that is, projective $\kappa[\epsilon]/(\epsilon^2)$-modules M equipped with an isomorphism $M/\epsilon M \simeq V$. Since any basis of V can be lifted to a basis for M, this groupoid has only one isomorphism class of objects (which is represented by the module $\kappa[\epsilon]/(\epsilon^2) \otimes_\kappa V$). We conclude that $\Omega^\infty T_{X^\wedge}$ is homotopy equivalent to the classifying space BG, where G is the group of automorphisms of $\kappa[\epsilon]/(\epsilon^2) \otimes_\kappa V$ which reduce to the identity automorphism modulo ϵ. Every such automorphism can be written uniquely as $1 + \epsilon M$, where $M \in \mathrm{End}(V)$ is an endomorphism of V. From this we deduce that the homology of the chain complex $\mathfrak{g}_*$ is isomorphic to $\mathrm{End}(V)$ in degree zero and vanishes in positive degrees. It also vanishes in negative degrees: this follows from the observation that each of the spaces $X^\wedge(\kappa \oplus \Sigma^n(\kappa))$ is connected (any basis for V can be lifted to a basis for any $(\kappa \oplus \Sigma^n(\kappa))$-module deforming V). It follows that $\mathfrak{g}_*$ is quasi-isomorphic to an ordinary Lie algebra $\mathfrak{g}$ over κ (concentrated in degree zero), whose underlying abelian group is isomorphic to $\mathrm{End}(V)$. With more effort, we can show that the isomorphism $\mathfrak{g} \simeq \mathrm{End}(V)$ is an isomorphism of Lie algebras: that is, the Lie bracket on $\mathfrak{g}$ can be identified with the usual commutator bracket $[A, B] = AB - BA$ of κ-linear endomorphisms of V (see Example 16.5.4.3).

However, there is more to the story. If $R \in \mathrm{CAlg}^{\mathrm{art}}_\kappa$, then any connective R-module M equipped with an equivalence $\kappa \otimes_R M \simeq V$ is automatically projective of rank n. We can therefore identify $X^\wedge(R)$ with the fiber product $\mathrm{LMod}^{\mathrm{cn}}_R \times_{\mathrm{LMod}^{\mathrm{cn}}_\kappa} \{V\}$. This description of $X^\wedge(R)$ makes no reference to the commutativity of R. We can therefore extend $X^\wedge$ to a functor $X^\wedge_+ : \mathrm{Alg}^{\mathrm{art}}_\kappa \to \mathcal{S}$, where $\mathrm{Alg}^{\mathrm{art}}_\kappa$ denotes the ∞-category of Artinian $\mathbb{E}_1$-algebras over κ (see Definition 14.0.0.1 below). The existence of the extension $X^\wedge_+$ is a special property enjoyed by the formal moduli problem $X^\wedge$. Since $X^\wedge$ is completely determined by the Lie algebra $\mathrm{End}(V)$, we should expect that the existence of $X^\wedge_+$ reflects a special property of $\mathrm{End}(V)$. In fact, there *is* something special about $\mathrm{End}(V)$: it is the underlying Lie algebra of an associative algebra. We will see that this is a general phenomenon: if $\mathfrak{g}_*$ is a differential graded Lie algebra and $Y : \mathrm{CAlg}^{\mathrm{art}}_\kappa \to \mathcal{S}$ is the associated formal moduli problem for $\mathbb{E}_\infty$-algebras over κ, then Y extends to a formal moduli problem $Y_+ : \mathrm{Alg}^{\mathrm{art}}_\kappa \to \mathcal{S}$ for $\mathbb{E}_1$-algebras over κ if and only if $\mathfrak{g}_*$ is quasi-isomorphic to the underlying Lie algebra of a (nonunital) differential graded algebra A_* (see Example 13.1.2.2).

Our main goal in this section is to prove an analogue of Theorem 13.0.0.2 in the setting of noncommutative geometry. Before we can state our result, we need to introduce a bit of terminology.

Definition 14.0.0.1. Let κ be a field. We let Alg_κ denote the ∞-category of $\mathbb{E}_1$-algebras over κ. We will say that an object $A \in \mathrm{Alg}_\kappa$ is *Artinian* if it satisfies the following conditions:

(a) The algebra A is connective: that is, $\pi_i A \simeq 0$ for $i < 0$.

(b) The algebra A is truncated: that is, we have $\pi_i A \simeq 0$ for $i \gg 0$.

(c) Each of the homotopy groups $\pi_i A$ is finite dimensional when regarded as a vector space over field κ.

(d) Let $\mathfrak{n}$ denote the radical of the ring $\pi_0 A$. Then the canonical map $\kappa \to (\pi_0 A)/\mathfrak{n}$ is an isomorphism.

We let $\mathrm{Alg}_\kappa^{\mathrm{art}}$ denote the full subcategory of Alg_κ spanned by the Artinian κ-algebras.

Warning 14.0.0.2. Let κ be a field and let A be a discrete κ-algebra. If A is Artinian in the sense of Definition 14.0.0.1, then A is left (and right) Artinian in the usual sense of noncommutative algebra: that is, the collection of left ideals of A satisfies the descending chain condition. However, the converse is false: for example, the ring $M_2(\kappa)$ of 2-by-2 matrices over κ does not satisfy condition (d) of Definition 14.0.0.1. One should instead regard Definition 14.0.0.1 as a noncommutative (and derived) analogue of the condition that a commutative κ-algebra A be a local Artinian ring with residue field κ: see Warning 12.1.2.6.

Remark 14.0.0.3. Let κ be a field and let $A \in \mathrm{Alg}_\kappa^{\mathrm{art}}$. It follows from conditions (a) and (d) of Definition 14.0.0.1 that the mapping space $\mathrm{Map}_{\mathrm{Alg}_\kappa}(A, \kappa)$ is contractible: that is, A admits an essentially unique augmentation. Consequently, the projection map $\mathrm{Alg}_\kappa^{\mathrm{art}} \times_{\mathrm{Alg}_\kappa} \mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Alg}_\kappa^{\mathrm{art}}$ is an equivalence of ∞-categories; here $\mathrm{Alg}_\kappa^{\mathrm{aug}} = (\mathrm{Alg}_\kappa)_{/\kappa}$ denotes the ∞-category of augmented $\mathbb{E}_1$-algebras over κ. Because of this, we will often abuse notation by identifying $\mathrm{Alg}_\kappa^{\mathrm{art}}$ with its inverse image in $\mathrm{Alg}_\kappa^{\mathrm{aug}}$.

Definition 14.0.0.4. Let κ be a field and let $X : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ be a functor. We will say that X is a *formal* $\mathbb{E}_1$*-moduli problem* if it satisfies the following conditions:

(1) The space $X(\kappa)$ is contractible.

(2) For every pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

in $\mathrm{Alg}_\kappa^{\mathrm{art}}$ for which the underlying maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, the diagram

$$\begin{array}{ccc} X(R) & \longrightarrow & X(R_0) \\ \downarrow & & \downarrow \\ X(R_1) & \longrightarrow & X(R_{01}) \end{array}$$

is a pullback square.

We let $\mathrm{Moduli}_\kappa^{(1)}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S})$ spanned by the formal $\mathbb{E}_1$-moduli problems.

We can now state our main result:

Theorem 14.0.0.5. *Let κ be a field. Then there is an equivalence of ∞-categories $\Psi : \mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Moduli}_\kappa^{(1)}$.*

Remark 14.0.0.6. Unlike Theorem 13.0.0.2, Theorem 14.0.0.5 does not require any assumptions on the characteristic of the field κ.

Like Theorem 13.0.0.2, Theorem 14.0.0.5 is a reflection of Koszul duality: this time in the setting of associative algebras. In §14.1, we will introduce the Koszul duality functor $\mathfrak{D}^{(1)} : (\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{aug}}$. Roughly speaking, if $\epsilon : A \to \kappa$ is an augmented $\mathbb{E}_1$-algebra over κ, then the Koszul dual $\mathfrak{D}^{(1)}(A)$ is the (derived) endomorphism algebra of κ as a (left) A-module. In §14.2, we will show that $\mathfrak{D}^{(1)}$ is a deformation theory (in the sense of Definition 12.3.3.2). We will then deduce Theorem 14.0.0.5 from Theorem 12.3.3.5, using the functor $\Psi : \mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Moduli}_\kappa^{(1)}$ given by

$$\Psi(A)(R) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(R), A).$$

For every field κ, there is an evident forgetful functor $\mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathrm{Alg}_\kappa^{\mathrm{art}}$. Composition with this forgetful functor determines a map $\theta : \mathrm{Moduli}_\kappa^{(1)} \to \mathrm{Moduli}_\kappa$. In §14.3, we will show that if the characteristic of κ is zero, then θ fits into a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Alg}_\kappa^{\mathrm{aug}} & \longrightarrow & \mathrm{Moduli}_\kappa^{(1)} \\ \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta} \\ \mathrm{Lie}_\kappa & \longrightarrow & \mathrm{Moduli}_\kappa, \end{array}$$

where the upper horizontal map is the equivalence of Theorem 14.0.0.5 and the lower horizontal map is the equivalence of Theorem 13.0.0.2. Here θ' is a functor which assigns to each augmented $\mathbb{E}_1$-algebra $\epsilon : A \to \kappa$ its augmentation ideal $\mathfrak{m}_A = \mathrm{fib}(\epsilon)$, equipped with the commutator bracket (Proposition 14.3.1.1).

If X is a formal $\mathbb{E}_1$-moduli problem over κ, then we can associate to X a pair of ∞-categories $\mathrm{QCoh}_L(X)$ and $\mathrm{QCoh}_R(X)$, which we call the *∞-categories of (left and right) quasi-coherent sheaves on X*. Roughly speaking, an object $\mathscr{F} \in \mathrm{QCoh}_L(X)$ is a rule which assigns to each point $\eta \in X(A)$ a left A-module $\mathscr{F}_\eta$, depending functorially on η (and $\mathrm{QCoh}_R(X)$ is defined similarly, using right modules in place of left). In §14.5, we will construct fully faithful embeddings

$$\mathrm{QCoh}_L(X) \hookrightarrow \mathrm{QCoh}^!_L(X) \qquad \mathrm{QCoh}_R(X) \hookrightarrow \mathrm{QCoh}^!_R(X),$$

where $\mathrm{QCoh}^!_L(X)$ and $\mathrm{QCoh}^!_R(X)$ are the (left and right) ∞-categories of Ind-*coherent sheaves on X*. Our construction requires a "fiberwise" version of the construction of opposite categories $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$, which we review in §14.4. In §14.6, we will use these constructions to formulate and prove a noncommutative analogue of Theorem 13.4.0.1: if $X = \Psi(A)$ is the formal moduli problem associated to an augmented $\mathbb{E}_1$-algebra A over κ, then there are canonical equivalences of ∞-categories

$$\mathrm{QCoh}^!_L(X) \simeq \mathrm{RMod}_A \qquad \mathrm{QCoh}^!_R(X) \simeq \mathrm{LMod}_A$$

(Theorem 14.6.0.1). In particular, this gives fully faithful embeddings

$$\mathrm{QCoh}_L(X) \hookrightarrow \mathrm{RMod}_A \qquad \mathrm{QCoh}_R(X) \hookrightarrow \mathrm{LMod}_A,$$

which are equivalences when restricted to connective objects (Proposition 14.6.3.1).

Contents

14.1 Koszul Duality for Associative Algebras

Let κ be a field, let A be an associative algebra over κ, and let M be a left A-module. The *commutant* B of A in $\mathrm{End}_\kappa(M)$ is defined to be the set of A-linear endomorphisms of M. Then B can be regarded as an associative algebra over κ, and M admits an action of the tensor product $A \otimes_\kappa B$. In many cases, one can show that the relationship between A and B is symmetric. For example, if A is a finite dimensional central simple algebra over κ and M is nonzero and of finite dimension over κ, then we can recover A as the commutant of B in $\mathrm{End}_\kappa(M)$.

In this section, we will discuss the operation of Koszul duality in the setting of (augmented) $\mathbb{E}_1$-algebras over a field κ. Roughly speaking, Koszul duality can be regarded as a derived version of the formation of commutants. Suppose that A is an $\mathbb{E}_1$-algebra over κ equipped with an augmentation $\epsilon : A \to \kappa$. Then ϵ determines an action of A on the κ-module $M = \kappa$. The *Koszul dual* of A is an $\mathbb{E}_1$-algebra B over κ which classifies A-linear maps from M to itself. We have commuting actions of A and B on M, which can be encoded by a map $\mu : A \otimes_\kappa B \to \kappa$ extending the augmentation ϵ. This suggests the following definition:

($*$) Let A be an augmented $\mathbb{E}_1$-algebra over a field κ. Then the *Koszul dual* of A is universal among $\mathbb{E}_1$-algebras B equipped with an augmentation $\mu : A \otimes_\kappa B \to \kappa$ extending the augmentation on A.

Our first goal in this section is to make ($*$) more precise, and show that it determines a (contravariant) functor $\mathfrak{D}^{(1)}$ from the ∞-category $\mathrm{Alg}^{\mathrm{aug}}_\kappa$ of augmented $\mathbb{E}_1$-algebras over κ to itself. Every augmentation $\mu : A \otimes_\kappa B \to \kappa$ restricts to augmentations on A and B, and is classified by a map of augmented $\mathbb{E}_1$-algebras $\alpha : B \to \mathfrak{D}^{(1)}(A)$. We will say that μ *exhibits B as a Koszul dual of A* if the map α is an equivalence. The main results of this section establish some basic formal properties of Koszul duality:

(a) Let $\mu : A \otimes_\kappa B \to \kappa$ be an augmentation which exhibits B as the Koszul dual of A. Under some mild hypotheses, there is a close relationship between the ∞-categories LMod_A and LMod_B of (left) modules over A and B, respectively (Theorem 14.1.3.2).

(*b*) Let $\mu : A \otimes_\kappa B \to \kappa$ be an augmentation which exhibits B as the Koszul dual of A. Under some mild hypotheses, it follows that μ also exhibits A as a Koszul dual of B (Corollary 14.1.3.3). In other words, the double commutant map $A \to \mathfrak{D}^{(1)}\mathfrak{D}^{(1)}(A)$ is often an equivalence.

14.1.1 The Self-Pairing of $\mathrm{Alg}_\kappa^{\mathrm{aug}}$

We begin by reviewing some terminology. If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, a *pairing* of $\mathcal{C}$ with $\mathcal{D}$ is an ∞-category $\mathcal{M}$ equipped with a right fibration $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ (see §HA.**??**). We say that such a pairing is *left representable* if, for each object $C \in \mathcal{C}$, there exists a final object of the fiber $\mathcal{M} \times_{\mathcal{C}} \{C\}$. In this case, we let $\mathfrak{D}_\lambda(C)$ denote the image of that final object in $\mathcal{D}$; the construction $C \mapsto \mathfrak{D}_\lambda(C)$ determines a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$, which we will refer to as the *duality functor* associated to the pairing $(\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D})$. Similarly, we see that λ is *right representable* if, for each $D \in \mathcal{D}$, the fiber $\mathcal{M} \times_{\mathcal{D}} \{D\}$ has a final object; in this case, we let $\mathcal{D}'_\lambda(C)$ denote the image of that final object in $\mathcal{C}$. If λ is both right and left representable, then the duality functors $\mathbb{D}_\lambda$ and $\mathbb{D}'_\lambda$ are mutually adjoint (see Construction HA.5.2.1.9).

Let us now specialize to the main example of interest.

Construction 14.1.1.1. Let κ be a field, let $\mathrm{Alg}_\kappa = \mathrm{Alg}(\mathrm{Mod}_\kappa)$ denote the ∞-category of associative algebra objects of Mod_κ, and let $\mathrm{Alg}_\kappa^{\mathrm{aug}} = (\mathrm{Alg}_\kappa)_{/\kappa}$ denote the ∞-category of augmented associative algebra objects of Mod_κ. We will regard Alg_κ as a symmetric monoidal ∞-category (where the tensor product operation is computed pointwise). Let $m : \mathrm{Alg}_\kappa \times \mathrm{Alg}_\kappa \to \mathrm{Alg}_\kappa$ denote the tensor product functor, and let $p_0, p_1 : \mathrm{Alg}_\kappa \times \mathrm{Alg}_\kappa \to \mathrm{Alg}_\kappa$ denote the projection onto the first and second factor, respectively. Since the unit object of Alg_κ is initial, we have natural transformations $p_0 \xrightarrow{\alpha_0} m \xleftarrow{\alpha_1} p_1$, which determine a map $\mathrm{Alg}_\kappa \times \mathrm{Alg}_\kappa \to \mathrm{Fun}(\Lambda^2_2, \mathrm{Alg}_\kappa)$. We let $\mathcal{M}^{(1)}$ denote the fiber product

$$(\mathrm{Alg}_\kappa \times \mathrm{Alg}_\kappa) \times_{\mathrm{Fun}(\Lambda^2_2, \mathrm{Alg}_\kappa)} \mathrm{Fun}(\Lambda^2_2, \mathrm{Alg}_\kappa^{\mathrm{aug}}).$$

More informally, the objects of $\mathcal{M}^{(1)}$ can be identified with triples (A, B, ϵ), where A and B are $\mathbb{E}_1$-algebras over κ, and $\epsilon : A \otimes_\kappa B \to \kappa$ is an augmentation on the tensor product $A \otimes_\kappa B$ (which then induces augmentations on A and B, respectively). Note that evaluation on the vertices $0, 1 \in \Lambda^2_2$ induces a right fibration $\lambda : \mathcal{M}^{(1)} \to \mathrm{Alg}_\kappa^{\mathrm{aug}} \times \mathrm{Alg}_\kappa^{\mathrm{aug}}$.

Proposition 14.1.1.2. *Let κ be a field and let $\lambda : \mathcal{M}^{(1)} \to \mathrm{Alg}_\kappa^{\mathrm{aug}} \times \mathrm{Alg}_\kappa^{\mathrm{aug}}$ be the pairing of ∞-categories described in Construction 14.1.1.1. Then λ is both left and right representable.*

Proof. We will prove that λ is left representable; the proof of right representability is similar. Fix an object $A \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$, and let $F : (\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathcal{S}$ be the functor given by

$$F(B) = \mathrm{fib}(\mathrm{Map}_{\mathrm{Alg}_\kappa}(A \otimes_\kappa B, \kappa) \to \mathrm{Map}_{\mathrm{Alg}_\kappa}(A, \kappa) \times \mathrm{Map}_{\mathrm{Alg}_\kappa}(B, \kappa)).$$

We wish to show that the functor F is representable by an object of $\mathrm{Alg}_\kappa^{\mathrm{aug}}$. Define $F' : \mathrm{Alg}_\kappa^{\mathrm{op}} \to \mathcal{S}$ by the formula

$$F'(B) = \mathrm{fib}(\mathrm{Map}_{\mathrm{Alg}_\kappa}(A \otimes_\kappa B, \kappa) \to \mathrm{Map}_{\mathrm{Alg}_\kappa}(A, \kappa)).$$

Corollary HA.5.3.1.15 implies that the functor F' is corepresented by an object $B_0 \in \mathrm{Alg}_\kappa$, given by a centralizer of the augmentation $\epsilon : A \to \kappa$. In particular, we have a point of $\eta \in F'(B_0)$, which determines an augmentation $\mu : A \otimes_\kappa B_0 \to \kappa$. Let us regard B_0 as an augmented algebra object via the composite map $B_0 \to A \otimes_\kappa B_0 \xrightarrow{\mu} \kappa$, so that η lifts to a point $\overline{\eta} \in F(B_0)$. To complete the proof, it will suffice to show that for each $B \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$, evaluation on $\overline{\eta}$ induces a homotopy equivalence $\theta : \mathrm{Map}_{\mathrm{Alg}_\kappa^{\mathrm{aug}}}(B, B_0) \to F(B)$. This map fits into a map of fiber sequences

$$\begin{array}{ccccc} \mathrm{Map}_{\mathrm{Alg}_\kappa^{\mathrm{aug}}}(B, B_0) & \longrightarrow & \mathrm{Map}_{\mathrm{Alg}_\kappa}(B, B_0) & \longrightarrow & \mathrm{Map}_{\mathrm{Alg}_\kappa}(B, \kappa) \\ \downarrow & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta''} \\ F(B) & \longrightarrow & F'(B) & \longrightarrow & \mathrm{Map}_{\mathrm{Alg}_\kappa}(B, \kappa), \end{array}$$

where θ' and θ'' are homotopy equivalences (in the first case, this follows from our assumption that η exhibits F' as the functor represented by B_0). □

Definition 14.1.1.3. Let κ be a field. We let $\mathfrak{D}^{(1)} : (\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{aug}}$ denote the functor obtained by applying Construction **??** to the left representable pairing $\lambda : \mathcal{M}^{(1)} \to \mathrm{Alg}_\kappa^{\mathrm{aug}} \times \mathrm{Alg}_\kappa^{\mathrm{aug}}$ of Construction 14.1.1.1. We will refer to the functor $\mathfrak{D}^{(1)}$ as *Koszul duality*.

Remark 14.1.1.4. Since the pairing $\lambda : \mathcal{M}^{(1)} \to \mathrm{Alg}_\kappa^{\mathrm{aug}} \times \mathrm{Alg}_\kappa^{\mathrm{aug}}$ of Construction 14.1.1.1 is both left and right representable, it determines two functors $(\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{aug}}$. It follows by symmetry considerations that these functors are (canonically) equivalent to one another; hence there is no risk of confusion if we denote them both by $\mathfrak{D}^{(1)} : (\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{aug}}$. Using Construction HA.5.2.1.9 we see that $\mathfrak{D}^{(1)}$ is self-adjoint: more precisely, the functor $\mathfrak{D}^{(1)} : (\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{aug}}$ is right adjoint to the induced map between opposite ∞-categories $\mathrm{Alg}_\kappa^{\mathrm{aug}} \to (\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}}$. More concretely, for any pair of objects $A, B \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$ we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}_\kappa^{\mathrm{aug}}}(A, \mathfrak{D}^{(1)}(B)) \simeq \mathrm{Map}_{\mathrm{Alg}_\kappa^{\mathrm{aug}}}(B, \mathfrak{D}^{(1)}(A)).$$

In fact, both of these spaces can be identified with the homotopy fiber of the canonical map

$$\mathrm{Map}_{\mathrm{Alg}_\kappa}(A \otimes_\kappa B, \kappa) \to \mathrm{Map}_{\mathrm{Alg}_\kappa}(A, \kappa) \times \mathrm{Map}_{\mathrm{Alg}_\kappa}(B, \kappa).$$

Remark 14.1.1.5. Let κ be a field and let $\mu : A \otimes_\kappa B \to \kappa$ be a morphism in Alg_κ, which we can identify with an object of the ∞-category $\mathcal{M}^{(1)}$ of Construction 14.1.1.1. We say that μ *exhibits B as a Koszul dual of A* if it is left universal in the sense of Definition **??**: that is, if it induces an equivalence $B \to \mathfrak{D}^{(1)}(A)$. Similarly, we say that μ *exhibits A as a Koszul dual of B* if it is right universal: that is, if it induces an equivalence $A \to \mathfrak{D}^{(1)}(B)$.

14.1.2 The Pairing on Modules

We now discuss a linearized version of Koszul duality.

Construction 14.1.2.1. Let κ be a field and let $\mu : A \otimes_\kappa B \to \kappa$ be a morphism in Alg_κ. Then extension of scalars along μ induces a functor

$$\mathrm{LMod}_A \times \mathrm{LMod}_B \to \mathrm{LMod}_\kappa \simeq \mathrm{Mod}_\kappa .$$

We let LPair_μ denote the fiber product $(\mathrm{LMod}_A \times \mathrm{LMod}_B) \times_{\mathrm{Mod}_\kappa} (\mathrm{Mod}_\kappa)_{/\kappa}$. We can think of the objects of LPair_μ as triples (M, N, ϵ), where M is a left module over A, N is a left module over B, and $\epsilon : M \otimes_\kappa N \to \kappa$ is a map of left modules over $A \otimes_\kappa B$. The projection map $\lambda : \mathrm{LPair}_\mu \to \mathrm{LMod}_A \times \mathrm{LMod}_B$ is a right fibration, so that we can regard LPair_μ as a pairing of ∞-categories.

Proposition 14.1.2.2. *Let κ be a field, let $\mu : A \otimes_\kappa B \to \kappa$ be a morphism in Alg_κ, and let $\lambda : \mathrm{LPair}_\mu \to \mathrm{LMod}_A \times \mathrm{LMod}_B$ be the pairing of ∞-categories of Construction 14.1.2.1. Then λ is both left and right representable.*

Proof. We will prove that λ is left representable; the proof of right representability is similar. Fix an object $M \in \mathrm{LMod}_A$; we wish to show that that the functor $N \mapsto \mathrm{Map}_{\mathrm{LMod}_{A \otimes_\kappa B}}(M \otimes_\kappa N, \kappa)$ is representable by an object of LMod_B. Let us denote this functor by F. According to Proposition HTT.5.5.2.2 , it will suffice to show that the functor F carries colimits in LMod_B to limits in $\mathcal{S}$. This follows from the observation that the construction $N \mapsto M \otimes_\kappa N$ determines a functor $\mathrm{LMod}_B \to \mathrm{LMod}_{A \otimes_\kappa B}$ which commutes with small colimits. □

Notation 14.1.2.3. Let κ be a field and let $\mu : A \otimes_\kappa B \to \kappa$ be a morphism in Alg_κ. Combining Proposition 14.1.2.2 with Construction **??**, we obtain duality functors

$$\mathrm{LMod}_A^{\mathrm{op}} \xrightarrow{\mathfrak{D}_\mu} \mathrm{LMod}_B \qquad \mathrm{LMod}_B^{\mathrm{op}} \xrightarrow{\mathfrak{D}'_\mu} \mathrm{LMod}_A .$$

By construction, we have canonical homotopy equivalences

$$\mathrm{Map}_{\mathrm{LMod}_A}(M, \mathfrak{D}'_\mu N) \simeq \mathrm{Map}_{\mathrm{LMod}_{A \otimes_\kappa B}}(M \otimes_\kappa N, \kappa) \simeq \mathrm{Map}_{\mathrm{LMod}_B}(N, \mathfrak{D}_\mu M).$$

Remark 14.1.2.4. Let κ be a field and let $\mu : A \otimes_\kappa B \to \kappa$ be a morphism in Alg_κ. The proof of Theorem HA.5.3.1.14 shows that μ exhibits B as a Koszul dual of A if and only if μ exhibits B as a classifying object for morphisms from A to κ in $\mathrm{Mod}_A^{\mathrm{Assoc}}(\mathrm{Mod}_\kappa) \simeq {}_A\mathrm{BMod}_A(\mathrm{Mod}_\kappa)$ (here we regard ${}_A\mathrm{BMod}_A(\mathrm{Mod}_\kappa)$ as left-tensored over the ∞-category Mod_κ). This is equivalent to the condition that ϵ exhibit B as a classifying object for morphisms from $\kappa \simeq A \otimes_A \kappa$ to itself in ${}_A\mathrm{BMod}_\kappa(\mathrm{Mod}_\kappa) \simeq \mathrm{LMod}_A$: that is, that μ induces an equivalence of left B-modules $B \to \mathfrak{D}_\mu \mathfrak{D}'_\mu(B) \simeq \mathfrak{D}_\mu(\kappa)$. Similarly, μ exhibits A as a Koszul dual of B if and only if it induces an equivalence $A \to \mathfrak{D}'_\mu(\kappa)$ of left A-modules.

14.1.3 Biduality

We would like to use Remark 14.1.2.4 to verify (in good cases) that the relation of Koszul duality is symmetric. For this, we need to understand the linear duality functors $\mathfrak{D}_\mu$ and $\mathfrak{D}'_\mu$ associated to a pairing $\mu : A \otimes_\kappa B \to \kappa$.

Definition 14.1.3.1. Let κ be a field. An object $A \in \mathrm{Alg}_\kappa$ is *coconnective* if the unit map $\kappa \to A$ exhibits κ as a connective cover A. Equivalently, A is coconnective if $\pi_0 A$ is a 1-dimensional vector space over κ generated by the unit element, and $\pi_n A \simeq 0$ for $n > 0$.

If $M \in \mathrm{Mod}_\kappa$, we will say that M is *locally finite* if each of the homotopy groups $\pi_n M$ is finite dimensional as a vector space over κ. We will say that an object $A \in \mathrm{Alg}_\kappa$ is *locally finite* if it is locally finite when regarded as an object of Mod_κ.

Our analysis of the Koszul duality functor rests on the following result, which we will prove at the end of this section:

Theorem 14.1.3.2. *Let κ be a field and let $\mu : A \otimes_\kappa B \to \kappa$ be a morphism in Alg_κ. Assume that A is coconnective and that μ exhibits B as a Koszul dual of A. Then:*

(1) *Let M be a left A-module such that $\pi_n M \simeq 0$ for $n > 0$. Then $\pi_n \mathfrak{D}_\mu(M) \simeq 0$ for $n < 0$.*

(2) *The $\mathbb{E}_1$-algebra B is connective.*

(3) *Let N be a connective B-module. Then $\pi_n \mathfrak{D}'_\mu(N) \simeq 0$ for $n > 0$.*

(4) *Let M be as in (1) and assume that M is locally finite. Then the canonical map $M \to \mathfrak{D}'_\mu \mathfrak{D}_\mu M$ is an equivalence in LMod_A.*

Corollary 14.1.3.3. *Let κ be a field, and let $A \in \mathrm{Alg}^{\mathrm{aug}}_\kappa$ be coconnective and locally finite. Then the canonical map $A \to \mathfrak{D}^{(1)}\mathfrak{D}^{(1)}(A)$ is an equivalence. In other words, if $\mu : A \otimes_\kappa B \to \kappa$ exhibits B as a Koszul dual of A, then μ also exhibits A as a Koszul dual of B.*

Proof. Let $\mu : A \otimes_\kappa B \to \kappa$ be a map which exhibits B as a Koszul dual of A. We wish to prove that μ exhibits A as the Koszul dual of B. According to Remark 14.1.2.4, it will suffice to show that the unit map $A \to \mathfrak{D}'_\mu \mathfrak{D}_\mu(A)$ is an equivalence of left A-modules. Since A is coconnective and locally finite, this follows from Theorem 14.1.3.2. □

14.1.4 Free Resolutions of Coconnective Modules

If A is a connective $\mathbb{E}_1$-ring, then the free A-modules of finite rank comprise compact projective generators for the ∞-category $\mathrm{LMod}^{\mathrm{cn}}_A$ of connective left A-modules. It follows that every connective A-module M can be written as the geometric realization of a simplicial object $M_\bullet$ of $\mathrm{LMod}_A^{\mathrm{cn}}$, where each M_k is free. Our next goal is to prove an analogous (but in some sense dual) result in the case where A is a coconnective $\mathbb{E}_1$-algebra over a field κ (see Definition 14.1.3.1). This result will play an important role in our proof of Theorem 14.1.3.2:

Proposition 14.1.4.1. *Let A be a coconnective $\mathbb{E}_1$-algebra over a field κ and let M be a left A-module. Then there exists a sequence of left A-modules*

$$0 = M(0) \to M(1) \to M(2) \to \cdots$$

with the following properties:

(a) *For each $n \geqslant 0$, there exists a κ-module spectrum $V(n)$ such that $\pi_i V(n) \simeq 0$ for $i \geqslant 0$ and a cofiber sequence of left A-modules*

$$A \otimes_\kappa V(n) \to M(n) \to M(n+1).$$

(b) *There exists a map $\theta : \varinjlim M(n) \to M$ which induces an isomorphism $\pi_m \varinjlim M(n) \to \pi_m M$ for $m \leqslant 0$.*

Remark 14.1.4.2. In the situation of Proposition 14.1.4.1, it follows easily by induction on n that $\pi_m M(n)$ vanishes for $m > 0$, so that the map $\theta : \varinjlim M(n) \to M$ is an equivalence if and only if $\pi_m M \simeq 0$ for $m > 0$.

Proof. Let M' be the underlying κ-module spectrum of M, and let $V(1) = \tau_{\leqslant 0} M'$. Since κ is a field, the canonical map $M' \to V(1)$ admits a section s. Set $M(1) = A \otimes_\kappa V(1)$, so that s determines a map of left A-modules $M(1) \to M$. By construction, the map $\pi_m M(1) \to \pi_m M$ is surjective for $m \leqslant 0$ and bijective for $m = 0$. We construct $M(n) \in (\mathrm{LMod}_A)_{/M}$ for $n > 1$ by induction on n. Assume that we have already constructed $M(n-1) \in (\mathrm{LMod}_A)_{/M}$, and that the map $e_m : \pi_m M(n-1) \to \pi_m M$ is bijective for $m = 0$ and surjective for $m < 0$. Let W_m denote the kernel of e_m (as a vector space over κ), and let $V(n) = \bigoplus_{m<0} \Sigma^m(W_m)$ (as a κ-module spectrum). We have an evident map of κ-modules $V(n) \to \mathrm{fib}(M(n-1) \to M)$, hence a map of left A-modules $f : (A \otimes_\kappa V(n)) \to \mathrm{fib}(M(n-1) \to M)$. Let $M(n)$ denote the cofiber of the map $A \otimes_\kappa V(n) \to M(n-1)$, so that f determines a map $M(n-1) \to M(n)$ in $(\mathrm{LMod}_A)_{/M}$ and we have a cofiber sequence

$$A \otimes_\kappa V(n) \to M(n-1) \to M(n).$$

For each $m \in \mathbf{Z}$, let $e'_m : \pi_m M(n) \to \pi_m M$ be the evident map. It is clear that e'_m is surjective for $m < 0$ (since e_m factors through e'_m). We claim that e'_m is bijective when

$m = 0$. To prove this, it suffices to show that the evident map $\pi_0 M(n-1) \to \pi_0 M(n)$ is bijective. We have a long exact sequence

$$\pi_1(A \otimes_\kappa V(n)) \to \pi_0 M(n-1) \xrightarrow{e'_n} \pi_0 M(n) \to \pi_0(A \otimes_\kappa V(n)) \xrightarrow{\phi} \pi_{-1} M(n-1).$$

Since A is coconnective and $\pi_m V(n) \simeq 0$ for $m > 0$, the vector space $\pi_1(A \otimes_\kappa V(n))$ is trivial and $\pi_0(A \otimes_\kappa V(n)) \simeq \pi_0 V(n) \simeq \ker(\pi_{-1} M(n-1) \to \pi_{-1} M)$. It follows that ϕ is injective so that e'_0 is an isomorphism.

It remains to prove that the map $\theta : \varinjlim M(n) \to M$ induces an isomorphism on π_m for $m \leqslant 0$. It is clear that the map $\vec{e}_m : \pi_m \varinjlim M(n) \to \pi_m M$ is surjective for $m \leqslant 0$. If η belongs to the kernel of $\vec{e}_m$, then η can be represented by an element of $\pi_m M(n-1)$ belonging to the kernel of e_m for some $n \gg 0$. By construction, the image of this class in $\pi_m M(n)$ vanishes, so that $\eta = 0$. □

Corollary 14.1.4.3. *Let A be a coconnective $\mathbb{E}_1$-algebra over a field κ, let M be a left A-module, and let N be a right A-module. Suppose that $\pi_i M \simeq \pi_i N \simeq 0$ for $i > 0$. Then $\pi_i(N \otimes_A M) \simeq 0$ for $i > 0$. Moreover, the map $(\pi_0 N) \otimes_\kappa (\pi_0 M) \to \pi_0(N \otimes_\kappa M)$ is injective.*

Proof. Let $\{M(n)\}_{n \geqslant 0}$ be as in the proof of Proposition 14.1.4.1, so that $M \simeq \varinjlim M(n)$ by Remark 14.1.4.2. Then $\pi_i(N \otimes_A M) \simeq \varinjlim \pi_i(N \otimes_A M(n))$, and we have $M(1) \simeq A \otimes_\kappa M$ so that

$$\pi_0(N \otimes_A M(1)) \simeq \pi_0(N \otimes_\kappa M) \simeq (\pi_0 N) \otimes_\kappa (\pi_0 M).$$

It will therefore suffice to show that $\pi_i(N \otimes_A M(1)) \simeq 0$ for each $i > 0$ and that the maps $\pi_0(N \otimes_A M(1)) \to \pi_0(N \otimes_A M(n))$ are injective, for which we use induction on n. When $n = 1$, the result is obvious. Otherwise, we have a cofiber sequence

$$A \otimes_\kappa V(n) \to M(n-1) \to M(n)$$

where $V(n) \in (\mathrm{Mod}_\kappa)_{\leqslant -1}$, whence a cofiber sequence of spectra

$$N \otimes_\kappa V(n) \to N \otimes_A M(n-1) \to N \otimes_A M(n).$$

The desired result now follows from the inductive hypothesis, since $\pi_i(N \otimes_\kappa V(n)) \simeq 0$ for $i \geqslant 0$. □

Corollary 14.1.4.4. *Let κ be a field and let $\phi : A \to B$ be a map of coconnective $\mathbb{E}_1$-algebras over κ. Let M be a left A-module such that $\pi_i M \simeq 0$ for $i > 0$. Then the homotopy groups $\pi_i(B \otimes_A M)$ vanish for $i > 0$, and the map $\pi_0 M \to \pi_0(B \otimes_A M)$ is injective.*

Corollary 14.1.4.5. *Let κ be a field and let $\phi : A \to B$ be a map of coconnective $\mathbb{E}_1$-algebras over κ. Let M be a left A-module such that $\pi_i M \simeq 0$ for $i > 0$. If $B \otimes_A M \simeq 0$, then $M \simeq 0$.*

14.1.5 Connectivity and Mapping Spaces

Let A be a connective $\mathbb{E}_1$-ring. Then the ∞-category LMod_A of left A-modules admits a t-structure $((\mathrm{LMod}_A)_{\geqslant 0}, (\mathrm{LMod}_A)_{\leqslant 0})$, where a left A-module M belongs to $(\mathrm{LMod}_A)_{\geqslant 0}$ if and only if the homotopy groups of M are concentrated in degrees $\geqslant 0$, and a left A-module N belongs to $(\mathrm{LMod}_A)_{\leqslant 0}$ if and only if the homotopy groups of N are concentrated in degrees $\leqslant 0$ (see Proposition **??**). In particular, if $M, N \in \mathrm{LMod}_A$ are left A-modules and there exists an integer m for which the groups $\pi_* M$ vanish for $* < m$ and the groups $\pi_* N$ vanish for $* \geqslant m$, then any morphism $M \to N$ is automatically nullhomotopic. We now prove an analogous (but dual) result in the case where A is a coconnective $\mathbb{E}_1$-algebra over a field.

Proposition 14.1.5.1. *Let A be a coconnective $\mathbb{E}_1$-algebra over a field κ. Let M and N be left A-modules. Assume that $\pi_m M \simeq 0$ for $m > 0$ and that $\pi_m N \simeq 0$ for $m \leqslant 0$. Then any map $f : M \to N$ is nullhomotopic.*

Proof. Let $\{M(n)\}_{n \geqslant 0}$ be as in the proof of Proposition 14.1.4.1, so that $M \simeq \varinjlim M(n)$ by Remark 14.1.4.2. We may therefore identify $\mathrm{Map}_{\mathrm{LMod}_A}(M, N)$ with the homotopy limit of the tower

$$\cdots \to \mathrm{Map}_{\mathrm{LMod}_A}(M(1), N) \to \mathrm{Map}_{\mathrm{LMod}_A}(M(0), N).$$

To prove that $\mathrm{Map}_{\mathrm{LMod}_A}(M, N)$ is connected, it will suffice to show that each of the mapping spaces $\mathrm{Map}_{\mathrm{LMod}_A}(M(n), N)$ is connected, and that each map

$$\pi_1 \mathrm{Map}_{\mathrm{LMod}_A}(M(n), N) \to \pi_1 \mathrm{Map}_{\mathrm{LMod}_A}(M(n-1), N)$$

is surjective. We proceed by induction on n. Using the cofiber sequence

$$A \otimes_\kappa V(n) \to M(n-1) \to M(n),$$

we obtain a fiber sequence of spaces

$$\mathrm{Map}_{\mathrm{LMod}_A}(M(n), N) \to \mathrm{Map}_{\mathrm{LMod}_A}(M(n-1), N) \to \mathrm{Map}_{\mathrm{LMod}_\kappa}(V(n), N).$$

It will therefore suffice to show that $\pi_1 \mathrm{Map}_{\mathrm{LMod}_\kappa}(V(n), N) \simeq 0$. Since κ is a field, this follows immediately from the our assumptions that $\pi_m V(n) \simeq 0$ for $m \geqslant 0$ and $\pi_m N \simeq 0$ for $m \leqslant 0$. $\square$

Corollary 14.1.5.2. *Let A be a coconnective $\mathbb{E}_1$-algebra over a field κ. Let M and N be left A-modules such that $\pi_m M \simeq 0$ for $m > 0$ and $\pi_m N \simeq 0$ for $m < 0$. Then the canonical map $\theta : \mathrm{Ext}^0_A(M, N) \to \mathrm{Hom}_\kappa(\pi_0 M, \pi_0 N)$ is surjective.*

Proof. We have an evident map of κ-module spectra $\pi_0 M \to M$, which determines a map of left A-modules $A \otimes_\kappa (\pi_0 M) \to M$. Let K denote the fiber of this map, so that we have a fiber sequence of spaces

$$\mathrm{Map}_{\mathrm{LMod}_A}(M, N) \xrightarrow{\phi} \mathrm{Map}_{\mathrm{Mod}_\kappa}(\pi_0 M, N) \to \mathrm{Map}_{\mathrm{LMod}_A}(K, N).$$

Since $\pi_m K \simeq 0$ for $m \geqslant 0$, Proposition 14.1.5.1 implies that the mapping space $\mathrm{Map}_{\mathrm{LMod}_A}(K, N)$ is connected. It follows that ϕ induces a surjection

$$\mathrm{Ext}^0_A(M, N) \to \mathrm{Ext}^0_A(\pi_0 M, N) \simeq \mathrm{Hom}_\kappa(\pi_0 M, \pi_0 N).$$

□

Corollary 14.1.5.3. *Let A be a coconnective $\mathbb{E}_1$-algebra over a field κ, and let M be a left A-module such that $\pi_m M \simeq 0$ for $m \neq 0$. Suppose we are given a map of $\mathbb{E}_1$-algebras $A \to \kappa$. Then M lies in the essential image of the forgetful functor $\theta : \mathrm{Mod}_\kappa \simeq \mathrm{LMod}_\kappa \to \mathrm{LMod}_A$.*

Proof. Let $V = \pi_0 M$, and regard V as a discrete κ-module spectrum. Corollary 14.1.5.2 implies that the evident isomorphism $\pi_0 \theta(V) \simeq \pi_0 M$ can be lifted to a map of left A-modules $\theta(V) \to M$, which is evidently an equivalence. □

14.1.6 The Proof of Theorem 14.1.3.2

Let κ be a field and let $\mu : A \otimes_\kappa B \to \kappa$ be a morphism in Alg_κ. Assume that A is coconnective and that μ exhibits B as a Koszul dual of A. We will verify each of the assertions of Theorem 14.1.3.2:

(1) Let M be a left A-module satisfying $\pi_m M \simeq 0$ for $m > 0$. We wish to show that $\pi_n \mathfrak{D}_\mu(M) \simeq 0$ for $n < 0$. Using Proposition 14.1.4.1 and Remark 14.1.4.2, we can write M as the colimit of a sequence of A-modules

$$0 = M(0) \to M(1) \to M(2) \to \cdots,$$

where each $M(n)$ fits into a cofiber sequence $A \otimes_\kappa V(n) \to M(n) \to M(n+1)$ for $V(n) \in (\mathrm{Mod}_\kappa)_{\leqslant -1}$. Then $\mathfrak{D}_\mu(M)$ is a limit of the tower $\{\mathfrak{D}_\mu(M(n))\}_{n \geqslant 0}$. It will therefore suffice to prove that $\pi_m \mathfrak{D}_\mu(M(n)) \simeq 0$ for $m < 0$ and that each of the maps $\pi_0 \mathfrak{D}_\mu(M(n)) \to \pi_0 \mathfrak{D}_\mu(M(n-1))$ is surjective. Using the fiber sequences

$$\mathfrak{D}_\mu(M(n)) \to \mathfrak{D}_\mu(M(n-1)) \to \mathfrak{D}_\mu(A \otimes_\kappa V(n)),$$

we are reduced to showing that the groups $\pi_m \mathfrak{D}_\mu(A \otimes_\kappa V(n))$ vanish for $m \leqslant 0$. Unwinding the definitions, we must show that if $m \leqslant 0$, then any map of $A \otimes_\kappa B$-modules from $(A \otimes_\kappa V(n)) \otimes_\kappa \Sigma^m(B)$ to κ is nullhomotopic. This is equivalent to the assertion that every map of κ-module spectra from $\Sigma^m V(n)$ into κ is nullhomotopic. Since κ is a field, this follows from the observation that $\pi_0 \Sigma^m V(n) \simeq \pi_{-m} V(n) \simeq 0$.

(2) Since $\mu : A \otimes_\kappa B \to \kappa$ exhibits B as a Koszul dual of A, the augmentation on B gives a map $\kappa \otimes_\kappa B \to \kappa$ which induces an equivalence $B \to \mathfrak{D}_\mu(\kappa)$. It follows from (1) that B is connective.

(3) Let N be a connective B-module. We wish to show that the homotopy groups $\pi_n \mathfrak{D}'_\mu(N)$ vanish for $n > 0$. Let $\mathcal{C}$ be the full subcategory of LMod_B spanned by those objects N for which $\pi_n \mathfrak{D}'_\mu(N) \simeq 0$ for $n > 0$. Since $\mathfrak{D}'_\mu : \mathrm{LMod}_B^{\mathrm{op}} \to \mathrm{LMod}_A$ preserves small limits, the ∞-category $\mathcal{C}$ is stable under small colimits in LMod_B. Consequently, to prove that $\mathcal{C}$ contains all connective left B-modules, it will suffice to show that $B \in \mathcal{C}$. This is clear, since $\mathfrak{D}'_\mu(B) \simeq \kappa$.

(4) Let M be a left A-module which is locally finite and satisfies $\pi_* M \simeq 0$ for $* > 0$. We wish to show that the canonical map $u_M : M \to \mathfrak{D}'_\mu \mathfrak{D}_\mu M$ is an equivalence in LMod_A. It follows from (1) that $\mathfrak{D}_\mu(M)$ is connective, so that $\pi_n \mathfrak{D}'_\mu \mathfrak{D}_\mu(M) \simeq 0$ for $n > 0$ by virtue of (3). Set $K_M = \mathrm{fib}(u_M)$, so that $\pi_n K_M \simeq 0$ for $n > 0$. We prove that $\pi_n K_M \simeq 0$ for all n, using descending induction on n. Using Proposition 14.1.4.1, we can choose a map of left A-modules $v : M' \to M$ which induces an isomorphism $\pi_m M' \to \pi_m M$ for $m < 0$ and satisfies $\pi_m M' \simeq 0$ for $m \geqslant 0$. Let M'' denote the cofiber of v, so that $\pi_m M'' \simeq 0$ for $m \neq 0$ and therefore Corollary 14.1.5.3 guarantees that M'' is a direct sum of (finitely many) copies of κ. The condition that μ exhibits B as a Koszul dual of A guarantees that $B \simeq \mathfrak{D}_\mu(\kappa)$ and therefore the unit map $u_\kappa : \kappa \simeq \mathfrak{D}'_\mu(B) \to \mathfrak{D}'_\mu \mathfrak{D}_\mu(\kappa)$ is an equivalence. It follows that $u_{M''}$ is an equivalence. The cofiber sequence $M \to M'' \to \Sigma M'$ induces an equivalence $K_M \simeq \Sigma^{-1} K_{\Sigma M'}$. The inductive hypothesis implies that $\pi_{n+1} K_{\Sigma M'} \simeq 0$, so that $\pi_n K_M \simeq 0$ as desired.

14.1.7 Koszul Duality and Tensor Products

In Chapter 15, we will need the following slightly stronger version of Corollary 14.1.3.3:

Proposition 14.1.7.1. *Let κ be a field and suppose given a finite collection of maps $\{\mu_i : A_i \otimes_\kappa B_i \to \kappa\}_{1 \leqslant i \leqslant m}$ in Alg_κ. Assume that each A_i is coconnective and locally finite and that each μ_i exhibits B_i as a Koszul dual of A_i. Let $A = \bigotimes_i A_i$, $B = \bigotimes_i B_i$, and let $\mu : A \otimes_\kappa B \to \kappa$ be the tensor product of the maps μ_i. Then μ exhibits A as the Koszul dual of B.*

Warning 14.1.7.2. In the situation of Proposition 14.1.7.1, it is not necessarily true that μ exhibits B as the Koszul dual of A. For example, suppose that $m = 2$ and that $A_1 = A_2 = \kappa \oplus \Sigma^{-1}(\kappa)$, endowed with the square-zero algebra structure. In this case, the Koszul dual of A_1 can be identified with the power series ring $\kappa[[x_1]]$, regarded as a discrete $\mathbb{E}_1$-algebra. Similarly, the Koszul dual of A_2 can be identified with $\kappa[[x_2]]$, and the Koszul dual of the tensor product $A_1 \otimes_\kappa A_2$ is given by $\kappa[[x_1, x_2]]$. The canonical map

$\theta : \kappa[[x_1]] \otimes_\kappa \kappa[[x_2]] \to \kappa[[x_1, x_2]]$ is not an isomorphism: however, Proposition 14.1.7.1 guarantees that θ induces an equivalence after applying the Koszul duality functor.

Proof. For $1 \leqslant i \leqslant m$, let $\mathfrak{D}_i : \mathrm{LMod}_{A_i}^{\mathrm{op}} \to \mathrm{LMod}_{B_i}$ be the duality functor determined by μ_i, and let $\mathfrak{D}'_\mu : \mathrm{LMod}_B^{\mathrm{op}} \to \mathrm{LMod}_A$ be the duality functor associated to μ. For every sequence of objects $\vec{M} = \{M_i \in \mathrm{LMod}_{A_i}\}$, we have a canonical map $u_{\vec{M}} : M_1 \otimes_\kappa \cdots \otimes_\kappa M_m \to \mathfrak{D}'_\mu(\mathfrak{D}_1 M_1 \otimes_\kappa \cdots \otimes_\kappa \mathfrak{D}_m M_m)$. We will prove the following:

$(*)$ If $\vec{M} = \{M_i\}_{1 \leqslant i \leqslant m} \in \prod_i \mathrm{LMod}_{A_i}$ is such that the homotopy groups $\pi_n M_i \simeq 0$ vanish for $i > 0$ and each M_i is locally finite, then $u_{\vec{M}}$ is an equivalence.

Fix $0 \leqslant m' \leqslant m$. We will show that assertion $(*)$ holds under the additional assumption that $M_i \simeq \kappa$ for $i > m'$. The proof proceeds by induction on m'. If $m' = 0$, then each $M_i \simeq \kappa$ and the desired result follows immediately from our assumption that each μ_i exhibits B_i as a Koszul dual of A_i. Let us therefore assume that $m' > 0$ and that condition $(*)$ holds whenever $M_i \simeq \kappa$ for $i < m'$.

Note that if $\vec{M}$ satisfies the hypotheses of $(*)$, then Theorem 14.1.3.2 guarantees that each $\mathfrak{D}_i(M_i)$ is connective and therefore that $\pi_n \mathfrak{D}'_\mu(\mathfrak{D}_1 M_1 \otimes_\kappa \cdots \otimes_\kappa \mathfrak{D}_m M_m) \simeq 0$ for $n > 0$. Let $K_{\vec{M}}$ denote the fiber of $u_{\vec{M}}$, so that $\pi_n K_{\vec{M}} \simeq 0$ for $n > 0$. We prove that $\pi_n K_{\vec{M}} \simeq 0$ for all n, using descending induction on n. Using Proposition 14.1.4.1, we can choose a map of left A-modules $v : M' \to M_{m'}$ which induces an isomorphism $\pi_p M' \to \pi_p M_{m'}$ for $p < 0$ and satisfies $\pi_p M' \simeq 0$ for $p \geqslant 0$. Let M'' denote the cofiber of v, so that $\pi_p M'' \simeq 0$ for $p \neq 0$ and therefore Corollary 14.1.5.3 guarantees that M'' is a direct sum of (finitely many) copies of κ. Let $\vec{M}''$ be the sequence of modules obtained from $\vec{M}$ by replacing $M_{m'}$ with M'', and let $\vec{N}$ be the sequence of modules obtained from $\vec{M}$ by replacing $M_{m'}$ with $\Sigma M'$. The inductive hypothesis implies that $K_{\vec{M}''} \simeq 0$. Using the cofiber sequence $M_{m'} \to M'' \to \Sigma M'$, we obtain an equivalence $K_{\vec{M}} \simeq \Sigma^{-1} K_{\vec{N}}$, so that $\pi_n K_{\vec{M}} \simeq \pi_{n+1} K_{\vec{N}}$ is trivial by the other inductive hypothesis. □

14.2 Formal Moduli Problems for Associative Algebras

Let κ be a field. In this section, we will use the Koszul duality functor $\mathfrak{D}^{(1)}$ of §14.1 to construct an equivalence of ∞-categories $\mathrm{Alg}_\kappa^{\mathrm{aug}} \simeq \mathrm{Moduli}_\kappa^{(1)}$, and thereby obtain a proof of Theorem 14.0.0.5. The main point is to show that the functor $\mathfrak{D}^{(1)}$ is a deformation theory (in the sense of Definition 12.3.3.2).

14.2.1 Associative Algebras as a Deformation Context

We begin by discussing a variant of Example 12.1.1.2:

Construction 14.2.1.1. Let κ be a field. Theorem HA.7.3.4.13 gives an equivalence between $\mathrm{Sp}(\mathrm{Alg}_\kappa^{\mathrm{aug}})$ and the ∞-category ${}_\kappa\mathrm{BMod}_\kappa(\mathrm{Mod}_\kappa) \simeq \mathrm{Mod}_\kappa$ of κ-module spectra. Let $E \in \mathrm{Sp}(\mathrm{Alg}_\kappa^{\mathrm{aug}})$ correspond to the unit object $\kappa \in \mathrm{Mod}_\kappa$ under this equivalence (so we have $\Omega^{\infty-n}E \simeq \kappa \oplus \Sigma^n(\kappa)$ for every integer n). We regard $(\mathrm{Alg}_\kappa^{\mathrm{aug}}, \{E\})$ as a deformation context (see Definition 12.1.1.1).

We begin by showing that the deformation context $(\mathrm{Alg}_\kappa^{\mathrm{aug}}, \{E\})$ of Construction 14.2.1.1 allows us to recover the notion of Artinian κ-algebra and formal $\mathbb{E}_1$-moduli problem via the general formalism laid out in §12.1.

Proposition 14.2.1.2. *Let κ be a field and let $(\mathrm{Alg}_\kappa^{\mathrm{aug}}, \{E\})$ be the deformation context of Construction 14.2.1.1. Then an object $A \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$ is Artinian (in the sense of Definition 12.1.2.4) if and only if its image in Alg_κ is Artinian (in the sense of Definition 14.0.0.1); that is, if and only if A satisfies the following conditions:*

(a) *The algebra A is connective: that is, $\pi_i A \simeq 0$ for $i < 0$.*

(b) *The algebra A is truncated: that is, we have $\pi_i A \simeq 0$ for $i \gg 0$.*

(c) *Each of the homotopy groups $\pi_i A$ is finite dimensional when regarded as a vector space over field κ.*

(d) *Let $\mathfrak{n}$ denote the radical of the ring $\pi_0 A$ (which is a finite-dimensional associative algebra over κ). Then the canonical map $\kappa \to (\pi_0 A)/\mathfrak{n}$ is an isomorphism.*

Proof. Suppose first that there there exists a finite sequence of maps

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq \kappa$$

where each A_i is a square-zero extension of A_{i+1} by $\Sigma^{n_i}(\kappa)$, for some $n_i \geqslant 0$. We prove that each A_i satisfies conditions (a) through (d) using descending induction on i. The case $i = n$ is obvious, so let us assume that $i < n$ and that A_{i+1} is known to satisfy conditions (a) through (d). We have a fiber sequence of κ-module spectra

$$\Sigma^{n_i}(\kappa) \to A_i \to A_{i+1}$$

which immediately implies that A_i satisfies (a), (b), and (c). To prove (d), we note that the map $\phi : \pi_0 A_i \to \pi_0 A_{i+1}$ is surjective and $\ker(\phi)^2 = 0$, so that the quotient of $\pi_0 A_i$ by its radical agrees with the quotient of $\pi_0 A_{i+1}$ by its radical.

Now suppose that A satisfies conditions (a) through (d). We will prove that A is Artinian by induction on the dimension of the κ-vector space $\pi_* A$. Let n be the largest integer for which $\pi_n A$ does not vanish. We first treat the case $n = 0$. We will abuse notation by identifying A with the underlying associative ring $\pi_0 A$. Let $\mathfrak{n}$ denote the radical of A. If

$\mathfrak{n} = 0$, then condition (d) implies that $A \simeq \kappa$ so there is nothing to prove. Otherwise, we can view $\mathfrak{n}$ as a nonzero module over the associative algebra $A \otimes_{\kappa} A^{\mathrm{op}}$. It follows that there exists a nonzero element $x \in \mathfrak{n}$ which is annihilated by $\mathfrak{n} \otimes_{\kappa} \mathfrak{n}$. Using (d) again, we deduce that the subspace $\kappa x \subseteq A$ is a two-sided ideal of A. Let A' denote the quotient ring $A/\kappa x$. Theorem HA.7.4.1.23 implies that A is a square-zero extension of A' by κ. The inductive hypothesis implies that A' is Artinian, so that A is also Artinian.

Now suppose that $n > 0$ and let $M = \pi_n A$. Then M is a nonzero bimodule over the finite dimensional κ-algebra $\pi_0 A$. It follows that there is a nonzero element $x \in M$ which is annihilated (on both sides) by the action of the radical $\mathfrak{n} \subseteq \pi_0 A$. Let M' denote the quotient of M by the bimodule generated by x (which, by virtue of (d), coincides with κx), and let $A'' = \tau_{\leqslant n-1} A$. It follows from Theorem HA.7.4.1.23 that there is a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ A'' & \longrightarrow & \kappa \oplus \Sigma^{n+1}(M). \end{array}$$

Set $A' = \kappa \times_{\kappa \oplus \Sigma^{n+1}(M')} A''$. Then $A \simeq \kappa \times_{\kappa \oplus \Sigma^{n+1}(\kappa)} A'$ so we have an elementary map $A \to A'$. Using the inductive hypothesis we deduce that A' is Artinian, so that A is also Artinian. □

We will also need a noncommutative analogue of Lemma 12.1.3.8:

Proposition 14.2.1.3. *Let κ be a field and let $f : A \to B$ be a morphism in $\mathrm{Alg}_{\kappa}^{\mathrm{art}}$. Then f is small (when regarded as a morphism in $\mathrm{Alg}_{\kappa}^{\mathrm{aug}}$) if and only if it induces a surjection of associative rings $\pi_0 A \to \pi_0 B$.*

Proof. The "only if" direction is obvious. For the converse, suppose that $\pi_0 A \to \pi_0 B$ is surjective, so that the fiber $I = \mathrm{fib}(f)$ is connective. We will prove that f is small by induction on the dimension of the graded vector space $\pi_* I$. If this dimension is zero, then $I \simeq 0$ and f is an equivalence. Assume therefore that $\pi_* I \neq 0$, and let n be the smallest integer such that $\pi_n I \neq 0$. Let $L_{B/A}$ denote the relative cotangent complex of B over A in the setting of $\mathbb{E}_1$-algebras, regarded as an object of ${}_B\mathrm{BMod}_B(\mathrm{Mod}_{\kappa})$. Remark HA.7.4.1.12 supplies a fiber sequence $L_{B/A} \to B \otimes_A B \to B$. In the ∞-category LMod_B, this sequence splits; we therefore obtain an equivalence of left B-modules

$$L_{B/A} \simeq \mathrm{cofib}(B \to B \otimes_A B) \simeq B \otimes_A \mathrm{cofib}(A \to B) \simeq B \otimes_A \Sigma(I).$$

The kernel of the map $\pi_0 A \to \pi_0 B$ is contained in the radical of $\pi_0 A$ and is therefore a nilpotent ideal. It follows that $\pi_{n+1} L_{B/A} \simeq \mathrm{Tor}_0^{\pi_0 B}(\pi_0 A, \pi_n I)$ is a nonzero quotient of of $\pi_n I$. Let us regard $\pi_{n+1} L_{B/A}$ as a bimodule over $\pi_0 B$, and let $\mathfrak{n}$ be the radical of $\pi_0 B$. Since $\mathfrak{n}$ is nilpotent, the two-sided submodule $\mathfrak{n}(\pi_{n+1} L_{B/A}) + (\pi_{n+1} L_{B/A})\mathfrak{n}$ does not coincide with $\pi_{n+1} L_{B/A}$. It follows that there exists a map of $\pi_0 B$-bimodules $\pi_{n+1} L_{B/A} \to \kappa$, which determines a map $L_{B/A} \to \Sigma^{n+1}(\kappa)$ in the ∞-category ${}_B\mathrm{BMod}_B(\mathrm{Mod}_\kappa)$. We can interpret this map as a derivation $B \to B \oplus \Sigma^{n+1}(\kappa)$. Let $B' = B \times_{B \oplus \Sigma^{n+1}(\kappa)} \kappa$ be the associated square-zero extension of B by $\Sigma^n(\kappa)$. Then f factors as a composition $A \xrightarrow{f'} B' \xrightarrow{f''} B$. Since the map f'' is elementary, it will suffice to show that f' is small, which follows from the inductive hypothesis. □

Corollary 14.2.1.4. *Let κ be a field and let $X : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ be a functor. Then X belongs to the full subcategory $\mathrm{Moduli}_\kappa^{(1)}$ of Definition 14.0.0.4 if and only if it is a formal moduli problem in the sense of Definition 12.1.3.1.*

Proof. The "if" direction follows immediately from Proposition 14.2.1.3. For the converse, suppose that X satisfies the conditions of Definition 14.0.0.4; we wish to show that X is a formal moduli problem. According to Proposition 12.1.3.2, it will suffice to show that for every $n > 0$ and every pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \kappa & \longrightarrow & \kappa \oplus \Sigma^n(\kappa) \end{array}$$

in the ∞-category $\mathrm{Alg}_\kappa^{\mathrm{art}}$, the associated diagram of spaces

$$\begin{array}{ccc} X(A) & \longrightarrow & X(B) \\ \downarrow & & \downarrow \\ X(\kappa) & \longrightarrow & X(\kappa \oplus \Sigma^n(\kappa)) \end{array}$$

is also a pullback square. This follows immediately from condition (2) of Definition 14.0.0.4. □

14.2.2 Koszul Duality as a Deformation Theory

We can now formulate our main result:

Theorem 14.2.2.1. *Let κ be a field. Then the Koszul duality functor*

$$\mathfrak{D}^{(1)} : (\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{aug}}$$

is a deformation theory (on the deformation context $(\mathrm{Alg}_\kappa^{\mathrm{aug}}, \{E\})$ of Construction 14.2.1.1).

Proof of Theorem 14.0.0.5. Let κ be a field, and let $\Psi : \mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S})$ be the functor given on objects by the formula $\Psi(A)(R) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(R), A)$. Combining

Theorems 14.2.2.1 and 12.3.3.5, we deduce that Ψ is a fully faithful embedding whose essential image is the full subcategory $\mathrm{Moduli}_{\kappa}^{(1)} \subseteq \mathrm{Fun}(\mathrm{Alg}_{\kappa}^{\mathrm{art}}, \mathcal{S})$ spanned by the formal $\mathbb{E}_1$-moduli problems. □

Proof of Theorem 14.2.2.1. The ∞-category $\mathrm{Alg}_{\kappa}^{\mathrm{aug}}$ is presentable by Corollary HA.3.2.3.5 , and $\mathfrak{D}^{(1)}$ admits a left adjoint by Remark 14.1.1.4. Let $\mathbf{B}_0 \subseteq \mathrm{Alg}_{\kappa}^{\mathrm{aug}}$ be the full subcategory spanned by those algebras which are coconnective and locally finite (see Definition 14.1.3.1). We will complete the proof that $\mathfrak{D}^{(1)}$ is a weak deformation theory by showing that the subcategory $\mathbf{B}_0$ satisfies conditions (a) thorugh (d) of Definition 12.3.1.1:

(a) For every object $A \in \mathbf{B}_0$, the unit map $A \to \mathfrak{D}^{(1)}\mathfrak{D}^{(1)}(A)$ is an equivalence. This follows from Corollary 14.1.3.3.

(b) The full subcategory $\mathbf{B}_0$ contains the initial object $\kappa \in \mathrm{Alg}_{\kappa}^{\mathrm{aug}}$. This is clear from the definitions.

(c) For each $n \geqslant 1$, there exists an object $K_n \in \mathbf{B}_0$ and an equivalence $\kappa \oplus \Sigma^n(\kappa) \simeq \mathfrak{D}^{(1)}(K_n)$. In fact, we can take K_n to be the free algebra $\bigoplus_{m \geqslant 0} V^{\otimes m}$ generated by $V = \Sigma^{-n-1}(\kappa)$ (this is a consequence of Proposition 15.3.2.1, but is also not difficult to verify by direct calculation).

(d) Let $n \geqslant 1$ and suppose we are given a pushout diagram σ :

$$\begin{array}{ccc} K_n & \longrightarrow & \kappa \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ A & \longrightarrow & A' \end{array}$$

in $\mathrm{Alg}_{\kappa}^{\mathrm{aug}}$, where K_n is as in (c). We must show that if $A \in \mathbf{B}_0$, then $A' \in \mathbf{B}_0$. Note that σ is also a pushout diagram in Alg_{κ}. We will make use of the fact that Alg_{κ} is the underlying ∞-category of the model category $\mathrm{Alg}_{\kappa}^{\mathrm{dg}}$ of differential graded algebras over κ (for a different argument which does not use the theory of model categories, we refer the reader to the proof of Theorem 15.3.3.1). Choose a cofibrant differential graded algebra A_* representing A, and let B_* denote the free differential graded algebra generated by a class x in degree $(-n-1)$. Since B_* is cofibrant and A_* is fibrant, the map $\phi : K_n \to A$ can be represented by a map $\phi_0 : B_* \to A_*$ of differential graded algebras, which is determined by the element $x' = \phi_0(x) \in A_{-n-1}$. Let B'_* denote the free differential graded algebra generated by the chain complex $E(-n)_*$ (see the proof of Proposition 13.1.3.2): in other words, B'_* is obtained from B_* by freely adjoining an element $y \in B'_{-n}$ satisfying $dy = x$. Then B'_* is quasi-isomorphic to the ground field κ.

Let $\psi_0 : B_* \to B'_*$ be the evident inclusion, and form a pushout diagram σ_0 :

$$\begin{array}{ccc} B_* & \xrightarrow{\psi_0} & B'_* \\ \downarrow{\scriptstyle \phi_0} & & \downarrow \\ A_* & \longrightarrow & A'_* \end{array}$$

in the category $\mathrm{Alg}^{\mathrm{dg}}_\kappa$. Since A_* is cofibrant and ψ_0 is a cofibration, the diagram σ_0 is also a homotopy pushout square, so that the image of σ_0 in Alg_κ is equivalent to the diagram σ. It follows that the differential graded algebra A'_* represents A'. We can describe A'_* explicitly as the differential graded algebra obtained from A_* by adjoining an element y' in degree $-n$ satisfying $dy' = x'$. As a chain complex, A'_* can be written as a union of an increasing family of subcomplexes

$$A_* = A'^0_* \subseteq A'^1_* \subseteq A'^2_* \subseteq \cdots ,$$

where A'^m_* denote the graded subspace of A'_* generated by products of the form $a_0 y a_1 y \cdots a_{m_1} y a_m$. The successive quotients for this filtration are given by $A'^m_* / A'^{m-1}_* \simeq A_*^{\otimes m+1}[-nm]$. It follows that the homology groups of A'_*/A_* can be computed by means of a (convergent) spectral sequence $\{E_r^{p,q}, d_r\}_{r \geqslant 2}$ with

$$E_2^{p,q} \simeq \begin{cases} 0 & \text{if } p \leqslant 0 \\ (\mathrm{H}_*(A_*)^{\otimes p+1})_{q+p+np} & \text{if } p \geqslant 1. \end{cases}$$

Since A is coconnective and $n > 0$, the groups $E_2^{p,q}$ vanish unless $p + q \leqslant -np < 0$. It follows that each homology group $\mathrm{H}_m(A'_*/A_*)$ admits a finite filtration by subquotients of the vector spaces $E_2^{p,q}$ with $p + q = m$, each of which is finite dimensional (since A is locally finite), and that the groups $\mathrm{H}_m(A'_*/A_*)$ vanish for $m \geqslant 0$. Using the long exact seuqence

$$\cdots \to \mathrm{H}_m(A_*) \to \mathrm{H}_m(A'_*) \to \mathrm{H}_m(A'_*/A_*) \to \mathrm{H}_{m-1}(A_*) \to \cdots ,$$

we deduce that $\mathrm{H}_m(A'_*)$ is finite dimensional for all m and isomorphic to $\mathrm{H}_m(A_*)$ for $m \geqslant 0$, from which it follows immediately that $A' \in \mathbf{B}_0$.

We now complete the proof of Theorem 14.2.2.1 by showing that the weak deformation theory $\mathfrak{D}^{(1)}$ satisfies axiom $(D4)$ of Definition 12.3.3.2. For $n \geqslant 1$, and $A \in \mathrm{Alg}^{\mathrm{aug}}_\kappa$, we have a canonical homotopy equivalence

$$\begin{aligned} \Psi(A)(\Omega^{\infty-n}(E)) &= \mathrm{Map}_{\mathrm{Alg}^{\mathrm{aug}}_\kappa}(\mathfrak{D}^{(1)}(\kappa \oplus \Sigma^n(\kappa)), A) \\ &\simeq \mathrm{Map}_{\mathrm{Alg}^{\mathrm{aug}}_\kappa}(K_n, A) \\ &\simeq \Omega^{\infty-n-1}\,\mathrm{fib}(A \to \kappa). \end{aligned}$$

These maps determine an equivalence from the functor $e : \mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Sp}$ with $\Sigma(I)$, where $I : \mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Sp}$ denotes the functor which assigns to each augmented algebra $\epsilon : A \to \kappa$ its augmentation ideal $\mathrm{fib}(\epsilon)$. This functor is evidently conservative, and preserves sifted colimits by virtue of Proposition HA.3.2.3.1 . □

Remark 14.2.2.2. Let κ be a field, let $\epsilon : A \to \kappa$ be an object of $\mathrm{Alg}_\kappa^{\mathrm{aug}}$, and let $X = \Psi(A)$ denote the formal $\mathbb{E}_1$-moduli problem associated to A via the equivalence of Theorem 14.0.0.5. The proof of Theorem 14.2.2.1 shows that the shifted tangent complex $\Sigma^{-1} X(E) \in \mathrm{Sp}$ can be identified with the augmentation ideal $\mathrm{fib}(\epsilon)$ of A.

14.2.3 Application: Prorepresentability

We close this section by proving a noncommutative analogue of Proposition 13.3.3.1.

Proposition 14.2.3.1. *Let κ be a field and let $X : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ be a formal $\mathbb{E}_1$-moduli problem over κ. The following conditions are equivalent:*

(1) *The functor X is prorepresentable (see Definition 12.5.3.1).*

(2) *Let $X(E)$ denote the tangent complex of X. Then $\pi_i X(E) \simeq 0$ for $i > 0$.*

(3) *The functor X has the form $\Psi(A)$, where $A \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$ is coconnective and $\Psi : \mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Moduli}_\kappa^{(1)}$ is the equivalence of Theorem 14.0.0.5.*

Proof. The equivalence of (2) and (3) follows from Remark 14.2.2.2. We next prove that (1) ⇒ (2). Since the construction $X \mapsto X(E)$ commutes with filtered colimits, it will suffice to show that $\pi_i X(E) \simeq 0$ for $i > 0$ in the case when $X = \mathrm{Spf}\, R$ is representable by an object $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$. In this case, we can write $X = \Psi(A)$ where $A = \mathbb{D}^{(1)}(R)$ belongs to the full subcategory $\mathbf{B}_0 \subseteq \mathrm{Alg}_\kappa^{\mathrm{aug}}$ appearing in the proof of Theorem 14.2.2.1. In particular, A is coconnective, so that X satisfies condition (3) (and therefore also condition (2)).

We now complete the proof by showing that (3) ⇒ (1). Let $A \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$ be coconnective. Choose a representative of A by a cofibrant differential graded algebra $A_* \in \mathrm{Alg}_\kappa^{\mathrm{dg}}$. Since A_* is cofibrant, we may assume that the augmentation of A is determined by an augmentation of A_*. We now construct a sequence of differential graded algebras

$$\kappa = A(-1)_* \to A(0)_* \to A(1)_* \to A(2)_* \to \cdots$$

equipped with maps $\phi(i) : A(i)_* \to A_*$. For each $n < 0$, choose a graded subspace $V_n \subseteq A_n$ consisting of cycles which maps isomorphically onto the homology $\mathrm{H}_n(A_*)$. We regard V_* as a differential graded vector space with trivial differential (which vanishes in nonnegative degrees). Let $A(0)_*$ denote the free differential graded Lie algebra generated by V_*, and $\phi(0) : A(0)_* \to A_*$ the evident map. Assume now that we have constructed a map

$\phi(i) : A(i)_* \to A_*$ extending $\phi(1)$. Since A is coconnective, the map $\theta : \mathrm{H}_n(A(i)_*) \to \mathrm{H}_*(A_*)$ is surjective. Choose a collection of cycles $x_\alpha \in A(i)_{n_\alpha}$ whose images form a basis for $\ker(\theta)$. Then we can write $\phi(i)(x_\alpha) = dy_\alpha$ for some $y_\alpha \in A_{n_\alpha+1}$. Let $A(i+1)_*$ be the differential graded algebra obtained from $A(i)_*$ by freely adjoining elements Y_α (in degrees $n_\alpha + 1$) satisfying $dY_\alpha = x_\alpha$. We let $\phi(i+1) : A(i+1)_* \to A_*$ denote the unique extension of $\phi(i)$ satisfying $\phi(i+1)(Y_\alpha) = y_\alpha$.

We now prove the following assertion for each integer $i \geqslant 0$:

$(*_i)$ The inclusion $V_{-1} \hookrightarrow A(i)_{-1}$ induces an isomorphism $V_{-1} \to \mathrm{H}_{-1}(A(i)_*)$, the unit map $\kappa \to A(i)_0$ is an isomorphism, and $A(i)_j \simeq 0$ for $j > 0$.

Assertion $(*_i)$ is obvious when $i = 0$. Let us assume that $(*_i)$ holds, and let θ be defined as above. Then θ is an isomorphism in degrees $\geqslant -1$, so that $A(i+1)_*$ is obtained from $A(i)_*$ by freely adjoining generators Y_α in degrees $\leqslant -1$. It follows immediately that $A(i+1)_j \simeq 0$ for $j > 0$ and that the unit map $\kappa \to A(i+1)_0$ is an isomorphism. Moreover, we can write $A(i+1)_{-1} \simeq A(i)_{-1} \oplus W$, where W is the subspace spanned by elements of the form Y_α where $n_\alpha = -2$. By construction, the differential on $A(i+1)_*$ induces a monomorphism from W to the quotient $A(i)_{-2}/dA(i)_{-1} \subseteq A(i+1)_{-2}/dA(i)_{-1}$, so that the differential graded algebras $A(i+1)_*$ and $A(i)_*$ have the same homology in degree -1.

Let A'_* denote the colimit of the sequence $\{A(i)_*\}_{i \geqslant 0}$. The evident map $\mathfrak{g}'_* \to \mathfrak{g}_*$ is surjective on homology (since the map $A(0)_* \to \mathfrak{g}_*$ is surjective on homology). If $\eta \in \ker(\mathrm{H}_*(A'_*) \to \mathrm{H}_*(A_*))$, then η is represented by a class $\overline{\eta} \in \ker(\mathrm{H}_*(A(i)_*) \to \mathrm{H}_*(A_*))$ for $i \gg 0$. By construction, the image of $\overline{\eta}$ vanishes in $\mathrm{H}_*(A(i+1)_*)$, so that $\eta = 0$. It follows that the map $A'_* \to A_*$ is a quasi-isomorphism. Since the collection of quasi-isomorphisms in $\mathrm{Alg}^{\mathrm{dg}}_\kappa$ is closed under filtered colimits, we conclude that A_* is a homotopy colimit of the sequence $\{A(i)_*\}_{i \geqslant 0}$ in the model category $\mathrm{Alg}^{\mathrm{dg}}_\kappa$. Let $A(i) \in \mathrm{Alg}^{\mathrm{aug}}_\kappa$ be the image of the differential graded algebra $A(i)_*$ (equipped with the augmentation determined by the map $\phi(i) : A(i)_* \to A_*$), so that $A \simeq \varinjlim A(i)$ in $\mathrm{Alg}^{\mathrm{aug}}_\kappa$. Setting $X(i) = \Psi(A(i)_*) \in \mathrm{Moduli}^{(1)}_\kappa$, we deduce that $X \simeq \varinjlim X(i)$. To prove that X is prorepresentable, it will suffice to show that each $X(i)$ is prorepresentable.

We now proceed by induction on i, the case $i = -1$ being trivial. To carry out the inductive step, we note that each of the Lie algebras $A(i+1)_*$ is obtained from $A(i)_*$ by freely adjoining a set of generators $\{Y_\alpha\}_{\alpha \in S}$ of degrees $n_\alpha + 1 \leqslant -1$, satisfying $dY_\alpha = x_\alpha \in A(i)_{n_\alpha}$. Choose a well-ordering of the set S. For each $\alpha \in S$, we let $A^{<\alpha}_*$ denote the Lie subalgebra of $A(i+1)_*$ generated by $A(i)_*$ and the elements Y_β for $\beta < \alpha$, and let $A^{\leqslant \alpha}_*$ be defined similarly. Set $X^{<\alpha} = \Psi(A^{<\alpha}_*)$ and $X^{\leqslant\alpha} = \Psi(A^{\leqslant\alpha}_*)$. For each integer n, let $B(n)_*$ be the free differential graded algebra generated by a class x in degree n and $B'(n)_*$ the free differential graded algebra generated by a class x in degree n and a class y in degree $n+1$ satisfying

$dy = x$. For each $\alpha \in S$, we have a homotopy pushout diagram of differential graded algebras

$$\begin{array}{ccc} B(n)_* & \longrightarrow & B'(n)_* \\ \downarrow & & \downarrow \\ A_*^{<\alpha} & \longrightarrow & A_*^{\leqslant\alpha}, \end{array}$$

hence a pushout diagram diagram of formal $\mathbb{E}_1$-moduli problems

$$\begin{array}{ccc} \mathrm{Spf}(\kappa \oplus \Sigma^{n_\alpha+1}(\kappa)) & \longrightarrow & \mathrm{Spf}(\kappa) \\ \downarrow & & \downarrow \\ X^{<\alpha} & \longrightarrow & X^{\leqslant\alpha}. \end{array}$$

It follows that the map $X(i) \to X(i+1)$ satisfies the criterion of Lemma 12.5.3.4. Since $X(i)$ is prorepresentable, we conclude that $X(i+1)$ is prorepresentable. □

14.3 Comparison with the Commutative Case

For every field κ, the forgetful functor $\mathrm{CAlg}_\kappa \to \mathrm{Alg}_\kappa$ carries Artinian $\mathbb{E}_\infty$-algebras over κ to Artinian $\mathbb{E}_1$-algebras over κ and therefore induces a forgetful functor $\theta : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathrm{Alg}_\kappa^{\mathrm{art}}$. If $X : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ is a formal $\mathbb{E}_1$-moduli problem over κ, then the composition $(X \circ \theta) : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ is a formal moduli problem over κ. Consequently, composition with θ determines a functor $\phi : \mathrm{Moduli}_\kappa^{(1)} \to \mathrm{Moduli}_\kappa$.

Suppose now that the field κ has characteristic zero. Theorems 14.0.0.5 and 13.0.0.2 supply equivalences of ∞-categories

$$\mathrm{Alg}_\kappa^{\mathrm{aug}} \simeq \mathrm{Moduli}_\kappa^{(1)} \qquad \mathrm{Lie}_\kappa \simeq \mathrm{Moduli}_\kappa,$$

so that we can identify ϕ with a functor $\phi' : \mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Lie}_\kappa$. Our goal in this section is to give an explicit description of the functor ϕ'.

Recall that the ∞-category Alg_κ of $\mathbb{E}_1$-algebras over κ can be identified with the underlying ∞-category of the model category $\mathrm{Alg}_\kappa^{\mathrm{dg}}$ of differential graded algebras over κ (Proposition HA.7.1.4.6). Let $(\mathrm{Alg}_\kappa^{\mathrm{dg}})_{/\kappa}$ denote the category of *augmented* differential graded algebras over κ. Then $(\mathrm{Alg}_\kappa^{\mathrm{dg}})_{/\kappa}$ inherits a model structure, and (because $\kappa \in \mathrm{Alg}_\kappa^{\mathrm{dg}}$ is fibrant) the underlying ∞-category of $(\mathrm{Alg}_\kappa^{\mathrm{dg}})_{/\kappa}$ can be identified with $\mathrm{Alg}_\kappa^{\mathrm{aug}}$. For every object $\epsilon : A_* \to \kappa$ of $(\mathrm{Alg}_\kappa^{\mathrm{dg}})_{/\kappa}$, we let $\mathfrak{m}_{A_*} = \ker(\epsilon)$ denote the augmentation ideal of A_*. Then $\mathfrak{m}_{A_*}$ inherits the structure of a nonunital differential graded algebra over κ. In particular, we can view $\mathfrak{m}_{A_*}$ as a differential graded Lie algebra over κ (see Example 13.1.2.2). The construction $A_* \mapsto \mathfrak{m}_{A_*}$ determines a functor $(\mathrm{Alg}_\kappa^{\mathrm{dg}})_{/\kappa} \to \mathrm{Lie}_\kappa^{\mathrm{dg}}$, which carries quasi-isomorphisms to quasi-isomorphisms. We therefore obtain an induced functor of ∞-categories $\psi : \mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Lie}_\kappa$. We will prove that the functors $\psi, \phi' : \mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Lie}_\kappa$ are equivalent to one another. We can state this result more precisely as follows:

Theorem 14.3.0.1. *Let κ be a field of characteristic zero. The diagram of ∞-categories*

$$\begin{array}{ccc} \mathrm{Alg}_\kappa^{\mathrm{aug}} & \xrightarrow{\psi} & \mathrm{Lie}_\kappa \\ \downarrow & & \downarrow \\ \mathrm{Moduli}_\kappa^{(1)} & \xrightarrow{\phi} & \mathrm{Moduli}_\kappa \end{array}$$

commutes (up to canonical homotopy). Here ϕ and ψ are the functors described above, and the vertical maps are the equivalences provided by Theorems 13.0.0.2 and 14.0.0.5.

14.3.1 Comparison of Koszul Duality Functors

To prove Theorem 14.3.0.1, we need to construct a homotopy between two functors $\mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Moduli}_\kappa \subseteq \mathrm{Fun}(\mathrm{CAlg}_\kappa^{\mathrm{art}}, \mathcal{S})$. Equivalently, we must construct a homotopy between the functors $F, F' : \mathrm{Alg}_\kappa^{\mathrm{aug}} \times \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ given by

$$F(A, R) = \mathrm{Map}_{\mathrm{Lie}_\kappa}(\mathfrak{D}(R), \psi(A)) \qquad F'(A, R) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(R), A).$$

Composing the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Lie}_\kappa$ with the equivalence of ∞-categories $\mathrm{Lie}_\kappa \simeq \mathrm{Moduli}_\kappa$, we obtain the functor $\mathrm{Spf} : (\mathrm{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Moduli}_\kappa$ of Example 12.1.3.3. It follows from Yoneda's lemma that this functor is fully faithful when restricted to $(\mathrm{CAlg}_\kappa^{\mathrm{art}})^{\mathrm{op}}$, so that $\mathfrak{D}$ induces an equivalence from $(\mathrm{CAlg}_\kappa^{\mathrm{art}})^{\mathrm{op}}$ onto its essential image $\mathcal{C} \subseteq \mathrm{Lie}_\kappa$. The inverse of this equivalence is given by $\mathfrak{g}_* \mapsto C^*(\mathfrak{g}_*)$. It follows that we can identify F and F' with functors $G, G' : \mathrm{Alg}_\kappa^{\mathrm{aug}} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, given by the formulas

$$G(A, \mathfrak{g}_*) = \mathrm{Map}_{\mathrm{Lie}_\kappa}(\mathfrak{g}_*, \psi(A)) \qquad G'(A, \mathfrak{g}_*) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{\mathrm{aug}}}(\mathfrak{D}^{(1)} C^*(\mathfrak{g}_*), A).$$

Note that the forgetful functor $(\mathrm{Alg}_\kappa^{\mathrm{dg}})_{/\kappa} \to \mathrm{Lie}_\kappa^{\mathrm{dg}}$ is a right Quillen functor, with left adjoint given by the universal enveloping algebra construction $\mathfrak{g}_* \mapsto U(\mathfrak{g}_*)$ of Remark 13.1.2.3. It follows that the functor ψ admits a left adjoint $\mathrm{Lie}_\kappa \to \mathrm{Alg}_\kappa^{\mathrm{aug}}$, which we will also denote by U. Then the functor $G : \mathrm{Alg}_\kappa^{\mathrm{aug}} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ can be described by the formula $G(A, \mathfrak{g}_*) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{\mathrm{aug}}}(U(\mathfrak{g}_*), A)$. To show that this functor is equivalent to G', it will suffice to show that there is a canonical equivalence $U(\mathfrak{g}_*) \simeq \mathfrak{D}^{(1)} C^*(\mathfrak{g}_*)$ for $\mathfrak{g}_* \in \mathcal{C}$. Note that $\mathfrak{g}_* \in \mathcal{C}$ guarantees that $U(\mathfrak{g}_*)$ is coconnective and locally finite, so that the biduality map $U(\mathfrak{g}_*) \to \mathfrak{D}^{(1)}\mathfrak{D}^{(1)} U(\mathfrak{g}_*)$ is an equivalence (Corollary 14.1.3.2). It will therefore suffice to show that there exists a canonical equivalence of augmented $\mathbb{E}_1$-algebras $\mathfrak{D}^{(1)} U(\mathfrak{g}_*) \simeq C^*(\mathfrak{g}_*)$. In fact, such an equivalence can be defined for *all* $\mathfrak{g}_* \in \mathrm{Lie}_\kappa$:

Proposition 14.3.1.1. *Let κ be a field of characteristic zero. Then the diagram of ∞-categories*

$$\begin{array}{ccc} (\mathrm{Lie}_\kappa)^{\mathrm{op}} & \xrightarrow{C^*} & \mathrm{CAlg}_\kappa^{\mathrm{aug}} \\ \downarrow{\scriptstyle U} & & \downarrow \\ (\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} & \xrightarrow{\mathfrak{D}^{(1)}} & \mathrm{Alg}_\kappa^{\mathrm{aug}} \end{array}$$

commutes up to canonical homotopy.

14.3.2 Twisted Arrow ∞-Categories

Before giving our proof of Proposition 14.3.1.1, we need to recall a bit of terminology (see §HA.5.2.1).

Construction 14.3.2.1. Let $\mathcal{C}$ be a category. We define a new category $\mathrm{TwArr}(\mathcal{C})$ as follows:

(*a*) An object of $\mathrm{TwArr}(\mathcal{C})$ is given by a triple (C, D, ϕ), where $\phi : C \to D$ is a morphism in $\mathcal{C}$.

(*b*) Given a pair of objects $(C, D, \phi), (C', D', \phi') \in \mathrm{TwArr}(\mathcal{C})$, a morphism from (C, D, ϕ) to (C', D', ϕ') consists of a pair of morphisms $\alpha : C \to C'$, $\beta : D' \to D$ for which the diagram

$$\begin{array}{ccc} C & \xrightarrow{\phi} & D \\ \downarrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} \\ C' & \xrightarrow{\phi'} & D' \end{array}$$

commutes.

(*c*) Given a pair of morphisms

$$(C, D, \phi) \xrightarrow{(\alpha,\beta)} (C', D', \phi') \xrightarrow{(\alpha',\beta')} (C'', D'', \phi'')$$

in $\mathrm{TwArr}(\mathcal{C})$, the composition of (α', β') with (α, β) is given by $(\alpha' \circ \alpha, \beta \circ \beta')$.

We will refer to $\mathrm{TwArr}(\mathcal{C})$ as the *twisted arrow category* of $\mathcal{C}$. The construction $(C, D, \phi) \mapsto (C, D)$ determines a forgetful functor $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ which exhibits $\mathrm{TwArr}(\mathcal{C})$ as fibered in sets over $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ (the fiber of λ over an object $(C, D) \in \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ can be identified with the set $\mathrm{Hom}_{\mathcal{C}}(C, D)$). In particular, λ is a pairing of ∞-categories. This pairing is both left and right representable, and the associated duality functors

$$\mathfrak{D}_\lambda : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \qquad \mathfrak{D}'_\lambda : \mathcal{C} \to \mathcal{C}$$

are equivalent to the identity.

Proposition 14.3.2.2. *Let κ be a field of characteristic zero and let $\mathcal{M}^{(1)} \to \mathrm{Alg}^{\mathrm{aug}}_\kappa \times \mathrm{Alg}^{\mathrm{aug}}_\kappa$ be the pairing of ∞-categories of Construction 14.1.1.1. There exists a left representable map of pairings of ∞-categories*

$$\begin{array}{ccc} \mathrm{TwArr}(\mathrm{Lie}^{\mathrm{dg}}_\kappa) & \xrightarrow{T} & \mathcal{M}^{(1)} \\ \downarrow{\scriptstyle\lambda} & & \downarrow \\ \mathrm{Lie}^{\mathrm{dg}}_\kappa \times (\mathrm{Lie}^{\mathrm{dg}}_\kappa)^{\mathrm{op}} & \xrightarrow{U \times C^*} & \mathrm{Alg}^{\mathrm{aug}}_\kappa \times \mathrm{Alg}^{\mathrm{aug}}_\kappa . \end{array}$$

Here U and C^* denote the (covariant and contravariant) functors from $\mathrm{Lie}_\kappa^{\mathrm{dg}}$ *to* $\mathrm{Alg}_\kappa^{\mathrm{aug}}$ *induced by the universal enveloping algebra and cohomological Chevalley-Eilenberg constructions, respectively.*

We first observe that Proposition 14.3.2.2 implies Proposition 14.3.1.1:

Proof of Proposition 14.3.1.1. As noted in Construction 14.3.2.1, the pairing of ∞-categories

$$\mathrm{TwArr}(\mathrm{Lie}_\kappa^{\mathrm{dg}}) \to \mathrm{Lie}_\kappa^{\mathrm{dg}} \times (\mathrm{Lie}_\kappa^{\mathrm{dg}})^{\mathrm{op}}$$

induces the identity functor $\mathrm{id} : (\mathrm{Lie}_\kappa^{\mathrm{dg}})^{\mathrm{op}} \to (\mathrm{Lie}_\kappa^{\mathrm{dg}})^{\mathrm{op}}$. Applying Proposition **??** to the morphism of pairings T of Proposition 14.3.2.2, we obtain an equivalence between the functors $C^* \circ \mathrm{id}, \mathfrak{D}^{(1)} \circ U : (\mathrm{Lie}_\kappa^{\mathrm{dg}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{aug}}$. Since the canonical map

$$\mathrm{Fun}(\mathrm{Lie}_\kappa^{\mathrm{op}}, \mathrm{Alg}_\kappa^{\mathrm{op}}) \to \mathrm{Fun}((\mathrm{Lie}_\kappa^{\mathrm{dg}})^{\mathrm{op}}, \mathrm{Alg}_\kappa^{\mathrm{aug}})$$

is fully faithful, we obtain an equivalence between the functors $C^*, \mathfrak{D}^{(1)} \circ U : \mathrm{Lie}_\kappa^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{aug}}$. □

14.3.3 The Proof of Proposition 14.3.2.2

Let $\mathfrak{g}_*$ be a differential graded Lie algebra over κ and let $\mathrm{Cn}(\mathfrak{g})_*$ be as in Construction 13.2.1.1. The universal enveloping algebra $U(\mathrm{Cn}(\mathfrak{g})_*)$ has the structure of a (differential graded) Hopf algebra, where the comultiplication is determined by the requirement that the image of $\mathrm{Cn}(\mathfrak{g})_*$ consists of primitive elements. In particular, we have a counit map $\epsilon : \mathrm{Cn}(\mathfrak{g})_* \to \kappa$. Let $\mathrm{End}(U\,\mathrm{Cn}(\mathfrak{g})_*)$ denote the chain complex of $U(\mathrm{Cn}(\mathfrak{g})_*)$-comodule maps from $U(\mathrm{Cn}(\mathfrak{g})_*)$ to itself. Since $U(\mathrm{Cn}(\mathfrak{g})_*)$ is cofree as a comodule over itself, composition with the counit map $\epsilon : U(\mathrm{Cn}(\mathfrak{g})_*) \to \kappa$ induces an isomorphism θ from $\mathrm{End}(U\,\mathrm{Cn}(\mathfrak{g})_*)$ to the κ-linear dual of $U\,\mathrm{Cn}(\mathfrak{g})_*$. We regard $\mathrm{End}(U\,\mathrm{Cn}(\mathfrak{g})_*)$ as endowed with the *opposite* of the evident differential graded Lie algebra structure, so that $U\,\mathrm{Cn}(\mathfrak{g})_*$ has the structure of a right module over $\mathrm{End}(U\,\mathrm{Cn}(\mathfrak{g})_*)$. Let $\mathrm{End}_{\mathfrak{g}}(U\,\mathrm{Cn}(\mathfrak{g})_*)$ denote the subcomplex of $\mathrm{End}(U\,\mathrm{Cn}(\mathfrak{g})_*)$ consisting of right $U(\mathfrak{g}_*)$-module maps, so that θ restricts to an isomorphism from $\mathrm{End}_{\mathfrak{g}}(U\,\mathrm{Cn}(\mathfrak{g})_*)$ to the κ-linear dual $C^*(\mathfrak{g}_*)$ of $C_*(\mathfrak{g}_*) \simeq U\,\mathrm{Cn}(\mathfrak{g})_* \otimes_{U(\mathfrak{g}_*)} \kappa$. It is not difficult to verify that this isomorphism is compatible with the multiplication on $C^*(\mathfrak{g})$ described in Construction 13.2.5.1. It follows that $U\,\mathrm{Cn}(\mathfrak{g})_*$ is equipped with a right action of $C^*(\mathfrak{g}_*)$, which is compatible with the right action of $U(\mathfrak{g}_*)$ on $U\,\mathrm{Cn}(\mathfrak{g})_*$. Let $M_*(\mathfrak{g}_*)$ denote the κ-linear dual of $U\,\mathrm{Cn}(\mathfrak{g})_*$. Then M_* is a contravariant functor, which carries a differential graded Lie algebra $\mathfrak{g}_*$ to a chain complex equipped with commuting right actions of $U(\mathfrak{g}_*)$ and $C^*(\mathfrak{g}_*)$. Moreover, the unit map $\kappa \to UE(\mathfrak{g}_*)$ determines a quasi-isomorphism $\epsilon_{\mathfrak{g}_*} : M_*(\mathfrak{g}_*) \to \kappa$.

Note that the initial object $\kappa \in \mathrm{Alg}_\kappa^{(1)}$ can be identified with a classifying object for endomorphisms of the unit object $\kappa \in \mathrm{Mod}_\kappa$. Using Theorem HA.4.7.1.34 and Proposition

HA.4.7.1.39 , we can identify $\mathrm{Alg}^{\mathrm{aug}}_{\kappa}$ with the fiber product $\mathrm{LMod}(\mathrm{Mod}_{\kappa}) \times_{\mathrm{Mod}_{\kappa}} \{\kappa\}$. Let $\mathcal{X} \subseteq (\mathrm{Mod}_{\kappa})_{/\kappa}$ denote the full subcategory spanned by the final objects, so that we have an equivalence of ∞-categories

$$\alpha : \mathcal{M}^{(1)} \simeq (\mathrm{Alg}_{\kappa} \times \mathrm{Alg}_{\kappa}) \times_{\mathrm{Alg}_{\kappa}} \mathrm{LMod}(\mathrm{Mod}_{\kappa}) \times_{\mathrm{Mod}_{\kappa}} \mathcal{X}.$$

We define a more rigid analogue of $\mathcal{M}^{(1)}$ as follows: let $\mathcal{Y} \subseteq (\mathrm{Vect}^{\mathrm{dg}}_{\kappa})_{\kappa/}$ be the full subcategory spanned by the quasi-isomorphisms of chain complexes $V_* \to \kappa$ and let $\mathcal{C}$ denote the category

$$\mathrm{Alg}^{\mathrm{dg}}_{\kappa} \times \mathrm{Alg}^{\mathrm{dg}}_{\kappa} \times_{\mathrm{Alg}^{\mathrm{dg}}_{\kappa}} \mathrm{LMod}(\mathrm{Vect}^{\mathrm{dg}}_{\kappa}) \times_{\mathrm{Vect}^{\mathrm{dg}}_{\kappa}} \mathcal{Y},$$

so that α determines a functor $T'' : \mathcal{C} \to \mathcal{M}^{(1)}$. We will define T as a composition $\mathrm{TwArr}(\mathrm{Lie}^{\mathrm{dg}}_{\kappa}) \xrightarrow{T'} \mathcal{C} \xrightarrow{T''} \mathcal{M}^{(1)}$. Here the functor T' assigns to each map $\gamma : \mathfrak{h}_* \to \mathfrak{g}_*$ of differential graded Lie algebras the object of $\mathcal{C}$ given by $(U(\mathfrak{h}_*), C^*(\mathfrak{g}_*), M_*(\mathfrak{g}_*), \epsilon_{\mathfrak{g}_*})$, where $M_*(\mathfrak{g}_*)$ is regarded as a left module over $U(\mathfrak{h}_*) \otimes_{\kappa} C^*(\mathfrak{g}_*)$ by combining the commuting left actions of $U(\mathfrak{g}_*)$ and $C^*(\mathfrak{g}_*)$ on $M_*(\mathfrak{g}_*)$ (and composing with the map γ).

We now claim that the diagram σ :

$$\begin{array}{ccc} \mathrm{TwArr}(\mathrm{Lie}^{\mathrm{dg}}_{\kappa}) & \xrightarrow{T} & \mathcal{M}^{(1)} \\ \downarrow{\lambda} & & \downarrow \\ \mathrm{Lie}^{\mathrm{dg}}_{\kappa} \times (\mathrm{Lie}^{\mathrm{dg}}_{\kappa})^{\mathrm{op}} & \xrightarrow{U \times C^*} & \mathrm{Alg}^{\mathrm{aug}}_{\kappa} \times \mathrm{Alg}^{\mathrm{aug}}_{\kappa} \end{array}$$

commutes up to canonical homotopy. Consider first the composition of T with the map $\mathcal{M}^{(1)} \to \mathrm{Alg}^{\mathrm{aug}}_{\kappa}$ given by projection onto the first factor. Unwinding the definitions, we see that this map is given by the composing the equivalence $\xi : \mathrm{LMod}(\mathrm{Mod}_{\kappa}) \times_{\mathrm{Mod}_{\kappa}} \mathcal{X} \simeq \mathrm{Alg}^{\mathrm{aug}}_{\kappa}$ with the functor $T'_0 : \mathrm{TwArr}(\mathrm{Lie}^{\mathrm{dg}}_{\kappa}) \to \mathrm{LMod}(\mathrm{Mod}_{\kappa}) \times_{\mathrm{Mod}_{\kappa}} \mathcal{X}$ given by

$$T'_0(\gamma : \mathfrak{h}_* \to \mathfrak{g}_*) = (U(\mathfrak{h}_*), M_*(\mathfrak{g}_*), \epsilon_{\mathfrak{g}_*}).$$

The counit map $U(\mathfrak{g}_*) \to \kappa$ determines a quasi-isomorphism of $U(\mathfrak{h}_*)$-modules $\kappa \to M_*(\mathfrak{g}_*)$, so that T'_0 is equivalent to the functor $\overline{T}'_0$ given by $\overline{T}'_0(\gamma : \mathfrak{h}_* \to \mathfrak{g}_*) = (U(\mathfrak{h}_*), \kappa, \mathrm{id}_{\kappa})$, which (after composing with ξ) can be identified with the map $\mathrm{TwArr}(\mathrm{Lie}^{\mathrm{dg}}_{\kappa}) \to \mathrm{Lie}^{\mathrm{dg}}_{\kappa} \xrightarrow{U} \mathrm{Alg}^{\mathrm{aug}}_{\kappa}$. Now consider the composition of T with the map $\mathcal{M}^{(1)} \to \mathrm{Alg}^{\mathrm{aug}}_{\kappa}$ given by projection onto the second factor. This functor is given by composing the equivalence ξ with the functor $T'_1 : \mathrm{TwArr}(\mathrm{Lie}^{\mathrm{dg}}_{\kappa}) \to \mathrm{LMod}(\mathrm{Mod}_{\kappa}) \times_{\mathrm{Mod}_{\kappa}} \mathcal{X}$ given by $T'_1(\gamma : \mathfrak{h}_* \to \mathfrak{g}_*) = (C^*(\mathfrak{g}), M_*(\mathfrak{g}_*), \epsilon_{\mathfrak{g}_*})$. Note that $\epsilon_{\mathfrak{g}}$ is a map of $C^*(\mathfrak{g})$-modules and therefore determines an equivalence of T'_1 with the functor $\overline{T}'_1$ given by $\overline{T}'_1(\gamma : \mathfrak{h}_* \to \mathfrak{g}_*) = (C^*(\mathfrak{g}), \kappa, \mathrm{id}_{\kappa})$. It follows that the composition of T'_1 with ξ can be identified with the composition

$$\mathrm{TwArr}(\mathrm{Lie}^{\mathrm{dg}}_{\kappa}) \to (\mathrm{Lie}^{\mathrm{dg}}_{\kappa})^{\mathrm{op}} \xrightarrow{C^*} \mathrm{Alg}^{\mathrm{aug}}_{\kappa}.$$

This proves the homotopy commutativity of the diagram σ. After replacing T by an equivalent functor, we can assume that the diagram σ is commutative.

It remains to show that σ determines a left representable map between pairings of ∞-categories. Let $\mathfrak{g}_*$ be a differential graded Lie algebra, and let $\mathrm{End}(M_*(\mathfrak{g}_*))$ denote the differential graded algebra of endomorphisms of the chain complex $M_*(\mathfrak{g}_*)$. Since $M_*(\mathfrak{g}_*)$ is quasi-isomorphic to κ, the unit map $\kappa \to \mathrm{End}(M_*(\mathfrak{g}_*))$ is a quasi-isomorphism of differential graded algebras. Unwinding the definitions, we must show that the map $\theta : U(\mathfrak{g}_*) \otimes_\kappa C^*(\mathfrak{g}_*) \to \mathrm{End}(M_*(\mathfrak{g}))$ exhibits $C^*(\mathfrak{g}_*)$ as Koszul dual (as an $\mathbb{E}_1$-algebra) to $U(\mathfrak{g}_*)$. Let A_* denote the differential graded algebra of endomorphisms of $U\,\mathrm{Cn}(\mathfrak{g})_*$ (as a chain complex). Then θ factors as a composition

$$U(\mathfrak{g}_*) \otimes_\kappa C^*(\mathfrak{g}_*) \xrightarrow{\theta'} A_* \xrightarrow{\theta''} \mathrm{End}(M_*(\mathfrak{g}_*))$$

where θ'' is a quasi-isomorphism. It will therefore suffice to show that θ' exhibits $C^*(\mathfrak{g}_*)$ as Koszul dual to $U(\mathfrak{g}_*)$. Since $UE_*(\mathfrak{g}_*)$ is a free $U(\mathfrak{g}_*)$-module, this is equivalent to the requirement that θ' induces a quasi-isomorphism $\phi : C^*(\mathfrak{g}_*) \to W_*$, where W_* is the differential graded algebra of right $U(\mathfrak{g}_*)$-module maps from $U\,\mathrm{Cn}(\mathfrak{g})_*$ to itself. This is clear, since ϕ admits a left inverse which given by composition with the quasi-isomorphism $U\,\mathrm{Cn}(\mathfrak{g})_* \to \kappa$.

14.4 Digression: Opposites of Cartesian Fibrations

For every ∞-category $\mathcal{C}$, the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$ is again an ∞-category. For many purposes, it is convenient to have a relative version of this construction. Suppose we are given a Cartesian fibration of simplicial sets $q : X \to S$, so that the fibers X_s of q are ∞-categories. Can we construct a new Cartesian fibration $q' : X' \to S$, whose fibers are X'_s are equivalent to the opposites of the ∞-categories X_s? In this section, we will consider two approaches to this question, the first abstract and the second more concrete.

14.4.1 Opposites of ∞-Categories

As explained in §HTT.3.3.2 , homotopy equivalence classes of Cartesian fibrations $q : X \to S$ are in bijection with homotopy classes of maps $S^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$. By general nonsense, it follows that every (functorial) procedure for converting one Cartesian fibration $q : X \to S$ into another Cartesian fibration $q' : X' \to S$ is given by a functor $\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$. Consequently, our problem is equivalent to that of showing there is a functor $\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ which carries each ∞-category to its opposite. The following theorem of Toën asserts that such a functor exists and is unique up to contractible ambiguity:

Theorem 14.4.1.1. *[Toën] Let $\mathcal{E}$ denote the full subcategory of* $\mathrm{Fun}(\mathcal{C}\mathrm{at}_\infty, \mathcal{C}\mathrm{at}_\infty)$ *spanned by those functors which are equivalences. Then $\mathcal{E}$ is equivalent to the (nerve of the) discrete category $\{\mathrm{id}, r\}$, where $r : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ is a functor which associates to every ∞-category its opposite.*

To prove Theorem 14.4.1.1, we will study an autoequivalence of $\mathcal{C}\mathrm{at}_\infty$ by analyzing its restriction to the full subcategory spanned by objects of the form $\mathrm{N}(P)$, where P is a partially ordered set. The first step is to show that this full subcategory admits an intrinsic description:

Proposition 14.4.1.2. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is equivalent to the nerve of a partially ordered set P.*

(2) *For every ∞-category $\mathcal{D}$ and every pair of functors $F, F' : \mathcal{D} \to \mathcal{C}$ such that $F(x) \simeq F'(x)$ for each object $x \in \mathcal{D}$, the functors F and F' are equivalent as objects of* $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

(3) *For every ∞-category $\mathcal{D}$, the map of sets*

$$\pi_0 \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{D}, \mathcal{C}) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(\pi_0 \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\Delta^0, \mathcal{D}), \pi_0 \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\Delta^0, \mathcal{C}))$$

is injective.

Proof. The implication (1) ⇒ (2) is obvious, and (3) is just a restatement of (2). Assume (2); we will show that (1) is satisfied. Let P denote the collection of equivalence classes of objects of $\mathcal{C}$, where $x \leqslant y$ if the space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is nonempty. There is a canonical functor $\mathcal{C} \to \mathrm{N}(P)$. To prove that this functor is an equivalence, it will suffice to show the following:

($*$) For every pair of objects $x, y \in \mathcal{C}$, the space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is either empty or contractible.

To prove ($*$), we may assume without loss of generality that $\mathcal{C}$ is the nerve of a fibrant simplicial category $\overline{\mathcal{C}}$. Let x and y be objects of $\overline{\mathcal{C}}$ such that the Kan complex $K = \mathrm{Map}_{\overline{\mathcal{C}}}(x, y)$ is nonempty. We define a new (fibrant) simplicial category $\overline{\mathcal{D}}$ so that $\overline{\mathcal{D}}$ consists of a pair of objects $\{x', y'\}$, with

$$\mathrm{Map}_{\overline{\mathcal{D}}}(x', y') \simeq K \quad \mathrm{Map}_{\overline{\mathcal{D}}}(x', x') \simeq \Delta^0 \simeq \mathrm{Map}_{\overline{\mathcal{D}}}(y', y') \quad \mathrm{Map}_{\overline{\mathcal{D}}}(y', x') \simeq \varnothing.$$

We let $\overline{F}, \overline{F}' : \overline{\mathcal{D}} \to \overline{\mathcal{C}}$ be simplicial functors such that $\overline{F}(x') = \overline{F}'(x') = x$, $\overline{F}(y') = \overline{F}'(y') = y$, where $\overline{F}$ induces the identity map from $\mathrm{Map}_{\overline{\mathcal{D}}}(x', y') = K = \mathrm{Map}_{\overline{\mathcal{C}}}(x, y)$ to itself, while $\overline{F}'$ induces a constant map from K to itself. Then $\overline{F}$ and $\overline{F}'$ induce functors F and F' from $\mathrm{N}(\overline{\mathcal{D}})$ to $\mathcal{C}$. It follows from assumption (2) that the functors F and F' are equivalent, which implies that the identity map from K to itself is homotopic to a constant map; this proves that K is contractible. □

Corollary 14.4.1.3. *Let $\sigma : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ be an equivalence of ∞-categories, and let $\mathcal{C}$ be an ∞-category (which we regard as an object of $\mathcal{C}\mathrm{at}_\infty$). Then $\mathcal{C}$ is equivalent to the nerve of a partially ordered set if and only if $\sigma(\mathcal{C})$ is equivalent to the nerve of a partially ordered set.*

Lemma 14.4.1.4. *Let $\sigma, \sigma' \in \{\mathrm{id}_{\mathbf{\Delta}}, r\} \subseteq \mathrm{Fun}(\mathbf{\Delta}, \mathbf{\Delta})$, where r denotes the reversal functor from $\mathbf{\Delta}$ to itself. Then*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathbf{\Delta},\mathbf{\Delta})}(\sigma, \sigma') = \begin{cases} \emptyset & \textit{if } \sigma = \sigma' \\ \{\mathrm{id}\} & \textit{if } \sigma = \sigma'. \end{cases}$$

Proof. Note that σ and σ' are both the identity at the level of objects. Let $\alpha : \sigma \to \sigma'$ be a natural transformation. Then, for each $n \geqslant 0$, $\alpha_{[n]}$ is a map from $[n]$ to itself. We claim that $\alpha_{[n]}$ is given by the formula

$$\alpha_{[n]}(i) = \begin{cases} i & \text{if } \sigma = \sigma' \\ n - i & \text{if } \sigma \neq \sigma'. \end{cases}$$

To prove this, we observe that a choice of $i \in [n]$ determines a map $[0] \to [n]$, which allows us to reduce to the case $n = 0$ (where the result is obvious) by functoriality.

It follows from the above argument that the natural transformation α is uniquely determined, if it exists. Moreover, α is a well-defined natural transformation if and only if each $\alpha_{[n]}$ is an order-preserving map from $[n]$ to itself; this is true if and only if $\sigma = \sigma'$. □

Proposition 14.4.1.5. *Let $\mathcal{P}$ denote the category of partially ordered sets, and let $\sigma : \mathcal{P} \to \mathcal{P}$ be an equivalence of categories. Then σ is isomorphic either to the identity functor $\mathrm{id}_{\mathcal{P}}$ or the functor r which carries every partially ordered set X to the same set with the opposite ordering.*

Proof. Since σ is an equivalence of categories, it carries the final object $[0] \in \mathcal{P}$ to itself (up to canonical isomorphism). It follows that for every partially ordered set X, we have a canonical bijection of sets

$$\eta_X : X \simeq \mathrm{Hom}_{\mathcal{P}}([0], X) \simeq \mathrm{Hom}_{\mathcal{P}}(\sigma([0]), \sigma(X)) \simeq \mathrm{Hom}_{\mathcal{P}}([0], \sigma(X)) \simeq \sigma(X).$$

We next claim that $\sigma([1])$ is isomorphic to $[1]$ as a partially ordered set. Since $\eta_{[1]}$ is bijective, the partially ordered set $\sigma([1])$ has precisely two elements. Thus $\sigma([1])$ is isomorphic either to $[1]$ or to a partially ordered set $\{x, y\}$ with two elements, neither larger than the other. In the second case, the set $\mathrm{Hom}_{\mathcal{P}}(\sigma([1]), \sigma([1]))$ has four elements. This is impossible, since σ is an equivalence of categories and $\mathrm{Hom}_{\mathcal{P}}([1], [1])$ has only three elements. Let $\alpha : \sigma([1]) \to [1]$ be an isomorphism (automatically unique, since the ordered set $[1]$ has no automorphisms in $\mathcal{P}$).

The map $\alpha \circ \eta_{[1]}$ is a bijection from the set $[1]$ to itself. We will assume that this map is the identity, and prove that σ is isomorphic to the identity functor $\mathrm{id}_{\mathcal{P}}$. The same argument, applied to $\sigma \circ r$, will show that if $\alpha \circ \eta_{[1]}$ is not the identity, then σ is isomorphic to r.

To prove that σ is equivalent to the identity functor, it will suffice to show that for every partially ordered set X, the map η_X is an isomorphism of partially ordered sets. In other words, we must show that both η_X and η_X^{-1} are maps of partially ordered sets. We will prove that η_X is a map of partially ordered sets; the same argument, applied to an inverse to the equivalence σ, will show that η_X^{-1} is a map of partially ordered sets. Let $x, y \in X$ satisfy $x \leqslant y$; we wish to prove that $\eta_X(x) \leqslant \eta_X(y)$ in $\sigma(X)$. The pair (x, y) defines a map of partially ordered sets $[1] \to X$. By functoriality, we may replace X by $[1]$, and thereby reduce to the problem of proving that $\eta_{[1]}$ is a map of partially ordered sets. This follows from our assumption that $\alpha \circ \eta_{[1]}$ is the identity map. □

Proof of Theorem 14.4.1.1. Let $\mathcal{C}$ be the full subcategory of $\mathcal{C}\mathrm{at}_\infty$ spanned by those ∞-categories which are equivalent to the nerves of partially ordered sets, and let $\mathcal{C}^0$ denote the full subcategory of $\mathcal{C}$ spanned by the objects $\{\Delta^n\}_{n \geqslant 0}$. Corollary 14.4.1.3 implies that every object $\sigma \in \mathcal{E}$ restricts to an equivalence from $\mathcal{C}$ to itself. According to Proposition 14.4.1.5, $\sigma|_{\mathcal{C}}$ is equivalent either to the identity functor, or to the restriction $r|_{\mathcal{C}}$. In either case, we conclude that σ also induces an equivalence from $\mathcal{C}^0$ to itself.

It follows from Proposition HA.A.7.10 that the full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}\mathrm{at}_\infty$ is dense (see Definition 20.4.1.7), so that the restriction functor $\mathcal{E} \to \mathrm{Fun}(\mathcal{C}^0, \mathcal{C}^0)$ is fully faithful (see Remark 20.4.1.6). In particular, any object $\sigma \in \mathcal{E}$ is determined by the restriction $\sigma|_{\mathcal{C}}$, so that σ is equivalent to either id or r by virtue of Proposition 14.4.1.5. Since $\mathcal{C}^0$ is equivalent to the nerve of the category $\mathbf{\Delta}$, Lemma 14.4.1.4 implies the existence of a fully faithful embedding from $\mathcal{E}$ to the nerve of the discrete category $\{\mathrm{id}, r\}$. To complete the proof, it will suffice to show that this functor is essentially surjective. In other words, we must show that there exists a functor $R : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ whose restriction to $\mathcal{C}$ is equivalent to r.

To carry out the details, it is convenient to replace $\mathcal{C}\mathrm{at}_\infty$ by an equivalent ∞-category with a slightly more elaborate definition. Recall that $\mathcal{C}\mathrm{at}_\infty$ is defined to be the simplicial nerve of a simplicial category $\mathcal{C}\mathrm{at}_\infty^\Delta$, whose objects are ∞-categories, where $\mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty^\Delta}(X, Y)$ is the largest Kan complex contained in $\mathrm{Fun}(X, Y)$. We would like to define R to be induced by the functor $X \mapsto X^{\mathrm{op}}$, but this is not a simplicial functor from $\mathcal{C}\mathrm{at}_\infty^\Delta$ to itself; instead we have a canonical isomorphism $\mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty^\Delta}(X^{\mathrm{op}}, Y^{\mathrm{op}}) \simeq \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty^\Delta}(X, Y)^{\mathrm{op}}$. However, if we let $\mathcal{C}\mathrm{at}_\infty^{\mathcal{T}\mathrm{op}}$ denote the topological category obtained by geometrically realizing the morphism spaces in $\mathcal{C}\mathrm{at}_\infty^\Delta$, then i induces an autoequivalence of $\mathcal{C}\mathrm{at}_\infty^{\mathcal{T}\mathrm{op}}$ as a topological category (via the natural homeomorphisms $|K| \simeq |K^{\mathrm{op}}|$, which is defined for every simplicial set K). We now define $\mathcal{C}\mathrm{at}'_\infty$ to be the topological nerve of $\mathcal{C}\mathrm{at}_\infty^{\mathcal{T}\mathrm{op}}$ (see Definition HTT.1.1.5.5). Then $\mathcal{C}\mathrm{at}'_\infty$ is an ∞-category equipped with a canonical equivalence $\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}'_\infty$, and the involution i induces an involution I on $\mathcal{C}\mathrm{at}'_\infty$, which carries each object $\mathcal{D} \in \mathcal{C}\mathrm{at}'_\infty$ to the opposite ∞-category $\mathcal{D}^{\mathrm{op}}$. We now define R to be the composition

$$\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}'_\infty \xrightarrow{I} \mathcal{C}\mathrm{at}'_\infty \to \mathcal{C}\mathrm{at}_\infty,$$

where the last map is a homotopy inverse to the equivalence $\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}'_\infty$. It is easy to see that R has the desired properties (moreover, we note that for every object $\mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$, the image $R\mathcal{D}$ is canonically equivalent with the opposite ∞-category $\mathcal{D}^{\mathrm{op}}$). □

14.4.2 Duals of Cartesian Fibrations

Let $q : X \to S$ be a Cartesian fibration of simplicial sets, let $\chi : S^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ be a diagram classifying q, and let $r : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ be the nontrivial equivalence of Theorem 14.4.1.1. Then $r \circ \chi$ classifies a Cartesian fibration $q' : X' \to S$, whose fibers are equipped with equivalences $X'_s \simeq X_s^{\mathrm{op}}$. The Cartesian fibration q' is determined uniquely up to equivalence. However, for many applications it is useful to have a more concrete description of X' (which is well-defined up to isomorphism rather than equivalence). We now sketch such a construction (for an alternative approach having a more combinatorial flavor, we refer the reader to [14]).

Construction 14.4.2.1. Let $p : X \to S$ be a map of simplicial sets. We define a new simplicial set $\mathrm{Dl}(p)$ equipped with a map $\mathrm{Dl}(p) \to S$ so that the following universal property is satisfied: for every map of simplicial sets $K \to S$, we have a bijection

$$\mathrm{Hom}_{(\mathcal{S}\mathrm{et}_\Delta)_{/S}}(K, \mathrm{Dl}(p)) \simeq \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(K \times_S X, \mathcal{S}).$$

Note that for each vertex $s \in S$, the fiber $\mathrm{Dl}(p)_s = \mathrm{Dl}(p) \times_S \{s\}$ is canonically isomorphic to the presheaf ∞-category $\mathrm{Fun}(X_s, \mathcal{S})$.

Assume that p is an inner fibration. We let $\mathrm{Dl}^0(p)$ the full simplicial subset of $\mathrm{Dl}(p)$ spanned by those vertices which correspond to corepresentable functors $X_s \to \mathcal{S}$, for some $s \in S$. If each of the ∞-categories X_s admits finite limits, we let $\mathrm{Dl}^{\mathrm{lex}}(p)$ denote the full simplicial subset of $\mathrm{Dl}(p)$ spanned by those vertices which correspond to left exact functors $X_s \to \mathcal{S}$, for some vertex $s \in S$,

Remark 14.4.2.2. Let $p : X \to S$ be an inner fibration and assume that each of the fibers X_s is an ∞-category which admits finite limits. Then for each vertex $s \in S$, we have a canonical isomorphism $\mathrm{Dl}^{\mathrm{lex}}(p)_s \simeq \mathrm{Ind}(X_s^{\mathrm{op}})$.

The next result summarizes some of the essential features of Construction 14.4.2.1.

Proposition 14.4.2.3. *Let $p : X \to S$ be a map of simplicial sets. Then:*

(1) *If p is a Cartesian fibration, then the map $\mathrm{Dl}(p) \to S$ is a coCartesian fibration. Moreover, for every edge $e : s \to s'$ in S, the induced functor*

$$\mathrm{Fun}(X_s, \mathcal{S}) \simeq \mathrm{Dl}(p)_s \to \mathrm{Dl}(p)_{s'} \simeq \mathrm{Fun}(X_{s'}, \mathcal{S})$$

is given by composition with the pullback functor $e^* : X_{s'} \to X_s$ determined by p.

(2) *If p is a coCartesian fibration, then the map $\mathrm{Dl}(p) \to S$ is a Cartesian fibration. Moreover, for every edge $e : s \to s'$ in S, the induced functor*

$$\mathrm{Fun}(X_{s'}, \mathcal{S}) \simeq \mathrm{Dl}(p)_{s'} \to \mathrm{Dl}(p)_s \simeq \mathrm{Fun}(X_s, \mathcal{S})$$

is given by composition with the functor $e_! : X_s \to X_{s'}$ determined by p.

(3) *Suppose p is a Cartesian fibration, that each fiber X_s of p admits finite limits and that for every edge $e : s \to s'$ in S, the pullback functor $e^* : X_{s'} \to X_s$ is left exact. Then the map $\mathrm{Dl}^{\mathrm{lex}}(p) \to S$ is a coCartesian fibration. Moreover, for every edge $e : s \to s'$ in S, the induced functor*

$$\mathrm{Ind}(X_s^{\mathrm{op}}) \simeq \mathrm{Dl}^{\mathrm{lex}}(p)_s \to \mathrm{Dl}^{\mathrm{lex}}(p)_{s'} \simeq \mathrm{Ind}(X_{s'}^{\mathrm{op}})$$

is given by composition with the pullback functor e^*.

(4) *If p is a coCartesian fibration, then the canonical map $q : \mathrm{Dl}(p) \to S$ is a coCartesian fibration, which restricts to a coCartesian fibration $\mathrm{Dl}^0(p) \to S$. If each fiber X_s admits finite limits, then q also restricts to a coCartesian fibration $\mathrm{Dl}^{\mathrm{lex}}(p) \to S$.*

Proof. Assertion (1) and (3) follow from Corollary HTT.3.2.2.12 . The implication (1) ⇒ (2) is immediate. We now prove (4). Assume that p is a coCartesian fibration. Then (2) implies that $\mathrm{Dl}(p) \to S$ is a Cartesian fibration, and that each edge $e : s \to s'$ induces a pullback functor $\mathrm{Dl}(p)_{s'} \to \mathrm{Dl}(p)_s$ which preserves small limits and filtered colimits Using Corollary HTT.5.5.2.9 , we deduce that this pullback functor admits a left adjoint $\mathrm{Fun}(X_s, \mathcal{S}) \to \mathrm{Fun}(X_{s'}, \mathcal{S})$, which is given by left Kan extension along the functor $e_! : X_s \to X_{s'}$. Corollary HTT.5.2.2.5 implies that the forgetful functor $q : \mathrm{Dl}(p) \to S$ is also a coCartesian fibration. Since the operation of left Kan extension carries corepresentable functors to corepresentable functors, we conclude that q restricts to a coCartesian fibration $q_0 : \mathrm{Dl}^0(p) \to S$ (and that a morphism in $\mathrm{Dl}^0(p)$ is q_0-coCartesian if and only if it is q-coCartesian). Now suppose that each fiber X_s of p admits finite limits. For each $s \in S$, the ∞-category $\mathrm{Dl}^{\mathrm{lex}}(p)_s = \mathrm{Ind}(X_s^{\mathrm{op}})$ can be identified with the full subcategory of $\mathrm{Dl}(p)_s = \mathcal{P}(X_s^{\mathrm{op}})$ generated by $\mathrm{Dl}^0(p)_s$ under filtered colimits. If $e : s \to s'$ is an edge of S, then the functor $e_! : \mathrm{Dl}(p)_s \to \mathrm{Dl}(p)_{s'}$ preserves small filtered colimits and carries $\mathrm{Dl}^0(p)_s$ into $\mathrm{Dl}^0(p)_{s'}$, and therefore carries $\mathrm{Dl}^{\mathrm{lex}}(p)_s$ into $\mathrm{Dl}^{\mathrm{lex}}(p)_{s'}$. It follows that q restricts to a coCartesian fibration $\mathrm{Dl}^{\mathrm{lex}}(p) \to S$. □

Proposition 14.4.2.4. *Let $q : X \to S$ be a Cartesian fibration of simplicial sets with essentially small fibers, so that q is classified by a functor $\chi : S^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$, and let $\chi' : S^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ classify the induced Cartesian fibration $q' : \mathrm{Dl}^0(q) \to S$. Then $\chi' \simeq \chi \circ r$, where $r : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ is the functor which associates to each ∞-category its opposite.*

Proof. It suffices to treat the universal case where $S = \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}}$ and χ is the identity map. Then we can identify χ' with a functor $\mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}}$; we wish to show that χ' is equivalent to r. We first show that χ'^2 is equivalent to the identity. For this, it suffices to construct an equivalence $X \to \mathrm{Dl}^0(q')$ of Cartesian fibrations over S; this equivalence is classified by the tautological map

$$X \times_S \mathrm{Dl}^0(q) \hookrightarrow X \times_S \mathrm{Dl}(q) \to \mathcal{S}.$$

It follows that χ' is an equivalence of $\mathcal{C}\mathrm{at}_\infty^{\mathrm{op}}$ with itself. Note that $\chi'(s) \simeq \mathrm{Dl}(q)^0_s \simeq X_s^{\mathrm{op}}$ for each $s \in S$. Choosing $s \in S$ so that X_s is not equivalent to its opposite, we conclude that χ' is not equivalent to the identity functor. Using Theorem 14.4.1.1, we deduce that χ' is equivalent to r. □

14.4.3 The Compactly Generated Case

We conclude this section with a result which will be useful in §14.5:

Proposition 14.4.3.1. *Let $p : X \to S$ be a coCartesian fibration of simplicial sets. Assume that:*

- *(a) For each $s \in S$, the fiber X_s is compactly generated.*
- *(b) For every morphism edge $e : s \to s'$ in S, the induced functor $X_s \to X_{s'}$ preserves compact objects and small colimits.*

Let X^c denote the full simplicial subset of X spanned by those vertices which are compact objects of X_s for some $s \in S$. Let $p^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ be the induced map between opposite simplicial sets, and let $p_c^{\mathrm{op}} : (X^c)^{\mathrm{op}} \to S^{\mathrm{op}}$ be the restriction of p^{op}. Then the restriction map $\phi : \mathrm{Dl}^0(p^{\mathrm{op}}) \subseteq \mathrm{Dl}^{\mathrm{lex}}(p^{\mathrm{op}}) \to \mathrm{Dl}^{\mathrm{lex}}(p_c^{\mathrm{op}})$ is an equivalence of coCartesian fibrations over S^{op}.

Proof. It follows from (a) and (b) that each of the functors $e_! : X_s \to X_{s'}$ admits a right adjoint e^*, so that p is a Cartesian fibration and therefore p^{op} is a coCartesian fibration. Applying Proposition 14.4.2.3, we conclude that the projection map $q : \mathrm{Dl}^0(p^{\mathrm{op}}) \to S^{\mathrm{op}}$ is a coCartesian fibration. It follows from (b) that the projection $X^c \to S$ is a coCartesian fibration whose fibers admit finite colimits, and that for every edge $e : s \to s'$ in S the induced functor $X^c_s \to X^c_{s'}$ preserves finite colimits. Applying Proposition 14.4.2.3 again, we deduce that the map $q' : \mathrm{Dl}^{\mathrm{lex}}(p_c^{\mathrm{op}}) \to S^{\mathrm{op}}$ is a coCartesian fibration. We next claim that ϕ

carries q-coCartesian morphisms to q'-coCartesian morphisms. Unwinding the definitions, this amounts to the following claim: if $\overline{e} : x \to x'$ is a p-Cartesian edge lifting $e : s \to s'$, then $\overline{e}$ induces an equivalence $h_{x'} \circ e_! \to h_x$ of functors $X_s^{\mathrm{op}} \to \mathcal{S}$, where $h_x : X_s^{\mathrm{op}} \to \mathcal{S}$ is the functor represented by x and $h_{x'} : X_{s'}^{\mathrm{op}} \to \mathcal{S}$ is the functor represented by x'. This is an immediate consequence of the definitions.

To complete the proof, it will suffice to show that for every vertex $s \in S$, the functor ϕ induces an equivalence of ∞-categories $\mathrm{Dl}^0(p^{\mathrm{op}})_s \to \mathrm{Dl}^{\mathrm{lex}}(p_c^{\mathrm{op}})$. That is, we must show that the composite functor

$$\psi : X_s \to \mathrm{Fun}(X_s^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}((X_s^c)^{\mathrm{op}}, \mathcal{S}) = \mathrm{Ind}(X_s^c)$$

is an equivalence of ∞-categories. This is clear, since ψ is right adjoint to the canonical map $\mathrm{Ind}(X_s^c) \to X_s$ (which is an equivalence by virtue of our assumption that X_s is compactly generated). □

14.5 Quasi-Coherent and Ind-Coherent Sheaves

Let κ be a field and let $X : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ be a formal moduli problem. In §13.4, we introduced a symmetric monoidal ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X. Our goal in this section is to study analogous definitions in the noncommutative setting. In this case, it is important to distinguish between left and right modules. Consequently, if $X : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ is a formal $\mathbb{E}_1$-moduli problem, then there are two natural analogues of the ∞-category $\mathrm{QCoh}(X)$. We will denote these ∞-categories by $\mathrm{QCoh}_L(X)$ and $\mathrm{QCoh}_R(X)$, and refer to them as the ∞-categories of (left and right) *quasi-coherent sheaves* on X. We will also study noncommutative counterparts of the fully faithful embedding $\mathrm{QCoh}(X) \to \mathrm{QCoh}^!(X)$ of Remark 13.4.6.6.

We will devote most of our attention to the case where $X = \mathrm{Spf}\, A$ is corepresented by an Artinian $\mathbb{E}_1$-algebra A over κ. At the end of this section, we will explain how to extrapolate our discussion to the general case (Construction 14.5.5.1).

14.5.1 Artinian Modules

Let κ be a field. We will say that an object $M \in \mathrm{Mod}_\kappa$ is *Artinian* if it is perfect as a κ-module: that is, if $\pi_* M$ has finite dimension over κ. If R is an $\mathbb{E}_1$-algebra over κ and M is a right or left module over R, we will say that M is *Artinian* if it is Artinian when regarded as an object of Mod_κ. We let $\mathrm{LMod}_R^{\mathrm{art}}$ denote the full subcategory of LMod_R spanned by the Artinian left R-modules, and $\mathrm{RMod}_R^{\mathrm{art}}$ the full subcategory of RMod_R spanned by the Artinian right R-modules.

Remark 14.5.1.1. Let κ be a field, let R be an augmented $\mathbb{E}_1$-algebra over κ. Assume that R is connective and that the kernel I of the augmentation map $\pi_0 R \to \kappa$ is a nilpotent ideal in $\pi_0 R$. Then an object $M \in \mathrm{LMod}_R$ is Artinian if and only if it belongs to the smallest stable full subcategory $\mathcal{C} \subseteq \mathrm{LMod}_R$ which contains $\kappa \simeq (\pi_0 R)/I$ and is closed under equivalence. The "if" direction is obvious (and requires no assumptions on R), since the full subcategory $\mathrm{LMod}_R^{\mathrm{art}} \subseteq \mathrm{LMod}_R$ is stable, closed under retracts, and contains κ. For the converse, suppose that M is Artinian; we prove that $M \in \mathcal{C}$ using induction on the dimension of the κ-vector space $\pi_* M$. If $M \simeq 0$ there is nothing to prove. Otherwise, there exists some largest integer n such that $\pi_n M$ is nonzero. Since I is nilpotent, there exists a nonzero element $x \in \pi_n M$ which is annihilated by I. Then multiplication by x determines a map of discrete R-modules $\kappa \to \pi_n M$, which in turn determines a fiber sequence of R-modules $\Sigma^n(\kappa) \to M \to M'$. The inductive hypothesis guarantees that $M' \in \mathcal{C}$ and it is clear that $\Sigma^n(\kappa) \in \mathcal{C}$, so that $M \in \mathcal{C}$ as desired.

We now show that if R is an $\mathbb{E}_1$-algebra over a field κ, then κ-linear duality determines a contravariant equivalence between the ∞-categories $\mathrm{LMod}_R^{\mathrm{art}}$ and $\mathrm{RMod}_R^{\mathrm{art}}$.

Proposition 14.5.1.2. *Let κ be a field and let R be an $\mathbb{E}_1$-algebra over κ. Define a functor $\lambda : \mathrm{RMod}_R^{\mathrm{op}} \times \mathrm{LMod}_R^{\mathrm{op}} \to \mathcal{S}$ by the formula $\lambda(M,N) = \mathrm{Map}_{\mathrm{Mod}_\kappa}(M \otimes_R N, \kappa)$. Then:*

(1) *For every right R-module M, let $\lambda_M : \mathrm{LMod}_R^{\mathrm{op}} \to \mathcal{S}$ be the restriction of λ to $\{M\} \times \mathrm{LMod}_R^{\mathrm{op}}$. Then λ_M is a representable functor.*

(1′) *Let $\mu : \mathrm{RMod}_R^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{LMod}_R^{\mathrm{op}}, \mathcal{S})$ be given by $\mu(M)(N) = \lambda(M,N)$. Then μ is homotopic to a composition $\mathrm{RMod}_R^{\mathrm{op}} \xrightarrow{\mu_0} \mathrm{LMod}_R \xrightarrow{j} \mathrm{Fun}(\mathrm{LMod}_R^{\mathrm{op}}, \mathcal{S})$, where j denotes the Yoneda embedding.*

(2) *For every left R-module M, let $\lambda_N : \mathrm{RMod}_R^{\mathrm{op}} \to \mathcal{S}$ be the restriction of λ to $\mathrm{RMod}_R^{\mathrm{op}} \times \{N\}$. Then λ_N is a representable functor.*

(2′) *Let $\mu' : \mathrm{LMod}_R^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{RMod}_R^{\mathrm{op}}, \mathcal{S})$ be given by $\mu'(N)(M) = \lambda(M,N)$. Then μ' is homotopic to a composition $\mathrm{LMod}_R^{\mathrm{op}} \xrightarrow{\mu'_0} \mathrm{RMod}_R \xrightarrow{j} \mathrm{Fun}(\mathrm{RMod}_R^{\mathrm{op}}, \mathcal{S})$, where j denotes the Yoneda embedding.*

(3) *The functors μ_0 and μ'_0 determine mutually inverse equivalences between the ∞-categories $\mathrm{RMod}_R^{\mathrm{art}}$ and $(\mathrm{LMod}_R^{\mathrm{art}})^{\mathrm{op}}$.*

Proof. We first note that (1′) and (2′) are reformulations of (1) and (2). We will prove (1); the proof of (2) is similar. Let $M \in \mathrm{RMod}_R$; we wish to show that λ_M is a representable functor. Since LMod_R is a presentable ∞-category, it will suffice to show that the functor λ_M preserves small limits (Proposition HTT.5.5.2.2). This is clear, since the functor $N \mapsto M \otimes_R N$ preserves small colimits.

Let $\mu_0 : \mathrm{RMod}_R^{\mathrm{op}} \to \mathrm{LMod}_R$ and $\mu_0' : \mathrm{LMod}_R^{\mathrm{op}} \to \mathrm{RMod}_R$ be as in $(1')$ and $(2')$. We note that μ_0' can be identified with the right adjoint to μ_0^{op}. Let $M \in \mathrm{RMod}_R$. For every integer n, we have canonical isomorphisms

$$\pi_n \mu_0(M) \simeq \pi_0 \mathrm{Map}_{\mathrm{RMod}_R}(\Sigma^n(R), \mu_0(M)) \simeq \pi_0 \mathrm{Map}_{\mathrm{Mod}_\kappa}(M \otimes_R \Sigma^n(R), \kappa) \simeq (\pi_{-n} M)^\vee,$$

where $(\pi_{-n}M)^\vee$ denotes the κ-linear dual of the vector space $\pi_{-n}M$. It follows that μ_0 carries $(\mathrm{RMod}_R^{\mathrm{art}})^{\mathrm{op}}$ into $\mathrm{LMod}_R^{\mathrm{art}}$. Similarly, μ_0' carries $(\mathrm{LMod}_R^{\mathrm{art}})^{\mathrm{op}}$ into $\mathrm{RMod}_R^{\mathrm{art}}$. To prove (3), it will suffice to show that for every pair of objects $M \in \mathrm{RMod}_R^{\mathrm{art}}$, $N \in \mathrm{LMod}_R^{\mathrm{art}}$, the unit maps $M \to \mu_0'\mu_0(M)$ and $N \to \mu_0\mu_0'(N)$ are equivalences in RMod_R and LMod_R, respectively. Passing to homotopy groups, we are reduced to proving that the biduality maps $\pi_n M \to ((\pi_n M)^\vee)^\vee$ and $\pi_n N \to ((\pi_n N)^\vee)^\vee$ are isomorphisms for every integer n. This follows from the finite-dimensionality of the vector spaces $\pi_n M$ and $\pi_n N$ over κ. $\square$

14.5.2 Ind-Coherent Modules

Let κ be a field and let $R \in \mathrm{Alg}_\kappa$. If R is Artinian (in the sense of Definition 14.0.0.1), then every perfect (left or right) R-module is perfect when viewed as a κ-module: that is, we can regard $\mathrm{LMod}_R^{\mathrm{art}}$ and $\mathrm{RMod}_R^{\mathrm{art}}$ as enlargements of the ∞-categories $\mathrm{LMod}_R^{\mathrm{perf}}$ and $\mathrm{RMod}_R^{\mathrm{perf}}$, respectively. We now introduce analogous enlargements of the ∞-categories LMod_R and RMod_R.

Definition 14.5.2.1. Let κ be a field and let $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$ be a Artinian $\mathbb{E}_1$-algebra over κ. We let $\mathrm{LMod}_R^!$ denote the full subcategory of $\mathrm{Fun}(\mathrm{RMod}_R^{\mathrm{art}}, \mathcal{S})$ spanned by the left exact functors, and $\mathrm{RMod}_R^!$ the full subcategory of $\mathrm{Fun}(\mathrm{LMod}_R^{\mathrm{art}}, \mathcal{S})$ spanned by the left exact functors. We will refer to $\mathrm{LMod}_R^!$ as the *∞-category of* Ind-*coherent left R-modules*, and $\mathrm{RMod}_R^!$ as the *∞-category of* Ind-*coherent right R-modules*.

Remark 14.5.2.2. Using the equivalence $\mathrm{RMod}_R^{\mathrm{art}} \simeq (\mathrm{LMod}_R^{\mathrm{art}})^{\mathrm{op}}$ of Proposition 14.5.1.2, we obtain equivalences of ∞-categories $\mathrm{LMod}_R^! \simeq \mathrm{Ind}(\mathrm{LMod}_R^{\mathrm{art}})$ and $\mathrm{RMod}_R^! \simeq \mathrm{Ind}(\mathrm{RMod}_R^{\mathrm{art}})$.

Our next goal is to study the dependence of the ∞-categories $\mathrm{LMod}_R^!$ and $\mathrm{RMod}_R^!$ on the choice of algebra $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$.

Construction 14.5.2.3. Let κ be a field and let $\mathrm{LMod}(\mathrm{Mod}_\kappa)$ and $\mathrm{RMod}(\mathrm{Mod}_\kappa)$ denote the ∞-categories of left and right module objects of the symmetric monoidal ∞-category Mod_κ. That is, $\mathrm{LMod}(\mathrm{Mod}_\kappa)$ is an ∞-category whose objects are pairs (R, M), where $R \in \mathrm{Alg}_\kappa$ and M is a left R-module, and $\mathrm{RMod}(\mathrm{Mod}_\kappa)$ is an ∞-category whose objects are pairs (R, M) where $R \in \mathrm{Alg}_\kappa$ and M is a left R-module. We let $\mathrm{LMod}^{\mathrm{art}}(\mathrm{Mod}_\kappa)$ denote the full subcategory of $\mathrm{LMod}(\mathrm{Mod}_\kappa)$ spanned by those pairs (R, M) where $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$ and $M \in \mathrm{LMod}_R^{\mathrm{art}}$, and define $\mathrm{RMod}^{\mathrm{art}}(\mathrm{Mod}_\kappa) \subseteq \mathrm{RMod}(\mathrm{Mod}_\kappa)$ similarly. We have evident forgetful functors

$$\mathrm{LMod}^{\mathrm{art}}(\mathrm{Mod}_\kappa) \xrightarrow{q} \mathrm{Alg}_\kappa^{\mathrm{art}} \xleftarrow{q'} \mathrm{RMod}^{\mathrm{art}}(\mathrm{Mod}_\kappa).$$

We define $\mathrm{RMod}^!(\mathrm{Mod}_\kappa) = \mathrm{Dl}^{\mathrm{lex}}(q)$ and $\mathrm{LMod}^!(\mathrm{Mod}_\kappa) = \mathrm{Dl}^{\mathrm{lex}}(q')$, so that we have evident forgetful functors

$$\mathrm{RMod}^!(\mathrm{Mod}_\kappa) \to \mathrm{Alg}_\kappa^{\mathrm{art}} \leftarrow \mathrm{LMod}^!(\mathrm{Mod}_\kappa).$$

Remark 14.5.2.4. It follows from Proposition 14.4.2.3 that the forgetful functors

$$\mathrm{LMod}^!(\mathrm{Mod}_\kappa) \to \mathrm{Alg}_\kappa^{\mathrm{art}} \leftarrow \mathrm{RMod}^!(\mathrm{Mod}_\kappa)$$

are coCartesian fibrations. For every object $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$, we can identify the fiber product $\mathrm{RMod}^!(\mathrm{Mod}_\kappa) \times_{\mathrm{Alg}_\kappa^{\mathrm{art}}} \{R\}$ with the ∞-category $\mathrm{RMod}_R^!$ of Definition 14.5.2.1, and the fiber product $\mathrm{LMod}^!(\mathrm{Mod}_\kappa) \times_{\mathrm{Alg}_\kappa^{\mathrm{art}}} \{R\}$ with the ∞-category $\mathrm{LMod}_R^!$ of Definition 14.5.2.1. If $f : R \to R'$ is a morphism in $\mathrm{Alg}_\kappa^{\mathrm{art}}$, then f induces functors $\mathrm{RMod}_R^! \to \mathrm{RMod}_{R'}^!$ and $\mathrm{LMod}_R^! \to \mathrm{LMod}_{R'}^!$, both of which we will denote by $f^!$.

14.5.3 Functoriality

Let κ be a field and let R be an Artinian κ-algebra (in the sense of Definition 14.0.0.1). Then the inclusion $\mathrm{RMod}_R^{\mathrm{perf}} \subseteq \mathrm{RMod}_R^{\mathrm{art}}$ induces a fully faithful embedding

$$\Phi_R : \mathrm{RMod}_R \simeq \mathrm{Ind}(\mathrm{RMod}_R^{\mathrm{perf}}) \hookrightarrow \mathrm{Ind}(\mathrm{RMod}_R^{\mathrm{art}}) \simeq \mathrm{RMod}_R^!.$$

However, this construction is badly behaved in some respects. For example, if $f : R \to R'$ is a morphism in $\mathrm{Alg}_\kappa^{\mathrm{art}}$, then the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{RMod}_R & \xrightarrow{f^*} & \mathrm{RMod}_{R'} \\ \downarrow{\scriptstyle \Phi_R} & & \downarrow{\scriptstyle \Phi_{R'}} \\ \mathrm{RMod}_R^! & \xrightarrow{f^!} & \mathrm{RMod}_{R'}^! \end{array}$$

generally does not commute up to homotopy (here f^* denotes the base change functor $M \mapsto M \otimes_R R'$). In what follows, we will instead consider the fully faithful embeddings $\Psi_R : \mathrm{RMod}_R \to \mathrm{RMod}_R^!$ given by the composition

$$\begin{aligned} \mathrm{RMod}_R &\simeq \mathrm{Ind}(\mathrm{RMod}_R^{\mathrm{perf}}) \\ &\simeq \mathrm{Ind}((\mathrm{LMod}_R^{\mathrm{perf}})^{\mathrm{op}}) \\ &\hookrightarrow \mathrm{Ind}((\mathrm{LMod}_R^{\mathrm{art}})^{\mathrm{op}}) \\ &\simeq \mathrm{Ind}(\mathrm{RMod}_R^{\mathrm{art}}) \\ &\simeq \mathrm{RMod}_R^!. \end{aligned}$$

Our next goal is to give a construction of this functor which is manifestly functorial in $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$.

Construction 14.5.3.1. Let κ be a field, and let $\lambda : \mathrm{RMod}(\mathrm{Mod}_\kappa) \times_{\mathrm{Alg}_\kappa} \mathrm{LMod}^{\mathrm{art}}(\mathrm{Mod}_\kappa) \to \mathcal{S}$ be the functor given by

$$\lambda(M, R, N) = \mathrm{Map}_{\mathrm{Mod}_\kappa}(\kappa, M \otimes_R N).$$

If we fix M and R, then the functor $N \mapsto \mathrm{Map}_{\mathrm{Mod}_\kappa}(\kappa, M \otimes_R N)$ is left exact. It follows that λ is determines a functor $\Psi : \mathrm{RMod}(\mathrm{Mod}_\kappa) \times_{\mathrm{Alg}_\kappa} \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathrm{RMod}^!(\mathrm{Mod}_\kappa)$.

Proposition 14.5.3.2. *Let κ be a field, and consider the diagram*

$$\begin{array}{ccc} \mathrm{RMod}(\mathrm{Mod}_\kappa) \times_{\mathrm{Alg}_\kappa} \mathrm{Alg}_\kappa^{\mathrm{art}} & \xrightarrow{\Psi} & \mathrm{RMod}^!(\mathrm{Mod}_\kappa) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathrm{Alg}_\kappa^{\mathrm{art}} & \end{array}$$

where p and q are the forgetful functors and Ψ is defined as in Construction 14.5.3.1. Then:

(1) *The functor Ψ carries p-coCartesian morphisms to q-coCartesian morphisms.*

(2) *For every object $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$, the induced functor $\Psi_R : \mathrm{RMod}_R \to \mathrm{RMod}^!_R$ preserves small colimits.*

(3) *The functor Ψ is fully faithful.*

Proof. We first prove (1). Let $\alpha : (M, R) \to (M', R')$ be a p-coCartesian morphism in $\mathrm{RMod}(\mathrm{Mod}_\kappa)$. We wish to prove that $\Psi(\alpha)$ is q-coCartesian. Unwinding the definitions, we must show that for every Artinian R'-module N, the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_\kappa}(\kappa, M \otimes_R N) \to \mathrm{Map}_{\mathrm{Mod}_\kappa}(\kappa, M' \otimes_{R'} N)$$

is an equivalence. This is clear, since the map $M \otimes_R N \to M' \otimes_{R'} N$ is an equivalence.

We now prove (2). Fix an object $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$. For every $N \in \mathrm{LMod}_R^{\mathrm{art}}$, the functor $M \mapsto \mathrm{Map}_{\mathrm{Mod}_\kappa}(\kappa, M \otimes_R N)$ commutes with filtered colimits and finite limits. It follows that Ψ_R commutes with filtered colimits and finite limits. Since Ψ_R is a left exact functor between stable ∞-categories, it is also right exact. We conclude that Ψ_R commutes with filtered colimits and finite colimits, and therefore with all small colimits.

We now prove (3). By virtue of (1), it will suffice to prove that for $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$ the functor $\Psi_R : \mathrm{RMod}_R \to \mathrm{RMod}^!_R$ is fully faithful. Using (2) and Proposition HTT.5.3.5.11 , we are reduced to proving that the restriction $\Psi_R|_{\mathrm{RMod}_R^{\mathrm{perf}}}$ is fully faithful. Note that if M be a perfect right R-module and $M^\vee$ its R-linear dual (regarded as a perfect left R-module), then $\Psi_R(M)$ is the functor corepresented by $M^\vee \in \mathrm{LMod}_R^{\mathrm{perf}} \subseteq \mathrm{LMod}_R^{\mathrm{art}}$. We can therefore identify $\Psi_R|_{\mathrm{RMod}_R^{\mathrm{perf}}}$ with the composition of fully faithful embeddings

$$\mathrm{RMod}_R^{\mathrm{perf}} \simeq (\mathrm{LMod}_R^{\mathrm{perf}})^{\mathrm{op}} \subseteq (\mathrm{LMod}_R^{\mathrm{art}})^{\mathrm{op}} \xhookrightarrow{j} \mathrm{RMod}^!_R,$$

(here j denotes the Yoneda embedding). □

Remark 14.5.3.3. Construction 14.5.3.1 and Proposition 14.5.3.2 have evident dual versions, which give a fully faithful embedding $\mathrm{LMod}(\mathrm{Mod}_\kappa) \times_{\mathrm{Alg}_\kappa} \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathrm{LMod}^!(\mathrm{Mod}_\kappa)$.

14.5.4 Connective Ind-Coherent Modules

Our next goal is to say something about the essential image of the functor $\Psi_R : \mathrm{RMod}_R \to \mathrm{RMod}_R^!$ of Proposition 14.5.3.2.

Definition 14.5.4.1. Let R be an Artinian $\mathbb{E}_1$-algebra over a field κ, and let $\epsilon : R \to \kappa$ be the augmentation. We will say that an object $M \in \mathrm{RMod}_R^!$ is *connective* if $\epsilon^! M$ is a connective object of $\mathrm{Mod}_\kappa \simeq \mathrm{Mod}_\kappa^!$. We let $\mathrm{RMod}_R^{!,\mathrm{cn}}$ denote the full subcategory of $\mathrm{RMod}_R^!$ spanned by the connective objects. Similarly, we define a full subcategory $\mathrm{LMod}_R^{!,\mathrm{cn}} \subseteq \mathrm{LMod}_R^!$.

Remark 14.5.4.2. Let R be an Artinian $\mathbb{E}_1$-algebra over a field κ. It follows from Proposition HTT.5.4.6.6 that $\mathrm{RMod}_R^{!,\mathrm{cn}}$ is an accessible subcategory of $\mathrm{RMod}_R^!$, which is evidently closed under small colimits and extensions. Applying Proposition HA.1.4.4.11 , we conclude that there exists a t-structure on the stable ∞-category $\mathrm{RMod}_R^!$ with $(\mathrm{RMod}_R^!)_{\geq 0} = \mathrm{RMod}_R^{!,\mathrm{cn}}$.

Proposition 14.5.4.3. *Let κ be a field, let $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$. Then the fully faithful embedding $\Psi_R : \mathrm{RMod}_R \to \mathrm{RMod}_R^!$ of Proposition 14.5.3.2 restricts to an equivalence of ∞-categories $\mathrm{RMod}_R^{\mathrm{cn}} \hookrightarrow \mathrm{RMod}_R^{!,\mathrm{cn}}$.*

Proof. Let $\epsilon : R \to \kappa$ be the augmentation map and let $M \in \mathrm{RMod}_R^!$. We wish to show that $\epsilon^! M$ is connective if and only if $M \simeq \Psi_R(M')$ for some $M' \in \mathrm{RMod}_R^{\mathrm{cn}}$. The "if" direction is clear: if $M \simeq \Psi_R(M')$, we have equivalences

$$\epsilon^! M \simeq \epsilon^! \Psi_R(M') \simeq \Psi_\kappa(\epsilon^* M') \simeq \kappa \otimes_R M'.$$

For the converse, assume that $\epsilon^! M$ is connective. Let $\mathcal{C} \subseteq \mathrm{RMod}_R^!$ denote the essential image of $\Psi_R|_{\mathrm{RMod}_R^{\mathrm{cn}}}$; we wish to prove that $M \in \mathcal{C}$. It follows from Remark 14.5.4.2 that $\mathcal{C}$ is closed under colimits and extensions in $\mathrm{RMod}_R^!$.

We begin by constructing a sequence of objects

$$0 = M(0) \to M(1) \to M(2) \to \cdots$$

in $\mathcal{C}$ and a compatible family of maps $\theta(i) : M(i) \to M$ with the following property:

- $(*)$ The groups $\pi_j \epsilon^! M(i)$ vanish unless $0 \leqslant j < i$, and the maps $\pi_j \epsilon^! M(j) \to \pi_j \epsilon^! M$ are isomorphisms for $0 \leqslant j < i$.

Assume that $i \geqslant 0$ and that we have already constructed a map $\theta(i)$ satisfying $(*)$. Let $M' = \mathrm{cofib}(\theta(i))$, so that $\pi_j \epsilon^! M' \simeq 0$ for $j < i$. Let us regard M' as a functor $\mathrm{LMod}_R^{\mathrm{art}} \to \mathcal{S}$, so that $M'(\Sigma^m(\kappa))$ is $(m+i)$-connective for every integer i. It follows by induction that for every m-connective object $N \in \mathrm{LMod}^{\mathrm{art}}_R$, the space $M'(N)$ is $(m+i)$-connective. In particular, $M'(N)$ is connected when N denotes the cofiber of the map $\Sigma^{-i}(R) \to \Sigma^{-i}(\kappa)$. Using the fiber sequence

$$M'(\Sigma^{-i}(R)) \to M'(\Sigma^{-i}(\kappa)) \to M'(\mathrm{cofib}(\Sigma^{-i}(R) \to \Sigma^{-i}(\kappa))),$$

we deduce that the map $\pi_0 M'(\Sigma^{-i}(R)) \to \pi_0 M'(\Sigma^{-i}(\kappa))$ is surjective. Let $K = \Psi_R(R) \in \mathrm{RMod}^!_R$. Then $\Sigma^i(K) = \Psi_R(\Sigma^i(R))$ can be identified with the functor corepresented by $\Sigma^{-i}(R)$. We have proven the following:

$(*')$ For every point $\eta \in \pi_i \epsilon^! M' \simeq \pi_0 M'(\Sigma^{-i}(\kappa))$, there exists a map $\Sigma^i(K) \to M$ such that η belongs to the image of the induced map $\kappa \simeq \pi_i \epsilon^! \Sigma^i(K) \to \pi_i \epsilon^! M'$.

Choose a basis $\{v_\alpha\}_{\alpha \in S}$ for the κ-vector space $\pi_i \epsilon^! M' \simeq \pi_i \epsilon^! M$. Applying $(*')$ repeatedly, we obtain a map $v : \bigoplus_{\alpha \in S} \Sigma^i(K) \to M'$. Let $M'' = \mathrm{cofib}(v)$ and let $M(i+1)$ denote the fiber of the composite map $M \to M' \to M''$. We have a fiber sequence

$$M(i) \to M(i+1) \to \bigoplus_{\alpha \in S} \Sigma^i(K).$$

Since $\mathcal{C}$ is closed under colimits and extensions (and contains $\Sigma^i(K) \simeq \Psi_R \Sigma^i(R)$), we conclude that $M(i+1) \in \mathcal{C}$. Using the long exact sequence of homotopy groups

$$\pi_j \epsilon^! M(i) \to \pi_j \epsilon^! M(i+1) \to \pi_j \epsilon^! \bigoplus_{\alpha \in S} \Sigma^i(K) \to \pi_{j-1} \epsilon^! M(i),$$

we deduce that the canonical map $M(i+1) \to M$ satisfies condition $(*)$.

Let $M(\infty) = \varinjlim M(i)$ and let $w : M(\infty) \to M$ be the tautological map. Since $\mathcal{C}$ is closed under colimits, we deduce that $M(\infty) \in \mathcal{C}$. Using $(*)$ (and the vanishing of the groups $\pi_j \epsilon^! M$ for $j < 0$), we deduce that w induces an equivalence $\epsilon^! M(\infty) \to \epsilon^! M$. Identifying M and $M(\infty)$ with left exact functors $\mathrm{LMod}^{\mathrm{art}}_R \to \mathcal{S}$, we conclude that w induces a homotopy equivalence $M(\infty)(\Sigma^j(\kappa)) \to M(\Sigma^j(\kappa))$ for every integer j. Since M and $M(\infty)$ are left exact, the collection of those objects $N \in \mathrm{LMod}^{\mathrm{art}}_R$ for which $M(\infty)(N) \to M(N)$ is a homotopy equivalence is closed under finite limits. Using Remark 14.5.1.1, we deduce that every object $N \in \mathrm{LMod}^{\mathrm{art}}_R$ has this property, so that w is an equivalence and $M \simeq M(\infty) \in \mathcal{C}$ as desired. □

Remark 14.5.4.4. Let R be an Artinian $\mathbb{E}_1$-algebra over a field κ. The natural t-structure on the ∞-category RMod_R is right complete. It follows from Proposition 14.5.4.3 that the fully faithful embedding $\Psi_R : \mathrm{RMod}_R \to \mathrm{RMod}^!_R$ induces an equivalence from RMod_R to the right completion of $\mathrm{RMod}^!_R$.

14.5.5 Sheaves on Formal Moduli Problems

We now consider global analogues of the preceding constructions.

Construction 14.5.5.1. Let κ be a field. The coCartesian fibrations

$$\mathrm{RMod}(\mathrm{Mod}_\kappa) \times_{\mathrm{Alg}_\kappa} \mathrm{Alg}^{\mathrm{art}}_\kappa \xrightarrow{p_R} \mathrm{Alg}^{\mathrm{art}}_\kappa \xleftarrow{q_R} \mathrm{RMod}^!(\mathrm{Mod}_\kappa)$$

are classified by functors $\chi, \chi_! : \mathrm{Alg}^{\mathrm{art}}_\kappa \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. Since $\widehat{\mathcal{C}\mathrm{at}}_\infty$ admits small limits, Theorem HTT.5.1.5.6 implies that χ and $\chi_!$ admit (essentially unique) factorizations as compositions

$$\mathrm{Alg}^{\mathrm{art}}_\kappa \xrightarrow{j} \mathrm{Fun}(\mathrm{Alg}^{\mathrm{art}}_\kappa, \mathcal{S})^{\mathrm{op}} \xrightarrow{\mathrm{QCoh}_R} \widehat{\mathcal{C}\mathrm{at}}_\infty \qquad \mathrm{Alg}^{\mathrm{art}}_\kappa \xrightarrow{j} \mathrm{Fun}(\mathrm{Alg}^{\mathrm{art}}_\kappa, \mathcal{S})^{\mathrm{op}} \xrightarrow{\mathrm{QCoh}^!_R} \widehat{\mathcal{C}\mathrm{at}}_\infty$$

where j denotes the Yoneda embedding and the functors QCoh_R and $\mathrm{QCoh}^!_R$ preserve small limits. Similarly, the coCartesian fibrations

$$\mathrm{LMod}(\mathrm{Mod}_\kappa) \times_{\mathrm{Alg}_\kappa} \mathrm{Alg}^{\mathrm{art}}_\kappa \to \mathrm{Alg}^{\mathrm{art}}_\kappa \leftarrow \mathrm{LMod}^!(\mathrm{Mod}_\kappa)$$

are classified by maps $\chi', \chi'_! : \mathrm{Alg}^{\mathrm{art}}_\kappa \to \widehat{\mathcal{C}\mathrm{at}}_\infty$, which admit factorizations

$$\mathrm{Alg}^{\mathrm{art}}_\kappa \xrightarrow{j} \mathrm{Fun}(\mathrm{Alg}^{\mathrm{art}}_\kappa, \mathcal{S})^{\mathrm{op}} \xrightarrow{\mathrm{QCoh}_L} \widehat{\mathcal{C}\mathrm{at}}_\infty \qquad \mathrm{Alg}^{\mathrm{art}}_\kappa \xrightarrow{j} \mathrm{Fun}(\mathrm{Alg}^{\mathrm{art}}_\kappa, \mathcal{S})^{\mathrm{op}} \xrightarrow{\mathrm{QCoh}^!_L} \widehat{\mathcal{C}\mathrm{at}}_\infty.$$

For each functor $X : \mathrm{Alg}^{\mathrm{art}}_\kappa \to \mathcal{S}$, we will refer to $\mathrm{QCoh}_L(X)$ and $\mathrm{QCoh}_R(X)$ as the ∞-categories of (left and right) *quasi-coherent sheaves on* X. Similarly, we will refer to $\mathrm{QCoh}^!_L(X)$ and $\mathrm{QCoh}^!_R(X)$ as the ∞-categories of (left and right) Ind-*coherent sheaves on* X.

Remark 14.5.5.2. For every functor $X : \mathrm{Alg}^{\mathrm{art}}_\kappa \to \mathcal{S}$, the ∞-categories $\mathrm{QCoh}_L(X)$, $\mathrm{QCoh}_R(X)$, $\mathrm{QCoh}^!_L(X)$, and $\mathrm{QCoh}^!_R(X)$ are presentable and stable.

Remark 14.5.5.3. Let κ be a field, let $X : \mathrm{Alg}^{\mathrm{art}}_\kappa \to \mathcal{S}$ be a functor which classifies a right fibration $\mathcal{X} \to \mathrm{Alg}^{\mathrm{art}}_\kappa$. Then $\mathrm{QCoh}_R(X)$ and $\mathrm{QCoh}^!_R(X)$ can be identified with the ∞-categories of coCartesian sections of the coCartesian fibrations

$$\mathcal{X} \times_{\mathrm{Alg}_\kappa} \mathrm{RMod}(\mathrm{Mod}_\kappa) \to \mathcal{X} \leftarrow \mathcal{X} \times_{\mathrm{Alg}^{\mathrm{art}}_\kappa} \mathrm{RMod}^!(\mathrm{Mod}_\kappa).$$

More informally, an object $\mathscr{F} \in \mathrm{QCoh}_R(X)$ is a rule which assigns to every point $\eta \in X(A)$ a right A-module $\mathscr{F}_\eta$, and to every morphism $f : A \to A'$ carrying η to $\eta' \in X(A')$ an equivalence $\mathscr{F}_{\eta'} \simeq \mathscr{F}_\eta \otimes_A A'$. Similarly, an object of $\mathscr{G} \in \mathrm{QCoh}^!_R(X)$ is a rule which assigns to every point $\eta \in X(A)$ an Ind-coherent right R-module $\mathscr{G}_\eta \in \mathrm{RMod}^!_A$, and to every morphism $f : A \to A'$ carrying η to $\eta' \in X(R')$ an equivalence $\mathscr{G}_{\eta'} \simeq f^! \mathscr{G}_\eta$. The ∞-categories $\mathrm{QCoh}_L(X)$ and $\mathrm{QCoh}^!_L(X)$ admit similar descriptions, using left modules in place of right modules.

Notation 14.5.5.4. By construction, the ∞-categories $\mathrm{QCoh}_R(X)$, $\mathrm{QCoh}_L(X)$, $\mathrm{QCoh}^!_R(X)$, and $\mathrm{QCoh}^!_L(X)$ depend contravariantly on the object $X \in \mathrm{Fun}(\mathrm{Alg}^{\mathrm{art}}_\kappa, \mathcal{S})$. If $\alpha : X \to Y$ is a natural transformation, we will denote the resulting functors by

$$\alpha^* : \mathrm{QCoh}_R(Y) \to \mathrm{QCoh}_R(X) \qquad \alpha^! : \mathrm{QCoh}^!_R(Y) \to \mathrm{QCoh}^!_R(X)$$

$$\alpha^* : \mathrm{QCoh}_L(Y) \to \mathrm{QCoh}_L(X) \qquad \alpha^! : \mathrm{QCoh}^!_L(Y) \to \mathrm{QCoh}^!_L(X).$$

Remark 14.5.5.5. Let κ be a field. The fully faithful embedding Ψ of Proposition 14.5.3.2 induces a natural transformation $\mathrm{QCoh}_R \to \mathrm{QCoh}^!_R$ of functors $\mathrm{Fun}(\mathrm{Alg}^{\mathrm{art}}_\kappa, \mathcal{S}) \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. For every functor $X : \mathrm{Alg}^{\mathrm{art}}_\kappa \to \mathcal{S}$, we obtain a fully faithful embedding $\mathrm{QCoh}_R(X) \to \mathrm{QCoh}^!_R(X)$ which preserves small colimits. Moreover, if $\alpha : X \to Y$ is a natural transformation of functors, we obtain a diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}_R(Y) & \longrightarrow & \mathrm{QCoh}^!_R(Y) \\ \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^!} \\ \mathrm{QCoh}_R(X) & \longrightarrow & \mathrm{QCoh}^!_R(X) \end{array}$$

which commutes up to canonical homotopy. Similarly, we have fully faithful embedding $\mathrm{QCoh}_L(X) \to \mathrm{QCoh}^!_L(X)$, which depend functorially on X in the same sense.

Notation 14.5.5.6. Let κ be a field and let $X : \mathrm{Alg}^{\mathrm{art}}_\kappa \to \mathcal{S}$ be a functor. We will say that an object $\mathscr{F} \in \mathrm{QCoh}_R(X)$ is *connective* if $\mathscr{F}_\eta \in \mathrm{RMod}_A$ is connective for every point $\eta \in X(A)$ (see Notation 14.5.5.3). We let $\mathrm{QCoh}_R(X)^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{QCoh}_R(X)$ spanned by the connective right quasi-coherent sheaves, and define a full subcategory $\mathrm{QCoh}_L(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}_L(X)$ similarly.

We will say that an object $\mathscr{G} \in \mathrm{QCoh}^!_R$ is *connective* if, for every point $\eta \in X(\kappa)$, the object $\mathscr{G}_\eta \in \mathrm{RMod}^!_\kappa \simeq \mathrm{Mod}_\kappa$ is connective. We let $\mathrm{QCoh}^!_R(X)^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{QCoh}^!_R(X)$ spanned by the connective objects, and define $\mathrm{QCoh}^!_L(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}^!_L(X)$ similarly.

Remark 14.5.5.7. Let $X : \mathrm{Alg}^{\mathrm{art}}_\kappa \to \mathcal{S}$ be a functor. The full subcategories

$$\mathrm{QCoh}_L(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}_L(X) \qquad \mathrm{QCoh}_R(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}_R(X)$$

$$\mathrm{QCoh}^!_L(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}^!_L(X) \qquad \mathrm{QCoh}^!_R(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}^!_R(X)$$

of Notation 14.5.5.6 are presentable and closed under extensions and small colimits. It follows from Proposition HA.1.4.4.11 that they determine t-structures on $\mathrm{QCoh}_L(X)$, $\mathrm{QCoh}_R(X)$, $\mathrm{QCoh}^!_L(X)$, and $\mathrm{QCoh}^!_R(X)$.

Using Proposition 14.5.4.3, we immediately deduce the following:

Proposition 14.5.5.8. *Let κ be a field and let $X : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ be a functor. Then the fully faithful embeddings*

$$\mathrm{QCoh}_L(X) \hookrightarrow \mathrm{QCoh}^!_L(X) \qquad \mathrm{QCoh}_R(X) \hookrightarrow \mathrm{QCoh}^!_R(X)$$

of Remark 14.5.5.5 induce equivalences of ∞-categories

$$\mathrm{QCoh}_L(X)^{\mathrm{cn}} \simeq \mathrm{QCoh}^!_L(X)^{\mathrm{cn}} \qquad \mathrm{QCoh}_R(X)^{\mathrm{cn}} \simeq \mathrm{QCoh}^!_R(X)^{\mathrm{cn}}.$$

14.6 Koszul Duality for Modules

Our goal in this section is to prove the following non-commutative analogue of Theorem 13.4.0.1:

Theorem 14.6.0.1. *Let κ be a field, let $A \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$, and let $X : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ be the formal $\mathbb{E}_1$-moduli problem associated to A (see Theorem 14.0.0.5). Then there are canonical equivalences of ∞-categories*

$$\mathrm{QCoh}^!_L(X) \simeq \mathrm{RMod}_A \qquad \mathrm{QCoh}^!_R(X) \simeq \mathrm{LMod}_A\,.$$

In particular, we have fully faithful embeddings

$$\mathrm{QCoh}_L(X) \hookrightarrow \mathrm{RMod}_A \qquad \mathrm{QCoh}_R(X) \hookrightarrow \mathrm{LMod}_A\,.$$

Remark 14.6.0.2 (Comparison with the Commutative Case)**.** Let κ be a field of characteristic zero and let $\theta : \mathrm{CAlg}_\kappa^{\mathrm{art}} \to \mathrm{Alg}_\kappa^{\mathrm{art}}$ denote the forgetful functor. Let $X : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ be a formal $\mathbb{E}_1$-moduli problem over κ, so that $X \circ \theta$ is a formal moduli problem over κ. For each $R \in \mathrm{CAlg}_\kappa^{\mathrm{art}}$, we have a canonical equivalence of ∞-categories $\mathrm{Mod}_R \simeq \mathrm{RMod}_{\theta(R)}$. Passing to the inverse limit over points $\eta \in X(\theta(R))$, we obtain a functor $\mathrm{QCoh}_R(X) \to \mathrm{QCoh}(X \circ \theta)$. According to Theorem 14.0.0.5, there exists an augmented $\mathbb{E}_1$-algebra A over κ such that X is given by the formula $X(R) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(R), A)$. Let $\mathfrak{m}_A$ denote the augmentation ideal of A. Regard $\mathfrak{m}_A$ as an object of Lie_κ, so that $X \circ \theta$ is given by the formula

$$(X \circ \theta)(R) = \mathrm{Map}_{\mathrm{Lie}_\kappa}(\mathfrak{D}(R), \mathfrak{m}_A)$$

(see Theorem 14.3.0.1). Theorems 13.4.0.1 and 14.6.0.1 determine fully faithful embeddings

$$\mathrm{QCoh}_R(X) \hookrightarrow \mathrm{LMod}_A \qquad \mathrm{QCoh}(X \circ \theta) \hookrightarrow \mathrm{Rep}_{\mathfrak{m}_A}\,.$$

We have an evident map of $\mathbb{E}_1$-algebras $U(\mathfrak{m}_A) \to A$, which determines a forgetful functor $\mathrm{LMod}_A \to \mathrm{LMod}_{U(\mathfrak{m}_A)} \simeq \mathrm{Rep}_{\mathfrak{m}_A}$. With some additional effort, one can show that the diagram

$$\begin{array}{ccc} \mathrm{QCoh}_R(X) & \longrightarrow & \mathrm{LMod}_A \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(X \circ \theta) & \longrightarrow & \mathrm{Rep}_{\mathfrak{m}_A} \end{array}$$

commutes up to canonical homotopy. That is, the algebraic models for quasi-coherent sheaves provided by Theorems 13.4.0.1 and 14.6.0.1 in the commutative and noncommutative settings are compatible with one another.

14.6.1 Reduction to the Representable Case

We will deduce Theorem 14.6.0.1 from the following:

Proposition 14.6.1.1. *Let κ be a field, and let $\chi_! : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the functor given by $\chi_!(R) = \mathrm{RMod}_R^!$ (see Construction 14.5.5.1), and let $\chi' : \mathrm{Alg}_\kappa^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ given by $\chi'(R) = \mathrm{LMod}_R$ (so that χ' classifies the Cartesian fibration $\mathrm{LMod}(\mathrm{Mod}_\kappa) \to \mathrm{Alg}_\kappa$). Then $\chi_!$ is homotopic to the composition*

$$\mathrm{Alg}_\kappa^{\mathrm{art}} \hookrightarrow \mathrm{Alg}_\kappa^{\mathrm{aug}} \xrightarrow{\mathfrak{D}^{(1)}} (\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{op}} \xrightarrow{\chi'} \widehat{\mathcal{C}\mathrm{at}}_\infty.$$

More informally: for every object $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$, there is a canonical equivalence of ∞-categories $\mathrm{RMod}_R^! \simeq \mathrm{LMod}_{\mathfrak{D}^{(1)}(R)}$.

Before giving the proof of Proposition 14.6.1.1, let us explain how it leads to a proof of Theorem 14.6.0.1.

Proof of Theorem 14.6.0.1. Let $X : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ be the formal $\mathbb{E}_1$-moduli problem associated to an object $A \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$. We will construct the equivalence $\mathrm{QCoh}_R^!(X) \simeq \mathrm{LMod}_A$; the construction of the equivalence $\mathrm{QCoh}_L^!(X) \simeq \mathrm{RMod}_A$ is similar. Let κ be a field and let $\mathrm{QCoh}_R^! : \mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Construction 14.5.5.1. Let $\Psi : \mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Moduli}_\kappa^{(1)}$ be the equivalence of ∞-categories provided by Theorem 14.0.0.5, and let Ψ^{-1} denote a homotopy inverse to Ψ. Let $L : \mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S}) \to \mathrm{Moduli}_\kappa^{(1)}$ denote a left adjoint to the inclusion functor $\mathrm{Moduli}_\kappa^{(1)} \subseteq \mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S})$ (see Remark 12.1.3.5), and let $\widehat{\mathfrak{D}}^{(1)} : \mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S}) \to \mathrm{Alg}_\kappa^{\mathrm{aug}}$ be the composition of $\Psi^{-1} \circ L$. The functor $\widehat{\mathfrak{D}}^{(1)}$ preserves small colimits, and the composition of $\widehat{\mathfrak{D}}^{(1)}$ with the Yoneda embedding $(\mathrm{Alg}_\kappa^{\mathrm{art}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S})$ can be identified with the Koszul duality functor $\mathfrak{D}^{(1)} : (\mathrm{Alg}_\kappa^{\mathrm{art}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{aug}}$. Let $\chi' : \mathrm{Alg}_\kappa^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Proposition 14.6.1.1 (given on objects by $\chi'(A) = \mathrm{LMod}_A$), and let $F : \mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the composite functor

$$\mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S})^{\mathrm{op}} \xrightarrow{\widehat{\mathfrak{D}}^{(1)}} (\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{op}} \xrightarrow{\chi'} \widehat{\mathcal{C}\mathrm{at}}_\infty.$$

Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S})$ spanned by the corepresentable functors. Proposition 14.6.1.1 implies that there is an equivalence of functors $\alpha_0 : F|_{\mathcal{C}^{\mathrm{op}}} \to \mathrm{QCoh}_R^!|_{\mathcal{C}^{\mathrm{op}}}$. Since $\mathrm{QCoh}_R^!$ is a right Kan extension of its restriction to $\mathcal{C}^{\mathrm{op}}$, the equivalence α_0 extends to a natural transformation $F \to \mathrm{QCoh}_R^!$. We will prove:

$(*)$ If $X : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ is a formal $\mathbb{E}_1$-moduli problem, then α induces an equivalence of ∞-categories $F(X) \to \mathrm{QCoh}^!_R(X)$.

Taking $X = \Psi(A)$ for $A \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$, we see that $(*)$ supplies an equivalence of ∞-categories

$$\beta_A : \mathrm{LMod}_A \simeq \chi'(\Psi^{-1}\Psi(A)) \simeq F(\Psi(A)) \to \mathrm{QCoh}^!_R(X).$$

It remains to prove $(*)$. Let $\mathcal{E} \subseteq \mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S})$ be the full subcategory spanned by those functors X for which α induces an equivalence of ∞-categories $F(X) \to \mathrm{QCoh}^!_R(X)$. The localization functor $L : \mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S}) \to \mathrm{Moduli}_\kappa^{(1)}$, the equivalence $\Psi^{-1} : \mathrm{Moduli}_\kappa^{(1)} \to \mathrm{Alg}_\kappa^{\mathrm{aug}}$, and the forgetful functor $\mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Alg}_\kappa$ preserve small colimits. Lemma 13.4.7.2 implies that the functor $\chi' : \mathrm{Alg}_\kappa^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ preserves sifted limits. It follows that the functor F preserves sifted limits. Since $\mathrm{QCoh}^!_R$ preserves small limits, the ∞-category $\mathcal{E}$ is closed under sifted colimits in $\mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S})$. Since $\mathcal{E}$ contains all corepresentable functors and is closed under filtered colimits, it contains all prorepresentable formal moduli problems (see Definition 12.5.3.1). Proposition 12.5.3.3 implies that every formal $\mathbb{E}_1$-moduli problem X can be obtained as the geometric realization of a simplicial object $X_\bullet$ of $\mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S})$, where each X_n is prorepresentable. Since $\mathcal{E}$ is closed under geometric realizations in $\mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S})$, we conclude that $X \in \mathcal{E}$ as desired. □

14.6.2 The Proof of Proposition 14.6.1.1

Let κ be a field, and let $\chi' : \mathrm{Alg}_\kappa^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be a functor which classifies the Cartesian fibration $p : \mathrm{LMod}(\mathrm{Mod}_\kappa) \to \mathrm{Alg}_\kappa$. Using Proposition 14.4.2.4, we see that χ' also classifies the coCartesian fibration $\mathrm{Dl}^0(p^{\mathrm{op}}) \to \mathrm{Alg}_\kappa^{\mathrm{op}}$. Let $\mathrm{LMod}^{\mathrm{perf}}(\mathrm{Mod}_\kappa)$ denote the full subcategory of $\mathrm{LMod}(\mathrm{Mod}_\kappa)$ spanned by those pairs (A, M), where $A \in \mathrm{Alg}_\kappa$ and M is a perfect left module over A. Let p_{perf} denote the restriction of p to $\mathrm{LMod}^{\mathrm{perf}}(\mathrm{Mod}_\kappa)$. Proposition 14.4.3.1 supplies an equivalence of $\mathrm{Dl}^0(p^{\mathrm{op}}) \simeq \mathrm{Dl}^{\mathrm{lex}}(p_{\mathrm{perf}}^{\mathrm{op}})$ of coCartesian fibrations over $\mathrm{Alg}_\kappa^{\mathrm{op}}$. By construction, $\chi_! : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classifies the coCartesian fibration $\mathrm{RMod}^!(\mathrm{Mod}_\kappa) = \mathrm{Dl}^{\mathrm{lex}}(q) \to \mathrm{Alg}_\kappa^{\mathrm{art}}$, where q denotes the Cartesian fibration $\mathrm{LMod}^{\mathrm{art}}(\mathrm{Mod}_\kappa) \to \mathrm{Alg}_\kappa^{\mathrm{art}}$. Consequently, Proposition 14.6.1.1 is a consequence of the following:

Proposition 14.6.2.1. *Let κ be a field and let $\mathfrak{D} : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathrm{Alg}_\kappa^{\mathrm{op}}$ denote the composition*

$$\mathrm{Alg}_\kappa^{\mathrm{art}} \hookrightarrow \mathrm{Alg}_\kappa^{\mathrm{aug}} \xrightarrow{\mathfrak{D}^{(1)}} (\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{op}},$$

where $\mathfrak{D}^{(1)}$ is the Koszul duality functor of Definition 14.1.1.3. Then there is a pullback *diagram of ∞-categories*

$$\begin{array}{ccc} \mathrm{LMod}^{\mathrm{art}}(\mathrm{Mod}_\kappa) & \xrightarrow{\mathfrak{D}'} & \mathrm{LMod}^{\mathrm{perf}}(\mathrm{Mod}_\kappa)^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathrm{Alg}_\kappa^{\mathrm{art}} & \xrightarrow{\mathfrak{D}} & \mathrm{Alg}_\kappa^{\mathrm{op}}. \end{array}$$

We now proceed to construct the diagram appearing in the statement of Proposition 14.6.2.1.

Construction 14.6.2.2. Fix a field κ. We let $\mathcal{M}^{(1)} \to \mathrm{Alg}_\kappa^{\mathrm{aug}} \times \mathrm{Alg}_\kappa^{\mathrm{aug}}$ be the pairing of ∞-categories defined in Construction 14.1.1.1. The objects of $\mathcal{M}^{(1)}$ are given by triple (A, B, ϵ), where $A, B \in \mathrm{Alg}_\kappa$ and $\epsilon : A \otimes_\kappa B \to \kappa$ is an augmentation on $A \otimes_\kappa B$ (which then determines augmentations on A and B). Set

$$\overline{\mathcal{M}} = \mathrm{LMod}(\mathrm{Mod}_\kappa) \times_{\mathrm{Alg}_\kappa} \mathcal{M}^{(1)} \times_{\mathrm{Alg}_\kappa} \mathrm{LMod}(\mathrm{Mod}_\kappa),$$

so that $\overline{\mathcal{M}}$ is an ∞-category whose objects can be identified with quintuples (A, B, ϵ, M, N), where $A, B \in \mathrm{Alg}_\kappa$, $\epsilon : A \otimes_\kappa B \to \kappa$ is an augmentation, $M \in \mathrm{LMod}_A$, and $N \in \mathrm{LMod}_B$. There is an evident functor $\chi : \overline{\mathcal{M}}^{\mathrm{op}} \to \mathcal{S}$, given on objects by the formula $\chi(A, B, \epsilon, M, N) = \mathrm{Map}_{\mathrm{LMod}_{A \otimes_\kappa B}}(M \otimes_\kappa N, \kappa)$. Then χ classifies a right fibration $\mathcal{M}^{\mathcal{LM}} \to \overline{\mathcal{M}}$. Let $\mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_\kappa)$ denote the fiber product $\mathrm{LMod}(\mathrm{Mod}_\kappa) \times_{\mathrm{Alg}_\kappa} \mathrm{Alg}_\kappa^{\mathrm{aug}}$, so that the forgetful functor $\mathcal{M}^{\mathcal{LM}} \to \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_\kappa) \times \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_\kappa)$ is a right fibration and therefore determines a pairing of $\mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_\kappa)$ with itself.

Proposition 14.6.2.3. *Let κ be a field, let $\lambda : \mathcal{M}^{(1)} \to \mathrm{Alg}_\kappa^{\mathrm{aug}} \times \mathrm{Alg}_\kappa^{\mathrm{aug}}$ be the pairing of Construction 14.1.1.1 and $\lambda' : \mathcal{M}^{\mathcal{LM}} \to \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_\kappa) \times \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_\kappa)$ the pairing of Construction 14.6.2.2. Then λ' is both left and right representable. Moreover, the forgetful functor $\mathcal{M}^{\mathcal{LM}} \to \mathcal{M}^{(1)}$ is both left and right representable.*

Proof. We will show that λ' is left representable and $\mathcal{M}^{\mathcal{LM}} \to \mathcal{M}^{(1)}$ is left representable; the corresponding assertions for right representability will follow by symmetry. Fix an object $A \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$ and a left A-module M. Let $B = \mathfrak{D}^{(1)}(A)$ be the Koszul dual of A and $\epsilon : A \otimes_\kappa B \to \kappa$ the canonical map. Proposition 14.1.2.2 implies that ϵ determines a duality functor $\mathfrak{D}_\epsilon : \mathrm{LMod}_A^{\mathrm{op}} \to \mathrm{LMod}_B$. We let $N = \mathfrak{D}_\epsilon(M)$, so that there is a canonical map of left $A \otimes_\kappa B$-modules $\mu : M \otimes_\kappa N \to \kappa$. The quintuple (A, B, ϵ, M, N) is an object of the ∞-category $\overline{\mathcal{M}}$ of Construction 14.6.2.2, and μ determines a lifting to an object $X \in \mathcal{M}^{\mathcal{LM}}$. We complete the proof by observing that X is left universal and has left universal image in $\mathcal{M}^{(1)}$. □

It follows from Proposition 14.6.2.3 that the pairing

$$\mathcal{M}^{\mathcal{LM}} \to \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_\kappa) \times \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_\kappa)$$

determines a duality functor $\mathfrak{D}^{\mathcal{LM}} : \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_\kappa) \to \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_\kappa)^{\mathrm{op}}$. Let $\mathfrak{D}'$ denote the composite map

$$\mathrm{LMod}^{\mathrm{art}}(\mathrm{Mod}_\kappa) \to \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_\kappa) \xrightarrow{\mathfrak{D}^{\mathcal{LM}}} \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_\kappa)^{\mathrm{op}} \to \mathrm{LMod}(\mathrm{Mod}_\kappa)^{\mathrm{op}}.$$

By construction, we have a commutative diagram σ :

$$\begin{array}{ccc} \mathrm{LMod}^{\mathrm{art}}(\mathrm{Mod}_\kappa) & \xrightarrow{\mathfrak{D}'} & \mathrm{LMod}(\mathrm{Mod}_\kappa)^{\mathrm{op}} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathrm{Alg}_\kappa^{\mathrm{art}} & \xrightarrow{\mathfrak{D}} & \mathrm{Alg}_\kappa^{\mathrm{op}} . \end{array}$$

We next claim that the functor $\mathfrak{D}'$ carries p-Cartesian morphisms to q-Cartesian morphisms. Unwinding the definitions, we must show that if $f : R \to R'$ is a morphism in $\mathrm{Alg}_\kappa^{\mathrm{art}}$ and M is an Artinian left R'-module, then the canonical map

$$\theta_M : \mathfrak{D}^{(1)}(R) \otimes_{\mathfrak{D}^{(1)}(R')} \mathfrak{D}_{\mu'}(M) \to \mathfrak{D}_\mu(M)$$

is an equivalence, where $\mathfrak{D}_\mu : \mathrm{LMod}_R^{\mathrm{op}} \to \mathrm{LMod}_{\mathfrak{D}^{(1)}(R)}$ and $\mathfrak{D}_{\mu'} : \mathrm{LMod}_{R'}^{\mathrm{op}} \to \mathrm{LMod}_{\mathfrak{D}^{(1)}(R')}$ are the duality functors determined by the pairings $\mu : R \otimes_\kappa \mathfrak{D}^{(1)}(R) \to \kappa$ and $\mu' : R' \otimes_\kappa \mathfrak{D}^{(1)}(R') \to \kappa$. The modules $M \in \mathrm{LMod}_{R'}$ for which θ_M is an equivalence span a stable subcategory of $\mathrm{LMod}_{R'}$ which includes κ, and therefore contains all Artinian R'-modules (Lemma 14.5.1.1).

To complete the proof of Proposition 14.6.2.1, it suffices to show that the functor $\mathfrak{D}'$ carries $\mathrm{LMod}^{\mathrm{art}}(\mathrm{Mod}_\kappa)$ into $\mathrm{LMod}^{\mathrm{perf}}(\mathrm{Mod}_\kappa)^{\mathrm{op}}$ and induces an equivalence of ∞-categories

$$\mathrm{LMod}^{\mathrm{art}}(\mathrm{Mod}_\kappa) \to \mathrm{LMod}^{\mathrm{perf}}(\mathrm{Mod}_\kappa)^{\mathrm{op}} \times_{\mathrm{Alg}_\kappa^{\mathrm{op}}} \mathrm{Alg}_\kappa^{\mathrm{art}} .$$

Using Corollary HTT.2.4.4.4 , we are reduced to proving that $\mathfrak{D}'$ induces an equivalence of ∞-categories from $\mathrm{LMod}_R^{\mathrm{art}}$ to the opposite of $\mathrm{LMod}_{\mathfrak{D}^{(1)}(R)}^{\mathrm{perf}}$, for every $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$. This is a consequence of Remark 14.5.1.1 together with the following more general assertion:

Proposition 14.6.2.4. *Let κ be a field and let $\mu : A \otimes_\kappa B \to \kappa$ be a map of $\mathbb{E}_1$-algebras over κ which exhibits B as a Koszul dual of A. Then the duality functor $\mathfrak{D}_\mu : \mathrm{LMod}_A^{\mathrm{op}} \to \mathrm{LMod}_B$ restricts to an equivalence $\mathcal{C} \to \mathrm{LMod}_B^{\mathrm{perf}}$, where $\mathcal{C}$ denotes the smallest stable subcategory of LMod_A which contains κ (regarded as a left A-module via the augmentation $A \to A \otimes_\kappa B \xrightarrow{\mu} \kappa$) and is closed under retracts.*

Proof. Let $\mathfrak{D}'_\mu : \mathrm{LMod}_B^{\mathrm{op}} \to \mathrm{LMod}_A$ be as in Notation 14.1.2.3, and let $\mathcal{D}$ denote the full subcategory of LMod_A spanned by those objects M for which the unit map $M \to \mathfrak{D}'_\mu \mathfrak{D}_\mu(M)$ is an equivalence in LMod_A. It is clear that $\mathcal{D}$ is a stable subcategory of LMod_A which is closed under retracts. Since μ exhibits B as a Koszul dual of A, the subcategory $\mathcal{D}$ contains κ so that $\mathcal{C} \subseteq \mathcal{D}$. It follows that the functor $\mathfrak{D}_\mu|_{\mathcal{C}}$ is fully faithful. Moreover, the essential image of $\mathfrak{D}_\mu|_{\mathcal{C}}$ is the smallest stable full subcategory of LMod_B which contains $\mathfrak{D}_\mu(\kappa) \simeq B$ and is closed under retracts: this is the full subcategory $\mathrm{LMod}_B^{\mathrm{perf}} \subseteq \mathrm{LMod}_B$. □

14.6.3 Connectivity Conditions

Let κ be a field, let $A \in \mathrm{Alg}^{\mathrm{aug}}_{\kappa}$, and let $X : \mathrm{Alg}^{\mathrm{art}}_{\kappa} \to \mathcal{S}$ be the formal $\mathbb{E}_1$-moduli problem associated to A (see Theorem 14.0.0.5). We conclude this section with a discussion of the exactness properties of equivalences $\mathrm{QCoh}^!_L(X) \simeq \mathrm{RMod}_A$ and $\mathrm{QCoh}^!_R(X) \simeq \mathrm{LMod}_A$ appearing in Theorem 14.6.0.1. Let $\mathrm{RMod}^{\mathrm{cn}}_A$ and $\mathrm{LMod}^{\mathrm{cn}}_A$ denote the full subcategories spanned by those right and left A-modules whose underlying spectra are connective. Note that the equivalences of Theorem 14.6.0.1 depend functorially on A, and when $A = \kappa$ they are equivalent to the identity functor from the ∞-category Mod_{κ} to itself. Let $*$ denote the final object of $\mathrm{Moduli}^{(1)}_{\kappa}$, so that we have a canonical map of formal moduli problems $* \to X$ (induced by the map of augmented $\mathbb{E}_1$-algebras $\kappa \to A$). It follows that Theorem 14.6.0.1 gives an equivalence

$$\begin{aligned} \mathrm{QCoh}^!_L(X)^{\mathrm{cn}} &\simeq \mathrm{QCoh}^!_L(X) \times_{\mathrm{QCoh}^!_L(*)} \mathrm{QCoh}^!_L(*)^{\mathrm{cn}} \\ &\simeq \mathrm{RMod}_A \times_{\mathrm{RMod}_{\kappa}} \mathrm{RMod}^{\mathrm{cn}}_{\kappa} \\ &\simeq \mathrm{RMod}^{\mathrm{cn}}_A \end{aligned}$$

and, by symmetry, an equivalence $\mathrm{QCoh}^!_R(X)^{\mathrm{cn}} \simeq \mathrm{LMod}^{\mathrm{cn}}_A$. Combining this observation with Proposition 14.5.5.8, we obtain the following result:

Proposition 14.6.3.1. *Let κ be a field, let $A \in \mathrm{Alg}^{\mathrm{aug}}_{\kappa}$, and let $X : \mathrm{Alg}^{\mathrm{art}}_{\kappa} \to \mathcal{S}$ be the formal $\mathbb{E}_1$-moduli problem associated to A (see Theorem 14.0.0.5). Then the fully faithful embeddings*

$$\mathrm{QCoh}_L(X) \hookrightarrow \mathrm{RMod}_A \qquad \mathrm{QCoh}_R(X) \hookrightarrow \mathrm{LMod}_A .$$

of Theorem 14.6.0.1 restrict to equivalences of ∞-categories

$$\mathrm{QCoh}_L(X)^{\mathrm{cn}} \simeq \mathrm{RMod}^{\mathrm{cn}}_A \qquad \mathrm{QCoh}_R(X)^{\mathrm{cn}} \simeq \mathrm{LMod}^{\mathrm{cn}}_A .$$

Warning 14.6.3.2. If A is an arbitrary $\mathbb{E}_1$-ring, then the full subcategories

$$\mathrm{LMod}^{\mathrm{cn}}_A = \mathrm{LMod}_A \times_{\mathrm{Sp}} \mathrm{Sp}^{\mathrm{cn}} \subseteq \mathrm{LMod}_A \qquad \mathrm{RMod}^{\mathrm{cn}}_A = \mathrm{RMod}_A \times_{\mathrm{Sp}} \mathrm{Sp}^{\mathrm{cn}} \subseteq \mathrm{RMod}_A$$

are presentable, closed under small colimits, and closed under extensions. It follows from Proposition HA.1.4.4.11 that the ∞-categories LMod_A and RMod_A admit t-structures with $(\mathrm{LMod}_A)_{\geqslant 0} = \mathrm{LMod}^{\mathrm{cn}}_A$ and $(\mathrm{RMod}_A)_{\geqslant 0} = \mathrm{RMod}^{\mathrm{cn}}_A$. However, it is often difficult to describe the subcategories $(\mathrm{LMod}_A)_{\leqslant 0} \subseteq \mathrm{LMod}_A$ and $(\mathrm{RMod}_A)_{\leqslant 0} \subseteq \mathrm{RMod}_A$. In particular, they generally do not coincide with the subcategories $\mathrm{LMod}_A \times_{\mathrm{Sp}} \mathrm{Sp}_{\leqslant 0} \subseteq \mathrm{LMod}_A$ and $\mathrm{RMod}_A \times_{\mathrm{Sp}} \mathrm{Sp}_{\leqslant 0} \subseteq \mathrm{RMod}_A$ unless the $\mathbb{E}_1$-ring A is connective.

Chapter 15

Moduli Problems for $\mathbb{E}_n$-Algebras

Let κ be a field. In Chapters 13 and 14 we studied the ∞-categories Moduli_κ and $\mathrm{Moduli}_\kappa^{(1)}$ consisting of formal moduli problems defined for commutative and associative algebras over κ, respectively. In the ∞-categorical context, there is a whole hierarchy of algebraic notions lying between these two extremes. Recall that the commutative ∞-operad can be identified with the colimit of a sequence

$$\mathrm{Assoc}^{\otimes} \simeq \mathbb{E}_1^{\otimes} \to \mathbb{E}_2^{\otimes} \to \mathbb{E}_3^{\otimes} \to \cdots ,$$

where $\mathbb{E}_n^{\otimes}$ denotes the Boardman-Vogt ∞-operad of little n-cubes (see Corollary HA.5.1.1.5). Consequently, the ∞-category CAlg_κ of $\mathbb{E}_\infty$-algebras over κ can be identified with the limit of a tower of ∞-categories

$$\cdots \to \mathrm{Alg}_\kappa^{(3)} \to \mathrm{Alg}_\kappa^{(2)} \to \mathrm{Alg}_\kappa^{(1)} \simeq \mathrm{Alg}_\kappa,$$

where $\mathrm{Alg}_\kappa^{(n)}$ denotes the ∞-category of $\mathbb{E}_n$-algebras over κ. Our goal in this section is to prove a generalization of Theorem 14.0.0.5 in the setting of $\mathbb{E}_n$-algebras, for an arbitrary integer $n \geqslant 0$. To formulate our result, we need a bit of terminology.

Definition 15.0.0.1. Let κ be a field, let $n \geqslant 1$, and let A be an $\mathbb{E}_n$-algebra over κ. We will say that A is *Artinian* if its image in Alg_κ is Artinian, in the sense of Definition 14.0.0.1. We let $\mathrm{Alg}_\kappa^{(n),\mathrm{art}}$ denote the full subcategory of $\mathrm{Alg}_\kappa^{\mathrm{art}}$ spanned by the Artinian $\mathbb{E}_n$-algebras over κ.

Warning 15.0.0.2. The terminology of Definition 15.0.0.1 is potentially confusing: when A is discrete and $n \geqslant 2$, it corresponds to the condition that A is a local Artinian ring with residue field κ, rather than merely an Artinian ring. See Warnings 12.1.2.6 and 14.0.0.2.

Remark 15.0.0.3. Let $n \geqslant 1$ and let A be an $\mathbb{E}_n$-algebra over κ. Then A is Artinian if and only if it is connective, $\pi_* A$ is finite-dimensional over κ, and the unit map $\kappa \to (\pi_0 A)/\mathfrak{m}$ is an isomorphism, where $\mathfrak{m}$ denotes the radical of $\pi_0 A$.

Remark 15.0.0.4. Let κ be a field and let A be an $\mathbb{E}_n$-algebra over κ, for $n \geqslant 0$. An *augmentation* on A is a map of $\mathbb{E}_n$-algebras $\epsilon : A \to \kappa$. We let $\mathrm{Alg}_\kappa^{(n),\mathrm{aug}} = (\mathrm{Alg}_\kappa^{(n)})_{/\kappa}$ denote the ∞-category of augmented $\mathbb{E}_n$-algebras over κ. Note that if $n \geqslant 1$ and $A \in \mathrm{Alg}_\kappa^{(n)}$ is Artinian, then the space $\mathrm{Map}_{\mathrm{Alg}_\kappa^{(n)}}(A, \kappa)$ of augmentations on A is contractible. It follows that the projection map $\mathrm{Alg}_\kappa^{(n),\mathrm{aug}} \times_{\mathrm{Alg}_\kappa^{(n)}} \mathrm{Alg}_\kappa^{(n),\mathrm{art}} \to \mathrm{Alg}_\kappa^{(n),\mathrm{art}}$ is an equivalence of ∞-categories. We will henceforth abuse notation by using this equivalence to identify $\mathrm{Alg}_\kappa^{(n),\mathrm{art}}$ with its inverse image in $\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$.

Remark 15.0.0.5. It will be convenient to have a version of Definition 14.0.0.1 also in the case $n = 0$. We therefore adopt the following convention: we will say that an augmented $\mathbb{E}_0$-algebra A over κ is *Artinian* if A is connective and $\pi_* A$ is a finite dimensional vector space over κ. We let $\mathrm{Alg}_\kappa^{(0),\mathrm{art}}$ denote the full subcategory of $\mathrm{Alg}_\kappa^{(0),\mathrm{aug}}$ spanned by the Artinian augmented $\mathbb{E}_0$-algebras over κ.

Notation 15.0.0.6. Let κ be a field, let $n \geqslant 0$, and let $\epsilon : A \to \kappa$ be an augmented $\mathbb{E}_n$-algebra over κ. We let $\mathfrak{m}_A$ denote the fiber of the map ϵ in the stable ∞-category Mod_κ. We will refer to $\mathfrak{m}_A$ as the *augmentation ideal* of A. The construction $(\epsilon : A \to \kappa) \mapsto \mathfrak{m}_A$ determines a functor $\mathfrak{m} : \mathrm{Alg}_\kappa^{(n),\mathrm{aug}} \to \mathrm{Mod}_\kappa$. In the case $n = 0$, this functor is an equivalence of ∞-categories.

Definition 15.0.0.7. Let κ be a field, let $n \geqslant 0$ be an integer and let $X : \mathrm{Alg}_\kappa^{(n),\mathrm{art}} \to \mathcal{S}$ be a functor. We will say that X is a *formal* $\mathbb{E}_n$*-moduli problem* if it satisfies the following conditions:

(1) The space $X(\kappa)$ is contractible.

(2) For every pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

in $\mathrm{Alg}_\kappa^{\mathrm{art}}$ for which the underlying maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, the diagram

$$\begin{array}{ccc} X(R) & \longrightarrow & X(R_0) \\ \downarrow & & \downarrow \\ X(R_1) & \longrightarrow & X(R_{01}) \end{array}$$

is a pullback square.

We let $\mathrm{Moduli}_\kappa^{(n)}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Alg}_\kappa^{(n),\mathrm{art}}, \mathcal{S})$ spanned by the formal $\mathbb{E}_n$-moduli problems.

Example 15.0.0.8. It is not difficult to show that a functor $X : \mathrm{Alg}_{\kappa}^{(0),\mathrm{art}} \to \mathcal{S}$ is a formal $\mathbb{E}_0$ moduli problem if and only if it is reduced and excisive (see Definition HA.1.4.2.1): that is, if and only if X carries the initial object of $\mathrm{Alg}_{\kappa}^{(0),\mathrm{art}}$ to a final object of $\mathcal{S}$, and carries pushout squares to pullback squares.

We can now state the main result of this section:

Theorem 15.0.0.9. *Let κ be a field and let $n \geqslant 0$ be an integer. Then there is an equivalence of ∞-categories $\Psi : \mathrm{Alg}_{\kappa}^{(n),\mathrm{aug}} \to \mathrm{Moduli}_{\kappa}^{(n)}$. Moreover, the diagram*

$$\begin{array}{ccc} \mathrm{Alg}_{\kappa}^{(n),\mathrm{aug}} & \xrightarrow{\Psi} & \mathrm{Moduli}_{\kappa}^{(n)} \\ \downarrow{\scriptstyle \mathfrak{m}} & & \downarrow{\scriptstyle \Sigma^{-n}T} \\ \mathrm{Mod}_{\kappa} & \longrightarrow & \mathrm{Sp} \end{array}$$

commutes up to canonical homotopy, where $T : \mathrm{Moduli}_{\kappa}^{(n)} \to \mathrm{Sp}$ denotes the tangent complex functor (so that $\Omega^{\infty - m} T_X \simeq X(\kappa \oplus \Sigma^m(\kappa))$ for $m \geqslant 0$) and $\mathfrak{m} : \mathrm{Alg}_{\kappa}^{(n),\mathrm{aug}} \to \mathrm{Mod}_{\kappa}$ denotes the augmentation ideal functor of Notation 15.0.0.6.

Example 15.0.0.10. When $n = 1$, Theorem 15.0.0.9 follows from Theorem 14.0.0.5 and Remark 14.2.2.2.

Remark 15.0.0.11. Suppose that κ is a field of characteristic zero. For each $n \geqslant 0$, there is an evident forgetful functor $\mathrm{CAlg}_{\kappa}^{\mathrm{art}} \to \mathrm{Alg}_{\kappa}^{(n),\mathrm{art}}$, which induces a forgetful functor $\theta : \mathrm{Moduli}_{\kappa}^{(n)} \to \mathrm{Moduli}_{\kappa}$. Using the equivalences

$$\mathrm{Lie}_{\kappa} \simeq \mathrm{Moduli}_{\kappa} \qquad \mathrm{Moduli}_{\kappa}^{(n)} \simeq \mathrm{Alg}_{\kappa}^{(n),\mathrm{aug}}$$

of Theorems 13.0.0.2 and 15.0.0.9, we can identify θ with a functor from $\mathrm{Alg}_{\kappa}^{(n),\mathrm{aug}}$ to Lie_{κ}. We can summarize the situation informally as follows: if A is an augmented $\mathbb{E}_n$-algebra over κ, then the shifted augmentation ideal $\Sigma^{n-1}\mathfrak{m}_A$ inherits the structure of a differential graded Lie algebra over κ. In particular, at the level of homotopy groups we obtain a Lie bracket operation

$$[,] : \pi_p \mathfrak{m}_A \times \pi_q \mathfrak{m}_A \to \pi_{p+q+n-1} \mathfrak{m}_A.$$

One can show that this Lie bracket is given by the *Browder operation* on $\mathfrak{m}_A$. If $\mathrm{Free} : \mathrm{Mod}_{\kappa} \to \mathrm{Alg}_{\kappa}^{(n)}$ denotes the free algebra functor (left adjoint to the forgetful functor $\mathrm{Alg}_{\kappa}^{(n)} \to \mathrm{Mod}_{\kappa}$), then the Browder operation is universally represented by the the map ϕ appearing in the cofiber sequence of augmented $\mathbb{E}_n$-algebras

$$\mathrm{Free}(\Sigma^{p+q+n-1}(\kappa)) \xrightarrow{\phi} \mathrm{Free}(\Sigma^p(\kappa) \oplus \Sigma^q(\kappa)) \to \mathrm{Free}(\Sigma^p(\kappa)) \otimes_{\kappa} \mathrm{Free}(\Sigma^q(\kappa))$$

supplied by Theorem HA.5.3.3.3 .

The appearance of the theory of $\mathbb{E}_n$-algebras on both sides of the equivalence

$$\mathrm{Alg}_\kappa^{(n),\mathrm{aug}} \simeq \mathrm{Moduli}_\kappa^{(n)} \subseteq \mathrm{Fun}(\mathrm{Alg}_\kappa^{(n),\mathrm{art}}, \mathcal{S})$$

is somewhat striking: it is a reflection of the Koszul self-duality of the little cubes operads $\mathbb{E}_n^{\otimes}$ (see [68]). In particular, there is a Koszul duality functor $\mathfrak{D}^{(n)} : (\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$. This functor is not difficult to define directly : if A is an augmented $\mathbb{E}_n$-algebra over κ, then $\mathfrak{D}^{(n)}(A)$ is universal among $\mathbb{E}_n$-algebras over κ for which the tensor product $A \otimes_\kappa \mathfrak{D}^{(n)}(A)$ admits an augmentation extending the augmentation on A. The equivalence Ψ appearing in the statement of Theorem 15.0.0.9 carries an augmented $\mathbb{E}_n$-algebra A to the functor X given by the formula $X(R) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}}(\mathfrak{D}^{(n)}(R), A)$.

In §15.3, we will prove that the Koszul duality functor $\mathfrak{D}^{(n)}$ is a deformation theory (in the sense of Definition 12.3.3.2), so that Theorem 15.0.0.9 is a consequence of Theorem 12.3.3.5. The main point is to produce a full subcategory $\mathbf{B}_0 \subseteq \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ which satisfies axiom $(D3)$ of Definition 12.3.1.1. We will define $\mathbf{B}_0$ to be the full subcategory of $\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ spanned by those augmented $\mathbb{E}_n$-algebras A which satisfy suitable finiteness and coconnectivity conditions. We will then need to prove two things:

- (a) The full subcategory $\mathbf{B}_0 \subseteq \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ has good closure properties.
- (b) For every object $A \in \mathbf{B}_0$, the biduality map $A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}(A)$ is an equivalence.

The verification of (a) comes down to connectivity properties of free algebras over the $\mathbb{E}_n$-operad. We will establish these properties in §15.1, using topological properties of configuration spaces of points in Euclidean space. We will prove (b) in §15.2. Our strategy is to use the description of Koszul duality in terms of iterated bar constructions (see §HA.5.2.3) to reduce to the case $n = 1$, which was analyzed in detail in Chapter 14.

Contents

15.1 Coconnective $\mathbb{E}_n$-Algebras

Let κ be a field and let $n \geqslant 0$ be an integer. Our goal in this section is to study some finiteness and coconnectivity conditions on $\mathbb{E}_n$-algebras over κ which will be relevant to our proof of Theorem 15.0.0.9.

Definition 15.1.0.1. Let κ be a field and let A be an $\mathbb{E}_0$-algebra over κ: that is, a κ-module equipped with a unit map $e : \kappa \to A$. Let m be an integer. We will say that A is *m-coconnective* if the homotopy groups $\pi_i \operatorname{cofib}(e)$ vanish for $i > -m$.

More generally, if A is an $\mathbb{E}_n$-ring equipped with a map of $\mathbb{E}_n$-rings $\kappa \to A$, we will say that A is *m-coconnective* if it is m-coconnective when regarded as an $\mathbb{E}_0$-algebra over κ (here we need not require that A is an $\mathbb{E}_n$-algebra over κ, though this will always be satisfied in cases of interest to us).

Remark 15.1.0.2. If A is an $\mathbb{E}_1$-algebra over a field κ, then A is coconnective (in the sense of Definition 14.1.3.1) if and only if it is 1-coconnective (in the sense of Definition 15.1.0.1).

Remark 15.1.0.3. If $m > 0$, then an $\mathbb{E}_n$-algebra A over κ is m-coconnective if and only if the unit map $\kappa \to A$ induces an isomorphism $\kappa \to \pi_0 A$ and the homotopy groups $\pi_i A$ vanish for $i > 0$ and $-m < i < 0$.

Notation 15.1.0.4. Let κ be a field and let $n \geqslant 0$ be an integer. We let $\operatorname{Free}^{(n)} : \operatorname{Mod}_\kappa \to \operatorname{Alg}_\kappa^{(n)}$ denote a left adjoint to the forgetful functor $\operatorname{Alg}_\kappa^{(n)} \to \operatorname{Mod}_\kappa$. For any object $V \in \operatorname{Mod}_\kappa$, the free algebra $\operatorname{Free}^{(n)}(V)$ is equipped with a canonical augmentation $\epsilon : \operatorname{Free}^{(n)}(V) \to \kappa$, corresponding to the zero morphism $V \to \kappa$ in $\operatorname{Mod}_\kappa$.

Our main result can be stated as follows:

Theorem 15.1.0.5. *Let κ be a field, let A be an $\mathbb{E}_n$-algebra over κ, and let $m \geqslant n$ be an integer. Suppose we are given a map $\phi : V \to A$ in $\operatorname{Mod}_\kappa$, where $\pi_i V \simeq 0$ for $i \geqslant -m$, and form a pushout diagram*

$$\begin{array}{ccc} \operatorname{Free}^{(n)}(V) & \xrightarrow{\phi'} & A \\ \downarrow{\scriptstyle \epsilon} & & \downarrow \\ \kappa & \longrightarrow & A' \end{array}$$

where ϕ' is the map of $\mathbb{E}_n$-algebras determined by ϕ and ϵ is the augmentation of Notation 15.1.0.4. If A is m-coconnective, then A' is also m-coconnective.

Our proof of Theorem 15.1.0.5 is somewhat indirect, and will require several digressions.

15.1.1 Dimensions of Configuration Spaces

Theorem 15.1.0.5 relies on having a good understanding of the free algebra functor $\mathrm{Free}^{(n)} : \mathrm{Mod}_\kappa \to \mathrm{Alg}_\kappa^{(n)}$. Recall that for $V \in \mathrm{Mod}_\kappa$, the underlying κ-module spectrum of $\mathrm{Free}^{(n)}(V)$ is given by $\bigoplus_{m \geqslant 0} \mathrm{Sym}^m_{\mathbb{E}_n}(V)$, where $\mathrm{Sym}^m_{\mathbb{E}_n}(V)$ is the colimit of a diagram indexed by the full subcategory $K_{m,n} \subseteq (\mathbb{E}_n^{\otimes})_{/\langle 1 \rangle}$ spanned by the active morphisms $\langle m \rangle \to \langle 1 \rangle$ in the ∞-operad $\mathbb{E}_n^{\otimes}$ (see Proposition HA.3.1.3.13). The geometric input to our proof of Theorem 15.1.0.5 is the following geometric fact about the Kan complexes $K_{m,n}$:

Proposition 15.1.1.1. *Let m and n be positive integers. Then the Kan complex $K_{m,n}$ defined above is homotopy equivalent to $\mathrm{Sing}(X)$, where X is a finite CW complex of dimension $\leqslant (m-1)(n-1)$.*

Remark 15.1.1.2. The Kan complex $K_{m,n}$ appearing in Proposition 15.1.1.1 is homotopy equivalent to the labelled configuration space $\mathrm{Conf}_m(\mathbf{R}^n)$ parametrizing m-tuples of distinct points in $\mathbf{R}^n$. This space is contractible if $m = 1$, and otherwise admits a fibration $\mathrm{Conf}_m(\mathbf{R}^n) \to \mathrm{Conf}_{m-1}(\mathbf{R}^n)$ whose fibers are homotopy equivalent to a bouquet of spheres having dimension $(n-1)$. One can use this observation to prove Proposition 15.1.1.1 using induction on m. We will give a different argument below, which uses induction on n rather than m.

Lemma 15.1.1.3. *Fix an integer $b \geqslant 0$, and let Q_b denote the set of sequences $(I_1, \ldots, I_b)$, where each $I_j \subseteq (-1, 1)$ is a closed interval, and we have $x < y$ whenever $x \in I_i$, $y \in I_j$, and $i < j$. We regard Q as a partially ordered set, where $(I_1, \ldots, I_b) \leqslant (I'_1, \ldots, I'_b)$ if $I_j \subseteq I'_j$ for $1 \leqslant i \leqslant j$. Then the nerve $\mathrm{N}(Q_b)$ is weakly contractible.*

Proof. The proof proceeds by induction on b. If $b = 0$, then Q_b has a single element and there is nothing to prove. Otherwise, we observe that "forgetting" the last coordinate induces a Cartesian fibration $q : \mathrm{N}(Q_b) \to \mathrm{N}(Q_{b-1})$. We will prove that the fibers of q are weakly contractible, so q is left cofinal (Lemma HTT.4.1.3.2) and therefore a weak homotopy equivalence. Fix an element $x = ([t_1, t'_1], [t_2, t'_2], \ldots, [t_{b-1}.t'_{b-1}]) \in Q_{b-1}$. Then $q^{-1}\{x\}$ can be identified with the nerve of the partially ordered set $Q' = \{(t_b, t'_b) : t'_{b-1} < t_b < t'_b < 1\}$, where $(t_b, t'_b) \leqslant (s_b, s'_b)$ if $t_b \geqslant s_b$ and $t'_b \leqslant s'_b$.

The map $(t_b, t'_b) \mapsto t'_b$ is a monotone map from Q' to the open interval $(t'_{b-1}, 1)$. This map determines a coCartesian fibration $q' : \mathrm{N}(Q') \to \mathrm{N}(t'_{b-1}, 1)$. The fiber of q' over a point s can be identified with the *opposite* of the nerve of the interval (t'_{b-1}, s), and is therefore weakly contractible. It follows that q' is a weak homotopy equivalence, so that $\mathrm{N}(Q')$ is weakly contractible as desired. □

Proof of Proposition 15.1.1.1. For every topological space X, let $\mathrm{Sym}^m(X)$ denote the quotient of X^m by the action of the symmetric group Σ_m, and let $\mathrm{Conf}_m(X)$ denote the subspace of $\mathrm{Sym}^m(X)$ corresponding to m-tuples of *distinct* points in X. Let $\square^n = (-1,1)^n$ denote an open cube of dimension n. Using Lemma HA.5.1.1.3 , we obtain a homotopy equivalence $K_{m,n} \simeq \mathrm{Sing}(\mathrm{Conf}_m(\square^n))$. It will therefore suffice to show that the configuration space $\mathrm{Conf}_m(\square^n)$ is homotopy equivalent to a finite CW complex of dimension $\leqslant (m-1)(n-1)$. If $n = 1$, then $\mathrm{Conf}_m(\square^n)$ is contractible and there is nothing to prove. We prove the result in general by induction on n. Let us therefore assume that $K_{m',n-1}$ is equivalent to the singular complex of a CW complex of dimension $\leqslant (m'-1)(n-2)$ for every integer $m' \geqslant 1$.

Let us identify $\square^n$ with a product $\square^{n-1} \times (-1,1)$, and let $p_0 : \square^n \to \square^{n-1}$ and $p_1 : \square^n \to (-1,1)$ be the projection maps. If $I \subseteq (-1,1)$ is a disjoint union of finitely many closed intervals (with nonempty interiors), we let $[t] \in \pi_0 I$ denote the connected component containing t for each $t \in I$. Then $\pi_0 I$ inherits a linear ordering, with $[t] < [t']$ if and only if $t < t'$ and $[t] \neq [t']$. Let P denote the partially ordered set of triples $(I, \sim, \mu)$, where $I \subseteq (-1,1)$ is a nonempty disjoint union of finitely many closed intervals, $\sim$ is an equivalence relation on $\pi_0 I$ such that $x < y < z$ and $x \sim z$ implies $x \sim y \sim z$, and $\mu : \pi_0 I \to \mathbf{Z}_{>0}$ is a positive integer-valued function such that $\sum_{x \in \pi_0 I} \mu(x) = m$. We regard $(I, \sim, \mu)$ as a partially ordered set, with $(I, \sim, \mu) \leqslant (I', \sim', \mu')$ if $I \subseteq I'$, $\mu'(x) = \sum_y \mu(y)$ where the sum is taken over all $y \in \pi_0 I$ contained in x, and $[s] \sim' [t]$ implies $[s] \sim [t]$ for all $s, t \in I$. For every pair $(I, \sim, \mu) \in P$, we let $Z(I, \sim, \mu)$ be the open subset of $\mathrm{Conf}_m(\square^n)$ consisting of subsets $S \subseteq \square^n$ which are contained in $\square^{n-1} \times I^{\circ}$ (here I° denotes the interior of I), have the property that if $s, s' \in S$ and $[p_1(s)] \sim [p_1(s')]$, then either $s = s'$ or $p_0(s) \neq p_0(s')$, and satisfy $\mu(x) = |\{s \in S : p_1(s) \in x\}|$ for $x \in \pi_0 I$.

We next claim:

$(*)$ The Kan complex $\mathrm{Sing}(\mathrm{Conf}_m(\square^n))$ is a homotopy colimit of the diagram of simplicial sets
$$\{\mathrm{Sing}(Z(I, \sim, \mu))\}_{(I, \sim, \mu) \in P}.$$

To prove this, it will suffice (by Theorem HA.A.3.1) to show that for each point $S \in \mathrm{Conf}_m(\square^n)$, the partially ordered set $P_S = \{(I, \sim, \mu) \in P : S \in Z(I, \sim, \mu)\}$ has weakly contractible nerve. Let Q denote the collection of all equivalence relations $\sim$ on S with the following properties:

(i) If $p_1(s) \leqslant p_1(s') \leqslant p_1(s'')$ and $s \sim s''$, then $s \sim s' \sim s''$.

(ii) If $s \sim s'$, then either $s = s'$ or $p_0(s) \neq p_0(s')$.

We regard Q as a partially ordered set with respect to refinement. Pullback of equivalence relations determines a forgetful functor $\phi : \mathrm{N}(P_S) \to \mathrm{N}(Q)^{\mathrm{op}}$. It is easy to see that μ is a Cartesian fibration. The simplicial set $\mathrm{N}(Q)$ is weakly contractible, since Q has a smallest element (given by the equivalence relation where $s \sim s'$ if and only if $s = s'$). We will complete the proof of $(*)$ by showing that the fibers of ϕ are weakly contractible, so that ϕ is left cofinal (Lemma HTT.4.1.3.2) and therefore a weak homotopy equivalence.

Fix an equivalence relation $\sim \in Q$. Unwinding the definitions, we see that $\phi^{-1}\{\sim\}$ can be identified with the nerve of the partially ordered set R consisting of those subsets $I \subseteq (-1, 1)$ satisfying the following conditions:

(a) The set I is a disjoint union of closed intervals in $(-1, 1)$.

(b) The set I contains $p_1(S)$, and p_1 induces a surjection $S \to \pi_0 I$.

(c) If $p_1(s)$ and $p_1(s')$ belong to the same connected component of I, then $s \sim s'$.

To see that $\mathrm{N}(R)$ is contractible, it suffices to observe that the partially ordered set R^{op} is filtered: it has a cofinal subset given by sets of the form $\bigcup_{s \in S}[p_1(s) - \epsilon, p_1(s) + \epsilon]$ for $\epsilon > 0$. This completes the proof of $(*)$.

We define a category $\mathcal{J}$ as follows:

- An object of $\mathcal{J}$ is a triple $([a], \sim, \mu)$ where $[a] \in \mathbf{\Delta}$, $\sim$ is an equivalence relation on $[a]$ such that $i < j < k$ and $i \sim k$ implies that $i \sim j \sim k$, and $\mu : [a] \to \mathbf{Z}_{>0}$ is a function satisfying $m = \sum_{0 \leqslant i \leqslant a} \mu(i)$.

- A morphism from $([a], \sim, \mu)$ to $([a'], \sim', \mu')$ in $\mathcal{J}$ is a nondecreasing map $\alpha : [a] \to [a']$ such that $\alpha(j) \sim \alpha(j')$ implies $j \sim j'$ and $\mu'(j) = \sum_{\alpha(i)=j} \mu(i)$.

There is an evident forgetful functor $q : P \to \mathcal{J}$, which carries a pair $(I, \sim, \mu)$ to $(\pi_0 I, \sim, \mu)$ where we abuse notation by identifying $\pi_0 I$ with the linearly ordered set $[a]$ for some $a \geqslant 0$. Let $Z' : \mathcal{J} \to \mathcal{S}\mathrm{et}_{\Delta}$ be a homotopy left Kan extension of Z along q. For each object $([a], \sim, \mu) \in \mathcal{J}$, we can identify $Z'([a], \sim, \mu)$ with the homotopy colimit of the diagram $Z|_{P_{[a],\sim,\mu}}$, where $P_{[a],\sim,\mu}$ denotes the partially ordered set of quadruples $(I, \lambda, \sim', \mu')$ where $I \subseteq (-1, 1)$ is a disjoint union of closed intervals, $\lambda : \pi_0 I \to [a]$ is nondecreasing surjection, $\sim'$ is an equivalence relation on $\pi_0 I$ such that $\lambda([t]) \sim \lambda([t'])$ implies $[t] \sim' [t']$, and $\mu' : \pi_0 I \to \mathbf{Z}_{>0}$ is a map satisfying $\mu(i) = \sum_{\lambda(x)=i} \mu'(x)$ for $0 \leqslant i \leqslant a$. Let $P'_{[a],\sim,\mu}$ be the subset of $P_{[a],\sim,\mu}$ consisting of those quadruples $(I, \lambda, \sim', \mu')$ where λ is a bijection and $\sim'$ is the pullback of $\sim$ along λ. The inclusion $\mathrm{N}(P'_{[a],\sim,\mu}) \to \mathrm{N}(P_{[a],\sim,\mu})$ admits a left adjoint and is therefore left cofinal. It follows that $Z'([a], \sim, \mu)$ can be identified with a homotopy colimit of the diagram $Z|_{P'_{[a],\sim,\mu}}$. Note that Z carries each morphism in $P'_{[a],\sim,\mu}$ to a homotopy equivalence of Kan complexes. Since $P'_{[a],\sim}$ is weakly contractible (Lemma 15.1.1.3), we conclude that the map $Z(I, \sim', \mu') \to Z'([a], \sim, \mu)$ is a weak homotopy equivalence for any $(I, \lambda, \sim', \mu') \in P'_{[a],\sim,\mu}$.

It follows from condition $(*)$ that $\mathrm{Sing}(K_{m,n})$ can be identified with a homotopy colimit of the diagram Z'. We may assume without loss of generality that Z' takes values in Kan complexes, so that Z' determines a functor of ∞-categories $\mathcal{J} \to \mathcal{S}$. We will abuse notation by denoting this map also by Z'. Note that if $([a], \sim, \mu) \in \mathcal{J}$, then $m = \sum_{0 \leqslant i \leqslant a} \mu(i) \geqslant a + 1$. For each object $([a], \sim, \mu) \in \mathcal{J}$, we define the *complexity* $d([a], \sim, \mu)$ to be the sum $|[a]/\sim| + \sum_{0 \leqslant i \leqslant a}(\mu(i) - 1)$. Since $[a]/\sim$ has at least one element, $d([a], \sim, \mu)$ is bounded below by 1 and bounded above by $|[a]| + \sum_{0 \leqslant i \leqslant a} \mu(i) - 1 = \sum_{0 \leqslant i \leqslant a} \mu(i) = m$. Note that for every nonidentity morphism $J \to J'$ in $\mathcal{J}$, we have $d(J) < d(J')$. From this, we deduce that every nondegenerate simplex in the nerve $\mathrm{N}(\mathcal{J})$, corresponding to a sequence of nonidentity morphisms $J_0 \to \cdots \to J_b$ in $\mathcal{J}$, is bounded in length by $b \leqslant m - 1$. It follows immediately that the simplicial set $\mathrm{N}(\mathcal{J})$ has only finitely many nondegenerate simplices. We will prove that for every finite simplicial subset $A \subseteq \mathrm{N}(\mathcal{J})$, the colimit of the diagram $Z'|_A$ is homotopy equivalent to the singular complex of finite CW complex of dimension $\leqslant (n-1)(m-1)$. This is obvious when $A = \emptyset$, and when $A = \mathrm{N}(\mathcal{J})$ it implies the desired result. To carry out the inductive step, assume that A is nonempty so that there is a pushout diagram

$$\begin{array}{ccc} \partial \Delta^b & \longrightarrow & \Delta^b \\ \downarrow & & \downarrow \sigma \\ A_0 & \longrightarrow & A \end{array}$$

for some smaller simplicial subset $A_0 \subseteq \mathrm{N}(\mathcal{J})$. The simplex σ carries the initial vertex $0 \in \Delta^b$ to an object $([a], \sim, \mu) \in \mathcal{J}$, and we have a pushout diagram

$$\begin{array}{ccc} Z'([a], \sim, \mu) \times \partial \Delta^b & \longrightarrow & Z'([a], \sim, \mu) \times \Delta^b \\ \downarrow & & \downarrow \\ \varinjlim(Z'|_{A_0}) & \longrightarrow & \varinjlim(Z'|_A) \end{array}$$

where $\varinjlim(Z'|_{A_0})$ is homotopy equivalent to the singular complex of a finite CW complex of dimension $\leqslant (n-1)(m-1)$. Let $I \subseteq (-1, 1)$ be a finite union of $a + 1$ closed intervals and identify $\pi_0 I$ with $[a]$, so that $Z'([a], \sim, \mu) \simeq Z(I, \sim, \mu)$ is homotopy equivalent to the product $\prod_{x \in [a]/\sim} \mathrm{Conf}_{\mu_x}(\square^{n-1})$, where $\mu_x = \sum_{i \in x} \mu(i)$. Using the inductive hypothesis, we deduce that $Z'([a], \sim, \mu)$ is homotopy equivalent to the nerve of a CW complex of dimension $(m-d)(n-2)$, where d is the cardinality of the quotient $[a]/\sim$. It follows that $\varinjlim(Z'|_A)$ is homotopy equivalent to the singular complex of a CW complex having dimension at most the maximum of $(m-1)(n-1)$ and $(m-d)(n-2)+b \leqslant (m-1)(n-2)+(m-1) = (m-1)(n-1)$, as desired. $\square$

Remark 15.1.1.4. Let $\mathcal{C}$ be a presentable ∞-category equipped with an $\mathbb{E}_n$-monoidal structure, and if $n > 0$ assume that the tensor product on $\mathcal{C}$ preserves coproducts separately in each variable.

Suppose now that $X \in \mathcal{C}$ is a coproduct of objects $\{X_s\}_{s \in S}$. Let $\mathcal{J}$ be the subcategory of $\mathcal{F}\mathrm{in}_*$ consisting of the object $\langle m \rangle$ and its automorphisms. Let $\mathcal{J}(S)$ be the category whose objects are maps of sets $\{1, \ldots, m\} \to S$, and whose morphisms are given by commutative diagrams

$$\begin{array}{ccc} \{1,\ldots,m\} & \xrightarrow{\nu} & \{1,\ldots,m\} \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

where ν is bijective. There is an evident forgetful functor $q : \mathcal{J}(S) \to \mathcal{J}$. Let $K_{m,n}$ be as in the statement of Proposition 15.1.1.1, and let $K_{m,n}(S)$ denote the fiber product $K_{m,n} \times_{\mathrm{N}(\mathcal{J})} \mathrm{N}(\mathcal{J}(S))$.

According to Proposition HA.3.1.3.13 , the free algebra functor $\mathrm{Free}^{(n)} : \mathcal{C} \to \mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{C})$ carries an object $X \in \mathcal{C}$ to the coproduct $\amalg_{m \geqslant 0} \mathrm{Sym}^{(n)}_{\mathbb{E}_n}(X)$, where $\mathrm{Sym}^{(n)}_{\mathbb{E}_n}(X)$ denotes the colimit of a diagram $\phi_X : K_{m,n} \to \mathcal{C}$.

Then ϕ_X can be identified with the left Kan extension (along q) of a functor $\phi_{\{X_s\}_{s \in S}} : \mathcal{J}(S) \to \mathcal{J}$, where $\overline{\beta}$ carries an operation $\gamma : \langle m \rangle \to \langle 1 \rangle$ in $\mathbb{E}_n^{\otimes}$ and a map $\alpha : \langle m \rangle^{\circ} \to S$ to the object $\gamma_!(X_{\alpha(1)}, \ldots, X_{\alpha(m)}) \in \mathcal{C}$ (here $\gamma_! : \mathcal{C}^m \to \mathcal{C}$ denotes the functor determined by the $\mathbb{E}_n$-monoidal structure on $\mathcal{C}$).

For every map $\mu : S \to \mathbf{Z}_{\geqslant 0}$ satisfying $\sum_{s \in S} \mu(s) = m < \infty$, let $\mathcal{J}(S, \mu)$ denote the full subcategory of $\mathcal{J}(S)$ spanned by those maps $\alpha : \langle m \rangle^{\circ} \to S$ such that $\alpha^{-1}\{s\}$ has cardinality $\mu(s)$, and let $K_{m,n}(S, \mu)$ denote the fiber product $K_{m,n}(S) \times_{\mathrm{N}(\mathcal{J}(S))} \mathrm{N}(\mathcal{J}(S, \mu))$. Then $K_{m,n}(S)$ is a disjoint union of the Kan complexes $K_{m,n}(S, \mu)$. It follows that $\mathrm{Sym}^m_{\mathbb{E}_n}(X)$ is a coproduct $\amalg_\mu \varinjlim \overline{\beta}|_{K_{m,n}(\mu)}$. Note that if $T \subseteq S$ and $\mu(s) = 0$ for $s \notin T$, then there is a canonical equivalence

$$\phi_{\{X_s\}_{s \in S}}|_{K_{m,n}(S,\mu)} \simeq \phi_{\{X_s\}_{s \in T}}|_{K_{m,n}(T,\mu|_T)}.$$

It follows that if the cardinality of S is larger than m, then $\mathrm{Sym}^m_{\mathbb{E}_n}(X)$ can be written as a coproduct of objects, each of which is a summand of $\mathrm{Sym}^m_{\mathbb{E}_n}(\amalg_{t \in S - \{s\}} X_t)$ for some $s \in S$.

15.1.2 Local Finiteness of Tensor Products

Let κ be a field. Recall that we say that an object $M \in \mathrm{Mod}_\kappa$ is *locally finite* if each homotopy group $\pi_i M$ is a finite-dimensional vector space over κ (Definition 14.1.3.1). We will need the following closure property of locally finite modules under tensor products:

Proposition 15.1.2.1. *Let A be a coconnective $\mathbb{E}_1$-algebra over a field κ such that $\pi_{-1}A \simeq 0$. Let M be a left A-module, let N be a right A-module. Suppose that A, M, and N are locally finite, and that $\pi_i M \simeq \pi_i N \simeq 0$ for $i > 0$. Then $N \otimes_A M$ is locally finite.*

Our proof of Proposition 15.1.2.1 will make use of the following:

Lemma 15.1.2.2. *Let A be a coconnective $\mathbb{E}_1$-algebra over a field κ such that $\pi_{-1}A \simeq 0$, and let M be a left A-module such that $\pi_i M \simeq 0$ for $i > 0$. Assume that A and M are locally finite. Then there exists a sequence of left A-modules*

$$0 = M(0) \to M(1) \to M(2) \to \cdots$$

with the following properties:

(1) *For each $n > 0$, there exists a locally finite object $V(n) \in (\mathrm{Mod}_\kappa)_{\leqslant -n}$ and a cofiber sequence of left A-modules $A \otimes_\kappa V(n) \to M(n-1) \to M(n)$.*

(2) *There exists an equivalence $\theta : \varinjlim M(n) \simeq M$.*

Proof. We construct $M(n)$ using induction on n, beginning with the case $n = 0$ where we set $M(0) = 0$. Assume that $M(n-1) \in (\mathrm{LMod}_A)_{/M}$ has been constructed, and let $V(n)$ denote the underlying κ-module of the fiber of the map $M(n-1) \to M$. We then define $M(n)$ to be the cofiber of the induced map $A \otimes_\kappa V(n) \to M(n-1)$. This construction produces a sequence of objects $M(0) \to M(1) \to M(2) \to \cdots$ in $(\mathrm{LMod}_A)_{/M}$, hence a map $\theta : \varinjlim M(n) \to M$. We claim that θ is an equivalence of left A-modules. To prove this, it suffices to show that θ is an equivalence in Mod_κ. As an object of Mod_κ, we can identify $\varinjlim M(n)$ with the direct limit of the sequence

$$M(0) \to M(0)/V(1) \to M(1) \to M(1)/V(2) \to \cdots$$

It therefore suffices to show that the map $\varinjlim M(i)/V(i+1) \to M$ is an equivalence in Mod_κ, which is clear (since each cofiber $M(i)/V(i+1)$ is equivalent to M). This proves (2). We next prove the following by a simultaneous induction on n:

(a_n) The map $M(n) \to M$ induces an isomorphism $\pi_i M(n) \to \pi_i M$ for $i > -n$ and an injection for $i = -n$.

(b_n) Each $M(n)$ is locally finite.

(a'_n) The κ-module $V(n+1)$ belongs to $(\mathrm{Mod}_\kappa)_{\leqslant -n-1}$.

(b'_n) The κ-module $V(n+1)$ is locally finite.

Assertions (a_0) and (b_0) are obvious, and the equivalences $(a_n) \Leftrightarrow (a'_n)$ and $(b_n) \Leftrightarrow (b'_n)$ follow from the existence of a long exact sequence

$$\cdots \to \pi_i V(n+1) \to \pi_i M(n) \to \pi_i M \to \pi_{i-1} V(n+1) \to \cdots$$

We will complete the proof by showing that (a'_n) and (b'_n) imply (a'_{n+1}) and (b_{n+1}). Assertion (b_{n+1}) follows from (b'_n) by virtue of the existence of an exact sequence

$$\cdots \to \pi_i(A \otimes_\kappa V(n+1)) \to \pi_i M(n) \to \pi_i M(n+1) \to \pi_{i-1}(A \otimes_\kappa V(n+1)) \to \cdots$$

To prove (a_{n+1}), we note that the identification $M \simeq M(n)/V(n)$ gives a fiber sequence

$$(A/\kappa) \otimes_\kappa V(n) \to M \xrightarrow{\lambda} M(n+1)$$

in Mod_κ, where λ is a right inverse to the A-module map $M(n+1) \to M$. We therefore have an equivalence $M(n+1) \simeq M \oplus ((A/\kappa) \otimes_\kappa \Sigma(V(n))$ in Mod_κ so that $V(n+1) \simeq \Sigma(A/\kappa) \otimes_\kappa V(n)$. Since $\pi_i A/\kappa \simeq 0$ for $i \geqslant -1$, it follows immediately that $(a'_n) \Rightarrow (a'_{n+1})$. □

Proof of Proposition 15.1.2.1. Let κ be a field, let A be a coconnective $\mathbb{E}_1$-algebra over κ satisfying $\pi_1 A \simeq 0$, and let $M \in \mathrm{LMod}_A$ and $N \in \mathrm{RMod}_A$. Suppose that A, M, and N are locally finite, and that $\pi_i M \simeq \pi_i N \simeq 0$ for $i > 0$. We wish to show that $N \otimes_A M$ is also locally finite. Choose $\{M(n)\}_{n \geqslant 0}$ be as in Lemma 15.1.2.2. Then $\pi_i(N \otimes_A M) \simeq \varinjlim \pi_i(N \otimes_A M(n))$. We have cofiber sequences $A \otimes_\kappa V(n) \to M(n-1) \to M(n)$ where $V(n) \in (\mathrm{Mod}_\kappa)_{\leqslant -n}$, hence also cofiber sequences $N \otimes_\kappa V(n) \to N \otimes_A M(n-1) \to N \otimes_A M(n)$ in Mod_κ. Since each $\pi_i V(n)$ is finite-dimensional, the homotopy groups of $N \otimes_\kappa V(n)$ are finite-dimensional. It follows by induction on n that $N \otimes_A M(n)$ is locally finite. Since $\pi_i(N \otimes_\kappa V(n)) \simeq 0$ for $i > -n$, the maps $\pi_i(N \otimes_A M(n-1)) \to \pi_i(N \otimes_A M(n))$ are bijective for $i > -n+1$. It follows that $\pi_i(N \otimes_A M) \simeq \varinjlim \pi_i(N \otimes_A M(n))$ is also a finite dimensional vector space over κ. □

15.1.3 The Algebra $\int A$

To prove Theorem 15.1.0.5, it will be convenient to introduce a device which can be used to translate certain questions about $\mathbb{E}_n$-algebras to questions about $\mathbb{E}_1$-algebras (which can then be analyzed using ideas from Chapter 14).

Construction 15.1.3.1. Let κ be a field and let A be an $\mathbb{E}_n$-algebra over κ. We let $\mathrm{Mod}_A^{\mathbb{E}_n} = \mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_\kappa)$ denote the ∞-category of $\mathbb{E}_n$-modules over A (see §HA.3.3.3). Note that $\mathrm{Mod}_A^{\mathbb{E}_n}$ is a presentable ∞-category (Theorem HA.3.4.4.2) and the forgetful functor $\theta : \mathrm{Mod}_A^{\mathbb{E}_n} \to \mathrm{Mod}_\kappa$ is conservative and preserves small limits and colimits (Corollaries HA.3.4.3.3 , HA.3.4.3.6 , and HA.3.4.4.6). It follows that $\mathrm{Mod}_A^{\mathbb{E}_n}$ is a stable ∞-category. The composite functor

$$\mathrm{Mod}_A^{\mathbb{E}_n} \xrightarrow{\theta} \mathrm{Mod}_\kappa \to \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}$$

preserves small limits and filtered colimits, and is therefore corepresentable by an object $M \in \mathrm{Mod}_A^{\mathbb{E}_n}$ (Proposition HTT.5.5.2.7). Since θ is conservative, the object M generates $\mathrm{Mod}_A^{\mathbb{E}_n}$ in the following sense: an object $N \in \mathrm{Mod}_A^{\mathbb{E}_n}$ vanishes if and only if the abelian groups $\mathrm{Ext}^n_{\mathrm{Mod}_A^{\mathbb{E}_n}}(M, N)$ vanish for every integer n. Applying Theorem HA.7.1.2.1 (and its proof), we see that there exists an $\mathbb{E}_1$-ring $\int A$ and an equivalence of ∞-categories $\mathrm{LMod}_{\int A} \simeq \mathrm{Mod}_A^{\mathbb{E}_n}$ carrying $\int A$ to the module M (this latter condition is equivalent to the requirement that the composition

$$\mathrm{LMod}_{\int A} \simeq \mathrm{Mod}_A^{\mathbb{E}_n} \to \mathrm{Mod}_k \to \mathrm{Sp}$$

is equivalent to the forgetful functor $\mathrm{LMod}_{\int A} \to \mathrm{Sp}$). The $\mathbb{E}_1$-ring $\int A$ can be characterized (up to equivalence) as the $\mathbb{E}_1$-ring of endomorphisms of M in the stable ∞-category $\mathrm{Mod}_A^{\mathbb{E}_n}$.

Remark 15.1.3.2 (Functoriality)**.** In the situation of Construction 15.1.3.1, let $\mathrm{Mod}^{\mathbb{E}_n}$ denote the ∞-category of pairs (A, M), where A is an $\mathbb{E}_n$-algebra over κ and M is an $\mathbb{E}_n$-module over A. We have a presentable fibration $\mathrm{Mod}^{\mathbb{E}_n} \to \mathrm{Alg}_\kappa^{(n)}$, classified by a functor $\chi : \mathrm{Alg}_\kappa^{(n)} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$; here $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denotes the ∞-category whose objects are presentable ∞-categories and whose morphisms are functors which preserve small colimits. Since each $\mathrm{Mod}_A^{\mathbb{E}_n}$ is stable, the functor χ factors as a composition

$$\mathrm{Alg}_\kappa^{(n)} \xrightarrow{\chi'} \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$$

(see Proposition HA.4.8.2.15). Since χ carries the inital object $\kappa \in \mathrm{Alg}_\kappa^{(n)}$ to Mod_κ (see Proposition HA.3.4.2.1), the canonical map $\mathrm{Sp} \to \mathrm{Mod}_\kappa$ allows us to factor χ through a functor $\chi'' : \mathrm{Alg}_\kappa^{(n)} \to \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Sp}/}$. According to Theorem HA.4.8.5.5 , the construction $B \mapsto \mathrm{LMod}_B$ determines a fully faithful embedding $\mathrm{Alg}(\mathrm{Sp}) \to \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Sp}/}$, and the argument of Construction 15.1.3.1 implies that the functor χ'' factors through the essential image of this embedding. It follows that we can regard the construction $A \mapsto \int A$ as a functor $\int : \mathrm{Alg}_\kappa^{(n)} \to \mathrm{Alg}(\mathrm{Sp})$.

Remark 15.1.3.3. Let κ be a field, and regard κ as an $\mathbb{E}_n$-algebra over itself. Then the forgetful functor $\mathrm{Mod}_\kappa^{\mathbb{E}_n} \to \mathrm{Mod}_\kappa$ is an equivalence of ∞-categories (Proposition HA.3.4.2.1), so that we have a canonical equivalence of $\mathbb{E}_1$-rings $\kappa \simeq \int \kappa$. For any $\mathbb{E}_n$-algebra A over κ, the unit map $\kappa \to A$ is a map of $\mathbb{E}_n$-algebras, and therefore induces a map of $\mathbb{E}_1$-rings $\kappa \simeq \int \kappa \to \int A$. In particular, the homotopy groups $\pi_* A$ can be regarded as vector spaces over the field κ.

With more effort, one can show that the map $\kappa \to \int A$ is central: that is, $\int A$ can be regarded as an $\mathbb{E}_1$-algebra over κ. We will not need this fact.

Example 15.1.3.4. If $n = 0$ and A is an $\mathbb{E}_n$-algebra over κ, then the forgetful functor $\mathrm{Mod}_A^{\mathbb{E}_n} \to \mathrm{Mod}_\kappa$ is an equivalence Proposition HA.3.3.3.19). It follows that the map $\kappa \to \int A$ of Remark 15.1.3.3 is an equivalence. That is, $\int : \mathrm{Alg}_\kappa^{(0)} \to \mathrm{Alg}^{(1)}$ can be identified with the constant functor taking the value κ.

Example 15.1.3.5. If $n = 1$ and $A \in \mathrm{Alg}_\kappa^{(n)}$, then there is a canonical equivalence of ∞-categories $\mathrm{Mod}_A^{\mathbb{E}_1} \simeq {}_A\mathrm{BMod}_A(\mathrm{Mod}_\kappa)$ (Theorem HA.4.4.1.28). Using Corollary HA.**??** , we obtain a canonical equivalence of $\mathbb{E}_1$-rings $\int A \simeq A \otimes_\kappa A^{\mathrm{rev}}$, where A^{rev} denotes the $\mathbb{E}_1$-algebra A equipped with the opposite multiplication.

More generally, for any integer $n \geqslant 1$, the inclusion of ∞-operads $\mathbb{E}_1^{\otimes} \to \mathbb{E}_n^{\otimes}$ determines a forgetful functor $\mathrm{Alg}_A^{\mathbb{E}_n} \to \mathrm{Alg}_A^{\mathbb{E}_1}$ which induces a map of $\mathbb{E}_1$-rings $A \otimes_\kappa A^{\mathrm{rev}} \to \int A$. We may therefore regard $\int A$ as an A-A bimodule object of Mod_κ.

Remark 15.1.3.6. Let A be an $\mathbb{E}_n$-algebra over a field κ. Theorem HA.7.3.5.1 supplies a fiber sequence $\Sigma^{n-1} L_{A/\kappa} \to \int A \to A$ of $\mathbb{E}_n$-modules over A, where $L_{A/\kappa}$ denotes the relative cotangent complex of A over κ in the setting of $\mathbb{E}_n$-algebras. If A is connective, then $L_{A/\kappa}$ is also connective. For $n \geqslant 1$, it follows that $\int A$ is also connective (this is also true for $n = 0$; see Example 15.1.3.4), so that the ∞-category $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_\kappa) \simeq \mathrm{Mod}_{\int A}$ inherits a t-structure from the t-structure on Mod_κ. For $n \geqslant 2$, we deduce also that the map $\pi_0 \int A \to \pi_0 A$ is an isomorphism, so that the forgetful functor $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_\kappa) \to \mathrm{LMod}_A$ induces an equivalence of abelian categories $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_\kappa)^{\heartsuit} \simeq \mathrm{LMod}_A^{\heartsuit}$.

Let κ be a field and let A be an $\mathbb{E}_n$-algebra over κ. One can show that the $\mathbb{E}_1$-ring $\int A$ is given by the topological chiral homology $\int_{\mathbf{R}^n - \{0\}} A$ defined in §HA.5.5.2). In what follows, we will not use this description directly. However, we will need the following consequence, for which we supply a direct proof:

Proposition 15.1.3.7. *Let κ be a field and let $n \geqslant 0$. The functor $\int : \mathrm{Alg}_\kappa^{(n)} \to \mathrm{Alg}^{(1)}$ preserves small sifted colimits.*

Proof. Let $\Theta : \mathrm{Alg}^{(1)} \to \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Sp}/}$ be the fully faithful embedding $B \mapsto \mathrm{LMod}_B$ of Theorem HA.4.8.5.5 . To prove that the functor $\int$ preseves small sifted colimits, it will suffice to show that the composite functor $\Theta \circ \int$ preserves small sifted colimits. Since every sifted simplicial set is contractible, it suffices to show that the induced map $\mathrm{Alg}_\kappa^{(n)} \to \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ preserves small sifted colimits (Proposition HTT.4.4.2.9). Using Theorem HA.7.3.4.13 , we see that this functor classifies the stable envelope of the Cartesian fibration $\theta : \mathrm{Fun}(\Delta^1, \mathrm{Alg}_\kappa^{(n)}) \to \mathrm{Fun}(\{1\}, \mathrm{Alg}_\kappa^{(n)})$, classified by the functor $\xi : \mathrm{Alg}_\kappa^{(n)} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ given informally by $A \mapsto (\mathrm{Alg}_\kappa^{(n)})_{/A}$. It will therefore suffice to show that ξ preserves small sifted colimits. Using Theorem HTT.5.5.3.18 , we are reduced to showing that the composite functor $\mathrm{Alg}_\kappa^{(n)} \xrightarrow{\xi} \mathcal{P}\mathrm{r}^{\mathrm{L}} \simeq \mathcal{P}\mathrm{r}^{\mathrm{R}^{\mathrm{op}}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty^{\mathrm{op}}$ preserves small sifted colimits. This functor also classifies the forgetful functor θ (this time as a Cartesian fibration). Fix a sifted ∞-category K and a colimit diagram $\overline{f} : K^{\triangleright} \to \mathrm{Alg}_\kappa^{(n)}$; we wish to show that the Cartesian fibration $\theta' : \mathrm{Fun}(\Delta^1, \mathrm{Alg}_\kappa^{(n)}) \times_{\mathrm{Fun}(\{1\}, \mathrm{Alg}_\kappa^{(n)})} K^{\triangleright}$ is classified by limit diagram $(K^{\triangleright})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. Let $A \in \mathrm{Alg}_\kappa^{(n)}$ denote the image under $\overline{f}$ of the cone point of $K^{\triangleright}$, and for each vertex $v \in K$ let $A_v = \overline{f}(v)$. According to Proposition **??**, it will suffice to verify the following:

(a) The pullback functors $q_v : (\mathrm{Alg}^{(n)}_\kappa)_{/A} \to (\mathrm{Alg}^{(n)}_\kappa)_{/A_v}$, given by $B \mapsto A_v \times_A B$, are jointly conservative. Since K is nonempty, it will suffice to show that for each $v \in K$, the pullback q_v is conservative. To this end, suppose we are given a map $\alpha : B \to B'$ in $(\mathrm{Alg}^{(n)}_\kappa)_{/A}$ such that $q_v(\alpha)$ is an equivalence. Since the fiber of α (as a map of spectra) is equivalent to the fiber of $q_v(\alpha)$ (by virtue of Corollary HA.3.2.2.5), we conclude that α is an equivalence as well.

(b) Let $h \in \mathrm{Fun}_{\mathrm{Alg}^{(n)}_\kappa}(K, \mathrm{Fun}(\Delta^1, \mathrm{Alg}^{(n)}_\kappa))$ be a map which carries each edge of K to a θ-Cartesian morphism in the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Alg}^{(n)}_\kappa)$, corresponding to a natural transformation $\{B_v \to A_v\}_{v \in K}$, and let $\overline{h} \in \mathrm{Fun}_{\mathrm{Alg}^{(n)}_\kappa}(K^{\triangleright}, \mathrm{Alg}^{(n)}_\kappa)$ be an θ-colimit diagram extending h; we wish to show that $\overline{h}$ carries each edge of $K^{\triangleright}$ to a θ-Cartesian morphism in $\mathrm{Fun}(\Delta^1, \mathrm{Alg}^{(n)}_\kappa)$. Unwinding the definitions, we must show that if $B = \varinjlim B_v$, then for each $v \in K$ the diagram σ :

$$\begin{array}{ccc} B_v & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_v & \longrightarrow & A \end{array}$$

is a pullback square in $\mathrm{Alg}^{(n)}_\kappa$. For each $v \in K$, let I_v denote the fiber of the map $B_v \to A_v$ in Mod_κ, and let I be the fiber of the map $B \to A$; we wish to show that each of the canonical maps $I_v \to I$ is an equivalence in Mod_κ. Our assumption on h guarantees that the diagram $v \mapsto I_v$ carries each edge of K to an equivalence in Mod_κ. It will therefore suffice to show that the canonical map $\varinjlim I_v \to I$ is an equivalence in Mod_κ. Since Mod_κ is stable, the formation of fibers commutes with colimits; it will therefore suffice to show that A and B are colimits of the diagrams $\{A_v\}_{v \in K}$ and $\{B_v\}_{v \in K}$ in Mod_κ, respectively. Since K is sifted, this follows because the forgetful functor $\mathrm{Alg}^{(n)}_\kappa \to \mathrm{Mod}_\kappa$ preserves sifted colimits (Proposition HA.3.2.3.1).

□

15.1.4 The Main Step

We are now ready to prove a special case of Theorem 15.1.0.5:

Proposition 15.1.4.1. *Let κ be a field, let A be an $\mathbb{E}_n$-algebra over κ, and let $m \geqslant n$ be an integer. Suppose we are given a map $\phi : V \to A$ in Mod_κ, where $\pi_i V \simeq 0$ for $i \geqslant -m$, and form a pushout diagram*

$$\begin{array}{ccc} \mathrm{Free}^{(n)}(V) & \xrightarrow{\phi'} & A \\ \downarrow{\scriptstyle \epsilon} & & \downarrow \\ \kappa & \longrightarrow & A' \end{array}$$

where ϕ' is the map of $\mathbb{E}_n$-algebras determined by ϕ and ϵ is the augmentation of Notation 15.1.0.4. Assume that A is m-coconnective and that $\int A$ is $(m+1-n)$-coconnective. Then A' is m-coconnective, and $\int A'$ is $(m+1-n)$-coconnective. Moreover, if A, $\int A$, and V are locally finite, then A' and $\int A'$ are locally finite (see Definition 14.1.3.1).

Remark 15.1.4.2. We will show in a moment that the hypothesis $\int A$ is $(m+1-n)$-connective is automatic (Proposition 15.1.5.1).

Proof of Proposition 15.1.4.1. We will assume $n > 0$ (otherwise the result is trivial). Let $\phi'_0 : \mathrm{Free}^{(n)}(V) \to A$ be the map of $\mathbb{E}_n$-algebras induced by the zero map $V \to A$, so that A' can be identified with the colimit of the coequalizer diagram $\mathrm{Free}^{(n)}(V) \underset{\phi'_0}{\overset{\phi'}{\rightrightarrows}} A$ classified by a functor $u_0 : \mathbf{\Delta}^{\mathrm{op}}_{s,\leqslant 1} \to \mathrm{Alg}^{(n)}_\kappa$. Let $u : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Alg}^{(n)}_\kappa$ be a left Kan extension of u_0 along the inclusion $\mathbf{\Delta}^{\mathrm{op}}_{s,\leqslant 1} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$, so that u determines a simplicial object $A_\bullet$ in $\mathrm{Alg}^{(n)}_\kappa$ with $A' \simeq |A_\bullet|$ and $A_p \simeq A \amalg \mathrm{Free}(V^p)$ for all $p \geqslant 0$. Let $R = \int A$ so that $\mathrm{LMod}_R \simeq \mathrm{Mod}^{\mathbb{E}_n}_A(\mathrm{Mod}_\kappa)$ is equipped with an $\mathbb{E}_n$-monoidal structure, where the tensor product is given by the relative tensor product over A. We will need the following estimate:

$(*)$ For each integer $a > 0$, the iterated tensor product $R^{\otimes a}$ belongs to $(\mathrm{Mod}_\kappa)_{\leqslant 0}$. Moreover, if A and $\int A$ are locally finite, then $R^{\otimes a}$ is locally finite.

Suppose first that $n = 1$, so that $R \simeq A \otimes_\kappa A^{\mathrm{rev}}$ (Example 15.1.3.5). Then $R^{\otimes a}$ can be identified with an iterated tensor product $A \otimes_\kappa A \otimes_\kappa \cdots \otimes_\kappa A$ and assertion $(*)$ is obvious. We may therefore assume that $n \geqslant 2$. In this case, A is $m \geqslant n \geqslant 2$-coconnective, so the desired result follows from Corollary 14.1.4.3 and Proposition 15.1.2.1.

Let $V' = R \otimes_\kappa V$ denote the image of V in $\mathrm{Mod}^{\mathbb{E}_n}_A$. For each $p \geqslant 0$, Corollary HA.3.4.1.5 allows us to identify A_p with the free $\mathbb{E}_n$-algebra generated by V' in the $\mathbb{E}_n$-monoidal ∞-category $\mathrm{Mod}^{\mathbb{E}_n}_A$. Using Proposition HA.3.1.3.13 , we obtain an equivalence $A_p \simeq \bigoplus_{a \geqslant 0} \mathrm{Sym}^a_{\mathbb{E}_n} V'^p$ (where the symmetric powers are computed in $\mathrm{Mod}^{\mathbb{E}_n}_A$). Let Q denote the cofiber of the map $A \to A'$ in the stable ∞-category $\mathrm{Mod}^{\mathbb{E}_n}_A(\mathrm{Mod}_\kappa)$, so that Q is given by the geometric realization of a simplicial object $Q_\bullet$ with $Q_p \simeq \bigoplus_{a>0} \mathrm{Sym}^a_{\mathbb{E}_n} V'^p$. To show that A' is m-coconnective, it will suffice to show that $\pi_i Q \simeq 0$ for $i > -m$.

Using Remark HA.1.2.4.4 , we obtain a spectral sequence $\{E^{p,q}_r\}_{r \geqslant 1}$ converging to $\pi_{p+q} Q$, where $E^{*,q}_1$ is the normalized chain complex associated to the simplicial κ-vector space $[p] \mapsto \bigoplus_{a>0} \pi_q \mathrm{Sym}^a_{\mathbb{E}_n}(V'^p)$. It follows from Remark 15.1.1.4 that the summand $\pi_q \mathrm{Sym}^a_{\mathbb{E}_n}(V'^p)$ lies in the image of the degeneracy maps of this simplicial vector space whenever $p > a$.

Note that $\mathrm{Sym}^a_{\mathbb{E}_n}(V'^p)$ is the colimit of a diagram $\phi_{V'} : K_{a,n} \to \mathrm{Mod}^{\mathbb{E}_n}_A(\mathrm{Mod}_\kappa)$ whose value on each vertex is given by $(V'^p)^{\otimes a}$, where $K_{a,n}$ is the Kan complex appearing in the statement of Proposition 15.1.1.1. Since $\pi_i V \simeq 0$ for $i > -m-1$, condition $(*)$ guarantees that $\pi_i (V'^p)^{\otimes a} \simeq 0$ for $i > (-m-1)a$, and that the homotopy groups of $(V'^p)^{\otimes a}$ are finitely generated vector spaces over κ provided that A and R are locally finite. Combining this with Proposition 15.1.1.1, we deduce that if $a > 0$, then $\pi_q \operatorname{Sym}^a_{\mathbb{E}_n}(V'^p)$ vanishes for

$$q > (a-1)(n-1) + (-m-1)a = 1 - 2a - m + (n-m)(a-1)$$

and thus for $q > 1 - 2a - m$.

If the vector space $E_1^{p,q}$ is nonzero, there must be an integer $a > 0$ such that $p \leqslant a$ and $q \leqslant 1 - 2a - m$, so that $p + q \leqslant 1 - a - m \leqslant -m$. This proves that $\pi_i Q \simeq 0$ for $i > p + q$, so that A' is m-coconnective. For any integer i, the inequality $i = p + q \leqslant 1 - a - m$ implies that a is bounded above by $1 - m - i$, so that $\pi_i Q$ admits a finite filtration whose associated graded vector space consists of subquotients of $\pi_{i-p} \operatorname{Sym}^a_{\mathbb{E}_n} V'^p$ where $a \leqslant 1 - m - i$ and $p \leqslant a$. It follows that if A and R are locally finite, then Q is also locally finite, so that A' is locally finite.

To complete the proof, we must show that $\int A'$ is $(1 + m - n)$-coconnective, and that $\int A'$ is locally finite if A and $\int A$ are locally finite. According to Proposition 15.1.3.7, $\int A'$ can be identified with the geometric realization of the simplicial $\mathbb{E}_1$-ring $\int A_\bullet$. Let B be an arbitrary $\mathbb{E}_n$-ring, let $W \in \operatorname{Mod}_\kappa$ and let $W' = (\int B) \otimes_\kappa W$ denote the image of W in $\operatorname{Mod}^{\mathbb{E}_n}_B \simeq \operatorname{LMod}_{\int B}$. Then the coproduct $B \amalg \operatorname{Free}^{(n)}(W)$ can be identified with the free $\mathbb{E}_n$-algebra in $\operatorname{Mod}^{\mathbb{E}_n}_B$ generated by W', which is given by $\bigoplus_{a \geqslant 0} \operatorname{Sym}^a_{\mathbb{E}_n}(W')$. If we let $Z(W)$ denote the cofiber in $\operatorname{Mod}_\kappa$ of the map $B \to B \amalg \operatorname{Free}^{(n)}(W)$, then we obtain an equivalence $\int B \simeq \varinjlim_{b \mapsto \infty} \Omega^b Z(\Sigma^b \kappa)$. Taking $B = A_p$, we obtain an equivalence

$$\int A_p \simeq \varinjlim_{b \mapsto \infty} \Omega^b (\bigoplus_{a \geqslant 0} \operatorname{Sym}^a_{\mathbb{E}_n}(V'^p \oplus \Sigma^b \kappa)) / (\bigoplus_{a \geqslant 0} \operatorname{Sym}^a_{\mathbb{E}_n}(V'^p)).$$

Remark 15.1.1.4 gives a canonical decomposition

$$\operatorname{Sym}^a_{\mathbb{E}_n}(V'^p \oplus \Sigma^b \kappa) \simeq \bigoplus_{a = a' + a''} F_{a', a''}(V'^p, \Sigma^b \kappa).$$

Note that $F_{a-1,1}$ is an exact functor of the second variable, and that if $a'' \geqslant 2$, then the colimit $\varinjlim \Omega^b F_{a', a''}(V'^p, \Sigma^b \kappa)$ vanishes. We therefore obtain an equivalence $\int A_p \simeq \bigoplus_{a > 0} F_{a-1,1}(V'^p, \kappa)$. Unwinding the definitions, we see that $F_{a-1,1}(X, Y)$ is given by the colimit of a diagram $\widetilde{K}_{a,n} \to \operatorname{Mod}^{\mathbb{E}_n}_A(\operatorname{Mod}_\kappa)$, which carries each vertex to the iterated tensor product $(V'^p)^{\otimes a-1} \otimes_A R$, here $\widetilde{K}_{a,n}$ is a finite-sheeted covering space of $K_{m,n}$ and therefore equivalent to the singular complex of a finite CW complex of dimension $\leqslant (a-1)(n-1)$ (Proposition 15.1.1.1). Since condition $(*)$ implies that $\pi_i (V'^p)^{\otimes a-1} \otimes_A R$ vanishes for $i > (-m-1)(a-1)$, we conclude that $\pi_q F_{a-1,1}(V'^p, \kappa) \simeq 0$ for $q > (-m-1)(a-1) + (n-1)(a-1) = (n-m-2)(a-1)$ and therefore for $q > 2 - 2a$. Moreover, if A, V, and $\int A$ are locally finite, then each $\pi_q F_{a-1,1}(V'^p, \kappa)$ is a finite dimensional vector space over κ.

Let Q' denote the spectrum given by the cofiber of the map $\int A \to \int A'$, so that Q' is the geometric realization of a simplicial spectrum $Q'_\bullet$ given by $Q'_p \simeq \bigoplus_{a \geqslant 2} F_{a-1,1}(V'^p, \kappa)$. Using

Remark HA.1.2.4.4 , we obtain a spectral sequence $\{E_r'^{p,q}\}_{r\geqslant 1}$ converging to $\pi_{p+q}Q'$, where $E_1'^{*,q}$ is the normalized chain complex associated to the simplicial κ-vector space $[p] \mapsto \bigoplus_{a\geqslant 2} \pi_q F_{a-1,1}(V'^p, \kappa)$. Arguing as above, we deduce that the summand $\pi_q F_{a-1,1}(V'^p, \kappa)$ lies in the image of the degeneracy maps of this vector space whenever $p \geqslant a$. It follows that if $E_1'^{p,q}$ is nonzero, then there exists an integer $a \geqslant 2$ such that $p < a$ and $q \leqslant 2-2a$, so that $p + q < 2 - a \leqslant 0$. It follows that $\pi_i Q' \simeq 0$ for $i \geqslant 0$, from which we immediately conclude that $\int A'$ is 1-coconnective. For any integer i, the inequality $i = p + q \leqslant 2 - a$ implies that a is bounded above by $2 - i$, so that $\pi_i Q'$ admits a finite filtration whose associated graded vector space consists of subquotients of $\pi_{i-p} F_{a-1,1}(V'^p, \kappa)$ where $2 \leqslant a \leqslant 2 - i$ and $p < a$. If A and $\int A$ are locally finite, then these subquotients are necessarily finite dimensional, so that each $\pi_i Q'$ is a finite dimensional vector space. It then follows that $\int A'$ is locally finite as desired. □

15.1.5 The Proof of Theorem 15.1.0.5

Let κ be a field and let A be an m-connective $\mathbb{E}_n$-algebra over κ, for some integers $m \geqslant n \geqslant 0$. If the algebra $\int A$ is $(m + 1 - n)$-coconnective, then the conclusion of Theorem 15.1.0.5 follows from Proposition 15.1.4.1. Consequently, to prove Theorem 15.1.0.5 in general, it will suffice to establish the following:

Proposition 15.1.5.1. *Let A be an $\mathbb{E}_n$-algebra over a field κ, and assume that A is m-coconnective for $m \geqslant n$. Then the $\mathbb{E}_1$-ring $\int A$ is $(m + 1 - n)$-coconnective.*

Proof. Let A be an $\mathbb{E}_n$-algebra over a field κ, and assume that A is m-coconnective for $m \geqslant n$. We wish to show that $\int A$ is $(m - n + 1)$-coconnective. The result is trivial if $n = 0$ (Example 15.1.3.4); we will therefore assume that $n \geqslant 1$. We construct a sequence of maps $A(0) \to A(1) \to \cdots$ in $(\mathrm{Alg}_\kappa^{(n)})_{/A}$ by induction. Let $A(0) = \kappa$. Assuming that $A(i)$ has already been defined, we let $V(i)$ denote the fiber of the map $A(i) \to A$ (in Mod_κ) and define $A(i + 1)$ so that there is a pushout square

$$\begin{array}{ccc} \mathrm{Free}^{(n)}(V(i)) & \xrightarrow{\phi'} & A(i) \\ \downarrow{\scriptstyle \epsilon} & & \downarrow \\ \kappa & \longrightarrow & A(i+1) \end{array}$$

as in the statement of Proposition 15.1.4.1. We prove the following statements by induction on i:

(a_i) The $\mathbb{E}_n$-algebra $A(i)$ is m-coconnective.

(b_i) The map $\pi_{-m}A(i) \to \pi_{-m}A$ is injective.

(c_i) The $\mathbb{E}_1$-algebra $\int A$ is $(m+1-n)$-coconnective.

(d_i) We have $\pi_j V(i) \simeq 0$ for $j \geqslant -m$.

It is clear that conditions (a_0), (b_0), and (c_0) are satisfied. Note that (a_i) and (b_i) imply (d_i) and that (a_i), (c_i) and (d_i) imply (a_{i+1}) and (c_{i+1}) by Proposition 15.1.4.1. It will therefore suffice to show that (a_i), (b_i), (c_i), and (d_i) imply condition (b_{i+1}). As in the proof of Proposition 15.1.4.1, we can identify $A(i+1)$ with the geometric realization of a simplicial object $A_\bullet$ of $\mathrm{Alg}_\kappa^{(n)}$, with $A_p \simeq A(i) \amalg \mathrm{Free}^{(n)}(V(i)^p)$. Let Q denote the cofiber of the map $A(i) \to A(i+1)$ (as an object of $\mathrm{Mod}_{A_i}^{\mathbb{E}_n}(\kappa)$) we have a canonical map $\phi : Q \to \mathrm{cofib}(A(i) \to A) \simeq \Sigma V(i)$. We wish to prove that ϕ induces an injection $\pi_{-n} Q \to \pi_{-n-1} V(i)$, which follows immediately by inspecting the spectral sequence $\{E_r^{p,q}\}_{r \geqslant 1}$ appearing in the proof of Proposition 15.1.4.1.

We now claim that the canonical map $\theta : \varinjlim A(i) \to A$ is an equivalence. Combining this with assertions (c_i) and Proposition 15.1.3.7, we conclude that $\int A \simeq \varinjlim \int A(i)$ is $(m+1-n)$-connective as desired. To prove that θ is an equivalence, we note that the image of $A(i)$ in Mod_κ can be identified with the colimit of the sequence

$$\kappa \simeq A(0) \to A(0)/V(0) \to A(1) \to A(1)/V(1) \to \cdots,$$

where each cofiber $A(i)/V(i)$ is equivalent to A. □

15.2 Koszul Duality for $\mathbb{E}_n$-Algebras

Our goal in this section is to study the operation of Koszul duality in the setting of augmented $\mathbb{E}_n$-algebras over a field κ. More precisely, we will construct a self-adjoint functor

$$\mathfrak{D}^{(n)} : (\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}.$$

Our main result asserts that for large class of augmented $\mathbb{E}_n$-algebras A, the unit map $A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)} A$ is an equivalence (Theorem 15.2.2.1).

15.2.1 The Definition of Koszul Duality

We begin with the definition of the Koszul duality functor $\mathfrak{D}^{(n)}$. Let A be an $\mathbb{E}_n$-algebra over a field κ. An *augmentation* on A is a map of $\mathbb{E}_n$-algebras $A \to \kappa$. We let $\mathrm{Aug}(A) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{(n)}}(A, \kappa)$ denote the space of augmentations on A. If we are given a pair of augmented $\mathbb{E}_n$-algebras $\epsilon : A \to \kappa$ and $\epsilon' : B \to \kappa$, we let $\mathrm{Pair}(A, B)$ denote the homotopy fiber of the map $\mathrm{Aug}(A \otimes_\kappa B) \to \mathrm{Aug}(A) \times \mathrm{Aug}(B)$, taken over the point (ϵ, ϵ'). More informally, we can describe $\mathrm{Pair}(A, B)$ as the space of augmentations on $A \otimes_\kappa B$ which extend the given augmentations on A and B. We will refer to the points of $\mathrm{Pair}(A, B)$ as *pairings* of A with B. The starting point for our discussion of Koszul duality is the following fact, which is a special case of Proposition HA.5.2.5.1 :

Proposition 15.2.1.1. *Let κ be a field, $n \geqslant 0$ an integer, and A be an augmented $\mathbb{E}_n$-algebra over κ. Then the construction $B \mapsto \mathrm{Pair}(A, B)$ determines a representable functor from $(\mathrm{Alg}_{\kappa}^{(n),\mathrm{aug}})^{\mathrm{op}}$ into $\mathcal{S}$. That is, there exists an augmented $\mathbb{E}_n$-algebra $\mathfrak{D}^{(n)}(A)$ and a pairing $\nu : A \otimes_{\kappa} \mathfrak{D}^{(n)}(A) \to \kappa$ with the following universal property: for every augmented $\mathbb{E}_n$-algebra B, composition with ν induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{Alg}_{\kappa}^{(n),\mathrm{aug}}}(B, \mathfrak{D}^{(n)}(A)) \to \mathrm{Pair}(A, B).$$

In the situation of Proposition, we will refer to $\mathfrak{D}^{(n)}(A)$ as the *Koszul dual* of A. The construction $A \mapsto \mathfrak{D}^{(n)}(A)$ determines a functor $\mathfrak{D}^{(n)} : (\mathrm{Alg}_{\kappa}^{(n),\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_{\kappa}^{(n),\mathrm{aug}}$, which we will refer to as *Koszul duality.*

Remark 15.2.1.2. If A is an augmented $\mathbb{E}_n$-algebra over a field κ, then the Koszul dual $\mathfrak{D}^{(n)}(A)$ can be identified (using the formalism of §HA.5.3.1) with a centralizer of the augmentation map $\epsilon : A \to \kappa$ (see Example HA.5.3.1.5).

Example 15.2.1.3. Suppose that $n = 0$. Then the construction $V \mapsto \kappa \oplus V$ defines an equivalence from the ∞-category Mod_{κ} of κ-module spectra to the ∞-category $\mathrm{Alg}_{\mathrm{aug}}^{(n)}$. If V and W are objects of Mod_{κ}, then a pairing of V with W is a κ-linear map

$$\phi : (\kappa \oplus V) \otimes_{\kappa} (\kappa \oplus W) \simeq \kappa \oplus V \oplus W \oplus (V \otimes_{\kappa} W) \to \kappa$$

equipped with homotopies $\phi|_{\kappa} \simeq \mathrm{id}$, $\phi|_V \simeq 0 \simeq \phi|_W$. It follows that we can identify $\mathrm{Pair}(\kappa \oplus V, \kappa \oplus W)$ with the space $\mathrm{Map}_{\mathrm{Mod}_{\kappa}}(V \otimes_{\kappa} W, \kappa)$. It follows that the Koszul duality functor $\mathfrak{D}^{(0)}$ is given by $\kappa \oplus V \mapsto \kappa \oplus V^{\vee}$, where $V^{\vee}$ is the κ-linear dual of V (with homotopy groups given by $\pi_i V^{\vee} \simeq \mathrm{Hom}_{\kappa}(\pi_{-i} V, \kappa)$).

Example 15.2.1.4. When $n = 1$, the Koszul duality functor $\mathfrak{D}^{(1)} : (\mathrm{Alg}_{\kappa}^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_{\kappa}^{\mathrm{aug}}$ agrees with the functor studied in §14.1.

Remark 15.2.1.5. The construction $A, B \mapsto \mathrm{Pair}(A, B)$ is symmetric in A and B. Consequently, for any pair of augmented $\mathbb{E}_n$-algebras A and B, we have homotopy equivalences

$$\mathrm{Hom}_{\mathrm{Alg}_{\kappa}^{(n),\mathrm{aug}}}(B, \mathfrak{D}^{(n)}(A)) \simeq \mathrm{Pair}(A, B) \simeq \mathrm{Pair}(B, A) \simeq \mathrm{Hom}_{\mathrm{Alg}_{\kappa}^{(n),\mathrm{aug}}}(A, \mathfrak{D}^{(n)}(B)).$$

In particular, the tautological pairing $A \otimes_{\kappa} \mathfrak{D}^{(n)}(A) \to \kappa$ can be identified with a point of $\mathrm{Pair}(\mathfrak{D}^{(n)}(A), A)$, which is classified by a *biduality map* $u_A : A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}(A)$. Our main goal in this section is to study conditions which guarantee that u_A is an equivalence.

15.2.2 Koszul Biduality

We can now formulate the main result of this section:

Theorem 15.2.2.1. *Let $n \geqslant 0$ and let A be an augmented $\mathbb{E}_n$-algebra over a field κ. If A is n-coconnective and locally finite, then the biduality map $u_A : A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}(A)$ is an equivalence of augmented $\mathbb{E}_n$-algebras over κ.*

Remark 15.2.2.2. In the case $n = 0$, Theorem 15.2.2.1 reduces to the statement that for every finite-dimensional vector space V over κ, the biduality map $V \to V^{\vee\vee}$ is an isomorphism (see Example 15.2.2.6 below). In the case $n = 1$, Theorem 15.2.2.1 reduces to Corollary 14.1.3.3.

Let A be an augmented $\mathbb{E}_n$-algebra over a field κ which is n-coconnective and locally finite. To establish Theorem 15.2.2.1, we must show that the biduality map $A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}(A)$ is an equivalence. This is equivalent to the requirement that, for every augmented $\mathbb{E}_n$-algebra B over κ, the canonical map

$$\mathrm{Map}_{\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}}(B, A) \to \mathrm{Map}_{\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}}(B, \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}(A)) \simeq \mathrm{Map}_{\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}}(\mathfrak{D}^{(n)}(A), \mathfrak{D}^{(n)}(B))$$

is a homotopy equivalence. Our strategy is to prove this using induction on n. To make the induction work, we will need to strengthen our inductive hypothesis. Note that the Koszul duality functor $\mathfrak{D}^{(n)} : (\mathrm{Alg}_\kappa^{(n)})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{(n)}$ is lax symmetric monoidal (see Remark HA.5.2.2.25). We will actually prove the following result, which immediately implies Theorem 15.2.2.1:

Proposition 15.2.2.3. *Let κ be a field, let $n \geqslant 0$ be an integer, and suppose we are given a finite collection $\{A_1, \ldots, A_m\}$ of augmented $\mathbb{E}_n$-algebras over κ. Let B be an arbitrary augmented $\mathbb{E}_n$-algebra over κ. If each A_i is n-coconnective and locally finite, then the canonical map*

$$\mathrm{Map}_{\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}}(B, A_1 \otimes_\kappa \cdots \otimes_\kappa A_m) \to \mathrm{Map}_{\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}}(\mathfrak{D}^{(n)}A_1 \otimes_\kappa \cdots \otimes_\kappa \mathfrak{D}^{(n)}A_m, \mathfrak{D}^{(n)}B)$$

is a homotopy equivalence.

Remark 15.2.2.4. The statement Proposition 15.2.2.3 can be reformulated as saying that the canonical map

$$A_1 \otimes_\kappa \cdots \otimes_\kappa A_m \to \mathfrak{D}^{(n)}(\mathfrak{D}^{(n)}A_1 \otimes_\kappa \cdots \otimes_\kappa \mathfrak{D}^{(n)}A_m)$$

is an equivalence of augmented $\mathbb{E}_n$-algebras over κ, provided that the $\mathbb{E}_n$-algebras A_i are coconnective and locally finite.

Warning 15.2.2.5. In the situation of Proposition 15.2.2.3, the tensor product $A = A_1 \otimes_\kappa \cdots \otimes_\kappa A_m$ is also a locally finite n-coconnective augmented $\mathbb{E}_n$-algebra over κ, so that (by virtue of Theorem 15.2.2.1) the biduality map $A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}A$ is an equivalence. It follows that the natural map

$$A_1 \otimes_\kappa \cdots \otimes_\kappa A_m \to \mathfrak{D}^{(n)}(\mathfrak{D}^{(n)}A_1 \otimes_\kappa \cdots \otimes_\kappa \mathfrak{D}^{(n)}A_m)$$

can be identified with the Koszul dual of a map

$$\theta : \mathfrak{D}^{(n)}A_1 \otimes_\kappa \cdots \otimes_\kappa \mathfrak{D}^{(n)}A_m \to \mathfrak{D}^{(n)}(A).$$

With some further assumptions, one can show that θ is an equivalence (and thereby deduce Proposition 15.2.2.3 from Theorem 15.2.2.1). For example, θ is an equivalence if each A_i is $(n+1)$-coconnective. However, θ is not an equivalence in general.

Example 15.2.2.6. For every vector space V over κ, let $V^\vee = \mathrm{Hom}_\kappa(V, \kappa)$ denote the dual vector space. For any object $A \in \mathrm{Alg}_\kappa^{(0),\mathrm{aug}}$, we have canonical isomorphisms $\pi_p \mathfrak{D}^{(0)}(A) \simeq (\pi_{-p}A)^\vee$ (see Example 15.2.1.3). It follows that if $\{A_i\}_{1 \leqslant i \leqslant m}$ is a finite collection of locally finite 0-connective objects of $\mathrm{Alg}_\kappa^{(0),\mathrm{aug}}$, then canonical map

$$\bigoplus_{p = p_1 + \cdots + p_m} \bigotimes_i (\pi_{p_i} A_i) \to \Big(\bigoplus_{p = p_1 + \cdots + p_m} (\bigotimes_i (\pi_{p_i} A_i)^\vee) \Big)^\vee$$

is an isomorphism for every integer p (since $\pi_{p_i} A_i \simeq 0$ for $p_i > 0$, each of the direct sums is essentially finite, and the desired result follows immediately because each $\pi_{p_i} A_i$ is a finite dimensional vector space over κ). Using Remark 15.2.2.4, we deduce that Proposition 15.2.2.3 is true when $n = 0$.

15.2.3 Categorical Generalities

Our proof of Proposition 15.2.2.3 will use the following general observation (which we formulate using the language of ∞-operads developed in [139]):

Proposition 15.2.3.1. *Let $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ be a lax symmetric monoidal functor between symmetric monoidal ∞-categories $p : \mathcal{C}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and $q : \mathcal{D}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Let $\mathcal{C}_0$ be a full subcategory of $\mathcal{C}$ satisfying the following condition:*

- $(*)$ *For every sequence of objects $\{C_i\}_{1 \leqslant i \leqslant m}$ of $\mathcal{C}_0$ and every object $C' \in \mathcal{C}$, the canonical map*

  $$\mathrm{Map}_{\mathcal{C}}(\bigotimes_i C_i, C') \to \mathrm{Map}_{\mathcal{D}}(\bigotimes_i F(C_i), F(C'))$$

  is a homotopy equivalence.

Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and suppose we are given a sequence of algebra objects $\{A_i \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})\}_{1 \leqslant i \leqslant n}$ such that, for each $X \in \mathcal{O}$ and $1 \leqslant i \leqslant n$, we have $A_i(X) \in \mathcal{C}_0$. Then for every object $B \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, the canonical map

$$\mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}(\bigotimes_i A_i, B) \to \mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})}(\bigotimes_i F(A_i), F(B))$$

is a homotopy equivalence.

Proof. Let $\Delta^1 \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ classify the unique active morphism $\langle n \rangle \to \langle 1 \rangle$, and let $\overline{\mathcal{C}} = \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\Delta^1, \mathcal{C}^{\otimes})$. In what follows, we will abuse notation by identifying $\mathcal{C}^{\otimes}_{\langle n \rangle}$ with $\mathcal{C}^n$. The ∞-category $\overline{\mathcal{C}}$ inherits a symmetric monoidal structure from $\mathcal{C}$, and we have symmetric monoidal forgetful functors $(\mathcal{C}^n)^{\otimes} \leftarrow \overline{\mathcal{C}}^{\otimes} \to \mathcal{C}^{\otimes}$. The sequence $(A_1, \ldots, A_n)$ can be identified with a $\mathcal{O}$-algebra object of $\mathcal{C}^n$, and B determines a map $\mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$. We let $\mathcal{C}'^{\otimes} = \overline{\mathcal{C}}^{\otimes} \times_{(\mathcal{C}^n)^{\otimes} \times \mathcal{C}^{\otimes}} \mathcal{O}^{\otimes}$, so that we have a fibration of ∞-operads $\mathcal{C}'^{\otimes} \to \mathcal{O}^{\otimes}$ and $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}')$ can be identified with the mapping space $\mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}(\bigotimes_i A_i, B)$. We define a fibration of ∞-operads $\mathcal{D}'^{\otimes} \to \mathcal{O}^{\otimes}$ similarly, so that $\mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})}(\bigotimes_i F(A_i), F(B)) \simeq \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}')$. We wish to show that F induces a homotopy equivalence of Kan complexes $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}') \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}')$. For this, it suffices to show that for every map of simplicial sets $K \to \mathcal{O}^{\otimes}$, the induced map $\theta : \mathrm{Fun}_{\mathcal{O}^{\otimes}}(K, \mathcal{C}'^{\otimes}) \to \mathrm{Fun}_{\mathcal{O}^{\otimes}}(K, \mathcal{D}'^{\otimes})$ is a homotopy equivalence of Kan complexes. Working simplex-by-simplex, we can assume that $K = \Delta^p$. Then the inclusion $K' = \Delta^{\{0,1\}} \amalg_{\{1\}} \cdots \amalg_{\{p-1\}} \Delta^{\{p-1,p\}} \hookrightarrow K$ is a categorical equivalence; we may therefore replace K by K'. Working simplex-by-simplex again, we can assume that $K = \Delta^p$ for $p = 0$ or $p = 1$. When $p = 0$, the desired result follows immediately from $(*)$. In the case $p = 1$, the map $\Delta^p \to \mathcal{O}^{\otimes}$ determines a morphism $\overline{\alpha} : X \to Y$ in $\mathcal{O}^{\otimes}$. Let $\alpha : \langle m \rangle \to \langle m' \rangle$ be the image of $\overline{\alpha}$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, so that $X \simeq \bigoplus_{j \in \langle m \rangle^{\circ}} X_j$ and $Y = \bigoplus_{j' \in \langle m' \rangle^{\circ}} Y_{j'}$ for some objects $X_j, Y_{j'} \in \mathcal{O}$. Unwinding the definitions, we see that $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\Delta^p, \mathcal{C}'^{\otimes})$ is given by the homotopy limit of the diagram

$$\begin{array}{ccc} \prod_{j \in \langle m \rangle^{\circ}} \mathrm{Map}_{\mathcal{C}}(\bigotimes_{1 \leqslant i \leqslant n} A_i(X_j), B(X_j)) & & \prod_{j' \in \langle m' \rangle^{\circ}} \mathrm{Map}_{\mathcal{C}}(\bigotimes_{1 \leqslant i \leqslant n} A_i(Y_{j'}), B(Y_{j'})) \\ & \searrow \quad \swarrow & \\ & \prod_{j' \in \langle m' \rangle^{\circ}} \mathrm{Map}_{\mathcal{C}}(\bigotimes_{1 \leqslant i \leqslant n, \alpha(j) = j'} A_i(X_j), B(Y_{j'})). & \end{array}$$

Similarly, $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\Delta^p, \mathcal{D}'^{\otimes})$ can be identified with the homotopy limit of the diagram

$$\begin{array}{ccc} \prod_{j \in \langle m \rangle^{\circ}} \mathrm{Map}_{\mathcal{D}}(\bigotimes_{1 \leqslant i \leqslant n} FA_i(X_j), FB(X_j)) & & \prod_{j' \in \langle m' \rangle^{\circ}} \mathrm{Map}_{\mathcal{D}}(\bigotimes_{1 \leqslant i \leqslant n} FA_i(Y_{j'}), FB(Y_{j'})) \\ & \searrow \quad \swarrow & \\ & \prod_{j' \in \langle m' \rangle^{\circ}} \mathrm{Map}_{\mathcal{D}}(\bigotimes_{1 \leqslant i \leqslant n, \alpha(j) = j'} FA_i(X_j), FB(Y_{j'})) & \end{array}$$

It now follows from $(*)$ that θ is a homotopy equivalence as desired. $\square$

15.2.4 Proof of Proposition 15.2.2.3

Let κ be a field, let $n \geqslant 0$ be an integer, and suppose we are given a finite collection $\{A_1, \ldots, A_m, B\}$ of augmented $\mathbb{E}_n$-algebras over κ. We wish to show that if each A_i is n-coconnective and locally finite, then the canonical map

$$\mathrm{Map}_{\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}}(B, A_1 \otimes_\kappa \cdots \otimes_\kappa A_m) \to \mathrm{Map}_{\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}}(\mathfrak{D}^{(n)} A_1 \otimes_\kappa \cdots \otimes_\kappa \mathfrak{D}^{(n)} A_m, \mathfrak{D}^{(n)} B)$$

is a homotopy equivalence. We will proceed by induction on n. When $n = 0$ the result is trivial (see Example 15.2.2.6), and when $n = 1$ it follows from Proposition 14.1.7.1. Let us therefore assume that $n \geqslant 1$ and that Proposition 15.2.2.3 is valid for $\mathbb{E}_n$-algebras; we wish to show that it is also valid for $\mathbb{E}_{n+1}$-algebras.

Using Theorem HA.5.1.2.2 , we can identify $\mathrm{Alg}_\kappa^{(n+1),\mathrm{aug}}$ with the ∞-category $\mathrm{Alg}(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})$. of algebra objects of $\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$. Applying Example HA.5.2.3.13 , we see that under this identification, the Koszul duality functor $\mathfrak{D}^{(n+1)}$ corresponds to the composition

$$\mathrm{Alg}(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}} \xrightarrow{G} \mathrm{Alg}((\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}}) \xrightarrow{G'} \mathrm{Alg}(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})$$

where G is given by the bar construction of §HA.5.2.2 (given on objects by $G(A) = \kappa \otimes_A \kappa$) and G' is induced by the (lax monoidal) functor $\mathfrak{D}^{(n)} : (\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$.

Assume that $A \in \mathrm{Alg}_\kappa^{(n+1),\mathrm{aug}}$ is $(n+1)$-coconnective and locally finite. We have a cofiber sequence of A-modules $A \to \kappa \to Q$ where $\pi_i Q \simeq 0$ for $i > -n$. Using Corollary 14.1.4.3, we deduce that $\pi_i(\kappa \otimes_A Q) \simeq 0$ for $i > -n$ so that $G(A)$ is n-coconnective. Moreover, Proposition 15.1.2.1 shows that $G(A)$ is locally finite (here we use our assumption that $n \geqslant 1$).

Using the inductive hypothesis together with Proposition 15.2.3.1, we deduce that for any sequence $\{A_i\}_{1 \leqslant i \leqslant m}$ of $(n+1)$-connective, locally finite objects of $\mathrm{Alg}_\kappa^{(n+1),\mathrm{aug}}$ and any object $C \in \mathrm{Alg}((\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}})$, the canonical map

$$\begin{array}{c} \mathrm{Map}_{\mathrm{Alg}((\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}})}(G(A_1) \otimes \cdots G(A_m), C) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Alg}(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})}((G'G)(A_1) \otimes \cdots \otimes (G'G)(A_m), G'(C)) \end{array}$$

is a homotopy equivalence. Consequently, to prove Proposition 15.2.2.3, it will suffice to show that for each $B \in \mathrm{Alg}_\kappa^{(n+1),\mathrm{aug}} \simeq \mathrm{Alg}(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})$, the functor G induces a homotopy equivalence

$$\begin{array}{c} \mathrm{Map}_{\mathrm{Alg}(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})}(B, A_1 \otimes \cdots \otimes A_m) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Alg}((\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}})}(G(A_1) \otimes \cdots \otimes G(A_m), G(B)). \end{array}$$

The formula $G(A) \simeq \kappa \otimes_A \kappa$ shows that G is a monoidal functor, so that $G(A_1)\otimes\cdots\otimes G(A_m) \simeq G(A)$ with $A \simeq A_1 \otimes \cdots \otimes A_m$. Note that A is locally finite and $(n+1)$-coconnective. Let F denote a left adjoint to G (given by the cobar construction; see Theorem HA.5.2.2.17). Using Remark 15.2.2.4, we are reduced to proving that the counit map $(F \circ G)(A) \to A$ is an equivalence in $\mathrm{Alg}(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}}$.

The monoidal ∞-category $\mathrm{Alg}_\kappa^{(0),\mathrm{aug}}$ admits geometric realizations and totalizations and the unit object is a zero object, so Theorem HA.5.2.2.17 implies that the bar and cobar constructions yield adjoint functors

$$\mathrm{Alg}((\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})^{\mathrm{op}}) \underset{G_0}{\overset{F_0}{\rightleftarrows}} \mathrm{Alg}(\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})^{\mathrm{op}}.$$

Let $\phi : \mathrm{Alg}_\kappa^{(n),\mathrm{aug}} \to \mathrm{Alg}_\kappa^{(0),\mathrm{aug}}$ denote the forgetful functor. Then ϕ is a (symmetric) monoidal functor which preserves geometric realizations of simplicial objects and totalizations of cosimplicial objects, so that ϕ is compatible with the bar and cobar constructions (Example HA.5.2.3.11). It will therefore suffice to show that the counit map $(F_0 \circ G_0)(\phi A) \to \phi(A)$ is an equivalence in $\mathrm{Alg}(\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})^{\mathrm{op}}$. Equivalently, it suffices to show that for each $B \in \mathrm{Alg}(\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})$, the canonical map

$$\mathrm{Map}_{\mathrm{Alg}(\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})}(B, \phi A) \to \mathrm{Map}_{\mathrm{Alg}((\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})^{\mathrm{op}})}(G_0(\phi A), G_0(B))$$

is a homotopy equivalence. Let $G_0' : \mathrm{Alg}((\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})^{\mathrm{op}}) \to \mathrm{Alg}(\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})$ be the functor given by composition with the lax monoidal functor $\mathfrak{D}^{(0)}$. Using Example 15.2.2.6 and Proposition 15.2.3.1, we deduce that G_0' induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}((\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})^{\mathrm{op}})}(G_0(\phi A), G_0(B)) \to \mathrm{Map}_{\mathrm{Alg}(\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})}((G_0'G_0)(\phi A), (G_0'G_0)(B)).$$

It will therefore suffice to show that the composite map

$$\begin{aligned} \mathrm{Map}_{\mathrm{Alg}(\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})}(B, \phi A) &\to \mathrm{Map}_{\mathrm{Alg}(\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})}((G_0'G_0)(\phi_A), (G_0'G_0)(B)) \\ &\simeq \mathrm{Map}_{\mathrm{Alg}_\kappa^{(1),\mathrm{aug}}}(\mathfrak{D}^{(1)}(\phi A), \mathfrak{D}^{(1)}(B)) \end{aligned}$$

is a homotopy equivalence. This follows from our inductive hypothesis, since ϕA is 1-connective (in fact, $(n+1)$-coconnective) and locally finite.

15.3 Deformation Theory for $\mathbb{E}_n$-Algebras

Let κ be a field. Our goal in this section is to prove Theorem 15.0.0.9, which asserts that the ∞-category $\mathrm{Moduli}_\kappa^{(n)}$ of formal $\mathbb{E}_n$-moduli problems over κ is equivalent to the ∞-category $\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ of augmented $\mathbb{E}_n$-algebras over κ. We first introduce a suitable deformation context, and show that our discussion fits into the general paradigm described in §12.1. We will then prove that the Koszul duality functor $\mathfrak{D}^{(n)} : (\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ of §15.2 is a deformation theory, in the sense of Definition 12.3.3.2 (Theorem 15.3.3.1). We will then use this result to deduce Theorem 15.0.0.9 from Theorem 12.3.3.5.

15.3.1 Augmented $\mathbb{E}_n$-Algebras as a Deformation Context

Let κ be a field and let $n \geqslant 0$ be an integer. Using Theorem HA.7.3.4.13 and Proposition HA.3.4.2.1 , we obtain equivalences of ∞-categories

$$\mathrm{Sp}(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}) \simeq \mathrm{Mod}_\kappa^{\mathbb{E}_n}(\mathrm{Mod}_\kappa) \simeq \mathrm{Mod}_\kappa .$$

In particular, we can identify the unit object $\kappa \in \mathrm{Mod}_\kappa$ with a spectrum object $E \in \mathrm{Sp}(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})$, given informally by the formula $\Omega^{\infty-m} E = \kappa \oplus \Sigma^m(\kappa)$. We regard the pair $(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}, \{E\})$ as a deformation context.

We will need the following generalization of Proposition 14.2.1.2:

Proposition 15.3.1.1. *Let κ be a field, let $n \geqslant 1$, and let $(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}, \{E\})$ be the deformation context defined above. Then an object $A \in \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ is Artinian in the sense of Definition 12.1.2.4 if and only if its image in $\mathrm{Alg}_\kappa^{(n)}$ is Artinian in the sense of Definition 15.0.0.1: that is, if and only if A satisfies the following conditions:*

- (a) *The algebra A is connective: that is, $\pi_i A \simeq 0$ for $i < 0$.*
- (b) *The algebra A is truncated: that is, we have $\pi_i A \simeq 0$ for $i \gg 0$.*
- (c) *Each of the homotopy groups $\pi_i A$ is finite dimensional when regarded as a vector space over field κ.*
- (d) *Let $\mathfrak{n}$ denote the radical of the ring $\pi_0 A$ (which is a finite-dimensional associative algebra over κ). Then the canonical map $\kappa \to (\pi_0 A)/\mathfrak{n}$ is an isomorphism.*

Remark 15.3.1.2. Proposition 15.3.1.1 is also valid in the case $n = 0$, provided that we adopt the convention of Remark 15.0.0.5. That is, an object $A \in \mathrm{Alg}_\kappa^{(0),\mathrm{aug}}$ is Artinian (in the sense of Definition 12.1.2.4) if and only if it connective and $\pi_* A$ is a finite-dimensional vector space over κ.

Proof. The "only if" direction follows from Proposition 14.2.1.2 (note that if A is Artinian as an augmented $\mathbb{E}_n$-algebra, then its image in $\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ is also Artinian). To prove the converse, suppose that $A \in \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ satisfies conditions (a) through (d). We wish to prove that there exists a finite sequence of maps $A = A_0 \to A_1 \to \cdots \to A_d \simeq \kappa$, where each A_i is a square-zero extension of A_{i+1} by $\Sigma^{m_i}(\kappa)$, for some $m_i \geqslant 0$. If $n = 1$, this follows from

Proposition 14.2.1.2. Let us therefore assume that $n \geqslant 2$. We proceed by induction on the dimension of the κ-vector space $\pi_* A$.

Let m be the largest integer for which $\pi_m A$ does not vanish. We first treat the case $m = 0$. We will abuse notation by identifying A with the underlying commutative ring $\pi_0 A$. Let $\mathfrak{n}$ denote the radical of A. If $\mathfrak{n} = 0$, then condition (d) implies that $A \simeq \kappa$ so there is nothing to prove. Otherwise, we can view $\mathfrak{n}$ as a nonzero module over the commutative ring A. It follows that there exists a nonzero element $x \in \mathfrak{n}$ which is annihilated by $\mathfrak{n}$. Using (d) again, we deduce that the subspace $\kappa x \subseteq A$ is an ideal of A. Let A' denote the quotient ring $A/\kappa x$. Theorem HA.7.4.1.26 implies that A is a square-zero extension of A' by κ. The inductive hypothesis implies that A' is Artinian, so that A is also Artinian.

Now suppose that $m > 0$ and let $M = \pi_m A$. Then M is a nonzero module over the finite dimensional κ-algebra $\pi_0 A$. It follows that there is a nonzero element $x \in M$ which is annihilated by the action of the radical $\mathfrak{n} \subseteq \pi_0 A$. Let M' denote the quotient of M by the submodule generated by x (which, by virtue of (d), coincides with κx), and let $A'' = \tau_{\leqslant n-1} A$. It follows from Theorem HA.7.4.1.26 that there is a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ A'' & \longrightarrow & \kappa \oplus \Sigma^{m+1}(M). \end{array}$$

Set $A' = \kappa \times_{\kappa \oplus \Sigma^{m+1}(M')} A''$. Then $A \simeq \kappa \times_{\kappa \oplus \Sigma^{m+1}(\kappa)} A'$, so we have an elementary map $A \to A'$. Using the inductive hypothesis we deduce that A' is Artinian, so that A is also Artinian. □

Proposition 15.3.1.3. *Let κ be a field and let $f : A \to B$ be a morphism in $\mathrm{Alg}_\kappa^{(n),\mathrm{art}}$. Then f is small (when regarded as a morphism in $\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$) if and only if it induces a surjection $\pi_0 A \to \pi_0 B$.*

Proof. If $n = 1$, the desired result follows from Proposition 14.2.1.3. We will assume that $n \geqslant 2$, and leave the case $n = 0$ to the reader. The "only if" direction follows from Proposition 14.2.1.3 (note that if f is small, then the induced map between the underlying $\mathbb{E}_1$-algebras is also small). We first treat the case where $B \simeq A \oplus \Sigma^j(M)$, for some $M \in \mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_\kappa)^\heartsuit$ and some $j \geqslant 1$. According to Remark 15.1.3.6, the abelian category $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_\kappa)^\heartsuit$ is equivalent to the category of modules over the commutative ring $\pi_0 B$. Since M is finite dimensional as a vector space over κ, it admits a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M,$$

where each of the successive quotients M_i / M_{i-1} is isomorphic to κ. This filtration determines a factorization of f as a composition

$$A \simeq A \oplus \Sigma^j(M_0) \to A \oplus \Sigma^j(M_1) \to \cdots \to A \oplus \Sigma^j(M_m) = B.$$

Each of the maps $A \oplus \Sigma^j(M_i) \to A \oplus \Sigma^j(M_{i+1})$ is elementary, so that f is small.

We now treat the general case. Note that the map $\pi_0 A \times_{\pi_0 B} B \to B$ is a pullback of the map $\pi_0 A \to \pi_0 B$ and therefore small (the map $\pi_0 A \to \pi_0 B$ is even small as a morphism of $\mathbb{E}_\infty$-algebras over κ, by Lemma 12.1.3.8). It will therefore suffice to show that the map $A \to \pi_0 A \times_{\pi_0 B} B$ is small. We will prove that each of the maps

$$\tau_{\leqslant j} A \times_{\tau_{\leqslant j} B} B \to \pi_0 A \times_{\pi_0 B} B$$

is small; taking $j \gg 0$ we will obtain the desired result. The proof proceeds by induction on j, the case $j = 0$ being trivial. Assume that $j > 0$; by the inductive hypothesis, we are reduced to proving that the map $\theta : \tau_{\leqslant j} A \times_{\tau_{\leqslant j} B} B \to \tau_{\leqslant j-1} A \times_{\tau_{\leqslant j-1} B} B$ is small.

Factor θ as a composition $\tau_{\leqslant j} A \times_{\tau_{\leqslant j} B} B \xrightarrow{\theta'} \tau_{\leqslant j} A \times_{\tau_{\leqslant j-1} B} B \xrightarrow{\theta''} \tau_{\leqslant j-1} A \times_{\tau_{\leqslant j-1} B} B$. The map θ'' is a pullback of the truncation map $u : \tau_{\leqslant j} A \to \tau_{\leqslant j-1} A$. It follows from Corollary HA.7.4.1.28 that u exhibits $\tau_{\leqslant j} A$ as a square-zero extension of $\tau_{\leqslant j-1} A$, so that we have a pullback square

$$\begin{array}{ccc} \tau_{\leqslant j} A & \xrightarrow{u} & \tau_{\leqslant j-1} A \\ \downarrow & & \downarrow \\ \tau_{\leqslant j-1} A & \xrightarrow{u_0} & (\tau_{\leqslant j-1} A) \oplus \Sigma^{j+1}(\pi_j A). \end{array}$$

Here the map u_0 is small by the argument given above, so that u is small and therefore θ'' is small. We will complete the proof by showing that θ' is small. Note that θ' is a pullback of the diagonal map $\delta : \tau_{\leqslant j} B \to \tau_{\leqslant j} B \times_{\tau_{\leqslant j-1} B} \tau_{\leqslant j} B$. Since $\tau_{\leqslant j} B$ is a square-zero extension of $\tau_{\leqslant j-1} B$ by $\Sigma^j(\pi_j B)$ (Corollary HA.7.4.1.28), the truncation map $\tau_{\leqslant j} B \to \tau_{\leqslant j-1} B$ is a pullback of the canonical map $\tau_{\leqslant j-1} B \to \tau_{\leqslant j-1} B \oplus \Sigma^{j+1}(\pi_j B)]$. It follows that δ' is a pullback of the map

$$\delta' : \tau_{\leqslant j-1} B \to \tau_{\leqslant j-1} B \times_{\tau_{\leqslant j-1} B \oplus \Sigma^{j+1}(\pi_j B)} \tau_{\leqslant j-1} B \simeq \tau_{\leqslant j-1} B \oplus \Sigma^j(\pi_j B).$$

Since $j \geqslant 1$, the first part of the proof shows that δ' is small. □

Corollary 15.3.1.4. *Let κ be a field, let $n \geqslant 0$ be an integer, and let and let $X : \mathrm{Alg}_\kappa^{(n),\mathrm{art}} \to \mathcal{S}$ be a functor. Then X belongs to the full subcategory $\mathrm{Moduli}_\kappa^{(n)}$ of Definition 15.0.0.7 if and only if it is a formal moduli problem in the sense of Definition 12.1.3.1.*

Proof. The "if" direction follows immediately from Proposition 15.3.1.3. For the converse, suppose that X satisfies the conditions of Definition 15.0.0.7; we wish to show that X is a formal moduli problem. According to Proposition 12.1.3.2, it will suffice to show that for every pullback diagram in $\mathrm{Alg}_\kappa^{(n),\mathrm{art}}$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \kappa & \longrightarrow & \kappa \oplus \Sigma^m(\kappa) \end{array}$$

satisfying $m > 0$, the associated diagram of spaces

$$\begin{array}{ccc} X(A) & \longrightarrow & X(B) \\ \downarrow & & \downarrow \\ X(\kappa) & \longrightarrow & X(\kappa \oplus \Sigma^m(\kappa)) \end{array}$$

is also a pullback square. This follows immediately from condition (2) of Definition 15.0.0.7. □

15.3.2 Digression: The Koszul Dual of a Free Algebra

Let κ be a field, and let $\mathrm{Free}^{(n)} : \mathrm{Mod}_\kappa \to \mathrm{Alg}_\kappa^{(n)}$ be a left adjoint to the forgetful functor (so that $\mathrm{Free}^{(n)}$ assigns to every κ-module spectrum V the free $\mathbb{E}_n$-algebra $\bigoplus_{n \geqslant 0} \mathrm{Sym}^n_{\mathbb{E}_\kappa}(V)$). Note that $\mathrm{Free}^{(n)}(0) \simeq \kappa$, so that $\mathrm{Free}^{(n)}$ determines a functor

$$\mathrm{Free}^{\mathrm{aug}} : \mathrm{Mod}_\kappa \simeq (\mathrm{Mod}_\kappa)_{/0} \to (\mathrm{Alg}_\kappa^{(n)})_{/\kappa} \simeq \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}.$$

Proposition 15.3.2.1. *Let κ be a field and let $\mathfrak{D}^{(n)} : (\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ be the Koszul duality functor. Then the composition $\mathfrak{D}^{(n)} \circ \mathrm{Free}^{\mathrm{aug}}$ is equivalent to the functor $\mathrm{Mod}_\kappa^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ given by $V \mapsto \kappa \oplus \Sigma^{-n}(V^\vee)$, where $V^\vee$ denotes the κ-linear dual of V.*

Proof. The functor $\mathfrak{D}^{(n)} \circ \mathrm{Free}^{\mathrm{aug}}$ admits a left adjoint and is therefore left exact. Since the ∞-category Mod_κ is stable, Proposition HA.1.4.2.22 implies that $\mathfrak{D}^{(n)} \circ \mathrm{Free}^{(n)}$ factors as a composition

$$\mathrm{Mod}_\kappa^{\mathrm{op}} \xrightarrow{T} \mathrm{Sp}(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}) \xrightarrow{\Omega^\infty} \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}.$$

Note that $\mathrm{Sp}(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})$ is equivalent to Mod_κ, and that under this equivalence the functor $\Omega^\infty : \mathrm{Sp}(\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}) \to \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ is given by the formation of square-zero extensions $V \mapsto \kappa \oplus V$ (Theorem HA.7.3.4.7). It follows that we can identify T with the functor $\mathrm{Mod}_\kappa^{\mathrm{op}} \to \mathrm{Mod}_\kappa$ given by the composition

$$\mathrm{Mod}_\kappa^{\mathrm{op}} \xrightarrow{\mathrm{Free}^{\mathrm{aug}}} (\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}} \xrightarrow{\mathfrak{D}^{(n)}} \mathrm{Alg}_\kappa^{(n),\mathrm{aug}} \xrightarrow{I} \mathrm{Mod}_\kappa,$$

where I denotes the functor which carries each augmented $\mathbb{E}_n$-algebra A to its augmentation ideal. The composition $I \circ \mathfrak{D}^{(n)}$ assigns to each augmented $\mathbb{E}_n$-algebra B its shifted tangent fiber $\underline{\mathrm{Map}}_{\mathrm{Mod}_B^{\mathbb{E}_n}}(\Sigma^n(L_{B/\kappa}), \kappa)$ (see Example HA.7.3.5.7), so that the composition $I \circ \mathfrak{D}^{(n)} \circ \mathrm{Free}^{(n)}$ is given by $V \mapsto (\Sigma^n(V))^\vee \simeq \Sigma^{-n}(V^\vee)$. □

15.3.3 Koszul Duality as a Deformation Theory

We can now formulate the main result of this section.

Theorem 15.3.3.1. *Let κ be a field and let $n \geqslant 0$ be an integer. Then the Koszul duality functor $\mathfrak{D}^{(n)} : (\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ is a deformation theory (in the sense of Definition 12.3.3.2).*

Proof. We show that the Koszul duality functor $\mathfrak{D}^{(n)} : \mathfrak{D}^{(n)} : (\mathrm{Alg}_\kappa^{(n),\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ satisfies axioms $(D1)$ through $(D4)$ of Definitions 12.3.1.1 and 12.3.3.2:

$(D1)$ The ∞-category $\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ is presentable: this follows from Corollary HA.3.2.3.5 .

$(D2)$ The functor $\mathfrak{D}^{(n)}$ admits a left adjoint. In fact, this left adjoint is given by the opposite of $\mathfrak{D}^{(n)}$ (see Remark 15.2.1.5).

$(D3)$ Let $\mathbf{B}_0 \subseteq \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ be the full subcategory spanned by those augmented $\mathbb{E}_n$-algebras A over κ, where A is coconnective and both A and $\int A$ are locally finite. We will verify that this subcategory satisfies the requirements of Definition 12.3.1.1:

- (a) For every object $A \in \mathbf{B}_0$, the biduality map $A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}(A)$ is an equivalence. This follows from Theorem 15.2.2.1.
- (b) The subcategory $\mathbf{B}_0$ contains the initial object $\kappa \in \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$.
- (c) For each $m \geqslant 1$, there exists an object $K_m \in \mathbf{B}_0$ and an equivalence $\alpha : \kappa \oplus \Sigma^m(\kappa) \simeq \mathfrak{D}^{(n)} K_m$. In fact, we can take K_m to be the free $\mathbb{E}_n$-algebra generated by $\Sigma^{-m-n}(\kappa)$. This belongs to $\mathbf{B}_0$ by virtue of Proposition 15.1.4.1, and Proposition 15.3.2.1 supplies the equivalence α.
- (d) For every pushout diagram

$$\begin{array}{ccc} K_m & \longrightarrow & A \\ \downarrow{\scriptstyle \epsilon} & & \downarrow \\ \kappa & \longrightarrow & A', \end{array}$$

  where $A \in \mathbf{B}_0$ and ϵ is the canonical augmentation on K_m, the object A' also belongs to $\mathbf{B}_0$. This follows immediately from Propositions 15.1.5.1 and 15.1.4.1.

$(D4)$ Arguing as in the proof of Theorem 15.0.0.9, we see that the functor $e : \mathrm{Alg}_\kappa^{(n),\mathrm{aug}} \to \mathrm{Sp}$ appearing in Definition 12.3.3.2 is given by $A \mapsto \Sigma^n \mathfrak{m}_A$, where $\mathfrak{m}_A$ denotes the augmentation ideal of A. This functor is obviously conservative, and preserves sifted colimits by Proposition HA.3.2.3.1 .

□

Proof of Theorem 15.0.0.9. Let κ be a field and let $n \geqslant 0$ be an integer. Define a functor $\Psi : \mathrm{Alg}_\kappa^{(n),\mathrm{aug}} \to \mathrm{Fun}(\mathrm{Alg}_\kappa^{(n),\mathrm{art}}, \mathcal{S})$ by the formula $\Psi(A)(R) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}}(\mathfrak{D}^{(n)}(R), A)$. Combining Theorem 15.3.3.1, Theorem 12.3.3.5, and Corollary 15.3.1.4, we deduce that

Ψ is a fully faithful embedding whose essential image is the full subcategory $\mathrm{Moduli}_\kappa^{(n)} \subseteq \mathrm{Fun}(\mathrm{Alg}_\kappa^{(n),\mathrm{art}}, \mathcal{S})$ spanned by the formal moduli problems. If $m \geqslant 0$, then Proposition 15.1.4.1 implies that $\mathrm{Free}^{\mathrm{aug}}(\Sigma^{-m-n}(\kappa))$ is n-coconnective and locally finite, so the the biduality map

$$\mathrm{Free}^{\mathrm{aug}}(\Sigma^{-m-n}(\kappa)) \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}\,\mathrm{Free}^{\mathrm{aug}}(\Sigma^{-m-n}(\kappa))$$

is an equivalence (Theorem 15.2.2.1). Using Proposition 15.3.2.1, we obtain canonical homotopy equivalences

$$\begin{aligned}
\Psi(A)(\kappa \oplus \Sigma^m(\kappa)) &\simeq \Psi(A)(\mathfrak{D}^{(n)}\,\mathrm{Free}^{\mathrm{aug}}(\Sigma^{-m-n}(\kappa)))\\
&\to \mathrm{Map}_{\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}}(\mathfrak{D}^{(n)}\mathfrak{D}^{(n)}\,\mathrm{Free}^{\mathrm{aug}}(\Sigma^{-m-n}(\kappa)), A)\\
&\to \mathrm{Map}_{\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}}(\mathrm{Free}^{\mathrm{aug}}(\Sigma^{-m-n}(\kappa)), A)\\
&\simeq \Omega^{\infty-m-n}\mathfrak{m}_A,
\end{aligned}$$

where $\mathfrak{m}_A$ denotes the augmentation ideal of A. These equivalences are natural in m, and therefore give rise to an equivalence of spectra $T_{\Psi(A)} \simeq \Sigma^n \mathfrak{m}_A$ (depending functorially on A). □

Example 15.3.3.2. Suppose that $n = 0$ in the situation of Theorem 15.0.0.9. Then the Koszul duality functor $\mathfrak{D}^{(0)} : (\mathrm{Alg}_\kappa^{(0),\mathrm{aug}})^{\mathrm{op}} \mapsto \mathrm{Alg}_\kappa^{(0),\mathrm{aug}}$ is given by $\kappa \oplus V \mapsto \kappa \oplus V^\vee$ (see Example 15.2.1.3). It follows that the functor $\Psi : \mathrm{Alg}_\kappa^{(0),\mathrm{aug}} \to \mathrm{Moduli}_\kappa^{(0)}$ is given by

$$\begin{aligned}
\Psi(\kappa \oplus W)(\kappa \oplus V) &= \mathrm{Map}_{\mathrm{Alg}_\kappa^{(0),\mathrm{aug}}}(\kappa \oplus V^\vee, \kappa \oplus W)\\
&\simeq \mathrm{Map}_{\mathrm{Mod}_\kappa}(V^\vee, W)\\
&\simeq \Omega^\infty(V \otimes_\kappa W).
\end{aligned}$$

Here the last equivalence follows from the observation that V is a dualizable object of Mod_κ.

We can summarize the situation as follows: every object $W \in \mathrm{Mod}_\kappa$ determines a formal $\mathbb{E}_0$-moduli problem, given by the formula $\kappa \oplus V \mapsto \Omega^\infty(V \otimes_\kappa W)$. Moreoever, every formal $\mathbb{E}_0$-moduli problem arises in this way, up to equivalence.

15.3.4 Application: Prorepresentable Formal $\mathbb{E}_n$-Moduli Problems

We close this section by proving a generalization of Proposition 14.2.3.1:

Proposition 15.3.4.1. *Let κ be a field and let $X : \mathrm{Alg}_\kappa^{(n),\mathrm{art}} \to \mathcal{S}$ be a formal $\mathbb{E}_n$-moduli problem over κ. The following conditions are equivalent:*

(1) *The functor X is prorepresentable (see Definition 12.5.3.1).*

(2) *Let $X(E)$ denote the tangent complex of X. Then $\pi_i X(E) \simeq 0$ for $i > 0$.*

(3) *The functor X has the form $\Psi(A)$, where $A \in \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ is n-coconnective coconnective and $\Psi : \mathrm{Alg}_\kappa^{(n),\mathrm{aug}} \to \mathrm{Moduli}_\kappa^{(n)}$ is the equivalence of Theorem 15.0.0.9.*

Lemma 15.3.4.2. *Let A be an augmented $\mathbb{E}_n$-algebra over a field κ. If A is connective, then the Koszul dual $\mathfrak{D}^{(n)}(A)$ is n-coconnective.*

Proof. Let $\mathrm{Mod}_A^{\mathbb{E}_n}$ denote the ∞-category of $\mathbb{E}_n$-modules over A in the ∞-category Mod_κ, and regard $\mathrm{Mod}_A^{\mathbb{E}_n}$ as tensored over Mod_κ. As an object of Mod_κ, we can identify $\mathfrak{D}^{(n)}(A)$ as a classifying object (in Mod_κ) for morphisms from A to κ in $\mathrm{Mod}_A^{\mathbb{E}_n}$ (see Example HA.5.3.1.5 and Theorem HA.5.3.1.30). Theorem HA.7.3.5.1 supplies fiber sequence

$$\int A \to A \to \Sigma^n L_{A/\kappa}$$

in the ∞-category $\mathrm{Mod}_A^{\mathbb{E}_n}$, where $L_{A/\kappa}$ denote the relative cotangent complex of A over κ as an $\mathbb{E}_n$-algebra. We therefore obtain a fiber sequence

$$\underline{\mathrm{Map}}_{\mathrm{Mod}_A^{\mathbb{E}_n}}(\Sigma^n L_{A/\kappa}, \kappa) \to \mathfrak{D}^{(n)}(A) \xrightarrow{\epsilon_A} \kappa$$

in the ∞-category Mod_κ. The map ϵ_A depends functorially on A and is an equivalence in the case $A = \kappa$, and can therefore be identified with the augmentation on $\mathfrak{D}^{(n)}(A)$. We may therefore identify the augmentation ideal $\mathfrak{m}_{\mathfrak{D}^{(n)}(A)}$ with a classifying object for morphisms from $\Sigma^n L_{A/\kappa}$ to κ in $\mathrm{Mod}_A^{\mathbb{E}_n}$. To prove that $\mathfrak{D}^{(n)}(A)$ is n-coconnective, it suffices to show that the mapping space

$$\mathrm{Map}_{\mathrm{Mod}_A^{\mathbb{E}_n}}(L_{A/\kappa}, \kappa) \simeq \mathrm{Map}_{\mathrm{Alg}_\kappa^{(n),\mathrm{aug}}}(A, \kappa[\epsilon]/(\epsilon^2))$$

is discrete. This is clear, since A is connective and $\kappa[\epsilon]/(\epsilon^2)$ is discrete. □

Proof of Proposition 15.3.4.1. The equivalence of (2) and (3) follows from the observation that for $X = \Psi(A)$, we have $\pi_i X(E) \simeq \pi_{i-n}\mathfrak{m}_A$, where $\mathfrak{m}_A$ is the augmentation ideal of A. We next prove that (1) $\Rightarrow$ (2). Since the construction $X \mapsto X(E)$ commutes with filtered colimits, we may reduce to the case where $X = \mathrm{Spf}\, R$ is representable by an object $R \in \mathrm{Alg}_\kappa^{(n),\mathrm{art}}$. Then R is connective and the desired result follows from Lemma 15.3.4.2.

We now complete the proof by showing that (3) $\Rightarrow$ (1). Let $A \in \mathrm{Alg}_\kappa^{(n),\mathrm{aug}}$ be n-coconnective, and choose a sequence of maps

$$\kappa = A(0) \to A(1) \to A(2) \to \cdots$$

as in the proof of Proposition 15.1.5.1. Then $A = \varinjlim A(i)$, so that $X \simeq \varinjlim X(i)$ with $X(i) = \Psi(A(i))$. To prove that X is prorepresentable, it will suffice to show that each $X(i)$ is prorepresentable. We proceed by induction on i, the case $i = 0$ being trivial.

Assume that $X(i)$ is prorepresentable. By construction, we have a pushout diagram

$$\begin{array}{ccc} \mathrm{Free}^{(n)}(V) & \longrightarrow & A(i) \\ \downarrow & & \downarrow \\ \kappa & \longrightarrow & A(i+1) \end{array}$$

where $\pi_j V \simeq 0$ for $j \geqslant -n$. For $m \geqslant n$, form a pushout diagram

$$\begin{array}{ccc} \mathrm{Free}^{(n)}(\tau_{\geqslant -m} V) & \longrightarrow & A(i) \\ \downarrow & & \downarrow \\ \kappa & \longrightarrow & A(i,m), \end{array}$$

so that $A(i+1) \simeq \varinjlim_m A(i,m)$. Then $X(i+1) \simeq \varinjlim_m \Psi(A(i,m))$, so we are reduced to proving that each $\Psi(A(i,m))$ is prorepresentable. We proceed by induction on m. If $m = n$, then $A(i,m) \simeq A(i)$ and the desired result follows from our inductive hypothesis. Assume that $m > n$ and that $\Psi(A(i,m-1))$ is prorepresentable. Let $W = \pi_{-m} V$, so that we have a pushout diagram

$$\begin{array}{ccc} \mathrm{Free}^{(n)}(\Sigma^{-m} W) & \longrightarrow & A(i,m-1) \\ \downarrow & & \downarrow \\ \kappa & \longrightarrow & A(i,m). \end{array}$$

Write W as a union of its finite-dimensional subspaces $\{W_\alpha\}$. For every finite dimensional subspace $W_\alpha \subseteq W$, form a pushout diagram

$$\begin{array}{ccc} \mathrm{Free}^{(n)}(\Sigma^{-m} W_\alpha) & \longrightarrow & A(i,m-1) \\ \downarrow & & \downarrow \\ \kappa & \longrightarrow & A(i,W_\alpha). \end{array}$$

Then $\Psi(A(i,m))$ is a filtered colimit of the objects $\Psi(A(i,W_\alpha))$. It will therefore suffice to show that each $\Psi(A(i,W_\alpha))$ is prorepresentable. We proceed by induction on the dimension of W_α; if that dimension is zero, then $A(i,W_\alpha) \simeq A(i,m-1)$ and the result is clear. If W_α has positive dimension, then we can choose a subspace W'_α of codimension 1. Then we have a pushout diagram

$$\begin{array}{ccc} \mathrm{Free}^{(n)}(\Sigma^{-m}(\kappa)) & \longrightarrow & A(i,W'_\alpha) \\ \downarrow & & \downarrow \\ \kappa & \longrightarrow & A(i,W_\alpha), \end{array}$$

hence a pushout diagram of formal moduli problems

$$\begin{array}{ccc} \mathrm{Spf}(\kappa \oplus \Sigma^{m-n}(\kappa)) & \longrightarrow & A(i, W'_\alpha) \\ \downarrow & & \downarrow \\ \mathrm{Spf}(\kappa) & \longrightarrow & \Psi(A(i, W_\alpha)). \end{array}$$

We conclude the proof by invoking Lemma 12.5.3.4. $\square$

Chapter 16

Examples of Formal Moduli Problems

In this section, we will illustrate our theory of formal moduli problems by considering some examples which arise naturally in algebraic geometry. We begin by considering the formal deformation theory of algebraic varieties. Let Z be an algebraic variety defined over a field κ. If R is a local Artin ring with residue field κ, then a *deformation* of Z over R is a pair (Z_R, α), where Z is a flat R-scheme and α is an isomorphism of Z with the fiber product $Z_R \times_{\operatorname{Spec} R} \operatorname{Spec} \kappa$ (see Example 11.5.0.7). The condition that Z_R be flat over R is equivalent to the requirement that the diagram

$$\begin{array}{ccc} Z & \longrightarrow & Z_R \\ \downarrow & & \downarrow \\ \operatorname{Spec} \kappa & \longrightarrow & \operatorname{Spec} R \end{array}$$

be a pullback square not only in the category of schemes, but also in the ∞-category of spectral algebraic spaces. This suggests the following more general definition: given a connective $\mathbb{E}_\infty$-ring R equipped with a map $R \to \kappa$, a *deformation* of Z over R is a spectral algebraic space Z_R over R, equipped with an equivalence $\mathsf{Z} \simeq \mathsf{Z}_R \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Sp\acute{e}t} \kappa$, where Z denotes the spectral algebraic space over κ determined by Z (see Proposition **??**). The collection of all deformations of Z over R can be organized into an ∞-groupoid $M(R)$. The construction $R \mapsto M(R)$ determines a functor $M : \mathrm{CAlg}^{\mathrm{art}}_\kappa \to \mathcal{S}$. In this situation, we will show that the functor M is a formal moduli problem over κ (in the sense of Definition

12.1.3.1). In other words, we will show that for every pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

in $\mathrm{CAlg}^{\mathrm{art}}_{\kappa}$, if the maps $\pi_0 R_0 \rightarrow \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, then $M(R)$ can be identified with the fiber product $M(R_0) \times_{M(R_{01})} M(R_1)$ in the ∞-category of spaces. More concretely, this means that giving a deformation Z_R of Z over R is equivalent to giving deformations Z_0 and Z_1 over R_0 and R_1 respectively, together with an equivalence between the resulting deformations of Z over R_{01}. We will verify this in §16.3, using the fact that any such deformation Z_R can be reconstructed as the pushout $\mathsf{Z}_0 \amalg_{\mathsf{Z}_{01}} \mathsf{Z}_1$, where $\mathsf{Z}_{01} = \mathrm{Sp\acute{e}t}\, R_{01} \times_{\mathrm{Sp\acute{e}t}\, R_0} \mathsf{Z}_0 \simeq \mathrm{Sp\acute{e}t}\, R_{01} \times_{\mathrm{Sp\acute{e}t}\, R_1} \mathsf{Z}_1$. The proof will use some general facts about pushouts of spectral Deligne-Mumford stacks along closed immersions which we establish in §16.1, together with an analysis of the ∞-category of quasi-coherent sheaves on a pushout which we carry out in §16.2.

Another important class of formal moduli problems arises from studying the deformation theory of (quasi)-coherent sheaves on algebraic varieties. Let $Z \rightarrow \mathrm{Spec}\, \kappa$ be as above, and suppose we are given a (discrete) quasi-coherent sheaf $\mathscr{F}$ on Z. If R is a local Artinian κ-algebra with residue field κ, a *deformation* of over R is a quasi-coherent sheaf $\mathscr{F}_R$ over the R-scheme $Z_R = \mathrm{Spec}\, R \times_{\mathrm{Spec}\, \kappa} Z$ which is flat over R, together with an isomorphism $\mathscr{F} \simeq \alpha^* \mathscr{F}_R$, where $\alpha : Z \rightarrow Z_R$ denotes the map of schemes determined by the augmentation $R \rightarrow \kappa$. Once again, the flatness of $\mathscr{F}_R$ over R admits a natural formulation in the language of spectral algebraic geometry: it is equivalent to the requirement that $\mathscr{F}$ be given by the pullback of $\mathscr{F}_R$ along α not only in the abelian category $\mathrm{QCoh}(Z)^{\heartsuit}$, but also in the stable ∞-category $\mathrm{QCoh}(Z)$ studied in Chapter I (here we abuse notation slightly by identifying Z with the associated spectral algebraic space). By virtue of Corollary **??**, we can identify deformations of $\mathscr{F}$ with inverse images of $\mathscr{F}$ under the extension-of-scalars functor $\mathrm{LMod}_R(\mathrm{QCoh}(Z)) \rightarrow \mathrm{LMod}_{\kappa}(\mathrm{QCoh}(Z)) \simeq \mathrm{QCoh}(Z)$, where we regard $\mathrm{QCoh}(Z)$ as a κ-linear ∞-category.

More generally, let κ be a field and let $\mathcal{C}$ be any prestable κ-linear ∞-category (see Definition D.1.4.1). We can associate to $\mathcal{C}$ several deformation-theoretic problems:

(a) Fix an object $C \in \mathcal{C}$, and let $R \in \mathrm{CAlg}^{\mathrm{art}}_{\kappa}$ be an Artinian $\mathbb{E}_{\infty}$-algebra over κ. A *deformation of* C *over* R is an object $C_R \in \mathrm{LMod}_R(\mathcal{C})$, together with an equivalence $C \simeq \kappa \otimes_R C_R$. Let $X(R)$ denote the ∞-category $\mathrm{LMod}_R(\mathcal{C}) \times_{\mathcal{C}} \{C\}$ of deformations of C over R.

(b) For $R \in \mathrm{CAlg}^{\mathrm{art}}_{\kappa}$, a *deformation of* $\mathcal{C}$ *over* R is a prestable R-linear ∞-category $\mathcal{C}_R$

equipped with an equivalence $\mathcal{C} \simeq \kappa \otimes_R \mathcal{C}_R$. Let $Y(R) = \mathrm{LinCat}_R^{\mathrm{PSt}} \times_{\mathrm{LinCat}_\kappa^{\mathrm{PSt}}} \{\mathcal{C}\}$ denote the ∞-category of deformations of $\mathcal{C}$ over R.

We will see later that for every object $R \in \mathrm{CAlg}_\kappa^{\mathrm{art}}$, the ∞-categories $X(R)$ and $Y(R)$ are essentially small Kan complexes. Consequently, we can view X and Y as functors from the ∞-category $\mathrm{CAlg}_\kappa^{\mathrm{art}}$ to the ∞-category $\mathcal{S}$ of spaces. Our goal in the latter part of this section is to analyze the behavior of these functors. We immediately encounter an obstacle: the functors X and Y need not be formal moduli problems, in the sense of Definition 12.1.3.1. Suppose, for example, that we are given a pullback diagram σ :

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

in $\mathrm{CAlg}_\kappa^{\mathrm{art}}$, where the maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective. If $C \in \mathcal{C}$ and C_R is a deformation of C over R, then we will see that C_R is uniquely determined by the objects $C_0 = R_0 \otimes_R C_R$, $C_1 = R_1 \otimes_R C_{R_1}$, together with the evident equivalence

$$R_{01} \otimes_{R_0} C_0 \simeq R_{01} \otimes_R C_R \simeq R_{01} \otimes_{R_1} C_1$$

(see Proposition 16.5.2.2). More precisely, the functor X described in (a) determines a fully faithful embedding (of Kan complexes)

$$X(R) \to X(R_0) \times_{X(R_{01})} X(R_1),$$

but this map need not be essentially surjective (see Warning 16.2.0.3). The functor Y described in (b) is even more problematic: the map

$$Y(R) \to Y(R_0) \times_{Y(R_{01})} Y(R_1)$$

need not be fully faithful in general, but always has discrete homotopy fibers (Proposition 16.6.2.1): that is, we can regard $Y(R)$ as a covering space of $Y(R_0) \times_{Y(R_{01})} Y(R_1)$. To accommodate these examples, it is useful to introduce a weaker version of the axiomatics developed in Chapter 12. For every integer $n \geqslant 0$, we will define the notion of a *n-proximate formal moduli problem* (Definition 16.4.1.5). When $n = 0$, we recover the notion of formal moduli problem introduced in Definition 12.1.3.1. The requirement that a functor Z be an n-proximate formal moduli problem becomes increasingly weak as n grows. Nonetheless, we will show that an n-proximate formal moduli problem Z is not far from being a formal moduli problem: namely, there exists an (essentially unique) formal moduli problem $Z^\wedge$ and a natural transformation $Z \to Z^\wedge$ such that, for every test algebra R, the map of spaces $Z(R) \to Z^\wedge(R)$ has $(n-1)$-truncated homotopy fibers (Theorem 16.4.2.1).

In §16.5, we will turn our attention to the functor X described above, which classifies the deformations of a fixed object $C \in \mathcal{C}$. We begin by observing that the definition of $X(R)$ does not require the assumption that R is commutative. Rather, the functor X is naturally defined on the ∞-category $\mathrm{Alg}_\kappa^{\mathrm{art}}$ of Artinian $\mathbb{E}_1$-algebras over κ. We may therefore regard the construction $R \mapsto X(R)$ as a functor $X : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$, which we will prove is a 1-proximate formal moduli problem (Corollary 16.5.3.2). Using Theorem 16.4.2.1, we can choose an embedding of X into a formal moduli problem $\overline{X} : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$. According to Theorem 14.0.0.5 (and its proof), the functor $\overline{X}$ is given by $\overline{X}(R) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(R), A)$, for some augmented $\mathbb{E}_1$-algebra A over κ. Our main result (Theorem 16.5.4.1) characterizes this algebra: the augmentation ideal $\mathfrak{m}_A$ can be identified (as a nonunital $\mathbb{E}_1$-algebra) with the endomorphism algebra of the object $C \in \mathcal{C}$.

Remark 16.0.0.1. Efimov, Lunts, and Orlov have made an extensive study of a variant of the deformation functor X described above. We refer the reader to [56], [57], and [58] for details. The global structure of moduli spaces of objects of (well-behaved) differential graded categories is treated in [212].

In §16.6, we will study the functor Y which classifies deformations of∞-category $\mathcal{C}$ itself. Once again, the definition of the space $Y(R)$ does not require the assumption that R is commutative: it requires only the ability to talk about prestable R-linear ∞-category. We can therefore regard the construction $R \mapsto Y(R)$ as a functor $Y : \mathrm{Alg}_\kappa^{(2),\mathrm{art}} \to \mathcal{S}$, which we will prove to be a 2-proximate formal moduli problem (Corollary 16.6.2.4). Using Theorem 16.4.2.1, we deduce the existence of a formal moduli problem $\overline{Y} : \mathrm{Alg}_\kappa^{(2),\mathrm{art}} \to \mathcal{S}$ and a natural transformation $Y \to \overline{Y}$ which induces a covering map $Y(R) \to \overline{Y}(R)$ for each $R \in \mathrm{Alg}_\kappa^{(2),\mathrm{art}}$. According to Theorem 15.0.0.9 (and its proof), the functor $\overline{Y}$ is given by $\overline{Y}(R) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{(2),\mathrm{aug}}}(\mathfrak{D}^{(2)}(R), A)$ for some augmented $\mathbb{E}_2$-algebra A over κ. Once again, our main result gives an explicit description of the algebra A: its augmentation ideal $\mathfrak{m}_A$ can be identified (as a nonunital $\mathbb{E}_2$-algebra) with the Hochschild cochain complex $\mathrm{HC}^*(\mathcal{C})$ of the ∞-category $\mathcal{C}$ (Theorem 16.6.3.8).

Remark 16.0.0.2. For a more extensive discussion of the deformation theory of differential graded categories, we refer the reader to [115]. See also [135] and [136].

Remark 16.0.0.3. It is possible to treat the functors X and Y introduced above simultaneously. Let $\mathbf{A}$ denote the ∞-category whose objects are pairs (A_1, A_2), where A_2 is an augmented $\mathbb{E}_2$-algebra over κ and A_1 is an $\mathbb{E}_1$-algebra over A_2 equipped with a map $A_1 \to \kappa$ of $\mathbb{E}_1$-algebras over A_2. We have spectrum objects $E_1, E_2 \in \mathrm{Sp}(\mathbf{A})$, given by

$$\Omega^{\infty-n} E_1 = (\kappa \oplus \Sigma^n(\kappa), \kappa) \qquad \Omega^{\infty-n} E_2 = (\kappa, \kappa \oplus \Sigma^n(\kappa)).$$

Let us regard $(\mathbf{A}, \{E_1, E_2\})$ as a deformation context (in the sense of Definition 12.1.1.1).

Let $\mathcal{C}$ be a κ-linear ∞-category and let $C \in \mathcal{C}$ be an object. Given a pair $(R_1, R_2) \in \mathbf{A}$, we let $Z(R_1, R_2)$ denote a classifying space for pairs $(C_1, \mathcal{C}_2)$, where $\mathcal{C}_2$ is an R_2-linear ∞-category deforming $\mathcal{C}$, and $C_1 \in \mathrm{LMod}_{R_1}(\mathcal{C}_2)$ is an object deforming C. The construction $(R_1, R_2) \mapsto Z(R_1, R_2)$ determines a 2-proximate formal moduli problem. Using Theorem 16.4.2.1 we can complete Z to a formal moduli problem $\overline{Z} : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$.

Using a generalization of the techniques studied in Chapters 14 and 15, one can combine the Koszul duality functors

$$\mathfrak{D}^{(1)} : (\mathrm{Alg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{\mathrm{aug}} \qquad \mathfrak{D}^{(2)} : (\mathrm{Alg}_\kappa^{(2),\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}_\kappa^{(2),\mathrm{aug}}$$

to obtain a deformation theory $\mathfrak{D} : (\mathbf{A})^{\mathrm{op}} \to \mathbf{A}$. Using Theorem 12.3.3.5, we see that the formal moduli problem $\overline{Z}$ is determined by an object $(A_1, A_2) \in \mathbf{A}$. One can show that the augmentation ideals $\mathfrak{m}_{A_1}$ and $\mathfrak{m}_{A_2}$ are given by the endomorphism algebra $\mathrm{End}(C)$ and the Hochschild cochain complex $\mathrm{HC}^*(\mathcal{C})$ of $\mathcal{C}$, respectively (note that $\mathrm{HC}^*(\mathcal{C})$ acts centrally on $\mathrm{End}(C)$).

At the cost of a bit of information, we can be much more concrete. The construction $R \mapsto \overline{Z}(R, R)$ determines a formal $\mathbb{E}_2$-moduli problem $F : \mathrm{Alg}_\kappa^{(2),\mathrm{aug}} \to \mathcal{S}$; for each $R \in \mathrm{Alg}_\kappa^{(2),\mathrm{art}}$ we have a fiber sequence $\overline{X}(R) \to F(R) \to \overline{Y}(R)$, where $\overline{X}$ and $\overline{Y}$ are the formal $\mathbb{E}_1$ and $\mathbb{E}_2$-moduli problems described above. Applying Theorem 15.0.0.9, we deduce that F is given by the formula $F(R) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{(2),\mathrm{aug}}}(\mathfrak{D}^{(2)}(R), A)$ for some augmented $\mathbb{E}_2$-algebra A over κ. Then the augmentation ideal $\mathfrak{m}_A$ can be identified with the fiber of the natural map $\mathrm{HC}^*(\mathcal{C}) \to \mathrm{End}(C)$.

Contents

16.1 Gluing along Closed Immersions

Let $f : (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be a morphism of spectral Deligne-Mumford stacks. Recall that f is said to be a *closed immersion* if the underlying geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ is a closed immersion of ∞-topoi and the map of structure sheaves $f^{-1}\mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$ induces an epimorphism on π_0 (see Definition **??**). In this section, we will study the operation of gluing spectral Deligne-Mumford stacks along closed immersions. Our main result can be formulated as follows:

Theorem 16.1.0.1. *Suppose we are given a pair of closed immersions of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ and $g : \mathsf{X} \to \mathsf{X}'$, and form a pushout diagram σ :*

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{f} & \mathsf{Y} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} \\ \mathsf{X}' & \xrightarrow{g'} & \mathsf{Y}' \end{array}$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ of spectrally ringed ∞-topoi. Then:

(1) *The pushout Y' is a spectral Deligne-Mumford stack.*

(2) *The diagram σ is also a pushout square in the subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}} \subseteq \infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ of locally spectrally ringed ∞-topoi (in particular, it is a pushout square in the ∞-category* SpDM *of spectral Deligne-Mumford stacks).*

(3) *The maps f' and g' are closed immersions.*

(4) *If X' and Y are affine, then Y' is also affine.*

Remark 16.1.0.2. Theorem 16.1.0.1 has an analogue in the setting of spectral schemes, which can be proven in more or less the same way; we leave the details to the reader.

16.1.1 Closed Immersions of ∞-Topoi

We begin with some generalities about gluing ∞-topoi along closed immersions. Recall that if U is an object of an ∞-topos $\mathcal{X}$, then $\mathcal{X}/U$ denotes the full subcategory of $\mathcal{X}$ spanned by those objects V for which the projection map $U \times V \to U$ is an equivalence (see §HTT.7.3.2). We refer to $\mathcal{X}/U$ as the *closed subtopos of $\mathcal{X}$ complementary to U.*

Proposition 16.1.1.1. *Let $\mathcal{X}$ be an ∞-topos, let U be a subobject of the final object of $\mathcal{X}$, and let $\mathcal{X}/U$ be the corresponding closed subtopos of $\mathcal{X}$. Suppose we are given a geometric morphism $f_* : \mathcal{X}/U \to \mathcal{Y}$, and form a pushout diagram*

$$\begin{array}{ccc} \mathcal{X}/U & \xrightarrow{g_*} & \mathcal{X} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f'_*} \\ \mathcal{Y} & \xrightarrow{g'_*} & \mathcal{Z} \end{array}$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi. Let $V = f'_*(U) \in \mathcal{Z}$. Then:

(1) *The functor f'_* induces an equivalence $\mathcal{X}_{/U} \simeq \mathcal{Z}_{/V}$.*

(2) *The functor g'_* is fully faithful and its essential image is $\mathcal{Z}/V$. In particular, g'_* is a closed immersion.*

Proof. According to Proposition HTT.6.3.2.3 , we can identify $\mathcal{Z}$ with the homotopy fiber product of $\mathcal{X}$ with $\mathcal{Y}$ over $\mathcal{X}/U$ (along the functors $g^* : \mathcal{X} \to \mathcal{X}/U$ and $f^* : \mathcal{Y} \to \mathcal{X}/U$ adjoint to g_* and f_*). Since g^*U is an initial object of $\mathcal{X}/U$, there exists an object $V' \in \mathcal{Z}$ such that $f'^*V' \simeq U$ and g'^*V' is an initial object of $\mathcal{Y}$. For each $Z \in \mathcal{Z}$, we have a homotopy pullback diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{Z}}(Z, V') & \longrightarrow & \mathrm{Map}_{\mathcal{X}}(f'^*Z, U) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{Y}}(g'^*Z, g'^*V') & \longrightarrow & \mathrm{Map}_{\mathcal{X}/U}(g^*f'^*Z, g^*U). \end{array}$$

It follows that $\mathrm{Map}_{\mathcal{Z}}(Z, V')$ is contractible if g'^*Z is an initial object of $\mathcal{Y}$, and empty otherwise. Consequently, the equivalence $f'^*V' \simeq U$ is adjoint to an equivalence $V' \to V$. We may therefore assume without loss of generality that $V' = V$.

We now prove (1). Note that the projection map $\mathcal{Z}_{/V} \to \mathcal{Z}$ is a fully faithful embedding, whose essential image is the full subcategory $\mathcal{Z}_0 \subseteq \mathcal{Z}$ spanned by those objects $Z \in \mathcal{Z}$ such that g'^*Z is an initial object of $\mathcal{Y}$. Similarly, the projection $\mathcal{X}_{/U} \to \mathcal{X}$ is a fully faithful embedding whose essential image is the full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ spanned by those objects X such that g^*X is an initial object of $\mathcal{X}/U$. It will therefore suffice to prove that the functor f'^* induces an equivalence of $\mathcal{Z}_0$ with $\mathcal{X}_0$, which follows immediately from Proposition HTT.6.3.2.3 .

We now prove (2). We first claim that g'_* carries $\mathcal{Y}$ into $\mathcal{Z}/V$. That is, we claim that if $Y \in \mathcal{Y}$, then the projection map $(g'_*Y) \times V \to V$ is an equivalence. Let $Z \in \mathcal{Z}$ be an object; we wish to show that the map

$$\mathrm{Map}_{\mathcal{Z}}(Z, g'_*Y \times V) \simeq \mathrm{Map}_{\mathcal{Y}}(g'^*Z, Y) \times \mathrm{Map}_{\mathcal{Z}}(Z, V) \to \mathrm{Map}_{\mathcal{Z}}(Z, V)$$

is a homotopy equivalence. If $Z \notin \mathcal{Z}_0$, then both sides are empty and the result is obvious. If $Z \in \mathcal{Z}_0$, then the result follows since $g'*Z$ is an initial object of $\mathcal{Y}$.

Now suppose $Z \in \mathcal{Z}/V$. We claim that the unit map $u : Z \to g'_*g'^*Z$ is an equivalence in $\mathcal{Z}$. To prove this, we show that for each $Z' \in \mathcal{Z}$, composition with u induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{Z}}(Z', Z) \to \mathrm{Map}_{\mathcal{Z}}(Z', g'_*g'^*Z) \simeq \mathrm{Map}_{\mathcal{Y}}(g'^*Z', g'^*Z).$$

This map fits into a homotopy pullback diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{Z}}(Z', Z) & \longrightarrow & \mathrm{Map}_{\mathcal{Y}}(g'^*Z', g'^*Z) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{X}}(f'^*Z', f'^*Z) & \longrightarrow & \mathrm{Map}_{\mathcal{X}/U}(g^*f'^*Z', g^*f'^*Z). \end{array}$$

It therefore suffices to show that the bottom horizontal map is a homotopy equivalence. In other words, we are reduced to showing that the unit map $f'^{*}Z \to g_* g^* f'^{*} Z$ is an equivalence in $\mathcal{X}$, which follows from the observation that $f'^{*} \in \mathcal{X}/U$.

The argument of the preceding paragraph shows that g'^{*} induces a fully faithful embedding $\mathcal{Z}/V \to \mathcal{Y}$. To complete the proof, we show the functor $g'^{*}|_{\mathcal{Z}/V}$ is essentially surjective. Fix an object $Y \in \mathcal{Y}$, and let $X = f^* Y \in \mathcal{X}/U$. Since g_* is fully faithful, the counit map $g^* g_* X \to X$ is an equivalence. It follows that there exists an object $Z \in \mathcal{Z}$ such that $g'^{*} Z \simeq Y$ and $f'^{*} Z \simeq g_* X$. It remains only to verify that $Z \in \mathcal{Z}/V$: that is, that the projection map $Z \times V \to V$ is an equivalence. Using (1), we are reduced to proving that the map $f'^{*}(Z \times V) \to f'^{*} V \simeq U$ is an equivalence. In other words, we are reduced to proving that $f'^{*} Z \simeq g_* X$ belongs to $\mathcal{X}/U$, which is clear. □

Corollary 16.1.1.2. *Suppose we are given a pushout diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g'_*} & \mathcal{Y} \\ \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} \\ \mathcal{X}' & \xrightarrow{g_*} & \mathcal{Y}' \end{array}$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi. If f_* and g_* are closed immersions, then f'_* and g'_* are closed immersions.

Lemma 16.1.1.3. *Let $\mathcal{X}$ be an ∞-topos containing a pair of (-1)-truncated objects U and V. Suppose that $U \times V$ is an initial object of $\mathcal{X}$. Then the coproduct $U \amalg V$ is also a (-1)-truncated object of $\mathcal{X}$.*

Proof. We must prove that the diagonal map

$$U \amalg V \to (U \amalg V) \times (U \amalg V) \simeq (U \times U) \amalg (U \times V) \amalg (V \times U) \amalg (V \times V)$$

is an equivalence. This map factors as as a composition

$$U \amalg V \to (U \times U) \amalg (V \times V) \to (U \times U) \amalg (U \times V) \amalg (V \times U) \amalg (V \times V)$$

where the first map is an equivalence since U and V are (-1)-truncated, and the second is an equivalence since $U \times V \simeq V \times U$ is an initial object of $\mathcal{X}$. □

Proposition 16.1.1.4. *Let $\mathcal{X}$ be an ∞-topos containing a pair of (-1)-truncated object U and V whose product $U \times V$ is an initial object of $\mathcal{X}$. Then the diagram*

$$\begin{array}{ccc} \mathcal{X}/(U \amalg V) & \longrightarrow & \mathcal{X}/U \\ \downarrow & & \downarrow \\ \mathcal{X}/V & \longrightarrow & \mathcal{X} \end{array}$$

is a pushout square in $\infty\mathcal{T}\mathrm{op}$.

Proof. Let $f^* : \mathcal{X} \to \mathcal{X}_{/U}$, $g^* : \mathcal{X} \to \mathcal{X}_{/V}$ and $h^* : \mathcal{X} \to \mathcal{X}_{/(U \amalg V)}$ be the associated pullback functors. Form a pushout diagram

$$\begin{array}{ccc} \mathcal{X}_{/(U \amalg V)} & \longrightarrow & \mathcal{X}_{/U} \\ \downarrow & & \downarrow \\ \mathcal{X}_{/V} & \longrightarrow & \mathcal{Y} \end{array}$$

in $\infty\mathcal{T}\mathrm{op}$; we wish to prove that the induced geometric morphism $\phi_* : \mathcal{Y} \to \mathcal{X}$ is an equivalence of ∞-topoi. We first claim that the pullback functor ϕ^* is conservative. In other words, we claim that a morphism $u : X \to X'$ in $\mathcal{X}$ is an equivalence provided that both $f^*(u)$ and $g^*(u)$ are equivalences. Indeed, either of these conditions guarantees that $h^*(u)$ is an equivalence, so (by Lemma HA.A.5.11) it will suffice to show that u induces an equivalence $X \times (U \amalg V) \to X' \times (U \amalg V)$. Since colimits in $\mathcal{X}$ are universal, it suffices to show that $u_U : X \times U \to X' \times U$ and $u_V : X \times V \to X' \times V$ are equivalences. We prove that u_U is an equivalence; the proof that u_V is an equivalence is similar. We have a commutative diagram

$$\begin{array}{ccc} X \times U & \xrightarrow{v} & (g_* g^* X) \times U \\ \downarrow & & \downarrow \\ X' \times U & \xrightarrow{v'} & (g_* g^* X') \times U. \end{array}$$

Since the right vertical map is an equivalence, it will suffice to show that v and v' are equivalences. We will prove that v is an equivalence; the roof that v' is an equivalence is similar. Since g_* is fully faithfully, the map $g^*(v)$ is an equivalence. Using Lemma HA.A.5.11 again, we are reduced to proving that the map $X \times U \times V \to (g_* g^* X) \times U \times V$ is an equivalence. This is clear, since both $X \times U \times V$ and $(g_* g^* X) \times U \times V$ are initial objects of $\mathcal{X}$.

To complete the proof that ϕ_* is an equivalence, it suffices to show that the counit map $\phi^* \phi_* Y \to Y$ is an equivalence for each object $Y \in \mathcal{Y}$. Using Proposition HTT.6.3.2.3 , we can identify the ∞-category $\mathcal{Y}$ with the fiber product $\mathcal{X}_{/U} \times_{\mathcal{X}_{/(U \amalg V)}} \mathcal{X}_{/V}$. Thus $Y \in \mathcal{Y}$ corresponds to a pair of objects $Y_0 \in \mathcal{X}_{/U}$, $Y_1 \in \mathcal{X}_{/V}$ together with an equivalence $g^* Y_0 \simeq f^* Y_1 = Y_{01} \in \mathcal{X}_{/(U \amalg V)}$. Unwinding the definitions, we see that the pushforward functor ϕ_* is given by the formula $\phi_* Y = f_* Y_0 \times_{h_* Y_{01}} g_* Y_1$. Since a morphism in $\mathcal{Y}$ is an equivalence if and only if its pullbacks to $\mathcal{X}_{/U}$ and $\mathcal{X}_{/V}$ are equivalences, we are reduced to proving that the projection maps

$$\alpha : f^*(f_* Y_0 \times_{h_* Y_{01}} g_* Y_1) \to Y_0 \qquad \beta : g^*(f_* Y_0 \times_{h_* Y_{01}} g_* Y_1) \to Y_1$$

are equivalences in $\mathcal{X}_{/U}$ and $\mathcal{X}_{/V}$, respectively. We prove that α is an equivalence; the proof for β is similar. The map α factors as a composition

$$f^*(f_* Y_0 \times_{h_* Y_{01}} g_* Y_1) \xrightarrow{\alpha'} f^* f_* Y_0 \xrightarrow{\alpha''} Y_0.$$

Since f_* is fully faithful, the counit map α'' is an equivalence; it will therefore suffice to show that α is an equivalence. Since f^* is left exact, the map α' is a pullback of $\gamma : f^* g_* Y_1 \to f^* h_* Y_{01}$. It will therefore suffice to show that γ is an equivalence in $\mathcal{X}/U$, which is an immediate translation of the condition that $Y_{01} \simeq f^* Y_1 \in \mathcal{X}/(U \amalg V) \subseteq \mathcal{X}/U$. $\square$

16.1.2 Spectrally Ringed ∞-Topoi

Let $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ denote the ∞-category of spectrally ringed ∞-topoi (see Construction 1.4.1.3). The ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ admits small colimits: given a diagram of spectrally ringed ∞-topoi $\{(\mathcal{X}_\alpha, \mathcal{O}_\alpha)\}$, the colimit is given by $(\mathcal{X}, \mathcal{O})$, where $\mathcal{X} \simeq \varinjlim \mathsf{X}_\alpha$ is the colimit of the diagram $\{\mathsf{X}_\alpha\}$ in the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi (which coincides with the *limit* of the associated diagram of left adjoint functors in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$), and $\mathcal{O}$ denotes the limit $\varprojlim \phi_{\alpha *} \mathcal{O}_\alpha$ in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$; here $\phi_{\alpha *} : \mathcal{X} \to \mathcal{X}_\alpha$ denotes the tautological map.

In particular, given a diagram of spectrally ringed ∞-topoi

$$(\mathcal{X}, \mathcal{O}_\mathcal{X}) \xrightarrow{f} (\mathcal{Z}, \mathcal{O}_\mathcal{Z}) \xleftarrow{g} (\mathcal{Y}, \mathcal{O}_\mathcal{Y}),$$

the pushout in $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ is given by $(\mathcal{X} \times_\mathcal{Z} \mathcal{Y}, g'_* \mathcal{O}_\mathcal{X} \times_{h_* \mathcal{O}_\mathcal{Z}} f'_* \mathcal{O}_\mathcal{Y})$ (here the fiber product $\mathcal{X} \times_\mathcal{Z} \mathcal{Y}$ is formed in $\widehat{\mathcal{C}\mathrm{at}}_\infty$), the functors f'_* and g'_* are the right adjoints to the projections of $\mathcal{X} \times_\mathcal{Z} \mathcal{Y}$ onto $\mathcal{Y}$ and $\mathcal{X}$, and h_* $\mathcal{Z} \to \mathcal{X} \times_\mathcal{Z} \mathcal{Y}$ is defined similarly. We will primarily be interested in the case where f and g are closed immersions:

Proposition 16.1.2.1. *Let $f : \mathsf{X} \to \mathsf{Y}$ and $g : \mathsf{X} \to \mathsf{X}'$ be closed immersions in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$, and form a pushout diagram σ :*

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{g} & \mathsf{Y} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathsf{X}' & \xrightarrow{g'} & \mathsf{Y}' \end{array}$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$. Then:

(1) *The diagram σ is also a pushout square in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$.*

(2) *The maps f' and g' are closed immersions.*

(3) *Suppose that the structure sheaves of Y and X' are strictly Henselian. Then the structure sheaf of Y' is also strictly Henselian.*

Proof. We first prove (2). Without loss of generality, it will suffice to show that f' is a closed immersion. Using Proposition 16.1.1.1, we see that f' induces a closed immersion at the level of the underlying ∞-topoi. To complete the proof, we must show that the map

$\theta : f'^{*} \mathscr{O}_{\mathsf{Y}'} \to \mathscr{O}_{\mathsf{Y}}$ has connective fiber. Set $h = f' \circ g \simeq g' \circ f$, so that $\mathscr{O}_{\mathsf{Y}'}$ is given by the fiber product

$$f'_* \mathscr{O}_{\mathsf{Y}} \times_{h_* \mathscr{O}_{\mathsf{X}}} g'_* \mathscr{O}_{\mathsf{X}'}.$$

Under this identification, θ is given by composing the equivalence $f'^{*} f'_* \mathscr{O}_{\mathsf{Y}} \to \mathscr{O}_{\mathsf{Y}}$ with the pullback of the projection map $\theta' : f'_* \mathscr{O}_{\mathsf{Y}} \times_{h_* \mathscr{O}_{\mathsf{X}}} g'_* \mathscr{O}_{\mathsf{X}'} \to f'_* \mathscr{O}_{\mathsf{Y}}$. It will therefore suffice to show that θ' has connective fiber. Equivalently, we must show that the map $g'_* \mathscr{O}_{\mathsf{X}'} \to h_* \mathscr{O}_{\mathsf{X}}$ has connective fiber. This is clear, since the map $\mathscr{O}_{\mathsf{X}'} \to f_* \mathscr{O}_{\mathsf{X}}$ has connective fiber (by virtue of our assumption that f is a closed immersion), and the pushforward functor g'_* is right t-exact (since g' is a closed immersion on the level of ∞-topoi). This completes the proof of (2).

We next claim that the structure sheaf $\mathscr{O}_{\mathsf{Y}'}$ is local. To prove this, it will suffice to show that $f'^{ast} \mathscr{O}_{\mathsf{Y}'}$ and $g'^{*} \mathscr{O}_{\mathsf{Y}'}$ are local, respectively. We will show that $f'^{*} \mathscr{O}_{\mathsf{Y}'}$ is local; the proof of the other case is similar. Write $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}})$ and $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$. Since g is a closed immersion, the geometric morphism g_* induces an equivalence $\mathcal{X} \simeq \mathcal{Y}/U$ for some (-1)-truncated object $U \in \mathcal{Y}$. Because $f'^{*} \mathscr{O}_{\mathsf{Y}'}|_U \simeq \mathscr{O}_{\mathsf{Y}}|_U$ is local by assumption, it will suffice to show that the restriction of $g^* f'^{*} \mathscr{O}_{\mathsf{Y}'} = h^* \mathscr{O}_{\mathsf{Y}'} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ is local. We have a pullback diagram

$$\begin{array}{ccc} h^* \mathscr{O}_{\mathsf{Y}'} & \longrightarrow & g^* \mathscr{O}_{\mathsf{Y}} \\ \downarrow & & \downarrow \\ f^* \mathscr{O}_{\mathsf{X}'} & \longrightarrow & \mathscr{O}_{\mathsf{X}} \end{array}$$

of CAlg-valued sheaves on X. Since f is a morphism in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$, the lower horizontal map in this diagram is local. It follows that the upper horizontal map is also local, so that the locality of $g^* \mathscr{O}_{\mathsf{Y}}$ implies that $h^* \mathscr{O}_{\mathsf{Y}'}$ is local (see Proposition 1.2.1.8), as desired.

We now complete the proof of (1) by showing that Y' is a pushout of X' and Y over X in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$. Equivalently, we must show that if $\mathsf{Z} = (\mathcal{Z}, \mathscr{O})$ is an object of $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ and $r : \mathsf{Y}' \to \mathsf{Z}$ is a morphism in $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$, then r is a morphism in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ if and only if $r \circ f'$ and $r \circ g'$ are morphisms in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$. That is, we must show that the map $r^* \mathscr{O} \to \mathscr{O}_{\mathsf{Y}'}$ is local if and only if the induced maps $(r \circ f')^* \mathscr{O} \to \mathscr{O}_{\mathsf{X}'}$ and $(r \circ g')^* \mathscr{O} \to \mathscr{O}_{\mathsf{Y}}$ are local. This follows immediately from the observation that a section of $\pi_0 \mathscr{O}_{\mathsf{Y}'}$ is invertible if and only if the resulting sections of $\pi_0 \mathscr{O}_{\mathsf{X}'}$ and $\pi_0 \mathscr{O}_{\mathsf{Y}}$ are invertible.

We now prove (3). Assume that $\mathscr{O}_{\mathsf{X}}$, $\mathscr{O}_{\mathsf{X}'}$, and $\mathscr{O}_{\mathsf{Y}}$ are strictly Henselian; we wish to prove that $\mathscr{O}_{\mathsf{Y}'}$ is strictly Henselian. That is, we wish to show that if R is a finitely generated commutative ring and $\{R \to R_\alpha\}$ is a finite collection of étale morphisms which determine a faithfully flat map $R \to \prod_\alpha R_\alpha$, then the induced map $\amalg_\alpha \mathcal{S}\mathrm{ol}_{R_\alpha}(\pi_0 \mathscr{O}_{\mathsf{Y}'}) \to \mathcal{S}\mathrm{ol}_R(\pi_0 \mathscr{O}_{\mathsf{Y}'})$ is an effective epimorphism. We have a fiber sequence $\mathscr{O}_{\mathsf{Y}'} \to g'_* \mathscr{O}_{\mathsf{X}'} \oplus f'_* \mathscr{O}_{\mathsf{Y}} \to h_* \mathscr{O}_{\mathsf{X}}$, which determines a short exact sequence of homotopy sheaves

$$h_*(\pi_1 \mathscr{O}_{\mathsf{X}}) \to \pi_0 \mathscr{O}_{\mathsf{Y}'} \xrightarrow{\nu} g'_*(\pi_0 \mathscr{O}_{\mathsf{X}'}) \times_{h_*(\pi_0 \mathscr{O}_{\mathsf{X}})} f'_*(\pi_0 \mathscr{O}_{\mathsf{Y}}) \to 0$$

In particular, ν is an epimorphism whose kernel is a square-zero ideal sheaf contained in $\pi_0 \mathscr{O}_{\mathsf{Y}'}$. For each index α, the assumption that R_α is étale over R guarantees that the diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{ol}_{R_\alpha}(\pi_0 \mathscr{O}_{\mathsf{Y}'}) & \longrightarrow & \mathcal{S}\mathrm{ol}_{R_\alpha}(g'_*(\pi_0 \mathscr{O}_{\mathsf{X}'}) \times_{h_*(\pi_0 \mathscr{O}_{\mathsf{X}})} f'_*(\pi_0 \mathscr{O}_{\mathsf{Y}})) \\ \downarrow & & \downarrow \\ \mathcal{S}\mathrm{ol}_{R}(\pi_0 \mathscr{O}_{\mathsf{Y}'}) & \longrightarrow & \mathcal{S}\mathrm{ol}_{R}(g'_*(\pi_0 \mathscr{O}_{\mathsf{X}'}) \times_{h_*(\pi_0 \mathscr{O}_{\mathsf{X}})} f'_*(\pi_0 \mathscr{O}_{\mathsf{Y}})) \end{array}$$

is a pullback square. We may therefore replace $\mathscr{O}_{\mathsf{Y}}$, $\mathscr{O}_{\mathsf{X}'}$, and $\mathscr{O}_{\mathsf{X}}$ by their truncations $\pi_0 \mathscr{O}_{\mathsf{Y}}$, $\pi_0 \mathscr{O}_{\mathsf{X}'}$, and $\pi_0 \mathscr{O}_{\mathsf{X}}$, and thereby reduce to the case where the structure sheaves of X, Y, and X' are discrete (from which it follows that the structure sheaf of Y' is also discrete).

We wish to prove that the map

$$\theta : \amalg_\alpha \mathcal{S}\mathrm{ol}_{R_\alpha}(g'_* \mathscr{O}_{\mathsf{X}'} \times_{h_* \mathscr{O}_{\mathsf{X}}} f'_* \mathscr{O}_{\mathsf{Y}}) \to \mathcal{S}\mathrm{ol}_{R}(g'_* \mathscr{O}_{\mathsf{X}'} \times_{h_* \mathscr{O}_{\mathsf{X}}} f'_* \mathscr{O}_{\mathsf{Y}})$$

is an effective epimorphism. Let $\mathcal{Y}'$ denote the underlying ∞-topos of Y', and let $\mathcal{U}$ denote the open subtopos of $\mathcal{Y}'$ complementary to the image of the closed immersion f'. Then the restriction of θ to $\mathcal{U}$ can be identified with the restriction of the effective epimorphism $\amalg_\alpha \mathcal{S}\mathrm{ol}_{R_\alpha}(\mathscr{O}_{\mathsf{X}'}) \to \mathcal{S}\mathrm{ol}_{R}(\mathscr{O}_{\mathsf{Y}'})$. A similar argument shows that θ is an effective epimorphism when restricted to the complement of the closed immersion g'. It will therefore suffice to show that $h^*(\theta)$ is an effective epimorphism. Unwinding the definitions, we can identify $h^*(\theta)$ with the canonical map

$$\amalg_\alpha \mathcal{S}\mathrm{ol}_{R_\alpha}(f^* \mathscr{O}_{\mathsf{X}'}) \times_{\mathcal{S}\mathrm{ol}_{R_\alpha}(\mathscr{O}_{\mathsf{X}})} \mathcal{S}\mathrm{ol}_{R_\alpha}(g^* \mathscr{O}_{\mathsf{Y}}) \to \mathcal{S}\mathrm{ol}_{R}(f^* \mathscr{O}_{\mathsf{X}'}) \times_{\mathcal{S}\mathrm{ol}_{R}(\mathscr{O}_{\mathsf{X}})} \mathcal{S}\mathrm{ol}_{R}(g^* \mathscr{O}_{\mathsf{Y}}).$$

It follows from Proposition 1.2.2.12 that this map is a pullback of $\amalg_\alpha \mathcal{S}\mathrm{ol}_{R_\alpha}(\mathscr{O}_{\mathsf{X}}) \to \mathcal{S}\mathrm{ol}_{R}(\mathscr{O}_{\mathsf{X}})$, hence an effective epimorphism (since $\mathscr{O}_{\mathsf{X}}$ is assumed to be strictly Henselisan). □

16.1.3 Pushouts in the Affine Case

We now establish a special case of Theorem 16.1.0.1.

Proposition 16.1.3.1. *Suppose we are given a pullback diagram of $\mathbb{E}_\infty$-rings*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B' & \xrightarrow{g} & B \end{array}$$

where A, B, and B' are connective and the maps f and g induce surjective ring homomorphisms $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$. Then:

(1) *The $\mathbb{E}_\infty$-ring A' is connective.*

(2) *The maps $\pi_0 A' \to \pi_0 A$ and $\pi_0 B' \to \pi_0 B$ are surjective.*

(3) *The induced diagram of spectral Deligne-Mumford stacks σ :*

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} B & \longrightarrow & \operatorname{Sp\acute{e}t} B' \\ \downarrow & & \downarrow \\ \operatorname{Sp\acute{e}t} A & \longrightarrow & \operatorname{Sp\acute{e}t} A' \end{array}$$

is a pushout diagram in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$.

Proof. Assertions (1) and (2) are obvious. For every $\mathbb{E}_\infty$-ring R, let $\mathcal{X}_R$ denote the underlying ∞-topos of Spét R (that is, the ∞-topos of étale sheaves on $\mathrm{CAlg}^{\text{ét}}_R$), and let $\mathscr{O}_R$ denote the structure sheaf of Spét R. By virtue of Proposition 16.1.2.1, it will suffice to show that σ is a pushout square in $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$. That is, we must show the following:

($3'$) The diagram of ∞-topoi

$$\begin{array}{ccc} \mathcal{X}_B & \longrightarrow & \mathcal{X}_{B'} \\ \downarrow & \overset{h_*}{\searrow} & \downarrow f_* \\ \mathcal{X}_A & \xrightarrow{g_*} & \mathcal{X}_{A'} \end{array}$$

is a pushout square in $\infty\mathcal{T}\mathrm{op}$.

($3''$) The diagram

$$\begin{array}{ccc} \mathscr{O}_{A'} & \longrightarrow & g_* \mathscr{O}_A \\ \downarrow & & \downarrow \\ f_* \mathscr{O}_{B'} & \longrightarrow & h_* \mathscr{O}_B \end{array}$$

is a pullback square in $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathsf{X}_{A'})$.

We first prove ($3'$). Using (2) and Proposition 3.1.4.1, we deduce the existence of equivalences

$$\mathcal{X}_{B'} \simeq \mathcal{X}_{A'}/U \qquad \mathcal{X}_A \simeq \mathcal{X}_{A'}/V \qquad \mathcal{X}_B \simeq \mathcal{X}_{A'}/W$$

for (-1)-truncated objects $U, V, W \in \mathcal{X}_{A'}$. Under the equivalence $\mathcal{X}_{A'} \simeq \mathcal{S}\mathrm{hv}^{\text{ét}}_{A'} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\text{ét}}_{A'}, \mathcal{S})$, the objects U, V, and W are given by the formulae

$$U(R) = \begin{cases} \Delta^0 & \text{if } R \otimes_{A'} B' \simeq 0 \\ \emptyset & \text{otherwise} \end{cases} \quad V(R) = \begin{cases} \Delta^0 & \text{if } R \otimes_{A'} A \simeq 0 \\ \emptyset & \text{otherwise} \end{cases} \quad W(R) = \begin{cases} \Delta^0 & \text{if } R \otimes_{A'} B \simeq 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

For any A-algebra R, we have a pullback diagram τ_R :

$$\begin{array}{ccc} R & \longrightarrow & R \otimes_{A'} A \\ \downarrow & & \downarrow \\ R \otimes_{A'} B' & \longrightarrow & R \otimes_{A'} B. \end{array}$$

It follows that if $R \otimes_{A'} A \simeq R \otimes_{A'} B' \simeq 0$, then $R \simeq 0$; this proves that $U \times V$ is an initial object of $\mathcal{X}_{A'}$. According to Lemma 16.1.1.3, the coproduct $U \amalg V$ in $\mathcal{X}_{A'}$ is (-1)-truncated. There is an evident map $j : U \amalg V \to W$. We will prove that j is an equivalence, so that assertion (3′) follows from Proposition 16.1.1.4. We must show that if R is an étale A'-algebra such that $W(R)$ is nonempty, then $(U \amalg V)(R)$ is nonempty. Indeed, if $W(R)$ is nonempty then we have $R \otimes_{A'} B \simeq 0$, so that the pullback diagram $\tau_{R'}$ shows that $R \simeq R_0 \times R_1$ with $R_0 = R \otimes_{A'} A$ and $R_1 = R \otimes_{A'} B'$). Then R_0 and R_1 are also étale A'-algebras, and we have $(U \amalg V)(R) \simeq (U \amalg V)(R_0) \times (U \amalg V)(R_1)$. It will therefore suffice to show that both $(U \amalg V)(R_0)$ and $(U \amalg V)(R_1)$ are nonempty. We will prove that $(U \amalg V)(R_0)$ is nonempty; the proof in the other case is similar. Since we have a map $U(R_0) \to (U \amalg V)(R_0)$, it is sufficient to show that $U(R_0)$ is nonempty: that is, that $R_0 \otimes_{A'} B' \simeq 0$. Let $I \subseteq \pi_0 A'$ be the kernel of the map $\pi_0 A' \to \pi_0 B'$, and $J \subseteq \pi_0 A'$ the kernel of the map $\pi_0 A' \to \pi_0 A$, so that the ideal $I + J$ is the kernel of the map $\pi_0 A' \to \pi_0 B$. We now compute

$$\begin{aligned} \pi_0(R_0 \otimes_{A'} B') &\simeq (\pi_0 R_0)/(I \pi_0 R_0) \\ &\simeq (\pi_0 R / J \pi_0 R)/I \pi_0 R_0 \\ &\simeq (\pi_0 R)/(I + J)\pi_0 R \\ &\simeq \pi_0(R \otimes_{A'} B) \\ &\simeq 0. \end{aligned}$$

It remains to prove (3″). Unwinding the definitions, we are reduced to the evident observation that if $R \in \mathrm{CAlg}^{\text{ét}}_{A'}$, then the diagram

$$\begin{array}{ccc} R & \longrightarrow & R \otimes_{A'} A \\ \downarrow & & \downarrow \\ R \otimes_{A'} B' & \longrightarrow & R \otimes_{A'} B \end{array}$$

is a pullback square of $\mathbb{E}_\infty$-rings. □

Example 16.1.3.2. Let X be an object of $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$, and suppose we are given a morphism $\epsilon : \mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{X}$, where κ is a field. Let $X : \mathrm{CAlg}^{\mathrm{art}}_{\kappa} \to \widehat{\mathcal{S}}$ denote the functor given by the formula

$$X(R) = \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\,R, \mathsf{X}) \times_{\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\,R, \mathsf{X})} \{\epsilon\}.$$

It follows from Proposition 16.1.3.1 that for every pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

in $\mathrm{CAlg}_\kappa^{\mathrm{art}}$ for which the ring homomorphisms $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, the associated diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \operatorname{Spét} R & \longleftarrow & \operatorname{Spét} R_0 \\ \uparrow & & \uparrow \\ \operatorname{Spét} R_1 & \longleftarrow & \operatorname{Spét} R_{01} \end{array}$$

is a pushout square in $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{loc}}$ (in fact, this is easier than Proposition 16.1.3.1, since the underlying ∞-topoi of these spectral Deligne-Mumford stacks are all the same). It follows that the diagram of spaces

$$\begin{array}{ccc} X(R) & \longrightarrow & X(R_0) \\ \downarrow & & \downarrow \\ X(R_1) & \longrightarrow & X(R_{01}) \end{array}$$

is a pullback square. Since $X(\kappa)$ is evidently contractible, it follows that the functor X is a formal moduli problem (modulo the technicality that the values of X might not be essentially small: this issue does not arise in most cases of interest, for example if X is a spectral Deligne-Mumford stack).

16.1.4 The Proof of Theorem 16.1.0.1

We now turn to the proof of our main result.

Lemma 16.1.4.1. *Suppose we are given a closed immersion $f : (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ of spectral Deligne-Mumford stacks, where $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is affine. Let $U \in \mathcal{X}$ be an object such that $(\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X}|_U)$ is also an affine Deligne-Mumford stack. Then there exists an equivalence $U \simeq f^* V$ for some affine object $V \in \mathcal{Y}$.*

Proof. Let $(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \simeq \operatorname{Spét} A$ for some connective $\mathbb{E}_\infty$-ring A. Then $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \simeq \operatorname{Spét} B$ for some connective A-algebra B such that $\pi_0 A \to \pi_0 B$ is surjective. Using Theorem 1.4.10.2, we see that the object $U \in \mathcal{X}$ corresponds to an étale B-algebra B'. Using Proposition B.1.1.3, we can write $B' \simeq A' \otimes_A B$ for some étale A-algebra A', which determines an object $V \in \mathcal{Y}$ with the desired properties. □

Proof of Theorem 16.1.0.1. Suppose we are given a pushout diagram σ :

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{f} & \mathsf{Y} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} \\ \mathsf{X}' & \xrightarrow{g'} & \mathsf{Y}' \end{array}$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ of spectrally ringed ∞-topoi. Assume that X, Y, and X' are spectral Deligne-Mumford stacks, and that f and g are closed immersions. It follows from Proposition 16.1.2.1 that f' and g' are closed immersions and that σ is also pushout diagram in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$. We wish to prove that Y' is a spectral Deligne-Mumford stack.

Write $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, $\mathsf{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, $\mathsf{X}' = (\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$, and $\mathsf{Y}' = (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'})$. The assertion that $(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'})$ is a spectral Deligne-Mumford stack is local on $\mathcal{Y}'$. Let $\mathcal{Y}'_0$ be the full subcategory of $\mathcal{Y}'$ spanned by those objects Y for which $(\mathcal{Y}'_{/Y}, \mathcal{O}_{\mathcal{Y}'}|_Y)$ is a spectral Deligne-Mumford stack; it will suffice to show that there exists an effective epimorphism $\amalg Y_i \to \mathbf{1}_{\mathcal{Y}'}$, where each $Y_i \in \mathcal{Y}'_0$ and $\mathbf{1}_{\mathcal{Y}'}$ denotes a final object of $\mathcal{Y}'$. Since f' and g' are closed immersions, we can write $\mathcal{X}' \simeq \mathcal{Y}'/U$ and $\mathcal{Y} \simeq \mathcal{Y}'/V$ for some (-1)-truncated objects $U, V \in \mathcal{Y}'$. Then

$$(\mathcal{Y}'_{/U}, \mathcal{O}_{\mathcal{Y}'}|_U) \simeq (\mathcal{Y}_{/f'^*U}, \mathcal{O}_{\mathcal{Y}}|_{f'^*U}) \qquad (\mathcal{Y}'_{/V}, \mathcal{O}_{\mathcal{Y}'}|_V) \simeq (\mathcal{X}'_{/g'^*V}, \mathcal{O}_{\mathcal{X}'}|_{g'^*V})$$

are spectral Deligne-Mumford stacks, so that $U, V \in \mathcal{Y}'_0$. Let $h^* = f^* \circ g'^* : \mathcal{Y}' \to \mathcal{X}$. Using Lemma 3.1.3.2, we are reduced to proving the existence of an effective epimorphism $\amalg X_i \to \mathbf{1}_{\mathcal{X}}$, where each X_i belongs to the essential image $h^* \mathcal{Y}'_0 \subseteq \mathcal{X}$ and $\mathbf{1}_{\mathcal{X}}$ denotes the final object of $\mathcal{X}$.

Let $\mathcal{X}_0$ be the full subcategory of $\mathcal{X}$ spanned by those objects $X \in \mathcal{X}$ with the following properties:

(a) There exists an object $Y \in \mathcal{Y}$ such that $f^*Y \simeq X$ and $(\mathcal{Y}_{/Y}, \mathcal{O}_{\mathcal{Y}}|_Y)$ is affine.

(b) There exists an object $X' \in \mathcal{X}'$ such that $g^*X' \simeq X$ and $(\mathcal{X}'_{/X'}, \mathcal{O}_{\mathcal{X}'}|'_X)$ is affine.

Using Proposition HTT.6.3.2.3 , we see that every object $X \in \mathcal{X}_0$ has the form h^*Y' for some $Y' \in \mathcal{Y}$ such that $(\mathcal{X}'_{/g'^*Y'}, \mathcal{O}_{\mathcal{X}'}|_{g'^*Y'})$ and $(\mathcal{Y}_{/f'^*Y'}, \mathcal{O}_{\mathcal{Y}}|_{f'^*Y'})$ are affine. It then follows from Proposition 16.1.3.1 that $Y' \in \mathcal{Y}_0$, so that $X \in h^* \mathcal{Y}'_0$. It will therefore suffice to show that there exists an effective epimorphism $\amalg X_i \to \mathbf{1}_{\mathcal{X}}$, where each $X_i \in \mathcal{X}_0$.

Since $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a spectral Deligne-Mumford stack, there exists an effective epimorphism $\amalg_i X_i \to \mathbf{1}_{\mathcal{X}}$ where each $X_i \in \mathcal{X}$ satisfies condition (i). Using our assumption that $(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a spectral Deligne-Mumford stack, we can choose for each index i an effective epimorphism $\theta_i : \amalg_j X'_{i,j} \to g_* X_i$ in $\mathcal{X}'$. Set $X_{i,j} = g^* X'_j$, so that the maps θ_i induce effective epimorphisms

$$\amalg_{i,j} X_{i,j} \to \amalg_i X_i \to \mathbf{1}_{\mathcal{X}}.$$

It will therefore suffice to show that each $X_{i,j}$ satisfies conditions (a) and (b). Condition (b) is evident from the construction, and condition (a) follows from Lemma 16.1.4.1 (together with our assumption that X_i satisfies (a)). This completes the proof that Y' is a spectral Deligne-Mumford stack.

We conclude the proof by observing that if X' and Y are affine (so that X is also affine, by virtue of Theorem 3.1.2.1), then Proposition 16.1.3.1 implies that Y' is also affine.

□

16.2 Clutching of Quasi-Coherent Sheaves

Suppose we are a diagram of spectral Deligne-Mumford stacks $\mathsf{X}_0 \overset{f}{\hookleftarrow} \mathsf{X}_{01} \overset{g}{\hookrightarrow} \mathsf{X}_1$, where f and g are closed immersions. In §16.1, we proved that there exists a pushout diagram

$$\begin{array}{ccc} \mathsf{X}_{01} & \xrightarrow{f} & \mathsf{X}_0 \\ \downarrow{g} & & \downarrow{g'} \\ \mathsf{X}_1 & \xrightarrow{f'} & \mathsf{X} \end{array}$$

in the ∞-category of spectral Deligne-Mumford stacks (Theorem 16.1.0.1). Our goal in this section is to study the ∞-category $\mathrm{QCoh}(\mathsf{X})$ of quasi-coherent sheaves on the pushout X. Note that every quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ determines objects $\mathscr{F}_0 = g'^* \mathscr{F} \in \mathrm{QCoh}(\mathsf{X}_0)$ and $\mathscr{F}_1 = f'^* \mathscr{F} \in \mathrm{QCoh}(\mathsf{X}_1)$, together with an equivalence $\alpha : f^* \mathscr{F}_0 \simeq g^* \mathscr{F}_1$ in the ∞-category $\mathrm{QCoh}(\mathsf{X}_{01})$. Our starting point is the observation that $\mathscr{F}$ is determined (up to canonical equivalence) by the triple $(\mathscr{F}_0, \mathscr{F}_1, \alpha)$. More precisely, we have the following:

Theorem 16.2.0.1. *Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks σ :*

$$\begin{array}{ccc} \mathsf{X}_{01} & \xrightarrow{f} & \mathsf{X}_0 \\ \downarrow{g} & & \downarrow{g'} \\ \mathsf{X}_1 & \xrightarrow{f'} & \mathsf{X}, \end{array}$$

where f and g are closed immersions. Then the induced diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X}_{01}) & \longleftarrow & \mathrm{QCoh}(\mathsf{X}_0) \\ \uparrow & & \uparrow \\ \mathrm{QCoh}(\mathsf{X}_1) & \longleftarrow & \mathrm{QCoh}(\mathsf{X}) \end{array}$$

determines a fully faithful embedding $\theta : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X}_0) \times_{\mathrm{QCoh}(\mathsf{X}_{01})} \mathrm{QCoh}(\mathsf{X}_1)$. Moreover, θ restricts to an equivalence of ∞-categories

$$\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \to \mathrm{QCoh}(\mathsf{X}_0)^{\mathrm{cn}} \times_{\mathrm{QCoh}(\mathsf{X}_{01})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X}_1)^{\mathrm{cn}}.$$

The assertion of Theorem 16.2.0.1 is local on X, so we can reduce to the case where $\mathsf{X} = \operatorname{Sp\acute{e}t} A$ is affine. In this case, Theorem 3.1.2.1 guarantees that X_0, X_1, and X_{01} are also affine, so that the diagram σ is induced by a diagram of connective $\mathbb{E}_\infty$-rings τ :

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01}. \end{array}$$

Since σ is a pushout diagram of closed immersions, τ is a pullback diagram of $\mathbb{E}_\infty$-rings, and the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjections. We can therefore reformulate Theorem 16.2.0.1 as follows:

Theorem 16.2.0.2. *Suppose we are given a pullback diagram of $\mathbb{E}_\infty$-rings τ :*

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01}. \end{array}$$

Then the induced functor $\theta : \mathrm{Mod}_A \to \mathrm{Mod}_{A_0} \times_{\mathrm{Mod}_{A_{01}}} \mathrm{Mod}_{A_1}$ is fully faithful. If τ is a pullback diagram of connective $\mathbb{E}_\infty$-rings and the map $\pi_0 A_0 \to \pi_0 A_{01}$ is surjective, then θ restricts to an equivalence of ∞-categories $\mathrm{Mod}_A^{\mathrm{cn}} \to \mathrm{Mod}_{A_0}^{\mathrm{cn}} \times_{\mathrm{Mod}_{A_{01}}^{\mathrm{cn}}} \mathrm{Mod}_{A_1}^{\mathrm{cn}}$.

Warning 16.2.0.3. The the situation of Theorem 16.2.0.2, the functor θ need not be an equivalence, even when τ is a diagram of connective $\mathbb{E}_\infty$-rings which induces surjections $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$. For example, let κ be a field and let $A_0 = \kappa[x]$, $A_{01} = \kappa$, and $A_1 = \kappa[y]$ (regarded as discrete $\mathbb{E}_\infty$-rings). Let M denote the A_0-module given by $\bigoplus_{m \in \mathbf{Z}} \Sigma^{2m} A_{01}$, and let N be the A_1-module given by $\bigoplus_{m \in \mathbf{Z}} \Sigma^{2m+1} A_{01}$. Then $A_{01} \otimes_{A_0} M$ and $A_{01} \otimes_{A_1} N$ are both equivalent to the A_{01}-module $P = \bigoplus_{i \in \mathbf{Z}} \Sigma^i A_{01}$. We may therefore choose an equivalence $\alpha : A_{01} \otimes_{A_0} M \simeq A_{01} \otimes_{A_1} N$. The triple (M, N, α) can be regarded as an object of the fiber product $\mathrm{Mod}_{A_0} \times_{\mathrm{Mod}_{A_{01}}} \mathrm{Mod}_{A_1}$ which does not lie in the essential image of the functor θ (since $M \times_P N \simeq 0$).

16.2.1 Clutching in the Stable Case

The first assertion of Theorem 16.2.0.2 is a consequence of the following more general claim:

Proposition 16.2.1.1. *Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. Suppose we are given a pullback diagram τ :*

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01} \end{array}$$

in Alg_R. *Then the induced functor*

$$F : \mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_{A_0}(\mathcal{C}) \times_{\mathrm{LMod}_{A_{01}}(\mathcal{C})} \mathrm{LMod}_{A_1}(\mathcal{C})$$

is fully faithful.

Proof. Let G denote a right adjoint to F and fix $M \in \mathrm{LMod}_A(\mathcal{C})$; we wish to show that the unit map $M \to (G \circ F)(M)$ is an equivalence. Unwinding the definitions, this is equivalent to showing that the map

$$M \to (A_0 \otimes_A M) \times_{A_{01} \otimes_A M} (A_1 \otimes_A M)$$

is an equivalence; that is, we must show that the diagram σ :

$$\begin{array}{ccc} A \otimes_A M & \longrightarrow & A_0 \otimes_A M \\ \downarrow & & \downarrow \\ A_1 \otimes_A M & \longrightarrow & A_{01} \otimes_A M \end{array}$$

is a pullback diagram in $\mathcal{C}$. Since $\mathcal{C}$ is stable, this is equivalent to showing that σ is a pushout diagram in $\mathcal{C}$. The relative tensor product over A preserves colimits, so we need only verify that τ is a pushout diagram in RMod_A. This equivalent to the requirement that τ be a pushout diagram of spectra. Since the ∞-category of spectra is stable, we reduce to our assumption that τ is a pullback diagram (in either Sp or Alg_R). □

16.2.2 Clutching in the Prestable Case

To complete the proof of Theorem 16.2.0.2, we will need a refinement of Proposition 16.2.1.1 in the setting of *prestable* ∞-categories.

Proposition 16.2.2.1. *Let R be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a prestable R-linear ∞-category. Suppose we are given a pullback diagram*

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow f \\ A_1 & \xrightarrow{g} & A_{01} \end{array}$$

in $\mathrm{Alg}_R^{\mathrm{cn}}$*. If $\mathcal{C}$ is separated and the map f induces a surjective ring homomorphism $\pi_0 A_0 \to \pi_0 A_{01}$, then the inducted functor*

$$f : \mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_{A_0}(\mathcal{C}) \times_{\mathrm{LMod}_{A_{01}}(\mathcal{C})} \mathrm{LMod}_{A_1}(\mathcal{C})$$

is an equivalence of ∞-categories.

Proof of Theorem 16.2.0.2. Apply Proposition 16.2.2.1 to the special case where $R = A$ and $\mathcal{C} = \mathrm{Mod}_A^{\mathrm{cn}}$. □

Proof of Proposition 16.2.2.1. Passing to stabilizations, we obtain a functor

$$F : \mathrm{LMod}_A(\mathrm{Sp}(\mathcal{C})) \to \mathrm{LMod}_{A_0}(\mathrm{Sp}(\mathcal{C})) \times_{\mathrm{LMod}_{A_{01}}(\mathrm{Sp}(\mathcal{C}))} \mathrm{LMod}_{A_1}(\mathrm{Sp}(\mathcal{C}))$$

which admits a right adjoint G. We first claim that G carries the fiber product

$$\mathrm{LMod}_{A_0}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0} \times_{\mathrm{LMod}_{A_{01}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}} \mathrm{LMod}_{A_1}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$$

into $\mathrm{LMod}_A(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$. To see this, suppose we are given objects $M \in \mathrm{LMod}_{A_0}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ and $N \in \mathrm{LMod}_{A_1}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ and equivalences $A_{01} \otimes_{A_0} M \simeq P \simeq A_{01} \otimes_{A_1} N$ in the ∞-category $\mathrm{LMod}_{A_{01}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$. We must show that the fiber product $M \times_P N$ belongs to $\mathrm{LMod}_A(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$. In the stable ∞-category $\mathrm{Sp}(\mathcal{C})$, we have a fiber sequence

$$M \times_P N \to M \oplus N \to P.$$

Since M, N, and P belong to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$, it will suffice to show that the map $\pi_0(M \oplus N) \to \pi_0 P$ is an epimorphism in the abelian category $\mathcal{C}^{\heartsuit}$. In fact, we claim that $\pi_0 M \to \pi_0 P$ is an epimorphism. Choose a cofiber sequence $I \to A_0 \xrightarrow{f} A_{01}$ in RMod_{A_0}. Since f induces a surjective ring homomorphism $\pi_0 A_0 \to \pi_0 A_{01}$, the module I is connective. We therefore have an induced cofiber sequence $I \otimes_{A_0} M \to M \to P$ in $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$, which proves that the map $\pi_0 M \to \pi_0 P$ is an epimorphism in $\mathcal{C}^{\heartsuit}$.

It follows from the above argument that F and G determine adjoint functors

$$\mathrm{LMod}_A(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0} \underset{G_{\geqslant 0}}{\overset{F_{\geqslant 0}}{\rightleftarrows}} \mathrm{LMod}_{A_0}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0} \times_{\mathrm{LMod}_{A_{01}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}} \mathrm{LMod}_{A_1}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0},$$

and the prestability of $\mathcal{C}$ guarantees that $F_{\geqslant 0}$ can be identified with our original functor f. We wish to prove that $F_{\geqslant 0}$ is an equivalence of ∞-categories. Proposition 16.2.1.1 implies that the functor F is fully faithful, so the functor $F_{\geqslant 0}$ is automatically fully faithful. To complete the proof, it will suffice to show that the functor $G_{\geqslant 0}$ is conservative. Let $\alpha : X \to Y$ be a morphism in the ∞-category

$$\mathrm{LMod}_{A_0}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0} \times_{\mathrm{LMod}_{A_{01}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}} \mathrm{LMod}_{A_1}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$$

such that $G_{\geqslant 0}(\alpha)$ is an equivalence. Then $G_{\geqslant 0}(\mathrm{cofib}(\alpha)) \simeq 0$. We will complete the proof by showing that $\mathrm{cofib}(\alpha) \simeq 0$, so that α is an equivalence.

Unwinding the definitions, we can identify $\mathrm{cofib}(\alpha)$ with a triple of objects

$$M \in \mathrm{LMod}_{A_0}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0} \qquad P \in \mathrm{LMod}_{A_{01}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0} \qquad N \in \mathrm{LMod}_{A_1}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$$

together with equivalences $A_{01} \otimes_A M \simeq P \simeq A_{01} \otimes_{A_1} N$. We will prove by induction on n that $M \in \mathrm{LMod}_A(\mathrm{Sp}(\mathcal{C}))_{\geqslant n}$ and $N \in \mathrm{LMod}_{B'}(\mathrm{Sp}(\mathcal{C}))_{\geqslant n}$. Provided that this is true for all n, our assumption that $\mathcal{C}$ is separated will imply that $M \simeq N \simeq 0$ so that $\mathrm{cofib}(\alpha) \simeq 0$ as desired.

In the case $n = 0$, there is nothing to prove. Assume therefore that $M \in \mathrm{LMod}_A(\mathcal{C})_{\geqslant n}$ and $N \in \mathrm{LMod}_A(\mathcal{C})_{\geqslant n}$. Since $G(\mathrm{cofib}(\alpha)) = M \times_P N \simeq 0$, we have an isomorphism

$\pi_n M \oplus \pi_n N \to \pi_n P$ in the abelian category $\mathcal{C}^{\heartsuit}$. The first part of the proof shows that the map $\pi_n M \to \pi_n P$ is an epimorphism, so that $\pi_n N \simeq 0$. It follows that $N \in \mathrm{LMod}_{A_1}(\mathrm{Sp}(\mathcal{C}))_{\geqslant n+1}$ and therefore $P \simeq A_{01} \otimes_{A_1} N$ belongs to $\mathrm{LMod}_{A_{01}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant n+1}$. In particular, we have $\pi_n P \simeq 0$, so that $\pi_n M \simeq \pi_n M \oplus \pi_n N \simeq \pi_n P \simeq 0$ and therefore $M \in \mathrm{LMod}_{A_0}(\mathrm{Sp}(\mathcal{C}))_{\geqslant n+1}$, as desired. □

We can use Proposition 16.2.2.1 to deduce a categorified version of Proposition 16.2.1.1, which serves as a sort of counterpart to the descent results of §D.4 and D.6:

Corollary 16.2.2.2. *For every connective $\mathbb{E}_2$-ring A, let $\mathrm{LinCat}_A^{\mathrm{sep}}$ denote the full subcategory of $\mathrm{LinCat}_A^{\mathrm{PSt}}$ spanned by those prestable A-linear ∞-categories $\mathcal{C}$ which are separated. Let*

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow f \\ A_1 & \xrightarrow{g} & A_{01} \end{array}$$

be a pullback diagram of connective $\mathbb{E}_2$-rings, and suppose that f induces a surjective ring homomorphism $\pi_0 A \to \pi_0 A_{01}$. Then the induced map

$$F : \mathrm{LinCat}_A^{\mathrm{sep}} \to \mathrm{LinCat}_{A_0}^{\mathrm{sep}} \times_{\mathrm{LinCat}_{A_{01}}^{\mathrm{sep}}} \mathrm{LinCat}_{A_1}^{\mathrm{sep}}$$

is fully faithful.

Proof. We note that F is the restriction of a functor

$$F' : \mathrm{LMod}_{\mathrm{LMod}_A^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{LMod}_{\mathrm{LMod}_{A_0}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \times_{\mathrm{LMod}_{\mathrm{LMod}_{A_{01}}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})} \mathrm{LMod}_{\mathrm{LMod}_{A_1}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$$

which admits a right adjoint G'. To prove that F is fully faithful, it will suffice to show that the unit map $u : \mathcal{C} \to (G' \circ F')(\mathcal{C})$ is an equivalence whenever $\mathcal{C} \in \mathrm{LMod}_{\mathrm{LMod}_A^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ is a separated prestable A-linear ∞-category. Unwinding the definitions, we can identify u with the equivalence $\mathcal{C} \to \mathrm{LMod}_{A_0}(\mathcal{C}) \times_{\mathrm{LMod}_{A_{01}}(\mathcal{C})} \mathrm{LMod}_{A_1}(\mathcal{C})$ of Proposition 16.2.2.1. □

16.2.3 Properties of Quasi-Coherent Sheaves

In the situation of Theorem 16.2.0.2, many important geometric properties of quasi-coherent sheaves $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ can be tested "piecewise":

Proposition 16.2.3.1. *Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks σ :*

$$\begin{array}{ccc} \mathsf{X}_{01} & \xrightarrow{i} & \mathsf{X}_0 \\ \downarrow j & & \downarrow j' \\ \mathsf{X}_1 & \xrightarrow{i'} & \mathsf{X}, \end{array}$$

where i and j are closed immersions. Let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, and set

$$\mathscr{F}_0 = j'^* \mathscr{F} \in \mathrm{QCoh}(\mathsf{X}_0) \qquad \mathscr{F}_1 = i'^* \mathscr{F} \in \mathrm{QCoh}(\mathsf{X}_1).$$

Then:

(1) *The quasi-coherent sheaf $\mathscr{F}$ is n-connective if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ are n-connective.*

(2) *The quasi-coherent sheaf $\mathscr{F}$ is almost connective if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ are almost connective.*

(3) *The quasi-coherent sheaf $\mathscr{F}$ has* Tor*-amplitude $\leqslant n$ if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ have* Tor*-amplitude $\leqslant n$.*

(4) *The quasi-coherent sheaf $\mathscr{F}$ is flat if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ are flat.*

(5) *The quasi-coherent sheaf is perfect to order n if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ are perfect to order n.*

(6) *The quasi-coherent sheaf $\mathscr{F}$ is almost perfect if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ are almost perfect.*

(7) *The quasi-coherent sheaf $\mathscr{F}$ is perfect if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ are perfect.*

(8) *The quasi-coherent sheaf $\mathscr{F}$ is locally free of finite rank if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ are locally free of finite rank.*

Proof. The "only if" directions follow immediately from Propositions 6.2.5.2 and 2.9.1.4. To prove the reverse directions, we may work locally on X and thereby reduce to the case where $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ is affine. Then σ is determined by a pullback square of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01} \end{array}$$

for which the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective. The quasi-coherent sheaf $\mathscr{F}$ corresponds to an A-module M. Set

$$M_0 = A_0 \otimes_A M \qquad M_{01} = A_{01} \otimes_A M \qquad M_1 = A_1 \otimes_A M.$$

To prove (1), we may assume without loss of generality that $n = 0$. The desired result then follows from the first step in the proof of Proposition 16.2.2.1. Assertion (2) is a consequence of (1). Assertion (4) follows from (1) and (3), assertion (6) follows from (5), assertion (7) follows from (3), (6), and Proposition HA.7.2.4.23 , and assertion (8) follows from (4), (6), and Proposition HA.7.2.4.20 .

We now prove (3). Assume that M_0 and M_1 have Tor-amplitude $\leqslant n$ over A_0 and A_1, respectively. We wish to show that M has Tor-amplitude $\leqslant n$ over A. Let N be a discrete A-module; we wish to show that $M \otimes_A N$ is n-truncated. Let $I_0 \subseteq \pi_0 A$ be the kernel of the map $\pi_0 A \to \pi_0 A_0$, and define $I_1 \subseteq \pi_0 A$ similarly. We have an exact sequence of discrete A-modules $0 \to I_0 N \to N \to N/I_0 N \to 0$. Note that $N/I_0 N$ admits the structure of an A_0-module, so that $M \otimes_A (N/I_0 N) \simeq M_0 \otimes_{A_0} (N/I_0 N)$ is n-truncated. We are therefore reduced to proving that $M \otimes_A I_0 N$ is n-truncated. We have a short exact sequence

$$0 \to I_0 I_1 N \to I_0 N \to I_0 N / I_0 I_1 N \to 0.$$

Since the quotient $I_0 N/I_0 I_1 N$ admits the structure of an A_1-module, we deduce that $M \otimes_A (I_0 N/I_0 I_1 N) \simeq M_1 \otimes_{A_1} (I_0 N/I_0 I_1 N)$ is n-truncated. We are therefore reduced to proving that $M \otimes_A I_0 I_1 N$ is n-truncated. Note that the ideal $I_0 I_1$ belongs to the kernel of the map $\pi_0 A \to \pi_0 A_0 \oplus \pi_0 A_1$, and therefore to the image of the map $\pi_1 A_{01} \to \pi_0 A$. It follows that the ideal $I_0 I_1$ has the structure of a module over $\pi_0 A_{01}$, and is therefore annihilated by I_0. We conclude that $I_0 I_1 N$ admits the structure of an A_0-module, so that $M \otimes_A I_0 I_1 N \simeq M_0 \otimes_{A_0} I_0 I_1 N$ is n-truncated as desired.

It remains to prove (5). Assume that M_0 and M_1 are perfect to order n; we wish to show that M is perfect to order n. Using (2), we deduce that M is almost connective. Replacing M by a shift, we may assume that M is connective. We now proceed by induction on n, the case $n < 0$ being trivial. If $n = 0$, we must show that $\pi_0 M$ is finitely generated as a module over $\pi_0 A$ (Proposition 2.7.2.1). Since M_0 and M_1 are perfect to order 0 and the maps $\pi_0 M \to \pi_0 M_0$ and $\pi_0 M \to \pi_0 M_1$, we can choose finitely many elements of $\pi_0 M$ whose images generate $\pi_0 M_0$ and $\pi_0 M_1$. This choice of elements determines a fiber sequence $M' \to A^m \to M$. Then $A_0 \otimes_A M'$ and $A_1 \otimes_A M'$ are connective, so that (1) implies that M' is connective. It follows that the map $\pi_0 A^m \to \pi_0 M$ is surjective, so that M is perfect to order 0 as desired.

Now suppose that $n > 0$. The argument above shows that $\pi_0 M$ is finitely generated, so we can choose a fiber sequence $M' \to A^m \to M$ where M' is connective. Using Proposition 2.7.2.1, we deduce that $A_0 \otimes_A M'$ and $A_1 \otimes_A M'$ are perfect to order $n-1$ as modules over A_0 and A_1, respectively. Invoking the inductive hypothesis, we conclude that M' is perfect to order $n-1$, so that M is perfect to order n by Proposition 2.7.2.1. □

16.3 Clutching for Spectral Deligne-Mumford Stacks

Suppose we are given a diagram of spectral Deligne-Mumford stacks $\mathsf{X}_0 \xleftarrow{i} \mathsf{X}_{01} \xrightarrow{j} \mathsf{X}_1$, where i and j are closed immersions. Theorem 16.1.0.1 implies that there exists a pushout

diagram

$$\begin{array}{ccc} \mathsf{X}_{01} & \xrightarrow{i} & \mathsf{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathsf{X}_1 & \longrightarrow & \mathsf{X} \end{array}$$

in the ∞-category of spectral Deligne-Mumford stacks, and Theorem 16.2.0.1 implies that there is a close relationship between quasi-coherent sheaves on X and quasi-coherent sheaves on the individual constituents X_0 and X_1 (together with a compatibility over X_{01}). In this section, we will discuss the following nonlinear analogue:

Theorem 16.3.0.1. *Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathsf{X}_{01} & \xrightarrow{i} & \mathsf{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathsf{X}_1 & \longrightarrow & \mathsf{X}, \end{array}$$

where i and j are closed immersions. Then the diagram

$$\begin{array}{ccc} \mathrm{SpDM}_{/\mathsf{X}} & \longrightarrow & \mathrm{SpDM}_{/\mathsf{X}_1} \\ \downarrow & & \downarrow \\ \mathrm{SpDM}_{/\mathsf{X}_0} & \longrightarrow & \mathrm{SpDM}_{/\mathsf{X}_{01}} \end{array}$$

is a pullback diagram of ∞-categories.

16.3.1 Gluing and Base Change

Theorem 16.3.0.1 is a consequence of the following simple observation:

Proposition 16.3.1.1. *Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks σ :*

$$\begin{array}{ccc} \mathsf{X}_{01} & \xrightarrow{i} & \mathsf{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathsf{X}_1 & \longrightarrow & \mathsf{X}, \end{array}$$

where i and j are closed immersions. Let $f : \mathsf{Y} \to \mathsf{X}$ be a map of spectral Deligne-Mumford stacks. Then the diagram σ' :

$$\begin{array}{ccc} \mathsf{Y} \times_{\mathsf{X}} \mathsf{X}_{01} & \xrightarrow{i'} & \mathsf{Y} \times_{\mathsf{X}} \mathsf{X}_0 \\ \downarrow{\scriptstyle j'} & & \downarrow \\ \mathsf{Y} \times_{\mathsf{X}} \mathsf{X}_1 & \longrightarrow & \mathsf{Y} \end{array}$$

is also a pushout square of spectral Deligne-Mumford stacks (note that the morphisms i' and j' are also closed immersions, by Corollary 3.1.2.3).

Proof. The assertion is local on both X and Y. We may therefore assume without loss of generality that $\mathsf{X} = \operatorname{Sp\acute{e}t} A$, so that σ is determined by a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & A_{01} \end{array}$$

where the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective. We may also assume that $\mathsf{Y} = \operatorname{Sp\acute{e}t} R$ is affine. In this case, σ' is determined by the diagram of connective $\mathbb{E}_\infty$-rings τ :

$$\begin{array}{ccc} R & \longrightarrow & R \otimes_A A_1 \\ \downarrow & & \downarrow \\ R \otimes_A A_0 & \longrightarrow & R \otimes_A A_{01}. \end{array}$$

Since the operation of tensor product with R is exact, τ is a pullback diagram. The desired result now follows from Proposition 16.1.3.1. □

Proof of Theorem 16.3.0.1. Suppose we are given a pushout diagram

$$\begin{array}{ccc} \mathsf{X}_{01} & \xrightarrow{i} & \mathsf{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathsf{X}_1 & \longrightarrow & \mathsf{X}, \end{array}$$

where i and j are closed immersions. We wish to prove that the canonical map

$$G : \mathrm{SpDM}_{/\mathsf{X}} \to \mathrm{SpDM}_{/\mathsf{X}_0} \times_{\mathrm{SpDM}_{/\mathsf{X}_{01}}} \mathrm{SpDM}_{/\mathsf{X}_1}$$

is an equivalence of ∞-categories. Let us identify objects of the codomain of G with triples $(\mathsf{Y}_0, \mathsf{Y}_1, \alpha)$, where $\mathsf{Y}_0 \in \mathrm{SpDM}_{/\mathsf{X}_0}$, $\mathsf{Y}_1 \in \mathrm{SpDM}_{/\mathsf{X}_1}$, and $\alpha : \mathsf{X}_{01} \times_{\mathsf{X}_0} \mathsf{Y}_0 \simeq \mathsf{X}_{01} \times_{\mathsf{X}_1} \mathsf{Y}_1$ is an equivalence in $\mathrm{SpDM}_{/\mathsf{X}_{01}}$. Given such a triple, let $\mathsf{Y}_{01} = \mathsf{X}_{01} \times_{\mathsf{X}_0} \mathsf{Y}_0$. We have a commutative diagram

$$\begin{array}{ccccc} \mathsf{Y}_0 & \xleftarrow{i'} & \mathsf{Y}_{01} & \xrightarrow{j'} & \mathsf{Y}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{X}_0 & \xleftarrow{i} & \mathsf{X}_{01} & \xrightarrow{j} & \mathsf{X}_1 \end{array}$$

where both squares are pullbacks. It follows that i' and j' are closed immersions, so that there exists a pushout diagram

$$\begin{array}{ccc} \mathsf{Y}_{01} & \xrightarrow{i'} & \mathsf{Y}_0 \\ \downarrow{\scriptstyle j'} & & \downarrow \\ \mathsf{Y}_1 & \longrightarrow & \mathsf{Y} \end{array}$$

of spectral Deligne-Mumford stacks (Theorem 16.1.0.1). The construction $(\mathsf{Y}_0, \mathsf{Y}_1, \alpha) \mapsto \mathsf{Y}$ determines a functor $F : \mathrm{SpDM}_{/\mathsf{X}_0} \times_{\mathrm{SpDM}_{/\mathsf{X}_{01}}} \mathrm{SpDM}_{/\mathsf{X}_1} \to \mathrm{SpDM}_{/\mathsf{X}}$ which is left adjoint to G. It follows from Proposition 16.3.1.1 that the counit map $v : F \circ G \to \mathrm{id}$ is an equivalence. We now prove that u is also an equivalence. Fix an object $(\mathsf{Y}_0, \mathsf{Y}_1, \alpha) \in \mathrm{SpDM}_{/\mathsf{X}_0} \times_{\mathrm{SpDM}_{/\mathsf{X}_{01}}} \mathrm{SpDM}_{/\mathsf{X}_1}$, so that we have a pushout diagram τ :

$$\begin{array}{ccc} \mathsf{Y}_{01} & \xrightarrow{i'} & \mathsf{Y}_0 \\ \downarrow{\scriptstyle j'} & & \downarrow \\ \mathsf{Y}_1 & \longrightarrow & \mathsf{Y} \end{array}$$

as above. We wish to prove that the induced maps

$$\phi : \mathsf{Y}_0 \to \mathsf{X}_0 \times_{\mathsf{X}} \mathsf{Y} \qquad \psi : \mathsf{Y}_1 \to \mathsf{X}_1 \times_{\mathsf{X}} \mathsf{Y}$$

are equivalences. We will prove that ϕ is an equivalence; the proof that ψ is an equivalence is similar. The assertion is local on X and Y; we may therefore assume that $\mathsf{X} \simeq \mathrm{Sp\acute{e}t}\, A$ and $\mathsf{Y} \simeq \mathrm{Sp\acute{e}t}\, R$ are affine. It follows that X_0, X_1, and X_{01} are affine; write

$$\mathsf{X}_0 \simeq \mathrm{Sp\acute{e}t}\, A_0 \qquad \mathsf{X}_{01} \simeq \mathrm{Sp\acute{e}t}\, A_{01} \qquad \mathsf{X}_1 \simeq \mathrm{Sp\acute{e}t}\, A_1$$

$$\mathsf{Y}_0 \simeq \mathrm{Sp\acute{e}t}\, R_0 \qquad \mathsf{Y}_{01} \simeq \mathrm{Sp\acute{e}t}\, R_{01} \qquad \mathsf{Y}_1 \simeq \mathrm{Sp\acute{e}t}\, R_1.$$

Then α determines an equivalence $\beta : A_{01} \otimes_{A_0} R_0 \simeq A_{01} \otimes_{A_1} R_1$, and we can identify the triple (R_0, R_1, β) with an object of $\mathrm{Mod}^{\mathrm{cn}}_{A_0} \times_{\mathrm{Mod}^{\mathrm{cn}}_{A_{01}}} \mathrm{Mod}^{\mathrm{cn}}_{A_1}$. Using Proposition 16.2.2.1, we deduce that (R_0, R_1, β) is determined by an object $M \in \mathrm{Mod}^{\mathrm{cn}}_A$. Then Proposition 16.1.3.1 supplies an equivalence $\theta : R \simeq R_0 \times_{R_{01}} R_1 \simeq (A_0 \times_{A_{01}} A_1) \otimes_A M \simeq M$. To prove that ϕ is an equivalence, it will suffice to show that it induces an equivalence of $\mathbb{E}_\infty$-rings $R_0 \otimes_R A \to A_0$. It now suffices to observe that the underlying map of spectra is given by the composition $R_0 \otimes_R A \overset{\theta}{\simeq} R_0 \otimes_R M \simeq A_0$. □

16.3.2 Properties Persistant Under Clutching

In the situation of Proposition 16.3.1.1, many important properties of morphism $f : \mathsf{Y} \to \mathsf{X}$ can be tested after pullback along the closed immersions $\mathsf{X}_0 \hookrightarrow \mathsf{X} \hookleftarrow \mathsf{X}_1$.

Proposition 16.3.2.1. *Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks σ :*

$$\begin{array}{ccc} \mathsf{X}_{01} & \xrightarrow{i} & \mathsf{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathsf{X}_1 & \longrightarrow & \mathsf{X}, \end{array}$$

where i and j are closed immersions. Let $f : \mathsf{Y} \to \mathsf{X}$ be a map of spectral Deligne-Mumford stacks. Set $\mathsf{Y}_0 = \mathsf{X}_0 \times_{\mathsf{X}} \mathsf{Y}$, $\mathsf{Y}_1 = \mathsf{X}_1 \times_{\mathsf{X}} \mathsf{Y}$, and let $f_0 : \mathsf{Y}_0 \to \mathsf{X}_0$ and $f_1 : \mathsf{Y}_1 \to \mathsf{X}_1$ be the projection maps. Then:

(1) *The map f is locally of finite generation to order n if and only if both f_0 and f_1 are locally of finite generation to order n (for any $n \geqslant 0$).*

(2) *The map f is locally almost of finite presentation if and only if both f_0 and f_1 are locally almost of finite presentation.*

(3) *The map f is locally of finite presentation if and only if f_0 and f_1 are locally of finite presentation.*

(4) *The map f is étale if and only if both f_0 and f_1 are étale.*

(5) *The map f is an equivalence if and only if both f_0 and f_1 are equivalences.*

(6) *The map f is an open immersion of and only if both f_0 and f_1 are open immersions.*

(7) *The map f is flat if and only if both f_0 and f_1 are flat.*

(8) *The map f is affine if and only if both f_0 and f_1 are affine.*

(9) *The map f is a closed immersion if and only if both f_0 and f_1 are closed immersions.*

(10) *The map f is separated if and only f_0 and f_1 are separated.*

(11) *The map f is n-quasi-compact if and only if f_0 and f_1 are n-quasi-compact, for $0 \leqslant n \leqslant \infty$.*

(12) *The map f is proper if and only if f_0 and f_1 are proper.*

Proof. The "only if" directions are obvious. The converse assertions are all local on X, so we may assume without loss of generality that $\mathsf{X} = \operatorname{Sp\acute{e}t} A$ is affine. In this case, the diagram σ is determined by a pullback square of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & A_{01}, \end{array}$$

where the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective.

We first prove (1). The assertion is local on Y, so we may assume that $\mathsf{Y} = \operatorname{Sp\acute{e}t} R$ is affine. By assumption, the $\mathbb{E}_\infty$-algebras $R_0 = A_0 \otimes_A R$ and $R_1 = A_1 \otimes_A R$ are of finite generation to order n over A_0 and A_1, respectively. We wish to show that R is of finite generation to order n over A. We first treat the case $n = 0$. Then $\pi_0 R_0$ and $\pi_0 R_1$ are finitely generated as commutative rings over $\pi_0 A$. Since the maps $\pi_0 R \to \pi_0 R_0$ and $\pi_0 R \to \pi_0 R_1$ are surjective, we can choose a finite collection $x_1, \ldots, x_n \in \pi_0 R$ whose images generate $\pi_0 R_0$ and $\pi_0 R_1$ over $\pi_0 A$. Let $B = A\{X_1, \ldots, X_n\}$ denote the $\mathbb{E}_\infty$-algebra over A freely generated by a collection of indeterminates $X_1, \ldots, X_n$, so that there is a map of $\mathbb{E}_\infty$-algebras $\phi : B \to R$ which is determined uniquely up to homotopy by the requirement that $\phi(X_i) = x_i \in \pi_0 R$ for $1 \leqslant i \leqslant k$. Let I denote the fiber of ϕ, and regard I as a R-module. Then $A_0 \otimes_A I$ and $A_1 \otimes_A I$ can be identified with the fibers of the induced maps

$$A_0\{X_1, \ldots, X_k\} \to R_0 \qquad A_1\{X_1, \ldots, X_k\} \to R_1,$$

and are therefore connective. It follows that I is connective, so that ϕ induces a surjection $\pi_0 B \simeq (\pi_0 A)[X_1, \ldots, X_k] \to \pi_0 R$. This proves that $\pi_0 R$ is finitely generated as a commutative ring over $\pi_0 A$, so that R is of finite generation to order 0 over A.

We next prove (1) in the case $n \geqslant 1$. Assume that R_0 and R_1 are of finite generation to order n over A_0 and A_1, respectively. Then the relative cotangent complexes

$$L_{R_0/A_0} \simeq R_0 \otimes_R L_{R/A} \qquad L_{R_1/A_1} \simeq R_1 \otimes_R L_{R/A}$$

are perfect to order n over R_0 and R_1, respectively. Using Proposition 16.2.3.1, we deduce that $L_{R/A}$ is perfect to order n as an R-module. Consequently, to show that R is of finite presentation to order n over A, it will suffice to show that $\pi_0 R$ is finitely presented as a commutative ring over $\pi_0 A$ (Proposition 4.1.2.1). Let $\phi : B \to R$ and $I = \operatorname{fib}(\phi)$ be defined as above. We have surjective maps

$$\pi_0 I \to \pi_0(A_0 \otimes_A I) \to \ker((\pi_0 A_0)[X_1, \ldots, X_k] \to \pi_0 R_0)$$
$$\pi_0 I \to \pi_0(A_1 \otimes_A I) \to \ker((\pi_0 A_1)[X_1, \ldots, X_k] \to \pi_0 R_1).$$

Since $\pi_0 R_0$ and $\pi_0 R_1$ are finitely presented as commutative rings over $\pi_0 A_0$ and $\pi_0 A_1$, respectively, we can choose a finite collection of elements $y_1, \ldots, y_m \in \pi_0 I$ whose images generate the ideals

$$\ker((\pi_0 A_0)[X_1, \ldots, X_k] \to \pi_0 R_0) \qquad \ker((\pi_0 A_1)[X_1, \ldots, X_k] \to \pi_0 R_1).$$

Let $A\{Y_1, \ldots, Y_m\}$ be the corresponding free $\mathbb{E}_\infty$-algebra over A, so that the choice of elements y_i determine a commutative diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A\{Y_1, \ldots, Y_m\} & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & R. \end{array}$$

Let B' denote the pushout $A \otimes_{A\{Y_1,\ldots,Y_m\}} B$, so that we obtain a map of $\mathbb{E}_\infty$-rings $\psi : B' \to R$. Then B' is of finite presentation over A, so that $\pi_0 B'$ is finitely presented as a commutative ring over $\pi_0 A$. By construction, ψ induces a surjection $\pi_0 B' \to \pi_0 R$. Let $J \subseteq \pi_0 B'$ denote the kernel of this surjection. To complete the proof, it will suffice to show that J is a finitely generated ideal in $\pi_0 B'$.

Let $B'_0 = A_0 \otimes_A B'$, and define B'_1 and B'_{01} similarly. We have a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} B' & \longrightarrow & B'_0 \\ \downarrow & & \downarrow \\ B'_1 & \longrightarrow & B'_{01}. \end{array}$$

By construction ψ induces isomorphisms $\pi_0 B'_0 \simeq \pi_0 R_0$ and $\pi_0 B'_1 \simeq \pi_0 R_1$. It follows that J is contained in the kernel of the map $\pi_0 B \to \pi_0 B'_0 \times \pi_0 B'_1$, which is the image of the boundary map $\pi_1 B'_{01} \to \pi_0 B'$. It follows that the action of $\pi_0 B'$ on J factors through the acton of $\pi_0 B'_{01}$. Since J belongs to the kernel of the map $\pi_0 B' \to \pi_0 B'_{01}$, we deduce that $J^2 = 0$. Consequently, the map $J \to \mathrm{Tor}_0^{\pi_0 B'}(\pi_0 R, J) \simeq \pi_0(R \otimes_{B'} \mathrm{fib}(\psi))$ is bijective. To prove that J is a finitely generated ideal in $\pi_0 B'$, it suffices to show that $\pi_0(R \otimes_{B'} \mathrm{fib}(\psi))$ is finitely generated as a module over $\pi_0 R$. Applying Theorem HA.7.4.3.1 , we obtain a canonical isomorphism $\pi_0(R \otimes_{B'} \mathrm{fib}(\psi)) \simeq \pi_1 L_{R/B'}$. Since $L_{R/B'}$ is 1-connective, it will suffice to show that $L_{R/B'}$ is perfect to order 1 as an R-module. We have a fiber seqence

$$R \otimes_{B'} L_{B'/A} \to L_{R/A} \to L_{R/B'}.$$

By construction, B' is finitely presented over A, so that $L_{B'/A}$ is perfect as a B'-module. Using Remark 2.7.0.7, we are reduced to showing that $L_{R/A}$ is perfect to order 1 as an R-module, which was already proven above. This completes the proof of (1).

Assertion (2) follows immediately from (1). We now prove (3). As before, we may suppose that $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, R$ is affine. Then R_0 and R_1 are locally of finite presentation over A_0 and A_1, respectively, so that the relative cotangent complexes $L_{R_0/A_0} \simeq R_0 \otimes_R L_{R/A}$ and $L_{R_1/A_1} \simeq R_1 \otimes_R L_{R/A}$ are perfect. Using Proposition 16.2.3.1, we deduce that $L_{R/A}$ is a perfect R-module. Since R is almost of finite presentation over A (by (2)), we deduce from Theorem HA.7.4.3.18 that R is locally of finite presentation over A.

We now prove (4). Once again we may suppose that $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, R$ is affine. If R_0 and R_1 are étale over A_0 and A_1, then we obtain

$$R_0 \otimes_R L_{R/A} \simeq L_{R_0/A_0} \simeq 0 \simeq L_{R_1/A_1} \simeq R_1 \otimes_R L_{R/A}.$$

Using Theorem 16.2.0.2 we deduce that $L_{R/A} \simeq 0$. Since R is almost of finite presentation over A (by (2)), it follows from Lemma B.1.3.3 that R is étale over A.

Assertion (5) follows immediately from Theorem 16.3.0.1. Assertion (6) follows from (4) and (5), since f is an open immersion if and only if f is étale and the diagonal map $\mathsf{Y} \to \mathsf{Y} \times_{\mathsf{X}} \mathsf{Y}$ is an equivalence. To prove (7), we may assume without loss of generality that $\mathsf{Y} = \operatorname{Sp\acute{e}t} R$ is affine. In this case, the desired result follows from Proposition 16.2.3.1. Assertion (8) follows from Theorem 16.1.0.1.

We now prove (9). Assume that f_0 and f_1 are closed immersions. It follows from (8) that f is affine, so that $\mathsf{Y} \simeq \operatorname{Sp\acute{e}t} R$ for some connective $\mathbb{E}_\infty$-ring R. We have a fiber sequence of A-modules $I \to A \to R$. The map f is a closed immersion if and only if the map $\pi_0 A \to \pi_0 R$ is surjective, which is equivalent to the requirement that I is connective. Since f_0 and f_1 are closed immersions, the tensor products $A_0 \otimes_A I$ and $A_1 \otimes_A I$ are connective. It follows from Proposition 16.2.3.1 that I is connective, as desired.

Assertion (10) follows by applying (9) to the diagonal map $\mathsf{Y} \to \mathsf{Y} \times_{\mathsf{X}} \mathsf{Y}$. To prove (11), it suffices to treat the case where $n < \infty$. We proceed by induction on n. If $n > 0$, it suffices to show that if we are given a pair of étale maps $\mathsf{U} \to \mathsf{Y} \leftarrow \mathsf{V}$ where U and V are affine, then $\mathsf{U} \times_{\mathsf{Y}} \mathsf{V}$ is $(n-1)$-quasi-compact. If Y_0 and Y_1 are n-quasi-compact, then the spectral Deligne-Mumford stacks

$$\mathsf{X}_0 \times_{\mathsf{X}} (\mathsf{U} \times_{\mathsf{Y}} \mathsf{V}) \qquad \mathsf{X}_1 \times_{\mathsf{X}} (\mathsf{U} \times_{\mathsf{Y}} \mathsf{V})$$

are $(n-1)$-quasi-compact, so that the desired result follows from the inductive hypothesis. It therefore suffices to treat the case $n = 0$. Suppose that Y can be described as the colimit of a diagram of open substacks $\{\mathsf{Y}_\alpha\}_{\alpha \in P}$ indexed by a filtered poset P. Since Y_0 is quasi-compact, there exists an index $\alpha \in P$ such that $\mathsf{Y}_\alpha \times_{\mathsf{Y}} \mathsf{Y}_0 \simeq \mathsf{Y}_0$. Enlarging α if necessary, we may suppose that $\mathsf{Y}_\alpha \times_{\mathsf{Y}} \mathsf{Y}_1 \simeq \mathsf{Y}_1$. It then follows from (5) that the open immersion $\mathsf{Y}_\alpha \hookrightarrow \mathsf{Y}$ is an equivalence.

We now prove (12). Suppose that f_0 and f_1 are proper. Then (1) implies that f is locally of finite type, (11) implies that f is quasi-compact, and (10) implies that f is separated. In order to prove that f is proper, it will suffice to show that f is universally closed. Replacing f by a pullback of f if necessary, we are reduced to showing that f is closed. Let $K \subseteq |\mathsf{X}|$ be a closed subset; we wish to show that the image of $f(K) \subseteq |\mathsf{Y}|$ is closed. This follows from the observation that $f(K)$ is given by the union of the images of the sets $K \times_{|\mathsf{X}|} |\mathsf{X}_0|$ and $K \times_{|\mathsf{X}|} |\mathsf{X}_1|$ along the closed maps

$$|\mathsf{X}_0| \xrightarrow{f_0} |\mathsf{Y}_0| \hookrightarrow |\mathsf{Y}| \hookleftarrow |\mathsf{Y}_1| \xleftarrow{f_1} |\mathsf{X}_1|.$$

□

Corollary 16.3.2.2. *Suppose we are given a pushout diagram of spectral Deligne-Mumford* *stacks*

$$\begin{array}{ccc} \mathsf{X}_{01} & \xrightarrow{i} & \mathsf{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathsf{X}_1 & \longrightarrow & \mathsf{X} \end{array}$$

where i and j are closed immersions. If X_0 and X_1 are separated, then X is also separated.

Proof. We wish to show that the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times \mathsf{X}$ is a closed immersion. Using Proposition 16.3.2.1, we may reduce to proving that each of the vertical maps appearing in the diagram

$$\begin{array}{ccccc} \mathsf{X}_0 & \longleftarrow & \mathsf{X}_{01} & \longrightarrow & \mathsf{X}_1 \\ \downarrow{\scriptstyle \delta_0} & & \downarrow{\scriptstyle \delta_{01}} & & \downarrow{\scriptstyle \delta_1} \\ \mathsf{X}_0 \times \mathsf{X} & \longleftarrow & \mathsf{X}_{01} \times \mathsf{X} & \longrightarrow & \mathsf{X}_1 \times \mathsf{X} \end{array}$$

is a closed immersion. We will prove that δ_0 is a closed immersion; the proof in the other two cases are similar. We can factor δ_0 as a composition

$$\mathsf{X}_0 \xrightarrow{\delta'} \mathsf{X}_0 \times \mathsf{X}_0 \xrightarrow{\delta''} \mathsf{X}_0 \times \mathsf{X}.$$

Here δ' is a closed immersion (by virtue of our assumption that X_0 is a separated spectral algebraic space), and δ'' is a pullback of the closed immersion $\mathsf{X}_0 \to \mathsf{X}$. □

16.3.3 Application: Deformations of Spectral Deligne-Mumford Stacks

For every connective $\mathbb{E}_\infty$-ring R, let $\mathrm{SpDM}_R = \mathrm{SpDM}_{/\operatorname{Sp\acute{e}t} R}$ denote the ∞-category of spectral Deligne-Mumford stacks over R. The ∞-category SpDM_R depends functorially on R. In particular, if we fix a spectral Deligne-Mumford stack X_0 defined over a field κ, then the construction $R \mapsto \mathrm{SpDM}_R \times_{\mathrm{SpDM}_\kappa} \{\mathsf{X}_0\}$ determines a functor $M : \mathrm{CAlg}^{\mathrm{art}}_\kappa \to \widehat{\mathcal{C}\mathrm{at}}_\infty$, which assigns to each Artinian κ-algebra R the ∞-category $M(R)$ of deformations of X_0 over R: that is, the ∞-category whose objects are pullback diagrams

$$\begin{array}{ccc} \mathsf{X}_0 & \longrightarrow & \mathsf{X}_R \\ \downarrow & & \downarrow \\ \operatorname{Sp\acute{e}t} \kappa & \longrightarrow & \operatorname{Sp\acute{e}t} R. \end{array}$$

If R is a nonzero connective $\mathbb{E}_\infty$-ring, then the ∞-category SpDM_R contains noninvertible morphisms, and is not essentially small. However, if R is an Artinian $\mathbb{E}_\infty$-algebra over κ, then the augmentation map $R \to \kappa$ determines a conservative functor $\mathrm{SpDM}_R \to \mathrm{SpDM}_\kappa$, so that $M(R)$ is a Kan complex.

Theorem 16.3.3.1. *Let κ be a field and let X_0 be a spectral Deligne-Mumford stack over κ. Then the construction $R \mapsto M(R)$ determines a formal moduli problem $M : \mathrm{CAlg}^{\mathrm{art}}_\kappa \to \mathcal{S}$.*

Proof. Let κ be a field and let $\mathsf{X}_0 \to \operatorname{Sp\acute{e}t} \kappa$ be a map of spectral Deligne-Mumford stacks over κ. For every pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

in $\mathrm{CAlg}_\kappa^{\mathrm{art}}$ where the maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, we obtain a pushout square of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \operatorname{Sp\acute{e}t} R_{01} & \longrightarrow & \operatorname{Sp\acute{e}t} R_0 \\ \downarrow & & \downarrow \\ \operatorname{Sp\acute{e}t} R_1 & \longrightarrow & \operatorname{Sp\acute{e}t} R \end{array}$$

where the morphisms are closed immersions. It then follows from Theorem 16.3.0.1 that the map $M(R) \to M(R_1) \times_{M(R_{01})} M(R_1)$ is a homotopy equivalence. It follows immediately from the definitions that $M(\kappa)$ is contractible.

To complete the proof, it will suffice to show that the ∞-groupoid $M(R)$ is essentially small for each object $R \in \mathrm{CAlg}_\kappa^{\mathrm{art}}$. Since R is Artinian, there exists a finite sequence of maps

$$R = R_0 \to R_1 \to \cdots \to R_n \simeq \kappa,$$

where each of the maps $R_i \to R_{i+1}$ fits into a pullback square

$$\begin{array}{ccc} R_i & \longrightarrow & R_{i+1} \\ \downarrow & & \downarrow \\ \kappa & \longrightarrow & \kappa \oplus \Sigma^{m_i}(\kappa). \end{array}$$

We will prove that each $M(R_i)$ is essentially small, using descending induction on i. When $i = n$, $M(R_i)$ is contractible and there is nothing to prove. To carry out the inductive step, we note that there is a fiber sequence $M(R_i) \to M(R_{i+1}) \to M(\kappa \oplus \Sigma^{m_i}(\kappa))$. Since $M(R_{i+1})$ is essentially small and $M(\kappa \oplus \Sigma^{m_i}(\kappa))$ is locally small, it follows immediately that $M(R_i)$ is essentially small. □

Remark 16.3.3.2. Let κ be a field of characteristic zero, and let X_0 be a spectral Deligne-Mumford stack over κ. It follows from Theorems 16.3.3.1 and 13.3.0.1 that there exists a differential graded Lie algebra $\mathfrak{g}_*$ over κ with the following universal property: for every Artinian $\mathbb{E}_\infty$-algebra R over κ, there is a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Lie}_\kappa}(\mathfrak{D}(R), \mathfrak{g}_*) \simeq \mathrm{SpDM}_R \times_{\mathrm{SpDM}_\kappa} \{\mathsf{X}_0\}.$$

That is, deformations of X_0 are "controlled" by the differential graded Lie algebra $\mathfrak{g}_*$. It is possible to describe the differential graded Lie algebra $\mathfrak{g}_*$ more explicitly: in good cases, it can be obtained as the global sections of the tangent complex of X over κ. We refer the reader to §19.4 for a more detailed discussion.

16.4 Approximations to Formal Moduli Problems

The notion of formal moduli problem introduced in Definition 12.1.3.1 is a very general one, which includes (as a special case) the formal completion of any reasonable algebro-geometric object at a point (see Example 16.1.3.2). However, there are also many functors $X : \mathrm{CAlg}^{\mathrm{cn}}_{\kappa} \to \mathcal{S}$ which do not quite satisfy the requirements of being a formal moduli problem, but are nevertheless of interest in deformation theory. The deformation functors that we will study in §16.5 and §16.6 are of this nature. In this section, we will introduce a generalization of the notion of formal moduli problem (see Definition 16.4.1.5) which incorporates these examples as well.

16.4.1 n-Proximate Formal Moduli Problems

We begin by introducing some terminology.

Definition 16.4.1.1. Let $n \geqslant -2$ be an integer. We will say that a diagram of spaces

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is *n-Cartesian* if the induced map $\phi : X' \to X \times_Y Y'$ is n-truncated (that is, the homotopy fibers of ϕ are n-truncated).

Example 16.4.1.2. If $n = -2$, then a commutative diagram of spaces is n-Cartesian if and only if it is a pullback square.

The following lemma summarizes some of the basic transitivity properties of Definition 16.4.1.1:

Lemma 16.4.1.3. *Let $n \geqslant -2$ be an integer, and suppose we are given a commutative diagram*

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & Z' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

in $\mathcal{S}$. If the right square is n-Cartesian, then the outer square is n-Cartesian if and only if the left square is n-Cartesian.

Using Lemma 16.4.1.3, we immediately deduce the following generalization of Proposition 12.1.3.2.

Proposition 16.4.1.4. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $X : \mathbf{A}^{\text{art}} \to \mathcal{S}$ be a functor. Let $n \geqslant 0$ be an integer. The following conditions are equivalent:*

(1) *Let σ :*

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \phi \\ A & \longrightarrow & B \end{array}$$

be a diagram in $\mathbf{A}^{\text{art}}$. If σ is a pullback diagram and ϕ is small, then $X(\sigma)$ is an $(n-2)$-Cartesian diagram in $\mathcal{S}$.

(2) *Let σ be as in (1). If σ is a pullback diagram and ϕ is elementary, then $X(\sigma)$ is an $(n-2)$-Cartesian diagram in $\mathcal{S}$.*

(3) *Let σ be as in (1). If σ is a pullback diagram and ϕ is the base point morphism $* \to \Omega^{\infty - m} E_\alpha$ for some $\alpha \in T$ and $m > 0$, then $X(\sigma)$ is an $(n-2)$-Cartesian diagram in $\mathcal{S}$.*

Definition 16.4.1.5. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $n \geqslant 0$ be an integer. We will say that a functor $X : \mathbf{A}^{\text{art}} \to \mathcal{S}$ is a *n-proximate formal moduli problem* if $X(*)$ is contractible and X satisfies the equivalent conditions of Proposition 16.4.1.4.

Example 16.4.1.6. A functor $X : \mathbf{A}^{\text{art}} \to \mathcal{S}$ is a 0-proximate formal moduli problem if and only if it is a formal moduli problem, in the sense of Definition 12.1.3.1.

Remark 16.4.1.7. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and suppose we are given a pullback diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in $\text{Fun}(\mathbf{A}^{\text{art}}, \mathcal{S})$. If X and Y' are n-proximate formal moduli problems and Y is an $(n+1)$-proximate formal moduli problem, then X' is an n-proximate formal moduli problem.

Definition 16.4.1.8. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathbf{A}^{\text{art}} \to \mathcal{S}$. We will say that f is *n-truncated* if the induced map $X(A) \to Y(A)$ is n-truncated, for each $A \in \text{Art}$.

16.4.2 Classification of n-Proximate Formal Moduli Problems

Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $n \geqslant 0$ be an integer. Roughly speaking, a functor $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ is an n-proximate formal moduli problem if the axioms of Definition 12.1.3.1 are *approximately* satisfied (where the degree of approximation required depends on n). The main result of this section asserts that, if $\mathbf{A}$ admits a deformation theory, then this is equivalent to the requirement that X can be approximated by a functor which satisfies the axioms of Definition 12.1.3.1 exactly.

Theorem 16.4.2.1. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context which admits a deformation theory, and let $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ be a functor such that $X(*)$ is contractible (here $*$ denotes the final object of $\mathbf{A}$). The following conditions are equivalent:*

(1) *The functor X is an n-proximate formal moduli problem.*

(2) *There exists an $(n-2)$-truncated map $f : X \to Y$, where Y is an n-proximate formal moduli problem.*

(3) *Let L denote a left adjoint to the inclusion* $\mathrm{Moduli}^{\mathbf{A}} \subseteq \mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ *(see Remark 12.1.3.5). Then the unit map $X \to LX$ is $(n-2)$-truncated.*

Remark 16.4.2.2. In the statement of Theorem 16.4.2.1, the implications $(3) \Rightarrow (2) \Rightarrow (1)$ do not require the assumption that $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ admits a deformation theory. We do not know if the implication $(1) \Rightarrow (3)$ holds in greater generality.

Remark 16.4.2.3. Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context which admits a deformation theory, and let $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ be an n-proximate formal moduli problem. It follows from Theorem 16.4.2.1 that there exists an $(n-2)$-truncated natural transformation $\alpha : X \to Y$, where Y is a formal moduli problem. In fact, the formal moduli problem Y (and the natural transformation α) are uniquely determined up to equivalence. To prove this, we note that α factors as a composition $X \xrightarrow{\beta} LX \xrightarrow{\gamma} Y$, where β is $(n-2)$-truncated and γ is a map between formal moduli problems. For each $\alpha \in T$ and each $m \geqslant 0$, we have homotopy equivalences

$$\Omega^n X(\Omega^{\infty - m - n} E_\alpha) \simeq \Omega^n LX(\Omega^{\infty - m - n} E_\alpha) \simeq LX(\Omega^{\infty - m} E_\alpha)$$

$$\Omega^n X(\Omega^{\infty - m - n} E_\alpha) \simeq \Omega^n Y(\Omega^{\infty - m - n} E_\alpha) \simeq Y(\Omega^{\infty - m} E_\alpha).$$

From this it follows that γ induces an equivalence $LX(\Omega^{\infty - m} E_\alpha) \to Y(\Omega^{\infty - m} E_\alpha)$. Since LX and Y are formal moduli problems, we conclude that γ is an equivalence.

16.4.3 Approximating the Tangent Complex

The main ingredient in our proof of Theorem 16.4.2.1 is to show that if $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ is an n-proximate formal moduli problem and $LX \in \mathrm{Moduli}^{\mathbf{A}}$ is the associated formal moduli problem, then the tangent complex of LX can be computed directly from X. To make this idea precise, we need to introduce a bit of notation. In what follows, let us identify $\mathrm{Sp} = \mathrm{Sp}(\mathcal{S})$ with the ∞-category of reduced and excisive functors $\mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$. Let $L_0 : \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}) \to \mathrm{Sp}$ denote a left adjoint to the inclusion. If $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ is any functor, then the composition $X \circ E_\alpha$ is a functor from $\mathcal{S}_*^{\mathrm{fin}}$ to $\mathcal{S}$, and therefore determines a spectrum $L_0(X \circ E_\alpha)$.

Remark 16.4.3.1. Suppose that $F : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ is a functor which preserves final objects. Using Example HA.6.1.1.28 , we see that $L_0 F \in \mathrm{Sp}$ is given by the formula $(L_0 F)(K) = \varinjlim_n \Omega^n L_0(\Sigma^n K)$. In particular, the functor L_0 is left exact when restricted to the full subcategory $\mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ spanned by those functors which preserve final objects.

We will need the following:

Proposition 16.4.3.2. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context which admits a deformation theory and let L denote a left adjoint to the inclusion functor $\mathrm{Moduli}^{\mathbf{A}} \subseteq \mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$. Suppose that $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ is an n-proximate formal moduli problem for some $n \geqslant 0$.*

For each $\alpha \in T$, the canonical map $X(E_\alpha) \to (LX)(E_\alpha)$ induces an equivalence of spectra $L_0(X \circ E_\alpha) \to (LX) \circ E_\alpha$.

Proof. The proof proceeds by induction on n. In this case $n = 0$, $X \simeq LX$ and $X \circ E_\alpha$ is a spectrum, so there is nothing to prove. Assume therefore that $n > 0$. Let $\mathcal{C}$ be the full subcategory of $\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ spanned by the $(n-1)$-proximate formal moduli problems, and let $\mathcal{C}_{/X}$ denote the fiber product $\mathcal{C} \times_{\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})} \mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})_{/X}$. By the inductive hypothesis, the map $L_0(Y \circ E_\alpha) \to (LY) \circ E_\alpha$ is an equivalence of spectra for each $Y \in \mathcal{C}_{/X}$. It will therefore suffice to prove the following assertions (for each $\alpha \in T$):

- (a) The spectrum $L_0(X \circ E_\alpha)$ is a colimit of the diagram $\{L_0(Y \circ E_\alpha)\}_{Y \in \mathcal{C}_{/X}}$ in the ∞-category Sp.
- (b) The spectrum $(LX) \circ E_\alpha$ is a colimit of the diagram $\{(LY) \circ E_\alpha\}_{Y \in \mathcal{C}_{/X}}$ in the ∞-category Sp.

To prove (a), we note that L_0 preserves colimits (being a left adjoint) and that the construction $Y \mapsto Y(E_\alpha)$ carries colimit diagrams in $\mathrm{Fun}_*(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ to colimit diagrams in $\mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$, where $\mathrm{Fun}_*(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ denotes the full subcategory of $\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ spanned by those functors which preserve final objects and $\mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ is defined similarly. It will therefore suffice to show that X is a colimit of the diagram $\mathcal{C}_{/X} \to \mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$. (and therefore also of the underlying functor $\mathcal{C}_{/X} \to \mathrm{Fun}_*(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$). We prove a more general assertion: namely, that the identity functor from $\mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$ to itself is a left Kan extension of the inclusion $\mathcal{C} \to \mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$. This follows from Proposition HTT.4.3.2.8 and Lemma HTT.5.1.5.3 , since $\mathcal{C}$ contains the corepresentable functor $\mathrm{Spf}\, R$ for each $R \in \mathbf{A}^{\mathrm{art}}$.

We now prove (b). Fix an index $\alpha \in T$, and let $F : \mathcal{C}_{/X} \to \mathrm{Sp}$ be the functor given by $Y \mapsto (LY) \circ E_\alpha$. Let $*$ denote the final object of $\mathbf{A}$ and let $X_0 = \mathrm{Spf}(*)$ denote the functor corepresented by $*$, so that X_0 is an initial object of $\mathrm{Fun}_*(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$. Let $X_\bullet$ denote the Čech nerve of the map $X_0 \to X$, and let $\mathcal{C}^0_{/X}$ denote the full subcategory of $\mathcal{C}_{/X}$ spanned by those maps $Y \to X$ which factor through X_0. We first prove:

$(*)$ The functors $L|_{\mathcal{C}_{/X}} : \mathcal{C}_{/X} \to \mathrm{Moduli}^{\mathbf{A}}$ and $F : \mathcal{C}_{/X} \to \mathrm{Sp}$ are left Kan extensions of $L|_{\mathcal{C}^0_{/X}}$ and $F|_{\mathcal{C}^0_{/X}}$.

To prove $(*)$, choose an object $Y \in \mathcal{C}_{/X}$, and let $\mathcal{C}^0_{/Y}$ be the full subcategory of $\mathcal{C}_{/Y}$ spanned by those morphisms $Z \to Y$ which factor through $Y_0 = X_0 \times_X Y$. We wish to prove that $LY \in \mathrm{Moduli}^{\mathbf{A}}$ and $FY \in \mathrm{Sp}$ are colimits of the diagrams $L|_{\mathcal{C}^0_{/Y}}$ and $F|_{\mathcal{C}^0_{/Y}}$, respectively. Let $Y_\bullet$ denote the simplicial object of $\mathcal{C}^0_{/Y}$ given by the Cech nerve of the map $Y_0 \to Y$, so that $Y_n \simeq X_n \times_X Y$. The construction $[n] \mapsto Y_n$ determines a left cofinal map $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}^0_{/Y}$; it will therefore suffice to show that the canonical maps

$$u : |LY_\bullet| \to LY \qquad v : |FY_\bullet| \to FY$$

are equivalences. Using Theorem 12.3.3.5 and condition (4) of Definition 12.3.3.2, we deduce that the construction $Z \mapsto Z \circ E_\alpha$ determines a functor $\mathrm{Moduli}^{\mathbf{A}} \to \mathrm{Sp}$ which commutes with sifted colimits. Consequently, to prove that u is an equivalence, it will suffice to show that v is an equivalence for every choice of index $\alpha \in T$. It follows from Remark 16.4.1.7 that each Y_m is an $(n-1)$-proximate formal moduli problem. Using the inductive hypothesis, we are reduced to showing that the canonical map $\theta : |L_0(Y_\bullet \circ E_\alpha)| \to L_0(Y(E_\alpha))$ is an equivalence of spectra. Note that $Y_\bullet \circ E_\alpha$ is the Čech nerve of the natural transformation $Y_0 \circ E_\alpha \to Y \circ E_\alpha$ in the ∞-category $\mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{S})$. Since the functor L_0 is left exact when restricted to $\mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{S})$ (Remark 16.4.3.1), we conclude that $L_0(Y_\bullet(E_\alpha))$ is a Čech nerve of the map $L_0(Y_0 \circ E_\alpha) \to L_0(Y \circ E_\alpha))$, so that θ is an equivalence as desired.

To prove (b), we must show that $(LX) \circ E_\alpha$ is a colimit of the diagram F. Since F is a left Kan extension of $F|_{\mathcal{C}^0_{/X}}$, it will suffice to show that $(LX) \circ E_\alpha)$ is a colimit of the diagram $F|_{\mathcal{C}^0_{/X}}$ (Lemma HTT.4.3.2.7). The simplicial object $X_\bullet$ determines a left cofinal map $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}^0_{/X}$. We are therefore reduced to proving that the map $|(LX_\bullet) \circ E_\alpha| \to (LX) \circ E_\alpha$ is an equivalence of spectra. Since the construction $Z \mapsto Z \circ E_\alpha$ determines a functor $\mathrm{Moduli}^{\mathbf{A}} \to \mathrm{Sp}$ which preserves sifted colimits, it will suffice to show that $|LX_\bullet| \simeq LX$ in $\mathrm{Moduli}^{\mathbf{A}}$. This is equivalent to the assertion that LX is a colimit of the diagram $L|_{\mathcal{C}^0_{/X}}$.

Using $(*)$ and Lemma HTT.4.3.2.7 , we are reduced to proving that LX is a colimit of the diagram $L|_{\mathcal{C}_{/X}}$. Since L preserves small colimits, this follows from the fact that X is a colimit of the inclusion functor $\mathcal{C}_{/X} \hookrightarrow \mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S})$. □

16.4.4 The Proof of Theorem 16.4.2.1

Our proof of Theorem 16.4.2.1 will require a few more preliminaries.

Lemma 16.4.4.1. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $X : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$ be an n-proximate formal moduli problem. For each $\alpha \in T$ and each $m \geqslant 0$, the canonical map $\theta : X(\Omega^{\infty-m} E_\alpha) \to \Omega^{\infty-m} L_0(X \circ E_\alpha))$ is $(n-2)$-truncated.*

Proof. We observe that θ is a filtered colimit of a sequence of morphisms

$$\theta_{m'} : X(\Omega^{\infty-m} E_\alpha) \to \Omega^{m'} X(\Omega^{\infty-m-m'} E_\alpha).$$

It will therefore suffice to show that each $\theta_{m'}$ is $(n-2)$-truncated. Each $\theta_{m'}$ is given by a composition of a finite sequence of morphisms $X(\Omega^{\infty-p} E_\alpha) \to \Omega X(\Omega^{\infty-p-1} E_\alpha)$, which is $(n-2)$-truncated by virtue of our assumption that X is an n-proximate formal moduli problem. □

Lemma 16.4.4.2. *Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $f : X \to Y$ be a natural transformation between n-proximate formal moduli problems $X, Y : \mathbf{A}^{\mathrm{art}} \to \mathcal{S}$. Assume that, for every index $\alpha \in T$ and each $m \geqslant 0$, the map of spaces $X(\Omega^{\infty-m} E_\alpha) \to Y(\Omega^{\infty-m} E_\alpha)$ is $(n-2)$-truncated. Then, for each $A \in \mathbf{A}^{\mathrm{art}}$, the map $X(A) \to Y(A)$ is $(n-2)$-truncated.*

Proof. Since A is Artinian, we can choose a sequence of elementary morphisms

$$A = A_0 \to A_1 \to \cdots \to A_p \simeq *.$$

We will prove that the map $\theta_i : X(A_i) \to Y(A_i)$ is $(n-2)$-truncated by descending induction on i. The case $i = p$ is clear (since θ is a morphism between contractible spaces and therefore a homotopy equivalence). Assume therefore that $i < p$ and that θ_{i+1} is $(n-2)$-truncated. Since the map $A_i \to A_{i+1}$ is elementary, we have a fiber sequence $A_i \to A_{i+1} \to \Omega^{\infty-m} E_\alpha$ in $\mathbf{A}^{\mathrm{art}}$. Let F be the homotopy fiber of the map $X(A_{i+1}) \to X(\Omega^{\infty-m} E_\alpha)$, and let F' be the homotopy fiber of the map $Y(A_{i+1}) \to Y(\Omega^{\infty-m} E_\alpha)$. We have a map of fiber sequences

$$\begin{array}{ccccc} F & \longrightarrow & X(A_{i+1}) & \longrightarrow & X(\Omega^{\infty-m} E_\alpha) \\ \downarrow \psi & & \downarrow \theta_{i+1} & & \downarrow \phi \\ F' & \longrightarrow & Y(A_{i+1}) & \longrightarrow & Y(\Omega^{\infty-m} E_\alpha). \end{array}$$

Since ϕ is $(n-2)$-truncated by assumption and θ_{i+1} is $(n-2)$-truncated by the inductive hypothesis, we conclude that ψ is $(n-2)$-truncated. The map θ_i factors as a composition

$X(A_i) \xrightarrow{\theta_i'} Y(A_i) \times_{F'} F \xrightarrow{\theta_i''} Y(A_i)$, where θ_i'' is a pullback of ψ and therefore $(n-2)$-truncated. It will therefore suffice to show that θ_i' is $(n-2)$-truncated. Since Y is an n-proximate formal moduli problem, the map $Y(A_i) \rightarrow F'$ is $(n-2)$-truncated, so the projection $Y(A_i) \times_{F'} F \rightarrow F$ is $(n-2)$-truncated. It will therefore suffice to show that the composite map $X(A_i) \xrightarrow{\theta_i'} Y(A_i) \times_{F'} F \rightarrow F$ is $(n-2)$-truncated, which follows from our assumption that X is an n-proximate formal moduli problem. □

Proof of Theorem 16.4.2.1. The implication (3) ⇒ (2) is obvious. We next show that (2) ⇒ (1). Let $X : \mathbf{A}^{\mathrm{art}} \rightarrow \mathcal{S}$ be a functor for which $X(*)$ is contractible and there exists an $(n-2)$-truncated map $f : X \rightarrow Y$, where Y is an n-proximate formal moduli problem. We wish to show that X is an n-proximate formal moduli problem. Choose a pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \phi \\ B' & \longrightarrow & B \end{array}$$

in $\mathbf{A}^{\mathrm{art}}$ where ϕ is small; we wish to show that left square in the diagram of spaces

$$\begin{array}{ccccc} X(A') & \longrightarrow & X(A) & \longrightarrow & Y(A) \\ \downarrow & & \downarrow & & \downarrow \\ X(B') & \longrightarrow & X(B) & \longrightarrow & Y(B) \end{array}$$

is $(n-2)$-Cartesian. Our assumption that f is $(n-2)$-truncated guarantees that the right square is $(n-2)$-Cartesian; it will therefore suffice to show that the outer square is $(n-2)$-Cartesian (Lemma 16.4.1.3). Using Lemma 16.4.1.3 again, we are reduced to showing that both the left and right squares in the diagram

$$\begin{array}{ccccc} X(A') & \longrightarrow & Y(A') & \longrightarrow & Y(A) \\ \downarrow & & \downarrow & & \downarrow \\ X(B') & \longrightarrow & Y(B') & \longrightarrow & Y(B) \end{array}$$

are $(n-2)$-Cartesian. For the left square, this follows from our assumption that f is $(n-2)$-truncated; for the right square, it follows from our assumption that Y is an n-proximate formal moduli problem.

We now complete the proof by showing that (1) ⇒ (3). Assume that $X : \mathbf{A}^{\mathrm{art}} \rightarrow \mathcal{S}$ is an n-proximate formal moduli problem and let $L : \mathrm{Fun}(\mathbf{A}^{\mathrm{art}}, \mathcal{S}) \rightarrow \mathrm{Moduli}^{\mathbf{A}}$ be a left adjoint to the inclusion; we wish to show that the canonical map $X \rightarrow LX$ is $(n-2)$-truncated. According to Lemma 16.4.4.2, it will suffice to show that the map $\phi : X(\Omega^{\infty-m} E_\alpha) \rightarrow LX(\Omega^{\infty-m} E_\alpha)$ is $(n-2)$-truncated for each $\alpha \in T$ and each $m \geqslant 0$. Using Proposition 16.4.3.2, we can identify

ϕ with the canonical map $X(\Omega^{\infty-m}E_\alpha) \to \Omega^{\infty-m}L_0(X \circ E_\alpha))$, which is $(n-2)$-truncated by Lemma 16.4.4.1. □

16.5 Deformations of Objects

Let X be an algebraic variety defined over a field κ and let $\mathscr{E}$ be an algebraic vector bundle on X. A *first order deformation* of $\mathscr{E}$ is an algebraic vector bundle $\overline{\mathscr{E}}$ over the scheme $\overline{X} = X \times_{\operatorname{Spec}\kappa} \operatorname{Spec}\kappa[\epsilon]/(\epsilon^2)$, together with an isomorphism $i^*\overline{\mathscr{E}} \to \mathscr{E}$ (where i denotes the closed immersion $X \hookrightarrow \overline{X}$). Standard arguments in deformation theory show that the collection of isomorphism classes of first-order deformations can be identified with the cohomology group $\mathrm{H}^1(X; \mathrm{End}(\mathscr{E}))$, while the automorphism of each first order deformation of $\mathscr{E}$ is given by $\mathrm{H}^0(X; \mathrm{End}(\mathscr{E}))$. Our goal in this section is to place these observations in a more general context:

- In the definition above, we can replace the ring of dual numbers $\kappa[\epsilon]/(\epsilon^2)$ by an arbitrary $R \in \mathrm{CAlg}^{\mathrm{art}}_\kappa$ to obtain a notion of a *deformation of $\mathscr{E}$ over R*. Let $\mathrm{ObjDef}_{\mathscr{E}}(R)$ denote a classifying space for deformations of $\mathscr{E}$ over R. Then $\mathrm{ObjDef}_{\mathscr{E}}$ can be regarded as a functor $\mathrm{CAlg}^{\mathrm{art}}_\kappa \to \mathcal{S}$. We will see below that this functor is a formal moduli problem.

- The isomorphisms

$$\mathrm{H}^0(X; \mathrm{End}(\mathscr{E})) \simeq \pi_1 \mathrm{ObjDef}_{\mathscr{E}}(\kappa[\epsilon]/(\epsilon^2)) \quad \mathrm{H}^1(X; \mathrm{End}(\mathscr{E})) \simeq \pi_0 \mathrm{ObjDef}_{\mathscr{E}}(\kappa[\epsilon]/(\epsilon^2))$$

  follow from an identification of the $\Gamma(X; \mathrm{End}(\mathscr{E}))$ with the (shifted) tangent complex $\Sigma^{-1}T_{\mathrm{ObjDef}_{\mathscr{E}}}$; here $\Gamma(X; \mathrm{End}(\mathscr{E}))$ denotes the *derived* global sections of the vector bundle $\mathrm{End}(\mathscr{E})$ on X.

- The definition of $\mathrm{ObjDef}_{\mathscr{E}}(R)$ does not require R to be commutative. Consequently, we can extend the domain of definition of $\mathrm{ObjDef}_{\mathscr{E}}$ to $\mathrm{Alg}^{\mathrm{art}}_\kappa$, and thereby regard $\mathrm{ObjDef}_{\mathscr{E}}$ as a formal $\mathbb{E}_1$-moduli problem (see Definition 14.0.0.4). We will see that the identification $\Gamma(X; \mathrm{End}(\mathscr{E})) \simeq \Sigma^{-1}T_{\mathrm{ObjDef}_{\mathscr{E}}}$ is multiplicative: that is, it can be regarded as an equivalence of nonunital $\mathbb{E}_1$-algebras (where $\Sigma^{-1}T_{\mathrm{ObjDef}_{\mathscr{E}}}$ is equipped with the nonunital $\mathbb{E}_1$-algebra structure given by Remark 14.2.2.2).

- The definition of the formal moduli problem $\mathrm{ObjDef}_{\mathscr{E}}$ depends only on the algebraic vector bundle $\mathscr{E}$ as an object of the stable ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X. We will therefore consider the more general problem of deforming an object E of an arbitrary stable κ-linear ∞-category $\mathcal{C}$.

16.5.1 Conventions

Let κ be a field and let Mod_κ denote the ∞-category of κ-modules. Recall that a *stable κ-linear ∞-category* is a Mod_κ-module object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ of presentable stable ∞-categories (see Variant D.1.5.1). Note that since Mod_κ is a symmetric monoidal ∞-category, there is no need to distinguish between left modules and right modules: any stable κ-linear ∞-category $\mathcal{C}$ can be regarded either as left-tensored or right-tensored over Mod_κ. In particular, if $A \in \mathrm{Alg}_\kappa$ is an $\mathbb{E}_1$-algebra over κ, then we can consider the ∞-categories $\mathrm{LMod}_A(\mathcal{C})$ and $\mathrm{RMod}_A(\mathcal{C})$ of left and right A-module objects of $\mathcal{C}$, respectively. Suppose we are given an object E of a stable κ-linear ∞-category $\mathcal{C}$. We will be interested in two different relationships that E might bear to an $\mathbb{E}_1$-algebra over κ:

(a) To every $\mathbb{E}_1$-algebra A over κ, we can associate a forgetful functor $\mu : \mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_\kappa(\mathcal{C}) \simeq \mathcal{C}$, given by restriction of scalars along the unit map $\kappa \to A$. We define a *left action of A on E* to be an object $\overline{E} \in \mathrm{LMod}_A(\mathcal{C})$ together with an equivalence $E \simeq \mu(\overline{E})$. There is a universal example of such a left action: giving a left action of A on E is equivalent to giving a morphism $A \to \mathrm{End}(E)$ in Alg_κ, where $\mathrm{End}(E)$ denotes the endomorphism algebra of E (see Corollary HA.4.7.1.41).

(b) To every augmented $\mathbb{E}_1$-algebra B over κ, we can associate a functor $\mathrm{RMod}_B(\mathcal{C}) \to \mathrm{RMod}_\kappa(\mathcal{C}) \simeq \mathcal{C}$, given by extension of scalars $C \mapsto C \otimes_B \kappa$ (along the augmentation map $B \to \kappa$). We define a *deformation of E over B* to be an object $E_B \in \mathrm{RMod}_B(\mathcal{C})$ together with an equivalence $E_B \otimes_B \kappa \simeq E$.

As we will see, these two notions are related by Koszul duality.

Remark 16.5.1.1. The reader might object that the distinction between right and left modules in the above discussion is artificial. For every algebra object $A \in \mathrm{Alg}_\kappa$, there is a canonical equivalence of ∞-categories $\mathrm{LMod}_A(\mathcal{C}) \simeq \mathrm{RMod}_{A^{\mathrm{rev}}}(\mathcal{C})$, where A^{rev} denotes the algebra A equipped with the opposite multiplication (see Remark HA.4.1.1.7). Consequently, we can replace left actions by right actions in (a), or right actions by left actions in (b). However, in what follows, it will be convenient not to make these replacements. When analyzing a fixed object $E \in \mathcal{C}$, algebras A which act on E itself will act on the left, and augmented algebras B over which we have a deformation of E will act on the right. This will help us to avoid confusion in the discussion which follows, and has the added benefit of being compatible with our conventions for the Koszul duality functor $\mathfrak{D}^{(1)}$ studied in Chapter 14.

16.5.2 Classifying Spaces of Deformations

We now observe that deformations of an object $E \in \mathcal{C}$ depend functorially on the ring they are defined over:

Construction 16.5.2.1. Let κ be a field, let $\mathcal{C}$ be a stable κ-linear ∞-category, and let $E \in \mathcal{C}$ be an object. Let $\mathrm{RMod}(\mathcal{C})$ denote the ∞-category of pairs (B, E_B) where $B \in \mathrm{Alg}_\kappa$ and E_B is a right B-module object of $\mathcal{C}$. The forgetful functor

$$q : \mathrm{RMod}(\mathcal{C}) \to \mathrm{Alg}_\kappa \qquad (B, E_B) \mapsto B$$

is a coCartesian fibration. Let $\mathrm{RMod}^{\mathrm{coCart}}(\mathcal{C})$ denote the subcategory of $\mathrm{RMod}(\mathcal{C})$ spanned by the q-coCartesian morphisms, so that q restricts to a left fibration $\mathrm{RMod}^{\mathrm{coCart}}(\mathcal{C}) \to \mathrm{Alg}_\kappa$. We will abuse notation by identifying E with an object of $\mathrm{RMod}^{\mathrm{coCart}}(\mathcal{C})$ (via the equivalence $\mathrm{RMod}_\kappa(\mathcal{C}) \to \mathcal{C}$). Set $\mathrm{Defor}[E] = \mathrm{RMod}^{\mathrm{coCart}}(\mathcal{C})_{/E}$. We will refer to $\mathrm{Defor}[E]$ as the *∞-category of deformations of E*.

There is an evident left fibration $\theta : \mathrm{Defor}[E] \to \mathrm{Alg}^{\mathrm{aug}}_\kappa$, whose fiber over an object $B \in \mathrm{Alg}^{\mathrm{aug}}_\kappa$ can be identified with the Kan complex $(\mathrm{RMod}_B(\mathcal{C}) \times_{\mathcal{C}} \{E\})^{\simeq}$ of deformations of E over B. Consequently, we can view the construction $B \mapsto (\mathrm{RMod}_B(\mathcal{C}) \times_{\mathcal{C}} \{E\})^{\simeq}$ determining a functor $\mathrm{ObjDef}^+_E : \mathrm{Alg}^{\mathrm{aug}}_\kappa \to \widehat{\mathcal{S}}$.

Let $\mathcal{C}$ be as in Construction 16.5.2.1. Proposition 16.2.1.1 implies that for every pullback diagram

$$\begin{array}{ccc} B & \longrightarrow & B_0 \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_{01} \end{array}$$

in Alg_κ, the induced functor $\mathrm{RMod}_B(\mathcal{C}) \to \mathrm{RMod}_{B_0}(\mathcal{C}) \times_{\mathrm{RMod}_{B_{01}}(\mathcal{C})} \mathrm{RMod}_{B_1}(\mathcal{C})$ is fully faithful. This immediately implies the following:

Proposition 16.5.2.2. *Let κ be a field, let $\mathcal{C}$ a stable κ-linear ∞-category, and let $E \in \mathcal{C}$ be an object. Then, for every pullback diagram*

$$\begin{array}{ccc} B & \longrightarrow & B_0 \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_{01} \end{array}$$

in $\mathrm{Alg}^{\mathrm{aug}}_\kappa$, the induced map $\mathrm{ObjDef}^+_E(B) \to \mathrm{ObjDef}^+_E(B_0) \times_{\mathrm{ObjDef}^+_E(B_{01})} \mathrm{ObjDef}^+_E(B_1)$ has (-1)-truncated homotopy fibers (that is, it induces a homotopy equivalence onto its essential image).

Corollary 16.5.2.3. *Let κ be a field, let $\mathcal{C}$ a stable κ-linear ∞-category, let $E \in \mathcal{C}$ an object, and let $\mathrm{ObjDef}^+_E : \mathrm{Alg}^{\mathrm{aug}}_\kappa \to \widehat{\mathcal{S}}$ be as in Construction 16.5.2.1. Then:*

(1) *The space $\mathrm{ObjDef}^+_E(\kappa)$ is contractible.*

(2) *Let $V \in \mathrm{Mod}_\kappa$. Then the space $\mathrm{ObjDef}^+_E(\kappa \oplus V)$ is essentially small.*

(3) *Let* $B \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$ *be Artinian. Then the space* $\mathrm{ObjDef}_E^+(B)$ *is essentially small.*

Proof. Assertion (1) is immediate from the definitions. To prove (2), we note that for each $B \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$, the space $\mathrm{ObjDef}_E^+(B)$ is locally small (when regarded as an ∞-category). We have a pullback diagram

$$\begin{array}{ccc} \kappa \oplus V & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ \kappa & \longrightarrow & \kappa \oplus \Sigma(V) \end{array}$$

so that Proposition 16.5.2.2 guarantees that $\mathrm{ObjDef}_E^+(\kappa \oplus V)$ is a summand of $\Omega\,\mathrm{ObjDef}_E^+(\kappa \oplus \Sigma(V))$, and therefore essentially small. We now prove (3). Assume that B is Artinian, so that there exists a finite sequence of maps $B \simeq B_0 \to B_1 \to \cdots \to B_n \simeq \kappa$ and pullback diagrams

$$\begin{array}{ccc} B_i & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ B_{i+1} & \longrightarrow & \kappa \oplus \Sigma^{m_i}(\kappa). \end{array}$$

Using (1), (2), and Proposition 16.5.2.2, we deduce that each $\mathrm{ObjDef}_E^+(B_i)$ is essentially small using descending induction on i. □

16.5.3 Deformations as a Moduli Problem

We now restrict our attention to deformations of an object $E \in \mathcal{C}$ over Artinian $\mathbb{E}_1$-algebras.

Notation 16.5.3.1. Let κ be a field, let $\mathcal{C}$ be a stable κ-linear ∞-category, and let $E \in \mathcal{C}$ an object. We let ObjDef_E denote the composite functor $\mathrm{Alg}_\kappa^{\mathrm{art}} \hookrightarrow \mathrm{Alg}_\kappa^{\mathrm{aug}} \xrightarrow{\mathrm{ObjDef}_E^+} \widehat{\mathcal{S}}$. It follows from Corollary 16.5.2.3 that the functor ObjDef_E takes essentially small values, and can therefore be regarded as a functor from $\mathrm{Alg}_\kappa^{\mathrm{art}}$ to $\mathcal{S}$.

More informally: the functor ObjDef_E assigns to each $B \in \mathrm{Alg}_\kappa^{\mathrm{art}}$ a classifying space for pairs (E_B, μ), where $E_B \in \mathrm{RMod}_B(\mathcal{C})$ and $\mu : E_B \otimes_B \kappa \to E$ is an equivalence in $\mathcal{C}$.

Combining Corollary 16.5.2.3 and Proposition 16.5.2.2, we obtain the following:

Corollary 16.5.3.2. *Let κ be a field, let $\mathcal{C}$ be a stable κ-linear ∞-category, and let $E \in \mathcal{C}$ be an object. Then the functor $\mathrm{ObjDef}_E : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ is a 1-proximate formal moduli problem (see Definition 16.4.1.5).*

Notation 16.5.3.3. Let κ be a field, and let $L : \mathrm{Fun}(\mathrm{Alg}_\kappa^{\mathrm{art}}, \mathcal{S}) \to \mathrm{Moduli}_\kappa^{(1)}$ denote a left adjoint to the inclusion. If $\mathcal{C}$ is a stable κ-linear ∞-category and $E \in \mathcal{C}$ is an object, we let $\mathrm{ObjDef}_E^\wedge$ denote the formal $\mathbb{E}_1$-moduli problem $L(\mathrm{ObjDef}_E)$. By construction, we have a natural transformation $\mathrm{ObjDef}_E \to \mathrm{ObjDef}_E^{\wedge}$. It follows from Theorem 16.4.2.1 that this natural transformation is (-1)-truncated: that is, it exhibits $\mathrm{ObjDef}_E(R)$ as a summand of $\mathrm{ObjDef}_E^{\wedge}(R)$, for each $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$. Moreover, $\mathrm{ObjDef}_E^{\wedge}$ is characterized up to equivalence (as a formal $\mathbb{E}_1$-moduli problem) by this property: see Remark 16.4.2.3.

16.5.4 Statement of the Main Theorem

Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category. For each object $E \in \mathcal{C}$, we let $\mathrm{End}(E) \in \mathrm{Mod}_\kappa$ denote the classifying object for endomorphisms of M: that is, $\mathrm{End}(E)$ is an object of Mod_κ equipped with a map $a : \mathrm{End}(E) \otimes E \to E$ in $\mathcal{C}$ having the following universal property: for every object $V \in \mathrm{Mod}_\kappa$, composition with a induces a homotopy equivalence $\mathrm{Map}_{\mathrm{Mod}_\kappa}(V, \mathrm{End}(E)) \simeq \mathrm{Map}_{\mathcal{C}}(V \otimes E, E)$. The existence of the object $\mathrm{End}(E)$ follows from Proposition HA.4.2.1.33 . Moreover, it follows from the results of §HA.4.7.1 show that we can regard $\mathrm{End}(E)$ as an object of Alg_κ and E as a left module over $\mathrm{End}(E)$. In what follows, it will be convenient to view $\mathrm{End}(E)$ as a *nonunital* $\mathbb{E}_1$-algebra over κ, which can be identified with the augmentation ideal of the augmented $\mathbb{E}_1$-algebra $\kappa \oplus \mathrm{End}(E)$.

We can now state our main result:

Theorem 16.5.4.1. *Let κ be a field, let $\mathcal{C}$ be a stable κ-linear ∞-category, let $E \in \mathcal{C}$ be an object, and let $\Psi : \mathrm{Alg}_\kappa^{\mathrm{aug}} \to \mathrm{Moduli}_\kappa^{(1)}$ be the equivalence of ∞-categories of Theorem 14.0.0.5. Then there is a canonical equivalence of formal $\mathbb{E}_1$-moduli problems $\mathrm{ObjDef}_E^{\wedge} \simeq \Psi(\kappa \oplus \mathrm{End}(E))$.*

Remark 16.5.4.2. In the situation of Theorem 16.5.4.1, for each Artinian $\mathbb{E}_1$-algebra B over κ, we have a (-1)-truncated map

$$\mathrm{ObjDef}_E(B) \hookrightarrow \mathrm{ObjDef}_E^{\wedge}(B) \simeq \mathrm{Map}_{\mathrm{Alg}_\kappa^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(B), \kappa \oplus \mathrm{End}(E)) \simeq \mathrm{Map}_{\mathrm{Alg}_\kappa}(\mathfrak{D}_B^{(1)}, \mathrm{End}(E)).$$

In other words, to every deformation E_B of E over B, we can associate a left action of the Koszul dual $\mathfrak{D}^{(1)}(B)$ on E which determines the deformation E_B up to a contractible space of choices. However, in some cases there exist actions of $\mathfrak{D}^{(1)}(B)$ on E which do not arise in this way (these actions correspond to a more general kind of deformation of E: see Remark 16.5.7.3).

Example 16.5.4.3. Let κ be a field and regard Mod_κ as a κ-linear ∞-category. Let V be a finite-dimensional vector space over κ and define ObjDef_V as above. We will see below that ObjDef_V is a formal $\mathbb{E}_1$-moduli problem (Proposition 16.5.7.1), so that $\mathrm{ObjDef}_V|_{\mathrm{CAlg}_\kappa^{\mathrm{art}}}$ is a formal moduli problem over κ. Assume now that κ has characteristic zero, and let $\Phi : \mathrm{Lie}_\kappa \to \mathrm{Moduli}_\kappa$ be the equivalence of Theorem 13.0.0.2. Combining Theorems 16.5.4.1 and 14.3.0.1, we deduce that $\mathrm{ObjDef}_V|_{\mathrm{CAlg}_\kappa^{\mathrm{art}}}$ corresponds, under the equivalence Φ, to the matrix algebra $\mathrm{End}(V)$ (equipped with its usual Lie bracket).

Remark 16.5.4.4. Let κ be a field, let $\mathcal{C}$ be a stable κ-linear ∞-category, and let $E \in \mathcal{C}$ be an object. For each $R \in \mathrm{Alg}_\kappa^{\mathrm{art}}$, Theorem HA.4.8.4.1 yields a canonical homotopy equivalence

$$\mathrm{ObjDef}_E(R) = \mathrm{RMod}_R(\mathcal{C})^{\simeq} \times_{\mathcal{C}} \{E\} \simeq \mathrm{Map}_{\mathrm{LinCat}_\kappa}(\mathrm{LMod}_R, \mathcal{C})^{\simeq} \times_{\mathcal{C}} \{E\}.$$

It follows that $\mathrm{ObjDef}_E(R)$ depends only on the κ-linear ∞-category LMod_R, together with the distinguished object $\kappa \in \mathrm{LMod}_R$ (supplied by the augmentation $R \to \kappa$), so that the construction $R \mapsto \mathrm{ObjDef}_E(R)$ enjoys some extra functoriality. This special feature of ObjDef_E is reflected in the structure of the associated formal $\mathbb{E}_1$-moduli problem $\mathrm{ObjDef}_E^\wedge$: according to Theorem 16.5.4.1, $\mathrm{ObjDef}_E^\wedge$ has the form $\Psi(A)$, where $A \in \mathrm{Alg}_\kappa^{\mathrm{aug}}$ is an augmented $\mathbb{E}_1$-algebra over κ whose augmentation ideal $\mathfrak{m}_A \simeq \mathrm{End}(E)$ is itself unital.

16.5.5 Construction of the Equivalence

The equivalence $\mathrm{ObjDef}_E^\wedge \simeq \Psi(\kappa \oplus \mathrm{End}(E))$ of Theorem 16.5.4.1 arises from a natural transformation $\mathrm{ObjDef}_E \to \Psi(\kappa \oplus \mathrm{End}(E))$. We will obtain this natural transformation from a somewhat elaborate construction.

Construction 16.5.5.1. Let κ be a field, let $\mathcal{C}$ be a stable κ-linear ∞-category, and let $E \in \mathcal{C}$ be an object. We let $\lambda : \mathcal{M}^{(1)} \to \mathrm{Alg}_\kappa^{\mathrm{aug}} \times \mathrm{Alg}_\kappa^{\mathrm{aug}}$ denote the pairing of Construction **??**, so that we can identify objects of $\mathcal{M}^{(1)}$ with triples (A, B, ϵ) where $A, B \in \mathrm{Alg}_\kappa$ and $\epsilon : A \otimes_\kappa B \to \kappa$ is an augmentation. We will also regard the ∞-category $\mathcal{C}$ as bitensored over Mod_κ.

Given an object $(A, B, \epsilon) \in \mathcal{M}^{(1)}$ and an object $(B, E_B, \mu) \in \mathrm{Defor}[E]$, we regard $A \otimes_\kappa E_B$ as an object of the ∞-category of bimodules ${}_A\mathrm{BMod}_{A \otimes_\kappa B}(\mathcal{C})$, so that $(A \otimes_\kappa E_B) \otimes_{A \otimes_\kappa B} \kappa$ can be identified with an object of $\mathrm{LMod}_A(\mathcal{C})$ whose image in $\mathcal{C}$ is given by $E_B \otimes_B \kappa \simeq E$. This observation determines a functor $\rho : \mathcal{M}^{(1)} \times_{\mathrm{Alg}_\kappa^{\mathrm{aug}}} \mathrm{Defor}[E] \to \mathrm{LMod}(\mathcal{C}) \times_{\mathcal{C}} \{E\}$, given on objects by

$$(A, B, \epsilon : A \otimes_\kappa B \to \kappa, E_B, \mu : E_B \otimes_B \kappa \simeq E) \mapsto (A \otimes E_B) \otimes_{A \otimes_\kappa B} \kappa$$

(see Corollary HA.4.7.1.40). Let $\mathrm{LMod}^{\mathrm{aug}}(\mathcal{C})$ denote the fiber product $\mathrm{Alg}_\kappa^{\mathrm{aug}} \times_{\mathrm{Alg}_\kappa} \mathrm{LMod}(\mathcal{C})$. Since an augmentation on $A \otimes_\kappa B$ induces an augmentation on A, we can lift ρ to a functor $\overline{\rho} : \mathrm{Defor}[E] \times_{\mathrm{Alg}_\kappa^{\mathrm{aug}}} \mathcal{M}^{(1)} \to \mathrm{Defor}[E] \times (\mathrm{LMod}^{\mathrm{aug}}(\mathcal{C}) \times_{\mathcal{C}} \{E\})$ which factors as a composition

$$\mathrm{Defor}[E] \times_{\mathrm{Alg}_\kappa^{\mathrm{aug}}} \mathcal{M}^{(1)} \xrightarrow{i} \widetilde{\mathcal{M}}^{(1)} \xrightarrow{\lambda'} \mathrm{Defor}[E] \times (\mathrm{LMod}^{\mathrm{aug}}(\mathcal{C}) \times_{\mathcal{C}} \{E\})$$

where i is an equivalence of ∞-categories and λ' is a categorical fibration. It is not difficult to see that λ' is a left representable pairing of ∞-categories, which induces a duality functor $\mathfrak{D}_E^{(1)} : \mathrm{Defor}[E]^{\mathrm{op}} \to \mathrm{LMod}^{\mathrm{aug}}(\mathcal{C}) \times_{\mathcal{C}} \{E\}$. Concretely, the functor $\mathfrak{D}_E^{(1)}$ assigns to each object $(B, E_A, \mu) \in \mathrm{Defor}[E]^{\mathrm{op}}$ the object $(\mathfrak{D}^{(1)}(B), E)$, where we regard E as a left $\mathfrak{D}^{(1)}(B)$-module object of $\mathcal{C}$ via the equivalence $E \simeq E_B \otimes_B \kappa \simeq (\mathfrak{D}^{(1)}(B) \otimes_\kappa E_B) \otimes_{\mathfrak{D}^{(1)}(B) \otimes_\kappa B} \kappa$.

Corollary HA.4.7.1.40 supplies an equivalence of ∞-categories $\eta : \mathrm{LMod}(\mathcal{C}) \times_{\mathcal{C}} \{E\} \simeq (\mathrm{Alg}_{\kappa})_{/\operatorname{End}(E)}$, hence also an equivalence $\mathrm{LMod}^{\mathrm{aug}}(\mathcal{C}) \times_{\mathcal{C}} \{E\} \simeq (\mathrm{Alg}_{\kappa}^{\mathrm{aug}})_{/\kappa \oplus \operatorname{End}(E)}$. We therefore obtain a diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Defor}[E]^{\mathrm{op}} & \longrightarrow & (\mathrm{Alg}_{\kappa}^{\mathrm{aug}})_{/\kappa\oplus\operatorname{End}(E)} \\ \downarrow & & \downarrow \\ (\mathrm{Alg}_{\kappa}^{\mathrm{aug}})^{\mathrm{op}} & \xrightarrow{\mathfrak{D}^{(1)}} & \mathrm{Alg}_{\kappa}^{\mathrm{aug}} \end{array}$$

which commutes up to canonical homotopy, where the vertical maps are right fibrations. This diagram determines a natural transformation $\beta^{+} : \mathrm{ObjDef}_E^{+} \to X$, where $X : \mathrm{Alg}_{\kappa}^{(1),\mathrm{art}} \to \mathcal{S}$ denotes the functor given on objects by $X(B) = \mathrm{Map}_{\mathrm{Alg}_{\kappa}^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(B), \kappa \oplus \operatorname{End}(E)) \simeq \mathrm{Map}_{\mathrm{Alg}_{\kappa}}(\mathfrak{D}^{(1)}(A), \operatorname{End}(E))$. Restricting to the case where B is Artinian, we obtain a natural transformation $\beta : \mathrm{ObjDef}_E \to \Psi(\kappa \oplus \operatorname{End}(E))$, where $\Psi : \mathrm{Alg}_{\kappa}^{\mathrm{aug}} \to \mathrm{Moduli}_{\kappa}^{(1)}$ is the equivalence of Theorem 14.0.0.5.

16.5.6 The Proof of Theorem 16.5.4.1

Using Construction 16.5.5.1, we can formulate a more precise version of Theorem 16.5.4.1:

Proposition 16.5.6.1. *Let κ be a field, let $\mathcal{C}$ be a stable κ-linear ∞-category, and let $E \in \mathcal{C}$ be an object. Then the natural transformation $\beta : \mathrm{ObjDef}_E \to \Psi(\kappa \oplus \operatorname{End}(E))$ of Construction 16.5.5.1 induces an equivalence of formal $\mathbb{E}_1$-moduli problems $\mathrm{ObjDef}_E^{\wedge} \simeq \Psi(\kappa \oplus \operatorname{End}(E))$. In other words, for every object $B \in \mathrm{Alg}_{\kappa}^{\mathrm{art}}$, the natural map*

$$\mathrm{ObjDef}_E(B) \to \Psi(\kappa \oplus \operatorname{End}(E))(B) \simeq \mathrm{Map}_{\mathrm{Alg}_{\kappa}^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(B), \kappa \oplus \operatorname{End}(E))$$

has (-1)-truncated homotopy fibers (see Remark 16.4.2.3).

To prove Proposition 16.5.6.1, it will suffice to show that the map $\mathrm{ObjDef}_E^{\wedge} \to \Psi(\kappa \oplus \operatorname{End}(E))$ induces an equivalence of tangent complexes (Proposition 12.2.2.6). Using the description of the tangent complex of $\mathrm{ObjDef}_E^{\wedge}$ supplied by Proposition 16.4.3.2, we are reduced to proving the following special case of Proposition 16.5.6.1:

Proposition 16.5.6.2. *Let κ be a field, let $\mathcal{C}$ be a stable κ-linear ∞-category, and let $E \in \mathcal{C}$ be an object. For each $m \geqslant 0$, the natural transformation $\beta : \mathrm{ObjDef}_E \to \Psi(\kappa \oplus \operatorname{End}(E))$ of Construction 16.5.5.1 induces a (-1)-truncated map*

$$\begin{aligned} \mathrm{ObjDef}_E(\kappa \oplus \Sigma^m(\kappa)) &\to \mathrm{Map}_{\mathrm{Alg}_{\kappa}^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(\kappa \oplus \Sigma^m(\kappa)), \kappa \oplus \operatorname{End}(E)) \\ &\simeq \mathrm{Map}_{\mathrm{Alg}_{\kappa}}(\mathfrak{D}^{(1)}(\kappa \oplus \Sigma^m(\kappa)), \operatorname{End}(E)). \end{aligned}$$

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{ObjDef}_E(\kappa \oplus \Sigma^m(\kappa)) & \longrightarrow & \mathrm{Map}_{\mathrm{Alg}_\kappa}(\mathfrak{D}^{(1)}(\kappa \oplus \Sigma^m(\kappa)), \mathrm{End}(E)) \\ \downarrow & & \downarrow \\ \Omega\,\mathrm{ObjDef}_E(\kappa \oplus \Sigma^{m+1}(\kappa)) & \xrightarrow{\theta} & \Omega\,\mathrm{Map}_{\mathrm{Alg}_\kappa}(\mathfrak{D}^{(1)}(\kappa \oplus \Sigma^{m+1}(\kappa)), \mathrm{End}(E)), \end{array}$$

where the left vertical map is (-1)-truncated by Corollary 16.5.3.2 and the right vertical map is a homotopy equivalence. It will therefore suffice to show that θ is a homotopy equivalence. Let $B = \kappa \oplus \Sigma^{m+1}(\kappa)$ and let $E_B = E \otimes_\kappa B \in \mathrm{RMod}_B(\mathcal{C})$. We can identify the domain of θ with the homotopy fiber of the map $\xi : \mathrm{Map}_{\mathrm{RMod}_B(\mathcal{C})}(E_B, E_B) \to \mathrm{Map}_{\mathcal{C})}(E, E)$. We have a fiber sequence $\Sigma^{m+1}E \to E_B \to E$ in $\mathrm{RMod}_B(\mathcal{C})$, where B acts on E via the augmentation $B \to \kappa$. It follows that the homotopy fiber of ξ is given by

$$\begin{aligned} \mathrm{Map}_{\mathrm{RMod}_B(\mathcal{C})}(E_B, \Sigma^{m+1}E) &\simeq \mathrm{Map}_{\mathcal{C}}(E_B \otimes_B \kappa, \Sigma^{m+1}E) \\ &\simeq \mathrm{Map}_{\mathcal{C}}(E, \Sigma^{m+1}E) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_\kappa}(\Sigma^{-m-1}(\kappa), \mathrm{End}(E)) \end{aligned}$$

The map θ is induced by a morphism $\nu : \Sigma^{-m-1}(\kappa) \to \mathfrak{D}^{(1)}(\kappa \oplus \Sigma^m(\kappa))$ in Mod_κ. Let $\mathrm{Free}^{(1)} : \mathrm{Mod}_\kappa \to \mathrm{Alg}_\kappa$ be a left adjoint to the forgetful functor, so that ν determines an augmentation $(\kappa \oplus \Sigma^m(\kappa)) \otimes_\kappa \mathrm{Free}^{(1)}(\Sigma^{-m-1}(\kappa)) \to \kappa$. This pairing exhibits $\kappa \oplus \Sigma^m(\kappa)$ as a Koszul dual of $\mathrm{Free}^{(1)}(\Sigma^{-m-1}(\kappa))$, and therefore also exhibits $\mathrm{Free}^{(1)}(\Sigma^{-m-1}(\kappa))$ as a Koszul dual of $\kappa \oplus \Sigma^m(\kappa)$ (see Theorem 14.1.3.2). It follows that θ is a homotopy equivalence, as desired. □

16.5.7 Connectivity Hypotheses

We conclude this section with a few observations concerning the discrepancy between the deformation functor ObjDef_E of Notation 16.5.3.1 and the associated formal $\mathbb{E}_1$-moduli problem $\mathrm{ObjDef}^{\wedge}_E$ of Notation 16.5.3.3. Under some mild additional hypotheses, one can show that these two functors are equivalent:

Proposition 16.5.7.1. *Let κ be a field, let $\mathcal{C}$ be a separated prestable κ-linear ∞-category. For every object $E \in \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$, the functor $\mathrm{ObjDef}_E : \mathrm{Alg}^{\mathrm{art}}_\kappa \to \mathcal{S}$ of Notation 16.5.3.1 is a formal $\mathbb{E}_1$-moduli problem.*

Proof. Let $B \in \mathrm{Alg}^{\mathrm{art}}_\kappa$. We first show that if $(E_B, \mu) \in \mathrm{ObjDef}_E(B)$, then $E_B \in \mathrm{RMod}_B(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$. Since B is Artinian, we can choose a finite sequence of maps

$$B \simeq B_0 \to B_1 \to \cdots \to B_n \simeq \kappa$$

and pullback diagrams

$$\begin{array}{ccc} B_i & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ B_{i+1} & \longrightarrow & \kappa \oplus \Sigma^{m_i}(\kappa) \end{array}$$

for some integers $m_i > 0$. We prove by descending induction on i that $E_B \otimes_B B_i$ belongs to $\mathrm{RMod}_{B_i}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$. In this case $i = n$, this follows from our assumption that $E \in \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. If $i < n$, it follows from the inductive hypothesis since we have a fiber sequence

$$\Sigma^{m_i - 1}(E) \to E_B \otimes_B B_i \to E_B \otimes_B B_{i+1}$$

in $\mathrm{Sp}(\mathcal{C})$.

Proposition 16.2.2.1 implies that if

$$\begin{array}{ccc} B & \longrightarrow & B_0 \\ \downarrow & & \downarrow f \\ B_1 & \xrightarrow{g} & B_{01} \end{array}$$

is a pullback diagram in $\mathrm{Alg}_\kappa^{\mathrm{art}}$ where the maps f and g induce surjections $\pi_0 B_0 \to \pi_0 B_{01} \leftarrow \pi_0 B_1$, then the functor

$$\mathrm{RMod}_B(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0} \to \mathrm{RMod}_{B_0}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0} \times_{\mathrm{RMod}_{B_{01}}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}} \mathrm{RMod}_{B_1}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$$

is an equivalence of ∞-categories. It follows immediately that

$$\begin{array}{ccc} \mathrm{ObjDef}_E(B) & \longrightarrow & \mathrm{ObjDef}_E(B_0) \\ \downarrow & & \downarrow \\ \mathrm{ObjDef}_E(B_1) & \longrightarrow & \mathrm{ObjDef}_E(B_{01}) \end{array}$$

is a pullback diagram in $\mathcal{S}$. □

Corollary 16.5.7.2. *Let κ be a field, let $\mathcal{C}$ be separated prestable κ-linear ∞-category, and let $E \in \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Then the natural transformation $\beta : \mathrm{ObjDef}_E \to \Psi(\kappa \oplus \mathrm{End}(E))$ of Construction 16.5.5.1 is an equivalence. In other words, $\mathrm{ObjDef}_E : \mathrm{Alg}_\kappa^{\mathrm{art}} \to \mathcal{S}$ is the formal $\mathbb{E}_1$-moduli problem which corresponds, under the equivalence of Theorem 14.0.0.5, to the augmented $\mathbb{E}_1$-algebra $\kappa \oplus \mathrm{End}(M)$.*

Proof. Combine Theorem 16.5.4.1, Corollary 16.5.3.2, and Theorem 16.4.2.1. □

Remark 16.5.7.3. Let κ be a field. For $B \in \mathrm{Alg}_\kappa^{\mathrm{art}}$, let $\mathrm{RMod}_B^!$ denote the ∞-category of Ind-coherent right B-modules over B (see §14.5). One can show that $\mathrm{RMod}_B^!$ has the structure of a κ-linear ∞-category (depending functorially on B). For any κ-linear ∞-category $\mathcal{C}$, let $\mathrm{RMod}^!_B(\mathcal{C})$ denote the relative tensor product $\mathcal{C} \otimes_{\mathrm{Mod}_\kappa} \mathrm{RMod}^!_B$. The equivalence $\mathrm{RMod}^!_B \simeq \mathrm{LMod}_{\mathfrak{D}^{(1)}(B)}$ of Proposition 14.6.1.1 is κ-linear, and (combined with Theorem HA.4.8.4.6) determines an equivalence $\mathrm{RMod}^!_B(\mathcal{C}) \simeq \mathrm{LMod}_{\mathfrak{D}^{(1)}(B)}(\mathcal{C})$. Let $E \in \mathcal{C}$ and let $X : \mathrm{Alg}^{\mathrm{art}}_\kappa \to \mathcal{S}$ be the formal $\mathbb{E}_1$-moduli problem associated to $\kappa \oplus \mathrm{End}(E)$. Then X is given by the formula

$$\begin{aligned} X(B) &= \mathrm{Map}_{\mathrm{Alg}^{\mathrm{aug}}_\kappa}(\mathfrak{D}^{(1)}(B), \kappa \oplus \mathrm{End}(E)) \\ &\simeq \mathrm{Map}_{\mathrm{Alg}_\kappa}(\mathfrak{D}^{(1)}(B), \mathrm{End}(E)) \\ &\simeq \mathrm{LMod}_{\mathfrak{D}^{(1)}(B)}(\mathcal{C}) \times_{\mathcal{C}} \{E\} \\ &\simeq \mathrm{RMod}^!_B(\mathcal{C}) \times_{\mathcal{C}} \{E\}. \end{aligned}$$

In other words, the formal $\mathbb{E}_1$-moduli problem X assigns to each $B \in \mathrm{Alg}^{\mathrm{art}}_\kappa$ the classifying space for pairs (E_B, μ), where $E_B \in \mathrm{RMod}^!_B(\mathcal{C})$ and μ is an equivalence of E with the image of E_B in the ∞-category $\mathcal{C}$. The (-1)-truncated map $\mathrm{ObjDef}_E(B) \hookrightarrow \mathrm{ObjDef}^\wedge_E(B) \simeq X(B)$ is induced by fully faithful embedding $\mathrm{RMod}_B(\mathcal{C}) \hookrightarrow \mathrm{RMod}^!_B(\mathcal{C})$, which are in turn determined by the fully faithful embeddings $\mathrm{RMod}_B \hookrightarrow \mathrm{RMod}^!_B$ of Proposition 14.5.3.2. From this point of view, we can view Proposition 16.5.7.1 as a generalization of Proposition 14.5.4.3: it asserts that the fully faithful embedding $\mathrm{RMod}_B(\mathcal{C}) \hookrightarrow \mathrm{RMod}^!_B(\mathcal{C})$ induces an equivalence on connective objects (where we declare an object of $\mathrm{RMod}^!_B(\mathcal{C})$ to be connective if its image in $\mathcal{C}$ is connective).

16.6 Deformations of Categories

Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category. In §16.5, we studied the problem of deforming a fixed object $E \in \mathcal{C}$. In this section, we will study the problem of the deforming the ∞-category $\mathcal{C}$ itself. For every Artinian $\mathbb{E}_2$-algebra B over κ, we will introduce a classifying space $\mathrm{CatDef}_{\mathcal{C}}(B)$ which parametrizes stable (right) B-linear ∞-categoriers $\mathcal{C}_B$ equipped with an equivalence $\mathcal{C} \simeq \mathcal{C}_B \otimes_{\mathrm{LMod}_B} \mathrm{Mod}_\kappa$. We will show that, modulo size issues, the construction $B \mapsto \mathrm{CatDef}_{\mathcal{C}}(B)$ is a 2-proximate formal $\mathbb{E}_2$-moduli problem (Corollary 16.6.2.4; in good cases, we can say even more: see Theorems **??** and 16.6.10.2). Using Theorem 16.4.2.1, we deduce that there is a 0-truncated natural transformation $\mathrm{CatDef}_{\mathcal{C}} \to \mathrm{CatDef}^\wedge_{\mathcal{C}}$, where $\mathrm{CatDef}^\wedge_{\mathcal{C}}$ is a formal $\mathbb{E}_2$-moduli problem (which is uniquely determined up to equivalence: see Remark 16.4.2.3). According to Theorem 15.0.0.9, the formal moduli problem $\mathrm{CatDef}^\wedge_{\mathcal{C}}$ is given by $R \mapsto \mathrm{Map}_{\mathrm{Alg}^{(2),\mathrm{aug}}_\kappa}(\mathfrak{D}^{(2)}(R), A)$ for an essentially unique augmented $\mathbb{E}_2$-algebra A over κ. The main result of this section identifies the augmentation ideal $\mathfrak{m}_A$ (as a nonunital $\mathbb{E}_2$-algebra) with the *κ-linear center* of the ∞-category $\mathcal{C}$ (Theorem 16.6.3.8): in other words, with the chain complex of Hochschild cochains on $\mathcal{C}$.

Remark 16.6.0.1. Our presentation in this section will follow that of §16.5: most of our results are direct parallels of analogous (but easier) statements about the deformation theory of an object E of a fixed κ-linear ∞-category $\mathcal{C}$.

16.6.1 Conventions on Deformations

Let R be an $\mathbb{E}_2$-ring. Then we can regard the ∞-category LMod_R of left R-modules as an associative algebra object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ of presentable stable ∞-categories. We define a *stable left R-linear ∞-category* to be a left module over LMod_R in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, and a *stable right R-linear ∞-category* to be a right module over LMod_R in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$.

Warning 16.6.1.1. Throughout the rest of this book, we use the term *stable R-linear ∞-category* to refer to a stable *left* R-linear ∞-category (see Variant D.1.5.1). However, in this section it will be important to distinguish between left and right actions of LMod_R. Of course, the difference is slight: the datum of a stable left R-linear ∞-category is equivalent to the datum of a stable right R^{rev}-linear ∞-category, where R^{rev} denotes the $\mathbb{E}_2$-ring obtained by reversing the multiplication on R (see Warning **??**). In particular, when R is an $\mathbb{E}_\infty$-ring, there is no difference between left and right R-linear ∞-categories; in this case, we will refer to either simply as a *stable R-linear ∞-category*.

Let $\phi : A \to B$ be a morphism of $\mathbb{E}_2$-rings. If $\mathcal{C}$ is a stable (left or right) B-linear ∞-category, then we can regard $\mathcal{C}$ as a stable (left or right) A-linear ∞-category by restriction of scalars along ϕ. In either case, the restriction of scalars functor admits a left adjoint, which we call *extension of scalars along ϕ*. If $\mathcal{D}$ is a stable right A-linear ∞-category, we denote its extension of scalars by $\mathcal{D} \otimes_A B = \mathcal{D} \otimes_{\mathrm{LMod}_A} \mathrm{LMod}_B$ or by $\mathrm{RMod}_B(\mathcal{D})$. If $\mathcal{D}$ is a stable left A-linear ∞-category, we denote its extension of scalars by $B \otimes_A \mathcal{D} = \mathrm{LMod}_B \otimes_{\mathrm{LMod}_A} \mathcal{D}$ or by $\mathrm{LMod}_B(\mathcal{D})$.

Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category. Suppose that B be an $\mathbb{E}_2$-algebra over κ equipped with an augmentation $\epsilon : B \to \kappa$. We define a *deformation of $\mathcal{C}$ over B* to be a pair $(\mathcal{C}_B, \mu)$, where $\mathcal{C}_B$ is a stable right B-linear ∞-category and $\mu : \mathcal{C}_B \otimes_B \kappa \to \mathcal{C}$ is an equivalence of stable κ-linear ∞-categories. Our first observation is that the collection of such deformations depends functorially on B.

Construction 16.6.1.2. Let κ be a field. We let $\mathrm{RCat}(\kappa)$ denote the ∞-category given by the fiber product $\mathrm{Alg}_\kappa^{(2)} \times_{\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{St}})} \mathrm{RMod}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$. We view the objects of $\mathrm{RCat}(\kappa)$ as pairs $(B, \mathcal{D})$, where B is an $\mathbb{E}_2$-algebra over κ and $\mathcal{D}$ is a stable right B-linear ∞-category. Projection onto the first factor furnishes a coCartesian fibration $q : \mathrm{RCat}(\kappa) \to \mathrm{Alg}_\kappa^{(2)}$. We let $\mathrm{RCat}(\kappa)^{\mathrm{coCart}}$ denote the subcategory of $\mathrm{RCat}(\kappa)$ spanned by the q-coCartesian morphisms (so a morphism from $(A, \mathcal{C})$ to $(B, \mathcal{D})$ in $\mathrm{RCat}(\kappa)^{\mathrm{coCart}}$ consists of a morphism $\phi : A \to B$ of $\mathbb{E}_2$-algebras over κ together with a B-linear equivalence $\mathcal{C} \otimes_A B \simeq \mathcal{D}$).

Let $\mathcal{C}$ be a stable κ-linear ∞-category and regard $(\kappa, \mathcal{C})$ as an object of $\mathrm{RCat}(\kappa)$. We let $\mathrm{Defor}[\mathcal{C}]$ denote the ∞-category $\mathrm{RCat}(\kappa)^{\mathrm{coCart}}_{/(\kappa,\mathcal{C})}$. We will refer to $\mathrm{Defor}[\mathcal{C}]$ as the *∞-category of deformations of $\mathcal{C}$*. This ∞-category is equipped with an evident left fibration

$$\rho : \mathrm{Defor}[\mathcal{C}] \to (\mathrm{Alg}^{(2)}_{\kappa})_{/\kappa} = \mathrm{Alg}^{(2),\mathrm{aug}}_{\kappa},$$

which is classified by a functor $\mathrm{CatDef}^{+}_{\mathcal{C}} : \mathrm{Alg}^{(2),\mathrm{aug}}_{\kappa} \to \widehat{\mathcal{S}}$. Unwinding the definitions, we see that if B is an augmented $\mathbb{E}_2$-algebra over κ, then we can identify $\mathrm{CatDef}^{+}_{\mathcal{C}}(B)$ with the underlying Kan complex of the ∞-category $\mathrm{RMod}_{\mathrm{LMod}_B}(\mathcal{P}\mathrm{r}^{\mathrm{St}}) \times_{\mathrm{Mod}_{\mathrm{Mod}_{\kappa}}(\mathcal{P}\mathrm{r}^{\mathrm{St}})} \{\mathcal{C}\}$ of deformations of $\mathcal{C}$ over B.

16.6.2 Deformations as a Formal Moduli Problem

We now analyze the behavior of Construction 16.6.1.2 with respect to fiber products of $\mathbb{E}_2$-algebras.

Proposition 16.6.2.1. *Let κ be a field, let $\mathcal{C}$ be a stable κ-linear ∞-category, and let $\mathrm{CatDef}^{+}_{\mathcal{C}} : \mathrm{Alg}^{(2),\mathrm{aug}}_{\kappa} \to \widehat{\mathcal{S}}$ be as in Construction 16.6.1.2. Then for every pullback diagram*

$$\begin{array}{ccc} B & \longrightarrow & B_0 \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_{01} \end{array}$$

in $\mathrm{Alg}^{(2),\mathrm{aug}}_{\kappa}$, the induced map

$$\theta : \mathrm{CatDef}^{+}_{\mathcal{C}}(B) \to \mathrm{CatDef}^{+}_{\mathcal{C}}(B_0) \times_{\mathrm{CatDef}^{+}_{\mathcal{C}}(B_{01})} \mathrm{CatDef}^{+}_{\mathcal{C}}(B_1)$$

is 0-truncated (that is, the homotopy fibers of θ are discrete).

Proof. If A is an $\mathbb{E}_2$-ring and $\mathcal{D}$ and $\mathcal{E}$ are stable right A-linear ∞-categories, we let $\mathrm{Fun}_A(\mathcal{D}, \mathcal{E})$ denote the ∞-category of right LMod_A-linear functors from $\mathcal{D}$ to $\mathcal{E}$.

If $\mathcal{E}$ is a stable right B-linear ∞-category, then Proposition 16.2.1.1 implies that the canonical map

$$\mathcal{E} \to (\mathcal{E} \otimes_B B_0) \times_{\mathcal{E} \otimes_B B_{01}} (\mathcal{E} \otimes_B B_1)$$

is fully faithful. Consequently, for any other stable right B-linear ∞-category $\mathcal{D}$, the induced map

$$\begin{aligned} \mathrm{Fun}_B(\mathcal{D}, \mathcal{E}) &\to \mathrm{Fun}_B(\mathcal{D}, \mathcal{E} \otimes_B B_0) \times_{\mathrm{Fun}_B(\mathcal{D}, \mathcal{E} \otimes_B B_{01})} \mathrm{Fun}_B(\mathcal{D}, \mathcal{E} \otimes_B B_1) \\ &\simeq \mathrm{Fun}_{B_0}(\mathcal{D} \otimes_B B_0, \mathcal{E} \otimes_B B_0) \times_{\mathrm{Fun}_{B_{01}}(\mathcal{D} \otimes_B B_{01}, \mathcal{E} \otimes_B B_{01})} \mathrm{Fun}_{B_1}(\mathcal{D} \otimes_B B_1, \mathcal{E} \otimes_B B_1) \end{aligned}$$

is fully faithful. Passing to the underlying Kan complexes and taking $\mathcal{D}$ and $\mathcal{E}$ to be deformations of $\mathcal{C}$ over B, we deduce that the commutative diagram

$$\begin{array}{ccc} \{\mathcal{D}\} \times_{\mathrm{CatDef}^+_{\mathcal{C}}(B)} \{\mathcal{E}\} & \longrightarrow & \{\mathcal{D}\} \times_{\mathrm{CatDef}^+_{\mathcal{C}}(B_0)} \{\mathcal{E}\} \\ \downarrow & & \downarrow \\ \{\mathcal{D}\} \times_{\mathrm{CatDef}^+_{\mathcal{C}}(B_1)} \{\mathcal{E}\} & \longrightarrow & \{\mathcal{D}\} \times_{\mathrm{CatDef}^+_{\mathcal{C}}(B_{01})} \{\mathcal{E}\} \end{array}$$

is (-1)-Cartesian (in the sense of Definition 16.4.1.1). Since this conclusion holds for every pair of deformations $\mathcal{D}$ and $\mathcal{E}$, we conclude that the diagram

$$\begin{array}{ccc} \mathrm{CatDef}^+_{\mathcal{C}}(B) & \longrightarrow & \mathrm{CatDef}^+_{\mathcal{C}}(B_0) \\ \downarrow & & \downarrow \\ \mathrm{CatDef}^+_{\mathcal{C}}(B_1) & \longrightarrow & \mathrm{CatDef}^+_{\mathcal{C}}(B_{01}) \end{array}$$

is 0-Cartesian, as desired. □

Corollary 16.6.2.2. *Let κ be a field, let $\mathcal{C}$ be a stable κ-linear ∞-category, and let $\mathrm{CatDef}^+_{\mathcal{C}} : \mathrm{Alg}^{(2),\mathrm{aug}}_{\kappa} \to \widehat{\mathcal{S}}$ be as in Construction 16.6.1.2. Then:*

(1) *The space $\mathrm{CatDef}^+_{\mathcal{C}}(\kappa)$ is contractible.*

(2) *Let $V \in \mathrm{Mod}_{\kappa}$. Then $\mathrm{CatDef}^+_{\mathcal{C}}(\kappa \oplus V)$ is locally small, when regarded as an ∞-category. In other words, each connected component of $\mathrm{CatDef}^+_{\mathcal{C}}(\kappa \oplus V)$ is essentially small.*

(3) *Let $B \in \mathrm{Alg}^{(2),\mathrm{aug}}_{\kappa}$ be Artinian. Then the space $\mathrm{CatDef}^+_{\mathcal{C}}(B)$ is locally small (that is, each connected component of $\mathrm{CatDef}^+_{\mathcal{C}}(B)$ is essentially small).*

Proof. Assertion (1) is immediate from the definitions. To prove (2), we note that for each $B \in \mathrm{Alg}^{(2),\mathrm{aug}}_{\kappa}$ and every point $\eta \in \mathrm{CatDef}^+_{\mathcal{C}}(B)$ corresponding to a pair $(\mathcal{C}_B, \mu)$, the space $\Omega^2(\mathrm{CatDef}^+_{\mathcal{C}}(B), \eta)$ can be identified with the homotopy fiber of the restriction map

$$\mathrm{Map}_{\mathrm{Fun}_B(\mathcal{C}_B, \mathcal{C}_B)}(\mathrm{id}, \mathrm{id}) \to \mathrm{Map}_{\mathrm{Fun}_{\kappa}(\mathcal{C}, \mathcal{C})}(\mathrm{id}, \mathrm{id})$$

and is therefore essentially small. We have pullback diagrams

$$\begin{array}{ccc} \kappa \oplus V & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ \kappa & \longrightarrow & \kappa \oplus \Sigma(V) \end{array} \qquad \begin{array}{ccc} \kappa \oplus \Sigma(V) & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ \kappa & \longrightarrow & \kappa \oplus \Sigma^2(V) \end{array}$$

so that Proposition 16.6.2.1 guarantees that the map $\mathrm{CatDef}^+_{\mathcal{C}}(\kappa \oplus V) \to \Omega^2 \mathrm{CatDef}^+_{\mathcal{C}}(\kappa \oplus \Sigma^2(V))$ has discrete homotopy fibers. It follows that each path component of $\mathrm{CatDef}^+_{\mathcal{C}}(\kappa \oplus V)$

is a connected covering space of the essentially small space $\Omega^2 \operatorname{CatDef}^+_{\mathcal{C}}(\kappa \oplus \Sigma^2(V))$, and is therefore essentially small.

We now prove (3). Assume that B is Artinian, so that there exists a finite sequence of maps $B \simeq B_0 \to B_1 \to \cdots \to B_n \simeq \kappa$ and pullback diagrams

$$\begin{array}{ccc} B_i & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ B_{i+1} & \longrightarrow & \kappa \oplus \Sigma^{m_i}(\kappa). \end{array}$$

We prove that each $\operatorname{CatDef}^+_{\mathcal{C}}(B_i)$ is locally small using descending induction on i. Using (1), (2), and the inductive hypothesis, we deduce that $X = \operatorname{CatDef}^+_{\mathcal{C}}(\kappa) \times_{\operatorname{CatDef}^+_{\mathcal{C}}(\kappa \oplus \Sigma^{m_i}(\kappa))} \operatorname{CatDef}^+_{\mathcal{C}}(B_{i+1})$ is locally small. Proposition 16.6.2.1 implies that the map $\operatorname{CatDef}^+_{\mathcal{C}}(B_{i+1}) \to X$ has discrete homotopy fibers. It follows that every connected component of $\operatorname{CatDef}^+_{\mathcal{C}}(B_{i+1})$ is a connected covering space of a path component of X, and is therefore essentially small. □

Notation 16.6.2.3. Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category. We let $\operatorname{CatDef}_{\mathcal{C}} : \operatorname{Alg}^{(2),\mathrm{art}}_{\kappa} \to \widehat{\mathcal{S}}$ denote the composition of the functor $\operatorname{CatDef}^+_{\mathcal{C}}$ of Construction 16.6.1.2 with the inclusion $\operatorname{Alg}^{(2),\mathrm{art}}_{\kappa} \hookrightarrow \operatorname{Alg}^{(2),\mathrm{aug}}_{\kappa}$.

Corollary 16.6.2.4. *Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category. Then there exists a formal $\mathbb{E}_2$-moduli problem $\operatorname{CatDef}^{\wedge}_{\mathcal{C}} : \operatorname{Alg}^{(2),\mathrm{art}}_{\kappa} \to \mathcal{S}$ and a natural transformation $\alpha : \operatorname{CatDef}_{\mathcal{C}} \to \operatorname{CatDef}^{\wedge}_{\mathcal{C}}$ which is 0-truncated. In particular, we can regard $\operatorname{CatDef}_{\mathcal{C}} : \operatorname{Alg}^{(2),\mathrm{art}}_{\kappa} \to \widehat{\mathcal{S}}$ as a 2-proximate formal $\mathbb{E}_2$-moduli problem (in a larger universe); see Theorem 16.4.2.1.*

Proof. Combining Corollary 16.6.2.2, Proposition 16.6.2.1, and Theorem 16.4.2.1, we deduce the existence of a formal moduli problem $\operatorname{CatDef}^{\wedge}_{\mathcal{C}} : \operatorname{Alg}^{(2),\mathrm{art}}_{\kappa} \to \widehat{\mathcal{S}}$ and a 0-truncated natural transformation $\alpha : \operatorname{CatDef}_{\mathcal{C}} \to \operatorname{CatDef}^{\wedge}_{\mathcal{C}}$. For each $m \geqslant 0$, we see that the space

$$\operatorname{CatDef}^{\wedge}_{\mathcal{C}}(\kappa \oplus \Sigma^m(\kappa)) \simeq \Omega^2 \operatorname{CatDef}^{\wedge}_{\mathcal{C}}(\kappa \oplus \Sigma^{m+2}(\kappa)) \simeq \Omega^2 \operatorname{CatDef}_{\mathcal{C}}(\kappa \oplus \Sigma^{m+2}(\kappa))$$

is essentially small (see the proof of Corollary 16.6.2.2). For an arbitrary object $B \in \operatorname{Alg}^{(2),\mathrm{art}}_{\kappa}$, we can choose a finite sequence of maps $B = B_0 \to B_1 \to \cdots \to B_n \simeq \kappa$ and pullback diagrams

$$\begin{array}{ccc} B_i & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ B_{i+1} & \longrightarrow & \kappa \oplus \Sigma^{m_i}(\kappa). \end{array}$$

Using that $\operatorname{CatDef}^{\wedge}_{\mathcal{C}}$ is a formal $\mathbb{E}_2$-moduli problem, we deduce that each $\operatorname{CatDef}^{\wedge}_{\mathcal{C}}(B_i)$ is essentially small by descending induction on i, so that $\operatorname{CatDef}^{\wedge}_{\mathcal{C}}(B)$ is essentially small. □

Remark 16.6.2.5. The formal $\mathbb{E}_2$-moduli problem $\operatorname{CatDef}^{\wedge}_{\mathcal{C}}$ appearing in Corollary 16.6.2.4 is unique (up to a contractible space of choices): see Remark 16.4.2.3.

16.6.3 The Main Theorem

Let κ be a field, let $\mathcal{C}$ be a stable κ-linear ∞-category, and let $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}$ be the formal $\mathbb{E}_2$-moduli problem of Corollary 16.6.2.4. Using Theorem 15.0.0.9 (and its proof), we see that there exists an (essentially unique) augmented $\mathbb{E}_2$-algebra A over κ for which the functor $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}$ is given by $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}(B) = \mathrm{Map}_{\mathrm{Alg}^{(2),\mathrm{aug}}_{\kappa}}(\mathfrak{D}^{(2)}(B), A)$. Our next goal is to make the relationship between A and $\mathcal{C}$ more explicit.

Definition 16.6.3.1. Let κ be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a stable κ-linear ∞-category. We let $\mathrm{LCat}(\kappa)$ denote the fiber product $\mathrm{Alg}^{(2)}_{\kappa} \times_{\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{St}})} \mathrm{LMod}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$ whose objects are pairs $(A, \mathcal{D})$, where A is an $\mathbb{E}_2$-algebra over κ and $\mathcal{D}$ is a stable left A-linear ∞-category (see Construction 16.6.1.2). The construction $(A, \mathcal{D}) \mapsto \mathcal{D}$ determines a forgetful functor $\mathrm{LCat}(\kappa) \to \mathrm{LinCat}^{\mathrm{St}}_{\kappa}$ (here $\mathrm{LinCat}^{\mathrm{St}}_{\kappa}$ denotes the ∞-category of stable κ-linear ∞-categories). We let $\mathrm{LCat}(\kappa)_{\mathcal{C}}$ denote the fiber product $\mathrm{LCat}(\kappa) \times_{\mathrm{LinCat}_{\kappa}} \{\mathcal{C}\}$: that is, the ∞-category whose objects are $\mathbb{E}_2$-algebras A together with an action of A on $\mathcal{C}$ (in the sense of §D.1). We will say that an object $(A, \mathcal{C}) \in \mathrm{LCat}(\kappa)_{\mathcal{C}}$ *exhibits A as the κ-linear center of $\mathcal{C}$* if $(A, \mathcal{C})$ is a final object of $\mathrm{LCat}(\kappa)_{\mathcal{C}}$.

Remark 16.6.3.2. In the situation of Definition 16.6.3.1 Corollary HA.4.7.1.42 implies that the forgetful functor $\mathrm{LMod}(\mathrm{LinCat}_{\kappa}) \times_{\mathrm{LinCat}_{\kappa}} \{\mathcal{C}\} \to \mathrm{Alg}(\mathrm{LinCat}_{\kappa})$ is a right fibration. It follows that the forgetful functor $q : \mathrm{LCat}(\kappa)_{\mathcal{C}} \to \mathrm{Alg}^{(2)}_{\kappa}$ is also a right fibration, so that an object $(A, \mathcal{C}) \in \mathrm{LCat}(\kappa)_{\mathcal{C}}$ is final if and only if the right fibration q is represented by the object $A \in \mathrm{Alg}^{(2)}_{\kappa}$. In other words, a κ-linear center A of $\mathcal{C}$ can be characterized by the following universal property: for every $A' \in \mathrm{Alg}^{(2)}_{\kappa}$, the space $\mathrm{Map}_{\mathrm{Alg}^{(2)}_{\kappa}}(A', A)$ can be identified with the space $\mathrm{LinCat}_{A'} \times_{\mathrm{LinCat}_{\kappa}} \{\mathcal{C}\}$ parametrizing actions of A' on $\mathcal{C}$.

Proposition 16.6.3.3. *Let κ be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a stable κ-linear ∞-category. Then there exists an object $(A, \mathcal{C}) \in \mathrm{LCat}(\kappa)_{\mathcal{C}}$ which exhibits A as a κ-linear center of $\mathcal{C}$.*

Proof. Let $\mathcal{E}$ be an endomorphism object of $\mathcal{C}$ in LinCat_{κ}: that is, $\mathcal{E}$ is the ∞-category of κ-linear functors from $\mathcal{C}$ to itself. We regard $\mathcal{E}$ as a monoidal ∞-category, so that $\mathcal{C}$ is a left $\mathcal{E}$-module object of LinCat_{κ}. According to Theorem HA.4.8.5.11 , the symmetric monoidal functor

$$\mathrm{Alg}_{\kappa} \to (\mathrm{LinCat}_{\kappa})_{\mathrm{Mod}_{\kappa}/} \qquad A \mapsto \mathrm{LMod}_A$$

admits a right adjoint G. It follows that G induces a right adjoint G' to the functor

$$\mathrm{Alg}^{(2)}_{\kappa} \simeq \mathrm{Alg}(\mathrm{Alg}_{\kappa}) \to \mathrm{Alg}((\mathrm{LinCat}_{\kappa})_{\mathrm{Mod}_{\kappa}/}) \simeq \mathrm{Alg}(\mathrm{LinCat}_{\kappa}).$$

Unwinding the definitions, we see that $A = G'(\mathcal{E})$ is a κ-linear center of $\mathcal{C}$. □

Remark 16.6.3.4. Let κ be an $\mathbb{E}_\infty$-ring and $\mathcal{C}$ a κ-linear ∞-category The proofs of Proposition 16.6.3.3 and Theorem HA.4.8.5.11 furnish a somewhat explicit description of the κ-linear center A of $\mathcal{C}$, at least as an $\mathbb{E}_1$-algebra over κ: it can be described as the endomorphism ring of the identity functor $\mathrm{id}_{\mathcal{C}} \in \mathcal{E}$, where $\mathcal{E}$ denotes the ∞-category of κ-linear functors from $\mathcal{C}$ to itself.

Example 16.6.3.5. Let κ be the sphere spectrum and let $\mathcal{C}$ be a presentable stable ∞-category, which we view as a stable κ-linear ∞-category. Then the center of $\mathcal{C}$ (in the sense of Construction D.1.5.4) coincides with the κ-linear center of $\mathcal{C}$.

Example 16.6.3.6. Let κ be an $\mathbb{E}_\infty$-ring, let $R \in \mathrm{Alg}_\kappa$ be an $\mathbb{E}_1$-algebra over κ, and let $\mathfrak{Z}(R) = \mathfrak{Z}_{\mathbb{E}_1}(R) \in \mathrm{Alg}_\kappa^{(2)}$ be a center of R (see Definition HA.5.3.1.12). Then $\mathfrak{Z}(R)$ is a κ-linear center of the ∞-category $\mathrm{LMod}_R(\mathrm{Mod}_\kappa)$.

Remark 16.6.3.7. Let κ be an $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be a stable κ-linear ∞-category, and let $A \in \mathrm{Alg}_\kappa^{(2)}$ denote the κ-linear center of $\mathcal{C}$. The homotopy groups $\pi_n A$ are often called the *Hochschild cohomology groups* of $\mathcal{C}$. In the special case where $\mathcal{C} = \mathrm{LMod}_R(\mathrm{Mod}_\kappa)$ for some $R \in \mathrm{Alg}_\kappa$, Example 16.6.3.6 allows us to identify the homotopy groups $\pi_* A$ with $\mathrm{Ext}^{-*}_{{}_R\mathrm{BMod}_R(\mathrm{Mod}_\kappa)}(R, R)$.

We are now ready to formulate the main result of this section.

Theorem 16.6.3.8. *Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category. Then the functor $\mathrm{CatDef}^\wedge_{\mathcal{C}} : \mathrm{Alg}_\kappa^{(2),\mathrm{art}} \to \mathcal{S}$ of Corollary 16.5.3.2 is given by*

$$\mathrm{ObjDef}^\wedge_{\mathcal{C}}(B) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{(2),\mathrm{aug}}}(\mathfrak{D}^{(2)}(B), \kappa \oplus \mathfrak{Z}(\mathcal{C})) \simeq \mathrm{Map}_{\mathrm{Alg}_\kappa^{(2)}}(\mathfrak{D}^{(2)}(B), \mathfrak{Z}(\mathcal{C})).$$

where $\mathfrak{Z}(\mathcal{C})$ denotes the κ-linear center of $\mathcal{C}$.

16.6.4 Construction of the Equivalence

Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category. Theorem 16.6.3.8 predicts the existence of a 0-truncated map $\mathrm{ObjDef}_{\mathcal{C}}(B) \to \mathrm{Map}_{\mathrm{Alg}_\kappa^{(2)}}(\mathfrak{D}^{(2)}(B), \mathfrak{Z}(\mathcal{C}))$, depending functorially on $B \in \mathrm{Alg}_\kappa^{(2),\mathrm{art}}$. Our proof begins with an explicit construction of this map.

Construction 16.6.4.1. Let κ be a field, and let $\lambda^{(2)} : \mathcal{M}^{(2)} \to \mathrm{Alg}_\kappa^{(2),\mathrm{aug}} \times \mathrm{Alg}_\kappa^{(2),\mathrm{aug}}$ be the Koszul duality pairing on $\mathbb{E}_2$-algebras (so that the objects of $\mathcal{M}^{(2)}$ can be identified with triples (A, B, ϵ), where A and B are $\mathbb{E}_2$-algebras over κ and $\epsilon : A \otimes_\kappa B \to \kappa$ is an augmentation).

Let $\mathcal{C}$ be a stable κ-linear ∞-category and let $\mathrm{Defor}[\mathcal{C}]$ be as in Construction 16.6.1.2. Given an object $(A, B, \epsilon) \in \mathcal{M}^{(2)}$ and an object $(B, \mathcal{C}_B, \mu) \in \mathrm{Defor}[\mathcal{C}]$, we regard the tensor product $A \otimes_\kappa \mathcal{C}_B$ as an object of the ∞-category

$${}_{\mathrm{LMod}_A}\mathrm{BMod}_{\mathrm{LMod}_{A \otimes_\kappa B}}(\mathrm{LinCat}_\kappa),$$

so that we can regard $(A \otimes_\kappa \mathcal{C}_B) \otimes_{A \otimes_\kappa B} \kappa$ as a stable left A-linear ∞-category whose image in LinCat_κ coincides with $\mathcal{C}$. This construction determines a functor $\mathcal{M}^{(2)} \times_{\mathrm{Alg}_\kappa^{(2),\mathrm{aug}}} \mathrm{Defor}[\mathcal{C}] \to \mathrm{LCat}(\kappa)_{\mathcal{C}}$, where $\mathrm{LCat}(\kappa)_{\mathcal{C}}$ as in Definition 16.6.3.1. The induced map

$$\lambda : \mathcal{M}^{(2)} \times_{\mathrm{Alg}_\kappa^{(2),\mathrm{aug}}} \mathrm{Defor}[\mathcal{C}] \to \mathrm{LCat}(\kappa)_{\mathcal{C}} \times \mathrm{Defor}[\mathcal{C}]$$

factors as a composition

$$\mathcal{M}^{(2)} \times_{\mathrm{Alg}_\kappa^{(2),\mathrm{aug}}} \mathrm{Defor}[\mathcal{C}] \xrightarrow{i} \widetilde{\mathcal{M}}^{(2)} \xrightarrow{\lambda'} \mathrm{LCat}(\kappa)_{\mathcal{C}} \times \mathrm{Defor}[\mathcal{C}]$$

where i is an equivalence of ∞-categories and λ' is a categorical fibration. It is not difficult to see that λ' is a left representable pairing of ∞-categories, which induces a duality functor $\mathfrak{D}_{\mathcal{C}}^{(2)} : \mathrm{Defor}[\mathcal{C}]^{\mathrm{op}} \to \mathrm{LCat}(\kappa)_{\mathcal{C}}$. Concretely, the functor $\mathfrak{D}_{\mathcal{C}}^{(2)}$ assigns to each object $(B, \mathcal{C}_B, \mu) \in \mathrm{Defor}[\mathcal{C}]^{\mathrm{op}}$ the object $(\mathfrak{D}^{(2)}(B), \mathcal{C})$, where we regard $\mathcal{C}$ as a stable left $\mathfrak{D}^{(2)}(B)$-linear ∞-category by means of the canonical equivalence $\mathcal{C} \simeq (A \otimes_\kappa \mathcal{C}_B) \otimes_{A \otimes_\kappa B} \kappa$.

Let $\mathfrak{Z}(\mathcal{C})$ denote the κ-linear center of $\mathcal{C}$ (Definition 16.6.3.1), so that we have a canonical equivalence of ∞-categories $\eta : \mathrm{LCat}(\kappa)_{\mathcal{C}} \simeq (\mathrm{Alg}_\kappa^{(2)})_{/\mathfrak{Z}(\mathcal{C})} \simeq (\mathrm{Alg}_\kappa^{(2),\mathrm{aug}})_{/\kappa \oplus \mathfrak{Z}(\mathcal{C})}$. Composing the equivalence η with the functor $\mathfrak{D}_{\mathcal{C}}^{(2)}$, we obtain a diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Defor}[\mathcal{C}]^{\mathrm{op}} & \xrightarrow{\eta \circ \mathfrak{D}_{\mathcal{C}}^{(2)}} & (\mathrm{Alg}_\kappa^{(2),\mathrm{aug}})_{/\kappa \oplus \mathfrak{Z}(\mathcal{C})} \\ \downarrow & & \downarrow \\ (\mathrm{Alg}_\kappa^{(2),\mathrm{aug}})^{\mathrm{op}} & \xrightarrow{\mathfrak{D}^{(2)}} & \mathrm{Alg}_\kappa^{(2),\mathrm{aug}} \end{array}$$

which commutes up to canonical homotopy, where the vertical maps are right fibrations. This diagram determines a natural transformation $\beta^+ : \mathrm{CatDef}_{\mathcal{C}}^+ \to X^+$, where $X^+ : \mathrm{Alg}_\kappa^{(2),\mathrm{aug}} \to \mathcal{S}$ denotes the functor given by the formula

$$X^+(B) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{(2),\mathrm{aug}}}(\mathfrak{D}^{(2)}(B), \kappa \oplus \mathfrak{Z}(\mathcal{C})) \simeq \mathrm{Map}_{\mathrm{Alg}_\kappa^{(2)}}(\mathfrak{D}^{(2)}(B), \mathfrak{Z}(\mathcal{C})).$$

16.6.5 The Proof of Theorem 16.6.3.8

By virtue of Remark 16.4.2.3, Theorem 16.6.3.8 is a formal consequence of the following more precise assertion:

Proposition 16.6.5.1. *Let κ be a field, let $\mathcal{C}$ be a stable κ-linear ∞-category, and let $\mathfrak{Z}(\mathcal{C}) \in \mathrm{Alg}_\kappa^{(2)}$ denote a κ-linear center of $\mathcal{C}$. Let $X : \mathrm{Alg}_\kappa^{(2),\mathrm{aug}} \to \mathcal{S}$ denote the functor given by the formula $X(B) = \mathrm{Map}_{\mathrm{Alg}_\kappa^{(2)}}(\mathfrak{D}^{(2)}(B), \mathfrak{Z}(\mathcal{C}))$. Then the natural transformation β^+ of Construction 16.6.4.1 induces a 0-truncated map $\beta : \mathrm{CatDef}_{\mathcal{C}} \to X$.*

Since the functor X appearing in Proposition 16.6.5.1 is a formal $\mathbb{E}_2$-moduli problem, the natural transformation $\beta : \mathrm{CatDef}_{\mathcal{C}} \to X$ factors as a composition $\mathrm{CatDef}_{\mathcal{C}} \to \mathrm{CatDef}^{\wedge}_{\mathcal{C}} \xrightarrow{\overline{\beta}} X$. We wish to show that $\overline{\beta}$ is an equivalence of formal $\mathbb{E}_2$-moduli problems (which implies Proposition 16.6.5.1, by virtue of Theorem 16.4.2.1). According to Proposition 12.2.2.6, it will suffice to show that $\overline{\beta}$ induces an equivalence of tangent complexes. Using the description of the tangent complex of $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}$ supplied by Proposition 16.4.3.2, we are reduced to proving the following special case of Proposition 16.6.5.1:

Proposition 16.6.5.2. *Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category. For each $m \geqslant 0$, the natural transformation β^+ of Construction 16.6.4.1 induces a 0-truncated map $\mathrm{CatDef}_{\mathcal{C}}(\kappa \oplus \Sigma^m(\kappa)) \to \mathrm{Map}_{\mathrm{Alg}^{(2)}_\kappa}(\mathfrak{D}^{(2)}(\kappa \oplus \Sigma^m(\kappa)), \mathfrak{Z}(\mathcal{C}))$.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{CatDef}_{\mathcal{C}}(\kappa \oplus \Sigma^m(\kappa)) & \longrightarrow & \mathrm{Map}_{\mathrm{Alg}^{(2)}_\kappa}(\mathfrak{D}^{(2)}(\kappa \oplus \Sigma^m(\kappa)), \mathfrak{Z}(\mathcal{C})) \\ \downarrow & & \downarrow \\ \Omega^2 \mathrm{CatDef}_{\mathcal{C}}(\kappa \oplus \Sigma^{m+2}(\kappa)) & \xrightarrow{\theta} & \Omega^2 \mathrm{Map}_{\mathrm{Alg}^{(2)}_\kappa}(\mathfrak{D}^{(2)}(\kappa \oplus \Sigma^{m+2}(\kappa)), \mathfrak{Z}(\mathcal{C})), \end{array}$$

where the left vertical map is 0-truncated by Corollary 16.6.2.4 and the right vertical map is a homotopy equivalence. It will therefore suffice to show that θ is a homotopy equivalence. Set $B = \kappa \oplus \Sigma^{m+2}(\kappa)$, let $\mathcal{C}_B = \mathcal{C} \otimes_\kappa B$, let $\mathcal{E}$ denote the ∞-category of κ-linear functors from $\mathcal{C}$ to itself, and let $\mathcal{E}_B$ denote the ∞-category of B-linear functors from $\mathcal{C}_B$ to itself, so that we have a canonical equivalence $\gamma : \mathcal{E}_B \simeq \mathrm{RMod}_B(\mathcal{E})$. Let $\mathrm{id}_{\mathcal{C}} \in \mathcal{E}$ denote the identity functor from $\mathcal{C}$ to itself. Under the equivalence γ, the identity functor from $\mathcal{C}_B$ to itself can be identified with the free module $\mathrm{id}_{\mathcal{C}} \otimes_\kappa B \in \mathrm{RMod}_B(\mathcal{E})$. Unwinding the definitions, we see that the domain of θ can be identified with the homotopy fiber of the map

$$\xi : \mathrm{Map}_{\mathcal{E}_B}(\mathrm{id}_{\mathcal{C}} \otimes_\kappa B, \mathrm{id}_{\mathcal{C}} \otimes_\kappa B) \simeq \mathrm{Map}_{\mathcal{E}}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}} \otimes_\kappa B) \to \mathrm{Map}_{\mathcal{E}}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}).$$

We have a canonical fiber sequence $\Sigma^{m+2}\,\mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}} \otimes_\kappa B \to \mathrm{id}_{\mathcal{C}}$ in the stable ∞-category $\mathcal{E}$, so that the homotopy fiber of ξ is given by

$$\mathrm{Map}_{\mathcal{E}}(\mathrm{id}_{\mathcal{C}}, \Sigma^{m+2}\,\mathrm{id}_{\mathcal{C}}) \simeq \mathrm{Map}_{\mathrm{Mod}_\kappa}(\Sigma^{-m-2}(\kappa), \mathfrak{Z}(\mathcal{C})).$$

The map θ is induced by a morphism $\nu : \Sigma^{-m-2}(\kappa) \to \mathfrak{D}^{(2)}(\kappa \oplus \Sigma^m(\kappa))$ in Mod_κ. Let $\mathrm{Free}^{(2)} : \mathrm{Mod}_\kappa \to \mathrm{Alg}^{(2)}_\kappa$ be a left adjoint to the forgetful functor, so that ν determines an augmentation $(\kappa \oplus \Sigma^m(\kappa)) \otimes_\kappa \mathrm{Free}^{(2)}(\Sigma^{-m-2}(\kappa)) \to \kappa$. The proof of Proposition 15.3.2.1 shows that this pairing exhibits $\mathrm{Free}^{(2)}(\Sigma^{-m-2}(\kappa))$ as the Koszul dual of $\kappa \oplus \Sigma^m(\kappa)$, from which it immediately follows that θ is a homotopy equivalence. □

16.6.6 Compactly Generated Deformations

Let κ be a field. To any stable κ-linear ∞-category $\mathcal{C}$, we can associate the 2-proximate formal $\mathbb{E}_2$-moduli problem $\mathrm{CatDef}_{\mathcal{C}} : \mathrm{Alg}_{\kappa}^{(2),\mathrm{art}} \to \widehat{\mathcal{S}}$ of Notation 16.6.2.3. By definition, $\mathrm{CatDef}_{\mathcal{C}}(B)$ parametrizes arbitrary deformations of $\mathcal{C}$ over B. In practice, we might not want to consider *all* deformations of $\mathcal{C}$: for example, if $\mathcal{C}$ is compactly generated, then we might want to restrict our attention to deformations which are compactly generated.

Notation 16.6.6.1. Let κ be a field and let $\mathcal{C}$ be a compactly generated stable κ-linear ∞-category. For every object $B \in \mathrm{Alg}_{\kappa}^{(2),\mathrm{art}}$, we let $\mathrm{CatDef}^c_{\mathcal{C}}(B)$ denote the summand of $\mathrm{CatDef}_{\mathcal{C}}(B)$ given by deformations $(\mathcal{C}_B, \mu : \mathcal{C}_B \otimes_B \kappa \simeq \mathcal{C})$ for which the ∞-category $\mathcal{C}_B$ is also compactly generated. It follows from Propositions C.6.2.1 and C.6.2.2 that this condition is compatible with extension of scalars, so that the construction $B \mapsto \mathrm{CatDef}^c_{\mathcal{C}}(B)$ determines a functor $\mathrm{CatDef}^c_{\mathcal{C}} : \mathrm{Alg}_{\kappa}^{(2),\mathrm{art}} \to \widehat{\mathcal{S}}$.

Proposition 16.6.6.2. *Let κ be a field and let $\mathcal{C}$ be a compactly generated stable κ-linear ∞-category. For every object $B \in \mathrm{Alg}_{\kappa}^{(2),\mathrm{art}}$, the space $\mathrm{CatDef}^c_{\mathcal{C}}(B)$ is essentially small. Consequently, we can view $\mathrm{CatDef}^c_{\mathcal{C}}$ as a $\mathcal{S}$-valued functor on $\mathrm{Alg}_{\kappa}^{(2),\mathrm{art}}$.*

Proof. Since B is Artinian, we can choose a finite sequence of maps

$$B \simeq B_0 \to B_1 \to \cdots \to B_n \simeq \kappa$$

and pullback diagrams

$$\begin{array}{ccc} B_i & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ B_{i+1} & \longrightarrow & \kappa \oplus \Sigma^{m_i}(\kappa) \end{array}$$

for some integers $m_i > 0$. We will show that each of the spaces $\mathrm{CatDef}^c_{\mathcal{C}}(B_i)$ is essentially small using descending induction on i. For $i < n$, Proposition 16.6.2.1 implies that the canonical map

$$\theta : \mathrm{CatDef}_{\mathcal{C}}(B_i) \to \mathrm{CatDef}^c_{\mathcal{C}}(\kappa) \times_{\mathrm{CatDef}^c_{\mathcal{C}}(\kappa \oplus \Sigma^{m_i}(\kappa))} \mathrm{CatDef}^c_{\mathcal{C}}(B_{i+1})$$

has discrete homotopy fibers, and the codomain of θ is essentially small by our inductive hypothesis (note that $\mathrm{CatDef}^c_{\mathcal{C}}(\kappa \oplus \Sigma^{m_i}(\kappa))$ is locally small by virtue of Corollary 16.6.2.2). It will therefore suffice to show that the homotopy fibers of θ are essentially small. Let η be a point of the codomain of θ, given by a compatible triple of deformations

$$\mathcal{C}_{B_{i+1}} \in \mathrm{CatDef}^c_{\mathcal{C}}(B_{i+1}) \qquad \mathcal{C}_{\kappa \oplus \Sigma^{m_i}(\kappa)} \in \mathrm{CatDef}^c_{\mathcal{C}}(\kappa \oplus \Sigma^{m_i}(\kappa)) \qquad \mathcal{C}_{\kappa} \in \mathrm{CatDef}^c_{\mathcal{C}}(\kappa).$$

By assumption, these ∞-categories are generated by their full subcategories of compact objects

$$\mathcal{C}^c_{B_{i+1}} \subseteq \mathcal{C}_{B_{i+1}} \qquad \mathcal{C}^c_{\kappa \oplus \Sigma^{m_i}(\kappa)} \subseteq \mathcal{C}_{\kappa \oplus \Sigma^{m_i}(\kappa)} \qquad \mathcal{C}^c_{\kappa} \subseteq \mathcal{C}_{\kappa}\,.$$

To show that the homotopy fiber of θ over the point η is essentially small, it will suffice to show there is only a bounded number of equivalence classes of full B_i-linear subcategories $\mathcal{C}_{B_i} \subseteq \mathcal{C}_{B_{i+1}} \times_{\mathcal{C}_{\kappa \oplus \Sigma^{m_i}(\kappa)}} \mathcal{C}_\kappa$ which are compactly generated and for which the projection maps induce equivalences

$$\mathcal{C}_{B_i} \otimes_{B_i} B_{i+1} \to \mathcal{C}_{B_{i+1}} \qquad \mathcal{C}_{B_i} \otimes_{B_i} \kappa \to \mathcal{C}_\kappa .$$

We conclude by observing that any such $\mathcal{C}_{B_i}$ is determined by its full subcategory of compact objects, which is contained in the (essentially small) ∞-category $\mathcal{C}^c_{B_{i+1}} \times_{\mathcal{C}^c_{\kappa \oplus \Sigma^{m_i}(\kappa)}} \mathcal{C}^c_\kappa$. □

Remark 16.6.6.3. In the situation of Proposition 16.6.6.2, the inclusion $\mathrm{CatDef}^c_{\mathcal{C}} \hookrightarrow \mathrm{CatDef}_{\mathcal{C}}$ is (-1)-truncated. Using Corollary 16.6.2.4 and Theorem 16.4.2.1, we deduce that $\mathrm{CatDef}^c_{\mathcal{C}}$ is a 2-proximate formal moduli problem. Moreover, the composite map $\mathrm{CatDef}^c_{\mathcal{C}} \to \mathrm{CatDef}_{\mathcal{C}} \to \mathrm{CatDef}^{\wedge}_{\mathcal{C}}$ is 0-truncated, so that we can view $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}$ as the formal $\mathbb{E}_2$-moduli problem associated to both $\mathrm{CatDef}^c_{\mathcal{C}}$ and $\mathrm{CatDef}_{\mathcal{C}}$.

16.6.7 Tame Compact Generation

Let κ be a field and let $\mathcal{C}$ be a compactly generated stable κ-linear ∞-category. One advantage of working with the functor $\mathrm{CatDef}^c_{\mathcal{C}}$ of Notation 16.6.6.1 is that it is generally "closer" to being a formal $\mathbb{E}_2$-moduli problem than the functor $\mathrm{CatDef}_{\mathcal{C}}$. Before making this precise, we need to introduce a bit of terminology.

Definition 16.6.7.1. Let $\mathcal{C}$ be a stable ∞-category. We will say that $\mathcal{C}$ is *tamely compactly generated* if it satisfies the following conditions:

(a) The ∞-category $\mathcal{C}$ is compactly generated (that is, $\mathcal{C}$ is generated under filtered colimits by the full subcategory $\mathcal{C}^c \subseteq \mathcal{C}$ spanned by the compact objects).

(b) For every pair of compact objects $C, D \in \mathcal{C}$, the groups $\mathrm{Ext}^n_{\mathcal{C}}(C, D)$ vanish for $n \gg 0$.

Example 16.6.7.2. Let X be a quasi-compact, quasi-separated spectral algebraic space. Then the ∞-category $\mathrm{QCoh}(\mathsf{X})$ is tamely compactly generated.

Theorem 16.6.7.3. *Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category which is tamely compactly generated. Then the functor* $\mathrm{CatDef}^c_{\mathcal{C}}$ *of Notation 16.6.6.1 is a* 1*-proximate formal moduli problem.*

Remark 16.6.7.4. In the situation of Theorem 16.6.7.3, any compactly generated deformation of $\mathcal{C}$ over an Artinian $\mathbb{E}_2$-algebra is automatically tamely compactly generated: see Proposition 16.6.9.2.

We will give a proof of Theorem 16.6.7.3 a bit later in this section. First, let us consider its consequences. Combining Theorems 16.6.7.3, 16.4.2.1, and 16.6.3.8, we obtain the following:

Corollary 16.6.7.5. *Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category which is tamely compactly generated. Then, for every object $B \in \mathrm{Alg}_{\kappa}^{(2),\mathrm{art}}$, the canonical map*

$$\mathrm{CatDef}^{c}_{\mathcal{C}}(B) \to \mathrm{CatDef}^{\wedge}_{\mathcal{C}}(B) \simeq \mathrm{Map}_{\mathrm{Alg}_{\kappa}^{(2)}}(\mathfrak{D}^{(2)}(B), \mathfrak{Z}(\mathcal{C}))$$

is (-1)-truncated.

Remark 16.6.7.6. If $\mathcal{C}$ is a stable κ-linear ∞-category which is tamely compactly generated, Corollary 16.6.7.5 asserts every compactly generated deformation of $\mathcal{C}$ over an algebra $B \in \mathrm{Alg}_{\kappa}^{(2),\mathrm{art}}$ is determined (up to essentially unique equivalence) by an action of the Koszul dual $\mathfrak{D}^{(2)}(B)$ on the ∞-category $\mathcal{C}$. Under much stronger assumptions, one can ensure that the inclusion $\mathrm{CatDef}^{c}_{\mathcal{C}} \hookrightarrow \mathrm{CatDef}^{\wedge}_{\mathcal{C}}$ is an equivalence, so that every action of $\mathfrak{D}^{(2)}(B)$ on $\mathcal{C}$ determines a deformation of $\mathcal{C}$ over B (see Theorem 16.6.10.2). In general, this is not the case: there are actions of $\mathfrak{D}^{(2)}(B)$ on $\mathcal{C}$ which do not arise from deformations of $\mathcal{C}$ (at least heuristically, these correspond to *curved* deformations of the ∞-category $\mathcal{C}$).

16.6.8 Piecewise Compactness

Before giving the proof of Theorem 16.6.7.3, we need to establish a few facts about compact objects of tamely compactly generated stable ∞-categories. To maintain terminological consistency with the rest of this book, we will use the term *stable R-linear ∞-category* to refer to a stable left R-linear ∞-category (as in Variant D.1.5.1), though in our intended application we will be interested in the case of right R-linear ∞-categories.

Notation 16.6.8.1. If $\mathcal{C}$ is a compactly generated stable ∞-category, we let $\mathcal{C}^{c}$ denote the full subcategory of $\mathcal{C}$ spanned by the compact objects.

The main fact we will need is the following:

Proposition 16.6.8.2. *Let A be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a stable A-linear ∞-category which is tamely compactly generated. Suppose we are given a pullback diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

of connective $\mathbb{E}_2$-rings which induces surjective maps $\pi_0 B \to \pi_0 B'$ and $\pi_0 A' \to \pi_0 B'$. Then the natural map $\theta^{c} : \mathcal{C}^{c} \to (B \otimes_A \mathcal{C})^{c} \times_{(B' \otimes_A \mathcal{C})^{c}} (A' \otimes_A \mathcal{C})^{c}$ is an equivalence of ∞-categories.

The proof of Proposition 16.6.8.2 will require some preliminaries. Recall that if R is an $\mathbb{E}_2$-ring and $\mathcal{C}$ is a stable R-linear ∞-category, then we can regard $\mathcal{C}$ as enriched over the monoidal ∞-category LMod_R. In particular, to every pair of objects $C, D \in \mathcal{C}$ we can associate a mapping object $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D) \in \mathrm{LMod}_R$, characterized by the universal property $\mathrm{Map}_{\mathrm{LMod}_R}(M, \underline{\mathrm{Map}}_{\mathcal{C}}(C, D)) \simeq \mathrm{Map}_{\mathcal{C}}(M \otimes_R C, D)$.

Lemma 16.6.8.3. *Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. If $C \in \mathcal{C}$ is compact, then the construction $D \mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ determines a colimit-preserving functor $\mathcal{C} \to \mathrm{LMod}_R$.*

Proof. It is clear that the construction $D \mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ commutes with limits and is therefore an exact functor. To prove that it preserves colimits, it suffices to show that it preserves filtered colimits. For this, it suffices to show that the construction $D \mapsto \Omega^{\infty}\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ preserves filtered colimits (as a functor from $\mathcal{C}$ to $\mathcal{S}$), which is equivalent to the requirement that C is compact. □

Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. Given an object $N \in \mathrm{LMod}_R$ and a pair of objects $C, D \in \mathcal{C}$, the canonical map $N \otimes_R \underline{\mathrm{Map}}_{\mathcal{C}}(C, D) \otimes_R C \to N \otimes_R D$ in $\mathcal{C}$ is classified by a morphism of left R-modules $\lambda : N \otimes \underline{\mathrm{Map}}_{\mathcal{C}}(C, D) \to \underline{\mathrm{Map}}_{\mathcal{C}}(C, N \otimes_R D)$.

Lemma 16.6.8.4. *Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be an R-linear ∞-category. Let $C, D \in \mathcal{C}$ and let $N \in \mathrm{LMod}_R$. If C is a compact object of $\mathcal{C}$, then the map $\lambda : N \otimes_R \underline{\mathrm{Map}}_{\mathcal{C}}(C, D) \to \underline{\mathrm{Map}}_{\mathcal{C}}(C, N \otimes_R D)$ is an equivalence.*

Proof. Using Lemma 16.6.8.3, we deduce that the functor $N \mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(C, N \otimes D)$ preserves small colimits. It follows that the collection of objects $N \in \mathrm{LMod}_R$ for which λ is an equivalence is closed under colimits in LMod_R. We may therefore suppose that $N \simeq \Sigma^n(R)$ for some integer n, in which case the result is obvious. □

Lemma 16.6.8.5. *Suppose we are given a morphism of $\mathbb{E}_2$-rings $R \to R'$, let $\mathcal{C}$ be a stable R-linear ∞-category, and let $\mathcal{C}' = R' \otimes_R \mathcal{C} \simeq \mathrm{LMod}_{R'}(\mathcal{C})$. Let $F : \mathcal{C} \to \mathcal{C}'$ be a left adjoint to the forgetful functor $G : \mathcal{C}' \to \mathcal{C}$, given on objects by $F(C) = R' \otimes_R C$. For every pair of objects $C, D \in \mathcal{C}$, the evaluation map $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D) \otimes C \to D$ induces a map*

$$(R' \otimes_R \underline{\mathrm{Map}}_{\mathcal{C}}(C, D)) \otimes_{R'} F(C) \simeq F(\underline{\mathrm{Map}}_{\mathcal{C}}(C, D) \otimes_R C) \to F(D),$$

which is classified by a morphism of left R'-modules $\alpha : R' \otimes_R \underline{\mathrm{Map}}_{\mathcal{C}}(C, D) \to \underline{\mathrm{Map}}_{\mathcal{C}'}(F(C), F(D))$. If $C \in \mathcal{C}$ is compact, then α is an equivalence.

Proof. The image of α under the forgetful functor $\mathrm{LMod}_{R'} \to \mathrm{LMod}_R$ coincides with the equivalence $R' \otimes_R \underline{\mathrm{Map}}_{\mathcal{C}}(C, D) \to \underline{\mathrm{Map}}_{\mathcal{C}}(C, R' \otimes_R D)$ of Lemma 16.6.8.4. □

Lemma 16.6.8.6. *Suppose we are given a pullback diagram of $\mathbb{E}_2$-rings*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

Let $\mathcal{C}$ be a stable A-linear ∞-category. Then an object $C \in \mathcal{C}$ is compact if and only if its images in $A' \otimes_A \mathcal{C}$ and $B \otimes_A \mathcal{C}$ are compact.

Proof. The "only if" direction is obvious, since the forgetful functors $\mathrm{LMod}_B(\mathcal{C}) \to \mathcal{C} \leftarrow \mathrm{LMod}_{A'}(\mathcal{C})$ preserve filtered colimits. For the converse, suppose that $C \in \mathcal{C}$ has compact images $C_{A'} \in (A' \otimes_A \mathcal{C})$ and $C_B \in (B \otimes_A \mathcal{C})$. Then the image of C in the fiber product $\mathcal{D} = (A' \otimes_A \mathcal{C}) \times_{(B' \otimes_A \mathcal{C})} (B \otimes_A \mathcal{C})$ is compact. Since the natural map $\mathcal{C} \to \mathcal{D}$ is fully faithful (Proposition 16.2.1.1) and preserves filtered colimits, we conclude that C is compact. □

Lemma 16.6.8.7. *Let $f : R \to R'$ be a morphism of connective $\mathbb{E}_2$-rings, let $\mathcal{C}$ be a stable R-linear ∞-category, and set $\mathcal{C}' = R' \otimes_R \mathcal{C} \simeq \mathrm{LMod}_{R'}(\mathcal{C})$. If $\mathcal{C}$ is tamely compactly generated, then $\mathcal{C}'$ is tamely compactly generated.*

Proof. Let $F : \mathcal{C} \to \mathcal{C}'$ denote the canonical map, given by $(C \in \mathcal{C}) \mapsto (R' \otimes_R C \in \mathrm{LMod}_{R'}(\mathcal{C}) \simeq \mathcal{C}')$. We note that $\mathcal{C}'$ is compactly generated: in fact, $\mathcal{C}'$ is generated under small colimits by the essential image of the composite functor map $\mathcal{C}^c \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{C}'$, which consists of compact objects (since F is left adjoint to a forgetful functor). It follows that the ∞-category $\mathcal{C}'^c$ is the smallest stable full subcategory of $\mathcal{C}'$ which contains $F(\mathcal{C}^c)$ and is closed under retracts. Let $\mathcal{X} \subseteq \mathcal{C}'$ be the full subcategory spanned by those objects C such that for every $D \in \mathcal{C}'^c$, we have $\mathrm{Ext}^n_{\mathcal{C}'}(C, D) \simeq 0$ for $n \gg 0$. It is easy to see that $\mathcal{X}$ is stable and closed under retracts. Consequently, to show that $\mathcal{C}'^c \subseteq \mathcal{X}$, it will suffice to show that $F(C_0) \in \mathcal{X}$ for each $C_0 \in \mathcal{C}^c$. Let us regard C_0 as fixed, and let $\mathcal{Y}$ be the full subcategory of $\mathcal{C}'$ spanned by those objects D for which the groups $\mathrm{Ext}^n_{\mathcal{C}'}(F(C_0), D)$ vanish for $n \gg 0$. Since $\mathcal{Y}$ is stable and closed under retracts, it will suffice to show that $F(D_0) \in \mathcal{Y}$ for each $D_0 \in \mathcal{C}^c$. In other words, we are reduced to proving that the homotopy groups $\pi_{-n}\underline{\mathrm{Map}}_{\mathcal{C}_B}(F(C_0), F(D_0))$ vanish for $n \gg 0$. Using Lemma 16.6.8.5, we are reduced to proving that the spectrum $R' \otimes_R \underline{\mathrm{Map}}_{\mathcal{C}}(C_0, D_0)$ is $(-n)$-connective for some $n \gg 0$. Using our assumption that R and R' are connective, we are reduced to showing that $\underline{\mathrm{Map}}_{\mathcal{C}}(C_0, D_0)$ is $(-n)$-connective for $n \gg 0$, which follows from our assumption that $\mathcal{C}$ is tamely compactly generated. □

Proof of Proposition 16.6.8.2. Suppose we are given a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

of connective $\mathbb{E}_2$-rings which induces surjective ring homomorphisms $\pi_0 B \to \pi_0 B'$ and $\pi_0 A' \to \pi_0 B'$, and let $\mathcal{C} = \mathcal{C}_A$ be a stable A-linear ∞-category which is tamely compactly generated. Define

$$\mathcal{C}_B = B \otimes_A \mathcal{C}_A \qquad \mathcal{C}_{B'} = B' \otimes_A \mathcal{C}_A \qquad \mathcal{C}_{A'} = A' \otimes_A \mathcal{C}_A,$$

and let $\theta : \mathcal{C}_A \to \mathcal{C}_B \times_{\mathcal{C}_{B'}} \mathcal{C}_{A'}$ be the fully faithful embedding of Proposition 16.2.1.1. Then θ restricts to a fully faithful embedding $\theta^c : \mathcal{C}^c_A \to \mathcal{C}^c_B \times_{\mathcal{C}^c_{B'}} \mathcal{C}^c_{A'}$, and we wish to prove that θ^c is essentially surjective.

Let us identify objects of $\mathcal{C}_B \times_{\mathcal{C}_{B'}} \mathcal{C}_{A'}$ with triples $(C_B, C_{A'}, \eta)$ where $C_B \in \mathcal{C}_B$, $C_{A'} \in \mathcal{C}_{A'}$, and η is an equivalence $B' \otimes_B C_B \simeq B' \otimes_{A'} C_{A'}$. Given such a triple, we will denote the object $B' \otimes_B C_B \simeq B' \otimes_{A'} C_{A'}$ by $C_{B'}$. Note that θ admits a right adjoint G, given by $(C_B, C_{A'}, \eta) \mapsto C_B \times_{C_{B'}} C_{A'}$. By virtue of Lemma 16.6.8.6, it will suffice to show that the counit transformation $v : \theta \circ G \to \mathrm{id}$ is an equivalence when restricted to objects of $\mathcal{C}^c_B \times_{\mathcal{C}^c_{B'}} \mathcal{C}^c_{A'}$. Choose such an object $(C_B, C_{A'}, \eta)$ (so that C_B and $C_{A'}$ are compact) and let $C_A = C_B \times_{C_{B'}} C_{A'}$; we wish to show that the canonical maps

$$\phi : B \otimes_A C_A \to C_B \qquad \phi' : A' \otimes_A C_A \to C_{A'}$$

are equivalences. We will show that ϕ is an equivalence; the argument that ϕ' is an equivalence is similar. Let $\mathcal{X} \subseteq \mathcal{C}_B$ be the full subcategory spanned by those objects $D_B \in \mathcal{C}_B$ such that ϕ induces an equivalence $\phi_0 : \underline{\mathrm{Map}}_{\mathcal{C}_B}(D_B, B \otimes_A C_A) \to \underline{\mathrm{Map}}_{\mathcal{C}_B}(D_B, C_B)$. We wish to show that $\mathcal{X} = \mathcal{C}_B$. Since $\mathcal{X}$ is closed under small colimits, it will suffice to show that $\mathcal{X}$ contains $B \otimes_A D_A$ for every compact object $D_A \in \mathcal{C}_A$. Let $D_{A'}$ and $D_{B'}$ be the images of D_A in $\mathcal{C}_{A'}$ and $\mathcal{C}_{B'}$, respectively. Using Lemma 16.6.8.4, we can identify ϕ_0 with the canonical map $B \otimes_A \underline{\mathrm{Map}}_{\mathcal{C}_A}(D_A, C_A) \to \underline{\mathrm{Map}}_{\mathcal{C}_B}(D_B, C_B)$. Note that we have a pullback diagram

$$\begin{array}{ccc} \underline{\mathrm{Map}}_{\mathcal{C}_A}(D_A, C_A) & \longrightarrow & \underline{\mathrm{Map}}_{\mathcal{C}_B}(D_B, C_B) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Map}}_{\mathcal{C}_{A'}}(D_{A'}, C_{A'}) & \longrightarrow & \underline{\mathrm{Map}}_{\mathcal{C}_{B'}}(D_{B'}, C_{B'}) \end{array}$$

and that Lemma 16.6.8.4 guarantees that the underlying maps

$$B' \otimes_{A'} \underline{\mathrm{Map}}_{\mathcal{C}_{A'}}(D_{A'}, C_{A'}) \to \underline{\mathrm{Map}}_{\mathcal{C}_{B'}}(D_{B'}, C_{B'}) \leftarrow B' \otimes_B \underline{\mathrm{Map}}_{\mathcal{C}_B}(D_B, C_B)$$

are equivalences. By virtue of Proposition 16.2.2.1, it will suffice to show that there exists an integer $n \gg 0$ such that $\underline{\mathrm{Map}}_{\mathcal{C}_B}(D_B, C_B)$ and $\underline{\mathrm{Map}}_{\mathcal{C}_{A'}}(D_{A'}, C_{A'})$ belong to $(\mathrm{LMod}_B)_{\geqslant -n}$ and $(\mathrm{LMod}_{A'})_{\geqslant -n}$, respectively. This follows from the fact that the ∞-categories $\mathcal{C}_B$ and $\mathcal{C}_{A'}$ are tamely compactly generated (Lemma 16.6.8.7). □

16.6.9 The Proof of Theorem 16.6.7.3

Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category which is tamely compactly generated. Let $\mathrm{CatDef}^c_{\mathcal{C}} : \mathrm{Alg}^{(2),\mathrm{art}}_\kappa \to \mathcal{S}$ be the functor defined in Notation 16.6.6.1. For $B \in \mathrm{Alg}^{(2),\mathrm{art}}_\kappa$ we let $\mathrm{CatDef}^{\mathrm{tcg}}_{\mathcal{C}}(B)$ denote the summand of $\mathrm{CatDef}^c_{\mathcal{C}}(B)$ consisting of those deformations $(\mathcal{C}_B, \mu : \mathcal{C}_B \otimes_B \kappa \simeq \mathcal{C})$ for which $\mathcal{C}_B$ is tamely compactly generated. It follows from Lemma 16.6.8.7 that the construction $B \mapsto \mathrm{CatDef}^{\mathrm{tcg}}_{\mathcal{C}}(B)$ determines a functor $\mathrm{CatDef}^{\mathrm{tcg}}_{\mathcal{C}} : \mathrm{Alg}^{(2),\mathrm{art}}_{\kappa} \to \mathcal{S}$. Theorem 16.6.7.3 is an immediate consequence of the following pair of assertions:

Proposition 16.6.9.1. *Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category which is tamely compactly generated. Then the functor $\mathrm{CatDef}^{\mathrm{tcg}}_{\mathcal{C}} : \mathrm{Alg}^{(2),\mathrm{art}}_{\kappa} \to \mathcal{S}$ is a 1-proximate formal $\mathbb{E}_2$-moduli problem.*

Proposition 16.6.9.2. *Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category which is tamely compactly generated. Then $\mathrm{CatDef}^{\mathrm{tcg}}_{\mathcal{C}} = \mathrm{CatDef}^{c}_{\mathcal{C}}$. In other words, if $\mathcal{C}_B$ is a deformation of $\mathcal{C}$ which is defined over an Artinian $\mathbb{E}_2$-algebra B and $\mathcal{C}_B$ is compactly generated, then it is automatically tamely compactly generated.*

Proof of Proposition 16.6.9.2. Let $\mathcal{D} \subseteq \mathcal{C}_B$ denote the full subcategory spanned by those objects D such that, for each $C \in \mathcal{C}^c_B$, the groups $\mathrm{Ext}^n_{\mathcal{C}_B}(C, D)$ vanish for $n \gg 0$. Note that if $C, D \in \mathcal{C}^c_B$, then

$$\mathrm{Ext}^n_{\mathcal{C}_B}(C, D \otimes_B \kappa) \simeq \mathrm{Ext}^n_{\mathrm{RMod}_{\kappa}(\mathcal{C}_B)}(C \otimes_B \kappa, D \otimes_B \kappa)$$

vanishes for $n \gg 0$ by virtue of our assumption that $\mathcal{C}$ is tamely compactly generated. It follows that $\mathcal{D}$ contains $D \otimes_B \kappa$ for every compact object $D \in \mathcal{C}_B$. Since B is Artinian, we can choose a finite sequence $B = B_0 \to \cdots \to B_n$ and pullback diagrams

$$\begin{array}{ccc} B_i & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ B_{i+1} & \longrightarrow & \kappa \oplus \Sigma^{m_i}(\kappa). \end{array}$$

In particular, we have fiber sequences of B-modules

$$B_i \to B_{i+1} \to \Sigma^{m_i}(\kappa).$$

It follows by descending induction on i that $D \otimes_B B_i$ belongs to $\mathcal{D}$ for every compact object $D \in \mathcal{C}_B$. Taking $i = 0$, we deduce that every compact object of $\mathcal{C}_B$ belongs to $\mathcal{D}$, so that $\mathcal{C}_B$ is tamely compactly generated as desired. □

Proof of Proposition 16.6.9.1. For every connective $\mathbb{E}_2$-ring R, let $\chi(R)$ denote the subcategory of $\mathrm{RMod}_{\mathrm{LMod}_R}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$ whose objects are stable right R-linear ∞-categories which are tamely compactly generated and whose morphisms are R-linear functors which preserve compact objects. It follows from Lemma 16.6.8.7 that the construction $R \mapsto \chi(R)$ determines a functor $\mathrm{Alg}^{(2),\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$. We will prove the following:

$(*)$ For every pullback diagram of connective $\mathbb{E}_2$-rings σ :

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

for which the maps $\pi_0 B \to \pi_0 B'$ and $\pi_0 A' \to \pi_0 B'$ are surjective, the induced functor $F : \chi(A) \to \chi(B) \times_{\chi(B')} \chi(A')$ is fully faithful.

Assuming $(*)$, we deduce that the induced map of spaces $\chi(A)^{\simeq} \to \chi(B)^{\simeq} \times_{\chi(B')^{\simeq}} \chi(A')^{\simeq}$ is (-1)-truncated. Specializing to the case where σ is a pullback diagram of Artinian $\mathbb{E}_2$-algebras over κ and passing to homotopy fibers over the point $\mathcal{C} \in \chi(\kappa)$, we conclude that the map $\mathrm{CatDef}^{\mathrm{tcg}}_{\mathcal{C}}(A) \to \mathrm{CatDef}^{\mathrm{tcg}}_{\mathcal{C}}(B) \times_{\mathrm{CatDef}^{\mathrm{tcg}}_{\mathcal{C}}(B')} \mathrm{CatDef}^{\mathrm{tcg}}_{\mathcal{C}}(A')$ is (-1)-truncated, so that $\mathrm{CatDef}^{\mathrm{tcg}}_{\mathcal{C}}$ is a (-1)-proximate formal moduli problem (see Proposition 16.4.1.4).

It remains to prove $(*)$. Let us identify objects of the fiber product $\chi(B) \times_{\chi(B')} \chi(A')$ with triples $(\mathcal{D}_B, \mathcal{D}_{A'}, \eta)$ where $\mathcal{D}_B$ is a stable right B-linear ∞-category which is tamely compactly generated, $\mathcal{D}_{A'}$ is a stable right A'-linear ∞-category which is tamely compactly generated, and η is a B'-linear equivalence $\mathcal{D}_B \otimes_B B' \simeq \mathcal{D}_{A'} \otimes_{A'} B'$. Given such a triple, we let $\mathcal{D}_{B'}$ denote the ∞-category $\mathcal{D}_B \otimes_B B' \simeq \mathcal{D}_{A'} \otimes_{A'} B'$. The functor F admits a right adjoint G, which carries a triple $(\mathcal{D}_B, \mathcal{D}_{A'}, \eta)$ to the full subcategory of $\mathcal{D}_B \times_{\mathcal{D}_{B'}} \mathcal{D}_{A'}$ generated under small colimits by $\mathcal{D}^c_B \times_{\mathcal{D}^c_{B'}} \mathcal{D}^c_{A'}$. We wish to show that the unit map $u : \mathrm{id} \to G \circ F$ is an equivalence. In other words, we wish to show that if $(\mathcal{D}_B, \mathcal{D}_{A'}, \eta) = F(\mathcal{D}_A)$ for some tamely compactly generated stable right A-linear ∞-category $\mathcal{D}_A$, then the canonical map $\mathcal{D}_A \to \mathcal{D}_B \times_{\mathcal{D}_{B'}} \mathcal{D}_{A'}$ is fully faithful and its essential image is the subcategory generated by $\mathcal{D}^c_B \times_{\mathcal{D}^c_{B'}} \mathcal{D}^c_A$ under small colimits. This follows immediately from Proposition 16.6.8.2. $\square$

16.6.10 Unobstructible Objects

We can improve further on Theorem 16.6.7.3 if we are willing to to impose some stronger conditions on the κ-linear ∞-category $\mathcal{C}$.

Definition 16.6.10.1. Let $\mathcal{C}$ be a presentable stable ∞-category. We will say that an object $C \in \mathcal{C}$ is *unobstructible* if C is compact and the groups $\mathrm{Ext}^n_{\mathcal{C}}(C, C)$ vanish for $n \geqslant 2$.

Theorem 16.6.10.2. *Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category. Assume that $\mathcal{C}$ is tamely compactly generated and that there exists a collection of unobstructible objects $\{C_\alpha\}$ which generates $\mathcal{C}$ under small colimits. Then the functor $\mathrm{CatDef}^{\mathrm{c}}_{\mathcal{C}} : \mathrm{Alg}^{(2),\mathrm{art}}_{\kappa} \to \mathcal{S}$ of Notation 16.6.6.1 is a formal $\mathbb{E}_2$-moduli problem.*

Corollary 16.6.10.3. *Let κ be a field and let $\mathcal{C}$ be a stable κ-linear ∞-category. Assume that $\mathcal{C}$ is tamely compactly generated and that there exists a collection of unobstructible* *objects $\{C_\alpha\}$ which generates $\mathcal{C}$ under small colimits. Then the composite map*

$$\mathrm{CatDef}^c_{\mathcal{C}} \to \mathrm{CatDef}_{\mathcal{C}} \to \mathrm{CatDef}^{\wedge}_{\mathcal{C}}$$

is an equivalence. Consequently, the functor $\mathrm{CatDef}^c_{\mathcal{C}}$ *is given by*

$$\mathrm{CatDef}^c_{\mathcal{C}}(B) = \mathrm{Map}_{\mathrm{Alg}^{(2)}_\kappa}(\mathfrak{D}^{(2)}(B), \mathfrak{Z}(\mathcal{C})),$$

where $\mathfrak{Z}(\mathcal{C})$ denotes the κ-linear center of $\mathcal{C}$.

Proof. Combine Theorems 16.6.10.2 and 16.6.3.8 with Remarks 16.6.6.3 and 16.4.2.3. □

Remark 16.6.10.4. In the situation of Corollary 16.6.10.3, for every Artinian $\mathbb{E}_2$-algebra B over κ we obtain an equivalence

$$\{\text{Compactly generated deformations of } \mathcal{C} \text{ over } B\} \simeq \{\text{Actions of } \mathfrak{D}^{(2)}(B) \text{ on } \mathcal{C}\}.$$

In other words, $\mathcal{C}$ has no "curved" deformations (see Remark **??**).

Warning 16.6.10.5. The hypotheses of Theorem 16.6.10.2 are very restrictive: many κ-linear ∞-categories of interest (such as the ∞-categories of quasi-coherent sheaves on most algebraic varieties of dimension $\geqslant 2$) cannot be generated by unobstructible objects.

The proof of Theorem 16.6.10.2 will require some preliminaries. Our first lemma gives an explanation for the terminology of Definition 16.6.10.1.

Lemma 16.6.10.6. *Let κ be a field, let $f : B \to B'$ be a small morphism between augmented $\mathbb{E}_2$-algebras over κ. Let $\mathcal{C}_B$ be a tamely compactly generated B-linear ∞-category, let $\mathcal{C}_{B'} = \mathcal{C}_B \otimes_B B'$, and let $\mathcal{C} = \mathcal{C}_B \otimes_B \kappa$. Suppose that $C \in \mathcal{C}_{B'}$ is a compact object whose image in $\mathcal{C}$ is unobstructible. Then there exists a compact object $C_B \in \mathcal{C}_B$ and an equivalence $C_{B'} \simeq C_B \otimes_B B'$ in $\mathcal{C}_{B'}$.*

Proof. Let $C \in \mathcal{C}$ denote the image of $C_{B'}$. Since f is small, we can choose a finite sequence of morphisms $B = B_0 \to \cdots \to B_n \simeq B'$ and pullback diagrams

$$\begin{array}{ccc} B_i & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ B_{i+1} & \longrightarrow & \kappa \oplus \Sigma^{m_i}(\kappa) \end{array}$$

in $\mathrm{Alg}^{(2),\mathrm{aug}}_\kappa$, where each $m_i \geqslant 1$. We prove by descending induction on i that $C_{B'}$ can be lifted to a compact object $C_i \in \mathcal{C}_B \otimes_B B_i$, the case $i = n$ being trivial. Assume that C_{i+1} has been constructed. Let $\mathcal{C}' = \mathcal{C} \otimes_\kappa (\kappa \oplus \Sigma^{m_i}(\kappa))$. According to Proposition 16.6.8.2, we have an equivalence of ∞-categories $\mathcal{C}^c_i \to \mathcal{C}^c_{i+1} \times_{\mathcal{C}'^c} \mathcal{C}^c$. Consequently, to show that C_{i+1} can be lifted to an object $C_i \in \mathcal{C}^c_i$, it will suffice to show that C_{i+1} and C have the same image in $\mathcal{C}'^c$. This is a special case of the following assertion:

(∗) Let $X, Y \in \mathcal{C}'$ be objects having the same $C \in \mathcal{C}$. If C is unobstructible, then there is an equivalence $X \simeq Y$ in $\mathcal{C}'$.

To prove (∗), we let $\mathrm{ObjDef}_C : \mathrm{Alg}_\kappa^{(1),\mathrm{art}} \to \mathcal{S}$ be defined as in Notation 16.5.3.1; we wish to prove that any two points of the space $\mathrm{ObjDef}_C(\kappa \oplus \Sigma^{m_i}(\kappa))$ belong to the same path component. According to Proposition 16.5.6.2, $\mathrm{ObjDef}_C(\kappa \oplus \Sigma^{m_i}(\kappa))$ can be identified with a summand of the mapping space $\mathrm{Map}_{\mathrm{Alg}_\kappa^{(1)}}(\mathfrak{D}^{(1)}(\kappa \oplus \Sigma^{m_i}(\kappa)), \mathrm{End}(C))$. Since the Koszul dual $\mathfrak{D}^{(1)}(\kappa \oplus \Sigma^{m_i}(\kappa))$ is the free associative algebra generated by $\Sigma^{-m_i-1}(\kappa)$, we have a canonical isomorphism

$$\pi_0 \mathrm{Map}_{\mathrm{Alg}_\kappa^{(1)}}(\mathfrak{D}^{(1)}(\kappa \oplus \Sigma_\kappa^{(m_i)}), \mathrm{End}(C)) \simeq \pi_{-m_i-1} \mathrm{End}(C) \simeq \mathrm{Ext}_{\mathcal{C}}^{m_i+1}(C, C).$$

These groups vanish by virtue of our assumption that C is unobstructible. □

Remark 16.6.10.7. In the situation of Lemma 16.6.10.6, if we assume that $\mathrm{Ext}^1_{\mathcal{C}}(C, C)$ vanishes, then lifting of $C_{B'}$ to $\mathcal{C}_B$ is unique up to equivalence: that is, C is *undeformable.*

Lemma 16.6.10.8. *Let κ be a field, let $B \in \mathrm{Alg}_\kappa^{(2),\mathrm{aug}}$ be an Artinian augmented $\mathbb{E}_2$-algebra over κ, let $\mathcal{C}_B$ be a tamely compactly generated B-linear ∞-category, and let $\mathcal{C} = \mathcal{C}_B \otimes_B \kappa$. Let $\{C_\alpha\}$ be a collection of objects of $\mathcal{C}$ which generates $\mathcal{C}$ under small colimits, and suppose that each C_α can be lifted to an object $\overline{C}_\alpha \in \mathcal{C}_B$ (so that $C_\alpha \simeq \overline{C}_\alpha \otimes_B \kappa$). Then the collection of objects $\{\overline{C}_\alpha\}$ generates $\mathcal{C}_B$ under small colimits.*

Proof. Let $\mathcal{E}$ be the full subcategory of $\mathcal{C}_B$ generated by $\{\overline{C}_\alpha\}$ under small colimits. Then $\mathcal{E}$ contains $\overline{C}_\alpha \otimes_B M$ for every connective left B-module M. Taking $M = \kappa$, we deduce that $\mathcal{E}$ contains the images of the objects $\{C_\alpha\}$ under the forgetful functor $\theta : \mathcal{C} \to \mathcal{C}_B$. Since θ preserves small colimits, it follows that $\mathcal{E}$ contains the essential image of θ. In particular, $\mathcal{E}$ contains $C \otimes_B \kappa$ for each object $C \in \mathcal{C}_B$. Since B is Artinian, we can choose a finite sequence

$$B = B_0 \to \cdots \to B_n \simeq \kappa$$

and pullback diagrams

$$\begin{array}{ccc} B_i & \longrightarrow & \kappa \\ \downarrow & & \downarrow \\ B_{i+1} & \longrightarrow & \kappa \oplus \Sigma^{m_i}(\kappa). \end{array}$$

It follows by descending induction on i that $\mathcal{E}$ contains $C \otimes_B B_i$ for each $C \in \mathcal{C}_B$. Taking $i = 0$, we deduce that $\mathcal{E} = \mathcal{C}_B$. □

Proof of Theorem 16.6.10.2. Theorem 16.6.7.3 implies that $\mathrm{CatDef}^c_{\mathcal{C}}$ is a 1-proximate formal moduli problem. Suppose we are given a pullback diagram

$$\begin{array}{ccc} B & \longrightarrow & B_0 \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_{01} \end{array}$$

in $\mathrm{Alg}^{(2),\mathrm{art}}_{\kappa}$ which induces surjective maps $\pi_0 B_0 \to \pi_0 B_{01} \leftarrow \pi_0 B_1$. Then the map

$$\theta : \mathrm{CatDef}^c_{\mathcal{C}}(B) \to \mathrm{CatDef}^c_{\mathcal{C}}(B_0) \times_{\mathrm{CatDef}^c_{\mathcal{C}}(B_{01})} \mathrm{CatDef}^c_{\mathcal{C}}(B_1)$$

is (-1)-truncated, and we wish to show that it is a homotopy equivalence. Fix a point of the fiber product $\mathrm{CatDef}^c_{\mathcal{C}}(B_0) \times_{\mathrm{CatDef}^c_{\mathcal{C}}(B_{01})} \mathrm{CatDef}_{\mathcal{C},\omega}(B_1)$, which determines a pair $(\mathcal{C}_{B_0}, \mathcal{C}_{B_1}, \eta)$ where $\mathcal{C}_{B_0}$ is a compactly generated stable right B_0-linear ∞-category, $\mathcal{C}_{B_1}$ is a compactly generated stable right B_1-linear ∞-category, and η is a B_{01}-linear equivalence $\mathcal{C}_{B_0} \otimes_{B_0} B_{01} \simeq \mathcal{C}_{B_1} \otimes_{B_1} B_{01}$.

Let $\mathcal{C}_{B_{01}}$ denote the ∞-category $\mathcal{C}_{B_0} \otimes_{B_0} B_{01} \simeq \mathcal{C}_{B_1} \otimes_{B_1} B_{01}$, and let $\mathcal{C}_B$ denote the full subcategory of $\mathcal{C}_{B_0} \times_{\mathcal{C}_{B_{01}}} \mathcal{C}_{B_1}$ generated under small colimits by $\mathcal{C}^c_{B_0} \times_{\mathcal{C}^c_{B_{01}}} \mathcal{C}^c_{B_1}$. We wish to show that $\mathcal{C}_B$ is a deformation of $\mathcal{C}$ over B satisfying $\theta(\mathcal{C}_B) \simeq (\mathcal{C}_{B_0}, \mathcal{C}_{B_1}, \eta)$. Unwinding the definitions, it suffices to show that the canonical maps $q : \mathcal{C}_B \otimes_B B_0 \to \mathcal{C}_{B_0}$ and $q' : \mathcal{C}_B \otimes_B B_1 \to \mathcal{C}_{B_1}$ are equivalences. We will show that q is an equivalence; the proof for q' is similar.

We first claim that q is fully faithful. Since the domain and codomain of q are compactly generated and the functor q preserves compact objects, it will suffice to show that q is fully faithful when restrict to compact objects. The collection of compact objects of $\mathcal{C}_B \otimes_B B_0$ is generated, under retracts and finite colimits, by the essential image of the free functor $F : \mathcal{C}_B \to \mathcal{C}_B \otimes_B B_0 \simeq \mathrm{RMod}_{B_0}(\mathcal{C}_B)$. It will therefore suffice to show that for every pair of compact objects $C, D \in \mathcal{C}_B$, the functor q induces an equivalence of left B_0-modules $\xi : \underline{\mathrm{Map}}_{\mathrm{RMod}_{B_0}(\mathcal{C}_B)}(F(C), F(D)) \to \underline{\mathrm{Map}}_{\mathcal{C}_{B_0}}(qF(C), qF(D))$. We can identify $\mathcal{C}^c_B$ with the fiber product $\mathcal{C}^c_{B_0} \times_{\mathcal{C}^c_{B_{01}}} \mathcal{C}^c_{B_1}$, so that C and D correspond to triples (C_0, C_1, γ) and (D_0, D_1, δ), where $C_0, D_0 \in \mathcal{C}^c_{B_0}$, $C_1, D_1 \in \mathcal{C}^c_{B_1}$, and $\gamma : C_0 \otimes_{B_0} B_{01} \simeq C_1 \otimes_{B_1} B_{01}$ and $\delta : D_0 \otimes_{B_0} B_{01} \simeq D_1 \otimes_{B_1} B_{01}$ are equivalences in $\mathcal{C}_{B_{01}}$. Set $C_{01} = C_0 \otimes_{B_0} B_{01} \simeq C_1 \otimes_{B_1} B_{01}$ and $D_{01} = D_0 \otimes_{B_0} B_{01} \simeq D_1 \otimes_{B_1} B_{01}$.

Using Lemma 16.6.8.5, we can identify ξ with the natural map $B_0 \otimes_B \underline{\mathrm{Map}}_{\mathcal{C}_B}(C, D) \to \underline{\mathrm{Map}}_{\mathcal{C}_{B_0}}(C_0, D_0)$. Here we have an equivalence

$$\underline{\mathrm{Map}}_{\mathcal{C}_B}(C, D) \simeq \underline{\mathrm{Map}}_{\mathcal{C}_{B_0}}(C_0, D_0) \times_{\underline{\mathrm{Map}}_{\mathcal{C}_{B_{01}}}(C_{01}, D_{01})} \underline{\mathrm{Map}}_{\mathcal{C}_{B_1}}(C_1, D_1).$$

Using Lemma 16.6.8.5 and Proposition 16.2.1.1, we are reduced to proving that $\underline{\mathrm{Map}}_{\mathcal{C}_{B_0}}(C_0, D_0)$ and $\underline{\mathrm{Map}}_{\mathcal{C}_{B_1}}(C_1, D_1)$ are n-connective for some integer n. This follows because $\mathcal{C}_{B_0}$ and $\mathcal{C}_{B_1}$ are tamely compactly generated (Proposition 16.6.9.2).

It remains to prove that the functor q is essentially surjective. Note that the essential image of q is closed under small colimits. Using Lemma 16.6.10.6, it will suffice to show that the essential image of q contains every object $C \in \mathcal{C}^c_{B_0}$ whose image in $\mathcal{C}$ is unobstructible. To prove this, it suffices to show that $C \otimes_{B_0} B_{01}$ can be lifted to a compact object of $\mathcal{C}_{B_1}$, which is a special case of Lemma 16.6.10.6. □

Part V

Representability Theorems

Let R be a commutative ring and let X be an R-scheme. Suppose that we want to give an explicit presentation of X. We might achieve this by choosing a covering of X by open subschemes $\{U_\alpha\}_{\alpha \in I}$, where each U_α is an affine scheme given by the spectrum of a commutative R-algebra A_α. Let us assume for simplicity that each intersection $U_\alpha \cap U_\beta$ is itself an affine scheme, which can be described as the spectrum of the localization $A_\alpha[x_{\alpha,\beta}^{-1}]$ for some element $x_{\alpha,\beta} \in A_\alpha$. To specify X, we need to supply the following data:

(a) For each $\alpha \in I$, a commutative R-algebra A_α. If X is of finite presentation over R, then the algebra A_α could be given by generators and relations as a quotient

$$R[x_1, \ldots, x_n]/(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_m))$$

for some collection of polynomials f_i.

(b) For every pair of indices $\alpha, \beta \in I$, a pair of elements $x_{\alpha,\beta} \in A_\alpha$ and $x_{\beta,\alpha} \in A_\beta$, together with an R-algebra isomorphism $\phi_{\alpha,\beta} : A_\alpha[x_{\alpha,\beta}^{-1}] \simeq A_\beta[x_{\beta,\alpha}^{-1}]$.

Moreover, the isomorphisms $\phi_{\alpha,\beta}$ should be the identity when $\alpha = \beta$, and satisfy the following cocycle condition:

(c) Given $\alpha, \beta, \gamma \in I$, the commutative ring $A_\gamma[x_{\gamma,\beta}^{-1}, \phi_{\beta,\gamma}(x_{\beta,\alpha})^{-1}]$ should be a localization of $A_\gamma[x_{\gamma,\alpha}^{-1}]$. Moreover, the composite map

$$A_\alpha \to A_\alpha[x_{\alpha,\gamma}^{-1}] \xrightarrow{\phi_{\alpha,\gamma}} A_\gamma[x_{\gamma,\alpha}^{-1}] \to A_\gamma[x_{\gamma,\beta}^{-1}, \phi_{\beta,\gamma}(x_{\beta,\alpha})^{-1}]$$

should be obtained by composing (localizations of) $\phi_{\alpha,\beta}$ and $\phi_{\beta,\gamma}$.

In §1.1, we introduced the notion of a *spectral scheme*. The notion of a spectral scheme is entirely analogous the classical notion of a scheme. However, the analogues of (a), (b), and (c) are much more complicated in the spectral setting. The data of an affine spectral scheme over a commutative ring R is equivalent to the data of an $\mathbb{E}_\infty$-algebra over R. These are often quite difficult to describe using generators and relations. For example, the polynomial ring $R[x]$ generally does not have a finite presentation as an $\mathbb{E}_\infty$-algebra over R (unless we assume that R has characteristic zero). These complications are amplified when we pass to the non-affine situation. In the spectral setting, (b) requires us to construct equivalences between $\mathbb{E}_\infty$-algebras, which are often difficult to specify concretely. Moreover, since $\mathbb{E}_\infty$-algebras form an ∞-category rather than an ordinary category, the analogue of the cocycle condition described in (c) is not a condition but an additional datum (namely, a homotopy between two $\mathbb{E}_\infty$-algebra maps $A_\alpha \to A_\gamma[x_{\gamma,\beta}^{-1}, \phi_{\beta,\gamma}(x_{\beta,\alpha})^{-1}]$ for every triple $\alpha, \beta, \gamma \in I$), which must be supplemented by “higher” coherence data involving four-fold intersections and beyond.

For these reasons, it can be difficult to provide “hands-on” constructions in the setting of spectral algebraic geometry.

Fortunately, there is another approach to describing a scheme X. Rather than trying to explicitly construct the commutative rings associated to some affine open covering of X, one can instead consider the functor h_X represented by X, given by the formula $h_X(R) = \mathrm{Hom}(\operatorname{Spec} R, X)$. The scheme X can be recovered from the functor h_X (up to canonical isomorphism). The situation for spectral schemes is again entirely analogous: every spectral scheme X determines a functor $h_{\mathsf{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and the construction $\mathsf{X} \mapsto h_{\mathsf{X}}$ determines a fully faithful embedding from the ∞-category of spectral schemes to the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ (Corollary **??**). In many case, it is easier to describe a spectral scheme (or spectral Deligne-Mumford stack) X by specifying the functor h_{X} than by specifying its structure sheaf $\mathscr{O}_{\mathsf{X}}$. This motivates the following general question:

Question 16.0.0.1. Given a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, under what circumstances is X representable by a spectral Deligne-Mumford stack?

In the setting of classical algebraic geometry, the analogous question is addressed by the following theorem of Artin:

Theorem 16.0.0.2 (Artin Representability Theorem)**.** *Let R be a Grothendieck ring (see Definition* **??***) and let X be functor from the category of commutative R-algebras to the category of sets. Then X is representable by an algebraic space X which is locally of finite presentation over R if the following conditions are satisfied:*

(1) *The diagonal map $X \to X \times_{\operatorname{Spec} R} X$ is representable by algebraic spaces (which must be quasi-compact schemes, if we wish to require that X is quasi-separated).*

(2) *The functor X is a sheaf for the étale topology.*

(3) *If B is a complete local Noetherian R-algebra with maximal ideal $\mathfrak{m}$, then the natural map $X(B) \to \varprojlim X(B/\mathfrak{m}^n)$ is bijective.*

(4) *The functor X admits an obstruction theory and a deformation theory, and satisfies Schlessinger's criteria for formal representability.*

(5) *The functor X commutes with filtered colimits.*

This result is of both philosophical and practical interest. Since conditions (1) through (5) are reasonable expectations for any functor X of a reasonably geometric nature, Theorem 16.0.0.2 provides evidence that the theory of algebraic spaces is natural and robust (in other words, that it exactly captures some intuitive notion of “geometricity”). On the other hand, if we are given a functor X, it is usually reasonably easy to check whether or not Artin's criteria are satisfied. Consequently, Theorem 16.0.0.2 can be used to construct a great number of moduli spaces.

Remark 16.0.0.3. We refer the reader to [4] for the original proof of Theorem 16.0.0.2. Note that in Artin's formulation, condition (3) is replaced by the weaker requirement that the map $X(B) \to \varprojlim X(B/\mathfrak{m}^n)$ has dense image (with respect to the inverse limit topology). Moreover, Artin's proof required a stronger assumption on the commutative ring R; for a careful discussion of the removal of this hypothesis, we refer the reader to [43].

Our goal in this paper is to prove an analogue of Theorem 16.0.0.2 in the setting of spectral algebraic geometry. Let R be an Noetherian $\mathbb{E}_\infty$-ring such that $\pi_0 R$ is a Grothendieck ring, and suppose we are given a functor $X : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$. Our main result (Theorem 16.0.1) supplies necessary and sufficient conditions for X to be representable by a spectral Deligne-Mumford n-stack which is locally almost of finite presentation over R. For the most part, these conditions are natural analogues of the hypotheses of Theorem 16.0.0.2. The main difference is in the formulation of condition (4). In the setting of Artin's original theorem, a deformation and obstruction theory are auxiliary constructs which are not uniquely determined by the functor X. The meaning of these conditions are clarified by working in the spectral setting: they are related to the problem of extending the functor X to $\mathbb{E}_\infty$-rings which are nondiscrete. In Chapter 17, we will make this idea more precise by studying various conditions on a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which are *necessary* for representability. We will be primarily interested in conditions of a deformation-theoretic nature: that is, conditions which describe the relationship between $X(\overline{A})$ and $X(A)$, where $\overline{A}$ is a *square-zero* extension of A by some A-module M (see Definition HA.7.4.1.6). For example:

(a) Any representable functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ admits a *cotangent complex* $L_X \in \mathrm{QCoh}(X)^{\mathrm{cn}}$ (see Definition 17.2.4.2), which can be used to describe the relationship between $X(\overline{A})$ and $X(A)$ in the case where $\overline{A}$ is the trivial square-zero extension $A \oplus M$ for some connnective A-module M.

(b) Any representable functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is *infinitesimally cohesive*: that is, it satisfies a gluing formula for infinitesimal thickenings (see Definition 17.3.1.5). In combination with (a), this allows us to describe the relationship between $X(\overline{A})$ and $X(A)$ for *arbitrary* square-zero-extensions (see Remark 17.3.1.8).

(c) Any representable functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is *nilcomplete* (see Definition 17.3.2.1): for any $A \in \mathrm{CAlg}^{\mathrm{cn}}$, the space $X(A)$ can be realized as the limit of the tower of spaces

$$\cdots \to X(\tau_{\leqslant 3} A) \to X(\tau_{\leqslant 2} A) \to X(\tau_{\leqslant 1} A) \xrightarrow{X} (\tau_{\leqslant 0} A) = X(\pi_0 A).$$

This is useful in combination with (a) and (b), since each truncation $\tau_{\leqslant n} A$ can be regarded as a square-zero extension of $\tau_{\leqslant n-1} A$.

(*d*) If $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is representable by a spectral Deligne-Mumford n-stack (for some $n < \infty$), then X is *integrable* (see Definition 17.3.4.1). If A is a complete local Noetherian ring with maximal ideal $\mathfrak{m}$, the space $X(A)$ can be realized as the limit of the tower

$$\cdots \to X(A/\mathfrak{m}^4) \to X(A/\mathfrak{m}^3) \to X(A/\mathfrak{m}^2) \xrightarrow{X} (A/\mathfrak{m}).$$

This is again useful in combination with (*a*) and (*b*), since each quotient $A/\mathfrak{m}^n$ can be realized as a square-zero extension of $A/\mathfrak{m}^{n-1}$.

Roughly speaking, assertions (*a*) and (*b*) correspond to Artin's criterion (4), while assertion (*d*) corresponds to Artin's criterion (3) (assertion (*c*) has no counterpart in classical algebraic geometry, but is a natural companion to (*a*) and (*b*)). In Chapter 18, we will use these ideas to establish a spectral analogue of Theorem 16.0.0.2:

Theorem 16.0.1 (Spectral Artin Representability Theorem)**.** Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor, and suppose we are given a natural transformation $X \to \operatorname{Spec} R$, where R is a Noetherian $\mathbb{E}_\infty$-ring and $\pi_0 R$ is a Grothendieck ring. Let $n \geqslant 0$. Then X is representable by a spectral Deligne-Mumford n-stack which is locally almost of finite presentation over R if and only if the following conditions are satisfied:

(1) For every discrete commutative ring A, the space $X(A)$ is n-truncated.

(2) The functor X is a sheaf for the étale topology.

(3) The functor X is nilcomplete, infinitesimally cohesive, and integrable.

(4) The functor X admits a connective cotangent complex L_X.

(5) The natural transformation f is locally almost of finite presentation (see Definition 17.4.1.1).

Theorem 18.3.0.1 can be regarded as a spectral analogue of Artin's representability criterion in classical algebraic geometry (see Theorem 16.0.0.2). However, it plays a more central role than its classical counterpart. Many moduli spaces of interest in classical algebraic geometry can be constructed directly, without resorting to an abstract representability theorem. In spectral algebraic geometry, direct constructions are more problematic: it is difficult to specify geometric objects by explicitly describing their coordinate rings, so the "functor of points" perspective becomes indispensable. In Chapter 19, we will illustrate this point by describing several applications of Theorem 18.3.0.1 (we will meet more examples in Part VIII, when we consider the analogue of Artin's theorem in the setting of *derived* algebraic geometry).

Chapter 17

Deformation Theory and the Cotangent Complex

Let A be a commutative ring and let B be a commutative algebra over A. We let $\Omega_{B/A}$ denote the module of Kähler differentials of B relative to A. More precisely, $\Omega_{B/A}$ is the B-module generated by symbols $\{dx\}_{x \in B}$ subject to the relations

$$d(x+y) = dx + dy \qquad d(xy) = xdy + ydx \qquad d\lambda = 0 \text{ if } \lambda \in A.$$

The theory of Kähler differentials can be relativized. To every morphism of schemes $f : X \to Y$, one can associate a quasi-coherent sheaf $\Omega_{X/Y}$ on X, called the *sheaf of relative Kähler differentials*, which is essentially characterized by the formula $\Omega_{X/Y}(U) \simeq \Omega_{\mathscr{O}_X(U)/\mathscr{O}_Y(V)}$ whenever $U \subseteq X$ is an affine open subset whose image $f(U)$ is contained in an affine open subset $V \subseteq Y$. The sheaf $\Omega_{X/Y}$ plays a central role in classical algebraic geometry: if f is smooth, then $\Omega_{X/Y}$ is a locally free sheaf which plays the role of a relative cotangent bundle for the morphism f.

The theory of Kähler differentials has an analogue in the setting of spectral algebraic geometry. To every morphism $\phi : A \to B$ of $\mathbb{E}_\infty$-rings, one can associate an object $L_{B/A} \in \mathrm{Mod}_B$, which we refer to as the *relative cotangent complex of* ϕ, which is universal among those B-modules M for which the projection map $B \oplus M \to B$ is equipped with a section (in the ∞-category CAlg_A); see §**??** for a more detailed discussion. Like the classical theory of Kähler differentials, this construction can be relativized. In §17.1, we associate to each morphism of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \mathsf{Y}$ a relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}}$, which we can regard as a quasi-coherent sheaf on X (Proposition 17.1.2.1). The relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}}$ can be regarded as kind of "relative cotangent bundle" of f, with the caveat that it is usually not a vector bundle: the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}}$ is locally free of finite rank if f is differentially smooth, and the converse holds under some mild finiteness hypotheses (Proposition 17.1.5.1).

Recall that a spectral Deligne-Mumford stack X is determined by the functor $h_{\mathsf{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ represented by X (given by $h_{\mathsf{X}}(R) = \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X})$; see Proposition 1.6.4.2). It follows that for any morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks, the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}} \in \mathrm{QCoh}(\mathsf{X})$ can be recovered from the associated natural transformation $h_{\mathsf{X}} \to h_{\mathsf{Y}}$. In §17.2, we make this observation explicit by associating a relative cotangent complex $L_{X/Y}$ to any (sufficiently nice) natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ (Definition 17.2.4.2). This "functor of points" approach has the advantage of making sense in many situations where the functors X and Y are not representable, or not yet known to be representable. This will be crucial in our discussion of representability questions in Chapter 18: the existence of a relative cotangent complex $L_{X/Y}$ is often the main ingredient in showing that a map $f : X \to Y$ is relatively representable (see Theorems 18.1.0.2 and 18.3.0.1).

Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. By definition, the relative cotangent complex $L_{X/Y}$ (if it exists) is a quasi-coherent sheaf on X with the property that, for every commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A & \xrightarrow{\eta} & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec}(A \oplus M) & \longrightarrow & Y, \end{array}$$

the space of dotted arrow rendering the diagram commutative can be identified with $\mathrm{Map}_{\mathrm{Mod}_A}(\eta^* L_{X/Y}, M)$. We can summarize the situation more informally with the following slogan:

$(*)$ Lifting of trivial square-zero extensions along a morphism $f : X \to Y$ are controlled by the relative cotangent complex $L_{X/Y}$.

For many applications, $(*)$ alone is not very useful: one would like to analyze more general lifting problems of the form σ :

$$\begin{array}{ccc} \operatorname{Spec} A & \xrightarrow{\eta} & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} \overline{A} & \longrightarrow & Y, \end{array}$$

where $\overline{A}$ is an *arbitrary* square-zero extension of A by a (connective) A-module M (see Definition HA.7.4.1.6). In this case, the general philosophy of deformation theory suggests that there should be an obstruction class $\alpha \in \mathrm{Ext}^1_A(\eta^* L_{X/Y}, M)$ which vanishes if only if the lifting problem depicted in the diagram σ admits a solution; moreover, if such a solution exists, then the space of solutions should form a torsor for the mapping space $\mathrm{Map}_{\mathrm{Mod}_A}(\eta^* L_{X/Y}, M)$. In §17.3, we will see that this expectation is correct under the additional assumption that X and Y are *infinitesimally cohesive* (see Definition 17.3.1.5 and Remark 17.3.1.8). We also introduce the related notions of *cohesive*, *nilcomplete*, and *integrable* functors $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ (Definitions 17.3.1.1, 17.3.2.1, and 17.3.4.1). Taken together, these conditions express the idea that X has a well-behaved deformation theory, and can be studied effectively using "infinitesimal" methods.

The deformation-theoretic ideas studied in §17.3 can be regarded as a partial answer to Question 16.0.0.1: they allow us to formulate a list of axioms on a natural transformation $f : X \to Y$ which are necessarily satisfied if f is relatively representable. In order to guarantee that these axioms are also sufficient, it is necessary to introduce some finiteness hypotheses on f. In §17.4, we introduce the notion of *local almost finite presentation* for a natural transformation $f : X \to Y$ (Definition 17.4.1.1). Under mild hypotheses, we prove that this agrees with the notion of almost finite presentation studied in Chapter 4 in the case where f is relatively representable (Corollary 17.4.2.2).

Recall that if Z is an algebraic variety over $\mathbf{C}$, then the *Zariski tangent space* to Z at a $\mathbf{C}$-valued point $z \in Z(\mathbf{C})$ can be defined as the fiber product $Z(\mathbf{C}[\epsilon]/(\epsilon^2)) \times_{Z(\mathbf{C})} \{z\}$. In §17.5, we consider an analogue of this construction in the setting of spectral algebraic geometry: if $f : X \to Y$ is an infinitesimally cohesive morphism of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, then we can associate a well-defined *tangent complex* $T_{X/Y}(\eta) \in \mathrm{Mod}_A$ to each point $\eta \in X(A)$ (Definition 17.5.1.1). Under the assumption that f admits a cotangent complex, the tangent complex $T_{X/Y}(\eta)$ can be described as the A-linear dual of $\eta^* L_{X/Y}$. In §17.5, we establish a partial converse: under appropriate hypotheses, the existence of $L_{X/Y}$ is equivalent to the assumption that the tangent complexes $T_{X/Y}(\eta)$ satisfy some mild finiteness conditions (Theorem 17.5.4.1).

Warning 17.0.0.1. Let $f : X \to Y$ be a morphism of schemes, which we will identify with the corresponding spectral Deligne-Mumford stacks. Our definition of the cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ is based on a globalization of *topological* André-Quillen homology, rather than classical André-Quillen homology. Consequently, it generally does not agree with usual cotangent complex studied in algebraic geometry (for example, in [99] and [100]), which we will denote by $L^{\mathrm{alg}}_{X/Y}$. There is a canonical map $\theta : L_{X/Y} \to L^{\mathrm{alg}}_{X/Y}$, which is an equivalence if X is a $\mathbf{Q}$-scheme. In general, θ induces isomorphisms $\pi_n L_{X/Y} \to \pi_n L^{\mathrm{alg}}_{X/Y}$ for $n \leqslant 1$ and an epimorphism when $n = 2$. The algebraic cotangent complex $L^{\mathrm{alg}}_{X/Y}$ plays a central role in the theory of *derived* algebraic geometry, which we will study in §VIII and §**??**.

Contents

17.1 The Cotangent Complex of a Spectrally Ringed ∞-Topos

In §HA.7.3 , we defined the relative cotangent complex $L_{B/A}$ of a morphism of $\mathbb{E}_\infty$-rings $\phi : A \to B$. In this section, we will study a generalization of the construction $\phi \mapsto L_{B/A}$, where we replace ϕ by a map of spectrally ringed ∞-topoi $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ (which reduces to the theory described in §HA.7.3 when we take $\mathcal{X} = \mathcal{Y} = \mathcal{S} = \mathcal{S}\mathrm{hv}(*)$).

17.1.1 Definitions

We begin by introducing some terminology. Let $(\mathcal{X}, \mathscr{A})$ be a spectrally ringed ∞-topos, and let $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\mathscr{A}//\mathscr{A}}$ denote the ∞-category of augmented $\mathscr{A}$-algebra objects of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. It follows from Theorem HA.7.3.4.7 that the ∞-category $\mathrm{Mod}_{\mathscr{A}}$ of $\mathscr{A}$-module objects of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ can be identified with the stabilization of the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\mathscr{A}//\mathscr{A}}$. In particular, there canonical map $\Omega^{\infty} : \mathrm{Mod}_{\mathscr{A}} \to \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\mathscr{A}//\mathscr{A}}$. If $\mathscr{M}$ is a $\mathscr{A}$-module object of $\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{X})$, we will denote its image under the functor Ω^{∞} by $\mathscr{A} \oplus \mathscr{M}$, and refer to it as the *trivial square-zero extension of $\mathscr{A}$ by $\mathscr{M}$*.

Definition 17.1.1.1. Let $(\mathcal{X}, \mathscr{A})$ be a spectrally ringed ∞-topos and let $\mathscr{M} \in \mathrm{Mod}_{\mathscr{A}}$. A *derivation of $\mathscr{A}$ into $\mathscr{M}$* is a section of the tautological map $\mathscr{A} \oplus \mathscr{M} \to \mathscr{A}$. We let $\mathrm{Der}(\mathscr{A}, \mathscr{M}) = \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{/\mathscr{A}}}(\mathscr{A}, \mathscr{A} \oplus \mathscr{M})$ denote the space of derivations of $\mathscr{A}$ into $\mathscr{M}$.

Remark 17.1.1.2. In the situation of Definition 17.1.1.1, a derivation of $\mathscr{A}$ into $\mathscr{M}$ can be identified with a pair (s, h), where $s : \mathscr{A} \to \mathscr{A} \oplus \mathscr{M}$ is a morphism of CAlg-valued sheaves on $\mathcal{X}$ and h is a homotopy from the composite map

$$\mathscr{A} \xrightarrow{s} \mathscr{A} \oplus \mathscr{M} \to \mathscr{A}$$

to the identity map $\mathrm{id}_{\mathscr{A}}$. It follows that, as a morphism of Sp-valued sheaves on $\mathcal{X}$, we can identify s with a product of the identity map $\mathrm{id} : \mathscr{A} \to \mathscr{A}$ and some auxiliary map $d : \mathscr{A} \to \mathscr{M}$. We will often abuse terminology by referring to the map d as a *derivation of $\mathscr{A}$ into $\mathscr{M}$*. Beware that this terminology is slightly misleading: the pair (s, h) is generally not determined by $d \in \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})}(\mathscr{A}, \mathscr{M})$, even up to homotopy. Roughly speaking, we can think of the pair (s, h) as witnessing the fact that the map $d : \mathscr{A} \to \mathscr{M}$ satisfies the Leibniz rule, up to coherent homotopy.

Definition 17.1.1.3 (The Cotangent Complex)**.** Let $\mathcal{X}$ be an ∞-topos. We let

$$L : \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X}) \to \mathrm{Mod}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$$

denote the absolute cotangent complex functor defined in §HA.7.3.2. To each object $\mathscr{A} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$, we let $L_{\mathscr{A}} \in \mathrm{Mod}_{\mathscr{A}}$ denote the image of $\mathscr{A}$ under the functor L. We refer to $L_{\mathscr{A}}$ as the *absolute cotangent complex of $\mathscr{A}$*. The object $L_{\mathscr{A}}$ can be characterized up to equivalence as follows: there exists a derivation $d \in \mathrm{Der}(\mathscr{A}, L_{\mathscr{A}})$ for which evaluation on d induces a homotopy equivalence $\mathrm{Map}_{\mathrm{Mod}_{\mathscr{A}}}(L_{\mathscr{A}}, \mathscr{M}) \to \mathrm{Der}(\mathscr{A}, \mathscr{M})$ for every object $\mathscr{M} \in \mathrm{Mod}_{\mathscr{A}}$. We will refer to d as the *universal derivation*.

Variant 17.1.1.4 (The Relative Cotangent Complex)**.** Let $\mathcal{X}$ be an ∞-topos and let $\phi : \mathscr{A} \to \mathscr{B}$ be a morphism of CAlg-valued sheaves on $\mathcal{X}$. Then the *relative cotangent complex $L_{\mathscr{B}/\mathscr{A}}$* is given by the cofiber of the map $\mathscr{B} \otimes_{\mathscr{A}} L_{\mathscr{A}} \to L_{\mathscr{B}}$ determined by ϕ (see §HA.7.3.3).

The formation of cotangent complexes is compatible with pullback, in the following sense:

Proposition 17.1.1.5. *Let $\phi^* : \mathcal{Y} \to \mathcal{X}$ be a geometric morphism of ∞-topoi, let $\mathscr{A} \in \mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y}))$, and let $d : \mathscr{A} \to \mathscr{A} \oplus L_{\mathscr{A}}$ be the universal derivation. Then the induced map*

$$\phi^* \mathscr{A} \to \phi^*(\mathscr{A} \oplus L_{\mathscr{A}}) \simeq \phi^* \mathscr{A} \oplus \phi^* L_{\mathscr{A}}$$

induces an equivalence $\phi^* L_{\mathscr{A}} \to L_{\phi^* \mathscr{A}}$.

Proof. Let $\mathscr{M} \in \mathrm{Mod}_{\phi^* \mathscr{A}}$; we wish to show that the pullback of d induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathrm{Mod}_{\phi^* \mathscr{A}}}(\phi^* L_{\mathscr{A}}, \mathscr{M}) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{/\phi^* \mathscr{A}}}(\phi^* \mathscr{A}, \phi^* \mathscr{A} \oplus \mathscr{M}).$$

Unwinding the definitions, we can identify θ with the composite map

$$\begin{aligned} \mathrm{Map}_{\mathrm{Mod}_{\mathscr{A}}}(L_{\mathscr{A}}, \phi_* \mathscr{M}) &\xrightarrow{\theta'} \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})_{/\mathscr{A}}}(\mathscr{A}, \mathscr{A} \oplus \phi_* \mathscr{M}) \\ &\xrightarrow{\theta''} \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})_{/\phi_* \phi^* \mathscr{A}}}(\mathscr{A}, \phi_*(\phi^* \mathscr{A} \oplus \mathscr{M})). \end{aligned}$$

The universal property of d implies that θ' is a homotopy equivalence, and θ'' is a homotopy equivalence because the diagram

$$\begin{array}{ccc} \mathscr{A} \oplus \phi_* \mathscr{M} & \longrightarrow & \phi_*(\phi^* \mathscr{A} \oplus \mathscr{M}) \\ \downarrow & & \downarrow \\ \mathscr{A} & \longrightarrow & \phi_* \phi^* \mathscr{A} \end{array}$$

is a pullback square. □

Example 17.1.1.6. Let $\mathcal{C}$ be a small ∞-category, and let $\mathcal{X} = \mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ be the ∞-category of presheaves on $\mathcal{C}$. We can identify $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ with the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ of presheaves of spectra on $\mathcal{C}$, and $\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$ with the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{CAlg})$ of presheaves of $\mathbb{E}_\infty$-rings on $\mathcal{C}$. If $\mathscr{A} : \mathcal{C}^{\mathrm{op}} \to \mathrm{CAlg}$ is such a presheaf, then Proposition 17.1.1.5 implies that $L_{\mathscr{A}}$ is given pointwise by the formula $L_{\mathscr{A}}(C) = L_{\mathscr{A}(C)}$ for $C \in \mathcal{C}$; here the right hand side denotes the $\mathscr{A}(C)$-module given by the absolute cotangent complex of the $\mathbb{E}_\infty$-ring $\mathscr{A}(C)$.

Example 17.1.1.7. Let $\mathcal{X}$ an arbitrary ∞-topos. Then there exists a small ∞-category $\mathcal{C}$ such that $\mathcal{X}$ is equivalent to an accessible left exact localization of $\mathcal{P}(\mathcal{C})$. Let us identify $\mathcal{X}$ with its image in $\mathcal{P}(\mathcal{C})$, and let $f^* : \mathcal{P}(\mathcal{C}) \to \mathcal{X}$ denote a left adjoint to the inclusion. Let $\mathscr{A}$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, so that we can identify $\mathscr{A}$ with a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{CAlg}$.

Combining Example 17.1.1.6 with Proposition 17.1.1.5, we deduce that $L_{\mathscr{A}} = f^* \mathscr{M}$, where $\mathscr{M} \in \mathrm{Mod}_{\mathscr{A}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}))$ is given by the formula $\mathscr{M}(C) = L_{\mathscr{A}(C)}$. In other words, $L_{\mathscr{A}}$ is the sheafification of the presheaf obtained from $\mathscr{A}$ by pointwise application of the algebraic cotangent complex functor $A \mapsto L_A$ defined in §HA.7.3.2 .

Definition 17.1.1.8. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectrally ringed ∞-topos. We let L_{X} denote the absolute cotangent complex $L_{\mathscr{O}_{\mathsf{X}}} \in \mathrm{Mod}_{\mathscr{O}_{\mathsf{X}}}$. We will refer to L_{X} as the *absolute cotangent complex* of X.

If $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}})$ is another spectrally ringed ∞-topos and $\phi : \mathsf{X} \to \mathsf{Y}$ is a morphism of spectrally ringed ∞-topoi, we let $L_{\mathsf{X}/\mathsf{Y}} \in \mathrm{Mod}_{\mathscr{O}_{\mathsf{X}}}$ denote the relative cotangent complex of the morphism $\phi^* \mathscr{O}_{\mathsf{Y}} \to \mathscr{O}_{\mathsf{X}}$ in $\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$; we refer to $L_{\mathsf{X}/\mathsf{Y}}$ as the *relative cotangent complex* of the morphism ϕ.

Remark 17.1.1.9. Let $\phi : \mathsf{X} \to \mathsf{Y}$ be a map of spectrally ringed ∞-topoi. If the structure sheaf of X is the pullback of the structure sheaf of Y, then the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}}$ vanishes. In particular, if ϕ is étale, then $L_{\mathsf{X}/\mathsf{Y}} \simeq 0$.

Remark 17.1.1.10. Suppose we are given morphisms of spectrally ringed ∞-topoi $\mathsf{X} \xrightarrow{\phi} \mathsf{Y} \xrightarrow{\psi} \mathsf{Z}$. Using Propositions 17.1.1.5 and HA.7.3.3.5 , we deduce that the diagram

$$\begin{array}{ccc} \phi^* L_{\mathsf{Y}/\mathsf{Z}} & \longrightarrow & L_{\mathsf{X}/\mathsf{Z}} \\ \downarrow & & \downarrow \\ \phi^* L_{\mathsf{Y}/\mathsf{Y}} & \longrightarrow & L_{\mathsf{X}/\mathsf{Y}} \end{array}$$

is a pushout square in the stable ∞-category $\mathrm{Mod}_{\mathscr{O}_{\mathsf{X}}}$. Since $L_{\mathsf{Y}/\mathsf{Y}} \simeq 0$, we obtain a fiber sequence $\phi^* L_{\mathsf{Y}/\mathsf{Z}} \to L_{\mathsf{X}/\mathsf{Z}} \to L_{\mathsf{X}/\mathsf{Y}}$.

17.1.2 The Cotangent Complex of a Spectral Deligne-Mumford Stack

When restricted to spectral Deligne-Mumford stacks, the constructions of §17.1 do not leave the world of spectral algebraic geometry.

Proposition 17.1.2.1. *Let X be a nonconnective spectral Deligne-Mumford stack. Then the cotangent complex L_{X} is a quasi-coherent sheaf on X.*

Corollary 17.1.2.2. *Let $\phi : \mathsf{X} \to \mathsf{Y}$ be a morphism of nonconnective spectral Deligne-Mumford stacks. Then the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}}$ is a quasi-coherent sheaf on X.*

The proof of Proposition 17.1.2.1 will require some preliminary observations.

Remark 17.1.2.3. Let $\phi : A \to B$ be an étale morphism of $\mathbb{E}_\infty$-rings. Then the relative cotangent complex $L_{B/A}$ vanishes (Corollary HA.7.5.4.5). It follows that, for every $\mathbb{E}_\infty$-ring R, every R-module M, and every map $\eta : L_R \to \Sigma M$, the diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}}(B, R^\eta) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(B, R) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}}(A, R^\eta) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}}(A, R) \end{array}$$

is a pullback square. In particular, taking $\eta = 0$, we obtain a pullback square

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}}(B, R \oplus M) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(B, R) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}}(A, R \oplus M) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}}(A, R). \end{array}$$

Remark 17.1.2.4. Let $\mathcal{X}$ be an ∞-topos, let $\mathscr{A}$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and let $\overline{\mathscr{A}}$ be a square-zero extension of $\mathscr{A}$. Using Remark 17.1.2.3, we deduce:

- (a) If $\mathscr{A}$ is local (Henselian, strictly Henselian), then $\overline{\mathscr{A}}$ is also local (Henselian, strictly Henselian).
- (b) Assume that $\mathscr{A}$ is local, and let $\mathscr{B}$ be another local sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then a morphism $\phi : \mathscr{B} \to \overline{\mathscr{A}}$ is local if and only if the composite map $\mathscr{B} \xrightarrow{\phi} \overline{\mathscr{A}} \to \mathscr{A}$ is local. In particular, the projection map $\overline{\mathscr{A}} \to \mathscr{A}$ is local.

Lemma 17.1.2.5. *Let A be an $\mathbb{E}_\infty$-ring and let $\mathsf{X} = (\mathcal{X}, \mathscr{O}) = \mathrm{Sp\acute{e}t}\, A$ denote the corresponding nonconnective spectral Deligne-Mumford stack. Then the cotangent complex $L_{\mathscr{O}}$ is a quasi-coherent sheaf on X. The equivalence $\mathrm{Mod}_A \simeq \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A)$ carries L_A to the absolute cotangent complex $L_{\mathscr{O}}$.*

Proof. The universal derivation $\mathscr{O} \to \mathscr{O} \oplus L_{\mathscr{O}}$ induces a morphism

$$A \simeq \Gamma(\mathcal{X}; \mathscr{O}) \to \Gamma(\mathcal{X}; \mathscr{O} \oplus L_{\mathscr{O}}) \simeq A \oplus \Gamma(\mathcal{X}; L_{\mathscr{O}})$$

in $\mathrm{CAlg}_{/A}$, which is classified by a map of A-modules $\epsilon : L_A \to \Gamma(\mathcal{X}; L_{\mathscr{O}})$. Let $\mathscr{M}$ denote a preimage of L_A under the equivalence $\mathrm{QCoh}(\mathsf{X}) \simeq \mathrm{Mod}_A$, so that ϵ determines a morphism $\epsilon' : \mathscr{M} \to L_{\mathscr{O}}$ in $\mathrm{Mod}_{\mathscr{O}}$. We will prove that ϵ' is an equivalence. To prove this, let $\mathscr{M}' \in \mathrm{Mod}_{\mathscr{O}}$ be arbitrary. We wish to show that composition with ϵ' induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}}}(L_{\mathscr{O}}, \mathscr{M}') \to \mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}}}(\mathscr{M}, \mathscr{M}') \simeq \mathrm{Map}_{\mathrm{Mod}_A}(L_A, \Gamma(\mathcal{X}; \mathscr{M}')).$$

Invoking the universal properties of $L_{\mathscr{O}}$ and L_A, we can identify θ with the map

$$\theta' : \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{/\mathscr{O}}}(\mathscr{O}, \mathscr{O} \oplus \mathscr{M}') \to \mathrm{Map}_{\mathrm{CAlg}_{/A}}(A, A \oplus \Gamma(\mathcal{X}; \mathscr{M}')).$$

It follows from Remark 17.1.2.4 (and the universal property of $\mathsf{X} = \operatorname{Spét} A$) that this map is a homotopy equivalence. □

Proof of Proposition 17.1.2.1. The assertion is local on X (Proposition 17.1.1.5). We may therefore assume without loss of generality that X is affine, in which case the result follows from Lemma 17.1.2.5. □

We note the following additional consequence of Lemma 17.1.2.5:

Proposition 17.1.2.6. *Suppose we are given a pullback square*

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ \mathsf{Y}' & \longrightarrow & \mathsf{Y} \end{array}$$

of nonconnective spectral Deligne-Mumford stacks. Then the canonical map

$$\phi^* L_{\mathsf{X}/\mathsf{Y}} \to L_{\mathsf{X}'/\mathsf{Y}'}$$

is an equivalence in $\operatorname{QCoh}(\mathsf{X}')$*.*

Proof. The assertion is local on Y; we may therefore assume without loss of generality that $\mathsf{Y} = \operatorname{Spét} A$ is affine. Similarly, we can assume that $\mathsf{Y}' = \operatorname{Spét} A'$ and $\mathsf{X}' = \operatorname{Spét} B$ are affine. Then $\mathsf{X}' \simeq \operatorname{Spét} B'$, where $B' = A' \otimes_A B$. Using Lemma 17.1.2.5, we are reduced to proving that the canonical map $B' \otimes_B L_{B/A} \to L_{B'/A'}$ is an equivalence of B'-modules, which is a special case of Proposition HA.7.3.3.7 . □

We close this section with a variant of Proposition 17.1.2.1:

Proposition 17.1.2.7. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack (Definition 8.1.3.1). Then the cotangent complex $L_{\mathscr{O}_{\mathfrak{X}}} \in \operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ is weakly quasi-coherent (see Definition 8.2.3.1).*

Proof. The assertion is local on $\mathfrak{X}$, so we may assume without loss of generality that $\mathfrak{X} = \operatorname{Spf} A$ for some adic $\mathbb{E}_\infty$-ring A. Let $f : \mathfrak{X} \to \operatorname{Spét} A$ be the canonical map and let us regard $\operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}}$ as a stable A-linear ∞-category. Then $f^* L_{\operatorname{Spét} A} \simeq L_A \otimes_A \mathscr{O}_{\mathfrak{X}}$ is weakly quasi-coherent, and we have a fiber sequence $f^* L_{\operatorname{Spét} A} \to L_{\mathsf{X}} \to L_{\mathsf{X}/\operatorname{Spét} A}$. Consequently, to show that L_{X} is weakly quasi-coherent, it will suffice to show that $L_{\mathsf{X}/\operatorname{Spét} A}$ is weakly quasi-coherent.

Let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition, so that we can identify the structure sheaf $\mathscr{O}_{\operatorname{Spf} A}$ with the CAlg-valued presheaf given by $(B \in \operatorname{CAlg}^{\text{ét}}_A) \mapsto B^{\wedge}_I$. Unwinding the definitions, we see that $L_{\mathfrak{X}/\operatorname{Spét} A}$ can be identified with the sheafification of the presheaf given by $B \mapsto L_{B^{\wedge}_I / B}$. Note that if $\{B_n\}_{n>0}$ is a tower of B-algebras satisfying the requirements of Lemma 8.1.2.2, then the unit map $B_n \to B_n \otimes_B B^{\wedge}_I$ is an equivalence for each $n > 0$, so that

$$B_n \otimes_B L_{B^{\wedge}_I / B} \simeq L_{(B_n \otimes_B B^{\wedge}_I)/B_n} \simeq 0.$$

Using Lemma 8.1.2.3, we deduce that the I-completion of $L_{B^{\wedge}_I / B}$ vanishes: that is, each of the relative cotangent complexes $L_{B^{\wedge}_I / B}$ is I-local when regarded as an A-module. It follows that the sheaf $L_{\mathfrak{X} / \operatorname{Sp\acute{e}t} A}$ is an I-local object of the stable A-linear ∞-category $\operatorname{Mod}_{\mathscr{O}_{\mathfrak{X}}}$, and is therefore weakly coherent by Proposition 8.3.2.2. □

In the situation of Proposition 17.1.2.7, the cotangent complex $L_{\mathfrak{X}}$ is generally not quasi-coherent. However, we can remedy this by passing to the completion:

Definition 17.1.2.8. Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. We let $L^{\wedge}_{\mathfrak{X}}$ denote the quasi-coherent sheaf on $\mathfrak{X}$ obtained by applying Construction 8.2.4.1 to the cotangent complex $L_{\mathfrak{X}}$ (which is weakly quasi-coherent by virtue of Proposition 17.1.2.7). We will refer to $L^{\wedge}_{\mathfrak{X}}$ as the *completed cotangent complex of* $\mathfrak{X}$.

More generally, if $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of formal spectral Deligne-Mumford stacks, we let $L^{\wedge}_{\mathfrak{X}/\mathfrak{Y}} \in \operatorname{QCoh}(\mathfrak{X})$ denote the quasi-coherent sheaf obtained by applying Construction 8.2.4.1 to the relative cotangent complex $L_{\mathfrak{X}/\mathfrak{Y}}$. Equivalently, we can described $L^{\wedge}_{\mathfrak{X}/\mathfrak{Y}}$ as the cofiber of the canonical map $f^{\star} L^{\wedge}_{\mathfrak{Y}} \to L^{\wedge}_{\mathfrak{X}}$.

Example 17.1.2.9. Let A be an adic $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal of definition. Then the equivalence $\operatorname{QCoh}(\operatorname{Spf} A) \simeq \operatorname{Mod}_A^{\operatorname{Cpl}(I)}$ of Corollary 8.2.4.15 carries the completed cotangent complex $L^{\wedge}_{\operatorname{Spf} A}$ to the A-module $(L_A)^{\wedge}_I$.

Example 17.1.2.10. Let X be a spectral Deligne-Mumford stack, let $K \subseteq |\mathsf{X}|$ be a cocompact closed subset, and let $\mathsf{X}^{\wedge}_K$ denote the formal completion of X along K (Definition 8.1.6.1). Then the completed cotangent complex $L^{\wedge}_{\mathsf{X}^{\wedge}_K / \mathsf{X}}$ vanishes. To prove this, we can work locally and thereby reduce to the case where $\mathsf{X} \simeq \operatorname{Sp\acute{e}t} A$ is affine, in which case the desired result follows from the proof of Proposition 17.1.2.7.

17.1.3 Square-Zero Extensions

We now discuss square-zero extensions which are not necessarily trivial.

Construction 17.1.3.1 (The Square-Zero Extension of a Derivation)**.** Let $\mathcal{X}$ be an ∞-topos, $\mathscr{A}$ a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and $\mathscr{M} \in \operatorname{Mod}_{\mathscr{A}}$. Let $\eta : L_{\mathscr{A}} \to \Sigma \mathscr{M}$ be a morphism of $\mathscr{A}$-modules, so that η determines a derivation $d_\eta : \mathscr{A} \to \mathscr{A} \oplus \Sigma \mathscr{M}$ (in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{/\mathscr{A}}$). Similarly, the zero map $L_{\mathscr{A}} \to \Sigma \mathscr{M}$ classifies a map $d_0 : \mathscr{A} \to \mathscr{A} \oplus \Sigma \mathscr{M}$.

Form a pullback diagram

$$\begin{array}{ccc} \mathscr{A}^{\eta} & \longrightarrow & \mathscr{A} \\ \downarrow & & \downarrow d_{\eta} \\ \mathscr{A} & \xrightarrow{d_0} & \mathscr{A} \oplus \Sigma \mathscr{M}. \end{array}$$

We will refer to $\mathscr{A}^{\eta}$ as the *square-zero extension of* $\mathscr{A}$ *determined by* η. By construction, there is a canonical fiber sequence $\mathscr{M} \to \mathscr{A}^{\eta} \to \mathscr{A}$ in the ∞-category of Sp-valued sheaves on $\mathcal{X}$.

In general, the passage from a morphism $\eta : L_{\mathscr{A}} \to \Sigma \mathscr{M}$ to the associated square-zero extension $\mathscr{A}^{\eta}$ involves a loss of information. We now study a situation where this is not the case:

Definition 17.1.3.2. Let $\mathcal{X}$ be an ∞-topos and let $n \geqslant 0$ be an integer. We will say that a morphism $\phi : \overline{\mathscr{A}} \to \mathscr{A}$ in $\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$ is an *n-small extension* if the following conditions are satisfied:

- (i) The sheaf $\mathscr{A}$ is connective.
- (ii) The fiber $\mathcal{I} = \mathrm{fib}(\phi)$ is n-connective (from which it follows that $\overline{\mathscr{A}}$ is also connective).
- (iii) The fiber $\mathcal{I}$ belongs to $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 2n}$.
- (iv) The multiplication on $\mathscr{A}$ induces a nullhomotopic map $\mathcal{I} \otimes \mathcal{I} \to \mathcal{I}$.

The following result is a special case of Theorem HA.7.4.1.26 :

Theorem 17.1.3.3. *Let $\mathcal{X}$ be an ∞-topos, let $\mathscr{A}$ be a connective sheaf of $\mathbb{E}_{\infty}$-rings on $\mathcal{X}$, and let $n \geqslant 0$ be an integer. Let $\mathcal{C}$ denote the full subcategory of $(\mathrm{Mod}_{\mathscr{A}})_{L_{\mathscr{A}}/}$ spanned by morphisms of the form $\eta : L_{\mathscr{A}} \to \Sigma \mathcal{I}$, where $\mathcal{I}$ is n-connective and $(2n)$-truncated. Then the construction $\eta \mapsto \mathscr{A}^{\eta}$ determines a fully faithful embedding from $\mathcal{C}$ to $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{/\mathscr{A}}$, whose essential image is the collection of n-small extensions $\overline{\mathscr{A}} \to \mathscr{A}$.*

We now show that the class of spectral Deligne-Mumford stacks is closed under construction of square-zero extensions described in §17.1.3.

Proposition 17.1.3.4. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack, $\mathscr{M}$ a connective quasi-coherent sheaf on $\mathcal{X}$, and $\eta : L_{\mathsf{X}} \to \Sigma \mathscr{M}$ a morphism in $\mathrm{QCoh}(\mathsf{X})$. Then the pair $(\mathcal{X}, \mathscr{O}^{\eta}_{\mathcal{X}})$ is also a spectral Deligne-Mumford stack. If X is locally Noetherian and $\mathscr{M}$ is almost perfect, then $(\mathcal{X}, \mathscr{O}^{\eta}_{\mathcal{X}})$ is also locally Noetherian.*

The proof of Proposition 17.1.3.4 will require some preliminaries.

Lemma 17.1.3.5. *Let A be a connective $\mathbb{E}_\infty$-ring, M a connective A-module, and $\eta : L_A \to \Sigma M$ a map of A-modules which determines a square-zero extension $A^\eta \to A$. Then the base change functor $\theta : \mathrm{CAlg}^{\acute{e}t}_{A^\eta} \to \mathrm{CAlg}^{\acute{e}t}_A$ is an equivalence of ∞-categories.*

Proof. We have a short exact sequence of abelian groups

$$\pi_0 A^\eta \to \pi_0 A \to \pi_{-1} M.$$

Since M is connective, the map $\pi_0 A^\eta \to \pi_0 A$ is a surjection. Using the structure theory of étale morphisms (Proposition B.1.1.3), we deduce that θ is essentially surjective. It remains to show that θ is fully faithful. Let $\overline{B}$ and $\overline{B}'$ be étale A^η-algebras, and set $B = A \otimes A^{\eta}\overline{B}$ and $B' = A \otimes_{A^\eta} \overline{B}'$. We wish to show that θ induces a homotopy equivalence

$$\phi : \mathrm{Map}_{\mathrm{CAlg}_{A^\eta/}}(\overline{B}', \overline{B}) \to \mathrm{Map}_{\mathrm{CAlg}_{A^\eta/}}(\overline{B}', B) \simeq \mathrm{Map}_{\mathrm{CAlg}_{A/}}(B', B).$$

We have a pullback diagram of A^η-algebras

$$\begin{array}{ccc} \overline{B} & \longrightarrow & B \\ \downarrow & & \downarrow \\ B & \longrightarrow & (A \oplus \Sigma M) \otimes_{A^\eta} \overline{B}. \end{array}$$

We note that the lower right corner can be identified with the square-zero extension $B \oplus \Sigma N$, where $N = B \otimes_A M$. It follows that ϕ is a pullback of the map

$$\phi_0 : \mathrm{Map}_{\mathrm{CAlg}_{A^\eta/}}(\overline{B}', B) \to \mathrm{Map}_{\mathrm{CAlg}_{A^\eta/}}(\overline{B}', B \oplus \Sigma N).$$

It will therefore suffice to show that ϕ_0 is a homotopy equivalence. The projection $B \oplus \Sigma N \to B$ induces a map

$$\psi : \mathrm{Map}_{\mathrm{CAlg}_{A^\eta/}}(\overline{B}', B \oplus \Sigma N) \to \mathrm{Map}_{\mathrm{CAlg}_{A^\eta/}}(\overline{B}', B)$$

which is left homotopy inverse to ϕ_0. We claim that ψ is a homotopy equivalence. To prove this, fix a map of A^η-algebras $f : \overline{B}' \to B$. We will show that the homotopy fiber of ψ over f is contractible. This homotopy fiber is given by $\mathrm{Map}_{\mathrm{Mod}_{\overline{B}'}}(L_{\overline{B}'/A^\eta}, \Sigma N)$, which is contractible by virtue of our assumption that $\overline{B}'$ is étale over A^η. □

Lemma 17.1.3.6. *Let A be a connective $\mathbb{E}_\infty$-ring, M a connective A-module, and $\eta : L_A \to \Sigma M$ a map of A-modules which determines a square-zero extension A^η of A. Then the induced map $\mathrm{Sp\acute{e}t}\, A \to \mathrm{Sp\acute{e}t}\, A^\eta$ induces an equivalence of the underlying ∞-topoi.*

Proof. According to Lemma 17.1.3.5, we have an equivalence of ∞-categories $\mathrm{CAlg}^{\acute{e}t}_{A^\eta} \simeq \mathrm{CAlg}^{\acute{e}t}_A$. Note that a morphism $\overline{f} : \overline{B}' \to \overline{B}$ in $\mathrm{CAlg}^{\acute{e}t}_{A^\eta}$ is faithfully flat if and only if its image $f : B' \to B$ in $\mathrm{CAlg}^{\text{ét}}_A$ is faithfully flat. The "only if" direction is obvious, and the "if" direction follows from the observation that the map of commutative rings $\pi_0 \overline{B} \to \pi_0 B$ is a surjection with nilpotent kernel, and therefore induces a bijection $|\operatorname{Spec} B| \to |\operatorname{Spec} \overline{B}|$. It follows that the equivalence $\mathrm{CAlg}^{\text{ét}}_{A^\eta} \simeq \mathrm{CAlg}^{\text{ét}}_A$ induces an equivalence after taking sheaves with respect to the étale topology, and therefore induces an equivalence between the underlying ∞-topoi of $\operatorname{Spét} A^\eta$ and $\operatorname{Spét} A$ (see Proposition 1.4.2.4). □

Lemma 17.1.3.7. *Let A be a connective $\mathbb{E}_\infty$-ring, let M be an A-module which is connective as a spectrum, and let $\mathscr{M}$ be the corresponding quasi-coherent sheaf on* $\operatorname{Spét} A = (\mathcal{X}, \mathscr{O})$*. Suppose we are given a map $\eta : L_{\mathscr{O}} \to \Sigma \mathscr{M}$ which determines a square-zero extension $\mathscr{O}^\eta$ of $\mathscr{O}$. Passing to global sections (and using Lemma 17.1.2.5), we obtain a map of A-modules $\eta_0 : L_A \to \Sigma M$ which determines a square-zero extension A^{η_0} of A. Then there is a canonical equivalence $(\mathcal{X}, \mathscr{O}^\eta) \simeq$* $\operatorname{Spét} A^{\eta_0}$ *(in the ∞-category* RTop *of spectrally ringed ∞-topoi).*

Proof. Remark 17.1.2.4 implies that $\mathscr{O}^\eta$ is strictly Henselian. Since A^{η_0} can be identified with the $\mathbb{E}_\infty$-ring of global sections of $\mathscr{O}^\eta$, the universal property of $\operatorname{Spét} A^{\eta_0}$ gives a map of spectrally ringed ∞-topoi $\phi : (\mathcal{X}, \mathscr{O}^\eta) \to \operatorname{Spét} A^{\eta_0}$. Lemma 17.1.3.6 implies that ϕ induces an equivalence at the level of the underlying ∞-topoi. Write $\operatorname{Spét} A^{\eta_0}$ as $(\mathcal{X}, \mathscr{O}')$. We can identify $\mathscr{O}'$ with the sheaf of $\mathbb{E}_\infty$-rings on $(\mathrm{CAlg}^{\text{ét}}_A)^{\mathrm{op}}$ given by a homotopy inverse of the equivalence $\mathrm{CAlg}^{\text{ét}}_{A^{\eta_0}} \to \mathrm{CAlg}^{\text{ét}}_A$ of Lemma 17.1.3.5. Then ϕ induces a map of sheaves $\mathscr{O}' \to \mathscr{O}^\eta$; we wish to show that this map is an equivalence. Unwinding the definitions, we are required to show that for every étale A-algebra B, if we let $\eta' : L_B \to B \otimes_A \Sigma M$ denote the map induced by η, then ϕ induces an equivalence of $\mathbb{E}_\infty$-rings $\mathscr{O}'(B) \to B^{\eta'}$. Using Lemma 17.1.3.5, we are reduced to proving that $B^{\eta'}$ is étale over A^{η_0}, and that the canonical map $A \otimes_{A^{\eta_0}} B^{\eta'} \to B$ is an equivalence. This is a special case of Proposition HA.7.4.2.5 . □

Proof of Proposition 17.1.3.4. The assertion is local on $\mathcal{X}$. We may therefore assume without loss of generality that $\mathsf{X} = \operatorname{Spét} A$ is affine, that $\mathscr{M}$ is the quasi-coherent sheaf associated to a connective A-module M, and we can identify η with an A-module map $\eta_0 : L_A \to \Sigma M$. It follows from Lemma 17.1.3.7 that $(\mathcal{X}, \mathscr{O}^\eta_{\mathcal{X}}) \simeq \operatorname{Spét} A^{\eta_0}$ is an affine spectral Deligne-Mumford stack. For every integer n, we have a short exact sequence of $\pi_0 A^{\eta_0}$-modules $\pi_n M \to \pi_n A^{\eta_0} \to \pi_n A$. If A is Noetherian and $M \in \mathrm{Mod}_A$ is almost perfect, then $\pi_n A$ and $\pi_n M$ are Noetherian objects of the abelian category $\mathrm{Mod}^\heartsuit_A$, and therefore also of the abelian category $\mathrm{Mod}^\heartsuit_{A^{\eta_0}}$. It follows that $\pi_n A^{\eta_0}$ is also a Noetherian object of $\mathrm{Mod}^\heartsuit_{A^{\eta_0}}$. In particular, $\pi_0 A^{\eta_0}$ is a Noetherian ring and each homotopy group $\pi_n A^{\eta_0}$ is a finitely generated module over $\pi_0 A^{\eta_0}$. It follows that A^{η_0} is a Noetherian $\mathbb{E}_\infty$-ring, so that the spectral Deligne-Mumford stack $(\mathcal{X}, \mathscr{O}^\eta_{\mathcal{X}}) \simeq \operatorname{Spét} A^{\eta_0}$ is locally Noetherian as desired. □

17.1.4 Connectivity Estimates

Construction 17.1.4.1. Let $\mathcal{X}$ be an ∞-topos and let $f : \mathscr{A} \to \mathscr{B}$ be a morphism of sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. The canonical map $\eta : L_{\mathscr{B}} \to L_{\mathscr{B}/\mathscr{A}}$ determines a square-zero extension $\mathscr{B}^\eta$ of $\mathscr{B}$ by $\Sigma^{-1} L_{\mathscr{B}/\mathscr{A}}$. Since the restriction of η to $L_{\mathscr{A}}$ vanishes, the associated square-zero extension of $\mathscr{A}$ is split: that is, the map f factors as a composition

$$\mathscr{A} \xrightarrow{f'} \mathscr{B}^\eta \xrightarrow{f''} \mathscr{B}.$$

In particular, we obtain a map of $\mathscr{A}$-modules $\mathrm{cofib}(f) \to \mathrm{cofib}(f'')$, which induces a map of $\mathscr{B}$-modules

$$\epsilon_f : \mathscr{B} \otimes_{\mathscr{A}} \mathrm{cofib}(f) \to \mathrm{cofib}(f'') \simeq L_{\mathscr{B}/\mathscr{A}}.$$

The following result is a special case of Theorem HA.7.4.3.12:

Theorem 17.1.4.2. *Let $\mathcal{X}$ be an ∞-topos, let $f : \mathscr{A} \to \mathscr{B}$ be a morphism between sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and let $\epsilon_f : \mathscr{B} \otimes_{\mathscr{A}} \mathrm{cofib}(f) \to L_{\mathscr{B}/\mathscr{A}}$ be defined as in Construction 17.1.4.1. Assume that $\mathscr{A}$ and $\mathscr{B}$ are connective and that the cofiber $\mathrm{cofib}(f)$ is n-connective (as a sheaf of spectra on $\mathcal{X}$). Then the morphism ϵ_f of Construction 17.1.4.1 is $(2n)$-connective: that is, $\mathrm{fib}(\epsilon_f)$ is a $2n$-connective sheaf of spectra on $\mathcal{X}$.*

Let us collect up some consequences of Theorem 17.1.4.2:

Corollary 17.1.4.3. *Let $\mathcal{X}$ be an ∞-topos and let $f : \mathscr{A} \to \mathscr{B}$ be a morphism of connective sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Assume that $\mathrm{cofib}(f)$ is n-connective for some $n \geqslant 0$. Then $L_{\mathscr{B}/\mathscr{A}}$ is n-connective. The converse holds provided that f induces an isomorphism $\pi_0 \mathscr{A} \to \pi_0 \mathscr{B}$.*

Proof. Let $\epsilon_f : \mathscr{B} \otimes_{\mathscr{A}} \mathrm{cofib}(f) \to L_{\mathscr{B}/\mathscr{A}}$ be as in Construction 17.1.4.1, so that we have a fiber sequence of $\mathscr{B}$-modules: $\mathrm{fib}(\epsilon_f) \to \mathscr{B} \otimes_{\mathscr{A}} \mathrm{cofib}(f) \xrightarrow{\epsilon_f} L_{\mathscr{B}/\mathscr{A}}$. To prove that $L_{\mathscr{B}/\mathscr{A}}$ is n-connective, it will suffice to show that $\mathscr{B} \otimes_{\mathscr{A}} \mathrm{cofib}(f)$ is n-connective and that $\mathrm{fib}(\epsilon_f)$ is $(n-1)$-connective. The first assertion is obvious, and the second follows from Theorem 17.1.4.2 (since $2n \geqslant n-1$).

To prove the converse, let us suppose that $\mathrm{cofib}(f)$ is *not* n-connective. We wish to show that $L_{\mathscr{B}/\mathscr{A}}$ is not n-connective. Let us assume that n is chosen as small as possible, so that $\mathrm{cofib}(f)$ is $(n-1)$-connective. By assumption, f induces an isomorphism $\pi_0 \mathscr{A} \to \pi_0 \mathscr{B}$, so we must have $n \geqslant 2$. Applying Theorem 17.1.4.2, we conclude that ϵ_f is $(2n-2)$-connective. Since $n \geqslant 2$, we deduce in particular that ϵ_f is n-connective, so that the map $\pi_{n-1}(\mathscr{B} \otimes_{\mathscr{A}} \mathrm{cofib}(f)) \to \pi_{n-1} L_{\mathscr{B}/\mathscr{A}}$ is an isomorphism. Since $\mathrm{cofib}(f)$ is $(n-1)$-connective and $\pi_0 \mathscr{A} \simeq \pi_0 \mathscr{B}$, the map $\pi_{n-1} \mathrm{cofib}(f) \to \pi_{n-1}(\mathscr{B} \otimes_{\mathscr{A}} \mathrm{cofib}(f))$ is an isomorphism. It follows that $\pi_{n-1} \mathrm{cofib}(f) \to \pi_{n-1} L_{\mathscr{B}/\mathscr{A}}$ is also an isomorphism, so that $\pi_{n-1} L_{\mathscr{B}/\mathscr{A}}$ is nonzero. □

Corollary 17.1.4.4. *Let $\mathcal{X}$ be a sheaf of $\mathbb{E}_\infty$-rings and let $\mathscr{A}$ be a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then the absolute cotangent complex $L_{\mathscr{A}}$ is connective.*

Proof. Let $\mathbf{1}$ denote the initial object of $\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$, and apply Corollary 17.1.4.3 to the unit map $\mathbf{1} \to \mathscr{A}$. □

Corollary 17.1.4.5. *Let $f : \mathscr{A} \to \mathscr{B}$ be a map of connective sheaves of $\mathbb{E}_\infty$-rings on an ∞-topos $\mathcal{X}$. Assume that $\mathrm{cofib}(f)$ is n-connective for $n \geqslant 0$. Then the induced map $L_f : L_{\mathscr{A}} \to L_{\mathscr{B}}$ has n-connective cofiber. In particular, the canonical map $\pi_0 L_{\mathscr{A}} \to \pi_0 L_{\pi_0 \mathscr{A}}$ is an isomorphism.*

Proof. The map L_f factors as a composition $L_{\mathscr{A}} \xrightarrow{g} \mathscr{B} \otimes_{\mathscr{A}} L_{\mathscr{A}} \xrightarrow{g'} L_{\mathscr{B}}$. We observe that $\mathrm{cofib}(g) \simeq \mathrm{cofib}(f) \otimes_{\mathscr{A}} L_{\mathscr{A}}$. Since the cotangent complex $L_{\mathscr{A}}$ is connective and $\mathrm{cofib}(f)$ is n-connective, we conclude that $\mathrm{cofib}(g)$ is n-connective. It will therefore suffice to show that $\mathrm{cofib}(g') \simeq L_{B/A}$ is n-connective. Let ϵ_f be as Construction 17.1.4.1, so we have a fiber sequence $\mathscr{B} \otimes_{\mathscr{A}} \mathrm{cofib}(f) \xrightarrow{\epsilon_f} L_{\mathscr{B}/\mathscr{A}} \to \mathrm{cofib}(\epsilon_f)$. It therefore suffices to show that $\mathscr{B} \otimes_{\mathscr{A}} \mathrm{cofib}(f)$ and $\mathrm{cofib}(\epsilon_f)$ are n-connective. The first assertion follows immediately from the n-connectivity of $\mathrm{cofib}(f)$, and the second from Theorem 17.1.4.2 since $2n + 1 \geqslant n$. □

17.1.5 Finiteness Conditions on $L_{\mathsf{X}/\mathsf{Y}}$

Let $\phi : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Many of the algebro-geometric properties of ϕ studied earlier in this book (such as the properties of being étale, fiber-smooth, differentially smooth, locally of finite presentation, and locally almost of finite presentation) can be formulated as properties of the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}}$:

Proposition 17.1.5.1. *Let $\phi : \mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}}) = \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Then:*

(1) *If the morphism ϕ is étale, then the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}}$ vanishes.*

(2) *If the morphism ϕ is locally of finite presentation, then the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}} \in \mathrm{QCoh}(\mathsf{X})$ is perfect.*

(3) *If the morphism ϕ is locally almost of finite presentation, then the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}} \in \mathrm{QCoh}(\mathsf{X})$ is almost perfect.*

(4) *If the morphism ϕ is locally of finite generation to order n (see Definition 4.2.0.1), then the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}} \in \mathrm{QCoh}(\mathsf{X})$ is perfect to order n.*

(5) *If the morphism ϕ is differentially smooth, then the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}} \in \mathrm{QCoh}(\mathsf{X})$ is locally free of finite rank.*

(6) *If the morphism ϕ is fiber-smooth, then ϕ is flat and, for every field κ and every morphism $\psi : \mathrm{Sp\acute{e}t}\,\kappa \to \mathsf{X}$, we have $\pi_1 \psi^* L_{\mathsf{X}/\mathsf{Y}} \simeq 0$.*

The converse assertions hold if we assume that ϕ exhibits $(\mathcal{X}, \pi_0 \mathscr{O}_{\mathsf{X}})$ as locally finitely 0-presented over $(\mathcal{Y}, \pi_0 \mathscr{O}_{\mathsf{Y}})$ (see Definition 4.2.3.1).

Proof. We may assume without loss of generality that $\mathsf{X} = \mathrm{Sp\acute{e}t}\, A$ and $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, B$ are affine. In this case, Lemma 17.1.2.5 allows us to identify $L_{\mathsf{X}/\mathsf{Y}}$ with the quasi-coherent sheaf associated to the A-module $L_{A/B}$. Assertion (1) now follows from Lemma B.1.3.3, assertions (2) and (3) from Theorem HA.7.4.3.18 , assertion (4) from Proposition 4.1.2.1, assertion (5) from the definition of differential smoothness (see Proposition 11.2.2.1), and assertion (6) from Corollary 11.2.4.2. □

Corollary 17.1.5.2. *Let $\phi : \mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}}) = \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. Then ϕ is an equivalence if and only if it satisfies the following conditions:*

(a) *The underlying map of 0-truncations $(\mathcal{X}, \pi_0 \mathscr{O}_{\mathsf{X}}) \to (\mathcal{Y}, \pi_0 \mathscr{O}_{\mathsf{Y}})$ is an equivalence.*

(b) *The relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}} \in \mathrm{QCoh}(\mathsf{X})$ vanishes.*

17.1.6 Application: Noetherian Approximation

We conclude this section by applying the theory of the cotangent complex to prove a technical result which was needed in §4.4.4. Recall that if $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ is a spectral Deligne-Mumford stack, then $\tau_{\leqslant n} \mathsf{X} = (\mathcal{X}, \tau_{\leqslant n} \mathscr{O}_{\mathsf{X}})$ denotes its n-truncation.

Proposition 17.1.6.1. *Let $n \geqslant 0$, and suppose we are given a commutative diagram σ :*

$$\begin{array}{ccc} \tau_{\leqslant n} \mathsf{X} & \xrightarrow{g} & \mathsf{Y} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathsf{X} & \xrightarrow{h} & \mathsf{Z} \end{array}$$

of quasi-compact, quasi-separated spectral algebraic spaces. Assume the following:

(i) *The morphism g is affine.*

(ii) *The morphism f is locally almost of finite presentation.*

(iii) *The spectral algebraic space Y is locally Noetherian.*

Then σ can be extended to a commutative diagram σ':

$$\begin{array}{ccc} \tau_{\leqslant n}\mathsf{X} & \xrightarrow{g} & \mathsf{Y} \\ \downarrow & & \downarrow{\scriptstyle i} \\ \mathsf{X} & \xrightarrow{\overline{g}} & \mathsf{Y}' \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f'} \\ \mathsf{X} & \longrightarrow & \mathsf{Z} \end{array}$$

where Y' is locally Noetherian and i induces an equivalence $\tau_{\leqslant n}\mathsf{Y} \to \tau_{\leqslant n}\mathsf{Y}'$.

Proof. Write $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}})$. Let $\mathscr{A} = f^{-1}\mathscr{O}_{\mathsf{Z}} \in \mathcal{S}\mathrm{hv}^{\mathrm{cn}}_{\mathrm{CAlg}}(\mathsf{Y})$ denote the pullback of the structure sheaf of Z along the morphism f. Let $\mathcal{C} = \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})_{\mathscr{A}/}$ denote the ∞-category of $\mathscr{A}$-algebra objects of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})$. Using the morphism h, we can regard $\mathscr{B} = g_* \mathscr{O}_{\mathsf{X}}$ as an object of $\mathcal{C}$. The diagram σ determines a morphism $u(n) : \mathscr{O}_{\mathsf{Y}} \to \tau_{\leqslant n}\mathscr{B}$ in the ∞-category $\mathcal{C}$. We will show that there exists a commutative diagram

$$\begin{array}{ccc} \cdots & & \cdots \\ \downarrow & & \downarrow \\ \mathscr{O}_{\mathsf{Y}}(n+2) & \xrightarrow{u(n+2)} & \tau_{\leqslant n+2}\mathscr{B} \\ \downarrow & & \downarrow \\ \mathscr{O}_{\mathsf{Y}}(n+1) & \xrightarrow{u(n+1)} & \tau_{\leqslant n+1}\mathscr{B} \\ \downarrow & & \downarrow \\ \mathscr{O}_{\mathsf{Y}}(n) & \xrightarrow{u(n)} & \tau_{\leqslant n}\mathscr{B} \end{array}$$

in the ∞-category $\mathcal{C}$ which satisfies the following conditions:

(*a*) The sheaf $\mathscr{O}_{\mathsf{Y}}(n)$ coincides with $\mathscr{O}_{\mathsf{Y}}$ and the morphism $u(n)$ coincides with u.

(*b*) Each of the pairs $(\mathcal{Y}, \mathscr{O}_{\mathsf{Y}}(m))$ is a locally Noetherian spectral algebraic space.

(*c*) Each of the maps $\mathscr{O}_{\mathsf{Y}}(m+1) \to \mathscr{O}_{\mathsf{Y}}(m)$ induces an equivalence of m-truncations.

Assuming this can be done, we define $\mathscr{O}_{\mathsf{Y}'} = \varprojlim \mathscr{O}_{\mathsf{Y}}(m) \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})$. It follows from (*b*) and (*c*) that the pair $\mathsf{Y}' = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}'})$ is a spectral algebraic space, and that the natural map $(\mathcal{Y}, \mathscr{O}_{\mathsf{Y}}(m)) \to (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}'})$ induces an equivalence on m-truncations for each $m \geqslant n$ (to prove this, we can work locally and thereby reduce to the case where Y is affine, in which case the result follows immediately from the Postnikov-completeness of the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$). In particular, we conclude from (*b*) that $\mathsf{Y}' = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}}(m)$ is locally Noetherian (since this can be tested at the level of m-truncations) and from (*c*) that the natural map $i : \mathsf{Y} \to \mathsf{Y}'$ induces an equivalence of n-truncations. Moreover, the inverse system $\{u(m)\}_{m \geqslant n}$ determines a map $u(\infty) : \mathscr{O}_{\mathsf{Y}'} \to \varprojlim \tau_{\leqslant m} \mathscr{B} \simeq \mathscr{B}$ which classifies a morphism $\overline{g} : \mathsf{X} \to \mathsf{Y}'$ having the desired properties.

It remains to construct the maps $u(m) : \mathscr{O}_{\mathsf{Y}}(m) \to \tau_{\leqslant m} \mathscr{B}$. We proceed by induction on m, with the case $m = n$ handled by (a). To carry out the inductive step, let us assume that $u(m)$ has been constructed for some $m \geqslant n$. It follows from assumptions (b), (c), and Remark 4.2.0.4 that the canonical map $\mathsf{Y} \to (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}}(m))$ is locally almost of finite presentation. Combining this observation with (ii) and Proposition 17.1.5.1, we deduce that the relative cotangent complexes $L_{\mathscr{O}_{\mathsf{Y}} / \mathscr{A}}, L_{\mathscr{O}_{\mathsf{Y}} / \mathscr{O}_{\mathsf{Y}}(m)} \in \mathrm{QCoh}(\mathsf{Y})$ are almost perfect. Using the fiber sequence

$$\mathscr{O}_{\mathsf{Y}} \otimes_{\mathscr{O}_{\mathsf{Y}}(m)} L_{\mathscr{O}_{\mathsf{Y}}(m) / \mathscr{A}} \to L_{\mathscr{O}_{\mathsf{Y}} / \mathscr{A}} \to L_{\mathscr{O}_{\mathsf{Y}} / \mathscr{O}_{\mathsf{Y}}(m)}$$

and Proposition 2.7.3.2, we deduce that $L_{\mathscr{O}_{\mathsf{Y}}(m) / \mathscr{A}}$ is almost perfect (when viewed as a quasi-coherent sheaf on the spectral algebraic space $(\mathcal{Y}, \mathscr{O}_{\mathsf{Y}}(m))$.

Set $\mathscr{F} = g_* \pi_{m+1} \mathscr{O}_{\mathsf{X}}$. Let us regard $\mathscr{F}$ as a $\mathscr{O}_{\mathsf{Y}}(m)$-module via the morphism $u(m)$, so that it is a quasi-coherent sheaf on the spectral Deligne-Mumford stack $(\mathsf{Y}, \mathscr{O}_{\mathsf{Y}}(m))$. Using Proposition 10.5.1.5, we can write $\mathscr{F}$ as the colimit of a filtered diagram $\{\mathscr{F}_\alpha\}$, where each $\mathscr{F}_\alpha \in \mathrm{QCoh}(\mathcal{Y}, \mathscr{O}_{\mathsf{Y}}(m))^{\heartsuit}$ is almost perfect. It follows from Theorem 17.1.4.2 that we can regard $\tau_{\leqslant m+1} \mathscr{B}$ as a square-zero extension of $\tau_{\leqslant m} \mathscr{B}$ by $\Sigma^{m+1} \mathscr{F}$: that is, we have an equivalence $\tau_{\leqslant m+1} \mathscr{B} \simeq (\tau_{\leqslant m} \mathscr{B})^{\eta}$ for some map $\eta : L_{\tau_{\leqslant m} \mathscr{B} / \mathscr{A}} \to \Sigma^{m+2} \mathscr{F}$. Let η' denote the composite map

$$L_{\mathscr{O}_{\mathsf{Y}}(m) / \mathscr{A}} \xrightarrow{u(m)} L_{\tau_{\leqslant m} \mathscr{B} / \mathscr{A}} \xrightarrow{\eta} \Sigma^{m+2} \mathscr{F} .$$

Since $L_{\mathscr{O}_{\mathsf{Y}}(m) / \mathscr{A}}$ is almost perfect, the morphism η' factors as a composition

$$L_{\mathscr{O}_{\mathsf{Y}}(m) / \mathscr{A}} \xrightarrow{\eta''} \Sigma^{m+2} \mathscr{F}_\alpha \to \Sigma^{m+2} \mathscr{F}$$

for some index α. We let $\mathscr{O}_{\mathsf{Y}}(m+1) = \mathscr{O}_{\mathsf{Y}}(m)^{\eta''}$ denote the square-zero extension of $\mathscr{O}_{\mathsf{Y}}(m)$ by $\Sigma^{m+1} \mathscr{F}_\alpha$ determined by η'', so that $u(m)$ lifts to a morphism

$$u(m+1) : \mathscr{O}_{\mathsf{Y}}(m+1) = \mathscr{O}_{\mathsf{Y}}(m)^{\eta''} \to (\tau_{\leqslant m} \mathscr{B})^{\eta} \simeq \tau_{\leqslant m+1} \mathscr{B} .$$

By construction, we have a canonical fiber sequence $\Sigma^{m+1} \mathscr{F}_\alpha \to \mathscr{O}_{\mathsf{Y}}(m+1) \xrightarrow{\rho} \mathscr{O}_{\mathsf{Y}}(m)$, so that the morphism ρ induces an equivalence on m-truncations. It follows from Proposition 17.1.3.4 that the pair $(\mathcal{Y}, \mathscr{O}_{\mathsf{Y}}(m+1))$ is a locally Noetherian spectral Deligne-Mumford stack (and therefore also a spectral algebraic space, since its m-truncation is a spectral algebraic space). □

17.2 The Cotangent Complex of a Functor

Let $\phi : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks, and let $X = h_{\mathsf{X}}$ and $Y = h_{\mathsf{Y}}$ be the functors represented by X and Y, respectively. The relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}}$ is a quasi-coherent sheaf on X (Proposition 17.1.2.1). According to Proposition 6.2.4.1, we can identify quasi-coherent sheaves on X with quasi-coherent sheaves on the functor X. In particular, we can think of $L_{\mathsf{X}/\mathsf{Y}}$ as a rule which assigns an A-module $\eta^* L_{\mathsf{X}/\mathsf{Y}} \in \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A) \simeq \mathrm{Mod}_A$ to each A-valued point $\eta : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$. Unwinding the definitions, we see that if N is a connective A-module, then we can identify A-module maps from $\eta^* L_{\mathsf{X}/\mathsf{Y}}$ into N with dotted arrows completing the diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}\, A & \xrightarrow{\eta} & \mathsf{X} \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Sp\acute{e}t}(A \oplus N) & \longrightarrow & \mathsf{Y}, \end{array}$$

where the lower horizontal map is given by the composition

$$\mathrm{Sp\acute{e}t}(A \oplus N) \to \mathrm{Sp\acute{e}t}\, A \xrightarrow{\eta} \mathsf{X} \to \mathsf{Y}.$$

The above analysis suggests the possibility of defining the relative cotangent complex for a general natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Our goal in this section is to develop the theory of the cotangent complex in this setting, and to show that it agrees with Definition 17.1.1.8 when we restrict to functors which are represented by spectral Deligne-Mumford stacks (Proposition 17.2.5.1).

17.2.1 Almost Representable Functors

Suppose we are given a natural transformation $f : X \to Y$ between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. In good cases, we would like to associate to f a relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$. $L_{X/Y} \in \mathrm{QCoh}(X)$. We can think of $L_{X/Y}$ as a rule which assigns to each point $\eta \in X(A)$ an A-module M_η, which is compatible with base change. This module M_η should have the following property: for every connective A-module N, $\mathrm{Map}_{\mathrm{Mod}_A}(M_\eta, N)$ is given by the fiber of the canonical map

$$X(A \oplus N) \to X(A) \times_{Y(A)} Y(A \oplus N)$$

(over the base point determined by η). In the special case where M_η is connective, this mapping property determines M_η up to a contractible space of choices (by the ∞-categorical version of Yoneda's lemma). However, for some applications this is unnecessarily restrictive: the cotangent complex of an Artin stack (over a field of characteristic zero, say) is usually not connective. We will therefore need a mechanism for recovering M_η given the functor that it corepresents on the ∞-category $\mathrm{Mod}_A^{\mathrm{cn}}$ of connective A-modules.

Notation 17.2.1.1. Recall that if $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories which admit final objects, then a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *reduced* if it preserves final objects. If $\mathcal{C}$ admits finite colimits and $\mathcal{D}$ admits finite limits, we say that F is *excisive* if it carries pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$. We let $\mathrm{Exc}_*(\mathcal{C},\mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ spanned by those functors which are reduced and excisive.

Lemma 17.2.1.2. *Let $\mathcal{C}$ be a stable ∞-category equipped with a right-bounded t-structure. Then the restriction functor $\theta : \mathrm{Exc}_*(\mathcal{C},\mathcal{S}) \to \mathrm{Exc}_*(\mathcal{C}_{\geqslant 0},\mathcal{S})$ is a trivial Kan fibration.*

Proof. Since θ is obviously a categorical fibration, it will suffice to show that θ is a categorical equivalence. Note that $\mathrm{Exc}_*(\mathcal{C},\mathcal{S})$ is the homotopy limit of the tower of ∞-categories

$$\cdots \to \mathrm{Exc}_*(\mathcal{C}_{\geqslant -2},\mathcal{S}) \to \mathrm{Exc}_*(\mathcal{C}_{\geqslant -1},\mathcal{S}) \to \mathrm{Exc}_*(\mathcal{C}_{\geqslant 0},\mathcal{S}).$$

It will therefore suffice to show that each of the restriction maps

$$\mathrm{Exc}_*(\mathcal{C}_{\geqslant -n},\mathcal{S}) \to \mathrm{Exc}_*(\mathcal{C}_{\geqslant 0},\mathcal{S})$$

is an equivalence of ∞-categories. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Exc}_*(\mathcal{C}_{\geqslant -n},\mathrm{Sp}) & \longrightarrow & \mathrm{Exc}_*(\mathcal{C}_{\geqslant 0},\mathrm{Sp}) \\ \downarrow & & \downarrow \\ \mathrm{Exc}_*(\mathcal{C}_{\geqslant -n},\mathcal{S}) & \longrightarrow & \mathrm{Exc}_*(\mathcal{C}_{\geqslant 0},\mathcal{S}) \end{array}$$

where the vertical maps (given by composition with $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$) are equivalences of ∞-categories (Proposition HA.1.4.2.22). It will therefore suffice to show that the forgetful functor

$$\theta : \mathrm{Exc}_*(\mathcal{C}_{\geqslant -n},\mathrm{Sp}) \to \mathrm{Exc}_*(\mathcal{C}_{\geqslant 0},\mathrm{Sp})$$

is an equivalence of ∞-categories. This is clear, since θ has a homotopy inverse given by the construction $F \mapsto \Omega^n \circ F \circ \Sigma^n$. □

Example 17.2.1.3. Let $\mathcal{C}$ be a stable ∞-category equipped with a right-bounded t-structure, let $C \in \mathcal{C}$ be an object, and let $F : \mathcal{C} \to \mathcal{S}$ be the functor corepresented by C. Then F is an excisive functor. It follows from Lemma 17.2.1.2 that F is determined by the restriction $F|_{\mathcal{C}_{\geqslant 0}}$, (up to a contractible space of choices). Combining this observation with Yoneda's lemma (Proposition HTT.5.1.3.2), we see that the object C can be recovered from $F|_{\mathcal{C}_{\geqslant 0}}$ (again up to a contractible space of choices). More precisely, the construction $C \mapsto F|_{\mathcal{C}_{\geqslant 0}}$ determines a fully faithful embedding $\mathcal{C}^{\mathrm{op}} \to \mathrm{Exc}_*(\mathcal{C}_{\geqslant 0},\mathcal{S})$.

Example 17.2.1.4. Let A be a connective $\mathbb{E}_\infty$-ring. Recall that an A-module M is said to be *almost connective* if it is n-connective for some n, and let $\mathrm{Mod}_A^{\mathrm{acn}}$ denote the full subcategory of Mod_A spanned by the A-modules which are almost connective. Example 17.2.1.3 determines a fully faithful embedding

$$\theta : (\mathrm{Mod}_A^{\mathrm{acn}})^{\mathrm{op}} \to \mathrm{Exc}_*(\mathrm{Mod}_A^{\mathrm{cn}},\mathcal{S}).$$

We will say that a functor $\mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ is *almost corepresentable* if it belongs to the essential image of the functor θ.

Proposition 17.2.1.5. *Let A be connective $\mathbb{E}_\infty$-ring and let $F : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Then F is almost corepresentable if and only if the following conditions are satisfied:*

(*a*) *The functor F is reduced and excisive.*

(*b*) *There exists an integer n such that the functor $M \mapsto \Omega^n F(M)$ commutes with small limits.*

(*c*) *The functor F is accessible: that is, F commutes with κ-filtered colimits for some regular cardinal κ.*

Proof. Assume that condition (a) is satisfied, so that F extends to an left exact $F^+ : \mathrm{Mod}_A^{\mathrm{acn}} \to \mathcal{S}$ (Lemma 17.2.1.2). Suppose that F^+ is represented by an almost connective A-module N. Choose n such that $\Sigma^n N$ is connective. Then the functor

$$M \mapsto \Omega^n F(M) \simeq F^+(\Omega^n M) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(N, \Omega^n M) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(\Sigma^n N, M)$$

is corepresented by the object $\Sigma^n N \in \mathrm{Mod}_A^{\mathrm{cn}}$, and therefore preserves small limits. If N is a κ-compact object of Mod_A, then F commutes with κ-filtered colimits, so that (c) is satisfied.

Conversely, suppose that (b) and (c) are satisfied. Choose $n \geqslant 0$ as in (b). Then the restriction $F^+|_{(\mathrm{Mod}_A)_{\geqslant -n}}$ is given by the composition

$$(\mathrm{Mod}_A)_{\geqslant -n} \xrightarrow{\Sigma^n} \mathrm{Mod}_A^{\mathrm{cn}} \xrightarrow{F} \mathcal{S}_* \xrightarrow{\Omega^n} \mathcal{S},$$

and therefore commutes with small limits. Using Proposition HTT.5.5.2.7 , we deduce that $F^+|_{(\mathrm{Mod}_A)_{\geqslant -n}}$ is corepresented by an object $N \in (\mathrm{Mod}_A)_{\geqslant -n}$. Using Lemma 17.2.1.2, we deduce that F^+ is corepresented by N, so that F is almost corepresentable. □

17.2.2 Digression: Local Representability

Let $\mathcal{C}$ be an ∞-category. It follows from the ∞-categorical version of Yoneda's Lemma (Proposition HTT.5.1.3.2) that an object $C \in \mathcal{C}$ is determined (up to canonical equivalence) by the functor $\mathrm{Map}_{\mathcal{C}}(C, \bullet) : \mathcal{C} \to \mathcal{S}$ corepresented by C. We will need a relative version of this assertion, which applies to an arbitrary coCartesian fibration $p : X \to S$.

Definition 17.2.2.1. Let $p : X \to S$ be a coCartesian fibration of simplicial sets. We will say that a map $F : X \to \mathcal{S}$ is *locally corepresentable* (with respect to p) if the following conditions are satisfied:

(1) For every vertex $s \in S$, there exists an object x of the ∞-category X_s and a point $\eta \in F(x)$ which corepresents the functor $F|_{X_s}$ in the following sense: for every object $y \in X_s$, evaluation on η induces a homotopy equivalence $\mathrm{Map}_{X_s}(x, y) \to F(y)$.

(2) Let $x \in X_s$ and $\eta \in F(x)$ be as in (1), let $e : x \to x'$ be a coCartesian edge of X covering an edge $s \to s'$ in S. Let $\eta' \in F(x')$ be the image of η under the map $F(x) \to F(x')$ determined by e. Then η' corepresents the functor $F|_{X_{s'}}$ (that is, for every $y \in X_{s'}$, evaluation on η' induces a homotopy equivalence $\mathrm{Map}_{X_{s'}}(x', y) \to F(y)$).

In the situation of Definition 17.2.2.1, condition (2) ensures that the object x_s representing the functors $F_s = F|_{X_s}$ can be chosen to depend functorially on $s \in S$. We can articulate this idea more precisely as follows:

Proposition 17.2.2.2 (Relative Yoneda Lemma)**.** *Let $p : X \to S$ be a coCartesian fibration of simplicial sets and let $\mathcal{C} \subseteq \mathrm{Fun}_S(S, X)$ denote the full subcategory of $\mathrm{Fun}_S(S, X)$ spanned by those maps $f : S \to X$ which carry each edge of S to a coCartesian edge of X. Then there is fully faithful embedding $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(X, \mathcal{S})$, whose essential image is the full subcategory of $\mathrm{Fun}(X, \mathcal{S})$ spanned by the locally corepresentable functors.*

Proof. Let $\chi : S \to \mathcal{C}\mathrm{at}_\infty$ be a map classifying the coCartesian fibration p (given informally by the formula $\chi(s) = X_s$), so that $\mathcal{C}$ can be identified with the limit of the diagram χ in the ∞-category $\mathcal{C}\mathrm{at}_\infty$ (Proposition HTT.3.3.3.1). Let χ' be the result of composing χ with the "opposition" functor $\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$.

Let $\mathrm{Dl}(p)$ and $\mathrm{Dl}^0(p)$ be defined as in Construction 14.4.2.1 (so that $\mathrm{Dl}(p) \to S$ is a Cartesian fibration whose fibers are given by $\mathrm{Dl}(p)_s = \mathrm{Fun}(X_s, \mathcal{S})$), and $\mathrm{Dl}^0(p)$ is the full simplicial subset whose fibers $\mathrm{Dl}^0(p)_s$ are the full subcategories of $\mathrm{Fun}(X_s, \mathcal{S})$ spanned by the corepresentable functors. Then the projection $q : \mathrm{Dl}^0(p) \to S$ is a coCartesian fibration classified by the map χ' (Proposition 14.4.2.4). We have an isomorphism of simplicial sets $\theta : \mathrm{Fun}_S(S, \mathrm{Dl}(p)) \simeq \mathrm{Fun}(X, \mathcal{S})$. A map $F : X \to \mathcal{S}$ is locally corepresentable if and only if $\theta^{-1}(F) : S \to \mathrm{Dl}(p)$ factors through $\mathrm{Dl}^0(p)$ and carries edges of S to q-coCartesian edges of $\mathrm{Dl}^0(p)$. Using Proposition HTT.3.3.3.1 , we can identify the limit $\varprojlim \chi'$ with the full subcategory of $\mathrm{Fun}(X, \mathcal{S})$ spanned by the locally corepresentable functors. We conclude the proof by observing that there is a canonical equivalence of ∞-categories $(\varprojlim \chi)^{\mathrm{op}} \simeq \varprojlim \chi'$. □

17.2.3 Local Almost Representability

Suppose that $f : X \to Y$ is a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and η is a point of $X(A)$. Example 17.2.1.4 shows that if there exists an almost connective A-module M_η which corepresents the functor

$$(N \in \mathrm{Mod}_A^{\mathrm{cn}}) \mapsto (\mathrm{fib}\, X(A \oplus N) \to X(A) \times_{Y(A)} Y(A \oplus N) \in \mathcal{S}),$$

then M_η is determined up to a contractible space of choices. However, we will need a stronger statement in what follows: namely, that M_η can be chosen to depend functorially on the pair (A, η). To prove this, we will need a variant of Proposition 17.2.2.2.

Definition 17.2.3.1. Let $p : X \to S$ be a coCartesian fibration of simplicial sets. Assume that:

(i) For each vertex $s \in S$, the ∞-category X_s is stable and equipped with a right-bounded t-structure $(X_{s,\geqslant 0}, X_{s,\leqslant 0})$.

(ii) For every edge $e : s \to s'$ in S, the associated functor $X_s \to X_{s'}$ is exact and right t-exact.

Let $X_{\geqslant 0}$ be the full simplicial subset of X spanned by those vertices which belong to $X_{s,\geqslant 0}$ for some vertex $s \in S$.

We will say that a map $F : X_{\geqslant 0} \to \mathcal{S}$ is *locally almost corepresentable* (with respect to p) if the following conditions are satisfied:

(1) For every vertex $s \in S$, the induced map $X_{s,\geqslant 0} \to \mathcal{S}$ is reduced and excisive.

(2) Let $F^+ : X \to \mathcal{S}$ be an extension of F such that $F^+|_{X_s}$ is reduced and excisive for each $s \in S$ (it follows from Lemma 17.2.3.4 that F^+ exists and is unique up to a contractible space of choices). Then F^+ is locally corepresentable (in the sense of Definition 17.2.2.1).

Proposition 17.2.3.2. *Let $p : X \to S$ be as in Definition 17.2.3.1, let $\mathcal{C}$ denote the full subcategory of* $\mathrm{Fun}_S(S, X)$ *spanned by those maps which carry each edge of S to a p-coCartesian edge of X. Then there is a fully faithful functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(X_{\geqslant 0}, \mathcal{S})$, whose essential image is the full subcategory of* $\mathrm{Fun}(X_{\geqslant 0}, \mathcal{S})$ *spanned by the locally almost corepresentable functors.*

Remark 17.2.3.3. In the situation of Proposition 17.2.3.2, the fully faithful functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(X_{\geqslant 0}, \mathcal{S})$ is left exact. In particular, the essential image of this functor is closed under finite limits.

Proposition 17.2.3.2 is an immediate consequence of Proposition 17.2.2.2 together with the following relative version of Lemma 17.2.1.2:

Lemma 17.2.3.4. *Let $p : X \to S$ be as in Definition 17.2.3.1, let $\mathcal{E} \subseteq \mathrm{Fun}(X, \mathcal{S})$ denote the full subcategory of* $\mathrm{Fun}(X, \mathcal{S})$ *spanned by those functors whose restriction to each fiber X_s is reduced and excisive, and define $\mathcal{E}_0 \subseteq \mathrm{Fun}(X_{\geqslant 0}, \mathcal{S})$ similarly. Then the restriction functor $\mathcal{E} \to \mathcal{E}_0$ is a trivial Kan fibration.*

Proof. Since $\mathcal{E} \to \mathcal{E}_0$ is obviously a categorical fibration, it will suffice to show that it is an equivalence of ∞-categories. For every map of simplicial sets $\phi : T \to S$, let $\mathcal{E}(T) \subseteq \mathrm{Fun}(X \times_S T, \mathcal{S})$ denote the full subcategory spanned by those functors $F : X \times_S T \to \mathcal{S}$ whose restriction to $X_{\phi(t)}$ is reduced and excisive for each vertex $t \in T$, and define $\mathcal{E}_0(T)$ similarly. There is an evident restriction map $\psi(T) : \mathcal{E}(T) \to \mathcal{E}_0(T)$. We will prove that this map is an equivalence of ∞-categories for every map $\phi : T \to S$. Note that $\phi(T)$ is the homotopy limit of a tower of functors $\psi(\mathrm{sk}^n T)$ for $n \geqslant 0$. We may therefore assume that T is a simplicial set of finite dimension n. We proceed by induction on n, the case $n = -1$ being vacuous. Let K be the set of n-simplices of T. We have a pushout diagram of simplicial sets

$$\begin{array}{ccc} K \times \partial\,\Delta^n & \longrightarrow & K \times \Delta^n \\ \downarrow & & \downarrow \\ \mathrm{sk}^{n-1} T & \longrightarrow & T, \end{array}$$

which gives rise to a homotopy pullback diagram of functors

$$\begin{array}{ccc} \psi(K \times \partial\,\Delta^n) & \longleftarrow & \psi(K \times \Delta^n) \\ \uparrow & & \uparrow \\ \psi(\mathrm{sk}^{n-1} T) & \longleftarrow & \psi(T). \end{array}$$

It will therefore suffice to prove that $\psi(\mathrm{sk}^{n-1} T)$, $\psi(K \times \partial\,\Delta^n)$, and $\psi(K \times \Delta^n)$ are equivalences. In the first two cases, this follows from the inductive hypothesis. In the third case, we can write $\psi(K \times \Delta^n)$ as a product of functors $\psi(\{v\} \times \Delta^n)$ indexed by the elements of K. We are therefore reduced to proving the Lemma in the case $S = \Delta^n$.

For $0 \leqslant i \leqslant n$, let X_i denote the fiber of p over the ith vertex of $S = \Delta^n$. Using Proposition HTT.3.2.2.7 , we can choose a composable sequence of maps

$$\theta : X_0^{\mathrm{op}} \to X_1^{\mathrm{op}} \to \cdots \to X_n^{\mathrm{op}}$$

and a categorical equivalence $M(\theta)^{\mathrm{op}} \to X$, where $M(\theta)$ denotes the mapping simplex of the diagram θ (see §HTT.3.2.2). Note that each of the maps in the above diagram is exact and right t-exact, so that θ restricts to a sequence of maps

$$\theta_0 : (X_{0,\geqslant 0})^{\mathrm{op}} \to \cdots \to (X_{n,\geqslant 0})^{\mathrm{op}}$$

and we have a categorical equivalence $M(\theta_0)^{\mathrm{op}} \to X_{\geqslant 0}$. For every simplicial subset $T \subseteq S = \Delta^n$, let $\mathcal{E}'(T)$ denote the full subcategory of $\mathrm{Fun}(T \times_S M(\theta)^{\mathrm{op}}, \mathcal{S})$ spanned by those functors whose restriction to each X_i is reduced and excisive, and define $\mathcal{E}'_0(T) \subseteq \mathrm{Fun}(T \times_S M(\theta_0)^{\mathrm{op}}, \mathcal{S})$ similarly. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}(T) & \longrightarrow & \mathcal{E}_0(T) \\ \downarrow & & \downarrow \\ \mathcal{E}'(T) & \longrightarrow & \mathcal{E}'_0(T) \end{array}$$

where the vertical maps are categorical equivalences. It follows from the inductive hypothesis that the restriction map $\mathcal{E}'(T) \to \mathcal{E}'_0(T)$ is a categorical equivalence for every proper simplicial subset $T \subseteq S$. To complete the proof, it will suffice to show that $\mathcal{E}'(S) \to \mathcal{E}'_0(S)$ is a categorical equivalence.

Let σ denote the face of $S = \Delta^n$ opposite the 0th vertex. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}'(S) & \longrightarrow & \mathcal{E}'_0(S) \\ \downarrow & & \downarrow \\ \mathcal{E}'(\sigma) & \longrightarrow & \mathcal{E}'_0(\sigma), \end{array}$$

where the bottom horizontal map is a categorical equivalence. It will therefore suffice to show that this diagram is a homotopy pullback square: that is, that the map

$$\rho : \mathcal{E}'(S) \to \mathcal{E}'_0(S) \times_{\mathcal{E}'_0(\sigma)} \mathcal{E}'(\sigma)$$

is a categorical equivalence. Let $\mathcal{C} = X_0$ and $\mathcal{C}_{\geqslant 0} = X_{0,\geqslant 0}$. Unwinding the definitions, we see that ρ is a pullback of the canonical map

$$\rho_0 : \mathrm{Fun}(S, \mathrm{Exc}_*(\mathcal{C}, \mathcal{S})) \to \mathrm{Fun}(S, \mathrm{Exc}_*(\mathcal{C}_{\geqslant 0}, \mathcal{S})) \times_{\mathrm{Fun}(\sigma, \mathrm{Exc}_*(\mathcal{C}_{\geqslant 0}, \mathcal{S}))} \mathrm{Fun}(\sigma, \mathrm{Exc}_*(\mathcal{C}, \mathcal{S})).$$

It follows from Lemma 17.2.1.2 that this map is a trivial Kan fibration. □

17.2.4 Functorial Definition of the Cotangent Complex

We now specialize the formal considerations of §17.2.1, §17.2.2, and §17.2.3 to the situation of interest.

Notation 17.2.4.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Let $\overline{\mathrm{CAlg}}^{\mathrm{cn}} \to \mathrm{CAlg}^{\mathrm{cn}}$ be a left fibration classified by X, and let Mod^X denote the fiber product $\mathrm{Mod}(\mathrm{Sp}) \times_{\mathrm{CAlg}} \overline{\mathrm{CAlg}}^{\mathrm{cn}}$. More informally, we let Mod^X denote the ∞-categories whose objects are triples (A, η, M), where A is a connective $\mathbb{E}_\infty$-ring, $\eta \in X(A)$ is a point, and $M \in \mathrm{Mod}_A$ is an A-module spectrum. Let $\mathrm{Mod}^X_{\mathrm{acn}}$ denote the full subcategory of Mod^X spanned by those triples (A, η, M) where M is almost connective (that is, M is n-connective for $n \ll 0$). The forgetful functor $q : \mathrm{Mod}^X_{\mathrm{acn}} \to \overline{\mathrm{CAlg}}^{\mathrm{cn}}$ is a coCartesian fibration. Moreover, the ∞-category of coCartesian sections of q is canonically equivalent to $\mathrm{QCoh}(X)^{\mathrm{acn}}$, the full subcategory of $\mathrm{QCoh}(X)$ spanned by the almost connective quasi-coherent sheaves on X (see Remark 6.2.2.7).

Let $\mathrm{Mod}^X_{\mathrm{cn}}$ denote the full subcategory of Mod^X spanned by those triples (A, η, M) where M is connective. Applying Proposition 17.2.3.2, we deduce that $\mathrm{QCoh}(X)^{\mathrm{acn}}$ is equivalent to the full subcategory of $\mathrm{Fun}(\mathrm{Mod}^X_{\mathrm{cn}}, \mathcal{S})^{\mathrm{op}}$ spanned by those functors $\mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ which are locally almost corepresentable (relative to q).

Definition 17.2.4.2 (The Relative Cotangent Complex)**.** Suppose we are given a natural transformation $\alpha : X \to Y$ between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We define a functor $F : \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ by the formula

$$F(A, \eta, M) = \mathrm{fib}(X(A \oplus M) \to X(A) \times_{Y(A)} Y(A \oplus M)),$$

where the fiber is taken over the point of $X(A) \times_{Y(A)} Y(A \oplus M)$ determined by η. We will say that α *admits a cotangent complex* if the functor F is locally almost corepresentable relative to q. In this case, we let $L_{X/Y} \in \mathrm{QCoh}(X)$ denote a preimage for F under the fully faithful embedding $\mathrm{QCoh}(X)^{\mathrm{aperf}} \to \mathrm{Fun}(\mathrm{Mod}^X_{\mathrm{cn}}, \mathcal{S})^{\mathrm{op}}$ given in Notation 17.2.4.1. We will refer to $L_{X/Y}$ as the *relative cotangent complex of X over Y*.

In the special case where Y is a final object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, we will say that X *admits a cotangent complex* if the essentially unique map $\alpha : X \to Y$ admits a cotangent complex. In this case, we will denote the relative cotangent complex $L_{X/Y}$ by L_X and refer to it as the *absolute cotangent complex* of X.

Remark 17.2.4.3. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and $F : \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ be functors. Unwinding the definitions, we see that F is locally almost corepresentable if and only if the following conditions are satisfied:

(a) For every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, the induced functor $F_\eta : \mathrm{Mod}^{\mathrm{cn}}_A \to \mathcal{S}$ is corepresented by an almost connective A-module M_η (which is uniquely determined up to contractible ambiguity: see Example 17.2.1.4).

(b) Let $\eta \in X(A)$ be as in (a), and suppose we are given a map of connective $\mathbb{E}_\infty$-rings $A \to A'$. Let $\eta' \in X(A')$ denote the image of η. Then the functor $F_{\eta'}$ is corepresented by $A' \otimes_A M_\eta$. More precisely, for every A'-module N, the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_{A'}}(A' \otimes_A M_\eta, N) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(M_\eta, N) \simeq F_\eta(N) \to F_{\eta'}(N)$$

is a homotopy equivalence.

We can rephrase condition (b) as follows:

(b') The functor F carries p-Cartesian morphisms in $\mathrm{Mod}^X_{\mathrm{cn}}$ to homotopy equivalences, where $p : \mathrm{Mod}^X_{\mathrm{cn}} \to \overline{\mathrm{CAlg}}^{\mathrm{cn}}$ denotes the projection map (here $\overline{\mathrm{CAlg}}^{\mathrm{cn}}$ is defined as in Notation 17.2.4.1).

Example 17.2.4.4. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Then X admits a cotangent complex if and only if the following conditions are satisfied:

(a) For every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, define $F_\eta : \mathrm{Mod}^{\mathrm{cn}}_A \to \mathcal{S}$ by the formula $F_\eta(N) = X(A \oplus N) \times_{X(A)} \{\eta\}$. Then the functor F_η is corepresented by an almost connective A-module M_η.

(b) For every map of connective $\mathbb{E}_\infty$-rings $A \to B$ and every connective B-module M, the diagram of spaces

$$\begin{array}{ccc} X(A \oplus M) & \longrightarrow & X(B \oplus M) \\ \downarrow & & \downarrow \\ X(A) & \longrightarrow & X(B) \end{array}$$

is a pullback square.

In this case, the absolute cotangent complex $L_X \in \mathrm{QCoh}(X)$ is described by the formula $\eta^* L_X = M_\eta \in \mathrm{Mod}_A$ for $\eta \in X(A)$.

Using Proposition 17.2.1.5, we can reformulate condition (a) as follows:

(a') For every point $\eta \in X(A)$, the functor $F_\eta : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ is reduced, excisive, and accessible. Moreover, there exists an integer $n \geqslant 0$ such that the functor $M \mapsto \Omega^n F_\eta(M)$ preserves small limits.

Remark 17.2.4.5. Fix an integer n and a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Then $\mathrm{QCoh}(X)_{\geqslant n}$ is a full subcategory of $\mathrm{QCoh}(X)^{\mathrm{acn}}$ which is closed under small colimits. The construction of Notation 17.2.4.1 determines a fully faithful embedding $\mathrm{QCoh}(X)^{\mathrm{op}}_{\geqslant n} \to \mathrm{Fun}(\mathrm{Mod}^X_{\mathrm{cn}}, \mathcal{S})$ which commutes with small limits. It follows that the essential image of this embedding is closed under small limits. From this we deduce the following:

($*$) Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the limit of a diagram of functors $\{X_\alpha : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}\}$. Assume that each X_α admits a cotangent complex which is n-connective. Then X admits a cotangent complex which is n-connective. Moreover, we have a canonical equivalence $L_X \simeq \varinjlim_\alpha f_\alpha^* L_{X_\alpha}$, where $f_\alpha : X \to X_\alpha$ denotes the projection map.

Remark 17.2.4.6. Suppose we are given a pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. If f admits a cotangent complex, then f' also admits a cotangent complex. Moreover, we have a canonical equivalence $L_{X'/Y'} \simeq g^* L_{X/Y}$ in the ∞-category $\mathrm{QCoh}(X')$.

Remark 17.2.4.6 admits the following converse:

Proposition 17.2.4.7. *Let $f : X \to Y$ be a morphism in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Suppose that, for every corepresentable functor $Y' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and every natural transformation $\phi : Y' \to Y$, the projection map $Y' \times_Y X \to Y'$ admits a cotangent complex. Then f admits a cotangent complex.*

Proof. Let $\mathrm{Mod}^X_{\mathrm{cn}}$ be as in Definition 17.2.4.2, and let $F : \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ be given by the formula $F(R, \eta, M) = \mathrm{fib}(X(R \oplus M) \to X(R) \times_{Y(R)} Y(R \oplus M))$. We wish to show that F is locally almost corepresentable. We will show that F satisfies conditions (a) and (b') of Remark 17.2.4.3.

To verify condition (a), let us fix a point $\eta \in X(R)$ and consider the functor $F_\eta : \mathrm{Mod}^{\mathrm{cn}}_R \to \mathcal{S}$ given by the restriction of F. Let $Y' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor corepresented by R. Then η determines a natural transformation $Y' \to Y$. Let $X' = Y' \times_Y X$, and let $F' : \mathrm{Mod}^{X'}_{\mathrm{cn}} \to \mathcal{S}$ be the functor given by the formula

$$F'(R_0, \eta_0, M_0) = \mathrm{fib}(X'(R_0 \oplus M_0) \to X'(R_0) \times_{Y'(R_0)} Y'(R_0 \oplus M_0)).$$

Since the projection map $X' \to Y'$ admits a cotangent complex, the functor F' is locally almost corepresentable and therefore satisfies condition (a) of Remark 17.2.4.3. We now observe that η lifts canonically to a point $\eta' \in X'(R)$. The restriction of F' to the fiber of $\mathrm{Mod}^{X'}_{\mathrm{cn}}$ over (R, η') agrees with F_η. It follows that F_η is corepresentable by an almost connective R-module, as desired.

We now verify condition (b'). Choose a morphism $\alpha : (R, \eta, M) \to (R', \eta', M')$ in $\mathrm{Mod}^X_{\mathrm{cn}}$ which induces an equivalence $R' \otimes_R M \to M'$. We wish to prove that $F(\alpha)$ is a homotopy equivalence. Let $F' : \mathrm{Mod}^{X'}_{\mathrm{cn}} \to \mathcal{S}$ be defined as above, and observe that α lifts canonically to a morphism $\overline{\alpha}$ in $\mathrm{Mod}^{X'}_{\mathrm{cn}}$. Since F' is locally almost corepresentable, it satisfies condition (b') of Remark 17.2.4.3. It follows that $F'(\overline{\alpha})$ is a homotopy equivalence. Since F' is the composition of F with the forgetful functor $\mathrm{Mod}^{X'}_{\mathrm{cn}} \to \mathrm{Mod}^X_{\mathrm{cn}}$, we deduce that $F(\alpha)$ is a homotopy equivalence. □

17.2.5 Comparison with the Geometric Definition

Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks. We now have two *a priori* different definitions of the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}}$: one given by Definition 17.1.1.8 (where we regard X and Y as spectrally ringed ∞-topoi) and one given by Definition 17.2.4.2 (where we identify X and Y with the associated functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$). We now show that these definitions are essentially equivalent (Corollary 17.2.5.4). We begin by considering the case $\mathsf{Y} = \mathrm{Spét}\, S$.

Proposition 17.2.5.1. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a formal spectral Deligne-Mumford stack, and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by $\mathfrak{X}$. Then X admits a cotangent complex L_X. Moreover, we can identify L_X with the image of the completed cotangent complex $L^{\wedge}_{\mathfrak{X}}$ under the equivalence of ∞-categories $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}} \simeq \mathrm{QCoh}(X)^{\mathrm{cn}}$ of Theorem 8.3.4.4.*

Proof. Let $\mathrm{Mod}^X_{\mathrm{cn}}$ denote the ∞-category introduced in Definition 17.2.4.2. Let $F : \mathrm{Mod}^X_{\mathrm{cn}}$ be the functor given by $F(R, \eta, M) = X(R \oplus M) \times_{X(R)} \{\eta\}$. For each object $(R, \eta, M) \in \mathrm{Mod}^X_{\mathrm{cn}}$, let us regard $\mathcal{S}\mathrm{hv}^{\text{ét}}_R$ as the underlying ∞-topos of both $\mathrm{Spét}\, R$ and $\mathrm{Spét}(R \oplus M)$, let $\widetilde{M}$ denote the quasi-coherent $\mathscr{O}_{\mathrm{Sp\acute{e}t}\, R}$-module corresponding to M, and regard η as a morphism of spectrally ringed ∞-topoi from $\mathrm{Sp\acute{e}t}\, R$ to $\mathfrak{X}$. Unwinding the definitions, we obtain canonical homotopy equivalences

$$\begin{aligned}
F(R, \eta, M) &\simeq \mathrm{Map}_{\mathrm{fSpDM}_{\mathrm{Sp\acute{e}t}\, R/}}(\mathrm{Sp\acute{e}t}(R \oplus M), \mathfrak{X}) \\
&\simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{S}\mathrm{hv}^{\mathrm{\acute{e}t}}_R)_{/\mathscr{O}_{\mathrm{Sp\acute{e}t}\, R}}}(\eta^{-1}\mathscr{O}_{\mathfrak{X}}, \mathscr{O}_{\mathrm{Sp\acute{e}t}(R \oplus M)}) \\
&\simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{S}\mathrm{hv}^{\mathrm{\acute{e}t}}_R)_{/\mathscr{O}_{\mathrm{Sp\acute{e}t}\, R}}}(\eta^{-1}\mathscr{O}_{\mathfrak{X}}, \mathscr{O}_{\mathrm{Sp\acute{e}t}\, R} \oplus \widetilde{M}) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\eta^{-1}\mathscr{O}_{\mathfrak{X}}}}(L_{\eta^{-1}\mathscr{O}_{\mathfrak{X}}}, \widetilde{M}) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\eta^{-1}\mathscr{O}_{\mathfrak{X}}}}(\eta^{-1} L_{\mathscr{O}_{\mathfrak{X}}}, \widetilde{M}) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathscr{O}_{\mathrm{Sp\acute{e}t}\, R}}}(\eta^{*} L_{\mathscr{O}_{\mathfrak{X}}}, \widetilde{M}) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R)}(\eta^{*} L^{\wedge}_{\mathfrak{X}}, \widetilde{M}) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_R}(\Gamma(\mathrm{Sp\acute{e}t}\, R; \eta^{*} L^{\wedge}_{\mathfrak{X}}), M).
\end{aligned}$$

□

Proposition 17.2.5.2. *Suppose we are given a commutative diagram*

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ X & & \xrightarrow{h} & & Z \end{array}$$

in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Assume that g and h admit cotangent complexes. Then f admits a cotangent complex. Moreover, we have a canonical fiber sequence $f^* L_{Y/Z} \to L_{X/Z} \to L_{X/Y}$ in the stable ∞-category $\mathrm{QCoh}(X)$.

Proof. Let $\mathrm{Mod}^X_{\mathrm{cn}}$ be the ∞-category introduced in Definition 17.2.4.2. We define functors $F', F, F'' : \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ by the formulae

$$F'(R, \eta, M) = \mathrm{fib}(X(R \oplus M) \to Y(R \oplus M) \times_{Y(M)} X(M))$$

$$F(R, \eta, M) = \mathrm{fib}(X(R \oplus M) \to Z(R \oplus M) \times_{Z(M)} X(M))$$

$$F''(R, \eta, M) = \mathrm{fib}(Y(R \oplus M) \to Z(R \oplus M) \times_{Z(M)} Y(M)),$$

so that we have a fiber sequence of functors $F' \to F \xrightarrow{\alpha} F''$. Let $\theta : (\mathrm{QCoh}(X)^{\mathrm{acn}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Mod}^X_{\mathrm{cn}}, \mathcal{S})$ be the fully faithful functor of Proposition 17.2.3.2. Since g and h admit cotangent complexes, we have equivalences $F \simeq \theta(L_{X/Z})$ and $F'' \simeq \theta(f^* L_{Y/Z})$. Since θ is fully faithful, the natural transformation α is induced by a map $\beta : f^* L_{Y/Z} \to L_{X/Z}$. It follows from Remark 17.2.3.3 that F' is equivalent to $\theta(\mathrm{cofib}(\beta))$. □

Corollary 17.2.5.3. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Suppose that X and Y admit cotangent complexes L_X and L_Y. Then f admits a cotangent complex. Moreover, we have a canonical fiber sequence $f^* L_Y \to L_X \to L_{X/Y}$ in the stable ∞-category $\mathrm{QCoh}(X)$.*

Corollary 17.2.5.4. *Let $\phi : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks, and let $f : X \to Y$ be the induced map between the functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ represented by X and Y. Then f admits a cotangent complex. Moreover, we can identify $L_{X/Y}$ with the image of the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}}$ under the equivalence of ∞-categories $\mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(X)$.*

Proof. Combine Corollary 17.2.5.3 with Proposition 17.2.5.1. □

17.3 Cohesive, Nilcomplete, and Integrable Functors

For every spectral Deligne-Mumford stack X, let $h_{\mathsf{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by X, given by the formula $h_{\mathsf{X}}(R) = \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Spét}\, R, \mathsf{X})$. According to Theorem **??**, the construction $\mathsf{X} \mapsto h_{\mathsf{X}}$ determines a fully faithful embedding from the ∞-category SpDM of spectral Deligne-Mumford stacks to the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ of $\mathcal{S}$-valued functors on $\mathrm{CAlg}^{\mathrm{cn}}$. This embedding h is useful in part because the target ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ has very strong closure properties: for example, it admits all small limits and colimits, while the ∞-category SpDM does not. Beware, however, that there are several *naturally occurring* colimits in the ∞-category SpDM which are not preserved by the functor h:

(*a*) Let $\{\mathsf{X}_\alpha\}$ be a (small) diagram of spectral Deligne-Mumford stacks in which the transition morphisms $\mathsf{X}_\alpha \to \mathsf{X}_\beta$ are étale. Then there exists a colimit $\mathsf{X} = \varinjlim \mathsf{X}_\alpha$ in the ∞-category of spectral Deligne-Mumford stacks. However, the canonical map $\varinjlim h_{\mathsf{X}_\alpha} \to h_{\mathsf{X}}$ is generally not an equivalence.

(*b*) Suppose we are given a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B' & \xrightarrow{g} & B, \end{array}$$

where f and g induce surjective ring homomorphisms $\pi_0 A \to \pi_0 B \leftarrow \pi_0 B'$. Then the associated diagram of closed immersions

$$\begin{array}{ccc} \mathrm{Spét}\, A' & \longleftarrow & \mathrm{Spét}\, A \\ \uparrow & & \uparrow \\ \mathrm{Spét}\, B' & \longleftarrow & \mathrm{Spét}\, B \end{array}$$

is a pushout square (see Proposition 16.1.3.1). However, this pushout square is usually not preserved by the functor $h : \mathrm{SpDM} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$.

(c) Let R be a connective $\mathbb{E}_\infty$-ring. Then the affine spectral algebraic space $\mathrm{Spét}\, R$ can be identified with the colimit of the diagram $\{\mathrm{Spét}\, \tau_{\leqslant n} R\}_{n \geqslant 0}$ (Proposition 17.3.2.3). However, this colimit is not preserved by the functor $h : \mathrm{SpDM} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ unless R is truncated.

(d) Let R be a complete local Noetherian ring with maximal ideal $\mathfrak{m}$. Then the affine spectral algebraic space $\mathrm{Spét}\, R$ can be identified with the colimit of the diagram $\{\mathrm{Spét}\, R/\mathfrak{m}^n\}$ in the ∞-category of spectral algebraic spaces (Proposition 17.3.4.2). However, this colimit is not preserved by the functor h unless $\mathfrak{m}$ is nilpotent.

In case (a), we can remedy the situation by replacing the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ by the full subcategory $\mathcal{S}\mathrm{hv}_{\mathrm{ét}}(\mathrm{CAlg}^{\mathrm{cn}})$ spanned by those functors which are sheaves with respect to the étale topology. Our goal in this section is to introduce the classes of *cohesive*, *nilcomplete*, and *integrable* functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ (see Definitions 17.3.1.1, 17.3.2.1, and 17.3.4.1), which provide analogous remedies for (b), (c), and (a slight variant of) (d).

17.3.1 Cohesive Functors

We begin by studying functors $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which are, in some sense, compatible with gluing along closed immersions.

Definition 17.3.1.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is *cohesive* if it satisfies the following condition:

($*$) For every pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B' & \xrightarrow{g} & B \end{array}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$ for which the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective, the induced diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow X(f) \\ X(B') & \xrightarrow{X(g)} & X(B) \end{array}$$

is a pullback square in $\mathcal{S}$.

Example 17.3.1.2. Let X be a locally spectrally ringed ∞-topos, and let $h_{\mathsf{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor given by $h_{\mathsf{X}}(R) = \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X})$ be the functor represented by X. Then h_{X} is cohesive (Proposition 16.1.3.1). In particular, any functor which is representable by a (formal) spectral Deligne-Mumford stack is cohesive.

Example 17.3.1.3. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a geometric stack, in the sense of Definition 9.3.0.1. Then X is cohesive: this follows by Theorem 9.3.0.3, Theorem 16.2.0.2, and Proposition 16.2.3.1.

Remark 17.3.1.4. Using Proposition 16.1.3.1, we can reformulate condition $(*)$ of Definition 17.3.1.1 as follows:

$(*')$ For every pushout diagram of spectral Deligne-Mumford stacks σ :

$$\begin{array}{ccc} \mathsf{Y}_{01} & \xrightarrow{i} & \mathsf{Y}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathsf{Y}_1 & \longrightarrow & \mathsf{Y}, \end{array}$$

where i and j are closed immersions and Y is affine, the associated diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(h_{\mathsf{Y}_{01}}, X) & \longleftarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(h_{\mathsf{Y}_0}, X) \\ \uparrow & & \uparrow \\ \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(h_{\mathsf{Y}_1}, X) & \longleftarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(h_{\mathsf{Y}}, X) \end{array}$$

is a pullback square of spaces.

If X is cohesive and satisfies étale descent, then condition $(*')$ holds more generally without the assumption that Y is affine.

In the study of deformation theory, it will be useful to consider the following variant of Definition 17.3.1.1:

Definition 17.3.1.5. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is *infinitesimally cohesive* if the following condition is satisfied:

$(*)$ Suppose we are given a pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B \end{array}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$, where the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjections of commutative rings whose kernels are nilpotent ideals in $\pi_0 A$ and $\pi_0 B'$. Then the induced diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow{\scriptstyle f} \\ X(B') & \xrightarrow{g} & X(B) \end{array}$$

is a pullback square in $\mathcal{S}$.

Remark 17.3.1.6. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. If X is cohesive, then X is infinitesimally cohesive. In particular, if X is representable by a (formal) spectral Deligne-Mumford stack, then X is infinitesimally cohesive.

Remark 17.3.1.7. Let $\{X_\alpha\}_{\alpha \in A}$ be a filtered diagram of functors from $\mathrm{CAlg}^{\mathrm{cn}}$ to $\mathcal{S}$ having colimit X. If each X_α is cohesive (infinitesimally cohesive), then so is X.

Remark 17.3.1.8. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Let R be a connective $\mathbb{E}_\infty$-ring, and let $\widetilde{R}$ be a square-zero extension of R by a connective R-module M, classified by a map of R-modules $d : L_R \to \Sigma M$. We then have a commutative diagram of spaces σ :

$$\begin{array}{ccc} X(\widetilde{R}) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ X(R) & \longrightarrow & X(R \oplus \Sigma M). \end{array}$$

Let η be a point of $X(R)$ and let $X(\widetilde{R})_\eta$ denote the fiber product $X(\widetilde{R}) \times_{X(R)} \{\eta\}$. Suppose that X admits a cotangent complex L_X, so that we can identify $\eta^* L_X$ with an R-module. Let ν denote the composite map $\eta^* L_X \to L_R \xrightarrow{d} \Sigma M$. Then the diagram σ determines a map $\theta : X(\widetilde{R})_\eta \to P$, where P denotes the space of paths from ν to the base point of the mapping space $\mathrm{Map}_{\mathrm{Mod}_R}(\eta^* L_X, \Sigma M)$. If X is infinitesimally cohesive, then σ is a pullback diagram, so that θ is a homotopy equivalence. In this case, η can be lifted to a point of $X(\widetilde{R})$ if and only if ν represents the zero element of the abelian group $\mathrm{Ext}^1_R(\eta^* L_X, M)$.

We can summarize Remark 17.3.1.8 informally as follows: if $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is an infinitesimally cohesive functor which admits a cotangent complex L_X, then L_X "controls" the deformation theory of the functor X. We will later prove that the converse holds under some additional assumptions (Proposition 17.3.6.1).

17.3.2 Nilcomplete Functors

We now study functors $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which are determined by their values on truncated $\mathbb{E}_\infty$-rings.

Definition 17.3.2.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is *nilcomplete* if, for every connective $\mathbb{E}_\infty$-ring R, the canonical map $X(R) \to \varprojlim X(\tau_{\leqslant n} R)$ is a homotopy equivalence.

Remark 17.3.2.2. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack, and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be an arbitrary functor. For each $n \geqslant 0$, let $\tau_{\leqslant n} \mathsf{X} = (\mathcal{X}, \tau_{\leqslant n} \mathcal{O}_{\mathcal{X}})$. We then have a canonical map

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(h_{\mathsf{X}}, X) \to \varprojlim \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(h_{\tau_{\leqslant n} \mathsf{X}}, X).$$

If X is affine and X is nilcomplete, then the map θ is a homotopy equivalence. It follows that if X is nilcomplete and satisfies étale descent, then θ is a homotopy equivalence for an arbitrary spectral Deligne-Mumford stack X.

Proposition 17.3.2.3. *Let $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ be a locally spectrally ringed ∞-topos, and assume that the sheaf $\mathcal{O}$ is connective. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be the functor represented by X (so that X is given by the formula $X(R) = \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Sp\acute{e}t}\, R, \mathsf{X})$). Then X is nilcomplete.*

Proof. Fix a connective $\mathbb{E}_\infty$-ring R and write $\mathrm{Sp\acute{e}t}\, R = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. We note that for every integer $n \geqslant 0$, we have an equivalence $\mathrm{Sp\acute{e}t}\, \tau_{\leqslant n} R \simeq (\mathcal{Y}, \tau_{\leqslant n} \mathcal{O}_{\mathcal{Y}})$. We wish to show that the canonical map

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \mathsf{X}) \to \varprojlim_n \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}}((\mathcal{Y}, \tau_{\leqslant n} \mathcal{O}_{\mathcal{Y}}), \mathsf{X})$$

is a homotopy equivalence. Note that a map of spectrally ringed ∞-topoi $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to \mathsf{X}$ is local if and only if the induced map $(\mathcal{Y}, \tau_{\leqslant 0} \mathcal{O}_{\mathcal{Y}}) \to \mathsf{X}$ is local; it will therefore suffice to show that the map

$$\theta : \mathrm{Map}_{\mathrm{RTop}}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \mathsf{X}) \to \varprojlim_n \mathrm{Map}_{\mathrm{RTop}}((\mathcal{Y}, \tau_{\leqslant n} \mathcal{O}_{\mathcal{Y}}), \mathsf{X})$$

is a homotopy equivalence. Let $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{Y})$ spanned by the geometric morphisms $f^* : \mathcal{X} \to \mathcal{Y}$. To prove that θ is a homotopy equivalence, it will suffice to show that it induces a homotopy equivalence after passing to the homotopy fiber over any geometric morphism $f^* \in \mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})^{\simeq}$. In other words, we must show that the canonical map $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})}(f^* \mathcal{O}, \mathcal{O}_{\mathcal{Y}}) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})}(f^* \mathcal{O}, \tau_{\leqslant n} \mathcal{O}_{\mathcal{Y}})$ is an equivalence. For this, it suffices to show that $\mathcal{O}_{\mathcal{Y}} \simeq \varprojlim \tau_{\leqslant n} \mathcal{O}_{\mathcal{Y}}$, which was established in the proof of Theorem 1.4.8.1. □

The following reformulation of Definition 17.3.2.1 is sometimes convenient:

Proposition 17.3.2.4. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

(1) *The functor X is nilcomplete.*

(2) *Suppose we are given a tower of connective $\mathbb{E}_\infty$-rings*

$$\cdots \to R(2) \to R(1) \to R(0)$$

satisfying the following condition: for every integer n, the tower of abelian groups

$$\cdots \to \pi_n R(2) \to \pi_n R(1) \to \pi_n R(0)$$

is eventually constant. Then the canonical map $X(\varprojlim R(n)) \to \varprojlim X(R(n))$ is a homotopy equivalence.

Proof. Let R be an arbitrary connective $\mathbb{E}_\infty$-ring. Then the Postnikov tower

$$\cdots \to \tau_{\leqslant 2} R \to \tau_{\leqslant 1} R \to \tau_{\leqslant 0} R$$

satisfies the hypothesis appearing in condition (2). It follows that (2) $\Rightarrow$ (1). For the converse, let us assume that X is nilcomplete and let

$$\cdots \to R(2) \to R(1) \to R(0)$$

be a tower of connective $\mathbb{E}_\infty$-rings satisfying the hypothesis of (2). Set $R = \varprojlim R(n)$. We have a commutative diagram

$$\begin{array}{ccc} X(R) & \longrightarrow & \varprojlim_n X(R(n)) \\ \downarrow & & \downarrow \\ \varprojlim_m X(\tau_{\leqslant m} R) & \longrightarrow & \varprojlim_{n,m} X(\tau_{\leqslant m} R(n)) \end{array}$$

Since X is nilcomplete, the vertical maps in this diagram are homotopy equivalences. Consequently, to show that the upper horizontal map is a homotopy equivalence, it suffices to show that the lower horizontal map is a homotopy equivalence. For this, it suffices to show that for every $m \geqslant 0$, the map $X(\tau_{\leqslant m} R) \to \varprojlim_n X(\tau_{\leqslant m} R(n))$ is a homotopy equivalence. This is clear, since the tower

$$\cdots \to \tau_{\leqslant m} R(2) \to \tau_{\leqslant m} R(1) \to \tau_{\leqslant m} R(0)$$

is eventually constant (with value $\tau_{\leqslant m} R$). □

The following result will be needed in §17.5:

Proposition 17.3.2.5. *Let R be a connective $\mathbb{E}_\infty$-ring, let $Y = \operatorname{Spec} R$, let $f : X \to Y$ be a morphism in $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, and let $X_0 : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ denote the functor given by $X_0(A) = \operatorname{fib}(X(A) \to Y(A))$. Suppose that the following conditions are satisfied:*

(*a*) *For every morphism* $A \to B$ *in* $\mathrm{CAlg}_R^{\mathrm{cn}}$ *every connective* B*-module* M*, the diagram*

$$\begin{array}{ccc} X_0(A \oplus M) & \longrightarrow & X_0(B \oplus M) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) \end{array}$$

is a pullback square.

(*b*) *For every truncated object* $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$ *and every point* $\eta \in X_0(A)$*, the functor* F_η *given by* $F_\eta(M) = X_0(A \oplus M) \times_{X_0(A)} \{\eta\}$ *is corepresentable by an almost connective* A*-module* $L_{X/Y}(\eta)$*.*

(*c*) *The functor* X *is nilcomplete.*

Then f *admits a cotangent complex.*

Proof. Using assumption (*a*) and the argument of Example 17.2.4.4, we are reduced to proving that for every $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$ and every point $\eta \in X_0(A)$, the functor F_η described in (*b*) is almost corepresentable. For every integer $n \geqslant 0$, let η_n denote the image of η in $X_0(\tau_{\leqslant n} A)$. Then assumption (*b*) guarantees the existence of almost connective objects $L_{X/Y}(\eta_n)$ corepresenting the functors F_{η_n}, and (*a*) gives equivalences

$$\tau_{\leqslant n-1} A \otimes_{\tau_{\leqslant n} A} L_{X/Y}(\eta_n) \simeq L_{X/Y}(\eta_{n-1})$$

for $n > 0$. Choose an integer m such that $L_X(\eta_0)$ is m-connective. It follows that each $L_{X/Y}(\eta_n)$ is m-connective, and that the maps $L_{X/Y}(\eta_n) \to L_{X/Y}(\eta_{n-1})$ are $(m+n)$-connective for $n > 0$. Let N denote the limit of the tower

$$\cdots \to L_{X/Y}(\eta_2) \to L_{X/Y}(\eta_1) \to L_{X/Y}(\eta_0)$$

in the ∞-category Mod_A. Then N is m-connective, and the canonical map $N \to L_{X/Y}(\eta_n)$ is $(m+n+1)$-connective for every integer n. Let M be a connective A-module. We may assume without loss of generality that $m \leqslant 0$. Using assumptions (*a*) and (*c*), we obtain homotopy equivalences

$$\begin{aligned}
F_\eta(M) &\simeq \varprojlim_k F_\eta(\tau_{\leqslant k+m} M) \\
&\simeq \varprojlim_k F_{\eta_k}(\tau_{\leqslant k+m} M) \\
&\simeq \varprojlim_k \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leqslant k} A}}(L_{X/Y}(\eta_k), \tau_{\leqslant k+m} M) \\
&\simeq \varprojlim_k \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leqslant k} A}}(\tau_{\leqslant k+m} L_{X/Y}(\eta_k), \tau_{\leqslant k+m} M) \\
&\simeq \varprojlim_k \mathrm{Map}_{\mathrm{Mod}_A}(\tau_{\leqslant k+m} L_{X/Y}(\eta_k), \tau_{\leqslant k+m} M) \\
&\simeq \varprojlim_k \mathrm{Map}_{\mathrm{Mod}_A}(N, \tau_{\leqslant k+m} M) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_A}(N, M)
\end{aligned}$$

depending functorially on M. It follows that the functor F_η is corepresented by the A-module N. □

17.3.3 Nilcompletion

The failure of a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ to be nilcomplete can be corrected.

Definition 17.3.3.1. Let $u : X \to \widehat{X}$ be a natural transformation between functors $X, \widehat{X} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will say that *$u$ exhibits $\widehat{X}$ as a nilcompletion of X* if the following conditions are satisfied:

(a) The functor $\widehat{X}$ is nilcomplete.

(b) For every nilcomplete functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, composition with u induces a homotopy equivalence $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\widehat{X}, Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X, Y)$.

It is immediate from the definition that if a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ admits a nilcompletion $\widehat{X}$, then $\widehat{X}$ is uniquely determined up to equivalence and depends functorially on X. For existence, we note the following:

Proposition 17.3.3.2. *Every functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ admits a nilcompletion $\widehat{X}$, given concretely by the formula $\widehat{X}(R) = \varprojlim_n X(\tau_{\leqslant n} R)$.*

Proof. Let $\mathcal{C} \subseteq \mathrm{CAlg}^{\mathrm{cn}}$ denote the full subcategory spanned by the truncated objects. Note that the canonical map $X(R) \to \widehat{X}(R)$ is a homotopy equivalence for $R \in \mathcal{C}$. It follows that $\widehat{X}$ is nilcomplete. Moreover, if $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is nilcomplete, then Y is a right Kan extension of $Y|_{\mathcal{C}}$. It follows that the canonical map $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\widehat{X}, Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X, Y)$ can be identified with $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{S})}(\widehat{X}|_{\mathcal{C}}, Y|_{\mathcal{C}}) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{S})}(X|_{\mathcal{C}}, Y|_{\mathcal{C}})$. □

Remark 17.3.3.3. In the situation of Proposition 17.3.3.2, we can also write $\widehat{X}(R) \simeq \varprojlim X(R_n)$ for *any* tower $\{R_n\}$ of truncated objects of $\mathrm{CAlg}^{\mathrm{cn}}_R$, provided that the maps $R \to R_n$ become m-connective for $n \gg m$.

We now show that the process of nilcompletion has a mild effect on the deformation theory of a functor X.

Proposition 17.3.3.4. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and let $\widehat{X}$ be its nilcompletion. If X is cohesive (infinitesimally cohesive), then $\widehat{X}$ has the same property.*

Proof. We show that if X is cohesive, then $\widehat{X}$ is cohesive; the proof for infinitesimally cohesive functors is the same. Suppose we are given a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B' & \xrightarrow{g} & B, \end{array}$$

where the ring homomorphisms $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective. We wish to show that the diagram σ :

$$\begin{array}{ccc} \widehat{X}(A') & \longrightarrow & \widehat{X}(A) \\ \downarrow & & \downarrow \\ \widehat{X}(B') & \longrightarrow & \widehat{X}(B) \end{array}$$

is a pullback square of spaces. Using the description of $\widehat{X}$ supplied by Proposition 17.3.3.2 (and Remark **??**), we deduce that σ is the limit of a tower of diagrams σ_n :

$$\begin{array}{ccc} X(\tau_{\leqslant n} A \times_{\tau_{\leqslant n} B} \tau_{\leqslant n} B') & \longrightarrow & X(\tau_{\leqslant n} A) \\ \downarrow & & \downarrow \\ X(\tau_{\leqslant n} B') & \longrightarrow & X(\tau_{\leqslant n} B), \end{array}$$

each of which is a pullback square by virtue of our assumption that X is cohesive. □

Proposition 17.3.3.5. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and let $\widehat{X}$ be its nilcompletion. Then the restriction map $\mathrm{QCoh}(\widehat{X}) \to \mathrm{QCoh}(X)$ induces an equivalence of ∞-categories $\mathrm{QCoh}(\widehat{X})^{\mathrm{acn}} \to \mathrm{QCoh}(X)^{\mathrm{acn}}$.*

Proof. Let $\mathcal{C} \subseteq \mathrm{CAlg}^{\mathrm{cn}}$ be the full subcategory spanned by the truncated objects. We prove, more generally, that for any natural transformation $f : X \to Y$ between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, if the induced map $X|_{\mathcal{C}} \to Y|_{\mathcal{C}}$ is an equivalence, then the restriction functor

$\mathrm{QCoh}(Y)^{\mathrm{acn}} \to \mathrm{QCoh}(X)^{\mathrm{acn}}$ is also an equivalence. For every functor $U : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, let U' be a left Kan extension of $U|_{\mathcal{C}}$. We then have a commutative diagram of pullback functors

$$\begin{array}{ccc} \mathrm{QCoh}(Y)^{\mathrm{acn}} & \xrightarrow{f^*} & \mathrm{QCoh}(X)^{\mathrm{acn}} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(Y')^{\mathrm{acn}} & \longrightarrow & \mathrm{QCoh}(X')^{\mathrm{acn}}, \end{array}$$

where the lower horizontal map is an equivalence. It will therefore suffice to show that the vertical maps are equivalences. In other words, we are reduced to proving that for every functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the restriction functor $u_Y^* : \mathrm{QCoh}(Y)^{\mathrm{acn}} \to \mathrm{QCoh}(Y')^{\mathrm{acn}}$ is an equivalence. Note that the construction $Y \mapsto u_Y^*$ carries colimits in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ to limits in $\mathrm{Fun}(\Delta^1, \widehat{\mathcal{C}\mathrm{at}_\infty})$. We can therefore reduce to the case where the functor $Y = h^R$ is the functor corepresented by an $\mathbb{E}_\infty$-ring R. In this case, we can identify u_Y^* with the natural map

$$\mathrm{Mod}_R^{\mathrm{acn}} \to \varprojlim_n \mathrm{Mod}_{\tau_{\leq n} R}^{\mathrm{acn}} \qquad M \mapsto \{\tau_{\leq n} R \otimes_R M\},$$

which is evidently an equivalence of ∞-categories. □

Proposition 17.3.3.6. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and let $\widehat{X}$ be its nilcompletion. Suppose that X admits a cotangent complex L_X. Then $\widehat{X}$ admits a cotangent complex $L_{\widehat{X}} \in \mathrm{QCoh}(\widehat{X})^{\mathrm{acn}}$.*

Proof. We will show that the functor $\widehat{X}$ satisfies conditions (a) and (b) of Remark 17.2.4.3. Let A be a connective $\mathbb{E}_\infty$-ring and let $\eta \in \widehat{X}(A)$. Then η determines a compatible sequence of points $\eta_n \in X(\tau_{\leq n} A)$. For each $n \geq 0$, we regard $\eta_n^* L_X$ as a module over $\tau_{\leq n} A$. Choose an integer m such that $\eta_0^* L_X$ is $(-m)$-connective; it then follows that $\eta_n^* L_X$ is $(-m)$-connective for each $n \geq 0$ (Proposition 2.7.3.2). Set $K = \varprojlim \eta_n^* L_X$, so that K is a $(-m)$-connected A-module and the canonical map $(\tau_{\leq n} A) \otimes_A K \to \eta_n^* L_X$ is an equivalence for each n. For every A-module M, we have canonical homotopy equivalences

$$\begin{aligned} \widehat{X}(A \oplus M) \times_{\widehat{X}(A)} \{\eta\} &\simeq \varprojlim_n X((\tau_{\leq n} A) \oplus (\tau_{\leq n} M)) \times_{X(\tau_{\leq n} A)} \{\eta_n\} \\ &\simeq \varprojlim_n \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leq n} A}}(\eta_n^* L_X, \tau_{\leq n} M) \\ &\simeq \varprojlim_n \mathrm{Map}_{\mathrm{Mod}_A}(K, \tau_{\leq n} M) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_A}(K, M). \end{aligned}$$

This completes the verification of condition (a). To prove (b), suppose we are given a morphism of connective $\mathbb{E}_\infty$-rings $A \to A'$. Let η'_n denote the image of η in $X(\tau_{\leq n} A')$ and set $K' = \varprojlim \eta'^*_n L_X$. We wish to show that the canonical map $\theta : A' \otimes_A K \to K'$ is an equivalence. To prove this, we note that for each $n \geqslant 0$, the n-truncation of θ agrees with the n-truncation of the canonical map $(\tau_{\leqslant n+m} A') \otimes_{\tau_{\leqslant n+m} A} \eta^*_{n+m} L_X \to \eta'^*_{n+m} L_X$, which is an equivalence by virtue of our assumption that X satisfies condition (b) of Remark 17.2.4.3. □

Remark 17.3.3.7. In the situation of Proposition 17.3.3.6, the canonical map $L_{\widehat{X}}|_X \to L_X$ is an equivalence: in other words, the cotangent complex $L_{\widehat{X}}$ is the image of L_X under the equivalence $\mathrm{QCoh}(X)^{\mathrm{acn}} \to \mathrm{QCoh}(\widehat{X})^{\mathrm{acn}}$ of Proposition 17.3.3.5. In particular, the relative cotangent complex $L_{X/\widehat{X}}$ vanishes.

17.3.4 Integrable Functors

Let A be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 A$, and let $\mathrm{Spf}\, A$ denote the formal spectrum of A (Construction 8.1.1.10). According to Theorem 8.5.3.1, the restriction map

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, A, \mathsf{X}) \to \mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Spf}\, A, \mathsf{X})$$

is a homotopy equivalence whenever X is a quasi-separated spectral algebraic space. If A is Noetherian and the ideal $I \subseteq \pi_0 A$ is maximal, then the same assertion holds more generally for any spectral Deligne-Mumford n-stack X (see Proposition 17.3.4.2 below). Motivated by this observation, we introduce the following:

Definition 17.3.4.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is *integrable* if the following condition is satisfied:

$(*)$ Let A be a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal $\mathfrak{m} \subseteq \pi_0 A$. Then the inclusion of functors $\mathrm{Spf}\, A \hookrightarrow \mathrm{Spec}\, A$ induces a homotopy equivalence

$$X(A) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spec}\, A, X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, A, X).$$

Here we abuse notation by identifying the formal spectrum $\mathrm{Spf}\, A$ with the functor that it represents (see Theorem 8.1.5.1).

Proposition 17.3.4.2. *Let X be a spectral Deligne-Mumford n-stack for some $n < \infty$ and let X denote the functor represented by X. Then X is integrable.*

Proof. Let A be a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal and let $\mathrm{Spf}\, A$ denote the formal spectrum of A, which we regard as an object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Choose a tower of A-algebras

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma 8.1.2.2, so that $\operatorname{Spf} A \simeq \varinjlim \operatorname{Spec} A_n$ (Proposition 8.1.5.2). Each of the maps $\pi_0 A_i \to \pi_0 A_0$ is surjective with nilpotent kernel, and therefore induces an equivalence of ∞-categories $\mathrm{CAlg}^{\text{ét}}_{A_i} \to \mathrm{CAlg}^{\text{ét}}_{A_0}$. For every functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ satisfying étale descent, let $Y(m) \in \mathcal{S}\mathrm{hv}^{\text{ét}}_{A_0}$ be the functor given by the composition

$$\mathrm{CAlg}^{\text{ét}}_{A_0} \simeq \mathrm{CAlg}^{\text{ét}}_{A_m} \to \mathrm{CAlg}^{\mathrm{cn}} \xrightarrow{Y} \mathcal{S},$$

and let $Y(\infty)$ denote the image of $(Y|_{\mathrm{CAlg}^{\text{ét}}_A}) \in \mathcal{S}\mathrm{hv}^{\text{ét}}_A$ under the pullback map $\mathcal{S}\mathrm{hv}^{\text{ét}}_A \to \mathcal{S}\mathrm{hv}^{\text{ét}}_{A_0}$. Then the canonical map $X(A) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\operatorname{Spf} A, X) \to \varprojlim_{m \geqslant 0} X(A_m)$ can be identified with the composition

$$X(A) \xrightarrow{\theta} X(\infty)(A_0) \xrightarrow{\theta'} \varprojlim_m X(m)(A_0).$$

Proposition B.3.1.4 implies that A is Henselian, so that θ is a homotopy equivalence by Proposition B.6.5.4. To prove that θ' is a homotopy equivalence, it will suffice to verify the following assertion:

$(*)$ Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is representable by a spectral Deligne-Mumford n-stack. Then the canonical map $\phi_Y : Y(\infty) \to \varprojlim Y(m)$ is an equivalence in the ∞-topos $\mathcal{S}\mathrm{hv}^{\text{ét}}_{A_0}$.

To prove $(*)$, choose an étale surjection $u : Y_0 \to Y$, where Y_0 is representable by a disjoint union of affine spectral Deligne-Mumford stacks. Let $Y_\bullet$ denote the Čech nerve of u, so that $Y \simeq |Y_\bullet|$. Then ϕ_Y can be identified with the composite map

$$Y(\infty) \simeq |Y_\bullet(\infty)| \xrightarrow{\phi} |\varprojlim_m Y_\bullet(m)| \xrightarrow{\phi'} \varprojlim_m Y(m).$$

We first claim that ϕ' is an equivalence. Note that the simplicial object $\varprojlim_m Y_\bullet$ is given by the Čech nerve of the map $v : \varprojlim Y_0(m) \to \varprojlim Y(m)$. Since $\mathcal{S}\mathrm{hv}^{\text{ét}}_{A_0}$ is an ∞-topos, the map ϕ' is an equivalence if and only if v is an effective epimorphism. Let B_0 be any étale A_0-algebra, so that B_0 admits an essentially unique lift to an étale A_m-algebra B_m for each m. Since u is étale, the canonical map $Y_0(B_m) \to Y_0(B_0) \times_{Y(B_0)} Y(B_m)$ is a homotopy equivalence for each m. It follows that v is a pullback of the map $Y_0 \to Y$, which is an effective epimorphism by virtue of our assumption that u is an étale surjective.

Using the above argument, we see that ϕ_Y is an equivalence if and only if ϕ is an equivalence. Consequently, to prove that ϕ_Y is an equivalence, it will suffice to show that ϕ_{Y_p} is an equivalence. We now proceed by induction on n. If $n > 1$, then each Y_p is representable by a spectral Deligne-Mumford $(n-1)$-stack, so that the desired result follows from the inductive hypothesis. If $n = 1$, then each Y_p is representable by a spectral algebraic space; it will therefore suffice to verify $(*)$ in the special case where Y is representable by a spectral algebraic space. In this case, for each $p \geqslant 0$, the canonical map $Y_p(R) \to Y_0(R)^p$ is injective for every discrete commutative ring R. It will therefore suffice to verify $(*)$ under the assumption that there exists a map $Y \to Z$ which induces a monomorphism $Y(R) \to Z(R)$ for every discrete commutative ring R, where Z is representable by a disjoint union of affine spectral Deligne-Mumford stacks. In this case, each Y_p is itself a disjoint union of affine spectral Deligne-Mumford stacks. It will therefore suffice to verify $(*)$ in the special case $Y = \coprod_\alpha Y_\alpha$, where each Y_α is corepresented by a connective $\mathbb{E}_\infty$-ring R_α.

Let B_0 be an étale A_0-algebra; we wish to show that the canonical map $\gamma : Y(\infty)(B_0) \to \varprojlim Y(m)(B_0)$ is a homotopy equivalence. Without loss of generality, we may suppose that the spectrum of B_0 is connected. In this case, γ is given by a disjoint union of maps $\gamma_\alpha : Y_\alpha(\infty)(B_0) \to Y_\alpha(m)(B_0)$. It will therefore suffice to show that each γ_α is a homotopy equivalence. Let B be a finite étale A-algebra satisfying $B_0 \simeq B \otimes_A A_0$ (see Proposition B.6.5.2). We are then reduced to showing that the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}}(R_\alpha, B) \to \varprojlim_m \mathrm{Map}_{\mathrm{CAlg}}(R_\alpha, \varprojlim(B \otimes_A A_m))$$

is a homotopy equivalence. To prove this, it suffices to show that B is given by the limit of the diagram $\{B \otimes_A A_m\}$. Since B is a finite flat A-module, this follows from the identification $A \simeq \varprojlim_m A_m$. □

17.3.5 An Integrability Criterion

In good cases, the integrability of a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ can be tested at the level of discrete commutative rings:

Proposition 17.3.5.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. The following conditions are equivalent:*

(a) *The functor X is integrable.*

(b) *For every complete local Noetherian ring A, the canonical map*

$$X(A) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\operatorname{Spec} A, X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\operatorname{Spf} A, X)$$

is a homotopy equivalence. Here $\operatorname{Spf} A$ denotes the formal spectrum of A (with respect to the maximal ideal of $\pi_0 A$), which we identify with an object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$.

(c) *For every complete local Noetherian ring A with maximal ideal $\mathfrak{m}$, the canonical map*

$$X(A) \to \varprojlim_n X(A/\mathfrak{m}^n)$$

is a homotopy equivalence.

The proof of Proposition 17.3.5.1 will require some preliminaries.

Lemma 17.3.5.2. *Let A be a Noetherian commutative ring, let $I \subseteq A$ be an ideal, and let M be a finitely generated discrete A-module. For each integer $n > 0$, the tower $\{\mathrm{Tor}_n^A(A/I^m, M)\}_{m \geqslant 0}$ is trivial as a* Pro-*object in the category of A-modules.*

Proof. We proceed by induction on n. Since M is finitely generated, we can choose a surjection $\phi : A^d \to M$. Since A is Noetherian, the kernel $K = \ker(\phi)$ is also finitely generated. For each integer m, we have an exact sequence

$$0 = \mathrm{Tor}_n^A(A/I^m, A^r) \to \mathrm{Tor}_n^A(A/I^m, M) \to \mathrm{Tor}_{n-1}^A(A/I^m, K) \xrightarrow{\psi_m} \mathrm{Tor}_{n-1}^A(A/I^m, A^r).$$

In particular, we obtain a monomorphism of Pro-objects

$$\{\mathrm{Tor}_n^A(A/I^m, M)\}_{m \geqslant 0} \hookrightarrow \{\mathrm{Tor}_{n-1}^A(A/I^m, K)\}_{m \geqslant 0}.$$

If $n > 1$, then the inductive hypothesis implies that $\{\mathrm{Tor}_{n-1}^A(A/I^m, K)\}_{m \geqslant 0}$ vanishes (as a Pro-objects), so that $\{\mathrm{Tor}_n^A(A/I^m, M)\}_{m \geqslant 0}$ also vanishes. If $n = 1$, we are reduced to proving that the maps ψ_m determine a monomorphism $\{K/I^m K\}_{m \geqslant 0} \to \{(A/I^m A)^r\}_{m \geqslant 0}$, which follows from the Artin-Rees lemma. □

Remark 17.3.5.3. Let B be a connective $\mathbb{E}_\infty$-ring, and suppose we are given a tower

$$\cdots \to M_2 \to M_1 \to M_0$$

of discrete B-modules. Then $\{M_m\}_{m \geqslant 0}$ is a zero object of $\mathrm{Pro}(\mathrm{Mod}_B)$ if and only if, for every integer k, there exists an integer $k' \geqslant k$ such that the map $M_{k'} \to M_k$ vanishes. In particular, if we are given a map of connective $\mathbb{E}_\infty$-rings $\phi : A \to B$, then the tower $\{M_m\}_{m \geqslant 0}$ vanishes as a Pro-object of Mod_B if and only if it vanishes as a Pro-object of Mod_A.

Lemma 17.3.5.4. *Let A be a Noetherian commutative ring, let $I \subseteq A$ be an ideal, and let M be a connective A-module such that each homotopy group of M is annihilated by the ideal I. Then, for every integer $n \geqslant 0$, the tower $\{\tau_{\leqslant n}(M \otimes_A A/I^m)\}_{m \geqslant 0}$ is equivalent to $\tau_{\leqslant n} M$ as a* Pro-*object of* Mod_A.

Proof. We may assume without loss of generality that M is truncated (since $\tau_{\leqslant k}(M \otimes_A A/I^m)$ does not change if we replace M by $\tau_{\leqslant k} M$). Writing M as as a successive extension of modules of the form $\Sigma^d(\pi_d M)$, we may reduce to the case where M admits the structure of a module over A/I.

Fix an integer $n \geqslant 0$, and let $\mathcal{C}$ denote the full subcategory of Mod_A spanned by those A-modules which are connective and n-truncated. We may assume without loss of generality that $M \in \mathcal{C}$. We view $\mathcal{C}$ as a symmetric monoidal ∞-category, with tensor product $\otimes^\tau$ given by $N \otimes^\tau N' = \tau_{\leqslant n}(N \otimes_A N')$. We wish to prove that the canonical map

$\theta_M : M \to \{M \otimes^{\tau} A/I^m\}_{m \geqslant 0}$ is an equivalence in $\mathrm{Pro}(\mathcal{C})$. Note that θ_M is given by tensoring M with $\theta_{A/I}$ over A/I. It will therefore suffice to show that $\theta_{A/I}$ is an equivalence when viewed as a morphism in $\mathrm{Pro}(\mathrm{Mod}_{A/I}(\mathcal{C}))$. Unwinding the definitions, we are reduced to proving that $\theta_{A/I}$ determines isomorphisms

$$\{\mathrm{Tor}_d^A(A/I, A/I^m)\}_{m \geqslant 0} \simeq \begin{cases} A/I & \text{if } d = 0 \\ 0 & \text{if } 0 < d \leqslant n. \end{cases}$$

in the abelian category of Pro-objects in discrete A/I-modules. This is obvious when $d = 0$, and follows from Lemma 17.3.5.2 (together with Remark 17.3.5.3) when $d > 0$. □

Remark 17.3.5.5. Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings, and suppose we are given a tower of morphisms $\{\phi_m : M_m \to N_m\}$ in the abelian category of discrete B-modules. Then $\{\phi_m\}$ is an equivalence of Pro-objects of Mod_B if and only if it is an equivalence of Pro-objects of Mod_A. This follows immediately from Remark 17.3.5.3, applied to the towers $\{\ker(\phi_m)\}$ and $\{\mathrm{coker}(\phi_m)\}$.

Lemma 17.3.5.6. *Let A be a Noetherian commutative ring, let $I \subseteq A$ be an ideal, let R be a connective A-algebra such that the image of I generates a nilpotent ideal in $\pi_0 R$, and let $n \geqslant 0$ be an integer. Then the tower $\{\tau_{\leqslant n}(R \otimes_A A/I^m)\}_{m \geqslant 0}$ is equivalent to $\tau_{\leqslant n} R$ as a* Pro-*object of* CAlg_R.

Proof. Replacing I by a power of I if necessary, we may assume that the image of I vanishes in $\pi_0 R$. We proceed by induction on n, the case $n = 0$ being trivial. To carry out the inductive step, let us suppose that the canonical map

$$\rho : \tau_{\leqslant n-1} R \to \{\tau_{\leqslant n-1}(R \otimes_A A/I^m)\}_{m \geqslant 0}$$

is an equivalence of Pro-objects of CAlg_R. For each integer m, set $K_m = \pi_n(R \otimes_A A/I^m)$ and set $K = \pi_n R$. Applying Theorem HA.7.4.1.26 , we obtain a pullback diagram σ :

$$\begin{array}{ccc} \tau_{\leqslant n} R & \longrightarrow & \tau_{\leqslant n-1} R \\ \downarrow & & \downarrow \\ \tau_{\leqslant n-1} R & \longrightarrow & (\tau_{\leqslant n-1} R) \oplus \Sigma^{n+1} K \end{array}$$

with a map into a tower of pullback diagrams σ_m :

$$\begin{array}{ccc} \tau_{\leqslant n}(R \otimes_A A/I^m) & \longrightarrow & \tau_{\leqslant n-1}(R \otimes_A A/I^m) \\ \downarrow & & \downarrow \\ \tau_{\leqslant n-1}(R \otimes_A A/I^m) & \longrightarrow & \tau_{\leqslant n-1}(R \otimes_A A/I^m) \oplus \Sigma^{n+1} K_m. \end{array}$$

Applying the inductive hypothesis, we are reduced to proving that the canonical map

$$\theta : (\tau_{\leqslant n-1} R) \oplus \Sigma^{n+1} K \to \{\tau_{\leqslant n-1}(R \otimes_A A/I^m) \oplus \Sigma^{n+1} K_m\}_{m \geqslant 0}$$

is an equivalence of Pro-objects of CAlg_R. The map θ factors as a composition

$$(\tau_{\leqslant n-1} R) \oplus \Sigma^{n+1} K \xrightarrow{\theta'} \{\tau_{\leqslant n-1} R \oplus \Sigma^{n+1} K_m\}_{m \geqslant 0} \xrightarrow{\theta''} \{\tau_{\leqslant n-1}(R \otimes_A A/I^m) \oplus \Sigma^{n+1} K_m\}_{m \geqslant 0}$$

where θ'' is a pullback of ρ and therefore an equivalence by the inductive hypothesis. To prove that θ' is an equivalence, it will suffice to show that the canonical map $\nu : K \to \{K_m\}_{m \geqslant 0}$ is an equivalence of Pro-objects in $\tau_{\leqslant n-1} R$. By virtue of Remark 17.3.5.5, it will suffice to show that ν is an equivalence of Pro-objects in Mod_A, which follows from Lemma 17.3.5.4. □

Lemma 17.3.5.7. *Let A be a Noetherian commutative ring, let $I \subseteq A$ be an ideal, and choose a tower of A-algebras*

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma **??**. *Then, for every integer $m \geqslant 0$, the tower $\{\tau_{\leqslant n} A_m\}_{m \geqslant 0}$ is equivalent (as a pro-object of* CAlg*) to the tower $\{A/I^m\}_{m \geqslant 0}$.*

Proof. It will suffice to show that the canonical map $\{\tau_{\leqslant n} A_m\}_{m \geqslant 0} \to \{A/I^m\}_{m \geqslant 0}$ is an equivalence of Pro-objects of $\tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}_A$. Let $R \in \tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}_A$; we wish to show that the canonical map

$$\theta : \varinjlim \mathrm{Map}_{\mathrm{CAlg}_A}(A/I^m, R) \to \varinjlim \mathrm{Map}_{\mathrm{CAlg}_A}(A_m, R)$$

is a homotopy equivalence. If the image of I does not generate a nilpotent ideal in $\pi_0 R$, then the domain and codomain of θ are both empty and there is nothing to prove. Let us therefore suppose that the image of I generates a nilpotent ideal in R; we wish to show that the direct limit $\varinjlim \mathrm{Map}_{\mathrm{CAlg}_A}(A/I^m, R)$ is contractible. Since R is n-truncated, we can rewrite this direct limit as $\varinjlim \mathrm{Map}_{\mathrm{CAlg}_R}(\tau_{\leqslant n}(R \otimes_A A/I^m), R)$, which is contractible by virtue of the fact that the tower $\{\tau_{\leqslant n}(R \otimes_A A/I^m)\}_{m \geqslant 0}$ is equivalent to R as a Pro-object of CAlg_R (Lemma 17.3.5.6). □

Proof of Proposition 17.3.5.1. The implication $(a) \Rightarrow (b)$ is obvious. We next prove that $(b) \Rightarrow (a)$. Let A be a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to the maximal ideal $\mathfrak{m} \subseteq \pi_0 A$. Choose a tower of A-algebras $\{A_m\}_{m > 0}$ satisfying the requirements of Lemma 8.1.2.2, so that $\mathrm{Spf}\, A$ represents the functor $\varinjlim_m \mathrm{Spec}\, A_m$. For every connective A-algebra B, we can regard B as an adic $\mathbb{E}_\infty$-ring (with ideal of definition $\mathfrak{m}(\pi_0 B)$), so that the formal spectrum $\mathrm{Spf}\, B$ represents the functor $\varinjlim_m \mathrm{Spec}(A_m \otimes_A B)$. Let θ_B denote the canonical map $X(B) \to \varprojlim_m X(A_m \otimes_A B) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, B, X)$. We wish to show that θ_A is a homotopy equivalence.

Consider the diagram

$$\begin{array}{ccc} X(A) & \longrightarrow & \varprojlim_n X(\tau_{\leqslant n} A) \\ \downarrow & & \downarrow \\ \varprojlim_m X(A_m) & \longrightarrow & \varprojlim_{m,n} X(A_m \otimes_A \tau_{\leqslant n} A). \end{array}$$

The upper horizontal map is a homotopy equivalence since X is nilcomplete, and the bottom horizontal map is a homotopy equivalence by Proposition 17.3.2.4. It follows that θ_A can be identified with the limit of the tower of maps $\{\theta_{\tau_{\leqslant n} A}\}_{n \geqslant 0}$. It will therefore suffice to show that each $\theta_{\tau_{\leqslant n} A}$ is a homotopy equivalence. We may therefore replace A by $\tau_{\leqslant n} A$ and thereby reduce to the case where A is n-truncated for some integer n.

If $n = 0$, then A is discrete and the desired result follows from (b). Let us therefore assume that $n > 0$. Let $A' = \tau_{\leqslant n-1} A$ and let $M = \Sigma^{n+1}(\pi_n A)$, so that A fits into a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A' \oplus M. \end{array}$$

Since X is infinitesimally cohesive, we obtain a pullback diagram

$$\begin{array}{ccc} \theta_A & \longrightarrow & \theta_{A'} \\ \downarrow & & \downarrow \\ \theta_{A'} & \longrightarrow & \theta_{A' \oplus M} \end{array}$$

in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{S})$. The inductive hypothesis implies that $\theta_{A'}$ is a homotopy equivalence. It will therefore suffice to show that $\theta_{A' \oplus M}$ is a homotopy equivalence. Using the inductive hypothesis, we are reduced to proving that the canonical map

$$\psi : X(A' \oplus M) \rightarrow X(A') \times_{\varprojlim X(A_m \otimes_A A')} \varprojlim X((A_m \otimes_A A') \oplus (A_m \otimes_A M)).$$

Using the assumption that X is infinitesimally cohesive, we can identify the right side with $\varprojlim_m X(A' \oplus (A_m \otimes_A M))$. To show that ψ is a homotopy equivalence, it will suffice to show that ψ induces a homotopy equivalence after passing to the fibers over any point $\eta \in X(A')$. Since X admits a cotangent complex, this is equivalent to the assertion that the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_{A'}}(\eta^* L_X, M) \rightarrow \varprojlim \mathrm{Map}_{\mathrm{Mod}_{A'}}(\eta^* L_X, A_m \otimes_A M).$$

For this, it suffices to show that the canonical map $M \rightarrow \varprojlim_m A_m \otimes_A M$ is an equivalence. Since M is connective, this is equivalent to the requirement that M is $\mathfrak{m}$-complete, where

$\mathfrak{m}$ denotes the maximal ideal of $\pi_0 A$ (Remark ??). This completeness follows from our assumption that A is complete, since M is an almost perfect A-module (Proposition 7.3.5.7). This completes the proof that $(b) \Rightarrow (a)$.

To prove that (b) and (c) are equivalent, it will suffice to show that for every complete local Noetherian ring A with maximal ideal $\mathfrak{m}$, the canonical map $\rho : X(A) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, A, X)$ is a homotopy equivalence if and only if the canonical map $X(A) \to \varprojlim_n X(A/\mathfrak{m}^n)$ is a homotopy equivalence. To prove this, choose $\{A_m\}_{m \geqslant 0}$ as above. Since X is nilcomplete, we can identify ρ with the composite map

$$X(A) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, A, X) \simeq \varprojlim_{m} X(A_m) \simeq \varprojlim_{m,n} X(\tau_{\leqslant n} A_m).$$

The desired result now follows from Lemma 17.3.5.7. □

17.3.6 A Differential Criterion for Infinitesimal Cohesiveness

Our next goal is to establish a converse to Remark 17.3.1.8:

Proposition 17.3.6.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a nilcomplete functor which admits a cotangent complex. The following conditions are equivalent:*

(1) *For every pullback diagram*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ B' & \longrightarrow & B, \end{array}$$

of connective $\mathbb{E}_\infty$-rings, if the map f induces a surjection of commutative rings $\pi_0 A \to \pi_0 B$ with nilpotent kernel, then the diagram of spaces

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ X(B') & \longrightarrow & X(B) \end{array}$$

is a pullback square.

(2) *The functor X is infinitesimally cohesive.*

(3) *Let R be a connective $\mathbb{E}_\infty$-ring, M a connective R-module, $\eta : L_R \to \Sigma M$ a derivation, and R^η the corresponding square-zero extension of R by M, so that we have a pullback square*

$$\begin{array}{ccc} R^\eta & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \oplus \Sigma M. \end{array}$$

Then the diagram

$$\begin{array}{ccc} X(R^{\eta}) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ X(R) & \longrightarrow & X(R \oplus \Sigma M) \end{array}$$

is a pullback square in $\mathcal{S}$.

Before giving the proof of Proposition 17.3.6.1, let us sketch a sample application:

Proposition 17.3.6.2. *Let $f : X \to Y$ be a natural transformations between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that X and Y are nilcomplete, infinitesimally cohesive and admit cotangent complexes, and that the relative cotangent complex $L_{X/Y}$ is $(n+2)$-connective for some integer $n \geqslant 0$. The following conditions are equivalent:*

(1) *For every commutative ring R (regarded as a discrete $\mathbb{E}_\infty$-ring), the map f induces a homotopy equivalence $X(R) \to Y(R)$.*

(2) *For every n-truncated connective $\mathbb{E}_\infty$-ring R, the map f induces a homotopy equivalence $X(R) \to Y(R)$.*

Proof. The implication (2) ⇒ (1) is obvious. Conversely, suppose that (1) is satisfied. We must show that for every n-truncated connective $\mathbb{E}_\infty$-ring A, the map f induces a homotopy equivalence $f_A : X(A) \to Y(A)$. The proof proceeds by induction on n. When $n = 0$, the desired result follows from (1). If $n > 0$, then $\tau_{\leqslant n} A$ is a square-zero extension of $\tau_{\leqslant n-1} A$ (Corollary HA.7.4.1.28). We therefore have a pullback square of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} \tau_{\leqslant n} A & \longrightarrow & \tau_{\leqslant n-1} A \\ \downarrow & & \downarrow \\ \tau_{\leqslant n-1} A & \longrightarrow & \tau_{\leqslant n-1} A \oplus M, \end{array}$$

where $M = \Sigma^{n+1}(\pi_n A)$. Since X and Y are infinitesimally cohesive, to prove that $f_{\tau_{\leqslant n} A}$ is a homotopy equivalence, it will suffice to show that $f_{\tau_{\leqslant n-1} A}$ is a homotopy equivalence and $f_{\tau_{\leqslant n-1} A \oplus M}$ is (-1)-truncated (that is, it is equivalent to the inclusion of a summand). In the first case, this follows from the inductive hypothesis. For the second case, consider the commutative diagram

$$\begin{array}{ccc} X(\tau_{\leqslant n-1} A \oplus M) & \longrightarrow & Y(\tau_{\leqslant n-1} A \oplus M) \\ \downarrow & & \downarrow \\ X(\tau_{\leqslant n-1} A) & \longrightarrow & Y(\tau_{\leqslant n-1} A). \end{array}$$

We wish to prove that the upper horizontal map is (-1)-truncated. Since the bottom horizontal map is a homotopy equivalence, it will suffice to prove that we obtain a (-1)-truncated map after passing to the homotopy fibers over any point $\eta \in X(\tau_{\leqslant n-1}A)$. Unwinding the definitions, we are reduced to proving that the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_{\tau_{\leqslant n-1}A}}(\eta^* L_X, M) \to \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leqslant n-1}A}}(\eta^* f^* L_Y, M)$$

is (-1)-truncated. Using the fiber sequence $\eta^* f^* L_Y \to \eta^* L_X \to \eta^* L_{X/Y}$, we are reduced to proving that $\eta^* L_{X/Y}$ is $(n+2)$-truncated, which follows from our hypothesis. □

Corollary 17.3.6.3. *Let $f : X \to Y$ be a natural transformations between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that X and Y are infinitesimally cohesive, nilcomplete, and admit cotangent complexes. Then f is an equivalence if and only if the following conditions are satisfied:*

(1) *For every commutative ring R (regarded as a discrete $\mathbb{E}_\infty$-ring), the map f induces a homotopy equivalence $X(R) \to Y(R)$.*

(2) *The relative cotangent complex $L_{X/Y}$ is trivial.*

Proof. It is clear that if f is an equivalence then conditions (1) and (2) are satisfied. Conversely, suppose that (1) and (2) are satisfied. We wish to show that for every connective $\mathbb{E}_\infty$-ring R, the canonical map $\theta : X(R) \to Y(R)$ is a homotopy equivalence. Since X and Y are nilcomplete, the map θ is a limit of maps $\theta_n : X(\tau_{\leqslant n}R) \to Y(\tau_{\leqslant n}R)$. It will therefore suffice to show that each θ_n is a homotopy equivalence, which follows from Proposition 17.3.6.2. □

We now turn to the proof of Proposition 17.3.6.1.

Lemma 17.3.6.4. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Suppose that f induces a surjection of commutative rings $\pi_0 A \to \pi_0 B$ whose kernel I is a nilpotent ideal of $\pi_0 A$. Then we can write A as the limit of a tower*

$$\cdots \to B(2) \to B(1) = B$$

in the ∞-category $\mathrm{CAlg}_{/B}$ with the following property: each $B(n+1)$ is a square-zero extension of $B(n)$ by a $B(n)$-module $\Sigma^{k_n} M$, where M is discrete and $k_n \geqslant 0$. Moreover, we can assume that the sequence of integers $\{k_n\}_{n \geqslant 0}$ tends to infinity as n grows.

Proof. Choose an integer m such that $I^m = 0$. For $k \leqslant m$, we define $B(k)$ by the formula $B \times_{\pi_0 B} (\pi_0 A / I^k)$. Since $\pi_0 A / I^{k+1}$ is a square-zero extension of $\pi_0 A / I^k$ by I^k / I^{k+1}, we deduce that $B(k+1)$ is a square-zero extension of $B(k)$ by the discrete module I^k / I^{k+1} for $0 < k < m$. We next define $B(k) \in \mathrm{CAlg}_A$ for $k > m$ using induction on k, so that the fiber of the map $A \to B(k)$ is $(k-m)$-connective. Assume that $B(k)$ has been defined for $k \geqslant m$, and let $M = \pi_{k-m} \operatorname{fib}(A \to B(k))$. Since the map $\pi_0 A \to \pi_0 B(k)$ is an isomorphism, Theorem HA.7.4.3.1 implies that $L_{B(k)/A}$ is $(k-m+1)$-connective and that there is a canonical isomorphism $\pi_{k-m+1} L_{B(k)/A} \simeq M$. In particular, there exists a map of $B(k)$-modules $\eta : L_{B(k)/A} \to \Sigma^{k-m+1} M$ which induces an isomorphism $\pi_{k-m+1} L_{B(k)/A} \simeq M$. Let $B(k+1) = B(k)^\eta$ denote the square-zero extension of $B(k)$ by $\Sigma^{k-m} M$ classified by η. We now observe that by construction, the canonical map $A \to B(k+1)$ has $(k-m+1)$-connective fiber. □

Proof of Proposition 17.3.6.1. The implications (1) ⇒ (2) and (2) ⇒ (3) are obvious (and do not require any assumptions on X). Let us prove that (3) ⇒ (1). Suppose we are given a pullback square of connective $\mathbb{E}_\infty$-rings σ :

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B, \end{array}$$

where the maps $\pi_0 A \to \pi_0 B$ is a surjection with nilpotent kernel. We wish to show that $X(\sigma)$ is a pullback square in $\mathcal{S}$. Choose a tower

$$\cdots \to B(3) \to B(2) \to B(1) = B$$

satisfying the requirements of Lemma 17.3.6.4. For each integer $n \geqslant 1$, let $B'(n) = B(n) \times_B B'$, so that we have a pullback square $\sigma(n)$:

$$\begin{array}{ccc} B'(n) & \longrightarrow & B(n) \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B. \end{array}$$

Since X is nilcomplete, Proposition 17.3.2.4 implies that $X(\sigma)$ is a limit of the tower $\{X(\sigma(n))\}_{n \geqslant 1}$. It will therefore suffice to show that each $X(\sigma(n))$ is a pullback square in $\mathcal{S}$. The proof proceeds by induction on n, the case $n = 1$ being trivial. If $n > 1$, we consider the commutative diagram

$$\begin{array}{ccc} X(B'(n)) & \longrightarrow & X(B(n)) \\ \downarrow & & \downarrow \\ X(B'(n-1)) & \longrightarrow & X(B(n-1)) \\ \downarrow & & \downarrow \\ X(B') & \longrightarrow & X(B). \end{array}$$

The inductive hypothesis implies that the lower square is a pullback diagram. To prove that the outer square is a pullback diagram, it suffices to show that the upper square is a pullback diagram. By hypothesis, $B(n)$ is a square-zero extension of $B(n-1)$ by a connective $B(n-1)$-module M. We therefore have a commutative diagram

$$\begin{array}{ccccc} X(B'(n)) & \longrightarrow & X(B(n)) & \longrightarrow & X(B(n-1)) \\ \downarrow & & \downarrow & & \downarrow \\ X(B'(n-1)) & \longrightarrow & X(B(n-1)) & \longrightarrow & X(B(n-1) \oplus \Sigma M) \end{array}$$

where the square on the right is a pullback diagram by virtue of assumption (3). To prove that the left square is a pullback, it will suffice to show that the outer rectangle is a pullback. Note that the bottom horizontal composite admits a factorization

$$B'(n-1) \to B'(n-1) \oplus \Sigma M \to B(n-1) \oplus \Sigma M.$$

We may therefore form a commutative diagram

$$\begin{array}{ccccc} B'(n) & \longrightarrow & R & \longrightarrow & B(n-1) \\ \downarrow & & \downarrow & & \downarrow \\ B'(n-1) & \longrightarrow & B'(n-1) \oplus \Sigma M & \longrightarrow & B(n-1) \oplus \Sigma M \\ & & \downarrow & & \downarrow \\ & & B'(n-1) & \longrightarrow & B(n-1) \end{array}$$

where every square is a pullback diagram. Since the vertical composition on the right is an equivalence, it follows that the vertical compositon in the middle is an equivalence: that is, we can identify R with $B(n-1)$. Applying the functor X, we obtain a diagram of spaces

$$\begin{array}{ccccc} X(B'(n)) & \longrightarrow & X(B'(n-1)) & \longrightarrow & X(B(n-1)) \\ \downarrow & & \downarrow & & \downarrow \\ X(B'(n-1)) & \longrightarrow & X(B'(n-1) \oplus \Sigma M) & \longrightarrow & X(B(n-1) \oplus \Sigma M) \\ & & \downarrow & & \downarrow \\ & & X(B'(n-1)) & \longrightarrow & X(B(n-1)). \end{array}$$

The upper left square is a pullback diagram by assumption (3). Since X admits a cotangent complex, the lower right square is also a pullback diagram (Example 17.2.4.4). Since the vertical composite maps are equivalences, the rectangle on the right is a pullback diagram. It follows that the upper left square is a pullback square, so that the upper rectangle is a pullback square as desired. □

17.3.7 Deformation-Theoretic Conditions on Morphisms

We now introduce relative versions of Definitions 17.3.1.1, 17.3.1.5, 17.3.2.1, and 17.3.4.1:

Definition 17.3.7.1. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will say that f is:

(a) *cohesive* if, for every pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

of connective $\mathbb{E}_\infty$-rings such that $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective, the cubical diagram of spaces

$$\begin{array}{ccccccc} X(A') & & \longrightarrow & & X(A) & & \\ \downarrow & \searrow & & & \downarrow & \searrow & \\ & & Y(A') & \longrightarrow & & & Y(A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X(B') & & \longrightarrow & & X(B) & & \\ & \searrow & \downarrow & & & \searrow & \downarrow \\ & & Y(B') & \longrightarrow & & & Y(B) \end{array}$$

is a limit.

(b) *infinitesimally cohesive* if, for every pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

of connective $\mathbb{E}_\infty$-rings such that $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjections with nilpotent kernel, the diagram of spaces

$$\begin{array}{ccccccc}
X(A') & & \longrightarrow & & X(A) & & \\
\downarrow & \searrow & & & \downarrow & \searrow & \\
& & Y(A') & \longrightarrow & & & Y(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X(B') & & \longrightarrow & & X(B) & & \\
& \searrow & \downarrow & & & \searrow & \downarrow \\
& & Y(B') & \longrightarrow & & & Y(B)
\end{array}$$

is a limit cube.

(c) *nilcomplete* if, for every connective $\mathbb{E}_\infty$-ring A, the diagram

$$\begin{array}{ccc}
X(A) & \longrightarrow & \varprojlim X(\tau_{\leqslant n} A) \\
\downarrow & & \downarrow \\
Y(A) & \longrightarrow & \varprojlim Y(\tau_{\leqslant n} A)
\end{array}$$

is a pullback square.

(d) *integrable* if, for every complete local Noetherian $\mathbb{E}_\infty$-ring A, the diagram

$$\begin{array}{ccc}
X(A) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\operatorname{Spf} A, X) \\
\downarrow & & \downarrow \\
Y(A) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\operatorname{Spf} A, Y)
\end{array}$$

is a pullback square; here $\operatorname{Spf} A$ denotes the formal spectrum of A, regarded as an object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$.

Example 17.3.7.2. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be an arbitrary functor, and let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the constant functor taking the value $* \in \mathcal{S}$. Then there is a unique natural transformation $f : X \to Y$ (up to homotopy). Moreover, the natural transformation f is cohesive (infinitesimally cohesive, nilcomplete, integrable) if and only if X is cohesive (infinitesimally cohesive, nilcomplete, integrable).

Remark 17.3.7.3. Suppose we are given a diagram

$$\begin{array}{ccccc}
& & Y & & \\
& \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\
X & & \xrightarrow{h} & & Z
\end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, where g is cohesive (infinitesimally cohesive, nilcomplete, integrable). Then f is cohesive (infinitesimally cohesive, nilcomplete, integrable) if and only if h is cohesive (infinitesimally cohesive, nilcomplete, integrable).

Taking Z to be the final object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, we deduce that if $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is cohesive (infinitesimally cohesive, nilcomplete, integrable), then a morphism $f : X \to Y$ is cohesive (infinitesimally cohesive, nilcomplete, integrable) if and only if X is cohesive (infinitesimally cohesive, nilcomplete, integrable).

Remark 17.3.7.4. Let $f : X \to Y$ be a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Suppose we are given a field κ and a point $\eta \in X(\kappa)$. We define a functor $X^{\wedge} : \mathrm{CAlg}^{\mathrm{sm}}_{\kappa} \to \mathcal{S}$ by the formula

$$X^{\wedge}(A) = \mathrm{fib}(X(A) \to Y(A) \times_{Y(\kappa)} X(\kappa)).$$

If the morphism f is infinitesimally cohesive, then the functor $X^{\wedge}$ is a formal moduli problem (in the sense of Definition 12.1.3.1) and therefore has a well-defined tangent complex $T_{X^{\wedge}} \in \mathrm{Mod}_{\kappa}$ (Definition 12.2.2.1). Unwinding the definitions, we see that this tangent complex is characterized up to equivalence by the formula

$$\mathrm{Map}_{\mathrm{Mod}_{\kappa}}(M^{\vee}, T_{X^{\wedge}}) \simeq X^{\wedge}(\kappa \oplus M) \simeq \mathrm{fib}(X(\kappa \oplus M) \to Y(\kappa \oplus M) \times_{Y(\kappa)} X(\kappa)),$$

where M varies over perfect connective κ-modules. If f admits a relative tangent cocomplex $L_{X/Y}$, then the right hand side can be identified with $\mathrm{Map}_{\mathrm{Mod}_{\kappa}}(\eta^* L_{X/Y}, M)$. We can therefore identify $T_{X^{\wedge}}$ with the κ-linear dual $(\eta^* L_{X/Y})^{\vee}$ of $\eta^* L_{X/Y}$.

17.3.8 Relativization and Fibers

Roughly speaking, a morphism $f : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ is cohesive (infinitesimally cohesive, nilcomplete, integrable) if and only if each fiber of f is cohesive (infinitesimally cohesive, nilcomplete, integrable). We now articulate this idea more precisely.

Notation 17.3.8.1. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Given a point $\eta \in Y(R)$, we let $X_{\eta} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ given on objects by the formula $X_{\eta}(A) = X(A) \times_{Y(A)} \{\eta_A\}$, where $\eta_A \in Y(A)$ denotes the image of η.

Suppose now that η induces an equivalence $\mathrm{Spec}\, R \to Y$: that is, for every $\mathbb{E}_{\infty}$-ring A, evaluation at η induces a homotopy equivalence $\mathrm{Map}_{\mathrm{CAlg}}(R, A) \to Y(A)$. In this case, the construction $X \mapsto X_{\eta}$ induces an equivalence of ∞-categories $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Y} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})$ (Corollary HTT.5.1.6.12).

Now let $F : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be an arbitrary functor. The above discussion shows that there is an equivalence $F \simeq X_{\eta}$ for some natural transformation $f : X \to \mathrm{Spec}\, R$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, which is determined uniquely up to a contractible space of choices. In this case, we will denote the functor X by $\underline{F}$. We will say that F is *cohesive* (*infinitesimally cohesive*, *nilcomplete*, *integrable*) if the functor $\underline{F}$ is cohesive (infinitesimally cohesive, nilcomplete, *integrable*). Since $\operatorname{Spec} R$ is cohesive and nilcomplete, we see from Remark 17.3.7.3 that F is cohesive (infinitesimally cohesive, nilcomplete, integrable) if and only if the natural transformation f is cohesive (infinitesimally cohesive, nilcomplete, integrable).

Remark 17.3.8.2. Let R be a connective $\mathbb{E}_\infty$-ring, let $F : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ be a functor, and let $\underline{F} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be as in Notation 17.3.8.1. If F classifies a left fibration $p : \mathcal{C} \to \mathrm{CAlg}_R^{\mathrm{cn}}$, then $\underline{F}$ classifies the left fibration given by the composite map $\mathcal{C} \xrightarrow{p} \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CAlg}^{\mathrm{cn}}$. More informally: we can identify $\underline{F}(A)$ with a classifying space for pairs (ϕ, η), where $\phi : R \to A$ is a map of $\mathbb{E}_\infty$-rings and $\eta \in F(A)$, where A is regarded as an R-algebra via the map ϕ.

Remark 17.3.8.3. Let R be a connective $\mathbb{E}_\infty$-ring and let $F : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Unwinding the definitions, we deduce:

- The functor F is cohesive if and only if, for every pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B \end{array}$$

  in $\mathrm{CAlg}_R^{\mathrm{cn}}$ for which the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective, the induced diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow{\scriptstyle f} \\ X(B') & \xrightarrow{g} & X(B) \end{array}$$

  is a pullback square in $\mathcal{S}$.

- The functor F is infinitesimally cohesive if and only if, for every pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B \end{array}$$

  in $\mathrm{CAlg}_R^{\mathrm{cn}}$ for which the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective with nilpotent kernel, the induced diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow{\scriptstyle f} \\ X(B') & \xrightarrow{g} & X(B) \end{array}$$

  is a pullback square in $\mathcal{S}$.

- The functor F is nilcomplete if and only if, for every connective R-algebra A, the canonical map $F(A) \to \varprojlim F(\tau_{\leqslant n} A)$ is a homotopy equivalence.

- The functor F is integrable if and only if, for every local Noetherian $\mathbb{E}_\infty$-algebra A over R which is complete with respect to its maximal ideal $\mathfrak{m}$, the canonical map $F(A) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S})}(Y, F)$, where $Y : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is the functor which assigns to each R-algebra B the full subcategory of $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B)$ spanned by those maps which annihilate some power of the maximal ideal $\mathfrak{m}$.

Proposition 17.3.8.4. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. The following conditions are equivalent:*

(1) *The map f is cohesive (infinitesimally cohesive, nilcomplete, integrable).*

(2) *For every pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, the map f' is cohesive (infinitesimally cohesive, nilcomplete, integrable).

(3) *For every pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$ where Y' is a corepresentable functor, the map f' is cohesive (infinitesimally cohesive, nilcomplete,integrable).

(4) *For every connective $\mathbb{E}_\infty$-ring R and every point $\eta \in Y(R)$, the functor $X_\eta : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is cohesive (infinitesimally cohesive, nilcomplete,integrable).*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious, and the equivalence (3) ⇔ (4) follows from Remark 17.3.7.3. We will complete the proof by showing that (3) ⇒ (1). For simplicity, let us treat the assertion concerning nilcomplete functors; the proofs in the other cases are essentially the same. Let R be a connective $\mathbb{E}_\infty$-ring; we wish to show that the diagram

$$\begin{array}{ccc} X(R) & \longrightarrow & \varprojlim X(\tau_{\leqslant n} R) \\ \downarrow & & \downarrow \\ Y(R) & \longrightarrow & Y(\tau_{\leqslant n} R) \end{array}$$

is a pullback square. Equivalently, we wish to show that for every point $\eta \in Y(R)$, the induced map

$$X(R) \times_{Y(R)} \{\eta\} \to \varprojlim (X(\tau_{\leqslant n} R) \times_{Y(\tau_{\leqslant n} R)} \{\eta\})$$

is a homotopy equivalence (here we abuse notation by identifying η with its image in $Y(\tau_{\leqslant n} R)$, for each $n \geqslant 0$). The point η determines a natural transformation $Y' \to Y$, where $Y' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is the functor corepresented by R. Since η lies in the essential image of the map $Y'(R) \to Y(R)$, we may replace f by the projection map $f' : X \times_Y Y' \to Y'$. In this case, the desired result follows from (3). □

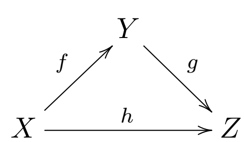
Corollary 17.3.8.5. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and suppose that f is relatively representable by spectral Deligne-Mumford stacks. Then f is cohesive, nilcomplete, and admits a cotangent complex. Moreover, the relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ is connective.*

Proof. The first two assertions follow from Proposition 17.3.8.4, Example 17.3.1.2, and Proposition 17.3.2.3. For the third, we combine Propositions 17.2.4.7 and 17.2.5.1. □

17.3.9 Cohesive Functors and the Relative Cotangent Complex

Using the notion of an infinitesimally cohesive morphism, we can formulate a useful variant of Proposition 17.2.5.2:

Proposition 17.3.9.1. *Let*



be a commutative diagram in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Assume that g is infinitesimally cohesive and admits a cotangent complex. Then f is infinitesimally cohesive and admits a cotangent complex if and only if h is infinitesimally cohesive and admits a cotangent complex.

Proof. The "if" direction follows immediately from Remark 17.3.7.3 and Proposition 17.2.5.2. For the converse, let us suppose that f is infinitesimally cohesive and admits a cotangent complex. Remark 17.3.7.3 implies that h is infinitesimally cohesive. We will complete the proof by showing that h admits a cotangent complex.

Let $\mathrm{Mod}^X_{\mathrm{cn}}$ be the ∞-category defined in Example **??**, and let $F : \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ be defined by the formula

$$F(R, \eta, M) = \mathrm{fib}(X(R \oplus M) \to X(R) \times_{Z(R)} Z(R \oplus M)).$$

We wish to prove that F is locally almost corepresentable. Define $F', F'' : \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ by the formulae

$$F'(R, \eta, M) = \mathrm{fib}(X(R \oplus M) \to X(R) \times_{Y(R)} Y(R \oplus M))$$

$$F''(R, \eta, M) = \mathrm{fib}(Y(R \oplus M) \to Y(R) \times_{Z(R)} Z(R \oplus M)),$$

so that we have a fiber sequence of functors $F' \to F \to F''$. Note that each of these functors is naturally pointed, so we get a fiber sequence $\Omega F \to \Omega F'' \to F'$. Since f and g admit cotangent complexes, the functors F' and F'' are locally almost corepresentable. It follows that ΩF is locally almost corepresentable (Remark 17.2.3.3). Since h is infinitesimally cohesive, the functor F is given by the formula $F(R, \eta, M) \simeq (\Omega F)(R, \eta, \Sigma M)$ and is therefore also locally almost corepresentable. □

Remark 17.3.9.2. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Suppose that f is infinitesimally cohesive and admits a cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$. Let A be a connective $\mathbb{E}_\infty$-ring and let $\overline{A}$ be a square-zero extension of A by a connective A-module M, so that we have a pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} \overline{A} & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \oplus \Sigma M. \end{array}$$

This diagram satisfies the hypotheses of case (b) of Definition 17.3.7.1, so that we obtain a limit diagram

$$\begin{array}{ccccccc} X(\overline{A}) & & \longrightarrow & & X(A) & & \\ \downarrow & \searrow & & & \downarrow & \searrow & \\ & & Y(\overline{A}) & \longrightarrow & & & Y(A) \\ X(A) & & \downarrow & \longrightarrow & X(A \oplus \Sigma M) & & \downarrow \\ & \searrow & & & & \searrow & \\ & & Y(A) & \longrightarrow & & & Y(A \oplus \Sigma M). \end{array}$$

We can think of this as giving us a pullback square

$$\begin{array}{ccc} X(\overline{A}) & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ X(A) \times_{Y(A)} Y(\overline{A}) & \longrightarrow & X(A \oplus \Sigma M) \times_{Y(A \oplus \Sigma M)} Y(A). \end{array}$$

Fixing a point $\eta \in X(A)$ having image $\eta_0 \in Y(A)$, we obtain a fiber sequence of spaces

$$\{\eta\} \times_{X(A)} X(\overline{A}) \to \{\eta_0\} \times_{Y(A)} Y(\overline{A}) \to \mathrm{Map}_{\mathrm{Mod}_A}(\eta^* L_{X/Y}, \Sigma M).$$

We can summarize the situation as follows: to every point $\overline{\eta}_0 \in Y(\overline{A})$ lifting η_0, we can associate a class $\sigma \in \mathrm{Ext}^1_A(\eta^* L_{X/Y}, M)$, which vanishes if and only if the lifting problem

$$\begin{array}{ccc} \mathrm{Spec}\, A & \xrightarrow{\eta} & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}\, \overline{A} & \xrightarrow{\overline{\eta}_0} & Y \end{array}$$

admits a solution.

Proposition 17.3.9.3. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Suppose that f is infinitesimally cohesive and that f admits a relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$. Let A be a connective $\mathbb{E}_\infty$-ring, and let $\overline{A}$ be a square-zero extension of A by a connective A-module M. Then:*

(1) *If $\eta^* L_{X/Y}$ is a projective A-module for each $\eta \in X(A)$, then the canonical map $X(\overline{A}) \to X(A) \times_{Y(A)} Y(\overline{A})$ is surjective on connected components.*

(2) *If $L_{X/Y}$ vanishes, then the canonical map $X(\overline{A}) \to X(A) \times_{Y(A)} Y(\overline{A})$ is a homotopy equivalence.*

In the case of representable functors, we have the following converse:

Proposition 17.3.9.4. *Let $\phi : \mathsf{X} \to \mathsf{Y}$ be a map of spectral Deligne-Mumford stacks representing functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and suppose that the underlying map $\tau_{\leqslant 0} \mathsf{X} \to \tau_{\leqslant 0} \mathsf{Y}$ is locally finitely 0-presented (see Definition 4.2.3.1). Then:*

(1) *The map ϕ is étale if and only if, for every connective $\mathbb{E}_\infty$-ring A and every square-zero extension $\overline{A}$ of A by a connective A-module, the map $X(\overline{A}) \to X(A) \times_{Y(A)} Y(\overline{A})$ is a homotopy equivalence.*

(2) *The map ϕ is differentially smooth if and only if, for every connective $\mathbb{E}_\infty$-ring A and every square-zero extension A^{η} of A by a connective A-module, the map $X(\overline{A}) \to X(A) \times_{Y(A)} Y(\overline{A})$ is surjective on connected components.*

(3) *The map ϕ is fiber-smooth if and only if ϕ is flat and, for every discrete $\mathbb{E}_\infty$-ring A and every square-zero extension A^{η} of A by a discrete A-module, the map $X(\overline{A}) \to X(A) \times_{Y(A)} Y(\overline{A})$ is surjective on connected components.*

Proof. We first prove (1). Applying Proposition 17.1.5.1, we see that ϕ is étale if and only if the relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}}$ vanishes. The "only if" direction now follows immediately from Proposition 17.3.9.3. To prove the converse, fix an étale map $\eta : \operatorname{Spét} A \to \mathsf{X}$. If the canonical map $X(A \oplus M) \to X(A) \times_{Y(A)} Y(A \oplus M)$ is a homotopy equivalence, then the mapping space $\operatorname{Map}_{\operatorname{Mod}_A}(\eta^* L_{\mathsf{X}/\mathsf{Y}}, M)$ is contractible. Taking $M = \eta^* L_{\mathsf{X}/\mathsf{Y}}$, we deduce that $\eta^* L_{\mathsf{X}/\mathsf{Y}}$ vanishes.

To prove the "if" directions of (2) and (3), we can reduce to the case where X and Y are affine, in which case the desired results follow from Propositions 11.2.2.1 and Corollary 11.2.4.2. The "only if" direction of (2) follows from Proposition 17.3.9.3 and 17.1.5.1. We will complete the proof by verifying the "only if" direction of (3). Suppose that ϕ is fiber-smooth, and that we are given a lifting problem

$$\begin{array}{ccc} \operatorname{Spét} A & \xrightarrow{\eta} & \mathsf{X} \\ \downarrow & \nearrow & \downarrow{\scriptstyle \phi} \\ \operatorname{Spét} \overline{A} & \longrightarrow & \mathsf{Y}, \end{array}$$

where A is a discrete $\mathbb{E}_\infty$-ring and $\overline{A}$ is a square-zero extension of A by a discrete A-module M. To prove that this lifting problem admits a solution, it will suffice to show that the group $\operatorname{Ext}^1_A(\eta^* L_{\mathsf{X}/\mathsf{Y}}, M)$ vanishes (see Remark 17.3.9.2). This follows from the fact that $\tau_{\leqslant 1} \eta^* L_{\mathsf{X}/\mathsf{Y}}$ is a projective A-module (see Proposition 17.1.5.1). $\square$

17.4 Finiteness Conditions on Morphisms

Let $f : X \to Y$ be a morphism of schemes. Recall that f is said to be *locally of finite presentation* if it satisfies either of the following equivalent conditions:

(a) For every pair of affine open subsets $\operatorname{Spec} A \simeq U \subseteq Y$ and $\operatorname{Spec} B \simeq V \subseteq X \times_Y U$, the morphism f exhibits B as a finitely presented A-algebra.

(b) For every filtered diagram of commutative rings $\{R_\alpha\}$ having colimit R, the diagram

$$\begin{array}{ccc} \varinjlim X(R_\alpha) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ \varinjlim Y(R_\alpha) & \longrightarrow & Y(R) \end{array}$$

is a pullback square.

In §4.2, we studied several finiteness conditions that can be placed on a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks, which are defined using a variant of (a)

(see Definition 4.2.0.1). Our goal here is to provide alternative definitions for these notions which are closer in spirit to (b), and have the virtue of making sense for arbitrary functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. The contents of this section can be summarized as follows:

- (i) In §17.4.1, we introduce several finiteness conditions on a natural transformation $f : X \to Y$ of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$: the conditions that f is *locally of finite presentation*, *locally almost of finite presentation*, and *locally of finite generation to order* n for $n \geqslant 0$ (Definition 17.4.1.1).

- (ii) In §17.4.2, we show that if $f : X \to Y$ is a functor which admits a cotangent complex $L_{X/Y}$, then there is a close relationship between finiteness conditions on f and finiteness conditions on $L_{X/Y}$. In particular, we show that if f is locally of finite presentation (locally almost of finite presentation, locally of finite generation to order n), then $L_{X/Y}$ is perfect (almost perfect, perfect to order n). Moreover, the converse assertions hold under some mild additional assumptions (see Proposition 17.4.2.1, Corollary 17.4.2.2, and Proposition 17.4.2.3).

- (iii) In §17.4.3, we show that if X and Y are (representable by) spectral Deligne-Mumford m-stacks for some $m < \infty$, then the finiteness conditions introduced in Definition 17.4.1.1 agree with the analogous finiteness conditions appearing in Definition 4.2.0.1 (Proposition 17.4.3.1).

17.4.1 Local Finite Presentation

We now introduce several finiteness conditions on a morphism $f : X \to Y$ in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$.

Definition 17.4.1.1. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will say that f is *locally of finite presentation* if the following condition is satisfied:

- (a) Let $\{A_\alpha\}$ be a filtered diagram of connective $\mathbb{E}_\infty$-rings with colimit A. Then the canonical map
$$\theta : \varinjlim X(A_\alpha) \to X(A) \times_{Y(A)} \varinjlim Y(A_\alpha)$$
is a homotopy equivalence.

We say that f is *locally almost of finite presentation* if it satisfies the following weaker condition:

- (b) Let $m \geqslant 0$, and let $\{A_\alpha\}$ be a filtered diagram of m-truncated connective $\mathbb{E}_\infty$-rings with colimit A. Then the canonical map
$$\theta : \varinjlim X(A_\alpha) \to X(A) \times_{Y(A)} \varinjlim Y(A_\alpha)$$

is a homotopy equivalence.

If $n \geqslant 0$, we say that f is *locally of finite generation to order* n if the following still weaker condition is satisfied:

(c) Let $\{A_\alpha\}$ be a filtered diagram of connective $\mathbb{E}_\infty$-rings having colimit A. Assume that each A_α is n-truncated and that the transition maps $\pi_n A_\alpha \to \pi_n A_\beta$ are monomorphisms. Then the canonical map

$$\theta : \varinjlim X(A_\alpha) \to X(A) \times_{Y(A)} \varinjlim Y(A_\alpha)$$

is a homotopy equivalence.

Remark 17.4.1.2. A morphism $f : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ is locally almost of finite presentation if and only if it is locally of finite generation to order n for every integer $n \geqslant 0$.

Remark 17.4.1.3. Suppose we are given a commutative diagram

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ X & & \xrightarrow{h} & & Z \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Suppose that g is locally of finite presentation (locally almost of finite presentation, locally of finite generation to order n). Then f is locally of finite presentation (locally almost of finite presentation, locally of finite generation to order n) if and only if h is locally of finite presentation (locally almost of finite presentation, locally of finite generation to order n).

Remark 17.4.1.4. Let $\{X_\alpha\}$ be a diagram in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ having colimit $X = \varinjlim_\alpha X_\alpha$, and let $f : X \to Y$ be a natural transformation. If each each of the composite maps $X_\alpha \to X \xrightarrow{f} Y$ is locally of finite presentation (locally almost of finite presentation, locally of finite generation to order n), then so is f.

We have the following counterpart of Proposition 17.3.8.4:

Proposition 17.4.1.5. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. The following conditions are equivalent:*

(1) *The map f is locally of finite presentation (locally almost of finite presentation, locally of finite generation to order n).*

(2) *For every pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$*, the map* f' *is locally of finite presentation (locally almost of finite presentation, locally of finite generation to order* n*).*

(3) *For every pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ *where* Y' *is a corepresentable functor, the map* f' *is locally of finite presentation (locally almost of finite presentation, locally of finite generation to order* n*).*

Remark 17.4.1.6. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Suppose that there exists a connective $\mathbb{E}_\infty$-ring R and a point $\eta \in Y(R)$ which exhibits Y as the functor corepresented by R. Let $X_\eta : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be as in Notation 17.3.8.1. Then:

(a) The map f is locally of finite presentation if and only if the functor X_η commutes with filtered colimits.

(b) The map f is locally almost of finite presentation if and only if, for every integer $m \geqslant 0$, the restriction $X_\eta|_{\tau_{\leqslant m} \mathrm{CAlg}^{\mathrm{cn}}_R}$ commutes with filtered colimits.

(c) The map f is locally of finite generation to order n if and only if, for every filtered diagram $\{A_\alpha\}$ of connective $\mathbb{E}_\infty$-rings having colimit A, if each A_α is n-truncated and each of the transition maps $\pi_n A_\alpha \to \pi_n A_\beta$ is a monomorphism, then the canonical map $\varinjlim X_\eta(A_\alpha) \to X_\eta(A)$ is a homotopy equivalence.

17.4.2 Finite Presentation and the Cotangent Complex

Let $f : X \to Y$ be a morphism in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. If f admits a cotangent complex, then the finiteness conditions appearing in Definition 17.4.1.1 can often be reformulated as conditions on $L_{X/Y} \in \mathrm{QCoh}(X)$.

Proposition 17.4.2.1. *Let* $f : X \to Y$ *be a natural transformation of functors* $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$*, and suppose that* f *admits a cotangent complex* $L_{X/Y}$*. Then:*

(1) *If f is locally of finite generation to order n, then the relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ is perfect to order n.*

(2) *Assume that f is infinitesimally cohesive and satisfies the following additional condition:*

(∗) *For every filtered diagram $\{A_\alpha\}$ of commutative rings having colimit A, the diagram of spaces*

$$\begin{array}{ccc} \varinjlim X(A_\alpha) & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ \varinjlim Y(A_\alpha) & \longrightarrow & Y(A) \end{array}$$

is a pullback square.

If the relative cotangent complex $L_{X/Y}$ is perfect to order n, then f is locally of finite generation to order n.

Corollary 17.4.2.2. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that f admits a cotangent complex. Then:*

(1) *If f is locally almost of finite presentation, then the relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ is almost perfect.*

(2) *Assume that f is infinitesimally cohesive and satisfies condition (∗) of Proposition 17.4.2.1. If $L_{X/Y}$ is almost perfect, then f is locally almost of finite presentation.*

Proof. Combine Proposition 17.4.2.1 with Remark 17.4.1.2 □

Proof of Proposition 17.4.2.1. Suppose first that f is locally of finite generation to order n. Choose a connective $\mathbb{E}_\infty$-ring A and a point $\eta \in X(A)$; we wish to show that $\eta^* L_{X/Y} \in \mathrm{Mod}_A$ is perfect to order n. We will verify that $\eta^* L_{X/Y}$ satisfies the third criterion of Proposition 2.7.0.4. Let $\{M_\alpha\}$ be a filtered diagram of discrete A-modules having colimit M, where the transition maps $M_\alpha \to M_\beta$ are monomorphisms; we wish to show that the canonical map $\theta : \varinjlim \mathrm{Map}_{\mathrm{Mod}_A}(\eta^* L_{X/Y}, \Sigma^n M_\alpha) \to \mathrm{Map}_{\mathrm{Mod}_A}(\eta^* L_{X/Y}, \Sigma^n M)$ is a homotopy equivalence. To prove this, we may replace A by $\tau_{\leq n} A$ and thereby reduce to the case where A is n-truncated. Unwinding the definitions, we see that θ can be obtained from the commutative diagram σ :

$$\begin{array}{ccc} \varinjlim X(A \oplus \Sigma^n M_\alpha) & \longrightarrow & X(A) \times_{Y(A)} \varinjlim Y(A \oplus \Sigma^n M_\alpha) \\ \downarrow & & \downarrow \\ X(A \oplus M) & \longrightarrow & X(A) \times_{Y(A)} Y(A \oplus \Sigma^n M_\alpha) \end{array}$$

by passing to homotopy fibers in the horizontal direction. We now conclude by observing that our hypothesis that f is locally of finite generation to order n guarantees that σ is a pullback square. This completes the proof of (1).

We now prove (2). Using Proposition 17.4.1.5, we can reduce to the case where the functor Y is corepresentable by a connective $\mathbb{E}_\infty$-ring R. The assumption that f is infinitesimally cohesive then implies that X is infinitesimally cohesive (Remark 17.3.7.3). Let $X_\eta : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be as in Remark 17.4.1.6. Assumption $(*)$ implies that the restriction of X_η to $\mathrm{CAlg}^{\heartsuit}_R$ commutes with filtered colimits. We will show that X_η satisfies condition (c) of Remark 17.4.1.6. Fix a diagram $\{A_\alpha\}$ in $\mathrm{CAlg}^{\mathrm{cn}}_R$ indexed by a filtered partially ordered set I, where where each A_α is n-truncated and the transition maps $\pi_n A_\alpha \to \pi_n A_\beta$ are monomorphisms. Set $A = \varinjlim_\alpha A_\alpha$. For $0 \leqslant i \leqslant n$, let $\theta_i : \varinjlim_\alpha X_\eta(A_\alpha) \to X_\eta(A)$ denote the canonical map. We will show that each of the maps θ_i is a homotopy equivalence. The proof proceeds by induction on i. In the case $i = 0$, this follows from assumption $(*)$.

To carry out the inductive step, let us assume that $i > 0$ and that θ_{i-1} is a homotopy equivalence. For each α, set $A'_\alpha = \tau_{\leqslant i-1} A_\alpha$ and $M_\alpha = \pi_i A_\alpha$, so that we have a filtered system of pullback diagrams

$$\begin{array}{ccc} \tau_{\leqslant i} A_\alpha & \longrightarrow & A'_\alpha \\ \downarrow & & \downarrow \\ A'_\alpha & \longrightarrow & A'_\alpha \oplus \Sigma^{i+1} M_\alpha. \end{array}$$

Set $A' = \varinjlim_\alpha A'_\alpha \simeq \tau_{\leqslant i-1} A$ and $M = \varinjlim_\alpha M_\alpha$, and let $\rho : \varinjlim X_\eta(A'_\alpha \oplus \Sigma^{i+1} M_\alpha) \to X_\eta(A' \oplus M)$ be the canonical map. Since the functor X_η is infinitesimally cohesive, we can identify θ_i with the fiber product $\theta_{i-1} \times_\rho \theta_{i-1}$ in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{S})$. Consequently, to show that θ_i is a homotopy equivalence, it will suffice to show that ρ has (-1)-truncated homotopy fibers.

We have a commutative diagram

$$\begin{array}{ccc} \varinjlim X_\eta(A'_\alpha \oplus \Sigma^{i+1} M_\alpha) & \xrightarrow{\rho} & X_\eta(A' \oplus \Sigma^{i+1} M) \\ \downarrow u & & \downarrow v \\ \varinjlim X_\eta(A'_\alpha) & \xrightarrow{\theta_{i-1}} & X_\eta(A'), \end{array}$$

where the lower horizontal map is a homotopy equivalence by virtue of our inductive hypothesis. It will therefore suffice to show that the map ρ induces a (-1)-truncated map from each homotopy fiber of u to the corresponding homotopy fiber of v. Fix a point of the space $\varinjlim X_\eta(A'_\alpha)$, which we can represent by an element $\overline{\eta} \in X_\eta(A'_\alpha)$ for some $\alpha \in I$. Unwinding the definitions, we see that the homotopy fiber of u over $\overline{\eta}$ can be identified with the direct limit $\varinjlim_{\beta \geqslant \alpha} \mathrm{Map}_{\mathrm{Mod}_{A'_\alpha}}(\overline{\eta}^* L_{X/Y}, \Sigma^{i+1} M_\beta)$, while the homotopy fiber of v

over $\theta_{i-1}(\overline{\eta})$ can be identified with $\mathrm{Map}_{\mathrm{Mod}_{A'_\alpha}}(\overline{\eta}^* L_{X/Y}, \Sigma^{i+1} M)$. We are therefore reduced to showing that the map

$$\varinjlim_{\beta \geqslant \alpha} \mathrm{Map}_{\mathrm{Mod}_{A'_\alpha}}(\overline{\eta}^* L_{X/Y}, \Sigma^{i+1} M_\beta) \to \mathrm{Map}_{\mathrm{Mod}_{A'_\alpha}}(\overline{\eta}^* L_{X/Y}, \Sigma^{i+1} M)$$

has (-1)-truncated homotopy fibers. In other words, we wish to show that the colimit $\varinjlim_{\beta \geqslant \alpha} \mathrm{Map}_{\mathrm{Mod}_{A'_\alpha}}(\overline{\eta}^* L_{X/Y}, K_\beta)$ is contractible, where K_β denotes the fiber of the canonical map $\Sigma^{i+1} M_\beta \to \Sigma^{i+1} M$. Since each K_β is n-truncated and the colimit $\varinjlim K_\beta$ vanishes, this follows from our assumption that $\overline{\eta}^* L_{X/Y}$ is perfect to order n as an A'_α-module. □

Proposition 17.4.2.3. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and assume that f admits a cotangent complex. Then:*

(1) *If f is locally of finite presentation, the relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ is perfect.*

(2) *Assume that f is nilcomplete, infinitesimally cohesive, and satisfies condition $(*)$ of Proposition 17.4.2.1. If $L_{X/Y}$ is perfect, then f is locally of finite presentation.*

Proof. We first prove (1). Choose a connective $\mathbb{E}_\infty$-ring A and a point $\eta \in X(A)$; we wish to show that $\eta^* L_{X/Y} \in \mathrm{Mod}_A$ is perfect. Since $L_{X/Y}$ is locally almost connective, we can choose an integer k such that $\eta^* L_{X/Y} \in (\mathrm{Mod}_A)_{\geqslant -k}$. To prove that $\eta^* L_{X/Y}$ is perfect, it will suffice to show that it is a compact object of $(\mathrm{Mod}_A)_{\geqslant -k}$. For this, we note that the functor corepresented by $\eta^* L_{X/Y}$ is given by

$$M \mapsto \Omega^k \operatorname{fib}(X(R \oplus \Sigma^k M) \to X(R) \times_{Y(R)} Y(R \oplus \Sigma^k M)),$$

which commutes with filtered colimits if f is locally of finite presentation.

We now prove (2). Using Proposition 17.4.1.5, we may assume without loss of generality that Y is corepresentable by a connective $\mathbb{E}_\infty$-ring R. Let $X_0 : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be the functor given by the formula $X_0(A) = \operatorname{fib}(X(A) \to Y(A))$, as in Remark 17.4.1.6. We wish to prove that X_0 commutes with filtered colimits.

Let $\{A_\alpha\}$ be a diagram of connective $\mathbb{E}_\infty$-rings indexed by a filtered partially ordered set P, and set $A = \varinjlim A_\alpha$. We wish to prove that the canonical map $\varinjlim X_0(A_\alpha) \to X_0(A)$ is a homotopy equivalence. For this, it suffices to show that for every point $\eta \in X_0(\pi_0 A)$, the induced map

$$\theta : \varinjlim X_0(A_\alpha) \times_{X_0(\pi_0 A)} \{\eta\} \to X_0(A) \times_{X_0(\pi_0 A)} \{\eta\}$$

is a homotopy equivalence. Since $X_0|_{\mathrm{CAlg}^\heartsuit_R}$ commutes with filtered colimits, we may assume that η is the image of a point $\eta_\alpha \in X_0(\pi_0 A_\alpha)$ for some $\alpha \in P$. For $\beta \geqslant \alpha$, let η_β denote the image of η in $X_0(\pi_0 A_\beta)$. Then we can identify θ with the canonical map

$$\varinjlim_{\beta \geqslant \alpha} X_0(A_\beta) \times_{X_0(\pi_0 A_\beta)} \{\eta_\beta\} \to X_0(A) \times_{X_0(\pi_0 A)} \{\eta\}.$$

To prove that this map is a homotopy equivalence, it will suffice to show that for every integer $n \geqslant 0$, the induced map

$$\theta_n : \tau_{\leqslant n} \varinjlim_{\beta \geqslant \alpha} (X_0(A_\beta) \times_{X_0(\pi_0 A_\beta)} \{\eta_\beta\}) \to \tau_{\leqslant n}(X_0(A) \times_{X_0(\pi_0 A)} \{\eta\})$$

is a homotopy equivalence.

For every map of $\mathbb{E}_\infty$-rings $A_\alpha \to B$, let η_B denote the image of η_α in $X_0(\pi_0 B)$. Our proof relies on the following assertion:

($\star$) There exists an integer $m \geqslant 0$ with the following property: for every map of connective $\mathbb{E}_\infty$-rings $A_\alpha \to B$, the canonical map

$$\tau_{\leqslant n}(X_0(B) \times_{X_0(\pi_0 B)} \{\eta_B\}) \to \tau_{\leqslant n}(X_0(\tau_{\leqslant m} B) \times_{X_0(\pi_0 B)} \{\eta_B\})$$

is a homotopy equivalence.

Let m satisfy the condition of ($\star$). We have a commutative diagram

$$\begin{array}{ccc} \tau_{\leqslant n} \varinjlim_{\beta \geqslant \alpha} (X_0(A_\beta) \times_{X_0(\pi_0 A_\beta)} \{\eta_\beta\}) & \xrightarrow{\theta_n} & \tau_{\leqslant n}(X_0(A) \times_{X_0(\pi_0 A)} \{\eta\}) \\ \downarrow & & \downarrow \\ \tau_{\leqslant n} \varinjlim_{\beta \geqslant \alpha} (X_0(\tau_{\leqslant m} A_\beta) \times_{X(\pi_0 A_\beta)} \{\eta_\beta\}) & \xrightarrow{\theta'_n} & \tau_{\leqslant n}(X_0(\tau_{\leqslant m} A) \times_{X_0(\pi_0 A)} \{\eta\}) \end{array}$$

where the vertical maps are homotopy equivalences. Consequently, to prove that θ_n is a homotopy equivalence, it suffices to show that θ'_n is a homotopy equivalence. This follows from the fact that the functor X is locally almost of finite presentation (which follows from Corollary 17.4.2.2).

It remains to prove ($\star$). Since $L_{X/Y}$ is perfect, $\eta_\alpha^* L_{X/Y}$ is a dualizable object of $\mathrm{Mod}_{\pi_0 A_\alpha}$. Let V denote a dual of $\eta_\alpha^* L_{X/Y}$, and choose an integer k such that V is k-connective. We claim that $m = n - k$ satisfies the condition of ($\star$). Choose a map of connective $\mathbb{E}_\infty$-rings $A_\alpha \to B$. We will prove that the map

$$X_0(B) \times_{X_0(\pi_0 B)} \{\eta_B\} \to X_0(\tau_{\leqslant m} B) \times_{X_0(\pi_0 B)} \{\eta_B\}$$

is $(n+1)$-connective.

Since the functor X is nilcomplete (Remark 17.3.7.3), $X_0(B) \times_{X_0(\pi_0 B)} \{\eta_B\}$ is the homotopy inverse limit of the tower $\{X_0(\tau_{\leqslant m'} B) \times_{X_0(\pi_0 B)} \{\eta_B\}\}_{m' \geqslant m}$. It will therefore suffice to show that the transition maps

$$\gamma_{m'} : X_0(\tau_{\leqslant m'+1} B) \times_{X_0(\pi_0 B)} \{\eta_B\} \to X_0(\tau_{\leqslant m'} B) \times_{X_0(\pi_0 B)} \{\eta_B\}$$

are $(n+1)$-connective for each $m' \geqslant m$.

Set $M = \pi_{m'+1} B$, so that there is a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} \tau_{\leqslant m'+1} B & \longrightarrow & \tau_{\leqslant m'} B \\ \downarrow & & \downarrow \\ \tau_{\leqslant m'} B & \longrightarrow & \tau_{\leqslant m'} B \oplus \Sigma^{m'+2} M. \end{array}$$

Since X is infinitesimally cohesive (Remark 17.3.7.3), this diagram gives us a pullback square of spaces

$$\begin{array}{ccc} X_0(\tau_{\leqslant m'+1} B) \times_{X_0(\pi_0 B)} \{\eta_B\} & \xrightarrow{\gamma(m')} & X_0(\tau_{\leqslant m'} B) \times_{X_0(\pi_0 B)} \{\eta_B\} \\ \downarrow & & \downarrow \\ X_0(\tau_{\leqslant m'} B) \times_{X_0(\pi_0 B)} \{\eta_B\} & \xrightarrow{\gamma'} & X_0(\tau_{\leqslant m'} B \oplus \Sigma^{m'+2} M) \times_{X_0(\pi_0 B)} \{\eta_B\}. \end{array}$$

It will therefore suffice to show that the map γ' is $(n+1)$-connective. We note that γ' has a left homotopy inverse

$$\epsilon : X_0(\tau_{\leqslant m'} B \oplus \Sigma^{m'+2} M) \times_{X_0(\pi_0 B)} \{\eta_B\} \to X_0(\tau_{\leqslant m'} B) \times_{X_0(\pi_0 B)} \{\eta_B\}.$$

Consequently, we are reduced to proving that the homotopy fibers of ϵ are $(n+2)$-connective. Choose a point of $X_0(\tau_{\leqslant m'} B) \times_{X_0(\pi_0 B)} \{\eta_B\}$, corresponding to a point $\eta' \in X_0(\tau_{\leqslant m'} B)$ lifting η_B. Unwinding the definitions, we see that the homotopy fiber of ϵ over this point is given by the mapping space

$$\begin{aligned} \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leqslant m'} B}}(\eta'^* L_{X/Y}, \Sigma^{m'+2} M) &\simeq \mathrm{Map}_{\mathrm{Mod}_{\pi_0 B}}(\eta_B^* L_{X/Y}, \Sigma^{m'+2} M) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{\pi_0 A_\alpha}}(\eta_\alpha^* L_{X/Y}, \Sigma^{m'+2} M) \\ &\simeq \Omega^\infty(V \otimes_{\pi_0 A_\alpha} \Sigma^{m'+2} M). \end{aligned}$$

Since V is k-connective, $V \otimes_{\pi_0 A_\alpha} \Sigma^{m'+2} M$ is $(k+m'+2)$-connective. It now suffices to observe that $k + m' + 2 \geqslant n + 2$, since $m' \geqslant m = n - k$. □

17.4.3 Relationship with Geometric Finiteness Conditions

We conclude this section by studying the relationship of finiteness conditions of Definition 17.4.1.1 with the analogous conditions on morphisms of spectral Deligne-Mumford stacks studied in Chapter 4.

Proposition 17.4.3.1. *Let X and Y be spectral Deligne-Mumford stacks representing functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, let $\phi : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks, and let $f : X \to Y$ be the natural transformation determined by ϕ. Assume that ϕ is a relative Deligne-Mumford m-stack for some integer $m \gg 0$, and let $f : X \to Y$ denote the induced map of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Then:*

(1) *For every integer $n \geqslant 0$, the morphism f is locally of finite generation to order n (in the sense of Definition 17.4.1.1) if and only if ϕ is locally of finite generation to order n (in the sense of Definition 4.2.0.1).*

(2) *The map f is locally almost of finite presentation (in the sense of Definition 17.4.1.1) if and only if ϕ is locally almost of finite presentation (in the sense of Definition 4.2.0.1).*

(3) *The map f is locally of finite presentation (in the sense of Definition 17.4.1.1) if and only if ϕ is locally of finite presentation (in the sense of Definition 4.2.0.1).*

The proof of Proposition 17.4.3.1 will require some preliminaries.

Lemma 17.4.3.2. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and let $n \geqslant 0$. Suppose that there exists an integer $m \geqslant -2$ which satisfies the following condition:*

($*_m$) *For every n-truncated object $A \in \mathrm{CAlg}^{\mathrm{cn}}$, the induced map $X(A) \to Y(A)$ has (m)-truncated homotopy fibers.*

Let $X', Y' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the sheafification of X and Y with respect to the étale topology. If f is locally of finite generation to order n, then the induced map $f' : X' \to Y'$ is also locally of finite presentation to order n.

Proof. We proceed by induction on m. In the case $m = -2$, the map $f : X(A) \to Y(A)$ is an equivalence whenever A is n-truncated, and the desired result is obvious. To carry out the inductive step, let us assume that $m > -2$ and that f is locally of finite generation to order n. It follows that the relative diagonal $\delta : X \to X \times_Y X$ is also locally of finite generation to order n. Since δ satisfies condition ($*_{m-1}$), our inductive hypothesis guarantees that the relative diagonal $\delta' : X' \to X' \times_{Y'} X'$ is locally of finite generation to order n.

Fix a connective $\mathbb{E}_\infty$-ring R and a point $\eta \in Y(R)$, define $X_\eta : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ as in Notation 17.3.8.1, and define X'_η similarly. Note that we can identify X'_η with the sheafification of X_η with respect to the étale topology. By virtue of Remark 17.4.1.6, it will suffice to proving the following:

(i) For every diagram $\{A_\alpha\}_{\alpha \in I}$ in $\mathrm{CAlg}^{\mathrm{cn}}_R$ indexed by a filtered partially ordered set I, where each A_α is n-truncated and each transition map $\pi_n A_\alpha \to \pi_n A_\beta$ is a monomorphism, the canonical map $\rho : \varinjlim X'_\eta(A_\alpha) \to X'_\eta(A)$ is a homotopy equivalence; here $A = \varinjlim_{\alpha \in I} A_\alpha$.

In the situation of (i), our hypothesis that δ' is locally of finite generation to order n guarantees that the relative diagonal of ρ is a homotopy equivalence: that is, the homotopy fibers of ρ are (-1)-truncated. It will therefore suffice to show that ρ is surjective on connected components.

Fix a point $x' \in X'_\eta(A)$. Since X'_η is the sheafification of X_η with respect to the étale topology, there exists a finite collection of étale maps $\{A \to A(i)\}_{1 \leqslant i \leqslant n}$ with the following properties:

- (a) The map $A \to \prod_{1 \leqslant i \leqslant n} A(i)$ is faithfully flat.
- (b) For $1 \leqslant i \leqslant n$, let $x'_i \in X'_\eta(A(i))$ denote the image of x. Then x'_i lifts (up to homotopy) to a point $x_i \in X_\eta(A(i))$.

Using the structure theory of étale morphisms (Proposition B.1.1.3), we may choose an index $\alpha \in I$ and a collection of étale morphisms $A_\alpha \to A_\alpha(i)$, together with equivalences $A(i) \simeq A_\alpha(i) \otimes_{A_\alpha} A$. Enlarging α, we may assume that the map $A_\alpha \to \prod A_\alpha(i)$ is faithfully flat.

For $\beta \geqslant \alpha$, let us define a functor $\mathscr{F}_\beta : \mathrm{CAlg}^{\text{ét}}_{A_\alpha} \to \mathcal{S}$ by the formua $\mathscr{F}_\beta(B) = X'_\eta(B \otimes_{A_\alpha} A_\beta)$. Define $\mathscr{F} : \mathrm{CAlg}^{\text{ét}}_{A_\alpha} \to \mathcal{S}$ by the formula $\mathscr{F}(B) = X'_\eta(B \otimes_{A_\alpha} A)$, so that we have an evident map $\psi : \varinjlim_{\beta \geqslant \alpha} \mathscr{F}_\beta \to \mathscr{F}$. Since the map δ' is locally of finite generation to order n, the map ψ is (-1)-truncated. To complete the proof, it will suffice to show that the point $x' \in X'_\eta(A) \simeq \mathscr{F}(A_\alpha)$ belongs to the essential image of the map $\psi(A_\alpha) : \varinjlim_{\beta \geqslant \alpha} \mathscr{F}_\beta(A_\alpha) \to \mathscr{F}(A_\alpha)$. Note that $\mathscr{F}$ and each $\mathscr{F}_\beta$ are m-truncated sheaves with respect to the étale topology on $\mathrm{CAlg}^{\text{ét}}_{A_\alpha}$, so that the colimit $\varinjlim_{\beta \geqslant \alpha} \mathscr{F}_\beta$ is also a sheaf with respect to the étale topology. Consequently, the assertion that x' belongs to the essential image of $\psi(A_\alpha)$ can be tested locally with respect to the étale topology on A_α. We may therefore replace A_α by $A_\alpha(i)$ and thereby reduce to the case where x' belongs to the essential image of the map $X_\eta(A) \to X'_\eta(A)$. In this case, the desired result follows from our assumption that f is locally of finite presentation to order n. □

Remark 17.4.3.3. In the statement of Lemma 17.4.3.2 (in in its applications given below), we can replace the étale topology by the Zariski or Nisnevich topologies: the proof carries over without essential change.

Recall that if R is an $\mathbb{E}_\infty$-ring, we let $\mathcal{S}\mathrm{hv}^{\text{ét}}_R$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\text{ét}}_R, \mathcal{S})$ spanned by those functors which are sheaves with respect to the étale topology.

Lemma 17.4.3.4. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathscr{F}$ be a truncated object of $\mathcal{S}\mathrm{hv}^{\text{ét}}_R$. For every map of $\mathbb{E}_\infty$-rings $R \to A$, let $\mathscr{F}_A$ denote the image of $\mathscr{F}$ in the ∞-category $\mathcal{S}\mathrm{hv}^{\text{ét}}_A$ (in other words, the pullback of $\mathscr{F}$ along the map $\mathrm{Spét}\, A \to \mathrm{Spét}\, R$). Then the functor $A \mapsto \mathscr{F}_A(A)$ commutes with filtered colimits.*

Proof. Choose an integer m such that $\mathscr{F}$ is n-truncated. Let $X : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be a left Kan extension of $\mathscr{F} : \mathrm{CAlg}^{\text{ét}}_R \to \mathcal{S}$. For every connective R-algebra A, we have

$X(A) \simeq \varinjlim_{R' \to A} \mathscr{F}(R')$, where the colimit is taken over the full subcategory of $\mathrm{CAlg}_{R//A}$ spanned by those R' which are étale over R. Since this ∞-category is filtered, we deduce that $X(A)$ is m-truncated for every object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$. Because every étale R-algebra is a compact object of CAlg_R, the functor X commutes with filtered colimits. Note that the functor $A \mapsto \mathscr{F}_A(A)$ is the sheafification of X with respect to the étale topology. The desired result now follows from Lemma 17.4.3.2. □

Lemma 17.4.3.5. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that f is a relative étale Deligne-Mumford n-stack: that is, for every morphism $Y' \to Y$ where Y' is representable by a connective $\mathbb{E}_\infty$-ring R, the fiber product $X \times_Y Y'$ is representable by a spectral Deligne-Mumford n-stack which is étale over* $\mathrm{Spét}\, R$*. Then f is locally of finite presentation.*

Proof. Using Propositon 17.4.1.5, we may suppose that Y is corepresentable. The desired result now follows by combining the criterion of Remark 17.4.1.6 with Lemma 17.4.3.4. □

Lemma 17.4.3.6. *Let $f : R \to A$ be a map of connective $\mathbb{E}_\infty$-rings, and assume that the fiber of f is m-connective for $m \geqslant 0$. If B is an n-truncated $\mathbb{E}_\infty$-ring, then the mapping space* $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B)$ *is $(n - m - 1)$-truncated.*

Proof. We define a diagram of objects $R = R(0) \to R(1) \to R(2) \to \cdots$ in the ∞-category $(\mathrm{CAlg}_R)_{/A}$ by induction. Assuming that $R(i)$ has been defined, let $K(i)$ denote the fiber of the map $R(i) \to A$, and form a pushout diagram

$$\begin{array}{ccc} \mathrm{Sym}^* K(i) & \longrightarrow & R \\ \downarrow & & \downarrow \\ R(i) & \longrightarrow & R(i+1). \end{array}$$

We first claim that the canonical map $\varinjlim R(i) \to A$ is an equivalence of $\mathbb{E}_\infty$-algebras over R. To prove this, it suffices to show that θ induces an equivalence in the ∞-category of R-modules. This is clear, since colimit $\{R(i)\}$ agrees with the colimit of

$$R(0) \to \mathrm{cofib}(K(0) \to R(0)) \to R(1) \to \mathrm{cofib}(K(1) \to R(1)\} \to \cdots,$$

which contains a cofinal subsequence taking the constant value A.

We next claim that each $K(i)$ is m-connective. Equivalently, we claim that each of the maps $\pi_j R(i) \to \pi_j A$ is surjective for $i = m$ and bijective for $i < m$. This is true by hypothesis when $i = 0$; we treat the general case using induction on i. Since $K(i)$ is

m-connective, we have equivalences

$$\begin{aligned}\tau_{\leqslant m-1}R(i+1) &\simeq \tau_{\leqslant m-1}(R\otimes_{\mathrm{Sym}^* K(i)} R(i))\\ &\simeq \tau_{\leqslant m-1}(\tau_{\leqslant m-1}R\otimes_{\tau_{\leqslant m-1}\mathrm{Sym}^* K(i)} \tau_{\leqslant m-1}R(i))\\ &\simeq \tau_{\leqslant m-1}(\tau_{\leqslant m-1}R\otimes_{\tau_{\leqslant m-1}R} \tau_{\leqslant m-1}R(i))\\ &\simeq \tau_{\leqslant m-1}R(i)\\ &\simeq \tau_{\leqslant m-1}A.\end{aligned}$$

The surjectivity of the map $\pi_m R(i) \to \pi_m A$ follows from the surjectivity of the map $\pi_m R \to \pi_m A$.

Note that $\mathrm{Map}_{\mathrm{CAlg}_R}(A,B)$ can be identified with the homotopy limit of the tower of spaces $\{\mathrm{Map}_{\mathrm{CAlg}_R}(R(i),B)\}_{i\geqslant 0}$, and that the mapping space $\mathrm{Map}_{\mathrm{CAlg}_R}(R,B)$ is contractible. To prove that $\mathrm{Map}_{\mathrm{CAlg}_R}(A,B)$ is $(n-1-m)$-truncated, it will suffice to show that each of the maps $\theta_i : \mathrm{Map}_{\mathrm{CAlg}_R}(R(i+1),B) \to \mathrm{Map}_{\mathrm{CAlg}_R}(R(i),B)$ is $(n-m-1)$-truncated. Note that θ_i is a pullback of the map

$$\theta'_i : * \simeq \mathrm{Map}_{\mathrm{CAlg}_R}(R,B) \to \mathrm{Map}_{\mathrm{CAlg}_R}(\mathrm{Sym}^* K(i),B) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(K(i),B).$$

It will therefore suffice to show that each of the mapping spaces $\mathrm{Map}_{\mathrm{Mod}_R}(K(i),B)$ is $(n-m)$-truncated. This follows from our assumption that $K(i)$ is m-connective and B is n-truncated. □

Proof of Proposition 17.4.3.1. The implication (1) ⇒ (2) is obvious. Note that f is locally of finite presentation if and only if it locally almost of finite presentation and $L_{X/Y}$ is perfect (Proposition 17.4.2.3). To prove that (2) ⇒ (3), it will suffice to show that ϕ is locally of finite presentation if and only if it is locally almost of finite presentation and $L_{\mathsf{X}/\mathsf{Y}}$ is perfect. This assertion is local on X and Y. We may therefore suppose that X and Y are affine, in which case the desired result follows from Theorem HA.7.4.3.18 .

It remains to prove (1). Using Proposition 17.4.1.5, we can reduce to the case where $\mathsf{Y} = \mathrm{Spét}\, R$ is affine, so that X is a spectral Deligne-Mumford m-stack. Suppose first that f is locally of finite generation to order n. Choose an étale map $u : \mathrm{Spét}\, A \to \mathsf{X}$; we wish to show that A is of finite generation to order n over R. Unwinding the definitions, we see that this is equivalent to the requirement that the composite map $\mathrm{Spec}\, A \to X \xrightarrow{f} Y = \mathrm{Spec}\, R$ is of finite generation to order n. This follows from Remark 17.4.1.3, since f is locally of finite generation to order n (by assumption) and g is locally of finite presentation (Lemma 17.4.3.5).

We now prove the converse. We first treat the case where X is a coproduct of affine spectral Deligne-Mumford stacks $\{\mathsf{X}_\alpha\}_{\alpha\in S}$, indexed by some finite set S. For every finite subset $T \subseteq S$, let $\mathsf{X}_T = \coprod_{\alpha\in T}\mathsf{X}_\alpha$, and let X_T denote the functor represented by X_T. If X is locally of finite generation to order n over Y, then each X_T has the same property. Since each X_T is affine, we conclude that the map $X_T \to Y$ is locally of of finite generation to order n, so that the induced map $\varinjlim_{T \subseteq S} X_T \to Y$ is locally of finite generation to order n (here the colimit is taken over the filtered partially ordered set of finite subsets of S). Note that X is the sheafification of $\varinjlim_{T \subseteq S} X_T$ with respect to the étale topology. It follows from Lemma 17.4.3.2 that X is locally of finite generation to order n over Y.

We now treat the general case. Choose an étale surjection $u : \mathsf{X}_0 \to \mathsf{X}$, where X_0 is a coproduct of affine spectral Deligne-Mumford stacks. Let $X_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by X_0. For every connective $\mathbb{E}_\infty$-ring A, let $X'(A)$ denote the essential image of the map $X_0(A) \to X(A)$. Since u is an étale surjection, the inclusion $X' \hookrightarrow X$ exhibits X as a sheafificaton of X' with respect to the étale topology. Using Lemma 17.4.3.2, we are reduced to proving that the map $X' \to Y$ is locally of finite generation to order n. Choose a filtered diagram $\{A_\alpha\}$ in $\mathrm{CAlg}^{\mathrm{cn}}$ having colimit A, where each A_α is n-truncated and each of the transition maps $\pi_n A_\alpha \to \pi_n A_\beta$ is a monomorphism; we wish to show that the map $\theta : \varinjlim X'(A_\alpha) \to X'(A) \times_{Y(A)} \varinjlim Y(A_\alpha)$ is a homotopy equivalence. The map θ fits into a commutative diagram

$$\begin{array}{ccc} \varinjlim X_0(A_\alpha) & \longrightarrow & X_0(A) \times_{Y(A)} \varinjlim Y(A_\alpha) \\ \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \psi} \\ \varinjlim X'(A_\alpha) & \xrightarrow{\theta} & X'(A) \times_{Y(A)} \varinjlim Y(A_\alpha). \end{array}$$

Lemma 17.4.3.5 implies that this diagram is a pullback square, and the fact that X_0 is a coproduct of affine spectral Deligne-Mumford stacks guarantees that θ' is a homotopy equivalence. We now complete the proof by observing that ψ is surjective on connected components (by construction). □

17.5 The Tangent Complex

Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Suppose that we wish to show that f admits a relative cotangent complex $L_{X/Y}$, in the sense of Definition 17.2.4.2. As a starting point, we note that the *dual* of $L_{X/Y}$ is well-defined under very mild hypotheses. In §17.5.1, we show that if f is infinitesimally cohesive, then to each point $\eta \in X(A)$ we can associate an A-module $T_{X/Y}(\eta)$, which we will refer to as the *relative tangent complex of X over Y at the point η* (Construction 17.5.1.1). In §17.5.2, we study conditions under which the formation of tangent complexes is compatible with extension of scalars (Proposition 17.5.2.1), so that we can regard the construction $\eta \mapsto T_{X/Y}(\eta)$ as a quasi-coherent sheaf on X. Specializing to the case $Y = \operatorname{Spec} R$ for some Noetherian $\mathbb{E}_\infty$-ring R which admits a dualizing complex, we obtain an existence criterion for the cotangent complex $L_{X/Y}$ (Theorem 17.5.4.1). The proof makes use of characterization of those functors $\mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ which are corepresentable by almost perfect R-modules (Theorem 17.5.3.1), which we establish using ideas from §6.6.

17.5.1 The Tangent Complex at a Point

In §18, we studied the tangent complex associated to a formal moduli problem X over a field κ. We now introduce a "global" variant of this construction.

Construction 17.5.1.1. Let $f : X \to Y$ be an infinitesimally cohesive natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. For every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, let $F_\eta : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ denote the functor given by the formula

$$F_\eta(M) = \mathrm{fib}(X(A \oplus M) \to Y(A \oplus M) \times_{Y(A)} X(A)),$$

where the fiber is taken over the point determined by η. Since f is infinitesimally cohesive, the canonical map $F_\eta(M) \to \Omega F_\eta(\Sigma M)$ is an equivalence for each $M \in \mathrm{Mod}_A^{\mathrm{cn}}$, so that the functor F_η is reduced and excisive (Proposition HA.1.4.2.13). Applying Lemma 17.2.1.2, we deduce that F_η admits an essentially unique extension to a left exact functor $F_\eta^+ : \mathrm{Mod}_A^{\mathrm{acn}} \to \mathcal{S}$. We can identify the restriction $F_\eta^+|_{\mathrm{Mod}_A^{\mathrm{perf}}}$ with an object of $\mathrm{Ind}((\mathrm{Mod}_A^{\mathrm{perf}})^{\mathrm{op}}) \simeq \mathrm{Ind}(\mathrm{Mod}_A^{\mathrm{perf}}) \simeq \mathrm{Mod}_A$ (see §HA.7.2.4). We will denote the corresponding A-module by $T_{X/Y}(\eta)$, and refer to it as the *relative tangent complex of f at the point η*. It is characterized by the following universal property: for every connective perfect A-module M, we have a canonical homotopy equivalence $F_\eta(M) \simeq \Omega^\infty(T_{X/Y}(\eta) \otimes_A M)$. In particular, we have a homotopy equivalence $\Omega^{\infty-n} T_{X/Y}(\eta) \simeq F_\eta(\Sigma^n A)$ for each $n \geqslant 0$.

Example 17.5.1.2. Let $f : X \to Y$ be an infinitesimally cohesive morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, and suppose that f admits a relative cotangent complex $L_{X/Y}$. For every point $\eta \in X(A)$, the functor F_η of Construction 17.5.1.1 is given by $F_\eta(M) = \mathrm{Map}_{\mathrm{Mod}_A}(\eta^* L_{X/Y}, M)$. It follows that the relative tangent complex $T_{X/Y}(\eta)$ can be identified with the A-linear dual $\underline{\mathrm{Map}}_A(\eta^* L_{X/Y}, A)$ of $\eta^* L_{X/Y}$.

Variant 17.5.1.3. In the situation of Construction 17.5.1.1, suppose we are given an almost connective A-module N. The functor $M \mapsto F_\eta^+(M \otimes_A N)$ is left exact, and therefore its restriction to perfect A-modules determines an A-module $T_{X/Y}(\eta; N)$ equipped with canonical equivalences $\Omega^{\infty-n} T_{X/Y}(\eta; N) \simeq F_\eta(\Sigma^n N)$. Note that we have $T_{X/Y}(\eta) \simeq T_{X/Y}(\eta; A)$. Moreover, the construction $N \mapsto T_{X/Y}(\eta; N)$ is an exact functor from Mod_A to itself.

Remark 17.5.1.4. Let $f : X \to Y$ be an infinitesimally cohesive morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, let $\phi : A \to A'$ be a map of connective $\mathbb{E}_\infty$-rings, let $\eta \in X(A)$ and let η' denote its image in $X(A')$. Define functors $F_\eta : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ and $F_{\eta'} : \mathrm{Mod}_{A'}^{\mathrm{cn}} \to \mathcal{S}$ as in Construction 17.5.1.1, and let $U : \mathrm{Mod}_{A'}^{\mathrm{acn}} \to \mathrm{Mod}_A^{\mathrm{acn}}$ denote the forgetful functor. Then ϕ induces a natural transformation of reduced excisive functors $F_\eta \circ (U|_{\mathrm{Mod}_A^{\mathrm{cn}}}) \to F_{\eta'}$, which extends to a natural transformation of left exact functors $F_\eta^+ \circ U \to F_{\eta'}^+$. For every A'-module N, we obtain a map of A-modules $T_{X/Y}(\eta; N) \to T_{X/Y}(\eta'; N)$. This map is an equivalence if f is cohesive and ϕ induces a surjection $\pi_0 A \to \pi_0 A'$.

Remark 17.5.1.5. Suppose we are given a pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, where the vertical maps are infinitesimally cohesive. Let A be a connective $\mathbb{E}_\infty$-ring, let $\eta \in X(A)$, and let η' denote the image of η in $X'(A)$. These is a canonical equivalence of A-modules $T_{X/Y}(\eta) \simeq T_{X'/Y'}(\eta')$.

17.5.2 Compatibility with Base Change

Let $f : X \to Y$ be an infinitesimally cohesive morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Suppose that $\phi : A \to A'$ is a morphism of connective $\mathbb{E}_\infty$-rings. Let η be a point of $X(A)$, and let η' denote the image of η in $X(A')$. Applying Remark 17.5.1.4 in the special case $N = A'$, we obtain a natural map A-modules

$$T_{X/Y}(\eta) = T_{X/Y}(\eta; A) \to T_{X/Y}(\eta; A') \to T_{X/Y}(\eta', A') = T_{X/Y}(\eta').$$

Extending scalars along ϕ, we obtain a morphism of A'-modules $\rho : A' \otimes_A T_{X/Y}(\eta) \to T_{X/Y}(\eta')$. Our goal in this section is to prove the following:

Proposition 17.5.2.1. *Let $f : X \to Y$ be a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, where Y is corepresentable by a Noetherian $\mathbb{E}_\infty$-ring R, and let $X_0 : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ denote the functor given by $X_0(A) = \mathrm{fib}(X(A) \to \mathrm{Map}_{\mathrm{CAlg}}(R, A))$. Assume that f is cohesive, nilcomplete, and locally almost of finite presentation. The following conditions are equivalent:*

(1) *For every morphism $\phi : A \to B$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$ and every connective B-module M, the diagram*

$$\begin{array}{ccc} X_0(A \oplus M) & \longrightarrow & X_0(B \oplus M) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) \end{array}$$

is a pullback square.

(2) *For every point $\eta \in X(A)$ and every flat morphism $\phi : A \to B$ carrying η to a point $\eta' \in X(B)$, the natural map $B \otimes_A T_{X/Y}(\eta) \to T_{X/Y}(\eta')$ is an equivalence, where η' denotes the image of η in $X(B)$.*

(3) *For every discrete integral domain A and every point $\eta \in X(A)$ which exhibits A as a finitely generated algebra over $\pi_0 R$, the natural map $A[x] \otimes_A T_{X/Y}(\eta) \to T_{X/Y}(\eta')$ is an equivalence, where η' denotes the image of $\eta \in X(A[x])$.*

(4) *For every discrete integral domain A with fraction field K, every point $\eta \in X(A)$ which exhibits A as a finitely generated algebra over $\pi_0 R$, and every extension field L of K, the canonical map $L \otimes_A T_{X/Y}(\eta) \to T_{X/Y}(\eta')$ is an equivalence, where η' denotes the image of η in $X(L)$.*

Remark 17.5.2.2. If f satisfies the equivalent conditions of Proposition 17.5.2.1, then the construction $(\eta \in X(A)) \mapsto (T_{X/Y}(\eta) \in \mathrm{Mod}_A)$ determines a quasi-coherent sheaf on the functor X, which we will denote by $T_{X/Y}$ and refer to as the *tangent complex of f*.

Proof of Proposition 17.5.2.1. We first show that (1) $\Rightarrow$ (2). Fix a point $\eta \in X(A)$, let B be a flat $\mathbb{E}_\infty$-algebra over A, and let $\eta' \in X(B)$ denote the image of η. Define $F_\eta : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ and $F_{\eta'} : \mathrm{Mod}_B^{\mathrm{cn}} \to \mathcal{S}$ as in Construction 17.5.1.1. To prove that the canonical map $B \otimes_A T_{X/Y}(\eta) \to T_{X/Y}(\eta')$ is an equivalence, it will suffice to show that for each $n \geqslant 0$ the map $\theta : \Omega^{\infty-n}(B \otimes_A T_{X/Y}(\eta)) \to \Omega^{\infty-n} T_{X/Y}(\eta')$ is a homotopy equivalence of spaces. Since B is flat over A, we can write B as a filtered colimit $\varinjlim P_\alpha$, where each P_α is a free A-module of finite rank (Theorem HA.7.2.2.15). We can then identify θ with the composite map

$$\begin{aligned} \Omega^{\infty-n}(B \otimes_A T_{X/Y}(\eta)) &\simeq \varinjlim \Omega^{\infty-n}(P_\alpha \otimes_A T_{X/Y}(\eta)) \\ &\simeq \varinjlim F_\eta(\Sigma^n P_\alpha) \\ &\xrightarrow{\theta'} F_\eta(B) \\ &\xrightarrow{\theta''} F_{\eta'}(B). \end{aligned}$$

The map θ' is a homotopy equivalence by virtue of our assumption that f is locally almost of finite presentation, and the map θ'' is a homotopy equivalence by virtue of assumption (1).

The implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are obvious. We next show that (3) $\Rightarrow$ (1). Choose a connective $\mathbb{E}_\infty$-ring R and a point $\eta \in Y(R)$, and let $X_0 : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ be as in Notation 17.3.8.1. We wish to show that for every morphism $\phi : A \to B$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$ and every connective B-module M, the diagram σ_M :

$$\begin{array}{ccc} X_0(A \oplus M) & \longrightarrow & X_0(B \oplus M) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) \end{array}$$

is a pullback square. Since X is nilcomplete, σ_M is the limit of the diagrams $\sigma_{\tau_{\leqslant n} M}$. It will therefore suffice to show that each $\sigma_{\tau_{\leqslant n} M}$ is a pullback diagram. We proceed by induction on n, the case $n = 0$ being trivial. If $n > 0$, we have a fiber sequence of B-modules $\tau_{\leqslant n} M \to \tau_{\leqslant n-1} M \to \Sigma^{n+1} N$ where $N \simeq \pi_n M$ is a discrete B-module. The square $\sigma_{\tau_{\leqslant n} M}$ fits into a larger diagram

$$\begin{array}{ccccc} X_0(A \oplus \tau_{\leqslant n} M) & \longrightarrow & X_0(B \oplus \tau_{\leqslant n} M) & \longrightarrow & X_0(B \oplus \tau_{\leqslant n-1} M) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) & \longrightarrow & X_0(B \oplus \Sigma^{n+1} N) \end{array}$$

The right square in this diagram is a pullback since X is cohesive. It will therefore suffice to show that the outer rectangle is a pullback diagram. That is, we must show that the outer rectangle in the diagram

$$\begin{array}{ccccc} X_0(A \oplus \tau_{\leqslant n} M) & \longrightarrow & X_0(A \oplus \tau_{\leqslant n-1} M) & \longrightarrow & X_0(B \oplus \tau_{\leqslant n-1} M) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A \oplus K) & \longrightarrow & X_0(B \oplus \Sigma^{n+1} N) \end{array}$$

is a pullback square. Since the left square is a pullback by virtue of our assumption that X is cohesive, it suffices to show that the right square is also a pullback. This square fits into a commutative diagram

$$\begin{array}{ccc} X_0(A \oplus \tau_{\leqslant n-1} M) & \longrightarrow & X_0(B \oplus \tau_{\leqslant n-1} M) \\ \downarrow & & \downarrow \\ X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(B \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) \end{array}$$

where the outer rectangle is a pullback square by the inductive hypothesis. It will therefore suffice to show that $\sigma_{\Sigma^{n+1} N}$ is a pullback diagram.

Since N is a module over $\pi_0 B$, $\sigma_{\Sigma^{n+1} N}$ fits into a commutative diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(B \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(\pi_0 B \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) & \longrightarrow & X_0(\pi_0 B) \end{array}$$

The right square is a pullback diagram by virtue of our assumption that X is cohesive. It will therefore suffice to show that the outer rectangle is a pullback. Equivalently, we must show that the outer rectangle in the diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(\pi_0 A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(\pi_0 B \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(\pi_0 A) & \longrightarrow & X_0(\pi_0 B) \end{array}$$

is a pullback square. Since the left square is a pullback (because X is cohesive), we are reduced to proving that the right square is a pullback. In other words, we may replace A by $\pi_0 A$ and B by $\pi_0 B$, and thereby reduce to the case where A and B are discrete.

Write A as a filtered colimit of subalgebras A_α which are finitely generated over $\pi_0 R$. Since f is locally almost of finite presentation, the functor X_0 commutes with filtered colimits when restricted to $(n+1)$-connective R-algebras. It will therefore suffice to show that each of the diagrams

$$\begin{array}{ccc} X_0(A_\alpha \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(B \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow \\ X_0(A_\alpha) & \longrightarrow & X_0(B). \end{array}$$

We may therefore replace A by A_α and thereby reduce to the case where A is finitely generated as an algebra over $\pi_0 R$. In particular, A is a Noetherian ring. Choose a surjection of commutative A-algebras $P \to B$, where P is a polynomial ring over A. We then have a commutative diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(P \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(B \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(P) & \longrightarrow & X_0(B). \end{array}$$

The right square is a pullback since X_0 is cohesive. It will therefore suffice to show that the left square is a homotopy pullback. Write $P \simeq \varinjlim P_\beta$, where each P_β is a polynomial ring over A on finitely many generators. It will therefore suffice to show that each of the diagrams

$$\begin{array}{ccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(P_\beta \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(P_\beta). \end{array}$$

is a pullback square. Write $P_\beta = A[x_1, \ldots, x_k]$. Working by induction on k, we can reduce to the case where $k = 1$: that is, we are given a discrete $A[x]$-module N, and we wish to show that the diagram τ :

$$\begin{array}{ccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A[x]) \end{array}$$

is a pullback square.

Let N_0 denote the underlying A-module of N, and regard $N_0[x]$ as an $A[x]$-module. We have a short exact sequence of discrete $A[x]$-modules $0 \to N_0[x] \to N_0[x] \to N \to 0$, hence a fiber sequence $\Sigma^{n+1} N \to \Sigma^{n+2} N_0[x] \to \Sigma^{n+2} N_0[x]$. It follows that τ fits into a commutative diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} N_0[x]) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} N_0[x]). \end{array}$$

Our assumption that X is cohesive guarantees that the right square is a pullback. It will therefore suffice to show that the outer rectangle is also a pullback. Equivalently, we must show that the outer rectangle in the diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(A \oplus \Sigma^{n+2} N_0[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} N_0[x]) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A \oplus \Sigma^{n+2} N_0[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} N_0[x]) \end{array}$$

is a pullback square. Since the left square is a pullback (because X is cohesive), it will suffice to show that the right square is also a pullback. This square fits into a larger diagram

$$\begin{array}{ccc} X_0(A \oplus \Sigma^{n+2} N_0[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} N_0[x]) \\ \downarrow & & \downarrow \\ X_0(A \oplus \Sigma^{n+2} N_0[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} N_0[x]) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A[x]). \end{array}$$

It will therefore suffice to show that the lower square and the outer rectangle in this diagram are pullback squares. For this, it suffices to verify the following general assertion: for every discrete A-module T, the diagram τ_T :

$$\begin{array}{ccc} X_0(A \oplus \Sigma^{n+2} T[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} T[x]) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A[x]) \end{array}$$

is a pullback square.

Since f is locally almost of finite presentation, the construction $T \mapsto \tau_T$ commutes with filtered colimits. Writing T as a filtered colimit of its finitely generated submodules, we are reduced to proving that τ_T is an equivalence when T is finitely generated over A. Since A is Noetherian, T is also Noetherian. Working by Noetherian induction, we can assume that for every nonzero submodule $T' \subseteq T$, the diagram $\tau_{T/T'}$ is a pullback square. If $T = 0$, there is nothing to prove. Otherwise, T has an associated prime: that is, we can choose a nonzero element $x \in T$ whose annihilator is a prime ideal $\mathfrak{p} \subseteq A$. Let T' denote the submodule of T generated by x. The diagram τ_T fits into a commutative square

$$\begin{array}{ccccc}
X_0(A \oplus \Sigma^{n+2} T[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} T[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} T/T'[x]) \\
\downarrow & & \downarrow & & \downarrow \\
X_0(A) & \longrightarrow & X_0(A[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+3} T'[x]).
\end{array}$$

Since X is cohesive, the right square is a pullback. It will therefore suffice to show that the outer rectangle is a pullback. This is equivalent to the outer rectangle in the diagram

$$\begin{array}{ccccc}
X_0(A \oplus \Sigma^{n+2} T[x]) & \longrightarrow & X_0(A \oplus \Sigma^{n+2} T/T'[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} T/T'[x]) \\
\downarrow & & \downarrow & & \downarrow \\
X_0(A) & \longrightarrow & X_0(A \oplus \Sigma^{n+3} T'[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+3} T'[x]).
\end{array}$$

Since the left square in this diagram is a pullback (by virtue of the assumption that X is cohesive), we are reduced to proving that the right square is also a pullback. To prove this, we consider the rectangular diagram

$$\begin{array}{ccc}
X_0(A \oplus \Sigma^{n+2} T/T'[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} T/T'[x]) \\
\downarrow & & \downarrow \\
X_0(A \oplus \Sigma^{n+3} T'[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+3} T'[x]) \\
\downarrow & & \downarrow \\
X_0(A) & \longrightarrow & X_0(A[x]).
\end{array}$$

The inductive hypothesis implies that the outer rectangle is a pullback diagram. It will therefore suffice to show that the lower square is also a pullback diagram.

Write $T' = A/\mathfrak{p}$, and consider the diagram

$$\begin{array}{ccccc}
X_0(A \oplus \Sigma^{n+3} T'[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+3} T'[x]) & \longrightarrow & X_0(A[x]/\mathfrak{p} \oplus \Sigma^{n+3} T'[x]) \\
\downarrow & & \downarrow & & \downarrow \\
X_0(A) & \longrightarrow & X_0(A[x]) & \longrightarrow & X_0(A[x]/\mathfrak{p}).
\end{array}$$

Since X is cohesive, the left square is a pullback. It will therefore suffice to show that the outer rectangle is a pullback. Equivalently, we must show that the outer rectangle in the diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+3}T'[x]) & \longrightarrow & X_0(A/\mathfrak{p} \oplus \Sigma^{n+3}T'[x]) & \longrightarrow & X_0(A[x]/\mathfrak{p} \oplus \Sigma^{n+3}T'[x]) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A/\mathfrak{p}) & \longrightarrow & X_0(A[x]/\mathfrak{p}) \end{array}$$

is a pullback. Here the left square is a pullback by virtue of our assumption that X is cohesive. We are therefore reduced to proving that the right square is a pullback diagram, which follows assumption (3).

We now complete the proof by showing that (4) $\Rightarrow$ (3). Suppose that (4) is satisfied. We will prove the following more general version of (3):

$(*)$ Let A be a commutative ring, let $\eta \in X(A)$ exhibit A as a finitely generated algebra over $\pi_0 R$, and let M be a finitely generated (discrete) R-module. Let η' denote the image of η in $X(A[x])$. Then the canonical map

$$\psi_M : A[x] \otimes_A T_{X/Y}(\eta; M) \simeq T_{X/Y}(\eta; M[x]) \to T_{X/Y}(\eta', M[x])$$

is an equivalence (see Variant 17.5.1.3).

To prove $(*)$, we first note that A is Noetherian, so that M is a Noetherian A-module. Working by Noetherian induction, we may suppose that $\psi_{M/M'}$ is an equivalence for every nonzero submodule $M' \subseteq M$. If $M = 0$ there is nothing to prove. Otherwise, M has an associated prime ideal: that is, there is an exact sequence $0 \to M' \to M \to M'' \to 0$ where $M' \simeq A/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq A$. Since $\psi_{M''}$ is an equivalence by virtue of the inductive hypothesis, we are reduced to proving that $\psi_{M'}$ is an equivalence. Since f is cohesive, we may replace A by $A/\mathfrak{p}$, and thereby reduce to the special case where A is an integral domain and $M = A$.

For each nonzero element $a \in A$, we have an exact sequence $0 \to aM \to M \to M/aM \to 0$. The inductive hypothesis implies that $\psi_{M/aM}$ is an equivalence. It follows that multiplication by a induces an equivalence from $\mathrm{cofib}(\psi_M)$ to itself. Let K denote the fraction field of A, so that $K \otimes_A \mathrm{cofib}(\psi_M)$ is equivalent to $\mathrm{cofib}(\psi_M)$. We are therefore reduced to proving that ψ_M induces an equivalence $K[x] \otimes_A T_{X/Y}(\eta) \to K[x] \otimes_{A[x]} T_{X/Y}(\eta')$.

Let $h(x) \in A[x]$ be a polynomial whose image $K[x]$ is irreducible. Let $B = A[x]/(h(x))$, and let $L = K[x]/(h(x))$ be the fraction field of B. Let η'_B denote the image of η in $X(B)$ and define η'_L similarly. Since X is infinitesimally cohesive, can identify $T_{X/Y}(\eta'_B)$ with the cofiber of the map $h(x) : T_{X/Y}(\eta') \to T_{X/Y}(\eta')$. Using condition (4), we can identify

$T_{X/Y}(\eta'_L)$ with the cofiber of $h(x)$ on $K[x] \otimes_{A[x]} T_{X/Y}(\eta')$. We therefore have a commutative diagram of fiber sequences

$$\begin{array}{ccccc} K[x] \otimes_A T_{X/Y}(\eta) & \xrightarrow{h(x)} & K[x] \otimes_A T_{X/Y}(\eta) & \longrightarrow & L \otimes_A T_{X/Y}(\eta) \\ \downarrow & & \downarrow & & \downarrow \\ K[x] \otimes_{A[x]} T_{X/Y}(\eta') & \xrightarrow{h(x)} & K[x] \otimes_{A[x]} T_{X/Y}(\eta') & \longrightarrow & T_{X/Y}(\eta'_L) \end{array}$$

where condition (4) implies that the right vertical map is an equivalence. It follows that multiplication by $h(x)$ acts invertibly on $\mathrm{cofib}(\psi_M)$.

Let K' denote the fraction field of the integral domain $A[x]$. The reasoning above shows that $\mathrm{cofib}(\psi_M) \simeq K' \otimes_{A[x]} \mathrm{cofib}(\psi_M)$. Consequently, to show that $\mathrm{cofib}(\psi_M) \simeq 0$, it will suffice to show that the horizontal map in the diagram

$$\begin{array}{ccc} K' \otimes_A T_{X/Y}(\eta) & \longrightarrow & K' \otimes_{A[x]} T_{X/Y}(\eta') \\ & \searrow \quad \swarrow & \\ & T_{X/Y}(\eta_{K'}) & \end{array}$$

is an equivalence, where $\eta_{K'}$ denotes the image of η in $X(K')$. It now suffices to observe that condition (4) implies that both of the vertical maps are equivalences. □

17.5.3 Digression: A Representability Criterion

Let R be a Noetherian $\mathbb{E}_\infty$-ring, let $f : X \to \operatorname{Spec} R$ be a natural transformation of functors, and let $X_0 : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ be the functor given by $X_0(A) = \mathrm{fib}(X(A) \to \mathrm{Map}_{\mathrm{CAlg}}(R, A))$. Arguing as in Example 17.2.4.4, we see that f admits a relative cotangent complex if and only if it satisfies the following pair of conditions:

(a) For every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, define $F_\eta : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ by the formula $F_\eta(M) = X_0(A \oplus M) \times_{X_0(A)} \{\eta\}$. Then F_η is almost corepresentable (see Example 17.2.1.4).

(b) For every map of connective $\mathbb{E}_\infty$-rings $A \to B$ and every connective B-module M, the diagram of spaces

$$\begin{array}{ccc} X_0(A \oplus M) & \longrightarrow & X_0(B \oplus M) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) \end{array}$$

is a pullback square.

Under some mild hypotheses, condition (b) is equivalent to the requirement that the tangent complexes of f are preserved by flat base change (Proposition 17.5.2.1). Our goal in this section is to prove a result which is helpful for verifying condition (a):

Theorem 17.5.3.1. *Let R be a Noetherian $\mathbb{E}_\infty$-ring which admits a dualizing module K and let $F : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Then F is corepresentable by an almost perfect (not necessarily connective) R-module if and only if the following conditions are satisfied:*

(1) *The functor F is reduced and excisive (and therefore admits an essentially unique extension to a left exact functor $F^+ : \mathrm{Mod}_R^{\mathrm{acn}} \to \mathcal{S}$, by Lemma 17.2.1.2).*

(2) *For every connective R-module M, the canonical map $F(M) \to \varprojlim F(\tau_{\leqslant n} M)$ is an equivalence.*

(3) *For every integer n, the restriction $F|_{(\mathrm{Mod}_R^{\mathrm{cn}})_{\leqslant n}}$ commutes with filtered colimits.*

(4) *There exists an integer $n \geqslant 0$ such that $F(M)$ is n-truncated for every discrete R-module M.*

(5) *For every R-module M which is truncated and almost perfect, the abelian group $\pi_0 F^+(M)$ is finitely generated as a module over $\pi_0 R$.*

Remark 17.5.3.2. Let R be a Noetherian $\mathbb{E}_\infty$-ring and let Mod_R^b denote the full subcategory of Mod_R spanned by those R-modules which are truncated and almost perfect. Then Mod_R^b is a stable subcategory of Mod_R, and the t-structure on Mod_R restricts to a bounded t-structure $((\mathrm{Mod}_R^b)_{\geqslant 0}, (\mathrm{Mod}_R^b)_{\leqslant 0})$ on Mod_R^b. It follows that the ∞-category $\mathrm{Ind}(\mathrm{Mod}_R^b)$ inherits a t-structure $(\mathrm{Ind}(\mathrm{Mod}_R^b)_{\geqslant 0}, \mathrm{Ind}(\mathrm{Mod}_R^b)_{\geqslant 0})$. Since Mod_R admits filtered colimits (and the t-structure on Mod_R is stable under filtered colimits), the inclusion $\mathrm{Mod}_R^b \to \mathrm{Mod}_R$ induces a t-exact functor $F : \mathrm{Ind}(\mathrm{Mod}_R^b) \to \mathrm{Mod}_R$. We claim that F induces an equivalence $F_{\leqslant 0} : \mathrm{Ind}(\mathrm{Mod}_R^b)_{\leqslant 0} \to (\mathrm{Mod}_R)_{\leqslant 0}$. Since every object of $(\mathrm{Mod}_R^b)_{\leqslant 0}$ is compact when viewed as an object of $(\mathrm{Mod}_R)_{\leqslant 0}$, Proposition HTT.5.3.5.11 implies that the functor $F_{\leqslant 0}$ is fully faithful. To complete the proof, it will suffice to show that $F_{\leqslant 0}$ is essentially surjective. Since the image of $F_{\leqslant 0}$ is closed under filtered colimits and every object $M \in (\mathrm{Mod}_R)_{\leqslant 0}$ can be written as the colimit of the diagram

$$\tau_{\geqslant 0} M \to \tau_{\geqslant -1} M \to \tau_{\geqslant -2} M \to \cdots,$$

it suffice to show that every object $M \in (\mathrm{Mod}_R)_{\leqslant 0} \cap (\mathrm{Mod}_R)_{\geqslant -n}$ belongs to the essential image of $F_{\leqslant 0}$. We proceed by induction on n. When $n = 0$, it suffices to observe that every discrete R-module can be written as a filtered colimit of its finitely generated submodules, which we can identify with objects of $(\mathrm{Mod}_R^b)_{\leqslant 0}$. If $n > 0$, then we have a fiber sequence $\Sigma^{-1}(\tau_{\leqslant -1} M) \xrightarrow{\alpha} \pi_0 M \to M$. The inductive hypothesis guarantees that α is the image of a morphism $\overline{\alpha}$ in $\mathrm{Ind}(\mathrm{Mod}_R^b)_{\leqslant 0}$. Note that the domain of $\overline{\alpha}$ belongs to $\mathrm{Ind}(\mathrm{Mod}_R^b)_{\leqslant -1}$ (in fact, it belongs to $\mathrm{Ind}(\mathrm{Mod}_R^b)_{\leqslant -2}$), so that $\mathrm{cofib}(\overline{\alpha}) \in \mathrm{Ind}(\mathrm{Mod}_R^b)_{\leqslant 0}$ Then $M \simeq \mathrm{cofib}(\alpha) = F(\mathrm{cofib}(\overline{\alpha}))$ belongs to the essential image of $F_{\leqslant 0}$, as desired.

Proof of Theorem 17.5.3.1. Suppose that the functor $F : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ is given by the formula $F(M) = \mathrm{Map}_{\mathrm{Mod}_R}(N, M)$ for some R-module N. Then conditions (1) and (2) are vacuous. Condition (4) follows from assumption that N is almost connective, and condition (3) from the condition that N is almost perfect. Moreover, if N is almost perfect, then condition (5) follows from Lemma 6.4.3.5.

Now suppose that conditions (1) through (5) are satisfied. We wish to prove that F^+ is corepresentable by an almost perfect R-module. Fix a dualizing module $K \in \mathrm{Mod}_R$. For every R-module M, we let $\mathbb{D}(M)$ denote the R-module $\underline{\mathrm{Map}}_R(M, K)$. It follows from Theorem 6.6.1.8 and Proposition 6.6.1.9 that the construction $M \mapsto \mathbb{D}(M)$ induces a contravariant equivalence from the ∞-category Mod_R^b to itself. We define a functor $G : (\mathrm{Mod}_R^b)^{\mathrm{op}} \to \mathcal{S}$ by the formula $G(M) = F^+(\mathbb{D}(M))$. Assumption (1) implies that the functor G is left exact, and can therefore be identified with an object of $\mathrm{Ind}(\mathrm{Mod}_R^b)$.

We first claim that F satisfies the following stronger version of (4):

(4′) There exists an integer n such that $F^+(M)$ is $(n+m)$-truncated whenever $M \in \mathrm{Mod}_R^{\mathrm{acn}}$ is m-truncated.

To prove (4′), we first apply Proposition HA.1.4.2.22 to factor F^+ as a composition $\mathrm{Mod}_R^{\mathrm{acn}} \xrightarrow{f} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}$, where f is exact. Let $M \in \mathrm{Mod}_R$ be m-truncated and k-connective; we will prove that the spectrum $f(M)$ is $(n+m)$-truncated. The proof proceeds by induction on m, the case $m < k$ being trivial. We have a fiber sequence $\Sigma^m(\pi_m M) \to M \to \tau_{\leqslant m-1} M$. Since the functor f is exact, to prove that $f(M)$ is $(n+m)$-truncated it will suffice to show that $f(\tau_{\leqslant m-1} M)$ and $f(\Sigma^m(\pi_m M))$ are $(n+m)$-truncated. In the first case, this follows from the inductive hypothesis. In the second, we must show that the spectrum $f(\pi_m M)$ is n-truncated. Since $n \geqslant 0$, this is equivalent to the assertion that the space $\Omega^\infty f(\pi_m M) \simeq F(\pi_m M)$ is n-truncated, which follows from (4).

Let n be an integer satisfying (4′). Choose an integer n' such that K is n'-truncated. If $M \in \mathrm{Mod}_R^b$ is $(n+n'+1)$-connective, then $\mathbb{D}(M)$ is $(-n-1)$-truncated so that condition (5) guarantees that $G(M) = F^+\,\mathbb{D}(M)$ is contractible (note that $F^+\,\mathbb{D}(M)$ is automatically nonempty). It follows that, as an object of $\mathrm{Ind}(\mathrm{Mod}_R^b)$, G belongs to $\mathrm{Ind}(\mathrm{Mod}_R^b)_{\leqslant n+n'}$. Applying Remark 17.5.3.2, we conclude that G is the image of an object $N \in (\mathrm{Mod}_R)_{\leqslant n+n'}$ under the right adjoint to the functor $\mathrm{Ind}(\mathrm{Mod}_R^b) \to \mathrm{Mod}_R$ appearing in Remark 17.5.3.2. Unwinding the definitions, we deduce that N represents the functor G: that is, we have homotopy equivalences $G(M) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(M, N)$ for $M \in \mathrm{Mod}_R^b$ which depend functorially on M. In particular, we obtain bijections

$$\pi_i N \simeq \pi_0 \mathrm{Map}_{\mathrm{Mod}_R}(\tau_{\leqslant n+n'}(\Sigma^i R), N) \simeq \pi_0 G(\tau_{\leqslant n+n'}(\Sigma^i R)) \simeq \pi_0 F^+\,\mathbb{D}(\tau_{\leqslant n+n'}\Sigma^i R)).$$

It follows from (5) that each homotopy group of N is finitely generated as a module over $\pi_0 R$. Using Proposition 6.6.1.9, we deduce that $\mathbb{D}(N)$ is an almost perfect R-module, and that we have functorial homotopy equivalences

$$F^+(M) = G(\mathbb{D}(M)) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(\mathbb{D}(M), N) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(\mathbb{D}(N), M).$$

Let $F' : \mathrm{Mod}_R^{\mathrm{acn}} \to \mathcal{S}$ be the functor corepresented by $\mathbb{D}(N)$. For every pair of integers a and b, let $\mathcal{C}(a,b)$ denote the full subcategory $(\mathrm{Mod}_R)_{\leqslant a} \cap (\mathrm{Mod}_R)_{\geqslant b}$, and let $\mathcal{C}_0(a,b)$ denote the full subcategory spanned by those R-modules $M \in \mathcal{C}(a,b)$ which are truncated and almost perfect. Let $\mathcal{C} = \bigcup_{a,b} \mathcal{C}(a,b)$ denote the full subcategory of Mod_R spanned by those R-modules which are truncated and almost connective. Arguing as in Remark 17.5.3.2, we deduce that the inclusion $\mathcal{C}_0(a,b) \to \mathcal{C}(a,b)$ extends to an equivalence $\mathrm{Ind}(\mathcal{C}_0(a,b)) \to \mathcal{C}(a,b)$. Since $\mathbb{D}(N)$ is almost perfect, $F'|_{\mathcal{C}(a,b)}$ commutes with filtered colimits. Condition (3) implies that $F^+|_{\mathcal{C}(a,b)}$ commutes with filtered colimits. Using Proposition HTT.5.3.5.10 , we deduce that the restriction map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}(a,b),\mathcal{S})}(F'|_{\mathcal{C}(a,b)}, F^+|_{\mathcal{C}(a,b)}) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_0(a,b),\mathcal{S})}(F'|_{\mathcal{C}_0(a,b)}, F^+|_{\mathcal{C}(a,b)})$$

is a homotopy equivalence. Passing to the homotopy inverse limit over pairs (a,b), we obtain a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(F'|_{\mathcal{C}}, F^+|_{\mathcal{C}}) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{Mod}_R^b,\mathcal{S})}(F'|_{\mathrm{Mod}_R^b}, F^+|_{\mathrm{Mod}_R^b}).$$

In particular, the equivalence $F'|_{\mathrm{Mod}_R^b} \simeq F^+|_{\mathrm{Mod}_R^b}$ lifts to a natural transformation $\alpha : F'|_{\mathcal{C}} \to F^+|_{\mathcal{C}}$. Every object $M \in \mathcal{C}$ belongs to $\mathcal{C}(a,b)$ for some pair of integers $a, b \in \mathbf{Z}$. Since $F'|_{\mathcal{C}(a,b)}$ and $F^+|_{\mathcal{C}(a,b)}$ both commute with filtered colimits, we deduce that $\alpha_M : F'(M) \to F^+(M)$ is a filtered colimit of equivalences $F'(M_0) \to F^+(M_0)$, where M_0 is truncated and almost perfect. It follows that α is an equivalence of functors.

To complete the proof that F^+ is corepresentable by an almost perfect R-module, it will suffice to show that α lifts to an equivalence between F^+ and F'. For this, it will suffice to show that F^+ and F' are both right Kan extensions of their restrictions to $\mathcal{C}$. We will need the following criterion:

$(*)$ Let $H : \mathrm{Mod}_R^{\mathrm{acn}} \to \mathcal{S}$ be a functor. Then H is a right Kan extension of $H|_{\mathcal{C}}$ if and only if, for every almost connective R-module M, the canonical map $H(M) \to \varprojlim H(\tau_{\leqslant n} M)$ is an equivalence.

It follows from $(*)$ that F' is a right Kan extension of $F'|_{\mathcal{C}}$, and $(*)$ together with (2) guarantee that F^+ is also a right Kan extension of $F'|_{\mathcal{C}}$.

To prove $(*)$, it will suffice to show that for every object $M \in \mathrm{Mod}_R^{\mathrm{acn}}$, the Postnikov tower $\{\tau_{\leqslant n} M\}_{n \geqslant 0}$ determines a right cofinal functor $\mathrm{N}(\mathbf{Z}_{\geqslant 0})^{\mathrm{op}} \to \mathcal{C} \times_{\mathrm{Mod}_R^{\mathrm{acn}}} (\mathrm{Mod}_R^{\mathrm{acn}})_{M/}$. This is equivalent to the assertion that for every object $N \in \mathcal{C}$, the canonical map

$$\varinjlim \mathrm{Map}_{\mathrm{Mod}_R}(\tau_{\leqslant n} M, N) \to \mathrm{Map}_{\mathrm{Mod}_R}(M, N)$$

is a homotopy equivalence. This is clear, since the assumption that N is truncated implies that the map $\mathrm{Map}_{\mathrm{Mod}_R}(\tau_{\leqslant n} M, N) \to \mathrm{Map}_{\mathrm{Mod}_R}(M, N)$ is a homotopy equivalence for $n \gg 0$. □

17.5.4 Application: Existence of the Cotangent Complex

Let R be an $\mathbb{E}_\infty$-ring and suppose we are given a morphism $f : X \to \operatorname{Spec} R$ in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Under some mild finiteness assumptions, the existence of a cotangent complex for f is equivalent to good behavior of the tangent complex of f. More precisely, we have the following result:

Theorem 17.5.4.1. *Let R be a Noetherian $\mathbb{E}_\infty$-ring which admits a dualizing module (Definition 6.6.1.1), let $Y = \operatorname{Spec} R$, and suppose we are given a morphism $f : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ which is cohesive, nilcomplete, and locally almost of finite presentation. Assume further that f satisfies the hypotheses of Proposition 17.5.2.1, and that there exists an integer q such that $X(A)$ is q-truncated for every discrete commutative ring A. Then the following conditions are equivalent:*

- (A) *The functor X admits a cotangent complex.*
- (B) *For every Noetherian $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, each homotopy group $\pi_n T_{X/Y}(\eta)$ is a finitely generated module over $\pi_0 A$.*
- (C) *Let A be an integral domain, let $\eta \in X(A)$ exhibit A as a finitely generated module over $\pi_0 R$. Then the homotopy groups $\pi_n T_{X/Y}(\eta)$ are finitely generated as modules over A.*

If these conditions are satisfied, then the relative cotangent complex $L_{X/Y}$ is almost perfect. Moreover, if X is integrable, then (A), (B), and (C) are equivalent to either of the following conditions:

- (D) *Let A be an integral domain and let $\eta \in X(A)$ exhibit A as a finitely generated module over $\pi_0 R$. For every integer n, there exists a finite collection of elements $x_1, x_2, \ldots, x_p \in \pi_n T_{X/Y}(\eta)$ and an element $a \in A$ such that, for every field K and every ring homomorphism $A[a^{-1}] \to K$ carrying η in $\eta_K \in X(K)$ the images of the elements $x_1, \ldots, x_p$ form a basis for the vector space $\pi_n T_{X/Y}(\eta_K)$.*
- (E) *Let A be an integral domain and let $\eta \in X(A)$ exhibit A as a finitely generated module over $\pi_0 R$. For every integer n, there exists a nonzero element $a \in A$ such that $(\pi_n T_{X/Y}(\eta))[a^{-1}]$ is a free $A[a^{-1}]$-module of finite rank.*

The proof of Theorem 17.5.4.1 will require some preliminaries.

Proof of Theorem 17.5.4.1. Note that if f admits a relative cotangent complex $L_{X/Y}$, then $L_{X/Y}$ is almost perfect (since f is locally almost of finite presentation; see Corollary 17.4.2.2).

Suppose first that (A) is satisfied; we will prove (B). Assumption (A) implies that the cotangent complex $L_{X/Y}$ exists and is almost perfect. For each point $\eta \in X(A)$, the tangent complex $T_{X/Y}(\eta)$ is given by the A-linear dual of $\eta^* L_{X/Y}$ (Example 17.5.1.2). In particular, we have isomorphisms $\pi_n T_{X/Y}(\eta) \simeq \mathrm{Ext}_A^{-n}(\eta^* L_{X/Y}, A)$, so that $\pi_n T_{X/Y}(\eta)$ is a finitely generated module over $\pi_0 A$ whenever A is Noetherian (Lemma 6.4.3.5).

The implication $(B) \Rightarrow (C)$ is obvious. We next prove that (C) implies (A). Suppose that condition (C) is satisfied. To prove that X admits a cotangent complex, it will suffice to show that the morphism f admits a cotangent complex (Proposition 17.3.9.1). Using Proposition 17.5.2.1, see that f satisfies conditions (a) and (c) of Proposition 17.3.2.5. It will therefore suffice to show that for every truncated object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$, the functor $F_\eta : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ given by $F_\eta(M) = X_0(A \oplus M) \times_{X_0(A)} \{\eta\}$ is almost corepresentable. Write A as the colimit of a filtered diagram $\{A_\alpha\}$ of connective $\mathbb{E}_\infty$-algebras which are of finite presentation over R. Choose $m \geqslant 0$ such that A is m-truncated, so that $A \simeq \varinjlim \tau_{\leqslant m} A_\alpha$. Since f is locally almost of finite presentation, we can assume that η is the image of a point $\eta_\alpha \in X_0(A_\alpha)$ for some index α. Using condition (a) of Proposition 17.3.2.5, we see that F_η factors as a composition $\mathrm{Mod}_A^{\mathrm{cn}} \to \mathrm{Mod}_{\tau_{\leqslant n} A_\alpha}^{\mathrm{cn}} \xrightarrow{F_{\eta_\alpha}} \mathcal{S}$. We may therefore replace A by A_α, and thereby reduce to the case where A is almost of finite presentation over R. Then A admits a dualizing module (Theorem 6.6.4.3). We will show that F_η is almost corepresentable by verifying conditions (1) through (5) of Theorem 17.5.3.1:

(1) The functor $F_\eta : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ is obviously reduced. Since X is infinitesimally cohesive, the canonical map $F_\eta(M) \to \Omega F_\eta(\Sigma M)$ is an equivalence for every connective A-module M, so that F_η is excisive by Proposition HA.1.4.2.13 .

(2) For every connective A-module M, we claim that the canonical map $F_\eta(M) \to \varprojlim F_\eta(\tau_{\leqslant n} M)$ is a homotopy equivalence. This follows immediately from the nilcompleteness of the functor X.

(3) We claim that F_η commutes with filtered colimits when restricted to $(\mathrm{Mod}_A)_{\leqslant n}$. This is an immediate consequence of our assumption that the map f is locally almost of finite presentation.

(4) Choose an integer n such that $X(B)$ is n-truncated for every commutative ring B. We claim that $F_\eta(M)$ is n-truncated for every discrete A-module M. Since X is cohesive, we can replace A by $\pi_0 A$ and thereby reduce to the case where A is discrete. Then $F_\eta(M)$ is the fiber of a map $X(A \oplus M) \to X(A) \times_{Y(A)} Y(A \oplus M)$ whose domain and codomain are n-truncated, and therefore itself n-truncated.

(5) Using Lemma 17.2.1.2, we can extend F_η to an excisive $F_\eta^+ : \mathrm{Mod}_A^{\mathrm{acn}} \to \mathcal{S}$. We wish to prove that for every object $M \in \mathrm{Mod}_A^b$, the abelian group $\pi_0 F_\eta^+(M)$ is finitely generated

as a discrete module over $\pi_0 A$. Given a fiber sequence $M' \to M \to M''$ in Mod_A^b, we obtain an exact sequence of $\pi_0 A$-modules $\pi_0 F_\eta^+(M') \to \pi_0 F_\eta^+(M) \to \pi_0 F_\eta^+(M'')$. Consequently, to prove that $\pi_0 F_\eta^+(M)$ is finitely generated, it suffices to prove the corresponding assertions for M' and M''. We may therefore reduce to the case where the module M is concentrated in a single degree k. Then $\pi_k M$ is a finitely generated module over the Noetherian ring $\pi_0 A$, and therefore admits a finite composition series with successive quotients of the form $(\pi_0 A)/\mathfrak{p}$, where $\mathfrak{p} \subseteq \pi_0 A$ is a prime ideal. We may therefore assume that $\pi_k M$ has the form $(\pi_0 A)/\mathfrak{p}$. Since X is cohesive, we can replace A by the integral domain $(\pi_0 A)/\mathfrak{p}$, so that $M \simeq \Sigma^k A$. In this case, $\pi_0 F_\eta^+(M) \simeq \pi_k T_{X/Y}(\eta)$ is finitely generated by virtue of assumption (C).

We next show that $(A) \Rightarrow (D)$. Let A be a Noetherian integral domain equipped with a point $\eta \in X(A)$, and let n be an integer. Corollary 17.4.2.2 implies that $\eta^* L_{X/Y}$ is an almost perfect A-module. In particular, the homotopy groups $\pi_m \eta^* L_{X/Y}$ are finitely generated A-modules, which vanish for $m \ll 0$. We may therefore choose a nonzero element $a \in A$ such that $(\pi_m \eta^* L_{X/Y})[a^{-1}]$ is a finitely generated free module over $A[a^{-1}]$ of rank r_m for $m \leqslant -n$. For each $m \leqslant -n$, choose a collection of elements $\{x_{i,m} \in \pi_m \eta^* L_{X/Y}\}_{1 \leqslant i \leqslant r_m}$ whose images form a basis for $(\pi_m \eta^* L_{X/Y})[a^{-1}]$ as a module over $A[a^{-1}]$. These choices determine a map

$$\tau_{\geqslant 1-n}(\eta^* L_{X/Y}) \oplus \bigoplus_{m \leqslant -n} (\Sigma^m A)^{r_m} \to \eta^* L_{X/Y}$$

which is an equivalence after inverting the element a. It follows that if M is an $A[a^{-1}]$-module, the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_A}(\eta^* L_{X/Y}, M) \to \mathrm{Map}_{\mathrm{Mod}_A}(\tau_{\geqslant 1-n} \eta^* L_{X/Y}, M) \times \prod_{m \leqslant -n} (\Omega^{\infty - m} M)^{r_m}.$$

In particular, given a ring homomorphism $A[a^{-1}] \to K$ carrying η to $\eta_K \in X(K)$, taking $M = \Sigma^{-n} K$ gives a vector space isomorphism $\pi_n T_{X/Y}(\eta_K) \simeq \mathrm{Ext}_A^{-n}(\eta^* L_{X/Y}, K) \simeq K^{r_n}$, given by evaluation on the elements $\{x_{i,n}\}_{1 \leqslant i \leqslant r_n}$.

We now show that $(D) \Rightarrow (E)$. Assume that X satisfies (D), let A be an integral domain, and let $\eta \in X(A)$ exhibit A as a finitely generated algebra over $\pi_0 R$. Since R admits a dualizing module, so does A (Theorem 6.6.4.3), so that A has finite Krull dimension d (Remark **??**). Using (D), we can choose $a \in A$ and, for $n - 1 \leqslant m \leqslant n + d + 1$, a finite collection of elements $\{y_{i,m} \in \pi_m T_{X/Y}(\eta)\}_{1 \leqslant i \leqslant r_m}$ with the following property: for every field K equipped with a map $A[a^{-1}] \to K$ carrying η to $\eta_K \in X(K)$, the images of the elements $\{y_{i,m}\}_{1 \leqslant i \leqslant r_m}$ form a basis for the K-vector space $\pi_m(T_{X/Y}(\eta_K))$. For every commutative $A[a^{-1}]$-algebra B, let η_B denote the image of η in $X(B)$, so that the elements $\{y_{i,m}\}$ determines a map of B-modules $\bigoplus_{n-1 \leqslant m \leqslant n+d+1} (\Sigma^m B)^{r_m} \to T_{X/Y}(\eta_B)$. Let us denote the fiber of this map by F_B. Note that if B is a field, then the homotopy groups $\pi_i F_B$ vanish for $n - 1 \leqslant i \leqslant n + d$. We will prove the following assertion:

$(*)$ Let B be a quotient ring of $A[a^{-1}]$ having Krull dimension $\leqslant d'$. Then the homotopy groups $\pi_i F_B$ vanish for $n-1 \leqslant i \leqslant n+d-d'$.

Taking $B = A[a^{-1}]$ and $d = d'$, we deduce that $\pi_{n-1} F_{A[a^{-1}]} \simeq \pi_n F_{A[a^{-1}]} \simeq 0$, so that the map

$$\pi_n(\bigoplus_{n-1 \leqslant m \leqslant n+d+1} (\Sigma^m A[a^{-1}])^{r_m}) \to \pi_n T_{X/Y}(\eta_{A[a^{-1}]})$$

is an isomorphism: that is, the images of the elements $\{y_{i,n}\}_{1 \leqslant i \leqslant r_n}$ comprise a basis for $(\pi_n T_{X/Y}(\eta))[a^{-1}] \simeq \pi_n T_{X/Y}(\eta_{A[a^{-1}]})$ as a module over $A[a^{-1}]$.

It remains to prove $(*)$. We proceed by Noetherian induction on B. If $B = 0$, there is nothing to prove. Otherwise, let $\mathfrak{p}$ be an associated prime of B, so that there exists a nonzero ideal $I \subseteq B$ which is isomorphic, as a B-module, to $B/\mathfrak{p}$. We then have an exact sequence of B-modules $0 \to B/\mathfrak{p} \to B \to B/I \to 0$ which determines a fiber sequence $F_{B/\mathfrak{p}} \to F_B \to F_{B/I}$. It follows from the inductive hypothesis that the homotopy groups $\pi_i F_{B/I}$ vanish for $n-1 \leqslant i \leqslant n+d-d'$. It will therefore suffice to show that the homotopy groups $\pi_i F_{B/\mathfrak{p}}$ vanish for $n-1 \leqslant i \leqslant n+d-d'$. Replacing B by $B/\mathfrak{p}$, we can reduce to the case where B is an integral domain. For every nonzero element $b \in B$, the quotient ring $B/(b)$ has Krull dimension $\leqslant d'-1$. Applying the inductive hypothesis, we deduce that the homotopy groups $\pi_i F_{B/(b)}$ vanish for $n-1 \leqslant i \leqslant n+1+d-d'$. Using the fiber sequence $F_B \xrightarrow{b} F_B \to F_{B/(b)}$, we deduce that multiplication by b induces an isomorphism from $\pi_i F_B$ to itself for $n-1 \leqslant i \leqslant n+d-d'$. It will therefore suffice to show that $K \otimes_B \pi_i F_B$ vanishes for $n-1 \leqslant i \leqslant n+d-d'$, where K denotes the fraction field of B. This follows from our construction, since Proposition 17.5.2.1 supplies an equivalence $K \otimes_B \pi_i F_B \simeq \pi_i(K \otimes_B F_B) \simeq \pi_i F_K$.

We now complete the proof by showing that if X is integrable, then condition (E) implies condition (C). Assume that condition (E) is satisfied. We will show that for every Noetherian commutative ring A, every point $\eta \in X(A)$ which exhibits A as a finitely generated algebra over $\pi_0 R$, and every finitely generated A-module M, the homotopy groups $\pi_n T_{X/Y}(\eta; M)$ are finitely generated A-modules. Proceeding by Noetherian induction, we may suppose that this condition is satisfied for every quotient M/M' of M by a nonzero submodule M'.

If $M \simeq 0$ there is nothing to prove. Otherwise, M has an associated prime ideal: that is, there exists a nonzero element $x \in M$ whose annihilator is a prime ideal $\mathfrak{p} \subseteq A$. Using our inductive hypothesis, we can replace M by Ax and thereby reduce to the case where M has the form $A/\mathfrak{p}$. Using our assumption that f is cohesive, we can replace A by $A/\mathfrak{p}$ and thereby reduce to the case where A is an integral domain and $M = A$. For every ideal $I \subseteq A$, let $F_I : \mathrm{Mod}^{\mathrm{cn}}_{A/I} \to \mathcal{S}$ denote the functor given by $F_I(M) = X_0(A/I \oplus M) \times_{X_0(A/I)} \{\eta_I\}$. Using the inductive hypothesis and the proof of the implication $(C) \Rightarrow (A)$, we see that F_I is corepresented by an almost perfect module over A/I for every nonzero ideal $I \subseteq A$.

Fix an integer n; we wish to show that $\pi_n T_{X/Y}(\eta)$ is finitely generated. Using condition

(E), we can choose a nonzero element $a \in A$ such that the modules $\pi_{n+1} T_{X/Y}(\eta')$ and $\pi_n T_{X/Y}(\eta')$ are finitely generated free modules over $A[a^{-1}]$, where η' denotes the image of η in $X(A[a^{-1}])$. Let $\widehat{A} = \varinjlim A/(a^n)$ denote the completion of A with respect to the principal ideal (a). We have a pullback diagram of A-modules

$$\begin{array}{ccc} A & \longrightarrow & A[a^{-1}] \\ \downarrow & & \downarrow \\ \widehat{A} & \longrightarrow & \widehat{A}[a^{-1}]. \end{array}$$

Let $\widehat{\eta}$ denote the image of η in $X(\widehat{A})$, and define $\widehat{\eta}' \in X(\widehat{A}[a^{-1}])$ similarly. Since X is cohesive, we have a pullback square of tangent complexes

$$\begin{array}{ccc} T_{X/Y}(\eta) & \longrightarrow & T_{X/Y}(\eta') \\ \downarrow & & \downarrow \\ T_{X/Y}(\widehat{\eta}) & \longrightarrow & T_{X/Y}(\widehat{\eta}') \end{array}$$

and therefore a long exact sequence of A-modules

$$\cdots \to \pi_{m+1} T_{X/Y}(\widehat{\eta}') \to \pi_m T_{X/Y}(\eta) \to \pi_m T_{X/Y}(\eta') \oplus \pi_m T_{X/Y}(\widehat{\eta}) \xrightarrow{\rho(m)} \pi_m T_{X/Y}(\widehat{\eta}') \to \cdots$$

We will prove that $\pi_n T_{X/Y}(\widehat{\eta})$ and $\pi_{n+1} T_{X/Y}(\widehat{\eta})$ are finitely generated modules over $\widehat{A}$. Since $\widehat{A}[a^{-1}]$ is flat over $\widehat{A}$ and $A[a^{-1}]$, we can then apply Proposition 17.5.2.1 and Corollary 7.4.2.3 to conclude that $\rho(m+1)$ is surjective and $\ker(\rho(m))$ is a finitely generated A-module, so that $\pi_n T_{X/Y}(\eta)$ is also finitely generated as an A-module.

For every integer $k \geqslant 0$, let $L_k \in \mathrm{Mod}_{A/(a^k)}$ corepresent the functor $F_{(a^k)}$. Since X is locally almost of finite presentation, each L_k is almost perfect. Corollary 8.3.5.6 supplies an equivalence of ∞-categories $\mathrm{Mod}_{\widehat{A}}^{\mathrm{aperf}} \simeq \varprojlim_k \mathrm{Mod}_{A/(a^k)}^{\mathrm{aperf}}$. Under this equivalence, we can identify the inverse system $\{L_k\}_{k \geqslant 0}$ with an almost perfect $\widehat{A}$-module $\widehat{L}$. For each $m \geqslant 0$, let η_k denote the image of η in $X(A/(a^k))$. Set $T = \underline{\mathrm{Map}}_{\widehat{A}}(\widehat{L}, \widehat{A})$, so that we have a canonical identification

$$\begin{aligned} \varprojlim_k T_{X/Y}(\eta_k) &\simeq \varprojlim_k \underline{\mathrm{Map}}_{A/(a^k)}(L_k, A/(a^k)) \\ &\simeq \varprojlim_k \underline{\mathrm{Map}}_{\widehat{A}}(\widehat{L}, A/(a^k)) \\ &\simeq \underline{\mathrm{Map}}_{\widehat{A}}(\widehat{L}, \widehat{A}) \\ &\simeq T. \end{aligned}$$

It follows from Lemma 6.4.3.5 that the homotopy groups of T are finitely generated modules over $\widehat{A}$. We will complete the proof by showing that the map $\rho : T_{X/Y}(\widehat{\eta}) \to T$ is an equivalence.

Since f is cohesive, the canonical map $\widehat{A}/(a) \otimes_A T_{X/Y}(\widehat{\eta}) \to T_{X/Y}(\eta_1)$ is an equivalence. It follows that ρ induces an equivalence after tensoring with $\widehat{A}/(a)$: that is, the homotopy groups of $\mathrm{fib}(\rho)$ are modules over $\widehat{A}[a^{-1}]$. Fix an integer m; we wish to show that $\pi_m \mathrm{fib}(\rho) \simeq 0$. For this, we study the exact sequence $\pi_{m+1} T \xrightarrow{\mu} \pi_m \mathrm{fib}(\rho) \xrightarrow{\nu} \pi_m T_{X/Y}(\widehat{\eta})$. We will show that $\nu = 0$, so that μ is surjective. It then follows that $\pi_m \mathrm{fib}(\rho)$ is a finitely generated module over $\widehat{A}$. Since a acts invertibly on $\pi_m \mathrm{fib}(\rho)$, it then follows from Nakayama's lemma that $\pi_m \mathrm{fib}(\rho) = 0$, as desired.

Choose an element $y_0 \in \pi_m T_{X/Y}(\widehat{\eta})$ belonging to the image of ν; we wish to show that $y_0 = 0$. Note that y_0 is a-divisible: that is, we can find elements $y_1, y_2, \ldots \in \pi_m T_{X/Y}(\widehat{\eta})$ such that $a y_{i+1} = y_i$. If $y_0 \neq 0$, then we can choose a maximal ideal $\mathfrak{m} \subseteq \widehat{A}$ such that the image of y_0 is nonzero in the localization $(\pi_m T_{X/Y}(\widehat{\eta}))_{\mathfrak{m}}$. Let B denote the completion of $\widehat{A}$ at the maximal ideal $\mathfrak{m}$, and let η_B denote the image of η in $X(B)$. Then B is faithfully flat over $\widehat{A}_{\mathfrak{m}}$, so that the image of y_0 is nonzero in $B \otimes_{\widehat{A}} \pi_m T_{X/Y}(\widehat{\eta}) \simeq \pi_m T_{X/Y}(\eta_B)$.

Let $\mathfrak{m}_B$ denote the maximal ideal of B, and choose a tower of $\mathbb{E}_\infty$-algebras $\{B_k\}_{k \geqslant 0}$ satisfying the requirements of Lemma **??**. Using Lemma 17.3.5.7, we see that for every pair of integers $p \geqslant q \geqslant 0$, we have an equivalence

$$\{\tau_{\leqslant p}(B_j \oplus \Sigma^q B_j)\}_{j \geqslant 0} \simeq \{B/\mathfrak{m}_B^j \oplus \Sigma^q B/\mathfrak{m}_B^j\}$$

of pro-objects of CAlg. Since f is nilcomplete and integrable, it follows that the canonical map $T_{X/Y}(\eta_B) \to \varprojlim_j T_{X/Y}(\eta_{B,j})$ is an equivalence, where $\eta_{B,j}$ denotes the image of η in $X(B/\mathfrak{m}_B^j)$. We therefore obtain an equivalence

$$\begin{aligned} T_{X/Y}(\eta_B) &\simeq \varprojlim_j \underline{\mathrm{Map}}_{B/\mathfrak{m}_B^j}((B/\mathfrak{m}_B^j) \otimes_{\widehat{A}} \widehat{L}, B/\mathfrak{m}_B^j) \\ &\simeq \varprojlim_j \underline{\mathrm{Map}}_B(B \otimes_{\widehat{A}} \widehat{L}, B/\mathfrak{m}_B^j) \\ &\simeq \underline{\mathrm{Map}}_B(B \otimes_{\widehat{A}} \widehat{L}, B). \end{aligned}$$

Since $\widehat{L}$ is almost perfect over $\widehat{A}$, $B \otimes_{\widehat{A}} \widehat{L}$ is almost perfect over B, so that the homotopy groups of $T_{X/Y}(\eta_B)$ are finitely generated B-modules by Lemma 6.4.3.5. Since the image of a is contained in the maximal ideal B, it follows from Nakayama's Lemma that $\pi_m T_{X/Y}(\eta_B)$ does not contain any nonzero a-divisible elements. It follows that the image of y_0 in $\pi_m T_{X/Y}(\eta_B) \simeq B \otimes_{\widehat{A}} \pi_m T_{X/Y}(\widehat{A})$ is zero, contrary to our earlier assumption. □

Chapter 18

Artin's Representability Theorem

Our goal in this Chapter is to address the following question: given a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, when is X representable by a spectral Deligne-Mumford stack? We have the following necessary conditions:

(*a*) If X is representable by a spectral Deligne-Mumford stack, then X has a well-behaved deformation theory. More precisely, X must be nilcomplete, infinitesimally cohesive, and must admit a cotangent complex (Propositions 17.3.2.3, 17.3.1.2, and 17.2.5.1).

(*b*) Let $\mathrm{CAlg}^{\heartsuit}$ denote the full subcategory of $\mathrm{CAlg}^{\mathrm{cn}}$ spanned by the discrete $\mathbb{E}_\infty$-rings (so that we can identify $\mathrm{CAlg}^{\heartsuit}$ with the ordinary category of commutative rings). If X is representable by a spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O})$, then the restriction $X_0 = X|_{\mathrm{CAlg}^{\heartsuit}}$ also representable by the 0-truncated Deligne-Mumford stack $(\mathcal{X}, \tau_{\leqslant 0}\, \mathcal{O})$.

In §18.1, we will prove that conditions (*a*) and (*b*) are also sufficient (Theorem 18.1.0.2). This can be regarded as an illustration of the heuristic principle

{Spectral Algebraic Geometry} = {Classical Algebraic Geometry}+{Deformation Theory}.

The proof is a straightforward application of the deformation-theoretic ideas developed in Chapter 17: it articulates the idea that any connective $\mathbb{E}_\infty$-ring A can be viewed as an "infinitesimal thickening" of the ordinary commutative ring $\pi_0 A$, since the Postnikov tower $\cdots \to \tau_{\leqslant 3} A \to \tau_{\leqslant 2} A \to \tau_{\leqslant 1 A} \to \tau_{\leqslant 0} A = \pi_0 A$ is a tower of square-zero extensions.

Using Theorem 18.1.0.2, we can reduce many representability questions in spectral algebraic geometry to the analogous questions in classical algebraic geometry. These classical questions can then be addressed using Artin's representability theorem (Theorem 16.0.0.2). However, this sort of reasoning is unnecessarily circuitous: the hypotheses of Artin's theorem are closely related to our condition (*a*), and are somewhat clarified in the setting of spectral algebraic geometry. The remainder of this chapter is devoted to proving the following analogue of Artin's theorem in the setting of spectral algebraic geometry:

(∗) Let R be a sufficiently nice $\mathbb{E}_\infty$-ring and suppose we are given a natural transformation of functors $f : X \to \operatorname{Spec} R$ which is locally almost of finite presentation (Definition 17.4.1.1). Then X is representable by a spectral Deligne-Mumford n-stack if and only if it is nilcomplete, infinitesimally cohesive, integrable, admits a connective cotangent complex, and the homotopy groups $\pi_m X(A)$ vanish for $A \in \mathrm{CAlg}^\heartsuit$ and $m > n$ (Theorem 18.3.0.1).

Our proof of (∗) will use Artin's strategy (together with some simplifications introduced by Conrad and de Jong in [43]), which can be broken naturally into two steps:

(*i*) Suppose first that we are given a κ-valued point $x \in X(\kappa)$, where κ is a finitely generated extension field of some residue field of R. We can then study the formal completion $\widehat{X}$ of X at the point x.Adapting an argument of Schlessinger to the spectral setting, we can choose a formally smooth map $\widehat{u} : \operatorname{Spf}(\widehat{A}) \to \widehat{X}$, where $\widehat{A}$ is a complete local Noetherian $\mathbb{E}_\infty$-ring.

(*ii*) Using approximation arguments, we show $\widehat{u}$ is "close" to being the formal completion of a map $u : \operatorname{Spec}(A) \to X$, where A is almost of finite presentation over R. The techniques used to prove (1) can then be applied to approximate u by an étale map $u' : \operatorname{Spec}(A') \to X$, which plays the role of an étale neighborhood of the point x.

We will carry out step (*i*) in §18.2. With an eye to future applications, we consider more generally the problem of describing a formal neighborhood of a B-valued point of a functor X, where B is a connective $\mathbb{E}_\infty$-ring which is not necessarily a field (or even Noetherian). Step (*ii*) is the subject of §18.3. Our proof makes essential use of Popescu's smoothing theorem, which we discuss in §**??**.

Contents

18.1 From Classical to Spectral Algebraic Geometry

To every spectral scheme $(X, \mathscr{O}_X)$, we can associate an ordinary scheme $(X, \pi_0 \mathscr{O}_X)$, which we refer to as the *underlying scheme* of $(X, \mathscr{O}_X)$ (see Remark 1.1.2.10). In this section, we will study the problem of inverting this construction:

$(*)$ Given a scheme $(X, \mathscr{O}_X)$, how can one describe the collection of all spectral schemes having underlying scheme $(X, \mathscr{O}_X)$?

The first observation is that this collection is always nonempty: every scheme $(X, \mathscr{O}_X)$ can be regarded as a spectral scheme, by identifying $\mathscr{O}_X$ with a discrete sheaf of $\mathbb{E}_\infty$-rings on X (Proposition 1.1.8.4). However, this representative might not be well-suited for a particular application. Suppose, for example, that the scheme $(X, \mathscr{O}_X)$ is given as the solution to a moduli problem. In other words, suppose that we are given a functor F from the category $\mathrm{CAlg}^\heartsuit$ of commutative rings to the category $\mathcal{S}\mathrm{et}$ of sets, and that X is characterized by the existence of a natural bijection $F(R) \simeq \mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} R, (X, \mathscr{O}_X))$. In many cases of interest, there is a natural way to extend F to a functor of ∞-categories $\overline{F} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. If $\overline{F}$ is represented by a spectral scheme $(Y, \mathscr{O}_Y)$, then the underlying scheme of $(Y, \pi_0 \mathscr{O}_Y)$ represents the original functor F, and is therefore isomorphic to $(X, \mathscr{O}_X)$ in the category of schemes. Moreover, the data of $(Y, \mathscr{O}_Y)$ is equivalent to the data of the functor $\overline{F}$ (Proposition 1.6.4.2). We may therefore rephrase question $(*)$ as follows:

$(*')$ Let $F : \mathrm{CAlg}^\heartsuit \to \mathcal{S}\mathrm{et}$ be a functor which is representable by a scheme, and let $\overline{F}$: $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor extending F. Under what circumstances is $\overline{F}$ representable by a spectral scheme?

Question $(*')$ is addressed by our first main result:

Theorem 18.1.0.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Then X is representable by a spectral scheme if and only if it is nilcomplete, infinitesimally cohesive, admits a cotangent complex, and the functor $X|_{\mathrm{CAlg}^\heartsuit}$ is representable by a scheme.*

By virtue of Corollary 1.6.7.4, Theorem 18.1.0.1 is an immediate consequence of the following more general result:

Theorem 18.1.0.2. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Then X is representable by a spectral Deligne-Mumford stack if and only if the following conditions are satisfied:*

(1) *There exists a spectral Deligne-Mumford stack Y representing a functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and an equivalence of functors $X|_{\mathrm{CAlg}^{\heartsuit}} \simeq Y|_{\mathrm{CAlg}^{\heartsuit}}$.*

(2) *The functor X admits a cotangent complex.*

(3) *The functor X is nilcomplete.*

(4) *The functor X is infinitesimally cohesive.*

Remark 18.1.0.3. A version of Theorem 18.1.0.2 appears in the third appendix of [214].

18.1.1 A Criterion for Étale Descent

Our first step is to show that any functor X which satisfies the hypotheses of Theorem 18.1.0.2 is a sheaf for the étale topology. This is a consequence of the following more general assertion:

Proposition 18.1.1.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. The following conditions are equivalent:*

(1) *The functor X is a sheaf with respect to the étale topology.*

(2) *The functor $X|_{\mathrm{CAlg}^{\heartsuit}}$ is a sheaf with respect to the étale topology.*

Proof. The implication (1) ⇒ (2) is obvious. To prove the converse, let us suppose that $X|_{\mathrm{CAlg}^{\heartsuit}}$ is a sheaf with respect to the étale topology. We wish to prove that, for every connective $\mathbb{E}_\infty$-ring R, the restriction $X_R = X|_{\mathrm{CAlg}_R^{\text{ét}}}$ is a sheaf with respect to the étale topology. Since X is nilcomplete, X_R is the limit of a tower of functors $\{X_R^n\}_{n \geqslant 0}$ given by the formula $X_R^n(A) = X(\tau_{\leqslant n} A)$. It will therefore suffice to show that each X_R^n is a sheaf with respect to the étale topology. Replacing R by $\tau_{\leqslant n} R$, we may suppose that R is n-truncated. We proceed by induction on n. When $n = 0$, the desired result follows from assumption (2). Let us therefore assume that $n > 0$, so that R is a square-zero extension of $R' = \tau_{\leqslant n-1} R$ by $M = \Sigma^n(\pi_n R)$. We have a pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ R' & \longrightarrow & R' \oplus \Sigma M. \end{array}$$

Define functors $Y_R, Z_R : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ by the formulas

$$Y_R(A) = X(A \otimes_R R') \qquad Z_R(A) = X(A \otimes_R R' \oplus M).$$

Since X is infinitesimally cohesive, we have a pullback diagram of functors

$$\begin{array}{ccc} X_R & \longrightarrow & Y_R \\ \downarrow & & \downarrow \\ Y_R & \longrightarrow & Z_R. \end{array}$$

It follows from the inductive hypothesis that Y_R is a sheaf with respect to the étale topology. To complete the proof, it will suffice to show that Z_R is a sheaf with respect to the étale topology. Applying Lemma D.4.3.2 to the projection map $Z_R \to Y_R$, we are reduced to proving the following:

(∗) For every étale R-algebra A and every point $\eta \in X(\tau_{\leqslant n-1} A)$, the construction

$$B \mapsto \mathrm{fib}(X(\tau_{\leqslant n-1} B \oplus (B \otimes_R M) \to X(\tau_{\leqslant n-1} B)))$$

defines an étale sheaf $\mathscr{F} : \mathrm{CAlg}^{\text{ét}}_A \to \mathcal{S}$.

Invoking the definition of the cotangent complex L_X, we see that the functor $\mathscr{F}$ is given by the formula $\mathscr{F}(B) = \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leqslant n-1} A}}(\eta^* L_X, B \otimes_R M)$. It follows from Corollary D.6.3.4 (and Proposition A.5.7.2) that $\mathscr{F}$ is a hypercomplete sheaf with respect to the flat topology. □

Remark 18.1.1.2. Proposition 18.1.1.1 has many variants which can be proven by the same argument. Suppose that the functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ infinitesimally cohesive, nilcomplete, and has a cotangent complex. Then X is a (hypercomplete) sheaf with respect to the Zariski topology (flat topology, Nisnevich topology) if and only if the restriction $X|_{\mathrm{CAlg}^{\heartsuit}}$ has the same property.

Corollary 18.1.1.3. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and which admits a cotangent complex. Let $\mathsf{Y} = (\mathcal{Y}, \mathscr{O})$ be a spectral Deligne-Mumford stack, let $\mathscr{F}$ be a connective quasi-coherent sheaf on Y, and let Y' denote the spectral Deligne-Mumford stack $(\mathcal{Y}, \mathscr{O} \oplus \mathscr{F})$. Let $Y, Y' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functors represented by Y and Y', respectively, and let $\alpha : Y \to Y'$ be the canonical map. Suppose we are given a map $\eta : Y \to X$. If $X|_{\mathrm{CAlg}^{\heartsuit}}$ is a sheaf with respect to the étale topology, then the canonical map*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{Y/}}(Y', X) \to \mathrm{Map}_{\mathrm{QCoh}(Y)}(\eta^* L_X, \alpha^* L_{Y'}) \to \mathrm{Map}_{\mathrm{QCoh}(Y)}(L_X, \mathscr{F})$$

is a homotopy equivalence.

Proof. It follows from Proposition 18.1.1.1 that X is a sheaf with respect to the étale topology. The assertion is therefore local on Y, so we may reduce to the case where Y is affine. In this case, the desired result follows immediately from the definition of L_X. □

Remark 18.1.1.4. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack, let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ be connective, and let $\eta : L_{\mathsf{X}} \to \Sigma \mathscr{F}$ be a map of quasi-coherent sheaves, classifying a square-zero extension $\mathcal{O}^\eta$ of $\mathcal{O}$. We have a commutative diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{O} \oplus \Sigma \mathscr{F}) & \longrightarrow & (\mathcal{X}, \mathcal{O}) \\ \downarrow & & \downarrow \\ (\mathcal{X}, \mathcal{O}) & \longrightarrow & (\mathcal{X}, \mathcal{O}^\eta), \end{array}$$

giving rise to a commutative diagram

$$\begin{array}{ccc} X^+ & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X^\eta \end{array}$$

in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Suppose that $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is an infinitesimally cohesive functor which is a sheaf with respect to the étale topology. Then the induced diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X^+, Y) & \longleftarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X, Y) \\ \uparrow & & \uparrow \\ \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X, Y) & \longleftarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X^\eta, Y) \end{array}$$

is a pullback square. To prove this, we use the fact that Y is a sheaf with respect to the étale topology to reduce to the case where X is affine, in which case it follows from the definition of an infinitesimally cohesive functor.

18.1.2 Approximations to Étale Morphisms

The main idea in the proof of Theorem 18.1.0.2 is to show that a map $Y_0 \to X$ which is close to being étale can be approximated by another map $Y \to X$ which is actually étale. We can articulate this more precisely as follows:

Proposition 18.1.2.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. Let Y_0 be a functor which is representable by a spectral Deligne-Mumford stack $(\mathcal{Y}, \mathcal{O}_0)$, and suppose we are given a map $f_0 : Y_0 \to X$ for which the relative cotangent complex $L_{Y_0/X}$ is 2-connective. Assume either that $(\mathcal{Y}, \mathcal{O}_0)$ is affine or that X satisfies étale descent. Then the map f_0 factors as a composition*

$$Y_0 \xrightarrow{g} Y \xrightarrow{f} X$$

where $L_{Y/X} \simeq 0$, Y is representable by a spectral Deligne-Mumford stack $(\mathcal{Y}, \mathcal{O})$, and g is induced by a 1-connective map $\mathcal{O} \to \mathcal{O}_0$.

Proof. We will give the proof under the assumption that X satisfies étale descent; the same argument works in general when $(\mathcal{Y}, \mathscr{O}_0)$ is affine. We will construct $\mathscr{O}$ as the inverse limit of a tower of CAlg-valued sheaves on $\mathcal{Y}$

$$\cdots \to \mathscr{O}_2 \to \mathscr{O}_1 \to \mathscr{O}_0,$$

where each pair $(\mathcal{Y}, \mathscr{O}_k)$ is a spectral Deligne-Mumford stack representing a functor $Y_k : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, equipped with a map $f_k : Y_k \to X$ for which the relative cotangent complex $L_{Y_i/X}$ is $(2^k + 1)$-connective.

Let us assume that $\mathscr{O}_k$ has been constructed and that the relative cotangent complex $L_{Y_k/X}$ is $(1 + 2^k)$-connective. Let $\mathscr{O}_{k+1}$ denote the square-zero extension of $\mathscr{O}_k$ classified by the map $u : L_{Y_k} \to L_{Y_k/X}$ in $\mathrm{QCoh}(Y_k) \simeq \mathrm{QCoh}(\mathcal{Y}, \mathscr{O}_k)$. Let Z_k be the functor represented by $(\mathcal{Y}, \mathscr{O}_k \oplus L_{Y_k/X})$. We have a pushout diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} (\mathcal{Y}, \mathscr{O}_k \oplus L_{Y_k/X}) & \longrightarrow & (\mathcal{Y}, \mathscr{O}_k) \\ \downarrow & & \downarrow \\ (\mathcal{Y}, \mathscr{O}_k) & \longrightarrow & (\mathcal{Y}, \mathscr{O}_{k+1}) \end{array}$$

(see Proposition 17.1.3.4), giving rise to a diagram of functors

$$\begin{array}{ccc} Z_k & \xrightarrow{\delta} & Y_k \\ \downarrow{\scriptstyle \delta'} & & \downarrow \\ Y_k & \longrightarrow & Y_{k+1}. \end{array}$$

We have a canonical nullhomotopy of the restriction of u to $f_k^* L_X$, which gives a homotopy between $f_k \circ \delta$ and $f_k \circ \delta'$ (Corollary **??**). Using Remark 18.1.1.4, we see that this homotopy gives rise to a map $f_{k+1} : Y_{k+1} \to X$ extending f_k. We wish to prove that the relative cotangent complex $L_{Y_{k+1}/X}$ is $(2^{k+1} + 1)$-connective. Let $i : Y_k \to Y_{k+1}$ denote the canonical map. Since $L_{Y_k/X}$ is $(2^k + 1)$-connective, the projection map $q : \mathscr{O}_{k+1} \to \mathscr{O}_k$ is 2^k connective, and therefore induces an isomorphism $\pi_0 \mathscr{O}_{k+1} \to \pi_0 \mathscr{O}_k$. It will therefore suffice to show that $i^* L_{Y_{k+1}/X}$ is $(2^{k+1} + 1)$-connective. This pullback fits into a fiber sequence

$$i^* L_{Y_{k+1}/X} \to L_{Y_k/X} \xrightarrow{\phi} L_{Y_k/Y_{k+1}}.$$

We will prove that the map ϕ is $(2^{k+1} + 1)$-connective. Unwinding the definitions, we see that ϕ factors as a composition

$$L_{Y_k/X} \simeq \mathrm{cofib}(q) \xrightarrow{\phi'} \mathscr{O}_k \otimes_{\mathscr{O}_{k+1}} \mathrm{cofib}(q) \xrightarrow{\epsilon_q} L_{Y_k/Y_{k+1}},$$

where ϵ_q is as in Lemma 17.1.4.2. Since $\mathrm{cofib}(q)$ is $2^k + 1$-connective, the map q is 2^k-connective, so that the map ϕ' is $(2^{k+1} + 1)$-connective. Lemma 17.1.4.2 implies that ϵ_q is $(2^{k+1} + 2)$-connective, so that the composition ϕ is $(2^{k+1} + 1)$-connective as desired.

Let $\mathscr{O}$ denote the sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{Y}$ given by $\varprojlim \mathscr{O}_k$. Since each $\mathscr{O}_k$ is hypercomplete, the inverse limit $\mathscr{O}$ is hypercomplete. For any affine object $U \in \mathcal{Y}$, we have a tower of connective $\mathbb{E}_\infty$-rings

$$\cdots \to \mathscr{O}_2(U) \to \mathscr{O}_1(U) \to \mathscr{O}_0(U),$$

where the map $\mathscr{O}_{k+1}(U) \to \mathscr{O}_k(U)$ is 2^k-connective. It follows that that the projection map $\mathscr{O}(U) \to \mathscr{O}_k(U)$ is 2^k-connective for each k, so that the projection map $\mathscr{O} \to \mathscr{O}_k$ induces an equivalence $\tau_{\leqslant 2^k - 1}\mathscr{O} \to \tau_{\leqslant 2^k-1}\mathscr{O}_k$ for each $k \geqslant 0$. Applying the criterion of Theorem 1.4.8.1, we deduce that $(\mathcal{Y}, \mathscr{O})$ is a spectral Deligne-Mumford stack. Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by Y. Using the fact that X is a nilcomplete étale sheaf, we deduce that the natural transformations $\beta_i : Y_i \to X$ induce a natural transformation $\beta : Y \to X$.

It is clear from the construction that the map $\mathscr{O} \to \mathscr{O}_0$ is 1-connective. We will complete the proof by showing that $L_{Y/X} \simeq 0$. Fix an integer k; we will show that $L_{Y/X}$ is 2^k-connective. Let $i : Y_k \to Y$ denote the canonical map; since $\mathscr{O} \to \mathscr{O}_k$ is an equivalence, it will suffice to show that $i^* L_{Y/X}$ is 2^k-connective. We have a fiber sequence

$$i^* L_{Y/X} \to L_{Y_k/X} \to L_{Y_k/Y}.$$

Since $L_{Y_k/X}$ is 2^k-connective, it will suffice to show that $L_{Y_k/Y}$ is (2^k+1)-connective. This follows from Corollary 17.1.4.3, since the map $\mathscr{O} \to \mathscr{O}_k$ is 2^k-connective. □

18.1.3 The Proof of Theorem 18.1.0.2

Before turning to the proof of Theorem 18.1.0.2, we record the following observation:

Lemma 18.1.3.1. *Let $\mathsf{X} = (\mathcal{X}, \mathscr{O})$ be a spectral Deligne-Mumford stack, and assume that $\mathscr{O}$ is discrete. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by X, and let $X' : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a left Kan extension of $X|_{\mathrm{CAlg}^{\heartsuit}}$. Then the canonical map $X' \to X$ exhibits X as a sheafification of X' with respect to the étale topology.*

Proof. For every object $U \in \mathcal{X}$, let X_U denote the functor represented by the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathscr{O}|_U)$, and let X'_U be a left Kan extension of $X_U|_{\mathrm{CAlg}^{\heartsuit}}$. Let $\mathcal{X}_0$ denote the full subcategory of $\mathcal{X}$ spanned by those objects for which the canonical map $X'_U \to X_U$ exhibits X_U as a sheafification of X'_U with respect to the étale topology. To complete the proof, it will suffice to show that $\mathcal{X}_0 = \mathcal{X}$. If U is affine, then X_U is corepresented by an object of $\mathrm{CAlg}^{\heartsuit}$, so the canonical map $X'_U \to X_U$ is an equivalence; it follows that $U \in \mathcal{X}_0$. Since $\mathcal{X}$ is generated by affine objects under small colimits (Lemma **??**), it will suffice to show that $\mathcal{X}_0$ is closed under small colimits. Suppose that $U \in \mathcal{X}$ is given as a colimit of a small diagram $\{U_\alpha\}$ of objects of $\mathcal{X}_0$. To prove that $U \in \mathcal{X}_0$, it will suffice to show that for every functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the canonical map

$$\begin{aligned} \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X_U, Y) &\xrightarrow{\theta_U} \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X'_U, Y) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \mathcal{S})}(X_U|_{\mathrm{CAlg}^{\heartsuit}}, Y|_{\mathrm{CAlg}^{\heartsuit}}) \end{aligned}$$

is a homotopy equivalence. This map fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X_U, Y) & \xrightarrow{\theta_U} & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \mathcal{S})}(X_U|_{\mathrm{CAlg}^{\heartsuit}}, Y|_{\mathrm{CAlg}^{\heartsuit}}) \\ \downarrow & & \downarrow \\ \varprojlim \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X_{U_\alpha}, Y) & \xrightarrow{\varprojlim \theta_{U_\alpha}} & \varprojlim \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \mathcal{S})}(X_{U_\alpha}|_{\mathrm{CAlg}^{\heartsuit}}, Y|_{\mathrm{CAlg}^{\heartsuit}}). \end{array}$$

Since each U_α belongs to $\mathcal{X}_0$, the lower horizontal map is a homotopy equivalence. It will therefore suffice to show that the vertical maps are homotopy equivalences. In other words, we are reduced to proving that X_U is a sheafification of $\varinjlim X_{U_\alpha}$ with respect to the étale topology. This follows from Lemma **??**. □

Proof of Theorem 18.1.0.2. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. Suppose that there exists a spectral Deligne-Mumford stack $(\mathcal{Y}, \mathcal{O}_0)$ which represents a functor Y_0 such that $X|_{\mathrm{CAlg}^{\heartsuit}} \simeq Y_0|_{\mathrm{CAlg}^{\heartsuit}}$. We wish to prove that X is representable by a spectral Deligne-Mumford stack.

Since Y_0 is a sheaf for the étale topology, $X|_{\mathrm{CAlg}^{\heartsuit}}$ is also a sheaf for the étale topology. Applying Proposition 18.1.1.1, we deduce that X is a sheaf for the étale topology. Replacing $\mathcal{O}_0$ by $\tau_{\leqslant 0}\, \mathcal{O}_0$, we may assume without loss of generality that the structure sheaf $\mathcal{O}_0$ is discrete. Let $Y'_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a left Kan extension of $Y_0|_{\mathrm{CAlg}^{\heartsuit}}$, so that the equivalence $Y_0|_{\mathrm{CAlg}^{\heartsuit}} \simeq X|_{\mathrm{CAlg}^{\heartsuit}}$ extends to a natural transformation $\alpha : Y'_0 \to X$. It follows from Lemma 18.1.3.1 that the canonical map $Y'_0 \to Y_0$ exhibits Y_0 as a sheafification of Y'_0 with respect to the étale topology. Since X is an étale sheaf, the map α factors as a composition $Y'_0 \to Y_0 \xrightarrow{f_0} X$.

We next prove:

$(*)$ The quasi-coherent sheaf $f_0^* L_X$ is connective, and the canonical map $\pi_0 L_{Y_0} \to \pi_0 f_0^* L_X$ is an isomorphism.

To prove $(*)$, choose an étale map $\eta : \mathrm{Spét}\, R \to (\mathcal{Y}, \mathcal{O}_0)$; we will show that $\eta^* f_0^* L_X$ is connective and the map $\pi_0 \eta^* L_X \to \pi_0 \eta^* f_0^* L_X$ is an isomorphism. Note that $\eta^* f_0^* L_X$ is almost connective; if it is not connective, then there exists a discrete R-module M and a nonzero map $\eta^* \beta_0^* L_X \to M[k]$ for some integer $k < 0$. It then follows that the mapping space $\mathrm{Map}_{\mathrm{Mod}_R}(\eta^* f_0^* L_X, M)$ is non-discrete. The quasi-coherent sheaf L_{Y_0} is connective, so that for any discrete R-module M $\mathrm{Map}_{\mathrm{Mod}_R}(\eta^* L_{Y_0}, M)$ is a discrete space, homotopy equivalent to the abelian group of R-module maps from $\pi_0 \eta^* L_{Y_0}$ to M. We are therefore reduced to proving that the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_R}(\eta^* f_0^* L_X, M) \to \mathrm{Map}_{\mathrm{Mod}_R}(\eta^* L_{Y_0}, M)$$

is a homotopy equivalence. This map is obtained by passing to vertical homotopy fibers in the diagram

$$\begin{array}{ccc} Y_0(R \oplus M) & \longrightarrow & X(R \oplus M) \\ \downarrow & & \downarrow \\ Y_0(R) & \longrightarrow & X(R). \end{array}$$

This diagram is a homotopy pullback square because the horizontal maps are homotopy equivalences (the map $f_0 : Y_0 \to X$ induces a homotopy equivalence after evaluation on any commutative ring R, by assumption).

Since X and Y_0 admit cotangent complexes, the morphism $f_0 : Y_0 \to X$ admits a cotangent complex, which fits into a fiber sequence $f_0^* L_X \to L_{Y_0} \to L_{Y_0/X}$ (see Corollary 17.2.5.3). Using $(*)$, we deduce that $L_{Y_0/X}$ is 1-connective. We will need the following slightly stronger assertion:

$(*')$ The relative cotangent complex $L_{Y_0/X}$ is 2-connective.

To prove $(*')$, we note that $(*)$ gives a short exact sequence $\pi_1 f_0^* L_X \to \pi_1 L_{Y_0} \to \pi_1 L_{Y_0/X} \to 0$ in the abelian category $\mathrm{QCoh}(Y_0)^{\heartsuit}$. Let $\mathscr{F} = \pi_1 L_{Y_0/X}$. If $\mathscr{F}$ is nonzero, then we obtain a nonzero map $\gamma : L_{Y_0} \to L_{Y_0/X} \to \Sigma \mathscr{F}$ whose restriction to $f_0^* L_X$ vanishes. Choose an étale map $\eta : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$ such that $M = \eta^* \mathscr{F}$ is nonzero. Then γ determines a derivation $L_R \to \Sigma M$ which classifies a square-zero extension R^γ of R by M. Since R and M are discrete, the $\mathbb{E}_\infty$-ring R^γ is discrete. Since the derivation γ is nonzero, the point $\eta \in Y_0(R)$ cannot be lifted to a point of $Y_0(R^\gamma)$. However, the restriction of γ to $f_0^* L_X$ vanishes, so that $f_0(\eta)$ can be lifted to a point of $X(R^\gamma)$. This is a contradiction, since the map $Y_0(R^\gamma) \to X(R^\gamma)$ is a homotopy equivalence.

Combining $(*')$ with Proposition 18.1.2.1, we deduce that there exists a sheaf of $\mathbb{E}_\infty$-rings $\mathscr{O}$ on $\mathcal{Y}$ equipped with a 1-connective map $q : \mathscr{O} \to \mathscr{O}_0$, such that $(\mathcal{Y}, \mathscr{O})$ represents a functor Y and f_0 factors as a composition $Y_0 \to Y \xrightarrow{f} X$ where $L_{Y/X} \simeq 0$. The map q induces an isomorphism $\pi_0 \mathscr{O} \to \pi_0 \mathscr{O}_0$, so that f induces a homotopy equivalence $Y(R) \to X(R)$ whenever R is discrete. Applying Corollary 17.3.6.3, we deduce that f is an equivalence, so that X is representable by the spectral Deligne-Mumford stack $(\mathcal{Y}, \mathscr{O})$. □

18.2 Schlessinger's Criterion

Let $i : \mathsf{X} \to \mathsf{Y}$ be a morphim of spectral Deligne-Mumford stacks, representing functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that i is a closed immersion which is locally almost of finite presentation. Then the image of i is a cocompact closed subset $K \subseteq |\mathsf{Y}|$. In §8.1.6, we defined the *formal completion* $\mathfrak{Y}$ of Y along K; this is a formal spectral Deligne-Mumford stack which represents the functor

$$\widehat{Y} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \qquad \widehat{Y}(R) = \{f : \mathrm{Sp\acute{e}t}(R) \to \mathsf{Y} : f(|\operatorname{Spec}(R)|) \subseteq K\}.$$

Note that a map $f : \mathrm{Sp\acute{e}t}(R) \to \mathsf{Y}$ has the property that $f(|\operatorname{Spec}(R)|)$ is contained in K if and only if the composite map $\mathrm{Sp\acute{e}t}(R^{\mathrm{red}}) \to \mathrm{Sp\acute{e}t}(R) \xrightarrow{f} \mathsf{Y}$ factors through i (in which case the factorization is unique). It follows that the functor $\widehat{Y}$ can be described directly in terms of X and Y via the formula

$$\widehat{Y}(R) = Y(R) \times_{Y(R^{\mathrm{red}})} X(R^{\mathrm{red}}).$$

In this section, we will study an analogous construction in the case where the functors X and Y are not representable (or not yet known to be representable). We begin in §18.2.1 by introducing the *relative de Rham space* $(X/Y)_{\mathrm{dR}}$ of an arbitrary morphism $i : X \to Y$ (Definition 18.2.1.1). In the case where i is locally almost of finite presentation (in the sense of Definition 17.4.1.1), the functor $(X/Y)_{\mathrm{dR}}$ is given by the preceding formula $(X/Y)_{\mathrm{dR}}(R) = Y(R) \times_{Y(R^{\mathrm{red}})} X(R^{\mathrm{red}})$ (Example 18.2.1.5). Our main goal is to address the following:

Question 18.2.0.1. Let $i : X \to Y$ be a natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Under what circumstances is the relative de Rham space $(X/Y)_{\mathrm{dR}}$ representable by a formal spectral Deligne-Mumford stack?

To obtain a satisfying answer to Question 18.2.0.1, it is convenient to impose some additional finiteness constraints. In §**??**, we introduce the notion of a *formal thickening* of formal spectral Deligne-Mumford stacks (Definition 18.2.2.1). A formal thickening is a closed immersion of formal spectral Deligne-Mumford stacks which is locally almost of finite presentation and induces an equivalence of reductions (which we make precise in §8.1.4). In §18.2.3, we address Question 18.2.0.1 by showing that if X is representable by a formal spectral Deligne-Mumford stack and we set $\widehat{Y} = (X/Y)_{\mathrm{dR}}$, then the map $X \to \widehat{Y}$ is (representable by) a formal thickening of formal spectral Deligne-Mumford stacks if and only if the functor $\widehat{Y}$ has a sufficiently well-behaved deformation theory, and the relative cotangent complex $L_{X/\widehat{Y}}$ is 1-connective and almost perfect (Theorem 18.2.3.1). For this result to be useful in practice, it is important to know that the formal thickenings of a fixed formal spectral Deligne-Mumford stack $\mathfrak{X}$ are "not too wild." We address this point in §18.2.4 by showing that if $i : \mathfrak{X} \to \mathfrak{Y}$ is a formal thickening, then $\mathfrak{X}$ is locally Noetherian if and only if $\mathfrak{Y}$ is locally Noetherian (Corollary 18.2.4.4).

The results of this section will play an important role in our proof of the Artin representability theorem in §18.3. Suppose that Y is a functor which is not yet known to be representable Roughly speaking, we would like to apply the results of this section to show that (under some reasonable hypotheses) the functor Y becomes representable after formally completing at any point. Here we encounter a technical obstacle: the points we are interested in are κ-valued points: that is, morphisms $i : \mathrm{Spec}(\kappa) \to Y$ where κ is a field and $\mathrm{Spec}(\kappa)$ denotes the functor corepresented by κ. In general, one does not expect the relative cotangent complex $L_{\mathrm{Spec}(\kappa)/Y}$ of such a morphism to be 1-connective. We will address this point in §18.2.5 by proving a more general version of Theorem 18.2.3.1 (at least in the affine case) which can be used to show that, even without the connectivity of $L_{X/Y}$, the functor $\widehat{Y}$ admits "smooth charts" by formal thickenings of X (Theorem 18.2.5.1).

18.2.1 The de Rham Space

Let X be a smooth algebraic variety defined over the field $\mathbf{C}$ of complex numbers. Let us abuse notation by identifying X with its functor of points, regarded as a functor from the category $\mathrm{CAlg}_{\mathbf{C}}^{\heartsuit}$ of commutative algebras over $\mathbf{C}$ to the category $\mathcal{S}\mathrm{et}$ of sets. In [192], Simpson introduced the *de Rham space* of X: this is another functor from $\mathrm{CAlg}_{\mathbf{C}}^{\heartsuit}$ to $\mathcal{S}\mathrm{et}$, given by the construction $R \mapsto X(R^{\mathrm{red}})$. The terminology is motivated by a relationship with the Grothendieck's theory of algebraic de Rham cohomology: the algebraic de Rham cohomology of X can be identified with the cohomology of the de Rham space of X (with coefficients in its structure sheaf). In this section, we introduce a slight variant on this construction:

Definition 18.2.1.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. The *absolute de Rham space of* X is the functor $X_{\mathrm{dR}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ given by the formula $X_{\mathrm{dR}}(R) = \varinjlim_I X(\pi_0(R)/I)$, where the direct limit is taken over the filtered system of all nilpotent ideals $I \subseteq \pi_0(R)$.

If $f : X \to Y$ is a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, we define the *relative de Rham space* $(X/Y)_{\mathrm{dR}}$ to be the fiber product $X_{\mathrm{dR}} \times_{Y_{\mathrm{dR}}} Y$, given concretely by the formula

$$(X/Y)_{\mathrm{dR}}(R) = \varinjlim_{I} X(\pi_0(R)/I) \times_{Y(\pi_0(R)/I)} Y(R);$$

here again the direct limit is taken over all nilpotent ideals of $\pi_0(R)$.

Remark 18.2.1.2. Let $f : X \to Y$ be a natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Then f factors as a composition

$$X \xrightarrow{f'} (X/Y)_{\mathrm{dR}} \xrightarrow{f''} Y.$$

Remark 18.2.1.3. Let R be a connective $\mathbb{E}_\infty$-ring and let $f : X \to \mathrm{Spec}(R)$ be a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, corresponding to a functor $X' : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$. Then the projection map $(X/\mathrm{Spec}(R))_{\mathrm{dR}} \to \mathrm{Spec}(R)$ corresponds to the functor $\mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ given by $A \mapsto \varinjlim_I X(\pi_0(A)/I)$, where the colimit is taken over all nilpotent ideals of the commutative ring $\pi_0(A)$.

Remark 18.2.1.4 (Compatibility with Base Change)**.** Suppose we are given a pullback diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Then the relative de Rham space $(X'/Y')_{\mathrm{dR}}$ can be identified with the fiber product $Y' \times_Y (X/Y)_{\mathrm{dR}}$.

Example 18.2.1.5. Let $f : X \to Y$ be a natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and suppose that f is locally of finite presentation to order 1 (Definition 17.4.1.1). Then, for every connective $\mathbb{E}_\infty$-ring R, the diagram

$$\begin{array}{ccc} \varinjlim_I X(\pi_0(R)/I) & \longrightarrow & X(R^{\mathrm{red}}) \\ \downarrow & & \downarrow \\ \varinjlim_I Y(\pi_0(R)/I) & \longrightarrow & Y(R^{\mathrm{red}}) \end{array}$$

is a pullback square, where the colimits are taken over the filtered system of all nilpotent ideals of $\pi_0(R)$. It follows that the relative de Rham space $(X/Y)_{\mathrm{dR}}$ is given concretely by the formula $(X/Y)_{\mathrm{dR}}(R) = Y(R) \times_{Y(R^{\mathrm{red}})} X(R^{\mathrm{red}})$.

Definition 18.2.1.6. Let $f : X \to Y$ be morphism of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will say that Y *is formally complete along* f if, for every commutative ring R, the canonical map $\varinjlim_I X(R/I) \to \varinjlim_I Y(R/I)$ is an equivalence.

Proposition 18.2.1.7. *Let $f : X \to Y$ be any natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and let $f' : X \to (X/Y)_{\mathrm{dR}}$ be as in Remark 18.2.1.2. Then $(X/Y)_{\mathrm{dR}}$ is formally complete along f'.*

Proof. Let R be a commutative ring. Unwinding the definitions, we can identify the colimit limit $\varinjlim_I (X/Y)_{\mathrm{dR}}(R/I)$ with $\varinjlim_{I \subseteq J} Y(R/I) \times_{Y(R/J)} X(R/J)$, where the latter colimit is indexed by the partially ordered set P of pairs of nilpotent ideals $I \subseteq J \subseteq R$. Under this identification, the canonical map $\varinjlim_I X(R/I) \to \varinjlim_I (X/Y)_{\mathrm{dR}}(R/I)$ is given by the natural map

$$\varinjlim_{(I,J) \in P_0} Y(R/I) \times_{Y(R/J)} X(R/J) \to \varinjlim_{(I,J) \in P} Y(R/I) \times_{Y(R/J)} X(R/J),$$

where P_0 denotes the subset of P consisting of those pairs (I, J) with $I = J$. The desired result now follows from the observation that the inclusion $P_0 \hookrightarrow P$ is left cofinal. $\square$

Corollary 18.2.1.8. *Let $f : X \to Y$ be any natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. The following conditions are equivalent:*

(1) *The functor Y is formally complete along f.*

(2) *The map f induces an equivalence $X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$.*

(3) *The projection map $(X/Y)_{\mathrm{dR}} \to Y$ is an equivalence.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows immediately from the definitions, the implication (2) $\Rightarrow$ (3) follows from the existence of a pullback diagram

$$\begin{array}{ccc} (X/Y)_{\mathrm{dR}} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_{\mathrm{dR}} & \longrightarrow & Y_{\mathrm{dR}}, \end{array}$$

and the implication (3) $\Rightarrow$ (1) follows from Proposition 18.2.1.7. $\square$

Corollary 18.2.1.9. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor, and let $\mathcal{C} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{X/}$ be the full subcategory spanned by those natural transformations $f : X \to Y$ for which Y is formally complete along X. Then the inclusion functor $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{X/}$ admits a right adjoint, given by the construction $Y \mapsto (X/Y)_{\mathrm{dR}}$.*

We now show that deformation-theoretic properties of a map $f : X \to Y$ tend to be inherited by the induced map $X \to (X/Y)_{\mathrm{dR}}$.

Proposition 18.2.1.10. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Then the de Rham space $X_{\mathrm{dR}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is nilcomplete, infintiesimally cohesive, and admits a cotangent complex, given by the zero object $0 \in \mathrm{QCoh}(X_{\mathrm{dR}})$.*

Proof. Let R be a connective $\mathbb{E}_\infty$-ring. Since each of the maps $X_{\mathrm{dR}}(R) \to X_{\mathrm{dR}}(\tau_{\leqslant n} R)$ is a homotopy equivalence, the canonical map $X_{\mathrm{dR}}(R) \to \varprojlim_n X_{\mathrm{dR}}(\tau_{\leqslant n} R)$ is a homotopy equivalence; this shows that X is infinitesimally cohesive. Given a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

where the underlying ring homomorphisms $\pi_0(R_0) \to \pi_0(R_{01}) \leftarrow \pi_0(R_1)$ are surjective with nilpotent kernel, the square

$$\begin{array}{ccc} X_{\mathrm{dR}}(R) & \longrightarrow & X_{\mathrm{dR}}(R_0) \\ \downarrow & & \downarrow \\ X_{\mathrm{dR}}(R_1) & \longrightarrow & X_{\mathrm{dR}}(R_{01}) \end{array}$$

is a homotopy pullback since each map is a homotopy equivalence. Finally, the existence and vanishing of the cotangent complex $L_{X_{\mathrm{dR}}}$ follows from the observation that the projection map $X_{\mathrm{dR}}(R \oplus M) \to X_{\mathrm{dR}}(R)$ is a homotopy equivalence for every connective R-module M. □

Corollary 18.2.1.11. *Let $f : X \to Y$ be a natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Then:*

(1) *The induced map $X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$ is nilcomplete, infinitesimally cohesive, and admits a relative cotangent complex, given by the zero object $0 \in \mathrm{QCoh}(X_{\mathrm{dR}})$.*

(2) *The projection map $(X/Y)_{\mathrm{dR}} \to Y$ is nilcomplete, infinitesimally cohesive, and admits a relative cotangent complex, given by the zero object $0 \in \mathrm{QCoh}((X/Y)_{\mathrm{dR}})$.*

(3) *The map f is nilcomplete if and only if the induced map $f' : X \to (X/Y)_{\mathrm{dR}}$ is nilcomplete.*

(4) *The map f is infinitesimally cohesive if and only if the induced map $f' : X \to (X/Y)_{\mathrm{dR}}$ is infinitesimally cohesive.*

(5) *Suppose that f admits a cotangent complex $L_{X/Y}$. Then the map $f' : X \to (X/Y)_{\mathrm{dR}}$ admits a cotangent complex, and the canonical map $L_{X/Y} \to L_{X/(X/Y)_{\mathrm{dR}}}$ is an equivalence in $\mathrm{QCoh}(X)$.*

(6) *Suppose that the map $f' : X \to (X/Y)_{\mathrm{dR}}$ is infinitesimally cohesive and admits a cotangent complex. Then f is infinitesimally cohesive and admits a cotangent complex.*

Proof. Assertion (1) follows from Proposition 18.2.1.10, Remark 17.3.7.3, and Proposition 17.2.5.2. Assertion (2) follows from (1), Proposition 17.3.8.4, and Remark 17.2.4.6. Assertions (3) and (4) follow from (2) together with Remark 17.3.7.3. Assertion (5) follows from (2) and Proposition 17.2.5.2. Assertion (6) follows from (2) and Proposition 17.3.9.1. □

Proposition 18.2.1.12. *Let $f : X \to Y$ be a natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that f is locally almost of finite presentation (locally of finite presentation, locally of finite presentation to order n for $n \geqslant 1$). Then the induced map $f' : X \to (X/Y)_{\mathrm{dR}}$ is locally almost of finite presentation (locally of finite presentation, locally of finite presentation to order n for $n \geqslant 1$).*

Proof. We will treat the case where f is locally almost of finite presentation; the other cases are the same. By virtue of Remark 17.4.1.3, it will suffice to show that the projection map

$(X/Y)_{\mathrm{dR}} \to Y$ is locally almost of finite presentation. Using the pullback square

$$\begin{array}{ccc} (X/Y)_{\mathrm{dR}} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_{\mathrm{dR}} & \longrightarrow & Y_{\mathrm{dR}}, \end{array}$$

we can further reduce to showing that the map $X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$ is locally almost of finite presentation (Proposition 17.4.1.5). Let $\{R_\alpha\}$ be a filtered diagram of n-truncated connective $\mathbb{E}_\infty$-rings; we wish to show that the diagram σ :

$$\begin{array}{ccc} \varinjlim X_{\mathrm{dR}}(R_\alpha) & \longrightarrow & X_{\mathrm{dR}}(\varinjlim R_\alpha) \\ \downarrow & & \downarrow \\ \varinjlim Y_{\mathrm{dR}}(R_\alpha) & \longrightarrow & Y_{\mathrm{dR}}(\varinjlim R_\alpha) \end{array}$$

is a pullback square. Using Example 18.2.1.5, we can identify σ with the diagram

$$\begin{array}{ccc} \varinjlim X(R_\alpha^{\mathrm{red}}) & \longrightarrow & X(\varinjlim R_\alpha^{\mathrm{red}}) \\ \downarrow & & \downarrow \\ \varinjlim Y(R_\alpha^{\mathrm{red}}) & \longrightarrow & Y(\varinjlim R_\alpha^{\mathrm{red}}), \end{array}$$

which is a pullback square by virtue of our assumption that f is locally almost of finite presentation. □

We close this section with a variant of Proposition 18.2.1.12:

Proposition 18.2.1.13. *Let $f : X \to Y$ be a natural transformation of functors $X, Y :$ $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that f is nilcomplete, infinitesimally cohesive, and admits a relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$. The following conditions are equivalent:*

(1) *The relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ is almost perfect.*

(2) *The induced map $X \to (X/Y)_{\mathrm{dR}}$ is locally almost of finite presentation.*

Warning 18.2.1.14. In the situation of Proposition 18.2.1.13, it is not necessarily true that the map $f : X \to Y$ is locally almost of finite presentation, or that the canonical map $(X/Y)_{\mathrm{dR}}(R) \to Y(R) \times_{Y(R^{\mathrm{red}})} X(R^{\mathrm{red}})$ is an equivalence for $R \in \mathrm{CAlg}^{\mathrm{cn}}$.

Proof of Proposition 18.2.1.13. Using Corollary 18.2.1.11, we can replace Y by the relative de Rham space $(X/Y)_{\mathrm{dR}}$ and thereby reduce to the case where Y is formally complete along f. In this case, the implication (2) ⇒ (1) follows from Proposition 17.4.2.1. We will prove the converse. Assume that f satisfies condition (1); we wish to show that f is locally almost of finite presentation. Without loss of generality we may assume that the functor $Y = \operatorname{Spec}(R)$ is corepresentable by a connective $\mathbb{E}_\infty$-ring R, so that X determines a functor $X' : \operatorname{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$. We wish to show that X' commutes with filtered colimits when restricted to $\tau_{\leqslant n} \operatorname{CAlg}_R^{\mathrm{cn}}$, for every nonnegative integer n. Using Proposition 17.4.2.1 we can reduce to the case $n = 0$. We first prove the following:

(∗) Let $\{A_s\}_{s \in S}$ be a diagram in $\operatorname{CAlg}_R^{\heartsuit}$ indexed by a filtered partially ordered set S, with transition maps $\alpha_{s,s'} : A_s \to A_{s'}$. Suppose we are given a collection of ideals $I_s \subseteq A_s$ such that $\alpha_{s,s'}(I_s) \subseteq I_{s'}$. If there exists an integer $n \geqslant 0$ such that $I_s^n = (0)$ for all $s \in S$, then the diagram

$$\begin{array}{ccc} \varinjlim X'(A_s) & \longrightarrow & X'(\varinjlim A_s) \\ \downarrow & & \downarrow \\ \varinjlim X'(A_s/I_s) & \longrightarrow & X'(\varinjlim A_s/I_s) \end{array}$$

is a pullback square.

Proceeding by induction on n, we can reduce to the case $n = 2$, so that each A_s is a square-zero extension of A_s/I_s. In this case, Theorem HA.7.4.1.26 supplies a pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A_s & \longrightarrow & A_s/I_s \\ \downarrow & & \downarrow \\ A_s/I_s & \longrightarrow & (A_s/I_s) \oplus \Sigma I_s, \end{array}$$

depending functorially on s. Invoking our assumption that f is infinitesimally cohesive, we are reduced to proving that the upper square in the diagram

$$\begin{array}{ccc} \varinjlim X'(A_s/I_s) & \longrightarrow & X'(\varinjlim A_s/I_s) \\ \downarrow & & \downarrow \\ \varinjlim X'((A_s/I_s) \oplus \Sigma I_s) & \longrightarrow & X'(\varinjlim (A_s/I_s) \oplus \Sigma I_s) \\ \downarrow & & \downarrow \\ \varinjlim X'(A_s/I_s) & \longrightarrow & X'(\varinjlim A_s/I_s) \end{array}$$

is a pullback. Since the vertical compositions are the identity, we are reduced to showing that the bottom square is a pullback. This follows from our assumption that $f : X \to Y$ admits a cotangent complex which is almost perfect.

For every commutative ring A, let $(-nilA)$ denote the (filtered) partially ordered set consisting of nilpotent ideals of A. Our assumption that Y is formally complete along f translates to the following assertion:

$(*')$ For every commutative ring A, the colimit $\varinjlim_{I \in (-nil A)} X'(A/I)$ is contractible.

Now suppose we are given a diagram $\{A_s\}_{s \in S}$ in $\mathrm{CAlg}_R^{\heartsuit}$ indexed by a filtered partially ordered set S, having colimit A; we wish to show that the canonical map $\theta : \varinjlim_{s \in S} X(A_s) \to X(A)$ is a homotopy equivalence. For each element $t \in S$ and every nilpotent ideal $I \subseteq A_t$, assertion $(*)$ guarantees that the diagram $\sigma_{t,I}$:

$$\begin{array}{ccc} \varinjlim_{s \in S} X'(A_s) & \xrightarrow{\theta} & X'(A) \\ \downarrow & & \downarrow \\ \varinjlim_{s \geqslant t} X'(A_s/IA_s) & \longrightarrow & X'(A/IA) \end{array}$$

is a pullback square. Passing to the colimit over t and I, we obtain a pullback square σ :

$$\begin{array}{ccc} \varinjlim_{s \in S} X'(A_s) & \xrightarrow{\theta} & X'(A) \\ \downarrow & & \downarrow \\ \varinjlim_{s \in S} \varinjlim_{I \in (-nil A_s)} X'(A_s) & \xrightarrow{\theta'} & \varinjlim_{I \in {}_{-}nil0(A)} X'(A/I), \end{array}$$

where ${}_{-}nil0(A) \subseteq (-nil A)$ is the subset consisting of those nilpotent ideals having the form JA, where J is a nilpotent ideal of some A_s. Consequently, to complete the proof of Proposition 18.2.1.13, it will suffice to show that the map θ' is a homotopy equivalence. In fact, we claim that the domain and codomain of θ are contractible. For the domain, this follows immediately from $(*')$. For the codomain, it will suffice (by virtue of $(*')$) to show that the canonical map $\varinjlim_{I \in {}_{-}nil0(A)} X'(A/I) \to \varinjlim_{I \in (-nil A)} X'(A/I)$ is a homotopy equivalence. In fact, we claim that the functor

$$(-nil A) \to \mathcal{S} \qquad I \mapsto X'(A/I)$$

is a left Kan extension of its restriction to the subset ${}_{-}nil0(A) \subseteq (-nil A)$. Fix a nilpotent ideal $J \subseteq A$; we wish to show that the canonical map $\rho : \varinjlim_{I \in {}_{-}nil0(A), I \subseteq J} X'(A/I) \to X'(A/J)$ is a homotopy equivalence. To prove this, we observe that every finitely generated nilpotent ideal of A belongs to ${}_{-}nil0(A)$, so that A/J can be written as the colimit of the (filtered) diagram $\{A/I\}_{I \in {}_{-}nil0(A), I \subseteq J}$. Invoking $(*)$, we deduce that ρ fits into a pullback square

$$\begin{array}{ccc} \varinjlim_{I \in {}_{-}nil0(A), I \subseteq J} X'(A/I) & \xrightarrow{\rho} & X'(A/J) \\ \downarrow & & \downarrow \\ \varinjlim_{I \in {}_{-}nil0(A), I \subseteq J} X'(A/J) & \xrightarrow{\rho'} & X'(A/J), \end{array}$$

where the map ρ' is obviously a homotopy equivalence. □

18.2.2 Formal Thickenings

We now study a special class of closed immersions between formal spectral Deligne-Mumford stacks.

Definition 18.2.2.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. We will say that f is a *formal thickening* if the following conditions are satisfied:

(1) The induced map $\mathfrak{X}^{\mathrm{red}} \to \mathfrak{Y}^{\mathrm{red}}$ is an equivalence (see §8.1.4).

(2) The morphism f is representable by closed immersions which are locally almost of finite presentation. In other words, for every pullback diagram

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \longrightarrow & \mathfrak{Y} \end{array}$$

where $\mathfrak{Y}'$ is a spectral Deligne-Mumford stack, the fiber product $\mathfrak{X}'$ is also a spectral Deligne-Mumford stack, and the morphism f' is a closed immersion which is locally almost of finite presentation.

Remark 18.2.2.2. In the situation of Definition 18.2.2.1, we can restate condition (1) as follows:

(1′) For every reduced commutative ring R, composition with f induces a homotopy equivalence
$$\mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Sp\acute{e}t}(R), \mathfrak{X}) \to \mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Sp\acute{e}t}(R), \mathfrak{Y}).$$

Remark 18.2.2.3. Suppose we are given a pullback diagram of formal spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}. \end{array}$$

If f is a formal thickening, then so is f'. The converse holds if g is an étale surjection.

Remark 18.2.2.4. Let $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{Z}$ be formal thickenings of formal spectral Deligne-Mumford stacks. Then the composite map $g \circ f : \mathfrak{X} \to \mathfrak{Z}$ is also a formal thickening.

Remark 18.2.2.5. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a formal thickening of formal spectral Deligne-Mumford stacks, and let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functors represented by $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Let us abuse notation by identifying f with the induced natural transformation

$X \to Y$. Then Y is formally complete along $f : X \to Y$, in the sense of Definition 18.2.1.6. To prove this, we observe that for every commutative ring R, our assumption that f is locally almost of finite presentation guarantees that we have a pullback diagram

$$\begin{array}{ccc} \varinjlim X(R/I) & \longrightarrow & X(R^{\mathrm{red}}) \\ \downarrow & & \downarrow \\ \varinjlim Y(R/I) & \longrightarrow & Y(R^{\mathrm{red}}), \end{array}$$

where the colimit is taken over all nilpotent ideals $I \subseteq R$. Since the map f induces an equivalence $\mathfrak{X}^{\mathrm{red}} \to \mathfrak{Y}^{\mathrm{red}}$, the right vertical map is a homotopy equivalence, so the left vertical map is a homotopy equivalence as well.

Proposition 18.2.2.6. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. Then f is a formal thickening if and only if it satisfies the following conditions:*

(1) *The morphism f is representable by affine spectral Deligne-Mumford stacks.*

(2) *The morphism f is locally almost of finite presentation.*

(3) *The completed cotangent complex $L^{\wedge}_{\mathfrak{X}/\mathfrak{Y}} \in \mathrm{QCoh}(\mathfrak{X})$ is 1-connective.*

(4) *The induced map $\mathfrak{X}^{\mathrm{red}} \to \mathfrak{Y}^{\mathrm{red}}$ is an equivalence.*

Proof. The necessity of conditions (1) through (4) is clear. For the converse, suppose that (1) through (4) are satisfied; we wish to show that f is representable by closed immersions. To prove this, it will suffice to show that for every map $\mathrm{Spét}(A) \to \mathfrak{Y}$, the projection map $\mathrm{Spét}(A) \times_{\mathfrak{Y}} \mathfrak{X} \to \mathrm{Spét}(A)$ is a closed immersion. It follows from (1) that we can write $\mathrm{Spét}(A) \times_{\mathfrak{Y}} \mathfrak{X} = \mathrm{Spét}(B)$ for some connective $\mathbb{E}_\infty$-ring B. Condition (2) guarantees that B is locally almost of finite presentation over A; in particular, $\pi_0 B$ is finitely generated as an algebra over $\pi_0 A$. Let $x_1, \ldots, x_n$ be a set of generators for $\pi_0 B$ as an algebra over $\pi_0 A$. It follows from (3) that we can arrange (after modifying each x_i by an element of $\pi_0 A$ if necessary) that each x_i is nilpotent in $\pi_0 B$. Then the map $\pi_0 A \to (\pi_0 B)/(x_1, \ldots, x_n)$ is a surjection, whose kernel is a nilpotent ideal $I \subseteq (\pi_0 A)$. We therefore obtain a homomorphism $u : (\pi_0 B) \to (\pi_0 A)/I$. Let v denote the composite map

$$(\pi_0 B) \xrightarrow{u} (\pi_0 A)/I \to (\pi_0 B)/I(\pi_0 B),$$

and let $v' : (\pi_0 B) \to (\pi_0 B)/I(\pi_0 B)$ be the tautological map. Then v and v' agree on $\pi_0 A$. Let $J \subseteq (\pi_0 B)/I(\pi_0 B)$ be the ideal generated by the elements $\{v(x_i) - v'(x_i)\}_{1 \leqslant i \leqslant n}$. The difference between v and v' determines a $(\pi_0 A)$-linear derivation of $(\pi_0 B)$ into J/J^2, which automatically trivial since condition (3) guarantees the vanishing of the module of Kähler differentials $\Omega_{\pi_0 B/\pi_0 A} \simeq \pi_0 L_{B/A}$. It follows that $J = J^2$. Since each x_i is nilpotent in $(\pi_0 B)$, the ideal J is nilpotent, so the equation $J = J^2$ implies that $J = 0$. It follows that $v = v'$, so that v is surjective. In particular, the map $\pi_0 A \to \pi_0 B$ induces a surjection modulo the ideal I. Since the ideal I is nilpotent, it follows from Nakayama's lemma that the map $\pi_0 A \to \pi_0 B$ is surjective, as desired. □

Proposition 18.2.2.7. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings which induces a surjection $\pi_0 A \to \pi_0 B$, and assume that f exhibits B as an almost perfect A-module. Let $I \subseteq \pi_0 A$ be a finitely generated ideal which contains the kernel of $\pi_0(f)$. Let us regard $\pi_0 A$ as equipped with the I-adic topology and $\pi_0 B$ as equipped with the $I(\pi_0 B)$-adic topology. Then the induced map $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is a formal thickening.*

Proof. We have a pullback diagram

$$\begin{array}{ccc} \mathrm{Spf}(B) & \longrightarrow & \mathrm{Sp\acute{e}t}(B) \\ \downarrow & & \downarrow \\ \mathrm{Spf}(A) & \longrightarrow & \mathrm{Sp\acute{e}t}(A). \end{array}$$

Our assumptions on f guarantee that the right vertical map is a closed immersion which is locally almost of finite presentation (see Corollary 5.2.2.2), so that the left vertical map is representable by closed immerions which are locally almost of finite presentation. We will complete the proof by verifying condition $(1')$ of Remark 18.2.2.2. Let R be a reduced commutative ring equipped with a map $g : \mathrm{Sp\acute{e}t}(R) \to \mathrm{Spf}(A)$; we wish to show that the homotopy fiber $F = \mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Sp\acute{e}t}(R), \mathrm{Spf}(B)) \times_{\mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Sp\acute{e}t}(R), \mathrm{Spf}(A))} \{g\}$ is contractible. Let us identify g with a ring homomorphism $\rho : \pi_0 A \to R$ which annihilates some power of I; we wish to show that ρ factors through $\pi_0 B$. This is clear: the assumption that R is reduced guarantees that ρ vanishes on I and therefore also on the kernel $\ker(\pi_0(f)) \subseteq I$. □

We now show that every formal thickening is locally of the form described by Proposition 18.2.2.7.

Proposition 18.2.2.8. *Let A be a complete adic $\mathbb{E}_\infty$-ring and let $f : \mathfrak{X} \to \mathrm{Spf}(A)$ be a formal thickening. Then there exists an equivalence $\mathsf{X} \simeq \mathrm{Spf}(B)$, where B is an $\mathbb{E}_\infty$-algebra which is almost perfect as an A-module, the map $\pi_0(A) \to \pi_0(B)$ is surjective, and we regard $\pi_0(B)$ as equipped with the $I\pi_0(B)$-adic topology; here $I \subseteq \pi_0(A)$ is an ideal of definition which contains the kernel of the map $\pi_0(A) \to \pi_0(B)$.*

Proof. Choose a tower of connective $\mathbb{E}_\infty$-algebras over A

$$\cdots \to A_4 \to A_3 \to A_2 \to A_1$$

which satisfies the requirements of Lemma 8.1.2.2. Since f is a formal thickening, each fiber product $\mathsf{X} \times_{\mathrm{Spf}(A)} \mathrm{Spét}(A_n)$ has the form $\mathrm{Spét}(B_n)$, where B_n is almost perfect as a module over A_n and the map $\pi_0 A_n \to \pi_0 B_n$ is a surjection. Set $B = \varprojlim B_n \simeq \Gamma(\mathfrak{X}; \mathscr{O}_{\mathfrak{X}})$. Using Proposition 8.4.1.2, we deduce that B is almost perfect as an A-module and the canonical maps $B \otimes_A A_n \to B_n$ are equivalences. Moreover, the fiber $\mathrm{fib}(A \to B)$ can be written as the limit of a tower

$$\cdots \to \mathrm{fib}(A_3 \to B_3) \to \mathrm{fib}(A_2 \to B_2) \to \mathrm{fib}(A_1 \to B_1)$$

of connective A-modules whose transition maps are surjective on π_0. It follows that the fiber of the map $A \to B$ is connective: that is, B is connective and the ring homomorphism $\pi_0 A \to \pi_0 B$ is surjective.

Choose a finite collection of elements x_i which generate $\pi_0 \mathrm{fib}(A_1 \to B_1)$ as a module over the commutative ring $\pi_0 A_1$. Each of the elements x_i can be lifted to an element $\overline{x}_i \in \pi_0 \mathrm{fib}(A \to B)$. Since f is a formal thickening, the canonical map $A_1 \to B_1$ induces an isomorphism $A_1^{\mathrm{red}} \simeq B_1^{\mathrm{red}}$. It follows that each each x_i has nilpotent image in A_1, so that each $\overline{x}_i$ has topologically nilpotent image in $\pi_0 A$. We can therefore choose a finitely generated ideal of definition $I \subseteq \pi_0 A$ which contains the image of each $\overline{x}_i$. Note that the elements $\overline{x}_i$ generate $\pi_0 \mathrm{fib}(A \to B)$ as a module over the commutative ring $\pi_0 A$, so that I contains the kernel of the map $\pi_0 A \to \pi_0 B$. Let us regard B as an adic $\mathbb{E}_\infty$-ring, with ideal of definition $I(\pi_0 B)$. We will complete the proof by showing that the canonical map $\mathfrak{X} \to \mathrm{Spf}(B)$ is an equivalence. To prove this, it will suffice to show that $\mathfrak{X}$ and $\mathrm{Spf}(B)$ represent the same functor on $\mathrm{CAlg}^{\mathrm{cn}}$: that is, that the canonical map

$$\theta : \mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Spét}(R), \mathfrak{X}) \to \mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Spét}(R), \mathrm{Spf}(B))$$

is a homotopy equivalence for every connective $\mathbb{E}_\infty$-ring B (see Theorem 8.1.5.1). Fix a map $u : \mathrm{Spét}(R) \to \mathrm{Spf}(A)$; we will show that the induced map

$$\begin{array}{c} \mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Spét}(R), \mathfrak{X}) \times_{\mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Spét}(R), \mathrm{Spf}(A))} \{u\} \\ \downarrow \theta_u \\ \mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Spét}(R), \mathrm{Spf}(B)) \times_{\mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Spét}(R), \mathrm{Spf}(A))} \{u\} \end{array}$$

is a homotopy equivalence. Without loss of generality, we may assume that u factors through $\mathrm{Spét}(A_n)$ for some $n \geqslant 0$. We are therefore reduced to showing that the map

$$\mathsf{X} \times_{\mathrm{Spf}(A)} \mathrm{Spét}(A_n) \to \mathrm{Spf}(B) \times_{\mathrm{Spf}(A)} \mathrm{Spét}(A_n)$$

is an equivalence of formal spectral Deligne-Mumford stacks. This is clear, since both sides can be identified with $\mathrm{Spét}(B_n)$. $\square$

Corollary 18.2.2.9. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a formal thickening of formal spectral Deligne-Mumford stacks. Then the completed relative cotangent complex $L^{\wedge}_{\mathfrak{X}/\mathfrak{Y}} \in \mathrm{QCoh}(\mathfrak{X})$ is 1-connective and almost perfect.*

Proof. The assertion is local on $\mathfrak{Y}$, so we may assume without loss of generality that $\mathfrak{Y} = \mathrm{Spf}(A)$ for some connective $\mathbb{E}_\infty$-ring A which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 A$. Applying Proposition 18.2.2.8, we can write $\mathfrak{X} \simeq \mathrm{Spf}(B)$, where B is almost perfect as an A-module and the map $\pi_0 A \to \pi_0 B$ is surjective. Using Example 17.1.2.9, we see that $L^{\wedge}_{\mathfrak{X}/\mathfrak{Y}}$ is the quasi-coherent sheaf associated to the I-completion of the relative cotangent complex $L_{B/A}$. It now suffices to observe that $L_{B/A}$ is 1-connective (since the map $\pi_0 A \to \pi_0 B$ is surjective) and almost perfect (since B is almost of finite presentation over A by virtue of Corollary 5.2.2.2). □

18.2.3 Existence of Formal Thickenings

We can now formulate the main result of this section.

Theorem 18.2.3.1. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that:*

(0) *The functor X is representable by a formal spectral Deligne-Mumford stack $\mathfrak{X}$.*

(1) *The functor Y is nilcomplete, infinitesimally cohesive, and admits a cotangent complex.*

(2) *The relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ which is 1-connective and almost perfect.*

(3) *The functor Y is a sheaf for the étale topology.*

Then the relative de Rham space $(X/Y)_{\mathrm{dR}}$ is representable by a formal thickening of $\mathfrak{X}$.

We will deduce Theorem 18.2.3.1 from the following more specific result, whose proof we postpone until §18.2.5:

Theorem 18.2.3.2. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, Assume that:*

(0) *The functor X is representable by an affine formal spectral Deligne-Mumford stack $\mathrm{Spf}(B)$.*

(1) *The functor Y is nilcomplete, infinitesimally cohesive, and admits a cotangent complex.*

(2) *The relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ is almost perfect and 1-connective.*

(3) *The functor Y is formally complete along f, in the sense of Definition 18.2.1.6.*

Then Y is (representable by) an affine formal spectral Deligne-Mumford stack $\mathrm{Spf}(A)$*, and the map* $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ *is a formal thickening (in the sense of Definition 18.2.2.1).*

Proof of Theorem 18.2.3.1 from Theorem 18.2.3.2. Let $f : X \to Y$ be any natural transformation satisfying conditions (0) through (3) of Theorem 18.2.3.1. Without loss of generality, we can replace Y by the relative de Rham space $(X/Y)_{\mathrm{dR}}$ and thereby reduce to the case where Y is formally complete along f (see Corollary 18.2.1.11). In this case, we wish to show that Y is representable by a formal thickening of $\mathfrak{X}$.

Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$. For each object $U \in \mathfrak{X}$, let X_U denote the functor represented by $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathcal{O}_{\mathfrak{X}}|_U)$, and define a functor $Y_U : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ by the formula $Y_U(R) = Y(R) \times_{Y(R^{\mathrm{red}})} X_U(R^{\mathrm{red}})$. Let us say that U is *good* if the functor Y_U is representable by a formal spectral Deligne-Mumford stack $\mathfrak{Y}_U$ which is a formal thickening of $\mathfrak{X}_U$. Note that f is locally almost of finite presentation (Proposition 18.2.1.13), so we can identify Y with the functor $Y_{\mathbf{1}}$ where $\mathbf{1}$ is the final object of $\mathcal{X}$ (Example 18.2.1.5). It will therefore suffice to show that the object $\mathbf{1} \in \mathcal{X}$ is good. We will complete the proof by showing that every object $U \in \mathcal{X}$ is good. We first note that Theorem 18.2.3.2 guarantees that every affine object of $\mathcal{X}$ is good. By virtue of Proposition 8.1.3.7, it will suffice to show that the collection of good objects of $\mathcal{X}$ is closed under colimits. Let $\{U_\alpha\}$ be a small diagram in $\mathcal{X}$ having colimit U where each U_α is good. Then $\{\mathfrak{Y}_{U_\alpha}\}$ is a diagram of formal spectral Deligne-Mumford stacks with étale transition maps, and therefore has a colimit $\mathfrak{Z}$ in the ∞-category of formal spectral Deligne-Mumford stacks. It is clear that the canonical map $\mathfrak{X}_U \to \mathfrak{Z}$ is a formal thickening (since this is a local condition). We complete the proof by observing that Y_U is the functor represented by $\mathfrak{Z}$: in fact, both functors can identified with the sheafification (with respect to the étale topology) of $\varinjlim_\alpha Y_{U_\alpha}$. □

We close this section by noting another consequence of Theorem 18.2.3.2:

Corollary 18.2.3.3. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a formal thickening of formal spectral Deligne-Mumford stacks. Then $\mathfrak{X}$ is affine if and only if $\mathfrak{Y}$ is affine.*

Proof. The "if" direction follows from Proposition 18.2.2.8. For the converse, suppose that $\mathfrak{X}$ is affine. Let X and Y denote the functors represented by $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, so that f determines a natural transformation $X \to Y$. Then f satisfies the hypotheses of Theorem 18.2.3.2 (condition (2) follows from Corollary 18.2.2.9). It follows that Y is representable by an affine formal spectral Deligne-Mumford stack $\mathfrak{Y}'$. Applying Theorem 8.1.5.1 we obtain an equivalence $\mathfrak{Y} \simeq \mathfrak{Y}'$, so that $\mathfrak{Y}$ is affine. □

18.2.4 The Noetherian Case

We now specialize to the study of locally Noetherian formal spectral Deligne-Mumford stacks.

Proposition 18.2.4.1. *Let $\mathfrak{X}$ be a formal spectral Deligne-Mumford stack. Then $\mathfrak{X}$ is locally Noetherian if and only if the following pair of conditions is satisfied:*

(1) *The reduction $\mathfrak{X}^{\mathrm{red}}$ is locally Noetherian.*

(2) *The canonical map $\mathfrak{X}^{\mathrm{red}} \to \mathfrak{X}$ is a formal thickening.*

The proof of Proposition 18.2.4.1 will require some preliminaries.

Lemma 18.2.4.2. *Let R be a commutative ring and let t be an element of R. Suppose that R/tR is Noetherian and that R is (t)-complete (in the sense of Definition 7.3.0.5). Then R is Noetherian.*

Proof. For each $m \geqslant 0$, let I_m denote the kernel of the map $R/tR \xrightarrow{t^k} t^k R/t^{k+1}R$. We then have $0 = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq R/tR$. The assumption that R/tR is Noetherian guarantees that this sequence stabilizes. Consequently, there exists an integer $m \geqslant 0$ with the following property: for any $k \geqslant m$ and any $y \in R$, if $t^k y$ belongs to $t^{k+1}R$, then we can write $t^m y = t^{m+1} y'$ for some $y' \in R$.

For each integer k, let J_k denote the kernel of the map $t^k : R \to R$. If $k \geqslant m$ and y belongs to J_k, then $t^k y = 0 \in t^{k+1}R$, so we can write $t^m y = t^{m+1} y'$ for some $y' \in R$. Note that y' belongs to J_{k+1} and that $y \equiv ty' \pmod{J_m}$. It follows that $J_k = tJ_{k+1} + J_m$. We have a short exact sequence of Pro-abelian groups

$$0 \to \{J_m\}_{k \geqslant m} \to \{J_k\}_{k \geqslant m} \to \{J_k/J_m\}_{k \geqslant m} \to 0,$$

where transition maps are given by multiplication by t. We therefore obtain a short exact sequence of abelian groups

$$\lim{}^1 \{J_m\}_{k \geqslant m} \to \lim{}^1 \{J_k\}_{k \geqslant m} \to \lim{}^1 \{J_k/J_m\}_{k \geqslant m}$$

where the first term vanishes because $\{J_m\}_{k \geqslant m}$ is trivial as a Pro-object, and the third term vanishes because the tower $\{J_k/J_m\}_{k \geqslant m}$ has surjective transition maps. It follows that $\varprojlim^1 \{J_k\}_{k \geqslant m} \simeq \varprojlim^1 \{\pi_1 \operatorname{cofib}(t^k : R \to R)\}_{k \geqslant m}$ vanishes. Consequently, the assumption that R is (t)-complete is equivalent to the assumption that R is classically (t)-adically complete: that is, it is isomorphic to the inverse limit of the tower $\{R/t^k R\}_{k \geqslant 0}$. Applying Proposition 7.3.8.1, we deduce that R is Noetherian. $\square$

Variant 18.2.4.3. Let R be a commutative ring containing an element t and let M be a discrete R-module. Assume that R and M are (t)-complete, that R/tR is a Noetherian ring, and that M/tM is a finitely generated module over R/tR. Then M is a finitely generated module over R.

Proof. Applying Lemma 18.2.4.2 to the direct sum $R \oplus M$, we deduce that $R \oplus M$ is a Noetherian ring. In particular, the ideal $M \subseteq R \oplus M$ is finitely generated as an $(R \oplus M)$-module, and therefore also as an R-module. □

Proof of Proposition 18.2.4.1. The assertion is local on $\mathfrak{X}$; we may therefore assume without loss of generality that $\mathfrak{X}$ is affine. Write $\mathfrak{X} = \mathrm{Spf}(A)$, where A is a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 A$. Set $R = (\pi_0 A)/I$. Assume first that $\mathfrak{X}$ is locally Noetherian, so that A is Noetherian. Then $\mathfrak{X}^{\mathrm{red}} \simeq \mathrm{Sp\acute{e}t}(R^{\mathrm{red}})$ is also locally Noetherian. Moreover, R^{red} is almost perfect as an A-module (Proposition HA.7.2.4.17), so that the map $\mathfrak{X}^{\mathrm{red}} \to \mathfrak{X}$ is a formal thickening by virtue of Proposition 18.2.2.7.

Now suppose that conditions (1) and (2) are satisfied; we will show that A is Noetherian. Note that condition (2) guarantees that R^{red} is almost perfect as an A-module, so that the kernel of the map $\pi_0 A \to R^{\mathrm{red}}$ is a finitely generated ideal $J \subseteq \pi_0 A$. We may therefore replace I by J and thereby reduce to the case where $R = R^{\mathrm{red}}$. In this case, assumption (1) guarantees that R is Noetherian. Write $I = (x_1, \ldots, x_n)$. We proceed by induction on n. Let us begin with the case $n = 0$, so that $R = \pi_0 A$. We wish to show that each homotopy group of A are finitely generated as a module over R. Let J denote the fiber of the map $A \to R$, so that J is 1-connective. It follows that each tensor power $J^{\otimes k}$ is k-connective, so that the canonical map $\pi_* A \to \pi_* \mathrm{cofib}(J^{\otimes k} \to A)$ is an isomorphism for $* < k$. It will therefore suffice to prove the following:

- $(*)$ For each integer k, each homotopy group of the cofiber $\mathrm{cofib}(J^{\otimes k} \to A)$ is finitely generated as an R-module.

We prove $(*)$ by induction on k, the case $k = 0$ being trivial. To carry out the inductive step, we observe that there is a cofiber sequence

$$J^{\otimes k} \otimes_A R \to \mathrm{cofib}(J^{\otimes (k+1)} \to A) \to \mathrm{cofib}(J^{\otimes k} \to A).$$

We are therefore reduced to proving that each homotopy group $J^{\otimes k} \otimes_A R$ Is finitely generated as an R-module: in other words, that $J^{\otimes k} \otimes_A R$ is almost perfect over R. This is clear, since assumption (2) guarantees that J is almost perfect as an A-module. This completes the proof in the case $n = 0$.

Let us now assume that $n > 0$. Let S denote the sphere spectrum, and let $S\{t\}$ denote the free $\mathbb{E}_\infty$-ring on one generator, so that there is a morphism of $\mathbb{E}_\infty$-rings $S\{t\} \to A$ given by $t \mapsto x_n$. We next prove the following:

- $(*)$ Let M be a connective $S\{t\}$-module. Assume that each homotopy group of M is finitely generated as an abelian group and that the action of t on $\pi_* M$ is locally nilpotent. Then each homotopy group of $A \otimes_{S\{t\}} M$ is Noetherian when regarded as a module over the commutative ring $\pi_0 A$.

Note that the canonical map $\pi_*(A \otimes_{S\{t\}} M) \to \pi_*(A \otimes_{S\{t\}} \tau_{\leqslant m} M)$ is an isomorphism for $* \leqslant m$. Consequently, it will suffice to prove $(*)$ under the additional assumption that M is truncated. Writing M as a successive extension of $S\{t\}$ concentrated in a single degree, we can further reduce to the case where M is discrete. In this case, our assumption that the action of t is locally nilpotent allows us to write M as a successive extension of modules on which the action of t is trivial. In this case, we have a short exact sequence of abelian groups

$$0 \to \mathbf{Z}^a \to \mathbf{Z}^b \to M \to 0,$$

which allows us to reduce to the case $M = \mathbf{Z}$. Set $A' = A \otimes_{S\{t\}} \mathbf{Z}$, so that the map $A \to R$ factors through A'. Note that A' is almost perfect as an A-module and is therefore I-complete, and that $I(\pi_0 A') = (x_1, \ldots, x_{n-1})$ is generated by fewer than n elements. Moreover, since A' and R are both almost of finite presentation over A, the map $A' \to R$ is almost of finite presentation. It follows that we can regard $\mathrm{Spf}(A')$ as a formal thickening of $\mathrm{Sp\acute{e}t}(R)$, so our inductive hypothesis guarantees that A' is Noetherian. In particular, each homotopy group of A' is Noetherian when viewed as a module over the commutative ring $\pi_0 A$, which proves $(*)$.

Note that $S\{t\}$ is Noetherian (Proposition HA.7.2.4.31). In particular, each homotopy group $\pi_m S\{t\}$ is finitely generated as a module of the polynomial ring $\pi_0 S\{t\} \simeq \mathbf{Z}[t]$. It follows that the kernel and cokernel of multiplication by t on $\pi_m S\{t\}$ are finitely generated abelian groups, so that $M = \mathrm{cofib}(t : S\{t\} \to S\{t\})$ satisfies the hypotheses of $(*)$. We therefore obtain the following:

$(*')$ Each homotopy group of the cofiber $\mathrm{cofib}(t : A \to A)$ is Noetherian when viewed as a module over the commutative ring $\pi_0 A$.

Note that each quotient $(\pi_m A)/t(\pi_m A)$ can be viewed as a submodule of $\pi_m \mathrm{cofib}(t : A \to A)$, and is therefore also Noetherian as a module over $\pi_0 A$. It is therefore finitely generated when viewed as a module over $(\pi_0 A)/t(\pi_0 A)$. Applying Lemma 18.2.4.2 and Variant 18.2.4.3, we conclude that $\pi_0 A$ is a Noetherian ring and that each $\pi_m A$ is a finitely generated module over $\pi_0 A$, so that A is Noetherian as desired. □

Corollary 18.2.4.4. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a formal thickening of formal spectral Deligne-Mumford stacks. Then $\mathfrak{X}$ is locally Noetherian if and only if $\mathfrak{Y}$ is locally Noetherian.*

Proof. Suppose first that $\mathfrak{X}$ is locally Noetherian. Applying Proposition 18.2.4.1, we deduce that $\mathfrak{X}^{\mathrm{red}}$ is locally Noetherian and that the natural map $i : \mathfrak{X}^{\mathrm{red}} \to \mathfrak{X}$ is a formal thickening. Then the canonical map $\mathfrak{Y}^{\mathrm{red}} \to \mathfrak{Y}$ can be identified with the composition $f \circ i$, which is a formal thickening by virtue of Remark 18.2.2.4. Applying Proposition 18.2.4.1, we conclude that $\mathfrak{Y}$ is locally Noetherian.

Conversely, suppose that $\mathfrak{Y}$ is locally Noetherian; we claim that $\mathfrak{X}$ is also locally Noetherian. This assertion is local on $\mathfrak{Y}$, so we may assume that $\mathfrak{Y}$ is affine. Write $\mathfrak{Y} = \mathrm{Spf}(A)$, where A is Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 A$. Applying Proposition 18.2.2.8, we deduce that $\mathfrak{X} \simeq \mathrm{Spf}(B)$, where B is almost of finite presentation over A and therefore also Noetherian (Proposition HA.7.2.4.31). □

18.2.5 Existence of Formal Charts

We will deduce Theorem 18.2.3.2 from the following variant:

Theorem 18.2.5.1. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, where X is corepresentable by a connective $\mathbb{E}_\infty$-ring B. Suppose that f admits a relative cotangent complex $L_{X/Y}$ and that we are given a morphism $\alpha : \mathscr{F} \to L_{X/Y}$ in $\mathrm{QCoh}(X)$, where $\mathscr{F}$ is perfect of* Tor*-amplitude $\leqslant 0$. Assume that:*

(1) *The functor Y is nilcomplete, infinitesimally cohesive, and admits a cotangent complex.*

(2) *The functor Y is formally complete along f.*

(3) *The cofiber* $\mathrm{cofib}(\alpha)$ *is* 1*-connective and almost perfect.*

Then the map f factors as a composition $X \xrightarrow{f'} U \xrightarrow{f''} Y$, where U is representable by an affine formal spectral Deligne-Mumford stack $\mathrm{Spf}(A)$*, the map* $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ *is a formal thickening, f'' is locally almost of finite presentation, and α can be identified with the natural map $f'^{*} L_{U/Y} \to L_{X/Y}$.*

Remark 18.2.5.2. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ satisfying conditions (1) and (2) of Theorem 18.2.5.1, and suppose that the relative cotangent complex $L_{X/Y}$ is almost perfect. Then $M = \Gamma(\mathfrak{X}; L_{X/Y}) \in \mathrm{Mod}_B$ is an almost perfect B-module (assuming that the adic $\mathbb{E}_\infty$-ring B is complete), so Corollary 2.7.2.2 guarantees that there is a fiber sequence $M' \to M \to M''$, where M' is perfect of Tor-amplitude $\leqslant 0$, and M'' is 1-connective. It follows that there exists a morphism $\alpha : \mathscr{F} \to L_{X/Y}$ in $\mathrm{QCoh}(X)$, where $\mathscr{F}$ is perfect of Tor-amplitude $\leqslant 0$ and $\mathrm{cofib}(\alpha)$ is 1-connective and almost perfect. Beware that the morphism α is not unique (for example, we can always add a copy of the structure sheaf $\mathscr{O}_X$ to $\mathscr{F}$).

To deduce Theorem 18.2.3.2 from Theorem 18.2.5.1, we will need the following general observation (which will also be useful in the proof of Theorem 18.2.5.1 itself):

Lemma 18.2.5.3. *Let $f : X \to Y$ be a natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that f is nilcomplete, infinitesimally cohesive, and admits a relative cotangent complex $L_{X/Y}$ which vanishes in* $\mathrm{QCoh}(X)$*. If Y is formally complete along f, then f is an equivalence.*

Proof. Since X and Y are infinitesimally cohesive and $L_{X/Y}$ vanishes, Proposition 17.3.9.3 implies the following:

$(*)$ Let $\widetilde{R} \to R$ be a square-zero extension of a connective $\mathbb{E}_\infty$-ring R by a connective R-module M. Then the diagram

$$\begin{array}{ccc} X(\widetilde{R}) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ Y(\widetilde{R}) & \longrightarrow & Y(R) \end{array}$$

is a pullback square. In particular, if $X(R) \to Y(R)$ is a homotopy equivalence, then so is $X(\widetilde{R}) \to Y(\widetilde{R})$.

Let R be a connective $\mathbb{E}_\infty$-ring; we wish to show that the map $X(R) \to Y(R)$ is a homotopy equivalence. Since X and Y are nilcomplete, we may assume without loss of generality that R is n-connective for some integer $n \geqslant 0$. We proceed by induction on n. If $n > 0$, then R is a square-zero extension of $\tau_{\leqslant n-1}R$, so that the desired result follows from $(*)$ together with our inductive hypothesis. We may therefore assume that $n = 0$: that is, that R is an ordinary commutative ring. Let $\{I_\alpha\}$ be the collection of all nilpotent ideals of R. We have a commutative diagram σ :

$$\begin{array}{ccc} X(R) & \longrightarrow & \varinjlim_\alpha X(R/I_\alpha) \\ \downarrow & & \downarrow \\ Y(R) & \longrightarrow & \varinjlim_\alpha Y(R/I_\alpha), \end{array}$$

and our hypothesis that Y is formally complete along f guarantees that the right vertical map is an equivalence. It will therefore suffice to show that σ is a pullback square. Writing σ as a filtered colimit of diagrams σ_α :

$$\begin{array}{ccc} X(R) & \longrightarrow & X(R/I_\alpha) \\ \downarrow & & \downarrow \\ Y(R) & \longrightarrow & Y(R/I_\alpha) \end{array}$$

we are reduced to showing that each σ_α is a pullback square, which follows from iterated application of $(*)$. $\square$

Proof of Theorem 18.2.3.2 from Theorem 18.2.5.1. Let $f : X \to Y$ be a natural transformation satisfying the hypotheses of Theorem 18.2.3.2, so that $X \simeq \mathrm{Spf}(B)$ for some complete connective adic $\mathbb{E}_\infty$-ring B. Using Lemma 8.1.2.2, we see that there exists an $\mathbb{E}_\infty$-algebra B' which is almost of finite presentation over B, where $\pi_0(B')$ is the quotient of $\pi_0(B)$ by an ideal of definition. Set $X' = \mathrm{Spec}(B)$, so that $L_{X'/X} \in \mathrm{QCoh}(X')$ is 1-connective and almost perfect. Using the fiber sequence $L_{X/Y}|_X \to L_{X'/Y} \to L_{X'/X}$, we conclude that $L_{X'/Y}$ Is also 1-connective and almost perfect. Taking $\mathscr{F} = 0$ and applying Theorem 18.2.5.1, we conclude that the map $X' \to X \to Y$ factors as a composition $X' \xrightarrow{f'} U \xrightarrow{f''} Y$, where $U \simeq \operatorname{Spf}(A)$ is a formal thickening of $X' \simeq \operatorname{Spét}(B')$, f'' is locally almost of finite presentation, and the restriction $L_{U/Y}|_{X'}$ vanishes. Since f'' is locally almost of finite presentation, $L_{U/Y} \in \operatorname{QCoh}(U)$ is almost perfect (Corollary 17.4.2.2); it follows that $L_{U/Y}$ itself vanishes. Applying Lemma 18.2.5.3 to the morphism f'', we deduce that f'' is an equivalence, so that $Y \simeq \operatorname{Spf}(A)$ is a formal thickening of $\operatorname{Spét}(B')$. Note that the kernel of the map $\pi_0(A) \to \pi_0(B')$ is an ideal of definition for the topology on $\operatorname{Spf}(\pi_0(A))$, so that the morphism $f : \operatorname{Spf}(B) \to \operatorname{Spf}(A)$ is representable (by affine spectral Deligne-Mumford stacks). Since f is locally almost of finite presentation (Proposition 18.2.1.13) and $L_{X/Y}$ is 1-connective, it follows from Proposition 18.2.2.6 that f is a formal thickening. □

We now turn to the proof of Theorem 18.2.5.1. We will need several auxiliary results.

Lemma 18.2.5.4. *Let R be a connective $\mathbb{E}_\infty$-ring, let $\widetilde{R}$ be a square-zero extension of R by a connective R-module I, let M be an R-module which is perfect of* Tor*-amplitude $\leqslant 0$, let N be an almost perfect $\widetilde{R}$-module, and let $u : M \to R \otimes_{\widetilde{R}} N$ be a morphism with 1-connective cofiber. Then we can lift u to a morphism of $\widetilde{R}$-modules $\widetilde{u} : \widetilde{M} \to N$, where $\widetilde{M}$ is perfect of* Tor*-amplitude $\leqslant 0$.*

Proof. Since M is perfect, there exists an integer k such that M is $(-k)$-connective. We proceed by induction on k. In the case $k = 0$, M is a projective R-module of finite rank. Using Corollary HA.7.2.2.19 , we can lift M to an $\widetilde{R}$-module $\widetilde{M}$ which is projective of finite rank. Since M and $\operatorname{cofib}(u)$ are connective, the tensor product $R \otimes_{\widetilde{R}} N$ is connective, which guarantees that N is connective (Proposition 2.7.3.2). It then follows from the projectivity of $\widetilde{M}$ that the composite map $\widetilde{M} \to M \to R \otimes_{\widetilde{R}} N$ can be lifted to a morphism of $\widetilde{R}$-modules $\widetilde{u} : \widetilde{M} \to N$.

We now carry out the inductive step. Assume that $k > 0$. Since M is perfect and $(-k)$-connective, the homotopy group $\pi_{-k} M$ is finitely generated as a module over $\pi_0 R$. We can therefore choose a map $f : \Sigma^{-k} R^d \to M$ which is surjective on π_{-k}. Note that since M and $\operatorname{cofib}(u)$ are $(-k)$-connective, the tensor product $R \otimes_{\widetilde{R}} N$ is $(-k)$-connective, so that N is $(-k)$-connective (Proposition 2.7.3.2). We can therefore lift the composite map

$$\Sigma^{-k} \widetilde{R}^d \to \Sigma^{-k} R^d \to M \xrightarrow{u} \widetilde{R} \otimes_R N$$

to a map $g : \Sigma^{-k} \widetilde{R}^d \to N$. Let v denote the induced map $\operatorname{cofib}(f) \to R \otimes_{\widetilde{R}} \operatorname{cofib}(g)$, so that $\operatorname{cofib}(v) \simeq \operatorname{cofib}(u)$ is 1-connected. Note that $\operatorname{cofib}(f)$ is $(1-k)$-connective, so our inductive hypothesis guarantees that we can lift v to a map $\widetilde{v} : K \to \operatorname{cofib}(g)$, where $P \in \operatorname{Mod}_{\widetilde{R}}$ is perfect of Tor-amplitude $\leqslant 0$. Form a pullback diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{u}} & N \\ \downarrow & & \downarrow \\ K & \xrightarrow{\widetilde{v}} & \operatorname{cofib}(f). \end{array}$$

It is now easy to see that $\widetilde{u} : \widetilde{M} \to N$ has the desired properties. □

The proof of Theorem 18.2.5.1 is based on the following:

Construction 18.2.5.5. Let $f : X \to Y$ and $\alpha : \mathscr{F} \to L_{X/Y}$ be as in the statement of Theorem 18.2.5.1, where $X = \operatorname{Spec}(B)$. We construct a sequence of natural transformations

$$\operatorname{Spec}(B_0) \to \operatorname{Spec}(B_1) \to \operatorname{Spec}(B_2) \to \cdots \to Y$$

and morphisms $\alpha_n : M_n \to L_{\operatorname{Spec}(B_n)/Y}$ in $\operatorname{QCoh}(\operatorname{Spec}(B_n)) \simeq \operatorname{Mod}_{B_n}$, where M_n is perfect of Tor-amplitude $\leqslant n$ and $\operatorname{cofib}(\alpha_n)$ is 1-connective, as follows:

- Set $B_0 = B$, equipped with the morphism $f_0 = f_0 : \operatorname{Spec}(B_0) \to Y$; we let M_0 denote the image of $\mathscr{F}$ under the equivalence $\operatorname{QCoh}(X) \simeq \operatorname{Mod}_{B_0}$ and $\alpha_0 : M_0 \to L_{\operatorname{Spec}(B_0)/Y}$ the image of α.

- Assume that $n \geqslant 0$ and that we have already specified a natural transformation $f_n : \operatorname{Spec}(B_n) \to Y$ together with a morphism of B_n-modules $\alpha_n : M_n \to L_{\operatorname{Spec}(B_n)/Y}$ with 1-connective cofiber. We let B_{n+1} denote the square-zero extension of B_n by $\Sigma^{-1} \operatorname{cofib}(\alpha_n)$ classified by the composite map $L_{\operatorname{Spec}(B_n)} \to L_{\operatorname{Spec}(B_n)/Y} \to \operatorname{cofib}(\alpha_n)$. The factorization of this composite map through the relative cotangent complex $L_{\operatorname{Spec}(B_n)/Y}$ determines a factorization of f_n as a composition

$$\operatorname{Spec}(B_n) \to \operatorname{Spec}(B_{n+1}) \xrightarrow{f_{n+1}} Y.$$

  By construction, the canonical map $L_{\operatorname{Spec}(B_n)/Y} \to \operatorname{cofib}(\alpha_n)$ factors as a composition

$$L_{\operatorname{Spec}(B_n)/Y} \to L_{B_n/B_{n+1}} \xrightarrow{\rho} \operatorname{cofib}(\alpha_n),$$

  We therefore obtain a fiber sequence

$$L_{\operatorname{Spec}(B_{n+1})/Y}|_{\operatorname{Spec}(B_n)} \to M_n \xrightarrow{\nu} \operatorname{fib}(\rho).$$

  It follows from Theorem HA.7.4.3.12 that ρ is 2-connective. Since M_n is perfect of Tor-amplitude $\leqslant 0$, the map ν is nullhomotopic. It follows that the map α_n factors as a composition We can therefore factor α_n as a composition

$$M_n \xrightarrow{\alpha_n'} L_{\operatorname{Spec}(B_{n+1})/Y}|_{\operatorname{Spec}(B_n)} \to L_{\operatorname{Spec}(B_n)/Y};$$

moreover, this factorization is unique up to homotopy (though not up to contractible choice). We have a fiber sequence $\mathrm{cofib}(\alpha'_n) \to \mathrm{cofib}(\alpha_n) \xrightarrow{\beta} L_{B_n/B_{n+1}}$, where β is surjective on π_1 (Theorem HA.7.4.3.12), so that $\mathrm{cofib}(\alpha'_n)$ is also 1-connective. Applying Lemma 18.2.5.4, we conclude that we can lift α'_n to a morphism of B_{n+1}-modules $\alpha_{n+1} : M_{n+1} \to L_{\mathrm{Spec}(B_{n+1})/Y}$, where M_{n+1} is perfect of Tor-amplitude $\leqslant 0$. By construction, we have an equivalence $B_n \otimes_{B_{n+1}} \mathrm{cofib}(\alpha_{n+1}) \simeq \mathrm{cofib}(\alpha_n)$, so that $\mathrm{cofib}(\alpha_{n+1})$ is 1-connective (Proposition 2.7.3.2).

We let $\overline{U} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor given by the direct limit $\varinjlim_n \mathrm{Spec}(B_n)$. Finally, we let U denote the nilcompletion of $\overline{U}$, given by the formula

$$U(R) = \varprojlim_m \overline{U}(\tau_{\leqslant m} R) = \varprojlim_m \varinjlim_n \mathrm{Map}_{\mathrm{CAlg}}(B_n, \tau_{\leqslant m} R).$$

In the situation of Construction 18.2.5.5, the map $f : X \to Y$ factors canonically as a composition $X \xrightarrow{f'} U \xrightarrow{f''} Y$. Our goal for the rest of this section is to prove that this factorization satisfies the requirements of Theorem 18.2.5.1. This will require a somewhat detailed analysis.

Lemma 18.2.5.6. *In the situation of Construction 18.2.5.5, the functor U is nilcomplete, cohesive, and admits a cotangent complex L_U. Moreover, the canonical map $L_{U/Y}|_X \to L_{X/Y}$ can be identified with $\alpha : \mathscr{F} \to L_{X/Y}$.*

Proof. The functor U is nilcomplete by construction. Each of the corepresentable functors $\mathrm{Spec}(B_n)$ preserves small limits, so that the functor $\overline{U} = \varinjlim \mathrm{Spec}(B_n)$ preserves pullback squares. In particular, $\overline{U}$ is cohesive, so that U is cohesive (Proposition 17.3.3.4). To show that U admits a cotangent complex, it will suffice to show that $\overline{U}$ admits a cotangent complex (Proposition 17.3.3.6). By virtue of Proposition 17.3.9.1, it will suffice to show that the map $\overline{U} \to Y$ admits a relative cotangent complex. We will prove this by verifying condition (a) of Remark 17.2.4.3 (condition (b) is automatic, since Y admits a cotangent complex and $\overline{U}$ preserves pullback squares). Fix a point $\eta \in \overline{U}(R)$, where R is a connective $\mathbb{E}_\infty$-ring, and define $F : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ by the formula $F(N) = \mathrm{fib}(\overline{U}(R \oplus N) \to Y(R \oplus N) \times_{Y(R)} \overline{U}(R)$; we wish to show that F is corepresented by an almost connective R-module. Lift η to a point $\eta_n \in \mathrm{Spec}(B_n)(R)$ for $n \gg 0$, which we identify with a morphism $B_n \to R$. Since each of the maps $\mathrm{Spec}(B_n) \to Y$ admits a cotangent complex, the functor F is given concretely by the formula

$$F(N) = \varinjlim_{n' \geqslant n} \mathrm{Map}_{\mathrm{Mod}_{B_{n'}}}(L_{\mathrm{Spec}(B_{n'})/Y}, N) \simeq \varinjlim_{n' \geqslant n} \mathrm{Map}_{\mathrm{Mod}_R}(R \otimes_{B_{n'}} L_{\mathrm{Spec}(B_{n'})/Y}, N).$$

In other words, $F(N)$ is represented by the Pro-object $\{R \otimes_{B_{n'}} L_{\mathrm{Spec}(B_{n'})/Y}\}_{n' \geqslant n}$ of Mod_R.

Let $\{\alpha_m : M_m \to L_{\operatorname{Spec}(B_m)/Y}\}_{m \geqslant 0}$ be as in Construction 18.2.5.5. Then the system $\{\alpha_m\}_{m \geqslant 0}$ induces a map of towers

$$\theta : \{R \otimes_{B_{n'}} M_{n'}\}_{n' \geqslant n} \to \{R \otimes_{B_{n'}} L_{\operatorname{Spec}(B_{n'})/Y}\}_{n' \geqslant n},$$

whose domain is a constant tower. Consequently, to complete the proof of the existence of $L_{\overline{U}/Y}$, it will suffice to show that θ is an equivalence of Pro-objects. For this, it will suffice to show that for each of the diagrams

$$\begin{array}{ccc} M_{n'} & \xrightarrow{\alpha'_{n'}} & L_{\operatorname{Spec}(B_{n'+1})/Y}|_{\operatorname{Spec}(B_{n'})} \\ \downarrow \mathrm{id} & \swarrow & \downarrow \\ M_{n'} & \xrightarrow{\alpha_{n'}} & L_{\operatorname{Spec}(B_{n'})/Y} \end{array}$$

appearing in Construction 18.2.5.5, there exists a dotted arrow as indicated, rendering the diagram commutative, which is immediate from the construction. □

Lemma 18.2.5.7. *In the situation of Construction 18.2.5.5, each relative cotangent complex* $L_{\operatorname{Spec}(B_n)/Y} \in \operatorname{QCoh}(\operatorname{Spec}(B_n)) \simeq \operatorname{Mod}_{B_n}$ *is almost perfect.*

Proof. We proceed by induction on n. In the case $n = 0$, this is one of the hypotheses of Theorem 18.2.5.1. To carry out the inductive step, let us assume that $n > 0$ and that $L_{\operatorname{Spec}(B_{n-1})/Y}$ is almost perfect. Then $\operatorname{cofib}(\alpha_{n-1})$ is almost perfect, so that B_n is a square-zero extension of B_{n-1} by a (connective) almost perfect module over B_{n-1}. Applying Corollary 5.2.2.5, we deduce that B_{n-1} is almost of finite presentation over B_n, so that L_{B_{n-1}/B_n} is almost perfect as a B_{n-1}-module. Using the fiber sequence

$$B_{n-1} \otimes_{B_n} L_{\operatorname{Spec}(B_n)/Y} \to L_{\operatorname{Spec}(B_{n-1})/Y} \to L_{B_{n-1}/B_n}$$

together with our inductive hypothesis, we conclude that $B_{n-1} \otimes_{B_n} L_{\operatorname{Spec}(B_n)/Y}$ is almost perfect as a B_{n-1}-module. Applying Proposition 2.7.3.2, we conclude that $L_{\operatorname{Spec}(B_n)/Y}$ is almost perfect as a B_n-module. □

Lemma 18.2.5.8. *In the situation of Construction 18.2.5.5, each of the natural transformations* $f_n : \operatorname{Spec}(B_n) \to Y$ *is locally almost of finite presentation.*

Proof. Since Y is formally complete along f, it is also formally complete along f_n. The desired result now follows from Lemma 18.2.5.7 and Proposition 18.2.1.13. □

Lemma 18.2.5.9. *In the situation of Construction 18.2.5.5, the natural transformation* $U \to Y$ *is locally almost of finite presentation.*

Proof. Since the functors U and $\overline{U}$ agree on truncated objects of $\mathrm{CAlg}^{\mathrm{cn}}$, it will suffice to show that the composite map $\overline{U} \to U \to Y$ is locally almost of finite presentation. This follows by combining Lemma 18.2.5.8 with Remark 17.4.1.4. □

Let us now specialize to the case where $\mathscr{F} = 0$.

Lemma 18.2.5.10. *In the situation of Construction 18.2.5.5, if $\mathscr{F} = 0$, then the map $U \to Y$ is an equivalence.*

Proof. Note that U and Y are infinitesimally cohesive and nilcomplete, and that the relative cotangent complex $L_{U/Y}$ vanishes (Lemma 18.2.5.6). Since Y is formally complete along the map $U \to Y$, the desired result follows from Lemma 18.2.5.3. □

Notation 18.2.5.11. In the situation of Construction 18.2.5.5, suppose that $\mathscr{F} = 0$. Let A denote the $\mathbb{E}_\infty$-ring $\Gamma(Y; \mathscr{O}_Y)$. It follows from Lemma 18.2.5.10 that we have a canonical equivalence

$$\begin{aligned} A &= \Gamma(Y; \mathscr{O}_Y) \\ &\simeq \Gamma(U; \mathscr{O}_U) \\ &\simeq \Gamma(\overline{U}; \mathscr{O}_U) \\ &\simeq \varprojlim \Gamma(\mathrm{Spec}(B_n); \mathscr{O}_{\mathrm{Spec}(B_n)}) \\ &\simeq \varprojlim B_n. \end{aligned}$$

In particular A is connective.

Lemma 18.2.5.12. *In the situation of Construction 18.2.5.5, suppose that $\mathscr{F} = 0$ and that $L_{X/Y}$ is m-connective for some positive integer m. Then, for each $n \geqslant 0$, we have the following:*

(a_n) *The relative cotangent complex $L_{\mathrm{Spec}(B_n)/Y}$ is m-connective.*

(b_n) *The fiber* $\mathrm{fib}(B_{n+1} \to B_n$ *is $(m-1)$-connective.*

Proof. Note that (a_0) equivalent to our hypothesis that $L_{X/Y}$ is m-connective. Moreover, the implication $(a_n) \Rightarrow (b_n)$ is immediate (since B_{n+1} is a square-zero extension of B_n by $\Sigma^{-1} L_{\mathrm{Spec}(B_n)/Y}$). To complete the proof, it will suffice to show that (a_n) and (b_n) imply (a_{n+1}). Assume that (a_n) and (b_n) are satisfied, and consider the fiber sequence

$$B_n \otimes_{B_{n+1}} L_{\mathrm{Spec}(B_{n+1})/Y} \to L_{\mathrm{Spec}(B_n)/Y} \xrightarrow{u} L_{B_n/B_{n+1}}.$$

Unwinding the definitions, we can identify u with the canonical map $\mathrm{cofib}(B_{n+1} \to B_n) \to L_{B_n/B_{n+1}}$, whose fiber is $(2m-1)$-connective by Corollary HA.7.4.3.6 . Applying Proposition 2.7.3.2, we conclude that $L_{\mathrm{Spec}(B_{n+1})/Y}$ is $(2m-1)$-connective, and therefore also m-connective (since $m \geqslant 1$). □

Lemma 18.2.5.13. *In the situation of Construction 18.2.5.5, suppose that $\mathscr{F} = 0$ and that $L_{X/Y}$ is m-connective for some $m > 0$. Then:*

(1) *For every integer $n \geqslant 0$, the fiber $\mathrm{fib}(A \to B_n)$ is $(m-1)$-connective.*

(2) *The canonical map $\theta : \mathrm{fib}(A \to B) \to \Sigma^{-1} L_{B/A} \to \Sigma^{-1} L_{\mathrm{Spec}(B)/Y}$ is surjective on π_{m-1}.*

Proof. To prove (1), we note that $\mathrm{fib}(A \to B)$ can be identified with the limit of the tower $\{\mathrm{fib}(B_k \to B_n)\}_{k \geqslant n}$. To show that this limit is $(m-1)$-connective, it will suffice to show that each transition map $\mathrm{fib}(B_k \to B_n) \to \mathrm{fib}(B_{k-1} \to B_n)$ has $(m-1)$-connective homotopy fiber. This homotopy fiber is given by $\mathrm{fib}(B_k \to B_{k-1})$, which is $(m-1)$-connective by Lemma 18.2.5.12.

To prove (2), we note that θ can be identified with the natural map $\mathrm{fib}(A \to B_0) \to \mathrm{fib}(B_1 \to B_0)$, whose homotopy fiber $\mathrm{fib}(A \to B_1)$ is $(m-1)$-connective by virtue of (1). □

Lemma 18.2.5.14. *Let R be a connective $\mathbb{E}_\infty$-ring, let M be an almost perfect A-module, and suppose we are given a tower*

$$\cdots \to N_3 \to N_2 \to N_1 \to N_0$$

of connective A-modules. Then the canonical map

$$\rho_M : M \otimes_A (\varprojlim N_k) \to \varprojlim (M \otimes_A N_k)$$

is an equivalence.

Proof. Since each N_k is connective, the limit $\varprojlim N_k$ is (-1)-connective. It follows that if M is n-connective, then the domain of ρ_M is $(n-1)$-connective. Similarly, if M is n-connective, then each tensor product $M \otimes_A N_k$ is also n-connective, so that the codomain $\varprojlim(M \otimes_A N_k)$ is $(n-1)$-connective. It follows that $\mathrm{cofib}(\rho_M)$ is $(n-1)$-connective.

For any integer n, the assumption that n is almost perfect guarantees that we can choose a fiber sequence $M' \to M \to M''$ where M' is perfect and M'' is n-connective (Corollary 2.7.2.2). We then have a cofiber sequence

$$\mathrm{cofib}(\rho_{M'}) \to \mathrm{cofib}(\rho_M) \to \mathrm{cofib}(\rho_{M''})$$

where the first term vanishes (since $\rho_{M'}$ is an equivalence by virtue of our assumption that M' is perfect) and the third term is $(n-1)$-connective. It follows that $\mathrm{cofib}(\rho_M)$ is $(n-1)$-connective. Since n can be chosen arbitrarily, it follows that $\mathrm{cofib}(\rho_M) \simeq 0$: that is, ρ_M is an equivalence. □

Lemma 18.2.5.15. *In the situation of Construction 18.2.5.5, suppose that $\mathscr{F} = 0$, and let m be a positive integer. Then f fits into a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{f(m)} & Y(m) & \longrightarrow & \operatorname{Spec}(A(m)) \\ & \searrow^{f} & \downarrow & & \downarrow \\ & & Y & \longrightarrow & \operatorname{Spec}(A), \end{array}$$

where the right square is a pullback, the ring homomorphism $\pi_0 A \to \pi_0 A(m)$ is surjective, the $\mathbb{E}_\infty$-algebra $A(m)$ is almost perfect as an A-module, and the relative cotangent complex $L_{X/Y(m)}$ is m-connective.

Proof. We proceed by induction on m. In the case $m = 1$, we can take $A(m) = A$ and $Y(m) = Y$. To carry out the inductive step, let us suppose that $m \geqslant 1$ and that we have constructed a diagram

$$\begin{array}{ccccc} X & \xrightarrow{f(m)} & Y(m) & \longrightarrow & \operatorname{Spec}(A(m)) \\ & \searrow^{f} & \downarrow & & \downarrow \\ & & Y & \longrightarrow & \operatorname{Spec}(A), \end{array}$$

satisfying the requirements of Lemma 18.2.5.15; we will show that it is possible to construct a similar diagram with m replaced by $m+1$. Note that Y can be identified with the nilcompletion of $\widehat{U} \simeq \varinjlim \operatorname{Spec}(B_n)$ (Lemma 18.2.5.10), so that $Y(m)$ can be identified with the nilcompletion of the fiber product

$$\operatorname{Spec}(A(m)) \times_{\operatorname{Spec}(A)} \varinjlim_n \operatorname{Spec}(B_n) \simeq \varinjlim_n \operatorname{Spec}(A(m) \otimes_A B_n).$$

Invoking Lemma 18.2.5.14, we deduce that the canonical map

$$\begin{aligned} A(m) &\simeq A(m) \otimes_A \varprojlim_n B_n \\ &\to \varprojlim_n (A(m) \otimes_A B_n) \\ &\simeq \Gamma(Y(m); \mathscr{O}_{Y(m)}). \end{aligned}$$

is an equivalence.

Since the homomorphism $\pi_0 A \to \pi_0 A(m)$ is surjective, the map $\operatorname{Spec}(A(m))(R) \to \operatorname{Spec}(A)(R)$ is a monomorphism for any ordinary commutative ring R, so the map $Y(m)(R) \to Y(R)$ is (-1)-truncated. If R is a reduced commutative ring, then the composite map

$$X(R) \to Y(m)(R) \to Y(R)$$

is a homotopy equivalence, so the map $X(R) \to Y(m)(R)$ must also be a homotopy equivalence. Note that $Y(m)$ and X are both locally almost of finite presentation over Y, so that $f(m) : X \to Y(m)$ is locally almost of finite presentation (Remark 17.4.1.3). It follows that the map $f(m) : X \to Y(m)$ also satisfies the hypotheses of Theorem 18.2.5.1 (with $\mathscr{F} = 0$). We may therefore replace Y by $Y(m)$ (and A by $A(m)$) and thereby reduce to the case where $L_{X/Y}$ is m-connective.

Since $L_{X/Y}$ is almost perfect (Proposition 17.4.2.1). Let us view $L_{X/Y}$ as a B-module. Then $\pi_m L_{X/Y}$ is finitely generated as a module over the commutative ring $\pi_0 B$, and therefore also as a module over the commutative ring $\pi_0 A$. It follows from Lemma 18.2.5.13 that the canonical map $\pi_{m-1} \operatorname{fib}(A \to B) \to \pi_m L_{B/A} \to \pi_m L_{\operatorname{Spec}(B)/Y}$ is surjective. We can therefore choose a free A-module P of finite rank and a morphism $u : \Sigma^{m-1} P \to \operatorname{fib}(A \to B)$ for which the composite map

$$\rho : \Sigma^m P \xrightarrow{\Sigma(u)} \Sigma \operatorname{fib}(A \to B) \to L_{B/A} \to L_{\operatorname{Spec}(B)/Y}$$

is surjective on π_m. The map u determines a commutative diagram of connective $\mathbb{E}_\infty$-rings σ :

$$\begin{array}{ccc} \operatorname{Sym}^*_A(\Sigma^{m-1} P) & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & B. \end{array}$$

Let $A(m+1)$ denote the relative tensor product $A \otimes_{\operatorname{Sym}^*_A(\Sigma^{m-1} P)} A$. By construction, $A(m+1)$ is of finite presentation as an $\mathbb{E}_\infty$-algebra over A and the natural map $\pi_0 A \to \pi_0 A(m+1)$ is surjective on π_0. It follows from Corollary 5.2.2.2 that $A(m+1)$ is almost perfect as an A-module. Moreover, the commutative diagram σ determines a homomorphism of A-algebras $A(m+1) \to B$, which allows us to construct a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f(m)} & Y(m+1) & \longrightarrow & \operatorname{Spec}(A(m+1)) \\ & \searrow^{f} & \downarrow & & \downarrow \\ & & Y & \longrightarrow & \operatorname{Spec}(A) \end{array}$$

where the right square is a pullback. Unwinding the definitions, we see that the relative cotangent complex $L_{X/Y(m+1)}$ can be identified with the cofiber of the map ρ, and is therefore $(m+1)$-connective by construction. □

Remark 18.2.5.16. In the situation of Lemma 18.2.5.15, the fiber $\operatorname{fib}(A(m) \to B)$ is $(m-1)$-connected. To prove this, we can argue as in the proof of Lemma 18.2.5.15 to reduce to the case $Y = Y(m)$ and $A = A(m)$, in which case the desired result follows from Lemma 18.2.5.13.

Lemma 18.2.5.17. *In the situation of Construction 18.2.5.5, suppose that $\mathscr{F} = 0$. Then the map $u : A \to B$ is almost of finite presentation.*

Proof. It will suffice to show that B is of finite generation to order n over A, for each $n \geqslant 0$. Write u as a composition $A \xrightarrow{u'} A(n+2) \xrightarrow{u''} B$, where $A(n+2)$ is defined as in Lemma 18.2.5.15. Then the map u' is almost of finite presentation (Corollary 5.2.2.2) and u'' induces an equivalence $\tau_{\leqslant n} A(n+2) \to \tau_{\leqslant n} B$ (Remark 18.2.5.16), and is therefore of finite generation to order n. Applying Proposition 4.1.3.1, we deduce that u is of finite generation to order n. □

In what follows, we will regard A as an adic $\mathbb{E}_\infty$-ring, by equipping $\pi_0 A$ with the I-adic topology for $I = \ker(\pi_0 A \to \pi_0 B)$. Note that I is a finitely generated ideal (this follows from Lemma 18.2.5.17).

Lemma 18.2.5.18. *In the situation of Construction 18.2.5.5, suppose that $\mathscr{F} = 0$. Then the canonical map $Y \to \operatorname{Spec}(A)$ induces an equivalence $g : Y \simeq \operatorname{Spf}(A)$.*

Proof. We first show that the map $Y \to \operatorname{Spec}(A)$ factors through $\operatorname{Spf}(A)$. Let η be an R-valued point of Y, for some connective $\mathbb{E}_\infty$-ring R; we wish to show that the induced map

$$\eta^* : A = \Gamma(Y; \mathscr{O}_Y) \to \Gamma(\operatorname{Spec}(R); \mathscr{O}_{\operatorname{Spec}(R)}) \simeq R$$

carries I to a nilpotent ideal in $\pi_0 R$. Replacing R by $\pi_0 R$, we can assume that R is discrete. In this case, Lemma 18.2.5.10 supplies an equivalence $Y(R) \simeq U(R) \simeq \widehat{U}(R) \simeq \varinjlim \operatorname{Map}_{\operatorname{CAlg}}(B_n, R)$. It follows that the map $\eta^* : A \to R$ factors through B_n for some $n \geqslant 0$. The desired result now follows from the observation that $I(\pi_0 B_n)$ is a nilpotent ideal in $\pi_0 B_n$.

We next claim that the map $g : Y \to \operatorname{Spf}(A)$ is locally almost of finite presentation. To prove this, we observe that Lemma 18.2.5.10 guarantees that Y agrees with $\overline{U} = \varinjlim \operatorname{Spec}(B_n)$ on truncated objects of $\operatorname{CAlg}^{\operatorname{cn}}$. By virtue of Remark 17.4.1.4, it suffices to show that the map $\operatorname{Spec}(B_n) \to \operatorname{Spf}(A)$ is locally almost of finite presentation. This follows from Lemma 18.2.5.17, applied to the map $f(n) : \operatorname{Spec}(B_n) \to Y$. Note that if R is a reduced commutative ring, then g induces a homotopy equivalence $Y(R) \to \operatorname{Spf}(A)(R)$; this follows by applying the two-out-of-three property to the commutative diagram

$$\begin{array}{ccccc} & & Y(R) & & \\ & \overset{\sim}{\nearrow} & & \searrow & \\ X(R) & & \xrightarrow{\sim} & & \operatorname{Spf}(A)(R). \end{array}$$

Consequently, to show that g is an equivalence, it will suffice to show that the relative cotangent complex $L_{Y/\operatorname{Spf}(A)} \simeq L_{Y/\operatorname{Spec}(A)}$ vanishes (Lemma 18.2.5.3).

Let m be any positive integer, and choose a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f(m)} & Y(m) & \longrightarrow & \operatorname{Spec}(A(m)) \\ & \searrow^{f} & \downarrow & & \downarrow \\ & & Y & \longrightarrow & \operatorname{Spec}(A), \end{array}$$

satisfying the requirements of Lemma 18.2.5.15. We then have equivalences

$$\begin{aligned} L_{Y/\operatorname{Spec}(A)}|_X &\simeq L_{Y(m)/\operatorname{Spec}(A(m))}|_X \\ &\simeq \operatorname{fib}(L_{X/Y(m)} \to L_{B/A(m)}). \end{aligned}$$

Here $L_{X/Y(m)}$ is m-connective by construction, and $L_{B/A(m)}$ is m-connective by virtue of Remark 18.2.5.16. It follows that $L_{Y/\operatorname{Spec}(A)}|_X$ is $(m-1)$-connective. Allowing m to vary, we conclude that $L_{Y/\operatorname{Spec}(A)}|_X \simeq 0$. By virtue of Proposition 2.7.3.2, this guarantees that $L_{Y/\operatorname{Spec}(A)}|_{\operatorname{Spec}(B_n)}$ vanishes for each $n \geqslant 0$, so that $L_{Y/\operatorname{Spec}(A)}|_{\overline{U}} \simeq 0$. Proposition 17.3.3.5 then implies that $L_{Y/\operatorname{Spec}(A)}|_U \simeq 0$, so that $L_{Y/\operatorname{Spec}(A)}$ vanishes by virtue of Lemma 18.2.5.10. □

Lemma 18.2.5.19. *Let $f : X \to Y$ be a morphism satisfying the hypotheses of Theorem* **??** *with $\mathscr{F} = 0$. Then Y is (representable by) a formal thickening of $X = \operatorname{Spec}(B)$.*

Proof. Combine Lemma 18.2.5.18 with Proposition 18.2.2.7. □

Proof of Theorem **??**. Let $f : \operatorname{Spec}(B) = X \to Y$ and $\alpha : \mathscr{F} \to L_{X/Y}$ be as in the statement of Theorem **??**. Applying Construction 18.2.5.5, we obtain a factorization of f as a composition $X \xrightarrow{f'} U \xrightarrow{f''} Y$. It follows from Lemma 18.2.5.6 that the morphism $f' : X \to U$ also satisfies the hypotheses of Theorem **??** (with the sheaf $\mathscr{F}$ replaced by zero). Applying Lemma 18.2.5.19, we conclude that U is (representable by) a formal thickening of X. We conclude by observing that $f'' : U \to Y$ is locally almost of finite presentation (Lemma 18.2.5.9) and that α can be identified with the map $L_{U/Y}|_X \to L_{X/Y}$ (Lemma 18.2.5.6). □

18.3 Artin's Representability Theorem

Let R be an excellent Noetherian ring and let X be a functor from the category of commutative R-algebras to the category of sets. In [4], Artin supplied necessary and sufficient conditions for X to be representable by an algebraic space which is locally of finite presentation over R. Our goal in this section is to prove the following analogue of Artin's result:

Theorem 18.3.0.1 (Spectral Artin Representability Criterion)**.** *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and suppose we are given a natural transformation $f : X \to \operatorname{Spec} R$, where R is a Noetherian $\mathbb{E}_\infty$-ring and $\pi_0 R$ is a Grothendieck ring. Let $n \geqslant 0$. Then X is representable by a spectral Deligne-Mumford n-stack which is locally almost of finite presentation over R if and only if the following conditions are satisfied:*

(1) *For every discrete commutative ring A, the space $X(A)$ is n-truncated.*

(2) *The functor X is a sheaf for the étale topology.*

(3) *The functor X is nilcomplete, infinitesimally cohesive, and integrable.*

(4) *The functor X admits a connective cotangent complex L_X.*

(5) *The natural transformation f is locally almost of finite presentation.*

18.3.1 Approximate Charts

Let $X \to \operatorname{Spec} R$ be a natural transformation satisfying the hypotheses of Theorem 18.3.0.1. We wish to show that X is representable by a spectral Deligne-Mumford stack: in other words, that there is a good supply of étale morphisms $u : \operatorname{Spec} A \to X$, where A is a connective $\mathbb{E}_\infty$-ring. We begin by looking for maps u which are "approximately" étale at some point of $|\operatorname{Spec} A|$.

Proposition 18.3.1.1. *Let R be an $\mathbb{E}_\infty$-ring, let $Y = \operatorname{Spec} R$, and suppose we are given a natural transformation $q : X \to Y$ of functors $X, Y \in \operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ satisfying the following conditions:*

(1) *The functor X is infinitesimally cohesive, nilcomplete, and integrable.*

(2) *The $\mathbb{E}_\infty$-ring R is Noetherian and $\pi_0 R$ is a Grothendieck commutative ring.*

(3) *The natural transformation q is locally almost of finite presentation.*

(4) *The map q admits a cotangent complex $L_{X/Y}$.*

Suppose we are given a field κ and a map $f : \operatorname{Spec} \kappa \to X$ which exhibits κ as a finitely generated field extension of some residue field of R. Then the map f factors as a composition $\operatorname{Spec} \kappa \to \operatorname{Spec} B \to X$, where B is almost of finite presentation over R and the vector space $\pi_1(\kappa \otimes_B L_{\operatorname{Spec} B/X})$ vanishes.

Remark 18.3.1.2. Proposition 18.3.1.1 does not require any connectivity assumption on the relative cotangent complex $L_{X/Y}$. Consequently, it can be used to prove a generalization of Theorem 18.3.0.1 to the setting of Artin stacks. We will return to this point in Part **??**.

Proof of Proposition 18.3.1.1. Since κ is a finitely generated field extension of some residue field of R, the relative cotangent complex $L_{\kappa/R}$ is an almost perfect κ-module. Since $L_{X/\operatorname{Spec} R}$ is almost perfect (Corollary 17.4.2.2), the fiber sequence $f^* L_{X/\operatorname{Spec} R} \to L_{\kappa/R} \to L_{\operatorname{Spec} \kappa/X}$ shows that the relative cotangent complex $L_{\operatorname{Spec}(\kappa)/X}$ is almost perfect.

Let $\widehat{X}$ denote the relative de Rham space of the morphism $\operatorname{Spec}(\kappa) \to X$, given by the formula $\widehat{X}(A) = X(A) \times_{X(A^{\mathrm{red}})} \operatorname{Spec}(\kappa)(A^{\mathrm{red}})$. Then we can identify the relative cotangent complex $L_{\operatorname{Spec}(\kappa)/X}$ with $L_{\operatorname{Spec}(\kappa)/\hat{X}}$ (see Corollary 18.2.1.11). Choose a fiber sequence

$$\mathscr{F} \xrightarrow{\alpha} L_{\operatorname{Spec}(\kappa)/\hat{X}} \to \mathscr{G}$$

in the ∞-category $\operatorname{QCoh}(\operatorname{Spec}(\kappa)) \simeq \operatorname{Mod}_\kappa$, where $\mathscr{F}$ is perfect of Tor-amplitude $\leqslant 0$ and $\mathscr{G}$ is 1-connective. Applying Theorem 18.2.5.1, we deduce that the canonical map $\operatorname{Spec}(\kappa) \to \widehat{X}$ factors as a composition $\operatorname{Spec}(\kappa) \to U \to \widehat{X}$, where U is representable by a formal spectral Deligne-Mumford stack $\operatorname{Spf}(A)$ which is a formal thickening of $\operatorname{Sp\acute{e}t}(\kappa)$, and α can be identified with the canonical map $L_{U/\hat{X}}|_{\operatorname{Spec}(\kappa)} \to L_{\operatorname{Spec}(\kappa)/\hat{X}}$. Here A denotes some connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 A$. Using Proposition 18.2.4.1, we deduce that A is a Noetherian local $\mathbb{E}_\infty$-ring with residue field κ. Since X is integrable and nilcomplete, the composite map $U \to \widehat{X} \to X$ induces a map $\overline{f} : \operatorname{Spec}(A) \to X$ which fits into a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(\kappa) & \longrightarrow & \operatorname{Spec}(A) \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle \overline{f}} & \\ & X. & \end{array}$$

By construction, we have $\pi_n(\kappa \otimes_A L_{\operatorname{Spec}(A)/X}) \simeq 0$ for $n > 0$.

Let $A' = \pi_0 A$. The canonical map $A \to A'$ is 1-connective, so that $L_{A'/A}$ is 2-connective (Corollary HA.7.4.3.2). Using the fiber sequence

$$\kappa \otimes_A L_{\operatorname{Spec}(A)/X} \to \kappa \otimes_{A'} L_{\operatorname{Spec}(A')/X} \to \kappa \otimes_{A'} L_{A'/A},$$

we deduce that $\pi_1(\kappa \otimes_{A'} L_{\operatorname{Spec}(A')/X}) \simeq 0$.

Let $\mathfrak{m}$ denote the maximal ideal of A', so that $A'/\mathfrak{m} \simeq \kappa$. Let A'' be a subalgebra of A' which is finitely generated over $\pi_0 R$ with the following properties:

- The subalgebra A'' contains generators of the field κ over $\pi_0 R$: that is, κ is the fraction field of the integral domain $A''/(A'' \cap \mathfrak{m})$.
- The subalgebra A'' contains a basis for the κ-vector space $\mathfrak{m}/\mathfrak{m}^2$.

Let $\mathfrak{p}$ denote the intersection $A'' \cap \mathfrak{m}$, and let $\widehat{A}''$ denote the completion of the Noetherian local ring $A''_{\mathfrak{p}}$. The conditions above guarantee that the map $v : A''_{\mathfrak{p}} \to A'$ induces an isomorphism of residue fields and a surjection of Zariski cotangent spaces. Since A' is complete with respect to its maximal ideal, v extends to a surjective map $\widehat{v} : \widehat{A}'' \to A'$. In particular, as an $\mathbb{E}_\infty$-algebra over $\widehat{A}''$, A' is almost of finite presentation.

Since A'' is finitely generated over $\pi_0 R$, it is a Grothendieck ring (Theorem **??**). It follows that the map $A''_{\mathfrak{p}} \to \widehat{A}''$ is geometrically regular. Applying Theorem **??**, we deduce that $\widehat{A}''$ can be written as a colimit $\varinjlim_{\alpha \in P} A''_\alpha$ indexed by a filtered partially ordered set P, where each A''_α is a smooth A''-algebra (in the sense of classical commutative algebra). Using Theorem 4.4.2.2 (and Proposition 4.6.1.1), we see that there exists an index α and an equivalence $A' \simeq \tau_{\leqslant 1}(C_\alpha \otimes_{A''_\alpha} \widehat{A}'')$, where C_α is an $\mathbb{E}_\infty$-ring which is finitely 1-presented over A''_α. For $\beta \geqslant \alpha$, let $C_\beta = \tau_{\leqslant 1}(C_\alpha \otimes_{A''_\alpha} A''_\beta)$, so that

$$\varinjlim_{\beta \geqslant \alpha} C_\beta \simeq \tau_{\leqslant 1}(C_\alpha \otimes_{A''_\alpha} \varinjlim_{\beta \geqslant \alpha} A''_\beta) \simeq \tau_{\leqslant 1}(C_\alpha \otimes_{A''_\alpha} \widehat{A}'') \simeq A'.$$

Since the map $X \to \operatorname{Spec}(R)$ is locally almost of finite presentation, the map $\operatorname{Spec}(A') \to X$ factors as a composition $\operatorname{Spec}(A') \to \operatorname{Spec}(A''_\beta) \xrightarrow{u'} X$ for some $\beta \geqslant \alpha$. Set $B = C_\beta$. We will complete the proof by showing that the relative cotangent complex of u' satisfies $\pi_1(\kappa \otimes_B L_{\operatorname{Spec}(B)/X}) \simeq 0$.

We have an exact sequence

$$\pi_2(\kappa \otimes_{A'} L_{A'/B}) \to \pi_1(\kappa \otimes_B L_{\operatorname{Spec}(B)/X}) \to \pi_1(\kappa \otimes_{A'} L_{\operatorname{Spec}(A')/X}).$$

Since the third term vanishes, it will suffice to show that $\pi_2(\kappa \otimes_{A'} L_{A'/B})$ vanishes. Let B' denote the tensor product $B \otimes_{A''_\beta} \widehat{A}''$, so that $A' \simeq \tau_{\leqslant 1} B'$. We have a short exact sequence

$$\pi_2(\kappa \otimes_{B'} L_{B'/B}) \to \pi_2(\kappa \otimes_{A'} L_{A'/B}) \to \pi_2(\kappa \otimes_{B'} L_{A'/B'}).$$

Since the third term vanishes by virtue of Corollary **??**, we are reduced to proving that the homotopy group $\pi_2(\kappa \otimes_{B'} L_{B'/B}) \simeq \pi_2(\kappa \otimes_{\widehat{A}''} L_{\widehat{A}''/A''_\beta})$ vanishes. For this, we use the exact sequence

$$\pi_2(\kappa \otimes_{\widehat{A}''} L_{\widehat{A}''/A''}) \to \pi_2(\kappa \otimes_{\widehat{A}''} L_{\widehat{A}''/A''_\beta}) \to \pi_1(\kappa \otimes_{A''_\beta} L_{A''_\beta/A''}).$$

Note that the tensor product $\kappa' = \widehat{A}'' \otimes_{A''} \kappa$ is equivalent to κ, so that $\kappa \otimes_{\widehat{A}''} L_{\widehat{A}''/A''} \simeq \kappa \otimes_{\kappa'} L_{\kappa'/\kappa} \simeq 0$. It will therefore suffice to show that the homotopy group $\pi_1(\kappa \otimes_{A''_\beta} L_{A''_\beta/A''})$ vanishes. Let $D = A''_\beta \otimes_{A''} \kappa$. Then D is a commutative algebra over κ which is smooth (in the sense of classical commutative algebra) and equipped with an augmentation $D \to \kappa$. We have a canonical isomorphism $\pi_1(\kappa \otimes_{A''_\beta} L_{A''_\beta/A''}) \simeq \pi_1(\kappa \otimes_D L_{D/\kappa})$. Note that the dual of the vector space $\pi_1(\kappa \otimes_D L_{D/\kappa})$ is the set of homotopy classes of D-module maps from $L_{D/\kappa}$ to $\Sigma\kappa$. This set classifies the collection of all isomorphism classes of square-zero extensions $0 \to I \to \widetilde{D} \to D \to 0$ (in the category of commutative algebras over κ) equipped with an isomorphism of D-modules $I \simeq \kappa$. Since D is smooth over κ, every such extension is automatically split. □

18.3.2 Refinement of Approximate Charts

Our next goal is to show that in the situation of Proposition 18.3.2.1, we can modify the map $\operatorname{Spec} B \to X$ to obtain a map which is formally étale, in the sense that the relative cotangent complex $L_{\operatorname{Spec} B/X}$ vanishes.

Proposition 18.3.2.1. *Let R be an $\mathbb{E}_\infty$-ring, let $Y = \operatorname{Spec} R$, and suppose we are given a natural transformation $q : X \to Y$ of functors $X, Y \in \operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Assume that the following conditions are satisfied:*

(1) *The functor X is infinitesimally cohesive, nilcomplete, and integrable.*

(2) *The $\mathbb{E}_\infty$-ring R is Noetherian and $\pi_0 R$ is a Grothendieck commutative ring.*

(3) *The natural transformation q is locally almost of finite presentation.*

(4) *The map q admits a connective cotangent complex $L_{X/Y}$.*

Suppose we are given an a connective $\mathbb{E}_\infty$-ring A and a map $f : \operatorname{Spec} A \to X$. Let $\mathfrak{p}$ be a prime ideal of $|\operatorname{Spec} A|$, and let $\kappa(\mathfrak{p})$ denote the residue field of A at the point p. Then there exists an étale A-algebra A' and a prime ideal $\mathfrak{p}'$ of A' lying over $\mathfrak{p}$ such that the induced map $\operatorname{Spec} A' \to X$ factors as a composition $\operatorname{Spec} A' \to \operatorname{Spec} B \to X$, where B is almost of finite presentation over R, and the relative cotangent complex $L_{\operatorname{Spec} B/X}$ vanishes.

Proof. Assume that $q : X \to Y = \operatorname{Spec} R$ satisfies hypotheses (1) through (4). Let $f : \operatorname{Spec} A \to X$ be an arbitrary map which exhibits A as almost of finite presentation over R, and let $\mathfrak{p}$ be a prime ideal of $\pi_0 A$. We wish to show that, after passing to an étale neighborhood of $\mathfrak{p}$, the map f factors as a composition $\operatorname{Spec} A \to \operatorname{Spec} B \xrightarrow{g} X$, where g exhibits B as almost of finite presentation over R and $L_{\operatorname{Spec} B/X} \simeq 0$. Note that, if these conditions are satisfied, then g induces a homotopy equivalence

$$(\operatorname{Spec} B)(A) \to (\operatorname{Spec} B)(\pi_0 A) \times_{X(\pi_0 A)} X(A).$$

We may therefore replace A by $\pi_0 A$, and thereby reduce to the case where A is discrete.

Write A as a filtered colimit of subalgebras A_α which are finitely generated as commutative rings over $\pi_0 R$. Since $X \to \operatorname{Spec} R$ is locally almost of finite presentation, the map f factors through $\operatorname{Spec} A_\alpha$ for some α. Replacing A by A_α and $\mathfrak{p}$ by $A_\alpha \cap \mathfrak{p}$, we may reduce to the case where A is finitely generated as a commutative ring over R.

Let κ denote the residue field of A at $\mathfrak{p}$, and consider the composite map $f_\kappa : \operatorname{Spec} \kappa \to \operatorname{Spec} A \xrightarrow{f} X$. Applying Proposition 18.3.1.1, we see that f_κ factors as a composition $\operatorname{Spec} \kappa \xrightarrow{j} \operatorname{Spec} B \xrightarrow{g} X$, where B is almost of finite presentation over R and the vector space $\pi_1(\kappa \otimes_B L_{\operatorname{Spec} B/X})$ vanishes. Since the relative cotangent complex $L_{X/Y}$ is connective, the fiber sequence $g^* L_{X/Y} \to L_{B/R} \to L_{\operatorname{Spec} B/X}$ shows that the map $\pi_0 L_{B/R} \to \pi_0 L_{\operatorname{Spec} B/X}$ is surjective. Since $\pi_0 L_{\operatorname{Spec} B/X}$ is a finitely generated module over $\pi_0 B$ (Corollary 17.4.2.2), we can find a finite sequence of elements $b_1, \ldots, b_m \in \pi_0 B$ such that the images of $db_1, db_2, \ldots, db_m$ in $\pi_0 L_{\operatorname{Spec} B/X}$ form a basis for the vector space $\pi_0(\kappa \otimes_B L_{\operatorname{Spec} B/X})$. The choice of these elements determines a map of B-modules $\psi : B^m \to L_{\operatorname{Spec} B/X}$. The κ-module $\kappa \otimes_B \operatorname{cofib}(\psi)$ is 2-connective. Replacing B by a localization if necessary, we may suppose that $\operatorname{cofib}(\psi)$ is 2-connective.

The map $\operatorname{Spec} B \to X$ is locally almost of finite presentation (Remark 17.4.1.3). Set $X' = \operatorname{Spec} \kappa \times_X \operatorname{Spec} B$, so that the projection map $q' : X' \to \operatorname{Spec} \kappa$ satisfies hypotheses (1), (2), and (3). The map f_κ determines a section s of q'. Applying Proposition 18.3.1.1 again, we deduce that s factors as a composition $\operatorname{Spec} \kappa \xrightarrow{\nu} \operatorname{Spec} C \xrightarrow{g'} X'$ where C is almost of finite presentation over κ and $\pi_1(\kappa \otimes_C L_{\operatorname{Spec} C/X'}) \simeq 0$. Using the fiber sequence $g'^* L_{X'/\kappa} \to L_{C/\kappa} \to L_{\operatorname{Spec} C/X'}$, we deduce that $\pi_1(\kappa \otimes L_{C/\kappa}) \simeq 0$. It follows that the ordinary scheme $\operatorname{Spec} \pi_0 C$ is smooth over κ at the point determined by ν. Replacing C by a localization if necessary, we may suppose that the ordinary scheme $\operatorname{Spec} \pi_0 C$ is smooth over κ.

For $1 \leqslant i \leqslant m$, let c_i denote the image of b_i in the commutative ring $\pi_0 C$. Since $\pi_1(\kappa \otimes_C L_{\operatorname{Spec} C/X'}) = 0$, the map $\pi_0(\kappa \otimes_B L_{\operatorname{Spec} B/X}) \to \pi_0(\kappa \otimes_C L_{\operatorname{Spec} C/\kappa})$ is injective: in other words, as functions on the affine scheme $\operatorname{Spec}(\pi_0 C)$, the c_i have linearly independent derivatives and therefore induce a smooth map of ordinary schemes $h : \operatorname{Spec} \pi_0 C \to \mathbf{A}^m_\kappa$. The image of h is a nonempty open subscheme U of the affine space $\mathbf{A}^m_\kappa$. Let $\kappa_0 \subseteq \kappa$ denote the prime field of κ. Then there exists a finite Galois extension κ'_0 of κ_0 such that U contains a point u which is rational over κ'_0. Let κ' be a separable extension of κ containing κ'_0, so that u defines a map of schemes $\operatorname{Spec} \kappa' \to U$. Since h defines a smooth surjection $\operatorname{Spec} \pi_0 C \to U$, we may (after enlarging κ' if necessary) assume that u factors as a composition $\operatorname{Spec} \kappa' \to \operatorname{Spec} \pi_0 C \to U$. This determines a new map $j' : \operatorname{Spec} \kappa' \to \operatorname{Spec} B$, whose composition with g agrees with the composition

$$\operatorname{Spec} \kappa' \to \operatorname{Spec} \kappa \to \operatorname{Spec} A \xrightarrow{f} X.$$

Since κ' is a finite separable extension of κ, we can write $\kappa' = \kappa[x]/(r(x))$ for some separable polynomial r. After localizing A, we can assume that r lifts to a separable polynomial $\overline{r}(x) \in (\pi_0 A)[x]$. Then $(\pi_0 A)[x]/(\overline{r}(x))$ is a finite étale extension of $\pi_0 A$. Using Theorem HA.7.5.0.6 , we can write $(\pi_0 A)[x]/(\overline{r}(x)) \simeq \pi_0 A'$, where A' is a finite étale A-algebra. Replacing A by A', κ by κ', and j by j', we can reduce to the case where j is given by a ring homomorphism $\pi_0 B \to \kappa$ which carries each b_i to an element $\lambda_i \in \kappa$ belongs to a subfield $\kappa'_0 \subseteq \kappa$ which is algebraic over the prime field κ_0.

Choose an integer N and a finite étale $\mathbf{Z}[N^{-1}]$-algebra D such that $\kappa'_0 \simeq \kappa_0 \otimes_{\mathbf{Z}} D$. Enlarging N if necessary, we may suppose that each λ_i can be lifted to an element $\overline{\lambda}_i \in D$. Replacing B by $B[N^{-1}]$ if necessary, we may suppose that N is invertible in $\pi_0 B$, so that

$(\pi_0 B) \otimes_{\mathbf{Z}[N^{-1}]} D$ is a finite étale extension of $\pi_0 B$. Applying Theorem HA.7.5.0.6 , we can write $(\pi_0 B) \otimes_{\mathbf{Z}[N^{-1}]} D \simeq \pi_0 \overline{B}$ for some finite étale extension $\overline{B}$ of B. Moreover, the embedding $\kappa'_0 \hookrightarrow \kappa'$ induces a map $\overline{B} \to \kappa$, which annihilates the elements $b_i - \overline{\lambda}_i \in \pi_0 \overline{B}$. Replacing B by $\overline{B}$ and the elements $b_i \in B$ by the differences $b_i - \overline{\lambda_i}$ (note that this does not change the differentials db_i), we may reduce to the case where the map $j : \operatorname{Spec} \kappa \to \operatorname{Spec} B$ annihilates each b_i.

Let $B_0 = B \otimes_{S\{b_1,\ldots,b_m\}} S$ denote the $\mathbb{E}_\infty$-algebra over B obtained by killing each b_i. We then have a fiber sequence $B_0 \otimes_B L_{\operatorname{Spec} B/X} \to L_{\operatorname{Spec} B_0/X} \to L_{B_0/B}$. We note that $L_{B_0/B} \simeq \Sigma(B_0^m)$, and the boundary map $\Omega L_{B_0/B} \to B_0 \otimes_B L_{\operatorname{Spec} B/X}$ is induced by the map $\psi : B^m \to L_{\operatorname{Spec} B/X}$ given by the elements b_i. Since $\operatorname{cofib}(\psi)$ is 2-connective, we deduce that $L_{\operatorname{Spec} B_0/X}$ is 2-connective. The map $j : \operatorname{Spec} \kappa \to \operatorname{Spec} B$ annihilates each b_i, and therefore factors through $\operatorname{Spec} B_0$. We may therefore replace B by B_0, and thereby reduce to the case where $L_{\operatorname{Spec} B/X}$ is 2-connective.

Proposition 18.1.2.1 implies that the map $g : \operatorname{Spec} B \to X$ factors as a composition $\operatorname{Spec} B \to \operatorname{Spec} \overline{B} \to X$, where $\pi_0 \overline{B} \simeq \pi_0 B$ and $L_{\overline{B}/X} \simeq 0$. It follows from Corollary 17.4.2.2 that $L_{B/\overline{B}} \simeq L_{\operatorname{Spec} B/X}$ is an almost perfect module over B. Since B is almost of finite presentation over R, $L_{B/R}$ is almost perfect. Using the fiber sequence $B \otimes_{\overline{B}} L_{\overline{B}/R} \to L_{B/R} \to L_{B/\overline{B}}$, we deduce that $B \otimes_{\overline{B}} L_{\overline{B}/R}$ is almost perfect as a B-module. It follows from Proposition 2.7.3.2 that $L_{\overline{B}/R}$ is almost perfect as an $\overline{B}$-module. Since $\pi_0 \overline{B} \simeq \pi_0 B$ is finitely presented as an commutative ring over $\pi_0 R$, Theorem HA.7.4.3.18 implies $\overline{B}$ is almost of finite presentation over R. We may therefore replace B by $\overline{B}$, and thereby reduce to the case where $L_{\operatorname{Spec} B/X} \simeq 0$.

We will now complete the proof by showing that there exists a finite étale A-algebra A' with $\kappa \otimes_A A' \neq 0$, such that the induced map $\operatorname{Spec} A' \to X$ factors through $\operatorname{Spec} B$.

Let $\widehat{A}$ denote the completion of the local ring $A_{\mathfrak{p}}$ at its maximal ideal, and let $\mathfrak{m}$ denote the maximal ideal of $\widehat{A}$, so that $\widehat{A}/\mathfrak{m} \simeq \kappa$. Each quotient $\widehat{A}/\mathfrak{m}^{n+1}$ is a square-zero extension of $\widehat{A}/\mathfrak{m}^n$. Since g is infinitesimally cohesive with $L_{\operatorname{Spec} B/X} \simeq 0$, it follows that each of the diagrams

$$\begin{array}{ccc} (\operatorname{Spec} B)(\widehat{A}/\mathfrak{m}^{n+1}) & \longrightarrow & (\operatorname{Spec} B)(\widehat{A}/\mathfrak{m}) \\ \downarrow & & \downarrow \\ X(\widehat{A}/\mathfrak{m}^{n+1}) & \longrightarrow & X(\widehat{A}/\mathfrak{m}^n). \end{array}$$

is a pullback square. Consequently, the diagram

$$\begin{array}{ccc} \varprojlim (\operatorname{Spec} B)(\widehat{A}/\mathfrak{m}^{n}) & \longrightarrow & (\operatorname{Spec} B)(\kappa) \\ \downarrow & & \downarrow \\ \varprojlim X(\widehat{A}/\mathfrak{m}^{n}) & \longrightarrow & X(\kappa) \end{array}$$

is also a pullback square. Since both X and $\operatorname{Spec} B$ are integrable, we obtain a pullback square

$$\begin{array}{ccc} (\operatorname{Spec} B)(\widehat{A}) & \longrightarrow & (\operatorname{Spec} B)(\kappa) \\ \downarrow & & \downarrow \\ X(\widehat{A}) & \longrightarrow & X(\kappa). \end{array}$$

It follows that the map $j : \operatorname{Spec} \kappa \to \operatorname{Spec} B$ admits an essentially unique factorization as a composition $\operatorname{Spec} \kappa \to \operatorname{Spec} \widehat{A} \xrightarrow{\widehat{j}} \operatorname{Spec} B$, where $\widehat{j}$ fits into a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} \widehat{A} & \xrightarrow{\widehat{j}} & \operatorname{Spec} B \\ \downarrow & & \downarrow g \\ \operatorname{Spec} A & \xrightarrow{f} & X. \end{array}$$

By assumption, $\pi_0 R$ is a Grothendieck ring. Since A is finitely generated as an algebra over $\pi_0 R$, it is also a Grothendieck ring (Theorem **??**), so that the map $A_{\mathfrak{p}} \to \widehat{A}$ is geometrically regular. Applying Theorem **??**, we can write $\widehat{A}$ as a filtered colimit $\varinjlim A_\alpha$, where each A_α is smooth over $A_{\mathfrak{p}}$ (in the sense of classical commutative algebra). Since B is almost of finite presentation over R and the natural transformation $X \to \operatorname{Spec} R$ is locally almost of finite presentation, the natural transformation g is locally almost of finite presentation. It follows that for some index α, there exists a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A_\alpha & \longrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow g \\ \operatorname{Spec} A & \longrightarrow & X. \end{array}$$

Since A_α is smooth over $A_{\mathfrak{p}}$, we can choose a smooth A-algebra $\overline{A}$ over A such that $A_\alpha \simeq \overline{A} \otimes_A A_{\mathfrak{p}}$. Then A_α can be written as a filtered colimit of A-algebras of the form $\overline{A}[a^{-1}]$, where $a \in A - \mathfrak{p}$. Using the fact that g is locally almost of finite presentation again, we obtain a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} \overline{A}[a^{-1}] & \longrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow g \\ \operatorname{Spec} A & \longrightarrow & X. \end{array}$$

for some $a \in A - \mathfrak{p}$. Note that $v : \operatorname{Spec} \overline{A}[a^{-1}] \to \operatorname{Spec} A$ is a smooth map of affine schemes whose image contains the prime ideal $\mathfrak{p}$. It follows that there exists an étale A-algebra A' such that $\kappa \otimes_A A' \neq 0$, and the map $\operatorname{Spec} A' \to \operatorname{Spec} A$ factors through v. Then the map $\operatorname{Spec} A' \to \operatorname{Spec} A \xrightarrow{f} X$ factors through the map $g : \operatorname{Spec} B \to X$, as desired. $\square$

18.3.3 The Proof of Artin Representability

We now turn to the proof of our main result. Let R be a connective $\mathbb{E}_\infty$-ring and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor equipped with a natural transformation $f : X \to \operatorname{Spec} R$. Suppose that X is representable by a spectral Deligne-Mumford n-stack X which is locally almost of finite presentation over R. In this case, it is easy to verify that the hypotheses of Theorem 18.3.0.1 are satisfied:

(1) For every discrete commutative ring A, the space $X(A)$ is n-truncated: this follows from our assumption that X is a spectral Deligne-Mumford n-stack (Definition 1.6.8.1).

(2) The functor X is a sheaf for the étale topology: this also follows immediately from the representability of X.

(3) The functor X is nilcomplete, infinitesimally cohesive, and integrable: this follows from Proposition 17.3.2.3, Example 17.3.1.2, and Proposition 17.3.4.2.

(4) The natural transformation f admits a connective cotangent complex $L_{X/\operatorname{Spec} R}$: see Proposition 17.2.5.1.

(5) The natural transformation f is locally almost of finite presentation: this follows from Proposition 17.4.3.1.

Now suppose that R is Noetherian, that $\pi_0 R$ is a Grothendieck ring, and that X satisfies conditions (1) through (5). We wish to prove that X is representable by a spectral Deligne-Mumford stack X. Then X is automatically a spectral Deligne-Mumford n-stack (by virtue of assumption (1)) which is locally almost of finite presentation over R (by condition (5) and Proposition 17.4.3.1). To prove the existence of X, we first note that hypothesis (1) can be restated as follows:

($1'_n$) For every discrete commutative ring A, the map $X(A) \to \mathrm{Map}_{\mathrm{CAlg}}(R, A)$ has n-truncated homotopy fibers.

Note that condition ($1'$) makes sense for all $n \geqslant -2$. We will show that for all $n \geqslant -2$, if $f : X \to \operatorname{Spec} R$ satisfies conditions ($1'_n$), (2), (3), (4), and (5), then X is representable by a spectral Deligne-Mumford stack X.

Our proof now proceeds by induction on n. We begin by treating the case $n = -2$. In this case, condition ($1'$) asserts that the map $X(A) \to \mathrm{Map}_{\mathrm{CAlg}}(R, A)$ is a homotopy equivalence for every discrete commutative ring A. In this case, the existence of X follows from Theorem 18.1.0.2.

Now suppose that $n \geqslant -1$. Let $\mathcal{S}\mathrm{hv}_{\text{ét}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ spanned by those functors which are sheaves for the étale topology. Let S be a set of representatives for all equivalences classes of maps $\operatorname{Spec} B_\alpha \to X$ for which $L_{\operatorname{Spec} B_\alpha / X} = 0$ and exhibit B_α as almost of finite presentation over R, and let X_0 denote the coproduct $\coprod_{\alpha \in S} \operatorname{Spec} B_\alpha$ formed in the ∞-category $\mathcal{S}\mathrm{hv}_{\text{ét}}$, and let $X_\bullet$ denote the simplicial object of $\mathcal{S}\mathrm{hv}_{\text{ét}}$ given by the Čech nerve of the map $X_0 \to X$. Note that each X_m is given as a coproduct (in the ∞-category $\mathcal{S}\mathrm{hv}_{\text{ét}}$)

$$\coprod_{(\alpha_1, \ldots, \alpha_m) \in S^m} X_{\alpha_1, \ldots, \alpha_m},$$

$$X_{\alpha_1, \ldots, \alpha_m} = (\operatorname{Spec} B_{\alpha_1}) \times_X \cdots \times_X (\operatorname{Spec} B_{\alpha_m}),$$

and therefore admits a map

$$X_{\alpha_1, \ldots, \alpha_m} \to \operatorname{Spec}(B_{\alpha_1} \otimes_R \cdots \otimes_R B_{\alpha_m})$$

satisfying condition $(1'_{n-1})$. Applying the inductive hypothesis, we deduce that each $X_{\alpha_1, \ldots, \alpha_m}$ is representable by a spectral Deligne-Mumford stack, so that each X_m is representable by a spectral Deligne-Mumford stack X_m. Proposition 17.4.3.1 implies that each X_m is locally almost of finite presentation over R, so that each of the transition maps $\mathsf{X}_m \to \mathsf{X}_{m'}$ is locally almost of finite presentation. By construction, we have $L_{X_0/X} \simeq 0$, which implies that each transition map $\mathsf{X}_m \to \mathsf{X}_{m'}$ has vanishing cotangent complex. Applying Proposition 17.1.5.1, we deduce that each of the maps $\mathsf{X}_m \to \mathsf{X}_{m'}$ is étale. It follows that the simplicial object $\mathsf{X}_\bullet$ admits a geometric realization $|\mathsf{X}_\bullet|$ in the ∞-category of spectral Deligne-Mumford stacks (Proposition **??**). Lemma **??** implies that $|\mathsf{X}_\bullet|$ represents the functor $|X_\bullet|$, where the geometric realization is formed in the ∞-category $\mathcal{S}\mathrm{hv}_{\text{ét}}$. To complete the proof that X is representable, it will suffice to show that the canonical map $|X_\bullet| \to X$ is an equivalence. Since $\mathcal{S}\mathrm{hv}_{\text{ét}}$ is an ∞-topos, this is equivalent to the requirement that the map $X_0 \to X$ is an effective epimorphism of étale sheaves, which follows from Proposition 18.3.2.1.

Chapter 19

Applications of Artin Representability

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor from the ∞-category of connective $\mathbb{E}_\infty$-rings to the ∞-category of spaces. In Chapter 18, we established a version of Artin's representability theorem (Theorem 18.3.0.1), which supplies conditions which are sufficient (and, under mild assumptions, also necessary) for the representability of X by a spectral Deligne-Mumford stack X. In this chapter, we will sketch some applications of Theorem 18.3.0.1:

- Let $\mathsf{X} \xrightarrow{f} \mathsf{S} \xrightarrow{g} \mathsf{S}'$ be morphisms of spectral algebraic spaces. In §19.1, we define the *Weil restriction* $\mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X})$. Using Theorem 18.3.0.1, we show that if f is quasi-separated and locally almost of finite presentation and g is proper, flat, and locally almost of finite presentation, then the Weil restriction $\mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X})$ is representable by a spectral algebraic space (which is quasi-separated and locally almost of finite presentation over S'); see Theorem 19.1.0.1.

- Let $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ be a morphism of spectral algebraic spaces which is equipped with a section $s : \mathrm{Sp\acute{e}t}\, R \to \mathfrak{X}$. In §19.2, we define the relative Picard functor $\mathscr{P}\mathrm{ic}^s_{\mathsf{X}/R}$ (Definition 19.2.0.2), which classifies line bundles $\mathscr{L}$ on $\mathfrak{X}$ equipped with a trivialization of $s^* \mathscr{L}$. Assuming that f is proper, flat, locally almost of finite presentation, geometrically reduced, and geometrically connected, we show that $\mathscr{P}\mathrm{ic}^s_{\mathsf{X}/R}$ is representable by a spectrally algebraic space which is locally of finite presentation over R (Theorem 19.2.0.5). We also study the representability of several related moduli problems (parametrizing quasi-coherent sheaves of various flavors).

- Let X be a spectral Deligne-Mumford stack and let $\mathfrak{X}$ denote the formal completion of X along a cocompact closed subset $K \subseteq |\mathsf{X}|$ be a cocompact closed subset. In §19.3, we study the relationship between modifications of X along K (that is, proper morphisms $\mathsf{X}' \to \mathsf{X}$ which induce an equivalence over the complement of K) and *formal modifications* of the formal completion $\mathfrak{X}$ (Definition 19.3.1.3). Under mild assumptions, we show that every formal modification of $\mathfrak{X}$ arises from an (essentially unique) modification of X (Theorem 19.3.1.8).

- In §19.4 we introduce a functor Var^+ whose R-valued points are spectral algebraic spaces $\mathfrak{X}$ which are proper, flat, and locally almost of finite presentation over R. The functor Var^+ is not representable by a spectral Deligne-Mumford stack (for two essentially different reasons: algebraic varieties can admit continuous families of automorphisms, and formal algebraic spaces can fail to be algebraizable). Nevertheless, we show that the functor Var^+ has a well-behaved deformation theory (which we will exploit in Chapter **??** to prove the representability of several closely related functors.

Contents

19.1 Existence of Weil Restrictions

Let X be an affine scheme defined over the complex numbers, and let $X(\mathbf{C})$ denote the collection of $\mathbf{C}$-points of X. Then $X(\mathbf{C})$ can be described as the set of $\mathbf{R}$-points of an affine $\mathbf{R}$-scheme Y. For example, if X is given as a closed subscheme of affine space $\mathbf{A}^n$ defined by a collection of polynomial equations $f_\alpha(z_1, \ldots, z_n) = 0$ with complex coefficients, then Y can be described as the closed subvariety of $\mathbf{A}^{2n}$ defined by the real polynomial equations

$$\Re(f_\alpha(x_1 + iy_1, \ldots, x_n + iy_n)) = 0 = \Im(f_\alpha(x_1 + iy_1, \ldots, x_n + iy_n)).$$

Here $\Re(w)$ and $\Im(w)$ denote the real and imaginary parts of a complex number w, respectively. The scheme Y is called the *Weil restriction* of X along the morphism $\operatorname{Spec}\mathbf{C} \to \operatorname{Spec}\mathbf{R}$. It is characterized by the following universal property: for every $\mathbf{R}$-scheme Z, there is a canonical bijection

$$\operatorname{Hom}(Z, Y) \simeq \operatorname{Hom}(Z \times_{\operatorname{Spec}\mathbf{R}} \operatorname{Spec}\mathbf{C}, X),$$

where the Hom-set on the left is computed in the category $\mathbf{R}$-schemes and the Hom-set on the right is computed in the category of $\mathbf{C}$-schemes.

In this section, we will study the operation of Weil restriction in the setting of spectral algebraic geometry. Suppose that we are given a map of spectral Deligne-Mumford stacks $f : \mathsf{S} \to \mathsf{S}'$ together with a map $\mathsf{X} \to \mathsf{S}$. A *Weil restriction* of X along f is another spectral Deligne-Mumford stack Y equipped with a map $\mathsf{Y} \to \mathsf{S}'$ and a commutative diagram

$$\begin{array}{ccc} \mathsf{Y} \times_{\mathsf{S}''} \mathsf{S} & \xrightarrow{\rho} & \mathsf{X} \\ & \searrow \quad \swarrow & \\ & \mathsf{S} & \end{array}$$

satisfying the following universal property: for every map of spectral Deligne-Mumford stacks $\mathsf{Z} \to \mathsf{S}'$, composition with ρ induces a homotopy equivalence

$$\operatorname{Map}_{\mathrm{SpDM}_{/\mathsf{S}'}}(\mathsf{Z}, \mathsf{Y}) \to \operatorname{Map}_{\mathrm{SpDM}_{/\mathsf{S}}}(\mathsf{Z} \times_{\mathsf{S}'} \mathsf{S}, \mathsf{X}).$$

In this case, the spectral Deligne-Mumford stack Y is determined up to canonical equivalence, and will be denoted by $\operatorname{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X})$. Our goal in this section is to prove the following result:

Theorem 19.1.0.1. *Let $f : \mathsf{S} \to \mathsf{S}'$ be a morphism of spectral Deligne-Mumford stacks which is proper, flat, and locally almost of finite presentation. Let $\mathsf{X} \to \mathsf{S}$ be a relative spectral* *algebraic space which is quasi-separated and locally almost of finite presentation. Then there exists a Weil restriction* $\mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X})$. *Moreover, the canonical map* $\mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X}) \to \mathsf{S}'$ *is a relative spectral algebraic space which is quasi-separated and locally almost of finite presentation.*

19.1.1 The Quasi-Affine Case

We begin by proving Theorem 19.1.0.1 in the special case where the morphism $\mathsf{X} \to \mathsf{S}$ is quasi-affine:

Proposition 19.1.1.1. *Let* $\phi : \mathsf{S} \to \mathsf{S}'$ *be a morphism of spectral Deligne-Mumford stacks which is proper, flat, and locally almost of finite presentation, and let* $\rho : \mathsf{X} \to \mathsf{S}$ *be a quasi-affine morphism of spectral Deligne-Mumford stacks. Then there exists a Weil restriction* $\mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X})$. *Moreover, the canonical map* $\rho' : \mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X}) \to \mathsf{S}'$ *is quasi-affine. Moreover, if* ρ *is affine, then* ρ' *is affine.*

In the case where $\rho : \mathsf{X} \to \mathsf{S}$ is affine, Proposition 19.1.1.1 is an immediate consequence of the following:

Proposition 19.1.1.2. *Let* $\phi : \mathsf{S} \to \mathsf{S}'$ *be a morphism between quasi-compact, quasi-separated spectral algebraic spaces which is proper, flat, and locally almost of finite presentation. Then the pullback functor* $\phi^* : \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}')) \to \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}))$ *admits a left adjoint* $\phi_\dagger$. *Moreover, the functor* $\phi_\dagger$ *carries connective objects of* $\mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}))$ *to connective objects of* $\mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}'))$.

Proof. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}))$ spanned by those objects $\mathscr{A}$ for which the functor

$$\mathscr{B} \mapsto \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}))}(\mathscr{A}, \phi^* \mathscr{B})$$

is corepresentable by an object $\phi_\dagger(\mathscr{A}) \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}'))$. To prove the existence of $\phi_\dagger$, it will suffice to show that $\mathcal{C} = \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}))$. Note that $\mathcal{C}$ is closed under small colimits in $\mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}))$. Let $\mathrm{Sym}^*_{\mathsf{S}} : \mathrm{QCoh}(\mathsf{S}) \to \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}))$ be a left adjoint to the forgetful functor, and define $\mathrm{Sym}^*_{\mathsf{S}'}$ similarly. It follows from Proposition HA.4.7.3.14 that $\mathcal{C}$ is generated under small colimits by the essential image of $\mathrm{Sym}^*_{\mathsf{S}}$. It will therefore suffice to show that $\mathcal{C}$ contains the essential image of $\mathrm{Sym}^*_{\mathsf{S}}$. In other words, it suffices to show that for each quasi-coherent sheaf $\mathscr{F}$ on S, the functor

$$\mathscr{B} \mapsto \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(\mathsf{Y}))}(\mathrm{Sym}^*_{\mathsf{Y}}(\mathscr{F}), \phi^* \mathscr{B}) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{Y})}(\mathscr{F}, \phi^* \mathscr{B})$$

is corepresentable.

According to Proposition 6.4.5.4, the pullback functor $\mathrm{QCoh}(\mathsf{S}) \to \mathrm{QCoh}(\mathsf{S}')$ admits a left adjoint ϕ_+. We then have

$$\mathrm{Map}_{\mathrm{QCoh}(\mathsf{S})}(\mathscr{F}, \phi^* \mathscr{B}) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{S}')}(\phi_+ \mathscr{F}, \mathscr{B}) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}'))}(\mathrm{Sym}^*_{\mathsf{S}'}(\phi_+(\mathscr{F})), \mathscr{B}).$$

It follows that $\mathrm{Sym}^*_{\mathsf{S}}(\mathscr{F})$ belongs to $\mathcal{C}$, and that we have a canonical equivalence

$$\phi_\dagger(\mathrm{Sym}^*_{\mathsf{S}}(\mathscr{F})) \simeq \mathrm{Sym}^*_{\mathsf{S}'}(\phi_+ \mathscr{F}) = \mathrm{Sym}^*_{\mathsf{S}'}(\phi_*(\mathscr{F} \otimes \omega_{\mathsf{S}/\mathsf{S}'})).$$

This completes the proof of the existence of $\phi_\dagger$.

Now suppose that $\mathscr{A} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{Y}))$ is connective; we wish to show that $\phi_\dagger(\mathscr{A})$ is connective. Let $\mathscr{B}$ be the connective cover of $\phi_\dagger(\mathscr{A})$, and let $u : \mathscr{B} \to \phi_\dagger(\mathscr{A})$ be the tautological map. Since ϕ is flat, the induced map $\phi^*(u) : \phi^*(\mathscr{B}) \to \phi^*\phi_\dagger(\mathscr{A})$ exhibits $\phi^*(\mathscr{B})$ as a connective cover of $\phi^*\phi_\dagger(\mathscr{A})$. Since $\mathscr{A}$ is connective, the unit map $\mathscr{A} \to \phi^*\phi_\dagger(\mathscr{A})$ factors through $\phi^*(\mathscr{B})$. It follows that the map u admits a section, so that $\phi_\dagger(\mathscr{A})$ is a retract of $\mathscr{B}$ and is therefore connective, as desired. □

Proof of Proposition 19.1.1.1. Let $\phi : \mathsf{S} \to \mathsf{S}'$ be a morphism of spectral Deligne-Mumford stacks which is proper, flat, and locally almost of finite presentation, and let $\rho : \mathsf{X} \to \mathsf{S}$ be quasi-affine. We wish to prove that the Weil restriction $\mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X})$ exists and is quasi-affine over S'. Working locally on S', we can reduce to the case where S' is affine. Then S is a quasi-compact, quasi-separated spectral algebraic space. Let us regard $\mathscr{A} = \tau_{\geq 0}\rho_* \mathscr{O}_{\mathsf{X}}$ as a connective object of $\mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}))$, so that $\mathscr{A}$ determines an affine morphism $\overline{\rho} : \overline{\mathsf{X}} \to \mathsf{S}$. The tautological map $\rho^* \mathscr{A} \to \mathscr{O}_{\mathsf{X}}$ classifies a map $j : \mathsf{X} \to \overline{\mathsf{X}}$, which is a quasi-compact open immersion by virtue of our assumption that ρ is quasi-affine. Let $\phi_\dagger(\mathscr{A}) \in \mathrm{CAlg}(\mathrm{QCoh}(\mathsf{S}')^{\mathrm{cn}})$ be as in Proposition 19.1.1.2. Then $\phi_\dagger(\mathscr{A})$ determines an affine spectral Deligne-Mumford stack $\overline{\mathsf{Y}}$ over S' equipped with a map $\overline{\psi} : \overline{\mathsf{Y}} \times_{\mathsf{S}'} \mathsf{S} \to \overline{\mathsf{X}}$ in $\mathrm{SpDM}_{/\mathsf{S}}$.

Let $K \subseteq |\overline{\mathsf{X}}|$ be the closed subset complementary to the image of j. Since the open immersion j is quasi-compact, the set K is constructible. Let $K' \subseteq |\overline{\mathsf{Y}} \times_{\mathsf{S}'} \mathsf{S}\,|$ be the inverse image of K, so that K' is also closed and constructible. Let $K'' \subseteq |\overline{\mathsf{Y}}|$ be the image of K' under the projection map $|\overline{\mathsf{Y}} \times_{\mathsf{S}'} \mathsf{S}\,| \to |\overline{\mathsf{Y}}|$. The map ϕ is proper and therefore universally closed, so that K'' is closed. Choose an open immersion $j' : \mathsf{Y} \hookrightarrow \overline{\mathsf{Y}}$ complementary to K''. Since ϕ is locally almost of finite presentation, the set K'' is constructible (Corollary 4.3.4.2), so that j' is quasi-compact. It follows that Y is quasi-affine over S'. By construction, the map $\overline{\psi}$ restricts to a map $\psi : \mathsf{Y} \times_{\mathsf{S}'} \mathsf{S} \to \mathsf{X}$. It follows immediately from the definitions that ψ' exhibits Y as the Weil restriction of X along ϕ. If the map ρ is affine, then $K = \varnothing$ so the open immersion j' is an equivalence, in which case Y is affine over X'. □

19.1.2 Weil Restriction of Functors

To prove Theorem 19.1.0.1 in general, it will be convenient to first work in the more general context of functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, neglecting issues of representability.

Notation 19.1.2.1. Fix a functor $Z : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will regard $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}$ as a symmetric monoidal ∞-category with respect to the operation of Cartesian product. Given objects $X, Y \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}$, we let $\underline{\mathrm{Map}}_{/Z}(X, Y)$ denote a classifying object for morphisms from X to Y (if such an object exists). More precisely, $\underline{\mathrm{Map}}_{/Z}(X,Y)$ denotes an object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}$ equipped with an evaluation map $e : X \times_Z \underline{\mathrm{Map}}_{/Z}(X,Y) \to Y$ with the following universal property: for every object $W \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}$, composition with e induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})_{/Z}}(W, \underline{\mathrm{Map}}_{/Z}(X,Y)) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})_{/Z}}(W \times_Z X, Y).$$

Recall that a morphism $f : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ is *representable* if, for every connective $\mathbb{E}_\infty$-ring R and every morphism $\operatorname{Spec} R \to Y$, the fiber product $\operatorname{Spec} R \times_Y X$ is representable by a spectral Deligne-Mumford stack.

Proposition 19.1.2.2. *Suppose we are given morphisms $X \to Z \leftarrow Y$ in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. *Assume that the functors Y and Z are sheaves with respect to the étale topology, and that the map $X \to Z$ is representable. Then a morphism object $\underline{\mathrm{Map}}_{/Z}(X,Y)$ exists. Moreover, $\underline{\mathrm{Map}}_{/Z}(X,Y)$ is also a sheaf with respect to the étale topology.*

Before proving Proposition 19.1.2.2, let us describe our application of interest:

Construction 19.1.2.3 (Weil Restriction)**.** Let $\mathcal{S}\mathrm{hv}_{\text{ét}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ spanned by those functors which are sheaves with respect to the étale topology. Suppose that $f : S \to S'$ is a representable morphism in $\mathcal{S}\mathrm{hv}_{\text{ét}}$. Then f determines a pullback functor $(\mathcal{S}\mathrm{hv}_{\text{ét}})_{/S'} \to (\mathcal{S}\mathrm{hv}_{\text{ét}})_{/S}$, given by $Z \mapsto Z \times_{S'} S$. Using Proposition 19.1.2.2, we deduce that this pullback functor admits a right adjoint $\mathrm{Res}_{S/S'} : (\mathcal{S}\mathrm{hv}_{\text{ét}})_{/S} \to (\mathcal{S}\mathrm{hv}_{\text{ét}})_{/S'}$, given by the formula $\mathrm{Res}_{S/S'}(X) = \underline{\mathrm{Map}}_{/S'}(S,X) \times_{\underline{\mathrm{Map}}_{/S'}(S,S)} S$. We will refer to $\mathrm{Res}_{S/S'}$ as the functor of *Weil restriction* along the map $S \to S'$.

Remark 19.1.2.4. In the special case where the functors S, S', and X are representable by spectral Deligne-Mumford stacks S, S', and X, the Weil restriction $\mathrm{Res}_{S/S'}(X)$ of Construction 19.1.2.3 is representable by the Weil restriction $\mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X})$ defined in the introduction to §19.1, provided that the latter Weil restriction exists. Consequently, Theorem 19.1.0.1 can be regarded as an assertion about the representability of the functor $\mathrm{Res}_{S/S'}(X)$.

Proof of Proposition 19.1.2.2. Fix a functor $Z : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and regard $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}$ as a full subcategory of the ∞-category $\mathcal{C} = \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/Z}$. The ∞-category $\mathcal{C}$ can be regarded as an ∞-topos in a larger universe, so that the Cartesian monoidal structure on $\mathcal{C}$ is closed. In particular, the morphism object $\underline{\mathrm{Map}}_{/Z}(X,Y)$ exists as an object of $\mathcal{C}$. To prove the first assertion, it will suffice to show that for every connective $\mathbb{E}_\infty$-ring R, the space $\underline{\mathrm{Map}}_{/Z}(X,Y)(R)$ is essentially small. Since $Z(R)$ is small, it will suffice to show that for every point $\eta \in Z(R)$, the homotopy fiber $\underline{\mathrm{Map}}_{/Z}(X,Y)(R) \times_{Z(R)} \{\eta\}$ is essentially small. The point η determines a map of functors $\operatorname{Spec} R \to Z$, and we can identify $\underline{\mathrm{Map}}_{/Z}(X,Y)(R) \times_{Z(R)} \{\eta\}$ with the mapping space $\mathrm{Map}_{\mathcal{C}}(\operatorname{Spec} R, \underline{\mathrm{Map}}_{/Z}(X,Y)) \simeq \mathrm{Map}_{\mathcal{C}}(\operatorname{Spec} R \times_Z X, Y)$. Since the morphism $X \to Z$ is representable, the functor $\operatorname{Spec} R \times_Z X$ is representable by a spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$. For every object $U \in \mathcal{X}$, let X_U denote the functor represented by the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathscr{O}_{\mathcal{X}}|_U)$. The construction $U \mapsto \operatorname{Map}_{\mathcal{C}}(X_U, Y)$ determines a functor $F : \mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{S}}$. To complete the proof, it will suffice to show that for each $U \in \mathcal{X}$, the space $F(U)$ is essentially small. Since Y is a sheaf with respect to the étale topology, the functor F preserves small limits. It will therefore suffice to show that $F(U)$ is essentially small when $U \in \mathcal{X}$ is affine. In this case, we can write $X_U = \operatorname{Spec} R'$, and $F(U)$ can be identified with a homotopy fiber of the map $Y(R') \to Z(R')$.

It remains to show that $\underline{\operatorname{Map}}_{/Z}(X, Y)$ is a sheaf with respect to the étale topology. Let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by those maps $W \to Z$ where $W : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is an étale sheaf, and let $L : \mathcal{C} \to \mathcal{C}_0$ be a left adjoint to the inclusion. We wish to show that $\underline{\operatorname{Map}}_{/Z}(X, Y)$ is L-local. Let $\alpha : W \to W'$ be a morphism in $\mathcal{C}$ such that $L(\alpha)$ is an equivalence; we wish to show that composition with α induces a homotopy equivalence $\theta : \operatorname{Map}_{\mathcal{C}}(W', \underline{\operatorname{Map}}_{/Z}(X, Y)) \to \operatorname{Map}_{\mathcal{C}}(W, \underline{\operatorname{Map}}_{/Z}(X, Y))$. Using the universal property of $\underline{\operatorname{Map}}_{/Z}(X, Y)$, we can identify θ with the canonical map $\operatorname{Map}_{\mathcal{C}}(W' \times_Z X, Y) \to \operatorname{Map}_{\mathcal{C}}(W \times_Z X, Y)$. Since $Y \in \mathcal{C}_0$, we are reduced to proving that $L(\beta)$ is an equivalence, where $\beta : W \times_Z X \to W' \times_Z X$ is the map induced by α. This follows from our assumption that $L(\alpha)$ is an equivalence, since the sheafification functor L is left exact. □

19.1.3 Deformation-Theoretic Properties of Weil Restrictions

We now show that, under mild hypotheses, the constructions of §19.1.2 preserve the deformation-theoretic properties studied in Chapter 17.

Proposition 19.1.3.1. *Suppose we are given morphisms $X \xrightarrow{f} Z \xleftarrow{g} Y$ in $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Assume that the functors Y and Z are sheaves with respect to the étale topology and that the map $f : X \to Z$ is representable. Then:*

(1) *If the map g is cohesive, then the induced map $\underline{\operatorname{Map}}_{/Z}(X, Y) \to Z$ is cohesive.*

(2) *If the map g is infinitesimally cohesive, then the induced map $\underline{\operatorname{Map}}_{/Z}(X, Y) \to Z$ is infinitesimally cohesive.*

(3) *If the map g is nilcomplete, then the induced map $\underline{\operatorname{Map}}_{/Z}(X, Y) \to Z$ is nilcomplete.*

(4) *Assume that f is representable by spectral algebraic spaces (that is, the map $X(R) \to Z(R)$ has discrete homotopy fibers for every discrete $\mathbb{E}_\infty$-ring R), quasi-compact, and quasi-separated. If g is locally of finite presentation, then the induced map $\underline{\operatorname{Map}}_{/Z}(X, Y) \to Z$ is locally of finite presentation.*

(5) *Assume that f is representable by spectral algebraic spaces, quasi-compact, quasi-separated, and locally of finite* Tor*-amplitude. If g is locally almost of finite presentation, then the induced map $\underline{\mathrm{Map}}_{/Z}(X, Y) \to Z$ is locally almost of finite presentation.*

Corollary 19.1.3.2. *Let $f : S \to S'$ be a representable morphism in $\mathcal{S}\mathrm{hv}_{\acute{e}t}$, and let $p : X \to S$ be an arbitrary morphism in $\mathcal{S}\mathrm{hv}_{\acute{e}t}$. If p is cohesive (infinitesimally cohesive, nilcomplete, locally almost of finite presentation), then the induced map $q : \mathrm{Res}_{S/S'}(X) \to Z$ is cohesive (infinitesimally cohesive, nilcomplete, locally almost of finite presentation).*

Proof. We will show that if p is cohesive, then q is cohesive; the proof in the other three cases is the same. We have a pullback diagram

$$\begin{array}{ccc} \mathrm{Res}_{S/S'}(X) & \longrightarrow & \underline{\mathrm{Map}}_{/S'}(S, X) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\ S' & \longrightarrow & \underline{\mathrm{Map}}_{/S'}(S, S). \end{array}$$

We may therefore reduce to proving that q' is cohesive. For this, it suffices to show that both of the projection maps $\underline{\mathrm{Map}}_{/S'}(S, S) \to S' \leftarrow \underline{\mathrm{Map}}_{/S'}(S, X)$ are cohesive (Remark 17.3.7.3). Using Proposition 19.1.3.1, we are reduced to showing that f and $f \circ p$ are cohesive. In the case of f, this follows from Proposition **??**. Because f and p are both cohesive, Remark 17.3.7.3 guarantees that $f \circ p$ is cohesive. □

Proof of Proposition 19.1.3.1. Using Propositions 17.2.4.7, 17.3.8.4, and Remark 17.4.1.6, we can reduce to the case where $Z = \mathrm{Spec}\, R$ is a corepresentable functor, so that X is representable by a spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For each object $U \in \mathcal{X}$, let X_U denote the functor represented by the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$. For the first three assertions, it will suffice to show that if Y is cohesive (infinitesimally cohesive, nilcomplete) then $\underline{\mathrm{Map}}_{/Z}(X_U, Y)$ has the same property, for each object $U \in \mathcal{X}$ (Remark 17.3.7.3). Since Y is a sheaf with respect to the étale topology, the construction $U \mapsto \underline{\mathrm{Map}}_{/Z}(X_U, Y)$ carries colimits in $\mathcal{X}$ to limits in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. It will therefore suffice to show that $\underline{\mathrm{Map}}_{/Z}(X_U, Y)$ is cohesive (infinitesimally cohesive, nilcomplete) in the special case where $U \in \mathcal{X}$ is affine, so that $X_U \simeq \mathrm{Spec}\, R'$ for some connective $\mathbb{E}_\infty$-ring R'. Let $F : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be the functor corresponding to Y under the equivalence of ∞-categories $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z} \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})$. Unwinding the definitions, we see that $\underline{\mathrm{Map}}_{/Z}(X_U, Y)$ corresponds to the functor $F_U : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ given by the formula $F_U(A) = F(R' \otimes_R A)$. We now consider each case in turn:

(1) To prove that $\underline{\mathrm{Map}}_{/Z}(X_U, Y)$ is cohesive, we must show that for every pullback diagram

τ :

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01} \end{array}$$

in $\mathrm{CAlg}_R^{\mathrm{cn}}$ for which the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective, the induced diagram σ :

$$\begin{array}{ccc} F_U(A) & \longrightarrow & F_U(A_0) \\ \downarrow & & \downarrow \\ F_U(A_1) & \longrightarrow & F_U(A_{01}) \end{array}$$

is a pullback square in $\mathcal{S}$. We can identify σ with the diagram

$$\begin{array}{ccc} F(R' \otimes_R A) & \longrightarrow & F(R' \otimes_R A_0) \\ \downarrow & & \downarrow \\ F(R' \otimes_R A_1) & \longrightarrow & F(R' \otimes_R A_{01}). \end{array}$$

This is a pullback square by virtue of our assumption that Y is cohesive, since the diagram of $\mathbb{E}_\infty$-rings τ' :

$$\begin{array}{ccc} R' \otimes_R A & \longrightarrow & R' \otimes_R A_0 \\ \downarrow & & \downarrow \\ R' \otimes_R A_1 & \longrightarrow & R' \otimes_R A_{01} \end{array}$$

is also a pullback square which induces surjections

$$\pi_0(R' \otimes_R A_0) \to \pi_0(R' \otimes_R A_{01}) \leftarrow (R' \otimes_R A_1).$$

(2) The argument is identical to that given in case (1), noting that if the diagram τ induces surjections $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ with nilpotent kernels, then τ' has the same property.

(3) Assume that Y is nilcomplete; we wish to show that $\underline{\mathrm{Map}}_{/Z}(X_U, Y)$ is nilcomplete. For this, it suffices to show that for every connective R-algebra A, the canonical map

$$F_U(A) \simeq F(R' \otimes_R A) \to \varprojlim F(R' \otimes_R \tau_{\leqslant n} A) \simeq \varprojlim F_U(\tau_{\leqslant n} A)$$

is an equivalence. This follows from Proposition 17.3.2.4.

We now prove (4). Assume that Y is locally of finite presentation over R, and that X is a quasi-compact, quasi-separated spectral algebraic space. Let us say that an object $U \in \mathcal{X}$

is *good* if the functor $F_U : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ commutes with filtered colimits. It is easy to see that the collection of good objects of $\mathcal{X}$ is closed under finite colimits; we wish to prove that the final object of $\mathcal{X}$ is good. Using Proposition 2.5.3.5 and Theorem 3.4.2.1, we are reduced to proving that every affine object $U \in \mathcal{X}$ is good. In this case, we can write $X_U = \operatorname{Spec} R'$ as above, so that F_U is given by the formula $F_U(A) = F(A \otimes_R R')$ and therefore commutes with filtered colimits as desired.

The proof of (5) is similar. It will suffice to show that if X is a quasi-compact, quasi-separated spectral algebraic space of Tor-amplitude $\leqslant d$ over R, then the map $\underline{\mathrm{Map}}_{/Z}(X, Y) \to Z$ is locally of finite presentation to order n. Let us say that a functor $G : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is *good* if it satisfies condition (b) of Remark 17.4.1.6, and let us say that an object $U \in \mathcal{X}$ is *good* if the functor F_U is good. The collection of good functors is closed under finite limits, so the collection of good objects of $\mathcal{X}$ is closed under finite colimits. We wish to prove that the final object of $\mathcal{X}$ is good. Invoking Proposition 2.5.3.5 and Theorem 3.4.2.1 again, we are reduced to proving that every affine object $U \in \mathcal{X}$ is good. In this case, we can write $X_U = \operatorname{Spec} R'$, where R' is flat over R. Then F_U is given by the formula $F_U(A) = F(A \otimes_R R')$. If $\{A_\alpha\}$ is a filtered diagram of m-truncated R-algebras having colimit A, then $\{A_\alpha \otimes_R R'\}$ is a filtered diagram of $(m+d)$-truncated R'-algebras having colimit $\{A \otimes_R R'\}$. If the functor F is good, we deduce that the map

$$\varinjlim F_U(A_\alpha) \simeq \varinjlim F(A_\alpha \otimes_R R') \to F(A \otimes_R R') \simeq F_U(A)$$

is a homotopy equivalence, so that F_U is also good. □

19.1.4 The Cotangent Complex of a Weil Restriction

We now prove an analogue of Proposition 19.1.3.1 for the relative cotangent complex.

Proposition 19.1.4.1. *Suppose we are given morphisms $X \xrightarrow{f} Z \xleftarrow{g} Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Assume that f is representable, proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude, and that Y and Z are sheaves the étale topology. Set $H = \underline{\mathrm{Map}}_{/Z}(X, Y)$ and let $e : X \times_Z H \to Y$ be the evaluation map, so that we have a commutative diagram*

$$\begin{array}{ccccc} H & \xleftarrow{\overline{f}} & X \times_Z H & \xrightarrow{e} & Y \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \overline{q}} & & \downarrow{\scriptstyle g} \\ Z & \xleftarrow{f} & X & \xrightarrow{f} & Z \end{array}$$

where the left square is a pullback. Then:

(1) *If g admits a relative cotangent complex $L_{Y/Z}$, then the projection map $q : H \to Z$ also admits a relative cotangent complex $L_{H/Z}$.*

(2) *The canonical map $g^* L_{Y/Z} \to L_{X \times_Z H/X} \simeq \overline{f}^* L_{H/Z}$ induces an equivalence $\overline{f}_+ g^* L_{Y/Z} \to L_{H/Z}$ in the ∞-category $\mathrm{QCoh}(H)$, where $\overline{f}_+$ is given as in Construction 6.4.5.1. In other words, we have a canonical equivalence $L_{H/Z} \simeq \overline{f}_*(\overline{q}^* \omega_{X/Z} \otimes e^* L_{Y/Z})$, where $\omega_{X/Z}$ denotes the relative dualizing sheaf of f (Definition 6.4.2.4).*

Proof. We will prove (1) by verifying conditions (a) and (b) of Remark 17.2.4.3. We first verify (a). Fix a connective $\mathbb{E}_\infty$-ring R and a point $\eta \in H(R)$, and consider the functor $U : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ given by the formula

$$U(M) = \mathrm{fib}(H(R \oplus M) \to H(R) \times_{Z(R)} Z(R \oplus M)).$$

Set $X_R = X \times_Z \operatorname{Spec} R$, so that the point η determines a map $\rho : X_R \to Y$. Let $f_R : X_R \to \operatorname{Spec} R$ be the projection map. Then ρ and f_R determine pullback functors

$$f_R^* : \mathrm{Mod}_R \simeq \mathrm{QCoh}(\operatorname{Spec} R) \to \mathrm{QCoh}(X_R) \qquad \rho^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X_R).$$

Unwinding the definitions and applying Proposition 6.4.5.3, we see that the functor U is given by the formula

$$U(M) = \mathrm{Map}_{\mathrm{QCoh}(X_R)}(\rho^* L_{Y/Z}, \pi^* M) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(f_{R+} \rho^* L_{Y/Z}, M).$$

It follows that U is almost corepresented by the object $f_{R+} \rho^* L_{Y/X}$ (which is almost connective by virtue of Proposition **??**). This completes the verification of condition (a) of Remark 17.2.4.3. Condition (b) follows from the second part of Proposition 6.4.5.4, and assertion (2) follows from the construction. □

Remark 19.1.4.2. In the situation of Proposition **??**, suppose that the relative cotangent complex $L_{Y/Z}$ is perfect (almost perfect). Then the relative cotangent complex $L_{H/Z}$ is also perfect (almost perfect). This follows from assertion (2) of Proposition **??** together with Remark 6.4.5.2.

Proposition 19.1.4.3. *Let $f : S \to S'$ be a morphism in $\mathcal{S}\mathrm{hv}_{\acute{e}t}$ which is representable, proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude. Let $g : X \to S$ be an arbitrary morphism in $\mathcal{S}\mathrm{hv}_{\acute{e}t}$ and let $\mathrm{Res}_{S/S'}(X)$ denote the Weil restriction of X along f, so that we have an evaluation map $e : S \times_{S'} \mathrm{Res}_{S/S'}(X) \to X$ which fits into a commutative diagram*

$$\begin{array}{ccccc}
\mathrm{Res}_{S/S'}(X) & \xleftarrow{\overline{f}} & S \times_{S'} \mathrm{Res}_{S/S'}(X) & \xrightarrow{e} & X \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \overline{q}} & & \downarrow{\scriptstyle g} \\
S' & \xleftarrow{f} & S & \xrightarrow{\mathrm{id}} & S.
\end{array}$$

Suppose then g admits a relative cotangent complex. Then:

(a) *The morphism q admits a relative cotangent complex.*

(b) *The canonical map*

$$e^* L_{X/S} \to L_{S \times_{S'} \mathrm{Res}_{S/S'}(X)/S} \simeq \overline{f}^* L_{\mathrm{Res}_{S/S'}(X)/S'}$$

induces an equivalence $\overline{f}_+ e^* L_{X/S} \to L_{\mathrm{Res}_{S/S'}(X)/S'}$ in the ∞-category $\mathrm{QCoh}(\mathrm{Res}_{S/S'}(X))$.

(c) *If $L_{X/S}$ is almost perfect, then $L_{\mathrm{Res}_{S/S'}(X)/S'}$ is almost perfect.*

Proof. Set $H = \underline{\mathrm{Map}}_{/S'}(S, X)$ and $H_0 = \underline{\mathrm{Map}}_{/S'}(S, S)$, so that we have a pullback diagram

$$\begin{array}{ccc} \mathrm{Res}_{S/S'}(X) & \xrightarrow{u} & H \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\ S' & \longrightarrow & H_0. \end{array}$$

The morphism $f : S \to S'$ is representable and therefore admits a cotangent complex $L_{S/S'}$. Moreover, g is infinitesimally cohesive (Corollary 17.3.8.5). Applying Proposition 17.3.9.1 (and our assumption that g admits a relative cotangent complex), we deduce that $(f \circ g)$ admits a relative cotangent complex $L_{X/S'}$. Let $\overline{e} : S \times_{S'} H \to X$ and $\overline{e}_0 : S \times_{S'} H_0 \to S$ be the evaluation maps. and let $p : H_0 \to S'$, $f_H : S \times_{S'} H \to H$, and $f_{H_0} : S \times_{S'} H_0 \to H_0$ be the projection maps.Applying Proposition **??**, we deduce that the morphisms p and $(p \circ q')$ admit relative cotangent complexes, given by $f_{H_0+} \overline{e}_0^* L_{S/S'}$ and $f_{H+} \overline{e}^* L_{X/S'}$, respectively. Applying Proposition 17.2.5.2, we deduce that the map q' admits a relative cotangent complex, $f_{H+} \overline{e}^* L_{X/S}$. Applying Remark 17.2.4.6, we deduce that q admits a relative cotangent complex, which we can identify with $u^* f_{H+} \overline{e}^* L_{X/S} \simeq \overline{f}_+ e^* L_{X/S}$. This proves (a), and a diagram chase shows that this identification coincides with the map $\overline{f}_+ e^* L_{X/S} \to L_{\mathrm{Res}_{S/S'}(X)/S'}$ appearing in (b). Assertion (c) follows from (b) and Remark 6.4.5.2. □

19.1.5 The Noetherian Case

We now study some cases in which we can use Artin's representability theorem (in the form of Theorem 18.3.0.1) to verify the representability of the functors given in Notation 19.1.2.1.

Proposition 19.1.5.1. *Let R be a Noetherian $\mathbb{E}_\infty$-ring such that $\pi_0 R$ is a Grothendieck ring and set $Z = \mathrm{Spec}\, R$. Suppose we are given natural transformations $X \to Z \leftarrow Y$ for some pair of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ satisfying the following conditions:*

- *The functor X is representable by a spectral algebraic space X which is proper, flat, and locally almost of finite presentation over R.*

- *The functor Y is representable by a spectral algebraic space Y which is quasi-separated and locally almost of finite presentation over R.*

Then the functor $\underline{\mathrm{Map}}_{/Z}(X, Y)$ is representable by a spectral Deligne-Mumford stack which is locally almost of finite presentation over R.

Remark 19.1.5.2. In the situation of Proposition 19.1.5.1, it follows immediately from the definitions that the spectral Deligne-Mumford stack representing $\underline{\mathrm{Map}}_{/Z}(X, Y)$ is a spectral algebraic space, and it follows from Theorem 19.1.0.1 that it has quasi-affine diagonal (and is therefore quasi-separated).

Proof. Set $F = \underline{\mathrm{Map}}_{/Z}(X, Y)$, and let $F_0 : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be given by the formula $F_0(A) = \mathrm{fib}(F(A) \to Z(A))$. We will show that F is representable by a spectral algebraic space by verifying conditions (1) through (5) of Theorem 18.3.0.1:

(1) If A is a discrete commutative ring, then the space $F(A)$ is discrete. To prove this, it will suffice to show that the fibers of the map $F(A) \to Z(A)$ are discrete (since $Z(A) \simeq \mathrm{Map}_{\mathrm{CAlg}}(R, A))$ is discrete. That is, we must show that if A is a discrete $\mathbb{E}_\infty$-algebra over R, then $F_0(A)$ is discrete. Unwinding the definitions, we have

$$F_0(A) = \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}}(\mathrm{Spec}\, A \times_Z X, Y).$$

Since X is flat over R, $\mathrm{Spec}\, A \times_Z X$ is representable by a spectral algebraic space $\mathrm{Spét}\, A \times_{\mathrm{Spét}\, R} \mathsf{X}$ which is flat over A, and therefore 0-truncated. The desired result now follows from Lemma 1.6.8.8.

(2) The functor F is a sheaf for the étale topology. This follows from Proposition 19.1.2.2.

(3) It follows from Proposition 19.1.3.1 that the forgetful functor $F \to Z$ is nilcomplete and infinitesimally cohesive (in fact, it is even cohesive). We claim that it is integrable. To prove this, suppose that A is a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect its maximal ideal. Let $\mathrm{Spf}\, A$ denote the formal spectrum of A, which we regard as an object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. We wish to show that the diagram

$$\begin{array}{ccc} F(A) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, A, F) \\ \downarrow & & \downarrow \\ Z(A) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, A, Z) \end{array}$$

is a pullback square. Unwinding the definitions, we must show that for every map of $\mathbb{E}_\infty$-algebra $R \to A$, the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}}(\mathrm{Spec}\, A \times_{\mathrm{Spec}\, R} X, Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}}(\mathrm{Spf}\, A \times_{\mathrm{Spec}\, R} X, Y)$$

is a homotopy equivalence. This follows immediately from Theorem 8.5.3.1.

(4) It follows from Proposition **??** that the natural transformation $F \to Z$ admits a relative cotangent complex $L_{F/Z}$. Moreover, if $\eta \in F(A)$ is a point classifying a map $u : \operatorname{Spét} A \times_{\operatorname{Spét} R} \mathsf{X} \to \mathsf{Y}$ and $p : \operatorname{Spét} A \times_{\operatorname{Spét} R} \mathsf{X} \to \operatorname{Spét} A$ denotes the projection map, then we have a canonical equivalence $\eta^* L_{F/Z} \simeq p_+ u^* L_{\mathsf{Y}/\operatorname{Spét} R}$. Since $u^* L_{\mathsf{Y}/\operatorname{Spét} R}$ is connective, it suffices to show that the functor p_+ is right t-exact. This is clear, since p_+ is left adjoint to the pullback functor p^* (which is left t-exact by virtue of our assumption that X is flat over R).

(5) The map $F \to Z$ is locally almost of finite presentation. This follows from Proposition 19.1.3.1, since the map $Y \to Z$ is locally almost of finite presentation.

□

19.1.6 The Proof of Theorem 19.1.0.1

We will deduce Theorem 19.1.0.1 from the following non-Noetherian analogue of Proposition 19.1.5.1:

Proposition 19.1.6.1. *Suppose we are given functors $X, Y, Z : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and natural transformations $f : X \to Z$, $g : Y \to Z$. Assume that f is representable, proper, flat, and locally almost of finite presentation. Assume that g is representable by spectral algebraic spaces, quasi-compact, quasi-separated, and locally almost of finite presentation. Then the map $\underline{\mathrm{Map}}_{/Z}(X, Y) \to Z$ is representable.*

Proof. We may assume without loss of generality that $Z = \operatorname{Spec} R$ for some connective $\mathbb{E}_\infty$-ring R, so that X is representable by a spectral algebraic space X and Y is representable by a quasi-separated spectral algebraic space Y. It follows from Propositions 19.1.3.1 and **??** that the functor $F = \underline{\mathrm{Map}}_{/Z}(X, Y)$ is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. According to Theorem 18.1.0.2, to show that F is representable by a spectral Deligne-Mumford stack, it will suffice to show that the restriction $F|_{\mathrm{CAlg}^{\heartsuit}}$ is representable by a spectral Deligne-Mumford stack, where $\mathrm{CAlg}^{\heartsuit}$ denotes the full subcategory of CAlg spanned by the discrete $\mathbb{E}_\infty$-rings. We may therefore replace R by $\pi_0 R$, and thereby reduce to the case where R is discrete. Write R as the union of finitely generated subrings R_α. Using Theorem 4.4.2.2, we can choose an index α and spectral Deligne-Mumford stacks X_α and Y_α which are finitely 0-presented over R_α, together with equivalences

$$\mathsf{X} \simeq \tau_{\leqslant 0}(\operatorname{Spét} R \times_{\operatorname{Spét} R_\alpha} \mathsf{X}_\alpha) \qquad \tau_{\leqslant 0}\, \mathsf{Y} \simeq \tau_{\leqslant 0}(\operatorname{Spét} R \times_{\operatorname{Spét} R_\alpha} \mathsf{Y}_\alpha).$$

Enlarging α if necessary, we can ensure that X_α is a spectral algebraic space which is proper and flat over R_α (Proposition 5.5.4.1 and Corollary 11.2.6.1 of [90]) and that Y_α is a spectral algebraic space. Then $\mathsf{X} \simeq \operatorname{Spét} R \times_{\operatorname{Spét} R_\alpha} \mathsf{X}_\alpha$. Set $\mathsf{Y}' = \operatorname{Spét} R \times_{\operatorname{Spét} R_\alpha} \mathsf{Y}_\alpha$, let Y' be the functor represented by Y', and set $F' = \underline{\mathrm{Map}}_{/Z}(X, Y')$. Then $F|_{\mathrm{CAlg}^{\heartsuit}} \simeq F'|_{\mathrm{CAlg}^{\heartsuit}}$.

Consequently, we are free to replace Y by Y' and reduce the case where $\mathsf{Y} = \operatorname{Spét} R \times_{\operatorname{Spét} R_\alpha} \mathsf{Y}_\alpha$. We may then replace R by R_α, thereby reducing to the case where R is finitely generated as a commutative ring. In particular, R is a Grothendieck ring (Theorem **??**), so that the desired result follows from Proposition 19.1.5.1. □

We are now in a position to prove our main result.

Proof of Theorem 19.1.0.1. Let $\phi : \mathsf{S} \to \mathsf{S}'$ be a morphism of spectral Deligne-Mumford stacks which is proper, flat, and locally almost of finite presentation, let $\psi : \mathsf{X} \to \mathsf{S}$ be a relative spectral algebraic space which is quasi-separated and locally almost of finite presentation. Then ϕ and ψ determine natural transformations $X \to S \to S'$ between functors $X, S, S' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We wish to show that $\mathrm{Res}_{S/S'}(X)$ is representable by a quasi-separated spectral algebraic space which is locally almost of finite presentation over S'. The assertion is local on S', so we may assume without loss of generality that $\mathsf{S}' \simeq \operatorname{Spét} R$ is affine. Write X as a union of quasi-compact open substacks X_α, representing functors $X_\alpha : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. It follows from Proposition 19.1.6.1 that each of the functors $\underline{\mathrm{Map}}_{/S'}(S, X_\alpha)$ and $\underline{\mathrm{Map}}_{/S'}(S, S)$ is representable by a quasi-separated spectral algebraic space, so that $\mathrm{Res}_{S'/S}(X_\alpha) = \underline{\mathrm{Map}}_{/S'}(S, X_\alpha) \times_{\underline{\mathrm{Map}}_{S'}(S,S)} S'$ is also representable by a quasi-separated spectral algebraic space $\mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X}_\alpha)$. Note that the transition maps $\mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X}_\alpha) \to \mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X}_\beta)$ are quasi-compact open immersions (see the proof of Proposition 19.1.1.1), so that the filtered colimit $\varinjlim_\alpha \mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X}_\alpha)$ is also a quasi-separated spectral algebraic space which represents the functor $\mathrm{Res}_{S/S'}(\varinjlim X_\alpha) = \mathrm{Res}_{S/S'}(X)$. To complete the proof, it will suffice to show that the colimit $\varinjlim_\alpha \mathrm{Res}_{\mathsf{S}/\mathsf{S}'}(\mathsf{X}_\alpha)$ is locally almost of finite presentation over S'. By virtue of Proposition 17.4.3.1, this is equivalent to the requirement that the projection map $\mathrm{Res}_{S/S'}(X) \to S'$ is locally almost of finite presentation, which follows from Corollary 19.1.3.2. □

19.2 The Picard Functor

Let X be a projective algebraic variety over a field κ. The *Picard group* of X is defined to be the group of isomorphism classes of line bundles on X. Under mild hypotheses, one can show that the Picard group of X itself has the structure of an algebraic variety over κ. More precisely, there exists a group scheme E over κ (usually not quasi-compact) whose group $E(\kappa)$ of κ-valued points is canonically isomorphic to the Picard group of X. Our goal in this section is to prove an analogous result in the setting of spectral algebraic geometry.

Construction 19.2.0.1 (The Picard Functor)**.** For every spectral Deligne-Mumford stack X, we let $\mathscr{P}\mathrm{ic}(\mathsf{X})$ denote the full subcategory of $\mathrm{QCoh}(\mathsf{X})^{\simeq}$ spanned by those quasi-coherent sheaves which are locally free of rank 1 (see Definition 2.9.4.1). The construction $\mathsf{X} \mapsto \mathscr{P}\mathrm{ic}(\mathsf{X})$

determines a contravariant functor from the ∞-category SpDM of spectral Deligne-Mumford stacks to the ∞-category of spaces.

Let R be a connective $\mathbb{E}_\infty$-ring and let $f : \mathsf{X} \to \operatorname{Spét} R$ be a map of spectral Deligne-Mumford stacks. We define a functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ by the formula

$$\mathscr{P}\mathrm{ic}_{\mathsf{X}/R}(R') = \mathscr{P}\mathrm{ic}(\operatorname{Spét} R' \times_{\operatorname{Spét} R} \mathsf{X}).$$

In general, it is not reasonable to expect the functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}/R}$ to be representable by a spectral Deligne-Mumford stack over R, because it does not have unramified diagonal: line bundles on X can admit continuous families of automorphisms. To address this issue, we introduce a rigidification of the functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}/R}$:

Definition 19.2.0.2. Let $f : \mathsf{X} \to \mathsf{Y} = \operatorname{Spét} R$ be a map of spectral Deligne-Mumford stacks, and suppose that f admits a section $x : \mathsf{Y} \to \mathsf{X}$. Then pullback along x determines a natural transformation of functors $\mathscr{P}\mathrm{ic}_{\mathsf{X}/R} \to \mathscr{P}\mathrm{ic}_{\mathsf{Y}/R}$. We will denote the fiber of this map by $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$.

More informally, the functor $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}$ assigns to every connective R-algebra R' a classifying space for pairs $(\mathscr{L}, \alpha)$, where $\mathscr{L}$ is a line bundle on $\operatorname{Spét} R' \times_{\operatorname{Spét} R} \mathsf{X}$ and α is an equivalence of R'-modules $R' \to x'^* \mathscr{L}$, where $x' : \operatorname{Spét} R' \to \operatorname{Spét} R' \times_{\operatorname{Spét} R} \mathsf{X}$ is the map determined by x.

Remark 19.2.0.3. In the situation of Definition 19.2.0.2, the fiber sequence

$$\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R} \to \mathscr{P}\mathrm{ic}_{\mathsf{X}/R} \to \mathscr{P}\mathrm{ic}_{\mathsf{Y}/R}$$

is canonically split by the map $\mathscr{P}\mathrm{ic}_{\mathsf{Y}/R} \mapsto \mathscr{P}\mathrm{ic}_{\mathsf{X}/R}$ given by pullback along the projection $f : \mathsf{X} \to \mathsf{Y}$. It follows that we have an equivalence of functors $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R} \times \mathscr{P}\mathrm{ic}_{\mathsf{Y}/R} \simeq \mathscr{P}\mathrm{ic}_{\mathsf{X}/R}$, given informally the formula $(\mathscr{L}, \mathscr{L}') \mapsto \mathscr{L} \otimes f^* \mathscr{L}'$.

Remark 19.2.0.4. Using Remark 19.2.0.3, one can show that the functor $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}$ of Definition 19.2.0.2 is independent of the section x, up to canonical equivalence. Moreover, the definition of $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}$ can be generalized to the case where f does not admit a section.

We are now ready to state our main result.

Theorem 19.2.0.5. *Let $f : \mathsf{X} \to \operatorname{Spét} R$ be a map of spectral algebraic spaces which is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. For any section $x : \operatorname{Spét} R \to \mathsf{X}$ of f, the functor $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}$ is representable by a spectral algebraic space which is quasi-separated and locally of finite presentation over R.*

The analogue of Theorem 19.2.0.5 in classical algebraic geometry was proven by Artin as an application of his representability criterion. It is possible to use Theorem 18.1.0.2 to deduce Theorem 19.2.0.5 from its classical analogue. We will give a slightly different argument at the end of this section, which appeals instead to Theorem 18.3.0.1. The main point is to show that the functor $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}$ has a well-behaved deformation theory.

19.2.1 Deformations of Modules

Let us regard the construction $R \mapsto \mathrm{Mod}_R$ as a functor from the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-rings to the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$ of (not necessarily small) ∞-categories. Our first goal is to study the deformation theoretic properties of this functor (and some of its variants). Since the functor in question takes values in $\widehat{\mathcal{C}\mathrm{at}}_\infty$, rather than the ∞-category $\mathcal{S}$ of spaces, we will need a mild extension of the terminology of Chapter 17.

Definition 19.2.1.1. Let $\mathcal{C}$ be an ∞-category and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}$ be a functor. We will say that X is *cohesive* (*infinitesimally cohesive*, *nilcomplete*, *integrable*) if, for every corepresentable functor $e : \mathcal{C} \to \mathcal{S}$, the composite functor $e \circ X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is cohesive (infinitesimally cohesive, nilcomplete, integrable); see Remark **??**.

We will say that a natural transformation $\alpha : X \to Y$ between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}$ is *cohesive* (*infinitesimally cohesive*, *nilcomplete*, *integrable*) if, for every corepresentable functor $e : \mathcal{C} \to \widehat{\mathcal{S}}$, the induced natural transformation $e \circ X \to e \circ Y$ is cohesive (infinitesimally cohesive, nilcomplete, integrable), in the sense of Definition 17.3.7.1.

Remark 19.2.1.2. Let $\alpha : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Then α is cohesive (infinitesimally cohesive, nilcomplete, integrable) in the sense of Definition 19.2.1.1 if and only if it is cohesive (infinitesimally cohesive, nilcomplete, integrable) in the sense of Definition 17.3.7.1.

Proposition 19.2.1.3. *Let $F : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the functor given by $R \mapsto \mathrm{Mod}_R^{\mathrm{acn}}$, where $\mathrm{Mod}_R^{\mathrm{acn}}$ denotes the full subcategory of Mod_R spanned by those objects which are almost connective (that is, n-connective for some integer n). Then the functor F is cohesive and nilcomplete.*

Corollary 19.2.1.4. *Let $F' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$ denote the functor given by $R \mapsto \mathrm{Mod}_R^{\mathrm{perf}}$. Then the functor F' is cohesive, nilcomplete, and commutes with filtered colimits.*

Proof. Let $F : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Proposition 19.2.1.3, so there is an evident natural transformation $\alpha : F' \to F$. The natural transformation α is cohesive (Proposition 16.2.3.1) and nilcomplete (Proposition 2.7.3.2). Since F is cohesive and nilcomplete, we conclude that F' is cohesive and nilcomplete. The assertion that F' commutes with filtered colimits follows from Corollary 4.5.1.3 and Lemma HA.7.3.5.11 . □

We now turn to the proof of Proposition 19.2.1.3. The assertion that the functor $R \mapsto \mathrm{Mod}_R^{\mathrm{acn}}$ is cohesive follows from Theorem 16.2.0.2. Nilcompleteness is a consequence of the following more general assertion:

Proposition 19.2.1.5. *For every connective $\mathbb{E}_1$-ring R, let $\mathrm{LMod}_R^{\mathrm{acn}}$ denote the full subcategory of LMod_R spanned by the almost connective objects. Then the canonical map*

$$\mathrm{LMod}_R^{\mathrm{acn}} \to \varprojlim \mathrm{LMod}_{\tau_{\leqslant n} R}^{\mathrm{acn}}$$

is an equivalence of ∞-categories.

Proof. We can identify objects of $\varprojlim \mathrm{LMod}^{\mathrm{acn}}_{\tau_{\leqslant n}R}$ with a sequences of objects $\{M_n \in \mathrm{LMod}^{\mathrm{acn}}_{\tau_{\leqslant n}R}\}$ together with equivalences $\alpha_n : M_n \simeq (\tau_{\leqslant n}R) \otimes_{\tau_{\leqslant n+1}R} M_{n+1}$. Using the assumption that M_{n+1} is almost connective, we deduce that M_n is nonzero if and only if M_{n+1} is nonzero. Suppose that M_0 is nonzero, so that M_n is nonzero for every integer n. Then there is some smallest integer $k(n)$ for which $\pi_{k(n)}M_n \neq 0$. Using the isomorphisms α_n, we deduce that all of the integers $k(n)$ are the same; let us denote this common value by k. For each integer n, the fiber of the canonical map $M_{n+1} \to M_n$ is given by the tensor product

$$\mathrm{fib}(\tau_{\leqslant n+1}R \to \tau_{\leqslant n}R) \otimes_{\tau_{\leqslant n+1}R} M_{n+1},$$

and is therefore $(k+n+1)$-connective. It follows that each of the towers of abelian groups $\{\pi_j M_n\}_{n \geqslant 0}$ are constant for $n \geqslant j-k$. Let $M = \varprojlim M_n$, so that the Milnor exact sequences

$$0 \to \lim{}^1\{\pi_{j+1}M_n\} \to \pi_j M \to \lim{}^0\{\pi_j M_n\} \to 0$$

specialize to give isomorphisms $\pi_j M \to \pi_j M_n$ for $n \geqslant j-k$. In particular, we deduce that M is k-connective. We conclude that the functor F admits a right adjoint $G : \varprojlim \mathrm{LMod}^{\mathrm{acn}}_{\tau_{\leqslant n}R} \to \mathrm{LMod}^{\mathrm{acn}}_R$, given by $\{M_n\}_{n \geqslant 0} \mapsto \varprojlim M_n$.

We next show that the unit map $\mathrm{id} \to G \circ F$ is an equivalence of functors from $\mathrm{LMod}^{\mathrm{acn}}_R$ to itself. Let M be a k-connective R-module; we wish to show that the canonical map $M \to \varprojlim (\tau_{\leqslant n}R) \otimes_R M$ is an equivalence. Fix an integer j, and consider the composition

$$\pi_j M \xrightarrow{\phi} \pi_j \varprojlim (\tau_{\leqslant n}R) \otimes_R M \xrightarrow{\psi} \pi_j (\tau_{\leqslant n}R) \otimes_R M.$$

Using the analysis above, we see that ψ is an isomorphism for $n \geqslant j-k$. It will therefore suffice to show that $\psi \circ \phi$ is an isomorphism for $n \geqslant j-k$. This follows from the existence of an exact sequence of abelian groups

$$\pi_j(\tau_{\geqslant n+1}R \otimes_R M) \to \pi_j M \to \pi_j(\tau_{\leqslant n}R \otimes_R M) \to \pi_{j-1}(\tau_{\geqslant n+1}R \otimes_R M),$$

since the k-connectivity of M implies that the abelian groups $\pi_j(\tau_{\geqslant n+1}R \otimes_R M)$ and $\pi_{j-1}(\tau_{\geqslant n+1}R \otimes_R M)$ are trivial.

To complete the proof, it will suffice to show that the functor G is conservative Since G is an exact functor between stable ∞-categories, we are reduced to proving that if $\{M_n\}_{n \geqslant 0}$ is an object of $\varprojlim \mathrm{LMod}^{\mathrm{acn}}_{\tau_{\leqslant n}R}$ such that $\varprojlim M_n \simeq 0$, then each M_n vanishes. Assume otherwise, and let k be defined as above. Then $\pi_k \varprojlim M_n \simeq \pi_k M_0 \neq 0$, and we obtain a contradiction. □

19.2.2 The Atiyah Class

Our next goal is to compute the cotangent complex of the functor appearing in Corollary 19.2.1.4. We begin with a more general discussion.

Notation 19.2.2.1. Let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack, let $\mathscr{E} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$, and let $\eta \in \mathrm{Der}(\mathscr{O}_{\mathsf{X}}, \Sigma\mathscr{E})$ (see Definition 17.1.1.1). We let $\mathscr{O}_{\mathsf{X}}^{\eta}$ denote the square-zero extension of $\mathscr{O}_{\mathsf{X}}$ be $\mathscr{E}$ determined by η (Construction 17.1.3.1), so that we have a pullback diagram

$$\begin{array}{ccc} \mathscr{O}_{\mathsf{X}}^{\eta} & \longrightarrow & \mathscr{O}_{\mathsf{X}} \\ \downarrow & & \downarrow \eta \\ \mathscr{O}_{\mathsf{X}} & \longrightarrow & \mathscr{O}_{\mathsf{X}} \oplus \Sigma\mathscr{E}. \end{array}$$

The pair $(\mathcal{X}, \mathscr{O}_{\mathsf{X}}^{\eta})$ is a spectral Deligne-Mumford stack (Proposition 17.1.3.4), which we will denote by X^{η}. In the special case where $\eta = 0$, we will denote X^{η} by $\mathsf{X}^{\mathcal{E}} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}} \oplus \mathscr{E})$.

In the situation of Notation 19.2.2.1, we have a pushout diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}^{\Sigma\mathscr{E}} & \xrightarrow{u} & \mathsf{X} \\ \downarrow & & \downarrow \\ \mathsf{X} & \longrightarrow & \mathsf{X}^{\eta} \end{array}$$

where the maps are closed immersions. Applying Theorem 16.2.0.1, we deduce that the diagram of pullback functors

$$\begin{array}{ccc} \mathrm{QCoh}(\mathsf{X}^{\eta})^{\mathrm{acn}} & \longrightarrow & \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}} & \longrightarrow & \mathrm{QCoh}(\mathsf{X}^{\Sigma\mathscr{E}})^{\mathrm{cn}} \end{array}$$

is a pullback square. Taking $\eta = 0$ and passing to homotopy fibers over some fixed object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}}$, we obtain equivalences

$$\begin{aligned} \mathrm{QCoh}(\mathsf{X}^{\mathscr{E}})^{\mathrm{acn}} \times_{\mathrm{QCoh}(\mathsf{X})} \{\mathscr{F}\} &\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X}^{\Sigma\mathscr{E}})}(u^*\mathscr{F}, u^*\mathscr{F}) \times_{\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F},\mathscr{F})} \{\mathrm{id}_{\mathscr{F}}\} \\ &\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, u_* u^*\mathscr{F}) \times_{\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F},\mathscr{F})} \{\mathrm{id}_{\mathscr{F}}\} \\ &\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \mathscr{F} \oplus (\Sigma\mathscr{E} \otimes \mathscr{F})) \times_{\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F},\mathscr{F})} \{\mathrm{id}_{\mathscr{F}}\} \\ &\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \Sigma(\mathscr{E} \otimes \mathscr{F})); \end{aligned}$$

here the first equivalence follows from the observation that any lifting of $\mathrm{id}_{\mathscr{F}}$ to an endomorphism of $u^*\mathscr{F}$ is automatically invertible. We can summarize the situation as follows:

Proposition 19.2.2.2 (First-Order Deformations of Quasi-Coherent Sheaves)**.** *Let* X *be a spectral Deligne-Mumford stack and suppose we are given quasi-coherent sheaves* $\mathscr{E} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$*,* $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}}$*. Then the* ∞*-category* $\mathrm{QCoh}(\mathsf{X}^{\mathscr{E}}) \times_{\mathrm{QCoh}(\mathsf{X})} \{\mathscr{F}\}$ *is a Kan complex, which is canonically homotopy equivalent to the mapping space* $\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \Sigma(\mathscr{E} \otimes \mathscr{F}))$*. In particular, we obtain a canonical bijection*

$$\pi_0 \,\mathrm{QCoh}(\mathsf{X}^{\mathscr{E}}) \times_{\mathrm{QCoh}(\mathsf{X})} \{\mathscr{F}\} \simeq \mathrm{Ext}^1_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \mathscr{E} \otimes \mathscr{F}).$$

In the situation of Proposition 19.2.2.2, suppose that $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ is perfect. Then $\mathscr{F}$ admits a dual $\mathscr{F}^\vee$ in $\mathrm{QCoh}(\mathsf{X})$, and we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F}, \Sigma(\mathscr{E} \otimes \mathscr{F})) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\Sigma^{-1}(\mathscr{F} \otimes \mathscr{F}^\vee), \mathscr{E}).$$

We now wish to apply this observation to the classification of perfect complexes.

Corollary 19.2.2.3. *Let* $\mathrm{Perf}^{\simeq} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ *denote the functor given by* $\mathrm{Perf}^{\simeq}(R) = (\mathrm{Mod}_R^{\mathrm{perf}})^{\simeq}$*. Then* $\mathrm{Perf}^{\simeq}$ *admits a (perfect) cotangent complex.*

Proof. We will show that conditions (a) and (b) of Example 17.2.4.4 are satisfied:

(a) Let R be a connective $\mathbb{E}_\infty$-ring and let $\eta \in \mathrm{Perf}^{\simeq}(R)$, corresponding to a perfect R-module N. Let $F : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ be the functor defined by the formula $F(M) = \mathrm{Perf}^{\simeq}(R \oplus M) \times_{\mathrm{Perf}^{\simeq}(R)} \{\eta\}$. Applying Proposition **??**, we see that F is almost corepresented by the perfect R-module $\Sigma^{-1}(N \otimes_R N^\vee)$.

(b) For every map of connective $\mathbb{E}_\infty$-rings $R \to R'$ and every connective R'-module M, we must show that the diagram of spaces

$$\begin{array}{ccc} \mathrm{Perf}^{\simeq}(R \oplus M) & \xrightarrow{\theta} & \mathrm{Perf}^{\simeq}(R' \oplus M) \\ \downarrow & & \downarrow \\ \mathrm{Perf}^{\simeq}(R) & \longrightarrow & \mathrm{Perf}^{\simeq}(R') \end{array}$$

is a pullback square. Choose a point $\eta \in \mathrm{Perf}(R)$ corresponding to a perfect R-module N, and let $\eta' \in \mathrm{Perf}(R')$ be its image (corresponding to the perfect R'-module $R' \otimes_R N$). We will prove that θ induces a homotopy equivalence after passing to the homotopy fibers over the points η and η', respectively. Using the proof of (a), we are reduced to showing that the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_R}(N \otimes_R N^\vee, \Sigma M) \to \mathrm{Map}_{\mathrm{Mod}_{R'}}(N' \otimes_{R'} N'^\vee, \Sigma M)$$

is a homotopy equivalence, which is clear.

□

Remark 19.2.2.4. In the situation of Corollary 19.2.2.3, we can be more precise. There is a "universal" perfect quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathrm{Perf}^{\simeq})$, and the proof of Corollary 19.2.2.3 shows that the cotangent complex $L_{\mathrm{Perf}^{\simeq}}$ can be identified with $\Sigma^{-1}(\mathscr{F} \otimes \mathscr{F}^{\vee})$.

Construction 19.2.2.5 (The Atiyah Class)**.** Let X be a spectral Deligne-Mumford stack, let $\mathscr{E} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$, let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}}$, and let $\eta \in \mathrm{Der}(\mathscr{O}_{\mathsf{X}}, \mathscr{E})$ be a derivation. Then we can identify η with a left homotopy inverse s of the evident map $\mathsf{X} \to \mathsf{X}^{\mathscr{E}}$. The pullback $s^* \mathscr{F}$ can be regarded as an object of the ∞-category $\mathrm{QCoh}(\mathsf{X}^{\mathscr{E}})^{\mathrm{acn}} \times_{\mathrm{QCoh}(\mathsf{X})} \{\mathscr{F}\}$, which (by virtue of Proposition 19.2.2.2) we can identify with a map $\rho_\eta : \mathscr{F} \to \Sigma(\mathscr{E} \otimes \mathscr{F})$ in the ∞-category $\mathrm{QCoh}(\mathsf{X})$.

Specializing to the case where $\mathscr{E} = L_{\mathsf{X}}$ and η is the universal derivation, we obtain a map $\rho_{\mathrm{univ}} : \mathscr{F} \to \Sigma(L_{\mathsf{X}} \otimes \mathscr{F})$. We will refer to ρ_{univ} as the *Atiyah class* of the quasi-coherent sheaf $\mathscr{F}$. In the special case where $\mathscr{F}$ is perfect, we can identify ρ_{univ} with a map $\Sigma^{-1}(\mathscr{F} \otimes \mathscr{F}^{\vee}) \to L_{\mathsf{X}}$, which can be understood as the "derivative" of the map $\mathsf{X} \to \mathrm{Perf}^{\simeq}$ which classifies $\mathscr{F}$.

Remark 19.2.2.6 (Obstruction Theoretic Interpretation of the Atiyah Class)**.** In the situation of Construction 19.2.2.5, suppose that $\mathscr{E} \in \mathrm{QCoh}(\mathsf{X})$ is 1-connective, so that the derivation $\eta \in \mathrm{Der}(\mathscr{O}_{\mathsf{X}}, \mathscr{E})$ determines a square-zero extension $\mathscr{O}_{\mathsf{X}}^{\eta}$ of $\mathscr{O}_{\mathsf{X}}$ by $\Sigma^{-1} \mathscr{E}$. Using Theorem 16.2.0.1, we see that the following conditions are equivalent:

(*a*) The map ρ_η of Construction 19.2.2.5 is nullhomotopic.

(*b*) The object $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}}$ can be lifted to an object of $\mathrm{QCoh}(\mathsf{X}^{\eta})$.

Conditions (*a*) and (*b*) are equivalent to the vanishing of a certain obstruction class $o(\eta) \in \mathrm{Ext}^2_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F} \otimes \mathscr{F}^{\vee}, \Sigma^{-1} \mathscr{E})$, which is given by the product of the Atiyah class $[\rho_{\mathrm{univ}}] \in \mathrm{Ext}^1_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{F} \otimes \mathscr{F}^{\vee}, L_{\mathsf{X}}))$ with the homotopy class $[\eta] \in \mathrm{Ext}^1_{\mathrm{QCoh}(\mathsf{X})}(L_{\mathsf{X}}, \Sigma^{-1} \mathscr{E})$ of the derivation η.

19.2.3 Diagrams of Perfect Complexes

For later applications, it will be convenient to study the deformation theory of diagrams $K \to \mathrm{Perf}(R)$, where K is a simplicial set.

Notation 19.2.3.1. For every simplicial set K, we let $\mathrm{Perf}_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor given by the formula $\mathrm{Perf}_K(R) = \mathrm{Fun}(K, \mathrm{Mod}_R^{\mathrm{perf}})^{\simeq}$. If $K = \Delta^0$, we will denote the functor Perf_K simply by Perf.

Example 19.2.3.2. When $K = \Delta^0$, the functor Perf_K of Notation 19.2.3.1 coincides with the functor $\mathrm{Perf}^{\simeq}$ of Corollary 19.2.2.3.

Proposition 19.2.3.3. *Let K be a simplicial set. Then the functor $\mathrm{Perf}_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is cohesive, nilcomplete, and admits a cotangent complex. If K is finite, then Perf_K commutes with filtered colimits and the cotangent complex L_{Perf_K} is perfect.*

Lemma 19.2.3.4. *Let R be a connective $\mathbb{E}_\infty$-ring, let M and N be R-modules, and define a functor $F : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ by the formula $F(R') = \mathrm{Map}_{\mathrm{Mod}_{R'}}(R' \otimes_R M, R' \otimes_R N) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(M, R' \otimes_R N)$. Let $\overline{F}$ denote the image of F under the equivalence of ∞-categories $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\operatorname{Spec} R}$. If M is almost connective and N is perfect, then the map $\alpha : \overline{F} \to \operatorname{Spec} R$ admits a cotangent complex. Moreover, there is a canonical equivalence*

$$L_{\overline{F}/\operatorname{Spec} R} \simeq \alpha^*(M \otimes N^\vee),$$

where $N^\vee$ denotes the R-linear dual of N.

Proof. For every connective R-algebra R' and every connective R'-module Q, the fiber of the canonical map $F(R' \oplus Q) \to F(R')$ is given by $\mathrm{Map}_{\mathrm{Mod}_R}(M, Q \otimes_R N) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(M \otimes N^\vee, Q)$. □

Proof of Proposition 19.2.3.3. The assertion that Perf_K is cohesive and nilcomplete follows from Corollary 19.2.1.4. We next show that Perf_K admits a cotangent complex. Writing K as a filtered colimit of finite simplicial sets (and using Remark 17.2.4.5), we can reduce to the case where K is finite. We proceed by induction on the dimension d of K and the number of nondegenerate d-simplices of K. If K is empty, there is nothing to prove. Otherwise, we can choose a pushout diagram of simplicial sets

$$\begin{array}{ccc} \partial \Delta^d & \longrightarrow & \Delta^d \\ \downarrow & & \downarrow \\ K' & \longrightarrow & K. \end{array}$$

Our inductive hypothesis implies that $\mathrm{Perf}_{K'}$ admits a cotangent complex. Using Proposition 17.3.9.1, we are reduced to showing that the restriction map $\mathrm{Perf}_K \to \mathrm{Perf}_{K'}$ admits a relative cotangent complex. We have a pullback diagram of functors

$$\begin{array}{ccc} \mathrm{Perf}_{\partial \Delta^d} & \xleftarrow{\beta} & \mathrm{Perf}_{\Delta^d} \\ \uparrow & & \uparrow \\ \mathrm{Perf}_{K'} & \longleftarrow & \mathrm{Perf}_K, \end{array}$$

so it will suffice to show that the map β admits a relative cotangent complex. If $d = 0$, this follows from Corollary 19.2.2.3. If $d = 1$, it follows from Lemma 19.2.3.4. For $d \geqslant 2$, we can choose a categorical equivalence $T \hookrightarrow \Delta^d$, where $T = \Delta^{\{0,1\}} \amalg_{\{1\}} \cdots \amalg_{\{d-1\}} \Delta^{\{d-1,d\}}$.

Our inductive hypothesis then implies that $\mathrm{Perf}_{\partial \Delta^d}$ and $\mathrm{Perf}_T \simeq \mathrm{Perf}_{\Delta^d}$ admit cotangent complexes, so that β admits a relative cotangent complex by virtue of Proposition 17.2.5.2.

To complete the proof, we observe that if K is finite then the functor Perf_K commutes with filtered colimits by virtue of Corollary 19.2.1.4. In this case, Proposition 17.4.2.3 implies that the cotangent complex L_{Perf_K} is perfect (this also follows from the proof given above). □

We now consider a relative version of Proposition 19.2.3.3.

Proposition 19.2.3.5. *Let R be a connective $\mathbb{E}_\infty$-ring and $f : \mathsf{X} \to \mathrm{Spét}\, R$ be a map of spectral Deligne-Mumford stacks. Define a functor $\mathrm{Perf}_{\mathsf{X}/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ by the formula*

$$\mathrm{Perf}_{\mathsf{X}/R}(R') = \mathrm{QCoh}(\mathrm{Spét}\, R' \times_{\mathrm{Spét}\, R} \mathsf{X})^{\mathrm{perf},\simeq}.$$

Let F denote the image of $\mathrm{Perf}_{\mathsf{X}/R}$ under the equivalence of ∞-categories $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\,\mathrm{Spec}\, R}$. Then:

(1) *The functor F is nilcomplete and cohesive.*

(2) *Assume that X is a quasi-compact, quasi-separated spectral algebraic space. Then the natural transformation $F \to \mathrm{Spec}\, R$ is locally of finite presentation.*

(3) *Assume that f is proper and locally almost of finite presentation. Then the functor F is integrable.*

(4) *Assume that f is proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude. Then the natural transformation $u : F \to \mathrm{Spec}\, R$ admits a perfect cotangent complex.*

Proof. Assertions (1), (2) and (4) follow from Proposition 19.1.3.1, Proposition **??**, Remark 19.1.4.2, and Proposition 19.2.3.3. To prove (3), suppose that X is a spectral algebraic space which is proper and locally almost of finite presentation over R; we wish to show that F is integrable. Let A is a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal and let $\mathrm{Spf}\, A$ denote the formal spectrum of A, which we regard as an object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. We wish to show that if $R \to A$ is a morphism of $\mathbb{E}_\infty$-rings, then the restriction functor

$$\theta : \mathrm{QCoh}(\mathrm{Spét}\, A \times_{\mathrm{Spét}\, R} \mathsf{X})^{\mathrm{perf}} \to \mathrm{QCoh}(\mathrm{Spf}\, A \times_{\mathrm{Spét}\, R} \mathsf{X})^{\mathrm{perf}}$$

is an equivalence of ∞-categories. Using Theorem 8.5.0.3 (together with Corollary 8.3.4.6 and Theorem 8.3.5.2), we deduce that the restriction functor

$$\overline{\theta} : \mathrm{QCoh}(\mathrm{Spét}\, A \times_{\mathrm{Spét}\, R} \mathsf{X})^{\mathrm{aperf}} \to \mathrm{QCoh}(\mathrm{Spf}\, A \times_{\mathrm{Spét}\, R} \mathsf{X})^{\mathrm{aperf}},$$

is an equivalence of symmetric monoidal ∞-categories. We now observe that θ is obtained from $\overline{\theta}$ by restricting to the dualizable objects of $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X})^{\mathrm{aperf}}$ and $\mathrm{QCoh}(\mathrm{Spf}\, A \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X})^{\mathrm{aperf}}$, respectively. $\square$

Remark 19.2.3.6. In the situation of Proposition 19.2.3.5, we describe the relative cotangent complex $L_{F/\operatorname{Spec} R}$ explicitly. Suppose we are given a point $\eta \in F(A)$, corresponding to a map of connective $\mathbb{E}_\infty$-rings $R \to A$ and a perfect object $\mathscr{F} \in \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X})$. Let $f' : \mathrm{Sp\acute{e}t}\, A \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathrm{Sp\acute{e}t}\, A$ denote the projection onto the first factor. Combining Remark 19.2.2.4 with Proposition 19.1.4.3, we see that $\eta^* L_{F/\operatorname{Spec} R} \in \mathrm{QCoh}(\operatorname{Spec} A) \simeq \mathrm{Mod}_A$ can be identified with $\Sigma^{-1} f'_+(\mathscr{F} \otimes \mathscr{F}^\vee) \simeq \Sigma^{-1} f'_*(\mathscr{F} \otimes \mathscr{F}^\vee)^\vee$, where f'_+ is the functor of Construction 6.4.5.1.

19.2.4 Vector Bundles

We now show that the results of §19.2.2 remain valid if we restrict our attention to perfect complexes which are locally free. First, we need a brief digression.

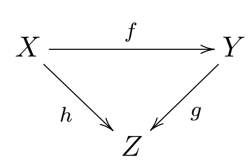
Definition 19.2.4.1. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We say that f is an *open immersion* if, for every map $\operatorname{Spec} R \to Y$, the fiber product $X \times_Y \operatorname{Spec} R$ is representable by a spectral Deligne-Mumford stack X_R, and the projection $\mathsf{X}_R \to \mathrm{Sp\acute{e}t}\, R$ is an open immersion (see Example 6.3.3.6). We note that f is an open immersion if and only the following conditions are satisfied:

(a) For every connective $\mathbb{E}_\infty$-ring R, the map $X(R) \to Y(R)$ induces a homotopy equivalence from $X(R)$ to a summand $Y_0(R) \subseteq Y(R)$.

(b) For every point $\eta \in Y(R)$, there exists an open subset $U \subseteq |\operatorname{Spec} R|$ with the following property: if $R \to R'$ is a map of connective $\mathbb{E}_\infty$-rings, then the image of η in $Y(R')$ belong to $Y_0(R')$ if and only if the map of topological spaces $|\operatorname{Spec} R'| \to |\operatorname{Spec} R|$ factors through U.

Remark 19.2.4.2. Let $f : X \to Y$ be an open immersion of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Then f is cohesive, nilcomplete, integrable, and admits a cotangent complex (Corollary 17.3.8.5). Moreover, the relative cotangent complex $L_{X/Y}$ is a zero object of $\mathrm{QCoh}(X)$. Using Proposition 17.4.3.1, we deduce that f is locally of finite presentation.

Proposition 19.2.4.3. *Suppose we are given a commutative diagram*



in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. *Assume that f is an open immersion. If g is cohesive (infinitesimally cohesive, nilcomplete, integrable, locally of finite presentation to order n, locally almost of finite presentation, locally of finite presentation), then h has the same property. If g admits a cotangent complex, then so does h; moreover, we have a canonical equivalence $L_{X/Z} \simeq f^* L_{Y/Z}$ in* $\mathrm{QCoh}(X)$.

Proof. The first assertions follow from Remark 19.2.4.2. The existence of a cotangent complex $L_{X/Z}$ follows from the existence of $L_{Y/Z}$ by virtue of the criterion supplied by Remark 17.2.4.3. □

Definition 19.2.4.4. Let $\mathrm{Perf} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor defined in Proposition 19.2.3.3. For every connective $\mathbb{E}_\infty$-ring R, we let $\mathrm{Vect}_{\simeq}(R)$ denote the summand of $\mathrm{Perf}(R)$ spanned by those perfect R-modules M which are locally free of finite rank over R, and $\mathrm{Vect}_n(R)$ the summand of $\mathrm{Perf}(R)$ spanned by those perfect R-modules which are locally free of rank n over R.

Using Proposition 2.9.3.2 and Lemma 2.9.3.4, we deduce the following:

Proposition 19.2.4.5. *For every integer $n \geqslant 0$, the inclusions $\mathrm{Vect}_n \hookrightarrow \mathrm{Vect}_{\simeq} \hookrightarrow \mathrm{Perf}$ are open immersions.*

Corollary 19.2.4.6. *The functor $\mathrm{Vect}_{\simeq} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is cohesive, nilcomplete, locally of finite presentation, and admits a perfect cotangent complex. Moreover, if $\eta \in \mathrm{Vect}_{\simeq}(R)$ classifies a locally free R-module M of finite rank, then $\eta^* L_{\mathrm{Vect}_{\simeq}} \in \mathrm{Mod}_R$ can be identified with the R-module $\Sigma^{-1}(M \otimes_R M^\vee)$.*

For every integer $n \geqslant 0$, the functor $\mathrm{Vect}_n : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is also cohesive, nilcomplete, locally of finite presentation and admits a perfect cotangent complex, given by the image of L_{Vect_n} in $\mathrm{QCoh}(\mathrm{Vect}_n)$.

Proof. Combine Proposition 19.2.3.3, Remark 19.2.2.4, Proposition 19.2.4.5, and Proposition 19.2.4.3. □

We now specialize to the study of locally free sheaves of rank 1.

Proposition 19.2.4.7. *Let R be a connective $\mathbb{E}_\infty$-ring and $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ be a map of spectral Deligne-Mumford stacks. Let F denote the image of the functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}/R}$ under the equivalence of ∞-categories $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\mathrm{Spec}\, R}$. Then:*

(1) *The functor F is nilcomplete and cohesive.*

(2) *Assume that X is a quasi-compact, quasi-separated spectral algebraic space. Then the natural transformation $F \to \mathrm{Spec}\, R$ is locally of finite presentation.*

(3) *Assume that* X *is a spectral algebraic space which is proper and locally almost of finite presentation over* R*. Then* F *is integrable.*

(4) *Assume that* X *is a spectral algebraic space which is proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude over* R*. Then the natural transformation* $F \to \operatorname{Spec} R$ *admits a perfect cotangent complex.*

Proof. Assertions (1), (2), and (4) follow from Proposition 19.1.3.1, Proposition **??**, Remark 19.1.4.2, and Corollary 19.2.4.6. To prove (3), let A be a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal and let $\operatorname{Spf} A$ denote the formal spectrum of A, which we regard as an object of $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. We must show that for any morphism of $\mathbb{E}_\infty$-rings $R \to A$, then the restriction map $\theta : \mathscr{P}\mathrm{ic}(\operatorname{Sp\acute{e}t} A \times_{\operatorname{Sp\acute{e}t} R} \mathsf{X}) \to \mathscr{P}\mathrm{ic}(\operatorname{Spf} A \times_{\operatorname{Sp\acute{e}t} R} \mathsf{X})$ is a homotopy equivalence. To prove this, we observe that Proposition 2.9.4.2 implies that θ is obtained from the symmetric monoidal forgetful functor

$$\overline{\theta} : \mathrm{QCoh}(\operatorname{Sp\acute{e}t} A \times_{\operatorname{Sp\acute{e}t} R} \mathsf{X})^{\mathrm{aperf,cn}} \to \mathrm{QCoh}(\operatorname{Spf} A \times_{\operatorname{Sp\acute{e}t} R} \mathsf{X})^{\mathrm{aperf,cn}}$$

by restricting to the subcategories spanned by invertible objects and equivalences between them. It now suffices to observe that $\overline{\theta}$ is an equivalence of symmetric monoidal ∞-categories (Theorems 8.5.0.3, 8.3.4.4, and 8.3.5.2). □

Remark 19.2.4.8. If $f : \mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to \operatorname{Sp\acute{e}t} R$ is proper, the inclusion $\mathscr{P}\mathrm{ic}_{\mathsf{X}/R} \hookrightarrow \mathrm{Perf}_{\mathsf{X}/R}$ is an open immersion of functors. In this case, we can deduce Proposition 19.2.4.7 from Propositions 19.2.3.5 and 19.2.4.3. Moreover, Remark 19.2.3.6 implies that the cotangent complex of the map $q : F \to \operatorname{Spec} R$ is given by $L_{F/\operatorname{Spec} R} \simeq \Sigma^{-1} q^*(f_* \mathscr{O}_{\mathcal{X}})^\vee$. In particular, the relative cotangent complex of F over $\operatorname{Spec} R$ is constant along the fibers of F. This is a reflection of the fact that the functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}/R}$ admits a group structure, given by the formation of tensor products.

19.2.5 Representability of the Picard Functor

We now turn to the proof of Theorem 19.2.0.5. In fact, we will prove the following slightly stronger result (which implies Theorem 19.2.0.5, by virtue of Proposition 8.6.4.1:

Proposition 19.2.5.1. *Let* $f : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ *be a morphism of spectral algebraic spaces which is flat, proper, and locally almost of finite presentation. Suppose that the cofiber of the unit map* $u : R \to f_* \mathscr{O}_{\mathsf{X}}$ *has* Tor*-amplitude* $\leqslant -1$*, and let* $x : \operatorname{Sp\acute{e}t} R \to \mathsf{X}$ *be a section of* f*. Then the functor* $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}$ *is representable by a spectral algebraic space which is quasi-separated and locally of finite presentation over* R*.*

Proof. Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the image of the functor $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}$ under the equivalence of ∞-categories $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S}) \simeq \operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\operatorname{Spec} R}$. Let $Y' = Y \times_{\operatorname{Spec} R} \operatorname{Spec}(\pi_0 R)$.

We will prove that Y' is representable by a quasi-separated spectral algebraic space Y' which is locally almost of finite presentation over $\pi_0 R$. Note that the functors Y and Y' agree on discrete $\mathbb{E}_\infty$-rings. Since the functor Y is nilcomplete, infinitesimally cohesive, and admits a cotangent complex (Proposition 19.2.4.7), it will then follow that Y is representable by a spectral Deligne-Mumford stack Y (Theorem 18.1.0.2). Note that that $\tau_{\leqslant 0}\mathsf{Y} \simeq \tau_{\leqslant 0}\mathsf{Y}'$. Since Y' is a quasi-separated spectral algebraic space, it follows immediately that Y is also a quasi-separated spectral algebraic space. Since $L_{Y/\operatorname{Spec} R}$ is perfect (Proposition 19.2.4.7), Proposition 17.4.2.3 shows that Y is locally of finite presentation over R. We may therefore replace R by $\pi_0 R$ and thereby reduce to the case where R is discrete.

Write R as the union of finitely generated subrings R_α. Using Theorem 4.4.2.2, we can choose an index α, a spectral Deligne-Mumford stack X_α which is finitely 0-presented over R_α, and an equivalence $\mathsf{X} \simeq \tau_{\leqslant 0}(\operatorname{Spét} R \times_{\operatorname{Spét} R_\alpha} \mathsf{X}_\alpha)$. Enlarging α if necessary, we may suppose that X_α is a spectral algebraic space which is proper and flat over R_α (Propositions 5.5.4.1 and 6.1.6.1). Let $f_\alpha : \mathsf{X}_\alpha \to \operatorname{Spét} R_\alpha$ denote the projection map and let M denote the cofiber of the unit map $R_\alpha \to f_{\alpha *} \mathscr{O}_{\mathsf{X}_\alpha}$. Then M is a perfect R_α-module (Theorem 6.1.3.2) and $R \otimes_{R_\alpha} M$ has Tor-amplitude $\leqslant -1$. Let $M^\vee$ be the R_α-linear dual of M, so that $R \otimes_{R_\alpha} M^\vee$ is 1-connective. Enlarging α if necessary, we may suppose that $M^\vee$ is 1-connective, so that M has Tor-amplitude $\leqslant 1$. We may therefore replace R by R_α and X by X_α, and thereby reduce to the case where R is finitely generated as a commutative ring. In particular, we may assume that R is a Grothendieck ring.

We now prove that Y is representable by a spectral algebraic space which is locally almost of finite presentation over R by verifying hypotheses (1) through (5) of Theorem 18.3.0.1. Hypothesis (2) is obvious, and hypotheses (3) and (5) follow immediately from Proposition 19.2.4.7. Let us check the remaining conditions:

(1) For every discrete commutative ring A, the space $Y(A)$ is discrete. Equivalently, we must show that if A is a discrete R-algebra, then the space $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}(A)$ is discrete. Let A be a discrete R-algebra and let $\mathscr{L}$ be a line bundle on $\mathsf{X}_A = \operatorname{Spét} A \times_{\operatorname{Spét} R} \mathsf{X}$. Then the mapping space $\operatorname{Map}_{\operatorname{QCoh}(\mathsf{X}_A)}(\mathscr{L}, \mathscr{L})$ is given by

$$\begin{aligned}
\operatorname{Map}_{\operatorname{QCoh}(\mathsf{X}_A)}(\mathscr{O}_{\mathsf{X}}, \mathscr{L} \otimes \mathscr{L}^\vee) &\simeq \operatorname{Map}_{\operatorname{QCoh}(\mathsf{X}_A)}(\mathscr{O}_{\mathsf{X}}, \mathscr{O}_{\mathsf{X}}) \\
&\simeq \operatorname{Map}_{\operatorname{Mod}_A}(A, A \otimes_R f_* \mathscr{O}_{\mathsf{X}}) \\
&\simeq \Omega^\infty A \oplus \Omega^\infty(A \otimes_R \operatorname{cofib}(u)).
\end{aligned}$$

Our assumption on the Tor-amplitude of $\operatorname{cofib}(u)$ guarantees that $\operatorname{Map}_{\operatorname{QCoh}(\mathsf{X}_A)}(\mathscr{L}, \mathscr{L})$ is homotopy equivalent to the discrete commutative ring $\pi_0 A \simeq \Omega^\infty A$. In particular, if we let $x' : \operatorname{Spét} A \to \mathsf{X}_A$ denote the map induced by x, then pullback along x' induces a homotopy equivalence $\operatorname{Map}_{\operatorname{QCoh}(\mathsf{X}_A)}(\mathscr{L}, \mathscr{L}) \to \operatorname{Map}_{\operatorname{QCoh}(\operatorname{Spec} A)}(x'^* \mathscr{L}, x'^* \mathscr{L})$. It follows that the space $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}(A)$ is discrete.

(4) The natural transformation $Y \to \operatorname{Spec} R$ admits a connective cotangent complex $L_{Y/\operatorname{Spec} R}$. Let $Z : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the image of the functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}/R}$ under the equivalence of ∞-categories

$$\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\operatorname{Spec} R},$$

and let Z_0 be the image of $\mathscr{P}\mathrm{ic}_{\mathrm{Sp\acute{e}t}\, R/R}$ under the same equivalence, so Y can be identified with the fiber of the map $Z \to Z_0$ determined by the section x. Using Remark 19.2.4.8, we deduce that the relative cotangent complexes $L_{Z/\operatorname{Spec} R}$ and $L_{Z_0/\operatorname{Spec} R}$ are given by the pullbacks of the R-modules $\Sigma^{-1}(f_* \mathscr{O}_{\mathsf{X}})^{\vee}$ and $\Sigma^{-1} R$, respectively. It follows that the relative cotangent complex $L_{Y/\operatorname{Spec} R}$ exists, and is given by the pullback of the R-linear dual of $\Sigma^1 \operatorname{fib}(f_* \mathscr{O}_{\mathsf{X}} \xrightarrow{x^*} R) \simeq \Sigma^1 \operatorname{cofib}(u)$. By assumption, $\operatorname{cofib}(u)$ has Tor-amplitude $\leqslant -1$, so that $\Sigma^{-1} \operatorname{cofib}(u)^{\vee}$ is connective and therefore $L_{Y/\operatorname{Spec} R}$ is connective as desired.

This completes the proof that the functor Y is representable by a spectral algebraic space Y which is locally almost of finite presentation over R. The above calculation shows that $L_{Y/\mathrm{Sp\acute{e}t}\, R}$ is perfect, so that Y is locally of finite presentation over R (Proposition 17.1.5.1). It remains to verify that Y is quasi-separated. Suppose we are given a pair of connective $\mathbb{E}_\infty$-rings A and B and maps $\mathrm{Sp\acute{e}t}\, A \xrightarrow{\phi} \mathsf{Y} \xleftarrow{\phi'} \mathrm{Sp\acute{e}t}\, B$; we wish to prove that the fiber product $\mathrm{Sp\acute{e}t}\, A \times_{\mathsf{Y}} \mathrm{Sp\acute{e}t}\, B$ is quasi-compact. Replacing R by $A \otimes_R B$, we may reduce to the case where $A = B = R$. Then ϕ and ϕ' determine line bundles $\mathscr{L}$ and $\mathscr{L}'$ on X equipped with trivializations of $x^* \mathscr{L}$ and $x^* \mathscr{L}'$. For every object $R' \in \mathrm{CAlg}^{\mathrm{cn}}_R$, let $\mathscr{L}_{R'}$ and $\mathscr{L}'_{R'}$ denote the pullbacks of $\mathscr{L}$ and $\mathscr{L}'$ to $\mathsf{X}_{R'}$. Define functors $F, F' : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ by the formulas

$$F(R') = \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X}_{R'})}(\mathscr{L}_{R'}, \mathscr{L}'_{R'}) \qquad F'(R') = \mathrm{Map}_{\mathrm{Mod}_{R'}}(R', R').$$

Since $f_*(\mathscr{L}' \otimes \mathscr{L}^{\vee})$ is perfect, we can identify $F(R')$ with $\mathrm{Map}_{\mathrm{Mod}_R}((f_* \mathscr{L}' \otimes \mathscr{L}^{\vee})^{\vee}, R')$. Note that $\mathscr{L}' \otimes \mathscr{L}^{\vee}$ is a line bundle on X. Since f is flat, the pushforward $f_*(\mathscr{L}' \otimes \mathscr{L}^{\vee})$ has Tor-amplitude $\leqslant 0$, so that $(f_* \mathscr{L}' \otimes \mathscr{L}^{\vee})^{\vee}$ is connective. It follows that the functor F is representable by the affine spectral Deligne-Mumford stack $\mathsf{Z} = \mathrm{Sp\acute{e}t}\, \mathrm{Sym}^*_R(f_*(\mathscr{L}' \otimes \mathscr{L}^{\vee})^{\vee})$ (see Proposition 19.1.1.1). Similarly, the functor F' is representable by the affine spectral Deligne-Mumford stack $\mathrm{Sp\acute{e}t}\, \mathrm{Sym}^*_R(R)$.

Let $g : \mathsf{Z} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathsf{X}$ be the projection onto the second factor. By construction, we have a canonical map of line bundles $\alpha : g^* \mathscr{L} \to g^* \mathscr{L}'$. Lemma 2.9.3.3 implies that there is a quasi-compact open immersion $\mathsf{U} \hookrightarrow \mathsf{Z} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ such that a map $h : \mathrm{Sp\acute{e}t}\, C \to \mathsf{Z} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ factors through U if and only if $h^* \operatorname{cofib}(\alpha)$ is 1-connective. Then U determines a constructible closed subset $K \subseteq |\, \mathsf{Z} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}\, |$. Since f is proper, the image of K is a constructible closed subset of Z, which determines a quasi-compact open immersion $\mathsf{Z}_0 \hookrightarrow \mathsf{Z}$. Unwinding the definitions, we see that Z_0 represents the subfunctor F_0 of F which carries an object $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$ to the summand of $F(R') = \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X}_{R'})}(\mathscr{L}_{R'}, \mathscr{L}'_{R'})$ consisting of equivalences of $\mathscr{L}_{R'}$ with $\mathscr{L}'_{R'}$. Unwinding the definitions, we obtain a pullback diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}\, A \times_{\mathsf{Y}} \mathrm{Sp\acute{e}t}\, B & \longrightarrow & \mathsf{Z}_0 \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}\, R & \longrightarrow & \mathsf{Z}'. \end{array}$$

It follows that $\mathrm{Sp\acute{e}t}\, A \times_{\mathsf{Y}} \mathrm{Sp\acute{e}t}\, B$ is quasi-affine (and in particular quasi-compact). □

In the situation of Proposition 19.2.5.1, if the map $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ is fiber smooth, then $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}$ satisfies the valuative criterion of properness:

Proposition 19.2.5.2. *Let $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ be a morphism of spectral algebraic spaces which is proper, fiber smooth, and geometrically connected. Then, for every object $V \in \mathrm{CAlg}_R$ which is a valuation ring with fraction field K, the map $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}(V) \to \mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}(K)$ is bijective.*

Proof. Without loss of generality we may assume that $V = R$ and the map $\mathrm{Sp\acute{e}t}\, V \to \mathrm{Sp\acute{e}t}\, R$ is the identity. Let $\mathcal{X}$ be the underlying ∞-topos of X. For every object $U \in \mathcal{X}$, let $\mathsf{X}_U = (\mathcal{X}_{/U}, \mathscr{O}_{\mathsf{X}}|_U)$. Set $\mathsf{X}_\eta = \mathrm{Sp\acute{e}t}\, K \times_{\mathrm{Sp\acute{e}t}\, V} \mathsf{X}$, and let $\mathcal{X}_\eta$ be the underlying ∞-topos of X_η, so that we have an open immersion of ∞-topoi $j_* : \mathcal{X}_0 \to \mathcal{X}$. Let $i_* : \mathcal{Y} \to \mathcal{X}$ be the complementary closed immersion, let t denote a uniformizer for the discrete valuation ring V and $\kappa = V/(t)$ its residue field.

Let $\mathscr{L}$ be a line bundle on X_η. We define a functor $F_{\mathscr{L}} : \mathcal{X}^{\mathrm{op}} \to \mathcal{S}$ by the formula

$$F_{\mathscr{L}}(U) = \mathscr{P}\mathrm{ic}(\mathsf{X}_U) \times_{\mathscr{P}\mathrm{ic}(\mathrm{Sp\acute{e}t}\, K \times_{\mathrm{Sp\acute{e}t}\, V} \mathsf{X}_U)} \{\mathscr{L}\}.$$

If U is affine, then $\mathsf{X}_U \simeq \mathrm{Sp\acute{e}t}\, A$ for some commutative ring A, and $\mathscr{L}$ determines an invertible module $M[\frac{1}{t}]$ over $A[\frac{1}{t}]$. Unwinding the definitions, we can identify $F_{\mathscr{L}}(U)$ with the set of all submodules $M \subseteq M[\frac{1}{t}]$ which are invertible over A and generate $M[\frac{1}{t}]$ as modules over $A[\frac{1}{t}]$. Note that the functor $F_{\mathscr{L}}$ is representable by a discrete object of $\mathcal{X}$, which we will denote by $F_{\mathscr{L}}$ (by a slight abuse of notation). Note that $j^* F_{\mathscr{L}}$ is a final object of $\mathcal{X}_\eta$, so we can write $F_{\mathscr{L}} \simeq i_* F'_{\mathscr{L}}$ for some object $F'_{\mathscr{L}} \in \mathcal{Y}$.

There is an evident action of the discrete group $\mathbf{Z} \simeq \mathrm{fib}(\mathscr{P}\mathrm{ic}(\mathrm{Sp\acute{e}t}\, V) \to \mathscr{P}\mathrm{ic}(\mathrm{Sp\acute{e}t}\, K))$ on $F'_{\mathscr{L}}$. We claim that this action exhibits $F'_{\mathscr{L}}$ as a $\mathbf{Z}$-torsor in the topos $\mathcal{Y}^{\heartsuit}$. To prove this, it will suffice to show that for every affine object $U \in \mathcal{X}$, the set $F_{\mathscr{L}}(U)$ is nonempty and is acted on transitively by the group $\mathrm{H}^0(|\,\mathrm{Sp\acute{e}t}\, \kappa \times_{\mathrm{Sp\acute{e}t}\, V} \mathsf{X}_U\,|; \mathbf{Z})$. Let $\mathsf{X}_U = \mathrm{Sp\acute{e}t}\, A$ and $M[\frac{1}{t}]$ be as above; we wish to show that $F_{\mathscr{L}}(U)$ is nonempty and acted on transitively by the group $\mathrm{H}^0(|\,\mathrm{Spec}\, A/tA|; \mathbf{Z})$. Note that A is smooth over V (in the sense of classical commutative algebra) and therefore regular. It follows that $M[\frac{1}{t}]$ can be identified with a fractional ideal of $A[\frac{1}{t}]$, given by a product $\prod_{1 \leqslant i \leqslant k} \mathfrak{p}_i^{m_i}$ for some integers m_i and some prime ideals $\mathfrak{p}_i \subseteq A[\frac{1}{t}]$ of height 1. Let Let $\mathfrak{p}'_i$ be the inverse image of $\mathfrak{p}$ in A, and let $\mathfrak{q}_1, \ldots, \mathfrak{q}_l$ be the set of height one prime ideals of A which contain tA. Then $F_{\mathscr{L}}(U)$ can be identified with the set of all fractional ideals of A having the form $\prod_{1 \leqslant i \leqslant k} \mathfrak{p}'^{m_i}_i \prod_{1 \leqslant j \leqslant l} \mathfrak{q}_j^{n_j}$. Note that since A is smooth over V, A/tA is smooth over the residue field of V, hence a finite product of integral domains, so that the group of continuous $\mathbf{Z}$-valued functions on $|\operatorname{Spec} A/tA|$ can be identified with $\mathbf{Z}^l$. The image of t in each localization $A_{\mathfrak{q}_j}$ is a uniformizer, so the action of $(a_1, \ldots, a_l) \in \mathbf{Z}^l$ on $F_{\mathscr{L}}(U)$ is given by

$$\prod_{1 \leqslant i \leqslant k} \mathfrak{p}'^{m_i}_i \prod_{1 \leqslant j \leqslant l} \mathfrak{q}_j^{n_j} \mapsto \prod_{1 \leqslant i \leqslant k} \mathfrak{p}'^{m_i}_i \prod_{1 \leqslant j \leqslant l} \mathfrak{q}_j^{a_j + n_j}.$$

It is now clear that this action is simply transitive.

To prove Proposition 19.2.5.2, we must show that the diagram

$$\begin{array}{ccc} \mathscr{P}\mathrm{ic}(\mathsf{X}) & \longrightarrow & \mathscr{P}\mathrm{ic}(\mathsf{X}_\eta) \\ \downarrow{\scriptstyle e^*} & & \downarrow \\ \mathscr{P}\mathrm{ic}(\operatorname{Sp\acute{e}t} V) & \longrightarrow & \mathscr{P}\mathrm{ic}(\operatorname{Sp\acute{e}t} K) \end{array}$$

is a pullback square. For this, it will suffice to show that for every line bundle $\mathscr{L}$ on X_η, the induced map

$$\theta : F_{\mathscr{L}}(\mathbf{1}) \to \mathscr{P}\mathrm{ic}(\operatorname{Sp\acute{e}t} V) \times_{\mathscr{P}\mathrm{ic}(\operatorname{Sp\acute{e}t} K)} \{\mathscr{L}\}$$

is a homotopy equivalence. Let $q_* : \mathcal{Y} \to \mathcal{S}$ be the global sections functor and q^* its left adjoint. The $\mathbf{Z}$-torsor $F'_{\mathscr{L}}$ is classified by a point $\eta \in \mathrm{H}^1(\mathcal{Y}; \mathbf{Z}) \simeq \pi_0 q_* q^* S^1$. Since $\operatorname{Sp\acute{e}t} \kappa \times_{\operatorname{Sp\acute{e}t} V} \mathsf{X}$ is smooth over κ and therefore normal, there exists a map of connected spaces $\gamma : K \to S^1$ with $\pi_1 K$ finite, and such that η belongs to the image of the map $\pi_0 q_* q^* K$ (Proposition **??**). The map γ is classified by an element in the group $\mathrm{H}^1(K; \mathbf{Z}) \simeq \mathrm{Hom}(\pi_1 K, \mathbf{Z}) \simeq 0$ and is therefore nullhomotopic, from which it follows that η is trivial. Then $F_{\mathscr{L}}(\{\mathbf{1}\})$ is nonempty and a torsor for the group $\mathrm{H}^0(|\operatorname{Sp\acute{e}t} \kappa \times_{\operatorname{Sp\acute{e}t} V} \mathsf{X}|; \mathbf{Z})$. Similarly, $\mathscr{P}\mathrm{ic}(\operatorname{Sp\acute{e}t} V) \times_{\mathscr{P}\mathrm{ic}(\operatorname{Sp\acute{e}t} K)} \{\mathscr{L}\}$ is a torsor for the group $\mathrm{H}^0(|\operatorname{Sp\acute{e}t} \kappa|; \mathbf{Z}) \simeq \mathbf{Z}$. The fiber $\operatorname{Sp\acute{e}t} \kappa \times_{\operatorname{Sp\acute{e}t} V} \mathsf{X}$ is connected, so the restriction map

$$\mathrm{H}^0(|\operatorname{Sp\acute{e}t} \kappa \times_{\operatorname{Sp\acute{e}t} V} \mathsf{X}|; \mathbf{Z}) \to \mathrm{H}^0(|\operatorname{Sp\acute{e}t} \kappa|; \mathbf{Z}) \simeq \mathbf{Z}$$

is an isomorphism, from which it follows that θ is a homotopy equivalence. □

Corollary 19.2.5.3. *Let $f : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ be a morphism of spectral algebraic spaces which is proper, fiber smooth, and geometrically connected and let Y be a closed subspace of $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/\operatorname{Sp\acute{e}t} R}$. Then the projection map $q : \mathsf{Y} \to \operatorname{Sp\acute{e}t} R$ is separated. If Y is quasi-compact, then q is proper.*

Proof. Combine Proposition 19.2.5.1, Proposition 19.2.5.2, Proposition **??**, and Corollary 5.3.1.2. □

19.2.6 Smooth Projective Space

Let S denote the sphere spectrum. In §**??**, we introduced a spectral algebraic space $\mathbf{P}^n_S$, which we referred to as *projective space of dimension* n *over* S (Construction 5.4.1.3). Let $\mathscr{O}$ denote the structure sheaf of $\mathbf{P}^n_S$. As in classical algebraic geometry, there is a tautological line bundle $\mathscr{O}(-1)$ on $\mathbf{P}^n_S$ (Construction 5.4.2.1) and map $\mathscr{O}(-1) \to \mathscr{O}^{n+1}$ (Construction 5.4.2.5) which locally exhibits $\mathscr{O}(-1)$ as a direct summand of $\mathscr{O}^{n+1}$. However, in contrast with classical algebraic geometry, the projective space $\mathbf{P}^n_S$ is not universal with respect to these properties: the universal property of Theorem 5.4.3.1 holds for ordinary commutative rings, but not for arbitrary connective $\mathbb{E}_\infty$-rings. In this section, we will construct a *different* spectral algebraic space, which we will denote by $\mathbf{P}^n_{\mathrm{sm}}$, which has the expected universal property (but suffers from other defects: see Remark **??**).

Construction 19.2.6.1 (Smooth Projective Space as a Functor)**.** Fix a nonnegative integer $n \geqslant 0$. For every connective $\mathbb{E}_\infty$-ring A, let $X(A)$ denote the subcategory of $(\mathrm{Mod}_A)_{/A^{n+1}}$ whose morphisms are equivalences and whose objects are maps $f : L \to A^{n+1}$ with the following property:

- (a) The map f admits a left homotopy inverse (that is, it exhibits L as a direct summand of A^{n+1}).
- (b) The A-module L is projective of rank 1.

Note that $X(A)$ is an essentially small Kan complex. We will regard the construction $A \mapsto X(A)$ as a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$.

Theorem 19.2.6.2. *Let $n \geqslant 0$ be a nonnegative integer. Then the functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ of Construction 19.2.6.1 is representable by a spectral algebraic space.*

Proof. Let $\mathrm{Perf}_{\Delta^1} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be as in Notation 19.2.3.1. For every connective $\mathbb{E}_\infty$-ring R, we will identify $\mathrm{Perf}_{\Delta^1}(R)$ with the Kan complex parametrizing maps $u : M \to N$, where M and N are perfect R-modules. Consider the map $\alpha : \mathrm{Perf}_{\Delta^1} \to \mathrm{Perf}$ given on R-valued points by $(u : M \to N) \mapsto N \in \mathrm{Mod}^{\mathrm{perf}}_R$. Let S denote the sphere spectrum, and let $\beta : \mathrm{Spec}\, S \to \mathrm{Perf}$ be the map classifiying the perfect S-module S^{n+1}. By definition, we can identify X with a subfunctor of the fiber product $\mathrm{Perf}_{\Delta^1} \times_{\mathrm{Perf}} \mathrm{Spec}\, S$, whose R-valued points are given by maps of connective perfect R-modules $(u : M \to R^{n+1})$ for which $\mathrm{cofib}(u)$ is locally free of rank n (which implies that M is locally free of rank 1). Using Proposition 19.2.4.5, we see that this identification determines an open immersion $j : X \hookrightarrow \mathrm{Perf}_{\Delta^1} \times_{\mathrm{Perf}} \mathrm{Spec}\, S$. Applying Propositions 19.2.3.3 and 19.2.4.3, we deduce that the functor X is cohesive, nilcomplete, and admits a perfect cotangent complex. Note that the restriction $X|_{\mathrm{CAlg}^{\heartsuit}}$ is representable by the spectral algebraic space $\mathbf{P}^n_S$ (Theorem 5.4.3.1). Applying Theorem 18.1.0.2, we deduce that the functor X is also representable by a spectral

Deligne-Mumford stack (which has the same 0-truncation as $\mathbf{P}^n_S$, and is therefore a schematic spectral algebraic space). □

Definition 19.2.6.3 (Smooth Projective Space)**.** Let $n \geqslant 0$ be a nonnegative integer. We let $\mathbf{P}^n_{\mathrm{sm}}$ denote a spectral algebraic space which represents the functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ (note that $\mathbf{P}^n_{\mathrm{sm}}$ is well-defined up to equivalence, by virtue of Proposition 1.6.4.2). We will refer to $\mathbf{P}^n_{\mathrm{sm}}$ as *smooth projective space of dimension* n (see Remark 19.2.6.5 below).

Remark 19.2.6.4. It is possible to prove Theorem 19.2.6.2 by explicitly constructing the smooth projective space $\mathbf{P}^n_{\mathrm{sm}}$, using a variant of Construction 5.4.1.3.

Remark 19.2.6.5. Let R be a connective $\mathbb{E}_\infty$-ring and let η be an R-valued point of $\mathbf{P}^n_{\mathrm{sm}}$, classifying a fiber sequence of R-modules $L \to R^{n+1} \to M$ where L and M are locally free of rank 1 and n, respectively. Then the pullback $\eta^* L_{\mathbf{P}^n_{\mathrm{sm}}}$ can be identified with the tensor product $L^{-1} \otimes_R M$ (this can be deduced from the results of §19.2.2, or directly from the definitions). In particular, the cotangent complex $L_{\mathbf{P}^n_{\mathrm{sm}}}$ is locally free of rank n. Applying Proposition 17.1.5.1, we deduce that $\mathbf{P}^n_{\mathrm{sm}}$ is differentially smooth over the sphere spectrum S.

Remark 19.2.6.6. Let $\mathbf{P}^n_S$ be as in Construction 5.4.1.3. The tautological map $\mathscr{O}(-1) \to \mathscr{O}^{n+1}$ on $\mathbf{P}^n_S$ determines a morphism of spectral algebraic spaces $e : \mathbf{P}^n_S \to \mathbf{P}^n_{\mathrm{sm}}$. For $n > 0$, this morphism is *not* an equivalence. However, it is not far off:

(a) The morphism e induces an equivalence from the 0-truncation of $\mathbf{P}^n_S$ to the 0-truncation of $\mathbf{P}^n_{\mathrm{sm}}$. In other words, the projective spaces $\mathbf{P}^n_S$ and $\mathbf{P}^n_{\mathrm{sm}}$ represent the same functor on the category of ordinary commutative rings (this is essentially a reformulation of Theorem 5.4.3.1).

(b) For every connective $\mathbb{E}_\infty$-ring R, the morphism e induces a map

$$e_R : \mathbf{P}^n_R \simeq \mathrm{Sp\acute{e}t}\, R \times_{\mathrm{Sp\acute{e}t}\, S} \mathbf{P}^n_S \xrightarrow{e} \mathrm{Sp\acute{e}t}\, R \times_{\mathrm{Sp\acute{e}t}\, S} \mathbf{P}^n_{\mathrm{sm}}.$$

If R is discrete, then this map exhibits $\mathbf{P}^n_R$ as the 0-truncation of $\mathrm{Sp\acute{e}t}\, R \times_{\mathrm{Sp\acute{e}t}\, S} \mathbf{P}^n_{\mathrm{sm}}$ (this follows from (a), since $\mathbf{P}^n_R$ is flat over R).

(c) If $R = \mathbf{Q}$ (or, more generally, if R is an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$), then the map e_R is an equivalence This follows from (b) and Remark 19.2.6.5, since every differentially smooth $\mathbf{Q}$-algebra is fiber smooth (Proposition 11.2.4.4) and therefore discrete.

Remark 19.2.6.7. Let us contrast the projective space $\mathbf{P}^n_S$ of Construction 5.4.1.3 with the smooth projective space $\mathbf{P}^n_{\mathrm{sm}}$ of Definition 19.2.6.3 (for $n > 0$):

(i) The projective space $\mathbf{P}^n_S$ is fiber smooth over the sphere spectrum S. In particular, it is flat over S. However, the smooth projective space $\mathbf{P}^n_{\mathrm{sm}}$ is not flat over S.

(ii) The smooth projective space $\mathbf{P}^n_{\mathrm{sm}}$ is differentially smooth over S. In particular, the cotangent complex $L_{\mathbf{P}^n_{\mathrm{sm}}} \in \mathrm{QCoh}(\mathbf{P}^n_{\mathrm{sm}})$ is locally free of finite rank. However, the cotangent complex $L_{\mathbf{P}^n_S} \in \mathrm{QCoh}(\mathbf{P}^n_S)$ is not perfect (though it is almost perfect).

(iii) On the projective space $\mathbf{P}^n_S$, the the line bundles $\mathscr{O}(d)$ have the "expected" cohomology (mirroring Serre's calculation in classical algebraic geometry): for example, if $d \geqslant 0$, then $\Gamma(\mathbf{P}^n_S; \mathscr{O}(d))$ is a locally free S-module of rank $\binom{n+d}{n}$ (Theorem 5.4.2.6). These line bundles extend to the smooth projective space $\mathbf{P}^n_{\mathrm{sm}}$, but their cohomology is substantially more complicated (even for $d = 0$).

(iv) The smooth projective space $\mathbf{P}^n_{\mathrm{sm}}$ is of finite presentation over S, while the projective space $\mathbf{P}^n_S$ is only almost of finite presentation over S.

(v) The projective space $\mathbf{P}^n_S$ is more rigid than its smooth counterpart $\mathbf{P}^n_{\mathrm{sm}}$. For example, there is no automorphism of $\mathbf{P}^1_S$ which exchanges the S-valued points corresponding to 0 and 1, while the smooth projective space $\mathbf{P}^1_{\mathrm{sm}}$ does admit such an automorphism.

(vi) The smooth projective space $\mathbf{P}^n_{\mathrm{sm}}$ represents an easily-described functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ (Construction 19.2.6.1). It is also possible to specify the functor represented by $\mathbf{P}^n_S$, but the description is more complicated.

19.3 Application: Existence of Dilatations

Let X be a Noetherian algebraic space, let $U \subseteq X$ be an open subspace, and let $\mathfrak{X}$ be the formal completion of X along the complement of U. Roughly speaking, one can think of X as obtained by gluing together U and $\mathfrak{X}$ along the "intersection" $\mathfrak{X}_U = U \cap \mathfrak{X}$ (which we view here as a purely heuristic object, though it can be given meaning in the setting of rigid analytic geometry). One prediction of this philosophy is that changing the algebraic space X while keeping U fixed should be equivalent to changing the formal algebraic space $\mathfrak{X}$ while keeping $\mathfrak{X}_U$ fixed. To make this idea more precise, it is convenient to introduce some terminology:

Definition 19.3.0.1. Let $f : X' \to X$ be a morphism of Noetherian algebraic spaces and let $K \subseteq |X|$ be a closed subset with inverse image $K' = f^{-1}(K)$. We will say that f is a *modification centered at* (K', K) if it is proper and the induced map $X' - K' \to X - K$ is an isomorphism. In this case, we will also say that X' is a *dilatation of* X *along* K or that X is a *contraction of* X' *along* K'.

In [5], Artin introduced the notion of a *formal modification* between formal algebraic spaces. Under mild hypotheses, Artin showed that if $\mathfrak{X}$ is the formal completion of a

Noetherian algebraic space X along a closed subset $K \subseteq |X|$, one has equivalences

$$\{ \text{ Dilatations of } X \text{ centered } K \} \xrightarrow{\sim} \{ \text{ Formal modifications } \mathfrak{X}' \to \mathfrak{X} \} \quad (19.1)$$

$$\{ \text{ Contractions of } X \text{ centered } K \} \xrightarrow{\sim} \{ \text{ Formal modifications } \mathfrak{X} \to \mathfrak{X}' \}. \quad (19.2)$$

Our goal in this section is to establish an analogue of (19.1) in the setting of spectral algebraic geometry. We begin in §?? by introducing the notion of a *formal modification* $f : \mathfrak{X}' \to \mathfrak{X}$ between formal spectral Deligne-Mumford stacks (Definition 19.3.1.3). Roughly speaking, the notion of formal modification abstracts those properties one would expect to see if f arose by formally completing a morphism of spectral Deligne-Mumford stacks $F : \mathsf{X}' \to \mathsf{X}$ which is proper, locally almost of finite presentation, and an equivalence over the complement of $|\,\mathfrak{X}\,| = K \subseteq |\,\mathsf{X}\,|$. In §??, we show that in this case, the spectral Deligne-Mumford stack X' can be recovered from X and the formal modification f: that is, the spectral analogue of (19.1) is fully faithful (see Proposition 19.3.2.1). The hard part is to show that it is essentially surjective: that is, that every formal modification $f : \mathfrak{X}' \to \mathfrak{X} = \mathsf{X}^{\wedge}_{K}$ arises as the formal completion of some map $F : \mathsf{X}' \to \mathsf{X}$. To prove this, we can work locally and thereby reduce to the case where $\mathsf{X} = \operatorname{Sp\acute{e}t} R$ for some connective $\mathbb{E}_{\infty}$-ring R. In this case, one might hope to recover X' as the Weil restriction of $\mathfrak{X}'$ along the canonical map $\operatorname{Spf} R \to \operatorname{Sp\acute{e}t} R$. In §19.3.3, we combine ideas of §?? with the spectral Artin representability theorem to show that this Weil restriction is representable by a spectral algebraic space X. The remaining difficulty is to show that X is proper over R, which we deduce from a general criterion established in §19.3.4.

19.3.1 Dilatations and Formal Modifications

We begin by adapting Definition 19.3.0.1 to the setting of spectral algebraic geometry.

Definition 19.3.1.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and let $K \subseteq |\,\mathsf{Y}\,|$ be a cocompact closed subset. We will say that X is a *dilatation of* Y *centered at* K if f is proper, locally almost of finite presentation, and the induced map $\mathsf{X} - f^{-1}(K) \to \mathsf{Y} - K$ is an equivalence. We let $\operatorname{Dil}_K(\mathsf{Y})$ denote the full subcategory of $\operatorname{SpDM}_{/\mathsf{Y}}$ spanned by those maps $f : \mathsf{X} \to \mathsf{Y}$ which are dilatations centered at K. We will refer to $\operatorname{Dil}_K(\mathsf{Y})$ as the *∞-category of dilatations of* Y *centered at* K.

If $f : \mathsf{X} \to \mathsf{Y}$ is any morphism of spectral Deligne-Mumford stacks and $K \subseteq |\,\mathsf{Y}\,|$ any cocompact closed subset, we obtain an induced map of formal completions $f^{\wedge}_{K} : \mathsf{X}^{\wedge}_{f^{-1}K} \to \mathsf{Y}^{\wedge}_{K}$. Our next goal is to describe some special features enjoyed by $f^{\wedge}_{K}$ in the special case where X is a dilatation of Y centered at K. We take our cue from the following simple observation:

Proposition 19.3.1.2. *Let* $f : \mathsf{X} \to \mathsf{Y}$ *be a morphism of spectral Deligne-Mumford stacks. Then* f *is an equivalence if and only if it satisfies the following conditions:*

(1) *The map f is proper and locally almost of finite presentation.*

(2) *The relative cotangent complex $L_{\mathsf{X}/\mathsf{Y}}$ vanishes.*

(3) *The unit map $\mathscr{O}_{\mathsf{Y}} \to f_* \mathscr{O}_{\mathsf{X}}$ is an equivalence.*

Proof. The necessity of conditions (1), (2), and (3) is obvious. For the converse, suppose that (1), (2), and (3) are satisfied. Combining (1) and (2) with Proposition 17.1.5.1, we deduce that f is étale. In particular, f is locally quasi-finite, so the properness of f guarantees that f is affine (Proposition 17.1.5.1). It now follows from (3) that f is an equivalence, as desired. □

Now suppose that $f : \mathsf{X} \to \mathsf{Y}$ is a formal modification of Y along $K \subseteq |\mathsf{Y}|$. Using Proposition 19.3.1.2, we deduce that the quasi-coherent sheaves $L_{\mathsf{X}/\mathsf{Y}}$ and $\mathrm{cofib}(\mathscr{O}_{\mathsf{Y}} \to f_* \mathscr{O}_{\mathsf{X}})$ are supported on the closed subsets $f^{-1}(K)$ and K, respectively. This motivates the following:

Definition 19.3.1.3. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal spectral Deligne-Mumford stacks. We will say that f is a *formal modification* if it satisfies the following conditions:

(1) The map f is representable, proper, and locally almost of finite presentation. In other words, for every map $\mathrm{Sp\acute{e}t}\, R \to \mathfrak{Y}$, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} \mathrm{Sp\acute{e}t}\, R$ is a spectral algebraic space which is proper and locally almost of finite presentation over R.

(2) The completed cotangent complex $L^{\wedge}_{\mathfrak{X}/\mathfrak{Y}} \in \mathrm{QCoh}(\mathfrak{X})$ (Definition 17.1.2.8) is nilcoherent (Definition 8.2.1.1).

(3) The cofiber of the unit map $\mathscr{O}_{\mathfrak{Y}} \to f_* \mathscr{O}_{\mathfrak{X}}$ is a nilcoherent object of $\mathrm{QCoh}(\mathfrak{Y})$.

Warning 19.3.1.4. Definition 19.3.1.3 is somewhat different from the notion of formal modification which appears [5]: in particular, we do not require *a priori* that f satisfy a "valuative criterion" for complete discrete valuation rings.

Proposition 19.3.1.5. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks which exhibits X as a dilatation of Y along a cocompact closed subset $K \subseteq |\mathsf{Y}|$. Set $\mathfrak{X} = \mathsf{X}^{\wedge}_{f^{-1}K}$ and $\mathfrak{Y} = \mathsf{Y}^{\wedge}_{K}$, so that we have a pullback diagram of formal spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{g} & \mathsf{X} \\ \downarrow{\scriptstyle f^{\wedge}} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y} & \xrightarrow{g'} & \mathsf{Y}. \end{array}$$

Then $f^{\wedge}$ is a formal modification.

Proof. Note that $L_{\mathsf{X}/\mathsf{Y}}$ and $\mathrm{cofib}(\mathscr{O}_{\mathsf{Y}} \to f_* \mathscr{O}_{\mathsf{X}})$ are almost perfect (Proposition 17.1.5.1 and Theorem 5.6.0.2). Applying Corollary 8.4.1.7, we obtain equivalences

$$L_{\mathfrak{X}/\mathfrak{Y}}^{\wedge} \simeq g^{\star} L_{\mathsf{X}/\mathsf{Y}} \simeq g^{*} L_{\mathsf{X}/\mathsf{Y}}$$

$$\mathrm{cofib}(\mathscr{O}_{\mathfrak{Y}} \to f_*^{\wedge} \mathscr{O}_{\mathfrak{X}}) \simeq g'^{\star} \mathrm{cofib}(\mathscr{O}_{\mathsf{Y}} \to f_* \mathscr{O}_{\mathsf{X}}) \simeq g'^{*} \mathrm{cofib}(\mathscr{O}_{\mathsf{Y}} \to f_* \mathscr{O}_{\mathsf{X}}).$$

Since $L_{\mathsf{X}/\mathsf{Y}}$ and $\mathrm{cofib}(\mathscr{O}_{\mathsf{Y}} \to f_* \mathscr{O}_{\mathsf{X}})$ are supported on $f^{-1}(K)$ and K, respectively, it follows from Proposition 8.3.2.6 that $L_{\mathfrak{X}/\mathfrak{Y}}^{\wedge}$ and $\mathrm{cofib}(\mathscr{O}_{\mathfrak{Y}} \to f_*^{\wedge} \mathscr{O}_{\mathfrak{X}})$ are nilcoherent, as desired. □

Remark 19.3.1.6. Suppose we are given a pullback diagram of formal spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}. \end{array}$$

where g (and therefore also g') is representable. If f is a formal modification, then so is f' (see Corollary 8.4.1.8).

Notation 19.3.1.7. Let $\mathfrak{Y}$ be a formal spectral Deligne-Mumford stack. We let $\mathrm{Dil}^{\mathrm{f}}(\mathfrak{Y})$ denote the full subcategory of $\mathrm{fSpDM}_{/\mathfrak{Y}}$ spanned by those maps $f : \mathfrak{X} \to \mathfrak{Y}$ where f is a formal modification. We will refer to $\mathrm{Dil}^{\mathrm{f}}(\mathfrak{Y})$ as the *∞-category of formal dilatations of* $\mathfrak{Y}$.

We can now formulate the main result of this section.

Theorem 19.3.1.8. *Let* Y *be a spectral Deligne-Mumford stack satisfying the following condition:*

(∗) *For every étale map* $\mathrm{Spét}\, R \to \mathsf{Y}$*, the* $\mathbb{E}_\infty$*-ring* R *is Noetherian and* $\pi_0 R$ *is a Grothendieck ring.*

Let K *be a closed subset of* $|\mathsf{Y}|$*. Then the construction* $\mathsf{X} \mapsto \mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}_K^{\wedge}$ *induces an equivalence of* ∞*-categories* $\rho : \mathrm{Dil}_K(\mathsf{Y}) \to \mathrm{Dil}^{\mathrm{f}}(\mathsf{Y}_K^{\wedge})$.

The proof of Theorem 19.3.1.8 will occupy our attention throughout this section. We proceed in three steps:

- We first show that the construction $\mathsf{X} \mapsto \mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}_K^{\wedge}$ is fully faithful (Proposition 19.3.2.1).
- We show that if $\mathsf{Y} = \mathrm{Spét}\, R$ is affine and $f_0 : \mathfrak{X} \to \mathsf{Y}_K^{\wedge}$ is a formal modification, then the Weil restriction $\mathrm{Res}_{\mathsf{Y}_K^{\wedge}/\mathsf{Y}}(\mathfrak{X})$ is representable by a spectral algebraic space X which is locally almost of finite presentation over R (Proposition 19.3.3.2).

- If $\mathsf{Y} = \operatorname{Sp\acute{e}t} R$ is affine and X is a spectral algebraic space which is locally almost of finite presentation over R, for which $\mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}^{\wedge}_{K} \to \mathsf{Y}^{\wedge}_{K}$ is a formal modification and the projection map $\mathsf{X} \times_{\mathsf{Y}} (\mathsf{Y} - K) \to \mathsf{Y} - K$ is an equivalence, then X is automatically proper over R (Proposition 19.3.4.1).

Assuming these results, we can give the proof of Theorem 19.3.1.8:

Proof of Theorem 19.3.1.8. Set $\mathfrak{Y} = \mathsf{Y}^{\wedge}_{K}$. The assertion that the construction $\mathsf{X} \mapsto \mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}^{\wedge}_{K}$ induces an equivalence of ∞-categories $\rho : \operatorname{Dil}_K(\mathsf{Y}) \to \operatorname{Dil}^{\mathrm{f}}(\mathsf{Y}^{\wedge}_{K})$ is local on Y. We may therefore assume without loss of generality that $\mathsf{Y} = \operatorname{Sp\acute{e}t} R$ is affine. The functor ρ is fully faithful by virtue of Proposition 19.3.2.1, so it will suffice to show that ρ is essentially surjective. Let $\mathfrak{X}$ be a formal modification of $\mathsf{Y}^{\wedge}_{K}$, which we will identify with an object of the ∞-category $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Applying Proposition 19.3.3.2, we deduce that the Weil restriction $\operatorname{Res}_{\mathsf{Y}^{\wedge}_{K}/\mathsf{Y}}(\mathfrak{X})$ is representable by a spectral algebraic space X which is locally almost of finite presentation over R. Unwinding the definitions, we obtain equivalences

$$\mathfrak{X} \simeq \mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}^{\wedge}_{K} \qquad \mathsf{X} \times_{\mathsf{Y}} (\mathsf{Y} - K) \simeq \mathsf{Y} - K.$$

It follows from Proposition 19.3.4.1 that X is proper over R, and is therefore a dilatation of $\operatorname{Sp\acute{e}t} R$ along K which is a preimage of $\mathfrak{X}$ under the functor ρ. □

19.3.2 Full Faithfulness

Our first step is to show that the functor $\rho : \operatorname{Dil}_K(\mathsf{Y}) \to \operatorname{Dil}^{\mathrm{f}}(\mathsf{Y}^{\wedge}_{K})$ appearing in the statement of Theorem 19.3.1.8 is fully faithful. This is an easy consequence of Proposition 9.2.4.4, and does not require any Noetherian hypotheses on Y:

Proposition 19.3.2.1. *Let Y be a spectral Deligne-Mumford stack, let $K \subseteq |\mathsf{Y}|$ be a cocompact closed subset, and let $\mathfrak{Y} = \mathsf{Y}^{\wedge}_{K}$ denote the formal completion of Y along K. Then the construction $\mathsf{X} \mapsto \mathsf{X} \times_{\mathsf{Y}} \mathfrak{Y}$ induces a fully faithful embedding of ∞-categories $\operatorname{Dil}_K(\mathsf{Y}) \to \operatorname{Dil}^{\mathrm{f}}(\mathfrak{Y})$.*

Note that the assertion of Proposition **??** is local on Y, so we can assume without loss of generality that $\mathsf{Y} = \operatorname{Sp\acute{e}t} R$ is affine. In this case, the subset $K \subseteq |\operatorname{Spec} R|$ can be written as the vanishing locus of a finitely generated ideal $I \subseteq \pi_0 R$. If R is I-complete, then Proposition **??** is an immediate consequence of Corollary 8.5.3.4 (which guarantees that the pullback functor $\mathsf{X} \mapsto \mathsf{X} \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Spf} R$ is fully faithful on *all* spectral algebraic spaces which are proper and almost of finite presentation over R, not only those which are dilatations centered at K). To handle the general case, we observe that there is a commutative diagram of ∞-categories

$$\begin{array}{ccc} \operatorname{Dil}_K(\operatorname{Sp\acute{e}t} R) & \longrightarrow & \operatorname{Dil}^{\mathrm{f}}(\operatorname{Spf} R) \\ \downarrow & & \downarrow \sim \\ \operatorname{Dil}_{K'}(\operatorname{Sp\acute{e}t} R^{\wedge}_{I}) & \longrightarrow & \operatorname{Dil}^{\mathrm{f}}(\operatorname{Spf} R^{\wedge}_{I}), \end{array}$$

where K' denotes the inverse image of K in $|\operatorname{Spec} R^{\wedge}_I|$. We are therefore reduced to proving the following:

Lemma 19.3.2.2. *Let R be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal defining a closed subset $K \subseteq |\operatorname{Spec} R|$, let $R^{\wedge}_I$ denote the I-completion of R, and let $K' \subseteq |\operatorname{Spec} R^{\wedge}_I|$ denote the inverse image of K. Then pullback along the map $\operatorname{Sp\acute{e}t} R^{\wedge}_I \to \operatorname{Sp\acute{e}t} R$ induces a fully faithful functor $\operatorname{Dil}_K(R) \to \operatorname{Dil}_{K'}(R^{\wedge}_I)$.*

Proof. Let X and Y be spectral Deligne-Mumford stacks over R, and set

$$\mathsf{X}' = \mathsf{X} \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Sp\acute{e}t} R^{\wedge}_I \qquad \mathsf{Y}' = \mathsf{Y} \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Sp\acute{e}t} R^{\wedge}_I .$$

We then have a canonical map

$$\theta_{\mathsf{X},\mathsf{Y}} : \operatorname{Map}_{\operatorname{SpDM}_R}(\mathsf{X}, \mathsf{Y}) \to \operatorname{Map}_{\operatorname{SpDM}_R}(\mathsf{X}', \mathsf{Y}) \simeq \operatorname{Map}_{\operatorname{SpDM}_{R^{\wedge}_I}}(\mathsf{X}', \mathsf{Y}').$$

To prove Lemma 19.3.2.2, we must show that $\theta_{\mathsf{X},\mathsf{Y}}$ is an equivalence whenever X and Y are dilatations of $\operatorname{Sp\acute{e}t} R$ along K. In fact, we will prove a more general assertion: the map $\theta_{\mathsf{X},\mathsf{Y}}$ is an equivalence whenever Y is a dilatation of $\operatorname{Sp\acute{e}t} R$ along K (and X is arbitrary). To show this, we can work locally on X, and thereby reduce to the case where $\mathsf{X} = \operatorname{Sp\acute{e}t} A$ is affine. Let U be the open substack of X complementary to the vanishing locus of I, and set $\mathsf{U}' = \mathsf{X}' \times_{\mathsf{X}} \mathsf{U}$. Applying Proposition 9.2.4.4, we deduce that the diagram

$$\begin{array}{ccc} \operatorname{Map}_{\operatorname{SpDM}_R}(\mathsf{X}, \mathsf{Y}) & \xrightarrow{\theta_{\mathsf{X},\mathsf{Y}}} & \operatorname{Map}_{\operatorname{SpDM}_R}(\mathsf{X}', \mathsf{Y}) \\ \downarrow & & \downarrow \\ \operatorname{Map}_{\operatorname{SpDM}_R}(\mathsf{U}, \mathsf{Y}) & \longrightarrow & \operatorname{Map}_{\operatorname{SpDM}_R}(\mathsf{U}', \mathsf{Y}) \end{array}$$

is a pullback square. It now suffices to observe that the bottom horizontal map is a homotopy equivalence, because the mapping spaces $\operatorname{Map}_{\operatorname{SpDM}_R}(\mathsf{U}, \mathsf{Y})$ and $\operatorname{Map}_{\operatorname{SpDM}_R}(\mathsf{U}', \mathsf{Y})$ are contractible. □

19.3.3 Weil Restriction along $\operatorname{Spf} R \to \operatorname{Sp\acute{e}t} R$

To deduce Theorem 19.3.1.8 from Proposition 19.3.2.1, we need to address the following:

Question 19.3.3.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a formal modification of formal spectral Deligne-Mumford stacks. Suppose that $\mathfrak{Y}$ is the formal completion of a spectral Deligne-Mumford stack Y along a cocompact closed subset $K \subseteq |\mathsf{Y}|$. Can f be obtained as the formal completion of a dilatation $\mathsf{X} \to \mathsf{Y}$ along K?

By virtue of Proposition 19.3.2.1, we know that the spectral Deligne-Mumford stack X is unique (up to equivalence) if it exists. Following Artin ([5]), we proceed in two steps: first, we construct X as an object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ (that is, we guess a candidate for the functor represented by the hypothetical dilatation X), and then argue that (under some mild assumptions) this functor is actually representable. This functor is easy to describe: it is given by the Weil restriction of $\mathfrak{X}$ along the map $\mathfrak{Y} \to \mathsf{Y}$. To simplify the discussion, let us assume that $\mathsf{Y} = \mathrm{Spét}\, R$ is affine. In this case, the representability is a consequence of the following assertion:

Proposition 19.3.3.2. *Let R be an adic $\mathbb{E}_\infty$-ring and let $f : \mathfrak{X} \to \mathrm{Spf}\, R$ be a morphism of formal spectral Deligne-Mumford stacks. Let us abuse notation by identifying $\mathfrak{X}$, $\mathrm{Spf}\, R$, and $\mathrm{Spét}\, R$ with the corresponding functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and let $X = \mathrm{Res}_{\mathrm{Spf}\, R / \mathrm{Spét}\, R}(\mathfrak{X})$ denote the Weil restriction of $\mathfrak{X}$ along the map $\mathrm{Spf}\, R \to \mathrm{Spét}\, R$ (see Construction 19.1.2.3). Suppose that the following conditions are satisfied:*

(a) The morphism f is representable by spectral algebraic spaces and locally almost of finite presentation (that is, for every map $\mathrm{Spét}\, A \to \mathrm{Spf}\, R$, the fiber product $\mathrm{Spét}\, A \times_{\mathrm{Spf}\, R} \mathfrak{X}$ is a spectral algebraic space which is locally almost of finite presentation over A.

(b) The $\mathbb{E}_\infty$-ring R is Noetherian and $\pi_0 R$ is a Grothendieck commutative ring.

(c) The completed cotangent complex $L^{\wedge}_{\mathfrak{X} / \mathrm{Spf}\, R} \in \mathrm{QCoh}(\mathfrak{X})$ is nilcoherent.

Then the functor X is representable by a spectral algebraic space which is locally almost of finite presentation over R.

The proof of Proposition 19.3.3.2 will require some preliminaries.

Lemma 19.3.3.3. *Let A be a commutative ring equipped with a ring homomorphism $\mathbf{Z}[t] \to A$. For each integer $n \geqslant 0$, set $A(n) = A \otimes_{\mathbf{Z}[t]} \mathbf{Z}[t]/(t^n)$. Suppose we are given an A-module M and a morphism $\phi : N \to A(n) \otimes_A M$ in $\mathrm{Mod}_{A(n)}$. If N admits the structure of an $A(m)$-module for some $m \leqslant n$, then the composite map*

$$N \to A(n) \otimes_A M \to A(n-m) \otimes_A M$$

is nullhomotopic (as a morphism in $\mathrm{Mod}_{A(n)}$).

Proof. Let us identify $A(n) \otimes_A M$ with the mapping object $\underline{\mathrm{Map}}_A(A(n), \Sigma M)$. Without loss of generality, it suffices to treat the universal case

$$\begin{aligned} N &= \underline{\mathrm{Map}}_{A(n)}(A(m), A(n) \otimes_A M) \\ &\simeq \underline{\mathrm{Map}}_{A(n)}(A(m), \underline{\mathrm{Map}}_A(A(n), M)) \\ &\simeq \underline{\mathrm{Map}}_A(A(m), \Sigma M). \end{aligned}$$

In this case, the cofiber $\mathrm{cofib}(\phi)$ can be identified with

$$\underline{\mathrm{Map}}_A(\mathrm{fib}(A(n) \to A(m)), \Sigma M) \simeq A(n-m) \otimes_A M.$$

□

Lemma 19.3.3.4. *Let A be a Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to an ideal $I \subseteq \pi_0 A$, and regard $\mathrm{Spf}\, A$ as an object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Suppose we are given a faithfully flat affine étale morphism $\phi : X \to \mathrm{Spf}\, A$. Then we can write $X \simeq \mathrm{Spf}\, C$, where C is a Noetherian $\mathbb{E}_\infty$ which is complete with respect to the ideal $J = I(\pi_0 C) \subseteq \pi_0 C$. Moreover, C is faithfully flat over A.*

Proof. Choose a tower of A-algebras $\{A_n\}_{n>0}$ satisfying the requirements of Lemma 8.1.2.2. Then we can write $X \times_{\mathrm{Spf}\, A} \mathrm{Spec}\, A_1 \simeq \mathrm{Spec}\, B_1$, where B_1 is an étale A_1-algebra. Using the structure theory of étale morphisms (Proposition B.1.1.3), we can choose an étale map $A \to B$ such that $B_1 \simeq A_1 \otimes_A B$. For each $n \geqslant 0$, set $B_n = A_n \otimes_A B$. Since each of the maps $\mathrm{CAlg}^{\text{ét}}_{A_{n+1}} \to \mathrm{CAlg}^{\text{ét}}_{A_n}$ is an equivalence of ∞-categories, we can choose a compatible family of equivalences $X \times_{\mathrm{Spf}\, A} \mathrm{Spec}\, A_n \simeq \mathrm{Spec}\, B_n$, so that $X \simeq \varinjlim_n X \times_{\mathrm{Spf}\, A} \mathrm{Spec}\, A_n$ can be identified with the formal spectrum of B with respect to the ideal $I' \subseteq \pi_0 B$ generated by the image of I. Let C be the I'-completion of B. Since B is étale over A, it is Noetherian. It follows that C is Noetherian and flat over B, hence flat over A. To complete the proof, it will suffice to show that C is faithfully flat over A. For this, it will suffice to prove that the image of the map $|\mathrm{Spec}\, C| \to |\mathrm{Spec}\, A|$ contains every maximal ideal of $\pi_0 A$. This follows from the commutativity of the diagram

$$\begin{array}{ccc} |\mathrm{Spec}\, B_0| & \longrightarrow & |\mathrm{Spec}\, C| \\ \downarrow & & \downarrow \\ |\mathrm{Spec}(\pi_0 A)/I| & \longrightarrow & |\mathrm{Spec}\, A|, \end{array}$$

since I is contained in every maximal ideal of $\pi_0 A$ by virtue of Remark 7.3.4.10. □

Proof of Proposition 19.3.3.2. Let R be an adic $\mathbb{E}_\infty$-ring with ideal of definition $I \subseteq \pi_0 R$, and let $\mathfrak{X} \to \mathrm{Spf}\, R$ be a morphism of formal spectral Deligne-Mumford stacks which satisfies hypotheses (a) through (c) of Proposition 19.3.3.2. In what follows, we will abuse terminology by identifying formal spectral Deligne-Mumford stacks with the functors that they represent. Let $X \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ denote the Weil restriction of (the functor represented by) $\mathfrak{X}$ along the map $\mathrm{Spf}\, R \to \mathrm{Sp\acute{e}t}\, R$. We wish to show that X is representable by a spectral algebraic space which is locally almost of finite presentation over R.

Choose a tower of R-algebras $\{R_n\}_{n>0}$ satisfying the requirements of Lemma 8.1.2.2, so that $\mathrm{Spf}\, R \simeq \varinjlim \mathrm{Spec}\, R_n$ (Proposition 8.1.5.2). For each integer $n \geqslant 0$, set $X(n)$ denote the Weil restriction of $\mathfrak{X} \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Sp\acute{e}t} R_n$ along the closed immersion $\operatorname{Sp\acute{e}t} R_n \to \operatorname{Sp\acute{e}t} R$. Let $X' : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ denote the image of X under the equivalence $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\operatorname{Spec} R} \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})$, given concretely by the formula

$$\begin{aligned} X'(A) &= \mathrm{fib}(X(A) \to \mathrm{Map}_{\mathrm{CAlg}}(R, A)) \\ &\simeq \mathrm{Map}_{\mathrm{fSpDM}_{/\operatorname{Spf} R}}(\operatorname{Sp\acute{e}t} A \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Spf} R, \mathfrak{X}) \end{aligned}$$

We will show that X is representable (by a spectral algebraic space which is locally almost of finite presentation over R) by verifying the hypotheses of Theorem 18.3.0.1.

(2) The functor X is a sheaf for the étale topology. This is an immediate consequence of Proposition 19.1.2.2.

(3) The functor X is nilcomplete, infinitesimally cohesive, and integrable. It follows from Corollary 19.1.3.2 that each of the functors $X(n)$ is nilcomplete and cohesive, so that $X \simeq \varprojlim X(n)$ is also nilcomplete and cohesive. To show that X is integrable, it will suffice to show that each of the maps $X(n) \to \operatorname{Spec} R$ is integrable. Let A be an $\mathbb{E}_\infty$-algebra over R which is local, Noetherian, and complete with respect to its maximal ideal $\mathfrak{m} \subseteq \pi_0 A$. We wish to show that the canonical map

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\operatorname{Spec} R}}(\operatorname{Spec} A, X(n)) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\operatorname{Spec} R}}(\operatorname{Spf} A, X(n))$$

is a homotopy equivalence. Note that $A_n = A \otimes_R R_n$ is again a complete local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal, and that $\operatorname{Spf} A_n \simeq \operatorname{Spf} A \times_{\operatorname{Spec} R} \operatorname{Spec} R_n$. We may therefore identify θ with the canonical map

$$\begin{array}{c} \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\operatorname{Spec} R_n}}(\operatorname{Spec} A_n, \mathfrak{X} \times_{\operatorname{Spf} R} \operatorname{Spec} R_n) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\operatorname{Spec} R_n}}(\operatorname{Spf} A_n, \mathfrak{X} \times_{\operatorname{Spf} R} \operatorname{Spec} R_n). \end{array}$$

This map is a homotopy equivalence, since $\mathfrak{X} \times_{\operatorname{Spf} R} \operatorname{Spec} R_n$ is representable by a spectral algebraic space (Proposition 17.3.4.2).

(4) The map $X \to \operatorname{Spec} R$ admits a connective cotangent complex $L_{X/\operatorname{Spec} R}$. We will prove this by verifying analogues of the conditions of Example 17.2.4.4. Suppose we are given a connective $\mathbb{E}_\infty$-ring A and a point $\eta \in X(A)$. Then η determines a map of $\mathbb{E}_\infty$-rings $R \to A$. Let $J \subseteq \pi_0 A$ be the ideal generated by the image of I, and let $\operatorname{Spf} A$ be the formal spectrum of A with respect to the ideal J. Then η determines a representable morphism of formal spectral Deligne-Mumford stacks $\eta_0 : \operatorname{Spf} A \to \mathfrak{X}$. Let us identify

the pullback $\eta_0^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R} \in \operatorname{QCoh}(\operatorname{Spf} A)$ with an almost perfect $A^{\wedge}_J$-module M_η. Let N be an arbitrary connective A-module. For each integer $n \geqslant 0$, set $A_n = R_n \otimes_R A$ and $N_n = R_n \otimes_R N$. Unwinding the definitions, we obtain a canonical homotopy equivalence

$$\begin{aligned}
\operatorname{fib}(X'(A \oplus N) \to X'(A)) &\simeq \varprojlim_n \operatorname{fib}(X'(n)(A \oplus N) \to X'(n)(A)) \\
&\simeq \varprojlim_n \operatorname{Map}_{\operatorname{fSpDM}_{\operatorname{Sp\acute{e}t} A_n // \operatorname{Spf} R}}(\operatorname{Sp\acute{e}t} A_n \oplus N_n, \mathfrak{X}) \\
&\simeq \varprojlim_n \operatorname{Map}_{\operatorname{Mod}_{A_n}}(A_n \otimes_A M_\eta, N_n) \\
&\simeq \varprojlim_n \operatorname{Map}_{\operatorname{Mod}_A}(M_\eta, N_n) \\
&\simeq \operatorname{Map}_{\operatorname{Mod}_A}(M_\eta, N^{\wedge}_J)
\end{aligned}$$

where the last equivalence follows from Lemma 8.1.2.3. Assumption (c) guarantees that M_η is J-nilpotent, so that the canonical map $\operatorname{Map}_{\operatorname{Mod}_A}(M_\eta, N) \to \operatorname{Map}_{\operatorname{Mod}_A}(M_\eta, N^{\wedge}_J)$ is a homotopy equivalence for every A-module N. It follows that the M_η corepresents the functor $N \mapsto \operatorname{fib}(X'(A \oplus N) \to X'(A))$.

To complete the proof of the existence of $L_{X/\operatorname{Spec} R}$, it will suffice to show that every map $A \to A'$ induces an equivalence $\alpha : A' \otimes_A M_\eta \to M_{\eta'}$, where η' denotes the image of η in $X(A')$. Let J' denote the ideal in $\pi_0 A'$ generated by I. Using the commutativity of the diagram

$$\begin{array}{ccc}
\operatorname{Mod}^{\operatorname{aperf}}_{A^{\wedge}_J} & \xrightarrow{\sim} & \operatorname{QCoh}(\operatorname{Spf} A)^{\operatorname{aperf}} \\
\downarrow & & \downarrow \\
\operatorname{Mod}^{\operatorname{aperf}}_{A'^{\wedge}_{J'}} & \xrightarrow{\sim} & \operatorname{QCoh}(\operatorname{Spf} A')^{\operatorname{aperf}},
\end{array}$$

we can identify α with the natural map

$$A' \otimes_A M_\eta \simeq (A' \otimes_A A^{\wedge}_J) \otimes_{A^{\wedge}_J} M_\eta \to A'^{\wedge}_{J'} \otimes_{A^{\wedge}_J} M_\eta.$$

The cofiber of this map is given by $K \otimes_{A^{\wedge}_J} M_\eta$, where $K = \operatorname{cofib}(A' \otimes_A A^{\wedge}_J \to A'^{\wedge}_{J'})$. This cofiber vanishes, since K is J-local and M_η is J-nilpotent.

(5) The map $f : X \to \operatorname{Spec} R$ is locally almost of finite presentation. Choose a set of elements $x_1, \ldots, x_d \in R$ which generate the ideal I. We will proceed by induction on d, the case $d = 0$ being trivial. The proof of (4) shows that the relative cotangent complex $L_{X/\operatorname{Spec} R}$ is almost perfect. By virtue of Corollary 17.4.2.2, it will suffice to show that the functor $X'|_{\operatorname{CAlg}^{\heartsuit}_R}$ commutes with filtered colimits. To prove this, we may replace R by $\pi_0 R$ and thereby reduce to the case where R is discrete. Let CAlg^c_R denote the full subcategory of $\operatorname{CAlg}^{\heartsuit}_R$ spanned by those discrete R-algebras which are finitely

presented over R; we wish to prove that $X'|_{\mathrm{CAlg}^{\heartsuit}_R}$ is a left Kan extension of $X'|_{\mathrm{CAlg}^{c}_R}$. Equivalently, we must show that for every discrete R-algebra A, the canonical map

$$\rho : \varinjlim X'(A_\alpha) \to X'(A)$$

is a homotopy equivalence, where the colimit is taken over all finitely generated R-subalgebras $A_\alpha \subseteq A$.

We will prove that the map ρ is k-connective for each $k \geqslant 0$, using induction on k. Let us first suppose that $k > 0$. To prove that ρ is k-connective, it will suffice to show that for every pair of points $\eta, \eta' \in \varinjlim X'(A_\alpha)$, the canonical map $\xi : \{\eta\} \times_{\varinjlim X'(A_\alpha)} \{\eta'\} \to \varinjlim \{\eta\} \times_{X'(A)} \{\eta'\}$ is $(k-1)$-connective. Choose a finitely generated subalgebra $A_0 \subseteq A$ such that η and η' are the images of points $\eta_0, \eta'_0 \in X'(A_0)$. Replacing R by A_0, we may reduce to the case where $A_0 = R$. In this case, we can identify η_0 and η'_0 with maps $\mathrm{Spf}\, R \to \mathfrak{X}$. Let $\mathfrak{Y}$ denote the fiber product $\mathrm{Spf}\, R \times_{\mathfrak{X}} \mathrm{Spf}\, R$, let $Y \simeq \mathrm{Spec}\, R \times_X \mathrm{Spec}\, R$ be the Weil restriction of $\mathfrak{Y}$ along the map $\mathrm{Spf}\, R \hookrightarrow \mathrm{Sp\acute{e}t}\, R$, and let $Y' : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be the image of Y under the equivalence of ∞-categories $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{\mathrm{Spec}\, R} \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})$. Then ξ can be identified with the canonical map $\varinjlim Y'(A_\alpha) \to Y'(A)$. Since the map $\mathfrak{Y}_0 \to \mathrm{Spf}\, R$ satisfies hypotheses (a) through (c), it follows from the inductive hypothesis that ξ is $(k-1)$-connective.

It remains to treat the case $k = 0$. Consider the map of commutative rings $\mathbf{Z}[t] \to R$ which carries t to x_1. For each integer $m \geqslant 0$, set $R(m) = R \otimes_{\mathbf{Z}[t]} \mathbf{Z}[t]/(t^m)$. For every connective R-algebra A, we set $A(m) = R(m) \otimes_R A$. We first prove:

- $(*)$ For every connective R-algebra A, the canonical map $\mu : X'(A) \to \varprojlim X'(A(m))$ is a homotopy equivalence.

To prove $(*)$, we can identify μ with the map

$$\varprojlim_n X'(R_n \otimes_R A) \to \varprojlim_m \varprojlim_n X'(R_n \otimes_R A(m)).$$

It will therefore suffice to each of the maps

$$X'(R_n \otimes_R A) \to \varprojlim_m X'(R_n \otimes_R A(m))$$

is a homotopy equivalence. Replacing A by $R_n \otimes_R A$, we may reduce to proving $(*)$ in the special case where the image of I generates a nilpotent ideal in $\pi_0 A$. In particular, we may assume that the image of x is a nilpotent ideal in $\pi_0 A$. Since the functor X is nilcomplete, we may also assume that A is r-truncated for some integer r, so that each $A(m)$ is $(r+1)$-truncated. We will complete the proof by showing that A is equivalent to $\{A(m)\}_{m \geqslant 0}$ as a Pro-object of $\tau_{\leqslant r+1} \mathrm{CAlg}^{\mathrm{cn}}_R$. To prove this, we show that for every $(r+1)$-truncated connective R-algebra B, the upper horizontal map in the diagram σ :

$$\begin{array}{ccc} \varinjlim_m \mathrm{Map}_{\mathrm{CAlg}_R}(A(m), B) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}_R}(A, B) \\ \downarrow & & \downarrow \\ \varinjlim_m \mathrm{Map}_{\mathrm{CAlg}}(\mathbf{Z}[t]/(t^m), B) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}}(\mathbf{Z}[t], B) \end{array}$$

is a homotopy equivalence. This follows from Lemma 17.3.5.7, which asserts that the lower horizontal map is a fully faithful embedding whose essential image is the collection of those maps $\mathbf{Z}[t] \to B$ which carry t to a nilpotent element of $\pi_0 B$.

Our next step is to prove the following:

$(*')$ There exists an integer $m \geqslant 0$ such that the map $\pi_0 X'(A) \to \pi_0 X'(A(m))$ is bijective, for every discrete R-algebra A.

Assume $(*')$ for the moment. Let $I(m)$ denote the ideal in $\pi_0 R(m)$ generated by the image of I. Then $I(m)$ has the same radical as the ideal generated by the images of the elements $x_2, \ldots, x_d \in R$. It follows from our inductive hypothesis that for every integer s, the composite functor $\mathrm{CAlg}^{\mathrm{cn}}_{R(m)} \to \mathrm{CAlg}^{\mathrm{cn}}_R \xrightarrow{X'} \mathcal{S}$ preserves filtered colimits when restricted to s-truncated objects of $\mathrm{CAlg}^{\mathrm{cn}}_{R(m)}$, for every integer s. Since $R(m)$ has Tor-amplitude $\leqslant 1$ as an R-module, it follows that the functor $A \mapsto X'(A(m))$ commutes with filtered colimits when restricted to discrete R-algebras. Combining this observation with $(*')$, we conclude that the functor $A \mapsto \pi_0 X'(A)$ commutes with filtered colimits when restricted to CAlg^0_R. It follows that the map ρ is bijective on connected components, and therefore 0-connective.

We now turn to the prove of $(*')$. By virtue of $(*)$, it will suffice to prove the following:

$(*'')$ There exists an integer $m \geqslant 0$ such that, for every discrete R-algebra A and every integer $n \geqslant m$, the map $X'(A(n+1)) \to X'(A(n))$ has connected homotopy fibers.

Since $\mathfrak{X}$ is quasi-compact, we can choose an étale surjection $u : \mathrm{Spf}\, B \to \mathfrak{X}$ for some complete adic $\mathbb{E}_\infty$-ring B. Let us identify $u^* L^\wedge_{\mathfrak{X}/\mathrm{Spf}\, R}$ with an almost perfect B-module using the equivalence $\mathrm{QCoh}(\mathrm{Spf}\, B)^{\mathrm{aperf}} \simeq \mathrm{Mod}^{\mathrm{aperf}}_B$. Choose a perfect B-module M and a 2-connective map $\lambda : M \to u^* L^\wedge_{\mathfrak{X}/\mathrm{Spf}\, R}$. Since $L^\wedge_{\mathfrak{X}/\mathrm{Spf}\, R}$ is nilcoherent, the map λ is annihilated by x_1^c for some integer $c \gg 0$.

Let $n \geqslant 0$ be an integer and let η be a point of $X'(A(n))$, which we can identify with a representable morphism $\operatorname{Spf} A(n) \to \mathfrak{X}$. Form a pullback diagram

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\eta'} & \operatorname{Spf} B \\ \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\ \operatorname{Spf} A(n) & \xrightarrow{\eta} & \mathfrak{X}. \end{array}$$

Since u is an affine étale surjection, it follows from Lemma 19.3.3.4 that we can choose an equivalence $\mathfrak{Y} \simeq \operatorname{Spf} C$, where C is a faithfully flat $A(n)$-algebra which is complete with respect to an ideal $J \subseteq \pi_0 C$. Pulling λ back along η', we conclude that there exists a perfect C-module N and a 2-connective map $N \to C \otimes_{A(n)} \eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R}$ which is annihilated by x_1^c. Since C is faithfully flat over $A(n)$, it follows that the homotopy groups $\pi_i \eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R}$ are annihilated by x_1^c for $0 \leqslant i \leqslant 2$.

We now show that $m = 3c+1$ satisfies the requirements of $(*'')$. As a first step, we show that for $n \geqslant m$, the homotopy fiber of the map $X'(A(n+1)) \to X'(A(n))$ is connected over every point $\eta \in X'(A(n))$. Note that $\mathbf{Z}[t]/(t^{n+1+3c})$ is a square-zero extension of $\mathbf{Z}[t]/(t^n)$ by the module $\mathbf{Z}[t]/(t^{1+3c})$. It follows that $A(n+3c+1)$ is a square-zero extension of $A(n)$ by $A(3c+1)$. In particular, the obstruction to lifting η to a point of $X'(A(n+3c+1))$ is measured by an element of the group $\operatorname{Ext}^1_{A(n)}(\eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R}, A(3c+1))$. To prove that η can be lifted to a point of $X(A(n+1))$, it will suffice to show that the image of this obstruction in $\operatorname{Ext}^1_{A(n)}(\eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R}, A(1))$ vanishes (see Remark 17.3.9.2). In fact, we claim that the map of abelian groups

$$\operatorname{Ext}^1_{A(n)}(\eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R}, A(3c+1)) \to \operatorname{Ext}^1_{A(n)}(\eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R}, A(1))$$

is zero.

Let v be an element of $\operatorname{Ext}^1_{A(n)}(\eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R}, A(3c+1))$. Since A is discrete, the suspension $\Sigma A(3c+1)$ is 2-truncated. We may therefore identify v with a map of $A(n)$-modules $v : \tau_{\leqslant 2} \eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R} \to \Sigma A(3c+1)$. We wish to show that the composite map

$$\tau_{\leqslant 2} \eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R} \xrightarrow{v} \Sigma A(3c+1) \to \Sigma A(1)$$

is nullhomotopic. Since $\pi_2 \eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R}$ is annihilated by x_1^c, it admits the structure of a module over $A(c)$. Applying Lemma 19.3.3.3, we deduce that the composite map

$$\Sigma^2(\pi_2 \eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R}) \to \tau_{\leqslant 2} \eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R} \xrightarrow{v} \Sigma A(3c+1) \to \Sigma A(2c+1)$$

is nullhomotopic. Consequently, there exists a commutative diagram

$$\begin{array}{ccc} \tau_{\leqslant 2} \eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R} & \xrightarrow{v} & \Sigma A(3c+1) \\ \downarrow & & \downarrow \\ \tau_{\leqslant 1} \eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R} & \xrightarrow{v'} & \Sigma A(2c+1). \end{array}$$

Since $\pi_1 \eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R}$ is annihilated by x_1^c, it also admits the structure of a module over $A(c)$. Applying Lemma 19.3.3.3 again, we deduce that the composite map

$$\Sigma(\pi_1 \eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R}) \to \tau_{\leqslant 1} \eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R} \xrightarrow{v'} \Sigma A(2c+1) \to \Sigma A(c+1)$$

is nullhomotopic. Consequently, there exists a commutative diaram

$$\begin{array}{ccc} \tau_{\leqslant 1} \eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R} & \xrightarrow{v'} & \Sigma A(2c+1) \\ \downarrow & & \downarrow \\ \tau_{\leqslant 0} \eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R} & \xrightarrow{v''} & \Sigma A(c+1). \end{array}$$

We are therefore reduced to proving that the composite map

$$\tau_{\leqslant 0} \eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R} \xrightarrow{v''} \Sigma A(c+1) \to \Sigma A(1)$$

is nullhomotopic. This follows from Lemma 19.3.3.3, since $\pi_0 \eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R}$ is annihilated by x_1^c and therefore admits the structure of a module over $A(c)$. This completes the proof that the homotopy fiber $X'(A(n+1)) \times_{X'(A(n))} \{\eta\}$ is nonempty.

Suppose that $\overline{\eta}$ and $\overline{\eta}'$ are points of the fiber product $X'(A(n+1)) \times_{X'(A(n))} \{\eta\}$; we will complete the proof of $(*'')$ by showing that $\overline{\eta}$ and $\overline{\eta}'$ belong to the same connected component of $X'(A(n+1)) \times_{X'(A(n))} \{\eta\}$. The first part of the proof shows that $\overline{\eta}$ and $\overline{\eta}'$ can be lifted to points of the fiber product $X'(A(n+2c+1)) \times_{X'(A(n))} \{\eta\}$. Since $A(n+2c+1)$ and $A(n+1)$ are square-zero extensions of $A(n)$ by $A(2c+1)$ and $A(1)$, the sets

$$\pi_0(X'(A(n+2c+1)) \times_{X'(A(n))} \{\eta\}) \qquad \pi_0(X'(A(n+1)) \times_{X'(A(n))} \{\eta\})$$

are torsors for the groups $\operatorname{Ext}^0_{A(n)}(\eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R}, A(2c+1))$ and $\operatorname{Ext}^0_{A(n)}(\eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R}, A(1))$, respectively. We are therefore reduced to showing that the map

$$\operatorname{Ext}^0_{A(n)}(\eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R}, A(2c+1)) \to \operatorname{Ext}^0_{A(n)}(\eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R}, A(1))$$

vanishes. Let w be an element of $\operatorname{Ext}^0_{A(n)}(\eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R}, A(2c+1))$. Since A is discrete, we can identify w with a map of $A(n)$-modules $\tau_{\leqslant 1} \eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R} \to A(2c+1)$. Since $\pi_1 L_{\hat{\mathfrak{X}} / \operatorname{Spf} R}$ is annihilated by x_1^c, Lemma 19.3.3.3 implies that the composite map

$$\Sigma^1(\pi_1 \eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R}) \to \tau_{\leqslant 1} \eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R} \xrightarrow{w} A(2c+1) \to A(c+1)$$

is nullhomotopic. Consequently, there exists a commutative diagram

$$\begin{array}{ccc} \tau_{\leqslant 1} \eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R} & \xrightarrow{w} & A(2c+1) \\ \downarrow & & \downarrow \\ \tau_{\leqslant 0} \eta^* L_{\hat{\mathfrak{X}} / \operatorname{Spf} R} & \xrightarrow{w'} & A(c+1). \end{array}$$

It will therefore suffice to show that the composite map

$$\tau_{\leqslant 0}\eta^* L^{\wedge}_{\mathfrak{X}/\operatorname{Spf} R} \xrightarrow{w'} A(c+1) \to A(1)$$

is nullhomotopic, which again follows from Lemma 19.3.3.3.

(1) We must show that for every commutative ring A, the space $X(A)$ is discrete. Equivalently, we must show that for every discrete R-algebra A, the space $X'(A)$ is discrete. Step (5) shows that the functor $A \mapsto X'(A)$ commutes with filtered colimits when restricted to discrete R-algebras. We may therefore assume that A is finitely generated as an R-algebra, and is therefore Noetherian. Set $J = IA \subseteq A$. For each integer $n \geqslant 1$, set $A_n = R_n \otimes_R A$. Then $X'(A) \simeq \varprojlim_n \operatorname{Map}_{\mathrm{fSpDM}_{/\operatorname{Spf} R}}(\operatorname{Sp\acute{e}t} A_n, \mathfrak{X})$. Since the functors represented by $\mathfrak{X}$ and $\operatorname{Spf} R$ are nilcomplete (Proposition 17.3.2.3), we obtain homotopy equivalences

$$\begin{aligned} X'(A) &\simeq \varprojlim_n \varprojlim_m \operatorname{Map}_{\mathrm{fSpDM}_{/\operatorname{Spf} R}}(\operatorname{Sp\acute{e}t} \tau_{\leqslant m} A_n, \mathfrak{X}) \\ &\simeq \varprojlim_m \varprojlim_n \operatorname{Map}_{\mathrm{fSpDM}_{/\operatorname{Spf} R}}(\operatorname{Sp\acute{e}t} \tau_{\leqslant m} A_n, \mathfrak{X}). \end{aligned}$$

It will therefore suffice to show that for each $m \geqslant 0$, the inverse limit

$$\varprojlim_n \operatorname{Map}_{\mathrm{fSpDM}_{/\operatorname{Spf} R}}(\operatorname{Sp\acute{e}t} \tau_{\leqslant m} A_n, \mathfrak{X})$$

is discrete. Lemma 17.3.5.7 implies that the towers $\{\tau_{\leqslant m} A_n\}_{n>0}$ and $\{A/J^n\}_{n>0}$ are equivalent as Pro-objects of CAlg, so we are reduced to showing that the limit $\varprojlim_n \operatorname{Map}_{\mathrm{fSpDM}_{/\operatorname{Spf} R}}(\operatorname{Sp\acute{e}t} A/J^n, \mathfrak{X})$ is discrete. In fact, each of the individual mapping spaces $\operatorname{Map}_{\mathrm{fSpDM}_{/\operatorname{Spf} R}}(\operatorname{Sp\acute{e}t} A/J^n, \mathfrak{X})$ is discrete, since $\mathfrak{X} \to \operatorname{Spf} R$ is representable by spectral algebraic spaces.

□

19.3.4 A Criterion for Properness

In order to apply Proposition 19.3.3.2 to Question 19.3.3.1, we need to verify that if $\mathfrak{X} \to \operatorname{Spf} R$ is a formal modification, then the Weil restriction $\operatorname{Res}_{\operatorname{Spf} R/\operatorname{Sp\acute{e}t} R}(\mathfrak{X})$ actually proper over R. We will deduce this from the following result, which may be of some independent interest:

Proposition 19.3.4.1. *Let R be a Noetherian $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be an ideal, and let $f : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ be a morphism in of spectral algebraic spaces which satisfies the following conditions:*

(a) *The morphism f is locally almost of finite presentation.*

(b) *The projection map* $f_0 : \mathfrak{X} = \mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spf} R \to \operatorname{Spf} R$ *is proper.*

(c) *The map* $\mathsf{X} \times_{\operatorname{Spec} R} \mathsf{U} \to \mathsf{U}$ *is an equivalence, where* U *denotes the open substack* $\operatorname{Spec} R$ *complementary to the vanishing locus of* I*.*

(d) *Then the cofiber of the unit map* $u : \mathscr{O}_{\operatorname{Spf} R} \to f_{0*} \mathscr{O}_{\mathfrak{X}}$ *is nilcoherent (as an object of* $\operatorname{QCoh}(\operatorname{Spf} R)$ *).*

Then X *is proper over* R*.*

The proof will require the following purely algebraic result:

Lemma 19.3.4.2. *Let* R *be a discrete valuation ring, let* $t \in R$ *be a generator of its maximal ideal, and let* B *be a finitely generated* R*-algebra. Assume that* B *is flat over* R *and that* $B[t^{-1}]$ *is finitely generated as an* $R[t^{-1}]$*-module. Then:*

(1) *The quotient ring* B/tB *is finitely generated as an* R/tR*-module.*

(2) *The set* $|\operatorname{Spec} B/tB|$ *is finite, and its topology is discrete.*

(3) *Let* $R^\wedge$ *denote the* (t)*-adic completion of* R*, and let* $B^\wedge$ *denote the* (t)*-adic completion of* B*. Then* $B^\wedge$ *is finitely generated as an* $R^\wedge$*-module.*

Proof. We first prove (1). Suppose that $B[t^{-1}]$ has dimension n as a vector space over the fraction field $R[t^{-1}]$. We claim that B/tB has dimension $\leqslant n$ as a vector space over the residue field R/tR. Suppose otherwise: then there exists a sequence of elements $b_0, \ldots, b_n \in B$ whose images in B/tB are linearly independent over R/tR. Since $B[t^{-1}]$ has dimension n over $R[t^{-1}]$, there exists a dependence relation $\sum \lambda_i b_i = 0$ where the coefficients λ_i belong to $R[t^{-1}]$, and not all of the λ_i vanish. Multiplying by an appropriate power of t, we can assume that each λ_i belongs to R. Let t^m be the largest power of t which divides each b_i. Since B is flat over R, the identity $t^m \sum (t^{-m}\lambda_i) b_i = 0$ implies that $\sum (t^{-m}\lambda_i) b_i = 0$ in B. The maximality of m guarantees that some coefficient $t^{-m}\lambda_i$ has nonzero image in B/tB, contradicting our assumption that the images of the elements b_i are linearly independent over R/tR.

Assertion (2) follows immediately from (1). If (1) is satisfied, then we can choose finitely many elements $x_1, \ldots, x_m \in B^\wedge$ whose images generate $B^\wedge/tB^\wedge \simeq B/tB$ as an $R^\wedge$-module. The cokernel of the induced map $\beta : (R^\wedge)^m \to B^\wedge$ is (t)-complete and satisfies $\operatorname{coker}(\beta)/t \operatorname{coker}(\beta) \simeq 0$, so that $\operatorname{coker}(\beta) \simeq 0$. It follows that β is a surjection, so that condition (3) is satisfied. □

Proof of Proposition 19.3.4.1. Let $f : \mathsf{X} \to \operatorname{Spét} R$ satisfy conditions (a) through (d), set $\mathsf{Y} = \mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét}(\pi_0 R)/I$ and let V denote the open substack of X complementary to Y. Choose a collection of étale maps $\{\operatorname{Spét} B_s \to \mathsf{X}\}_{s \in S}$ which are jointly surjective. Assumption

(c) implies that the projection $\mathsf{V} \to \mathsf{U}$ is an equivalence, so that V is quasi-compact. Assumption (b) implies that Y is proper over the commutative ring $(\pi_0 R)/I$, and therefore quasi-compact. We may therefore choose a finite subsets $S_0, S_1 \subseteq S$ such that the collections of maps

$$\{\operatorname{Sp\acute{e}t} B_s \times_{\mathsf{X}} \mathsf{V} \to \mathsf{V}\}_{s \in S_0} \qquad \{\operatorname{Sp\acute{e}t} B_s \times_{\mathsf{X}} \mathsf{Y} \to \mathsf{Y}\}_{s \in S_1}$$

are jointly surjective. It follows that the collection of maps $\{\operatorname{Sp\acute{e}t} B_s \to \mathsf{X}\}_{s \in S_0 \cup S_1}$ is jointly surjective, so that X is quasi-compact. Applying the same argument to each fiber product $\operatorname{Sp\acute{e}t} B_s \times_{\mathsf{X}} \operatorname{Sp\acute{e}t} B_t$ (which we regard as a spectral algebraic space which is locally almost of finite presentation over $B_s \otimes_R B_t$), we conclude that X is quasi-separated.

Let $X : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by X. Since X is a spectral algebraic space, we note that $X(A)$ is discrete whenever $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$ is discrete. In this case, we will abuse notation by identifying $X(A)$ with the set $\pi_0 X(A)$. By virtue of Corollary 5.3.1.2, it will suffice to verify the following:

$(*)$ For every object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$ which is a discrete valuation ring with residue field K, the map $X(A) \to X(K)$ is bijective.

To prove $(*)$, we can replace X by $\operatorname{Sp\acute{e}t} A \times_{\operatorname{Sp\acute{e}t} R} \mathsf{X}$ and thereby reduce to the case where R is a discrete valuation ring and $A = R$. If $I = 0$, then $\operatorname{Spf} R \simeq \operatorname{Sp\acute{e}t} R$ and the desired result is an immediate consequence of (b). If $I = R$, then $\mathsf{U} \simeq \operatorname{Sp\acute{e}t} A$ and the desired result follows from (c). We may therefore assume (replacing I by its radical if necessary) that I is the maximal ideal $\mathfrak{m}_R \subseteq R$. Let $t \in \mathfrak{m}_R$ be a generator. Note that assumption 9c) guarantees that $X(K)$ is contractible; we will complete the proof by showing that $X(R)$ is contractible.

Let $\overline{\mathscr{O}}$ denote the image (in the abelian category $\mathrm{QCoh}(\mathsf{X})^{\heartsuit}$) of the canonical map $\pi_0 \mathscr{O}_{\mathsf{X}} \to \pi_0 \mathscr{O}_{\mathsf{X}}[t^{-1}]$. Then $\overline{\mathscr{O}}$ is a commutative algebra object of $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$, and therefore determines an affine morphism $\overline{\mathsf{X}} \to \mathsf{X}$. Let $\overline{X}' : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by $\overline{\mathsf{X}}$. Note that the map $\overline{X}(R) \to X(R)$ is a homotopy equivalence. It will therefore suffice to show that $\overline{X}(R)$ is contractible.

Let $\mathcal{I}$ denote the fiber of the composite map $\mathscr{O}_{\mathsf{X}} \to \pi_0 \mathscr{O}_{\mathsf{X}} \to \overline{\mathscr{O}}$. Since X is locally Noetherian, each homotopy group $\pi_n \mathcal{I}$ is almost perfect (when regarded as an object of $\mathrm{QCoh}(\mathsf{X})^{\heartsuit} \subseteq \mathrm{QCoh}(\mathsf{X})$). It follows from (c) that $\mathscr{O}_{\mathsf{X}}[t^{-1}]$ is a discrete object of $\mathrm{QCoh}(\mathsf{X})$, so that $\mathcal{I}[t^{-1}] \simeq 0$. Using the quasi-compactness of X, we conclude that for every integer $n \geqslant 0$, there exists an integer k_n such that $\pi_n \mathcal{I}$ is annihilated by t^{k_n}, so that $\tau_{\leqslant n} \mathcal{I}$ is annihilated by $t^{k_0 + k_1 + \cdots + k_n}$. Using Proposition 2.5.4.4 and Theorem 3.4.2.1, we deduce that there exists an integer m such that the pushforward functor $f_* : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Mod}_R$ carries $\mathrm{QCoh}(\mathsf{X})_{\geqslant 0}$ into $(\mathrm{Mod}_R)_{\geqslant -m}$. It follows that for each integer n, $\tau_{\leqslant n}(f_* \mathcal{I}) \simeq \tau_{\leqslant n} f_* (\tau_{\leqslant n+m} \mathcal{I})$ is annihilated by $t^{k_0 + \cdots + k_{m+n}}$, so that the restriction of $f_* \mathcal{I}$ to $\mathrm{QCoh}(\operatorname{Spec} R)$ is $\mathfrak{m}_R$-nilpotent. Let $\overline{\mathscr{O}}_{\mathfrak{X}}$ denote the pullback of $\overline{\mathscr{O}}$ to $\mathrm{QCoh}(\mathfrak{X})$, so that we have maps

$$\mathscr{O}_{\operatorname{Spf} R} \xrightarrow{u} f_{0*} \mathscr{O}_{\mathsf{X}} \xrightarrow{u'} f_{0*} \overline{\mathscr{O}}_{\mathfrak{X}}$$

in QCoh(Spf R). Then $\mathrm{cofib}(u)$ is nilcoherent by assumption (d) and $\mathrm{cofib}(u')$ is nilcoherent by Proposition 8.3.2.6, so that $\mathrm{cofib}(u' \circ u)$ is also nilcoherent. We may therefore replace X by $\overline{\mathsf{X}}$, and thereby reduce to the case where X is flat over R.

We now complete the proof by showing that the map $\mathsf{X} \to \mathrm{Spét}\, R$ is an equivalence. For every étale map $\phi : \mathrm{Spét}\, B \to \mathsf{X}$, the $\mathbb{E}_\infty$-ring B is a flat over R. It follows from (c) that $B[t^{-1}]$ is étale over K and is therefore isomorphic to a finite product of separable extensions of K. Since B is R-flat, $B \simeq 0$ if and only if $B[t^{-1}] \simeq 0$. In particular, every nonempty open substack of X has nonempty interesection with $\mathsf{V} \simeq \mathrm{Spét}\, K$. Since V has no nonempty open substacks other than itself, every nonempty open substack of X must contain V.

Using Theorem 3.4.2.1, we can choose a scallop decomposition

$$\varnothing = \mathsf{V}_0 \hookrightarrow \mathsf{V}_1 \hookrightarrow \cdots \hookrightarrow \mathsf{V}_n = \mathsf{X}$$

for the quasi-compact, quasi-separated spectral algebraic space X. Without loss of generality, we may assume that V_1 is nonempty and therefore contains V. For $2 \leqslant i \leqslant n$, we have an excision square

$$\begin{array}{ccc} \mathsf{W}_i & \longrightarrow & \mathrm{Spét}\, B_i \\ \downarrow & & \downarrow \\ \mathsf{V}_{i-1} & \longrightarrow & \mathsf{V}_i, \end{array}$$

where $\mathrm{Spét}\, B_i$ is étale over X and W_i is a quasi-compact open of $\mathrm{Spét}\, B_i$ determined by an open set $W_i \subseteq |\mathrm{Spec}\, B_i|$. By construction, W_i contains the open subset $W_i^\circ = |\mathrm{Spec}\, B_i[t^{-1}]| \subseteq |\mathrm{Spec}\, B_i|$. Using Lemma 19.3.4.2, we see that there are only finitely many prime ideals of B containing t, none of which is contained in another. We may therefore write W_i as the union of W_i° and finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ of B_i containing t. Using prime avoidance, we can choose an element $b_i \in \bigcup_{1 \leqslant j \leqslant k} \mathfrak{m}_j$ which is not contained in any other prime ideal of B containing t. We then have an excision square

$$\begin{array}{ccc} \mathrm{Spét}\, B_i[t^{-1}] & \longrightarrow & \mathrm{Spét}\, B_i[b_i^{-1}] \\ \downarrow & & \downarrow \\ \mathsf{W}_i & \longrightarrow & \mathrm{Spét}\, B_i. \end{array}$$

We may therefore replace each B_i by $B_i[b_i^{-1}]$, and thereby reduce to the case where $\mathsf{W}_i \simeq \mathrm{Spét}\, B_i[t^{-1}]$. In this case, each of the maps $\mathsf{W}_i \to \mathsf{V}_{i-1}$ factors through V. We therefore have a pushout diagram:

$$\begin{array}{ccc} \coprod_{2 \leqslant i \leqslant n} \mathrm{Spét}\, B_i[t^{-1}] & \longrightarrow & \coprod_{2 \leqslant i \leqslant n} \mathrm{Spét}\, B_i \\ \downarrow & & \downarrow \\ \mathsf{V} & \longrightarrow & \mathsf{X}. \end{array}$$

Set $B = \prod_{2 \leqslant i \leqslant n} B_i$. Let $R^\wedge$ and $B^\wedge$ denote the (t)-adic completions of R and B, respectively. Using Lemma 19.3.4.2, we see that $B^\wedge$ is finitely generated as an $R^\wedge$-module. Since B is flat over R, the multiplication map $t : B \to B$ is injective. It follows that the map $t : B^\wedge \to B^\wedge$ is also injective, so that $B^\wedge$ is flat over $R^\wedge$. Since $R^\wedge$ is a discrete valuation ring, $B^\wedge$ is a free module of finite rank over $R^\wedge$. Unwinding the definitions, we note that condition (d) asserts that the unit map $u : R^\wedge \to B^\wedge$ induces an equivalence $R^\wedge[t^{-1}] \to B^\wedge[t^{-1}]$. It follows that $B^\wedge$ is a free module of rank 1 over $R^\wedge$, and that the map u is injective. In particular, $B^\wedge$ is faithfully flat as an $R^\wedge$-module, so that $\operatorname{coker}(u)$ is flat over $R^\wedge$. Since $\operatorname{coker}(u)[t^{-1}] \simeq 0$, we conclude that u is surjective and therefore an isomorphism.

We next claim that B is étale over R. Since B is finitely presented over R, it will suffice to show that $L_{B/R} \simeq 0$ (Lemma B.1.3.3). Suppose otherwise: then there is some smallest integer d such that $\pi_d L_{B/R} \neq 0$. Note that B is almost perfect as an object of $\mathrm{CAlg}_R^{\mathrm{cn}}$, so that $L_{B/R}$ is almost perfect and $\pi_d L_{B/R}$ is finitely generated as a B-module. We have already established that $B[t^{-1}]$ is étale over $R[t^{-1}]$, so that $L_{B/R}[t^{-1}] \simeq L_{B[t^{-1}]/R[t^{-1}]} \simeq 0$. It follows that $\pi_d L_{B/R}$ is annihilated by t^k for $k \gg 0$. Then

$$0 \neq \pi_d L_{B/R} \simeq \pi_d(B/t^k B \otimes_B L_{B/R}) \simeq \pi_d(L_{(B/t^k B)/(R/t^k R)}).$$

This is a contradiction, since the map $R \to B$ induces an isomorphism after t-adic completion (and therefore an isomorphism $R/t^k R \to B/t^k B$).

Since the map $\operatorname{Spét} B \to \mathsf{X}$ is an étale surjection, we conclude that X is étale over R. Consequently, to show that $\mathsf{X} \to \operatorname{Spét} R$ is an equivalence, it will suffice to show that the maps

$$\mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét} R[t^{-1}] \to \operatorname{Spét} R[t^{-1}]$$

$$\mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét}(R/tR) \to \operatorname{Spét}(R/tR)$$

are equivalences (Lemma HA.A.5.11 and Theorem 3.1.2.1). In the first case, this follows immediately from assumption (c). In the second, we note that the existence of the excision square σ supplies an identification $\mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét}(R/tR) \simeq \operatorname{Spét}(B/tB)$, so that the desired assertion follows from the fact that the unit map $R/tR \to B/tB$ is an isomorphism. □

19.4 Moduli of Algebraic Varieties

Let X be a proper smooth algebraic variety over the field $\mathbf{C}$ of complex numbers. Recall that a *first order deformation* of X is a flat $\mathbf{C}[\epsilon]/(\epsilon^2)$-scheme $\overline{X}$ equipped with an isomorphism $X \simeq \operatorname{Spec} \mathbf{C} \times_{\operatorname{Spec} \mathbf{C}[\epsilon]/(\epsilon^2)} \overline{X}$. A classical result of deformation theory establishes a bijection from the set of isomorphism classes of first-order deformations of X to the cohomology group $\mathrm{H}^1(X; T_X)$, where T_X denotes the tangent bundle of X. We can describe the situation more informally as follows:

($*$) Let $\mathcal{M}$ denote the "moduli space" of proper algebraic varieties, so that X determines a $\mathbf{C}$-valued point η of $\mathcal{M}$. Then the tangent space to $\mathcal{M}$ at η can be identified with $\mathrm{H}^1(X; T_X)$.

To articulate ($*$) more precisely, it is convenient to introduce a formal definition of the "moduli space" $\mathcal{M}$:

Construction 19.4.0.1 (Moduli of Spectral Algebraic Spaces)**.** Let SpDM denote the ∞-category of spectral Deligne-Mumford stacks, and let $\mathcal{C}$ denote the full subcategory of the fiber product

$$(\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}} \times_{\mathrm{Fun}(\{1\}, \mathrm{SpDM})} \mathrm{Fun}(\Delta^1, \mathrm{SpDM})$$

spanned by morphisms $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$, where f is proper, flat, and locally almost of finite presentation. We let $\mathrm{Var}^+ : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be a functor which classifies the Cartesian fibration $\mathcal{C} \to (\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}}$ (given by projection onto the first factor).

More informally, if R is a connective $\mathbb{E}_\infty$-ring, then $\mathrm{Var}^+(R)$ is the ∞-category spectral algebraic spaces which are proper, flat, and locally almost of finite presentation over R.

Our goal in this section is to study the functor Var^+ of Construction 19.4.0.1. Our main results can be summarized as follows:

Theorem 19.4.0.2. *Let* $\mathrm{Var}^+ : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ *be as in Construction 19.4.0.1. Then:*

(1) *The functor* Var^+ *is cohesive and nilcomplete.*

(2) *The functor* Var^+ *is locally almost of finite presentation: that is,* Var^+ *commutes with filtered colimits when restricted to* $\tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}$*, for every integer* $n \geqslant 0$.

(3) *For every connective* $\mathbb{E}_\infty$*-ring* R*, the* ∞*-category* $\mathrm{Var}^+(R)$ *is essentially small. Consequently, we can identify* Var^+ *with a functor* $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$.

(4) *For every simplicial set* K*, let* $\mathrm{Var}^+_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ *denote the functor given by the formula* $\mathrm{Var}^+_K(R) = \mathrm{Fun}(K, \mathrm{Var}^+(R))^{\simeq}$*. Then the functor* Var^+_K *admits a* (-1)*-connective cotangent complex (which can be described explicitly using Kodaira-Spencer theory; see §19.4.3).*

(5) *Suppose that* K *is a simplicial set having only finitely many simplices of each dimension. Then the functor* Var^+_K *is locally almost of finite presentation and the cotangent complex to* Var^+_K *is almost perfect.*

Warning 19.4.0.3. The functor Var^+ of Theorem 19.4.0.2 is not integrable. If A is a local Noetherian ring which is complete with respect to its maximal ideal $\mathfrak{m}$, then Corollary 8.5.3.3 implies that the functor $\mathrm{Var}^+(A) \to \varprojlim \mathrm{Var}^+(A/\mathfrak{m}^n)$ is a fully faithful embedding. However, it is generally not essentially surjective: that is, not every proper flat representable morphism $\mathfrak{X} \to \mathrm{Spf}\, A$ is algebraizable in the sense of Remark 8.5.3.6.

Remark 19.4.0.4. Let K be a simplicial set having only finitely many simplices of each dimension and let $\mathrm{Var}^+_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be defined as in Theorem 19.4.0.2. Then Var^+_K is *almost* representable by a spectral Deligne-Mumford stack which is locally almost of finite presentation over the sphere spectrum S: it satisfies all of the hypotheses of Theorem 18.3.0.1 *except* for integrability (Warning 19.4.0.3). There are several variants of Construction 19.4.0.1 which give rise to representable functors (for example, we could restrict our attention to morphisms $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ of relative dimension $\leqslant 1$). We will return to this point in §**??**.

19.4.1 Nilcompleteness and Cohesiveness

The first assertion of Theorem 19.4.0.2 hold in much greater geneality.

Notation 19.4.1.1. Let $n \geqslant 0$ be a nonnegative integer. Recall that a spectral Deligne-Mumford stack X is said to be *n-truncated* if the structure sheaf $\mathscr{O}_{\mathsf{X}}$ is n-truncated. We let $\mathrm{SpDM}^{\leqslant n}$ denote the full subcategory of SpDM spanned by the n-truncated spectral Deligne-Mumford stacks. The inclusion functor $\mathrm{SpDM}^{\leqslant n} \hookrightarrow \mathrm{SpDM}$ admits a right adjoint $\tau_{\leqslant n} : \mathrm{SpDM} \to \mathrm{SpDM}^{\leqslant n}$, given on objects by the formula $\tau_{\leqslant n}(\mathcal{X}, \mathscr{O}_{\mathsf{X}}) = (\mathcal{X}, \tau_{\leqslant n}\, \mathscr{O}_{\mathsf{X}})$.

Proposition 19.4.1.2. *Let X be a spectral Deligne-Mumford stack. Then the pullback functors*

$$(\mathsf{Y} \in \mathrm{SpDM}_{/\mathsf{X}}) \mapsto (\mathsf{Y} \times_{\mathsf{X}} \tau_{\leqslant n} \mathsf{X} \in \mathrm{SpDM}_{/\tau_{\leqslant n} \mathsf{X}})$$

induce an equivalence of ∞-categories $\mathrm{SpDM}_{/\mathsf{X}} \to \varprojlim \mathrm{SpDM}_{/\tau_{\leqslant n} \mathsf{X}}$.

Proof. For every spectral Deligne-Mumford stack X, let $\mathrm{SpDM}^{\leqslant n}_{/\mathsf{X}}$ denote the full subcategory of $\mathrm{SpDM}_{/\mathsf{X}}$ spanned by those morphisms $\mathsf{Y} \to \mathsf{X}$ where Y is n-truncated. We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{SpDM}_{/\mathsf{X}} & \longrightarrow & \varprojlim_n \mathrm{SpDM}_{/\tau_{\leqslant n} \mathsf{X}} \\
\downarrow & & \downarrow \\
\varprojlim_m \mathrm{SpDM}^{\leqslant m}_{/\mathsf{X}} & \longrightarrow & \varprojlim_n \varprojlim_m \mathrm{SpDM}^{\leqslant m}_{/\mathsf{X}_n},
\end{array}$$

where the horizontal maps are given by pullback and the vertical maps are given by truncation. The lower horizontal map is an equivalence, since the truncated pullback functor $\mathrm{SpDM}^{\leqslant m}_{/\mathsf{X}} \to \mathrm{SpDM}^{\leqslant m}_{/\tau_{\leqslant n} \mathsf{X}}$ is an equivalence for $m \leqslant n$. It follows from Proposition 1.4.9.1 that the vertical maps are equivalences, so that the upper horizontal map is an equivalence as well. □

Corollary 19.4.1.3. *The construction $R \mapsto \mathrm{SpDM}_{/\,\mathrm{Sp\acute{e}t}\, R}$ determines a cohesive and nil-complete functor $\overline{\mathrm{Var}}^+ : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$.*

Proof. The nilcompleteness follows from Proposition 19.4.1.2, and the cohesiveness is a consequence of Theorem 16.3.0.1. □

19.4.2 Deformation-Invariant Properties of Morphisms $\mathsf{X} \to \operatorname{Sp\acute{e}t} R$

Corollary 19.4.1.3 does not apply directly in the situation of Theorem 19.4.0.2, because the classification of proper flat spectral algebraic spaces is different from the classification of general spectral Deligne-Mumford stacks. Nevertheless, our next result guarantees that the deformation theory of the former is controlled by the deformation theory of the latter:

Proposition 19.4.2.1. *Let $\overline{\mathrm{Var}}^{+} : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ be as in Corollary 19.4.1.3, and let $\overline{\mathrm{Var}}^{+}_{0} : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ be the functor which assigns to each connective $\mathbb{E}_{\infty}$-ring R the full subcategory of $X(R) \simeq \mathrm{SpDM}_{/\operatorname{Sp\acute{e}t} R}$ spanned by those maps $f : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ satisfying any one of the following conditions:*

(1) *The map f is locally of finite generation to order n (where $n \geqslant 0$ is some fixed integer).*

(2) *The map f is locally almost of finite presentation.*

(3) *The map f is locally of finite presentation.*

(4) *The map f is n-quasi-compact (where $0 \leqslant n \leqslant \infty$).*

(5) *The map f is separated.*

(6) *The map f is flat.*

(7) *The map f is proper.*

Then the inclusion $j : \overline{\mathrm{Var}}^{+}_{0} \to \overline{\mathrm{Var}}^{+}$ is cohesive and nilcomplete.

Remark 19.4.2.2. In the situation of Proposition 19.4.2.1, let $\overline{\mathrm{Var}}^{+}_{0} : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ be the functor which assigns to a connective $\mathbb{E}_{\infty}$-ring R the full subcategory of $\overline{\mathrm{Var}}^{+}(R) = \mathrm{SpDM}_{/\operatorname{Sp\acute{e}t} R}$ spanned by those maps of spectral Deligne-Mumford stacks $f : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ which satisfy some combination of the conditions (1) through (7) appearing in Proposition 19.4.2.1. Then the inclusion $j : \overline{\mathrm{Var}}^{+}_{0} \to \overline{\mathrm{Var}}^{+}$ is cohesive and nilcomplete. It then follows from Corollary 19.4.1.3 that the functor $\overline{\mathrm{Var}}^{+}_{0}$ is cohesive and nilcomplete.

The proof of Proposition 19.4.2.1 will require some preliminaries.

Lemma 19.4.2.3. *Let $f : \mathsf{X} \to \operatorname{Sp\acute{e}t} R$ be a map of spectral Deligne-Mumford stacks, and let $f_0 : \mathsf{X} \times_{\operatorname{Sp\acute{e}t} R} \operatorname{Sp\acute{e}t} \pi_0 R \to \operatorname{Sp\acute{e}t} \pi_0 R$ be the projection onto the second factor. Suppose that f_0 satisfies one of the following conditions:*

(1) *The map f_0 is locally of finite generation to order n (where $n \geqslant 0$ is some fixed integer).*

(2) *The map f_0 is locally almost of finite presentation.*

(3) *The map f_0 is locally of finite presentation.*

(4) *The map* f_0 *is* n*-quasi-compact (where* $0 \leqslant n \leqslant \infty$*).*

(5) *The map* f_0 *is separated.*

(6) *The map* f_0 *is flat.*

(7) *The map* f_0 *is proper.*

Then f *has the same property.*

Proof. We first treat cases (1), (2), and (3). The assertions are local on X, so we may assume that $\mathsf{X} = \operatorname{Spét} A$ is affine. Let $A' = A \otimes_R \pi_0 R$. Assertion (1) is obvious in the case $n = 0$ (see Remark 4.1.1.4). In all other cases, we may assume that f_0 is of finite generation to order 1, so that $\pi_0 A \simeq \pi_0 A'$ is finitely presented as a commutative ring over $\pi_0 R$. It will therefore suffice to show that if $L_{A'/\pi_0 R}$ is perfect to order n (almost perfect, perfect) as a module over A', then $L_{A/R}$ is perfect to order n (almost perfect, perfect) as a module over A (see Proposition 4.1.2.1 and Theorem HA.7.4.3.18). This follows from Proposition 2.7.3.2.

Case (4) is easy, since the underlying ∞-topoi of X and $\mathsf{X}' = \mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét} \pi_0 R$ are the same. To treat case (5), we must show that the diagonal map $\delta : \mathsf{X} \to \mathsf{X} \times_{\operatorname{Spét} R} \mathsf{X}$ is a closed immersion of spectral Deligne-Mumford stacks. Since δ admits a left homotopy inverse (given by the projection onto either fiber), it suffices to show that δ induces a closed immersion between the underlying ∞-topoi of X and $\mathsf{X} \times_{\operatorname{Spét} R} \mathsf{X}$. This follows from our assumption on f_0, since the underlying ∞-topos of $\mathsf{X} \times_{\operatorname{Spét} R} \mathsf{X}$ is equivalent to the underlying ∞-topos of $\mathsf{X}' \times_{\operatorname{Spét} \pi_0 R} \mathsf{X}'$.

We now consider case (6). The assertion is local on X, so we may assume that $\mathsf{X} = \operatorname{Spét} A$ is affine. We wish to show that A is flat over R. Let M be a discrete R-module; we wish to show that $A \otimes_R M$ is discrete. This is clear, since M has the structure of a module over $\pi_0 R$, and the tensor product $A \otimes_R \simeq A' \otimes_{\pi_0 R} M$ is discrete by virtue of our assumption that A' is flat over $\pi_0 R$.

It remains to treat case (7). Assume that f_0 exhibits X' as a proper spectral algebraic space over $\pi_0 R$. It follows from (1), (4) and (5) that X is a quasi-compact separated spectral algebraic space which is locally of finite type over R. It will therefore suffice to show that, for every map of connective $\mathbb{E}_\infty$-rings $R \to R'$, the map of topological spaces $\phi : |\,\mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét} R'| \to |\operatorname{Spec} R'|$ is closed. This follows from our assumption on f_0, since ϕ can be identified with the map of topological spaces

$$|\,\mathsf{X}' \times_{\operatorname{Spét} \pi_0 R} \operatorname{Spét}(\pi_0 R \otimes_R R')| \to |\operatorname{Spec}(\pi_0 R \otimes_R R')|.$$

□

Proof of Proposition 19.4.2.1. The nilcompleteness of j is a consequence of Lemma 19.4.2.3. We will prove that j is cohesive. In cases (1) through (6), this follows from Proposition

16.3.2.1. Let us consider (7). Suppose we are given a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

where the maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective. Fix a map $f : \mathsf{X} \to \operatorname{Spét} R$, and assume that $\mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét} R_0$ and $\mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét} R_1$ are spectral algebraic spaces which are proper over R_0 and R_1, respectively. It follows from (1), (4), and (5) that X is a quasi-compact separated spectral algebraic space which is locally of finite type over R. To complete the proof, it will suffice to show that for every connective R-algebra R', the map of topological spaces $|\mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét} R'| \to |\operatorname{Spec} R'|$ is closed. Replacing R by R', it suffices to show that the map $|\mathsf{X}| \to |\operatorname{Spec} R|$ is closed. Fix a closed subset $K \subseteq |\mathsf{X}|$. Then K is the image of a closed immersion $i : \mathsf{Y} \to \mathsf{X}$ (where we can take the spectral Deligne-Mumford stack Y to be reduced, if so desired). Let $\mathsf{Y}_0 = \mathsf{Y} \times_{\mathsf{X}} \mathsf{X}_0$, and define Y_1 and Y_{01} similarly. We have a pushout diagram of spectral Deligne-Mumford stacks and closed immersions

$$\begin{array}{ccc} \mathsf{Y}_{01} & \longrightarrow & \mathsf{Y}_0 \\ \downarrow & & \downarrow \\ \mathsf{Y}_1 & \longrightarrow & \mathsf{Y}, \end{array}$$

hence a pushout diagram of topological spaces

$$\begin{array}{ccc} |\mathsf{Y}_{01}| & \longrightarrow & |\mathsf{Y}_0| \\ \downarrow & & \downarrow \\ |\mathsf{Y}_1| & \longrightarrow & |\mathsf{Y}|. \end{array}$$

It follows that the image of K in $|\operatorname{Spec} R|$ is the union of the images of the maps

$$|\mathsf{Y}_0| \to |\mathsf{X}_0| \to |\operatorname{Spec} R_0| \hookrightarrow |\operatorname{Spec} R|$$

$$|\mathsf{Y}_1| \to |\mathsf{X}_1| \to |\operatorname{Spec} R_1| \hookrightarrow |\operatorname{Spec} R|.$$

Each of these sets is closed, since X_0 is proper over R_0 and X_1 is proper over R_1. □

19.4.3 Kodaira-Spencer Theory

We now consider the classification of first-order deformations of spectral Deligne-Mumford stacks. We will parallel the discussion of §19.2.2, where we studied the analogous (but easier) problem of classifying first-order deformations of quasi-coherent sheaves. In particular, we make use of Notation 19.2.2.1: if $\mathsf{Y} = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}})$ is a spectral Deligne-Mumford stack and we are given a derivation $\eta \in \mathrm{Der}(\mathscr{O}_{\mathsf{Y}}, \Sigma \mathscr{E})$, we let $\mathsf{Y}^{\eta} = (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}}^{\eta})$ denote the "infinitesimal thickening" of Y determined by η; when $\eta = 0$, we denote this thickening simply by $\mathsf{Y}^{\mathscr{E}}$. Note that we have a pushout diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{Y}^{\Sigma \mathscr{E}} & \longrightarrow & \mathsf{Y} \\ \downarrow & & \downarrow \\ \mathsf{Y} & \longrightarrow & \mathsf{Y}^{\eta} \end{array}$$

where the maps are closed immersions. Applying Theorem 16.3.0.1, we deduce that the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{SpDM}_{/\mathsf{Y}^{\eta}} & \longrightarrow & \mathrm{SpDM}_{/\mathsf{Y}} \\ \downarrow & & \downarrow \\ \mathrm{SpDM}_{/\mathsf{Y}} & \longrightarrow & \mathrm{SpDM}_{/\mathsf{Y}^{\Sigma \mathscr{E}}} \end{array}$$

is a pullback square. Taking $\eta = 0$ and passing to homotopy fibers over some fixed object $\mathsf{X} \in \mathrm{SpDM}_{/\mathsf{Y}}$, we obtain equivalences

$$\begin{aligned} \mathrm{SpDM}_{/\mathsf{Y}^{\mathscr{E}}} \times_{\mathrm{SpDM}_{/\mathsf{Y}}} \{\mathsf{X}\} &\simeq \mathrm{Map}_{\mathrm{SpDM}_{/\mathsf{Y}^{\Sigma \mathscr{E}}}}(\mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}^{\Sigma \mathscr{E}}, \mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}^{\Sigma \mathscr{E}}) \times_{\mathrm{Map}_{\mathrm{SpDM}_{/\mathsf{Y}}}(\mathsf{X}, \mathsf{X})} \{\mathrm{id}_{\mathsf{X}}\} \\ &\simeq \mathrm{Map}_{\mathrm{SpDM}_{/\mathsf{Y}}}(\mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}^{\Sigma \mathscr{E}}, \mathsf{X}) \times_{\mathrm{Map}_{\mathrm{SpDM}_{/\mathsf{Y}}}(\mathsf{X}, \mathsf{X})} \{\mathrm{id}_{\mathsf{X}}\} \\ &\simeq \mathrm{Map}_{\mathrm{SpDM}_{\mathsf{X}//\mathsf{Y}}}(\mathsf{X}^{f^* \Sigma \mathscr{E}}, \mathsf{X}) \\ &\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(L_{\mathsf{X}/\mathsf{Y}}, \Sigma f^* \mathscr{E}); \end{aligned}$$

here the first equivalence follows from the observation that any lifting of id_{X} to an endomorphism of $\mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}^{\Sigma \mathscr{E}}$ is automatically invertible (Corollary 1.4.7.4). We can summarize the situation as follows:

Proposition 19.4.3.1 (Classification of First-Order Deformations)**.** *Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks and let $\mathscr{E} \in \mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}$. Then the ∞-category $\mathrm{SpDM}_{/\mathsf{Y}^{\mathscr{E}}} \times_{\mathrm{SpDM}_{/\mathsf{Y}}} \{\mathsf{X}\}$ is a Kan complex, which is canonically homotopy equivalent to the mapping space $\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(L_{\mathsf{X}/\mathsf{Y}}, \Sigma f^* \mathscr{E})$. In particular, we obtain a canonical bijection*

$$\pi_0(\mathrm{SpDM}_{/\mathsf{Y}^{\mathscr{E}}} \times_{\mathrm{SpDM}_{/\mathsf{Y}}} \{\mathsf{X}\}) \simeq \mathrm{Ext}^1_{\mathrm{QCoh}(\mathsf{X})}(L_{\mathsf{X}/\mathsf{Y}}, f^* \mathscr{E}).$$

In the situation of Proposition 19.4.3.1, suppose that the morphism f is proper, locally almost of finite presentation, and locally of finite Tor-amplitude. Then we can consider the relative dualizing sheaf $\omega_{\mathsf{X}/\mathsf{Y}}$ (Definition 6.4.2.4) and the functor $f_+ : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ of Construction 6.4.5.1. Applying Proposition 6.4.5.3, we obtain a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(L_{\mathsf{X}/\mathsf{Y}}, \Sigma f^* \mathscr{E}) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{Y})}(f_+ L_{\mathsf{X}/\mathsf{Y}}, \Sigma \mathscr{E}) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{Y})}(f_*(\omega_{\mathsf{X}/\mathsf{Y}} \otimes L_{\mathsf{X}/\mathsf{Y}}), \Sigma \mathscr{E}).$$

Notation 19.4.3.2. For any simplicial set K, we let $\mathrm{Var}^+_K : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be the functor introduced in Theorem 19.4.0.2 (given by the formula $\mathrm{Var}^+_K(R) = \mathrm{Fun}(K, \mathrm{Var}^+(R))^{\simeq}$). In the special case $K = \Delta^0$, we will denote Var^+_K by $\mathrm{Var}^+_{\simeq}$ (so that $\mathrm{Var}^+_{\simeq}(R)$ is the underlying Kan complex of the ∞-category of spectral algebraic spaces which are proper, flat, and locally almost of finite presentation over R).

Corollary 19.4.3.3. *The functor* $\mathrm{Var}^+_{\simeq}$ *admits a cotangent complex which is almost perfect and* (-1)*-connective.*

Proof. Let η be an R-valued point of $\mathrm{Var}^+_{\simeq}$, classifying a morphism $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ which is proper, flat, and locally almost of finite presentation. For every connective R-module M, Propositions 19.4.3.1 and 19.4.2.1 supply a canonical homotopy equivalence

$$\mathrm{Var}^+_{\simeq}(R \oplus M) \times_{\mathrm{Var}^+(R)} \{\eta\} \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\, R}, \Sigma f^* M) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(\Sigma^{-1} f_+ L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\, R}, M).$$

Consequently, the functor $\mathrm{Var}^+_{\simeq}$ satisfies condition (a) of Example 17.2.4.4, and condition (b) follows from the compatibility of f_+ with base change (see Proposition 6.4.5.4). It follows that $\mathrm{Var}^+_{\simeq}$ admits a cotangent complex $L_{\mathrm{Var}^+_{\simeq}}$, satisfying $\eta^* L_{\mathrm{Var}^+_{\simeq}} = \Sigma^{-1} f_+ L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\, R}$. Since the quasi-coherent sheaf $L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\, R}$ is connective and almost perfect, the R-module $\Sigma^{-1} f_+ L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\, R}$ is (-1)-connective (by virtue of our assumption that f is flat) and almost perfect (Remark 6.4.5.2). □

Remark 19.4.3.4. In the situation of Corollary 19.4.3.3, there is a representable morphism of functors $f : \mathcal{E} \to \mathrm{Var}^+_{\simeq}$ which is proper, flat, and locally almost of finite presentation (and universal with respect to those properties). In more invariant terms, the calculation of Corollary 19.4.3.3 shows that there is a canonical equivalence

$$L_{\mathrm{Var}^+_{\simeq}} \simeq \Sigma^{-1} f_+ L_{\mathcal{E}/\mathrm{Var}^+_{\simeq}} \simeq \Sigma^{-1} f_*(\omega_{\mathcal{E}/\mathrm{Var}^+_{\simeq}} \otimes L_{\mathcal{E}/\mathrm{Var}^+_{\simeq}}).$$

Remark 19.4.3.5. In the situation of Corollary 19.4.3.3, suppose that $\eta \in \mathrm{Var}^+_{\simeq}(R)$ classifies a morphism $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ which is locally of finite presentation, so that the cotangent complex $L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\, R}$ is perfect. Let T_{X} denote the dual of $L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\, R}$. Then we have a canonical equivalence

$$\eta^* L_{\mathrm{Var}^+_{\simeq}} \simeq \Sigma^{-1} f_+ L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\, R} \simeq \Sigma^{-1}(f_* T_{\mathsf{X}})^{\vee}.$$

In particular, extensions of X along the closed immersion $\mathrm{Sp\acute{e}t}\, R \hookrightarrow \mathrm{Sp\acute{e}t}(R \oplus \Sigma^n R)$ are classified by elements of

$$\mathrm{Ext}^1_R((f_* T_{\mathsf{X}})^{\vee}, \Sigma^n R) \simeq \mathrm{Ext}^{n+1}_R((f_* T_{\mathsf{X}}^{\vee}, R) \simeq \mathrm{Ext}^{n+1}_R(R, (f_* T_{\mathsf{X}})) \simeq \mathrm{Ext}^{n+1}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{O}_{\mathsf{X}}, T_{\mathsf{X}}).$$

Construction 19.4.3.6 (The Kodaira-Spencer Map)**.** Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne-Mumford stacks, let $\mathscr{E} \in \mathrm{QCoh}(\mathsf{Y})^{\mathrm{cn}}$, and let $\eta \in \mathrm{Der}(\mathscr{O}_{\mathsf{Y}}, \mathscr{E})$. Then we can identify η with a left homotopy inverse of the evident closed immersion $\mathsf{Y} \to \mathsf{Y}^{\mathscr{E}}$. The pullback $\mathsf{X} \times_{\mathsf{Y}} \mathsf{Y}^{\mathscr{E}}$ (along the morphism s) determines an object of the ∞-category $\mathrm{SpDM}_{/\mathsf{Y}^{\mathscr{E}}} \times_{\mathrm{SpDM}_{/\mathsf{Y}}} \{\mathsf{X}\}$, which (by virtue of Proposition 19.4.3.1) we can identify with a map $\rho_\eta : L_{\mathsf{X}/\mathsf{Y}} \to \Sigma f^* \mathscr{E}$ in the ∞-category $\mathrm{QCoh}(\mathsf{X})$. In the special case where f is proper, locally almost of finite presentation, and locally of finite Tor-amplitude, we can identify ρ_η with a map

$$\rho'_\eta : f_+ L_{\mathsf{X}/\mathsf{Y}} = f_*(\omega_{\mathsf{X}/\mathsf{Y}} \otimes L_{\mathsf{X}/\mathsf{Y}}) \to \Sigma \mathscr{E}.$$

Specializing to the case where $\mathscr{E} = L_{\mathsf{Y}}$ and η is the universal derivation, we obtain maps

$$\rho_{\mathrm{univ}} : L_{\mathsf{X}/\mathsf{Y}} \to \Sigma f^* L_{\mathsf{Y}} \qquad \rho'_{\mathrm{univ}} : f_*(\omega_{\mathsf{X}/\mathsf{Y}} \otimes L_{\mathsf{X}/\mathsf{Y}}) \to \Sigma L_{\mathsf{Y}}.$$

Unwinding the definitions, we see that ρ_{univ} is the boundary map associated to the fiber sequence $f^* L_{\mathsf{Y}} \to L_{\mathsf{X}} \to L_{\mathsf{X}/\mathsf{Y}}$. When f is proper, locally almost of finite presentation, and locally of finite Tor-amplitude, then we will refer to the map ρ'_{univ} as the *Kodaira-Spencer map*. When f is flat, it can be regarded as the "derivative" of the map $\mathsf{Y} \to \mathrm{Var}^+_{\simeq}$ which classifies f.

Remark 19.4.3.7 (Obstruction Theoretic Interpretation of the Kodaira-Spencer Map)**.** In the situation of Construction 19.4.3.6, suppose that $\mathscr{E} \in \mathrm{QCoh}(\mathsf{Y})$ is 1-connective, so that the derivation $\eta \in \mathrm{Der}(\mathscr{O}_{\mathsf{Y}}, \mathscr{E})$ determines a square-zero extension $\mathscr{O}^\eta_{\mathsf{Y}}$ of $\mathscr{O}_{\mathsf{Y}}$ by $\Sigma^{-1} \mathscr{E}$. Using Theorem 16.3.0.1, we see that the following conditions are equivalent:

(*a*) The map ρ_η of Construction 19.4.3.6 is nullhomotopic.

(*b*) The spectral Deligne-Mumford stack X can be "extended" over the closed immersion $\mathsf{Y} \hookrightarrow \mathsf{Y}^\eta$: that is, there exists a pullback diagram

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{f} & \mathsf{X}' \\ \downarrow & & \downarrow \\ \mathsf{Y} & \longrightarrow & \mathsf{Y}^\eta. \end{array}$$

When f is proper, locally almost of finite presentation, and locally of finite Tor-amplitude, then conditions (*a*) and (*b*) are equivalent to the vanishing of a certain obstruction class $o(\eta) \in \mathrm{Ext}^2_{\mathrm{QCoh}(\mathsf{Y})}(f_*(\omega_{\mathsf{X}/\mathsf{Y}} \otimes L_{\mathsf{X}/\mathsf{Y}}), \Sigma^{-1} \mathscr{E})$, which is given by the product of the Kodaira-Spencer class $[\rho'_{\mathrm{univ}}] \in \mathrm{Ext}^1_{\mathrm{QCoh}(\mathsf{Y})}(f_*(\omega_{\mathsf{X}/\mathsf{Y}} \otimes L_{\mathsf{X}/\mathsf{Y}}), L_{\mathsf{Y}}))$ with the homotopy class $[\eta] \in \mathrm{Ext}^1_{\mathrm{QCoh}(\mathsf{Y})}(L_{\mathsf{Y}}, \Sigma^{-1} \mathscr{E})$ of the derivation η.

19.4.4 The Proof of Theorem 19.4.0.2

Let $\mathrm{Var}^+ : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the functor which assigns to each connective $\mathbb{E}_\infty$-ring R the full subcategory $\mathrm{Var}^+(R) \subseteq \mathrm{SpDM}_{/\mathrm{Sp\acute{e}t}\, R}$ spanned by those maps $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ which are proper, flat, and locally almost of finite presentation. We verify each of the assertions of Theorem 19.4.0.2 in turn:

(1) The functor Var^+ is cohesive and nilcomplete: this follows from Proposition 19.4.2.1 and Remark 19.4.2.2.

(2) The functor Var^+ is locally almost of finite presentation: this follows from Theorem 4.4.2.2 and Proposition 5.5.4.1.

(3) For every connective $\mathbb{E}_\infty$-ring R, the ∞-category $\mathrm{Var}^+(R)$ is essentially small. Since Var^+ is nilcomplete, we may assume that R is n-truncated for some $n \gg 0$. In this case, the desired result follows from Theorem 4.4.2.2.

(4) For every simplicial set K, the functor $\mathrm{Var}^+_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ admits a (-1)-connective cotangent complex, where Var^+_K is defined by the formula $\mathrm{Var}^+_K(R) = \mathrm{Fun}(K, \mathrm{Var}^+(R))^{\simeq}$. In the case $K = \Delta^0$, this follows from Corollary 19.4.3.3. We now treat the case of a general simplicial set K. Writing K as the union of its finite simplicial subsets and applying Remark 17.2.4.5, we may reduce to the case where K is finite. We will prove more generally that for any inclusion $K' \subseteq K$ of finite simplicial sets, the restriction map $F_K \to F_{K'}$ admits a (-1)-connective relative cotangent complex. We proceed by induction on the dimension of K. Using Proposition 17.3.9.1 repeatedly, we can reduce to the case where K is obtained from K' by adjoining a single nondegenerate simplex, so that we have a pushout diagram of simplicial sets

$$\begin{array}{ccc} \partial\,\Delta^n & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow \\ K' & \longrightarrow & K \end{array}$$

and therefore a pullback diagram of functors

$$\begin{array}{ccc} \mathrm{Var}^+_K & \longrightarrow & \mathrm{Var}^+_{K'} \\ \downarrow & & \downarrow \\ \mathrm{Var}^+_{\Delta^n} & \longrightarrow & \mathrm{Var}^+_{\partial\,\Delta^n} . \end{array}$$

We may therefore replace K by Δ^n and K' by $\partial\,\Delta^n$. If $n \geqslant 2$, we have a commutative diagram of functors

$$\begin{array}{ccccc} & & \mathrm{Var}^+_{K'} & & \\ & \nearrow & & \searrow & \\ \mathrm{Var}^+_K & & \longrightarrow & & \mathrm{Var}^+_{\Lambda^n_1} \end{array}$$

where the lower horizontal map is an equivalence. Since the diagonal map on the right admits a (-1)-connective cotangent complex by the inductive hypothesis, Proposition 17.2.5.2 implies that the restriction map $\mathrm{Var}^+_K \to \mathrm{Var}^+_{K'}$ admits a cotangent complex which is a pullback of $\Sigma L_{\mathrm{Var}^+_{K'} / \mathrm{Var}^+_{\Lambda^n_1}}$, and therefore connective. We may therefore assume that $n \leqslant 1$. If $n = 0$, then we are in the case $K = \Delta^0$ treated above. Let us therefore assume that $n = 1$. According to Proposition 17.2.4.7, it will suffice to show that for every pullback diagram of functors

$$\begin{array}{ccc} U & \longrightarrow & \mathrm{Var}^+_{\Delta^1} \\ \downarrow{\scriptstyle q} & & \downarrow \\ \mathrm{Spec}\, R & \xrightarrow{\eta} & \mathrm{Var}^+_{\partial \Delta^0}, \end{array}$$

the natural transformation q admits a relative cotangent complex (which is (-1)-connective). The map η classifies a pair of spectral Deligne-Mumford stacks X and Y which are proper, flat, and locally almost of finite presentation over R. Unwinding the definitions, we see that the functor U is given by the formula $U(R') = \mathrm{Map}_{\mathrm{SpDM}_{/\,\mathrm{Spét}\, R'}}(\mathsf{X} \times_{\mathrm{Spét}\, R} R', \mathsf{Y} \times_{\mathrm{Spét}\, R} R')$. The existence of a relative cotangent complex of q now follows from Proposition ??. Moreover, if $g \in U(R')$, then $L_{U/\,\mathrm{Spét}\, R}(g)$ can be identified with the R'-module given by $f_+ g^* L_{\mathsf{Y}\,/\,\mathrm{Spét}\, R}$, where $f : \mathsf{X} \times_{\mathrm{Spét}\, R} \mathrm{Spét}\, R' \to \mathrm{Spét}\, R'$ denotes the projection onto the second factor. Since f is flat, the functor f_+ is right t-exact. From this we deduce that $L_{U/\,\mathrm{Spét}\, R}$ is connective (and, in particular, (-1)-connective).

(5) Let K be a simplicial set with finitely many simplices of each dimension. Fix an integer n; we wish to show that the Var^+_K commutes with filtered colimits when restricted to $\tau_{\leqslant n}\, \mathrm{CAlg}^{\mathrm{cn}}$. Let R be an n-truncated connective $\mathbb{E}_\infty$-ring. We may assume without loss of generality that $n \geqslant 1$, so that every object of $\mathrm{Var}^+(R)$ is n-localic (Corollary 1.6.8.6). Note that if X is a spectral algebraic space which is flat over R, then the structure sheaf of X is also n-truncated. It follows from Lemma 1.6.8.8 that $\mathrm{Var}^+(R)$ is equivalent to an $(n+1)$-category (that is, the mapping spaces in $\mathrm{Var}^+(R)$ as n-truncated). Consequently, the restriction map $\mathrm{Var}^+_K \to \mathrm{Var}^+_{\mathrm{sk}^{n+2} K}$ is an equivalence of functors. To prove that Var^+_K commutes with filtered colimits when restricted to $\tau_{\leqslant n}\, \mathrm{CAlg}^{\mathrm{cn}}$, we may replace K by the skeleton $\mathrm{sk}^{n+2} K$ and thereby reduce to the case where the simplicial set K is finite. The desired result then follows immediately from (2). The assertion that the cotangent complex $L_{\mathrm{Var}^+_K}$ is almost perfect follows from Corollary 17.4.2.2 (and can also be proven by direct calculation; for example, see Remark 19.4.3.4).

Remark 19.4.4.1. Let κ be a field. For every κ-linear ∞-category $\mathcal{C}$ and every pair of objects $C, D \in \mathcal{C}$, we let $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D) \in \mathrm{Mod}_\kappa$ denote a classifying object for morphisms from

C to D (see Construction **??**). Suppose that $f : \mathsf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \operatorname{Spét} \kappa$ be a proper flat morphism of spectral algebraic spaces, so that f is classified by a point $\eta \in \operatorname{Var}^+_{\simeq}(\kappa)$. Let $\widehat{\operatorname{Var}}^+_{\simeq}$ denote the restriction of $\operatorname{Var}^+_{\simeq}$ to the ∞-category $\operatorname{CAlg}^{\mathrm{sm}}_{\kappa}$ of small κ-algebras. Since the functor $\operatorname{Var}^+_{\simeq}$ is infinitesimally cohesive, the functor $\widehat{\operatorname{Var}}^+_{\simeq}$ is a formal moduli problem over κ (see Proposition 12.1.2.9). Using Remark 17.3.7.4 and Corollary 19.4.3.3, we see that the tangent complex of $\widehat{\operatorname{Var}}^+_{\simeq}$ is given by

$$\begin{aligned} T_{\widehat{\operatorname{Var}}^+_{\simeq}} &\simeq \underline{\operatorname{Map}}_{\operatorname{Mod}_{\kappa}}(\eta^* L_{\operatorname{Var}^+_{\simeq}}, \kappa) \\ &\simeq \Sigma \underline{\operatorname{Map}}_{\operatorname{Mod}_{\kappa}}(f_+ L_{\mathsf{X}/\operatorname{Spét}\kappa}, \kappa) \\ &\simeq \Sigma \underline{\operatorname{Map}}_{\operatorname{QCoh}(\mathsf{X})}(L_{\mathsf{X}/\operatorname{Spét}\kappa}, \mathcal{O}_{\mathcal{X}}). \end{aligned}$$

If κ is a field of characteristic zero, then Theorem 13.0.0.2 implies that the shifted tangent complex $\Sigma^{-1} T_{\widehat{\operatorname{Var}}^+_{\simeq}} \simeq \underline{\operatorname{Map}}_{\operatorname{QCoh}(\mathsf{X})}(L_{\mathsf{X}/\operatorname{Spét}\kappa}, \mathcal{O}_{\mathcal{X}})$ is quasi-isomorphic to a differential graded Lie algebra $\mathbf{g}_*$, which determines the formal moduli problem $\widehat{\operatorname{Var}}^+_{\simeq}$ up to equivalence. Note that since the cotangent complex $L_{\mathsf{X}/\operatorname{Spét}\kappa}$ is connective, the homologies of the Lie algebra $\mathbf{g}_*$ are concentrated in nonpositive degrees. Moreover, the zeroth homology group of $\mathbf{g}_*$ can be identified with the space of vector fields on X (that is, the set of κ-linear derivations of the structure sheaf $\mathcal{O}_{\mathcal{X}}$ into itself). With more effort, one can show that induced Lie algebra structure on this set is given by the classical Lie bracket of tangent fields.

19.4.5 Open Loci

If R is a connective $\mathbb{E}_{\infty}$-ring, then the ∞-category $\operatorname{Var}^+(R)$ introduced in Construction 19.4.0.1 is rather unwieldy: it contains *all* spectral algebraic spaces which are proper, flat, and locally almost of finite presentation over R. In practice, it is often convenient to restrict attention to some subcategory of $\operatorname{Var}^+(R)$ spanned by algebraic spaces which satisfy additional requirements. We consider here a few simple examples; we will meet others in §**??**.

Notation 19.4.5.1. Let P be some property of morphisms $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks. For every connective $\mathbb{E}_{\infty}$-ring R, we let $\operatorname{Var}^+_P(R)$ denote the subspace of $\operatorname{Var}^+_{\simeq}(R)$ spanned by those maps $\mathsf{X} \to \operatorname{Spét} R$ which have the property P.

Proposition 19.4.5.2. *Let P be any one of the following properties of a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne-Mumford stacks:*

(a) *The property of having relative dimension $\leqslant d$ (Definition 3.3.1.1.*

(b) *The property of being fiber smooth (Definition 11.2.5.5).*

(c) *The property of being geometrically reduced (Definition **??**).*

(d) *The property of being finite flat (Definition 5.2.3.1).*

(e) *The property of being finite étale (Definition 3.3.2.3).*

(f) *The property of being geometrically reduced and geometrically connected (Proposition 8.6.4.1).*

Then the inclusion $\mathrm{Var}^+_P \hookrightarrow \mathrm{Var}^+_{\simeq}$ *is an open immersion of functors (Definition 19.2.4.1).*

Proof. Let R be a connective $\mathbb{E}_\infty$-ring and let $\eta \in \mathrm{Var}^+_{\simeq}(R)$ be a point which classifies a morphism $f : \mathsf{X} \to \mathrm{Spét}\, R$ which is proper, flat, and locally almost of finite presentation. Set $F = \mathrm{Var}^+_P \times_{\mathrm{Var}^+_{\simeq}} \mathrm{Spec}\, R \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. By definition, the value of F on a connective $\mathbb{E}_\infty$-ring A can be identified with the summand of $\mathrm{Map}_{\mathrm{CAlg}}(R, A)$ spanned by those maps for which the projection $f_A : \mathsf{X}_A = \mathrm{Spét}\, A \times_{\mathrm{Spét}\, R} \mathsf{X} \to \mathrm{Spét}\, A$ has the property P. We wish to show that F is representable by an open subspace of $\mathrm{Spét}\, A$.

We first treat cases (a), (b), and (c). Let $\mathsf{U} \subseteq \mathsf{X}$ be the largest open substack over which $f|_{\mathsf{U}}$ has the property P. Using Propositions 4.2.2.1, 11.2.5.10, and 8.6.3.1, we see for $A \in \mathrm{CAlg}^{\mathrm{cn}}_R$, the projection map $f_A : \mathsf{X}_A \to \mathrm{Spét}\, A$ has the property P if and only if the projection $\mathsf{X}_A \to \mathsf{X}$ factors through U. Let $K \subseteq |\mathsf{X}|$ be the closed subset complementary to U. Then f_A has the property P if and only if the underlying map $|\mathrm{Spec}\, A| \to |\mathrm{Spec}\, R|$ factors through $|\mathrm{Spec}\, R| - f(K)$. The desired result now follows from the fact that $f(K) \subseteq |\mathrm{Spec}\, R|$ is closed (by virtue of our assumption that f is proper).

Note that (d) is a special case of (a) (since a morphism $f : \mathsf{X} \to \mathsf{Y}$ which is proper, flat, and locally almost of finite presentation is finite flat if and only if it has relative dimension $\leqslant 0$; see Proposition 5.2.3.3). Moreover, case (e) follows from (c) and (d) (since a morphism is finite étale if and only if is finite flat and geometrically reduced). To handle the case (f), we first use (c) to reduce to the case where the projection $f : \mathsf{X} \to \mathrm{Spét}\, R$ is geometrically reduced. Let $M \in \mathrm{Mod}_R$ denote the cofiber of the unit map $R \to \Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}})$. Since f is proper, flat, and locally almost of finite presentation, the module M is perfect (Theorem 6.1.3.2). Using Proposition 8.6.4.1, we see that for $A \in \mathrm{CAlg}^{\mathrm{cn}}_R$, the projection map $f_A : \mathsf{X}_A \to \mathrm{Spét}\, A$ is geometrically connected if and only if $M_A = A \otimes_R M$ has Tor-amplitude $\leqslant 1$, or equivalently if the dual $M^\vee_A = A \otimes_R M^\vee$ is 1-connective. The desired result now follows from Lemma 2.9.3.3. □

Definition 19.4.5.3. Let R be a connective $\mathbb{E}_\infty$-ring. A *variety over* R is a spectral algebraic space X equipped with a map $f : \mathsf{X} \to \mathrm{Spét}\, R$ which is proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. We let $\mathrm{Var}(R)$ denote the full subcategory of $\mathrm{Var}^+(R)$ spanned by those objects which are varieties over R.

For each integer $d \geqslant 0$, we let $\mathrm{Var}^{+,\leqslant d}(R)$ denote the full subcategory of $\mathrm{Var}^+(R)$ spanned by those morphisms $f : \mathsf{X} \to \mathrm{Spét}\, R$ which are proper, flat, locally almost of finite presentation, and of relative dimension $\leqslant d$. We let $\mathrm{Var}^{\leqslant d}(R)$ denote the intersection $\mathrm{Var}^{+,\leqslant d}(R) \cap \mathrm{Var}(R)$.

Remark 19.4.5.4. Suppose we are given a pullback diagram of spectral algebraic spaces

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Sp\acute{e}t}\, R' & \longrightarrow & \mathrm{Sp\acute{e}t}\, R. \end{array}$$

If f exhibits X as a variety over R, then f' exhibits X' as a variety over R'. Consequently, we can regard the construction $R \mapsto \mathrm{Var}(R)$ as a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$, so that the inclusion $\mathrm{Var}(R) \hookrightarrow \mathrm{Var}^+(R)$ depends functorially on R. Similarly, for each $d \geqslant 0$, we can regard the constructions $R \mapsto \mathrm{Var}^{+,\leqslant d}(R)$ and $R \mapsto \mathrm{Var}^{\leqslant d}(R)$ as functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$.

Variant 19.4.5.5. For every simplicial set K, we let $\mathrm{Var}_K, \mathrm{Var}_K^{\leqslant d}, \mathrm{Var}_K^{+,\leqslant d} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functors given by

$$\mathrm{Var}_K(R) = \mathrm{Fun}(K, \mathrm{Var}(R))^{\simeq} \qquad \mathrm{Var}_K^{\leqslant d}(R) = \mathrm{Fun}(K, \mathrm{Var}^{\leqslant d}(R))^{\simeq} \qquad \mathrm{Var}_K^{+,\leqslant d}(R) = \mathrm{Fun}(K, \mathrm{Var}^{+,\leqslant d}(R))^{\simeq}.$$

In the special case where $K = \Delta^0$, we will denote these functors by $\mathrm{Var}_{\simeq}$, $\mathrm{Var}_{\simeq}^{\leqslant d}$, and $\mathrm{Var}_{\simeq}^{+,\leqslant d}$, respectively.

Proposition 19.4.5.6. *Let K be a simplicial set having finitely many vertices. Then the inclusion maps $\mathrm{Var}_K, \mathrm{Var}_K^{\leqslant d}, \mathrm{Var}_K^{+,\leqslant d} \hookrightarrow \mathrm{Var}_K^+$ are open immersions.*

Proof. Without loss of generality, we may assume that $K = \Delta^0$, in which case the desired result follows from Proposition 19.4.5.2. □

Corollary 19.4.5.7. *For every integer $n \geqslant 0$, the functors $\mathrm{Var}, \mathrm{Var}^{\leqslant d}, \mathrm{Var}^{+,\leqslant d} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$ commute with filtered colimits when restricted to $\tau_{\leqslant n}\,\mathrm{CAlg}^{\mathrm{cn}}$.*

Proof. We must show that for each $m \geqslant 0$, the functors $\mathrm{Var}_{\Delta^m}, \mathrm{Var}_{\Delta^m}^{\leqslant d}, \mathrm{Var}_{\Delta^m}^{+,\leqslant d} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ are locally almost of finite presentation. This follows from Proposition 19.4.5.6 and Theorem 19.4.0.2. □

Corollary 19.4.5.8. *The functor $\mathrm{Var}_{\simeq}, \mathrm{Var}_{\simeq}^{\leqslant d}, \mathrm{Var}_{\simeq}^{+,\leqslant d} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ are cohesive, nilcomplete, locally almost of finite presentation, and admits cotangent complexes which are almost perfect and (-1)-connective.*

Proof. Combine Proposition 19.2.4.3, Proposition 19.4.5.6, and Theorem 19.4.0.2. □

Part VI

Structured Spaces

Chapter 20

Fractured ∞-Topoi

Let X be a scheme (which we assume for simplicity to be quasi-compact and quasi-separated). Let Sch_X denote the category whose objects are X-schemes of finite presentation, and let $\mathrm{Sch}_X^{\text{ét}}$ denote the full subcategory of Sch_X spanned by those schemes which are étale over X. Associated to these categories are two different notions of étale sheaf on X:

- One can consider sheaves which are defined only on étale X-schemes: that is, functors $\mathscr{F} : (\mathrm{Sch}_X^{\text{ét}})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ which satisfy descent with respect to the étale topology. The category of such functors is a Grothendieck topos, which we will denote by $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{Sch}_X^{\text{ét}})$ and refer to as the *small étale topos* of X.
- One can consider sheaves which are defined on *all* X-schemes of finite presentation: that is, functors $\mathscr{F} : \mathrm{Sch}_X^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ which satisfy descent for the étale topology. The category of such functors is also a Grothendieck topos, which we will denote by $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{Sch}_X)$ and refer to as the *big étale topos* of X.

Both of these topoi play a useful role in organizing algebro-geometric information about X. We can think of the small étale topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{Sch}_X^{\text{ét}})$ as sort of generalized topological space which "enhances" the Zariski topology on X (which can be recovered as the localic reflection of $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{Sch}_X^{\text{ét}})$). The big étale topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{Sch}_X)$ has a somewhat different role to play: it can be viewed as a *classifying topos* for strictly Henselian (sheaves of) $\mathscr{O}_X$-algebras (see Proposition **??**).

Warning 20.0.0.1. The terminology introduced above is not standard: most authors define the *big étale site* of X to be the category of *all* X-schemes, rather than X-schemes of finite presentation. However, the category Sch_X defined above is closer in spirit to the "big sites" of interest to us in this book, and has the added virtue of being an essentially small category (so that we can refer to $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{Sch}_X)$ as a Grothendieck topos without appealing to any set-theoretic legerdemain).

The category $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{Sch}_X)$ is a prototypical example of what is sometimes called a *gros topos*: an object $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{Sch}_X)$ can be viewed not as a single sheaf, but as a family of sheaves $\{\mathscr{F}_Y \in \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathrm{Sch}_Y^{\text{ét}})\}_{Y \in \mathrm{Sch}_X}$ which are compatible in the sense that every morphism $f : Y \to Z$ in Sch_X determines a map of sheaves $f^* \mathscr{F}_Z \to \mathscr{F}_Y$ (which is an isomorphism when f is étale). Our goal in this chapter is to introduce some language which formalizes this idea, and to simultaneously introduce some organizational principles which will be useful for formulating various generalizations of algebraic geometry. Since most of our intended applications are higher-categorical in nature, we work in the context of ∞-topoi rather than ordinary Grothendieck topoi (this extra generality actually simplifies certain aspects of the theory).

We begin in §20.1 by introducing the notion of a *fractured ∞-topos*. Roughly speaking, this is an ∞-topos $\mathcal{X}$ equipped with a (usually non-full) subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ of *corporeal objects*, satisfying certain axioms (see Definition 20.1.2.1). A prototypical example of a fractured ∞-topos is the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$ of $\mathcal{S}$-valued sheaves on the big étale site of a scheme X: in this case, the relevant subcategory $\mathcal{S}\mathrm{hv}_{\mathcal{S}}^{\mathrm{corp}}(\mathrm{Sch}_X)$ of corporeal objects is equivalent to $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X^{\circ})$, where Sch_X° denotes the subcategory of Sch_X containing all objects, but only étale morphisms between X-schemes. This is a special case of general construction which we study in §20.6: given a Grothendieck site $\mathcal{G}$ equipped with a distinguished class of "admissible" morphisms (satisfying some conditions which we will axiomatize in §20.2), the ∞-category $\mathcal{S}\mathrm{hv}(\mathcal{G})$ of $\mathcal{S}$-valued sheaves on $\mathcal{G}$ can be regarded as a fractured ∞-topos, where the subcategory of corporeal objects can be identified with sheaves on the subcategory $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ spanned by the admissible morphisms.

In §20.3, we consider the following question: given a fractured ∞-topos $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$, to what extent can $\mathcal{X}^{\mathrm{corp}}$ be recovered from $\mathcal{X}$? Our main observation is that, under a mild additional assumption, we do not need to remember the *objects* of the subcategory $\mathcal{X}^{\mathrm{corp}}$, only its morphisms. More precisely, we show that $\mathcal{X}^{\mathrm{corp}}$ determines a class of $\mathcal{X}^{\mathrm{corp}}$-*admissible morphisms* in $\mathcal{X}$ (Definition 20.3.1.1), satisfying the axiomatics of §20.2. Moreover, if the collection of corporeal objects of $\mathcal{X}$ is closed under retracts, we show that the subcategory $\mathcal{X}^{\mathrm{corp}}$ can be recovered from the class of $\mathcal{X}^{\mathrm{corp}}$-admissible morphisms (Proposition 20.3.3.11). This gives an alternate approach to the theory of fractured ∞-topoi based on the notion of a *geometric admissibility structure* (Definition 20.3.4.1), which is closely related to earlier work of Joyal-Moerdijk ([109]) and Dubuc ([52]) on "classes of étale morphisms" in a Grothendieck topos.

For our applications in this book, all of the fractured ∞-topoi we will need are supplied by the construction of §20.6: that is, they have the form $\mathcal{S}\mathrm{hv}(\mathcal{G})$, where $\mathcal{G}$ is a geometric site. However, it seems unlikely that *all* fractured ∞-topoi can be obtained by this construction, because of the technical nuisance that not every left exact localization of a presheaf ∞-category arises from a Grothendieck topology. In §20.5, we remedy the situation by defining the notion of a *presentation* of a fractured ∞-topos (Definition 20.5.3.1), and proving that every fractured ∞-topos admits a presentation (Theorem 20.5.3.4). The analysis of this notion relies on some general facts about exact and dense functors to ∞-topoi, which we discuss in §20.4.

Remark 20.0.0.2. Many of the ideas described in this chapter have been developed in work of Carchedi (see [38]). In particular, we are indebted to him for the observation that the corporeal objects of a fractured ∞-topos can again be regarded as an ∞-topos (Proposition 20.1.3.3).

Contents

20.1 Fracture Subcategories

Let X be a scheme (which we assume, for simplicity, to be quasi-compact and quasi-separated), let Sch_X denote the category of finitely presented X-schemes, and let $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$ denote the ∞-topos of $\mathcal{S}$-valued sheaves on the category Sch_X (where we regard Sch_X as equipped with the étale topology). Let us abuse notation by not distinguishing between an object $Y \in \mathrm{Sch}_X$ and the representable functor $h_Y \in \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$, given by the formula $h_Y(Z) = \mathrm{Hom}_{\mathrm{Sch}_X}(Z, Y)$; that is, we identify Sch_X with its image under the Yoneda embedding $h : \mathrm{Sch}_X \hookrightarrow \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$. The ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$ is generated by this image under small colimits: every sheaf $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$ can be recovered as the colimit $\varinjlim_{\eta \in \mathscr{F}(Y)} Y$, indexed by those pairs (Y, η) where $Y \in \mathrm{Sch}_X$ and $\eta \in \mathscr{F}(Y)$. We can describe the situation more informally as follows: every sheaf $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$ can be obtained by "gluing together" schemes of finite presentation over X. To carry out this gluing, we are forced to stray far from algebraic geometry: every diagram in the category Sch_X admits a colimit in $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$, but such colimits are rarely representable by geometric objects such as schemes. However, if $\{Y_\alpha\}$ is a diagram in Sch_X for which all of the the transition maps $Y_\alpha \to Y_\beta$ are assumed to be étale, then the situation is better: in this case, the colimit $\varinjlim \{Y_\alpha\}$ is representable by a higher Deligne-Mumford stack which is locally of finite presentation over X. Let us refer to an object $\mathscr{F} \in \mathrm{Sch}_{\mathcal{S}}(\mathrm{Sch}_X)$ as *corporeal* if it can obtained in this way. Let $\mathcal{S}\mathrm{hv}_{\mathcal{S}}^{\mathrm{corp}}(\mathrm{Sch}_X)$ denote the subcategory of $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$ spanned by corporeal objects and étale morphisms between them. This subcategory has several pleasant features:

(a) The ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{S}}^{\mathrm{corp}}(\mathrm{Sch}_X)$ is itself an ∞-topos (it is equivalent to the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X^{\text{ét}})$: see Theorem 20.6.3.4, or Corollary 6.2.2 of **??**).

(b) The inclusion functor $j_! : \mathcal{S}\mathrm{hv}_{\mathcal{S}}^{\mathrm{corp}}(\mathrm{Sch}_X) \hookrightarrow \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$ commutes with fiber products, and admits a conservative right adjoint j^* which preserves small colimits.

(c) For every morphism $U \to V$ in the ∞-category $\mathcal{X}^{\mathrm{corp}}$, the diagram

$$\begin{array}{ccc} j^*U & \longrightarrow & j^*V \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

is a pullback square in $\mathcal{X}$ (here the vertical maps are induced by the counit transformation $j_! j^* \to \mathrm{id}$).

Roughly speaking, the adjunction $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_{\mathcal{S}}(\mathrm{Sch}_X) \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$ encodes the idea that the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$ is a "gros" ∞-topos, which can be decomposed into the small étale ∞-topoi $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}^{\mathrm{\acute{e}t}}_Y)$, where Y ranges over the objects of Sch_X. Our goal in this section is to introduce an axiomatic framework for studying variants of this example. This framework is based on the concept of a *fracture subcategory* (Definition 20.1.2.1) of an ∞-topos $\mathcal{X}$: that is, a subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ which satisfies analogues of conditions (b) and (c) (assertion (a) is then a formal consequence: see Proposition 20.1.3.3).

Remark 20.1.0.1. It may not be immediately obvious that assertions (a), (b), and (c) are true: in other words, that $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_{\mathcal{S}}(\mathrm{Sch}_X)$ is a fracture subcategory of the ∞-topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$. This is a special case of a more general result that we will prove in §20.6: see Theorem 20.6.3.4.

20.1.1 Replete Subcategories

We begin with a brief categorical digression. Recall that if $\mathcal{C}$ is an ∞-category, then a simplicial subset $\mathcal{C}_0 \subseteq \mathcal{C}$ is said to be a *subcategory* of $\mathcal{C}$ if there exists a pullback square

$$\begin{array}{ccc} \mathcal{C}_0 & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ (\mathrm{h}\mathcal{C})_0 & \longrightarrow & \mathrm{h}\mathcal{C}, \end{array}$$

where $(\mathrm{h}\mathcal{C})_0$ is a subcategory of the homotopy category $\mathrm{h}\mathcal{C}$ (in the sense of ordinary category theory): see §HTT.1.2.11 .

Proposition 20.1.1.1. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a subcategory. The following conditions are equivalent:*

(a) *The inclusion map $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is a categorical fibration.*

(b) *For every morphism $f : X \to Y$ in $\mathcal{C}$ which is an equivalence, if X belongs to $\mathcal{C}_0$, then f belongs to $\mathcal{C}_0$.*

Proof. This is an immediate consequence of Corollary HTT.2.4.6.5 . □

Definition 20.1.1.2. Let $\mathcal{C}$ be an ∞-category. We will say that a subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is *replete* if it satisfies the equivalent conditions of Proposition 20.1.1.1.

Remark 20.1.1.3. Let $\mathcal{C}$ be an ∞-category. Then a subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is replete if and only if the homotopy category $\mathrm{h}\mathcal{C}_0$ is a replete subcategory of $\mathrm{h}\mathcal{C}$.

Remark 20.1.1.4. Let $\mathcal{C}$ be an ∞-category. If $\mathcal{C}_0$ is a replete subcategory of $\mathcal{C}$, then the inclusion functor $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is conservative.

Proposition 20.1.1.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. The following conditions are equivalent:*

- (a) *The functor F induces an equivalence from $\mathcal{C}$ to a replete subcategory $\mathcal{D}_0 \subseteq \mathcal{D}$.*
- (b) *For every pair of objects $C, C' \in \mathcal{C}$, the induced map $\mathrm{Map}_{\mathcal{C}}(C, C) \to \mathrm{Map}_{\mathcal{D}}(FC, FC')$ induces a homotopy equivalence from $\mathrm{Map}_{\mathcal{C}}(C, D)$ to a summand $\mathrm{Map}_{\mathcal{D}}(FC, FC')_0 \subseteq \mathrm{Map}_{\mathcal{D}}(FC, FC')$. Moreover, every equivalence $\alpha : FC \to FD$ in the ∞-category $\mathcal{D}$ is contained in the summand $\mathrm{Map}_{\mathcal{D}}(FC, FC')_0$.*

Proof. The implication $(a) \Rightarrow (b)$ is immediate. Conversely, suppose that (b) is satisfied. Let D and D' be objects of $\mathcal{D}$ belonging to the essential image of F, so that we can choose equivalences $\alpha : FC \to D$ and $\beta : D' \to FC'$ for some objects $C, C' \in \mathcal{C}$. Let us say that a morphism $\gamma : D \to D'$ is *good* if the composition $\beta \circ \gamma \circ \alpha$ belongs to the summand $\mathrm{Map}_{\mathcal{D}}(FC, FC')_0 \subseteq \mathrm{Map}_{\mathcal{D}}(FC, FC')$ described in (b). The final assumption of (b) guarantees that this condition does not depend on the choice of equivalences α and β. The collection of good morphisms contains all equivalences and is stable under composition. Let $\mathcal{D}_0 \subseteq \mathcal{D}$ be the subcategory spanned by those objects which belong to the essential image of F and whose morphisms are good. Then $\mathcal{D}_0$ is replete and the functor F induces an equivalence $\mathcal{C} \to \mathcal{D}_0$. □

20.1.2 Fractured ∞-Topoi

We are now ready to introduce the main objects of interest in this section.

Definition 20.1.2.1. Let $\mathcal{X}$ be an ∞-topos. We will say that a subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ is a *fracture subcategory* if it satisfies the following conditions:

- (0) The subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ is replete (Definition 20.1.1.2).
- (1) The ∞-category $\mathcal{X}^{\mathrm{corp}}$ admits fiber products, which are preserved by the inclusion functor $j_! : \mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$.

(2) The inclusion functor $j_! : \mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$ admits a right adjoint $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$. Moreover, the functor j^* is conservative and preserves small colimits.

(3) For every morphism $U \to V$ in $\mathcal{X}^{\mathrm{corp}}$, the diagram

$$\begin{array}{ccc} j^*U & \longrightarrow & j^*V \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

(determined by the counit map $j_! j^* \to \mathrm{id}$) is a pullback square in $\mathcal{X}$.

A *fractured ∞-topos* is a pair $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$, where $\mathcal{X}$ is an ∞-topos and $\mathcal{X}^{\mathrm{corp}}$ is a fracture subcategory of $\mathcal{X}$. In this case, we will say that an object $X \in \mathcal{X}$ is *corporeal* if it belongs to the subcategory $\mathcal{X}^{\mathrm{corp}}$.

Example 20.1.2.2. Let $\mathcal{X}$ be an ∞-topos. Then $\mathcal{X}$ is a fracture subcategory of itself.

Example 20.1.2.3. Let $\mathcal{X}$ be an ∞-topos, and let $F : \mathcal{X} \times \mathcal{X} \to \mathrm{Fun}(\Delta^1, \mathcal{X})$ be the functor which carries a pair of objects $X, Y \in \mathcal{X}$ to the map $X \hookrightarrow X \amalg Y$. Then F induces an equivalence from $\mathcal{X} \times \mathcal{X}$ to a fracture subcategory $\mathrm{Fun}(\Delta^1, \mathcal{X})^{\mathrm{corp}} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{X})$.

Remark 20.1.2.4. Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos and let $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ be a right adjoint to the inclusion. Then the functor j^* preserves small limits. In particular, if $\mathbf{1}$ denotes a final object of $\mathcal{X}$, then $j^*\mathbf{1}$ is a final object of $\mathcal{X}^{\mathrm{corp}}$. It follows that the ∞-category $\mathcal{X}^{\mathrm{corp}}$ admits finite limits.

Warning 20.1.2.5. In the situation of Definition 20.1.2.1, we do not assume that the inclusion functor $j_!$ preserves final objects: in fact, this condition is never satisfied except in the trivial case $\mathcal{X}^{\mathrm{corp}} = \mathcal{X}$.

Warning 20.1.2.6. The commutative diagram appearing in requirement (3) of Definition 20.1.2.1 is *not* a commutative diagram in the ∞-category $\mathcal{X}^{\mathrm{corp}}$. Every object in the diagram is corporeal, but the vertical maps generally do not belong to $\mathcal{X}^{\mathrm{corp}}$.

Notation 20.1.2.7. Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos. We let $\mathbf{1}^{\mathrm{corp}}$ denote the final object of $\mathcal{X}^{\mathrm{corp}}$ (which is usually not final as an object of $\mathcal{X}$).

Remark 20.1.2.8. Let $\mathcal{X}$ be an ∞-topos, let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a subcategory, and let $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ be a right adjoint to the inclusion functor. Then $\mathcal{X}^{\mathrm{corp}}$ admits a final object $\mathbf{1}^{\mathrm{corp}}$ (obtained by applying the functor j^* to a final object of $\mathcal{X}$). For every morphism $U \to V$ in $\mathcal{X}^{\mathrm{corp}}$, we obtain a commutative diagram

$$\begin{array}{ccccc} j^*U & \longrightarrow & j^*V & \longrightarrow & j^*\mathbf{1}^{\mathrm{corp}} \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & V & \longrightarrow & \mathbf{1}^{\mathrm{corp}} \end{array}$$

in the ∞-category $\mathcal{X}$. Consequently, to prove that the square on the left is a pullback diagram, it suffices to check that the square on the right and the outer rectangle are pullback diagrams. We can therefore replace condition (3) of Definition 20.1.2.1 with the following *a priori* weaker condition:

(3′) For every object $U \in \mathcal{X}^{\mathrm{corp}}$, the induced diagram

$$\begin{array}{ccc} j^*U & \longrightarrow & j^*\mathbf{1}^{\mathrm{corp}} \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathbf{1}^{\mathrm{corp}} \end{array}$$

is a pullback square in $\mathcal{X}$.

Proposition 20.1.2.9. *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos and let $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ be a right adjoint to the inclusion. Then, for every morphism $U \to V$ in $\mathcal{X}^{\mathrm{corp}}$, the diagram σ :*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ j^*U & \longrightarrow & j^*V \end{array}$$

(induced by the unit map $\mathrm{id}_{\mathcal{X}^{\mathrm{corp}}} \to j^*|_{\mathcal{X}^{\mathrm{corp}}}$) is a pullback square in $\mathcal{X}^{\mathrm{corp}}$ (and therefore also in $\mathcal{X}$).

Proof. Let $j_! : \mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$ denote the inclusion map. The unit and counit for the adjunction $\mathcal{X}^{\mathrm{corp}} \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathcal{X}$ determine a commutative diagram

$$\begin{array}{ccc} j_!U & \longrightarrow & j_!V \\ \downarrow & & \downarrow \\ j_!j^*j_!U & \longrightarrow & j_!j^*j_!V \\ \downarrow & & \downarrow \\ j_!U & \longrightarrow & j_!V \end{array}$$

in the ∞-category $\mathcal{X}$, where the vertical composite maps are homotopic to the identity. It follows that the outer rectangle is a pullback square in $\mathcal{X}$. Requirement (3) of Definition 20.1.2.1 guarantees that the lower square is also a pullback, so the upper square is a pullback as well. In other words, the diagram σ is a pullback square in the ∞-category $\mathcal{X}$. Since the functor $j_!$ is conservative (Remark 20.1.1.4) and preserves fiber products (requirement (1) of Definition 20.1.2.1), it follows that σ is also a pullback square in $\mathcal{X}^{\mathrm{corp}}$. □

Example 20.1.2.10. Applying Proposition 20.1.2.9 in the special case where $V = \mathbf{1}^{\mathrm{corp}}$, we deduce that every object $U \in \mathcal{X}^{\mathrm{corp}}$ fits into a canonical fiber sequence $U \to j^*U \to j^*\mathbf{1}^{\mathrm{corp}}$ (in the ∞-category $\mathcal{X}^{\mathrm{corp}}$).

20.1.3 Properties of Fracture Subcategories

Our next goal is to show that any fracture subcategory of an ∞-topos is itself an ∞-topos.

Proposition 20.1.3.1. *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos. Then for every corporeal object $X \in \mathcal{X}^{\mathrm{corp}}$, the inclusion map $F : \mathcal{X}^{\mathrm{corp}}_{/X} \hookrightarrow \mathcal{X}_{/X}$ is fully faithful.*

Proof. Let $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ be a right adjoint to the inclusion. The functor F admits a right adjoint G, given by the formula $G(Y) = j^*Y \times_{j^*Y} Y$. It follows from Proposition 20.1.2.9 that the counit map $\mathrm{id} \to G \circ F$ is an equivalence, so that F is fully faithful. □

Remark 20.1.3.2. It follows from Proposition 20.1.3.1 that a fracture subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ can be identified with a full subcategory $\mathcal{Y} \subseteq \mathcal{X}_{/\mathbf{1}^{\mathrm{corp}}}$ for some (uniquely determined) object $\mathbf{1}^{\mathrm{corp}} \in \mathcal{X}$. Condition (1) of Definition 20.1.2.1 guarantee that $\mathcal{Y}$ is closed under finite limits, and condition (2) guarantees that the inclusion $\mathcal{Y} \hookrightarrow \mathcal{X}_{/\mathbf{1}^{\mathrm{corp}}}$ admits a right adjoint. Beware that $\mathcal{Y}$ is generally not a fracture subcategory of $\mathcal{X}_{/\mathbf{1}^{\mathrm{corp}}}$ (Warning 20.1.3.5).

Proposition 20.1.3.3. *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos. Then the ∞-category $\mathcal{X}^{\mathrm{corp}}$ is an ∞-topos.*

Proof. Since the forgetful functor $\mathcal{X}^{\mathrm{corp}}_{/\mathbf{1}^{\mathrm{corp}}} \to \mathcal{X}^{\mathrm{corp}}$ is an equivalence, it will suffice to show that $\mathcal{X}^{\mathrm{corp}}_{/\mathbf{1}^{\mathrm{corp}}}$ is an ∞-topos. Remark **??** shows that $\mathcal{X}^{\mathrm{corp}}_{/\mathbf{1}^{\mathrm{corp}}}$ is a full subcategory of $\mathcal{X}_{/\mathbf{1}^{\mathrm{corp}}}$ which is closed under small colimits and finite limits. It will therefore suffice to show that $\mathcal{X}^{\mathrm{corp}}_{/\mathbf{1}^{\mathrm{corp}}}$ is presentable (as in the proof of Proposition HTT.6.3.6.2). Let $G : \mathcal{X}_{/\mathbf{1}^{\mathrm{corp}}} \to \mathcal{X}^{\mathrm{corp}}_{/\mathbf{1}^{\mathrm{corp}}}$ be a right adjoint to the inclusion, given by $G(X) = j^*X \times_{j^*\mathbf{1}^{\mathrm{corp}}} \mathbf{1}^{\mathrm{corp}}$ (as in the proof of Proposition 20.1.3.1). Since the functor $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ preserves small colimits (which are then preserved by the inclusion $\mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$) and colimits in $\mathcal{X}$ are universal, it follows that the functor G preserves small colimits. Consequently, if $\{X_\alpha\}$ is a small collection of objects which generates $\mathcal{X}_{/\mathbf{1}^{\mathrm{corp}}}$ under small colimits, then the objects $G(X_\alpha)$ generate $\mathcal{X}^{\mathrm{corp}}_{/\mathbf{1}^{\mathrm{corp}}}$ under small colimits. It now suffices to observe that we can choose a regular cardinal κ such that each $G(X_\alpha)$ is a κ-compact object of $\mathcal{X}_{/\mathbf{1}^{\mathrm{corp}}}$ (and therefore also a κ-compact object of $\mathcal{X}^{\mathrm{corp}}_{/\mathbf{1}^{\mathrm{corp}}}$). □

Corollary 20.1.3.4. *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos. Then $\mathcal{X}$ is generated under small colimits by corporeal objects.*

Proof. Let $\mathcal{C} \subseteq \mathcal{X}$ be the smallest full subcategory which contains $\mathcal{X}^{\mathrm{corp}}$ and is closed under small colimits. Since $\mathcal{X}^{\mathrm{corp}}$ is accessible (Proposition 20.1.3.3), we can choose an essentially small subcategory $\mathcal{X}^{\mathrm{corp}}_0 \subseteq \mathcal{X}^{\mathrm{corp}}$ which generates $\mathcal{X}^{\mathrm{corp}}$ under small colimits. Then $\mathcal{C}$ is the smallest full subcategory of $\mathcal{X}$ which contains $\mathcal{X}^{\mathrm{corp}}_0$ and is closed under small colimits. It follows that the ∞-category $\mathcal{C}$ is presentable. Applying Corollary HTT.5.5.2.9 , we deduce that the inclusion $\mathcal{C} \hookrightarrow \mathcal{X}$ admits a right adjoint $G : \mathcal{X} \to \mathcal{C}$. Let $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$

be a right adjoint to the inclusion, so that j^* factors through G. Our assumption that $\mathcal{X}^{\mathrm{corp}}$ is a fracture subcategory of $\mathcal{X}$ guarantees that j^* is conservative, so the functor G is also conservative. It follows that G is an equivalence of ∞-categories, so that $\mathcal{X} = \mathcal{C}$ as desired. □

Warning 20.1.3.5. Let $\mathcal{X}$ be an ∞-topos. Then a fracture subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ is never a full subcategory of $\mathcal{X}$, except in the trivial situation where $\mathcal{X}^{\mathrm{corp}} = \mathcal{X}$ (Example 20.1.2.2). To see this, we note that if the inclusion $j_! : \mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ were fully faithful, then the collection of corporeal objects of $\mathcal{X}$ would be closed under small colimits. We could then apply Corollary 20.1.3.4 to deduce that every object of $\mathcal{X}$ is corporeal, so that $\mathcal{X}^{\mathrm{corp}} = \mathcal{X}$.

Remark 20.1.3.6. Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos and let $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ be a right adjoint to the inclusion. Then j^* preserves all limits (since it is a right adjoint) and all colimits (by assumption). It follows that j^* admits a further right adjoint $j_* : \mathcal{X}^{\mathrm{corp}} \to \mathcal{X}$ which is a geometric morphism of ∞-topoi.

Warning 20.1.3.7. In the situation of Remark 20.1.3.6, the functor $j_* : \mathcal{X}^{\mathrm{corp}} \to \mathcal{X}$ is generally neither faithful (that is, it need not induce an equivalence of $\mathcal{X}^{\mathrm{corp}}$ with a subcategory of $\mathcal{X}$) nor conservative.

Remark 20.1.3.8. Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos, let $j_! : \mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$ denote the inclusion and let $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ be right adjoint to $j_!$. Then the adjunction $\mathcal{X}^{\mathrm{corp}} \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathcal{X}$ is monadic: that is, it induces an equivalence of ∞-categories $\mathcal{X} \simeq \mathrm{LMod}_T(\mathcal{X}^{\mathrm{corp}})$, where T denotes the monad $j^* \circ j_!$. This follows from the Barr-Beck theorem (Theorem HA.4.7.3.5), since the functor j^* is conservative and preserves small colimits.

20.2 Admissibility Structures

Let $\phi : A \to B$ be a homomorphism of commutative rings. We will say that ϕ is *local* if it carries noninvertible elements of A to noninvertible elements of B. At the other extreme, we say that ϕ is *localizing* if it induces an isomorphism $A[S^{-1}] \simeq B$, where S is some collection of elements of A. An arbitrary ring homomorphism $\phi : A \to B$ admits an essentially unique factorization $A \xrightarrow{\phi'} A' \xrightarrow{\phi''} B$, where ϕ' is localizing and ϕ'' is local: namely, we can take $A' = A[S^{-1}]$, where S is the collection of all elements $a \in A$ for which $\phi(a)$ is an invertible element of B. We can summarize the situation by saying that the collections of local and localizing morphisms define a factorization system on the category $\mathrm{CAlg}^{\heartsuit}$ of commutative rings.

The category of commutative rings is compactly generated: every commutative ring A can be realized as a filtered direct limit of subrings which are finitely generated (and therefore also finitely presented) over $\mathbf{Z}$. Let $\mathrm{CAlg}^{\heartsuit}_c$ denote the category of finitely presented commutative rings, so that $\mathrm{CAlg}^{\heartsuit} \simeq \mathrm{Ind}(\mathrm{CAlg}^{\heartsuit}_c)$. The factorization system described above above does not restrict to a factorization system on $\mathrm{CAlg}^{\heartsuit}_c$. For example, the reduction map $\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ factors as a composition $\mathbf{Z} \xrightarrow{\phi'} \mathbf{Z}_{(2)} \xrightarrow{\phi''} \mathbf{Z}/2\mathbf{Z}$, where ϕ' is localizing and ϕ'' is local; here the commutative rings $\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z}$ are finitely generated, but the localization $\mathbf{Z}_{(2)}$ is not. Nevertheless, the factorization system on $\mathrm{CAlg}^{\heartsuit}$ is *visible* at the level of finitely presented commutative rings in the following sense: every localizing ring homomorphism $\phi : A \to B$ can be written as a filtered colimit of localizing homomorphisms between finitely presented commutative rings (that is, homomorphisms of the form $R \to R[t^{-1}]$, where R is finitely presented).

In this section, we introduce the notion of an *admissibility structure* (Definition **??**), which axiomatizes the essential features of the class of localizing morphisms in the category $(\mathrm{CAlg}^{\heartsuit}_c)^{\mathrm{op}}$ (passage to the opposite category here is a matter of convenience). Our main results can be summarized as follows:

(a) Let $\mathcal{G}$ be a small ∞-category. Then every admissibility structure on $\mathcal{G}$ determines (and is determined by) a factorization system on the ∞-category $\mathrm{Pro}(\mathcal{G})$ (Theorem 20.2.2.5).

(b) Let $\mathcal{G}$ be a small ∞-category. Then every admissibility structure on $\mathcal{G}$ determines (and is determined by) a fracture subcategory of the presheaf ∞-topos $\mathcal{P}(\mathcal{G}) = \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$ (Theorem 20.2.4.1).

Most of this chapter is devoted to expanding on (b): we will later see that *all* fractured ∞-topoi can be obtained as localizations of fractured ∞-topoi having the form $\mathcal{P}(\mathcal{G})$ (see Theorem 20.5.3.4). Moreover, in §20.3 we will see that every fractured ∞-topos $\mathcal{X}$ can itself be regarded as an ∞-category equipped with an admissibility structure (Proposition 20.3.1.3), and this admissibility structure *almost* determines the fracture subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ (Proposition 20.3.3.11).

Remark 20.2.0.1. The relationship between assertions (a) and (b) will appear as a major theme of Chapter 21.

20.2.1 Definitions

We begin by introducing some terminology.

Definition 20.2.1.1. Let $\mathcal{G}$ be an ∞-category. An *admissibility structure* on $\mathcal{G}$ is a collection of morphisms of $\mathcal{G}$, which we will refer to as *admissible morphisms*, satisfying the following axioms:

(1) Every equivalence in $\mathcal{G}$ is admissible.

(2) Let $f : U \to X$ be an admissible morphism in $\mathcal{G}$, and let $g : X' \to X$ be an arbitrary morphism in $\mathcal{G}$. Then there exists a pullback diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X, \end{array}$$

and the morphism f' is admissible.

(3) Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & Z & \end{array}$$

be a commutative diagram in $\mathcal{G}$, where g is admissible. Then f is admissible if and only if h is admissible.

(4) The collection of admissible morphisms is closed under retracts (when viewed as a full subcategory of the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{G})$).

If $\mathcal{G}$ is an ∞-category equipped with an admissibility structure, we let $\mathcal{G}^{\mathrm{ad}}$ denote the subcategory of $\mathcal{G}$ containing all objects, whose morphisms are the admissible morphisms in $\mathcal{G}$. We will generally abuse terminology by identifying the admissibility structure on $\mathcal{G}$ with the full subcategory $\mathcal{G}^{\mathrm{ad}}$.

Remark 20.2.1.2. It follows from condition (4) of Definition 20.2.1.1 (or from conditions (1) and (3)) that if $f : U \to V$ is an admissible morphism in $\mathcal{G}$, then any morphism homotopic to f is also admissible.

Remark 20.2.1.3. Let $\mathcal{G}$ be an ∞-category and let $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ be an admissibility structure on $\mathcal{G}$. Then $\mathcal{G}^{\mathrm{ad}}$ is a replete subcategory of $\mathcal{G}$ (Definition 20.1.1.2): this is a restatement of condition (1) of Definition 20.2.1.1.

Notation 20.2.1.4. Let $\mathcal{G}$ be an ∞-category equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$. For each object $X \in \mathcal{G}$, we let $\mathcal{G}^{\mathrm{ad}}_{/X}$ denote the full subcategory of $\mathcal{G}_{/X}$ spanned by the admissible morphisms $U \to X$. Note the ∞-category $\mathcal{G}^{\mathrm{ad}}_{/X}$ can also be described as the overcategory $(\mathcal{G}^{\mathrm{ad}})_{/X}$ (since any morphism in $\mathcal{G}^{\mathrm{ad}}_{/X}$ is itself admissible, by condition (3) of Definition 20.2.1.1).

Example 20.2.1.5. Let $\mathrm{CAlg}^{\heartsuit}_c$ denote the category of finitely generated commutative rings. Then the category $(\mathrm{CAlg}^{\heartsuit}_c)^{\mathrm{op}}$ admits an admissibility structure, whose admissible morphisms are given by those ring homomorphisms $\phi : A \to B$ which exhibit B as isomorphic to a localization $A[t^{-1}]$ for some $t \in A$.

Example 20.2.1.6. Let $\mathcal{T}\mathrm{op}$ denote the category of topological spaces. Then the collection of all open immersions determines an admissibility structure on $\mathcal{T}\mathrm{op}$.

Example 20.2.1.7. Let X be a scheme and let Sch_X be the category of finitely presented X-schemes. Then the collection of étale morphisms determines an admissibility structure on Sch_X.

Example 20.2.1.8. Let $\mathcal{G}$ and $\mathcal{C}$ be ∞-categories which admit finite limits, and suppose we are given a natural transformation $f \to g$ between left-exact functors $f, g : \mathcal{G} \to \mathcal{C}$. Then there is an admissibility structure on $\mathcal{G}$ whose admissible morphisms are those maps $U \to X$ for which the diagram

$$\begin{array}{ccc} f(U) & \longrightarrow & g(U) \\ \downarrow & & \downarrow \\ f(X) & \longrightarrow & g(X) \end{array}$$

is a pullback square in $\mathcal{C}$.

Remark 20.2.1.9. Let $\mathcal{G}$ be an ∞-category with an admissibility structure. Then, for every object $X \in \mathcal{G}$, the ∞-category $\mathcal{G}_{/X}$ inherits an admissibility structure, where we say that a morphism in $\mathcal{G}_{/X}$ is admissible if its image in $\mathcal{G}$ is admissible.

Warning 20.2.1.10. When used in combination with Remark 20.2.1.9, Notation 20.2.1.4 presents some opportunity for confusion. Let $\mathcal{G}$ be an ∞-category equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ and let X be an object of $\mathcal{G}$. Then Remark 20.2.1.9 supplies an admissibility structure $(\mathcal{G}_{/X})^{\mathrm{ad}} \subseteq \mathcal{G}_{/X}$ on the ∞-category $\mathcal{G}_{/X}$. Beware that the subcategory $(\mathcal{G}_{/X})^{\mathrm{ad}}$ does *not* coincide with the subcategory $\mathcal{G}^{\mathrm{ad}}_{/X} \subseteq \mathcal{G}_{/X}$ appearing in Notation 20.2.1.4: the former contains all objects of $\mathcal{G}_{/X}$, while the latter contains only those objects which are given by admissible morphisms $U \to X$.

Remark 20.2.1.11. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Suppose that $\mathcal{C}$ admits pullbacks and that f preserves pullbacks. Then every admissibility structure on $\mathcal{D}$ determines an admissibility structure on $\mathcal{C}$, where we say that a morphism u in $\mathcal{C}$ is admissible if its image $F(u)$ is an admissible morphism of $\mathcal{D}$.

Remark 20.2.1.12. Let $\mathcal{G}$ be an ∞-category equipped with an admissibility structure. Then, for every object $X \in \mathcal{G}$, the ∞-category $\mathcal{G}^{\mathrm{ad}}_{/X}$ admits finite limits, which are preserved by the inclusion $\mathcal{G}^{\mathrm{ad}}_{/X} \hookrightarrow \mathcal{G}_{/X}$. To prove this, it suffices to observe that $\mathcal{G}^{\mathrm{ad}}_{/X}$ contains the final object of $\mathcal{G}_{/X}$ (by virtue of condition (1) of Definition 20.2.1.1) and that it is closed under fiber products. Suppose that we are given morphisms $\alpha : Y_0 \to Y$ and $\beta : Y_1 \to Y$ in $\mathcal{G}^{\mathrm{ad}}_{/X}$. Using condition (3) of Definition 20.2.1.1, we see that α and β are admissible, so that (by virtue of (2)) we can choose a pullback square

$$\begin{array}{ccc} Y_{01} & \xrightarrow{\beta'} & Y_0 \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ Y_1 & \xrightarrow{\beta} & Y \end{array}$$

where α' and β' are also admissible. Condition (3) implies that the composition

$$Y_{01} \xrightarrow{\alpha'} Y_1 \xrightarrow{\beta} Y \to X$$

is admissible, so that $Y_{01} \in \mathcal{G}^{\mathrm{ad}}_{/X}$.

20.2.2 Admissibility Structures and Factorization Systems

We now study the relationship between admissibility structures (in the sense of Definition 20.2.1.1) and factorization systems (in the sense of Definition HTT.5.2.8.8). Our first observation is that if $\mathcal{G}$ is an ∞-category which admits pullback squares, then every factorization system determines an admissibility structure:

Proposition 20.2.2.1. *Let $\mathcal{G}$ be an ∞-category which admits pullbacks, and suppose we are given a factorization system (S_L, S_R) on $\mathcal{G}$ (see Definition HTT.5.2.8.8). Then the collection of morphisms S_R is an admissibility structure on $\mathcal{G}$.*

Proof. Applying Proposition HTT.**??** , we immediately deduce that S_R satisfies conditions (1), (3), and (4) of Definition 20.2.1.1, together with the following weaker form of (2);

(2′) For every pullback diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X \end{array}$$

in $\mathcal{G}$, if f belongs to S_R, then f' also belongs to S_R.

Our assumption that $\mathcal{C}$ admits pullbacks guarantees that (2) and (2′) are equivalent. □

Our next goal is to establish a weak converse to Proposition **??**: we will show that every admissibility structure on $\mathcal{G}$ determines a factorization system not on $\mathcal{G}$, but on the ∞-category $\mathrm{Pro}(\mathcal{G})$ of Pro-objects of $\mathcal{G}$ (Theorem 20.2.2.5). We begin by introducing some terminology.

Definition 20.2.2.2. Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure, let $\mathrm{Pro}(\mathcal{G})$ denote the ∞-category of Pro-objects of $\mathcal{G}$ (see Definition A.8.1.1) and let $j : \mathcal{G} \to \mathrm{Pro}(\mathcal{G})$ denote the Yoneda embedding. We will say that a morphism $f : U \to X$ in the ∞-category $\mathrm{Pro}(\mathcal{G})$ is *proadmissible* if there exists a small filtered diagram $p : \mathcal{I} \to \mathrm{Fun}(\Delta^1, \mathcal{G})$ such that each $p(I)$ is an admissible morphism of $\mathcal{G}$, and f is a limit of the composite diagram

$$\mathcal{I} \xrightarrow{p} \mathrm{Fun}(\Delta^1, \mathcal{G}) \xrightarrow{j} \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{G})).$$

For each object $X \in \mathrm{Pro}(\mathcal{G})$, we let $\mathrm{Pro}(\mathcal{G})^{\text{pro-ad}}_{/X}$ denote the full subcategory of $\mathrm{Pro}(\mathcal{G})_{/X}$ spanned by the proadmissible morphisms $Y \to X$.

Remark 20.2.2.3. Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure. The composition $\mathrm{Fun}(\Delta^1, \mathcal{G}) \times \Delta^1 \to \mathcal{G} \xrightarrow{j} \mathrm{Pro}(\mathcal{G})$ classifies a map $\mathrm{Fun}(\Delta^1, \mathcal{G}) \to \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{G}))$. According to Proposition HTT.5.3.5.15 , this map induces an equivalence of ∞-categories $\phi : \mathrm{Pro}(\mathrm{Fun}(\Delta^1, \mathcal{G})) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{G}))$. Let $\mathrm{Fun}^{\mathrm{ad}}(\Delta^1, \mathcal{G})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{G})$ spanned by the admissible morphisms of $\mathcal{G}$. The inclusion of $\mathrm{Fun}^{\mathrm{ad}}(\Delta^1, \mathcal{G})$ into $\mathrm{Fun}(\Delta^1, \mathcal{G})$ induces a fully faithful embedding $\mathrm{Pro}(\mathrm{Fun}^{\mathrm{ad}}(\Delta^1, \mathcal{G})) \to \mathrm{Pro}(\mathrm{Fun}(\Delta^1, \mathcal{G}))$. Composing with the equivalence ϕ, we obtain a fully faithful embedding $\mathrm{Pro}(\mathrm{Fun}^{\mathrm{ad}}(\Delta^1, \mathcal{G})) \hookrightarrow \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{G}))$, whose essential image is spanned by the proadmissible morphisms of $\mathrm{Pro}(\mathcal{G})$.

Remark 20.2.2.4. Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure, let $j : \mathcal{G} \to \mathrm{Pro}(\mathcal{G})$ denote the Yoneda embedding, and let $f : U \to X$ be a morphism in $\mathcal{G}$. It follows from Remark 20.2.2.3 (and condition (4) of Definition 20.2.1.1) that $j(f)$ is proadmissible if and only if f is an admissible morphism in $\mathcal{G}$.

We can now formulate the main result of this section:

Theorem 20.2.2.5. *Let $\mathcal{G}$ be an essentially small ∞-category which is equipped with an admissibilty structure, let S denote the collection of all proadmissible morphisms in* $\mathrm{Pro}(\mathcal{G})$*, and let ${}^{\perp}S$ be the collection of all morphisms in* $\mathrm{Pro}(\mathcal{G})$ *which are left orthogonal to every morphism in S (see Definition HTT.5.2.8.1). Then $({}^{\perp}S, S)$ determines a factorization system on* $\mathrm{Pro}(\mathcal{G})$*. In other words, every morphism $f : X \to Z$ admits a factorization*

$$X \xrightarrow{f'} Y \xrightarrow{f''} Z,$$

where $f' \in {}^{\perp}S$ and f'' is proadmissible.

Remark 20.2.2.6. Let $\mathcal{G}$ be an essentially small ∞-category which admits finite limits, and suppose we are given a factorization system (S_L, S_R) on $\mathrm{Pro}(\mathcal{G})$. Then $\mathrm{Pro}(\mathcal{G})$ admits finite limits, so that S_R is an admissibility structure on $\mathrm{Pro}(\mathcal{G})$ (Proposition 20.2.2.1). Applying

Remark 20.2.1.11, we obtain an admissibility structure on $\mathcal{G}$, where a morphism in $\mathcal{G}$ is admissible if its image under the Yoneda embedding $j : \mathcal{G} \hookrightarrow \mathrm{Pro}(\mathcal{G})$ belongs to S_R. Note that since S_R is closed under limits in $\mathrm{Pro}(\mathcal{G})$ (see Proposition HTT.**??**), it follows that S_R contains every proadmissible morphism in $\mathrm{Pro}(\mathcal{G})$. If the reverse inclusion holds (that is, if S_R is generated under filtered limits by morphisms of the form $f : U \to X$, where U and X belong to the essential image of j), then (S_L, S_R) coincides with the factorization system appearing in the statement of Theorem 20.2.2.5.

Remark 20.2.2.7. Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$. and regard $\mathrm{Pro}(\mathcal{G})$ as equipped with the factorization system of Theorem 20.2.2.5. If $\mathcal{G}$ admits finite limits, then the construction of Remark 20.2.2.6 produces another admissibility structure on $\mathcal{G}$, determined by the collection of morphisms f in $\mathcal{G}$ whose image under the Yoneda embedding $j : \mathcal{G} \to \mathrm{Pro}(\mathcal{G})$ is proadmissible. It follows from Remark **??** that this admissibility structure coincides with $\mathcal{G}^{\mathrm{ad}}$.

Remark 20.2.2.8. Let $\mathcal{G}$ be an essentially small ∞-category which admits finite limits. Combining Remarks 20.2.2.6 and 20.2.2.7, we see that the construction of Theorem 20.2.2.5 establishes a bijection between the following data:

- Admissibility structures on $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$.
- Factorization systems (S_L, S_R) on $\mathrm{Pro}(\mathcal{G})$ for which every morphism $f : X \to Y$ in S_R can be written as a filtered limit of morphisms $f_\alpha : X_\alpha \to Y_\alpha$ which belong to S_R, where X_α and Y_α belong to the essential image of the Yoneda embedding $j : \mathcal{G} \to \mathrm{Pro}(\mathcal{G})$.

Corollary 20.2.2.9. *Let $\mathcal{G}$ be an ∞-category equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$, and suppose we are given morphisms $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathcal{G}$. If $g \circ f$ and $h \circ g$ are admissible, then f is admissible.*

Proof. Without loss of generality, we may assume that $\mathcal{G}$ is small. Let us abuse notation by identifying $\mathcal{G}$ with a full subcategory of the ∞-category $\mathrm{Pro}(\mathcal{G})$. Let $({}^{\perp}S, S)$ denote the factorization system of Theorem 20.2.2.5. By virtue of Remark 20.2.2.4, it will suffice to show that the morphism f is right orthogonal to every morphism $u : A \to B$ in ${}^{\perp}S$. Fix a map $A \to W$ in the ∞-category $\mathrm{Pro}(\mathcal{G})$; we wish to show that f induces a homotopy equivalence $f' : \mathrm{Map}_{\mathrm{Pro}(\mathcal{G})_{A/}}(B, W) \to \mathrm{Map}_{\mathrm{Pro}(\mathcal{G})_{A/}}(B, X)$. Consider the diagram

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{G})_{A/}}(B, W) \xrightarrow{f'} \mathrm{Map}_{\mathrm{Pro}(\mathcal{G})_{A/}}(B, X) \xrightarrow{g'} \mathrm{Map}_{\mathrm{Pro}(\mathcal{G})_{A/}}(B, Y) \xrightarrow{h'} \mathrm{Map}_{\mathrm{Pro}(\mathcal{G})_{A/}}(B, Z).$$

Since $g \circ f$ and $h \circ g$ are admissible, the maps $g' \circ f'$ and $h' \circ g'$ are homotopy equivalences. It follows that g' admits left and right homotopy inverses, and is therefore a homotopy equivalence. Then g' and $g' \circ f'$ are both homotopy equivalences, so that f' is a homotopy equivalence as desired. $\square$

20.2.3 The Proof of Theorem 20.2.2.5

Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure. Our proof of Theorem 20.2.2.5 will use a variant of Quillen's "small object" argument. We begin with some general observations about the class of proadmissible morphisms in $\mathrm{Pro}(\mathcal{G})$.

Remark 20.2.3.1. Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure and let $X \in \mathrm{Pro}(\mathcal{G})$. Then the ∞-category $\mathrm{Pro}(\mathcal{G})_{/X}$ admits small filtered limits, and the full subcategory $\mathrm{Pro}(\mathcal{G})^{\text{pro-ad}}_{/X} \subseteq \mathrm{Pro}(\mathcal{G})_{/X}$ is closed under small filtered limits.

Lemma 20.2.3.2. *Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure and let X be an object of $\mathrm{Pro}(\mathcal{G})$. A morphism $f : U \to X$ is proadmissible if and only if, as an object of $\mathrm{Pro}(\mathcal{G})_{/X}$, U can be identified with a small filtered limit of morphisms $U_\alpha \to X$ which fit into pullback diagrams*

$$\begin{array}{ccc} U_\alpha & \longrightarrow & X \\ \downarrow & & \downarrow \\ j(U'_\alpha) & \xrightarrow{j(f'_\alpha)} & j(X'_\alpha), \end{array}$$

where $j : \mathcal{G} \to \mathrm{Pro}(\mathcal{G})$ denotes the Yoneda embedding and each f'_α is an admissible morphism of $\mathcal{G}$.

Proof. Suppose first that f is proadmissible. Then f can be written as a filtered limit of morphisms $j(f'_\alpha) : j(U'_\alpha) \to j(X'_\alpha)$, where f'_α is an admissible morphism of $\mathcal{G}$. For each index α, we can write X as the filtered limit of a diagram $\{j(X''_\beta)\}$, where $X''_\beta \in \mathcal{G}_{/X_\alpha}$. Since each f'_α is admissible, we can choose pullback diagrams

$$\begin{array}{ccc} U''_\beta & \longrightarrow & X''_\beta \\ \downarrow & & \downarrow \\ U'_\alpha & \xrightarrow{f'_\alpha} & X'_\alpha \end{array}$$

in the ∞-category $\mathcal{G}$. It follows that $U_\alpha \simeq \varprojlim_\beta j(U''_\beta)$ is a fiber product $j(U'_\alpha) \times_{j(X'_\alpha)} X$, and that $U \simeq \varprojlim_\alpha U_\alpha$.

To prove the converse, it will suffice (by virtue of Remark 20.2.3.1) to show that for every pullback diagram

$$\begin{array}{ccc} U_\alpha & \xrightarrow{f_\alpha} & X \\ \downarrow & & \downarrow \\ j(U'_\alpha) & \xrightarrow{j(f'_\alpha)} & j(X'_\alpha), \end{array}$$

with f'_α admissible, the map f_α is proadmissible. Let $\{X''_\beta\}$ be as above. Then f_α is a filtered limit of maps $j(f''_\beta)$, where $f''_\beta : X''_\beta \times_{X'_\alpha} U'_\alpha \to X''_\beta$ is the projection onto the first factor (and therefore admissible, since the collection of admissible morphisms is stable under pullback). □

Corollary 20.2.3.3. *Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure, let $f : U \to X$ be a proadmissible morphism in* $\mathrm{Pro}(\mathcal{G})$*, and let $g : X' \to X$ be an arbitrary morphism in* $\mathrm{Pro}(\mathcal{G})$*. Then there exists a pullback diagram*

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

in $\mathrm{Pro}(\mathcal{G})$*. Moreover, the map f' is proadmissible.*

Lemma 20.2.3.4. *Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure. Then the collection of proadmissible morphisms in* $\mathrm{Pro}(\mathcal{G})$ *is closed under composition.*

Proof. Choose proadmissible morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathrm{Pro}(\mathcal{G})$; we wish to show that $g \circ f$ is proadmissible. By virtue of Lemma 20.2.3.2, we may assume that f is a filtered limit of morphisms $\{f_\alpha : X_\alpha \to Y\}$, where each f_α is the pullback $j(f'_\alpha)$ for some admissible morphism f'_α in $\mathcal{G}$. It suffices to show that each composition $g \circ f_\alpha$ is admissible. Replacing f by f_α, we may assume that there exists a pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle h} \\ j(X') & \xrightarrow{j(f')} & j(Y') \end{array}$$

where f' is an admissible morphism of $\mathcal{G}$.

Applying Lemma 20.2.3.2 again, we may assume that g is the limit of a diagram of morphisms $\{g_\beta : Y_\beta \to Z\}_{\beta \in B}$ indexed by a filtered partially ordered set B, such that each g_β is a pullback of $j(g'_\beta)$ for some admissible morphism $g'_\beta : Y'_\beta \to Z'_\beta$ in $\mathcal{G}$. Since $j(Y')$ is a cocompact object of $\mathrm{Pro}(\mathcal{G})$, we may assume that h is factors as a composition $Y \to j(Y_{\beta_0}) \xrightarrow{j(h')} j(Y')$ for some $\beta_0 \in B$. It follows that we can identify X with the limit of the diagram $\{j(Y_\beta) \times_{j(Y')} j(X')\}_{\beta \geq \beta_0}$. It will therefore suffice to show that each of the composite maps

$$j(Y_\beta) \times_{j(Y')} j(X') \to j(Y_\beta) \to j(Z_\beta)$$

is proadmissible. In fact, the map $Y_\beta \times_{Y'} X' \to Z_\beta$ is an admissible morphism in $\mathcal{G}$, since the collection of admissible morphisms is stable under pullback and composition. □

Corollary 20.2.3.5. *Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure and let $X \in \mathrm{Pro}(\mathcal{G})$. Then the ∞-category $\mathrm{Pro}(\mathcal{G})^{\text{pro-ad}}_{/X}$ admits small products, and the inclusion functor $\mathrm{Pro}(\mathcal{G})^{\text{pro-ad}}_{/X} \hookrightarrow \mathrm{Pro}(\mathcal{G})_{/X}$ preserves small products.*

Proof. By virtue Remark 20.2.3.1, it will suffice to treat the case of finite products. Proceeding by induction on the number of factors (and noting that $\mathrm{Pro}(\mathcal{G})^{\text{ad}}_{/X}$ contains X as a final object), we are reduced to proving that if $U \to X$ and $V \to X$ are proadmissible morphisms, then there exists a fiber product $U \times_X V$ in $\mathrm{Pro}(\mathcal{G})$, and the map $U \times_X V \to X$ is proadmissible. This follows immediately from Lemma 20.2.3.4 and Corollary 20.2.3.3. □

Proof of Theorem 20.2.2.5. Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure and let $f : X \to Z$ be a morphism in $\mathrm{Pro}(\mathcal{G})$; we wish to show that f factors as a composition $X \xrightarrow{f'} Y \xrightarrow{f''} Z$, where f'' is proadmissible and f' is left orthogonal to every proadmissible morphism. We begin by inductively constructing a sequence of maps $\{g_i : X \to Y_i\}_{i \geqslant 0}$, using induction on i. Set $g_0 = f$ and $Y_0 = Z$. To carry out the inductive step, suppose that $g_n : X \to Y_n$ has already been constructed. Let $\{\sigma_\alpha\}$ be a set of representatives for all homotopy equivalence classes of diagrams

$$\begin{array}{ccc} X & \longrightarrow & j(U_\alpha) \\ \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle j(h_\alpha)} \\ Y_n & \longrightarrow & j(V_\alpha), \end{array}$$

where h_α is an admissible morphism in $\mathcal{G}$. It follows from Corollary 20.2.3.3 that for each index α, the fiber product $Y'_{n,\alpha} = Y_n \times_{j(V_\alpha)} j(U_\alpha)$ exists, and the projection map $Y'_{n,\alpha} \to Y_n$ is proadmissible. Let Y_{n+1} denote the product of the objects $Y'_{n,\alpha}$, formed in the ∞-category $\mathrm{Pro}(\mathcal{G})_{/Y_n}$, and let $g_{n+1} : X \to Y_{n+1}$ be the product of the maps determined by the diagrams σ_α. We have a tower of morphisms

$$\cdots \to Y_3 \to Y_2 \to Y_1 \to Y_0 = Z$$

in $\mathrm{Pro}(\mathcal{G})_{X/}$, and Corollary 20.2.3.5 implies that each of the morphisms in this diagram is proadmissible. Set $Y = \varprojlim_n Y_n$, so that the map f factors as a composition $X \xrightarrow{f'} Y \xrightarrow{f''} Z$. The morphism f'' is a filtered limit of proadmissible morphisms in $\mathrm{Pro}(\mathcal{G})$, and therefore proadmissible. To complete the proof, it will suffice to show that f' is left orthogonal to all proadmissible morphisms.

Let T be the collection of all morphisms in $\mathrm{Pro}(\mathcal{G})$ which are right orthogonal to f'. We wish to prove that T contains all proadmissible morphisms. Since T is closed under filtered limits, it will suffice to show that T contains $j(h)$ for every admissible morphism $h : U \to V$ in $\mathcal{G}$. Let us therefore assume that we are given a map $X \to j(U)$ in $\mathrm{Pro}(\mathcal{G})$; we wish to prove that composition with f' induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathrm{Pro}(\mathcal{G})_{X/}}(Y, j(U)) \to \mathrm{Map}_{\mathrm{Pro}(\mathcal{G})_{X/}}(Y, j(V)).$$

We will prove that θ is n-connective using induction on n. Assume first that $n > 0$, so that θ has nonempty homotopy fibers. To prove that θ is n-connective, it will suffice to show that the diagonal map

$$\delta : \mathrm{Map}_{\mathrm{Pro}(\mathcal{G})_{X/}}(Y, j(U)) \to \mathrm{Map}_{\mathrm{Pro}(\mathcal{G})_{X/}}(Y, j(U)) \times_{\mathrm{Map}_{\mathrm{Pro}(\mathcal{G})_{X/}}(Y, j(V))} \mathrm{Map}_{\mathrm{Pro}(\mathcal{G})_{X/}}(Y, j(V)).$$

is $(n-1)$-connective. This follows by applying the inductive hypothesis after replacing h by the map $h' : U \to U \times_V U$ (which is an admissible morphism in $\mathcal{G}$, since it fits into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h'} & U \times_V U \\ & {\scriptstyle h}\searrow \quad \swarrow & \\ & V & \end{array}$$

where the vertical maps are admissible).

It remains to treat the case $n = 0$. For this, we must show that every morphism $\rho : Y \to j(V)$ in $\mathrm{Pro}(\mathcal{G})_{X/}$ factors through $j(X)$. Since $j(V)$ is cocompact object of $\mathrm{Pro}(\mathcal{G})$, we can assume that the map ρ factors through Y_n for some integer n. In this case, it follows from our construction that there exists a commutative diagram

$$\begin{array}{ccc} Y_{n+1} & \longrightarrow & j(U) \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & j(V) \end{array}$$

in $\mathrm{Pro}(\mathcal{G})_{X/}$. □

20.2.4 Admissibility Structures and Fracture Subcategories

Our next goal is to establish a relationship between the notion of admissibility structure and the theory of fractured ∞-topoi introduced in §20.1.

Theorem 20.2.4.1. *Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$. Let $\mathcal{P}(\mathcal{G}) = \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$ denote the ∞-category of presheaves on $\mathcal{G}$, let $\mathcal{P}(\mathcal{G}^{\mathrm{ad}}) = \mathrm{Fun}((\mathcal{G}^{\mathrm{ad}})^{\mathrm{op}}, \mathcal{S})$ be defined similarly, and let $j_! : \mathcal{P}(\mathcal{G}^{\mathrm{ad}}) \to \mathcal{P}(\mathcal{G})$ denote the functor given by left Kan extension along the inclusion map $j : \mathcal{G}^{\mathrm{ad}} \hookrightarrow \mathcal{G}$. Then the functor $j_!$ induces an equivalence from $\mathcal{P}(\mathcal{G}^{\mathrm{ad}})$ onto a fracture subcategory $\mathcal{P}(\mathcal{G})^{\mathrm{corp}} \subseteq \mathcal{P}(\mathcal{G})$.*

The proof of Theorem 20.2.4.1 will be given in §20.2.5. We begin by establishing some preliminaries. Our starting point is the following mild generalization of Proposition HTT.6.1.5.2:

Proposition 20.2.4.2. *Let $\mathcal{G}$ be a small ∞-category which admits pullbacks, let $j : \mathcal{G} \to \mathcal{P}(\mathcal{G})$ denote the Yoneda embedding, let $\mathcal{X}$ be an ∞-topos, and let $F : \mathcal{P}(\mathcal{G}) \to \mathcal{X}$ be a functor which preserves small colimits. The following conditions are equivalent:*

- (a) *The functor F preserves pullbacks.*
- (b) *The composite functor $f = F \circ j$ preserves pullbacks.*

Proof. Let $\alpha : \mathscr{F} \to \mathscr{G}$ be a morphism in $\mathcal{P}(\mathcal{G})$. We will say that α is *good* if for every pullback square

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathscr{G}' & \longrightarrow & \mathscr{G} \end{array}$$

in $\mathcal{P}(\mathcal{G})$, the induced diagram

$$\begin{array}{ccc} F(\mathscr{F}') & \longrightarrow & F(\mathscr{F}) \\ \downarrow & & \downarrow{\scriptstyle F(\alpha)} \\ F(\mathscr{G}') & \xrightarrow{\beta} & F(\mathscr{G}) \end{array}$$

is a pullback square in $\mathcal{X}$. Note that the collection of good morphisms in $\mathcal{P}(\mathcal{G})$ is closed under composition (see Lemma HTT.4.4.2.1).

Using the fact that colimits in $\mathcal{X}$ are universal (and the fact that the functor F preserves small colimits), we see that a morphism $\alpha : \mathscr{F} \to \mathscr{G}$ is good if and only if the following *a priori* weaker condition is satisfied:

$(*)$ For every object $C \in \mathcal{G}$ and every morphism $j(C) \to \mathscr{G}$, the induced diagram

$$\begin{array}{ccc} F(j(C) \times_{\mathscr{G}} \mathscr{F}) & \longrightarrow & F(\mathscr{F}) \\ \downarrow & & \downarrow{\scriptstyle F(\alpha)} \\ F(j(C)) & \xrightarrow{\beta} & F(\mathscr{G}) \end{array}$$

is a pullback square in $\mathcal{X}$.

Let us say that an object $\mathscr{G} \in \mathcal{P}(\mathcal{G})$ is *good* if every morphism $\alpha : \mathscr{F} \to \mathscr{G}$ in $\mathcal{P}(\mathcal{G})$ is good. Since colimits in $\mathcal{X}$ are universal and the functor F preserves small colimits, the collection of good morphisms $\mathscr{F} \to \mathscr{G}$ is closed under colimits in the ∞-category $\mathcal{P}(\mathcal{G})_{/\mathscr{G}}$. Using the fact that the ∞-category $\mathcal{P}(\mathcal{G})$ is generated under small colimits by the essential image of the Yoneda embedding j, we deduce that $\mathscr{G}$ is good if and only if it satisfies the following:

$(*')$ For every pair of objects $C, D \in \mathcal{G}$ and every pair of maps $j(C) \rightarrow \mathscr{G} \leftarrow j(D)$ the diagram

$$\begin{array}{ccc} F(j(C) \times_{\mathscr{G}} j(D)) & \longrightarrow & F(j(D)) \\ \downarrow & & \downarrow \\ F(j(C)) & \longrightarrow & F(\mathscr{G}) \end{array}$$

is a pullback square in $\mathcal{X}$.

Note that condition (a) is equivalent to the assertion that every object of $\mathcal{P}(\mathcal{G})$ is good, and condition (b) is equivalent to the assertion that every object in the essential image of the Yoneda embedding $j : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{G})$ is good (this follows from the observation that j is fully faithful and preserves pullbacks). To show that these conditions are equivalent, it will suffice to show that the collection of good objects of $\mathcal{P}(\mathcal{G})$ is closed under small colimits. Using Proposition HTT.4.4.3.3 , we are reduced to showing that the collection of good objects of $\mathcal{P}(\mathcal{G})$ is closed under coequalizers and small coproducts.

We first consider the case of coproducts. Let $\{\mathscr{G}_i\}_{i \in I}$ be a family of good objects of $\mathcal{P}(\mathcal{G})$ indexed by a (small) set I and let $\{\phi_i : \mathscr{G}_i \rightarrow \mathscr{G}\}_{i \in I}$ be a family of morphisms which exhibit $\mathscr{G}$ as a coproduct of the family $\{\mathscr{G}_i\}_{i \in I}$. Suppose we are given a pullback diagram

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & j(D) \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ j(C) & \xrightarrow{\beta} & \mathscr{G} \end{array}$$

in $\mathcal{P}(\mathcal{G})$. The map α factors as a composition $j(D) \xrightarrow{\alpha'} \mathscr{G}_i \xrightarrow{\phi_i} \mathscr{G}$ for some $i \in I$. By assumption, the morphism α' is good; it therefore suffices to prove that ϕ_i is good. By a similar argument, we can replace β by a map $\phi_j : \mathscr{G}_j \rightarrow \mathscr{G}$ for some $j \in I$. We are now required to show that if

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathscr{G}_i \\ \downarrow & & \downarrow{\scriptstyle \phi_i} \\ \mathscr{G}_j & \xrightarrow{\phi_j} & \mathscr{G} \end{array}$$

is a pullback diagram in $\mathcal{P}(\mathcal{G})$, then

$$\begin{array}{ccc} F(\mathscr{F}) & \longrightarrow & F(\mathscr{G}_i) \\ \downarrow & & \downarrow{\scriptstyle F(\phi_i)} \\ F(\mathscr{G}_j) & \xrightarrow{F(\phi_j)} & F(\mathscr{G}) \end{array}$$

is a pullback diagram in $\mathcal{X}$. This follows from the assumption that F preserves coproducts and the fact that coproducts in $\mathcal{X}$ are disjoint (see Lemma HTT.6.1.5.1).

We now complete the proof by showing that the collection of good objects of $\mathcal{P}(\mathcal{G})$ is stable under the formation of coequalizers. Let

$$\mathscr{G}'' \rightrightarrows \mathscr{G}' \xrightarrow{s} \mathscr{G}$$

be a coequalizer diagram in $\mathcal{P}(\mathcal{G})$, and suppose that the presheaves $\mathscr{G}'$ and $\mathscr{G}''$ are good. We must show that any pullback diagram

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & j(D) \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ j(C) & \xrightarrow{\beta} & \mathscr{G} \end{array}$$

remains a pullback diagram after applying the functor F. Note that the map α factors as a composition $j(D) \xrightarrow{\alpha'} \mathscr{G}' \xrightarrow{s} \mathscr{G}$. Since $\mathscr{G}'$ is good, the morphism α' is good. It will therefore suffice to show that α'' is good: that is, we must show that every pullback diagram

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathscr{G}' \\ \downarrow & & \downarrow{\scriptstyle s} \\ j(C) & \xrightarrow{\beta} & \mathscr{G} \end{array}$$

remains a pullback diagram after applying F. Applying the same argument to β, we are reduced to showing that the canonical map $F(\mathscr{G}' \times_{\mathscr{G}} \mathscr{G}') \to F(\mathscr{G}') \times_{F(\mathscr{G})} F(\mathscr{G}')$ is a pullback diagram in $\mathcal{X}$. This follows from Proposition HTT.6.1.4.2 (together with our assumption that $\mathscr{G}'$ is good). □

Corollary 20.2.4.3. *Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$, and let $j_! : \mathcal{P}(\mathcal{G}^{\mathrm{ad}}) \to \mathcal{P}(\mathcal{G})$ denote the functor given by left Kan extension along the inclusion. Then the functor $j_!$ preserves pullbacks.*

Proof. Using Proposition 20.2.4.2, we are reduced to showing that the composite functor $\mathcal{G}^{\mathrm{ad}} \hookrightarrow \mathcal{G} \xrightarrow{h} \mathcal{P}(\mathcal{G})$ preserves pullback squares, where h denotes the Yoneda embedding. Since h preserves all pullback squares which exist in $\mathcal{G}$, it will suffice to show that the inclusion $\mathcal{G}^{\mathrm{ad}} \hookrightarrow \mathcal{G}$ preserves pullbacks, which follows from Remark 20.2.1.12. □

Proposition 20.2.4.4. *Let $\mathcal{G}$ be an essentially small ∞-category, let $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ be an admissibility structure on $\mathcal{G}$, let $j^* : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{G}^{\mathrm{ad}})$ be the restriction map and let $j_! : \mathcal{P}(\mathcal{G}^{\mathrm{ad}}) \to \mathcal{P}(\mathcal{G})$ be a left adjoint to j^* (given by left Kan extension along the inclusion $\mathcal{G}^{\mathrm{ad}} \hookrightarrow \mathcal{G}$). Then, for every morphism $\alpha : \mathscr{F} \to \mathscr{G}$ in $\mathcal{P}(\mathcal{G})$, the diagram σ:*

$$\begin{array}{ccc} j_! j^* j_! \mathscr{F} & \longrightarrow & j_! j^* j_! \mathscr{G} \\ \downarrow & & \downarrow \\ j_! \mathscr{F} & \longrightarrow & j_! \mathscr{G} \end{array}$$

is a pullback square in $\mathcal{P}(\mathcal{G})$ *(where the vertical maps are induced by the counit* $j_! j^* \to \mathrm{id}$*).*

Proof. Let us regard the presheaf $\mathscr{G}$ as fixed. Since the functors j^* and $j_!$ preserve small colimits, the collection of those objects $\mathscr{F} \in \mathcal{P}(\mathcal{G}^{\mathrm{ad}})_{/\mathscr{G}}$ for which σ is a pullback square is closed under small colimits. We may therefore assume without loss of generality that the presheaf $\mathscr{F}$ is representable. Write $\mathscr{G}$ as a colimit $\varinjlim \mathscr{G}_\alpha$, where each of the presheaves $\mathscr{G}_\alpha \in \mathcal{P}(\mathcal{G}^{\mathrm{ad}})$ is representable. Without loss of generality, we may assume that $\mathscr{F} = \mathscr{G}_{\alpha_0}$ for some index α_0 (as an object of the ∞-category $\mathcal{P}(\mathcal{G}^{\mathrm{ad}})_{/\mathscr{G}}$). It will therefore suffice to show that each of the diagrams

$$\begin{array}{ccc} j_! j^* j_! \mathscr{G}_\alpha & \longrightarrow & j_! j^* j_! \mathscr{G} \\ \downarrow & & \downarrow \\ j_! \mathscr{G}_\alpha & \longrightarrow & j_! \mathscr{G} \end{array}$$

is a pullback square. Applying Theorem HTT.6.1.3.9 , we are reduced to showing that each of the diagrams

$$\begin{array}{ccc} j_! j^* j_! \mathscr{G}_\alpha & \longrightarrow & j_! j^* j_! \mathscr{G}_\beta \\ \downarrow & & \downarrow \\ j_! \mathscr{G}_\alpha & \longrightarrow & j_! \mathscr{G}_\beta \end{array}$$

is a pullback square. In other words, it will suffice to verify Proposition 20.2.4.4 in the special case where $\mathscr{F}$ and $\mathscr{G}$ are representable by objects $X, Y \in \mathcal{G}^{\mathrm{ad}}$ (so that the map $\mathscr{F} \to \mathscr{G}$ is induced by an admissible morphism $u : X \to Y$).

Let U be an object of $\mathcal{G}$; we wish to show that the diagram of spaces $\sigma(U)$:

$$\begin{array}{ccc} (j_! j^* j_! \mathscr{F})(U) & \longrightarrow & (j_! j^* j_! \mathscr{G})(U) \\ \downarrow & & \downarrow \\ (j_! \mathscr{F})(U) & \longrightarrow & (j_! \mathscr{G})(U) \end{array}$$

is a pullback. Unwinding the definitions, we can identify $\sigma(U)$ with the diagram

$$\begin{array}{ccc} \varinjlim_{X' \in (\mathcal{G}_{/X})^{\mathrm{ad}}} \mathrm{Map}_{\mathcal{G}}(U, X') & \longrightarrow & \varinjlim_{Y' \in (\mathcal{G}_{/Y})^{\mathrm{ad}}} \mathrm{Map}_{\mathcal{G}}(U, Y') \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{G}}(U, X) & \longrightarrow & \mathrm{Map}_{\mathcal{G}}(U, Y). \end{array}$$

We are therefore reduced to showing that the canonical map

$$\varinjlim_{X' \in (\mathcal{G}_{/X})^{\mathrm{ad}}} \mathrm{Map}_{\mathcal{G}}(U, X') \to \varinjlim_{Y' \in (\mathcal{G}_{/Y})^{\mathrm{ad}}} \mathrm{Map}_{\mathcal{G}}(U, Y' \times_Y X)$$

is a homotopy equivalence. To verify this, it suffices to establish that construction $Y' \mapsto Y' \times_Y X$ determines a left cofinal functor $u^* : (\mathcal{G}_{/Y})^{\mathrm{ad}} \to (\mathcal{G}_{/X})^{\mathrm{ad}}$. This is clear, since the functor u^* admits a left adjoint (given by composition with u). □

Proposition 20.2.4.5. *Let $\mathcal{G}$ be an essentially small ∞-category, let $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ be an admissibility structure on $\mathcal{G}$, let $j^* : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{G}^{\mathrm{ad}})$ be the restriction map and let $j_! : \mathcal{P}(\mathcal{G}^{\mathrm{ad}}) \to \mathcal{P}(\mathcal{G})$ be a left adjoint to j^* (given by left Kan extension along the inclusion $\mathcal{G}^{\mathrm{ad}} \hookrightarrow \mathcal{G}$). Then:*

(a) *For every presheaf $\mathscr{F} \in \mathcal{P}(\mathcal{G}^{\mathrm{ad}})$, the unit map $u_{\mathscr{F}} : \mathscr{F} \to j^* j_! \mathscr{F}$ is (-1)-truncated.*

(b) *For every morphism $\mathscr{F} \to \mathscr{G}$ in $\mathcal{P}(\mathcal{G}^{\mathrm{ad}})$, the diagram*

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{u_{\mathscr{F}}} & j^* j_! \mathscr{F} \\ \downarrow & & \downarrow \\ \mathscr{G} & \xrightarrow{u_{\mathscr{G}}} & j^* j_! \mathscr{G} \end{array}$$

is a pullback square.

Proof. We first prove (a). Fix an object $X \in \mathcal{G}$; we wish to show that the unit map $u_{\mathscr{F}}(X) : \mathscr{F}(X) \to (j^* j_! \mathscr{F})(X) = (j_! \mathscr{F})(X)$ is an equivalence. Set $\mathcal{C} = \mathcal{G}^{\mathrm{ad}} \times_{\mathcal{G}} \mathcal{G}_{X/}$; we wish to show that the canonical map $\mathscr{F}(X) \to \varinjlim_{Y \in \mathcal{C}^{\mathrm{op}}} \mathscr{F}(Y)$ is (-1)-truncated. The objects of $\mathcal{C}$ can be identified with morphisms $f : X \to Y$ in the ∞-category $\mathcal{G}$. Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ spanned by those objects for which f is admissible, and let $\mathcal{C}_1$ be the full subcategory of $\mathcal{C}$ spanned by those objects for which f is not admissible. Using condition (3) of Definition 20.2.1.1, we see that $\mathrm{Map}_{\mathcal{C}}(Y, Z)$ is empty if $Y \in \mathcal{C}_0$ and $Z \in \mathcal{C}_1$, or if $Y \in \mathcal{C}_1$ and $Z \in \mathcal{C}_0$. It follows that we can identify the colimit $\varinjlim_{Y \in \mathcal{C}^{\mathrm{op}}} \mathscr{F}(Y)$ with the coproduct of $\varinjlim_{Y \in \mathcal{C}_0^{\mathrm{op}}} \mathscr{F}(Y)$ and $\varinjlim_{Y \in \mathcal{C}_1^{\mathrm{op}}} \mathscr{F}(Y)$. We are therefore reduced to proving that the canonical map $\mathscr{F}(X) \to \varinjlim_{Y \in \mathcal{C}_0^{\mathrm{op}}} \mathscr{F}(Y)$ is (-1)-truncated. In fact, this map is an equivalence, since the identity map $\mathrm{id} : X \to X$ is an initial object of $\mathcal{C}_0$. This completes the proof of (a).

We now prove (b). Fix an object $X \in \mathcal{G}$ and define $\mathcal{C}$ as above; we wish to show that the diagram of spaces

$$\begin{array}{ccc} \mathscr{F}(X) & \longrightarrow & \varinjlim_{Y \in \mathcal{C}^{\mathrm{op}}} \mathscr{F}(Y) \\ \downarrow & & \downarrow \\ \mathscr{G}(X) & \longrightarrow & \varinjlim_{Y \in \mathcal{C}^{\mathrm{op}}} \mathscr{G}(Y) \end{array}$$

is a pullback square. Arguing as above, we can rewrite this diagram as

$$\begin{array}{ccc} \varinjlim_{Y \in \mathcal{C}_0^{op}} \mathscr{F}(Y) & \longrightarrow & (\varinjlim_{Y \in \mathcal{C}_0^{op}} \mathscr{F}(Y)) \amalg (\varinjlim_{Y \in \mathcal{C}_1^{op}} \mathscr{F}(Y)) \\ \downarrow & & \downarrow \\ \varinjlim_{Y \in \mathcal{C}_0^{op}} \mathscr{G}(Y) & \longrightarrow & (\varinjlim_{Y \in \mathcal{C}_0^{op}} \mathscr{G}(Y)) \amalg (\varinjlim_{Y \in \mathcal{C}_1^{op}} \mathscr{G}(Y)), \end{array}$$

so that the desired result is evident. □

20.2.5 The Proof of Theorem 20.2.4.1

Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$, which we regard as fixed throughout this section. Let $j^* : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{G}^{\mathrm{ad}})$ denote the restriction map and let $j_! : \mathcal{P}(\mathcal{G}^{\mathrm{ad}}) \to \mathcal{P}(\mathcal{G})$ be a left adjoint to j^* (given by left Kan extension along the inclusion $\mathcal{G}^{\mathrm{ad}} \hookrightarrow \mathcal{G}$). Our proof of Theorem 20.6.3.4 will require a bit of (temporary) terminology. For any pair of objects $\mathscr{F}, \mathscr{G} \in \mathcal{P}(\mathcal{G})$), it follows from Proposition 20.2.4.5 that the canonical map

$$\mathrm{Map}_{\mathcal{P}(\mathcal{G}^{\mathrm{ad}})}(\mathscr{F}, \mathscr{G}) \to \mathrm{Map}_{\mathcal{P}(\mathcal{G}^{\mathrm{ad}})}(\mathscr{F}, j^* j_! \mathscr{G}) \simeq \mathrm{Map}_{\mathcal{P}(\mathcal{G})}(j_! \mathscr{F}, j_! \mathscr{G})$$

has (-1)-truncated homotopy fibers. We will say that a morphism $f : j_! \mathscr{F} \to j_! \mathscr{G}$ is *admissible* if it belongs to the essential image of this map. Our first goal is to show that this notion of admissible morphism satisfies analogues of the axioms of Definition 20.2.1.1.

Lemma 20.2.5.1. *Let $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{H}$ be objects of $\mathcal{P}(\mathcal{G}^{\mathrm{ad}})$, and suppose we are given morphisms $f : j_! \mathscr{F} \to j_! \mathscr{G}$ and $g : j_! \mathscr{G} \to j_! \mathscr{H}$. Then:*

(a) If f and g are admissible, then the composition $g \circ f$ is admissible.

(b) If g and $g \circ f$ are admissible, then f is admissible.

(c) If f and $g \circ f$ are admissible and f is an effective epimorphism, then g is admissible.

Proof. Assertion (a) is obvious. We next prove (b). Assume that g and $g \circ f$ are admissible; we wish to show that f is admissible. Let us identify f with a map $\widehat{f} : \mathscr{F} \to j^* j_! \mathscr{G}$; we wish to show that $\widehat{f}$ factors through the unit map $u_{\mathscr{G}} : \mathscr{G} \to j^* j_! \mathscr{G}$. Without loss of generality, we can assume that $g = j_!(g_0)$ for some morphism $g_0 : \mathscr{G} \to \mathscr{H}$. Proposition 20.2.4.5 guarantees that the diagram

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{u_{\mathscr{G}}} & j^* j_! \mathscr{G} \\ \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle j^*(g)} \\ \mathscr{H} & \xrightarrow{u_{\mathscr{H}}} & j^* j_! \mathscr{H} \end{array}$$

is a pullback square. Consequently, it will suffice to show that the composite map $j^*(g) \circ \widehat{f}$ factors through $u_{\mathscr{H}}$, which is equivalent to our assumption that $g \circ f$ is admissible.

We now prove (c). Assume that f is admissible and an effective epimorphism, so that we can write $f = j_!(f_0)$ for some morphism $f_0 : \mathscr{F} \to \mathscr{G}$. Note that the morphism f_0 is a pullback of $j^*(f)$ (Proposition 20.2.4.5), and is therefore also an effective epimorphism. Let us identify g with a map $\widehat{g} : \mathscr{G} \to j^* j_! \mathscr{H}$. Form a commutative diagram

$$\begin{array}{ccccc} \mathscr{F}_0 & \longrightarrow & \mathscr{G}_0 & \longrightarrow & \mathscr{H} \\ \downarrow & & \downarrow & & \downarrow u_{\mathscr{H}} \\ \mathscr{F} & \xrightarrow{f_0} & \mathscr{G} & \xrightarrow{\widehat{g}} & j^* j_! \mathscr{H}, \end{array}$$

where both squares are pullbacks. It follows from Proposition 20.2.4.5 that the vertical maps in this diagram are (-1)-truncated. If $g \circ f$ is admissible, then the left vertical map admits a section and is therefore an equivalence. Since f_0 is an effective epimorphism, this guarantees that the middle vertical map is also an equivalence, so that $\widehat{g}$ factors through $u_{\mathscr{H}}$. We then conclude that g is admissible, as desired. □

Remark 20.2.5.2. Suppose we are given a collection of morphisms $\{f_i : j_! \mathscr{F}_i \to j_! \mathscr{G}\}_{i \in I}$, indexed by a set I. Since the functor $j_!$ commutes with coproducts, we can amalgamate the morphisms f_i to obtain a map $f : j_!(\coprod_{i \in I} \mathscr{F}_i) \to j_! \mathscr{G}$. The morphism f is admissible if and only if each f_i is admissible.

Lemma 20.2.5.3. *Let $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{G}'$ be objects of $\mathcal{P}(\mathcal{G}^{\mathrm{ad}})$. Suppose we are given morphisms $f : j_! \mathscr{F} \to j_! \mathscr{G}$ and $g : j_! \mathscr{G}' \to j_! \mathscr{G}$ in the ∞-category $\mathcal{P}(\mathcal{G})$, where f is admissible. Then there exists an object $\mathscr{F}' \in \mathcal{P}(\mathcal{G}^{\mathrm{ad}})$ and a pullback diagram*

$$\begin{array}{ccc} j_! \mathscr{F}' & \xrightarrow{f'} & j_! \mathscr{G}' \\ \downarrow g' & & \downarrow g \\ j_! \mathscr{F} & \xrightarrow{f} & j_! \mathscr{G} \end{array}$$

in $\mathcal{P}(\mathcal{G})$, where f' is admissible.

Remark 20.2.5.4. In the situation of Lemma 20.2.5.3, suppose that g is also admissible. Applying Lemma 20.2.5.1 we deduce that $g \circ f' \simeq f \circ g'$ is admissible, so that g' is also admissible.

Proof of Lemma 20.2.5.3. Set $\mathscr{F}' = \mathscr{G}' \times_{j^* j_! \mathscr{G}} j^* j_! \mathscr{F}$. We then have a commutative diagram

$$\begin{array}{ccc} j_! \mathscr{F}' & \longrightarrow & j_! \mathscr{G}' \\ \downarrow & & \downarrow \\ j_! j^* j_! \mathscr{F} & \longrightarrow & j_! j^* j_! \mathscr{G} \\ \downarrow & & \downarrow \\ j_! \mathscr{F} & \longrightarrow & j_! \mathscr{G} \end{array}$$

where the upper horizontal map is admissible by construction, the upper square is a pullback diagram because the functor $j_!$ preserves pullbacks (Corollary 20.2.4.3), and the bottom square is a pullback diagram by virtue of Proposition 20.2.4.4. It follows that the outer rectangle has the desired properties. □

Lemma 20.2.5.5. *Let $\mathscr{F}_0$, $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$ be objects of $\mathcal{P}(\mathcal{G}^{\mathrm{ad}})$, and suppose we are given morphisms*

$$j_! \mathscr{F}_0 \xrightarrow{f} j_! \mathscr{F}_1 \xrightarrow{g} j_! \mathscr{F}_2 \xrightarrow{h} j_! \mathscr{F}_3 .$$

If $g \circ f$ and $h \circ g$ are admissible, then f is admissible.

Proof. Let $h : \mathcal{G}^{\mathrm{ad}} \to \mathcal{P}(\mathcal{G}^{\mathrm{ad}})$ denote the Yoneda embedding. Then we can choose a collection of object $\{Z_i\}_{i \in I}$ in $\mathcal{G}^{\mathrm{ad}}$ and an effective epimorphism $\rho : \amalg_{i \in I} h(Z_i) \to \mathscr{F}_3$. Using Lemma 20.2.5.3 repeatedly, we can form a commutative diagram

$$\begin{array}{ccc} j_!(\amalg_{i \in I} \mathscr{F}_{0,i}) & \xrightarrow{\phi_0} & j_! \mathscr{F}_0 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ j_!(\amalg_{i \in I} \mathscr{F}_{1,i}) & \xrightarrow{\phi_1} & j_! \mathscr{F}_1 \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ j_!(\amalg_{i \in I} \mathscr{F}_{2,i}) & \xrightarrow{\phi_2} & j_! \mathscr{F}_2 \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ j_!(\amalg_{i \in I} h(Z_i)) & \xrightarrow{j_!(\rho)} & j_! \mathscr{F}_3 \end{array}$$

where each square is a pullback and the horizontal maps are admissible. Applying Remark 20.2.5.4, we see that the composite maps $g' \circ f'$ and $h' \circ g'$ are both admissible. Note that since $j_!$ preserves small colimits, the morphism $j_!(\rho)$ is an effective epimorphism; it follows that ϕ_0 is an admissible effective epimorphism. Using Lemma 20.2.5.1, we see that f is admissible if and only if $f \circ \phi_0 \simeq \phi_1 \circ f'$ is admissible, if and only if f' is admissible. We may therefore replace the right column in the above diagram by the left column, and thereby reduce to the case where $\mathscr{F}_3$ is a coproduct of objects of the form $h(Z)$ for $Z \in \mathcal{G}^{\mathrm{ad}}$. Using Remark 20.2.5.2, we can reduce further to the case where $\mathscr{F}_3 = h(Z)$ for some $Z \in \mathcal{G}^{\mathrm{ad}}$.

Applying similar arguments to $\mathscr{F}_2$, $\mathscr{F}_1$, and $\mathscr{F}_0$, we can assume that $\mathscr{F}_2 = h(Y)$, $\mathscr{F}_1 = h(X)$, and $\mathscr{F}_0 = h(W)$. Then the morphisms f, g, and h are induced by maps and that the morphisms f, g, and h are induced by maps $W \xrightarrow{f_0} X \xrightarrow{g_0} Y \xrightarrow{h_0} Z$, where $g_0 \circ f_0$ and $h_0 \circ g_0$ are admissible. Applying Corollary 20.2.2.9, we conclude that f_0 is also an admissible morphism of $\mathcal{G}$, so that $f \simeq j_! h(f_0)$ is an admissible morphism in $\mathcal{P}(\mathcal{G})$. □

Lemma 20.2.5.6. *Let $\mathscr{F}$ and $\mathscr{G}$ be objects of $\mathcal{P}(\mathcal{G})$ and let $f : j_! \mathscr{F} \to j_! \mathscr{G}$ be an equivalence in the ∞-category $\mathcal{P}(\mathcal{G})$. Then f is homotopic to $f_!(f_0)$ for some equivalence $f_0 : \mathscr{F} \to \mathscr{G}$ in the ∞-category $\mathcal{P}(\mathcal{G}^{\mathrm{ad}})$.*

Proof. Let f^{-1} denote a homotopy inverse to f. Applying Lemma 20.2.5.5 to the diagram

$$j_! \mathscr{F} \xrightarrow{f} j_! \mathscr{G} \xrightarrow{f^{-1}} j_! \mathscr{F} \xrightarrow{f} j_! \mathscr{G},$$

we deduce that f is admissible: that is, we can write $f = j_!(f_0)$ for some morphism f_0 in $\mathcal{P}(\mathcal{G}^{\mathrm{ad}})$. It follows from Proposition 20.2.4.5 that the morphism f_0 is a pullback of $j^*(f)$, and is therefore an equivalence. □

Proof of Theorem 20.6.3.4. It follows from Proposition 20.2.4.5 and Lemma 20.2.5.6 that the functor $j_! : \mathcal{P}(\mathcal{G}^{\mathrm{ad}}) \to \mathcal{P}(\mathcal{G})$ induces an equivalence from $\mathcal{P}(\mathcal{G}^{\mathrm{ad}})$ to a replete subcategory $\mathcal{P}(\mathcal{G})^{\mathrm{corp}} \subseteq \mathcal{P}(\mathcal{G})$. To complete the proof, it will suffice to show that $\mathcal{P}(\mathcal{G})^{\mathrm{corp}}$ is a fracture subcategory of $\mathcal{P}(\mathcal{G})$: that is, that it satisfies conditions (1), (2), and (3) of Definition 20.1.2.1. Condition (1) follows from Corollary 20.2.4.3, condition (2) from Corollary **??**, and condition (3) from Proposition 20.2.4.4. □

20.3 Geometric Admissibility Structures

Let X be a quasi-compact, quasi-separated scheme, let Sch_X denote the category of X-schemes of finite presentation, and let $\mathrm{Sch}_X^{\text{ét}} \subseteq \mathrm{Sch}_X$ denote the full subcategory spanned by the étale X-schemes. We regard both Sch_X and $\mathrm{Sch}_X^{\text{ét}}$ as Grothendieck sites, and denote the associated ∞-topoi by $\mathcal{S}\mathrm{hv}(\mathrm{Sch}_X)$ and $\mathcal{S}\mathrm{hv}(\mathrm{Sch}_X^{\text{ét}})$, respectively. These ∞-topoi are closely related to one another. For each functor $\mathscr{F} : \mathrm{Sch}_X^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ and each object $Y \in \mathrm{Sch}_X$, let $\mathscr{F}|_Y$ denote the composition of $\mathscr{F}$ with the forgetful functor $(\mathrm{Sch}_Y^{\text{ét}})^{\mathrm{op}} \to \mathrm{Sch}_X^{\mathrm{op}}$. Then $\mathscr{F}$ is a sheaf for the étale topology on Sch_X if and only if each $\mathscr{F}|_Y$ is a sheaf with respect to the étale topology on $\mathrm{Sch}_Y^{\text{ét}}$. In this case, $\mathscr{F}$ can be functorially recovered from the sheaves $\mathscr{F}|_Y$ together with the natural transition maps

$$\tau_f : f^* \mathscr{F}|_Y \to \mathscr{F}|_Z$$

defined for each morphism $f : Z \to Y$ in Sch_X. We say that a sheaf $\mathscr{F} \in \mathcal{S}\mathrm{hv}(\mathrm{Sch}_X)$ is *Cartesian* if τ_f is an isomorphism for every morphism f in Sch_X. The construction $\mathscr{F} \mapsto \mathscr{F}|_X$ determines an equivalence of ∞-categories $\mathcal{S}\mathrm{hv}(\mathrm{Sch}_X)^{\mathrm{cart}} \to \mathcal{S}\mathrm{hv}(\mathrm{Sch}_X^{\text{ét}})$, where $\mathcal{S}\mathrm{hv}(\mathrm{Sch}_X)^{\mathrm{cart}}$ denotes the full subcategory of $\mathcal{S}\mathrm{hv}(\mathrm{Sch}_X)$ spanned by the Cartesian sheaves (note that if $\mathscr{F}$ is Cartesian, then for any $f : Y \to X$ we have $\mathscr{F}(Y) = \Gamma(Y; f^* \mathscr{F}|_X))$.

The condition that a sheaf $\mathscr{F} \in \mathcal{S}\mathrm{hv}(\mathrm{Sch}_X)$ is Cartesian can be relativized. Given a morphism $\alpha : \mathscr{F} \to \mathscr{G}$ in $\mathcal{S}\mathrm{hv}(\mathrm{Sch}_X)$, let us say that α is *Cartesian* if, for every morphism $f : Z \to Y$ in Sch_X, the diagram

$$\begin{array}{ccc} f^* \mathscr{F}|_Y & \longrightarrow & \mathscr{F}|_Z \\ \downarrow & & \downarrow \\ f^* \mathscr{G}|_Y & \longrightarrow & \mathscr{G}|_Z \end{array}$$

is a pullback square (in the ∞-category $\mathcal{S}\mathrm{hv}(\mathrm{Sch}_Z^{\text{ét}})$). In this section, we introduce the notion of a *geometric admissibility structure* on an ∞-topos $\mathcal{X}$ (Definition 20.3.4.1), which abstracts the essential properties enjoyed by the class of Cartesian morphisms in $\mathcal{S}\mathrm{hv}(\mathrm{Sch}_X)$. Our main result (Theorem 20.3.4.4) asserts that the datum of a geometric admissibility structure on an ∞-topos $\mathcal{X}$ is essentially equivalent to the datum of a fracture subcategory of $\mathcal{X}$ (more precisely, it is equivalent to the datum of a *complete* fracture subcategory of $\mathcal{X}$: see Definition 20.3.3.9).

Remark 20.3.0.1. The framework developed in this section is closely related to the notion of a *class of étale morphisms* studied by Joyal-Moerdijk ([109]) and Dubuc ([52]). More precisely, our notion of *local admissibility structure* (Definition 20.3.2.1) can be regarded as an ∞-categorical version of a class of étale morphisms; our notion of *geometric admissibility structure* is then obtained by adding an additional axiom (Definition 20.3.4.1).

20.3.1 Admissible Morphisms in a Fractured ∞-Topos

In §20.2, we introduced the notion of an admissibility structure on an ∞-category (Definition 20.2.1.1). So far, we have been primarily interested in admissibility structures on *small* ∞-categories $\mathcal{G}$, and their relationships with structures on larger ∞-categories built from $\mathcal{G}$ (such as factorization systems on $\mathrm{Pro}(\mathcal{G})$ and fracture subcategories of $\mathcal{P}(\mathcal{G})$). We now study a class of admissibility structures which live naturally on *large* ∞-categories (more specifically, on fractured ∞-topoi).

Definition 20.3.1.1. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fracture subcategory. We will say that a morphism $f : U \to X$ in $\mathcal{X}$ is $\mathcal{X}^{\mathrm{corp}}$-*admissible* if, for every pullback

diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X \end{array}$$

where X' is corporeal, the morphism f' also belongs to the subcategory $\mathcal{X}^{\mathrm{corp}}$ (so, in particular, the object U' is also corporeal).

Example 20.3.1.2. Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos, let $X \in \mathcal{X}^{\mathrm{corp}}$ be a corporeal object of $\mathcal{X}$, and let $f : U \to X$ be a morphism in $\mathcal{X}$. The following conditions are equivalent:

(a) The morphism f is $\mathcal{X}^{\mathrm{corp}}$-admissible.

(b) The morphism f belongs to $\mathcal{X}^{\mathrm{corp}}$.

The implication $(a) \Rightarrow (b)$ is trivial, and the converse follows from our assumption that the inclusion $\mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$ preserves pullbacks.

Proposition 20.3.1.3. *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos. Then the collection of $\mathcal{X}^{\mathrm{corp}}$-admissible morphisms determines an admissibility structure on the ∞-topos $\mathcal{X}$.*

Proof. We verify each requirement of Definition 20.2.1.1 in turn:

(1) Let $f : U \to X$ be an equivalence in $\mathcal{X}$; we wish to show that f is $\mathcal{X}^{\mathrm{corp}}$-admissible. Consider any pullback square

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X \end{array}$$

in the ∞-category $\mathcal{X}$. Then f' is an equivalence. If X' is corporeal, then f' belongs to $\mathcal{X}^{\mathrm{corp}}$ (by virtue of the fact that $\mathcal{X}^{\mathrm{corp}}$ is a replete subcategory of $\mathcal{X}$).

(2) Let $f : U \to X$ be a $\mathcal{X}^{\mathrm{corp}}$-admissible morphism in $\mathcal{X}$ and let $g : X' \to X$ be an arbitrary morphism in $\mathcal{X}$. The ∞-category $\mathcal{X}$ admits small limits, so we can form a pullback square

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X. \end{array}$$

We wish to show that f' is also $\mathcal{X}^{\mathrm{corp}}$-admissible. To prove this, consider any pullback diagram

$$\begin{array}{ccc} U'' & \longrightarrow & U' \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} \\ X'' & \longrightarrow & X' \end{array}$$

where X'' is corporeal; we wish to show that f'' belongs to $\mathcal{X}^{\mathrm{corp}}$. This follows from our assumption that f is $\mathcal{X}^{\mathrm{corp}}$-admissible, since the outer rectangle in the diagram

$$\begin{array}{ccccc} U'' & \longrightarrow & U' & \longrightarrow & U \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X'' & \longrightarrow & X' & \longrightarrow & X \end{array}$$

is also a pullback square.

(3) Suppose we are given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{X}$, where g is $\mathcal{X}^{\mathrm{corp}}$-admissible. We wish to show that f is $\mathcal{X}^{\mathrm{corp}}$-admissible if and only if $g \circ f$ is $\mathcal{X}^{\mathrm{corp}}$-admissible. Assume first that f is $\mathcal{X}^{\mathrm{corp}}$-admissible. Consider any map $Z' \to Z$, where Z' is corporeal, and form a diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Z' & \longrightarrow & Z \end{array}$$

where both squares are pullbacks. Our assumption that g is $\mathcal{X}^{\mathrm{corp}}$-admissible guarantees that g' belongs to $\mathcal{X}^{\mathrm{corp}}$. In particular, Y' is corporeal. Applying our assumption that f is $\mathcal{X}^{\mathrm{corp}}$-admissible, we deduce that f' belongs to $\mathcal{X}^{\mathrm{corp}}$. It follows that $g' \circ f'$ belongs to $\mathcal{X}^{\mathrm{corp}}$, as desired.

Now suppose that $g \circ f$ is $\mathcal{X}^{\mathrm{corp}}$-admissible, and consider any map $Y' \to Y$, where Y' is corporeal. We then have a commutative diagram

$$\begin{array}{ccccc} X \times_Y Y' & \longrightarrow & Y' & & \\ \downarrow & & \downarrow{\scriptstyle t} & \searrow^{\mathrm{id}} & \\ X \times_Z Y' & \xrightarrow{s} & Y \times_Z Y' & & \\ & \searrow_{u} & & \searrow_{v} & \\ & & & & Y'. \end{array}$$

Our assumptions that g and $(g \circ f)$ are $\mathcal{X}^{\mathrm{corp}}$-admissible guarantee that the morphisms u and v belong to $\mathcal{X}^{\mathrm{corp}}$. Applying Proposition 20.1.3.1, we deduce that s and t also belong to $\mathcal{X}^{\mathrm{corp}}$. The upper left square is a pullback diagram in $\mathcal{X}$ and therefore also in $\mathcal{X}^{\mathrm{corp}}$ (since the inclusion $\mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$ is conservative and preserves pullbacks), so that the projection map $X \times_Y Y' \to Y'$ also belongs to $\mathcal{X}^{\mathrm{corp}}$.

(4) Suppose that $f : U \to X$ is a retract of some morphism $g : V \to Y$ in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{X})$, where g is $\mathcal{X}^{\mathrm{corp}}$-admissible. We wish to show that f is also $\mathcal{X}^{\mathrm{corp}}$-admissible. Choose any map $X' \to X$, where X' is corporeal. We then have a commutative diagram

$$\begin{array}{ccccc} U \times_X X' & \longrightarrow & V \times_Y X' & \longrightarrow & U \times_X X' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \downarrow \\ X' & \xrightarrow{\mathrm{id}} & X' & \xrightarrow{\mathrm{id}} & X', \end{array}$$

where the horizontal composite maps are the identity. Our assumption that g is $\mathcal{X}^{\mathrm{corp}}$-admissible guarantees that g' belongs to $\mathcal{X}^{\mathrm{corp}}$; we wish to show that f' also belongs to $\mathcal{X}^{\mathrm{corp}}$. To prove this, it suffices to show that the full subcategory $\mathcal{X}^{\mathrm{corp}}_{/X'} \subseteq \mathcal{X}_{/X'}$ is closed under retracts. This is clear, since the ∞-category $\mathcal{X}^{\mathrm{corp}}_{/X'}$ is idempotent complete (in fact, it is an ∞-topos: see Proposition 20.1.3.3).

□

20.3.2 Local Admissibility Structures

We now axiomatize some of the special features enjoyed by the admissibility structures obtained from Proposition 20.3.1.3.

Definition 20.3.2.1. Let $\mathcal{X}$ be an ∞-topos. We will say that an admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ is *local* if it satisfies the following conditions:

(a) If $f : U \to X$ is an arbitrary morphism in $\mathcal{X}$ and there exists an effective epimorphism $\amalg_\alpha X_\alpha \to X$ such that each of the induced maps $X_\alpha \times_X U \to X_\alpha$ is admissible, then f is admissible (in other words, the collection of admissible morphisms is *local* in the sense of Definition HTT.6.1.3.8).

(b) For each object $X \in \mathcal{X}$, the ∞-category $\mathcal{X}^{\mathrm{ad}}_{/X}$ is presentable and the inclusion $\mathcal{X}^{\mathrm{ad}}_{/X} \hookrightarrow \mathcal{X}_{/X}$ preserves small colimits.

Example 20.3.2.2. Let $\mathcal{X}$ be an ∞-topos. Then $\mathcal{X}$ has a local admissibility structure where we declare that *all* morphisms in $\mathcal{X}$ are admissible.

Example 20.3.2.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi, and suppose we are given geometric morphisms $f^*, g^* : \mathcal{X} \to \mathcal{Y}$ and a natural transformation $\alpha : f^* \to g^*$. Then α determines a local admissibility structure on $\mathcal{X}$, where we declare that a morphism $X \to X'$ in $\mathcal{X}$ is *admissible* if the diagram

$$\begin{array}{ccc} f^* X & \xrightarrow{\alpha(X)} & g^* X \\ \downarrow & & \downarrow \\ f^* X' & \xrightarrow{\alpha(X')} & g^* X' \end{array}$$

is a pullback square in $\mathcal{Y}$ (see Example 20.2.1.8).

Example 20.3.2.4. Let $\mathcal{X}$ be an ∞-topos, so that $\mathrm{Fun}(\Delta^1, \mathcal{X})$ is also an ∞-topos. The ∞-topos $\mathrm{Fun}(\Delta^1, \mathcal{X})$ admits a local admissibility structure, where we declare that a morphism $f \to f'$ in $\mathrm{Fun}(\Delta^1, \mathcal{X})$ is *admissible* if it corresponds to a pullback diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X \end{array}$$

in the ∞-topos $\mathcal{X}$. This is a special case of Example 20.3.2.3; it can also be deduced by applying Proposition 20.3.1.3 to the fracture subcategory $\mathrm{Fun}(\Delta^1, \mathcal{X})^{\mathrm{corp}} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{X})$ of Example 20.1.2.3.

Remark 20.3.2.5. Let $\mathcal{X}$ be an ∞-topos, let I be a (small) set, and suppose that for each $\alpha \in I$ we are given a local admissibility structure $\mathcal{X}^{\mathrm{ad}}_{\alpha} \subseteq \mathcal{X}$. Then the intersection $\bigcap_{\alpha \in I} \mathcal{X}^{\mathrm{ad}}_{\alpha}$ is also a local admissibility structure on $\mathcal{X}$.

Remark 20.3.2.6. Let $\mathcal{X}$ be an ∞-topos equipped with a local admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$. For every object $X \in \mathcal{X}$, the ∞-category $\mathcal{X}_{/X}$ is an ∞-topos. Since $\mathcal{X}^{\mathrm{ad}}_{/X}$ is a presentable full subcategory of $\mathcal{X}_{/X}$ which is closed under small colimits (by definition) and finite limits (Remark 20.2.1.12), it follows that $\mathcal{X}^{\mathrm{ad}}_{/X}$ is also an ∞-topos. Moreover, the inclusion functor $\iota^* : \mathcal{X}^{\mathrm{ad}}_{/X} \hookrightarrow \mathcal{X}_{/X}$ is a geometric morphism of ∞-topoi (that is, it preserves small colimits and finite limits).

We will prove in a moment that if $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ is a fractured ∞-topos, then the admissibility structure of Proposition 20.3.1.3 is local (Corollary 20.3.2.8). To prove this, we need the following reformulation of Definition 20.3.1.1:

Proposition 20.3.2.7. *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos, let $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ be a right adjoint to the inclusion, and let $f : U \to X$ be a morphism. The following conditions are equivalent:*

(1) *The morphism f is $\mathcal{X}^{\mathrm{corp}}$-admissible.*

(2) *The diagram σ :*

$$\begin{array}{ccc} j^*U & \longrightarrow & U \\ \downarrow{\scriptstyle j^*f} & & \downarrow{\scriptstyle f} \\ j^*X & \longrightarrow & X \end{array}$$

is a pullback square in $\mathcal{X}$.

Proof. Suppose first that (2) is satisfied, and consider any morphism $g : X' \to X$. If X' is corporeal, then g factors as a composition $X' \to g' j^* X \to X$, where g' belongs to $\mathcal{X}^{\mathrm{corp}}$. Extend σ to a commutative diagram

$$\begin{array}{ccccc} U' & \longrightarrow & j^*U & \longrightarrow & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle j^*f} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g'} & j^*X & \longrightarrow & X \end{array}$$

where the left square is a pullback. If condition (2) is satisfied, then the outer rectangle is also a pullback square. Consequently, to verify condition (1), it will suffice to show that the morphism f' belongs to $\mathcal{X}^{\mathrm{corp}}$. This is clear, since g' and j^*f belong to $\mathcal{X}^{\mathrm{corp}}$ (since the inclusion $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ preserves pullback squares).

We now show that (1) implies (2). Suppose that f is $\mathcal{X}^{\mathrm{corp}}$-admissible. Let us say that an object $X' \in \mathcal{X}_{/X}$ is *good* if the projection map $U \times_X X' \to X'$ satisfies condition (2): that is, if the left square in the diagram

$$\begin{array}{ccccc} j^*(U \times_X X') & \longrightarrow & U \times_X X' & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ j^*X' & \longrightarrow & X' & \longrightarrow & X \end{array}$$

is a pullback square. Since the right square is a pullback, this is equivalent to the requirement that the outer square is a pullback. Because colimits in $\mathcal{X}$ are universal and the functor j^* preserves colimits, the collection of good objects $X' \in \mathcal{X}_{/X}$ is closed under small colimits.

To complete the proof, it will suffice to show that the object $X \in \mathcal{X}_{/X}$ is good. In fact, we claim that every object $X' \in \mathcal{X}_{/X}$ is good. Since $\mathcal{X}$ is generated under small colimits by corporeal objects (Corollary 20.1.3.4), we may assume without loss of generality that X' is corporeal (when regarded as an object of X). In this case, assumption (1) guarantees that the projection map $U \times_X X' \to X'$ belongs to $\mathcal{X}^{\mathrm{corp}}$. The desired result then follows from condition (3) of Definition 20.1.2.1. □

Corollary 20.3.2.8. *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos, and let $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ be the subcategory of $\mathcal{X}$ spanned by the $\mathcal{X}^{\mathrm{corp}}$-admissible morphisms. Then $\mathcal{X}^{\mathrm{ad}}$ is a local admissibility structure on $\mathcal{X}$.*

Proof. We first verify condition (a) of Definition 20.3.2.1. Suppose we are given a pullback diagram

$$\begin{array}{ccc} \amalg U_\alpha & \longrightarrow & U \\ \downarrow{\scriptstyle \amalg f_\alpha} & & \downarrow{\scriptstyle f} \\ \amalg X_\alpha & \xrightarrow{g} & X \end{array}$$

in the ∞-category $\mathcal{X}$, where g is an effective epimorphism and each f_α is $\mathcal{X}^{\mathrm{corp}}$-admissible; we wish to show that f is also $\mathcal{X}^{\mathrm{corp}}$-admissible. Let $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ denote a right adjoint to the inclusion. Consider the diagram

$$\begin{array}{ccccc} \amalg j^* U_\alpha & \longrightarrow & j^* U & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ \amalg j^* X_\alpha & \xrightarrow{j^* g} & j^* X & \longrightarrow & X. \end{array}$$

We wish to show that the square on the right is a pullback (Proposition 20.3.2.7). Since the functor j^* preserves small limits and colimits, the square on the left is a pullback diagram, and the map $j^*(g)$ is an effective epimorphism. It will therefore suffice to show that the outer rectangle is a pullback diagram. To see this, consider the the diagram

$$\begin{array}{ccccc} \amalg j^* U_\alpha & \longrightarrow & \amalg U_\alpha & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ \amalg j^* X_\alpha & \longrightarrow & \amalg X_\alpha & \xrightarrow{g} & X. \end{array}$$

The right square is a pullback by definition, and the left square is a pullback by virtue of our assumption that each f_α is $\mathcal{X}^{\mathrm{corp}}$-admissible (Proposition 20.3.2.7). It follows that the outer rectangle is also a pullback diagram, as desired.

We now prove (b). Let $\widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the ∞-category of (not necessarily small) ∞-categories and let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ whose objects are presentable ∞-categories and whose morphisms are functors which preserve small colimits. The construction $X \mapsto \mathcal{X}_{/X}$ determines a functor $F : \mathcal{X}^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ (classifying the Cartesian fibration $\mathrm{Fun}(\Delta^1, \mathcal{X}) \to \mathrm{Fun}(\{1\}, \mathcal{X})$, and the construction $X \mapsto \mathcal{X}^{\mathrm{ad}}_{/X}$ determines a functor $F_0 : \mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ equipped with a natural transformation $\iota : F_0 \to F$. Since $\mathcal{X}$ is an ∞-topos, the functor F preserves small limits. Using (a), we see that F_0 also preserves small limits (see Lemma HTT.6.1.3.7). Let us say that an object $X \in \mathcal{X}$ is *good* if $\mathcal{X}^{\mathrm{ad}}_{/X}$ is presentable and the inclusion $\mathcal{X}^{\mathrm{ad}}_{/X} \hookrightarrow \mathcal{X}$ preserves small colimits: that is, if the canonical map $\iota(X) : F_0(X) \to F(X)$ is a morphism in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Since the inclusion functor $\mathcal{P}\mathrm{r}^{\mathrm{L}} \hookrightarrow \widehat{\mathcal{C}\mathrm{at}}_\infty$ preserves small limits (Proposition HTT.5.5.3.13), the collection of good objects of $\mathcal{X}$ is closed under small colimits. To prove (b), we must show that every object $X \in \mathcal{X}$ is good. Using Corollary 20.1.3.4, we can reduce to the case where X is corporeal. Using Example 20.3.1.2, we are reduced to showing that the ∞-category $\mathcal{X}^{\mathrm{corp}}_{/X}$ is presentable and that the inclusion $\mathcal{X}^{\mathrm{corp}}_{/X} \to \mathcal{X}_{/X}$ preserves small colimits. The first assertion follows from Proposition 20.1.3.3, and the second from the fact that the inclusion $\mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$ preserves small colimits (since it admits a right adjoint j^*). □

20.3.3 Corporeal Objects

Let $\mathcal{X}$ be an ∞-topos. According to Corollary 20.3.2.8, every fracture subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ determines a local admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$. We now consider the problem of recovering $\mathcal{X}^{\mathrm{corp}}$ from $\mathcal{X}^{\mathrm{ad}}$.

Definition 20.3.3.1. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ be a local admissibility structure on $\mathcal{X}$ (Definition 20.3.2.1). For each object $X \in \mathcal{X}$, the inclusion functor $\mathcal{X}^{\mathrm{ad}}_{/X} \hookrightarrow \mathcal{X}_{/X}$ is a functor between presentable ∞-categories which preserves small colimits, and therefore admits a right adjoint $\rho_X : \mathcal{X}_{/X} \to \mathcal{X}^{\mathrm{ad}}_{/X}$. We will say that the object X is $\mathcal{X}^{\mathrm{ad}}$-*corporeal* if the functor ρ_X preserves small colimits.

Example 20.3.3.2. Let $\mathcal{X}$ be an ∞-topos equipped with a fracture subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$, let $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ be a right adjoint to the inclusion functor, and let $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ be the subcategory of $\mathcal{X}^{\mathrm{corp}}$-admissible morphisms. If $X \in \mathcal{X}^{\mathrm{corp}}$ is a corporeal object of $\mathcal{X}$, then we have $\mathcal{X}^{\mathrm{ad}}_{/X} = \mathcal{X}^{\mathrm{corp}}_{/X}$ (Example 20.3.1.2). It follows that the functor $\rho_X : \mathcal{X}_{/X} \to \mathcal{X}^{\mathrm{ad}}_{/X}$ of Definition 20.3.3.1 is given by the formula $\rho_X(U) = j^*(U) \times_{j^*(X)} X$. Since the functor j^* preserves small colimits, we deduce that X is $\mathcal{X}^{\mathrm{ad}}$-corporeal.

Warning 20.3.3.3. In the situation of Example 20.3.3.2, it is generally not true that every $\mathcal{X}^{\mathrm{ad}}$-corporeal object of $\mathcal{X}$ is corporeal. However, this is not far from being true: see Proposition 20.3.3.11 below.

Example 20.3.3.4. Let $\mathcal{X}$ be an ∞-topos equipped with the local admissibility structure $\mathcal{X}^{\mathrm{ad}} = \mathcal{X}$ of Example 20.3.2.2 (so that every morphism in $\mathcal{X}$ is admissible). Then every object of $\mathcal{X}$ is $\mathcal{X}^{\mathrm{ad}}$-corporeal.

Example 20.3.3.5. Let $\mathcal{X}$ be an ∞-topos, and let us regard the ∞-topos $\mathrm{Fun}(\Delta^1, \mathcal{X})$ as equipped with the local admissibility structure of Example 20.3.2.4. For every pair of objects X and Y, the inclusion $X \hookrightarrow X \amalg Y$ is a $\mathrm{Fun}(\Delta^1, \mathcal{X})^{\mathrm{ad}}$-corporeal object of $\mathrm{Fun}(\Delta^1, \mathcal{X})$.

Remark 20.3.3.6. Let $\mathcal{X}$ be an ∞-topos equipped with a local admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$, and suppose we are given a small collection of objects $X_\alpha \in \mathcal{X}$ having coproduct $X \in \mathcal{X}$. Then the equivalence of ∞-categories $\mathcal{X}_{/X} \simeq \prod_\alpha \mathcal{X}_{/X_\alpha}$ restricts to an equivalence $\mathcal{X}^{\mathrm{ad}}_{/X} \simeq \prod_\alpha \mathcal{X}^{\mathrm{ad}}_{/X_\alpha}$. It follows that X is $\mathcal{X}^{\mathrm{ad}}$-corporeal if and only if each X_α is $\mathcal{X}^{\mathrm{ad}}$-corporeal.

Remark 20.3.3.7. Let $\mathcal{X}$ be an ∞-topos equipped with a local admissibility structure, let $X \in \mathcal{X}$ be an object, and let $\rho_X : \mathcal{X}_{/X} \to \mathcal{X}^{\mathrm{ad}}_{/X}$ be a right adjoint to the inclusion. For every admissible morphism $U \to V$ in $\mathcal{X}_{/X}$, the associated diagram σ :

$$\begin{array}{ccc} \rho_X U & \longrightarrow & \rho_X V \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

is a pullback square in $\mathcal{X}$. To prove this, let $W = \rho_X V \times_V U$, and let $\mathcal{D}$ denote the full subcategory of $\mathcal{X}_{/W}$ spanned by those maps $W' \to W$ for which the composite map $W' \to W \to X$ is admissible. Since the map $U \to V$ is admissible, the projection map $W \to \rho_X V \in \mathcal{X}^{\mathrm{ad}}_{/X}$ is admissible, so that we can regard W as a final object of $\mathcal{D}$. The diagram σ determines a map $\rho_X U \to W$. To show that σ is a pullback square, it will suffice to show that this map exhibits $\rho_X U$ as a final object object of $\mathcal{D}$. Fix an object $D \in \mathcal{D}$; we wish to show that the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{X}}(D, \rho_X U) & \longrightarrow & \mathrm{Map}_{\mathcal{X}}(D, U) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{X}}(D, \rho_X V) & \longrightarrow & \mathrm{Map}_{\mathcal{X}}(D, V) \end{array}$$

is a pullback square. This is clear, since the horizontal maps are homotopy equivalences.

Proposition 20.3.3.8. *Let $\mathcal{X}$ be an ∞-topos equipped with a local admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$. Then:*

(1) *The collection of $\mathcal{X}^{\mathrm{ad}}$-corporeal objects of $\mathcal{X}$ is closed under retracts.*

(2) *Let $f : U \to X$ be an admissible morphism of $\mathcal{X}$. If X is $\mathcal{X}^{\mathrm{ad}}$-corporeal, then U is $\mathcal{X}^{\mathrm{ad}}$-corporeal. The converse holds if f is an effective epimorphism.*

Proof. For each object $X \in \mathcal{X}$, let $\rho_X : \mathcal{X}_{/X} \to \mathcal{X}^{\mathrm{ad}}_{/X}$ denote a right adjoint to the inclusion map. Let $f : Y \to X$ be a morphism in $\mathcal{X}$, and let $f^* : \mathcal{X}_{/X} \to \mathcal{X}_{/Y}$ be the functor given $f^* U = U \times_X Y$. For each $U \in \mathcal{X}_{/X}$, we have an evident map $\rho_X(U) \to U$, which induces a map $u : f^* \rho_X(U) \to f^* U$. Since $f^* \rho_X(U)$ belongs to $\mathcal{X}^{\mathrm{ad}}_{/Y}$, this map factors through $\rho_Y f^* U$. Moreover, this factorization depends functorially on U, and determines a natural transformation of functors $f^* \circ \rho_X \to \rho_Y \circ f^*$.

We now prove (1). Suppose that we are given a commutative diagram

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ X & & \xrightarrow{\mathrm{id}} & & X \end{array}$$

in $\mathcal{X}$, where Y is $\mathcal{X}^{\mathrm{ad}}$-corporeal. We wish to prove that X is $\mathcal{X}^{\mathrm{ad}}$-corporeal. Then the identity transformation from ρ_X to itself factors as a composition

$$\rho_X \simeq f^* g^* \rho_X \to f^* \rho_Y g^* \to \rho_X f^* g^* \simeq \rho_X,$$

so that ρ_X is a retract of the composite functor $f^* \rho_Y g^*$. Since Y is $\mathcal{X}^{\mathrm{ad}}$-corporeal, the functor ρ_Y preserves small colimits, so that $f^* \rho_Y g^*$ also preserves small colimits. It follows that ρ_X preserves small colimits, so that X is also $\mathcal{X}^{\mathrm{ad}}$-corporeal.

We now prove (2). Let $f : U \to X$ be an admissible morphism in $\mathcal{X}$. Then a map $V \to U$ is admissible if and only if the composite map $V \to U \to X$ is admissible. In particular, for any map $V \to U$, we have an equivalence $\rho_U(V) \simeq \rho_X(V)$ in the ∞-category $\mathcal{X}$. In other words, we have a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{X}_{/U} & \xrightarrow{\rho_U} & \mathcal{X}^{\mathrm{ad}}_{/U} \\ \downarrow & & \downarrow \\ \mathcal{X}_{/X} & \xrightarrow{\rho_X} & \mathcal{X}^{\mathrm{ad}}_{/X}, \end{array}$$

where the vertical maps are given by composition with f. If X is $\mathcal{X}^{\mathrm{ad}}$-corporeal, it follows that the composite functor $\mathcal{X}_{/U} \xrightarrow{\rho_U} \mathcal{X}^{\mathrm{ad}}_{/U} \to \mathcal{X}^{\mathrm{ad}}_{/X}$ commutes with small colimits. Since composition with f determines a conservative functor $\mathcal{X}^{\mathrm{ad}}_{/U} \to \mathcal{X}^{\mathrm{ad}}_{/X}$ which preserves small colimits, we conclude that U is also $\mathcal{X}^{\mathrm{ad}}$-corporeal.

Conversely, suppose that U is $\mathcal{X}^{\mathrm{ad}}$-corporeal. For each object $V \in \mathcal{X}_{/X}$, we have an equivalence

$$\rho_X(V) \times_X U = \rho_X(V) \times_X \rho_X(U) \simeq \rho_X(V \times_X U) \simeq \rho_U(V \times_X U),$$

so that the functor $V \mapsto \rho_X(V) \times_X U$ preserves small colimits. If f is an effective epimorphism, then the functor $V \mapsto \rho_X(V)$ also preserves small colimits, so that X is $\mathcal{X}^{\mathrm{ad}}$-corporeal. □

We now establish a weak converse to Example 20.3.3.2. First, we need a definition.

Definition 20.3.3.9. Let $\mathcal{X}$ be an ∞-topos. We will say that a fracture subcategory $\mathcal{X}^{\mathrm{corp}}$ is *complete* if, for every object $X \in \mathcal{X}$ which belongs to $\mathcal{X}^{\mathrm{corp}}$, any retract of X also belongs to $\mathcal{X}^{\mathrm{corp}}$. We will say that a fractured ∞-topos $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ is *complete* if the fracture subcategory $\mathcal{X}^{\mathrm{corp}}$ is complete.

Warning 20.3.3.10. Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos. Then the fracture subcategory $\mathcal{X}^{\mathrm{corp}}$ is itself an ∞-topos (Proposition 20.1.3.3), and is therefore idempotent complete. However, this does not imply that $\mathcal{X}^{\mathrm{corp}}$ is complete in the sense of Definition 20.3.3.9. The inclusion $\mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$ is not fully faithful, so it is possible for there to exist a corporeal object $X \in \mathcal{X}$ equipped with a (coherently) idempotent map $e : X \to X$ which is not contained in $\mathcal{X}^{\mathrm{corp}}$.

Proposition 20.3.3.11. *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos and let $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ be the subcategory spanned by the $\mathcal{X}^{\mathrm{corp}}$-admissible morphisms. The following conditions are equivalent:*

(1) *The fracture subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ is complete (in the sense of Definition 20.3.3.9).*

(2) *An object $X \in \mathcal{X}$ is contained in $\mathcal{X}^{\mathrm{corp}}$ if and only if it is $\mathcal{X}^{\mathrm{ad}}$-corporeal.*

We will deduce Proposition 20.3.3.11 from the following more refined result:

Proposition 20.3.3.12. *Let $\mathcal{X}$ be an ∞-topos equipped with a local admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ and let $\mathcal{X}_0 \subseteq \mathcal{X}$ be a full subcategory with the following property:*

(i) *Every object of $\mathcal{X}_0$ is $\mathcal{X}^{\mathrm{ad}}$-corporeal.*

(ii) *For every object $X \in \mathcal{X}$, there exists an effective epimorphism $\amalg X_\alpha \to X$, where each U_α belongs to $\mathcal{X}_0$.*

Let X be an arbitrary object of $\mathcal{X}$. Then the following conditions are equivalent:

(a) *The object X is $\mathcal{X}^{\mathrm{ad}}$-corporeal.*

(b) *There exists an admissible effective epimorphism $\amalg U_\alpha \to X$, where each U_α is a retract (in the ∞-category $\mathcal{X}$) of an object V_α which admits an admissible morphism $V_\alpha \to X_\alpha$ for some $X_\alpha \in \mathcal{X}_0$.*

Proof. The implication $(b) \Rightarrow (a)$ follows from Remark 20.3.3.6 and Proposition 20.3.3.8. Conversely, suppose that (a) is satisfied. Using assumption (ii), we can choose an effective epimorphism $u : \amalg X_\alpha \to X$, where each X_α belongs to $\mathcal{X}_0$. Let $X_\bullet$ denote the Čech nerve of u, so that X can be identified with the geometric realization $|X_\bullet|$. Let $\rho_X : \mathcal{X}_{/X} \to \mathcal{X}^{\mathrm{ad}}_{/X}$ be a right adjoint to the inclusion, and for each index α set $U_\alpha = \rho_X(X_\alpha)$. Since X is $\mathcal{X}^{\mathrm{ad}}$-corporeal, the functor ρ_X preserves small colimits. We therefore have an equivalence $X \simeq \rho_X(X) \simeq \rho_X(|X_\bullet|) \simeq |\rho_X(X_\bullet)|$. In particular, the canonical map $\amalg U_\alpha \simeq \rho_X(X_0) \simeq\to X$ is an effective epimorphism. Moreover, each U_α is a retract of the fiber product $V_\alpha = U_\alpha \times_X X_\alpha$. We conclude by observing that each of the projection maps $V_\alpha \to X_\alpha$ is admissible (since it is a pullback of the map $U_\alpha \to X$). □

Proof of Proposition 20.3.3.11. The implication $(2) \Rightarrow (1)$ follows from Proposition 20.3.3.8. Conversely, suppose that the fracture subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ is complete. Every object $X \in \mathcal{X}^{\mathrm{corp}}$ is $\mathcal{X}^{\mathrm{ad}}$-corporeal by virtue of Example 20.3.3.2. To prove the converse, suppose that $X \in \mathcal{X}$ is $\mathcal{X}^{\mathrm{ad}}$-corporeal. Applying Proposition 20.3.3.12, we deduce that there exists a $\mathcal{X}^{\mathrm{corp}}$-admissible effective epimorphism $f : \amalg U_\alpha \to X$, where each U_α is a retract of some object V_α which admits a $\mathcal{X}^{\mathrm{corp}}$-admissible morphism $g_\alpha : V_\alpha \to X_\alpha$, where X_α is corporeal. The $\mathcal{X}^{\mathrm{corp}}$-admissibility of g_α then guarantees that each V_α is corporeal (Example 20.3.1.2). Invoking our assumption that $\mathcal{X}^{\mathrm{corp}}$ is complete, we conclude that each U_α is corporeal. Let $X_\bullet$ denote the Čech nerve of f. Since the inclusion $\mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$ preserves small colimits, the object $X_0 = \amalg U_\alpha$ is corporeal. Because f is $\mathcal{X}^{\mathrm{corp}}$-admissible, each X_k admits a $\mathcal{X}^{\mathrm{corp}}$-admissible morphism $X_k \to X_0$, and is therefore corporeal (Example 20.3.1.2). It follows that $X_\bullet$ can be regarded as a simplicial object of the subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$. Since the inclusion $\mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$ preserves small colimits, we conclude that $X \simeq |X_\bullet|$ also belongs to $\mathcal{X}^{\mathrm{corp}}$. □

20.3.4 Geometric Admissibility Structures

Our next goal is to prove a converse of Corollary 20.3.2.8. For this, we need to introduce an additional assumption.

Definition 20.3.4.1. Let $\mathcal{X}$ be an ∞-topos. A *geometric admissibility structure* on $\mathcal{X}$ is a local admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ which satisfies the following additional condition: the ∞-category $\mathcal{X}$ is generated under small colimits by $\mathcal{X}^{\mathrm{ad}}$-corporeal objects.

Example 20.3.4.2. Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos and let $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ be the subcategory of $\mathcal{X}^{\mathrm{corp}}$-admissible morphisms. Then every object $X \in \mathcal{X}^{\mathrm{corp}}$ is $\mathcal{X}^{\mathrm{ad}}$-corporeal (Example 20.3.3.2). Since $\mathcal{X}$ is generated under small colimits by corporeal objects (Corollary 20.1.3.4), it follows from Corollary 20.3.2.8 that $\mathcal{X}^{\mathrm{ad}}$ is a geometric admissibility structure on $\mathcal{X}$.

Example 20.3.4.3. Let $\mathcal{X}$ be an ∞-topos, and let us regard the ∞-topos $\mathrm{Fun}(\Delta^1, \mathcal{X})$ as equipped with the admissibility structure of Example 20.3.2.4 (so that admissible morphisms in $\mathrm{Fun}(\Delta^1, \mathcal{X})$ are pullback diagrams in $\mathcal{X}$). According to Example 20.3.3.5, the inclusion $X \hookrightarrow X \amalg Y$ is $\mathrm{Fun}(\Delta^1, \mathcal{X})^{\mathrm{ad}}$-corporeal for every pair of objects $X, Y \in \mathcal{X}$. It is not difficult to see that such objects generate $\mathrm{Fun}(\Delta^1, \mathcal{X})$ under small colimits (in fact, under pushouts), so that $\mathrm{Fun}(\Delta^1, \mathcal{X})^{\mathrm{ad}}$ is a geometric admissibility structure.

We can now formulate our main result:

Theorem 20.3.4.4. *Let $\mathcal{X}$ be an ∞-topos equipped with a geometric admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$. Let $\mathcal{X}^{\mathrm{corp}}$ denote the full subcategory of $\mathcal{X}^{\mathrm{ad}}$ spanned by the $\mathcal{X}^{\mathrm{ad}}$-corporeal objects. Then $\mathcal{X}^{\mathrm{corp}}$ is a complete fracture subcategory of $\mathcal{X}$.*

Remark 20.3.4.5. Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ be as in Theorem 20.3.4.4 and let $f : U \to X$ be a morphism in $\mathcal{X}$. Then f is admissible if and only if it is $\mathcal{X}^{\mathrm{corp}}$-admissible. The "only if" direction is obvious (since the collection of admissible morphisms in $\mathcal{X}$ is closed under fiber products). Conversely, suppose that f is $\mathcal{X}^{\mathrm{corp}}$-admissible, and choose an effective epimorphism $\amalg X_\alpha \to X$, where each X_α is $\mathcal{X}^{\mathrm{ad}}$-corporeal. Then each projection map $U \times_X X_\alpha \to X_\alpha$ is admissible, so that f is admissible by virtue of our assumption that the admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ is local.

Remark 20.3.4.6. Let $\mathcal{X}$ be an ∞-topos. Combining Corollary 20.3.2.8, Theorem 20.3.4.4, Remark 20.3.4.5, and Corollary **??**, we obtain a bijective correspondence

$$\begin{array}{c}\{\text{ Geometric admissibility structures } \mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X} \text{ }\} \\ \downarrow \sim \\ \{\text{ Complete fracture subcategories } \mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}. \text{ }\}\end{array}$$

Remark 20.3.4.7. Let $\mathcal{X}$ be an ∞-topos, let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fracture subcategory which is not complete, let $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ be the subcategory of $\mathcal{X}^{\mathrm{corp}}$-admissible morphisms, and let $\mathcal{X}'^{\mathrm{corp}} \subseteq \mathcal{X}^{\mathrm{ad}}$ be the full subcategory of $\mathcal{X}^{\mathrm{ad}}$ spanned by the $\mathcal{X}^{\mathrm{ad}}$-corporeal objects. Then $\mathcal{X}'^{\mathrm{corp}}$ is a fracture subcategory of $\mathcal{X}$ which contains $\mathcal{X}^{\mathrm{corp}}$ as a full subcategory (Example 20.3.3.2). By virtue of Remark 20.3.4.6, it is characterized by the following two properties:

- The fracture subcategory $\mathcal{X}'^{\mathrm{corp}} \subseteq \mathcal{X}$ is complete.
- A morphism $f : U \to X$ in $\mathcal{X}$ is $\mathcal{X}^{\mathrm{corp}}$-admissible if and only if it is $\mathcal{X}'^{\mathrm{corp}}$-admissible.

We will refer to $\mathcal{X}'^{\mathrm{corp}}$ as the *completion* of the fracture subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$. Note that Proposition 20.3.3.12 supplies necessary and sufficient conditions for an object $X \in \mathcal{X}$ to belong to $\mathcal{X}'^{\mathrm{corp}}$.

The proof of Theorem 20.3.4.4 will require some preliminaries.

Lemma 20.3.4.8. *Let $\mathcal{X}$ be an ∞-topos equipped with a geometric admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$. Then:*

(1) *The ∞-category $\mathcal{X}^{\mathrm{ad}}$ admits small colimits.*

(2) *The inclusion functor $\mathcal{X}^{\mathrm{ad}} \hookrightarrow \mathcal{X}$ preserves small colimits.*

(3) *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}^{\mathrm{ad}}$ denote the full subcategory spanned by the $\mathcal{X}^{\mathrm{ad}}$-corporeal objects. Then $\mathcal{X}^{\mathrm{corp}}$ is closed under small colimits in $\mathcal{X}^{\mathrm{ad}}$.*

Proof. Let $\{X_\alpha\}$ be a small diagram in the ∞-category $\mathcal{X}^{\mathrm{ad}}$, and let X denote the colimit $\varinjlim X_\alpha$, formed in the ∞-category $\mathcal{X}$. Our first goal is to show that each of the canonical maps $u_\alpha : X_\alpha \to X$ is admissible. Using our assumption that the admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ is geometric, we can choose an effective epimorphism $X' \to X$, where X' is $\mathcal{X}^{\mathrm{ad}}$-corporeal. To show that u_α is admissible, it will suffice to show that the projection map $X_\alpha \times_X X' \to X'$ is admissible. We may therefore replace X by X' (and $\{X_\alpha\}$ by the diagram $\{X_\alpha \times_X X'\}$) and thereby reduce to the case where X is $\mathcal{X}^{\mathrm{ad}}$-corporeal.

Let $\rho_X : \mathcal{X}_{/X} \to \mathcal{X}^{\mathrm{ad}}_{/X}$ be a right adjoint to the inclusion, so that we have a morphism of diagrams $\{X_\alpha\} \to \{\rho_X X_\alpha\}$ in the ∞-category $\mathcal{X}_{/X}$. Using Remark 20.3.3.7, we see that this morphism is Cartesian: that is, each of the transition maps $X_\alpha \to X_\beta$ determines a pullback square

$$\begin{array}{ccc} \rho_X X_\alpha & \longrightarrow & \rho_X X_\beta \\ \downarrow & & \downarrow \\ X_\alpha & \longrightarrow & X_\beta \end{array}$$

is a pullback square in $\mathcal{X}_{/X}$. It follows from Theorem HTT.6.1.3.9 that for each index α, the diagram

$$\begin{array}{ccc} \rho_X X_\alpha & \longrightarrow & \varinjlim_\beta \rho_X X_\beta \\ \downarrow & & \downarrow \phi \\ X_\alpha & \longrightarrow & \varinjlim_\beta X_\beta \end{array}$$

is also a pullback square. Since X is corporeal, the functor ρ_X preserves small colimits, so we can identify ϕ with the counit map $\rho_X(X) \to X$. This map is an equivalence (since the identity map $\mathrm{id}_X : X \to X$ is admissible). It follows that the map $\rho_X X_\alpha \to X_\alpha$ is also an equivalence: that is, the map $X_\alpha \to X$ is admissible.

To complete the proofs of (1) and (2), it will suffice to show that X is a colimit of the diagram $\{X_\alpha\}$ in the ∞-category $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$: that is, that a morphism $f : X \to Y$ is admissible if and only if each of the composite maps $X_\alpha \xrightarrow{u_\alpha} X \xrightarrow{f} Y$ is admissible. The "only if" direction follows from the admissibility of u_α (since the collection of admissible morphisms is closed under composition), and the converse follows from the fact that the full subcategory $\mathcal{X}^{\mathrm{ad}}_{/Y} \subseteq \mathcal{X}_{/Y}$ is closed under small colimits (see Definition 20.3.2.1).

We now prove (3). Let $\{X_\alpha\}$ and X be as above, and suppose that each X_α is $\mathcal{X}^{\mathrm{ad}}$-corporeal. The admissibility of each u_α guarantees that the induced map $u : \amalg X_\alpha \to X$ is admissible. Note that $\amalg X_\alpha$ is $\mathcal{X}^{\mathrm{ad}}$-corporeal (Remark 20.3.3.6). Since u is an effective epimorphism, Proposition 20.3.3.8 implies that X is also $\mathcal{X}^{\mathrm{ad}}$-corporeal. □

Lemma 20.3.4.9. *Let $\mathcal{X}$ be an ∞-topos equipped with a geometric admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ and let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}^{\mathrm{ad}}$ be the full subcategory spanned by the $\mathcal{X}^{\mathrm{ad}}$-corporeal objects. Then $\mathcal{X}^{\mathrm{corp}}$ is a presentable ∞-category.*

Proof. It follows from Lemma 20.3.4.8 that the ∞-category $\mathcal{X}^{\mathrm{corp}}$ admits small colimits. Since the admissibility structure $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ is geometric, the ∞-category $\mathcal{X}$ is generated under small colimits by the essential image of the inclusion $\mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$. Using the presentability of $\mathcal{X}$, we can choose a small collection of $\mathcal{X}^{\mathrm{ad}}$-corporeal objects $\{X_i\}_{i \in I}$ which generate $\mathcal{X}$ under small colimits. For each $i \in I$, let $\mathcal{C}_i$ denote the fiber product

$$\mathcal{X} \times_{\mathrm{Fun}(\Delta^{\{0,2\}}, \mathcal{X})} \mathrm{Fun}(\Delta^2, \mathcal{X}) \times_{\mathrm{Fun}(\{1\}, \mathcal{X})} \mathcal{X}^{\mathrm{ad}}_{/X_i}$$

whose objects are commutative diagrams

$$\begin{array}{ccc} & U & \\ \nearrow & & \searrow \\ V & \xrightarrow{\mathrm{id}_V} & V \end{array}$$

in the ∞-category $\mathcal{X}$, where U is equipped with an admissible morphism $U \to X_i$. Let $F_i : \mathcal{C}_i \to \mathcal{X}$ denote the projection map onto the first factor. Because the collection of admissible morphisms in $\mathcal{X}$ is closed under retracts, the functor F_i factors through the subcategory $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$. Similarly, the collection of $\mathcal{X}^{\mathrm{ad}}$-corporeal objects of $\mathcal{X}$ is closed under retracts (Proposition 20.3.3.8), so that F_i factors through the subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}^{\mathrm{ad}}$. Since the ∞-categories $\mathrm{Fun}(\Delta^m, \mathcal{X})$ and $\mathcal{X}^{\mathrm{ad}}_{/X_i}$ are presentable, the ∞-category $\mathcal{C}_i$ is presentable (Proposition **??**). We can therefore choose a small collection of objects $\{C_{i,j}\}_{j \in I_j}$ which generate $\mathcal{C}_i$ under small colimits. Each of the ∞-categories $\mathcal{X}^{\mathrm{ad}}_{/F_i(C_{i,j})}$ is presentable, and is therefore generated by a small collection of objects $\{Y_{i,j,k}\}_{k \in K_{i,j}}$. Note that each $Y_{i,j,k}$ is a $\mathcal{X}^{\mathrm{ad}}$-corporeal object of $\mathcal{X}$ (Proposition 20.3.3.8). Let $\mathcal{E} \subseteq \mathcal{X}^{\mathrm{corp}}$ be the smallest full subcategory which contains each of the objects $Y_{i,j,k}$ and is closed under small colimits. Choose a regular cardinal κ such that each $Y_{i,j,k}$ is κ-compact as an object of $\mathcal{X}$. Then each $Y_{i,j,k}$ is also κ-compact as an object of $\mathcal{E}$, so that the ∞-category $\mathcal{E}$ is κ-accessible. We will complete the proof by showing that $\mathcal{E} = \mathcal{X}^{\mathrm{corp}}$.

Fix an object $X \in \mathcal{X}^{\mathrm{corp}}$; we wish to show that X belongs to $\mathcal{E}$. Applying Proposition 20.3.3.12, we can choose an admissible effective epimorphism $u : \amalg U_\alpha \to X$, where each U_α is a retract of some object V_α which admits an admissible morphism $V_\alpha \to X_i$ for some index i. Let $X_\bullet$ be the Čech nerve of u, so that X can be identified with the geometric realization $|X_\bullet|$. Since $\mathcal{E}$ is closed under small colimits, it will suffice to show that each X_m belongs to $\mathcal{E}$. Writing X_m as a coproduct of objects of the form $U_{\alpha_0} \times_X \cdots \times_X U_{\alpha_m}$, we are reduced to proving the following:

- $(*)$ If W is an object of $\mathcal{X}^{\mathrm{corp}}$ which admits an admissible morphism $W \to U_\alpha$ for some index α, then W belongs to $\mathcal{E}$.

Choose a commutative diagram σ :

$$\begin{array}{ccc} & V_\alpha & \\ \nearrow & & \searrow \\ U_\alpha & \xrightarrow{\mathrm{id}} & U_\alpha, \end{array}$$

where V_α admits an admissible morphism $V_\alpha \to X_i$ for some $i \in I$. Then σ determines an object $C \in \mathcal{C}_i$ satisfying $F_i(C) = U_\alpha$. Let us say that an object $C' \in (\mathcal{C}_i)_{/C}$ is *good* if the fiber product $W \times_{U_\alpha} F_i(C')$ belongs to $\mathcal{E}$. We wish to show that the object $C \in (\mathcal{C}_i)_{/C}$ is good.

Since $\mathcal{E}$ is closed under small colimits in $\mathcal{X}^{\mathrm{corp}}$, the collection of good objects of $(\mathcal{C}_i)_{/C}$ is also closed under small colimits. It will therefore suffice to show that any map $C_{i,j} \to C$ in the ∞-category $\mathcal{C}_i$ exhibits $C_{i,j}$ as a good object of $(\mathcal{C}_i)_{/C}$. This is clear, since $W \times_{U_\alpha} F_i(C_{i,j})$ is an object of the ∞-category $\mathcal{X}^{\mathrm{ad}}_{/F_i(C_{i,j})}$, which is generated under small colimits by the objects $Y_{i,j,k}$. $\square$

Proof of Theorem 20.3.4.4. Let $\mathcal{X}$ be an ∞-topos, let $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ be a geometric admissibility structure, and let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}^{\mathrm{ad}}$ be the full subcategory spanned by the $\mathcal{X}^{\mathrm{ad}}$-corporeal objects. We will show that $\mathcal{X}^{\mathrm{corp}}$ is a fracture subcategory of $\mathcal{X}$ (automatically complete, by virtue of Proposition 20.3.3.8) by verifying each of the requirements of Definition 20.1.2.1:

(0) The subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ is replete; this follows immediately from the definitions.

(1) The ∞-category $\mathcal{X}^{\mathrm{corp}}$ admits fiber products, which are preserved by the inclusion functor $j_! : \mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$. Suppose we are given morphisms $f : U \to X$ and $g : X' \to X$ in the ∞-category $\mathcal{X}^{\mathrm{corp}}$. Form a pullback square σ :

$$\begin{array}{ccc} U' & \xrightarrow{g'} & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

in the ∞-category $\mathcal{X}$. Since the collection of admissible morphisms of $\mathcal{X}$ is stable under pullbacks, we deduce that f' and g' are admissible. Using Proposition 20.3.3.8, we see that U' is $\mathcal{X}^{\mathrm{ad}}$-corporeal. Finally, we observe that a morphism $h : Y \to U'$ is admissible if and only if the composite maps $g' \circ h$ and $f' \circ h$ are admissible, so that σ is also a pullback square in $\mathcal{X}^{\mathrm{corp}}$.

(2) The inclusion functor $j_! : \mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$ admits a right adjoint $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ which is conservative and preserves small colimits. Since $\mathcal{X}$ and $\mathcal{X}^{\mathrm{corp}}$ are presentable (Lemma 20.3.4.9) and the functor $j_!$ preserves small colimits (Lemma 20.3.4.8), the existence of the functor j^* follows from the adjoint functor theorem (Corollary HTT.5.5.2.9). The functor j^* is conservative by virtue of our assumption that $\mathcal{X}$ is generated under small colimits by $\mathcal{X}^{\mathrm{ad}}$-corporeal objects. Let $\{X_\alpha\}$ be a small diagram in the ∞-topos $\mathcal{X}$ having a colimit X; we wish to show that the canonical map $\phi : \varinjlim j^* X_\alpha \to j^* X$ is an equivalence. Let $\rho : \mathcal{X}_{/j^* X} \to \mathcal{X}^{\mathrm{ad}}_{/j^* X}$ be a right adjoint to the inclusion. Since j^* is conservative, the functor ρ preserves small colimits. Note that we have canonical equivalences $j^* X_\alpha \simeq \rho(X_\alpha \times_X j^* X)$ (this follows by comparing universal properties), so we can identify ϕ with the natural map $\varinjlim \rho(X_\alpha \times_X j^* X) \to \rho(j^* X) \simeq j^* X$. This map is an equivalence, since colimits in $\mathcal{X}$ are universal and the functor ρ commutes with small colimits (by virtue of the fact that $j^* X$ is corporeal).

(3) For every morphism $U \to V$ in $\mathcal{X}^{\mathrm{corp}}$, the diagram

$$\begin{array}{ccc} j^*U & \longrightarrow & j^*V \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

(determined by the counit map $j_!j^* \to \mathrm{id}$) is a pullback square in $\mathcal{X}$. Equivalently, we must show that the canonical map $j^*U \to U \times_V j^*V$. This follows by comparing the universal property of both sides as objects of the ∞-category $\mathcal{X}^{\mathrm{ad}}_{/j^*V}$.

□

20.4 Exactness and Density

Let $\mathcal{X}$ be a topos and let P be some property of objects of $\mathcal{X}$. Then the following conditions are equivalent:

(a) For every object $X \in \mathcal{X}$, there exists an effective epimorphism $\amalg_\alpha U_\alpha \to X$ where each U_α has the property P.

(b) The category $\mathcal{X}$ is generated under small colimits by objects having the property P.

(c) For every object $X \in \mathcal{X}$, the natural map $\varinjlim_{U \in \mathcal{X}^0_{/X}} U \to X$ is an isomorphism, where $\mathcal{X}^0_{/X}$ denotes the full subcategory of $\mathcal{X}_{/X}$ spanned by those objects which have the property P.

The implications $(c) \Rightarrow (b) \Rightarrow (a)$ are immediate, while the implication $(a) \Rightarrow (c)$ requires more work (see Corollary 20.4.5.3 below). If P satisfies any one of these equivalent conditions, then we say that P holds *locally* on $\mathcal{X}$.

In the setting of ∞-topoi, the analogous equivalence does not hold:

Counterexample 20.4.0.1. Let $Q = \prod_{n \in \mathbf{Z}} [0,1]$ denote the Hilbert cube and let $\mathcal{X} = \mathcal{S}\mathrm{hv}(Q)$ be the ∞-topos of $\mathcal{S}$-valued sheaves on Q. For each open set $U \subseteq Q$, let $\mathscr{F}_U \in \mathcal{X}$ denote the sheaf given by $\mathscr{F}_U(V) = \begin{cases} * & \text{if } V \subseteq U \\ \varnothing & \text{otherwise.} \end{cases}$ Let $\mathcal{U}$ be the collection of all open subsets of Q which are homeomorphic to a product $[0,1) \times Q$. One can show that $\mathcal{U}$ forms a basis for the topology of Q (see [40]), so that the collection of objects $\{\mathscr{F}_U\}_{U \in \mathcal{U}}$ satisfies condition (a). However, we claim that it does not satisfy condition (b). To see this, note that the formation of compactly supported cochain complexes determines a functor $U \mapsto C^*_c(U; \mathbf{Z})$ which extends to a colimit-preserving map $F : \mathcal{X} \to \mathrm{Mod}_{\mathbf{Z}}$ (this is essentially equivalent to the existence of Mayer-Vietoris sequences in compactly supported cohomology). The Hilbert cube Q is compact and contractible, so for each $U \in \mathcal{U}$ we have

$$\mathrm{H}^*_c(U; \mathbf{Z}) \simeq \mathrm{H}^*_c([0,1); \mathbf{Z}) \simeq 0.$$

It follows that the functor F vanishes on the full subcategory of $\mathcal{X}$ generated by $\{\mathscr{F}_U\}_{U \in \mathcal{U}}$ under small colimits. However, the functor F does not vanish identically: its value on the final object of $\mathcal{X}$ is given by $C^*_c(Q; \mathbf{Z}) \simeq \mathbf{Z}$.

In this section, we will show that Counterexample 20.4.0.1 is essentially an infinite-dimensional phenomenon: conditions (a), (b), and (c) are all equivalent if we restrict our attention to the case where the ∞-topos $\mathcal{X}$ is hypercomplete (Proposition 20.4.5.1).

20.4.1 Dense Functors

We begin with some general remarks.

Definition 20.4.1.1. Let $F : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories. We will say that F is *dense* if it exhibits identity functor $\mathrm{id}_{\mathcal{C}}$ is a left Kan extension of F along itself.

Remark 20.4.1.2. A functor $F : \mathcal{C} \to \mathcal{D}$ is dense if and only if, for every object $D \in \mathcal{D}$, the evident diagram $(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D})^{\triangleright} \to \mathcal{D}^{\triangleright}_{/D} \to \mathcal{D}$ exhibits D as a colimit of the diagram $(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}) \to \mathcal{C} \xrightarrow{f} \mathcal{D}$. In particular, this implies that every object of $\mathcal{D}$ can be obtained as the colimit of a diagram which factors through $\mathcal{C}$. Moreover, if $\mathcal{C}$ is small and $\mathcal{D}$ is locally small, then the diagram can be assumed small.

Remark 20.4.1.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a dense functor between ∞-categories and let $q : K \to \mathcal{D}$ be any diagram in $\mathcal{D}$. Then the induced map $F' : \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/q} \to \mathcal{D}_{/q}$ is also dense. To prove this, it suffices to observe that for each object $D \in \mathcal{D}_{/q}$, the projection map $\pi : \mathcal{D}_{/q} \to \mathcal{D}$ induces an equivalence of ∞-categories $(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/q}) \times_{\mathcal{D}_{/q}} (\mathcal{D}_{/q})_{/D} \simeq \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/\pi(C)}$, and that a diagram in $\mathcal{D}_{/q}$ is a colimit provided that its image in $\mathcal{D}$ is a colimit (Proposition HTT.1.2.13.8).

Remark 20.4.1.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and between ∞-categories and let $q : \mathcal{M} \to \Delta^1$ be a correspondence associated to F (so that $\mathcal{M} \times_{\Delta^1} \{0\} \simeq \mathcal{C}$, $\mathcal{M} \times_{\Delta^1} \{1\} \simeq \mathcal{D}$, and q is a coCartesian fibration which determines the map $F : \mathcal{M} \times_{\Delta^1} \{0\} \to \mathcal{M} \times_{\Delta^1} \{1\}$). Then F is dense if and only if the identity map $\mathrm{id}_{\mathcal{M}}$ is a q-left Kan extension of $\mathrm{id}_{\mathcal{M}}|_{\mathcal{C}}$.

Remark 20.4.1.5. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories where $\mathcal{C}$ is small, $\mathcal{D}$ is locally small, and $\mathcal{D}$ admits small colimits. Using Theorem HTT.5.1.5.6 , we may assume without loss of generality that f factors as a composition $\mathcal{C} \xrightarrow{j} \mathcal{P}(\mathcal{C}) \xrightarrow{F} \mathcal{D}$, where $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ denotes the ∞-category of presheaves on $\mathcal{C}$, the functor j is the Yoneda embedding, and F preserves small colimits. Corollary HTT.5.2.6.5 implies that F admits a right adjoint G, given by the composition

$$\mathcal{D} \xrightarrow{j'} \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\circ f} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) = \mathcal{P}(\mathcal{C}),$$

where j' denotes the Yoneda embedding for $\mathcal{D}$; moreover, the transformation

$$f = F \circ j \to (F \circ (G \circ F)) \circ j \simeq (F \circ G) \circ f$$

exhibits $(F \circ G)$ as a left Kan extension of f along itself. It follows that f is dense if and only if the counit map $F \circ G \to \mathrm{id}_{\mathcal{C}}$ is an equivalence of functors. This is equivalent to the requirement that the functor G is fully faithful.

In other words, the functor $f : \mathcal{C} \to \mathcal{D}$ is dense if and only if the induced map $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ exhibits $\mathcal{D}$ as a localization of $\mathcal{P}(\mathcal{C})$. In particular, the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ is dense for any small ∞-category $\mathcal{C}$.

Remark 20.4.1.6. Let $f : \mathcal{C} \to \mathcal{D}$ be as in Remark 20.4.1.5, and let $\mathcal{E}$ be an ∞-category which admits small colimits. Let $\mathrm{LFun}(\mathcal{D}, \mathcal{E})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{E})$ spanned by those functors which preserve small colimits. If f is dense, then composition with f induces a fully faithful functor $\mathrm{LFun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$. This follows from Theorem HTT.5.1.5.6 , Proposition HTT.5.5.4.20 , and Remark 20.4.1.5.

We now specialize Definition 20.4.1.1 to the case of inclusion functors.

Definition 20.4.1.7. Let $\mathcal{C}$ be an ∞-category. We will say that a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is *dense* in $\mathcal{C}$ if the inclusion functor $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is dense, in the sense of Definition 20.4.1.1.

Example 20.4.1.8. The full subcategory of $\mathcal{S}$ spanned by its final object is dense in $\mathcal{S}$ (this follows immediately from Remark 20.4.1.5).

Example 20.4.1.9. Let us identify $\mathbf{\Delta}$ with the full subcategory of $\mathcal{C}\mathrm{at}_\infty$ spanned by the objects $\{\Delta^n\}_{n \geqslant 0}$. Then $\mathbf{\Delta}$ is dense in $\mathcal{C}\mathrm{at}_\infty$: this follows from Proposition HA.A.7.10 .

Remark 20.4.1.10. A full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is dense if and only if the identity functor $\mathrm{id}_{\mathcal{C}}$ is a left Kan extension of $\mathrm{id}_{\mathcal{C}} |_{\mathcal{C}_0}$. It follows from Proposition HTT.4.3.2.8 that if $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}$ are full subcategories and $\mathcal{C}_0$ is dense, then $\mathcal{C}_1$ is also dense.

Remark 20.4.1.11. Let $f : \mathcal{D} \to \mathcal{C}$ be a dense functor between ∞-categories, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the essential image of f. Then $\mathcal{C}_0$ is dense in $\mathcal{C}$. To prove this, let $q : \mathcal{M} \to \Delta^1$ be a correspondence associated to f (as in Remark 20.4.1.4), and let $\mathcal{M}_0$ be the full subcategory of $\mathcal{M}$ spanned by the objects of $\mathcal{D} \simeq \mathcal{M} \times_{\Delta^1} \{0\}$ and $\mathcal{C}_0 \subseteq \mathcal{C} \simeq \mathcal{M} \times_{\Delta^1} \{1\}$. Since f is dense, the identity functor $\mathrm{id}_{\mathcal{M}}$ is a q-left Kan extension of its restriction to $\mathcal{D}$, and therefore also a q-left Kan extension of $\mathrm{id}_{\mathcal{M}} |_{\mathcal{M}_0}$. Let C be an arbitrary object of $\mathcal{C}$, so that C can be identified with a q-colimit of the forgetful functor $\mathcal{M}_0 \times_{\mathcal{M}} \mathcal{M}_{/C} \to \mathcal{M}$. To prove that $\mathrm{id}_{\mathcal{C}}$ is a left Kan extension of $\mathrm{id}_{\mathcal{C}} |_{\mathcal{C}_0}$ at C, it will suffice to show that C is also a q-colimit of the diagram $\mathcal{C}_0 \times_{\mathcal{M}} \mathcal{M}_{/C} \to \mathcal{M}$. For this, it suffices to show that the inclusion

$$\iota : \mathcal{C}_0 \times_{\mathcal{M}} \mathcal{M}_{/C} \hookrightarrow \mathcal{M}_0 \times_{\mathcal{M}} \mathcal{M}_{/C}$$

is left cofinal. This is clear, since the functor ι admits a left adjoint.

20.4.2 Local Left Exactness

Let $f : \mathcal{C} \to \mathcal{X}$ be a dense functor, where the ∞-category $\mathcal{C}$ is small and the ∞-category $\mathcal{X}$ is presentable. Then f exhibits $\mathcal{X}$ as a localization of $\mathcal{P}(\mathcal{C})$ (Remark 20.4.1.5). We now study the condition that this localization be left exact. Note that in this case, the ∞-category $\mathcal{X}$ is necessarily an ∞-topos.

Definition 20.4.2.1. Let $\mathcal{C}$ be a small ∞-category and let $\mathcal{X}$ be an ∞-topos. We will say that a functor $f : \mathcal{C} \to \mathcal{X}$ is *locally left exact* if it is homotopic to a composition

$$\mathcal{C} \xrightarrow{j} \mathcal{P}(\mathcal{C}) \xrightarrow{F} \mathcal{X}$$

where j is the Yoneda embedding and F is a functor which preserves small colimits and finite limits.

We let $\mathrm{Fun}^{\mathrm{llex}}(\mathcal{C}, \mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{X})$ spanned by those functors which are locally left exact.

Remark 20.4.2.2. In the situation of Definition 20.4.2.1, the functor f is automatically homotopic to a composition $\mathcal{C} \xrightarrow{j} \mathcal{P}(\mathcal{C}) \xrightarrow{F} \mathcal{X}$ where j is the Yoneda embedding and F preserves small colimits. Moreover, the functor F is well-defined up to a contractible space of choices (Theorem HTT.5.1.5.6). The functor f is locally left exact if and only if F is left exact.

Warning 20.4.2.3. If $\mathcal{C}$ does not admit finite limits, then locally left exact functors need not be left exact; see Remark 21.2.3.11.

Remark 20.4.2.4. Let $\mathcal{C}$ be a small ∞-category. Then the Yoneda embedding $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ preserves all limits which exist in $\mathcal{C}$ (Proposition HTT.5.1.3.2). Consequently, if $\mathcal{X}$ is an ∞-topos and $f : \mathcal{C} \to \mathcal{X}$ is a locally left exact functor, then f preserves all finite limits which exist in $\mathcal{C}$.

Remark 20.4.2.5. Let $\mathcal{X}$ be an ∞-topos, let $\mathcal{C}$ be a small ∞-category, and let $f : \mathcal{C} \to \mathcal{X}$ be a functor which is dense and locally left exact. Write f as a composition $\mathcal{C} \xrightarrow{j} \mathcal{P}(\mathcal{C}) \xrightarrow{F} \mathcal{X}$, where j is the Yoneda embedding and F preserves small colimits. Then F is left exact and admits a fully faithful right adjoint G. For each object $C \in \mathcal{C}$, the functor f induces a map

$f_C : \mathcal{C}_{/C} \to \mathcal{X}_{/fC}$, which extends to a left exact functor $F_C : \mathcal{P}(\mathcal{C})_{/jC} \simeq \mathcal{P}(\mathcal{C}_{/C}) \to \mathcal{X}_{/fC}$. The functor F_C admits a right adjoint $G_C : \mathcal{X}_{/fC} \to \mathcal{P}(\mathcal{C})_{/jC}$, given concretely by the formula $G_C(X) = G(X) \times_{G(fC)} jC$. Note that the counit map $v_C : (F_C \circ G_C)(X) \to X$ is given by the left vertical composition in the diagram

$$\begin{array}{ccc} F(G(X) \times_{G(fC)} jC) & \longrightarrow & F(jC) \\ \downarrow & & \downarrow w \\ (F \circ G)(X) & \longrightarrow & F(G(fC)) \\ \downarrow v & & \\ X. & & \end{array}$$

Since G is fully faithful, the maps v and w are equivalences. The left exactness of F guarantees that the square is a pullback, so that v_C is also an equivalence. Allowing X to vary, we deduce that G_C is fully faithful. Consequently, the functor $f_C : \mathcal{C}_{/C} \to \mathcal{X}_{/fC}$ is also dense and locally left exact.

Warning 20.4.2.6. The terminology of Definition 20.4.2.1 is not standard. Many authors use the term *flat* to refer to the property of being locally left exact (see §21.2.3 for an explanation of this terminology).

Remark 20.4.2.7. If $\mathcal{X}$ is an ∞-topos, then filtered colimits in $\mathcal{X}$ are left exact (Example HTT.7.3.4.7). For any small ∞-category $\mathcal{C}$, it follows that the full subcategory $\mathrm{Fun}^*(\mathcal{P}(\mathcal{C}), \mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{X})$ is closed under filtered colimits, where $\mathrm{Fun}^*(\mathcal{P}(\mathcal{C}), \mathcal{X})$ is spanned by those functors which are left exact and preserve small colimits. Composing with the Yoneda embedding $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$, we conclude that the collection of locally left exact functors from $\mathcal{C}$ to $\mathcal{X}$ is closed under filtered colimits.

Example 20.4.2.8. Let $\mathcal{C}$ be an ∞-category, let $C \in \mathcal{C}$ be an object, and let $h^C : \mathcal{C} \to \mathcal{S}$ denote the functor corepresented by C. Then h^C is homotopic to the composition $\mathcal{C} \xrightarrow{j} \mathcal{P}(\mathcal{C}) \xrightarrow{e} \mathcal{S}$, where $e : \mathcal{P}(\mathcal{C}) \to \mathcal{S}$ denotes the functor given by evaluation at C. Since the functor e preserves small limits and colimits, we conclude that h^C is locally left exact.

In the case where $\mathcal{X} = \mathcal{S}$ is the ∞-category of spaces, we can make Definition 20.4.2.1 more explicit.

Proposition 20.4.2.9. *Let $\mathcal{C}$ be a small ∞-category, and let $f : \mathcal{C} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

(1) *Let $\mathcal{C}' \to \mathcal{C}$ be a left fibration classified by f. Then the ∞-category $\mathcal{C}'^{\mathrm{op}}$ is filtered.*

(2) *The functor f belongs to $\mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S})$.*

(3) *The functor f is locally left exact.*

Proof. The equivalence of (1) and (2) follows immediately from the definitions (see §HTT.5.3.5). We next prove that (2) $\Rightarrow$ (3). Assume that $f \in \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S})$. According to Corollary HTT.5.3.5.4 , we can write f as a filtered colimit of corepresentable functors from $\mathcal{C}$ to $\mathcal{S}$. Using Remark 20.4.2.7 and Example 20.4.2.8, we deduce that f is locally left exact.

We complete the proof by showing that (3) $\Rightarrow$ (1). Let K be a finite simplicial set and let $q : K \to \mathcal{C}'$ be a diagram; we wish to prove that q can be extended to a map $\overline{q} : K^{\triangleleft} \to \mathcal{C}'$. Let $q_0 : K \to \mathcal{C}$ be the composition of q with the projection map $\mathcal{C}' \to \mathcal{C}$. Then q can be identified with a vertex of the Kan complex $\mathrm{Fun}_{\mathcal{C}}(K, \mathcal{C}')$, which can be viewed as a limit of the diagram $(f \circ q_0) : K \to \mathcal{S}$ (see Corollary HTT.3.3.3.3). Without loss of generality, we may assume that f is given by the composition $\mathcal{C} \xrightarrow{j} \mathcal{P}(\mathcal{C}) \xrightarrow{F} \mathcal{S}$, where j is the Yoneda embedding and F preserves small colimits. Assumption (3) implies that F preserves finite limits, so that

$$\lim(f \circ q_0) = \lim(F \circ j \circ q_0) = F \lim(j \circ q_0).$$

Write $\lim(j \circ q_0) \in \mathcal{P}(\mathcal{C})$ as a colimit of representable presheaves $\mathscr{F}_\alpha$. Then we can identify q with a point of the colimit $\varinjlim_\alpha F(\mathscr{F}_\alpha)$. It follows that there exists an object $C \in \mathcal{C}$ and a morphism $u : j(C) \to \lim(j \circ q_0)$ such that q lifts to a point $\eta \in F(j(C)) = f(C)$. We conclude by observing that the map u determines an extension $\overline{q}_0 : K^{\triangleleft} \to \mathcal{C}$ extending q_0, and the point η determines a lifting of $\overline{q}_0$ to a map $\overline{q} : K^{\triangleleft} \to \overline{\mathcal{C}}$ extending q. □

20.4.3 A Criterion for Local Left Exactness

The following result can be regarded as a generalization of Proposition HTT.6.1.5.2 :

Proposition 20.4.3.1. *Let $\mathcal{X}$ be an ∞-topos, let $\mathcal{C}$ be a small ∞-category, and let $f : \mathcal{C} \to \mathcal{X}$ be a functor. The following conditions are equivalent:*

(1) *The functor f is locally left exact.*

(2) *For every finite simplicial set K and every map $q : K \to \mathcal{C}$, the canonical map*

$$\varinjlim f|_{\mathcal{C}_{/q}} \to \varprojlim (f \circ q)$$

is an equivalence in the ∞-topos $\mathcal{X}$.

Corollary 20.4.3.2. *Let $\mathcal{C}$ be a small ∞-category which admits finite limits, and let $\mathcal{X}$ be an ∞-topos. Then a functor $f : \mathcal{C} \to \mathcal{X}$ is locally left exact if and only if it is left exact.*

Corollary 20.4.3.3. *Let $\mathcal{X}$ be an ∞-topos, let $\mathcal{C}$ be a small ∞-category, and let $f : \mathcal{C} \to \mathcal{X}$ be a functor. If f is dense and fully faithful, then it is locally left exact.*

Proof. Let K be a finite simplicial set and let $q : K \to \mathcal{C}$ be a diagram, and let $X \in \mathcal{X}$ be a limit of the diagram $f \circ q : K \to \mathcal{X}$. We have an equivalence of ∞-categories

$$\mathcal{C} \times_{\mathcal{X}} \mathcal{X}_{/X} \simeq \mathcal{C} \times_{\mathcal{X}} \mathcal{X}_{/fq} \simeq \mathcal{C}_{/q},$$

so that our assumption that f is dense guarantees that $X = \varprojlim fq$ is a colimit of $f|_{\mathcal{C}_{/q}}$. The desired result now follows from Proposition 20.4.3.1. □

Warning 20.4.3.4. Corollary 20.4.3.3 is not necessarily true if the functor f is not fully faithful. For example, if $\mathcal{C}$ is any weakly contractible ∞-category, then the constant map

$$\mathcal{C} \to \{*\} \hookrightarrow \mathcal{S}$$

is dense, but is usually not locally left exact (since colimits indexed by $\mathcal{C}^{\mathrm{op}}$ need not commute with finite limits).

Proof of Proposition 20.4.3.1. The implication (1) $\Rightarrow$ (2) follows immediately from the definitions. To prove the converse, we proceed as in the proof of Proposition HTT.6.1.5.2 . Using Theorem HTT.5.1.5.6 , we can assume without loss of generality that f is given by a composition $\mathcal{C} \xrightarrow{j} \mathcal{P}(\mathcal{C}) \xrightarrow{F} \mathcal{X}$, where j denotes the Yoneda embedding and the functor F preserves small colimits. Assume that (2) is satisfied; we wish to prove that F is left exact. Taking $K = \varnothing$ in (2), we deduce that F preserves final objects. It will therefore suffice to show that F preserves pullbacks. Let us say that an object $Z \in \mathcal{P}(\mathcal{C})$ is *good* if the following condition is satisfied:

(∗) For every pullback diagram σ :

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

in $\mathcal{P}(\mathcal{C})$, the image $F(\sigma)$ is a pullback diagram in $\mathcal{X}$.

Since $\mathcal{P}(\mathcal{C})$ is generated under small colimits by the essential image of j and colimits are universal in both $\mathcal{X}$ and $\mathcal{P}(\mathcal{C})$, (∗) is equivalent to the following apparently weaker condition:

(∗′) For every pullback diagram σ :

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

in $\mathcal{P}(\mathcal{C})$, if both X and Y belong to the essential image of j, then $F(\sigma)$ is a pullback diagram in $\mathcal{X}$.

We wish to show that every object of $\mathcal{P}(\mathcal{C})$ is good. It follows from condition (2) that for each object $C \in \mathcal{C}$, the representable functor $j(C) \in \mathcal{P}(\mathcal{C})$ satisfies $(*')$, and is therefore good. We will complete the proof by showing that the collection of good objects of $\mathcal{P}(\mathcal{C})$ is closed under small colimits. By virtue of Proposition HTT.4.4.3.3 , it will suffice to prove that the collection of good objects of $\mathcal{P}(\mathcal{C})$ is closed under the formation of coequalizers and small coproducts.

We first consider the case of coproducts. Let $\{Z_i\}_{i \in I}$ be small collection of good objects of $\mathcal{P}(\mathcal{C})$ having coproduct $Z \in \mathcal{P}(\mathcal{C})$. We claim that Z satisfies $(*')$. Suppose we are given a pullback diagram σ :

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & Z \end{array}$$

in $\mathcal{P}(\mathcal{C})$, where X and Y belong to the essential image of j. Then α and β determine maps $\alpha_0 : X \rightarrow Z_i$ and $\beta_0 : X \rightarrow Z_j$, for some uniquely determined pair of indices $i, j \in I$. If $i \neq j$, then $W \simeq \emptyset$ and $F(\sigma)$ is a pullback square by virtue of the fact that coproducts in $\mathcal{X}$ are disjoint. If $i = j$, then we can identify W with the fiber product $X \times_{Z_i} Y$, so that $F(\sigma)$ is a pullback diagram by virtue of our assumption that Z_i is good.

We now complete the proof by showing that the collection of good objects of $\mathcal{P}(\mathcal{C})$ is stable under the formation of coequalizers. Let

$$P \rightrightarrows Q \xrightarrow{s} Z$$

be a coequalizer diagram in $\mathcal{P}(\mathcal{C})$, and suppose that P and Q are good. Let σ :

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & Z \end{array}$$

be a pullback diagram, where X and Y belong to the essential image of j. Then α factors as a composition $X \xrightarrow{\overline{\alpha}} Q \xrightarrow{s} Z$. We may therefore identify σ with the outer rectangle in a commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \overline{\alpha} \\ Q \times_Z Y & \longrightarrow & Q \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z. \end{array}$$

Since Q is good, the functor F carries the upper square in this diagram to a pullback square in $\mathcal{X}$. It will therefore suffice to show that F carries the lower square to a pullback as well.

We may therefore replace X by Q and thereby reduce to the case $\alpha = s$. Similarly, we can assume that $\beta = s$. In this case, the desired result follows from Proposition HTT.6.1.4.2 . □

Remark 20.4.3.5. In the statement of Proposition 20.4.3.1, it suffices to assume that condition (2) holds in the special cases $K = \varnothing$ and $K = \Lambda^2_2$.

20.4.4 Comparison with Left Exactness

We now investigate the relationship between left exactness and local left exactness.

Proposition 20.4.4.1. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between small ∞-categories, and let $j_{\mathcal{D}} : \mathcal{D} \to \mathcal{P}(\mathcal{D})$ be the Yoneda embedding. The following conditions are equivalent:*

(1) *The functor f is left exact.*

(2) *The functor $j_{\mathcal{D}} \circ f$ is locally left exact.*

Corollary 20.4.4.2. *Let $f : \mathcal{C} \to \mathcal{D}$ be a left exact functor between small ∞-categories, let $\mathcal{X}$ be an ∞-topos, and let $g : \mathcal{D} \to \mathcal{X}$ be a locally left exact functor. Then the composition $(g \circ f) : \mathcal{C} \to \mathcal{X}$ is locally left exact.*

Proof. We have a commutative diagram

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} & & \\ \downarrow{\scriptstyle j_{\mathcal{C}}} & & \downarrow{\scriptstyle j_{\mathcal{D}}} & \searrow{\scriptstyle g} & \\ \mathcal{P}(\mathcal{C}) & \xrightarrow{F} & \mathcal{P}(\mathcal{D}) & \xrightarrow{G} & \mathcal{X} \end{array}$$

where the vertical maps denote Yoneda embeddings, and the functors F and G preserve small colimits. Since f is left exact, Proposition 20.4.4.1 implies that F is left exact. Since g is locally left exact, the functor G is left exact. It follows that $G \circ F$ is left exact, so that $g \circ f$ is locally left exact. □

Corollary 20.4.4.3. *Let $\mathcal{C}$ be a small ∞-category and let $\mathcal{X}$ be an ∞-topos. If $f : \mathcal{C} \to \mathcal{X}$ is left exact, then f is locally left exact.*

Proof. Let $\mathcal{X}_0$ be a full subcategory of $\mathcal{X}$ which contains the essential image of f. Enlarging $\mathcal{X}_0$ if necessary, we may suppose that $\mathcal{X}_0$ admits finite limits. Then f factors as a composition

$$\mathcal{C} \xrightarrow{f_0} \mathcal{X}_0 \xrightarrow{g} \mathcal{X},$$

where $f_0 = f$ and g is the inclusion functor. The assumption that f is left exact implies that f_0 is left exact, and g is locally left exact by virtue of Corollary 20.4.3.2. It follows from Corollary 20.4.4.2 that f is locally left exact. □

Warning 20.4.4.4. The converse of Corollary 20.4.4.3 is valid when $\mathcal{C}$ admits finite limits (Remark 20.4.2.2), but not in general.

Proof of Proposition 20.4.4.1. Choose a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow{\scriptstyle j_{\mathcal{C}}} & & \downarrow{\scriptstyle j_{\mathcal{D}}} \\ \mathcal{P}(\mathcal{C}) & \xrightarrow{F} & \mathcal{P}(\mathcal{D}), \end{array}$$

where F preserves small colimits. Then $j_{\mathcal{D}} \circ f$ is locally left exact if and only if F is left exact. This is equivalent to the assertion that, for every object $D \in \mathcal{D}$, the composite functor

$$\mathcal{P}(\mathcal{C}) \xrightarrow{F} \mathcal{P}(\mathcal{D}) \xrightarrow{e} \mathcal{S}$$

is left exact, where $e : \mathcal{P}(\mathcal{D}) \to \mathcal{S}$ is given by evaluation at D. Since e preserves small colimits, this is equivalent to the requirement that the functor $(e \circ j_{\mathcal{D}} \circ f) : \mathcal{C} \to \mathcal{S}$ be locally left exact. Note that $e \circ j_{\mathcal{D}}$ is the functor corepresented by D, so that the composite functor $e \circ j_{\mathcal{D}} \circ f$ is classified by the left fibration $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$. Using Proposition 20.4.2.9, we see that $j_{\mathcal{D}} \circ f$ is locally left exact if and only if the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is filtered, for each object $D \in \mathcal{D}$, which is equivalent to the assumption that f is left exact. □

20.4.5 The Hypercomplete Case

We now return to the question raised at the beginning of this section.

Proposition 20.4.5.1. *Let $\mathcal{X}$ be a hypercomplete ∞-topos and let $\mathcal{C} \subseteq \mathcal{X}$ be an essentially small full subcategory. The following conditions are equivalent:*

- (a) *For every object $X \in \mathcal{X}$, there exists an effective epimorphism $\amalg_\alpha U_\alpha \to X$ where each U_α belongs to $\mathcal{C}$.*
- (b) *The category $\mathcal{X}$ is generated under small colimits by $\mathcal{C}$.*
- (c) *The subcategory $\mathcal{C}$ is dense in $\mathcal{X}$.*

Remark 20.4.5.2. In the situation of Proposition 20.4.5.1, if $\mathcal{C} \subseteq \mathcal{X}$ satisfies any of the equivalent conditions (a) through (c), then the inclusion $\mathcal{C} \hookrightarrow \mathcal{X}$ is locally left exact (Corollary 20.4.3.3). It follows that f exhibits $\mathcal{X}$ as a left exact localization of the ∞-category $\mathcal{P}(\mathcal{C})$.

Corollary 20.4.5.3. *Let $\mathcal{X}$ be an n-topos for $0 \leqslant n < \infty$ and let $\mathcal{C} \subseteq \mathcal{X}$ be an essentially small full subcategory. The following conditions are equivalent:*

- (a) *For every object $X \in \mathcal{X}$, there exists an effective epimorphism $\amalg_\alpha U_\alpha \to X$ where each U_α belongs to $\mathcal{C}$.*

(b) *The category $\mathcal{X}$ is generated under small colimits by $\mathcal{C}$.*

(c) *The subcategory $\mathcal{C}$ is dense in $\mathcal{X}$.*

Proof. The implications $(c) \Rightarrow (b) \Rightarrow (a)$ are immediate. Choose an ∞-topos $\mathcal{Y}$ for which we can identify $\mathcal{X}$ with the full subcategory $\tau_{\leqslant n-1}\mathcal{Y}$ of $(n-1)$-truncated objects of $\mathcal{Y}$. Replacing $\mathcal{Y}$ by its hypercompletion if necessary (which does not change the class of $(n-1)$-truncated objects of $\mathcal{Y}$), we may assume that $\mathcal{Y}$ is hypercomplete. Proposition 20.4.5.3 implies that $\mathcal{C}$ is dense in $\mathcal{Y}$, so that the functor

$$G : \mathcal{Y} \to \mathcal{P}(\mathcal{C}) \qquad G(Y)(C) = \mathrm{Map}_{\mathcal{Y}}(C, Y)$$

is fully faithful. It follows that the restriction $G|_{\mathcal{X}}$ is also fully faithful, so that $\mathcal{C}$ is a dense subcategory of $\mathcal{X}$. □

We will deduce Proposition 20.4.5.1 from the following lemma:

Lemma 20.4.5.4. *Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{C} \subseteq \mathcal{X}$ be an essentially small full subcategory which satisfies condition (a) of Proposition 20.4.5.1. Then the colimit $\varinjlim_{U \in \mathcal{C}} U$ is an ∞-connective object of $\mathcal{X}$.*

Proof. We will prove that the colimit $X = \varinjlim_{U \in \mathcal{C}} U$ is n-connective for each $n \geqslant 0$. We proceed by induction on n. We first note that X is 0-connective (since the final object $\mathbf{1} \in \mathcal{X}$ admits an effective epimorphism $\amalg_{U \in \mathcal{C}} U \to \mathbf{1}$); this completes the proof in the case $n = 0$. To handle the inductive step, it will suffice to show that the diagonal map $\delta : X \to X \times X$ is $(n-1)$-connective. Since colimits in $\mathcal{X}$ distribute over finite products, we can write the product $X \times X$ as a colimit $\varinjlim_{(V,W) \in \mathcal{C} \times \mathcal{C}} V \times W$. Rewriting X as an iterated colimit $\varinjlim_{(V,W) \in \mathcal{C} \times \mathcal{C}} \varinjlim_{U \to V \times W} U$, we can obtain δ as a colimit of maps $\delta_{V,W} : \varinjlim_{U \in \mathcal{C} \times_{\mathcal{X}} \mathcal{X}_{/V \times W}} U \to V \times W$. It will therefore suffice to show that each $\delta_{V,W}$ is $(n-1)$-connective. This follows from the inductive hypothesis, applied to the ∞-topos $\mathcal{X}_{/V \times W}$. □

Proof of Proposition 20.4.5.1. The implications $(c) \Rightarrow (b) \Rightarrow (a)$ are trivial; we will show that $(a) \Rightarrow (c)$. To establish (c), we must show that for every object $X \in \mathcal{X}$, the canonical map $\eta : \varinjlim_{U \in \mathcal{C} \times_{\mathcal{X}} \mathcal{X}_{/X}} U \to X$ is an equivalence. Since $\mathcal{X}$ is hypercomplete, it will suffice to show that η is ∞-connective. This follows from Lemma 20.4.5.4, applied to the ∞-topos $\mathcal{X}_{/X}$. □

20.5 Presentations of Fractured ∞-Topoi

Recall that an ∞-category $\mathcal{X}$ is an ∞-topos if and only if $\mathcal{X}$ can obtained as an accessible left exact localization of a presheaf ∞-category $\mathcal{P}(\mathcal{G})$, where $\mathcal{G}$ is a small ∞-category (Theorem HTT.6.1.0.6). It is convenient to break this assertion down into three parts:

(a) For any small ∞-category $\mathcal{G}$, the presheaf ∞-category $\mathcal{P}(\mathcal{G}) = \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$ is an ∞-topos.

(b) If $\mathcal{Y}$ is an ∞-topos and $\mathcal{X} \subseteq \mathcal{Y}$ is an accessible left exact localization of $\mathcal{Y}$, then $\mathcal{X}$ is also an ∞-topos.

(c) Every ∞-topos $\mathcal{X}$ can be obtained by combining (a) and (b): more precisely, there is an equivalence from $\mathcal{X}$ to an accessible left exact localization of $\mathcal{P}(\mathcal{G})$, for some small ∞-category $\mathcal{G}$.

In §20.2, we established an analogue of (a) in the setting of fractured ∞-topoi. More precisely, we proved that for any small ∞-category $\mathcal{G}$ equipped with an admissibility structure, the presheaf ∞-category $\mathcal{P}(\mathcal{G})$ can be regarded as a fractured ∞-topos (Theorem 20.2.4.1). Our goal in this section is to establish "fractured" analogues of (b) and (c). Given a fractured ∞-topos $\mathcal{Y}^{\mathrm{corp}} \subseteq \mathcal{Y}$ we introduce the notion of a *fractured localization* $\mathcal{X}_0 \subseteq \mathcal{Y}^{\mathrm{corp}}$ (Definition **??**). Our first main result is that any fractured localization $\mathcal{X}_0 \subseteq \mathcal{Y}^{\mathrm{corp}}$ determines an accessible left exact localization $\mathcal{X} \subseteq \mathcal{Y}$ and a fracture subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ which is equivalent to $\mathcal{X}_0$ (Theorem 20.5.1.2). We then show that every fractured ∞-topos $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ can be obtained by applying this construction to a fractured ∞-topos of the form $\mathcal{P}(\mathcal{G})$, where $\mathcal{G}$ is a small ∞-category equipped with an admissibility structure (Theorem **??**).

20.5.1 Fractured Localizations

We begin by introducing some terminology.

Definition 20.5.1.1. Let $\mathcal{Y}^{\mathrm{corp}} \subseteq \mathcal{Y}$ be a fractured ∞-topos. We say that a full subcategory $\mathcal{X}_0 \subseteq \mathcal{Y}^{\mathrm{corp}}$ is a *fractured localization* of $\mathcal{Y}^{\mathrm{corp}}$ if the following conditions are satisfied:

(a) The inclusion $\mathcal{X}_0 \hookrightarrow \mathcal{Y}^{\mathrm{corp}}$ admits a left exact left adjoint $L_0 : \mathcal{Y}^{\mathrm{corp}} \to \mathcal{X}_0$. Moreover, L_0 is accessible when regarded as a functor from $\mathcal{Y}^{\mathrm{corp}}$ to itself (so that $\mathcal{X}_0$ is an ∞-topos: see Theorem HTT.6.1.0.6).

(b) Let $j^* : \mathcal{Y} \to \mathcal{Y}^{\mathrm{corp}}$ be a right adjoint to the inclusion, and form a pullback diagram of ∞-categories σ :

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow{\scriptstyle \overline{j}^*} & & \downarrow{\scriptstyle j^*} \\ \mathcal{X}_0 & \longrightarrow & \mathcal{Y}^{\mathrm{corp}} . \end{array}$$

Then the diagram σ is left adjointable. In other words, for each object $Y \in \mathcal{Y}$, the canonical map $L_0 j^* Y \to \overline{j}^* LY$ is an equivalence, where $L : \mathcal{Y} \to \mathcal{X}$ denotes a left adjoint to the inclusion (which exists by virtue of Theorem HTT.5.5.3.18).

We can now formulate our main result:

Theorem 20.5.1.2. *Let $\mathcal{Y}^{\mathrm{corp}} \subseteq \mathcal{Y}$ be a fractured ∞-topos, let $\mathcal{X}_0 \subseteq \mathcal{Y}^{\mathrm{corp}}$ be a fractured localization of $\mathcal{Y}$, and form a pullback diagram of ∞-categories*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow{\scriptstyle \overline{j}^*} & & \downarrow{\scriptstyle j^*} \\ \mathcal{X}_0 & \longrightarrow & \mathcal{Y}^{\mathrm{corp}}. \end{array}$$

Then:

(1) *The inclusion functor $\mathcal{X} \hookrightarrow \mathcal{Y}$ admits an (accessible) left exact left adjoint $L : \mathcal{Y} \to \mathcal{X}$. In particular, the ∞-category $\mathcal{X}$ is an ∞-topos.*

(2) *The functor $\overline{j}^*$ admits a left adjoint $\overline{j}_! : \mathcal{X}_0 \hookrightarrow \mathcal{X}$.*

(3) *The functor $\overline{j}_!$ induces an equivalence from $\mathcal{X}_0$ to a fracture subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$.*

Warning 20.5.1.3. In the statement of Theorem 20.5.1.2, the ∞-categories $\mathcal{X}_0$ and $\mathcal{X}^{\mathrm{corp}}$ do not coincide as subcategories of $\mathcal{Y}$ (though they are canonically equivalent to one another via the functor $\overline{j}_!$).

The remainder of this section is devoted to the proof of Theorem 20.5.1.2. We begin by observing that Theorem HTT.5.5.3.18 guarantees that the ∞-category $\mathcal{X}$ is presentable, that the inclusion $\mathcal{X} \hookrightarrow \mathcal{Y}$ admits a left adjoint $L : \mathcal{Y} \to \mathcal{X}$, and that the functor $\overline{j}^*$ admits a left adjoint $\overline{j}_!$. Our assumption that $\mathcal{X}_0 \subseteq \mathcal{Y}^{\mathrm{corp}}$ is a fractured localization guarantees that the diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{L} & \mathcal{X} \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle \overline{j}^*} \\ \mathcal{Y}^{\mathrm{corp}} & \xrightarrow{L_0} & \mathcal{X}_0 \end{array}$$

commutes up to canonical homotopy, where $L_0 : \mathcal{Y}^{\mathrm{corp}} \to \mathcal{X}_0$ is left adjoint to the inclusion. Since L_0 and j^* are left exact, the composite functor $\overline{j}^* \circ L$ is left exact. The functor $\overline{j}^*$ preserves small limits (since it admits a left adjoint) and is conservative (since j^* is conservative), so the functor L is left exact. This completes the proof of assertions (1) and (2) of Theorem 20.5.1.2.

The proof of (3) will require some preliminary observations.

Lemma 20.5.1.4. *The functor $\overline{j}_! : \mathcal{X}_0 \to \mathcal{X}$ preserves pullbacks.*

Proof. The commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow{\scriptstyle \overline{j}^*} & & \downarrow{\scriptstyle j^*} \\ \mathcal{X}_0 & \longrightarrow & \mathcal{Y} \end{array}$$

yields a commutative diagram of left adjoints

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{L} & \mathcal{Y} \\ \uparrow{\scriptstyle \overline{j}_!} & & \uparrow{\scriptstyle j_!} \\ \mathcal{X}_0 & \xleftarrow{L_0} & \mathcal{Y}^{\mathrm{corp}}, \end{array}$$

where $j_!$ denotes the inclusion functor. In other words, we can identify $\overline{j}_!$ with the restriction $(L \circ j_!)|_{\mathcal{X}_0}$. Since L is left exact and $j_!$ preserves pullbacks, it follows that $\overline{j}_!$ preserves pullbacks. □

Lemma 20.5.1.5. *For each object $X \in \mathcal{X}_0$, the unit map $\overline{u}_X : X \to \overline{j}^* \overline{j}_! X$ is (-1)-truncated.*

Proof. Unwinding the definitions, we see that $\overline{u}_X$ can be obtained by applying the functor L_0 to the morphism $u_X : X \to j^* j_! X$ of $\mathcal{Y}^{\mathrm{corp}}$. Since the morphism u_X is (-1)-truncated (Proposition 20.1.3.1) and L_0 is left exact, we conclude that $\overline{u}_X$ is (-1)-truncated. □

Notation 20.5.1.6. Let U and V be objects of $\mathcal{X}_0$. It follows from Lemma 20.5.1.5 that the canonical map

$$\mathrm{Map}_{\mathcal{X}_0}(U, V) \to \mathrm{Map}_{\mathcal{X}_0}(U, \overline{j}^* \overline{j}_! V) \simeq \mathrm{Map}_{\mathcal{X}}(\overline{j}_! U, \overline{j}_! V)$$

is (-1)-truncated. We will say that a morphism $f : \overline{j}_! U \to \overline{j}_! V$ is *admissible* if it belongs to the essential image of θ: that is, if f is homotopic to $\overline{j}_!(f_0)$, for some morphism $f_0 : U \to V$.

Lemma 20.5.1.7. *Suppose we are given objects $X_0, X_1, X_2, X_3 \in \mathcal{X}_0$ and morphisms $\overline{j}_! X_0 \xrightarrow{f} \overline{j}_! X_1 \xrightarrow{g} \overline{j}_! X_2 \xrightarrow{h} \overline{j}_! X_3$ in the ∞-category $\mathcal{X}$. If $g \circ f$ and $h \circ g$ are admissible, then f is admissible.*

Proof. Since the functor L_0 is essentially surjective, we can assume without loss of generality that $X_i = L_0 Y_i$ for $0 \leqslant i \leqslant 3$, for some objects $Y_i \in \mathcal{Y}^{\mathrm{corp}}$. We now proceed in several steps:

- Regard h as a morphism from $\overline{j}_! L_0(Y_2) \simeq LY_2$ to $\overline{j}_! L_0(Y_3) \simeq LY_3$. Replacing Y_2 by the pullback $Y_2 \times_{\overline{j}^* LY_3} j^* Y_3$, we can reduce to the case where h is homotopic to $L(h')$, for some map $h' : Y_2 \to Y_3$ in the ∞-category $\mathcal{Y}$.

- Regard g as a morphism from $\overline{j}_! L_0(Y_1) \simeq LY_1$ to $\overline{j}_! L_0(Y_2) \simeq LY_2$. Replacing Y_1 by the pullback $Y_1 \times_{\overline{j}^* LY_2} j^* Y_2$, we can reduce to the case where g is homotopic to $L(g')$, for some map $g' : Y_1 \to Y_2$ in the ∞-category $\mathcal{Y}$.

- Replacing Y_1 by the fiber product $Y_1 \times_{j^* Y_3} Y_3$ (which does not change the image of Y_1 under the functor L_0, by virtue of our assumption that $h \circ g$ is admissible), we can arrange that the composition $h' \circ g'$ is a morphism of $\mathcal{Y}^{\mathrm{corp}}$.

- Regard f as a morphism from $\overline{j}_! L_0(Y_0) \simeq LY_0$ to $\overline{j}_! L_0(Y_1) \simeq LY_1$. Replacing Y_0 by the pullback $Y_0 \times_{\overline{j}^* LY_1} j^* Y_1$, we can reduce to the case where f is homotopic to $L(f')$, for some map $f' : Y_0 \to Y_1$ in the ∞-category $\mathcal{Y}$.

- Replacing Y_0 by the fiber product $Y_0 \times_{j^* Y_2} Y_2$ (which does not change the image of Y_0 under the functor L_0, by virtue of our assumption that $g \circ f$ is admissible), we can arrange that the composition $g' \circ f'$ is a morphism of $\mathcal{Y}^{\mathrm{corp}}$.

Let us regard the ∞-category $\mathcal{Y}$ as equipped with the admissibility structure of Proposition 20.3.1.3, so that $g' \circ f'$ and $h' \circ g'$ are admissible morphisms in $\mathcal{Y}$. Applying Corollary 20.2.2.9, we deduce that f' is also admissible: that is, it is contained in $\mathcal{Y}^{\mathrm{corp}}$ (Example 20.3.1.2). It follows that $f \simeq L(f') \simeq j_!(L_0(f'))$ is an admissible morphism in $\mathcal{X}$, in the sense of Notation 20.5.1.6. □

Lemma 20.5.1.8. *Let U and V be objects of $\mathcal{X}_0$. Then every equivalence $f : \overline{j}_! U \simeq \overline{j}_! V$ is admissible.*

Proof. Let f^{-1} be a homotopy inverse to f, and apply Lemma 20.5.1.7 to the diagram

$$\overline{j}_! U \xrightarrow{f} \overline{j}_! V \xrightarrow{f^{-1}} \overline{j}_! U \xrightarrow{f} \overline{j}_! V.$$

□

Lemma 20.5.1.9. *The functor $\overline{j}_! : \mathcal{X}_0 \to \mathcal{X}$ induces an equivalence from $\mathcal{X}_0$ to a replete subcategory $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$.*

Proof. Combine Lemma 20.5.1.5, Lemma 20.5.1.8, and Proposition 20.1.1.5. □

Proof of Theorem 20.5.1.2. Let $\mathcal{X}^{\mathrm{corp}}$ be as in Lemma 20.5.1.9; we wish to show that $\mathcal{X}^{\mathrm{corp}}$ is a fracture subcategory of $\mathcal{X}$. To prove this, it suffices to verify that the functor $\overline{j}_! : \mathcal{X}_0 \to \mathcal{X}$ satisfies analogues of conditions (1), (2), and (3) of Definition 20.1.2.1:

(1) The functor $\overline{j}_!$ preserves pullbacks: this follows from Lemma 20.5.1.4.

(2) The functor $\overline{j}_!$ admits a right adjoint $\overline{j}^*$, which is conservative and preserves small limits. The existence of $\overline{j}^*$ follows from our construction. The statement that $\overline{j}^*$ is conservative follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow{\scriptstyle \overline{j}^*} & & \downarrow{\scriptstyle j^*} \\ \mathcal{X}_0 & \longrightarrow & \mathcal{Y}^{\mathrm{corp}}, \end{array}$$

since the functor j^* is conservative. To show that $\overline{j}^*$ preserves small colimits, we use the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{L} & \mathcal{X} \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle \overline{j}^*} \\ \mathcal{Y}^{\mathrm{corp}} & \xrightarrow{L_0} & \mathcal{X}_0 \end{array}$$

together with the fact that j^* preserves small colimits.

(3) Let $f : U \to V$ be a morphism in $\mathcal{X}_0$; we wish to show that the induced diagram σ :

$$\begin{array}{ccc} \overline{j}_!\overline{j}^*\overline{j}_!U & \longrightarrow & \overline{j}_!\overline{j}^*\overline{j}_!V \\ \downarrow & & \downarrow \\ \overline{j}_!U & \longrightarrow & \overline{j}_!V \end{array}$$

is a pullback square in $\mathcal{X}$. Let us identify U and V with their images in $\mathcal{Y}^{\mathrm{corp}}$, so that σ is obtained by applying the functor L to the diagram σ_0

$$\begin{array}{ccc} j_!j^*j_!U & \longrightarrow & j_!j^*j_!V \\ \downarrow & & \downarrow \\ j_!U & \longrightarrow & j_!V. \end{array}$$

The diagram σ_0 is a pullback square in $\mathcal{Y}$ by virtue of our assumption that $\mathcal{Y}^{\mathrm{corp}} \subseteq \mathcal{Y}$ is a fracture subcategory. Since the functor $L : \mathcal{Y} \to \mathcal{X}$ is left exact, it follows that $\sigma \simeq L\sigma_0$ is a pullback square in $\mathcal{X}$.

□

20.5.2 Recognition of Fractured Localizations

Let $\mathcal{Y}^{\mathrm{corp}} \subseteq \mathcal{Y}$ be a fractured ∞-topos, let $\mathcal{X}_0 \subseteq \mathcal{Y}^{\mathrm{corp}}$ be a fractured localization, and define $\mathcal{X} = \mathcal{X}_0 \times_{\mathcal{Y}^{\mathrm{corp}}} \mathcal{Y}$ and $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ as in Theorem 20.5.1.2. We then have a commutative diagram of ∞-categories σ :

$$\begin{array}{ccc} \mathcal{Y}^{\mathrm{corp}} & \longrightarrow & \mathcal{Y} \\ \downarrow{\scriptstyle L^{\mathrm{corp}}} & & \downarrow{\scriptstyle L} \\ \mathcal{X}^{\mathrm{corp}} & \longrightarrow & \mathcal{X}, \end{array}$$

where the horizontal maps are inclusions, the functor L is left adjoint to the inclusion $\mathcal{X} \hookrightarrow \mathcal{Y}$, and L_0 is obtained by composing a left adjoint to the inclusion $\mathcal{X}_0 \hookrightarrow \mathcal{Y}^{\mathrm{corp}}$ with the equivalence $\mathcal{X}_0 \simeq \mathcal{X}^{\mathrm{corp}}$. Moreover, the condition that $\mathcal{X}_0 \subseteq \mathcal{X}$ is a fractured localization guarantees that the diagram σ is right adjointable. We now establish a converse to this assertion:

Proposition 20.5.2.1. *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ and $\mathcal{Y}^{\mathrm{corp}} \subseteq \mathcal{Y}$ be fractured ∞-topoi. Suppose we are given a commutative diagram of ∞-categories σ :*

$$\begin{array}{ccc} \mathcal{Y}^{\mathrm{corp}} & \longrightarrow & \mathcal{Y} \\ \downarrow{\scriptstyle L^{\mathrm{corp}}} & & \downarrow{\scriptstyle L} \\ \mathcal{X}^{\mathrm{corp}} & \longrightarrow & \mathcal{X}, \end{array}$$

where σ is right adjointable and the functor L^{corp} exhibits $\mathcal{X}^{\mathrm{corp}}$ as a left exact localization of $\mathcal{Y}^{\mathrm{corp}}$ (that is, the functor L^{corp} is left exact and has a fully faithful right adjoint). Then:

(1) *The functor L exhibits $\mathcal{X}$ as a left exact localization of $\mathcal{Y}$ (that is, the functor L is left exact and has a fully faithful right adjoint).*

(2) *Let $U^{\mathrm{corp}} : \mathcal{X}^{\mathrm{corp}} \to \mathcal{Y}^{\mathrm{corp}}$ be a right adjoint to L^{corp} and let $\mathcal{X}_0 \subseteq \mathcal{Y}$ be the essential image of U^{corp}. Then $\mathcal{X}_0$ is a fractured localization of $\mathcal{Y}^{\mathrm{corp}}$.*

(3) *Taking right adjoints of the functors appearing in the diagram σ yields a pullback diagram of ∞-categories σ^R:*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{U} & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X}^{\mathrm{corp}} & \xrightarrow{U^{\mathrm{corp}}} & \mathcal{Y}^{\mathrm{corp}}. \end{array}$$

In other words, $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ is canonically equivalent to the fractured ∞-topos obtained by applying Theorem 20.5.1.2 to the fractured localization $\mathcal{X}_0 \subseteq \mathcal{Y}^{\mathrm{corp}}$.

Remark 20.5.2.2. In the statement of Proposition 20.5.2.1, we do not assume *a priori* that the functor L is left exact (the left exactness of L is part of the conclusion, not the hypothesis).

Proof of Proposition 20.5.2.1. Let $j^* : \mathcal{Y} \to \mathcal{Y}^{\mathrm{corp}}$ and $j'^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ denote right adjoints to the inclusion functors. Our assumption that σ is right adjointable supplies a commutative diagram of ∞-categories σ':

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{j^*} & \mathcal{Y}^{\mathrm{corp}} \\ \downarrow L & & \downarrow L^{\mathrm{corp}} \\ \mathcal{X} & \xrightarrow{j'^*} & \mathcal{X}^{\mathrm{corp}} . \end{array}$$

The functor L^{corp} admits a right adjoint, and therefore preserves small colimits. Since j^* preserves small colimits, we conclude that $L^{\mathrm{corp}} \circ j^* \simeq j'^* \circ L$ preserves small colimits. The functor j'^* is conservative and preserves small colimits, so L preserves small colimits. Consequently, the diagram σ^R appearing in statement (3) is well-defined.

Let $U : \mathcal{X} \to \mathcal{Y}$ denote a right adjoint to L. We next claim that U is fully faithful. To prove this, it will suffice to show that for each object $X \in \mathcal{X}$, the counit map $v_X : (L \circ U)(X) \to X$ is an equivalence. Since the functor j'^* is conservative, it will suffice to show that $j'^*(v_X)$ is an equivalence in the ∞-category $\mathcal{X}^{\mathrm{corp}}$. Using the commutativity of the diagrams σ^R and σ', we can identify $j'^*(v_X)$ with the counit map $(L^{\mathrm{corp}} \circ U^{\mathrm{corp}})(j'^* X) \to j'^* X$, which is an equivalence by virtue of our assumption that U^{corp} is fully faithful.

We now prove (3). Since U and U^{corp} are fully faithful, it will suffice to show that if $Y \in \mathcal{Y}$ has the property that $j^* Y$ belongs to the essential image of U^{corp}, then Y belongs to the essential image of U. To show that Y belongs to the essential image of U, it will suffice to show that the unit map $u_Y : Y \to (U \circ L)(Y)$ is an equivalence in $\mathcal{Y}$. Because the functor j^* is conservative, we are reduced to showing that $j^*(u_Y)$ is an equivalence in $\mathcal{Y}^{\mathrm{corp}}$. Using the commutativity of the diagrams σ^R and σ', we can identify $j^*(u_Y)$ with the unit map $j^* Y \to (U^{\mathrm{corp}} \circ L^{\mathrm{corp}}(j^* Y)$ in the ∞-category $\mathcal{Y}^{\mathrm{corp}}$. This map is an equivalence, since $j^* Y$ belongs to the essential image of the functor U^{corp} by virtue of our assumption that $Y \in \mathcal{X}'$.

Using (3), we see that (2) is equivalent to the left adjointability of the diagram σ^R. This follows from our assumption that σ is right adjointable. Assertion (1) now follows from Theorem 20.5.1.2. □

20.5.3 Presentations of Fractured ∞-Topoi

We now show that every fractured ∞-topos can be obtained by combining the constructions of Theorems **??** and 20.5.1.2.

Definition 20.5.3.1. Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos. A *presentation of* $\mathcal{X}$ is a functor $h : \mathcal{G} \to \mathcal{X}$, where $\mathcal{G}$ is an essentially small ∞-category equipped with an admissibility structure and h satisfies the following conditions:

(1) The functor h carries $\mathcal{G}^{\mathrm{ad}}$ into $\mathcal{X}^{\mathrm{corp}}$ (in particular, h carries each object of $\mathcal{G}$ to a corporeal object of $\mathcal{X}$.

Let $F : \mathcal{P}(\mathcal{G}) \to \mathcal{X}$ be the colimit-preserving functor determined by h and let $\mathcal{P}(\mathcal{G})^{\mathrm{corp}} \subseteq \mathcal{P}(\mathcal{G})$ be the fracture subcategory of Theorem **??**. It follows from (a) that the functor F carries $\mathcal{P}(\mathcal{G})^{\mathrm{corp}}$ into $\mathcal{X}^{\mathrm{corp}}$, so we obtain a commutative diagram σ :

$$\begin{array}{ccc} \mathcal{P}(\mathcal{G})^{\mathrm{corp}} & \longrightarrow & \mathcal{P}(\mathcal{G}) \\ \downarrow{\scriptstyle F^{\mathrm{corp}}} & & \downarrow{\scriptstyle F} \\ \mathcal{X}^{\mathrm{corp}} & \longrightarrow & \mathcal{X} . \end{array}$$

(2) The diagram σ satisfies the hypotheses of Proposition 20.5.2.1. That is, the diagram σ is right adjointable, and the functor F^{corp} exhibits $\mathcal{X}^{\mathrm{corp}}$ as a left exact localization of $\mathcal{P}(\mathcal{G})^{\mathrm{corp}}$.

Remark 20.5.3.2. Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos and let $h : \mathcal{G} \to \mathcal{X}$ be a presentation of $\mathcal{X}$. Then the functor h is dense and locally left exact: this follows from Proposition 20.5.2.1.

Remark 20.5.3.3. Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos and let $h : \mathcal{G} \to \mathcal{X}$ be a presentation of $\mathcal{X}$. Applying Proposition 20.5.2.1, we conclude that the underlying map $h|_{\mathcal{G}^{\mathrm{ad}}} : \mathcal{G}^{\mathrm{ad}} \to \mathcal{X}^{\mathrm{corp}}$ induces a fully faithful embedding $\mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{P}(\mathcal{G}^{\mathrm{ad}}) \simeq \mathcal{P}(\mathcal{G})^{\mathrm{corp}}$, whose essential image is a fractured localization in the sense of Definition 20.5.1.1. Moreover, the fractured ∞-topos $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ can then be recovered (up to canonical equivalence) by applying the construction of Theorem 20.5.1.2 to this fractured localization.

We can now formulate our main result:

Theorem 20.5.3.4. *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos. Then $\mathcal{X}$ admits a presentation $h : \mathcal{G} \to \mathcal{X}$, where $\mathcal{G}$ is an essentially small ∞-category equipped with an admissibility structure. Moreover, we can arrange that h is fully faithful, and that a morphism in $\mathcal{G}$ is admissible if and only if its image in $\mathcal{X}$ is admissible.*

Before giving the proof of Theorem 20.5.3.4, let us spell out Definition 20.5.3.1 in more detail.

Lemma 20.5.3.5. *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos, let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$, and let $h : \mathcal{G} \to \mathcal{X}$ be a functor. Then h is presentation of $\mathcal{X}$ if and only if it satisfies the following conditions:*

(a) *The functor h carries $\mathcal{G}^{\mathrm{ad}}$ into $\mathcal{X}^{\mathrm{corp}}$.*

(b) *The induced functor $h^{\mathrm{ad}} : \mathcal{G}^{\mathrm{ad}} \to \mathcal{X}^{\mathrm{corp}}$ is locally left exact (Definition 20.4.2.1).*

(c) *The functor $h^{\mathrm{ad}} : \mathcal{G}^{\mathrm{ad}} \to \mathcal{X}^{\mathrm{corp}}$ is dense (Definition 20.4.1.1).*

(d) *For every object $X \in \mathcal{G}$, the canonical map $\varinjlim_{Y \in (\mathcal{G}_{/X})^{\mathrm{ad}}} f(Y) \to f(X)$ in the ∞-topos $\mathcal{X}$ induces an equivalence $\varinjlim_{Y \in (\mathcal{G}_{/X})^{\mathrm{ad}}} f(Y) \to j^* f(X)$, where $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ is right adjoint to the inclusion functor.*

Proof. Condition (a) is clearly necessary, so assume that it is satisfied. Let $H : \mathcal{P}(\mathcal{G}) \to \mathcal{X}$ be the colimit preserving functor determined by h and define $H^{\mathrm{ad}} : \mathcal{P}(\mathcal{G}^{\mathrm{ad}}) \to \mathcal{X}^{\mathrm{corp}}$ similarly. Then the functor $H^{\mathrm{corp}} : \mathcal{P}(\mathcal{G})^{\mathrm{corp}} \to \mathcal{X}^{\mathrm{corp}}$ can be identified with the composition of H^{ad} with a homotopy inverse to the equivalence $\mathcal{P}(\mathcal{G}^{\mathrm{ad}}) \simeq \mathcal{P}(\mathcal{G})^{\mathrm{corp}}$ given by left Kan extension along the inclusion $\mathcal{G}^{\mathrm{ad}}$. Consequently, the functor H^{corp} is left exact if and only if condition (b) is satisfied, and H^{corp} has a fully faithful right adjoint if and only if condition (c) is satisfied. To complete the proof, it will suffice to show that (d) is equivalent to the right adjointability of the diagram σ :

$$\begin{array}{ccc} \mathcal{P}(\mathcal{G}^{\mathrm{ad}}) & \longrightarrow & \mathcal{P}(\mathcal{G}) \\ \downarrow{\scriptstyle F^{\mathrm{ad}}} & & \downarrow{\scriptstyle F} \\ \mathcal{X}^{\mathrm{corp}} & \longrightarrow & \mathcal{X}. \end{array}$$

Let $U : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{G}^{\mathrm{ad}})$ denote the restriction functor and let $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ denote a right adjoint to the inclusion, so that σ determines a natural transformation of functors $\alpha : F^{\mathrm{ad}} \circ U \to j^* \circ F$. Note that the domain and codomain of α are colimit-preserving functors, so that α is an equivalence if and only if it induces an equivalence after evaluation on the representable functor $\mathrm{Map}_{\mathcal{G}}(\bullet, X)$ for each $X \in \mathcal{G}$. The equivalence of this assertion with (d) now follows by unwinding the definitions. □

Proof of Theorem 20.5.3.4. Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a fractured ∞-topos. Then $\mathcal{X}^{\mathrm{corp}}$ is a presentable ∞-category, so we can choose an essentially small dense subcategory $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{X}^{\mathrm{corp}}$. Enlarging $\mathcal{G}^{\mathrm{ad}}$ if necessary, we may assume that it has the following closure property

(∗) For every morphism $f : U \to X$ in $\mathcal{G}^{\mathrm{ad}}$ and every morphism $g : X' \to X$, where X' belongs to $\mathcal{G}^{\mathrm{ad}}$, the fiber product $X' \times_X U$ belongs to $\mathcal{G}^{\mathrm{ad}}$.

Let $\mathcal{G}$ be the full subcategory of $\mathcal{X}$ having the same objects as $\mathcal{G}^{\mathrm{ad}}$. Using (∗), we deduce that $\mathcal{G}^{\mathrm{ad}}$ is an admissibility structure on $\mathcal{G}$ (see Proposition 20.3.1.3). We will complete the proof by showing that the inclusion map $\iota : \mathcal{G} \to \mathcal{X}$ is a presentation of $\mathcal{X}$. To prove this, it suffices to verify conditions (a), (b), (c), and (d) of Lemma 20.5.3.5. Conditions (a) and (c) are automatic, and (b) follows from Corollary 20.4.3.3. To verify (d), choose an object $X \in \mathcal{G}$. Let $j^* : \mathcal{X} \to \mathcal{X}^{\mathrm{corp}}$ be a right adjoint to the inclusion, so that we have a canonical equivalence $(\mathcal{G}_{/X})^{\mathrm{ad}} \simeq \mathcal{G}^{\mathrm{ad}} \times_{\mathcal{X}^{\mathrm{corp}}} \mathcal{X}^{\mathrm{corp}}_{/j^*X}$. We are therefore reduced to showing that the canonical map $\varinjlim_{Y \in \mathcal{G}^{\mathrm{ad}} \times_{\mathcal{X}^{\mathrm{corp}}} \mathcal{X}^{\mathrm{corp}}_{/j^*X}} Y \to j^*X$ is an equivalence, which follows from our assumption that $\mathcal{G}^{\mathrm{ad}}$ is dense in $\mathcal{X}^{\mathrm{corp}}$. □

Remark 20.5.3.6. The proof of Theorem 20.5.3.4 shows that if $\mathcal{G}$ is a full subcategory of $\mathcal{X}$ spanned by corporeal objects such that $\mathcal{G}^{\mathrm{ad}} = \mathcal{X}^{\mathrm{corp}} \cap \mathcal{G}$ is dense in $\mathcal{X}^{\mathrm{corp}}$ and satisfies condition $(*)$, then $\mathcal{G}$ is dense in $\mathcal{X}$.

Remark 20.5.3.7. In the proof of Theorem 20.5.3.4, the subcategory $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{X}^{\mathrm{corp}}$ can be chosen as large as we like: in particular, we can arrange that it contains any small collection of corporeal objects of $\mathcal{X}$.

20.5.4 Finite Limits of Corporeal Objects

In the situation of Theorem 20.5.3.4, we can arrange that the properties of the ∞-category $\mathcal{G}$ mirror the properties of the fractured ∞-topos $\mathcal{X}$. For example, we have the following:

Proposition 20.5.4.1. *Let $\mathcal{X}^{\mathrm{corp}} \subseteq \mathcal{X}$ be a complete fractured ∞-topos (Definition 20.3.3.9). The following conditions are equivalent:*

(1) *Let $\mathcal{X}_{\mathrm{corp}}$ be the full subcategory of $\mathcal{X}$ spanned by the corporeal objects (that is, the essential image of the inclusion $\mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$). Then $\mathcal{X}_{\mathrm{corp}}$ is closed under finite limits.*

(2) *There exists a presentation $h : \mathcal{G} \to \mathcal{X}$, where $\mathcal{G}$ is an essentially small ∞-category which admits finite limits, the functor h is fully faithful, and a morphism in $\mathcal{G}$ is admissible if and only if its image belongs to $\mathcal{X}^{\mathrm{corp}}$.*

(3) *There exists a presentation $h : \mathcal{G} \to \mathcal{X}$, where the ∞-category $\mathcal{G}$ admits finite limits.*

Proof. The implication (1) ⇒ (2) follows from the proof of Theorem 20.5.3.4 (where we replace condition $(*)$ by the requirement that $\mathcal{G}$ is closed under finite limits), and the implication (2) ⇒ (3) is trivial. We will show that (3) ⇒ (1). Suppose that $h : \mathcal{G} \to \mathcal{X}$ is a presentation and that $\mathcal{G}$ admits finite limits. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the essential image of h. Then $\mathcal{X}_0 \subseteq \mathcal{X}$ is dense (Remark 20.4.1.11) and consists of corporeal objects of $\mathcal{X}$. Note that the functor h is left exact (Lemma 20.5.3.5 and Remark 20.4.2.2), and therefore preserves final objects. In particular, the final object of $\mathcal{X}$ is corporeal. We will complete the proof by showing that the full subcategory $\mathcal{X}_{\mathrm{corp}} \subseteq \mathcal{X}$ is closed under fiber products.

Let us say that a morphism in $\mathcal{X}$ is *admissible* if it is $\mathcal{X}^{\mathrm{corp}}$-admissible (Definition 20.3.1.1). Our assumption that $\mathcal{X}$ is complete guarantees that an object $X \in \mathcal{X}$ is corporeal if and only if it is $\mathcal{X}^{\mathrm{ad}}$-corporeal, where $\mathcal{X}^{\mathrm{ad}} \subseteq \mathcal{X}$ is the subcategory spanned by the admissible morphisms (Proposition 20.3.3.11). Suppose we are given a pullback diagram σ :

$$\begin{array}{ccc} X_{01} & \longrightarrow & X_0 \\ \downarrow & & \downarrow v \\ X_1 & \xrightarrow{w} & X \end{array}$$

in the ∞-topos $\mathcal{X}$, where X_0, X_1, and X are corporeal; we wish to prove that X_{01} is corporeal. Using Proposition 20.3.3.12, we deduce that there exists an admissible effective epimorphism $\amalg U_\alpha \to X$ in $\mathcal{X}$, where each U_α is a retract of an object V_α which admits an admissible morphism $V_\alpha \to W_\alpha$ for some W_α in $\mathcal{X}_0$. We then have an admissible effective epimorphism $\amalg U_\alpha \times_X X_{01} \to X_{01}$. Consequently, to prove that X_{01} is corporeal, it will suffice to show that each fiber product $U_\alpha \times_X X_{01}$ is corporeal (Proposition 20.3.3.8). Replacing X by U_α (and the objects X_i by the fiber products $X_i \times_X U_\alpha$), we may reduce to the case where X is a retract of an object V which admits an admissible morphism $V \to W$ for $W \in \mathcal{X}_0$. Then $X_{01} \simeq X_0 \times_X X_1$ is a retract of $X_0 \times_V X_1$. Since the collection of corporeal objects of $\mathcal{X}$ is closed under retracts (Proposition 20.3.3.8), we can replace X by V and thereby reduce to the case where there exists an admissible morphism $X \to W$, where $W \in \mathcal{X}_0$. We have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\delta} & X \times_W X \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

where the vertical maps are admissible, so that δ is also an admissible morphism. It follows that the map

$$X_{01} \simeq X_0 \times_X X \times_X X_1 \to X_0 \times_X (X \times_W X) \times_X X_1 \simeq X_0 \times_W X_1$$

is also admissible. Using Proposition 20.3.3.8 again, we are reduced to proving that $X_0 \times_W X_1$ is corporeal. We may therefore replace X by W and thereby reduce to the case where $X \in \mathcal{X}_0$, so that $X \simeq h(Z)$ for some object $Z \in \mathcal{G}$.

We now apply similar reasoning to the object X_0. Invoking Proposition 20.3.3.12, we can choose an admissible effective epimorphism $\amalg U'_\beta \to X_0$ where each U'_β is a retract of an object V'_β which admits an admissible morphism $V'_\beta \to W'_\beta$ for some $W'_\beta \in \mathcal{X}_0$. We then have an admissible effective epimorphism $\amalg U'_\beta \times_X X_1 \to X_{01}$. Using Proposition 20.3.3.8, we are reduced to proving that each fiber product $U'_\beta \times_X X_1$ is corporeal. Since U'_β is a retract of V'_β, the composite map $U'_\beta \to X_0 \to X$ factors through a map $V'_\beta \to X$ for which $U'_\beta \times_X X_1$ is a retract of $V'_\beta \times_X X_1$. Write $W'_\beta = h(Y)$ for some $Y \in \mathcal{G}$. Since h is a presentation of $\mathcal{X}$, the object $V'_\beta \in \mathcal{X}$ can be written as a colimit of objects of the form $h(Y')$ for $Y' \in \mathcal{G}^{\mathrm{ad}}_{/Y}$ (this follows from the density of the functor $\mathcal{G}^{\mathrm{ad}}_{/Y} \to \mathcal{X}^{\mathrm{corp}}_{/h(Y)}$; see Remark 20.4.2.5). In particular, there exists an admissible effective epimorphism $\amalg h(Y_\gamma) \to V'_\beta$, hence an admissible effective epimorphism $\amalg h(Y_\gamma) \times_X X_1 \to V'_\beta \times_X X_1$. We may therefore replace X_0 by $h(Y_\gamma)$ and thereby reduce to the case where X_0 has the form $h(Z_0)$ for some $Z_0 \in \mathcal{G}$.

The morphism $h(Z_0) = X_0 \xrightarrow{v} X = h(Z)$ need not be the image under h of a morphism $Z_0 \to Z$ in $\mathcal{G}$. However, our assumption that $h : \mathcal{G} \to \mathcal{X}$ is a presentation guarantees that we can can choose a collection of admissible morphisms $\{Z_\gamma \to Z_0\}$ in $\mathcal{G}$ which determine an effective epimorphism $\amalg h(Z_\gamma) \to h(Z_0)$, for which each of the composite maps $h(Z_\gamma) \to h(Z_0) = X_0 \to X = h(Z)$ is induced by a morphism $Z_\gamma \to Z$ in the ∞-category $\mathcal{G}$. We then have an admissible effective epimorphism $\amalg h(Z_\gamma) \to h(Z_0)$, which gives rise to an admissible effective epimorphism

$$\amalg h(Z_\gamma) \times_X X_1 \to X_0 \times_X X_1.$$

Using Proposition 20.3.3.8, are reduced to proving that each of the fiber products $h(Z_\gamma) \times_X X_1$ is corporeal. We may therefore replace X_0 by $h(Z_\gamma)$ and thereby reduce to the case where $v : X_0 \to X$ has the form $h(v_0)$ for some morphism $v_0 : Z_0 \to Z$ in the ∞-category $\mathcal{G}$. Similarly, we can assume that w has the form $h(w_0)$ for some morphism $w_0 : Z_1 \to Z$ in $\mathcal{G}$. Since the functor h is left exact, it follows that $X_{01} = h(Z_0) \times_{h(Z)} h(Z_1) \simeq h(Z_0 \times_Z Z_1)$ belongs to the essential image of h, and is therefore corporeal. □

20.6 Geometric Sites

Let X be a quasi-compact, quasi-separated scheme, and let Sch_X denote the category whose objects are X-schemes of finite presentation, endowed with the étale topology. In §20.1, we asserted without proof that the ∞-category $\mathcal{S}\mathrm{hv}(\mathrm{Sch}_X) = \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Sch}_X)$ can be regarded as a fractured ∞-topos (in the sense of Definition 20.1.2.1). Our goal in this section is to prove a more general form of this assertion. We begin by introducing the notion of a *geometric site* (Definition 20.6.2.1): an essentially small ∞-category $\mathcal{G}$ which is equipped with a Grothendieck topology τ and an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ satisfying an appropriate compatibility. Our main result asserts that for any geometric site $\mathcal{G}$, the ∞-category $\mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$ inherits the structure of a fractured ∞-topos, whose subcategory of corporeal objects can be identified with $\mathcal{S}\mathrm{hv}_\tau(\mathcal{G}^{\mathrm{ad}})$ (Theorem 20.6.3.4). This result can be applied to the category $\mathcal{G} = \mathrm{Sch}_X$ by taking τ to be the topology generated by étale coverings and $\mathcal{G}^{\mathrm{ad}}$ to be the subcategory $\mathrm{Sch}_X^{\text{ét}} \subseteq \mathrm{Sch}_X$ of étale morphisms. We will consider several variants of this example in §20.6.4 (and will meet others in Part VII).

20.6.1 Restrictions of Grothendieck Topologies

Throughout this section, we assume that the reader is familiar with the theory of Grothendieck topologies on ∞-categories (see Definition HTT.6.2.2.1 or §A.3). If $\mathcal{G}$ is an ∞-category, we will typically use the symbol τ (or variants thereof) to denote a Grothendieck topology on $\mathcal{G}$. We will say that a collection of morphisms $\{f_\alpha : U_\alpha \to X\}$ in $\mathcal{G}$ is a *τ-covering* if it generates a covering sieve on X: that is, if the full subcategory $\mathcal{G}^{(0)}_{/X} \subseteq \mathcal{G}_{/X}$ spanned by those morphisms $f : U \to X$ which factor through some f_α is a covering sieve with respect to τ. We will denote the ∞-category of $\mathcal{S}$-valued sheaves on $\mathcal{G}$ by $\mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$ to indicate its dependence on τ.

Proposition 20.6.1.1. *Let $f : \mathcal{G}' \to \mathcal{G}$ be a functor between ∞-categories and let τ be a Grothendieck topology on $\mathcal{G}$. Assume that $\mathcal{G}'$ admits pullbacks and that the functor f preserves pullbacks. Then there exists a Grothendieck topology $f^*\tau$ on $\mathcal{G}'$ which is characterized by the following property: a sieve $\mathcal{G}'^{(0)}_{/X} \subseteq \mathcal{G}'_{/X}$ is a covering sieve for $f^*\tau$ if and only if the collection of morphisms $\{f(U) \to f(X)\}_{U \in \mathcal{G}'^{(0)}_{/X}}$ is a τ-covering of X.*

Remark 20.6.1.2. In the statement of Proposition 20.6.1.1, we do not require that the ∞-category $\mathcal{G}$ admits finite limits (in fact, this condition is not satisfied in many examples of interest).

Proof of Proposition 20.6.1.1. We show that the collection of $(f^*\tau)$-covering sieves satisfies the conditions of Definition HTT.6.2.2.1:

(1) For each object $X \in \mathcal{G}'$, the full subcategory $\mathcal{G}'_{/X} \subseteq \mathcal{G}'_{/X}$ is a $(f^*\tau)$-covering sieve. This is clear, since the identity morphism $\mathrm{id} : f(X) \to f(X)$ is a τ-covering.

(2) If $u : X \to Y$ is a morphism in $\mathcal{G}'$ and $\mathcal{G}'^{(0)}_{/Y}$ is a $(f^*\tau)$-covering sieve on Y, then $u^* \mathcal{G}'^{(0)}_{/Y} = \mathcal{G}^{(0)}_{/Y} \times_{\mathcal{G}'_{/Y}} \mathcal{G}'_{/X}$ is a $(f^*\tau)$ covering sieve on X. To prove this, choose a collection of objects $U_\alpha \in \mathcal{G}'^{(0)}_{/Y}$ for which the maps $\{f(U_\alpha) \to f(Y)\}$ generate a τ-covering sieve $\mathcal{G}^{(0)}_{/f(Y)}$ on $f(Y) \in \mathcal{G}'$. Since τ is a Grothendieck topology, the pullback $\mathcal{G}^{(0)}_{/f(Y)} \times_{\mathcal{G}_{/f(Y)}} \mathcal{G}_{/f(X)}$ is also a τ-covering sieve. Our assumption that f preserves finite limits guarantees that this sieve is generated by the maps $\{f(X \times_Y U_\alpha) \to f(X)\}$. It follows that the projection maps $\{X \times_Y U_\alpha \to X\}$ generate a $(f^*\tau)$-covering of X, so that $u^* \mathcal{G}'^{(0)}_{/Y}$ is a $(f^*\tau)$-covering sieve.

(3) Let Y be an object of $\mathcal{G}'$, let $\mathcal{G}'^{(0)}_{/Y} \subseteq \mathcal{G}'_{/Y}$ be a $(f^*\tau)$-covering sieve on Y, and let $\mathcal{G}'^{(1)}_{/Y} \subseteq \mathcal{G}'_{/Y}$ be another sieve with the property that for each object $u : X \to Y$ in $\mathcal{G}'^{(0)}_{/Y}$, the pullback $u^* \mathcal{G}'^{(1)}_{/Y}$ is a $(f^*\tau)$-covering sieve on X. We must show that $\mathcal{G}'^{(1)}_{/Y}$ is a $(f^*\tau)$-covering sieve on Y. We begin by choosing maps $u_\alpha : U_\alpha \to Y$ which belong to $\mathcal{G}'^{(0)}_{/Y}$ for which the induced maps $f(U_\alpha) \to f(Y)$ are a τ-covering of $f(Y) \in \mathcal{G}$. Since each $u_\alpha^* \mathcal{G}'^{(1)}_{/Y}$ is a $(f^*\tau)$-covering sieve, we can choose a collection of maps $V_{\alpha,\beta} \to U_\alpha$ for which the composite maps $V_{\alpha,\beta} \to U_\alpha \to Y$ belong to $\mathcal{G}'^{(1)}_{/Y}$, and for each α the collection of maps $\{f(V_{\alpha,\beta}) \to f(U_\alpha)\}$ is a τ-covering of $f(U_\alpha)$. Since τ is a Grothendieck topology, it follows that the collection of composite maps $f(V_{\alpha,\beta}) \to f(U_\alpha) \to f(Y)$ is a τ-covering of $f(Y)$, so that $\mathcal{G}'^{(1)}_{/Y}$ is a covering sieve as desired.

□

Proposition 20.6.1.3. *Let $f : \mathcal{G}' \to \mathcal{G}$ be a functor between ∞-categories and let τ be a Grothendieck topology on $\mathcal{G}$. Assume that the ∞-category $\mathcal{G}'$ admits pullbacks and that the functor f preserves pullbacks, and let $T : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{G}')$ be the functor given by precomposition with f. Then the functor T carries $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ into $\mathcal{S}\mathrm{hv}_{f^*\tau}(\mathcal{G}')$.*

Proof. Let $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$. We wish to show that $T(\mathscr{F})$ is a sheaf with respect to the Grothendieck topology $f^*\tau$. Fix an object $X \in \mathcal{G}'$ and a sieve $\mathcal{G}'^{(0)}_{/X}$ which is covering with respect to $(f^*\tau)$; we wish to show that the canonical map $T(\mathscr{F})(X) \to \varprojlim_{U \in \mathcal{G}'^{(0)}_{/X}} T(\mathscr{F})(U)$ is a homotopy equivalence.

Choose a collection of morphisms $\{U_\alpha \to X\}_{\alpha \in I}$ which belong to $\mathcal{G}'^{(0)}_{/X}$, such that the maps $f(U_\alpha) \to f(X)$ generate a τ-covering sieve in $\mathcal{G}_{/f(X)}$. Let $\mathcal{G}'^{(1)}_{/X} \subseteq \mathcal{G}_{/X}$ be the sieve generated by the morphisms $U_\alpha \to X$. We will complete the proof by showing the following:

- (a) The canonical map $\rho : T(\mathscr{F})(X) \to \varprojlim_{U \in \mathcal{G}'^{(1)}_{/X}} T(\mathscr{F})(U)$ is a homotopy equivalence.
- (b) The functor $T(\mathscr{F})|_{\mathcal{G}'^{(0)}_{/X}}$ is a right Kan extension of its restriction to $\mathcal{G}'^{(1)}_{/X}$.

Note that if $u : U \to X$ is a morphism belonging the sieve $\mathcal{G}'^{(0)}_{/X}$, then $T(\mathscr{F})|_{\mathcal{G}'^{(0)}_{/X}}$ is a right Kan extension of $T(\mathscr{F})|_{\mathcal{G}'^{(1)}_{/X}}$ at U if and only if the canonical map $T(\mathscr{F})(U) \to \varprojlim_{V \in u^* \mathcal{G}'^{(1)}_{/X}} T(\mathscr{F})(V)$ is an equivalence. The sieve $u^* \mathcal{G}'^{(1)}_{/X}$ is generated by the fiber products $U_\alpha \times_X U$ whose images in $\mathcal{G}$ generate a covering sieve with respect to τ (since the functor f preserves pullbacks). Consequently, assertion (b) can be deduced from (a) (applied after replacing X by U and $\mathcal{G}'^{(1)}_{/X}$ by $u^* \mathcal{G}'^{(1)}_{/X}$).

It remains to prove (a). Let $\mathcal{G}^{(1)}_{/f(X)}$ denote the sieve in $\mathcal{G}_{/f(X)}$ generated by the maps $f(U_\alpha) \to f(X)$. Then the map ρ can identified with the composition $\mathscr{F}(f(X)) \xrightarrow{\rho'} \varprojlim_{V \in \mathcal{G}^{(1)}_{/f(X)}} \mathscr{F}(V) \xrightarrow{\rho''} \varprojlim_{U \in \mathcal{G}'^{(1)}_{/X}} \mathscr{F}(f(U))$. Our assumption that $\mathscr{F}$ is a sheaf with respect to the Grothendieck topology τ guarantees that ρ' is a homotopy equivalence. We are therefore reduced to showing that ρ'' is a homotopy equivalence. For this, it will suffice to show that f induces a left cofinal map $\mathcal{G}'^{(1)}_{/X} \to \mathcal{G}^{(1)}_{/f(X)}$. Using the criterion of Theorem **??**, we must show that for each morphism $V \to f(X)$ belonging the sieve $\mathcal{G}^{(1)}_{/X}$, the ∞-category $\mathcal{C} = \mathcal{G}'^{(1)}_{/X} \times_{\mathcal{G}^{(1)}_{/f(X)}} (\mathcal{G}^{(1)}_{/f(X)})_{V/}$ is weakly contractible. In fact, the ∞-category $\mathcal{C}^{\mathrm{op}}$ is sifted: it is nonempty (by the definition of the sieve $\mathcal{G}^{(1)}_{/f(X)}$) and admits pushouts (by virtue of our assumption that f preserves fiber products). □

20.6.2 Grothendieck Topologies and Admissibility Structures

We are now ready to introduce our main objects of interest.

Definition 20.6.2.1. Let $\mathcal{G}$ be an ∞-category equipped with a Grothendieck topology τ. We will say that an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ is *compatible with* τ if, for every object $X \in \mathcal{G}$ and every covering sieve $\mathcal{G}^{(0)}_{/X} \subseteq \mathcal{G}_{/X}$ with respect to τ, there exists a τ-covering $\{f_\alpha : U_\alpha \to X\}$ where each f_α is an admissible morphism which belongs to the sieve $\mathcal{G}^{(0)}_{/X}$. In this case, we will also say that the Grothendieck topology τ is *compatible with the admissibility structure* $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$, or that $\mathcal{G}^{\mathrm{ad}}$ *and* τ *are compatible*.

A *geometric site* is a triple $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$, where $\mathcal{G}$ is an essentially small ∞-category, $\mathcal{G}^{\mathrm{ad}}$ is an admissibility structure on $\mathcal{G}$, and τ is a Grothendieck topology on $\mathcal{G}$ which is compatible with $\mathcal{G}^{\mathrm{ad}}$.

Remark 20.6.2.2. If $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ is a geometric site, we will often abuse terminology and simply refer to the ∞-category $\mathcal{G}$ as a geometric site; in this case, we implicitly assume that the admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ and the Grothendieck topology τ have been specified. Beware that there is some danger of confusion here, because we will often consider different geometric sites which have the same underlying ∞-category (see Examples 20.6.4.1 and 20.6.4.2, or Examples 20.6.4.4 and 20.6.4.5).

Remark 20.6.2.3. Let $\mathcal{G}$ be an ∞-category equipped with a Grothendieck topology τ and a compatible admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$. If $\mathcal{G}'^{\mathrm{ad}}$ is an admissibility structure on $\mathcal{G}$ which contains $\mathcal{G}^{\mathrm{ad}}$, then $\mathcal{G}'^{\mathrm{ad}}$ is also compatible with τ.

Example 20.6.2.4. Let $\mathcal{G}$ be an ∞-category which admits pullbacks. Then $\mathcal{G}^{\mathrm{ad}} = \mathcal{G}$ is an admissibility structure on $\mathcal{G}$ which is compatible with every Grothendieck topology on $\mathcal{G}$.

Example 20.6.2.5. Let $\mathcal{G}$ be an ∞-category and let τ be the trivial topology on $\mathcal{G}$ (so that a sieve $\mathcal{G}^{(0)}_{/X} \subseteq \mathcal{G}_{/X}$ is a τ-covering if and only if $\mathcal{G}^{(0)}_{/X} = \mathcal{G}_{/X}$). Then τ is compatible with every admissibility structure on $\mathcal{G}$.

Example 20.6.2.6. Let $\mathcal{T}\mathrm{op}$ denote the category whose objects are topological spaces X and whose morphisms are continuous maps $f : X \to Y$. We regard $\mathcal{T}\mathrm{op}$ as equipped with a Grothendieck topology τ, where a collection of morphisms $\{f_\alpha : U_\alpha \to X\}$ is a τ-covering if and only if, for every point $x \in X$, there exists an open set $U \subseteq X$ containing x, an index α, and a map $g : U \to U_\alpha$ such that the composite map $U \xrightarrow{g} U_\alpha \xrightarrow{f_\alpha} X$ is the identity. The collection of open immersions between topological spaces is an admissibility structure on $\mathcal{T}\mathrm{op}$ which is compatible with the Grothendieck topology τ. We can also equip $\mathcal{T}\mathrm{op}$ with the admissibility structure consisting of all local homeomorphisms between topological spaces; it follows from Remark 20.6.2.3 that this admissibility structure is also compatible with τ.

Remark 20.6.2.7. One can give many variants on Example 20.6.2.6 by restricting our attention to topological spaces which have additional properties or are equipped with additional structures. For example, we can replace $\mathcal{T}\mathrm{op}$ by the category whose objects are topological manifolds and whose morphisms are continuous maps, or by the category whose objects are smooth manifolds and whose morphisms are smooth maps. We will study the latter example in detail in §**??**.

Construction 20.6.2.8. Let $\mathcal{G}$ be an ∞-category equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}}$. Then the ∞-category $\mathcal{G}^{\mathrm{ad}}$ admits fiber products and the inclusion functor $j : \mathcal{G}^{\mathrm{ad}} \hookrightarrow \mathcal{G}$ preserves fiber products (see Remark 20.2.1.12). Applying Proposition 20.6.1.1, we see that every Grothendieck topology τ induces a Grothendieck topology $j^*\tau$ on the ∞-category $\mathcal{G}^{\mathrm{ad}}$. We will denote this Grothendieck topology by τ^{ad}. Applying Proposition 20.6.1.3, we deduce that precomposition with j determines a functor $j^* : \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G}) \to \mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}})$.

Construction 20.6.2.8 can be applied to an arbitrary Grothendieck topology τ on $\mathcal{G}$. We will generally be interested in this construction on in the case where τ is compatible with the admissibility structure $\mathcal{G}^{\mathrm{ad}}$. In this case, passage from τ to τ^{ad} involves no loss of information:

Proposition 20.6.2.9. *Let $\mathcal{G}$ be an ∞-category equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$. Then Construction 20.6.2.8 determines an injective map*

$$\{ \text{ Grothendieck topologies on } \mathcal{G} \text{ compatible with } \mathcal{G}^{\mathrm{ad}} \} \to \{ \text{ Grothendieck topologies on } \mathcal{G}^{\mathrm{ad}} \}.$$

The image of this map consists of those Grothendieck topologies τ_0 on $\mathcal{G}^{\mathrm{ad}}$ with the following property:

- $(*)$ *Let $f : X \to Y$ be a morphism in $\mathcal{G}$ and let $\mathcal{G}^{\mathrm{ad}(0)}_{/Y} \subseteq \mathcal{G}^{\mathrm{ad}}_{/Y}$ be a τ_0-covering sieve on Y. Then the collection of morphisms $\{U \times_Y X \to X\}_{U \in \mathcal{G}^{\mathrm{ad}(0)}_{/Y}}$ generates a τ_0-covering sieve on X.*

Proof. Let τ be a Grothendieck topology on $\mathcal{G}$ and let $\mathcal{G}^{(0)}_{/X} \subseteq \mathcal{G}_{/X}$ be sieve. If $\mathcal{G}^{(0)}_{/X} \cap \mathcal{G}^{\mathrm{ad}}_{/X} \subseteq \mathcal{G}^{\mathrm{ad}}_{/X}$ is a τ^{ad}-covering sieve, then $\mathcal{G}^{(0)}_{/X}$ is a τ-covering sieve. If τ is compatible with $\mathcal{G}^{\mathrm{ad}}$, then the converse holds. It follows immediately that a Grothendieck topology τ compatible with $\mathcal{G}^{\mathrm{ad}}$ can be recovered from τ^{ad}, so that the construction $\tau \mapsto \tau^{\mathrm{ad}}$ is injective.

Now suppose that τ is any Grothendieck topology on $\mathcal{G}$; we show that τ^{ad} satisfies condition $(*)$. Let $f : X \to Y$ be a morphism in $\mathcal{G}$ and let $\mathcal{G}^{\mathrm{ad}(0)}_{/Y} \subseteq \mathcal{G}^{\mathrm{ad}}_{/Y}$ be a τ^{ad}-covering sieve. Then the collection of maps $\{U \to Y\}_{U \in \mathcal{G}^{\mathrm{ad}(0)}_{/Y}}$ generates a τ-covering of Y. Since τ is a Grothendieck topology on $\mathcal{G}$, it follows that collection of maps $\{U \times_Y X \to X\}_{U \in \mathcal{G}^{\mathrm{ad}(0)}_{/Y}}$ generates a τ-covering of Y (in the ∞-category $\mathcal{G}$), and therefore also a τ^{ad}-covering of Y (in the ∞-category $\mathcal{G}^{\mathrm{ad}}$).

Finally, let τ_0 be any Grothendieck topology on $\mathcal{G}^{\mathrm{ad}}$ which satisfies condition $(*)$. We attempt to define a Grothendieck topology τ on $\mathcal{G}$ as follows: a sieve $\mathcal{G}^{(0)}_{/X} \subseteq \mathcal{G}_{/X}$ is a

τ-covering if and only if the intersection $\mathcal{G}^{(0)}_{/X} \cap \mathcal{G}^{\mathrm{ad}}_{/X} \subseteq \mathcal{G}^{\mathrm{ad}}_{/X}$ is a τ_0-covering. To show that τ is a Grothendieck topology, we verify that it satisfies the requirements of Definition HTT.6.2.2.1:

(1) For every object $X \in \mathcal{G}$, the subcategory $\mathcal{G}_{/X} \subseteq \mathcal{G}_{/X}$ is a τ-covering sieve (since $\mathcal{G}^{\mathrm{ad}}_{/X} \subseteq \mathcal{G}^{\mathrm{ad}}_{/X}$ is a τ_0-covering sieve).

(2) Let $f : Y \to X$ be a morphism in $\mathcal{G}$ and let $\mathcal{G}^{(0)}_{/X} \subseteq \mathcal{G}_{/X}$ be a τ-covering sieve on X; we must show that the fiber product $\mathcal{G}_{/Y} \times_{\mathcal{G}_{/X}} \mathcal{G}^{(0)}_{/X}$ is a τ-covering sieve on Y. This follows immediately from assumption $(*)$.

(3) Let $\mathcal{G}^{(0)}_{/X} \subseteq \mathcal{G}_{/X}$ be a τ-covering sieve on an object $X \in \mathcal{G}$ and let $\mathcal{G}^{(1)}_{/X}$ be another sieve with the property that for each object $Y \to X$ of $\mathcal{G}^{(0)}_{/X}$, the fiber product $\mathcal{G}^{(1)}_{/X} \times_{\mathcal{G}_{/X}} \mathcal{G}_{/Y}$ is a τ-covering sieve of Y. We must show that $\mathcal{G}^{(1)}_{/X}$ is a τ-covering sieve of X. The assumption that $\mathcal{G}^{(0)}_{/X}$ is a τ-covering implies that it contains admissible morphisms $\{U_\alpha \to X\}$ which determine a τ_0-covering of X. Since each $\mathcal{G}^{(1)}_{/X} \times_{\mathcal{G}_{/X}} \mathcal{G}_{/U_\alpha}$ is a τ-covering of U_α, we can choose admissible morphisms $V_{\alpha,\beta} \to U_\alpha$ which generate a τ_0-covering sieve, for which the composite maps $V_{\alpha,\beta} \to U_\alpha \to X$ belong to $\mathcal{G}^{(1)}_{/X}$. Then the collection of admissible maps $\{V_{\alpha,\beta} \to X\}$ are contained in $\mathcal{G}^{(1)}_{/X}$ and generate a τ_0-covering sieve on X, so that $\mathcal{G}^{(1)}_{/X} \subseteq \mathcal{G}_{/X}$ is a τ-covering sieve as desired.

We now complete the proof by observing that the Grothendieck topology τ is compatible with $\mathcal{G}^{\mathrm{ad}}$, and that the Grothendieck topology τ^{ad} (obtained by applying Construction **??** to τ) coincides with τ_0. □

20.6.3 From Geometric Sites to Fractured Localizations

Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site and let $\mathscr{F} : \mathcal{G}^{\mathrm{op}} \to \mathcal{S}$ be a $\mathcal{S}$-valued presheaf on $\mathcal{G}$. If $\mathscr{F}$ is a sheaf with respect to the Grothendieck topology τ, then Proposition 20.6.1.3 guarantees that $\mathscr{F}|_{\mathcal{G}^{\mathrm{ad\,op}}}$ is a sheaf with respect to the Grothendieck topology τ^{ad} of Construction 20.6.2.8. We now establish the converse:

Proposition 20.6.3.1. *Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site and let $T : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{G}^{\mathrm{ad}})$ denote the restriction functor. Then the diagram of ∞-categories*

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_\tau(\mathcal{G}) & \longrightarrow & \mathcal{P}(\mathcal{G}) \\ \downarrow & & \downarrow T \\ \mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}}) & \longrightarrow & \mathcal{P}(\mathcal{G}^{\mathrm{ad}}) \end{array}$$

is a pullback square. In other words, an object $\mathscr{F} \in \mathcal{P}(\mathcal{G})$ is a sheaf with respect to τ if and only if $\mathscr{F}|_{\mathcal{G}^{\mathrm{ad}\,\mathrm{op}}}$ is a sheaf with respect to τ^{ad}.

Proof. Assume that $T(\mathscr{F})$ is a sheaf with respect to τ^{ad}. Let X be an object of $\mathcal{G}$ and let $\mathcal{G}^{(0)}_{/X} \subseteq \mathcal{G}_{/X}$ be a τ-covering sieve; we wish to show that the canonical map $\mathscr{F}(X) \xrightarrow{\rho} \varprojlim_{Y \in \mathcal{G}^{(0)}_{/X}} \mathscr{F}(Y)$ is a homotopy equivalence. Set $\mathcal{G}^{\mathrm{ad}(0)}_{/X} = \mathcal{G}^{(0)}_{/X} \times_{\mathcal{G}_{/X}} \mathcal{G}^{\mathrm{ad}}_{/X}$, and let $\mathcal{G}^{(1)}_{/X} \subseteq \mathcal{G}_{/X}$ denote the sieve generated by $\mathcal{G}^{\mathrm{ad}(0)}_{/X}$. Since τ is compatible with the admissibility structure $\mathcal{G}^{\mathrm{ad}}$, the sieve $\mathcal{G}^{\mathrm{ad}(0)}_{/X}$ is covering with respect to the topology τ^{ad}. Consequently, our assumption that $T(\mathscr{F})$ is a sheaf with respect to τ^{ad} guarantees that the composite map

$$\mathscr{F}(X) \xrightarrow{\rho} \varprojlim_{Y \in \mathcal{G}^{(0)}_{/X}} \mathscr{F}(Y) \xrightarrow{\rho'} \varprojlim_{Y \in \mathcal{G}^{(1)}_{/X}} \mathscr{F}(Y) \xrightarrow{\rho''} \varprojlim_{U \in \mathcal{G}^{\mathrm{ad}(0)}_{/X}} \mathscr{F}(U)$$

is a homotopy equivalence. Consequently, to show that ρ is a homotopy equivalence, it will suffice to show that ρ' and ρ'' are homotopy equivalences. The map ρ'' is a homotopy equivalence because the inclusion $\mathcal{G}^{\mathrm{ad}(0)}_{/X} \hookrightarrow \mathcal{G}^{(1)}_{/X}$ is left cofinal (for every object $Y \in \mathcal{G}^{(1)}_{/X}$, the fiber product $(\mathcal{G}^{(0)}_{/X}) \times_{\mathcal{G}_{/X}} \mathcal{G}_{Y//X}$ is nonempty and admits finite products, and is therefore weakly contractible). To show that ρ' is a homotopy equivalence, it will suffice to show that $\mathscr{F}|_{\mathcal{G}^{(0)\,\mathrm{op}}_{/X}}$ is a right Kan extension of its restriction to $\mathscr{F}|_{\mathcal{G}^{(1)\,\mathrm{op}}_{/X}}$. Fix an object $Y \in \mathcal{G}^{(0)}_{/X}$ and define

$$\mathcal{G}^{(1)}_{/Y} = \mathcal{G}^{(1)}_{/X} \times_{\mathcal{G}_{/X}} \mathcal{G}_{/Y} \qquad \mathcal{G}^{\mathrm{ad}(0)}_{/Y} = \mathcal{G}^{\mathrm{ad}(0)}_{/X} \times_{\mathcal{G}^{\mathrm{ad}}_{/X}} \mathcal{G}^{\mathrm{ad}}_{/Y},$$

so that $\mathscr{F}$ determines maps

$$\mathscr{F}(Y) \xrightarrow{\theta} \varprojlim_{Z \in \mathcal{G}^{(1)}_{/Y}} \mathscr{F}(Z) \xrightarrow{\theta'} \varprojlim_{V \in \mathcal{G}^{\mathrm{ad}(0)}_{/Y}} \mathscr{F}(V).$$

To prove that $\mathscr{F}|_{\mathcal{G}^{(0)\,\mathrm{op}}_{/X}}$ is a right Kan extension of its restriction to $\mathscr{F}|_{\mathcal{G}^{(1)\,\mathrm{op}}_{/X}}$ at Y, it will suffice to show that θ is a homotopy equivalence. The argument above shows that the inclusion $\mathcal{G}^{\mathrm{ad}(0)}_{/Y} \hookrightarrow \mathcal{G}^{(1)}_{/Y}$ is left cofinal, so that θ' is a homotopy equivalence. We are therefore reduced to showing that $\theta' \circ \theta$ is a homotopy equivalence, which follows from our assumption that $T(\mathscr{F})$ is a sheaf with respect to the topology τ^{ad} (since the sieve $\mathcal{G}^{\mathrm{ad}(0)}_{/Y} \subseteq \mathcal{G}^{\mathrm{ad}}_{/Y}$ is a τ^{ad}-covering). □

Proposition 20.6.3.2. *Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site, let $T : \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}((\mathcal{G}^{\mathrm{ad}})^{\mathrm{op}}, \mathcal{S})$ denote the restriction functor, and let $j^* : \mathcal{S}\mathrm{hv}_\tau(\mathcal{G}) \to \mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}})$ denote the restriction of T (Proposition 20.6.1.3). Then the diagram of ∞-categories*

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_\tau(\mathcal{G}) & \longrightarrow & \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S}) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle T} \\ \mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}}) & \longrightarrow & \mathrm{Fun}((\mathcal{G}^{\mathrm{ad}})^{\mathrm{op}}, \mathcal{S}) \end{array}$$

is left adjointable. In other words, the restriction functor $\mathscr{F} \mapsto \mathscr{F}|_{\mathcal{G}^{\mathrm{ad}}}$ *commutes with sheafification with respect to* τ*.*

Proof. Let $L : \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ denote a left adjoint to the inclusion (given by sheafification with respect to τ), and define $L^{\mathrm{ad}} : \mathrm{Fun}((\mathcal{G}^{\mathrm{ad}})^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}})$ similarly. We wish to show that, for every object $\mathscr{F} \in \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$, the canonical map $L^{\mathrm{ad}}(T(\mathscr{F})) \to T(L\mathscr{F})$ is an equivalence. Proposition 20.6.1.3 shows that $T(L\mathscr{F})$ is a sheaf with respect to τ^{ad}. It will therefore suffice to show that the functor T carries L-equivalences to L^{ad}-equivalences.

For each object $X \in \mathcal{G}$, let $h_X : \mathcal{G}^{\mathrm{op}} \to \mathcal{S}$ denote the functor represented by X. Given a sieve $\mathcal{G}^{(0)}_{/X} \subseteq \mathcal{G}_{/X}$, we let $h_{\mathcal{G}^{(0)}_{/X}}$ denote the corresponding subobject of h_X (given by the colimit $\varinjlim_{U \in \mathcal{G}^{(0)}_{/X}} h_U$). The collection of L-equivalences in $\mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$ is generated, as a strongly saturated collection of morphisms, by monomorphisms of the form $h_{\mathcal{G}^{(0)}_{/X}} \hookrightarrow h_X$, where $\mathcal{G}^{(0)}_{/X} \subseteq \mathcal{G}_{/X}$ is a τ-covering sieve. It will therefore suffice to show that for every such sieve, the map $T(h_{\mathcal{G}^{(0)}_{/X}}) \to T(h_X)$ is an L^{ad}-equivalence.

For each object $U \in \mathcal{G}^{\mathrm{ad}}$, let $h^{\mathrm{ad}}_U \in \mathrm{Fun}((\mathcal{G}^{\mathrm{ad}})^{\mathrm{op}}, \mathcal{S})$ denote the functor represented by U. The presheaf ∞-category $\mathrm{Fun}((\mathcal{G}^{\mathrm{ad}})^{\mathrm{op}}, \mathcal{S})$ is generated under small colimits by objects of the form h^{ad}_U. It will therefore suffice to show that, for every morphism $u : h^{\mathrm{ad}}_U \to T(h_X)$ (which we can identify with a map $U \to X$ in the ∞-category $\mathcal{G}$), the projection map $\pi : T(h_{\mathcal{G}^{(0)}_{/X}}) \times_{T(h_X)} h^{\mathrm{ad}}_U \to h^{\mathrm{ad}}_U$ is a τ^{ad}-equivalence. Unwinding the definitions, we see that π is the monomorphism classified by the sieve $\mathcal{G}^{\mathrm{ad}(0)}_{/U} \subseteq \mathcal{G}^{\mathrm{ad}}_{/U}$ given by those admissible morphisms $V \to U$ for which the composite map $V \to U \to X$ belongs the sieve $\mathcal{G}^{\mathrm{ad}}_{/X}$. To complete the proof, it suffices to observe that this sieve is covering with respect to the topology τ^{ad}. Unwinding the definitions, we need to show that the sieve $u^* \mathcal{G}^{(0)}_{/X} \subseteq \mathcal{G}_{/U}$ contains a τ-covering sieve which is generated by admissible morphisms. This follows from our assumption that the Grothendieck topology τ is compatible with the admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$. □

Combining Propositions 20.6.3.1 and 20.6.3.2, we obtain the following:

Proposition 20.6.3.3. *Let* $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ *be a geometric site. Let us regard the presheaf ∞-category* $\mathcal{P}(\mathcal{G})$ *as a fractured ∞-topos (see Theorem 20.2.4.1), and let us abuse notation by identifying the fracture subcategory* $\mathcal{P}(\mathcal{G})^{\mathrm{corp}} \subseteq \mathcal{P}(\mathcal{G})$ *with* $\mathcal{P}(\mathcal{G}^{\mathrm{ad}})$*. Then the full subcategory* $\mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}}) \subseteq \mathcal{P}(\mathcal{G}^{\mathrm{ad}})$ *is a fractured localization of* $\mathcal{P}(\mathcal{G}^{\mathrm{ad}})$ *(in the sense of Definition 20.5.1.1).*

Combining Proposition 20.6.3.3 and Theorem 20.5.1.2 (and using the identification $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G}) = \mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}}) \times_{\mathcal{P}(\mathcal{G}^{\mathrm{ad}})} \mathcal{P}(\mathcal{G})$ of Proposition 20.6.3.1), we obtain the following generalization of Theorem 20.2.4.1:

Theorem 20.6.3.4. *Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site and let $j^* : \mathcal{S}\mathrm{hv}_\tau(\mathcal{G}) \to \mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}})$ be the restriction functor. Then the functor j^* admits a left adjoint $j_! : \mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}}) \to \mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$. Moreover, $j_!$ induces an equivalence from $\mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}})$ to a fracture subcategory $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_\tau(\mathcal{G}) \subseteq \mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$.*

20.6.4 Examples of Geometric Sites

In this book, our primary interest is in geometric sites which are related to classical and spectral algebraic geometry.

Example 20.6.4.1 (Classical Algebraic Geometry: Zariski Topology)**.** Let Aff denote the category of affine schemes of finite type over $\mathbf{Z}$ (so that Aff is equivalent to the *opposite* of the category of finitely presented commutative rings). We regard Aff as equipped with the admissibility structure $\mathrm{Aff}^{\mathrm{Zar}} \subseteq \mathrm{Aff}$ of Example 20.2.1.5 (whose admissible morphisms are open immersions of the form $\operatorname{Spec} R[t^{-1}] \to \operatorname{Spec} R$, where R is a finitely presented commutative ring and $t \in R$ is an element (see Example 20.2.1.5). Let τ_{Zar} denote the Zariski topology on Aff, so that a collection of morphisms $\{f_\alpha : U_\alpha \to X\}$ is a τ_{Zar}-covering if and only if, for every point $x \in X$, there exists a Zariski-open set $U \subseteq X$ containing x, an index α, and a map of schemes $g : U \to U_\alpha$ for which the composite map $U \xrightarrow{g} U_\alpha \xrightarrow{f_\alpha} X$ is the inclusion. The admissibility structure $\mathrm{Aff}^{\mathrm{Zar}}$ is compatible with the Grothendieck topology τ_{Zar}. We therefore obtain a geometric site $(\mathrm{Aff}, \mathrm{Aff}^{\mathrm{Zar}}, \tau_{\mathrm{Zar}})$, which we will refer to as the *classical Zariski site*.

Example 20.6.4.2 (Classical Algebraic Geometry: Étale Topology)**.** Let Aff denote (the nerve of) the category of affine schemes of finite type over $\mathbf{Z}$ (so that Aff is equivalent to the *opposite* of the category of finitely presented commutative rings. We regard Aff as equipped with the admissibility structure $\mathrm{Aff}^{\text{ét}} \subseteq \mathrm{Aff}$ whose morphisms are étale maps between affine schemes of finite type over $\mathbf{Z}$. Let $\tau_{\text{ét}}$ denote the étale topology on Aff, so that a collection of morphisms $\{f_\alpha : U_\alpha \to X\}$ is a $\tau_{\text{ét}}$-covering if and only if, for every point $x \in X$, there exists an étale map $h : U \to X$ whose image contains x and an index α for which h factors as a composition $U \to U_\alpha \xrightarrow{f_\alpha} X$. The Grothendieck topology $\tau_{\text{ét}}$ is compatible with the admissibility structure $\mathrm{Aff}^{\text{ét}}$. We therefore obtain a geometric site $(\mathrm{Aff}, \mathrm{Aff}^{\text{ét}}, \tau_{\text{ét}})$, which we will refer to as the *classical étale site*.

Remark 20.6.4.3. Many variations on the preceding examples are possible. For example, we could work replace Aff by the category of affine R-schemes of finite presentation where R is some commutative ring, or allow schemes which are not affine, or relax the assumptions that our schemes are of finite type.

We now consider spectral variants of Examples 20.6.4.1 and 20.6.4.2.

Example 20.6.4.4 (Spectral Algebraic Geometry: Zariski Topology)**.** Let $\mathrm{Aff}_{\mathrm{Sp}}$ denote the opposite of the ∞-category CAlg_c of compact $\mathbb{E}_\infty$-rings. For each object $R \in \mathrm{CAlg}_c$, we will denote the corresponding object in $\mathrm{Aff}_{\mathrm{Sp}}$ by $\mathrm{Spét}\, R$ (which we can view as a nonconnective spectral scheme or nonconnective spectral Deligne-Mumford stack). The category $\mathrm{Aff}_{\mathrm{Sp}}$ admits an admissibility structure $\mathrm{Aff}^{\mathrm{Zar}}_{\mathrm{Sp}} \subseteq \mathrm{Aff}_{\mathrm{Sp}}$, whose morphisms are those of the form $\mathrm{Spét}\, R \to \mathrm{Spét}\, R[t^{-1}]$ for $t \in \pi_0 R$. The ∞-category $\mathrm{Aff}_{\mathrm{Sp}}$ also admits a Grothendieck topology τ_{Zar}, where a sieve on an object $\mathrm{Spec}\, R$ is covering if it contains a family of maps of the form $\{\mathrm{Spét}\, R[t_\alpha^{-1}] \to \mathrm{Spét}\, R\}$ for which the elements $t_\alpha \in \pi_0 R$ generate the unit ideal. The Grothendieck topology τ_{Zar} is compatible with the admissibility structure $\mathrm{Aff}^{\mathrm{Zar}}_{\mathrm{Sp}}$. We therefore obtain a geometric site $(\mathrm{Aff}_{\mathrm{Sp}}, \mathrm{Aff}^{\mathrm{Zar}}_{\mathrm{Sp}}, \tau_{\mathrm{Zar}})$, which we will refer to as the *spectral Zariski site*.

Example 20.6.4.5 (Spectral Algebraic Geometry: Étale Topology)**.** The ∞-category $\mathrm{Aff}_{\mathrm{Sp}}$ of Example 20.6.4.4 admits another Grothendieck topology $\tau_{\mathrm{ét}}$, where a sieve on an object $\mathrm{Spét}\, R$ is a $\tau_{\mathrm{ét}}$-covering if it contains a finite collection of étale morphisms $\mathrm{Spét}\, R_\alpha \to \mathrm{Spét}\, R$ for which the induced map $R \to \prod R_\alpha$ is faithfully flat. The Grothendieck topology $\tau_{\mathrm{ét}}$ is compatible with the admissibility structure $\mathrm{Aff}^{\mathrm{ét}}_{\mathrm{Sp}} \subseteq \mathrm{Aff}_{\mathrm{Sp}}$ spanned by the étale morphisms. We therefore obtain a geometric site $(\mathrm{Aff}_{\mathrm{Sp}}, \mathrm{Aff}^{\mathrm{ét}}_{\mathrm{Sp}}, \tau_{\mathrm{ét}})$, which we will refer to as the *spectral étale site*.

20.6.5 Modifications of Admissibility Structures

Let $\mathcal{G}$ be an essentially small ∞-category and let τ be a Grothendieck topology on $\mathcal{G}$. According to Theorem 20.6.3.4, every admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ which is compatible with τ determines a fracture subcategory $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_\tau(\mathcal{G}) \subseteq \mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$ (which can be identified with $\mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}})$). Let $h : \mathcal{G} \to \mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$ be the composition of the Yoneda embedding $\mathcal{G} \hookrightarrow \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$ with the sheafification functor $\mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$, and define $h^{\mathrm{ad}} : \mathcal{G}^{\mathrm{ad}} \to \mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}})$ similarly. The diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{G}^{\mathrm{ad}} & \xrightarrow{h^{\mathrm{ad}}} & \mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}}) \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow{h} & \mathcal{S}\mathrm{hv}_\tau(\mathcal{G}) \end{array}$$

commutes, up to canonical equivalence. In particular:

(*i*) For every object $X \in \mathcal{G}$, the sheaf $h(X)$ belongs to $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_\tau(\mathcal{G})$.

(*ii*) For every admissible morphism $U \to X$ of $\mathcal{G}$, the induced map $h(U) \to h(X)$ is a morphism of $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_\tau(\mathcal{G})$.

The converse of (ii) is false in general: if $\alpha : U \to X$ is a morphism in $\mathcal{G}$ for which $h(\alpha)$ belongs to $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_{\tau}(\mathcal{G})$, then α need not be admissible. In fact, many different admissibility structures on $\mathcal{G}$ can give rise to the same fracture subcategory of $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$:

Proposition 20.6.5.1. *Let $\mathcal{G}$ be an essentially small ∞-category equipped with a Grothendieck topology τ. Suppose we are given admissibility structures $\mathcal{G}^{\mathrm{ad}}_0 \subseteq \mathcal{G} \supseteq \mathcal{G}^{\mathrm{ad}}_1$ satisfying the following conditions:*

(1) *The admissibility structures $\mathcal{G}^{\mathrm{ad}}_0$ and $\mathcal{G}^{\mathrm{ad}}_1$ are both compatible with τ (in the sense of Definition 20.6.2.1), and $\mathcal{G}^{\mathrm{ad}}_0 \subseteq \mathcal{G}^{\mathrm{ad}}_1$.*

(2) *For every morphism $U \to X$ which belongs to $\mathcal{G}^{\mathrm{ad}}_1$, there exists a collection of morphisms $\{V_\alpha \to U\}$ which belong to $\mathcal{G}^{\mathrm{ad}}_0$ and generate a τ-covering of U, and for which the composite maps $V_\alpha \to U \to X$ also belong to $\mathcal{G}^{\mathrm{ad}}_0$.*

Then $\mathcal{G}^{\mathrm{ad}}_0$ and $\mathcal{G}^{\mathrm{ad}}_1$ determine the same fracture subcategory $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_{\tau}(\mathcal{G}) \subseteq \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$.

Example 20.6.5.2. Let $(\mathrm{Aff}, \mathrm{Aff}^{\mathrm{Zar}}, \tau_{\mathrm{Zar}})$ denote the classical Zariski site of Example 20.6.4.1. Then the category Aff admits another admissibility structure $\mathrm{Aff}'^{\mathrm{Zar}}$, where a morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ in Aff belongs to $\mathrm{Aff}'^{\mathrm{Zar}}$ if and only if it is an open immersion of affine schemes. Then $(\mathrm{Aff}, \mathrm{Aff}'^{\mathrm{Zar}}, \tau_{\mathrm{Zar}})$ is also a geometric site. Moreover, Proposition 20.6.5.1 implies that the admissibility structures $\mathrm{Aff}^{\mathrm{Zar}}$ and $\mathrm{Aff}'^{\mathrm{Zar}}$ determine the same fracture subcategory of $\mathcal{S}\mathrm{hv}_{\tau_{\mathrm{Zar}}}(\mathrm{Aff})$. A similar remark applies to the spectral Zariski site of Example 20.6.4.4.

Proof of Proposition 20.6.5.1. Replacing $\mathcal{G}$ by $\mathcal{G}^{\mathrm{ad}}_1$, we can reduce to the case where $\mathcal{G}^{\mathrm{ad}}_1 = \mathcal{G}$. Set $\mathcal{G}^{\mathrm{ad}} = \mathcal{G}^{\mathrm{ad}}_0$, so that $\mathcal{G}^{\mathrm{ad}}$ determines a fracture subcategory $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_{\tau}(\mathcal{G}) \subseteq \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$. We wish to show that $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_{\tau}(\mathcal{G}) = \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$.

Let $h : \mathcal{G} \to \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ denote the sheafified Yoneda embedding (that is, the composition of the Yoneda embedding $\mathcal{G} \to \mathcal{P}(\mathcal{G})$ with the sheafification functor $\mathcal{P}(\mathcal{G}) \to \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$). For each object $X \in \mathcal{G}$, the sheaf $h(X)$ is corporeal. Proposition 20.1.3.1 implies that the inclusion $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_{\tau}(\mathcal{G})_{/h(X)} \hookrightarrow \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})_{/h(X)}$ is a fully faithful embedding. Unwinding the definitions, we see that $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_{\tau}(\mathcal{G})_{/h(X)}$ can be identified with the full subcategory of $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})_{/h(X)}$ generated under small colimits by objects of the form $h(U)$, where $U \to X$ is admissible (that is, it belongs to $\mathcal{G}^{\mathrm{ad}}$). Using assumption (2), we deduce that $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_{\tau}(\mathcal{G})_{/h(X)} = \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})_{/h(X)}$.

Let us say that an object $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ is *good* if it is corporeal and the inclusion $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_{\tau}(\mathcal{G})_{/\mathscr{F}} \hookrightarrow \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})_{/\mathscr{F}}$ is an equality. To show that $\mathcal{S}\mathrm{hv}^{\mathrm{corp}}_{\tau}(\mathcal{G}) = \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$, it will suffice to show that every object of $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ is good. The preceding argument shows that $h(X)$ is good for each $X \in \mathcal{G}$. We will complete the proof by showing that the collection of good objects of $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ is closed under small colimits.

Suppose we are given a small diagram $\{\mathscr{F}_\alpha\}$ in $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ having a colimit $\mathscr{F}$, where each $\mathscr{F}_\alpha$ is good; we wish to show that $\mathscr{F}$ is good. Choose an arbitrary morphism $\rho : \mathscr{F}' \to \mathscr{F}$

in $\mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$. Then we can write $\mathscr{F}' = \varinjlim(\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}_\alpha)$. Note that each transition map $\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}_\alpha \to \mathscr{F}' \times_{\mathscr{F}} \mathscr{F}_\beta$ belongs to the subcategory $\mathcal{S}\mathrm{hv}_\tau^{\mathrm{corp}}(\mathcal{G}) \subseteq \mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$ (by virtue of our assumption that $\mathscr{F}_\beta$ is good). It follows that $\mathscr{F}'$ is corporeal, and is the colimit of the diagram $\{\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}_\alpha\}$ in the ∞-category $\mathcal{S}\mathrm{hv}_\tau^{\mathrm{corp}}(\mathcal{G})$. Specializing to the case $\rho = \mathrm{id}_{\mathscr{F}}$, we deduce that $\mathscr{F}$ is also corporeal. To complete the proof, it will suffice to show that the morphism ρ belongs to $\mathcal{S}\mathrm{hv}_\tau^{\mathrm{corp}}(\mathcal{G})$. Equivalently, we must show that each of the composite maps $\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}_\alpha \to \mathscr{F}' \xrightarrow{\rho} \mathscr{F}$ belongs to $\mathcal{S}\mathrm{hv}_\tau^{\mathrm{corp}}(\mathcal{G})$. This follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathscr{F}' \times_{\mathscr{F}} \mathscr{F}_\alpha & \longrightarrow & \mathscr{F}' \\ \downarrow & & \downarrow \rho \\ \mathscr{F}_\alpha & \longrightarrow & \mathscr{F}, \end{array}$$

since the left vertical map belongs to $\mathcal{S}\mathrm{hv}_\tau^{\mathrm{corp}}(\mathcal{G})$ by virtue of our assumption that $\mathscr{F}_\alpha$ is good, and the bottom horizontal map belongs to $\mathcal{S}\mathrm{hv}_\tau^{\mathrm{corp}}(\mathcal{G})$ because $\mathscr{F}$ is a colimit of the diagram $\{\mathscr{F}_\alpha\}$ in the ∞-category $\mathcal{S}\mathrm{hv}_\tau^{\mathrm{corp}}(\mathcal{X})$. □

Chapter 21

Structure Sheaves

Let X be a topological space. There are several different ways of thinking about a sheaf $\mathscr{O}_X$ of commutative rings on X:

(A) We can view $\mathscr{O}_X$ as a sheaf on X taking values in the category $\mathrm{CAlg}^{\heartsuit}$ of commutative rings. From this point of view, $\mathscr{O}_X$ is a functor $\mathcal{U}(X)^{\mathrm{op}} \to \mathrm{CAlg}^{\heartsuit}$, which satisfies the usual sheaf axioms; here $\mathcal{U}(X)$ denotes the partially ordered set of open subsets of X.

(B) We can view $\mathscr{O}_X$ as a *commutative ring object* of the topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X)$: that is, as an object of $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X)$ equipped with addition and multiplication maps

$$+, \times : \mathscr{O}_X \times \mathscr{O}_X \to \mathscr{O}_X$$

satisfying the usual commutative ring axioms (which can be translated into the requirement that certain diagrams commute in the category $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X)$).

(C) We can identify $\mathscr{O}_X$ with a geometric morphism of topoi $\pi_* : \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X) \to \mathcal{E}$, where $\mathcal{E}$ is the *classifying topos for commutative rings* (given concretely by the formula $\mathcal{E} = \mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}_c, \mathcal{S}\mathrm{et})$, where $\mathrm{CAlg}^{\heartsuit}_c$ is the category of finitely presented commutative rings).

Our goal in this chapter is to discuss the relationship between (A), (B), and (C) in an ∞-categorical setting, where we replace the topological space X by an ∞-topos $\mathcal{X}$ and the category $\mathrm{CAlg}^{\heartsuit}$ of commutative rings by some other ∞-category $\mathcal{C}$.

We begin by considering the relationship between (A) and (C). Let $\mathcal{C}$ be an arbitrary ∞-category. Recall that a *$\mathcal{C}$-valued sheaf* on an ∞-topos $\mathcal{X}$ is a functor $\mathscr{F} : \mathcal{X}^{\mathrm{op}} \to \mathcal{C}$ which preserves small limits (Definition 1.3.1.4). Our goal in §21.1 is to give a criterion for the existence of a *universal* $\mathcal{C}$-valued sheaf: that is, a $\mathcal{C}$-valued sheaf $\mathscr{F}_0$ on an ∞-topos $\mathcal{E}$ having the property that *any* $\mathcal{C}$-valued sheaf on any ∞-topos $\mathcal{X}$ can be written as a pullback $f^* \mathscr{F}_0$, for any essentially unique geometric morphism $f : \mathcal{X} \to \mathcal{E}$. To fix ideas, let us assume that the ∞-category $\mathcal{C}$ is presentable (which guarantees that there is a good notion of pullback for $\mathcal{C}$-valued sheaves). The main result of §21.1 is that there exists a universal $\mathcal{C}$-valued sheaf if and only if $\mathcal{C}$ is *compactly assembled*: that is, if and only if $\mathcal{C}$ is a retract (in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories) of a compactly generated ∞-category (Theorem **??**).

We will be primarily interested in studying $\mathcal{C}$-valued sheaves in the case where the ∞-category $\mathcal{C}$ is compactly generated. Under this assumption we can write $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$, where $\mathcal{G}$ is an essentially small ∞-category which admits finite limits. In this case, the theory of $\mathcal{C}$-valued sheaves can be reformulated directly in terms of the ∞-category $\mathcal{G}$: for any ∞-topos $\mathcal{X}$, there is a fully faithful embedding $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{G}, \mathcal{X})$, whose essential image consists of those functors from $\mathcal{G}$ to $\mathcal{X}$ which preserve finite limits. In §21.2, we will refer to such functors as *$\mathcal{G}$-objects of $\mathcal{X}$* and regard them as objects of a full subcategory $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{G}, \mathcal{X})$ (moreover, we consider also the case where $\mathcal{G}$ does not admit finite limits, in which case we define $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ to consist of the *locally* left exact functors in the sense of Definition 20.4.2.1). We regard the equivalence $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \simeq \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ as an ∞-categorical analogue of the equivalence between (A) and (B): it allows us to encode the datum of a $\mathcal{C}$-valued sheaf on $\mathcal{X}$ in terms of diagrams in the ∞-topos $\mathcal{X}$ itself. For our purposes, the main virtue of this equivalence is it provides a convenient language for discussing locality conditions on $\mathcal{C}$-valued sheaves and morphisms between $\mathcal{C}$-valued sheaves. For example:

- To every Grothendieck topology τ on $\mathcal{G}$, we can associate a full subcategory $\mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{X})$, whose objects we refer to as *τ-local* $\mathcal{G}$-objects of $\mathcal{X}$ (see Definition 21.2.1.1).
- To every admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$, we can associate a (non-full) subcategory $\mathrm{Obj}_{\mathcal{G}}^{\mathrm{loc}}(\mathcal{X})$ containing all objects, whose morphisms are *local* morphisms between $\mathcal{G}$-objects (see Definition 21.2.4.1).

Specializing to the case where $\mathcal{C}$ is the category of commutative rings (or the ∞-category CAlg of $\mathbb{E}_{\infty}$-rings), this formalism recovers the usual notions of locality for sheaves of rings (or of $\mathbb{E}_{\infty}$-rings) and morphisms between them, as studied in §1).

Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$. In §21.4, we show that the collection of local morphisms determines a factorization system on the ∞-category $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$: that is, every morphism $\mathcal{O} \to \mathcal{O}'$ in $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ admits an essentially unique factorization as a composition $\mathcal{O} \xrightarrow{\alpha} \mathcal{O}'' \xrightarrow{\beta} \mathcal{O}'$, where β is local and α is *localizing* (that is, it is left orthogonal to all local morphisms of $\mathcal{G}$-objects). Moreover, if the $\mathcal{G}$-objects $\mathcal{O}$ and $\mathcal{O}'$ are themselves local with respect to some Grothendieck topology τ which is compatible with $\mathcal{G}^{\mathrm{ad}}$, then $\mathcal{O}''$ is also local with respect to τ: consequently, we also obtain a factorization system on the ∞-category $\mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{X})$. This is a special case of a more general assertion (Theorem 21.3.0.1), whose formulation uses the language of fractured ∞-topoi developed in Chapter 20.

Let $\mathcal{G}$ be a small ∞-category equipped with a Grothendieck topology τ and a compatible admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ (so that the triple $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ is a *geometric site*, in the sense of Definition 20.6.2.1). We define a *$\mathcal{G}$-structured ∞-topos* to be a pair $(\mathcal{X}, \mathscr{O})$, where $\mathcal{X}$ is an ∞-topos and $\mathscr{O}$ is a $\mathcal{G}$-object of $\mathcal{X}$. In §21.4, we organize the collection of $\mathcal{G}$-structured ∞-topoi into an ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$, in which a morphism from $(\mathcal{X}, \mathscr{O})$ to $(\mathcal{X}', \mathscr{O}')$ is given by a pair (f_*, α), where $f_* : \mathcal{X} \to \mathcal{X}'$ is a geometric morphism of ∞-topoi and $\alpha : f^* \mathscr{O}' \to \mathscr{O}$ is a local morphism of $\mathcal{G}$-objects on $\mathcal{X}$. The theory of $\mathcal{G}$-structured ∞-topoi can be regarded as a generalization of the theory of locally spectrally ringed ∞-topoi (hence also of locally ringed spaces, modulo slight caveats), in the sense that the latter can be recovered from the former by choosing the geometric site $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ appropriately (see Examples 21.4.1.13, 21.4.1.14, 21.4.1.17, 21.4.1.18, 21.4.1.19, and 21.4.1.20). We conclude this chapter by studying some of the formal properties enjoyed by the ∞-categories $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ (and of a slightly more general construction, which depends on a fractured ∞-topos rather than a geometric site).

Contents

21.1 $\mathcal{C}$-Valued Sheaves

Let $\mathcal{C}$ be an ∞-category. Recall that a *$\mathcal{C}$-valued sheaf* on an ∞-topos $\mathcal{X}$ is a functor $\mathscr{F} : \mathcal{X}^{\mathrm{op}} \to \mathcal{C}$ which preserves small limits (Definition 1.3.1.4). The collection of all $\mathcal{C}$-valued sheaves on $\mathcal{X}$ can be organized into an ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ which depends functorially on $\mathcal{X}$. More precisely, every geometric morphism of ∞-topoi $\mathcal{X} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{Y}$ determines a direct image functor $f_*^{\mathcal{C}} : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{Y}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$, given on objects by the formula

$$(f_*^{\mathcal{C}} \mathscr{F})(X) = \mathscr{F}(f^* X).$$

If the ∞-category $\mathcal{C}$ is presentable, then one can use the adjoint functor theorem to show that the functor $f_*^{\mathcal{C}}$ admits a left adjoint $f^*_{\mathcal{C}} : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{Y})$ (see Proposition **??**). Moreover, the construction $(f^*, \mathscr{F}) \mapsto f^*_{\mathcal{C}} \mathscr{F}$ determines a functor $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y}) \times \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{Y})$. This motivates the following:

Definition 21.1.0.1. Let $\mathcal{C}$ be a compactly generated ∞-category, let $\mathcal{E}$ be an ∞-topos, and let $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{E})$ be a $\mathcal{C}$-valued sheaf on $\mathcal{E}$. We will say that $\mathscr{F}$ is *universal* if, for every ∞-topos $\mathcal{X}$, the construction $f^* \mapsto f^*_{\mathcal{C}} \mathscr{F}$ induces an equivalence of ∞-categories

$$\mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}).$$

In this case, we will say that $\mathscr{F}$ *exhibits $\mathcal{E}$ as a classifying topos for $\mathcal{C}$-valued sheaves.*

Our primary goal in this section is to answer the following:

Question 21.1.0.2. Let $\mathcal{C}$ be a presentable ∞-category. Under what conditions does there exist a classifying ∞-topos $\mathcal{E}$ for $\mathcal{C}$-valued sheaves, in the sense of Definition 21.1.0.1?

Our first objective will be to construct a suitable candidate for the classifying ∞-topos $\mathcal{E}$. Let $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ spanned by those functors which preserve small filtered colimits. In §21.1.1, we show that $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ is an ∞-topos. This is a special case of a more general result (Proposition 21.1.1.2), which does not require the presentability of $\mathcal{C}$.

In general, it is easy to describe geometric morphisms *from* the ∞-topos $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ to another ∞-topos $\mathcal{X}$ (see Corollary 21.1.1.6). However, to classify geometric morphisms in the other direction, we need an additional hypothesis on $\mathcal{C}$. In §21.1.2, we introduce the class of *compactly assembled* ∞-categories (Definition 21.1.2.1), which is an enlargement of the class of compactly generated ∞-categories. The main result of this section, which we prove in §21.1.3, provides a classification of colimit-preserving functors from $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ to $\mathcal{X}$, where $\mathcal{X}$ is an ∞-topos (see Proposition 21.1.3.3 and Theorem 21.1.4.6). We will apply this result in §21.1.4 to give a complete answer to Question 21.1.0.2: if $\mathcal{C}$ is a presentable ∞-category, then there exists a classifying ∞-topos for $\mathcal{C}$-valued sheaves if and only if $\mathcal{C}$ is compactly assembled (Corollary 21.1.4.9). Moreover, if the classifying ∞-topos exists, then it is given by $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ (Theorem 21.1.4.3).

In §21.1.5, we combine the analyses of §21.1.1 and §21.1.4 to show that the construction $\mathcal{C} \mapsto \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ is fully faithful when restricted to compactly assembled ∞-categories (Corollary 21.1.5.3). Moreover, the ∞-topoi having the form $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ where $\mathcal{C}$ is presentable and compactly assembled (that is, those ∞-topoi which classify $\mathcal{C}$-valued sheaves) admit a simple characterization (Proposition 21.1.5.4).

In this book, we will be primarily interested in studying $\mathcal{C}$-valued sheaves in situations where the ∞-category $\mathcal{C}$ is compactly generated. Under this assumption, one can construct a classifying topos for $\mathcal{C}$-valued sheaves more directly (we will return to this point in §21.2). However, there are interesting examples of (presentable) compactly assembled ∞-categories which are not compactly generated. In §21.1.6, we show that an ∞-topos $\mathcal{X}$ is compactly assembled if and only if it is *exponentiable* in the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi (Theorem 21.1.6.12). Notable examples include the ∞-topoi $\mathcal{S}\mathrm{hv}(X)$ where X is a coherent topological space (in which case $\mathcal{S}\mathrm{hv}(X)$ is compactly generated; see the proof of Proposition 21.1.7.8) or where X is a locally compact Hausdorff space (in which case $\mathcal{S}\mathrm{hv}(X)$ is compactly assembled but might not be compactly generated; see Proposition 21.1.7.1 and Remark 21.1.7.2).

Remark 21.1.0.3. Our notion of *compactly assembled ∞-category* is essentially the ∞-categorical version of the notion of *continuous category* studied by Johnstone and Joyal in [104] (except that our definition includes an accessibility hypothesis; see Remark 21.1.2.12). Most of the results in this section can be regarded as ∞-categorical analogues of 1-categorical results which appear in [104].

21.1.1 The ∞-Topoi $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$

We begin with some general remarks.

Notation 21.1.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories which admit small filtered colimits. We let $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors $F : \mathcal{C} \to \mathcal{D}$ which preserve small filtered colimits.

Let $\mathcal{C}_0$ be an essentially small ∞-category and let $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ be the ∞-category of Ind-objects of $\mathcal{C}_0$. Then $\mathcal{C}$ is freely generated by $\mathcal{C}_0$ under (small) filtered colimits. More precisely, there is a fully faithful embedding $j : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ with the following universal property: for any ∞-category $\mathcal{D}$ which admits small filtered colimits, composition with j induces an equivalence of ∞-categories $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_0, \mathcal{D})$ (Proposition HTT.5.3.5.10). In particular, taking $\mathcal{D} = \mathcal{S}$, we deduce that $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ is equivalent to the ∞-category $\mathrm{Fun}(\mathcal{C}_0, \mathcal{S}) = \mathcal{P}(\mathcal{C}_0^{\mathrm{op}})$ of presheaves on $\mathcal{C}_0^{\mathrm{op}}$, and is therefore an ∞-topos. In fact, this is a special case of a more general phenomenon:

Proposition 21.1.1.2. *Let $\mathcal{C}$ be an accessible ∞-category which admits small filtered colimits and let $\mathcal{X}$ be an ∞-topos. Then the ∞-category $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{X})$ is also an ∞-topos.*

Proof. For any regular cardinal κ, let $\mathrm{Fun}^{\kappa}(\mathcal{C}, \mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{X})$ spanned by those functors which preserve small κ-filtered colimits. It follows immediately from the definitions that the subcategory $\mathrm{Fun}^{\kappa}(\mathcal{C}, \mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{X})$ is closed under small colimits. Since filtered colimits in $\mathcal{X}$ are left exact, it is also closed under finite limits. Taking $\kappa = \omega$, this proves (1).

Choose a regular cardinal κ for which $\mathcal{C}$ is κ-accessible and let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by the κ-compact objects, so that the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ extends to an equivalence of ∞-categories $\mathrm{Ind}_{\kappa}(\mathcal{C}_0) \simeq \mathcal{C}$. Applying Proposition HTT.5.3.5.10 , we deduce that the restriction functor $\mathrm{Fun}^{\kappa}(\mathcal{C}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}_0, \mathcal{X})$ is an equivalence of ∞-categories. Applying Proposition **??**, we deduce that the ∞-category $\mathrm{Fun}^{\kappa}(\mathcal{C}, \mathcal{X})$ is an ∞-topos.

By definition, $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{X})$ is the full subcategory of $\mathrm{Fun}^{\kappa}(\mathcal{C}, \mathcal{X})$ spanned by those functors $F : \mathcal{C} \to \mathcal{X}$ which satisfy the following condition:

(a) For every small filtered ∞-category $\mathcal{J}$ and every functor $\rho : \mathcal{J} \to \mathcal{C}$, the canonical map $\varinjlim_{J \in \mathcal{J}} F(\rho(J)) \to F(\varinjlim_{J \in \mathcal{J}} \rho(J))$ is an equivalence in $\mathcal{X}$.

Using Proposition HTT.5.3.1.18 , we see that (a) is equivalent to the following *a priori* weaker condition:

(b) For every small filtered partially ordered set A and every functor $\rho : A \to \mathcal{C}$, the canonical map $\theta^F_{A,\rho} : \varinjlim_{\alpha \in A} F(\rho(\alpha)) \to F(\varinjlim_{\alpha \in A} \rho(\alpha))$ is an equivalence in $\mathcal{X}$.

For a fixed partially ordered set A, let (b_A) denote the assertion that $\theta^F_{A,\rho}$ is an equivalence for every functor $\rho : A \to \mathcal{C}$. Note that we can write A as a κ-filtered union of κ-small filtered partially ordered sets (Lemma HTT.5.4.2.8). If the functor F belongs to $\mathrm{Fun}^{\kappa}(\mathcal{C}, \mathcal{X})$, then each map $\theta_{A,\rho}$ can be written as a κ-filtered colimit of maps $\theta_{A_0, \rho|_{A_0}}$, where $A_0 \subseteq A$ is filtered and κ-small. Consequently, the functor F belongs to $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{X})$ if and only if it satisfies condition (b_A) whenever A is κ-small.

Note that if we fix a filtered partially ordered set A, then the ∞-category $\mathrm{Fun}(A, \mathcal{C})$ is accessible (Proposition **??**). Consequently, there exists a small collection of functors $\{\rho_i : A \to \mathcal{C}\}_{i \in I(A)}$ which generates $\mathrm{Fun}(A, \mathcal{C})$ under τ-filtered colimits, where τ is some regular cardinal that we can assume is $\geqslant \kappa$. Our assumption that $F \in \mathrm{Fun}^{\kappa}(\mathcal{C}, \mathcal{X})$ then guarantees that the construction $\rho \mapsto \theta^F_{A,\rho}$ commutes with τ-filtered colimits. Consequently, condition (b_A) is equivalent to the following *a priori* weaker condition:

(c_A) For each $i \in I(A)$, the canonical map $\theta^F_{A,\rho_i} : \varinjlim_{\alpha \in A} F(\rho_i(\alpha)) \to F(\varinjlim_{\alpha \in A} \rho_i(\alpha))$ is an equivalence in $\mathcal{X}$.

Let I denote the set of pairs (A, ρ), where A ranges over a set of representatives for isomorphism classes of κ-small filtered partially ordered sets and $\rho : A \to \mathcal{C}$ belongs to $I(A)$. For each pair $(A, \rho) \in I$, the construction $F \mapsto \theta^F_{A,\rho}$ determines a functor $\theta_{A,\rho} : \mathrm{Fun}^{\kappa}(\mathcal{C}, \mathcal{X}) \to \mathrm{Fun}(\Delta^1, \mathcal{X})$ which preserves small colimits and finite limits. Let $\mathrm{Fun}'(\Delta^1, \mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{X})$ spanned by the equivalences in $\mathcal{X}$ (so that the diagonal embedding $\mathcal{X} \to \mathrm{Fun}'(\Delta^1, \mathcal{X})$ is an equivalence). The above argument shows that we have a pullback diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{X}) & \longrightarrow & \mathrm{Fun}^{\kappa}(\mathcal{C}, \mathcal{X})\theta_{A,\rho} \\ \downarrow & & \downarrow \prod_{(A,\rho) \in I} \\ \prod_{(A,\rho) \in I} \mathrm{Fun}'(\Delta^1, \mathcal{X}) & \longrightarrow & \prod_{(A,\rho) \in I} \mathrm{Fun}(\Delta^1, \mathcal{X}). \end{array}$$

Applying Proposition HTT.6.3.2.2 , we deduce that $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{X})$ is an ∞-topos. □

Remark 21.1.1.3. We will be primarily interested in the special case of Proposition 21.1.1.2 where $\mathcal{X} = \mathcal{S}$ is the ∞-topos of spaces (which often does not lose very much information; see Remark 21.1.2.4 below). The proof of Proposition 21.1.1.2 shows that $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ can be realized as an accessible left exact localization of the presheaf ∞-category $\mathcal{P}(\mathcal{C}_0^{\mathrm{op}}) = \mathrm{Fun}(\mathcal{C}_0, \mathcal{S})$ for an essentially small full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$. In general, we do not know if this localization is determined by a Grothendieck topology on $\mathcal{C}_0$ (or if the ∞-category $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ admits *any* presentation as an ∞-topos of sheaves for a Grothendieck topology).

Our next goal is to show that the ∞-topoi appearing in Proposition 21.1.1.2 can be characterized by a universal mapping property.

Notation 21.1.1.4. Let $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\omega}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$ whose objects are accessible ∞-categories which admit small filtered colimits and whose morphisms are functors which preserve small filtered colimits.

Proposition 21.1.1.5. *Let $\mathcal{X}$ be an ∞-topos. Then:*

(1) *The construction* $\mathcal{Y} \mapsto \mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})$ *determines a functor* $G : \infty\mathcal{T}\mathrm{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty^\omega$.

(2) *The functor* G *admits a left adjoint, given on objects by* $\mathcal{C} \mapsto \mathrm{Fun}^\omega(\mathcal{C}, \mathcal{X})$.

Proof. Note that for any ∞-topos $\mathcal{Y}$, the ∞-category $\mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})$ is closed under small filtered colimits in $\mathrm{Fun}(\mathcal{Y}, \mathcal{X})$. In particular, $\mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})$ admits small filtered colimits, and precomposition with any geometric morphism $f^* : \mathcal{Y} \to \mathcal{Z}$ determines a functor $\mathrm{Fun}^*(\mathcal{Z}, \mathcal{X}) \to \mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})$ which preserves small filtered colimits. Consequently, to prove (1), it suffices to show that every ∞-category of the form $\mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})$ is accessible, which is the content of Proposition HTT.6.3.1.13 . Assertion (2) follows from the observation that for $\mathcal{Y} \in \infty\mathcal{T}\mathrm{op}$ and $\mathcal{C} \in \widehat{\mathcal{C}\mathrm{at}}_\infty^\omega$, we have a canonical isomorphism of simplicial sets

$$\mathrm{Fun}^\omega(\mathcal{C}, \mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})) \simeq \mathrm{Fun}^*(\mathcal{Y}, \mathrm{Fun}^\omega(\mathcal{C}, \mathcal{X}));$$

both sides can be identified with the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \mathcal{Y}, \mathcal{X})$ spanned by those functors $F : \mathcal{C} \times \mathcal{Y} \to \mathcal{X}$ which preserve small filtered colimits in the first variable, small colimits in the second variable, and finite limits in the second variable. □

Corollary 21.1.1.6. *For every ∞-topos $\mathcal{Y}$, let* $\mathrm{Pt}(\mathcal{Y}) = \mathrm{Fun}^*(\mathcal{Y}, \mathcal{S})$ *denote the ∞-category of points of $\mathcal{Y}$. Then:*

(1) *The construction* $\mathcal{Y} \mapsto \mathrm{Pt}(\mathcal{Y})$ *determines a functor* $\mathrm{Pt} : \infty\mathcal{T}\mathrm{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty^\omega$.

(2) *The functor* Pt *has a left adjoint, given on objets by* $\mathcal{C} \mapsto \mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$.

21.1.2 Compactly Assembled ∞-Categories

We now introduce the main objects of interest in this section.

Definition 21.1.2.1. Let $\mathcal{C}$ be an ∞-category. We will say that $\mathcal{C}$ is *compactly assembled* if there exists a small ∞-category $\mathcal{C}_0$ such that $\mathcal{C}$ is a retract of $\mathrm{Ind}(\mathcal{C}_0)$ in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty^\omega$. In other words, there exist functors $U : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}_0)$ and $V : \mathrm{Ind}(\mathcal{C}_0) \to \mathcal{C}$ which preserve small filtered colimits such that the composition $V \circ U$ is equivalent to the identity functor $\mathrm{id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$.

Remark 21.1.2.2. In the statement of Definition 21.1.2.1, it is not necessary to assume *a priori* that the ∞-category $\mathcal{C}$ is accessible. Suppose that $\mathcal{C}$ admits small filtered colimits, and that there exists a small ∞-category $\mathcal{C}_0$ and a pair of functors

$$U : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}_0) \qquad V : \mathrm{Ind}(\mathcal{C}_0) \to \mathcal{C}$$

which preserve small filtered colimits such that $V \circ U \simeq \mathrm{id}_\mathcal{C}$. It follows that $\mathcal{C}$ can be identified with a limit of the tower of ∞-categories

$$\cdots \to \mathrm{Ind}(\mathcal{C}_0) \xrightarrow{V \circ U} \mathrm{Ind}(\mathcal{C}_0) \xrightarrow{V \circ U} \mathrm{Ind}(\mathcal{C}_0) \xrightarrow{V \circ U} \mathrm{Ind}(\mathcal{C}_0),$$

and is therefore accessible by virtue of Proposition HTT.5.4.7.3 .

Example 21.1.2.3. For any small ∞-category $\mathcal{C}_0$, the ∞-category $\mathrm{Ind}(\mathcal{C}_0)$ is compactly assembled. In particular, any compactly generated ∞-category is compactly assembled.

Remark 21.1.2.4. Let $\mathcal{C}$ be a compactly assembled ∞-category and let $\mathcal{X}$ be a presentable ∞-category. Then $\mathcal{X}$ is tensored over the ∞-category $\mathcal{S}$ of spaces via a functor $\otimes : \mathcal{S} \times \mathcal{X} \to \mathcal{X}$ which preserves small colimits separately in each variable. We define a functor

$$\lambda : \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}) \times \mathcal{X} \to \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{X}) \qquad \lambda(F, X)(C) = F(C) \otimes X.$$

Then λ preserves small colimits separately in each variable, and therefore induces a map of presentable ∞-categories $\mu : \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}) \otimes \mathcal{X} \to \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{X})$. We claim that μ is an equivalence of ∞-categories. To prove this, we observe that μ depends functorially on $\mathcal{C}$, so we can reduce to the case where $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ for some small ∞-category $\mathcal{C}_0$. In this case, the desired result follows from the observation that μ can be computed as the composition of equivalences

$$\begin{aligned} \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}) \otimes \mathcal{X} &\simeq \mathrm{Fun}(\mathcal{C}_0, \mathcal{S}) \otimes \mathcal{X} \\ &\simeq \mathrm{RFun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{C}_0, \mathcal{S})) \\ &\simeq \mathrm{Fun}(\mathcal{C}_0, \mathrm{RFun}(\mathcal{X}^{\mathrm{op}}, \mathcal{S})) \\ &\simeq \mathrm{Fun}(\mathcal{C}_0, \mathcal{X}) \\ &\simeq \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{X}). \end{aligned}$$

In particular, if $\mathcal{X}$ is an ∞-topos, then the ∞-topos $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{X})$ of Proposition 21.1.1.2 can be identified with the product of $\mathcal{X}$ with $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ in the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi.

Our first goal is to show that, if $\mathcal{C}$ is a compactly assembled ∞-category, then there is a *canonical* way to realize $\mathcal{C}$ as a retract of an ∞-category of the form $\mathrm{Ind}(\mathcal{C}_0)$. Essentially, the idea is to take $\mathcal{C}_0$ to be $\mathcal{C}$ itself. However, we need to take some care, since the ∞-category $\mathcal{C}$ will usually not be small.

Definition 21.1.2.5. Let $\mathcal{C}$ be an ∞-category (which is not assumed to be small) and let $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{S}})$ be the Yoneda embedding (which carries an object $C \in \mathcal{C}$ to the representable functor $\mathrm{Map}_{\mathcal{C}}(\bullet, C)$). We let $\mathrm{Ind}(\mathcal{C})$ denote the smallest full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{S}})$ which contains the essential image of j and is closed under the formation of small filtered colimits. We will refer to $\mathrm{Ind}(\mathcal{C})$ as *the ∞-category of* Ind-*objects of* $\mathcal{C}$.

Remark 21.1.2.6. When the ∞-category $\mathcal{C}$ is small, the ∞-category $\mathrm{Ind}(\mathcal{C})$ appearing in Definition 21.1.2.5 agrees with the ∞-category $\mathrm{Ind}(\mathcal{C})$ of Definition HTT.5.3.5.1 (this is the content of Proposition HTT.5.3.5.3 and Corollary **??**).

Remark 21.1.2.7. Let $\mathcal{C}$ be a locally small ∞-category: that is, an ∞-category for which the mapping spaces $\mathrm{Map}_{\mathcal{C}}(X, Y)$ are (essentially) small for every pair of objects $C, D \in \mathcal{C}$.

In this case, the essential image of the inclusion $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{S}})$ contains all representable functors and is closed under small filtered colimits. Consequently, we can (by slight abuse of notation) regard $\mathrm{Ind}(\mathcal{C})$ as a full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$.

Remark 21.1.2.8. Let $\mathcal{C}$ be an ∞-category which is not assumed to be small. Using Remark HTT.5.3.5.9 , we see that the ∞-category $\mathrm{Ind}(\mathcal{C})$ can be characterized by the following universal property:

- The ∞-category $\mathrm{Ind}(\mathcal{C})$ admits small filtered colimits.
- Let $\mathcal{D}$ be any ∞-category which admits small filtered colimits, and let $\mathrm{Fun}^{\omega}(\mathrm{Ind}(\mathcal{C}), \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathcal{D})$ spanned by those functors which preserve small filtered colimits. Then composition with the Yoneda embedding $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ induces an equivalence of ∞-categories $\mathrm{Fun}^{\omega}(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

We can summarize the situation more informally as follows: if $\mathcal{D}$ is an ∞-category which admits small filtered colimits, then any functor $f : \mathcal{C} \to \mathcal{D}$ admits an essentially unique extension to a functor $F : \mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ which preserves small filtered colimits. We will refer to the functor F as the Ind-*extension* of the functor f.

Example 21.1.2.9. Let $f : \mathcal{C} \to \mathcal{D}$ be any morphism of ∞-categories, and let us abuse notation by identifying $\mathcal{C}$ and $\mathcal{D}$ with full subcategories of $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Ind}(\mathcal{D})$, respectively. Applying Remark 21.1.2.8 to the composite functor $\mathcal{C} \to \mathcal{D} \hookrightarrow \mathrm{Ind}(\mathcal{D})$, we deduce that f admits an essentially unique extension to a functor $F : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ which preserves small filtered colimits. Concretely, the functor F is given by left Kan extension along the induced map $f^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. In particular, if the functor f is fully faithful, then F is also fully faithful.

We can now give a characterization of compactly assembled ∞-categories:

Theorem 21.1.2.10. *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is compactly assembled if and only if it satisfies the following conditions:*

(1) *The ∞-category $\mathcal{C}$ is accessible.*

(2) *The ∞-category $\mathcal{C}$ admits small filtered colimits.*

(3) *Let $G : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ be the* Ind-*extension of the identity functor* $\mathrm{id}_{\mathcal{C}}$. *Then G admits a left adjoint $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.*

Remark 21.1.2.11. Let $\mathcal{C}$ be an arbitrary ∞-category, and let $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ be the Yoneda embedding. Then $\mathcal{C}$ admits small filtered colimits if and only if the functor $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ admits a left adjoint G. In this case, the functor G coincides with the Ind-extension of the identity map $\mathrm{id} : \mathcal{C} \to \mathcal{C}$. Consequently, hypotheses (2) and (3) of Theorem 21.1.2.10 can be summarized by saying that the functor j admits a left adjoint, which itself admits a further left adjoint.

Remark 21.1.2.12. In [104], Johnstone and Joyal define a *continuous category* to be a category $\mathcal{C}$ which satisfies conditions (2) and (3) of Theorem 21.1.2.10 (and then characterize such categories as those which satisfy an analogue of Definition 21.1.2.1, without a smallness condition on $\mathcal{C}_0$). Consequently, our notion of compactly assembled ∞-category (Definition 21.1.2.5) is essentially an ∞-categorical analogue of the notion of continuous category (except that we also require accessibility).

Example 21.1.2.13. Let $\mathcal{C}_0$ be a small ∞-category and let $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$. Then $\mathcal{C}$ obviously satisfies conditions (1) and (2) of Theorem 21.1.2.10. We claim that it also satisfies condition (3): that is, the functor $G : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ given by the Ind-extension of $\mathrm{id}_{\mathcal{C}}$ admits a left adjoint $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$. Concretely, the functor F can be described as the Ind-extension of the iterated Yoneda embedding $J : \mathcal{C}_0 \to \mathrm{Ind}(\mathcal{C}_0) \to \mathrm{Ind}(\mathrm{Ind}(\mathcal{C}_0)) = \mathrm{Ind}(\mathcal{C})$. To prove that this functor has the desired universal property, we must show that there exists a homotopy equivalence $\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(F(C), X) \simeq \mathrm{Map}_{\mathcal{C}}(C, G(X))$, depending functorially on $C \in \mathcal{C}$ and $X \in \mathrm{Ind}(\mathcal{C})$. Note that both sides are compatible with filtered colimits in C, so it suffices to treat construct such an equivalence when $C \in \mathcal{C}$ is compact. In this case, the the functors $X \mapsto \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(F(C), X)$ and $X \mapsto \mathrm{Map}_{\mathcal{C}}(C, G(X))$ both commute with filtered colimits in X, so we can also reduce to the case where X is a compact object of $\mathrm{Ind}(\mathcal{C})$: that is, it arises from an object $D \in \mathcal{C}$. In this case, both mapping spaces can be identified with $\mathrm{Map}_{\mathcal{C}}(C, D)$.

The proof of Theorem 21.1.2.10 depends on the following:

Lemma 21.1.2.14. *Let $G : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which is a retract, in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$, of another functor $G' : \mathcal{C}' \to \mathcal{D}'$. If G' admits a left adjoint and $\mathcal{C}$ is idempotent complete, then G also admits a left adjoint.*

Proof. Since G is a retract of G', we can choose a commutative diagram of ∞-categories

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{U} & \mathcal{C}' & \xrightarrow{V} & \mathcal{C} \\ \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle G'} & & \downarrow{\scriptstyle G} \\ \mathcal{D} & \xrightarrow{U'} & \mathcal{D}' & \xrightarrow{V'} & \mathcal{D} \end{array}$$

where the outer rectangle is the identity transformation from G to itself in $\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$. Let F' denote a left adjoint to G', and fix an object $D \in \mathcal{D}$. For every object $C \in \mathcal{C}$, we have canonical maps

$$\begin{aligned}
\mathrm{Map}_{\mathcal{D}}(D, GC) &\rightarrow \mathrm{Map}_{\mathcal{D}'}(U'DU'GC) \\
&\simeq \mathrm{Map}_{\mathcal{D}}(U'D, G'UC) \\
&\simeq \mathrm{Map}_{\mathcal{D}'}(F'U'D, UC) \\
&\rightarrow \mathrm{Map}_{\mathcal{C}}(VF'U'D, VUC) \\
&\simeq \mathrm{Map}_{\mathcal{C}}(VF'U'D, C) \\
&\rightarrow \mathrm{Map}_{\mathcal{D}}(GVF'U'D, GC) \\
&\simeq \mathrm{Map}_{\mathcal{D}}(V'G'F'U'D, GC) \\
&\rightarrow \mathrm{Map}_{\mathcal{D}}(V'U'D, GC) \\
&\simeq \mathrm{Map}_{\mathcal{D}}(D, GC).
\end{aligned}$$

These maps depend functorially on C, and the composition is homotopic to the identity (functorially in C). It follows that the functor $C \mapsto \mathrm{Map}_{\mathcal{D}}(D, GC)$ is a retract (in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$) of the functor corepresented by $VF'U'D$. Our assumption that $\mathcal{C}$ is idempotent complete guarantees that the collection of corepresentable functors is closed under retracts in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$, and therefore contains the functor $C \mapsto \mathrm{Map}_{\mathcal{D}}(D, GC)$. Using the criterion of Proposition HTT.5.2.4.2 , we deduce that G has a left adjoint. □

Before moving on to the proof of Theorem 21.1.2.10, we note the following easy consequence of Lemma 21.1.2.14:

Corollary 21.1.2.15. *Let $\mathcal{C}$ be an ∞-category which is a retract of another ∞-category $\mathcal{C}'$ (that is, there exist functors $U : \mathcal{C} \rightarrow \mathcal{C}'$ and $V : \mathcal{C}' \rightarrow \mathcal{C}$ such that $V \circ U$ is equivalent to the identity functor $\mathrm{id}_{\mathcal{C}}$). Suppose that $\mathcal{C}'$ admits K-indexed colimits, for some simplicial set K. If $\mathcal{C}$ is idempotent complete, then it also admits K-indexed colimits.*

Remark 21.1.2.16. In the statement of Corollary 21.1.2.15, we do not require any compatibility of the functors U and V with K-indexed colimits.

Proof of Corollary 21.1.2.15. Note that an ∞-category $\mathcal{D}$ admits K-indexed colimits if and only if the diagonal map $\delta_{\mathcal{D}} : \mathcal{D} \rightarrow \mathrm{Fun}(K, \mathcal{D})$ has a left adjoint. It will therefore suffice to show that if $\delta_{\mathcal{C}'}$ has a left adjoint and $\mathcal{C}$ is idempotent complete, then the functor $\delta_{\mathcal{C}}$ also admits a left adjoint. This is a special case of Lemma 21.1.2.14 (since the functor $\delta_{\mathcal{C}}$ is a retract of $\delta_{\mathcal{C}'}$). □

Proof of Theorem 21.1.2.10. Suppose first that $\mathcal{C}$ is a compactly assembled ∞-category. Then $\mathcal{C}$ admits small filtered colimits, and $\mathcal{C}$ is accessible by virtue of Remark 21.1.2.2. Let $G : \mathrm{Ind}(\mathcal{C}) \rightarrow \mathcal{C}$ be the Ind-extension of the identity functor; we wish to show that G admits a left adjoint. Write $\mathcal{C}$ as a retract (in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\omega}$) of $\mathcal{C}' = \mathrm{Ind}(\mathcal{C}_0)$, where $\mathcal{C}_0$ is a small ∞-category. Then G is a retract (in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}^{\omega}_{\infty}$, and therefore also in $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$) of the functor $G' : \mathrm{Ind}(\mathcal{C}') \to \mathcal{C}'$ given by the Ind-extension of the identity functor $\mathrm{id}_{\mathrm{Ind}(\mathcal{C}_0)}$. By virtue of Lemma 21.1.2.14, it will suffice to show that the functor G' admits a left adjoint (note that the ∞-category $\mathrm{Ind}(\mathcal{C})$ is automatically idempotent complete, since it admits small filtered colimits), which was established in Example 21.1.2.13.

Now suppose that $\mathcal{C}$ is an ∞-category satisfying conditions (1), (2), and (3) of Theorem 21.1.2.10; we wish to show that $\mathcal{C}$ is compactly assembled. Let $G : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ be the Ind-extension of the identity functor $\mathrm{id}_{\mathcal{C}}$, and let $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ be a left adjoint to G (whose existence is guaranteed by assumption (3)). Note that G has a fully faithful right adjoint (given by the Yoneda embedding $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$), so that the functor F is also fully faithful. In particular, the composition $G \circ F$ is equivalent to the identity (via the counit map $\mathrm{id}_{\mathcal{C}} \to G \circ F$).

If $\mathcal{C}_0 \subseteq \mathcal{C}$ is an essentially small full subcategory, we will abuse notation by identifying $\mathrm{Ind}(\mathcal{C}_0)$ with its essential image in $\mathrm{Ind}(\mathcal{C})$. Note that the union $\bigcup_{\mathcal{C}_0} \mathrm{Ind}(\mathcal{C}_0)$ contains the essential image of the Yoneda embedding $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ and is closed under small filtered colimits, and therefore coincides with $\mathrm{Ind}(\mathcal{C})$. Since $\mathcal{C}$ is accessible, there exists a small collection of objects $\{X_\alpha\}$ which generates $\mathcal{C}$ under small filtered colimits. Choose an essentially small full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ for which each $F(X_\alpha)$ belongs to the subcategory $\mathrm{Ind}(\mathcal{C}_0) \subseteq \mathrm{Ind}(\mathcal{C})$. Since the functor F commutes with small filtered colimits, it follows that F factors through a functor $F_0 : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}_0)$. Let $G_0 = G|_{\mathrm{Ind}(\mathcal{C}_0)}$. Then $G_0 \circ F_0$ is equivalent to the identity functor $\mathrm{id}_{\mathcal{C}}$, so that F_0 and G_0 exhibit $\mathcal{C}$ as a retract of $\mathrm{Ind}(\mathcal{C}_0)$ in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}^{\omega}_{\infty}$. □

Corollary 21.1.2.17. *Let $\mathcal{C}$ be an idempotent complete ∞-category. The following conditions are equivalent:*

- (a) *The ∞-category $\mathcal{C}$ is compactly assembled.*
- (b) *There exists a pair of adjoint functors $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$ where $\mathcal{D}$ is an ∞-category of the form $\mathrm{Ind}(\mathcal{D}_0)$ for some small ∞-category $\mathcal{D}_0$, the functor g preserve small filtered colimits, and the unit map $\mathrm{id}_{\mathcal{C}} \to g \circ f$ is an equivalence.*

Proof. Suppose that (a) is satisfied and let $G : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ be an Ind-extension of the identity functor $\mathrm{id}_{\mathcal{C}}$. Then G admits a left adjoint $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$. The proof of Theorem 21.1.2.10 shows that F factors through $\mathrm{Ind}(\mathcal{C}_0)$ for some essentially small subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$. Setting $G_0 = G|_{\mathrm{Ind}(\mathcal{C}_0)}$, we see that the adjunction $\mathcal{C} \underset{G_0}{\overset{F}{\rightleftarrows}} \mathrm{Ind}(\mathcal{C}_0)$ satisfies the requirements of (b) (note that the unit map $\mathrm{id}_{\mathcal{C}} \to G_0 \circ F$ is an equivalence because the functor F is fully faithful: this is equivalent to the full-faithfulness of the Yoneda embedding $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$, since both functors are adjoint to G).

Now suppose we are given an adjunction $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$ satisfying the requirements of (b). Then f and g exhibit $\mathcal{C}$ as a retract of $\mathcal{D}$. Since $\mathcal{C}$ is idempotent-complete, Corollary 21.1.2.15 implies that $\mathcal{C}$ admits small filtered colimits. The functor f automatically preserves small filtered colimits (since it is a left adjoint). Applying Remark 21.1.2.2, we see that $\mathcal{C}$ is accessible, so that f and g exhibit $\mathcal{C}$ as a retract of $\mathcal{D}$ in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty^\omega$. □

Corollary 21.1.2.18. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

- (a) *The ∞-category $\mathcal{C}$ is presentable and compactly assembled.*
- (b) *The ∞-category $\mathcal{C}$ is presentable. Moreover, in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories, $\mathcal{C}$ can be written as the retract of a compactly generated ∞-category $\mathcal{C}'$.*
- (c) *The ∞-category $\mathcal{C}$ is accessible and admits small filtered colimits. Moreover, in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty^\omega$, $\mathcal{C}$ can be written as a retract of a compactly generated ∞-category $\mathcal{C}'$.*

Proof. Assume first that (a) is satisfied, and let $G : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ be an Ind-extension of the identity functor. Applying Theorem 21.1.2.10, we deduce that G admits a left adjoint $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$. Moreover, the proof of Theorem 21.1.2.10 shows that F factors through $F_0 : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}_0)$, for some essentially small full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$. Enlarging $\mathcal{C}_0$ if necessary, we may assume that $\mathcal{C}_0$ admits finite colimits. In this case, the functors $F_0 : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}_0)$ and $G|_{\mathrm{Ind}(\mathcal{C}_0)}$ exhibit $\mathcal{C}$ as a retract of $\mathrm{Ind}(\mathcal{C}_0)$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories, so that condition (b) is satisfied.

The implication $(b) \Rightarrow (c)$ is obvious. We will complete the proof by showing that $(c) \Rightarrow (a)$. Assume that $\mathcal{C}'$ is a compactly generated ∞-category and that $\mathcal{C}$ is a retract of $\mathcal{C}'$ in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty^\omega$. Then $\mathcal{C}$ is compactly assembled. In particular, it is accessible, and therefore idempotent complete. To show that it is presentable, it suffices to show that $\mathcal{C}$ admits small colimits. This follows from Corollary 21.1.2.15, since the ∞-category $\mathcal{C}'$ admits small colimits. □

Corollary 21.1.2.19. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories, and let $\mathcal{C} \otimes \mathcal{D}$ denote their tensor product (in the sense of §HA.4.8.1). If $\mathcal{C}$ and $\mathcal{D}$ are compactly assembled, then $\mathcal{C} \otimes \mathcal{D}$ is also compactly assembled.*

Proof. By virtue of Corollary 21.1.2.18, we can write $\mathcal{C}$ and $\mathcal{D}$ as retracts (in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$) of compactly generated ∞-categories $\mathcal{C}'$ and $\mathcal{D}'$. Then $\mathcal{C} \otimes \mathcal{D}$ is a retract of $\mathcal{C}' \otimes \mathcal{D}'$, and is therefore compactly assembled (since $\mathcal{C}' \otimes \mathcal{D}'$ is also compactly generated). □

21.1.3 Another Universal Property of $\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$

Let $\mathcal{C}$ be a compactly assembled ∞-category. Our goal in this section is to characterize $\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$ by a universal property in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories:

that is, to classify colimit-preserving functors $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}) \to \mathcal{X}$, when $\mathcal{X}$ is a presentable ∞-category. We begin by treating an easy special case:

Proposition 21.1.3.1. *Let $\mathcal{C}$ be an ∞-category which has the form $\mathrm{Ind}(\mathcal{C}_0)$, for some small ∞-category $\mathcal{C}$. For any presentable ∞-category $\mathcal{X}$, there is a canonical equivalence of ∞-categories*

$$\rho_{\mathcal{C}} : \mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \simeq \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X}).$$

Here $\mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ spanned by those functors which preserve small filtered limits.

Proof. Let $\mathcal{C}_c \subseteq \mathcal{C}$ be the full subcategory spanned by the compact objects of $\mathcal{C}$. Note that the Yoneda embedding $j : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ carries $\mathcal{C}_c^{\mathrm{op}}$ into $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$. We therefore have restriction functors

$$\mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}_c^{\mathrm{op}}, \mathcal{X}) \leftarrow \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X}).$$

Using Proposition HTT.5.3.5.10 and Theorem HTT.5.1.5.6 , we see that these restriction functors are equivalences of ∞-categories. □

Remark 21.1.3.2. In the statement of Proposition 21.1.3.1, we do not need the full strength of our assumption that $\mathcal{X}$ is presentable: it is enough to assume that $\mathcal{X}$ admits small colimits and small filtered limits. The same observation applies to the all of the other results we prove in this section.

The main result of this section is the following generalization of Proposition 21.1.3.1:

Proposition 21.1.3.3. *Let $\mathcal{X}$ be a presentable ∞-category. For every compactly assembled ∞-category $\mathcal{C}$, there exists an equivalence of ∞-categories $\rho_{\mathcal{C}} : \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X}) \simeq \mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$. Moreover, these equivalences can be chosen to have the following properties:*

(1) *When $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ for some small ∞-category $\mathcal{C}$, the functor $\rho_{\mathcal{C}}$ agrees with the equivalence constructed in the proof of Proposition 21.1.3.1.*

(2) *Suppose we are given compactly assembled ∞-categories $\mathcal{C}$ and $\mathcal{D}$ and a pair of adjoint functors $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$ where f and g preserve small filtered colimits. Let $T_0 : \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}) \to \mathrm{Fun}^{\omega}(\mathcal{D}, \mathcal{S})$ be the functor given by precomposition with g. Then the diagram of ∞-categories*

$$\begin{array}{ccc} \mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) & \xrightarrow{\rho_{\mathcal{C}}} & \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X}) \\ \downarrow{\circ f^{\mathrm{op}}} & & \downarrow{\circ T_0} \\ \mathrm{Fun}_{\omega}(\mathcal{D}^{\mathrm{op}}, \mathcal{X}) & \xrightarrow{\rho_{\mathcal{D}}} & \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{D}, \mathcal{S}), \mathcal{X}) \end{array}$$

commutes (up to canonical homotopy).

Remark 21.1.3.4. In the situation of Proposition 21.1.3.3, the equivalence $\rho_{\mathcal{C}}$ is characterized by properties (1) and (2). To see this, note that we can choose an adjunction $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$ where $\mathcal{D} = \mathrm{Ind}(\mathcal{D}_0)$ for some small ∞-category $\mathcal{D}_0$, and the functor f is fully faithful (Corollary 21.1.2.17). In this case, property (2) guarantees that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) & \xrightarrow{\rho_{\mathcal{C}}} & \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X}) \\ \downarrow{\circ f^{\mathrm{op}}} & & \downarrow{\circ T_0} \\ \mathrm{Fun}_{\omega}(\mathcal{D}^{\mathrm{op}}, \mathcal{X}) & \xrightarrow{\rho_{\mathcal{D}}} & \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{D}, \mathcal{S}), \mathcal{X}) \end{array}$$

where the vertical maps are fully faithful embeddings. Consequently, the equivalence $\rho_{\mathcal{C}}$ is determined by the equivalence $\rho_{\mathcal{D}}$, which is determined by property (1).

The rest of this section is devoted to the proof of Proposition 21.1.3.3. The essential difficulty that we need to overcome is that the equivalence $\rho_{\mathcal{C}} : \mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \simeq \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X})$ of Proposition 21.1.3.1 was constructed using the auxiliary ∞-category $\mathrm{Fun}(\mathcal{C}_c^{\mathrm{op}}, \mathcal{X})$, and is therefore not obviously functorial in $\mathcal{C}$ (with respect to functors that do not preserve compact objects). Our first goal will be to rephrase the proof of Proposition 21.1.3.1 in a way that does not make explicit reference to compact objects of $\mathcal{C}$ (and can therefore be generalized to situations where $\mathcal{C}$ is not generated by compact objects).

Notation 21.1.3.5. Let $\mathcal{C}$ be a compactly assembled ∞-category. We let $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ spanned by the corepresentable functors (that is, the essential image of the Yoneda embedding $j : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$). We let $\mathrm{Fun}^{+}(\mathcal{C}, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ spanned by those functors $U : \mathcal{C} \to \mathcal{S}$ which belong either to $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}, \mathcal{S})$ or $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$, and let $\mathrm{Fun}^{\circ}(\mathcal{C}, \mathcal{S})$ denote the intersection $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}, \mathcal{S}) \cap \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$. Note that $\mathrm{Fun}^{\circ}(\mathcal{C}, \mathcal{S})$ is the full subcategory spanned by those functors $U : \mathcal{C} \to \mathcal{S}$ which are corepresented by *compact* objects of $\mathcal{C}$, and is therefore equivalent to the ∞-category $\mathcal{C}_c^{\mathrm{op}}$. Beware that, in general, the ∞-category $\mathrm{Fun}^{\circ}(\mathcal{C}, \mathcal{S})$ may be empty.

Remark 21.1.3.6 (Functoriality)**.** Suppose we are given a pair of adjoint functors $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are compactly assembled and the functors f and g preserve small filtered colimits. Then precomposition with g induces a functor $\mathrm{Fun}(\mathcal{C}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{S})$, which carries $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ into $\mathrm{Fun}^{\omega}(\mathcal{D}, \mathcal{S})$ (since the functor g preserves small filtered colimits) and $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}, \mathcal{S})$ into $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{D}, \mathcal{S})$ (since the functor g admits a left adjoint). We therefore obtain a functor $T : \mathrm{Fun}^+(\mathcal{C}, \mathcal{S}) \to \mathrm{Fun}^+(\mathcal{D}, \mathcal{S})$. Note that the diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathrm{Fun}^+(\mathcal{C}, \mathcal{S}) \\ \downarrow{\scriptstyle f^{\mathrm{op}}} & & \downarrow{\scriptstyle T} \\ \mathcal{D}^{\mathrm{op}} & \longrightarrow & \mathrm{Fun}^+(\mathcal{D}, \mathcal{S}) \end{array}$$

commutes up to canonical homotopy (here the horizontal maps are given by the Yoneda embeddings of $\mathcal{C}$ and $\mathcal{D}$, respectively).

Lemma 21.1.3.7. *Let $\mathcal{C}_0$ be a small ∞-category, let $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$, and let $\mathrm{Fun}^{\circ}(\mathcal{C}, \mathcal{S})$ be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ spanned by those functors which are corepresentable by* compact *objects of $\mathcal{C}$. Then:*

(*i*) *For every object $V \in \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$, the inclusion functor*

$$\mathrm{Fun}^{\circ}(\mathcal{C}, \mathcal{S}) \times_{\mathrm{Fun}^+(\mathcal{C},\mathcal{S})} \mathrm{Fun}^+(\mathcal{C}, \mathcal{S})_{/V} \hookrightarrow \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}, \mathcal{S}) \times_{\mathrm{Fun}^+(\mathcal{C},\mathcal{S})} \mathrm{Fun}^+(\mathcal{C}, \mathcal{S})_{/V}$$

is left cofinal.

(*ii*) *For every object $U \in \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}, \mathcal{S})$, the inclusion functor*

$$\mathrm{Fun}^{\circ}(\mathcal{C}, \mathcal{S}) \times_{\mathrm{Fun}^+(\mathcal{C},\mathcal{S})} \mathrm{Fun}^+(\mathcal{C}, \mathcal{S})_{U/} \hookrightarrow \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}) \times_{\mathrm{Fun}^+(\mathcal{C},\mathcal{S})} \mathrm{Fun}^+(\mathcal{C}, \mathcal{S})_{U/}$$

is right cofinal.

Proof. By virtue of Proposition HTT.4.1.3.1 , assertions (*i*) and (*ii*) are both equivalent to the following:

(*iii*) Suppose we are given a morphism $\alpha : U \to V$ in $\mathrm{Fun}^+(\mathcal{C}, \mathcal{S})$, where $U \in \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}, \mathcal{S})$ and $V \in \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$. Let $\mathcal{E}$ be the full subcategory of $\mathrm{Fun}^+(\mathcal{C}, \mathcal{S})_{U//V}$ spanned by those objects whose image in $\mathrm{Fun}^+(\mathcal{C}, \mathcal{S})$ belongs to $\mathrm{Fun}^{\circ}(\mathcal{C}, \mathcal{S})$. Then $\mathcal{E}$ is weakly contractible.

Choose an object $C \in \mathcal{C}$ which corepresents the functor U, so that α is classified by a point $\eta \in V(C)$. Using Proposition HTT.**??** , we see that the weak homotopy type of $\mathcal{E}$ can be computed as a colimit $\varinjlim_{C_0 \to C}(V(C_0) \times_{V(C)} \{\eta\})$, where the colimit is taken over the full subcategory of $\mathcal{C}_{/C}$ spanned by those maps $C_0 \to C$ where C_0 is compact. The desired result now follows from our assumption that the functor V commutes with small filtered colimits. □

Lemma 21.1.3.8. *Let $\mathcal{C}_0$ be a small ∞-category, let $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$, and define $\mathrm{Fun}^{\circ}(\mathcal{C}, \mathcal{S})$ as in Lemma 21.1.3.7. Suppose we are given a functor $F : \mathrm{Fun}^+(\mathcal{C}, \mathcal{S}) \to \mathcal{X}$, where the ∞-category $\mathcal{X}$ admits small colimits. The following conditions are equivalent:*

(a) *The functor F is a left Kan extension of its restriction $F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})}$.*

(b) *The functor $F|_{\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})}$ is a left Kan extension of its restriction $F|_{\mathrm{Fun}^{\circ}(\mathcal{C},\mathcal{S})}$.*

(c) *The functor $F|_{\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})}$ preserves small colimits.*

Moreover, every functor $F_0 : \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S}) \to \mathcal{X}$ admits a left Kan extension $F : \mathrm{Fun}^{+}(\mathcal{C},\mathcal{S})$.

Proof. The equivalence of (a) and (b) follows from Lemma 21.1.3.7, and the equivalence of (b) and (c) follows from Lemma HTT.5.1.5.5 (since the inclusion $\mathrm{Fun}^{\circ}(\mathcal{C},\mathcal{S}) \hookrightarrow \mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})$ is equivalent to the Yoneda embedding $\mathcal{C}_c^{\mathrm{op}} \hookrightarrow \mathcal{P}(\mathcal{C}_c^{\mathrm{op}})$, where $\mathcal{C}_c \subseteq \mathcal{C}$ is the full subcategory spanned by the compact objects). To prove the final assertion, it will suffice (by virtue of Lemma 21.1.3.7) to show that every functor $\mathrm{Fun}^{\circ}(\mathcal{C},\mathcal{S}) \to \mathcal{X}$ admits a left Kan extension to $\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})$, which also follows from Lemma HTT.5.1.5.5. □

Lemma 21.1.3.9. *Let $\mathcal{C}_0$ be a small ∞-category, let $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$, and define $\mathrm{Fun}^{\circ}(\mathcal{C},\mathcal{S})$ as in Lemma 21.1.3.7. Suppose we are given a functor $F : \mathrm{Fun}^{+}(\mathcal{C},\mathcal{S}) \to \mathcal{X}$, where the ∞-category $\mathcal{X}$ admits small filtered limits. The following conditions are equivalent:*

(a) *The functor F is a right Kan extension of its restriction $F|_{\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})}$.*

(b) *The functor $F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})}$ is a right Kan extension of its restriction to $F|_{\mathrm{Fun}^{\circ}(\mathcal{C},\mathcal{S})}$.*

(c) *The functor $F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})}$ preserves small filtered limits.*

Moreover, every functor $F_0 : \mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S}) \to \mathcal{X}$ admits a right Kan extension $F : \mathrm{Fun}^{+}(\mathcal{C},\mathcal{S}) \to \mathcal{X}$.

Proof. The equivalence of (a) and (b) follows from Lemma 21.1.3.7, and the equivalence of (b) and (c) follows from Lemma HTT.5.3.5.8 (since the inclusion $\mathrm{Fun}^{\circ}(\mathcal{C},\mathcal{S}) \hookrightarrow \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})$ is equivalent to the opposite of the inclusion $\mathcal{C}_c \hookrightarrow \mathrm{Ind}(\mathcal{C}_c) \simeq \mathcal{C}$). To prove the final assertion, it will suffice (by virtue of Lemma 21.1.3.7) to show that every functor $\mathrm{Fun}^{\circ}(\mathcal{C},\mathcal{S}) \to \mathcal{X}$ admits a right Kan extension to $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})$, which also follows from Lemma HTT.5.3.5.8. □

Lemma 21.1.3.10. *Suppose we are given an adjunction $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are compactly assembled ∞-categories, and let $T : \mathrm{Fun}^{+}(\mathcal{C},\mathcal{S}) \to \mathrm{Fun}^{+}(\mathcal{D},\mathcal{S})$ be as in Remark 21.1.3.6. Suppose we are given a functor $F : \mathrm{Fun}^{+}(\mathcal{D},\mathcal{S}) \to \mathcal{X}$, where $\mathcal{X}$ is a presentable ∞-category. Then:*

(i) *If F is a left Kan extension of $F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{D},\mathcal{S})}$, then $F \circ T$ is a left Kan extension of $(F \circ T)|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})}$.*

(ii) *If the functor $F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{D},\mathcal{S})}$ commutes with small filtered limits, then $(F \circ T)|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})}$ commutes with small filtered limits.*

(iii) *If F is a right Kan extension of $F|_{\mathrm{Fun}^{\omega}(\mathcal{D},\mathcal{S})}$, then $F \circ T$ is a right Kan extension of $(F \circ T)|_{\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})}$.*

(iv) *If the functor $F|_{\mathrm{Fun}^{\omega}(\mathcal{D},\mathcal{S})}$ commutes with small colimits, then the functor $(F \circ T)|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})}$ commutes with small colimits.*

Proof. Assertion (ii) follows from the observation that the functor f^{op} preserves small filtered limits. Similarly, assertion (iv) follows from the observation that the functor $T|_{\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})} \to \mathrm{Fun}^{\omega}(\mathcal{D},\mathcal{S})$ preserves small colimits (since colimits can be computed pointwise on both sides). We will prove (i); the proof of (iii) is similar. Assume that $F : \mathrm{Fun}^{+}(\mathcal{D},\mathcal{S}) \to \mathcal{X}$ is a left Kan extension of $F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{D},\mathcal{S})}$, and fix an object $V \in \mathrm{Fun}^{+}(\mathcal{C},\mathcal{S})$. We will show that $F \circ T$ is a left Kan extension of $(F \circ T)|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})}$ at V. Without loss of generality we may assume that $V \in \mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})$ (otherwise the result is trivial); we wish to show that the canonical map $\rho : \varinjlim_{U \in \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})_{/V}} (F \circ T)(U) \to (F \circ T)(V)$ is an equivalence. Since F is a left Kan extension $F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{D},\mathcal{S})}$ at $T(V)$, we can identify the codomain of ρ with the colimit $\varinjlim_{V' \in \mathrm{Fun}^{\mathrm{rep}}(\mathcal{D},\mathcal{S})_{/T(V)}} F(V')$. It will therefore suffice to show that the functor T induces a left cofinal map

$$\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})_{/V} \to \mathrm{Fun}^{\mathrm{rep}}(\mathcal{D},\mathcal{S})_{/T(V)}.$$

In fact, this functor admits a left adjoint (induced by the functor g^{op}). □

Lemma 21.1.3.11. . *Let $\mathcal{C}$ be a compactly assembled ∞-category and let $\mathcal{X}$ be a presentable ∞-category. Then:*

(a) *Let $F_0 : \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S}) \to \mathcal{X}$ be any functor. Then F_0 admits a left Kan extension $F : \mathrm{Fun}^{+}(\mathcal{C},\mathcal{S}) \to \mathcal{X}$, and the restriction $F|_{\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})} : \mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S}) \to \mathcal{X}$ preserves small colimits. Moreover, if F_0 preserves small filtered limits, then F is a right Kan extension of $F|_{\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})}$.*

(b) *Let $F_1 : \mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S}) \to \mathcal{X}$ be any functor. Then F_1 admits a right Kan extension $F : \mathrm{Fun}^{+}(\mathcal{C},\mathcal{S}) \to \mathcal{X}$, and the restriction $F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})} : \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S}) \to \mathcal{X}$ preserves small filtered limits. Moreover, if F_1 preserves small colimits, then F is a left Kan extension of $F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})}$.*

Proof. We will prove (a); the proof of (b) is similar. Since $\mathcal{C}$ is compactly assembled, the canonical map $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint f. Applying Corollary 21.1.2.17, we can choose an adjunction $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$ where $\mathcal{D} = \mathrm{Ind}(\mathcal{D}_0)$ for some small ∞-category $\mathcal{D}_0$, the functors f and g preserve small filtered colimits, and f is fully faithful. Let $\overline{F}_0$ denote the composition

$$\mathrm{Fun}^{\mathrm{rep}}(\mathcal{D},\mathcal{S}) \simeq \mathcal{D}^{\mathrm{op}} \xrightarrow{g^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})$$

so that $F_0 \simeq \overline{F}_0 \circ T$, where $T : \mathrm{Fun}^+(\mathcal{C}, \mathcal{S}) \to \mathrm{Fun}^+(\mathcal{D}, \mathcal{S})$ is the functor given by composition with g. Applying Lemma 21.1.3.8, we deduce that $\overline{F}_0$ admits a left Kan extension $\overline{F} : \mathrm{Fun}^+(\mathcal{D}, \mathcal{S}) \to \mathcal{X}$, where $\overline{F}|_{\mathrm{Fun}^\omega(\mathcal{D},\mathcal{S})}$ preserves small colimits. Set $F = \overline{F} \circ T$. It follows from Lemma 21.1.3.10 that F is a left Kan extension of $F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})} \simeq F_0$, and that $F|_{\mathrm{Fun}^\omega(\mathcal{C},\mathcal{S})}$ preserves small colimits. If the functor F_0 preserves small filtered limits, then the functor $\overline{F}_0$ has the same property, so that $\overline{F}$ is a right Kan extension of its restriction $\overline{F}|_{\mathrm{Fun}^\omega(\mathcal{D},\mathcal{S})}$ (Lemma 21.1.3.9). Applying Lemma 21.1.3.10, we deduce that F is a right Kan extension of its restriction to $\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$. □

Lemma 21.1.3.12. *Let $\mathcal{C}$ be a compactly assembled ∞-category, let $\mathcal{X}$ be a presentable ∞-category, and let $F : \mathrm{Fun}^+(\mathcal{C}, \mathcal{S}) \to \mathcal{X}$ be a functor. The following conditions are equivalent:*

- (a) *The functor $F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})}$ preserves small filtered limits, and F is a left Kan extension of $F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C},\mathcal{S})}$.*
- (b) *The functor $F|_{\mathrm{Fun}^\omega(\mathcal{C},\mathcal{S})}$ preserves small colimits, and F is a right Kan extension of $F|_{\mathrm{Fun}^{omega}(\mathcal{C},\mathcal{S})}$.*

Proof. Apply Lemma **??**. □

Notation 21.1.3.13. Let $\mathcal{X}$ be a presentable ∞-category and let $\mathcal{C}$ be a compactly assembled ∞-category. We let $\mathcal{M}[\mathcal{C}, \mathcal{X}]$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Fun}^+(\mathcal{C}, \mathcal{S}), \mathcal{X})$ spanned by those functors $F : \mathrm{Fun}^+(\mathcal{C}, \mathcal{S}) \to \mathcal{X}$ which satisfy the equivalent conditions (a) and (b) of Lemma 21.1.3.12.

Remark 21.1.3.14. Let $\mathcal{X}$ be a presentable ∞-category and let $\mathcal{C}$ be a compactly assembled ∞-category. We have evident restriction maps

$$\mathrm{Fun}_\omega(\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}, \mathcal{S}), \mathcal{X}) \leftarrow \mathcal{M}[\mathcal{C}, \mathcal{X}] \to \mathrm{LFun}(\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S}), \mathcal{X}).$$

It follows from Lemma 21.1.3.11 and Proposition HTT.4.3.2.15 that these maps are trivial Kan fibrations.

Remark 21.1.3.15. Let $\mathcal{X}$ be a presentable ∞-category and let $\mathcal{C}$ be an ∞-category of the form $\mathrm{Ind}(\mathcal{C}_0)$, where $\mathcal{C}_0$ is a small ∞-category. Then we have a commutative diagram of restriction functors

$$\begin{array}{ccccc} \mathcal{M}[\mathcal{C}, \mathcal{X}] & \longrightarrow & \mathrm{Fun}_\omega(\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}, \mathcal{S}), \mathcal{X}) & \longrightarrow & \mathrm{Fun}_\omega(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{LFun}(\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S}), \mathcal{X}) & \longrightarrow & \mathrm{Fun}(\mathrm{Fun}^\circ(\mathcal{C}, \mathcal{S}), \mathcal{X}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}_c^{\mathrm{op}}, \mathcal{X}). \end{array}$$

where the maps in the left square are trivial Kan fibrations and the right horizontal maps are equivalences (given by composition with the equivalences $\mathcal{C}^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}, \mathcal{S})$ and $\mathcal{C}_c^{\mathrm{op}} \simeq \mathrm{Fun}^{\circ}(\mathcal{C}, \mathcal{S})$). Unwinding the definition, we see that the equivalence $\rho_{\mathcal{C}} : \mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \simeq \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X})$ appearing in the proof of Proposition 21.1.3.1 can be written as a composition of the right vertical map with a homotopy inverse of the bottom composition. Using the commutativity of the diagram, we see that $\rho_{\mathcal{C}}$ can also be identified with the composition

$$\mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \xleftarrow{\sim} \mathrm{Fun}_{\omega}(\mathrm{Fun}_{\mathrm{rep}}(\mathcal{C}, \mathcal{S}), \mathcal{X}) \xleftarrow{\sim} \mathcal{M}[\mathcal{C}, \mathcal{X}] \xrightarrow{\sim} \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X}).$$

Proof of Proposition 21.1.3.3. For any compactly assembled ∞-category $\mathcal{C}$ and presentable ∞-category $\mathcal{X}$, define $\rho_{\mathcal{C}} : \mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \to \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X})$ to be a composition of the restriction map $\mathcal{M}[\mathcal{C}, \mathcal{X}] \to \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X})$ with a homotopy inverse to the restriction map $\mathcal{M}[\mathcal{C}, \mathcal{X}] \to \mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$. It follows from Proposition **??** that $\rho_{\mathcal{C}}$ is a well-defined equivalence of ∞-categories. By virtue of Remark 21.1.3.15, it agrees with the equivalence constructed in the proof of Proposition 21.1.3.1 in the case where $\mathcal{C}$ has the form $\mathrm{Ind}(\mathcal{C}_0)$, for some small ∞-category $\mathcal{C}_0$. If $\mathcal{D}$ is another compactly assembled ∞-category and we are given an adjunction $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$, where f and g preserve small filtered colimits, then the compatibility of $\rho_{\mathcal{C}}$ and $\rho_{\mathcal{D}}$ follows from the commutativity of the diagram of restriction functors

$$\begin{array}{ccccc}
\mathrm{Fun}_{\omega}(\mathrm{Fun}_{\mathrm{rep}}(\mathcal{D}, \mathcal{S}), \mathcal{X}) & \longleftarrow & \mathcal{M}[\mathcal{D}, \mathcal{X}] & \longrightarrow & \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{D}, \mathcal{S}), \mathcal{X}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Fun}_{\omega}(\mathrm{Fun}_{\mathrm{rep}}(\mathcal{C}, \mathcal{S}), \mathcal{X}) & \longleftarrow & \mathcal{M}[\mathcal{C}, \mathcal{X}] & \longrightarrow & \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X}),
\end{array}$$

where the vertical maps are given by composition with the functor T of Remark 21.1.3.6. □

Remark 21.1.3.16 (Wavy Arrows)**.** Let $\mathcal{C}$ be a compactly assembled ∞-category, let $\mathcal{X} = \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$, and let

$$\rho_{\mathcal{C}} : \mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \to \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X})$$

be the equivalence of Proposition 21.1.3.3. Then $\rho_{\mathcal{C}}^{-1}$ carries the identity functor $\mathrm{id}_{\mathcal{X}}$ to an object of $\mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{C}, \mathcal{S}))$, which we can identify with a bifunctor $W : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$. Unwinding the definitions, we see that W classifies the functor $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ which is left adjoint to the tautological map $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$. More concretely, the functor W is given by the formula $W(C, D) = \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(C, F(D))$ (where we abuse notation by identifying an object $C \in \mathcal{C}$ with its image under the Yoneda embedding $\mathcal{C} \to \mathrm{Ind}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$). Following Johnstone-Joyal, we refer to $W(C, D)$ as the *space of wavy arrows from C to D.*

Example 21.1.3.17. Let $\mathcal{C}_0$ be a small ∞-category and let $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$, and let D be an object of $\mathcal{C}$ which is represented by a filtered diagram $\{D_\alpha\}$ in the ∞-category $\mathcal{C}_0$. Unwinding the definitions, we see that for any $C \in \mathcal{C}$, the space of wavy arrows $W(C, D)$ can be identified with the direct limit $\varinjlim_\alpha \mathrm{Map}_{\mathcal{C}}(C, D_\alpha)$. In particular, there is a tautological map $W(C, D) \to \mathrm{Map}_{\mathcal{C}}(C, D)$, which is an equivalence when C is a compact object of $\mathcal{C}$. More generally, it is useful to think of $W(C, D)$ as a space of "compactly supported" morphisms from C to D (beware, however, that the natural map $W(C, D) \to \mathrm{Map}_{\mathcal{C}}(C, D)$ need not be the inclusion of a summand).

21.1.4 Existence of Classifying ∞-Topoi

Let $\mathcal{C}$ be an accessible ∞-category which admits small filtered colimits. In §21.1.1, we prove that the ∞-category $\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$ is an ∞-topos (Proposition 21.1.1.2) and described the functor *corepresented* by $\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$ on the ∞-category $\infty\mathcal{T}\mathrm{op}$. Our goal in this section is to show that, under the assumption that $\mathcal{C}$ is compactly assembled, we can also describe the functor *represented* by $\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$. Note that, for any ∞-topos $\mathcal{X}$, we can regard the ∞-category $\mathrm{Fun}^*(\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S}), \mathcal{X})$ of geometric morphisms from $\mathcal{X}$ to $\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$ as a full subcategory of the ∞-category $\mathrm{LFun}(\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S}), \mathcal{X})$ of *all* colimit-preserving functors from $\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$ to $\mathcal{X}$. Consequently, Proposition 21.1.3.3 supplies a fully faithful embedding

$$\phi : \mathrm{Fun}^*(\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S}), \mathcal{X}) \to \mathrm{Fun}_\omega(\mathcal{C}^{\mathrm{op}}, \mathcal{X}).$$

The essential image of this embedding can be described as follows:

Proposition 21.1.4.1. *Let $\mathcal{C}$ be a compactly assembled ∞-category, let $\mathcal{X}$ be an ∞-topos, and let $U : \mathcal{C}^{\mathrm{op}} \to \mathcal{X}$ be a functor which preserves small filtered limits. The following conditions are equivalent:*

(1) *The functor U belongs to the essential image of the fully faithful embedding ϕ : $\mathrm{Fun}^*(\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S}), \mathcal{X}) \to \mathrm{Fun}_\omega(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ described above.*

(2) *For every sufficiently large regular cardinal κ, the functor $U|_{\mathcal{C}_\kappa^{\mathrm{op}}} : \mathcal{C}_\kappa^{\mathrm{op}} \to \mathcal{X}$ is locally left exact (see Definition 20.4.2.1); here $\mathcal{C}_\kappa$ denotes the full subcategory of $\mathcal{C}$ spanned by the κ-compact objects.*

Proof. Let $G : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ be the Ind-extension of the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. Applying Theorem 21.1.2.10, we deduce that G admits a left adjoint $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$. Moreover, the proof of Theorem **??** shows that F factors through a functor $F_0 : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}_\kappa)$ for κ sufficiently large. Let $V : \mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S}) \to \mathcal{X}$ be the image of U under the equivalence $\rho_{\mathcal{C}}$ of Proposition 21.1.3.3. We will complete the proof by showing that V is left exact if and only if $U|_{\mathcal{C}_\kappa^{\mathrm{op}}}$ is locally left exact.

Let $G_0 = G|_{\mathrm{Ind}(\mathcal{C}_\kappa)}$, so that we have an adjunction $\mathcal{C} \underset{G_0}{\overset{F_0}{\rightleftarrows}} \mathrm{Ind}(\mathcal{C}_\kappa)$. Set $\overline{U} = U \circ G_0$, so that $U \simeq \overline{U} \circ F_0$. Let $\overline{V} : \mathrm{Fun}^{\omega}(\mathrm{Ind}(\mathcal{C}_\kappa), \mathcal{S}) \to \mathcal{X}$ be the image of $\overline{U}$ under the equivalence $\rho_{\mathrm{Ind}(\mathcal{C}_\kappa)}$ of Proposition 21.1.3.1. By construction, $\overline{V}$ can be identified with the (essentially unique) colimit-preserving functor $\mathrm{Fun}(\mathcal{C}_\kappa, \mathcal{S}) \to \mathcal{X}$ whose composition with the Yoneda embedding $\mathcal{C}_\kappa^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(\mathcal{C}_\kappa, \mathcal{S})$ coincides with the restriction $U|_{\mathcal{C}_\kappa^{\mathrm{op}}}$. Consequently, the functor $\overline{V}$ is left exact if and only if $U|_{\mathcal{C}_\kappa^{\mathrm{op}}}$ is locally left exact. Moreover, the compatibility of Proposition 21.1.3.3 shows that $\overline{V}$ is given by the composition

$$\mathrm{Fun}^{\omega}(\mathrm{Ind}(\mathcal{C}_\kappa), \mathcal{S}) \xrightarrow{T_0} \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}) \xrightarrow{V} \mathcal{X},$$

where T_0 is given by precomposition with G_0. Since T_0 exhibits $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ as a left exact localization of the ∞-topos $\mathrm{Fun}^{\omega}(\mathrm{Ind}(\mathcal{C}_\kappa), \mathcal{S})$, we see that $\overline{V}$ is left exact if and only if V is left exact. □

Corollary 21.1.4.2. *Let $\mathcal{C}$ be a compactly assembled presentable ∞-category and let $\mathcal{X}$ be an ∞-topos. Then a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{X}$ belongs to the essential image of ϕ :* $\mathrm{Fun}^*(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X}) \to \mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ *if and only if F preserves small limits.*

Proof. Suppose first that F preserves small limits. For any regular cardinal κ, the full subcategory $\mathcal{C}_\kappa \subseteq \mathcal{C}$ spanned by the κ-compact objects is closed under finite colimits. Consequently, the functor $F|_{\mathcal{C}_\kappa^{\mathrm{op}}}$ is left exact, and therefore locally left exact (Corollary 20.4.3.2). Applying Proposition 21.1.4.1, we deduce that F belongs to the essential image of ϕ. Conversely, if F belongs to the essential image of ρ, then Proposition 21.1.4.1 and Corollary 20.4.3.2 guarantee that F is left exact when restricted to $\mathcal{C}_\kappa$ for all sufficiently large κ, and is therefore left exact. Since F also preserves small filtered limits, it preserves all small limits (Proposition **??**). □

For every pair of presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, let $\mathrm{RFun}(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$ spanned by those functors which preserve small limits. Then $\mathrm{RFun}(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$ is a model for the tensor product $\mathcal{C} \otimes \mathcal{D}$ of §HA.4.8.1 (see Proposition HA.4.8.1.17). In particular, the ∞-categories $\mathrm{RFun}(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$ and $\mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ are canonically equivalent to one another. More concretely, we have canonical equivalences

$$\mathrm{RFun}(\mathcal{D}^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{RFun}(\mathcal{D}^{\mathrm{op}}, \mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})) \simeq \mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathrm{RFun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S})) \simeq \mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}).$$

Specializing to the case where $\mathcal{D} = \mathcal{X}$ is an ∞-topos, we obtain an equivalence $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) = \mathrm{RFun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$. Consequently, Corollary 21.1.4.2 supplies an equivalence of ∞-categories $\mathrm{Fun}^*(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$. In fact, we have the following stronger result:

Theorem 21.1.4.3. *Let $\mathcal{C}$ be a compactly assembled presentable ∞-category. Then the ∞-category $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ is a classifying ∞-topos for $\mathcal{C}$-valued sheaves, in the sense of Definition 21.1.0.1.*

To deduce Theorem 21.1.4.3, Proposition 21.1.3.3 alone is not sufficient: we will need to understand the ∞-category $\mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X})$ as a *functor* of $\mathcal{X}$. Here we encounter a subtlety: for a fixed ∞-category $\mathcal{C}$, the construction $\mathcal{X} \mapsto \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}), \mathcal{X})$ is *a priori* functorial with respect to *colimit*-preserving functors between presentable ∞-categories, while the construction $\mathcal{X} \mapsto \mathrm{Fun}_{\omega}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ is *a priori* functorial with respect to functors that preserve (filtered) inverse limits. To address this discrepancy, we need a brief digression.

Notation 21.1.4.4. Let $U : \mathcal{X} \to \mathcal{Y}$ be a functor of ∞-categories. We say that U is a *presentable fibration* if U is both a Cartesian fibration and a coCartesian fibration, and the fiber $\mathcal{X}_Y = U^{-1}\{Y\}$ is presentable for each object $Y \in \mathcal{Y}$.

Let $U : \mathcal{X} \to \mathcal{Y}$ be a presentable fibration and let $\mathcal{C}$ be some other ∞-category. We let $\mathrm{Fun}(\mathcal{C}, \mathcal{X} / \mathcal{Y})$ denote the fiber product

$$\mathrm{Fun}(\mathcal{C}, \mathcal{X}) \times_{\mathrm{Fun}(\mathcal{C}, \mathcal{Y})} \mathcal{Y},$$

so that we have a projection map $\mathrm{Fun}(\mathcal{C}, \mathcal{X} / \mathcal{Y}) \to \mathcal{Y}$ whose fiber over an object Y can be identified with the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{X}_Y)$. In particular, the objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{X} / \mathcal{Y})$ can be identified with pairs (Y, F), where Y is an object of $\mathcal{Y}$ and $F : \mathcal{C} \to \mathcal{X}_Y$ is a functor.

If the ∞-category $\mathcal{C}$ admits small colimits, we let $\mathrm{LFun}(\mathcal{C}, \mathcal{X} / \mathcal{Y})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{X} / \mathcal{Y})$ spanned by those objects (Y, F) for which the functor $F : \mathcal{C} \to \mathcal{X}_Y$ preserves small colimits. Similarly, if the ∞-category $\mathcal{C}$ admits small filtered limits, we let $\mathrm{Fun}_{\omega}(\mathcal{C}, \mathcal{X} / \mathcal{Y})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{X} / \mathcal{Y})$ spanned by those objects (Y, F) for which the functor $F : \mathcal{C} \to \mathcal{X}_Y$ preserves small filtered limits.

Remark 21.1.4.5. Let $U : \mathcal{X} \to \mathcal{Y}$ be a presentable fibration of ∞-categories. Then:

(*a*) Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category whose objects are presentable ∞-categories and whose morphisms are functors which preserve small colimits. Then, as a coCartesian fibration, U is classified by a functor $\chi : \mathcal{Y} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$, given on objects by $\chi(Y) = \mathcal{X}_Y$. If $\mathcal{C}$ is an ∞-category which admits small colimits, then the projection map $\mathrm{LFun}(\mathcal{C}, \mathcal{X} / \mathcal{Y}) \to \mathcal{Y}$ is also a coCartesian fibration, classified by the functor $Y \mapsto \mathrm{LFun}(\mathcal{C}, \chi(Y)) = \mathrm{LFun}(\mathcal{C}, \mathcal{X}_Y)$.

(*b*) Let $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ denote the ∞-category whose objects are presentable ∞-categories and whose morphisms are functors which are accessible and preserve small limits. Then, as a Cartesian fibration, U is classified by a functor $\chi' : \mathcal{Y}^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{R}}$, given on objects by $\chi'(Y) = \mathcal{X}_Y$. If $\mathcal{C}$ is an ∞-category which admits small filtered limits, then the projection map $\mathrm{Fun}_{\omega}(\mathcal{C}, \mathcal{X} / \mathcal{Y}) \to \mathcal{Y}$ is also a Cartesian fibration, classified by the functor $Y \mapsto \mathrm{Fun}_{\omega}(\mathcal{C}, \chi(Y)) = \mathrm{LFun}_{\omega}(\mathcal{C}, \mathcal{X}_Y)$.

We can now formulate a more refined version of Proposition 21.1.3.3:

Theorem 21.1.4.6. *Let $\mathcal{C}$ be compactly assembled ∞-category and let $U : \mathcal{X} \to \mathcal{Y}$ be a presentable fibration. Then there exists an equivalence of ∞-categories*

$$\mathrm{Fun}_\omega(\mathcal{C}^{\mathrm{op}}, \mathcal{X} / \mathcal{Y}) \simeq \mathrm{LFun}(\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S}), \mathcal{X} / \mathcal{Y})$$

which is compatible with the projection to $\mathcal{Y}$.

Remark 21.1.4.7. Let $U_0 : \overline{\mathcal{P}\mathrm{r}^{\mathrm{L}}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ be a coCartesian fibration which is classified by the inclusion functor $\mathcal{P}\mathrm{r}^{\mathrm{L}} \hookrightarrow \widehat{\mathcal{C}\mathrm{at}}_\infty$ (so that the objects of $\overline{\mathcal{P}\mathrm{r}^{\mathrm{L}}}$ can be identified with pairs $(\mathcal{C}, C)$, where $\mathcal{C}$ is a presentable ∞-category and C is an object of $\mathcal{C}$). Then U_0 is a *universal* presentable fibration: every presentable fibration $U : \mathcal{X} \to \mathcal{Y}$ fits into an essentially unique (homotopy) pullback diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \overline{\mathcal{P}\mathrm{r}^{\mathrm{L}}} \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U_0} \\ \mathcal{Y} & \longrightarrow & \mathcal{P}\mathrm{r}^{\mathrm{L}}, \end{array}$$

where the lower vertical map classifies U as a coCartesian fibration. Consequently, to prove Theorem 21.1.4.6 for an arbitrary presentable fibration U, it suffices to treat the case $U = U_0$. Moreover, we can then arrange that the equivalence whose existence is asserted by Theorem 21.1.4.6 is compatible with base change in $\mathcal{Y}$.

Remark 21.1.4.8. In the situation of Theorem 21.1.4.6, it follows that the projection map $\mathrm{LFun}(\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S}), \mathcal{X} / \mathcal{Y}) \to \mathcal{Y}$ is a Cartesian fibration (which can also be deduced easily from the adjoint functor theorem) and that the projection map $\mathrm{Fun}_\omega(\mathcal{C}^{\mathrm{op}}, \mathcal{X} / \mathcal{Y}) \to \mathcal{Y}$ is a coCartesian fibration (which is somewhat less obvious).

Proof of Theorem 21.1.4.6. Let $\mathcal{M}[\mathcal{C}, \mathcal{X} / \mathcal{Y}]$ denote the full subcategory of the ∞-category $\mathrm{Fun}(\mathrm{Fun}^+(\mathcal{C}, \mathcal{S}), \mathcal{X} / \mathcal{Y})$ whose objects are pairs (Y, F) where $Y \in \mathcal{Y}$ and $F : \mathrm{Fun}^+(\mathcal{C}, \mathcal{S}) \to \mathcal{X}_Y$ is a functor which belongs to $\mathcal{M}[\mathcal{C}, \mathcal{X}_Y]$ (see the proof of Proposition 21.1.3.3). Composition with the Yoneda embedding $j : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}^+(\mathcal{C}, \mathcal{S})$ and the inclusion $\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S}) \hookrightarrow \mathrm{Fun}^+(\mathcal{C}, \mathcal{S})$ determine forgetful functors

$$\mathrm{Fun}_\omega(\mathcal{C}^{\mathrm{op}}, \mathcal{X} / \mathcal{Y}) \xleftarrow{\phi} \mathcal{M}[\mathcal{C}, \mathcal{X} / \mathcal{Y}] \xrightarrow{\psi} \mathrm{LFun}(\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S}), \mathcal{X} / \mathcal{Y})$$

which are compatible with the projection to $\mathcal{Y}$. To prove Theorem 21.1.4.6, it will suffice to show that ϕ and ψ are equivalences of ∞-categories.

We will show that ψ is an equivalence; the proof for ϕ is similar. Let $\overline{\mathcal{M}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Fun}^+(\mathcal{C}, \mathcal{S}), \mathcal{X} / \mathcal{Y})$ spanned by those pairs (Y, F), where $F : \mathrm{Fun}^+(\mathcal{C}, \mathcal{S}) \to \mathcal{X}_Y$ is a right Kan extension of its restriction to $\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$. Let $q : \mathrm{Fun}(\mathrm{Fun}^+(\mathcal{C}, \mathcal{S}), \mathcal{X} / \mathcal{Y}) \to$

$\mathcal{Y}$ and $p : \mathrm{Fun}(\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S}),\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ denote the the projection maps, and set $\overline{q} = q|_{\overline{\mathcal{M}}}$. Note that if $\alpha : M \to M'$ is a q-Cartesian morphism in $\mathrm{Fun}(\mathrm{Fun}^{+}(\mathcal{C},\mathcal{S}),\mathcal{X}/\mathcal{Y})$ where $M' \in \overline{\mathcal{M}}$, then M also belongs to $\overline{\mathcal{M}}$; it follows that α can be regarded as a $\overline{q}$-Cartesian morphism in the ∞-category $\overline{\mathcal{M}}$. The map ψ fits into a pullback diagram

$$\begin{array}{ccc} \mathcal{M}[\mathcal{C},\mathcal{X}/\mathcal{Y}] & \xrightarrow{\psi} & \mathrm{LFun}(\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S}),\mathcal{X}/\mathcal{Y}) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}} & \xrightarrow{\overline{\psi}} & \mathrm{Fun}(\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S}),\mathcal{X}/\mathcal{Y}), \end{array}$$

where $\overline{\psi}$ carries $\overline{q}$-Cartesian morphisms of the ∞-category $\overline{\mathcal{M}}$ to p-Cartesian morphisms of the ∞-category $\mathrm{Fun}(\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S}),\mathcal{X}/\mathcal{Y})$. Consequently, to prove that ψ is an equivalence of ∞-categories, it will suffice to show that $\overline{\psi}$ is an equivalence of ∞-categories. By virtue of Corollary HTT.2.4.4.4 , it suffices to check that $\overline{\psi}$ induces an equivalence after passing to the fiber over any object $Y \in \mathcal{Y}$. In this case, the desired result follows from Proposition HTT.4.3.2.15 and Lemma **??**. □

Proof of Theorem 21.1.4.3. Let $\infty\mathcal{T}\mathrm{op}$ denote the ∞-category of ∞-topoi. Recall that we have defined $\infty\mathcal{T}\mathrm{op}$ to be the *opposite* of the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$ whose objects are ∞-topoi and whose morphisms are functors which preserve small colimits and finite limits. In particular, the inclusion map $\infty\mathcal{T}\mathrm{op}^{\mathrm{op}} \hookrightarrow \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ classifies a coCartesian fibration $\overline{\infty\mathcal{T}\mathrm{op}}^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{op}}$, which is a presentable fibration in the sense of Notation 21.1.4.4. Set $\mathcal{E} = \mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})$ and let $\mathrm{Fun}^{*}(\mathcal{E},\widehat{\infty\mathcal{T}\mathrm{op}}^{\mathrm{op}}/\infty\mathcal{T}\mathrm{op}^{\mathrm{op}})$ be the full subcategory of $\mathrm{Fun}(\mathcal{E},\widehat{\infty\mathcal{T}\mathrm{op}}^{\mathrm{op}}/\infty\mathcal{T}\mathrm{op}^{\mathrm{op}})$ whose objects are pairs $(\mathcal{X},f^{*})$ where $\mathcal{X}$ is an ∞-topos and $f^{*} \in \mathrm{Fun}^{*}(\mathcal{E},\mathcal{X})$. Similarly, let $\mathrm{RFun}(\mathcal{C}^{\mathrm{op}},\widehat{\infty\mathcal{T}\mathrm{op}}^{\mathrm{op}}/\infty\mathcal{T}\mathrm{op}^{\mathrm{op}})$ be the full subcategory $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\widehat{\infty\mathcal{T}\mathrm{op}}^{\mathrm{op}}/\infty\mathcal{T}\mathrm{op}^{\mathrm{op}})$ spanned by those pairs $(\mathcal{X},F)$ where $\mathcal{X}$ is an ∞-topos and $F \in \mathrm{RFun}(\mathcal{C}^{\mathrm{op}},\mathcal{X})$. Then the projection maps $\mathrm{Fun}^{*}(\mathcal{E},\widehat{\infty\mathcal{T}\mathrm{op}}^{\mathrm{op}}/\infty\mathcal{T}\mathrm{op}^{\mathrm{op}}) \to \infty\mathcal{T}\mathrm{op}^{\mathrm{op}}$ and $\mathrm{RFun}(\mathcal{C}^{\mathrm{op}},\widehat{\infty\mathcal{T}\mathrm{op}}^{\mathrm{op}}/\infty\mathcal{T}\mathrm{op}^{\mathrm{op}}) \to \infty\mathcal{T}\mathrm{op}^{\mathrm{op}}$ are coCartesian fibrations which are classified by the functors $\mathcal{X} \mapsto \mathrm{Fun}^{*}(\mathcal{E},\mathcal{X})$ and $\mathcal{X} \mapsto \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$, respectively. Using Theorem 21.1.4.6 and Corollary 21.1.4.2, we deduce that the functors $\mathcal{X} \mapsto \mathrm{Fun}^{*}(\mathcal{E},\mathcal{X})$ and $\mathcal{X} \mapsto \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ are equivalent. □

We can now give a definitive answer to Question 21.1.0.2:

Corollary 21.1.4.9. *Let $\mathcal{C}$ be a presentable ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is compactly assembled.*

(2) *There exists a classifying topos for $\mathcal{C}$-valued sheaves.*

Proof. The implication (1) ⇒ (2) follows from Theorem 21.1.4.3. For the converse, suppose that $\mathcal{C}$ is a presentable ∞-category and that there exists a universal $\mathcal{C}$-valued sheaf $\mathscr{F} \in$

$\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{E})$ for some ∞-topos $\mathcal{E}$. Without loss of generality, we may assume that $\mathcal{E}$ is an accessible left exact localization of an ∞-topos of presheaves $\mathcal{P}(\mathcal{G})$, for some small ∞-category $\mathcal{G}$. Let $\iota_* : \mathcal{E} \hookrightarrow \mathcal{P}(\mathcal{G})$ be the inclusion functor, so that ι_* induces a fully faithful embedding $\iota_*^{\mathcal{C}} : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{E}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{P}(\mathcal{G}))$ with a left adjoint $\iota_{\mathcal{C}}^* : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{P}(\mathcal{G})) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{E})$ Then $\iota_*^{\mathcal{C}} \mathscr{F}$ is a $\mathcal{C}$-valued sheaf on the presheaf ∞-topos $\mathcal{P}(\mathcal{G})$, so we can write $\iota_*^{\mathcal{C}} \mathscr{F} \simeq f_{\mathcal{C}}^* \mathscr{F}$ for some essentially unique geometric morphism $f_* : \mathcal{P}(\mathcal{G}) \to \mathcal{E}$. Since the functor $\iota_*^{\mathcal{C}}$ is fully faithful, we have equivalences $\mathscr{F} \simeq \iota_{\mathcal{C}}^* \iota_*^{\mathcal{C}} \mathscr{F} \simeq \iota_{\mathcal{C}}^* f_{\mathcal{C}}^* \mathscr{F}$. Invoking again our assumption that $\mathscr{F}$ is universal, we deduce that the composition $\iota^* \circ f^*$ is equivalent to the identity functor from $\mathcal{E}$ to itself. In other words, ι^* and f^* exhibit $\mathcal{E}$ as a retract (in the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi) of the presheaf ∞-category $\mathcal{P}(\mathcal{G})$. Applying the functor Pt of Corollary 21.1.1.6, we conclude that $\mathcal{C} = \mathrm{Pt}(\mathcal{E})$ is a retract of $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}}) \simeq \mathrm{Pt}(\mathcal{P}(\mathcal{G}))$ in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\omega}$, so that $\mathcal{C}$ is compactly assembled. □

21.1.5 Points of $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$

Let $\mathcal{C}$ be an accessible ∞-category which admits small filtered colimits. Then the ∞-topos $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ is closed under small colimits and finite limits in the larger ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$. It follows that, for each object $C \in \mathcal{C}$, the evaluation functor

$$\mathrm{ev}_C : \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}) \to \mathcal{S} \qquad \mathrm{ev}_C(F) = F(C)$$

preserves small colimits and finite limits: that is, it is a point of the ∞-topos $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$. We regard the construction $C \mapsto \mathrm{ev}_C$ as a functor of ∞-categories $\mathrm{ev} : \mathcal{C} \to \mathrm{Pt}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}))$.

Proposition 21.1.5.1. *Let $\mathcal{C}$ be a compactly assembled ∞-category. Then the functor $\mathrm{ev} : \mathcal{C} \to \mathrm{Pt}(\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}))$ is an equivalence of ∞-categories.*

Proof. Let us denote the evaluation functor ev by $\mathrm{ev}^{\mathcal{C}}$ to emphasize its dependence on $\mathcal{C}$. We observe that the construction $\mathcal{C} \mapsto \mathrm{ev}^{\mathcal{C}}$ determines a functor $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\omega} \to \mathrm{Fun}(\Delta^1, \widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\omega})$. It follows that if $\mathcal{C}$ is a retract of $\mathcal{D}$ in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\omega}$ and the functor $e^{\mathcal{D}}$ is an equivalence, then $e^{\mathcal{C}}$ is also an equivalence. It will therefore suffice to show that the functor $e^{\mathcal{C}}$ is an equivalence in the special case where $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ for some small ∞-category $\mathcal{C}_0$. In this case, the ∞-topos $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ can be identified with the ∞-category of presheaves $\mathcal{P}(\mathcal{C}_0^{\mathrm{op}})$, so the desired result follows from Proposition **??**. □

Corollary 21.1.5.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be accessible ∞-categories which admit small filtered colimits. If $\mathcal{D}$ is compactly assembled, then the canonical map*

$$F : \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^*(\mathrm{Fun}^{\omega}(\mathcal{D}, \mathcal{S}), \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}))$$

is an equivalence of ∞-categories.

Proof. The functor F fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{D}) & \xrightarrow{F} & \mathrm{Fun}^{*}(\mathrm{Fun}^{\omega}(\mathcal{D},\mathcal{S}),\mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})) \\ & \searrow \quad \swarrow & \\ & \mathrm{Fun}^{\omega}(\mathcal{C},\mathrm{Pt}(\mathrm{Fun}^{\omega}(\mathcal{D},\mathcal{S}))) & \end{array}$$

where the left vertical map is an equivalence by Proposition 21.1.5.1 and the right vertical map is an isomorphism of simplicial sets (see Corollary 21.1.1.6). □

Corollary 21.1.5.3. *Let $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\mathrm{ca}}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$ whose objects are compactly assembled ∞-categories and whose morphisms are functors which preserve small filtered colimits. Then the construction $\mathcal{C} \mapsto \mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})$ determines a fully faithful embedding $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\mathrm{ca}} \hookrightarrow \infty\mathcal{T}\mathrm{op}$, whose essential image consists of those ∞-topoi which can be written as a retract of a presheaf ∞-category $\mathcal{P}(\mathcal{G})$, for some small ∞-category $\mathcal{G}$.*

Proof. The first assertion follows from Corollary 21.1.5.2. For the second, we observe that $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\mathrm{ca}}$ is the idempotent completion of the full subcategory spanned by objects of the form $\mathrm{Ind}(\mathcal{C}_0)$, and that the construction $\mathcal{C} \mapsto \mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})$ carries $\mathrm{Ind}(\mathcal{C}_0)$ to $\mathcal{P}(\mathcal{C}_0^{\mathrm{op}})$. □

If we restrict our attention to presentable compactly assembled ∞-categories, we can say more.

Proposition 21.1.5.4. *Let $\mathcal{E}$ be an ∞-topos. The following conditions are equivalent:*

(1) *There exists a presentable compactly assembled ∞-category $\mathcal{C}$ and an equivalence $\mathcal{E} \simeq \mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})$.*

(2) *For every geometric morphism of ∞-topoi $f^{*} : \mathcal{X} \to \mathcal{Y}$, the induced map $\mathrm{Fun}^{*}(\mathcal{E},\mathcal{X}) \xrightarrow{f^{*}\circ} \mathrm{Fun}^{*}(\mathcal{E},\mathcal{Y})$ admits a right adjoint G. Moreover, if the direct image functor $f_{*} : \mathcal{Y} \to \mathcal{X}$ is fully faithful, then so is G.*

(3) *Let $f^{*} : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi for which the direct image functor $f_{*} : \mathcal{Y} \to \mathcal{X}$ is fully faithful. Then the functor $\mathrm{Fun}^{*}(\mathcal{E},\mathcal{X}) \xrightarrow{f^{*}\circ} \mathrm{Fun}^{*}(\mathcal{E},\mathcal{Y})$ is essentially surjective. In other words, every lifting problem of the form*

$$\begin{array}{ccc} & \mathcal{X} & \\ {\scriptstyle f_{*}} \nearrow & & \searrow \\ \mathcal{Y} & \longrightarrow & \mathcal{E} \end{array}$$

admits a solution in the ∞-category $\infty\mathcal{T}\mathrm{op}$.

(4) *Let $f_* : \mathcal{E} \to \mathcal{X}$ be a geometric morphism of ∞-topoi which is fully faithful (at the level of underlying ∞-categories). Then f_* admits a left homotopy inverse (in the ∞-category $\infty\mathcal{T}\mathrm{op}$).*

Remark 21.1.5.5. Johnstone defines a topos $\mathcal{E}$ to be *injective* if it satisfies the 1-categorical analogue of condition (3); see [103] and [104].

Proof of Proposition 21.1.5.4. Suppose first that (1) is satisfied. Then $\mathcal{E}$ is a classifying ∞-topos for $\mathcal{C}$-valued sheaves (Theorem 21.1.4.3). In particular, we have equivalences $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$, depending functorially on the ∞-topos $\mathcal{X}$. We now observe that if $f^* : \mathcal{X} \to \mathcal{Y}$ is a geometric morphism of ∞-topoi, then the associated pullback functor $f^*_{\mathcal{C}} : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{Y})$ is defined as the left adjoint of a direct image functor $f^{\mathcal{C}}_* : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{Y}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$, which is given by precomposition with f^*. If the usual direct image functor $f_* : \mathcal{Y} \to \mathcal{X}$ is fully faithful, then f^* is a localization functor, so the functor $f^{\mathcal{C}}_*$ is also fully faithful. That completes the proof that (1) $\Rightarrow$ (2). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate. We will conclude by showing that (4) $\Rightarrow$ (1). Let $\mathcal{E}$ be an ∞-topos satisfying (4), and write $\mathcal{E}$ as an accessible left exact localization of $\mathcal{P}(\mathcal{G})$, where $\mathcal{G}$ is a small ∞-category which admits finite limits. Applying condition (4), we deduce that $\mathcal{E}$ is a retract of $\mathcal{P}(\mathcal{G})$ in the ∞-category $\infty\mathcal{T}\mathrm{op}$. Applying the criterion of Corollary 21.1.5.3, we deduce that there is an equivalence $\mathcal{E} \simeq \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ for some compactly assembled ∞-category $\mathcal{C}$. We will complete the proof by showing that $\mathcal{C}$ is presentable. Using Proposition 21.1.5.1, we can identify $\mathcal{C}$ with the ∞-category $\mathrm{Pt}(\mathcal{E})$ of points of $\mathcal{E}$. We now observe that $\mathrm{Pt}(\mathcal{E})$ is a retract of $\mathrm{Pt}(\mathcal{P}(\mathcal{G})) \simeq \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$. Since $\mathcal{G}$ admits finite limits, the ∞-category $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ is compactly generated, so that $\mathcal{C} = \mathrm{Pt}(\mathcal{E})$ is presentable by virtue of Corollary 21.1.2.18. □

21.1.6 Application: Exponentiability of ∞-Topoi

Let X and Y be topological spaces, and let $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, Y)$ denote the set of continuous maps from X to Y. It is often convenient to equip the set $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, Y)$ with the structure of a topological space. Particularly useful is the *compact-open topology*, which has a subbasis consisting of open sets having the form

$$\{f \in \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, Y) : f(K) \subseteq U\}$$

where K is a compact subset of X and U is an open subset of Y. In good cases, this is characterized by a universal mapping property.

Proposition 21.1.6.1. *Let X, Y, and Z be topological spaces, and assume that X is a locally compact Hausdorff space. Then there is a canonical bijection*

$$\{\textit{Continuous functions } F : Z \to \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, Y)\} \simeq \{\textit{Continuous functions } f : Z \times X \to Y\},$$

which carries a function F to the function f given by the formula $f(z, x) = F(z)(x)$.

Proofs of Proposition 21.1.6.1 can be found in standard texts on point-set topology; see for example [159] (we will sketch a very indirect proof in Remark 21.1.7.7, at least for the special case where Y is sober). In this section, we will study an analogous construction in an ∞-categorical setting, replacing the category $\mathcal{T}\mathrm{op}$ of topological spaces with the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi. Our goal is to answer the following:

Question 21.1.6.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi. Under what circumstances can the collection of geometric morphisms $f_* : \mathcal{X} \to \mathcal{Y}$ be identified with the points of another ∞-topos $\mathcal{Y}^{\mathcal{X}}$?

Let us begin by making Question 21.1.6.2 more precise.

Notation 21.1.6.3. Let $\infty\mathcal{T}\mathrm{op}$ denote the ∞-category of ∞-topoi. Then $\infty\mathcal{T}\mathrm{op}$ admits finite products (Proposition HTT.6.3.4.6). Beware that if $\mathcal{X}$ and $\mathcal{Y}$ are ∞-topoi, then the product of $\mathcal{X}$ and $\mathcal{Y}$ in $\infty\mathcal{T}\mathrm{op}$ is *not* given by the usual Cartesian product of ∞-categories $\mathcal{X} \times \mathcal{Y}$ (instead, $\mathcal{X} \times \mathcal{Y}$ is the *coproduct* of $\mathcal{X}$ with $\mathcal{Y}$ in $\infty\mathcal{T}\mathrm{op}$). To avoid confusion, we will denote the product of $\mathcal{X}$ and $\mathcal{Y}$ in $\infty\mathcal{T}\mathrm{op}$ by $\mathcal{X} \otimes \mathcal{Y}$. Note that we can identify $\mathcal{X} \otimes \mathcal{Y}$ with the tensor product of $\mathcal{X}$ with $\mathcal{Y}$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories (with respect to the symmetric monoidal structure described in §HA.4.8.1 ; see Example HA.4.8.1.19). Less symmetrically, the product $\mathcal{X} \otimes \mathcal{Y}$ can be identified with the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{X}}(\mathcal{Y})$ of $\mathcal{X}$-valued sheaves on $\mathcal{Y}$, or with the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{Y}}(\mathcal{X})$ of $\mathcal{Y}$-valued sheaves on $\mathcal{X}$.

Definition 21.1.6.4. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{E}$ be ∞-topoi. We will say that a geometric morphism $e_* : \mathcal{E} \otimes \mathcal{X} \to \mathcal{Y}$ *exhibits $\mathcal{E}$ as an exponential of $\mathcal{Y}$ by $\mathcal{X}$* if the following universal property is satisfied: for any ∞-topos $\mathcal{Z}$, composition with e_* induces a homotopy equivalence

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{Z}, \mathcal{E}) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{Z} \times \mathcal{X}, \mathcal{Y}).$$

Remark 21.1.6.5. Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi. It follows immediately from the definitions that if there exists a geometric morphism $e_* : \mathcal{E} \otimes \mathcal{X} \to \mathcal{Y}$ which exhibits $\mathcal{E}$ as an exponential of $\mathcal{Y}$ by $\mathcal{X}$, then the ∞-topos $\mathcal{E}$ and the geometric morphism e_* are determined uniquely up to equivalence and depend functorially on both $\mathcal{X}$ and $\mathcal{Y}$. We will indicate this dependence by writing $\mathcal{E} = \mathcal{Y}^{\mathcal{X}}$. Beware that $\mathcal{Y}^{\mathcal{X}}$ is *not* the same as the functor ∞-category $\mathrm{Fun}(\mathcal{X}, \mathcal{Y})$.

Remark 21.1.6.6. Let $\mathcal{X}$ be an ∞-topos, and let $\mathcal{C}$ denote the full subcategory of $\infty\mathcal{T}\mathrm{op}$ spanned by those ∞-topoi $\mathcal{Y}$ for which there exists an exponential $\mathcal{Y}^{\mathcal{X}}$. Then $\mathcal{C}$ is closed under small limits. Moreover, the construction $\mathcal{Y} \mapsto \mathcal{Y}^{\mathcal{X}}$ determines a functor $\mathcal{C} \to \infty\mathcal{T}\mathrm{op}$ which preserves small limits.

Remark 21.1.6.7. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{E}$ be ∞-topoi, let $e_* : \mathcal{E} \otimes \mathcal{X} \to \mathcal{Y}$ be a geometric morphism, and let $e^* : \mathcal{Y} \to \mathcal{E} \otimes \mathcal{X}$ be left adjoint of e_*. For any ∞-topos $\mathcal{Z}$, composition with e^* induces a functor $\theta_{\mathcal{Z}} : \mathrm{Fun}^*(\mathcal{E}, \mathcal{Z}) \to \mathrm{Fun}^*(\mathcal{Y}, \mathcal{Z} \otimes \mathcal{X})$. Unwinding the definitions, we see that e_* exhibits $\mathcal{E}$ as an exponential of $\mathcal{X}$ by $\mathcal{Y}$ if and only if the following condition is satisfied, for any ∞-topos $\mathcal{Z}$:

$(a_{\mathcal{Z}})$ The functor $\theta_{\mathcal{Z}}$ induces a homotopy equivalence of Kan complexes $\mathrm{Fun}^*(\mathcal{E},\mathcal{Z})^{\simeq} \to \mathrm{Fun}^*(\mathcal{Y},\mathcal{Z}\otimes\mathcal{X})^{\simeq}$

However, it is not difficult to see that e_* exhibits $\mathcal{E}$ as an exponential of $\mathcal{X}$ by $\mathcal{Y}$ if and only if the following *a priori* stronger condition is satisfied, for any ∞-topos $\mathcal{Z}$:

$(b_{\mathcal{Z}})$ The functor $\theta_{\mathcal{Z}}$ is an equivalence of ∞-categories.

The implication $(b_{\mathcal{Z}}) \Rightarrow (a_{\mathcal{Z}})$ is immediate. Conversely, $(b_{\mathcal{Z}})$ can be deduced from $(a_{\mathcal{Z}})$ and $(a_{\mathrm{Fun}(\Delta^1,\mathcal{Z})})$.

Remark 21.1.6.8. Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi, and suppose that there exists an exponential $\mathcal{Y}^{\mathcal{X}}$. Then points of the ∞-topos $\mathcal{Y}^{\mathcal{X}}$ can be identified with geometric morphisms from $\mathcal{X}$ to $\mathcal{Y}$. More precisely, applying Remark 21.1.6.7 in the case $\mathcal{Z} = \mathcal{S}$, we obtain an equivalence of ∞-categories $\mathrm{Pt}(\mathcal{Y}^{\mathcal{X}}) \simeq \mathrm{Fun}^*(\mathcal{X},\mathcal{Y})$.

Example 21.1.6.9. Let $\mathcal{C}$ be a compactly assembled presentable ∞-category and let $\mathcal{Y} = \mathrm{Fun}^{\omega}(\mathcal{C},\mathcal{S})$ be a classifying ∞-topos for $\mathcal{C}$-valued sheaves (see Theorem 21.1.4.3). For any ∞-topoi $\mathcal{X}$ and $\mathcal{Z}$, we have canonical equivalences

$$\begin{aligned}\mathrm{Fun}^*(\mathcal{Y},\mathcal{X}\otimes\mathcal{Z}) &\simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}\otimes\mathcal{Z}) \\ &\simeq \mathcal{C}\otimes(\mathcal{X}\otimes\mathcal{Z}) \\ &\simeq (\mathcal{C}\otimes\mathcal{X})\otimes\mathcal{Z} \\ &\simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}\otimes\mathcal{X}}(\mathcal{Z}).\end{aligned}$$

It follows that an exponential $\mathcal{Y}^{\mathcal{X}}$ (if it exists) can be identified with a classifying ∞-topos for $(\mathcal{C}\otimes\mathcal{X})$-valued sheaves. In particular, the exponential $\mathcal{Y}^{\mathcal{X}}$ exists if and only if the tensor product $\mathcal{C}\otimes\mathcal{X} \simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ is compactly assembled. In this case, we can identify $\mathcal{Y}^{\mathcal{X}}$ with the ∞-topos $\mathrm{Fun}^{\omega}(\mathcal{C}\otimes\mathcal{X},\mathcal{S})$.

Example 21.1.6.10. Let $\mathcal{X}$ be any ∞-topos, and let $\mathcal{Y} = \mathrm{Fun}^{\omega}(\mathcal{S},\mathcal{S})$ be the classifying ∞-topos for $\mathcal{S}$-valued sheaves. Applying Example 21.1.6.9 in the special case $\mathcal{C} = \mathcal{S}$, we deduce that an exponential $\mathcal{Y}^{\mathcal{X}}$ exists if and only if the ∞-topos $\mathcal{X}$ is compactly assembed. If this condition is satisfied, then the exponential $\mathcal{Y}^{\mathcal{X}}$ can be identified with the ∞-topos $\mathrm{Fun}^{\omega}(\mathcal{X},\mathcal{S})$ which classifies $\mathcal{X}$-valued sheaves.

Definition 21.1.6.11. Let $\mathcal{X}$ be an ∞-topos. We will say that $\mathcal{X}$ is *exponentiable* if, for any ∞-topos $\mathcal{Y}$, there exists an ∞-topos $\mathcal{E}$ and a geometric morphism $e_* : \mathcal{E}\otimes\mathcal{X} \to \mathcal{Y}$ which exhibits $\mathcal{E}$ as an exponential of $\mathcal{Y}$ by $\mathcal{X}$.

In other words, an ∞-topos $\mathcal{X}$ is exponentiable if the exponential $\mathcal{Y}^{\mathcal{X}}$ exists for any ∞-topos $\mathcal{Y}$. We can now formulate Question 21.1.6.2 more precisely: which ∞-topoi are exponentiable?

Theorem 21.1.6.12. *Let $\mathcal{X}$ be an ∞-topos. The following conditions are equivalent:*

(1) *The ∞-topos $\mathcal{X}$ is exponentiable (Definition 21.1.6.11).*

(2) *There exists an exponential $\mathcal{Y}^{\mathcal{X}}$, where $\mathcal{Y} = \mathrm{Fun}^{\omega}(\mathcal{S}, \mathcal{S})$ is the classifying topos for $\mathcal{S}$-valued sheaves.*

(3) *There exists a classifying ∞-topos for $\mathcal{X}$-valued sheaves (Definition 21.1.0.1).*

(4) *The ∞-category $\mathcal{X}$ is compactly assembled (Definition 21.1.2.1).*

Example 21.1.6.13. Let $\mathcal{X} = \mathcal{P}(\mathcal{G})$ be the ∞-topos of presheaves on a small ∞-category $\mathcal{G}$. Then $\mathcal{X}$ is exponentiable (since it is compactly generated, and therefore compactly assembled).

Example 21.1.6.14. Let $\mathcal{C}$ be a compactly assembled ∞-category and let $\mathcal{X} = \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ be the ∞-topos of Proposition 21.1.1.2. Then $\mathcal{X}$ is exponentiable (since it is a retract of a presheaf ∞-topos; see Corollary 21.1.5.3).

Example 21.1.6.15. Let $\mathcal{C}$ be a compactly assembled presentable ∞-category and let $\mathcal{X}$ be a classifying ∞-topos for $\mathcal{C}$-valued sheaves. Then $\mathcal{X}$ is exponentiable (by virtue of Theorem 21.1.4.3, this is a special case of Example 21.1.6.14).

The proof of Theorem 21.1.6.12 will make use of the following simple observation:

Lemma 21.1.6.16. *Let $\mathcal{Y}$ be an arbitrary ∞-topos. Then there exists a pullback diagram*

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{Y}_0 \\ \downarrow & & \downarrow \\ \mathcal{Y}_1 & \longrightarrow & \mathcal{Y}_{01} \end{array}$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}$, where $\mathcal{Y}_0$, $\mathcal{Y}_1$, and $\mathcal{Y}_{01}$ are ∞-topoi of presheaves on small ∞-categories which admit finite limits.

Proof. By virtue of Proposition HTT.6.1.5.3 , we can assume that $\mathcal{Y}$ is an accessible left exact localization of $\mathcal{P}(\mathcal{C})$, where $\mathcal{C}$ is a small ∞-category which admits finite limits. Then we can write $\mathcal{Y} = S^{-1}\mathcal{P}(\mathcal{C})$, where S is some small collection of morphisms in $\mathcal{P}(\mathcal{C})$ (Proposition HTT.5.5.4.2). Let $\mathcal{D} \subseteq \mathcal{P}(\mathcal{C})$ be an essentially small full subcategory which contains the domain and codomain of every morphism in S. Let $L : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$ be the localization functor whose essential image is $\mathcal{Y}$ and let $\iota : \mathcal{D} \hookrightarrow \mathcal{P}(\mathcal{C})$ be the inclusion functor. Then the tautological natural transformation $\iota \to L \circ \iota$ determines a left exact functor $f : \mathcal{D} \times \Delta^1 \to \mathcal{P}(\mathcal{C})$. Applying Proposition HTT.6.1.5.2 , we see that f admits an essentially unique extension to a left exact functor $F^* : \mathcal{P}(\mathcal{D} \times \Delta^1) \to \mathcal{P}(\mathcal{C})$ which preserves small colimits. Let F_* be the right adjoint of F^*, and let $G_* : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{D} \times \Delta^1)$ be the functor given by precomposition with the projection map $\mathcal{D} \times \Delta^1 \to \mathcal{D}$. It is now easy to verify that the inclusion $\mathcal{Y} \hookrightarrow \mathcal{P}(\mathcal{C})$ extends to a pullback diagram

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{P}(\mathcal{C}) \\ \downarrow & & \downarrow F_* \\ \mathcal{P}(\mathcal{D}) & \xrightarrow{G_*} & \mathcal{P}(\mathcal{D} \times \Delta^1) \end{array}$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}$. □

Proof of Theorem 21.1.6.12. The implication (1) ⇒ (2) is obvious, the equivalence (2) ⇒ (3) follows from Example 21.1.6.10, and the equivalence (3) ⇒ (4) follows from Corollary 21.1.4.9. We will complete the proof by showing that (4) ⇒ (1). Assume that $\mathcal{X}$ is compactly assembled; we wish to show that the exponential $\mathcal{Y}^{\mathcal{X}}$ exists for every ∞-topos $\mathcal{Y}$. Using Remark 21.1.6.6 and Lemma 21.1.6.16, we can reduce to the case where $\mathcal{Y}$ has the form $\mathcal{P}(\mathcal{G})$, where $\mathcal{G}$ is a small ∞-category which admits finite limits. In particular, we can assume that $\mathcal{Y} \simeq \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S})$ is a classifying ∞-topos of $\mathcal{C}$-valued sheaves, where $\mathcal{C} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ is an ∞-category which is presentable and compactly assembled. In this case, Example 21.1.6.9 shows that the ∞-topos $\mathrm{Fun}^{\omega}(\mathcal{C} \otimes \mathcal{X}, \mathcal{S})$ has the desired universal property. □

21.1.7 Example: Locally Compact Topological Spaces

We now relate Theorem 21.1.6.12 to classical point-set topology.

Proposition 21.1.7.1. *Let X be a locally compact Hausdorff space. Then the ∞-topos $\mathcal{S}\mathrm{hv}(X)$ is exponentiable.*

Remark 21.1.7.2. Let X be a locally compact Hausdorff space. By virtue of Theorem 21.1.6.12, Proposition 21.1.7.1 is equivalent to the assertion that the ∞-category $\mathcal{S}\mathrm{hv}(X)$ is compactly assembled. Beware that $\mathcal{S}\mathrm{hv}(X)$ is usually not compactly generated. For example, if $X = \mathbf{R}$, then the ∞-category $\mathcal{S}\mathrm{hv}(X)$ does not contain any non-initial compact objects (note that if $\mathscr{F} \in \mathcal{S}\mathrm{hv}(X)$ is compact, then $\tau_{\leqslant -1} \mathscr{F}$ can be identified with a compact object in the category of open subsets of $\mathbf{R}$: that is, a compact open subset of $\mathbf{R}$).

Proof of Proposition 21.1.7.1. Let $G : \mathrm{Ind}(\mathcal{S}\mathrm{hv}(X)) \to \mathcal{S}\mathrm{hv}(X)$ be the Ind-extension of the identity functor $\mathrm{id}_{\mathcal{S}\mathrm{hv}(X)}$; to show that $\mathcal{S}\mathrm{hv}(X)$ is compactly assembled, it will suffice to show that G admits admits a left adjoint (Theorem 21.1.2.10). Let us say that an object $\mathscr{F} \in \mathcal{S}\mathrm{hv}(X)$ is *good* if the functor $\mathrm{Map}_{\mathcal{S}\mathrm{hv}(X)}(\mathscr{F}, G(\bullet))$ is corepresentable by an object of $\mathrm{Ind}(\mathcal{S}\mathrm{hv}(X))$. We wish to show that every object of $\mathcal{S}\mathrm{hv}(X)$ is good.

For each open set $U \subseteq X$, let $h_U \in \mathcal{S}\mathrm{hv}(X)$ denote the sheaf represented by U (given by the formula $h_U(V) = \begin{cases} * & \text{if } V \subseteq U \\ \varnothing & \text{if } V \nsubseteq U \end{cases}$). Note that the ∞-category $\mathcal{S}\mathrm{hv}(X)$ is generated under small colimits by objects of the form h_U. Since the ∞-category $\mathrm{Ind}(\mathcal{S}\mathrm{hv}(X))$ admits small colimits, the collection of good objects of $\mathcal{S}\mathrm{hv}(X)$ is closed under small colimits. Consequently, to complete the proof of Proposition 21.1.7.1, it will suffice to show that each of the sheaves h_U is good.

For open sets $U, V \subseteq X$, we write $V \Subset U$ if V is contained in a compact subset of U. For fixed U, the collection of open sets V satisfying $V \Subset U$ is filtered by inclusion. We let $h_{\Subset U}$ denote the Ind-object of $\mathcal{S}\mathrm{hv}(X)$ given by the filtered diagram $\{h_V\}_{V \Subset U}$. We will complete the proof by showing that $h_{\Subset U} \in \mathrm{Ind}(\mathcal{S}\mathrm{hv}(X))$ corepresents the functor $\mathrm{Map}_{\mathcal{S}\mathrm{hv}(X)}(h_U, G(\bullet))$. More precisely, we will show that for any filtered diagram $\{\mathscr{F}_\alpha\}$ in the ∞-category $\mathcal{S}\mathrm{hv}(X)$, the canonical map

$$\theta : (\varinjlim_{\alpha} \mathscr{F}_\alpha)(U) = \mathrm{Map}_{\mathcal{S}\mathrm{hv}(X)}(h_U, G(\{\mathscr{F}_\alpha\})) \to \mathrm{Map}_{\mathrm{Ind}(\mathcal{S}\mathrm{hv}(X))}(h_{\Subset U}, \{\mathscr{F}_\alpha\}) \simeq \varprojlim_{V \Subset U} \varinjlim_{\alpha} \mathscr{F}_\alpha(V)$$

is a homotopy equivalence.

For each compact set $K \subseteq X$ and each sheaf $\mathscr{F} \in \mathcal{S}\mathrm{hv}(X)$, let $\mathscr{F}(K)$ denote the direct limit $\varinjlim_V \mathscr{F}(V)$, where V ranges over all open neighborhoods of K. Using a cofinality argument, we can rewrite the codomain of θ as a limit $\varprojlim_{K \subseteq U} \varinjlim_\alpha \mathscr{F}_\alpha(K)$, so that θ factors as a composition

$$(\varinjlim_{\alpha} \mathscr{F}_\alpha)(U) \xrightarrow{\theta'} \varprojlim_{K \subseteq U} (\varinjlim_{\alpha} \mathscr{F}_\alpha)(K) \xrightarrow{\theta''} \varprojlim_{K \subseteq U} \varinjlim_{\alpha} (\mathscr{F}_\alpha(K)).$$

It now follows from Theorem HTT.7.3.4.9 that the map θ' is a homotopy equivalence, and from Corollary HTT.7.3.4.11 that the map θ'' is a homotopy equivalence. □

Proposition 21.1.7.1 asserts that, if X is a locally compact Hausdorff space and $\mathcal{Y}$ is an arbitrary ∞-topos, then there exists an exponential $\mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}$ in the ∞-category of ∞-topoi. In particular, if Y is another topological space, then we can form the exponential $\mathcal{S}\mathrm{hv}(Y)^{\mathcal{S}\mathrm{hv}(X)}$. We now study the relationship of this exponential with the compact-open topology on the mapping space $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, Y)$.

Lemma 21.1.7.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi for which the exponential $\mathcal{Y}^{\mathcal{X}}$ exists, and let $n \geqslant 0$ be an integer. If $\mathcal{Y}$ is n-localic, then $\mathcal{Y}^{\mathcal{X}}$ is n-localic.*

Proof. Let $f^* : \mathcal{Z} \to \mathcal{Z}'$ be a geometric morphism of ∞-topoi which induces an equivalence $\tau_{\leqslant n-1}\,\mathcal{Z} \to \tau_{\leqslant n-1}\,\mathcal{Z}'$; we wish to show that the induced map

$$\theta : \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{Z}, \mathcal{Y}^{\mathcal{X}}) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{Z}', \mathcal{Y}^{\mathcal{X}})$$

is a homotopy equivalence. Invoking the definition of the exponential $\mathcal{Y}^{\mathcal{X}}$, we can identify θ with the induced map

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{Z} \otimes \mathcal{X}, \mathcal{Y}) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{Z}' \otimes \mathcal{X}, \mathcal{Y}).$$

To show that this map is a homotopy equivalence, it will suffice to show that f^* induces an equivalence $\tau_{\leqslant n-1}(\mathcal{Z} \otimes \mathcal{X}) \to \tau_{\leqslant n-1}(\mathcal{Z}' \otimes \mathcal{X})$. This is clear, since both sides can be identified with $(\tau_{\leqslant n-1} \mathcal{S}) \otimes \mathcal{Z} \otimes \mathcal{X} \simeq (\tau_{\leqslant n-1} \mathcal{S}) \otimes \mathcal{Z}' \otimes \mathcal{X}$. □

For any ∞-topos $\mathcal{X}$, let $|\mathcal{Y}|$ denote the associated topological space (Definition 1.5.4.3).
.

Proposition 21.1.7.4. *Let $\mathcal{Y}$ be a 0-localic ∞-topos, and let X be a locally compact Hausdorff space. Then:*

(1) *The exponential $\mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}$ exists.*

(2) *The ∞-topos $\mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}$ is 0-localic.*

(3) *The underlying topological space $|\mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}|$ can be identified with the mapping space $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, |\mathcal{Y}|)$, equipped with the compact-open topology.*

Proof. Assertion (1) follows from Proposition 21.1.7.1 and assertion (2) from Lemma 21.1.7.3. To prove (3), we observe that for any topological space Z, we have canonical maps

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(Z, |\mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}|) &\xleftarrow{\sim} \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{S}\mathrm{hv}(Z), \mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}) \\
&\simeq \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{S}\mathrm{hv}(Z) \otimes \mathcal{S}\mathrm{hv}(X), \mathcal{Y}) \\
&\xrightarrow{\sim} \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{S}\mathrm{hv}(Z \times X), \mathcal{Y}) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(Z \times X, |\mathcal{Y}|);
\end{aligned}$$

here the first map is an equivalence by (2), the second by the universal property of the exponential, the third by virtue of Proposition HTT.7.3.1.11 (which uses the local compactness of X), and the fourth by our assumption that $\mathcal{Y}$ is 0-localic. Composing these maps, we obtain bijections

$$\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(Z, |\mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}|) \simeq \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(Z \times X, |\mathcal{Y}|)$$

depending functorially on Z. Applying Proposition 21.1.6.1, we deduce that $|\mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}|$ is homeomorphic to the mapping space $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, |\mathcal{Y}|)$ (with its compact-open topology). □

Corollary 21.1.7.5. *Let X be a locally compact Hausdorff space and let Y be a sober topological space. Then there is a canonical homeomorphism $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, Y) \simeq |\mathcal{S}\mathrm{hv}(Y)^{\mathcal{S}\mathrm{hv}(X)}|$ (where the left hand side is equipped with the compact-open topology).*

Warning 21.1.7.6. In the situation of Corollary 21.1.7.5, the homeomorphism $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, Y) \simeq |\mathcal{S}\mathrm{hv}(Y)^{\mathcal{S}\mathrm{hv}(X)}|$ determines a geometric morphism of ∞-topoi

$$e_* : \mathcal{S}\mathrm{hv}(\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, Y)) \to \mathcal{S}\mathrm{hv}(Y)^{\mathcal{S}\mathrm{hv}(X)}.$$

Beware that this geometric morphism is generally *not* an equivalence. The ∞-topos $\mathcal{S}\mathrm{hv}(Y)^{\mathcal{S}\mathrm{hv}(X)}$ is 0-localic, but the underlying locale generally does not have enough points. For example, if X consists of two points, then e_* can be identified with the canonical map $\mathcal{S}\mathrm{hv}(Y \times Y) \to \mathcal{S}\mathrm{hv}(Y) \otimes \mathcal{S}\mathrm{hv}(Y)$, which need not be an equivalence without additional assumptions on Y.

Remark 21.1.7.7. In the proof of Proposition 21.1.7.4, we constructed a homeomorphism $|\,\mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}\,| \simeq \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, |\,\mathcal{Y}\,|)$ by appealing to the universal property of the compact-open topology (Proposition 21.1.6.1). However, this is not necessary. Specializing the bijection $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(Z, |\,\mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}\,|) \simeq \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(Z \times X, |\,\mathcal{Y}\,|)$ to the case where Z is a point, we see that there is an isomorphism of *sets*

$$\beta : \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, |\,\mathcal{Y}\,|) \simeq |\,\mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}\,|,$$

which carries a continuous function $f : X \to |\,\mathcal{Y}\,|$ to the point of $|\,\mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}\,|$ determined by the associated geometric morphism $f_* : \mathcal{S}\mathrm{hv}(X) \to \mathcal{Y}$. There is a unique topology on $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, |\,\mathcal{Y}\,|)$ for which the map β is a homeomorphism: namely, one declares that a set $U \subseteq \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, |\,\mathcal{Y}\,|)$ is open if and only if there exists a (-1)-truncated object $V \in \mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)}$ such that $(f \in U) \Leftrightarrow (f_*$ factors through $(\mathcal{Y}^{\mathcal{S}\mathrm{hv}(X)})_{/V})$. By unwinding the proofs of Theorem 21.1.6.12 and Proposition 21.1.7.4, one can show directly that this topology coincides with the compact-open topology. The argument of Proposition 21.1.7.4 then provides a bijection

$$\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(Z, \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, |\,\mathcal{Y}\,|)) \simeq \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(Z \times X, |\,\mathcal{Y}\,|)$$

for every topological space Z. Specializing to the case $\mathcal{Y} = \mathcal{S}\mathrm{hv}(Y)$ for a sober topological space Y, we obtain another proof of Proposition 21.1.6.1 (at least for the special case where the target space Y is sober).

We close this section by mentioning a variant of Proposition 21.1.7.1:

Proposition 21.1.7.8. *Let X be a coherent topological space. Then the ∞-topos $\mathcal{S}\mathrm{hv}(X)$ is exponentiable.*

Proof. For each open subset $U \subseteq X$, let $h_U \in \mathcal{S}\mathrm{hv}(X)$ denote the sheaf represented by U. It follows from Corollary HTT.7.3.5.4 that if $U \subseteq X$ is quasi-compact, then the construction $\mathscr{F} \mapsto \mathscr{F}(U)$ determines a functor $\mathcal{S}\mathrm{hv}(X) \to \mathcal{S}$ which commutes with filtered colimits. This functor is corepresented by h_U, so that h_U is a compact object of $\mathcal{S}\mathrm{hv}(X)$. Since the sheaves h_U (where U is quasi-compact) generate the ∞-category $\mathcal{S}\mathrm{hv}(X)$ under colimits, we deduce that $\mathcal{S}\mathrm{hv}(X)$ is compact generated. Applying Theorem 21.1.6.12, we conclude that $\mathcal{S}\mathrm{hv}(X)$ is an exponentiable ∞-topos. $\square$

21.2 $\mathcal{G}$-Objects

Let X be a topological space and suppose that we wish to study sheaves on X taking values in some category $\mathcal{C}$. Let us suppose that the objects of $\mathcal{C}$ can be understood as "sets with extra structure" (such as abelian groups or commutative rings), so that $\mathcal{C}$ is equipped with a forgetful functor $T : \mathcal{C} \to \mathcal{S}\mathrm{et}$. If the functor T preserves inverse limits, then every $\mathcal{C}$-valued sheaf $\mathscr{O}$ on X determines a $\mathcal{S}\mathrm{et}$-valued sheaf $\mathscr{O}^T$ on X, given by the formula $\mathscr{O}^T(U) = T(\mathscr{O}(U))$. In practice, we often abuse notation by identifying $\mathscr{O}$ with $\mathscr{O}^T$, implicitly remembering that the set-valued sheaf $\mathscr{O}^T$ is equipped with "extra structure" from its origin as a $\mathcal{C}$-valued sheaf. For example, if $\mathcal{C}$ is the category of groups (and $T : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is the usual forgetful functor), then $\mathscr{O}^T$ can be regarded as an group object of the topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X)$: that is, it is equipped with a multiplication map $m : \mathscr{O}^T \times \mathscr{O}^T \to \mathscr{O}^T$, an inversion map $i : \mathscr{O}^T \to \mathscr{O}^T$, and a unit map $e : \mathbf{1}_X \to \mathscr{O}^T$ (here $\mathbf{1}_X$ denotes a final object of the topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X)$) which satisfy the usual group axioms, which are encoded by requiring the commutativity of the diagrams

$$\begin{array}{ccc} \mathscr{O}^T \times \mathbf{1}_X & \xrightarrow{\mathrm{id} \times e} & \mathscr{O}^T \times \mathscr{O}^T \\ & {\scriptstyle \sim} \searrow \quad \swarrow {\scriptstyle m} & \\ & \mathscr{O}^T & \end{array} \qquad \begin{array}{ccc} \mathbf{1}_X \times \mathscr{O}^T & \xrightarrow{e \times \mathrm{id}} & \mathscr{O}^T \times \mathscr{O}^T \\ & {\scriptstyle \sim} \searrow \quad \swarrow {\scriptstyle m} & \\ & \mathscr{O}^T & \end{array}$$

$$\begin{array}{ccc} \mathscr{O}^T & \xrightarrow{\mathrm{id} \times i} & \mathscr{O}^T \times \mathscr{O}^T \\ \downarrow & & \downarrow{\scriptstyle m} \\ \mathbf{1} & \xrightarrow{e} & \mathscr{O}^T \end{array} \qquad \begin{array}{ccc} \mathscr{O}^T & \xrightarrow{i \times \mathrm{id}} & \mathscr{O}^T \times \mathscr{O}^T \\ \downarrow & & \downarrow{\scriptstyle m} \\ \mathbf{1} & \xrightarrow{e} & \mathscr{O}^T \end{array}$$

$$\begin{array}{ccc} \mathscr{O}^T \times \mathscr{O}^T \times \mathscr{O}^T & \xrightarrow{m \times \mathrm{id}} & \mathscr{O}^T \times \mathscr{O}^T \\ \downarrow{\scriptstyle \mathrm{id} \times m} & & \downarrow{\scriptstyle m} \\ \mathscr{O}^T \times \mathscr{O}^T & \xrightarrow{m} & \mathscr{O}^T . \end{array}$$

In the setting of spectral algebraic geometry (and several of its variants), we need to consider sheaves on a topological space X (or some variant thereof) which take values in an ∞-category $\mathcal{C}$, rather than an ordinary category. In practice, the ∞-category $\mathcal{C}$ can usually be viewed as "spaces with extra structure," meaning that it is equipped with a well-behaved forgetful functor $T : \mathcal{C} \to \mathcal{S}$ (for example, $\mathcal{C}$ might be the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-rings, and T the 0th space functor $\Omega^\infty : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$). In this case, we can again associate to any $\mathcal{C}$-valued sheaf $\mathscr{O}$ a $\mathcal{S}$-valued sheaf $\mathscr{O}^T$, given by the formula $\mathscr{O}^T(U) = T(\mathscr{O}(U))$. However, it becomes more difficult to describe the "extra structure" inherited by $\mathscr{O}^T$. In the higher categorical setting, one must stipulate the existence *witness* to the commutativity of every diagram; these witnesses then satisfy higher-order coherence conditions which are prohibitively difficult to articulate directly. We therefore adopt an alternate approach, where we consider a family of forgetful functors $\{T : \mathcal{C} \to \mathcal{S}\}_{T \in \mathcal{G}}$ indexed by another ∞-category $\mathcal{G}$; the "extra structure" will then be encoded by the functorial dependence of the sheaf $\mathscr{O}^T$ on the object $T \in \mathcal{G}$.

Definition 21.2.0.1. Let $\mathcal{G}$ be an essentially small ∞-category and let $\mathcal{X}$ be an ∞-topos. A *$\mathcal{G}$-object* of $\mathcal{X}$ is a functor $\mathcal{G} \to \mathcal{X}$ which is locally left exact (see Definition 20.4.2.1). We let $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{G}, \mathcal{X})$ spanned by the $\mathcal{G}$-objects of $\mathcal{X}$.

Remark 21.2.0.2. The ∞-category $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ was already studied in Chapter 20, where it was denoted by $\mathrm{Fun}^{\mathrm{llex}}(\mathcal{G}, \mathcal{X})$ (Definition 20.4.2.1). The shift in notation reflects a slightly different perspective: in this chapter, we wish to emphasize the heuristic that an object of $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ should be viewed as an "object" of $\mathcal{X}$ equipped with extra structure.

Notation 21.2.0.3. In the situation of Definition 21.2.0.1, we will typically use the letter $\mathscr{O}$ to denote $\mathcal{G}$-objects of $\mathcal{X}$, and we will write $\mathscr{O}^T$ to denote the value of $\mathscr{O}$ on an object $T \in \mathcal{G}$.

We will be primarily interested in the special case of Definition 21.2.0.1 where the ∞-category $\mathcal{G}$ admits finite limits. In this case, a $\mathcal{G}$-object of $\mathcal{X}$ is simply a functor $\mathscr{O} : \mathcal{G} \to \mathcal{X}$ which preserves finite limits (see Corollary 20.4.3.2). In §21.2.2, we will show that if $\mathcal{G}$ admits finite limits, then the datum of a $\mathcal{G}$-object is equivalent to the datum of a $\mathcal{C}$-valued sheaf, where $\mathcal{C} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ (Proposition 21.2.2.1). This is a consequence of the following pair of assertions:

(*a*) The presheaf ∞-category $\mathcal{P}(\mathcal{G}) = \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$ is a classifying ∞-topos for $\mathcal{C} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$-valued sheaves (see Proposition **??**).

(*b*) The presheaf ∞-category $\mathcal{P}(\mathcal{G}) = \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$ is a classifying ∞-topos for $\mathcal{G}$-objects.

Here assertion (*a*) depends on the fact that the ∞-category $\mathcal{G}$ admits finite limits, but assertion (*b*) does not. Consequently, the theory of $\mathcal{G}$-objects offers a bit more flexibility than the theory of $\mathcal{C}$-valued sheaves: we give a concrete illustration of this point in §21.2.3. Note that not every ∞-topos arises as a classifying topos in the sense of (*a*) or (*b*) (the ∞-topoi that classify $\mathcal{C}$-valued sheaves are rather special, even without the assumption that $\mathcal{C}$ is compactly generated: see Proposition 21.1.5.4). A general ∞-topos $\mathcal{E}$ need not be equivalent to an ∞-category of presheaves $\mathcal{P}(\mathcal{G})$. However, every ∞-topos $\mathcal{E}$ can be realized as an (accessible) left exact localization of an ∞-category of the form $\mathcal{P}(\mathcal{G})$. In this case, for any ∞-topos $\mathcal{X}$, one has a fully faithful embedding $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \simeq \mathrm{Fun}^*(\mathcal{P}(\mathcal{G}), \mathcal{X}) \simeq \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$, whose essential image can be viewed as the collection of $\mathcal{G}$-objects of $\mathcal{X}$ which are in some sense "local" with respect to the chosen realization of $\mathcal{E}$. In §21.2.1, we make this heuristic more precise by introducing the notion of a *τ-local $\mathcal{G}$-object*, where τ is a Grothendieck topology on $\mathcal{G}$, and showing that the τ-local $\mathcal{G}$-object are classified by the ∞-topos $\mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$ (Proposition 21.2.1.13).

In §21.2.5, we specialize to the setting of algebraic geometry. When $\mathcal{G} = \mathrm{Aff}$ is the category of affine schemes of finite type over $\mathbf{Z}$, then the notion of local $\mathcal{G}$-object recovers the notion of local commutative ring object when $\mathcal{G}$ is equipped with the Zariski topology of Example 20.6.4.1, and the notion of a strictly Henselian commutative ring object when $\mathcal{G}$ is equipped with the étale topology of Example **??** (see Theorem 21.2.5.1). Replacing Aff by a suitable ∞-categorical analogue (see Examples 20.6.4.4 and **??**), we obtain a similar description of the theory of local and strictly Henselian CAlg-valued sheaves (Theorem 21.2.5.3). Moreover, we show that the notions of *local* morphism between $\mathrm{CAlg}^{\heartsuit}$-valued and CAlg-valued sheaves can be recovered as a special case of the more general notion of *local morphism of $\mathcal{G}$-object* which is determined by a choice of admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$, which we study in §21.2.4 (see Definition 21.2.4.1).

21.2.1 Local $\mathcal{G}$-Objects and Classifying ∞-Topoi

We begin by introducing a slight elaboration on Definition 21.2.0.1.

Definition 21.2.1.1. Let $\mathcal{G}$ be an essentially small ∞-category equipped with a Grothendieck topology τ and let $\mathcal{X}$ be an ∞-topos. We will say that a $\mathcal{G}$-object $\mathscr{O} : \mathcal{G} \to \mathcal{X}$ is *τ-local* if it satisfies the following condition:

(∗) For every collection of morphisms $\{T_\alpha \to T\}$ in $\mathcal{G}$ which generate a τ-covering of the object T, the induced map $\amalg \mathscr{O}^{T_\alpha} \to \mathscr{O}^T$ is an effective epimorphism in the ∞-topos $\mathcal{X}$.

We let $\mathrm{Obj}^\tau_{\mathcal{G}}(\mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{G}, \mathcal{X})$ spanned by the τ-local $\mathcal{G}$-objects of $\mathcal{X}$.

Example 21.2.1.2. In the situation of Definition 21.2.1.1, suppose that the Grothendieck topology τ is trivial: that is, a collection of morphisms $\{u_\alpha : T_\alpha \to T\}$ generates a τ-covering if and only if some u_α admits a section. Then every $\mathcal{G}$-object is τ-local: that is, we have $\mathrm{Obj}^\tau_{\mathcal{G}}(\mathcal{X}) = \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ for any ∞-topos $\mathcal{X}$.

Variant 21.2.1.3. If X is a topological space, then we let $\mathrm{Obj}^\tau_{\mathcal{G}}(X)$ denote the ∞-category $\mathrm{Obj}^\tau_{\mathcal{G}}(\mathcal{S}\mathrm{hv}(X))$ of $\mathcal{G}$-objects of the ∞-topos $\mathcal{S}\mathrm{hv}(X)$. In the case where the topology τ is trivial, we denote this ∞-category simply by $\mathrm{Obj}_{\mathcal{G}}(X)$.

Example 21.2.1.4. Let X be a topological space, let $\mathcal{U}(X)$ denote the partially ordered set of open subsets of X, and let $\mathcal{G}$ be an essentially small ∞-category which admits finite limits. Then $\mathrm{Obj}_{\mathcal{G}}(X)$ can be identified with the full subcategory of $\mathrm{Fun}(\mathcal{G} \times \mathcal{U}(X)^{\mathrm{op}}, \mathcal{S})$ spanned by those functors

$$\mathscr{O} : \mathcal{G} \times \mathcal{U}(X)^{\mathrm{op}} \to \mathcal{S} \qquad (T, U) \mapsto \mathscr{O}^T(U)$$

satisfying the following pair of requirements:

(a) For every fixed object $T \in \mathcal{G}$, the functor $U \mapsto \mathscr{O}^T(U)$ is a $\mathcal{S}$-valued sheaf on X.

(b) For each fixed open set $U \subseteq X$, the functor $T \mapsto \mathscr{O}^T(U)$ preserves finite limits.

If $\mathcal{G}$ is equipped with a Grothendieck topology τ, then we can identify $\mathrm{Obj}^{\tau}_{\mathcal{G}}(X)$ with the full subcategory of $\mathrm{Obj}_{\mathcal{G}}(X)$ spanned by those functors which satisfy the following additional requirement:

(c) For every τ-covering $\{T_\alpha \to T\}$ in $\mathcal{G}$ and every point $s \in \mathscr{O}^T(U)$, there exists an open covering $\{U_\beta\}$ of U such that each $s|_{U_\beta} \in \mathscr{O}^T(U_\beta)$ can be lifted to a point of $\mathscr{O}^{T_\alpha}(U_\beta)$, for some index α (which might depend on β).

Remark 21.2.1.5. Let $\mathcal{G}$ and X be as in Example 21.2.1.4. In practice, we will take $\mathcal{G}$ to be some relatively concrete class of "test objects" (such as affine schemes of finite type over $\mathbf{Z}$). If $\mathscr{O}$ is an object of the ∞-category $\mathrm{Obj}^{\tau}_{\mathcal{G}}(X)$, then it is useful to think of $\mathscr{O}^T(U)$ as a parameter space for "regular functions from U to T." Conditions (a), (b), and (c) of Example 21.2.1.4 encode natural expectations for how such a theory of "regular function" should behave.

Construction 21.2.1.6 (Pullbacks of $\mathcal{G}$-Objects)**.** Let $\mathcal{G}$ be an essentially small ∞-category equipped with a Grothendieck topology τ and suppose we are given a geometric morphism of ∞-topoi $\pi^* : \mathcal{X} \to \mathcal{Y}$. It follows immediately from the definitions that composition with π^* determines a functor $\mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{X}) \to \mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{Y})$, which we will refer to as the *pullback functor*. We will denote this functor either by $\pi^*_{\mathcal{G}}$ (when we need to emphasize its dependence on $\mathcal{G}$ or distinguish it from the usual pullback functor π^*) or simply by π^* (when its meaning is clear on context). Concretely, we have $(\pi^*_{\mathcal{G}}\, \mathscr{O})^T = \pi^*(\mathscr{O}^T)$ for each $\mathscr{O} \in \mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{X})$ and each $T \in \mathcal{G}$.

Notation 21.2.1.7. In the situation of Construction 21.2.1.6, suppose we are given a $\mathcal{G}$-object $\mathscr{O} \in \mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{X})$ and an object $U \in \mathcal{X}$. We let $\mathscr{O}|_U$ denote the pullback of $\mathscr{O}$ along the étale geometric morphism $\mathcal{X}_{/U} \to \mathcal{X}$. More concretely, we can describe $\mathscr{O}|_U$ as the functor from $\mathcal{G}$ to $\mathcal{X}_{/U}$ given by the formula $(\mathscr{O}|_U)^T = \mathscr{O}^T \times U$.

Remark 21.2.1.8 (Pushforwards of $\mathcal{G}$-Objects)**.** In the situation of Construction 21.2.1.6, suppose that $\mathcal{G}$ admits finite limits and that the Grothendieck topology τ is trivial. In this case, we can identify $\mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{X})$ and $\mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{Y})$ with the ∞-categories $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X})$ and $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{Y})$ of left exact functors from $\mathcal{G}$ to the ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$, respectively. It follows that composition with the direct image functor $\pi_* : \mathcal{Y} \to \mathcal{X}$ determines functor $\mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{Y}) \to \mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{X})$. We will denote this functor either by $\pi^{\mathcal{G}}_*$ (when we wish to distinguish it from the pushforward functor $\pi_* : \mathcal{Y} \to \mathcal{X}$) by simply by π_* (when there is no danger of confusion). Concretely, it is given by the formula $(\pi^{\mathcal{G}}_*\, \mathscr{O})^T = \pi_*(\mathscr{O}^T)$ for each $\mathscr{O} \in \mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{X})$ and each $T \in \mathcal{G}$. Note that the functor $\pi^{\mathcal{G}}_*$ is right adjoint to the pullback functor $\pi^*_{\mathcal{G}} : \mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{X}) \to \mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{Y})$ of Construction 21.2.1.6.

Warning 21.2.1.9. The hypotheses of Remark 21.2.1.8 are necessary. If we drop either the assumption that $\mathcal{G}$ admits finite limits or that the Grothendieck topology τ is trivial, then the pullback functor $\pi_{\mathcal{G}}^* : \mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{X}) \to \mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{Y})$ need not admit a right adjoint.

We now have the following variant of Definition 21.1.0.1:

Definition 21.2.1.10. Let $\mathcal{G}$ be an essentially small ∞-category equipped with a Grothendieck topology τ. Suppose that $\mathcal{E}$ is an ∞-topos and that $\mathcal{O} \in \mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{E})$ is a τ-local $\mathcal{G}$-object of $\mathcal{E}$. We will say that $\mathcal{O}$ is *universal* if, for every ∞-topos $\mathcal{X}$, the construction $f^* \mapsto f^* \mathcal{O}$ induces an equivalence of ∞-categories

$$\mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \to \mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{X}).$$

In this case, we will say that *$\mathcal{O}$ exhibits $\mathcal{E}$ as a classifying topos for τ-local $\mathcal{G}$-objects.*

It follows immediately from the definitions that, if there exists a classifying ∞-topos $\mathcal{E}$ for τ-local $\mathcal{G}$-objects, then $\mathcal{E}$ is determined uniquely up to equivalence. We now prove existence by an explicit construction.

Definition 21.2.1.11. Let $\mathcal{G}$ be an essentially small ∞-category equipped with a Grothendieck topology τ. We define the *sheafified Yoneda embedding* $h : \mathcal{G} \to \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ to be the composition

$$\mathcal{G} \xrightarrow{j} \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{L} \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G}),$$

where j is the Yoneda embedding for $\mathcal{G}$ and L is a left adjoint to the inclusion $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G}) \hookrightarrow \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$ (that is, L is the sheafification functor associated to τ).

Warning 21.2.1.12. The terminology of Definition 21.2.1.11 is potentially misleading: the sheafified Yoneda embedding $h : \mathcal{G} \to \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ need not be fully faithful (it is fully faithful if and only if the topology τ is *subcanonical*: that is, if and only if the functor j takes values in $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$).

Proposition 21.2.1.13. *Let $\mathcal{G}$ be an essentially small ∞-category equipped with a Grothendieck topology τ. Then the sheafified Yoneda embedding $h : \mathcal{G} \to \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ is a universal τ-local $\mathcal{G}$-object of the ∞-topos $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$. In particular, $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ is a classifying ∞-topos for τ-local $\mathcal{G}$-objects.*

Proof. Let $\mathcal{X}$ be an arbitrary ∞-topos; we wish to show that composition with the composite functor

$$\mathcal{G} \xrightarrow{j} \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{L} \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$$

induces an equivalence of ∞-categories $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G}), \mathcal{X}) \to \mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{X})$. When τ is the trivial topology, this follows from the universal property of the presheaf ∞-category $\mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$ (see Theorem HTT.5.1.5.6) together with the definition of local left exactness. The general case then follows from universal property of $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ given in Lemma HTT.6.2.3.20 . □

21.2.2 Comparison with $\mathcal{C}$-Valued Sheaves

We now show that, if $\mathcal{C}$ is a compactly generated ∞-category, then the theory of $\mathcal{C}$-valued sheaves (in the sense of Definition 1.3.1.4) is equivalent to the theory of $\mathcal{G}$-objects (in the sense of Definition 21.2.0.1), where $\mathcal{G} = \mathcal{C}_c^{\mathrm{op}}$ is the opposite of the ∞-category of compact objects of $\mathcal{C}$. More generally, we have the following:

Proposition 21.2.2.1. *Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{G}$ be an essentially small ∞-category which admits finite limits. Then there is a canonical equivalence of ∞-categories $\rho : \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})}(\mathcal{X}) \to \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$.*

Proof. Set $\mathcal{C} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$, so that composition with the Yoneda embedding $\mathcal{G}^{\mathrm{op}} \hookrightarrow \mathcal{C}$ induces an equivalence of ∞-categories $\mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}) \simeq \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$. It follows that the ∞-topos $\mathcal{E} = \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$ is a classifying ∞-topos for both $\mathcal{C}$-valued sheaves (Theorem 21.1.4.3) and for $\mathcal{G}$-objects (Proposition 21.2.1.13). In particular, for any ∞-topos $\mathcal{X}$, we have equivalences $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) \simeq \mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$. □

Remark 21.2.2.2. The equivalence ρ of Proposition 21.2.2.1 is characterized informally by the formula

$$\mathrm{Map}_{\mathcal{X}}(X, (\rho \mathscr{F})^T) = \mathrm{Map}_{\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})}(T, \mathscr{F}(X))$$

for $X \in \mathcal{X}$ and $T \in \mathcal{G}$ (here we abuse notation by identifying T with its image in $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$).

Variant 21.2.2.3. Let $\mathcal{G}$ be an essentially small ∞-category which admits finite limits and let X be a topological space. Combining Propositions 21.2.2.1 and 1.3.1.7, we obtain an equivalence of ∞-categories $\mathrm{Obj}_{\mathcal{G}}(X) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})}(X)$.

Remark 21.2.2.4 (Functoriality)**.** Suppose we are given a geometric morphism of ∞-topoi

$$\mathcal{X} \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathcal{Y}.$$

Then:

- For every ∞-category $\mathcal{C}$, there is a direct image functor $\pi_*^{\mathcal{C}} : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{Y}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$, given on objects by the formula $(\pi_*^{\mathcal{C}} \mathscr{F})(X) = \mathscr{F}(\pi^* X)$. In general, the functor $\pi_*^{\mathcal{C}}$ does not admit a left adjoint.

- For every essentially small ∞-category $\mathcal{G}$ equipped with a Grothendieck topology τ, there is a pullback functor $\pi_{\mathcal{G}}^* : \mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{X}) \to \mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{Y})$, given on objects by the formula $(\pi_{\mathcal{G}}^* \mathscr{O})^T = \pi^*(\mathscr{O}^T)$ (Construction 21.2.1.6). In general, the functor $\pi_{\mathcal{G}}^*$ does not admit a right adjoint.

In the special case where $\mathcal{G}$ admits finite limits and the Grothendieck topology τ is trivial, then the functor $\pi_{\mathcal{G}}^*$ *does* admit a right adjoint $\pi_*^{\mathcal{G}}$, given by pointwise composition with π_* (Remark 21.2.1.8). In this case, Proposition 21.2.2.1 supplies equivalences

$$\rho_{\mathcal{X}} : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \simeq \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) \qquad \rho_{\mathcal{Y}} : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{Y}) \simeq \mathrm{Obj}_{\mathcal{G}}(\mathcal{Y}),$$

where $\mathcal{C}$ denotes the compactly generated ∞-category $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$. In this case, the direct image functor $\pi_*^{\mathcal{C}}$ admits a left adjoint $\pi_{\mathcal{C}}^*$ (Proposition **??**). The diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) & \xrightarrow{\rho_{\mathcal{X}}} & \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) \\ \downarrow{\scriptstyle \pi_{\mathcal{C}}^*} & & \downarrow{\scriptstyle \pi_{\mathcal{G}}^*} \\ \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{Y}) & \xrightarrow{\rho_{\mathcal{Y}}} & \mathrm{Obj}_{\mathcal{G}}(\mathcal{Y}) \end{array}$$

commutes by construction. Passing to adjoints, we deduce that the diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{Y}) & \xrightarrow{\rho_{\mathcal{Y}}} & \mathrm{Obj}_{\mathcal{G}}(\mathcal{Y}) \\ \downarrow{\scriptstyle \pi_*^{\mathcal{C}}} & & \downarrow{\scriptstyle \pi_*^{\mathcal{G}}} \\ \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) & \xrightarrow{\rho_{\mathcal{X}}} & \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) \end{array}$$

also commutes (this also follows from the concrete description of $\rho_{\mathcal{X}}$ and $\rho_{\mathcal{Y}}$ supplied in Remark 21.2.2.2).

The rest of this section is devoted to giving a second proof of Proposition 21.2.2.1, which avoids appealing to the theory of classifying ∞-topoi (and yields some information even in the case where $\mathcal{G}$ does not admit finite limits).

Notation 21.2.2.5. Recall that a functor $f : \mathcal{C} \to \mathcal{D}$ between ∞-categories is said to be *left exact* if, for every object $D \in \mathcal{D}$, the ∞-category $(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/})^{\mathrm{op}}$ is filtered (see Proposition HTT.5.3.2.5). We let $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the left exact functors. According to Proposition HTT.5.3.2.9 , if $\mathcal{C}$ admits finite limits, then a functor $f : \mathcal{C} \to \mathcal{D}$ is left exact if and only if it preserves finite limits.

Proposition 21.2.2.6. *Let $\mathcal{X}$ be an ∞-topos, let $\mathcal{G}$ be an essentially small ∞-category, and set $\mathcal{C} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$. Then there is an equivalence of ∞-categories $\rho : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X})$, characterized by the formula*

$$\mathrm{Map}_{\mathcal{X}}(X, \rho(\mathscr{F})(T)) = \mathrm{Map}_{\mathcal{C}}(T, \mathscr{F}(X))$$

for $X \in \mathcal{X}$ and $T \in \mathcal{G}$ (here again we abuse notation by identifying T with its image in $\mathcal{C}$).

Corollary 21.2.2.7. *Let $\mathcal{X}$ be an ∞-topos, let $\mathcal{G}$ be an essentially small ∞-category, and set $\mathcal{C} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$. Then the functor ρ of Proposition 21.2.2.6 determines a fully faithful embedding $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \hookrightarrow \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$.*

Remark 21.2.2.8. In the situation of Corollary 21.2.2.7, if the ∞-category $\mathcal{G}$ admits finite limits, then fully faithful embedding $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \hookrightarrow \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ coincides with the equivalence of Proposition 21.2.2.1 (this is a reformulation of Remark 21.2.2.2).

The proof of Proposition 21.2.2.6 is essentially formal. Let us begin by reviewing some notation.

Notation 21.2.2.9. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. We let $\mathrm{LFun}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which admit right adjoints, and $\mathrm{RFun}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the full subcategory spanned by those functors which admit left adjoints. By virtue of Proposition HTT.5.2.6.2 , the formation of adjoint functors supplies an equivalence of ∞-categories $\mathrm{RFun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{LFun}(\mathcal{D}, \mathcal{C})^{\mathrm{op}}$.

Remark 21.2.2.10. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Using the evident isomorphism

$$\mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{LFun}(\mathcal{C}, \mathcal{D}^{\mathrm{op}})^{\mathrm{op}},$$

we can formulate the equivalence of Notation 21.2.2.9 in the following more symmetric form:

$$\mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{RFun}(\mathcal{D}^{\mathrm{op}}, \mathcal{C}).$$

Remark 21.2.2.11. Let $\mathcal{C}$ be a locally small ∞-category, and let $\mathcal{D}$ be a presentable ∞-category. Using Corollary HTT.5.5.2.9 and Remark HTT.5.5.2.10 , we see that a functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{C}$ admits a left adjoint if and only if it preserves small limits. In particular, if $\mathcal{X}$ is an ∞-topos, we have $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) = \mathrm{RFun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$ (see Definition 1.3.1.4), and therefore an equivalence $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \simeq \mathrm{LFun}(\mathcal{C}, \mathcal{X}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$.

Proposition 21.2.2.12. *Let $\mathcal{G}$ be an essentially small ∞-category, let $\mathcal{X}$ be a presentable ∞-category, and let $j : \mathcal{G} \to \mathrm{Pro}(\mathcal{G})$ denote the Yoneda embedding. Let $G : \mathrm{Pro}(\mathcal{G}) \to \mathcal{X}$ be a functor which preserves filtered limits. The following conditions are equivalent:*

(1) *The functor G admits a left adjoint F.*

(2) *The composite functor $g = G \circ j$ is left exact.*

Proof. Let us regard $\mathrm{Pro}(\mathcal{G})$ as a full subcategory of $\mathrm{Fun}(\mathcal{G}, \mathcal{S})^{\mathrm{op}} = \mathcal{P}(\mathcal{G}^{\mathrm{op}})^{\mathrm{op}}$. Using Theorem HTT.5.1.5.6 , we can assume without loss of generality that g factors as a composition

$$\mathcal{G} \xrightarrow{\overline{j}} \mathcal{P}(\mathcal{G}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\overline{G}} \mathcal{X},$$

where $\overline{G}$ is a functor which preserves small limits and $\overline{j}$ is the Yoneda embedding. Using Corollary HTT.5.5.2.9 and Remark HTT.5.5.2.10 , we deduce that $\overline{G}$ admits a left adjoint $\overline{F}$. Unwinding the definitions, we see that $\overline{F}$ carries an object $X \in \mathcal{X}$ to a functor $\overline{F}(X) : \mathcal{G} \to \mathcal{S}$

which classifies the left fibration $\mathcal{X}_{/X} \times_{\mathcal{X}} \mathcal{G} \to \mathcal{G}$. It follows that condition (1) is equivalent to the requirement that $\overline{F}$ takes values in the full subcategory $\mathrm{Pro}(\mathcal{G}) \subseteq \mathcal{P}(\mathcal{G}^{\mathrm{op}})^{\mathrm{op}}$, in which case we can identify $\overline{F}$ with a functor $F : \mathcal{X} \to \mathrm{Pro}(\mathcal{G})$ which is left adjoint to G. Conversely, suppose that G admits a left adjoint F. Then the unit map $u : \mathrm{id}_{\mathcal{X}} \to G \circ F = \overline{G} \circ F$ induces a natural transformation $\alpha : \overline{F} \to F$. To prove (1), it will suffice to show that α is an equivalence. Let $X \in \mathcal{X}$ be an object; we wish to show that α induces an equivalence $\alpha_X : \overline{F}X \to FX$. Equivalently, we must show that for each object $C \in \mathrm{Fun}(\mathcal{G}, \mathcal{S})^{\mathrm{op}}$, the induced map

$$\theta : \mathrm{Map}_{\mathcal{G}}(FX, C) \to \mathrm{Map}_{\mathcal{G}}(\overline{F}X, C)$$

is a homotopy equivalence. Writing C as a limit of representable functors, we may reduce to the case where C belongs to the essential image of $\overline{j}$, so that (in particular) $C \in \mathrm{Pro}(\mathcal{G})$. In this case, the domain and codomain of θ can both be identified with the mapping space $\mathrm{Map}_{\mathcal{X}}(X, GC)$. □

Proof of Proposition 21.2.2.6. Let $\mathcal{G}$ be an essentially small ∞-category, set $\mathcal{C} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}$, and let $\mathcal{X}$ be an ∞-topos. The functor ρ appearing in Proposition 21.2.2.6 is given by the composition

$$\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \xrightarrow{\theta} \mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{G}, \mathcal{X}),$$

where θ is the equivalence of Remark 21.2.2.11. We wish to show that this composition is a fully faithful embedding, whose essential image coincides with the full subcategory $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{G}, \mathcal{X})$. The full faithfulness is a consequence of Proposition HTT.5.3.5.10 , and the description of the essential image follows from Proposition 21.2.2.12. □

21.2.3 Digression: Flatness

Let $\mathcal{G}$ be an essentially small ∞-category and set $\mathcal{C} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$. For any ∞-topos $\mathcal{X}$, Proposition 21.2.2.6 supplies a fully faithful functor

$$\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X}) \subseteq \mathrm{Fun}^{\mathrm{llex}}(\mathcal{G}, \mathcal{X}) = \mathrm{Obj}_{\mathcal{G}}(\mathcal{X});$$

here $\mathrm{Fun}^{\mathrm{llex}}(\mathcal{G}, \mathcal{X})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{G}, \mathcal{X})$ spanned by the *locally* left exact functors. This functor is an equivalence when the ∞-category $\mathcal{G}$ admits finite limits (see Proposition HTT.6.1.5.2), but not in general. In this section, we study an example which illustrates the difference.

Notation 21.2.3.1. Let $\mathcal{X}$ be an ∞-topos. For any spectrum M, we let $\underline{M} \in \mathrm{Sp}(\mathcal{X})$ denote the constant sheaf (of spectra) on $\mathcal{X}$ with value M. Note that the functor $M \mapsto \underline{M}$ determines a symmetric monoidal functor $\mathrm{Sp} \to \mathrm{Sp}(\mathcal{X})$. In particular, if R is an $\mathbb{E}_1$-ring, then $\underline{R}$ is an associative algebra object of $\mathrm{Sp}(\mathcal{X})$; if M is a right module over R, then $\underline{M}$ inherits the structure of a right module over $\underline{R}$.

Definition 21.2.3.2. Let R be a connective $\mathbb{E}_1$-ring, let $\mathcal{X}$ be an ∞-topos, and let $\mathscr{F}$ be a left $\underline{R}$-module object of $\mathrm{Sp}(\mathcal{X})$. We will say that $\mathscr{F}$ is *R-flat* if the construction $M \mapsto \underline{M} \otimes_{\underline{R}} \mathscr{F}$ determines a t-exact functor $\mathrm{RMod}_R \to \mathrm{Sp}(\mathcal{X})$. We let $\mathrm{LMod}^{\flat}_{\underline{R}}$ denote the full subcategory of $\mathrm{LMod}_{\underline{R}}$ spanned by those left $\underline{R}$-modules which are R-flat.

Remark 21.2.3.3. In the situation of Definition 21.2.3.2, the functor $M \mapsto \underline{M} \otimes_{\underline{R}} \mathscr{F}$ is right t-exact if and only if $\mathscr{F}$ is connective. In particular, every R-flat left $\underline{R}$-module is connective.

Example 21.2.3.4. Let R be a connective $\mathbb{E}_\infty$-ring, let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}})$ be a spectral Deligne-Mumford stack equipped with a morphism $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$, and let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$. Then we can regard $\mathscr{F}$ as a sheaf of $\underline{R}$-modules on $\mathcal{X}$. Moreover, for every R-module M, the tensor product $\underline{M} \otimes_{\underline{R}} \mathscr{F}$ is also a quasi-coherent sheaf on X, whose value on an affine object $U \in \mathcal{X}$ is given by the formula

$$(\underline{M} \otimes_{\underline{R}} \mathscr{F})(U) \simeq M \otimes_R \mathscr{F}(U).$$

In particular, we see that the following conditions are equivalent:

- (i) The quasi-coherent sheaf $\mathscr{F}$ is R-flat (in the sense of Definition 21.2.3.2).
- (ii) For every affine $U \in \mathcal{X}$, the spectrum $\mathscr{F}(U)$ is flat as an R-module (in the sense of Definition HA.7.2.2.10).

Example 21.2.3.5. Let R be a connective $\mathbb{E}_\infty$-ring and let $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ be a map of spectral Deligne-Mumford stacks. Then f is flat (in the sense of Definition 2.8.2.1) if and only it the structure sheaf $\mathscr{O}_{\mathsf{X}}$ is R-flat (in the sense of Definition 21.2.3.2).

Example 21.2.3.6. Let R be a connective $\mathbb{E}_1$-ring and let $\mathcal{X} = \mathcal{S}$ be the ∞-topos of spaces, so that the formation of global sections supplies an equivalence $\Gamma : \mathrm{LMod}_{\underline{R}}(\mathcal{X}) \simeq \mathrm{LMod}_R$. Then a left $\underline{R}$-module $\mathscr{F}$ is flat (in the sense of Definition 21.2.3.2) if and only if the associated left R-module $\Gamma(\mathscr{F})$ is flat (in the sense of Definition HA.7.2.2.10).

Example 21.2.3.7. Let R be a connective $\mathbb{E}_1$-ring and let $\mathcal{X}$ be an ∞-topos with enough points (that is, the collection of geometric morphisms functors $x^* \in \mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$ are jointly conservative). Then a left $\underline{R}$-module $\mathscr{F} \in \mathrm{LMod}_{\underline{R}}(\mathrm{Sp}(\mathcal{X}))$ is R-flat (in the sense of Definition 21.2.3.2) if and only if, for each point $x^* \in \mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$, the stalk $x^* \mathscr{F} \in \mathrm{LMod}_R$ is flat as a left R-module (in the sense of Definition HA.7.2.2.10).

Variant 21.2.3.8. Let R be a connective $\mathbb{E}_1$-ring and let $\mathcal{X}$ be an ∞-topos. We will say that a left $\underline{R}$-module $\mathscr{F}$ is *$\underline{R}$-flat* if the functor

$$(\mathscr{G} \in \mathrm{RMod}_{\underline{R}}) \mapsto \mathscr{G} \otimes_{\underline{R}} \mathscr{F}$$

determines a t-exact functor $\mathrm{RMod}_{\underline{R}} \to \mathrm{Sp}(\mathcal{X})$.

Note that if $\mathscr{F}$ is $\underline{R}$-flat, then it is R-flat (in the sense of Definition 21.2.3.2): this follows from the t-exactness of the constant sheaf functor $M \mapsto \underline{M}$. If the ∞-topos $\mathcal{X}$ has enough points (or, more generally, if $\mathcal{X}$ is hypercomplete), then the converse holds. We do not know if the converse is true in general.

We can now formulate the main result of this section:

Proposition 21.2.3.9. *Let R be a connective $\mathbb{E}_1$-ring, let $\mathrm{LMod}_R^{\mathrm{proj}}$ denote the full subcategory of LMod_R spanned by the finitely generated projective left R-modules, and set $\mathcal{G} = (\mathrm{LMod}_R^{\mathrm{proj}})^{\mathrm{op}}$. Then, for any ∞-topos $\mathcal{X}$, there is a canonical equivalence of ∞-categories $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) \simeq \mathrm{LMod}_{\underline{R}}^{\flat}$.*

Example 21.2.3.10. In the situation of Proposition 21.2.3.9, suppose that $\mathcal{X} = \mathcal{S}$ is the ∞-topos of spaces. Invoking Proposition 20.4.2.9, we can identify $\mathcal{G}$-objects of $\mathcal{X}$ with Ind-objects of the ∞-category $\mathcal{G}^{\mathrm{op}} = \mathrm{LMod}_R^{\mathrm{proj}}$. In this case, Proposition 21.2.3.9 reduces to Lazard's theorem (Theorem HA.7.2.2.15), which supplies an equivalence of ∞-categories $\mathrm{LMod}_R^{\flat} \simeq \mathrm{Ind}(\mathrm{LMod}_R^{\mathrm{proj}})$.

Remark 21.2.3.11. In the situation of Proposition **??**, Corollary 21.2.2.7 supplies a fully faithful functor

$$\mathcal{S}\mathrm{hv}_{\mathrm{LMod}_R^{\flat}}(\mathcal{X}) = \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})}(\mathcal{X}) \hookrightarrow \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) \simeq \mathrm{LMod}_{\underline{R}}^{\flat}.$$

This functor is usually not an equivalence. For example, if $\mathcal{X}$ has enough points, then we can identify objects of the right hand side with left $\underline{R}$-modules $\mathscr{F}$ having the property that each stalk $x^* \mathscr{F}$ is a flat left R-module (Example 21.2.3.7), while objects of the left hand side are left $\underline{R}$-modules $\mathscr{F}$ having the property that the connective cover $\tau_{\geqslant 0} \mathscr{F}(U)$ is flat left R-module for each $U \in \mathcal{X}$. The latter condition is much more difficult to satisfy (except in some special cases, such as where R is a Dedekind domain).

Proof of Proposition 21.2.3.9. Using R-linear duality, we can identify $\mathcal{G} = (\mathrm{LMod}_R^{\mathrm{proj}})^{\mathrm{op}}$ with the ∞-category $\mathrm{RMod}_R^{\mathrm{proj}}$ of finitely generated projective *right* modules over R.

Since the ∞-category $\mathrm{LMod}_R^{\mathrm{cn}}$ is projectively generated (Corollary HA.7.1.4.15), we have an equivalence of ∞-categories $\mathrm{LMod}_R^{\mathrm{cn}} \simeq \mathrm{Fun}^{\pi}(\mathcal{G}, \mathcal{S})$ where $\mathrm{Fun}^{\pi}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve finite products. Passing to sheaves on $\mathcal{X}$, we obtain an equivalence of ∞-categories

$$\theta : \mathrm{LMod}_{\underline{R}}^{\mathrm{cn}} \simeq \mathcal{S}\mathrm{hv}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{X}) \to \mathrm{Fun}^{\pi}(\mathcal{G}, \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})) \simeq \mathrm{Fun}^{\pi}(\mathcal{G}, \mathcal{X}).$$

Unwinding the definitions, we see that θ is given on objects by the formula $\theta(\mathscr{F})(M) = \Omega^{\infty}(\underline{M} \otimes_{\underline{R}} \mathscr{F})$.

Fix an object $\mathscr{F} \in \mathrm{LMod}_{\underline{R}}^{\mathrm{cn}}$ and set $T = \theta(\mathscr{F})$. To complete the proof, it will suffice to show that the following conditions are equivalent:

(a) The sheaf $\mathscr{F}$ is R-flat, in the sense of Definition 21.2.3.2.

(b) The functor

$$T : \mathrm{RMod}_R^{\mathrm{proj}} \to \mathcal{X} \qquad T(M) = \Omega^\infty(\underline{M} \otimes_{\underline{R}} \mathscr{F})$$

is locally left exact.

Note that the ∞-category $\mathrm{RMod}_R^{\mathrm{cn}}$ is also projectively generated (Corollary HA.7.1.4.15), we have a fully faithful embedding $g : \mathrm{Mod}_R^{\mathrm{cn}} \hookrightarrow \mathcal{P}(\mathcal{G})$. Let f denote a left adjoint to g. The functor T admits an essentially unique extension to a functor $T^+ : \mathrm{RMod}_R^{\mathrm{cn}} \to \mathcal{X}$ which preserves sifted colimits; concretely, this extension is given by the same formula $T^+(M) = T(M) = \Omega^\infty(\underline{M} \otimes_{\underline{R}} \mathscr{F})$. Note that the composite functor $(T^+ \circ f) : \mathcal{P}(\mathcal{G}) \to \mathcal{X}$ is a colimit-preserving extension of the functor T. Consequently, assertion (b) is equivalent to the assertion that $T^+ \circ f$ is left exact.

Choose an essentially small full subcategory $\mathcal{C} \subseteq \mathrm{RMod}_R^{\mathrm{cn}}$ which is closed under finite limits and contains $\mathrm{RMod}_R^{\mathrm{proj}}$. Then $g(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{G})$ contains the essential image of the Yoneda embedding and is therefore dense in $\mathcal{P}(\mathcal{G})$. It follows that $g|_{\mathcal{C}}$ induces a localization functor $L : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{G})$. Let $T_{\mathcal{C}}$ denote the composite functor

$$\mathcal{P}(\mathcal{C}) \xrightarrow{L} \mathcal{P}(\mathcal{G}) \xrightarrow{f} \mathrm{RMod}_R^{\mathrm{cn}} \xrightarrow{T^+} \mathcal{X}\,.$$

Since g is left exact, this localization functor is left exact (Proposition HTT.6.1.5.2). Consequently, the left exactness of $T^+ \circ f$ is equivalent to the left exactness of $T_{\mathcal{C}}$. Using Proposition HTT.6.1.5.2 again, we see that this is equivalent to the left exactness of its composition with the Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C})$, which is the functor $T^+|_{\mathcal{C}}$. In other words, condition (b) can be reformulated as follows:

$(b_{\mathcal{C}})$ The functor $T^+|_{\mathcal{C}}$ is left exact.

Note that, if condition (b) is satisfied, then condition $(b_{\mathcal{C}})$ is satisfied for *every* essentially small full subcategory $\mathcal{C} \subseteq \mathrm{RMod}_R^{\mathrm{cn}}$ which is closed under finite limits and contains $\mathrm{RMod}_R^{\mathrm{proj}}$. It follows that (b) is equivalent to the following *a priori* stronger condition:

(c) The functor

$$T^+ : \mathrm{RMod}_R^{\mathrm{cn}} \to \mathcal{X} \qquad T^+(M) = \Omega^\infty(\underline{M} \otimes_{\underline{R}} \mathscr{F})$$

is left exact.

We now observe that (c) is equivalent to the left t-exactness of the functor $\underline{M} \mapsto \underline{M} \otimes_{\underline{R}} \mathscr{F}$. Since this functor is automatically right t-exact (by virtue of our assumption that $\mathscr{F}$ is connective; see Remark 21.2.3.3), we deduce that $(c) \Leftrightarrow (a)$ as desired. $\square$

21.2.4 Local Morphisms

Let $\mathcal{C}$ be a compactly generated ∞-category, and let $\mathcal{G} = \mathcal{C}_c^{\mathrm{op}}$ denote the opposite of the ∞-category of compact objects of $\mathcal{C}$. For any topological space X, Proposition 21.2.2.1 supplies an equivalence $\mathrm{Obj}_{\mathcal{G}}(X) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}}(X)$. If the ∞-category $\mathcal{G}$ is equipped with a Grothendieck topology τ, then the full subcategory $\mathrm{Obj}_{\mathcal{G}}^{\tau}(X) \subseteq \mathrm{Obj}_{\mathcal{G}}(X)$ corresponds to a full subcategory of $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(X)$, whose objects might be referred to as "τ-local" $\mathcal{C}$-valued sheaves on X. In §21.2.5, we will show that when $\mathcal{C} = \mathrm{CAlg}^{\heartsuit}$ is the category of commutative rings and τ is the Zariski topology of Example 20.6.4.1, then this condition reduces to the usual notion of a sheaf of local rings on X (see Proposition 21.2.5.1). For algebro-geometric applications, we would like to restrict our attention further to the (non-full) subcategory where we allow only *local* maps between local sheaves of commutative rings. This requires an additional structure on $\mathcal{G}$.

Definition 21.2.4.1. Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ (see Definition **??**). Let $\mathcal{X}$ be an ∞-topos and suppose we are given a morphism $\alpha : \mathscr{O} \to \mathscr{O}'$ between $\mathcal{G}$-objects $\mathscr{O}, \mathscr{O}' \in \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$. We will say that α is *local* if, for every morphism $U \to V$ in $\mathcal{G}^{\mathrm{ad}}$, the diagram

$$\begin{array}{ccc} \mathscr{O}^U & \xrightarrow{\alpha(U)} & \mathscr{O}'^U \\ \downarrow & & \downarrow \\ \mathscr{O}^V & \xrightarrow{\alpha(V)} & \mathscr{O}'^V \end{array}$$

is a pullback square in $\mathcal{X}$. We let $\mathrm{Obj}_{\mathcal{G}}^{\mathrm{loc}}(\mathcal{X})$ denote the (non-full) subcategory of $\mathrm{Fun}(\mathcal{G}, \mathcal{X})$ whose objects are locally left exact functors and whose morphisms are local natural transformations. If τ is a Grothendieck topology on $\mathcal{G}$, we let $\mathrm{Obj}_{\mathcal{G}}^{\tau,\mathrm{loc}}(\mathcal{X})$ denote the intersection $\mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{X}) \cap \mathrm{Obj}_{\mathcal{G}}^{\mathrm{loc}}(\mathcal{X})$.

Remark 21.2.4.2. In practice, we will use the notation $\mathrm{Obj}_{\mathcal{G}}^{\tau,\mathrm{loc}}$ only in situations where the Grothendieck topology τ is compatible with the admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$, in the sense of Definition 20.6.2.1. In this case, an object $\mathscr{O} \in \mathrm{Obj}_{\mathcal{G}}^{\mathrm{loc}}(\mathcal{X})$ belongs to the full subcategory $\mathrm{Obj}_{\mathcal{G}}^{\tau,\mathrm{loc}}(\mathcal{X})$ if and only if, for every collection of *admissible* morphisms $\{U_\alpha \to U\}$ in $\mathcal{G}$ which generate a τ-covering of U, the induced map $\amalg \mathscr{O}^{U_\alpha} \to \mathscr{O}^U$ is an effective epimorphism in $\mathcal{X}$.

Variant 21.2.4.3. In the situation of Definition 21.2.4.1, suppose that $\mathcal{X} = \mathcal{S}\mathrm{hv}(X)$ for some topological space X. Then we will denote the ∞-categories $\mathrm{Obj}_{\mathcal{G}}^{\mathrm{loc}}(\mathcal{X})$ and $\mathrm{Obj}_{\mathcal{G}}^{\tau,\mathrm{loc}}(\mathcal{X})$ by $\mathrm{Obj}_{\mathcal{G}}^{\mathrm{loc}}(X)$ and $\mathrm{Obj}_{\mathcal{G}}^{\tau,\mathrm{loc}}(X)$, respectively.

Definition 21.2.4.1 can be reformulated (and slightly generalized) using the language of fractured ∞-topoi developed in Chapter 20.

Definition 21.2.4.4. Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos, let $\mathcal{X}$ be an arbitrary ∞-topos, and let $\alpha : f^* \to g^*$ be a morphism in the ∞-category $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$. We will say that α is *local* if the following condition is satisfied: for every morphism $f : U \to V$ in the ∞-topos $\mathcal{E}^{\mathrm{corp}}$, the diagram

$$\begin{array}{ccc} f^*U & \xrightarrow{\alpha(U)} & g^*U \\ \downarrow & & \downarrow \\ f^*V & \xrightarrow{\alpha(V)} & g^*V \end{array}$$

is a pullback square in the ∞-topos $\mathcal{X}$. We let $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X})$ denote the subcategory of $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ containing all objects, whose morphisms are given by local natural transformations.

Remark 21.2.4.5. Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos and let $\mathcal{X}$ be an arbitrary ∞-topos. Then the collection of local morphisms in $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ contains all equivalences and is closed under composition (which we have implicitly invoked by defining the subcategory $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X}) \subseteq \mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$).

Warning 21.2.4.6. In the situation of Definition 21.2.4.4, the ∞-category $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X})$ depends not only on $\mathcal{E}$ and $\mathcal{X}$, but on the choice of a fracture subcategory $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$.

Example 21.2.4.7 (The Trivial Case)**.** In the situation of Definition 21.2.4.4, suppose that $\mathcal{E}^{\mathrm{corp}} = \mathcal{E}$ (see Example 20.1.2.2). Then a natural transformation $\alpha : f^* \to g^*$ is local if and only if it is an equivalence. The "if" direction is obvious; to prove the converse, we note that $\mathcal{E}^{\mathrm{corp}}$ contains the projection map $U \to \mathbf{1}$ for every object $U \in \mathcal{E}$, so that the locality of α guarantees that the diagram

$$\begin{array}{ccc} f^*U & \xrightarrow{\alpha(U)} & g^*U \\ \downarrow & & \downarrow \\ f^*\mathbf{1} & \longrightarrow & g^*\mathbf{1} \end{array}$$

is a pullback square. Since the domain and codomain of the bottom horizontal map are final objects of $\mathcal{X}$ (by virtue of the left exactness of f^* and g^*), we conclude that the bottom horizontal map is an equivalence and therefore $\alpha(U) : f^*U \to g^*U$ is an equivalence as well.

Proposition 21.2.4.8. *Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos, let $\mathcal{X}$ be an arbitrary ∞-topos, and let $\alpha : f^* \to g^*$ be a morphism in the ∞-category $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$. The following conditions are equivalent:*

(a) *The morphism α is local, in the sense of Definition 21.2.4.4.*

(b) *For every $\mathcal{E}^{\mathrm{corp}}$-admissible morphism $f : U \to V$ in $\mathcal{E}$, the diagram*

$$\begin{array}{ccc} f^*U & \xrightarrow{\alpha(U)} & g^*U \\ \downarrow & & \downarrow \\ f^*V & \xrightarrow{\alpha(V)} & g^*V \end{array}$$

is a pullback square in the ∞-topos $\mathcal{X}$.

Proof. The implication $(b) \Rightarrow (a)$ is obvious. Converse, suppose that (a) is satisfied. Let S be the collection of all morphisms $U \to V$ in $\mathcal{E}$ for which the diagram

$$\begin{array}{ccc} f^*U & \xrightarrow{\alpha(U)} & g^*U \\ \downarrow & & \downarrow \\ f^*V & \xrightarrow{\alpha(V)} & g^*V \end{array}$$

is a pullback square. Then S is a local admissibility structure on $\mathcal{E}$ (Example 20.3.2.3). We wish to show that every $\mathcal{E}^{\mathrm{corp}}$-admissible morphism $f : U \to V$ is contained in S. This assertion is local on V, so we may assume without loss of generality that V is corporeal. In this case, the morphism f belongs to $\mathcal{E}^{\mathrm{corp}}$ (Example 20.3.1.2), so the desired result follows from assumption (a). □

Remark 21.2.4.9. Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos and let $\mathcal{X}$ be an arbitrary ∞-topos. The subcategory $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X}) \subseteq \mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ depends only on the collection of $\mathcal{E}^{\mathrm{corp}}$-admissible morphisms in $\mathcal{E}$. Consequently, replacing the fracture subcategory $\mathcal{E}^{\mathrm{corp}}$ by its completion (Remark 20.3.4.7) does not change the class of local morphisms in $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$.

Proposition 21.2.4.10. *Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos and let $h : \mathcal{G} \to \mathcal{E}$ be a presentation of $\mathcal{E}$ (see Definition 20.5.3.1). Then, for any ∞-topos $\mathcal{X}$, composition with h induces a fully faithful embedding*

$$\theta : \mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \hookrightarrow \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}).$$

Moreover, a morphism in $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ *is local (in the sense of Definition 21.2.4.4) if and only if its image under θ is local (in the sense of Definition 21.2.4.1). In particular, θ induces a fully faithful embedding*

$$\theta_{\mathrm{loc}} : \mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X}) \hookrightarrow \mathrm{Obj}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X}).$$

Proof. The functor h is dense and locally left exact (Remark 20.5.3.2), and therefore exhibits $\mathcal{E}$ as an accessible left exact localization of the presheaf ∞-category $\mathcal{P}(\mathcal{G})$. Consequently, to show that θ is fully faithful, we may assume without loss of generality that $\mathcal{E} = \mathcal{P}(\mathcal{G})$, in which case the functor θ is an equivalence of ∞-categories (by virtue of Theorem HTT.5.1.5.6 and the definition of local left exactness). To complete the proof, we must show a morphism $\alpha : f^* \to g^*$ in $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ is local if and only if $\theta(\alpha)$ is local in $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$. The "only if" direction is obvious (since the functor h carries admissible morphisms in $\mathcal{G}$ to admissible morphisms in $\mathcal{E}$). Conversely, suppose that $\theta(\alpha)$ is local, and let $u : U \to V$ be an admissible morphism in $\mathcal{E}$; we wish to show that the diagram σ :

$$\begin{array}{ccc} f^*U & \xrightarrow{\alpha(U)} & g^*U \\ \downarrow & & \downarrow \\ f^*V & \xrightarrow{\alpha(V)} & g^*V \end{array}$$

is a pullback square in $\mathcal{X}$. Using Example 20.3.2.3, we see that this assertion can be tested locally on V. We may therefore assume without loss of generality that $V = h(V_0)$ for some object $V_0 \in \mathcal{G}$. Let us regard V and V_0 as fixed. The functor h induces a left exact localization $\mathcal{P}(\mathcal{G}^{\mathrm{ad}}) \to \mathcal{E}^{\mathrm{corp}}$, which therefore restricts to a left exact localization $\mathcal{P}(\mathcal{G}^{\mathrm{ad}}_{/V_0}) \to (\mathcal{E}^{\mathrm{corp}})_{/V}$. It follows that we can write U as a colimit (in the ∞-category $(\mathcal{X}^{\mathrm{corp}})_{/V}$) of objects of the form $h(U_0)$, for $U_0 \in \mathcal{G}^{\mathrm{ad}}_{/V_0}$. Using the fact that colimits are universal in the ∞-topos $\mathcal{X}$, we can reduce to the case where the map $U \to V$ is obtained by applying the functor h to some admissible morphism $U_0 \to V_0$ in $\mathcal{G}$, in which case the desired result follows from our assumption that $\theta(\alpha)$ is local. □

Example 21.2.4.11. Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site and let $h : \mathcal{G} \to \mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$ be the sheafified Yoneda embedding (Definition 21.2.1.11). Then h is a presentation of the fractured ∞-topos $\mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$, in the sense of Definition 20.5.3.1: that is the essential content of Proposition 20.6.3.2.

Corollary 21.2.4.12. *Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site (see Definition 20.6.2.1) and let $\mathcal{X}$ be an ∞-topos. Then composition with the sheafified Yoneda embedding $\mathcal{G} \to \mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$ induces equivalences of ∞-categories*

$$\theta : \mathrm{Fun}^*(\mathcal{S}\mathrm{hv}_\tau(\mathcal{G}), \mathcal{X}) \hookrightarrow \mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{X}) \qquad \theta_{\mathrm{loc}} : \mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{S}\mathrm{hv}_\tau(\mathcal{G}), \mathcal{X}) \hookrightarrow \mathrm{Obj}^{\tau,\mathrm{loc}}_{\mathcal{G}}(\mathcal{X}).$$

Proof. Combine Proposition 21.2.1.13, Proposition 21.2.4.10, and Example 21.2.4.11. □

21.2.5 Examples: Sheaves of Rings

Let Aff denote the category whose objects are affine schemes of finite type over $\mathbf{Z}$ (that is, schemes of the form $\mathrm{Spec}\, R$, where R is a finitely presented commutative ring). We can then regard the category Aff as equipped with either the Zariski topology τ_{Zar} of Example 20.6.4.1, or the étale topology $\tau_{\text{ét}}$ of Example **??**. For any ∞-topos $\mathcal{X}$, we denote the corresponding ∞-categories of Aff-objects by $\mathrm{Obj}^{\mathrm{Zar}}_{\mathrm{Aff}}(\mathcal{X})$ and $\mathrm{Obj}^{\text{ét}}_{\mathrm{Aff}}(\mathcal{X})$, respectively, so that we have inclusion functors

$$\mathrm{Obj}^{\text{ét}}_{\mathrm{Aff}}(\mathcal{X}) \subseteq \mathrm{Obj}^{\mathrm{Zar}}_{\mathrm{Aff}}(\mathcal{X}) \subseteq \mathrm{Obj}_{\mathrm{Aff}}(\mathcal{X}).$$

These ∞-categories can be described more concretely as follows:

Proposition 21.2.5.1. *Let $\mathcal{X}$ be an ∞-topos. Then:*

(1) *There is a canonical equivalence of ∞-categories $\psi : \mathrm{Obj}_{\mathrm{Aff}}(\mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^\heartsuit}(\mathcal{X})$, characterized informally by the formula $\mathrm{Map}_{\mathcal{X}}(U, \mathscr{O}^{\operatorname{Spec} R}) = \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit}(R, \psi(\mathscr{O})(U))$.*

(2) *An object $\mathscr{O} \in \mathrm{Obj}_{\mathrm{Aff}}(\mathcal{X})$ belongs to the full subcategory $\mathrm{Obj}^{\mathrm{Zar}}_{\mathrm{Aff}}(\mathcal{X}) \subseteq \mathrm{Obj}_{\mathrm{Aff}}(\mathcal{X})$ if and only if $\psi(\mathscr{O})$ is local, when regarded as a commutative ring object of the topos $\mathcal{X}^\heartsuit$ (see Definition 1.2.1.4).*

(3) *A morphism $\alpha : \mathscr{O} \to \mathscr{O}'$ in the ∞-category $\mathrm{Obj}^{\mathrm{Zar}}_{\mathrm{Aff}}(\mathcal{X})$ is local (with respect to the admissibility structure of Example 20.6.4.1) if $\psi(\alpha) : \psi(\mathscr{O}) \to \psi(\mathscr{O}')$ is local when viewed as a morphism of commutative ring objects of $\mathcal{X}^\heartsuit$ (in the sense of Definition 1.2.1.4).*

(4) *An object $\mathscr{O} \in \mathrm{Obj}_{\mathrm{Aff}}(\mathcal{X})$ belongs to the full subcategory $\mathrm{Obj}^{\text{ét}}_{\mathrm{Aff}}(\mathcal{X}) \subseteq \mathrm{Obj}_{\mathrm{Aff}}(\mathcal{X})$ if and only if $\psi(\mathscr{O})$ is strictly Henselian, when regarded as a commutative ring object of the topos $\mathcal{X}^\heartsuit$ (see Definition 1.2.2.5).*

(5) *A morphism $\alpha : \mathscr{O} \to \mathscr{O}'$ in the ∞-category $\mathrm{Obj}^{\text{ét}}_{\mathrm{Aff}}(\mathcal{X})$ is local (with respect to the admissibility structure of Example **??**) if $\psi(\alpha) : \psi(\mathscr{O}) \to \psi(\mathscr{O}')$ is local when viewed as a morphism of commutative ring objects of $\mathcal{X}^\heartsuit$.*

Proof. Assertion (1) is a special case of Proposition 21.2.2.1 (and Remark 21.2.2.2), assertion (4) is immediate from the definitions, and assertion (5) follows from Proposition 1.2.2.12. We now prove (2). Let $\mathscr{O}$ be an object of $\mathrm{Obj}_{\mathrm{Aff}}(\mathcal{X})$ and set $\mathscr{A} = \psi(\mathscr{O})$, which we regard as a commutative ring object of the topos $\mathcal{X}^\heartsuit$. Unwinding the definitions, we see that $\mathscr{O}$ belongs to $\mathrm{Obj}^{\mathrm{Zar}}_{\mathrm{Aff}}(\mathcal{X})$ if and only if the following condition is satisfied:

($*$) For every finitely presented commutative ring R and every collection of elements $\{t_i\}_{1 \leqslant i \leqslant n}$ which generate the unit ideal in R, the induced map

$$\amalg_{1 \leqslant i \leqslant n} \mathscr{O}^{\operatorname{Spec} R[t_i^{-1}]} \to \mathscr{O}^{\operatorname{Spec} R}$$

is an effective epimorphism in $\mathcal{X}$.

On the other hand, $\mathscr{A}$ is local (when regarded as a commutative ring object of $\mathcal{X}^\heartsuit$) if and only if it satisfies the following pair of conditions:

(a) The sheaf $\mathscr{O}^{\varnothing}$ is an initial object of $\mathcal{X}$.

(b) The canonical map

$$\mathscr{O}^{\operatorname{Spec}\mathbf{Z}[x,x^{-1}]} \amalg \mathscr{O}^{\operatorname{Spec}\mathbf{Z}[x,(1-x)^{-1}]} \to \mathscr{O}^{\operatorname{Spec}\mathbf{Z}[x]}$$

is an effective epimorphism in $\mathcal{X}$.

We first note that conditions (a) and (b) follow immediately from ($*$) (condition (a) follows by applying ($*$) in the case where $R = 0$ and $n = 0$; condition (b) follows by applying ($*$) in the case $R = \mathbf{Z}[x]$ and $n = 2$). Conversely, suppose that conditions (a) and (b) are satisfied. We will establish ($*$) using induction on n. If $n = 0$, then the unit ideal of R coincides with the zero ideal, so that $R = 0$ and ($*$) follows from (a). To handle the inductive step, suppose that $n > 0$ and we are given a collection of elements $\{t_1, \ldots, t_n\}$ which generate the unit ideal in R. We can then choose elements $c_1, \ldots, c_n \in R$ satisfying $c_1t_1 + c_2t_2 + \cdots + c_nt_n = 1$. Set $x = c_1t_1 + \cdots + c_{n-1}t_{n-1}$, so that $1 - x = c_nt_n$. Since $\rho\,\mathscr{O}$ is left exact, we have a pullback diagram

$$\begin{array}{ccc} \mathscr{O}^{\operatorname{Spec} R[x^{-1}]} \amalg \mathscr{O}^{\operatorname{Spec} R[(1-x)^{-1}]} & \longrightarrow & \mathscr{O}^{\operatorname{Spec} R} \\ \downarrow & & \downarrow \\ \mathscr{O}^{\operatorname{Spec}\mathbf{Z}[x,x^{-1}]} \amalg \mathscr{O}^{\operatorname{Spec}\mathbf{Z}[x,(1-x)^{-1}])} & \longrightarrow & \mathscr{O}^{\operatorname{Spec}\mathbf{Z}[x]} . \end{array}$$

It follows from assumption (b) that the lower horizontal map is an effective epimorphism, so that the upper horizontal map is an effective epimorphism as well. Since the map $\operatorname{Spec} R[(1-x)^{-1}] \to \operatorname{Spec} R$ factors through $\operatorname{Spec} R[t_n^{-1}]$, it will suffice to show that the map

$$\amalg_{1 \leqslant i \leqslant n-1} \mathscr{O}^{\operatorname{Spec} R[x^{-1}]} \times_{\mathscr{O}^{\operatorname{Spec}\mathbf{Z}[x]}} \mathscr{O}^{\operatorname{Spec}\mathbf{Z}[x,x^{-1}]} \to \mathscr{O}^{\operatorname{Spec} R} \times_{\mathscr{O}^{\operatorname{Spec}\mathbf{Z}[x]}} \mathscr{O}^{\operatorname{Spec}\mathbf{Z}[x,x^{-1}]}$$

is an effective epimorphism. This follows from our inductive hypothesis (and the left exactness of the functor $\mathscr{O} : \operatorname{Aff} \to \mathcal{X}$), since the images of the elements $\{t_i\}_{1 \leqslant i \leqslant n-1}$ generate the unit ideal in the commutative ring $R[x^{-1}]$.

We now prove (3). Let $\alpha : \mathscr{O} \to \mathscr{O}'$ be a morphism in $\operatorname{Obj}^{\operatorname{Zar}}_{\operatorname{Aff}}(\mathcal{X})$. Unwinding the definitions, we see that α is local (in the sense of Definition 21.2.4.1) if and only if, for every finitely generated commutative ring R and every element $x \in R$, the upper square in the commutative diagram

$$\begin{array}{ccc} \mathscr{O}^{\operatorname{Spec} R[x^{-1}]} & \longrightarrow & \mathscr{O}^{\operatorname{Spec} R} \\ \downarrow & & \downarrow \\ \mathscr{O}'^{\operatorname{Spec} R[x^{-1}]} & \longrightarrow & \mathscr{O}'^{\operatorname{Spec} R} \\ \downarrow & & \downarrow \\ \mathscr{O}'^{\operatorname{Spec}\mathbf{Z}[x,x^{-1}]} & \longrightarrow & \mathscr{O}'^{\operatorname{Spec}\mathbf{Z}[x]} \end{array}$$

is a pullback. Since the bottom square is a pullback (by virtue of the left exactness of $\mathscr{O}'$), the locality of α is equivalent to the requirement that the outer rectangle of the diagram

$$\begin{array}{ccc} \mathscr{O}^{\operatorname{Spec} R[x^{-1}]} & \longrightarrow & \mathscr{O}^{\operatorname{Spec} R} \\ \downarrow & & \\ \mathscr{O}^{\operatorname{Spec} \mathbf{Z}[x,x^{-1}]} & \longrightarrow & \mathscr{O}^{\operatorname{Spec} \mathbf{Z}[x]} \\ \downarrow & & \downarrow \\ \mathscr{O}'^{\operatorname{Spec} \mathbf{Z}[x,x^{-1}]} & \longrightarrow & \mathscr{O}'^{\operatorname{Spec} \mathbf{Z}[x]} \end{array}$$

is a pullback square. Here the top square is automatically a pullback (by virtue of the left exactness of $\mathscr{O}$). Consequently, the morphism α is local if and only if the commutative diagram

$$\begin{array}{ccc} \mathscr{O}^{\operatorname{Spec} \mathbf{Z}[x,x^{-1}]} & \longrightarrow & \mathscr{O}^{\operatorname{Spec} \mathbf{Z}[x]} \\ \downarrow & & \downarrow \\ \mathscr{O}'^{\operatorname{Spec} \mathbf{Z}[x,x^{-1}]} & \longrightarrow & \mathscr{O}'^{\operatorname{Spec} \mathbf{Z}[x]} \end{array}$$

is a pullback square, which is equivalent to the assertion that $\psi(\alpha)$ is local (as a morphism of commutative ring objects of $\mathcal{X}^{\heartsuit}$). □

Warning 21.2.5.2. In part (3) of Proposition 21.2.5.1, the assumption that $\mathscr{O}$ and $\mathscr{O}'$ belong to $\operatorname{Obj}^{\operatorname{Zar}}_{\operatorname{Aff}}(\mathcal{X})$ is superfluous: the same assertion holds for *any* morphism $\alpha : \mathscr{O} \to \mathscr{O}'$ in $\operatorname{Obj}_{\operatorname{Aff}}(\mathcal{X})$. However, the analogous assertion for part (5) is false. For a general morphism $\alpha : \mathscr{O} \to \mathscr{O}'$ in $\operatorname{Obj}_{\operatorname{Aff}}(\mathcal{X})$ (or even in $\operatorname{Obj}^{\operatorname{Zar}}_{\operatorname{Aff}}(\mathcal{X})$), the condition of being local with respect to the admissibility structure of Example 20.6.4.1 (which has admissible morphisms of the form $\operatorname{Spec} R[t^{-1}] \hookrightarrow \operatorname{Spec} R$) is weaker than the condition of being local with respect to the admissibility structure of Example **??** (where every étale morphism in Aff is admissible).

We now formulate an analogue of Proposition 21.2.5.1, where we replace sheaves taking values in the ordinary category $\operatorname{CAlg}^{\heartsuit}$ of commutative rings with sheaves taking values in the ∞-category CAlg of *all* $\mathbb{E}_\infty$-rings. Let CAlg_c denote the full subcategory of CAlg spanned by the compact object, and let $\operatorname{Aff}_{\operatorname{Sp}}$ denote the opposite of the ∞-category CAlg_c; we denote the objects of $\operatorname{Aff}_{\operatorname{Sp}}$ by $\operatorname{Spec} R$, where R is a compact $\mathbb{E}_\infty$-ring. We can then regard the ∞-category $\operatorname{Aff}_{\operatorname{Sp}}$ as equipped with either the Zariski topology $\tau_{\operatorname{Zar}}$ of Example 20.6.4.4, or the étale topology $\tau_{\text{ét}}$ of Example **??**. For any ∞-topos $\mathcal{X}$, we denote the corresponding ∞-categories of $\operatorname{Aff}_{\operatorname{Sp}}$-objects by $\operatorname{Obj}^{\operatorname{Zar}}_{\operatorname{Aff}_{\operatorname{Sp}}}(\mathcal{X})$ and $\operatorname{Obj}^{\text{ét}}_{\operatorname{Aff}_{\operatorname{Sp}}}(\mathcal{X})$, respectively, so that we have inclusion functors

$$\operatorname{Obj}^{\text{ét}}_{\operatorname{Aff}_{\operatorname{Sp}}}(\mathcal{X}) \subseteq \operatorname{Obj}^{\operatorname{Zar}}_{\operatorname{Aff}_{\operatorname{Sp}}}(\mathcal{X}) \subseteq \operatorname{Obj}_{\operatorname{Aff}_{\operatorname{Sp}}}(\mathcal{X}).$$

We now have the following result:

Proposition 21.2.5.3. *Let $\mathcal{X}$ be an ∞-topos. Then:*

(1) *There is a canonical equivalence of ∞-categories $\psi : \mathrm{Obj}_{\mathrm{Aff}_{\mathrm{Sp}}}(\mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$, characterized informally by the formula $\mathrm{Map}_{\mathcal{X}}(U, \mathcal{O}^{\mathrm{Spec}\, R}) = \mathrm{Map}_{\mathrm{CAlg}^{\heartsuit}}(R, \psi(\mathcal{O})(U))$.*

(2) *An object $\mathcal{O} \in \mathrm{Obj}_{\mathrm{Aff}_{\mathrm{Sp}}}(\mathcal{X})$ belongs to the full subcategory $\mathrm{Obj}^{\mathrm{Zar}}_{\mathrm{Aff}_{\mathrm{Sp}}}(\mathcal{X}) \subseteq \mathrm{Obj}_{\mathrm{Aff}_{\mathrm{Sp}}}(\mathcal{X})$ if and only if the sheaf of $\mathbb{E}_\infty$-rings $\psi(\mathcal{O})$ is local (in the sense of Definition 1.4.2.1): that is, if and only if $\pi_0\psi(\mathcal{O})$ is local when regarded as a commutative ring object of the topos $\mathcal{X}^{\heartsuit}$.*

(3) *A morphism $\alpha : \mathcal{O} \to \mathcal{O}'$ in the ∞-category $\mathrm{Obj}^{\mathrm{Zar}}_{\mathrm{Aff}_{\mathrm{Sp}}}(\mathcal{X})$ is local (with respect to the admissibility structure of Example 20.6.4.4) if $\psi(\alpha) : \psi(\mathcal{O}) \to \psi(\mathcal{O}')$ is local when viewed as a morphism of* CAlg*-valued sheaves on $\mathcal{X}$ (in the sense of Definition 1.4.2.1).*

(4) *An object $\mathcal{O} \in \mathrm{Obj}_{\mathrm{Aff}_{\mathrm{Sp}}}(\mathcal{X})$ belongs to the full subcategory $\mathrm{Obj}^{\text{ét}}_{\mathrm{Aff}_{\mathrm{Sp}}}(\mathcal{X}) \subseteq \mathrm{Obj}_{\mathrm{Aff}_{\mathrm{Sp}}}(\mathcal{X})$ if and only if the sheaf of $\mathbb{E}_\infty$-rings $\psi(\mathcal{O})$ is strictly Henselian (in the sense of Definition 1.4.2.1): that is, if and only if $\pi_0\psi(\mathcal{O})$ is strictly Henselian when regarded as a commutative ring object of the topos $\mathcal{X}^{\heartsuit}$.*

(5) *A morphism $\alpha : \mathcal{O} \to \mathcal{O}'$ in the ∞-category $\mathrm{Obj}^{\text{ét}}_{\mathrm{Aff}_{\mathrm{Sp}}}(\mathcal{X})$ is local (with respect to the admissibility structure of Example* **??***) if $\psi(\alpha) : \psi(\mathcal{O}) \to \psi(\mathcal{O}')$ is local when viewed as a morphism of* CAlg*-valued sheaves on $\mathcal{X}$ (in the sense of Definition 1.4.2.1)*

Proof. Assertion (1) is a special case of Proposition 21.2.2.1 (and Remark 21.2.2.2), and assertion (4) and (5) follow from Lemma 1.4.3.9. Assertions (2) and (3) can be established by repeating the proof of Proposition 21.2.5.1, without essential change. □

Warning 21.2.5.4. Assertion (3) of Proposition 21.2.5.3 holds more generally for *any* morphism $\alpha : \mathcal{O} \to \mathcal{O}'$ in $\mathrm{Obj}_{\mathrm{Aff}_{\mathrm{Sp}}}(\mathcal{X})$, but assertion (5) requires that $\mathcal{O}$ and $\mathcal{O}'$ belong to $\mathrm{Obj}^{\text{ét}}_{\mathrm{Aff}_{\mathrm{Sp}}}(\mathcal{X})$ (as with Warning 21.2.5.2).

21.3 Factorization Systems and Fractured ∞-Topoi

As noted in the introduction to §20.2, every commutative ring homomorphism $\phi : A \to B$ factors canonically as a composition

$$A \xrightarrow{\phi'} A[S^{-1}] \xrightarrow{\phi''} B,$$

where the homomorphism ϕ'' is local (that is, it carries noninvertible elements of $A[S^{-1}]$ to noninvertible elements of B) and the map ϕ' is localizing; moreover, we can take S to be the set of elements $a \in A$ for which $\phi(a) \in B$ is invertible. More generally, if $\phi : \mathscr{A} \to \mathscr{B}$ is a map of *sheaves* of commutative rings on a topological space X, then f admits a factorization

$$\mathscr{A} \xrightarrow{\phi'} \mathscr{A}' \xrightarrow{\phi''} \mathscr{B},$$

which reduces to the previous factorization after passing to stalks at any point $x \in X$. Our goal in this section is to show that an analogous phenomenon occurs in the setting of $\mathcal{E}$-structured ∞-topoi, for any fractured ∞-topos $\mathcal{E}$. More precisely, we will prove the following:

Theorem 21.3.0.1. *Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos. Then:*

(1) *For every ∞-topos $\mathcal{X}$, there exists a factorization system $(S_L^{\mathcal{X}}, S_R^{\mathcal{X}})$ on the ∞-category $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$, where $S_R^{\mathcal{X}}$ is the collection of local morphisms in $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ (Definition 21.2.4.4).*

(2) *The factorization system of (1) depends functorially on $\mathcal{X}$. In other words, for every geometric morphism of ∞-topoi $f^* : \mathcal{X} \to \mathcal{Y}$, composition with f^* carries $S_L^{\mathcal{X}}$ into $S_L^{\mathcal{Y}}$ and $S_R^{\mathcal{X}}$ into $S_R^{\mathcal{Y}}$.*

Remark 21.3.0.2. The functorial dependence of the factorization system $(S_L^{\mathcal{X}}, S_R^{\mathcal{X}})$ on the fractured ∞-topos $\mathcal{E}$ is more subtle; we will discuss it in §**??**.

Combining Theorem 21.3.0.1 with Proposition **??**, we obtain the following:

Corollary 21.3.0.3. *Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site (Definition 20.6.2.1). Then:*

(1) *For every ∞-topos $\mathcal{X}$, there exists a factorization system $(S_L^{\mathcal{X}}, S_R^{\mathcal{X}})$ on the ∞-category $\mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{X})$, where where $S_R^{\mathcal{X}}$ is the collection of local morphisms in $\mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{X})$ (Definition 21.2.4.1).*

(2) *The factorization system of (1) depends functorially on $\mathcal{X}$. In other words, for every geometric morphism of ∞-topoi $f^* : \mathcal{X} \to \mathcal{Y}$, composition with f^* carries $S_L^{\mathcal{X}}$ into $S_L^{\mathcal{Y}}$ and $S_R^{\mathcal{X}}$ into $S_R^{\mathcal{Y}}$.*

Our proof of Theorem 21.3.0.1 is quite involved, and will occupy our attention throughout this section. We will proceed in several steps:

(a) The first step was already carried out in §20.2.2: there we showed that for any admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ determined a factorization system on the ∞-category of $\mathrm{Pro}(\mathcal{G})$ (Theorem 20.2.2.5). After passing to opposite ∞-categories, this recovers the factorization system described in Corollary 21.3.0.3 in the special case where $\mathcal{X} = \mathcal{S}$ is the ∞-topos of spaces and the topology τ on $\mathcal{G}$ is trivial.

(*b*) In §21.3.1, we show that Corollary 21.3.0.3 holds whenever the ∞-category $\mathcal{G}$ admits finite limits and the Grothendieck topology τ is trivial (Propositions 21.3.1.1 and 21.3.1.5). Our proof will proceed by reducing to the case where $\mathcal{X}$ is an ∞-topos of presheaves, in which case the desired result can be deduced from (*a*).

(*c*) In §**??**, we prove that Corollary 21.3.0.3 holds whenever the Grothendieck topology τ is trivial, without the assumption that $\mathcal{G}$ admits finite limits (Propositions 21.3.3.1 and 21.3.3.2). The proof proceeds by reduction to (*b*) using an embedding of $\mathcal{G}$ into an ∞-category which admits finite limits. In carrying out this reduction, we will use some general facts about ∞-topoi of paths, which we discuss in §21.3.2.

(*d*) In §**??**, we will show Theorem 21.3.0.1 (and therefore Corollary 21.3.0.3) holds in general. Our strategy is to use the existence of a presentation $h : \mathcal{G} \to \mathcal{E}$ (Theorem 20.5.3.4) to reduce to case where $\mathcal{E}$ has the form $\mathcal{P}(\mathcal{G})$, which is covered by case (*c*).

Remark 21.3.0.4. For the examples studied in this book, we do not need the full strength of Theorem 21.3.0.1: the fractured ∞-topoi of interest to us all have the form $\mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$, where $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ is a geometric site for which the ∞-category $\mathcal{G}$ admits finite limits. At this level of generality, step (*c*) is superfluous and step (*d*) is slightly easier to handle.

21.3.1 The Case $\mathcal{G}$ Admits Finite Limits

We begin by establishing the following special case of Corollary 21.3.0.3.

Proposition 21.3.1.1. *Let $\mathcal{G}$ be an essentially small which admits finite limits and is equipped with an admissibility structure. Let $\mathcal{X}$ be an arbitrary ∞-topos. Then there exists a factorization system (S_L, S_R) on the ∞-category $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X})$ which is characterized by the following requirement: a morphism $\alpha : \mathscr{O} \to \mathscr{O}'$ in $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ belongs to S_R if and only if, for every admissible morphism $f : U \to V$ in $\mathcal{G}$, the diagram*

$$\begin{array}{ccc} \mathscr{O}^U & \longrightarrow & \mathscr{O}'^U \\ \downarrow & & \downarrow \\ \mathscr{O}^V & \longrightarrow & \mathscr{O}'^V \end{array}$$

is a pullback square in the ∞-topos $\mathcal{X}$.

Remark 21.3.1.2. In the situation of Proposition 21.3.1.1, Proposition 21.2.2.1 supplies an equivalence of ∞-categories $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$, where $\mathcal{C} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$. Consequently, we can view Proposition 21.3.1.1 as supplying a factorization system on the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ of $\mathcal{C}$-valued sheaves on $\mathcal{X}$.

The proof of Proposition 21.3.1.1 will require a few general remarks about factorization systems.

Lemma 21.3.1.3. *Let $\mathcal{C}$ be an ∞-category, $\mathcal{C}^0 \subseteq \mathcal{C}$ a localization of $\mathcal{C}$, and Y an object of $\mathcal{C}^0$. Then $\mathcal{C}^0_{/Y}$ is a localization of $\mathcal{C}_{/Y}$. Moreover, a morphism $f : X \to X'$ in $\mathcal{C}_{/Y}$ exhibits X' as a $\mathcal{C}^0_{/Y}$-localization of X if and only if f exhibits X' as a $\mathcal{C}^0$-localization of X in the ∞-category $\mathcal{C}$.*

Proof. We first prove the "if" direction of the last assertion. Choose a morphism $Y' \to Y$ in $\mathcal{C}$, where $Y' \in \mathcal{C}^0$. We have a map of homotopy fiber sequences

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}_{/Y}}(X', Y') & \xrightarrow{\phi} & \mathrm{Map}_{\mathcal{C}_{/Y}}(X, Y') \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}}(X', Y') & \xrightarrow{\phi'} & \mathrm{Map}_{\mathcal{C}}(X, Y') \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}}(X', Y) & \xrightarrow{\phi''} & \mathrm{Map}_{\mathcal{C}}(X, Y). \end{array}$$

Since the maps ϕ' and ϕ'' are homotopy equivalences, we conclude that ϕ is a homotopy equivalence as desired.

We now show that $\mathcal{C}^0_{/Y}$ is a localization of $\mathcal{C}_{/Y}$. In view of Proposition HTT.5.2.7.8 and the above argument, it will suffice to show that for every map $h : X \to Y$, there exists a factorization

$$\begin{array}{ccccc} & & X' & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ X & & \xrightarrow{h} & & Y \end{array}$$

where f exhibits X' as a $\mathcal{C}^0$-localization of X. The existence of f follows from Proposition HTT.5.2.7.8 (applied to the ∞-category $\mathcal{C}$), and the ability to complete the diagram follows from the assumption that $Y \in \mathcal{C}^0$.

To complete the proof, we observe that *any* $\mathcal{C}^0_{/Y}$-localization $X \to X''$ must be equivalent to the morphism $X \to X'$ constructed above, so that $X \to X''$ also exhibits X'' as a $\mathcal{C}^0$-localization of X. □

Recall that if $f : A \to B$ and $g : X \to Y$ are morphisms in an ∞-category $\mathcal{C}$, we write $f \perp g$ if, for every commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ B & \longrightarrow & Y, \end{array}$$

the space $\mathrm{Map}_{\mathcal{C}_{A//Y}}(B, X)$ of dotted arrows rendering the diagram commutative is contractible (Definition HTT.5.2.8.1).

Lemma 21.3.1.4. *Let $\mathcal{C}$ be an ∞-category equipped with a factorization system (S_L, S_R), and let $L : \mathcal{C} \to \mathcal{C}$ be a localization functor such that $LS_R \subseteq S_R$. Then the full subcategory $L\mathcal{C} \subseteq \mathcal{C}$ admits a factorization system (S'_L, S'_R), where:*

(1) *A morphism f' in $L\mathcal{C}$ belongs to S'_L if and only if f' is a retract of Lf, for some $f \in S_L$.*

(2) *A morphism g in $L\mathcal{C}$ belongs to S'_R if and only if $g \in S_R$.*

Proof. Clearly S'_L and S'_R are stable under the formation of retracts. Let $h : X \to Z$ be a morphism in $L\mathcal{C}$; we wish to show that h factors as a composition $X \xrightarrow{f'} Y' \xrightarrow{g'} Z$ where $f' \in S'_L$ and $g' \in S'_R$. First, choose a factorization of h as a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then $Lf \in S'_L$, $Lg \in S'_R$, and $h \simeq Lh \simeq Lg \circ Lf$.

It remains to show that $f' \perp g'$, for $f' \in S'_L$ and $g' \in S'_R$. Without loss of generality, we may suppose $f' = Lf$ for some $f \in S_L$. Choose a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & LA & \longrightarrow & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Lf} & & \downarrow{\scriptstyle g'} \\ B & \longrightarrow & LB & \longrightarrow & Y. \end{array}$$

We wish to show that the mapping space $\mathrm{Map}_{\mathcal{C}_{LA//Y}}(LB, X)$ is weakly contractible. We have a commutative diagram of fiber sequences

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}_{LA//Y}}(LB, X) & \xrightarrow{\phi} & \mathrm{Map}_{\mathcal{C}_{A//Y}}(B, X) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}_{/Y}}(LB, X) & \xrightarrow{\phi'} & \mathrm{Map}_{\mathcal{C}_{/Y}}(B, X) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}_{/Y}}(LA, X) & \xrightarrow{\phi''} & \mathrm{Map}_{\mathcal{C}_{/Y}}(A, X). \end{array}$$

Using Lemma 21.3.1.3, we deduce that ϕ' and ϕ'' are homotopy equivalences. It follows that ϕ is also a homotopy equivalence. We are therefore reduced to proving that $\mathrm{Map}_{\mathcal{C}_{A//Y}}(B, X)$ is contractible, which follows from the orthogonality relation $f \perp g'$. □

Proof of Proposition 21.3.1.1. In the special case where $\mathcal{X} = \mathcal{S}$, the desired result follows immediately from Theorem 20.2.2.5. To treat the general case, we may assume without loss of generality that the ∞-topos $\mathcal{X}$ has the form $L\mathcal{P}(\mathcal{C})$, where $\mathcal{C}$ is a small ∞-category and L

is a left exact localization functor on the presheaf ∞-category $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. Using Corollary HTT.5.2.8.18 , we deduce that the ∞-category $\mathrm{Obj}_{\mathcal{G}}(\mathcal{P}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Obj}_{\mathcal{G}}(\mathcal{S}))$ admits a factorization system $(\overline{S}_L, \overline{S}_R)$, where a morphism α belongs to $\overline{S}_R$ if and only if, for every admissible morphism $U \to V$ in $\mathcal{G}$, the diagram

$$\begin{array}{ccc} \mathscr{O}^U & \longrightarrow & \mathscr{O}'^U \\ \downarrow & & \downarrow \\ \mathscr{O}^X & \longrightarrow & \mathscr{O}'^X \end{array}$$

is a pullback square in the ∞-category $\mathcal{P}(\mathcal{C})$.

Composition with L induces a localization functor from $\mathrm{Obj}_{\mathcal{G}}(\mathcal{P}(\mathcal{C}))$ to itself; since L is left exact, this functor carries $\overline{S}_R$ to itself. It follows from Lemma 21.3.1.4 that $(\overline{S}_L, \overline{S}_R)$ induces a factorization system (S_L, S_R) on the ∞-category $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ having the desired properties. □

We conclude this section with a discussion of the functorial behavior of the factorization systems described in Proposition 21.3.1.1.

Proposition 21.3.1.5. *Let $\mathcal{G}$ be an essentially small which admits finite limits and is equipped with an admissibility structure, let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi, and let $(S_L^{\mathcal{X}}, S_R^{\mathcal{Y}})$ and $(S_L^{\mathcal{X}}, S_R^{\mathcal{Y}})$ be the factorization systems on $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ and $\mathrm{Obj}_{\mathcal{G}}(\mathcal{Y})$ given by Proposition 21.3.1.1. For every geometric morphism $f^* \in \mathrm{Fun}^*(\mathcal{Y}, \mathcal{Z})$, composition with f^* carries $S_L^{\mathcal{X}}$ into $S_L^{\mathcal{Y}}$ and $S_R^{\mathcal{X}}$ into $S_R^{\mathcal{Y}}$.*

Our proof will use the following general fact, which is an immediate consequence of Remark HTT.5.2.8.7 and Proposition HTT.5.2.8.11 .

Lemma 21.3.1.6. *Suppose given a pair of adjoint functors*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}.$$

Let (S_L, S_R) be a factorization system on $\mathcal{C}$, and (S'_L, S'_R) a factorization system on $\mathcal{D}$. The following conditions are equivalent:

(1) *The functor F carries S_L to S'_L.*

(2) *The functor G carries S'_R to S_R.*

Proof of Proposition 21.3.1.5. Let f_* be a right adjoint to f^*. Then composition with the functors f^* and f_* determines an adjunction

$$\mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) \underset{f_*^{\mathcal{G}}}{\overset{f_{\mathcal{G}}^*}{\rightleftarrows}} \mathrm{Obj}_{\mathcal{G}}(\mathcal{Y});$$

see Remark 21.2.1.8. It follows immediately from the definitions that the functor $f^*_{\mathcal{G}}$ carries $S^{\mathcal{X}}_R$ into $S^{\mathcal{Y}}_R$, and that the functor $f^{\mathcal{G}}_*$ carries $S^{\mathcal{X}}_R$ into $S^{\mathcal{Y}}_R$. Applying Lemma 21.3.1.6, we deduce that $f^*_{\mathcal{G}}$ also carries $S^{\mathcal{X}}_L$ into $S^{\mathcal{Y}}_L$. □

21.3.2 Digression: ∞-Topoi of Paths

Let $\infty\mathcal{T}\mathrm{op}$ denote the ∞-category of ∞-topoi. Then $\infty\mathcal{T}\mathrm{op}$ is naturally *cotensored* over the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of small ∞-categories. More precisely, for every ∞-topos $\mathcal{X}$ and any (small) simplicial set K, there exists another ∞-topos $\mathcal{X}^K$ and a map $\theta : K \to \mathrm{Fun}^*(\mathcal{X}, \mathcal{X}^K)$ with the following universal property: for every ∞-topos $\mathcal{Y}$, composition with θ induces an equivalence of ∞-categories

$$\mathrm{Fun}^*(\mathcal{X}^K, \mathcal{Y}) \to \mathrm{Fun}(K, \mathrm{Fun}^*(\mathcal{X}, \mathcal{Y}))$$

(see Proposition HTT.6.3.4.9). Note that $\mathcal{X}^K$ is determined up to (canonical) equivalence by $\mathcal{X}$ and K.

Warning 21.3.2.1. The ∞-topos $\mathcal{X}^K$ defined is *not* equivalent to the functor ∞-category $\mathrm{Fun}(K, \mathcal{X})$: the latter is instead characterized by the dual universal property property

$$\mathrm{Fun}^*(\mathcal{Y}, \mathrm{Fun}(K, \mathcal{X})) \simeq \mathrm{Fun}(K, \mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})).$$

Remark 21.3.2.2 (Relationship with Exponentials)**.** Let K be a small simplicial set. For any ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$, we have canonical equivalences

$$\begin{aligned}\mathrm{Fun}^*(\mathcal{X}^K, \mathcal{Y}) &\simeq \mathrm{Fun}(K, \mathrm{Fun}^*(\mathcal{X}, \mathcal{Y}))\\ &\simeq \mathrm{Fun}^*(\mathcal{X}, \mathrm{Fun}(K, \mathcal{Y}))\\ &\simeq \mathrm{Fun}^*(\mathcal{X}, \mathrm{Fun}(K, \mathcal{S}) \otimes \mathcal{Y}).\end{aligned}$$

It follows that $\mathcal{X}^K$ can be identified with an exponential of $\mathcal{Y}$ by the presheaf ∞-category $\mathrm{Fun}(K, \mathcal{S}) = \mathcal{P}(K^{\mathrm{op}})$, in the sense of Definition 21.1.6.4. Consequently, the existence of $\mathcal{X}^K$ (for an arbitrary ∞-topos $\mathcal{X}$) is equivalent to the exponentiability of the ∞-topos $\mathrm{Fun}(K, \mathcal{S})$. We may therefore regard this existence as a special case of Theorem 21.1.6.12 (note that the ∞-topos $\mathrm{Fun}(K, \mathcal{S})$ is compactly generated, and therefore compact assembled).

Definition 21.3.2.3. Let $\mathcal{X}$ be an ∞-topos. We will refer to $\mathcal{X}^{\Delta^1}$ as the *path ∞-topos* of $\mathcal{X}$.

Our goal in this section is to describe an explicit construction of the path ∞-topos $\mathcal{X}^{\Delta^1}$ in the special case where $\mathcal{X}$ is an ∞-topos of presheaves (Theorem 21.3.2.5).

Construction 21.3.2.4. Let $\mathcal{G}$ be a small ∞-category and let K be a simplicial set. Composition with the evaluation map $\mathrm{Fun}(K, \mathcal{G}) \times K \to \mathcal{G}$ induces a map

$$\begin{aligned} \mathcal{P}(\mathcal{G}) &= \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S}) \\ &\to \mathrm{Fun}(\mathrm{Fun}(K, \mathcal{G})^{\mathrm{op}} \times K^{\mathrm{op}}, \mathcal{S}) \\ &\simeq \mathrm{Fun}(K^{\mathrm{op}}, \mathrm{Fun}(\mathrm{Fun}(K, \mathcal{G})^{\mathrm{op}}, \mathcal{S})) \\ &= \mathrm{Fun}(K^{\mathrm{op}}, \mathcal{P}(\mathrm{Fun}(K, \mathcal{G}))). \end{aligned}$$

This functor preserves all small limits and colimits and therefore determines a map

$$e : K^{\mathrm{op}} \to \mathrm{Fun}^*(\mathcal{P}(\mathcal{G}), \mathcal{P}(\mathrm{Fun}(K, \mathcal{G}))),$$

which we can identify with a geometric morphism of ∞-topoi $\pi_* : \mathcal{P}(\mathrm{Fun}(K, \mathcal{G})) \to \mathcal{P}(\mathcal{G})^{K^{\mathrm{op}}}$.

Our result can now be stated as follows:

Theorem 21.3.2.5. *Let $\mathcal{G}$ be a small ∞-category. Then the geometric morphism π_* : $\mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})) \to \mathcal{P}(\mathcal{G})^{\Delta^1}$ is an equivalence of ∞-topoi. In other words, for any ∞-topos $\mathcal{Y}$, the natural map*

$$\mathrm{Fun}^*(\mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})), \mathcal{Y}) \to \mathrm{Fun}(\Delta^1, \mathrm{Fun}^*(\mathcal{P}(\mathcal{G}), \mathcal{Y}))$$

is an equivalence of ∞-categories.

Remark 21.3.2.6. In the special case $\mathcal{Y} = \mathcal{S}$, Theorem 21.3.2.5 asserts that the canonical map $\mathrm{Pro}(\mathrm{Fun}(\Delta^1, \mathcal{G})) \to \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{G}))$ is an equivalence of ∞-categories, which is a special case of Proposition HTT.5.3.5.15 .

Warning 21.3.2.7. In general, the geometric morphism $\pi_* : \mathcal{P}(\mathrm{Fun}(K, \mathcal{G})) \to \mathcal{P}(\mathcal{G})^{K^{\mathrm{op}}}$ of Construction 21.3.2.4 is not an equivalence. For example, taking $\mathcal{S}$-valued points, π_* induces the natural map $\mathrm{Pro}(\mathrm{Fun}(K, \mathcal{G})) \to \mathrm{Fun}(K, \mathrm{Pro}(\mathcal{G}))$. This map need not be an equivalence, even when K is finite (see Warning HTT.5.3.5.16). However, our proof of Theorem 21.3.2.5 can be generalized to show that π_* is an equivalence whenever K is the nerve of a finite partially ordered set (this is a generalization of Proposition HTT.5.3.5.15).

Proof of Theorem 21.3.2.5. We define functors $e_0, e_1 : \mathcal{G} \to \mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G}))$ by the formulae

$$e_0(X)(f : Y_0 \to Y_1) = \mathrm{Map}_{\mathcal{G}}(Y_0, X) \qquad e_1(X)(f : Y_0 \to Y_1) = \mathrm{Map}_{\mathcal{G}}(Y_1, X).$$

There is an evident natural transformation of functors $e_1 \to e_0$, which we can identify with a map $e : \Delta^1 \times \mathcal{G} \to \mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G}))$. Choose a Cartesian fibration $\mathcal{M} \to \Delta^1$ with

$$\mathcal{M}_0 = \{0\} \times_{\Delta^1} \mathcal{M} = \mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G}))$$

$$\mathcal{M}_1 = \{1\} \times_{\Delta^1} \mathcal{M} = \Delta^1 \times \mathcal{G},$$

which is associated to the functor $e : \Delta^1 \times \mathcal{G} \to \mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G}))$. Let $\mathcal{M}^\circ$ denote the full subcategory of $\mathcal{M}$ spanned by the objects of $\mathcal{M}_1$ together with the essential image of the Yoneda embedding $\mathrm{Fun}(\Delta^1, \mathcal{G}) \to \mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})) \simeq \mathcal{M}_0$. We will prove that, for a functor $F : \mathcal{M} \to \mathcal{Y}$, the following conditions are equivalent:

(a) The restriction $F|_{\mathcal{M}_0}$ belongs to $\mathrm{Fun}^*(\mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})), \mathcal{Y})$ and F is a left Kan extension of $F|_{\mathcal{M}_0}$.

(b) The functor F is a left Kan extension of $F|_{\mathcal{M}^\circ}$, the functor $F|_{\mathcal{M}^\circ}$ is a right Kan extension of $F|_{\mathcal{M}_1}$, and the restriction of F to $\{i\} \times \mathcal{G} \subseteq \mathcal{M}_1$ is locally left exact for $i \in \{0, 1\}$.

Assume this for the moment. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(\mathcal{M}, \mathcal{Y})$ spanned by those functors which satisfy the equivalent conditions (a) and (b). Using Proposition HTT.4.3.2.15 and characterization (a), we see that restriction to the full subcategory $\mathcal{M}_0 \subseteq \mathcal{M}$ determines a trivial Kan fibration $\phi : \mathcal{C} \to \mathrm{Fun}^*(\mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})), \mathcal{Y})$. Similarly, using characterization (b) and Proposition HTT.4.3.2.15 , we see that restriction to the full subcategory $\mathcal{M}_1 \subseteq \mathcal{M}$ determines a trivial Kan fibration $\psi : \mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathrm{Fun}^{\mathrm{llex}}(\mathcal{G}, \mathcal{Y}))$.

Composition with j determines an equivalence of ∞-categories $\rho : \mathrm{Fun}^*(\mathcal{P}(\mathcal{G}), \mathcal{Y}) \to \mathrm{Fun}^{\mathrm{llex}}(\mathcal{G}, \mathcal{Y})$. It will therefore suffice to show that the composite map

$$\mathrm{Fun}^*(\mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})), \mathcal{Y}) \xrightarrow{\alpha} \mathrm{Fun}(\Delta^1, \mathrm{Fun}^*(\mathcal{P}(\mathcal{G}), \mathcal{Y})) \xrightarrow{\rho} \mathrm{Fun}(\Delta^1, \mathrm{Fun}^{\mathrm{llex}}(\mathcal{G}, \mathcal{Y}))$$

is an equivalence of ∞-categories. We now complete the proof by observing that this composition is given by $\psi \circ s$, where $s : \mathrm{Fun}^*(\mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})), \mathcal{Y}) \to \mathcal{C}$ is a section of the trivial Kan fibration ϕ.

It remains to prove that (a) and (b) are equivalent. Suppose first that F satisfies (a); we will prove that F also satisfies (b). For this, we must establish three things:

- To prove that F is a left Kan extension of $F|_{\mathcal{M}^\circ}$, it will suffice to show that $F_0 = F|_{\mathcal{M}_0}$ is a left Kan extension of its restriction to the essential image of the Yoneda embedding $j' : \mathrm{Fun}(\Delta^1, \mathcal{G}) \to \mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G}))$. This follows from our assumption that F_0 preserves small colimits (Lemma HTT.5.1.5.5).

- Fix an object of $\mathrm{Fun}(\Delta^1, \mathcal{G})$, which we can identify with a morphism $u : C \to D$ in the ∞-category $\mathcal{G}$. We will prove that the functor F is a right Kan extension of $F_1 = F|_{\mathcal{M}_1}$ at $j'(u) \in \mathcal{M}_0$. To prove this, let $\mathcal{D} = \mathcal{M}_1 \times_{\mathcal{M}} \mathcal{M}_{j'(u)/}$; we wish to show that the canonical map $F(j'(f)) \to \varprojlim F|_{\mathcal{D}}$ is an equivalence in $\mathcal{Y}$. Note that there is a right cofinal map $\Lambda^2_2 \to \mathcal{D}$, corresponding to the diagram

$$(1, C) \to (1, D) \leftarrow (0, D)$$

in the ∞-category $\Delta^1 \times \mathcal{G}$. It will therefore suffice to show that the diagram

$$\begin{array}{ccc} F_0(j'(u)) & \longrightarrow & F_1(1,C) \\ \downarrow & & \downarrow \\ F_1(0,D) & \longrightarrow & F_1(1,D) \end{array}$$

is a pullback square in $\mathcal{Y}$. Since F is a left Kan extension of F_0, we can write $F_1 = F_0 \circ e$. We are therefore reduced to proving that the diagram

$$\begin{array}{ccc} F_0(j'(u)) & \longrightarrow & F_0(e_0(C)) \\ \downarrow & & \downarrow \\ F_0(e_1(D)) & \longrightarrow & F_0(e_0(D)) \end{array}$$

is a pullback square in $\mathcal{Y}$. Since the functor F_0 is left exact, this follows from the observation that the diagram

$$\begin{array}{ccc} j'(u) & \longrightarrow & e_0(C) \\ \downarrow & & \downarrow \\ e_1(D) & \longrightarrow & e_0(D) \end{array}$$

is a pullback square in $\mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G}))$. (More concretely, this means that for every object $u' : C' \to D'$ in $\mathrm{Fun}(\Delta^1, \mathcal{G})$, the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Fun}(\Delta^1,\mathcal{G})}(u', u) & \longrightarrow & \mathrm{Fun}_{\mathcal{G}}(C', C) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathcal{G}}(D', D) & \longrightarrow & \mathrm{Fun}_{\mathcal{G}}(C', D) \end{array}$$

is a pullback square of spaces).

- We must show that for $i \in \{0, 1\}$, the restriction of F_1 to $\{i\} \times \mathcal{G}$ is locally left exact. This follows from the observation that this restriction factors as a composition

$$\mathcal{G} \xrightarrow{j} \mathcal{P}(\mathcal{G}) \xrightarrow{E^*_{1-i}} \mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})) \xrightarrow{F} \mathcal{Y},$$

where j is locally left exact, E^*_i denotes the map of presheaf ∞ categories induced by composition with the functor $\mathrm{Fun}(\Delta^1, \mathcal{G}) \to \mathcal{G}$ given by evaluation on i (which preserves small limits and colimits), and $F|_{\mathcal{M}_0}$ is a left exact functor which preserves small colimits.

Now suppose that F satisfies (b). Let f_0 and f_1 denote the restrictions of F_1 to $\{0\} \times \mathcal{G}$ and $\{1\} \times \mathcal{G}$, respectively, so that $f_0, f_1 : \mathcal{G} \to \mathcal{Y}$ are locally left exact functors. We wish to show that F satisfies (a). Our proof breaks into three parts:

- We show that the functor F is a left Kan extension of F_0 at each object of $\Delta^1 \times \mathcal{G}$ having the form $(0, C)$. Equivalently, we must show that the canonical map $F_0(e_1(C)) \to f_0(C)$ is an equivalence. This is clear, since $e_1(C) \simeq j'(\mathrm{id}_C)$ belongs to the essential image of j', and the functor $F|_{\mathcal{M}^\circ}$ is a right Kan extension of $F|_{\mathcal{M}_1}$.

- We show that the functor F is a left Kan extension of F_0 at each object of $\Delta^1 \times \mathcal{G}$ having the form $(1, X)$. Equivalently, we must show that the canonical map $\beta : F_0(e_0(C)) \to f_1(C)$ is an equivalence in $\mathcal{Y}$. Since F is a left Kan extension of $F|_{\mathcal{M}^\circ}$, we can identify the domain of θ with the colimit of the composite map

$$\theta : \mathrm{Fun}(\Delta^1, \mathcal{G}) \times_{\mathrm{Fun}(\{0\}, \mathcal{G})} \mathcal{G}_{/C} \to \mathrm{Fun}(\Delta^1, \mathcal{G}) \xrightarrow{j'} \mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})) \xrightarrow{F_0} \mathcal{Y}.$$

  Let us identify objects of the domain of θ with pairs (u, v), where $u : D \to C$ and $v : D \to E$ are morphisms in $\mathcal{G}$. Since $F|_{\mathcal{M}^\circ}$ is a right Kan extension of $F|_{\mathcal{M}_1}$, we see that the functor θ is given informally by the formula $\theta(u, v) = f_0(E) \times_{f_1(E)} f_1(D)$. Evaluation at the final vertex of Δ^1 determines a map $p : \mathrm{Fun}(\Delta^1, \mathcal{G}) \times_{\mathrm{Fun}(\{0\}, \mathcal{G})} \mathcal{G}_{/C} \to \mathcal{G}$. Note that p is a coCartesian fibration, whose fiber over an object $E \in \mathcal{G}$ is equivalent to the ∞-category $\mathcal{G}_{/C} \times_{\mathcal{G}} \mathcal{G}_{/E}$. Let $\theta' : \mathcal{G} \to \mathcal{Y}$ denote the functor given by $\theta'(E) = f_0(E) \times f_1(C)$, so that we have an evident natural transformation $\theta \to \theta' \circ p$. We claim that this natural transformation exhibits θ' as a left Kan extension of θ along p. To prove this, it suffices (since p is a coCartesian fibration) to show that the canonical map

$$\varinjlim_{D \in \mathcal{G}_{/C} \times_{\mathcal{G}} \mathcal{G}_{/E}} f_0(E) \times_{f_1(E)} f_(D) \to f_0(E) \times f_1(C)$$

  is an equivalence, for each $E \in \mathcal{E}$. Since colimits in $\mathcal{Y}$ are universal, it will suffice to prove that the map

$$\varinjlim_{D \in \mathcal{G}_{/C} \times_{\mathcal{G}} \mathcal{G}_{/E}} f_1(D) \to f_1(E) \times f_1(C)$$

  is an equivalence, which follows from our assumption that f_1 is locally left exact. It follows that we can identify β with the canonical map $\varinjlim_{E \in \mathcal{G}} \theta'(E) \to f_1(C)$. To prove that this map is an equivalence, it will suffice (again using the fact that colimits in $\mathcal{Y}$ are universal) to show that the colimit $\varinjlim_{E \in \mathcal{G}} f_0(E)$ is a final object of $\mathcal{Y}$. This follows from our assumption that f_0 is locally left exact.

- We show that the functor F_0 belongs to $\mathrm{Fun}^*(\mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G}), \mathcal{Y})$. First, our assumption that F is a left Kan extension of $F|_{\mathcal{M}^\circ}$ implies that F_0 is a left Kan extension of its restriction to the essential image of j', so that F_0 preserves small colimits (Lemma HTT.5.1.5.5). It will therefore suffice to show that the functor $g = F_0 \circ j'$ is a locally left exact functor from $\mathrm{Fun}(\Delta^1, \mathcal{G})$ into $\mathcal{Y}$. Note that g is given by the formula

$$g(u : C \to D) = f_0(D) \times_{f_1(D)} f_1(C).$$

We will show that the functor g satisfies the criterion of Proposition 20.4.3.1. Let K be a finite simplicial set and suppose that $q : K \to \mathrm{Fun}(\Delta^1, \mathcal{G})$ is a diagram; we wish to prove that the canonical map

$$\varinjlim g|_{\mathrm{Fun}(\Delta^1,\mathcal{G})_{/q}} \to \varprojlim g \circ q$$

is an equivalence in $\mathcal{Y}$. For $i \in \{0, 1\}$, let $q_i : K \to \mathcal{G}$ be the composition of q with the evaluation functor $\mathrm{Fun}(\Delta^1, \mathcal{G}) \to \mathrm{Fun}(\{i\}, \mathcal{G}) \simeq \mathcal{G}$. Then evaluation at $\{1\}$ determines a map $\rho : \mathrm{Fun}(\Delta^1, \mathcal{G})_{/q} \to \mathcal{G}_{/q_1}$. Let $h : \mathcal{G}_{/q_1} \to \mathcal{Y}$ be the functor given by

$$h(D) = f_0(D) \times_{\varprojlim(f_1 \circ q_1)} \varprojlim(f_1 \circ q_0),$$

so that we have an evident map $g \to h \circ \rho$. We claim that this map exhibits h as a left Kan extension of g along ρ. Since ρ is a coCartesian fibration, this is equivalent to the assertion that for each object $D \in \mathcal{G}_{/q_1}$, the canonical map

$$\varinjlim g|_{\mathrm{Fun}(\Delta^1,\mathcal{G})_{/q}} \times_{\mathrm{Fun}(\Delta^1,\mathcal{G})_{/q_1}} \mathrm{Fun}(\Delta^1, \mathcal{G})_{/C} \to h(D)$$

is an equivalence. Since colimits are universal in $\mathcal{Y}$, this follows from our assumption that the functor f_1 is locally left exact (Proposition 20.4.3.1). We are therefore reduced to proving that the canonical map

$$\varinjlim h \to \varprojlim(g \circ q) \simeq \varprojlim(f_0 \circ q_1) \times_{\varprojlim(f_1 \circ q_1)} \varprojlim(f_1 \circ q_0)$$

is an equivalence. Since colimits in $\mathcal{Y}$ are universal, we are reduced to proving that the canonical map $\varinjlim_{D \in \mathcal{G}_{/q_1}} f_0(D) \to \varprojlim(f_0 \circ q_1)$ is an equivalence in $\mathcal{Y}$, which follows from our assumption that f_0 is locally left exact (Proposition 20.4.3.1).

□

21.3.3 The Case of a General $\mathcal{G}$

Our next goal is to prove analogues of Propositions 21.3.1.1 and 21.3.1.5 in the case where the ∞-category $\mathcal{G}$ need not admit finite limits.

Proposition 21.3.3.1. *Let $\mathcal{G}$ be an essentially small ∞-category which is equipped with an admissibility structure $\mathcal{G}^{\mathrm{ad}} \subseteq \mathcal{G}$ and let $\mathcal{X}$ be an arbitrary ∞-topos. Then there exists a* *factorization system* (S_L, S_R) *on the* ∞*-category* $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) = \mathrm{Fun}^{\mathrm{llex}}(\mathcal{G}, \mathcal{X})$ *which is characterized by the following requirement: a morphism* $\alpha : \mathscr{O} \to \mathscr{O}'$ *in* $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ *belongs to* S_R *if and only if, for every admissible morphism* $f : U \to V$ *in* $\mathcal{G}$*, the diagram*

$$\begin{array}{ccc} \mathscr{O}^U & \longrightarrow & \mathscr{O}'^U \\ \downarrow & & \downarrow \\ \mathscr{O}^V & \longrightarrow & \mathscr{O}'^V \end{array}$$

is a pullback square in $\mathcal{X}$*.*

Proof. We first choose a functor $\theta : \mathcal{G} \to \mathcal{G}^+$ with the following universal property:

(*i*) The ∞-category $\mathcal{G}^+$ admits finite limits.

(*ii*) For every pullback diagram σ :

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

in $\mathcal{G}$, where the vertical maps are admissible, the image $\theta(\sigma)$ is a pullback diagram in $\mathcal{G}^+$.

(*iii*) For any ∞-category $\mathcal{C}$ which admits finite limits, composition with θ induces a fully faithful embedding $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^+, \mathcal{C}) \to \mathrm{Fun}(\mathcal{G}, \mathcal{C})$, whose essential image is spanned by those $\rho : \mathcal{G} \to \mathcal{C}$ such that $\rho(\sigma)$ is a pullback square for every pullback square σ :

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

in $\mathcal{G}$ for which the vertical maps are admissible.

It follows from Proposition HTT.5.3.6.2 that there exists a functor $\theta : \mathcal{G} \to \mathcal{G}^+$ satisfying these conditions and that θ is fully faithful. Let Q be the smallest collection of morphisms in $\mathcal{G}^+$ which contains $\theta(f)$ for every admissible morphism f in $\mathcal{G}$ and satisfies the axioms of Definition 20.2.1.1 (so that Q determines an admissibility structure on $\mathcal{G}^+$).

Every locally left exact functor $\mathcal{G} \to \mathcal{X}$ preserves all pullback squares which exist in $\mathcal{G}$ (Remark 20.4.2.4). Using (*ii*), we obtain a fully faithful embedding $\rho : \mathrm{Fun}^{\mathrm{llex}}(\mathcal{G}, \mathcal{X}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^+, \mathcal{X})$. Let (S_L^+, S_R^+) be the factorization system on $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^+, \mathcal{X})$ given by Proposition 21.3.1.1. Let S_L and S_R be the inverse images of S_L^+ and S_R^+ under the functor ρ. To complete the proof, it will suffice to verify the following:

(a) The pair (S_L, S_R) is a factorization system on $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) = \mathrm{Fun}^{\mathrm{llex}}(\mathcal{G}, \mathcal{X})$.

(b) A morphism $\mathscr{O} \to \mathscr{O}'$ in $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ belongs to S_R if and only if every admissible morphism $f : U \to V$ in $\mathcal{G}$ determines a pullback square

$$\begin{array}{ccc} \mathscr{O}^U & \longrightarrow & \mathscr{O}'^U \\ \downarrow & & \downarrow \\ \mathscr{O}^V & \longrightarrow & \mathscr{O}'^V \end{array}$$

in the ∞-topos $\mathcal{X}$.

We first prove (b). The "only if" direction follows immediately from the definitions. To prove the converse, suppose that $\alpha : \mathscr{O} \to \mathscr{O}'$ is a morphism in $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ having the property that, for each admissible morphism $f : U \to V$ in $\mathcal{G}$, the diagram

$$\begin{array}{ccc} \mathscr{O}^U & \longrightarrow & \mathscr{O}'^U \\ \downarrow & & \downarrow \\ \mathscr{O}^V & \longrightarrow & \mathscr{O}'^V \end{array}$$

is a pullback square. Let Q' be the collection of all morphisms $U^+ \to V^+$ in $\mathcal{G}^+$ for which the diagram

$$\begin{array}{ccc} \mathscr{O}^{U^+} & \longrightarrow & \mathscr{O}'^{U^+} \\ \downarrow & & \downarrow \\ \mathscr{O}^{V^+} & \longrightarrow & \mathscr{O}'^{V^+} \end{array}$$

is a pullback square. Our hypothesis on α guarantees that Q' contains $\theta(f)$ for every admissible morphism f in $\mathcal{G}$. Using the left exactness of the functors $\mathscr{O}$ and $\mathscr{O}'$, we see that Q' determines an admissibility structure on $\mathcal{G}^+$ (Example 20.2.1.8). The minimality of Q implies that $Q \subseteq Q'$, so that $\alpha \in S_R$.

To prove (a), it will suffice to verify the following:

$(*)$ Let $\beta : \mathscr{O}' \to \mathscr{O}''$ be a morphism in $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^+, \mathcal{X})$, so that β factors as a composition

$$\mathscr{O}' \xrightarrow{\beta_L} \mathscr{O} \xrightarrow{\beta_R} \mathscr{O}''$$

where $\beta_L \in S_L^+$ and $\beta_R \in S_R^+$. If $\mathscr{O}'$ and $\mathscr{O}''$ belong to the essential image of ρ, then $\mathscr{O}$ also belongs to the essential image of ρ.

To prove $(*)$, set $\mathscr{O}'_0 = \mathscr{O}' \circ \theta$ and $\mathscr{O}''_0 = \mathscr{O}'' \circ \theta$, so that β determines a map $\beta_0 : \mathscr{O}'_0 \to \mathscr{O}''_0$ in $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$. Using Theorem 21.3.2.5, we see that β_0 determines a geometric morphism of ∞-topoi $g^* : \mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})) \to \mathcal{X}$, so that we can write $\beta_0 = g^*(\overline{\beta}_0)$ for some morphism $\overline{\beta}_0$

in $\mathrm{Fun}^{\mathrm{llex}}(\mathcal{G}, \mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})))$. Using (iii), we can assume that $\overline{\beta}_0$ is the restriction of a map $\overline{\beta} : \overline{\mathscr{O}}' \to \overline{\mathscr{O}}''$ in $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^+, \mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})))$. Let $(\overline{S}_L, \overline{S}_R)$ denote the factorization system on $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^+, \mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G})))$ given by Proposition 21.3.1.1, so that $\overline{\beta}$ factors as a composition $\overline{\mathscr{O}}' \xrightarrow{\overline{\beta}_L} \overline{\mathscr{O}} \xrightarrow{\overline{\beta}_R} \overline{\mathscr{O}}''$ with $\overline{\beta}_L \in \overline{S}_L$ and $\overline{\beta}_R \in \overline{S}_R$. Using Proposition 21.3.1.5, we can assume without loss of generality that the factorization described in $(*)$ is given by

$$\mathscr{O}' \xrightarrow{g^*\overline{\beta}_L} g^*\overline{\mathscr{O}} \xrightarrow{g^*\overline{\beta}_R} \mathscr{O}''.$$

It will therefore suffice to show that $g^*\overline{\mathscr{O}}$ belongs to the essential image of ρ: that is, that the composite functor $g^* \circ \overline{\mathscr{O}} \circ \theta : \mathcal{G} \to \mathcal{X}$ is locally left exact. To prove this, we are free to replace $\mathcal{X}$ by the ∞-topos $\mathcal{P}(\mathrm{Fun}(\Delta^1, \mathcal{G}))$. We may therefore assume without loss of generality that $\mathcal{X}$ has the form $\mathcal{P}(\mathcal{C})$, for some small ∞-category $\mathcal{C}$.

Let $\mathscr{O}$ be as in $(*)$; we wish to show that the functor $\mathscr{O} \circ \theta : \mathcal{G} \to \mathcal{P}(\mathcal{C})$ is locally left exact. Equivalently, we wish to show that for each object $C \in \mathcal{C}$, the composite functor

$$\mathcal{G} \xrightarrow{\theta} \mathcal{G}^+ \xrightarrow{\mathscr{O}} \mathcal{P}(\mathcal{C}) \xrightarrow{e_C} \mathcal{S}$$

is locally left exact, where e_C denotes the functor given by evaluation at C. Using Proposition 21.3.1.5 again, we are reduced to proving $(*)$ in the special case where $\mathcal{Y} = \mathcal{S}$. Let (S', S'') be the factorization system on $\mathrm{Obj}_{\mathcal{G}}(\mathcal{S})$ supplied by Theorem 20.2.2.5, so that β_0 factors as a composition

$$\mathscr{O}'_0 \xrightarrow{\beta'_0} \mathscr{O}_0 \xrightarrow{\beta''_0} \mathscr{O}''_0$$

where $\beta'_0 \in S'$ and $\beta''_0 \in S''$. To complete the proof, it will suffice to show that $\rho(\beta'_0) \in S^+_L$ and $\rho(\beta''_0) \in S^+_R$. The second of these assertions follows from (b) (and the description of the factorization system (S', S'') supplied by Theorem 20.2.2.5). To prove the first, it suffices to observe that the functor $\theta : \mathcal{G} \to \mathcal{G}^+$ induces a map $\mathrm{Pro}(\mathcal{G}) \to \mathrm{Pro}(\mathcal{G}^+)$ which carries proadmissible morphisms in $\mathrm{Pro}(\mathcal{G})$ to proadmissible morphisms in $\mathrm{Pro}(\mathcal{G}^+)$. □

We close this section by noting that the factorization system of Proposition 21.3.3.1 depends functorially on $\mathcal{X}$:

Proposition 21.3.3.2. *Let $\mathcal{G}$ be an essentially small ∞-category equipped with an admissibility structure, let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi, and let $(S^{\mathcal{X}}_L, S^{\mathcal{X}}_R)$ and $(S^{\mathcal{Y}}_L, S^{\mathcal{Y}}_R)$ denote the factorization systems on the ∞-categories $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ and $\mathrm{Obj}_{\mathcal{G}}(\mathcal{Y})$ given by Proposition 21.3.3.1. For every geometric morphism $f^* \in \mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$, composition with f^* carries $S^{\mathcal{X}}_L$ into $S^{\mathcal{Y}}_L$ and $S^{\mathcal{X}}_R$ into $S^{\mathcal{Y}}_R$.*

Proof. Let $\theta : \mathcal{G} \to \mathcal{G}^+$ be as in the proof of Proposition 21.3.3.1. Then we have a homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) & \xrightarrow{f^*} & \mathrm{Obj}_{\mathcal{G}}(\mathcal{Y}) \\ \downarrow \rho_{\mathcal{X}} & & \downarrow \rho_{\mathcal{Y}} \\ \mathrm{Obj}_{\mathcal{G}^+}(\mathcal{X}) & \xrightarrow{f^*} & \mathrm{Obj}_{\mathcal{G}^+}(\mathcal{Y}). \end{array}$$

We can then write

$$S_L^{\mathcal{X}} = \rho_{\mathcal{Y}}^{-1} S_L^{+,\mathcal{X}} \qquad S_R^{\mathcal{X}} = \rho_{\mathcal{X}}^{-1} S_R^{+,\mathcal{X}}$$

$$S_L^{\mathcal{Y}} = \rho_{\mathcal{Y}}^{-1} S_L^{+,\mathcal{Y}} \qquad S_R^{\mathcal{Y}} = \rho_{\mathcal{Y}}^{-1} S_R^{+,\mathcal{Y}},$$

where $(S_L^{+,\mathcal{X}}, S_R^{+,\mathcal{X}})$ and $(S_L^{+,\mathcal{Y}}, S_R^{+,\mathcal{Y}})$ are the factorization systems on $\mathrm{Obj}_{\mathcal{G}^+}(\mathcal{X})$ and $\mathrm{Obj}_{\mathcal{G}^+}(\mathcal{Y})$ supplied by Proposition 21.3.1.1. It will therefore suffice to show that composition with f^* carries $S_L^{+,\mathcal{X}}$ into $S_L^{+,\mathcal{Y}}$ and $S_R^{+,\mathcal{X}}$ into $S_R^{+,\mathcal{Y}}$, which follows from Proposition 21.3.1.5. □

21.3.4 The Case of a Fractured ∞-Topos

We now turn to the proof of Theorem 21.3.0.1 for a general fractured ∞-topos $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$. We will need the following:

Lemma 21.3.4.1. *Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos and let $h : \mathcal{G} \to \mathcal{E}$ be a presentation of $\mathcal{E}$ (see Definition 20.5.3.1). For any ∞-topos $\mathcal{X}$, let $\mathrm{LFun}(\mathcal{E}, \mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{X})$ spanned by those functors which preserve small colimits, so that composition with h determines a fully faithful embedding $\iota : \mathrm{LFun}(\mathcal{E}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{G}, \mathcal{X})$. Suppose we are given a natural transformation $\alpha : f \to g$ between functors $f, g : \mathcal{G} \to \mathcal{X}$ satisfying the following conditions:*

(∗) *For every admissible morphism $U \to V$ in $\mathcal{G}$, the diagram*

$$\begin{array}{ccc} f(U) & \longrightarrow & g(U) \\ \downarrow & & \downarrow \\ f(V) & \longrightarrow & g(V) \end{array}$$

is a pullback square in $\mathcal{X}$.

If g belongs to the essential image of ι, then so does f.

Proof. For each object $V \in \mathcal{G}$, let $F_V : \mathcal{P}(\mathcal{G}^{\mathrm{ad}}_{/V}) \to \mathcal{X}$ be the colimit-preserving extension of $f|_{\mathcal{G}^{\mathrm{ad}}_{/V}}$, and define $G_V : \mathcal{P}(\mathcal{G}^{\mathrm{ad}}_{/V}) \to \mathcal{X}$ similarly. The assumption that g belongs to the essential image of ι guarantees that G_V factors (up to homotopy) through the localization functor $\mathcal{P}(\mathcal{G}^{\mathrm{ad}}_{/V}) \to \mathcal{E}^{\mathrm{corp}}_{/hV}$, and we wish to show that F_V has the same property (for each $V \in \mathcal{G}$). This is clear, since assumption $(*)$ guarantees that F_V is given by the formula

$$F_V(\mathscr{F}) = G_V(\mathscr{F}) \times_{g(V)} f(V).$$

□

Proof of Theorem 21.3.0.1. Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos. Using Theorem 20.5.3.4, we can choose an essentially small full subcategory $\mathcal{G} \subseteq \mathcal{E}$ for which the inclusion $\mathcal{G} \hookrightarrow \mathcal{E}$ is a presentation of $\mathcal{E}$ (where a morphism in $\mathcal{G}$ is admissible if it is $\mathcal{E}^{\mathrm{corp}}$-admissible when regarded as a morphism in $\mathcal{E}$). The inclusion functor extends to a geometric morphism of ∞-topoi $H^* : \mathcal{P}(\mathcal{G}) \to \mathcal{E}$ whose right adjoint H_* is fully faithful. For every ∞-topos $\mathcal{X}$, composition with H^* induces a fully faithful embedding $\iota : \mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \hookrightarrow \mathrm{Fun}^*(\mathcal{P}(\mathcal{G}), \mathcal{X})$, whose essential image is spanned by those functors which carry H^*-equivalences in $\mathcal{P}(\mathcal{G})$ to equivalences in $\mathcal{X}$. Let (S_L, S_R) be the factorization system on $\mathrm{Fun}^*(\mathcal{P}(\mathcal{G}), \mathcal{X}) \simeq \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ given by Proposition 21.3.3.1. To prove the first assertion of Theorem 21.3.0.1, it will suffice to verify the following:

(a) A morphism α in $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ is local if and only if $\iota(\alpha)$ belongs to S_R.

(b) The factorization system (S_L, S_R) restricts to a factorization system on $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$. In other words, if we are given a morphism $\beta : \mathscr{O}' \to \mathscr{O}''$ in $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ which factors as a composition
$$\iota\,\mathscr{O}' \xrightarrow{\beta_L} \mathscr{O} \xrightarrow{\beta_R} \iota\,\mathscr{O}''$$
where $\beta_L \in S_L$ and $\beta_R \in S_R$, then $\mathscr{O} \in \mathrm{Fun}^*(\mathcal{P}(\mathcal{G}), \mathcal{X})$ belongs to the essential image of ι.

Assertion (a) follows from Corollary 21.2.4.12 and (b) follows from Lemma 21.3.4.1.

We now complete the proof of Theorem 21.3.0.1 by showing that the factorization system constructed above depends functorially on the ∞-topos $\mathcal{X}$. Let $f^* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi and let $(S^{\mathcal{X}}_L, S^{\mathcal{X}}_R)$ and $(S^{\mathcal{Y}}_L, S^{\mathcal{Y}}_R)$ be the factorization systems constructed in the first part of the proof. We wish to show that composition with f^* carries $S^{\mathcal{X}}_L$ into $S^{\mathcal{Y}}_L$ and $S^{\mathcal{X}}_R$ into $S^{\mathcal{Y}}_R$. This follows from Proposition 21.3.3.2 by virtue of the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) & \longrightarrow & \mathrm{Fun}^*(\mathcal{E}, \mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) & \longrightarrow & \mathrm{Obj}_{\mathcal{G}}(\mathcal{Y}). \end{array}$$

□

Corollary 21.3.4.2. *Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos and suppose that the collection of corporeal objects of $\mathcal{X}$ is closed under finite limits. Let $\mathcal{X}$ be an arbitrary ∞-topos and let $\mathscr{O} \in \mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$. Then the ∞-category $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X})_{/\mathscr{O}}$ is presentable.*

Proof. Note that if we are given a commutative diagram

$$\begin{array}{ccc} \mathscr{O}'' & \xrightarrow{\alpha} & \mathscr{O}' \\ & \searrow \quad \swarrow & \\ & \mathscr{O} & \end{array}$$

in the ∞-category $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X})$ where the vertical maps are local, then α is automatically local as well. We may therefore identify $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X})_{/\mathscr{O}}$ with the full subcategory of $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})_{/\mathscr{O}}$ spanned by the local morphisms $\mathscr{O}' \to \mathscr{O}$.

Using Proposition 20.5.4.1, we can choose a presentation $h : \mathcal{G} \to \mathcal{E}$ where the ∞-category $\mathcal{G}$ admits finite limits. Then composition with h induces a fully faithful embedding $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \to \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$. Set $\mathscr{O}_0 = (\mathscr{O} \circ h) \in \mathrm{Obj}_{\mathcal{G}}(\mathcal{Y})$. Using Lemma 21.3.4.1, we can identify $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X})_{/\mathscr{O}}$ with the full subcategory of $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})_{/\mathscr{O}_0}$ spanned by those maps $\mathscr{O}'_0 \to \mathscr{O}_0$ which satisfy condition $(*)$ of Lemma 21.3.4.1. This ∞-category is evidently accessible and Proposition 21.3.1.1 implies that it is a localization of $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})_{/\mathscr{O}_0}$. Since the collection of presentable ∞-categories is stable under accessible localization and passage to overcategories (Theorem HTT.5.5.1.1 and Proposition HTT.5.5.3.11), we are reduced to proving that $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ is a presentable ∞-category, which follows from the identification $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})}(\mathcal{X})$ of Proposition 21.2.2.1 (alternatively, apply Lemmas HTT.5.5.4.17 , HTT.5.5.4.18 , and HTT.5.5.4.19). □

Remark 21.3.4.3. In the situation of Corollary 21.3.4.2, let $U \in \mathcal{E}$ be an object and let $\mathscr{O}^U \in \mathcal{X}$ denote the value of $\mathscr{O}$ on U. Then evaluation at U induces a functor

$$e_U : \mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X})_{/\mathscr{O}} \to \mathcal{X}_{/\mathscr{O}^U}.$$

If $U \in \mathcal{E}$ is corporeal, then the functor e_U preserves limits. To prove this, it suffices to observe that we can choose a presentation $h : \mathcal{G} \to \mathcal{E}$ and an object $U_0 \in \mathcal{G}$ with $U = h(U_0)$; in this case, the functor e_U factors as a composition

$$\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X})_{/\mathscr{O}} \xrightarrow{\phi} \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})_{/\mathscr{O}_0} \xrightarrow{e_{U_0}} \mathcal{X}_{/\mathscr{O}^U}$$

where e_{U_0} is given by evaluation at U_0 (which clearly preserves small limits) and the functor ϕ exhibits $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})_{/\mathscr{O}}$ as a localization of $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})_{/\mathscr{O}_0}$ (by the proof of Corollary 21.3.4.2).

Remark 21.3.4.4 (Representability of $\mathcal{X} \mapsto (S^{\mathcal{X}}_L, S^{\mathcal{X}}_R)$)**.** Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos and let $\mathcal{E}^{\Delta^1}$ be the path ∞-topos for $\mathcal{E}$ (Definition 21.3.2.3). Then the ∞-topos

$\mathcal{E}^{\Delta^1}$ represents a functor $U : \infty\mathcal{T}\mathrm{op} \to \widehat{\mathcal{S}}$, whose value on an ∞-topos $\mathcal{X}$ can be regarded as a classifying space for morphisms in the ∞-category $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ (more precisely, we have a homotopy equivalence $U(\mathcal{X}) = \mathrm{Fun}(\Delta^1, \mathrm{Fun}^*(\mathcal{E}, \mathcal{X}))^{\simeq})$. For every ∞-topos $\mathcal{X}$, let $U_L(\mathcal{X}) \subseteq U(\mathcal{X})$ be the summand consisting of morphisms in $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ which belong to $S_L^{\mathcal{X}}$, and define $U_R(\mathcal{X}) \subseteq U(\mathcal{X})$ similarly. It follows from Theorem 21.3.0.1 that we can regard U_L and U_R as functors from $\infty\mathcal{T}\mathrm{op}$ to the ∞-category $\widehat{\mathcal{S}}$.

Note that any morphism $u : f^* \to g^*$ in $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ factors in an essentially unique way as a composition

$$f^* \xrightarrow{u'} h^* \xrightarrow{u''} g^*,$$

where $u' \in S_L^{\mathcal{X}}$ and $u'' \in S_R^{\mathcal{X}}$. Moreover, Theorem 21.3.0.1 guarantees that this factorization depends functorially on $\mathcal{X}$. It follows that the constructions $u \mapsto u'$ and $u \mapsto u''$ determine left homotopy inverses to the inclusions $U_L \hookrightarrow U$ and $U_R \hookrightarrow U$. In particular, the functors U_L and U_R are both retracts of U. Since the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi is idempotent complete (it admits all small limits and colimits), it follows that the functors U_L and U_R are representable by ∞-topoi $\mathcal{E}_L^{\Delta^1}$ and $\mathcal{E}_R^{\Delta^1}$, respectively.

21.4 Structured Spaces

Recall that a *locally ringed space* is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a local sheaf of commutative rings on X. The collection of locally ringed spaces can be organized into a category $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}}$, where a morphism from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ in $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}}$ is given by a pair (f, α), where $f : X \to Y$ is a continuous map of topological spaces and $\alpha : f^* \mathscr{O}_Y \to \mathscr{O}_X$ is a local map between sheaves of commutative rings on X (Definition 1.1.5.1).

In Chapter 1, we considered several variants of the category $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}}$:

(*a*) Replacing the ordinary category of commutative rings by the ∞-category CAlg of $\mathbb{E}_\infty$-rings, we obtained the *∞-category of locally spectrally ringed spaces* $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ (Definition 1.1.5.3).

(*b*) Replacing the ordinary category of topological spaces $\mathcal{T}\mathrm{op}$ by the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi, we obtained the *∞-category of locally spectrally ringed ∞-topoi* $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ (Definition 1.4.2.1).

(*c*) Replacing the Zariski topology by the étale topology, we obtained a full subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}} \subseteq \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ of locally spectrally ringed ∞-topoi with strictly Henselian structure sheaves (Definition 1.4.2.1).

Our goal in this section is to develop a formalism which encompasses all of these examples and several variants thereof, and thereby lay a foundation for discussing variants of the theory of spectral algebraic geometry. We begin in §21.4.1 by associating to every geometric site $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ an ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ of *locally $\mathcal{G}$-structured ∞-topoi* (Construction 21.4.1.15), whose objects are pairs $(\mathcal{X}, \mathscr{O})$ where $\mathcal{X}$ is an ∞-topos and $\mathscr{O}$ is a τ-local $\mathcal{G}$-object of $\mathcal{X}$. The ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ specializes to the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ of locally spectrally ringed ∞-topoi when $\mathcal{G} = \mathrm{Aff}_{\mathrm{Sp}}$ is the spectral Zariski site of Example 20.6.4.4 (Example 21.4.1.19; see also Examples 21.4.1.17, 21.4.1.18, and 21.4.1.20 for related statements).

In general, many different geometric sites $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ can give rise to the same theory of (local) $\mathcal{G}$-objects: for any ∞-topos $\mathcal{X}$, the ∞-category $\mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{X})$ depends only on the ∞-topos $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ and the subcategory $\mathrm{Obj}^{\tau,\mathrm{loc}}_{\mathcal{G}}(\mathcal{X}) \subseteq \mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{X})$ depends only on the tracture subcategory $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})^{\mathrm{corp}} \subseteq \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ of Theorem 20.6.3.4. In §21.4.2, we adopt a more invariant perspective and introduce an ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ associated to any fractured ∞-topos $\mathcal{E}$ (Definition 21.4.2.6), which extends and (slightly) generalizes the constructions of §21.4.1.

The remainder of this section is devoted to studying formal properties enjoyed by the ∞-categories $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ and $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ of §21.4.1 and §21.4.2. In §21.4.3 we show that these ∞-categories admit small filtered limits (Theorem 21.4.3.1), and in §21.4.4 and 21.4.5 we show that they admit colimits for some restricted classes of diagrams (Propositions 21.4.4.9 and 21.4.5.1). One special case is particularly notable: in §21.4.6 we introduce subcategories $\infty\mathcal{T}\mathrm{op}^{\text{ét}}(\mathcal{G}) \subseteq \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ and $\infty\mathcal{T}\mathrm{op}^{\text{ét}}(\mathcal{E}) \subseteq \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ of *étale morphisms* (Variant 21.4.6.3 and Definition 21.4.6.1), which admit all small colimits (Proposition 21.4.6.4); these subcategories will play an important role in Chapter **??**.

To apply the formalism of this section in practice, one typically begins by choosing a geometric site $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ consisting of "simple" geometric objects of some kind (such as affine schemes of finite type over $\mathbf{Z}$), and obtains an ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ consisting of more general objects of the same type (such as locally ringed spaces, or more general ∞-topoi with a sheaf of local rings). In §21.4.7 we make this heuristic more precise by constructing a functor $\mathcal{G} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$, which carries admissible morphisms in $\mathcal{G}$ to étale morphisms in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ (Remark 21.4.7.2). More generally, to any fractured ∞-topos $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$, we introduce a functor $\mathrm{Re} : \mathcal{E}_{\mathrm{corp}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ which we refer to as *corporeal realization* (Construction 21.4.7.1); here $\mathcal{E}_{\mathrm{corp}}$ denotes the full subcategory of $\mathcal{E}$ spanned by the corporeal objects. We show that the corporeal realization functor Re is always fully faithful (Proposition 21.4.7.10), and its value on a corporeal object $X \in \mathcal{E}$ can be characterized by a universal mapping property (Theorem 21.4.7.7).

21.4.1 The ∞-Category $\infty\mathcal{T}\mathrm{op}(\mathcal{G})$

Let $\mathcal{C}$ be an arbitrary ∞-category. In §1.1.2, we introduced the ∞-category $\mathcal{T}\mathrm{op}_{\mathcal{C}}$, whose objects are pairs $(X, \mathscr{F})$ where X is a topological space and $\mathscr{F}$ is a $\mathcal{C}$-valued sheaf on X

(Construction 1.1.2.2). We now consider some variants of this construction.

Construction 21.4.1.1 (The ∞-Category $\infty\mathcal{T}\mathrm{op}_{\mathcal{C}}$)**.** Let $\mathcal{C}$ be an arbitrary ∞-category. For every geometric morphism of ∞-topoi $f^* : \mathcal{X} \to \mathcal{Y}$, precomposition with f^* determines a direct image functor $f_*^{\mathcal{C}} : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{Y}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ (see Remark 21.2.2.4). We can therefore view the construction $\mathcal{X} \mapsto \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ as a functor $\infty\mathcal{T}\mathrm{op} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$. We let $U : \infty\mathcal{T}\mathrm{op}_{\mathcal{C}} \to \infty\mathcal{T}\mathrm{op}$ be a coCartesian fibration classified by the functor $\mathcal{X} \mapsto \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})^{\mathrm{op}}$. We will refer to $\infty\mathcal{T}\mathrm{op}_{\mathcal{C}}$ as *the ∞-category of ∞-topoi with a $\mathcal{C}$-valued sheaf.*

Remark 21.4.1.2. Let $\mathcal{C}$ be an arbitrary ∞-category. We can describe the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathcal{C}}$ of Construction 21.4.1.1 more informally as follows:

(i) The objects of $\infty\mathcal{T}\mathrm{op}_{\mathcal{C}}$ are pairs $(\mathcal{X}, \mathscr{F})$, where $\mathcal{X}$ is an ∞-topos and $\mathscr{F}$ is a $\mathcal{C}$-valued sheaf on $\mathcal{X}$.

(ii) A morphism from $(\mathcal{X}, \mathscr{F})$ to $(\mathcal{Y}, \mathscr{G})$ in the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathcal{C}}$ is given by a geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{Y}$ together with a natural transformation $\beta : \mathscr{G} \to f_*^{\mathcal{C}} \mathscr{F} = \mathscr{F} \circ f^*$.

Remark 21.4.1.3 (Comparison with $\mathcal{T}\mathrm{op}_{\mathcal{C}}$)**.** Let $\mathcal{C}$ be an ∞-category which admits small limits. For any topological space X, Proposition 1.3.1.7 supplies an equivalence of ∞-categories $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{S}\mathrm{hv}(X)) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(X)$. This equivalence depends functorially on X, and therefore determines a pullback diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{T}\mathrm{op}_{\mathcal{C}} & \longrightarrow & \infty\mathcal{T}\mathrm{op}_{\mathcal{C}} \\ \downarrow & & \downarrow \\ \mathcal{T}\mathrm{op} & \longrightarrow & \infty\mathcal{T}\mathrm{op}. \end{array}$$

Beware that this is generally false if $\mathcal{C}$ does not admit small limits.

Example 21.4.1.4 (Sheaves of $\mathbb{E}_\infty$-Rings)**.** Let $\mathcal{C} = \mathrm{CAlg}$ be the ∞-category of $\mathbb{E}_\infty$-rings. Then $\infty\mathcal{T}\mathrm{op}_{\mathcal{C}}$ is the ∞-category of spectrally ringed ∞-topoi (Construction 1.4.1.3), and $\mathcal{T}\mathrm{op}_{\mathcal{C}}$ is the ∞-category of spectrally ringed spaces (Definition 1.1.2.5).

We will be primarily interested in the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathcal{C}}$ in the case where the ∞-category $\mathcal{C}$ is compactly generated. In this case, we can reformulate Construction 21.4.1.1 using the ideas of §21.2.

Construction 21.4.1.5 (The ∞-Category $\infty\mathcal{T}\mathrm{op}(\mathcal{G})$)**.** Let $\mathcal{G}$ be an essentially small ∞-category. A *$\mathcal{G}$-structured ∞-topos* is a pair $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, where $\mathcal{X}$ is an ∞-topos and $\mathscr{O}_{\mathcal{X}} : \mathcal{G} \to \mathcal{X}$ is a locally left exact functor (which we will denote by $(T \in \mathcal{G}) \mapsto (\mathscr{O}_{\mathcal{X}}^{T} \in \mathcal{X})$). In this case, we refer to $\mathscr{O}_{\mathcal{X}}$ as the *structure sheaf* of $\mathcal{X}$.

Note that if $f^* : \mathcal{X} \to \mathcal{Y}$ is a geometric morphism of ∞-topoi, then composition with f^* determines a functor $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) \to \mathrm{Obj}_{\mathcal{G}}(\mathcal{Y})$ (see Remark 21.2.2.4). We can therefore view the construction $\mathcal{X} \mapsto \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ as a functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. We let $U : \infty\mathcal{T}\mathrm{op}(\mathcal{G}) \to \infty\mathcal{T}\mathrm{op}$ be a Cartesian fibration classified by the functor $\mathcal{X} \mapsto \mathrm{Obj}_{\mathcal{G}}(\mathcal{G})^{\mathrm{op}}$. We will refer to $\infty\mathcal{T}\mathrm{op}(\mathcal{G})$ as *the ∞-category of $\mathcal{G}$-structured ∞-topoi.*

Remark 21.4.1.6. Let $\mathcal{G}$ be an essentially small ∞-category. We can describe the ∞-category $\infty\mathcal{T}\mathrm{op}(\mathcal{G})$ of Construction 21.4.1.5 more informally as follows:

(*i*) The objects of $\infty\mathcal{T}\mathrm{op}(\mathcal{G})$ are $\mathcal{G}$-structured ∞-topoi $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$.

(*ii*) A morphism from $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ to $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ in the ∞-category $\infty\mathcal{T}\mathrm{op}(\mathcal{G})$ is given by a geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{Y}$ together with a natural transformation $\alpha : f^* \circ \mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$.

Here we can replace (*ii*) with the following variant:

(*ii'*) A morphism from $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ to $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ in the ∞-category $\infty\mathcal{T}\mathrm{op}(\mathcal{G})$ is given by a geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{Y}$ together with a natural transformation $\beta : \mathscr{O}_{\mathcal{Y}} \to f_* \circ \mathscr{O}_{\mathcal{X}}$.

Beware, however, that the functor $f_* \circ \mathscr{O}_{\mathcal{X}}$ appearing in (*ii'*) might fail to be locally left exact (unless we assume that $\mathcal{G}$ admits finite limits).

Variant 21.4.1.7. Let $\mathcal{G}$ be an essentially small ∞-category. A *$\mathcal{G}$-structured space* is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a $\mathcal{G}$-object of the ∞-topos $\mathcal{S}\mathrm{hv}(X)$. The collection of all $\mathcal{G}$-structured spaces can be organized into an ∞-category $\mathcal{T}\mathrm{op}(\mathcal{G}) = \mathcal{T}\mathrm{op} \times_{\infty\mathcal{T}\mathrm{op}} \infty\mathcal{T}\mathrm{op}(\mathcal{G})$. We will refer to $\mathcal{T}\mathrm{op}(\mathcal{G})$ as *the ∞-category of $\mathcal{G}$-structured spaces.*

The relationship between Constructions 21.4.1.1 and 21.4.1.5 can be summarized by the following relative version of Proposition 21.2.2.1, whose proof we leave to the reader:

Proposition 21.4.1.8. *Let $\mathcal{G}$ be an essentially small ∞-category which admits finite limits and let $\mathcal{C} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$. Then there is a canonical equivalence of ∞-categories $\rho : \infty\mathcal{T}\mathrm{op}_{\mathcal{C}} \to \infty\mathcal{T}\mathrm{op}(\mathcal{G})$ given on objects by the construction $(\mathcal{X}, \mathscr{F}) \mapsto (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, where $\mathscr{O}_{\mathcal{X}}$ is characterized by the formula $\mathrm{Map}_{\mathcal{X}}(X, \mathscr{O}^T_{\mathcal{X}}) \simeq \mathrm{Map}_{\mathcal{C}}(T, \mathscr{F}(X))$ for $X \in \mathcal{X}$ and $T \in \mathcal{G}$.*

Remark 21.4.1.9. Even without the assumption that $\mathcal{G}$ admits finite limits, we still have a fully faithful embedding $\infty\mathcal{T}\mathrm{op}_{\mathcal{C}} \hookrightarrow \infty\mathcal{T}\mathrm{op}(\mathcal{G})$.

Corollary 21.4.1.10. *Let $\mathcal{G}$ be an essentially small ∞-category which admits finite limits. Then the forgetful functor $\infty\mathcal{T}\mathrm{op}(\mathcal{G}) \to \infty\mathcal{T}\mathrm{op}$ is a coCartesian fibration.*

Proof. Apply Proposition 21.4.1.8 (noting that the forgetful functor $\infty\mathcal{T}\mathrm{op}_{\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})} \to \infty\mathcal{T}\mathrm{op}$ is a coCartesian fibration by construction); alternatively, this follows directly from Remark 21.2.1.8. □

Corollary 21.4.1.11. *Let $\mathcal{C}$ be a compactly generated ∞-category. Then the forgetful functor $\infty\mathcal{T}\mathrm{op}_{\mathcal{C}} \to \infty\mathcal{T}\mathrm{op}$ is a Cartesian fibration.*

Proof. Apply Proposition 21.4.1.8 in the case where $\mathcal{G} = \mathcal{C}_c^{\mathrm{op}}$, noting that the forgetful functor $\infty\mathcal{T}\mathrm{op}(\mathcal{G}) \to \infty\mathcal{T}\mathrm{op}$ is a Cartesian fibration by construction. □

Corollary 21.4.1.12. *Let $\mathcal{G}$ be an essentially small ∞-category which admits finite limits and let $\mathcal{C} = \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$. Then there is a canonical equivalence of ∞-categories $\rho : \mathcal{T}\mathrm{op}_{\mathcal{C}} \to \mathcal{T}\mathrm{op}(\mathcal{G})$ given on objects by the construction $(X, \mathscr{F}) \mapsto (X, \mathscr{O}_X)$, where $\mathscr{O}_{\mathcal{X}}$ is characterized by the formula $\mathscr{O}_X^T(U) = \mathrm{Map}_{\mathcal{C}}(T, \mathscr{F}(U))$ for $U \subseteq X$ and $T \in \mathcal{G}$.*

Proof. Combine Proposition 21.4.1.8 with Remark **??**. □

Example 21.4.1.13 (Ringed Spaces)**.** Let Aff denote the category of affine schemes of finite type of over **Z**. Using Corollary 21.4.1.12, we obtain an equivalence $\mathcal{T}\mathrm{op}(\mathrm{Aff}) \simeq \mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}$, where $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}$ denotes the category of ringed spaces (Definition 1.1.1.1).

Example 21.4.1.14 (Spectrally Ringed Spaces)**.** Let $\mathrm{Aff}_{\mathrm{Sp}}$ denote the opposite of the ∞-category CAlg_c of compact $\mathbb{E}_\infty$-rings. Then Proposition 21.2.2.1 supplies an equivalence of $\infty\mathcal{T}\mathrm{op}(\mathrm{Aff}_{\mathrm{Sp}})$ with the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ of spectrally ringed ∞-topoi (Construction 1.4.1.3). Similarly, Corollary 21.4.1.12 supplies an equivalence of $\mathcal{T}\mathrm{op}(\mathrm{Aff}_{\mathrm{Sp}})$ with the ∞-category $\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ of spectrally ringed spaces (Definition 1.1.2.5).

We now consider a variant of Construction 21.4.1.5.

Construction 21.4.1.15. Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site (Definition 20.6.2.1). We let $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ denote the (non-full) subcategory of $\infty\mathcal{T}\mathrm{op}(\mathcal{G})$ described as follows:

- An object $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ of $\infty\mathcal{T}\mathrm{op}(\mathcal{G})$ belongs to $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ if and only if $\mathscr{O}_{\mathcal{X}}$ belongs to the full subcategory $\mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{X}) \subseteq \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$ (see Definition 21.2.1.1).

- A morphism $(f, \alpha) : (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ in $\infty\mathcal{T}\mathrm{op}(\mathcal{G})$ belongs to $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ if and only if $\mathscr{O}_{\mathcal{X}}$ and $\mathscr{O}_{\mathcal{Y}}$ belong to $\mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{X})$ and $\mathrm{Obj}_{\mathcal{G}}^{\tau}(\mathcal{Y})$, and the morphism $\alpha : f^* \mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$ is local (in the sense of Definition 21.2.4.1).

We will refer to $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ as *the ∞-category of locally $\mathcal{G}$-structured ∞-topoi.* We let $\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ denote the fiber product $\mathcal{T}\mathrm{op} \times_{\infty\mathcal{T}\mathrm{op}} \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$, which we refer to as *the ∞-category of locally $\mathcal{G}$-structured spaces.*

In the case where $\mathcal{G}^{\mathrm{ad}} = \mathcal{G}^{\simeq}$ is the trivial admissibility structure on $\mathcal{G}$ (that is, only equivalences in $\mathcal{G}$ are admissible), then we will denote $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ and $\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ by $\infty\mathcal{T}\mathrm{op}^{\tau}(\mathcal{G})$ and $\mathcal{T}\mathrm{op}^{\tau}(\mathcal{G})$, respectively.

Remark 21.4.1.16. In the situation of Construction 21.4.1.15, the forgetful functor $U : \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G}) \to \infty\mathcal{T}\mathrm{op}$ is a Cartesian fibration, classified by the functor $\mathcal{X} \mapsto \mathrm{Obj}_{\mathcal{G}}^{\tau,\mathrm{loc}}(\mathcal{X})$. Beware that U is usually *not* a coCartesian fibration, even when $\mathcal{G}$ admits finite limits.

Example 21.4.1.17 (Algebraic Geometry: Zariski Topology)**.** Let $(\mathrm{Aff}, \mathrm{Aff}^{\mathrm{Zar}}, \tau_{\mathrm{Zar}})$ be the geometric site of Example 20.6.4.1, and set $\infty\mathcal{T}\mathrm{op}^{\mathrm{Zar}}(\mathrm{Aff}) = \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathrm{Aff})$. Using Propositions 21.2.5.1 and 21.4.1.8, we can identify the objects of $\infty\mathcal{T}\mathrm{op}^{\mathrm{Zar}}(\mathrm{Aff})$ with pairs $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, where $\mathcal{X}$ is an ∞-topos and $\mathscr{O}_{\mathcal{X}}$ is a local commutative ring object of the underlying topos $\mathcal{X}^{\heartsuit}$. More precisely, we have a pullback diagram

$$\begin{array}{ccc} \infty\mathcal{T}\mathrm{op}^{\mathrm{Zar}}(\mathrm{Aff}) & \longrightarrow & 1\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}} \\ \downarrow & & \downarrow \\ \infty\mathcal{T}\mathrm{op} & \xrightarrow{\heartsuit} & 1\mathcal{T}\mathrm{op} \end{array}$$

where $1\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}}$ is the 2-category of locally ringed topoi (Definition 1.2.1.4). In particular, we can identify $\mathcal{T}\mathrm{op}^{\mathrm{Zar}}(\mathrm{Aff}) = \mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathrm{Aff})$ with the category $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}}$ of locally ringed spaces (see Definition 1.1.5.1).

Example 21.4.1.18 (Algebraic Geometry: Étale Topology)**.** Let $(\mathrm{Aff}, \mathrm{Aff}^{\text{ét}}, \tau_{\text{ét}})$ be the geometric site of Example **??** and set $\infty\mathcal{T}\mathrm{op}^{\text{ét}}(\mathrm{Aff}) = \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathrm{Aff})$. Using Propositions 21.2.5.1 and 21.4.1.8, we can identify the objects of $\infty\mathcal{T}\mathrm{op}^{\text{ét}}(\mathrm{Aff})$ with pairs $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, where $\mathcal{X}$ is an ∞-topos and $\mathscr{O}_{\mathcal{X}}$ is a strictly Henselian commutative ring object of the underlying topos $\mathcal{X}^{\heartsuit}$. More precisely, we have a pullback diagram

$$\begin{array}{ccc} \infty\mathcal{T}\mathrm{op}^{\text{ét}}(\mathrm{Aff}) & \longrightarrow & 1\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^{\heartsuit}} \\ \downarrow & & \downarrow \\ \infty\mathcal{T}\mathrm{op} & \xrightarrow{\heartsuit} & 1\mathcal{T}\mathrm{op} \end{array}$$

where $1\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}^{\heartsuit}}$ is the 2-category of locally ringed topoi with strictly Henselian structure sheaf (Definition 1.2.2.5).

Example 21.4.1.19 (Spectral Algebraic Geometry: Zariski Topology)**.** Let $(\mathrm{Aff}_{\mathrm{Sp}}, \mathrm{Aff}^{\mathrm{Zar}}_{\mathrm{Sp}}, \tau_{\mathrm{Zar}})$ be the geometric site of Example 20.6.4.4 and set $\infty\mathcal{T}\mathrm{op}^{\mathrm{Zar}}(\mathrm{Aff}_{\mathrm{Sp}}) = \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathrm{Aff}_{\mathrm{Sp}})$. Using Propositions 21.2.5.3 and 21.4.1.8, we can identify the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{Zar}}(\mathrm{Aff}_{\mathrm{Sp}})$ with the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ of locally spectrally ringed ∞-topoi (see Definition 1.4.2.1). In particular, we can identify $\mathcal{T}\mathrm{op}^{\mathrm{Zar}}(\mathrm{Aff}_{\mathrm{Sp}}) = \mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathrm{Aff}_{\mathrm{Sp}})$ with the ∞-category of locally spectrally ringed spaces (see Definition 1.1.5.3).

Example 21.4.1.20 (Spectral Algebraic Geometry: Étale Topology)**.** Let $(\mathrm{Aff}_{\mathrm{Sp}}, \mathrm{Aff}^{\text{ét}}_{\mathrm{Sp}}, \tau_{\text{ét}})$ be the geometric site of Example **??** and set $\infty\mathcal{T}\mathrm{op}^{\text{ét}}(\mathrm{Aff}_{\mathrm{Sp}}) = \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathrm{Aff}_{\mathrm{Sp}})$. Using

Propositions 21.2.5.3 and 21.4.1.8, we can identify the ∞-category $\infty\mathcal{T}\mathrm{op}^{\text{ét}}(\mathrm{Aff}_{\mathrm{Sp}})$ with the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ of locally spectrally ringed ∞-topoi with strictly Henselian structure sheaf (see Definition 1.4.2.1).

21.4.2 The ∞-Category $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$

The constructions of §21.4.1 can be reformulated (and slightly generalized) by working directly with classifying ∞-topoi, rather than (geometric) sites.

Construction 21.4.2.1 (The ∞-Category $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$)**.** Let $\mathcal{E}$ be an ∞-topos. For every ∞-topos $\mathcal{X}$, we let $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{X})$ whose objects are functors which preserve small colimits and finite limits. Note that if $f^* : \mathcal{X} \to \mathcal{Y}$ belongs to $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$, then composition with f^* determines a functor $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \to \mathrm{Fun}^*(\mathcal{E}, \mathcal{Y})$. We can therefore view the construction $\mathcal{X} \mapsto \mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ as a functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$. We let $U : \infty\mathcal{T}\mathrm{op}(\mathcal{E}) \to \infty\mathcal{T}\mathrm{op}$ be a Cartesian fibration classified by the functor $\mathcal{X} \mapsto \mathrm{Fun}^*(\mathcal{E}, \mathcal{X})^{\mathrm{op}}$.

Warning 21.4.2.2. We notation $\infty\mathcal{T}\mathrm{op}(\mathcal{U})$ has now been assigned two different meanings: that of Construction 21.4.1.5 in the case where $\mathcal{U}$ is an essentially small ∞-category, and that of Construction 21.4.2.1 when $\mathcal{U}$ is an ∞-topos. There is some slight danger of confusion, because it is possible for an ∞-category $\mathcal{U}$ to satisfy both of these conditions. This happens only when $\mathcal{U}$ is a contractible Kan complex, and in this case Constructions 21.4.1.5 and 21.4.2.1 do not agree (the ∞-category $\infty\mathcal{T}\mathrm{op}(\mathcal{U})$ of Construction 21.4.1.5 is equivalent to $\infty\mathcal{T}\mathrm{op}$, while the ∞-category $\infty\mathcal{T}\mathrm{op}(\mathcal{U})$ is a contractible Kan complex).

Remark 21.4.2.3. Let $\mathcal{E}$ be an ∞-topos. Then the ∞-category $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ can be viewed as a *lax* version of the overcategory $\infty\mathcal{T}\mathrm{op}_{/\mathcal{E}}$. The objects of $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ can be identified with $\pi_* : \mathcal{X} \to \mathcal{E}$ in the ∞-category $\infty\mathcal{T}\mathrm{op}$, with morphisms from $\pi_* : \mathcal{X} \to \mathcal{E}$ to $\pi'_* : \mathcal{X}' \to \mathcal{E}$ given by diagrams

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f_*} & \mathcal{X}' \\ & {\scriptstyle \pi_*}\searrow \quad \swarrow{\scriptstyle \pi'_*} & \\ & \mathcal{E} & \end{array}$$

which commute up to a possibly non-invertible natural transformation $\alpha : \pi_* \to \pi'_* \circ f_*$. In particular, the ∞-category $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ contains the usual overcategory $\infty\mathcal{T}\mathrm{op}_{/\mathcal{E}}$ as a (non-full) subcategory; see Proposition 21.4.4.5 for more details.

Example 21.4.2.4. Let $\mathcal{G}$ be an essentially small ∞-category equipped with a Grothendieck topology τ, and let $\mathcal{E} = \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ be the associated ∞-topos. Then composition with the sheafified Yoneda embedding $h : \mathcal{G} \to \mathcal{E}$ induces an equivalence $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G}), \mathcal{X}) \to \mathrm{Obj}^{\tau}_{\mathcal{G}}(\mathcal{X})$ for every ∞-topos $\mathcal{X}$ (Proposition 21.2.1.13), which depends functorially on $\mathcal{X}$. It follows that h induces an equivalence of ∞-categories $\infty\mathcal{T}\mathrm{op}(\mathcal{E}) \to \infty\mathcal{T}\mathrm{op}^{\tau}(\mathcal{G})$, where $\infty\mathcal{T}\mathrm{op}^{\tau}(\mathcal{G})$ is defined as in Construction 21.4.1.15.

Example 21.4.2.5. Let $\mathcal{C}$ be a small ∞-category which admits finite colimits, and let $\mathcal{E} = \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ denote the ∞-topos of presheaves on $\mathcal{C}^{\mathrm{op}}$. Combining Example 21.4.2.4 with Proposition 21.4.1.8, we obtain an equivalence of ∞-categories $\infty\mathcal{T}\mathrm{op}(\mathcal{E}) \simeq \infty\mathcal{T}\mathrm{op}_{\mathrm{Ind}(\mathcal{C})}$,

Definition 21.4.2.6. Let $\mathcal{E}$ be an ∞-topos equipped with a fracture subcategory $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$. We let $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ denote the (non-full) subcategory of $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ whose morphisms are given by diagrams of ∞-topoi

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f_*} & \mathcal{X}' \\ & {\scriptstyle \pi_*}\searrow \quad \swarrow{\scriptstyle \pi'_*} & \\ & \mathcal{E} & \end{array}$$

which commute up to a natural transformation $\pi_* \to \pi'_* \circ f_*$ for which the induced map of left adjoints $f^* \circ \pi'^* \to \pi^*$ is local, in the sense of Definition 21.2.4.4.

Remark 21.4.2.7. Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos. Then the Cartesian fibration $\infty\mathcal{T}\mathrm{op}(\mathcal{E}) \to \infty\mathcal{T}\mathrm{op}$ restricts to a Cartesian fibration $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E}) \to \infty\mathcal{T}\mathrm{op}$, which is classified by the functor

$$\infty\mathcal{T}\mathrm{op}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty} \qquad \mathcal{X} \mapsto \mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X}).$$

Warning 21.4.2.8. In the situation of Construction 21.4.2.1, the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ depends not only on $\mathcal{E}$, but also on the choice of a fracture subcategory $\mathcal{E}^{\mathrm{corp}}$ (however, it depends only on the *completion* of $\mathcal{E}^{\mathrm{corp}}$, in the sense of Remark 20.3.4.7: see Remark 21.2.4.9.

Example 21.4.2.9. Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site (Definition 20.6.2.1) and regard $\mathcal{E} = \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ as equipped with the fracture subcategory $\mathcal{E}^{\mathrm{corp}} = \mathcal{S}\mathrm{hv}^{\mathrm{corp}}_{\tau}(\mathcal{G})$ of Theorem 20.6.3.4. Then composition with the sheafified Yoneda embedding $h : \mathcal{G} \to \mathcal{E}$ induces an equivalence $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G}), \mathcal{X}) \to \mathrm{Obj}^{\tau,\mathrm{loc}}_{\mathcal{G}}(\mathcal{X})$ for every ∞-topos $\mathcal{X}$ (Corollary 21.2.4.12), which depends functorially on $\mathcal{X}$. It follows that h induces an equivalence of ∞-categories $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E}) \to \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$, where $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ is defined as in Construction 21.4.1.15.

21.4.3 Filtered Limits in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$

Let $\infty\mathcal{T}\mathrm{op}$ be the ∞-category of ∞-topoi. Then the ∞-category $\infty\mathcal{T}\mathrm{op}$ admits all small limits (Corollary **??**). Moreover, *filtered* limits in the ∞-category $\infty\mathcal{T}\mathrm{op}$ are particularly easy to understand: they can be computed at the level of the underlying ∞-categories (in other words, the forgetful functor $\infty\mathcal{T}\mathrm{op} \hookrightarrow \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ preserves filtered limits: see Theorem HTT.6.3.3.1). In this section, we will show that an analogous statement holds for the variants of $\infty\mathcal{T}\mathrm{op}$ described in §21.4.1 and §21.4.2. Our main result can be stated as follows:

Theorem 21.4.3.1. (1) *Let $\mathcal{E}$ be an ∞-topos. Then the ∞-category $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ of Construction 21.4.2.1 admits small filtered limits, which are preserved by the forgetful functor $\infty\mathcal{T}\mathrm{op}(\mathcal{E}) \to \infty\mathcal{T}\mathrm{op}$.*

(2) *Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos. Then the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ of Definition 21.4.2.6 admits small filtered limits, which are preserved by the inclusion functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E}) \hookrightarrow \infty\mathcal{T}\mathrm{op}(\mathcal{E})$ (and therefore also by the forgetful functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E}) \to \infty\mathcal{T}\mathrm{op}$.*

(3) *Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site. Then the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ of Construction 21.4.1.15 admits small filtered limits, which are preserved by the forgetful functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G}) \to \infty\mathcal{T}\mathrm{op}$.*

(4) *Let $\mathcal{C}$ be a compactly generated ∞-category. Then the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathcal{C}}$ of Construction 21.4.1.1 admits small filtered limits, which are preserved by the forgetful functor $\infty\mathcal{T}\mathrm{op}_{\mathcal{C}} \to \infty\mathcal{T}\mathrm{op}$.*

We will deduce Theorem 21.4.3.1 from the following:

Proposition 21.4.3.2. *Let $\mathcal{E}$ and $\mathcal{X}$ be ∞-topoi. Then:*

(*i*) *The ∞-category $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ admits small filtered colimits, which are preserved by the inclusion functor $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{E}, \mathcal{X})$.*

(*ii*) *Suppose that $\mathcal{E}$ is equipped with a fracture subcategory $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$. Then the ∞-category $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X})$ admits small filtered colimits, which are preserved by the inclusion $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X}) \hookrightarrow \mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$.*

Let us assume Proposition 21.4.3.2 for the moment, and gather some consequences.

Corollary 21.4.3.3. *Let $\mathcal{E}$ be an ∞-topos. Then, for any geometric morphism of ∞-topoi $f^* : \mathcal{X} \to \mathcal{Y}$, composition with f^* induces a functor $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \to \mathrm{Fun}^*(\mathcal{E}, \mathcal{Y})$ which preserves small filtered colimits. If $\mathcal{E}$ is equipped with a fracture subcategory $\mathcal{E}^{\mathrm{corp}}$, then the induced map $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X}) \to \mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{Y})$ also preserves small filtered colimits.*

Warning 21.4.3.4. If $\mathcal{E}$ and $\mathcal{X}$ are ∞-topoi, then the ∞-category $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ is always accessible (Proposition HTT.6.3.1.13). However, if $\mathcal{E}$ is equipped with a fracture subcategory $\mathcal{E}^{\mathrm{corp}}$, then the subcategory of local morphisms $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X})$ need not be accessible. For example, if $\mathcal{E}^{\mathrm{corp}} = \mathcal{E}$, then $\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X}) \simeq \mathrm{Fun}^*(\mathcal{E}, \mathcal{X})^{\simeq}$ is accessible if and only if the ∞-category $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ is essentially small.

Corollary 21.4.3.5. *Let $\mathcal{E}$ be an ∞-topos and let $\mathcal{I}$ be a small filtered ∞-category. Then:*

(1) *Every commutative diagram*

$$\begin{array}{ccc} \mathcal{I}^{\mathrm{op}} & \longrightarrow & \infty\mathcal{T}\mathrm{op}(\mathcal{E}) \\ \big\downarrow & \nearrow & \big\downarrow U \\ (\mathcal{I}^{\mathrm{op}})^{\triangleleft} & \longrightarrow & \infty\mathcal{T}\mathrm{op} \end{array}$$

admits a completion as indicated, where the dotted arrow is a U-limit diagram in $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ (here U denotes the forgetful functor).

(2) *Suppose that $\mathcal{E}$ is equipped with a fracture subcategory $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$. Then every commutative diagram*

$$\begin{array}{ccc} \mathcal{I}^{\mathrm{op}} & \longrightarrow & \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E}) \\ \big\downarrow & \nearrow & \big\downarrow U^{\mathrm{loc}} \\ (\mathcal{I}^{\mathrm{op}})^{\triangleleft} & \longrightarrow & \infty\mathcal{T}\mathrm{op} \end{array}$$

admits a completion as indicated, where the dotted arrow is a U^{loc}-limit diagram in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$. Moreover, this completion is also a U-limit diagram.

Proof. Combine Proposition 21.4.3.2, Corollary 21.4.3.3, and Corollary HTT.4.3.1.11 . □

Proof of Theorem 21.4.3.1. Assertions (1) and (2) follow from Corollary 21.4.3.5, Theorem HTT.6.3.3.1 , and Proposition HTT.4.3.1.5 . The implication (2) ⇒ (3) follows from Example 21.4.2.9, and the implication (3) ⇒ (4) follows from Proposition **??**. □

Proof of Proposition 21.4.3.2. Let A be a (small) filtered partially ordered set and let $\{f^*_\alpha\}_{\alpha \in A}$ be a diagram in the ∞-category $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ having colimit f^* in $\mathrm{Fun}(\mathcal{E}, \mathcal{X})$. To prove Proposition 21.4.3.2, we must establish the following:

(a) The functor f^* belongs to $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$.

(b) Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fracture subcategory and suppose that, for each $\alpha \leqslant \beta$ in A, the associated natural transformation $f^*_\alpha \to f^*_\beta$ is local. Then each of the natural transformations $f^*_\alpha \to f^*$ is local.

(c) In the situation of (b), suppose we are given a natural transformation $u : f^* \to g^*$, where $g^* \in \mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$. Suppose further that each of the induced transformations $u_\alpha : f^*_\alpha \to g^*$ is local. Then u is local.

Assertion (a) follows immediately from Lemma HTT.5.5.2.3 and Example HTT.7.3.4.7 . To prove (b), consider an arbitrary $U \to V$ in $\mathcal{E}^{\mathrm{corp}}$. We wish to show that the diagram σ :

$$\begin{array}{ccc} f^*_\alpha(U) & \longrightarrow & f^*(U) \\ \big\downarrow & & \big\downarrow \\ f^*_\alpha(V) & \longrightarrow & f^*(V) \end{array}$$

is a pullback square in $\mathcal{X}$. This is clear, since σ is a filtered colimit of pullback squares

$$\begin{array}{ccc} f_\alpha^*(U) & \longrightarrow & f_\beta^*(U) \\ \downarrow & & \downarrow \\ f_\alpha^*(V) & \longrightarrow & f_\beta^*(V) \end{array}$$

for $\beta \geqslant \alpha$. The proof of (c) is similar, using the fact that each diagram

$$\begin{array}{ccc} f^*(U) & \longrightarrow & g^*(U) \\ \downarrow & & \downarrow \\ f^*(V) & \longrightarrow & g^*(V) \end{array}$$

is a filtered colimit of diagrams of the form

$$\begin{array}{ccc} f_\alpha^*(U) & \longrightarrow & g^*(U) \\ \downarrow & & \downarrow \\ f_\alpha^*(V) & \longrightarrow & g^*(V) \end{array}$$

for $\alpha \in A$. □

21.4.4 Cartesian Colimits in $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$

We now consider the problem of constructing colimits of ∞-topoi equipped with structure sheaves. We begin with an easy observation.

Proposition 21.4.4.1. *Let K be a (small) simplicial set and let $\mathcal{C}$ be an ∞-category which admits K^{op}-indexed limits. Then the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathcal{C}}$ admits K-indexed colimits. Moreover, the forgetful functor $U : \infty\mathcal{T}\mathrm{op}_{\mathcal{C}} \to \infty\mathcal{T}\mathrm{op}$ preserves K-indexed colimits.*

Proof. Note that the ∞-category $\infty\mathcal{T}\mathrm{op}$ admits K-indexed colimits (Proposition HTT.6.3.2.3). By virtue of Proposition HTT.4.3.1.5 , it will suffice to show that for commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & \infty\mathcal{T}\mathrm{op}_{\mathcal{C}} \\ \downarrow & \nearrow & \downarrow U \\ K^{\triangleright} & \longrightarrow & \infty\mathcal{T}\mathrm{op} \end{array}$$

admits an extension as indicated which is a U-colimit diagram. Using the criterion of Corollary HTT.4.3.1.11 , we are reduced to proving the following:

(i) For every ∞-topos $\mathcal{X}$, the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ admits K^{op}-indexed limits.

(*ii*) For every geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{Y}$, the direct image functor $f_*^{\mathcal{C}} : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{Y})$ (given by pointwise composition with f^*) preserves K^{op}-indexed limits.

Both assertions follow immediately from the observation that $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ and $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{Y})$ are closed under K^{op}-indexed limits in the ∞-categories $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$ and $\mathrm{Fun}(\mathcal{Y}^{\mathrm{op}}, \mathcal{C})$, respectively. □

Let us review the proof of Proposition 21.4.4.1 in more informal language. To fix ideas, let us specialize to the case where $\mathcal{C} = \mathrm{CAlg}$ is the ∞-category of $\mathbb{E}_\infty$-rings. Suppose that we are given a diagram $\{(\mathcal{X}_I, \mathcal{O}_I)\}_{I \in \mathcal{I}}$ in the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ of spectrally ringed ∞-topoi, indexed by some (small) ∞-category $\mathcal{I}$. Then the underlying diagram $\{\mathcal{X}_I\}$ admits a colimit $\mathcal{X}$ in the ∞-category $\infty\mathcal{T}\mathrm{op}$. In particular, for each $I \in \mathcal{I}$ we have a tautological geometric morphism $f_{I*} : \mathcal{X}_I \to \mathcal{X}$. Then the colimit of the diagram $\{(\mathcal{X}_I, \mathcal{O}_I)\}_{I \in \mathcal{I}}$ is then given by the pair $(\mathcal{X}, \varprojlim_{I \in \mathcal{I}^{\mathrm{op}}} f_{I*} \mathcal{O}_I)$. Our analysis then depends on the following features of the theory of CAlg-valued sheaves:

(*a*) Every $\mathcal{C}$-valued sheaf $\mathcal{O}_I$ on $\mathcal{X}_I$ admits a direct image $f_{I*} \mathcal{O}_I$ on $\mathcal{X}$.

(*b*) The ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ of CAlg-valued sheaves on $\mathcal{X}$ admits small limits (which are preserved by further direct image functors).

Beware that if we replace $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ by the subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ of *locally* spectrally ringed ∞-topoi, then the analogues of both (*a*) and (*b*) fail: local sheaves of $\mathbb{E}_\infty$-rings are not stable under either direct images or inverse limits. Consequently, it is not reasonable to expect to expect an analogue of Proposition 21.4.4.1 for the ∞-categories $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ and $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ introduced in §21.4.1 and §21.4.2. We will therefore restrict our attention to the study of a special class of diagrams, where in some sense the structure sheaf is not varying.

Definition 21.4.4.2. Let $\mathcal{E}$ be an ∞-topos and let $U : \infty\mathcal{T}\mathrm{op}(\mathcal{E}) \to \infty\mathcal{T}\mathrm{op}$ be the Cartesian fibration of Construction 21.4.2.1. We will say that a morphism in $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ is *Cartesian* if it is U-Cartesian. We let $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E})$ denote the (non-full) subcategory of $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ spanned by the U-Cartesian morphisms.

Remark 21.4.4.3. Let $\mathcal{E}$ be an ∞-topos, and regard $\mathcal{E}$ as a fracture subcategory of itself (Example 20.1.2.2). Then the subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E}) \subseteq \infty\mathcal{T}\mathrm{op}(\mathcal{E})$ of Definition 21.4.4.2 coincides with the subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E}) \subseteq \infty\mathcal{T}\mathrm{op}(\mathcal{E})$ of Definition 21.4.2.6.

Remark 21.4.4.4. Let $\mathcal{E}$ be an ∞-topos equipped with an arbitrary fracture subcategory $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$. Then we have an inclusion $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E}) \subseteq \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$.

Proposition 21.4.4.5. *Let $\mathcal{E}$ be an ∞-topos. Then the forgetful functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E}) \to \infty\mathcal{T}\mathrm{op}$ is a right fibration classified by the object $\mathcal{E} \in \infty\mathcal{T}\mathrm{op}$. In other words, there is an* *equivalence of ∞-categories $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E}) \simeq \infty\mathcal{T}\mathrm{op}_{/\mathcal{E}}$ (compatible with the projection to $\infty\mathcal{T}\mathrm{op}$).*

Proof. The statement that the forgetful functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E}) \to \infty\mathcal{T}\mathrm{op}$ is a right fibration is a special case of Corollary HTT.2.4.2.5 . To show that it is classified by the ∞-topos $\mathcal{E}$, it will suffices to observe that the object $(\mathcal{E}, \mathrm{id}_{\mathcal{E}})$ is a final object of $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E})$ (see Proposition HTT.4.4.4.5), which follows easily from the definitions. □

Variant 21.4.4.6. Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site (Definition 20.6.2.1) and let $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ be the ∞-category of Construction 21.4.1.15. We let $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{G})$ denote the (non-full) subcategory of $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ spanned by the U-Cartesian morphisms, where $U : \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G}) \to \infty\mathcal{T}\mathrm{op}(\mathcal{G})$ is the forgetful functor.

Remark 21.4.4.7. Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site and let $\mathcal{E} = \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ be the associated ∞-topos. Then the equivalence $\infty\mathcal{T}\mathrm{op}(\mathcal{E}) \simeq \infty\mathcal{T}\mathrm{op}^{\tau}(\mathcal{G})$ of Example 21.4.2.4 restricts to an equivalence of ∞-categories $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E}) \simeq \infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{G})$. In particular, the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{G})$ depends only on the ∞-category $\mathcal{G}$ and the Grothendieck topology τ, and not on the choice of admissibility structure $\mathcal{G}^{\mathrm{ad}}$.

Remark 21.4.4.8. In the situation of Variant 21.4.4.6, a morphism from $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ to $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ can be identified with a pair (f_*, α), where $f_* : \mathcal{X} \to \mathcal{Y}$ is a geometric morphism of ∞-topoi and $\alpha : f^* \mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$ is a local morphism of $\mathcal{G}$-objects. The pair (f_*, α) belongs to $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{G})$ if and only if the morphism α is an equivalence.

We can now state the main result of this section:

Proposition 21.4.4.9. *Let $\mathcal{E}$ be an ∞-topos. Then:*

(1) *The ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E})$ admits small colimits.*

(2) *The forgetful functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E}) \to \infty\mathcal{T}\mathrm{op}$ preserves small colimits.*

(3) *The inclusion functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E}) \hookrightarrow \infty\mathcal{T}\mathrm{op}(\mathcal{E})$ preserves small colimits.*

(4) *For each fracture subcategory $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$, the inclusion $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E}) \hookrightarrow \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ preserves small colimits.*

Warning 21.4.4.10. In the situation of Proposition 21.4.4.9, the ∞-categories $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ and $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ need not admit small colimits.

Corollary 21.4.4.11. *Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site. Then the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{G})$ admits small colimits, which are preserved by the inclusion functors $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G}) \hookleftarrow \infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{G}) \hookrightarrow \infty\mathcal{T}\mathrm{op}^{\tau}(\mathcal{G})$ and by the forgetful functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{G}) \to \infty\mathcal{T}\mathrm{op}$.*

Proof. Combine Proposition 21.4.4.9 with Remark 21.4.4.7. □

We will deduce Proposition 21.4.4.9 from the following general categorical principle, whose proof we defer to the end of this section:

Lemma 21.4.4.12. *Let $p : X \to S$ be a coCartesian fibration of simplicial sets classified by a diagram $\chi : S \to \mathcal{C}\mathrm{at}_\infty$. Let $q : K^{\triangleleft} \to X$ be a diagram with the following properties:*

- (a) *The composition $\chi \circ p \circ q : K^{\triangleleft} \to \mathcal{C}\mathrm{at}_\infty$ is a limit diagram.*
- (b) *The diagram q carries each edge of $K^{\triangleleft}$ to a p-coCartesian morphism in X.*

Then q is a p-limit diagram.

Proof of Proposition 21.4.4.9. According to Proposition HTT.6.3.2.3 , the ∞-category $\infty\mathcal{T}\mathrm{op}$ admits small colimits. It follows that, for any ∞-topos $\mathcal{E}$, the ∞-category $\infty\mathcal{T}\mathrm{op}_{/\mathcal{E}}$ also admits small colimits, which are preserved by the forgetful functor $\infty\mathcal{T}\mathrm{op}_{/\mathcal{E}} \to \infty\mathcal{T}\mathrm{op}$ (see Proposition HTT.1.2.13.8). Assertions (1) and (2) of Proposition 21.4.4.9 now follow from Proposition 21.4.4.5. To prove (3) and (4), it will suffice (by virtue of Lemma 21.4.4.12) to establish the following:

- $(3')$ For any ∞-topos $\mathcal{E}$, the construction $\mathcal{X} \mapsto \mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$ determines a functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which preserves small limits.
- $(4')$ For any fractured subcategory $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$, the construction $\mathcal{X} \mapsto \mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{E}, \mathcal{X})$ determines a functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which preserves small limits.

Since the forgetful functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ preserves small limits (Proposition HTT.6.3.2.1), it follows immediately from the definitions that the construction $\mathcal{X} \mapsto \mathrm{Fun}(\mathcal{E}, \mathcal{X})$ preserves small limits. We may therefore deduce $(3')$ and $(4')$ from the following elementary observations:

- $(3'')$ Let $\mathcal{X}$ be the colimit of a diagram $\{\mathcal{X}_\alpha\}$ in $\infty\mathcal{T}\mathrm{op}$. Then a functor $f^* : \mathcal{E} \to \mathcal{X}$ preserves small colimits and finite limits if and only if, for each index α, the composite functor
$$\mathcal{E} \xrightarrow{f^*} \mathcal{X} \to \mathcal{X}_\alpha$$
preserves small colimits and finite limits.
- $(4'')$ Let $\mathcal{X}$ be the colimit of a diagram $\{\mathcal{X}_\alpha\}$ in $\infty\mathcal{T}\mathrm{op}$, and let $u : f^* \to g^*$ be a morphism in $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$. Then u is local (with respect to some fracture subcategory $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$) if and only if, for each index α, the image of u in $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X}_\alpha)$ is local.

□

Proof of Lemma 21.4.4.12. Using Corollary HTT.3.3.1.2, we may reduce to the case where S is an ∞-category (so that X is also an ∞-category). Choose a categorical equivalence $K \to K'$ which is a monomorphism of simplicial sets, where K' is an ∞-category. Since X is an ∞-category, the map q factors through $K'^{\triangleleft}$. We may therefore replace K by K' and thereby reduce to the case where K is an ∞-category. In view of Corollary HTT.4.3.1.15, we may replace S by $K^{\triangleleft}$ (and X by the pullback $X \times_S K^{\triangleleft}$) and thereby reduce to the case where $p \circ q$ is an isomorphism.

Consider the map $\pi : K^{\triangleleft} \to (\Delta^0)^{\triangleleft} \simeq \Delta^1$. Since K is an ∞-category, the map π is a Cartesian fibration of simplicial sets. Let $p' : \mathcal{C} \to \Delta^1$ be the "pushforward" of the coCartesian fibration p, so that $\mathcal{C}$ is characterized by the universal mapping property

$$\mathrm{Hom}_{\Delta^1}(Y, \mathcal{C}) \simeq \mathrm{Hom}_{K^{\triangleleft}}(Y \times_{\Delta^1} K^{\triangleleft}, X).$$

Corollary HTT.3.2.2.12 implies that p' is a coCartesian fibration, associated to some functor f from $\mathcal{C}_0 = \mathcal{C} \times_{\Delta^1} \{0\}$ to $\mathcal{C}_1 = \mathcal{C} \times_{\Delta^1} \{1\}$. We can identify $\mathcal{C}_0$ with the fiber of p over the cone point of $K^{\triangleleft}$, and $\mathcal{C}_1$ with the ∞-category of sections of p over K. Let $\mathcal{C}'_1$ denote the full subcategory of $\mathcal{C}_1$ spanned by the coCartesian sections. Combining Corollary HTT.3.2.2.12, Proposition HTT.3.3.3.1, and assumption (a), we deduce that f determines an equivalence $\mathcal{C}_0 \to \mathcal{C}'_1$.

Let $q_0 = q|_K$. We can identify q_0 with an object $C \in \mathcal{C}_1$, and q with a morphism $\alpha : C' \to C$ in $\mathcal{C}$. We then have a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_0 \times_X X_{/q} & \longrightarrow & \mathcal{C}_0 \times_X X_{/q_0} \\ \downarrow & & \downarrow \\ \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/\alpha} & \longrightarrow & \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/C}. \end{array}$$

We wish to show that the upper horizontal map is a categorical equivalence. Since the vertical maps are isomorphisms, it will suffice to show that the lower horizontal map is a categorical equivalence. In other words, we wish to show that for every object $C_0 \in \mathcal{C}_0$, composition with α induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}_0}(C_0, C') \to \mathrm{Map}_{\mathcal{C}}(C_0, C)$. We have a commutative diagram (in the homotopy category of spaces)

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}_0}(C_0, C') & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(C_0, C) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}_1}(fC_0, fC') & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(fC_0, C). \end{array}$$

Here the right vertical map is a homotopy equivalence. Since f is fully faithful, the left vertical map is also a homotopy equivalence. It therefore suffices to show that the bottom horizontal map is a homotopy equivalence: in other words, that α induces an equivalence $fC' \to C$. This is simply a translation of condition (b). $\square$

21.4.5 Clutching for Structured Spaces

Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos. In §21.4.4, we proved that every small diagram in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{Cart}}(\mathcal{E})$ admits a colimit in the larger ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ (Proposition 21.4.4.9). The existence of colimits for general diagrams in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ is more subtle. However, we do have the following partial result:

Proposition 21.4.5.1. *Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos, and suppose we are given a diagram*

$$(\mathcal{X}_0, \mathcal{O}_0) \leftarrow (\mathcal{X}_{01}, \mathcal{O}_{01}) \rightarrow (\mathcal{X}_1, \mathcal{O}_1)$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$. Assume that the underlying geometric morphisms $\mathcal{X}_0 \xleftarrow{f_*} \mathcal{X}_{01} \xrightarrow{g_*} \mathcal{X}_1$ are closed immersions of ∞-topoi. Then there exists a pushout square σ :

$$\begin{array}{ccc} (\mathcal{X}_{01}, \mathcal{O}_{01}) & \longrightarrow & (\mathcal{X}_0, \mathcal{O}_0) \\ \downarrow & & \downarrow \\ (\mathcal{X}_1, \mathcal{O}_1) & \longrightarrow & (\mathcal{X}, \mathcal{O}) \end{array}$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$. Moreover:

(1) *The underlying diagram of ∞-topoi*

$$\begin{array}{ccc} \mathcal{X}_{01} & \xrightarrow{f_*} & \mathcal{X}_0 \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g'_*} \\ \mathcal{X}_1 & \xrightarrow{f'_*} & \mathcal{X} \end{array}$$

is a pushout square in $\infty\mathcal{T}\mathrm{op}$; let us denote the diagonal composition by $h_* : \mathcal{X}_{01} \to \mathcal{X}$.

(2) *For every corporeal object $T \in \mathcal{E}$, the natural map*

$$\mathcal{O}^T \to g'_* \, \mathcal{O}_0^T \times_{h_* \, \mathcal{O}_{01}^T} f'_* \, \mathcal{O}_1^T$$

is an equivalence in the ∞-topos $\mathcal{X}$.

(3) *The diagram σ is also a pushout square in $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$.*

Corollary 21.4.5.2. *Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site and suppose we are given a diagram*

$$(\mathcal{X}_0, \mathcal{O}_0) \leftarrow (\mathcal{X}_{01}, \mathcal{O}_{01}) \rightarrow (\mathcal{X}_1, \mathcal{O}_1)$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$, where the underlying geometric morphisms $\mathcal{X}_0 \xleftarrow{f_*} \mathcal{X}_{01} \xrightarrow{g_*} \mathcal{X}_1$ are closed immersions. Then there exists a pushout square σ:

$$\begin{array}{ccc} (\mathcal{X}_{01}, \mathcal{O}_{01}) & \longrightarrow & (\mathcal{X}_0, \mathcal{O}_0) \\ \downarrow & & \downarrow \\ (\mathcal{X}_1, \mathcal{O}_1) & \longrightarrow & (\mathcal{X}, \mathcal{O}) \end{array}$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$. Moreover:

(1) *The underlying diagram of ∞-topoi*

$$\begin{array}{ccc} \mathcal{X}_{01} & \xrightarrow{f_*} & \mathcal{X}_0 \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g'_*} \\ \mathcal{X}_1 & \xrightarrow{f'_*} & \mathcal{X} \end{array}$$

is a pushout square in $\infty\mathcal{T}\mathrm{op}$; let us denote the diagonal composition by $h_* : \mathcal{X}_{01} \to \mathcal{X}$.

(2) *For every object $T \in \mathcal{G}$, the natural map*

$$\mathscr{O}^T \to g'_* \mathscr{O}_0^T \times_{h_* \mathscr{O}_{01}^T} f'_* \mathscr{O}_1^T$$

is an equivalence in the ∞-topos $\mathcal{X}$.

(3) *The diagram σ is also a pushout square in $\infty\mathcal{T}\mathrm{op}(\mathcal{G})$.*

Proof. Set $\mathcal{E} = \mathcal{S}\mathrm{hv}_\tau(\mathcal{G})$, and regard $\mathcal{E}$ as equipped with the fracture subcategory described in Theorem 20.6.3.4. Then the sheafified Yoneda embedding $h : \mathcal{G} \to \mathcal{E}$ is a presentation of $\mathcal{E}$. Using Proposition 20.5.4.1, we deduce that the collection of corporeal objects of $\mathcal{E}$ is closed under finite limits. The desired result now follows from Proposition **??** and Example **??**. □

The rest of this section is devoted to the proof of Proposition 21.4.5.1. We begin by treating the special case where $\mathcal{X}_0 = \mathcal{X}_{01} = \mathcal{X}_1$.

Lemma 21.4.5.3. *Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos, let $h : \mathcal{G} \to \mathcal{E}$ be a presentation of $\mathcal{E}$ (Definition 20.5.3.1). Let $\mathcal{X}$ be an ∞-topos, so that composition with h determines a fully faithful embedding $\iota : \mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \to \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$. Suppose we are given a diagram*

$$\mathscr{O}_0 \to \mathscr{O}_{01} \leftarrow \mathscr{O}_1$$

in the ∞-category $\mathrm{Obj}_{\mathcal{G}}^{\mathrm{loc}}(\mathcal{X})$, and set $\mathscr{O} = \mathscr{O}_0 \times_{\mathscr{O}_{01}} \mathscr{O}_1$ (where the fiber product is formed in the ∞-category $\mathrm{Fun}(\mathcal{G}, \mathcal{X})$). If $\mathscr{O}_0$, $\mathscr{O}_{01}$, and $\mathscr{O}_1$ belong to the essential image of ι, then so does ι. Moreover, the diagram

$$\begin{array}{ccc} \mathscr{O} & \longrightarrow & \mathscr{O}_0 \\ \downarrow & & \downarrow \\ \mathscr{O}_1 & \longrightarrow & \mathscr{O}_{01} \end{array}$$

is a pullback square in $\mathrm{Obj}_{\mathcal{G}}^{\mathrm{loc}}(\mathcal{X})$.

Proof. Since the map $\mathscr{O}_1 \to \mathscr{O}_{01}$ is local, the pullback map $\mathscr{O} \to \mathscr{O}_0$ satisfies the hypothesis of Lemma 21.3.4.1. It follows that $\mathscr{O}$ belongs to the essential image of ι and that the map $\mathscr{O} \to \mathscr{O}_0$ is local. Similarly, the projection map $\mathscr{O} \to \mathscr{O}_1$ is local. To complete the proof, it will suffice to show that for any other object $\mathscr{O}' \in \mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$, a natural transformation $\alpha : \mathscr{O}' \to \mathscr{O}$ is local if and only if composite maps

$$\alpha_0 : \mathscr{O}' \to \mathscr{O} \to \mathscr{O}_0 \qquad \alpha_1 : \mathscr{O}' \to \mathscr{O} \to \mathscr{O}_1$$

are both local. The "only if" direction is trivial. For the converse, note that if α_0 is local and $U \to T$ is an admissible morphism in $\mathcal{G}$, then we have a commutative diagram

$$\begin{array}{ccccc} \mathscr{O}'^U & \longrightarrow & \mathscr{O}^U & \longrightarrow & \mathscr{O}_0^U \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{O}'^T & \longrightarrow & \mathscr{O}^T & \longrightarrow & \mathscr{O}_0^T \end{array}$$

where the outer rectangle and the right square are pullbacks, so that the left square is a pullback as well. □

To reduce Proposition 21.4.5.1 to the situation of Lemma 21.4.5.3, we will need the following:

Lemma 21.4.5.4. *Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos, let $h : \mathcal{G} \to \mathcal{E}$ be a presentation of $\mathcal{E}$, and let $\mathcal{X}$ be an ∞-topos containing a (-1)-truncated object U. Let $i^* : \mathcal{X} \to \mathcal{X} \backslash U$ and $j^* : \mathcal{X} \to \mathcal{X}_{/U}$ be the closed and open immersions associated to U. Then composition with h determines fully faithful embeddings*

$$\iota : \mathrm{Fun}^*(\mathcal{E}, \mathcal{X}) \to \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}) \quad \iota' : \mathrm{Fun}^*(\mathcal{E}, \mathcal{X} \backslash U) \to \mathrm{Obj}_{\mathcal{G}}(\mathcal{X} \backslash U) \quad \iota'' : \mathrm{Fun}^*(\mathcal{E}, \mathcal{X}_{/U}) \to \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}_{/U}).$$

Then:

(1) *A functor $\mathscr{O} : \mathcal{G} \to \mathcal{X}$ belongs to the essential image of ι if and only if $i^* \mathscr{O}$ and $j^* \mathscr{O}$ belong to the essential images of ι' and ι'', respectively.*

(2) *Let $\alpha : \mathscr{O} \to \mathscr{O}'$ be a morphism in $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X})$. Then α is local if and only if $i^*\alpha$ is a local morphism of $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X} \backslash U)$ and $j^*\alpha$ is a local morphism of $\mathrm{Obj}_{\mathcal{G}}(\mathcal{X}_{/U})$.*

Proof. We will prove (1); the proof of (2) is similar. The "only if" direction is obvious. Conversely, assume that $i^* \mathscr{O}$ and $j^* \mathscr{O}$ belong to the essential images of ι' and ι'', respectively. Extend $\mathscr{O}$ to a colimit-preserving functor $F : \mathcal{P}(\mathcal{G}) \to \mathcal{X}$, and the presentation h to a colimit-preserving functor $H : \mathcal{P}(\mathcal{G}) \to \mathcal{E}$. Our assumption guarantees that the functors i^*F and j^*F are left exact, so that F is left exact by virtue of Lemma **??**. To complete the proof, it will suffice to show that the functor F factors (up to homotopy) through the localization functor H. In other words, we must show that if α is a morphism in $\mathcal{P}(\mathcal{G})$ for which $H(\alpha)$ is an equivalence, then $F(\alpha)$ is also an equivalence. By assumption, the functors i^*F and j^*F factor (up to homotopy) through H, so that $i^*F(\alpha)$ and $j^*F(\alpha)$ are equivalences. Applying Lemma **??**, we deduce that $F(\alpha)$ is an equivalence, as desired. □

Proof of Proposition 21.4.5.1. Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos. Using Theorem 20.5.3.4, we can choose a presentation $\rho : \mathcal{G} \to \mathcal{E}$. Suppose we are given a pushout diagram of ∞-topoi

$$(\mathcal{X}_0, \mathcal{O}_0) \leftarrow (\mathcal{X}_{01}, \mathcal{O}_{01}) \rightarrow (\mathcal{X}_1, \mathcal{O}_1)$$

in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$, where the underlying geometric morphisms $\mathcal{X}_0 \xleftarrow{f_*} \mathcal{X}_{01} \xrightarrow{g_*} \mathcal{X}_1$ are closed immersions with complementary open immersions $j_* : \mathcal{U}_0 \to \mathcal{X}_0$ and $j'_* : \mathcal{U}_1 \to \mathcal{X}_1$. Form a pushout diagram of ∞-topoi

$$\begin{array}{ccc} \mathcal{X}_{01} & \xrightarrow{f_*} & \mathcal{X}_0 \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g'_*} \\ \mathcal{X}_1 & \xrightarrow{f'_*} & \mathcal{X} \end{array}$$

and denote the diagonal composition by $h_* : \mathcal{X}_{01} \to \mathcal{X}$. Set $\overline{\mathcal{O}}_0 = \mathcal{O}_0 \circ \rho \in \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}_0)$, and define $\overline{\mathcal{O}}_1 \in \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}_1)$ and $\overline{\mathcal{O}}_{01} \in \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}_{01})$ similarly. Let $\overline{\mathcal{O}} : \mathcal{G} \to \mathcal{X}$ be the functor given by the formula

$$\overline{\mathcal{O}}^T = g'_* \overline{\mathcal{O}}_0^T \times_{h_* \overline{\mathcal{O}}_{01}^T} f'_* \overline{\mathcal{O}}_1^T.$$

It follows from Lemma 21.4.5.3 that the pullback

$$h^* \overline{\mathcal{O}} \simeq f^* \overline{\mathcal{O}}_0 \times_{\overline{\mathcal{O}}_{01}} g^* \overline{\mathcal{O}}_1$$

belongs to the essential image of the embedding $\mathrm{Fun}^*(\mathcal{E}, \mathcal{X}_{01}) \to \mathrm{Obj}_{\mathcal{G}}(\mathcal{X}_{01})$. Moreover, $j^* g'^* \overline{\mathcal{O}} \simeq j^* \overline{\mathcal{O}}_0$ and $j'^* f'^* \overline{\mathcal{O}} \simeq j'^* \overline{\mathcal{O}}_1$ belong to the essential images of the embeddings $\mathrm{Fun}^*(\mathcal{E}, \mathcal{U}) \to \mathrm{Obj}_{\mathcal{G}}(\mathcal{U})$ and $\mathrm{Fun}^*(\mathcal{E}, \mathcal{U}') \to \mathrm{Obj}_{\mathcal{G}}(\mathcal{U}')$. Applying Lemma 21.4.5.4 twice, we deduce that $\overline{\mathcal{O}} \simeq \mathcal{O} \circ \rho$ for some $\mathcal{O} \in \mathrm{Fun}^*(\mathcal{E}, \mathcal{X})$. By construction, we have a commutative diagram σ :

$$\begin{array}{ccc} (\mathcal{X}_{01}, \mathcal{O}_{01}) & \longrightarrow & (\mathcal{X}_0, \mathcal{O}_0) \\ \downarrow & & \downarrow \\ (\mathcal{X}_1, \mathcal{O}_1) & \longrightarrow & (\mathcal{X}, \mathcal{O}) \end{array}$$

in the ∞-category $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$.

We now claim that σ is a pushout square in $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$. To prove this, let $\mathsf{Y} = (\mathcal{Y}, \mathcal{O}')$ be an arbitrary object of $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$. Set $\mathsf{X} = (\mathcal{X}, \mathcal{O}) \in \infty\mathcal{T}\mathrm{op}(\mathcal{E})$, and define $\mathsf{X}_0, \mathsf{X}_1, \mathsf{X}_{01} \in \infty\mathcal{T}\mathrm{op}(\mathcal{E})$

similarly. We then have a commutative diagram of spaces τ :

$$\begin{array}{ccc} \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}(\mathcal{E})}(\mathsf{X},\mathsf{Y}) & \longrightarrow & \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}(\mathcal{E})}(\mathsf{X}_0,\mathsf{Y}) \times_{\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}(\mathcal{E})}(\mathsf{X}_{01},\mathsf{Y})} \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}(\mathcal{E})}(\mathsf{X}_1,\mathsf{Y}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X},\mathcal{Y}) & \longrightarrow & \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}_0,\mathcal{Y}) \times_{\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}_{01},\mathcal{Y})} \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}_1,\mathcal{Y}) \end{array}$$

and we wish to show that the upper horizontal map is a homotopy equivalence. The lower horizontal map is a homotopy equivalence by construction, so it will suffice to show that τ is a homotopy pullback square: that is, that τ induces a homotopy equivalence after passing to the vertical homotopy fibers over any point $\eta_*; \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X},\mathcal{Y})$. Equivalently, we wish to show that the diagram of spaces τ':

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Fun}^*(\mathcal{E},\mathcal{X})}(\eta^*\,\mathcal{O}',\mathcal{O}) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}^*(\mathcal{E},\mathcal{X}_0)}(g'^*\eta^*\,\mathcal{O}',\mathcal{O}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{Fun}^*(\mathcal{E},\mathcal{X}_1)}(f'^*\eta^*\,\mathcal{O}',\mathcal{O}) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}^*(\mathcal{E},\mathcal{X}_{01})}(h^*\eta^*\,\mathcal{O}',\mathcal{O}) \end{array}$$

is a pullback square. Set $\overline{\mathcal{O}}' = \mathcal{O}' \circ \rho \in \mathrm{Obj}_{\mathcal{G}}(\mathcal{Y})$. Using our assumption that $\rho : \mathcal{G} \to \mathcal{E}$ is a presentation, we can rewrite the diagram τ' as

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Fun}(\mathcal{G},\mathcal{X})}(\eta^*\overline{\mathcal{O}}',\overline{\mathcal{O}}) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathcal{G},\mathcal{X})}(\eta^*\overline{\mathcal{O}}',g'_*\overline{\mathcal{O}}_0) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{Fun}(\mathcal{G},\mathcal{X})}(\eta^*\overline{\mathcal{O}}',f'_*\overline{\mathcal{O}}_1) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathcal{G},\mathcal{X})}(\eta^*\overline{\mathcal{O}}',h_*\overline{\mathcal{O}}_{01}), \end{array}$$

which is a pullback square by assumption.

We now claim that the diagram σ is also a pullback square in the subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$. To prove this, it will suffice to show that a morphism $(\mathcal{X},\mathcal{O}) \to (\mathcal{Y},\mathcal{O}')$ in $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ belongs to $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ if and only if the composite maps $(\mathcal{X}_i,\mathcal{O}_i) \to (\mathcal{X},\mathcal{O}) \to (\mathcal{Y},\mathcal{O}')$ belong to $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ for $i = 0$ and $i = 1$. This follows immediately from Lemmas 21.4.5.3 and 21.4.5.4.

We have now proven assertions (1) and (3) of Proposition **??**, along with the following weaker version of (2):

(2′) For every object $T \in \mathcal{E}$ which belongs to the essential image of $\rho : \mathcal{G} \to \mathcal{E}$, the natural map

$$\mathcal{O}^T \to g'_*\,\mathcal{O}_0^T \times_{h_*\,\mathcal{O}_{01}^T} f'_*\,\mathcal{O}_1^T$$

is an equivalence in the ∞-topos $\mathcal{X}$.

To conclude that assertion (2) holds in general, it suffices to observe that every corporeal object of $\mathcal{E}$ belongs to the essential image of *some* presentation $\rho' : \mathcal{G}' \to \mathcal{E}$ (see Remark 20.5.3.7).

□

21.4.6 Étale Morphisms of Structured ∞-Topoi

We now specialize our attention to a particularly useful class of Cartesian morphisms.

Definition 21.4.6.1. Let $\mathcal{E}$ be an ∞-topos. We will say that a morphism f in the ∞-category $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ is *étale* if it is Cartesian (in the sense of Definition 21.4.4.2) and the image of f under the forgetful functor $\infty\mathcal{T}\mathrm{op}(\mathcal{E}) \to \infty\mathcal{T}\mathrm{op}$ is an étale morphism of ∞-topoi. We let $\infty\mathcal{T}\mathrm{op}^{\text{ét}}(\mathcal{E})$ denote the (non-full) subcategory of $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ spanned by the étale morphisms.

Remark 21.4.6.2. Let $\mathcal{E}$ be an ∞-topos. Every étale morphism in $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ is Cartesian (Remark 21.4.4.4), and therefore belongs to $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ for any fracture subcategory $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$.

Variant 21.4.6.3. Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site. We will say that a morphism $f : (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is *étale* if it is Cartesian (in the sense of Variant 21.4.4.6) and the underlying geometric morphism $\mathcal{Y} \to \mathcal{X}$ is étale. Equivalently, f is étale if it determines an equivalence $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \simeq (\mathcal{X}_{/X}, \mathscr{O}_{\mathcal{X}}|_X)$ for some object $X \in \mathcal{X}$ (see Notation 21.2.1.7). We let $\infty\mathcal{T}\mathrm{op}^{\text{ét}}(\mathcal{G})$ denote the (non-full) subcategory of $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ spanned by the étale morphisms.

Some of basic properties of the class of étale morphisms are summarized in the following result:

Proposition 21.4.6.4. *Let $\mathcal{E}$ be an ∞-topos. Then:*

(1) *Every equivalence in the ∞-category $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$ is étale.*

(2) *Suppose given a commutative diagram*

$$\begin{array}{ccccc} & & (\mathcal{X}', \mathscr{O}_{\mathcal{X}'}) & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) & & \xrightarrow{\quad h \quad} & & (\mathcal{X}'', \mathscr{O}_{\mathcal{X}''}) \end{array}$$

in $\infty\mathcal{T}\mathrm{op}(\mathcal{E})$, where g is étale. Then f is étale if and only if h is étale.

(3) *The ∞-category $\infty\mathcal{T}\mathrm{op}^{\text{ét}}(\mathcal{E})$ admits small colimits.*

(4) *Small colimits in* $\infty\mathcal{T}\mathrm{op}^{\acute{e}t}(\mathcal{E})$ *are preserved by the forgetful functor* $\infty\mathcal{T}\mathrm{op}^{\acute{e}t}(\mathcal{E})$*, the inclusion functor* $\infty\mathcal{T}\mathrm{op}^{\acute{e}t}(\mathcal{E}) \hookrightarrow \infty\mathcal{T}\mathrm{op}(\mathcal{E})$*, and the inclusion functor* $\infty\mathcal{T}\mathrm{op}^{\acute{e}t}(\mathcal{E}) \hookrightarrow \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ *for any choice of fracture subcategory* $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$*.*

(5) *Fix an object* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \infty\mathcal{T}\mathrm{op}(\mathcal{E})$*. Then the* ∞*-category* $(\infty\mathcal{T}\mathrm{op}^{\acute{e}t}(\mathcal{E}))_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}$ *is canonically equivalent to* $\mathcal{X}$*.*

Proof. Assertion (1) is obvious. Assertion (2) follows from Proposition HTT.2.4.1.7 and Corollary HTT.6.3.5.9. Assertions (3) and (4) follow from Proposition 21.4.4.9 and Theorem HTT.6.3.5.13. Assertion (5) follows from Remark HTT.6.3.5.10. □

Remark 21.4.6.5. Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site and let $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a morphism in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$. The condition that f is étale is local in the following sense: if the ∞-topos $\mathcal{Y}$ admits a covering by objects $U_\alpha \in \mathcal{Y}$ for which each of the induced maps $(\mathcal{Y}_{/U_\alpha}, \mathcal{O}_{\mathcal{Y}}|_{U_\alpha}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is étale, then f is étale. To prove this, we let $\mathcal{Y}_0$ denote the full subcategory of $\mathcal{X}$ spanned by those objects U for which the map $(\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}}|_U) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is étale. Proposition 21.4.6.4 implies that $\mathcal{Y}_0$ is stable under the formation of small colimits. In particular, $U_0 = \amalg U_\alpha$ belongs to $\mathcal{Y}_0$. Let $U_\bullet$ be the simplicial object of $\mathcal{Y}$ given by the Čech nerve of the effective epimorphism $U_0 \to \mathbf{1}_{\mathcal{Y}}$. Since $\mathcal{Y}_0$ is a sieve, we deduce that each $U_n \in \mathcal{Y}_0$. Then $|U_\bullet| \simeq \mathbf{1}_{\mathcal{Y}} \in \mathcal{Y}_0$, so that f is étale as desired.

Remark 21.4.6.6. Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site and let $f : (\mathcal{X}_{/X}, \mathcal{O}_{\mathcal{X}}|_X) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be an étale morphism in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$. Suppose we are given some other morphism $g : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$. Combining the fact that f is U-Cartesian (where $U : \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G}) \to \infty\mathcal{T}\mathrm{op}$ is the forgetful functor) with the universal property of the ∞-topos $\mathcal{X}_{/X}$ given by Remark HTT.6.3.5.7, we deduce the existence of a canonical fiber sequence

$$\Gamma(\mathcal{Y}; g^*X) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathcal{X}, \mathcal{O}_{\mathcal{X}})),$$

where $\Gamma(\mathcal{Y}; \bullet) : \mathcal{Y} \to \mathcal{S}$ denotes the global sections functor.

Remark 21.4.6.7. Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site and let $g : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a morphism in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$. For any object $X \in \mathcal{X}$, the diagram

$$\begin{array}{ccc} (\mathcal{Y}_{/g^*X}, \mathcal{O}_{\mathcal{Y}}|_{g^*X}) & \longrightarrow & (\mathcal{X}_{/X}, \mathcal{O}_{\mathcal{X}}|_X) \\ \downarrow & & \downarrow \\ (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) & \longrightarrow & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \end{array}$$

is a pullback square in $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{G})$ (whose vertical maps are étale). This follows immediately from the universal property of Remark 21.4.6.6.

21.4.7 Corporeal Realization

Let Aff denote the category of affine schemes of finite type over $\mathbf{Z}$ and let $\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}(\mathrm{Aff})$ denote the ∞-topos of $\mathcal{S}$-valued sheaves with respect to the Zariski topology on Aff. Example 21.4.1.17 supplies a functor

$$\theta : \mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}} \to \infty\mathcal{T}\mathrm{op}(\mathrm{Aff}),$$

where $\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}}$ denotes the category of locally ringed topological spaces (Definition 1.1.5.1) and $\infty\mathcal{T}\mathrm{op}(\mathrm{Aff})$ denotes the ∞-category of $\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}(\mathrm{Aff})$-structured ∞-topoi. Since every scheme X can be regarded as a locally ringed space, we can compose θ with the inclusion map $\mathrm{Aff} \hookrightarrow \mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^{\heartsuit}}$ to obtain a functor $\mathrm{Aff} \hookrightarrow \infty\mathcal{T}\mathrm{op}(\mathrm{Aff})$. Our goal in this section is to show that a similar phenomenon occurs if we replace $\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}(\mathrm{Aff})$ by an arbitrary fractured ∞-topos.

Construction 21.4.7.1 (Corporeal Realization)**.** Let $\mathcal{E}$ be a fractured ∞-topos. For each object $X \in \mathcal{E}$, we let $\mathscr{O}_X$ denote a right adjoint to the forgetful functor $\mathcal{E}^{\mathrm{ad}}_{/X} \to \mathcal{E}$. Note that the functor $\mathscr{O}_X$ factors as a composition

$$\mathcal{E} \xrightarrow{\times X} \mathcal{E}_{/X} \xrightarrow{\rho} \mathcal{E}^{\mathrm{ad}}_{/X},$$

where ρ_X is a right adjoint to the inclusion $\mathcal{X}^{\mathrm{ad}}_{/X} \hookrightarrow \mathcal{X}_{/X}$ (see Definition 20.3.3.1). Since colimits in $\mathcal{X}$ are universal, the functor $\mathcal{E} \xrightarrow{\times X} \mathcal{E}_{/X}$ always preserves small colimits. If $X \in \mathcal{E}$ is corporeal, then the functor ρ_X also preserves small colimits (Example 20.3.3.2), so that $\mathscr{O}_X : \mathcal{E} \to \mathcal{E}^{\mathrm{ad}}_{/X}$ preserves small colimits and can therefore be regarded as an object of the ∞-category $\mathrm{Fun}^*(\mathcal{E}, \mathcal{E}^{\mathrm{ad}}_{/X})$ (note that the functor $\mathscr{O}_X$ automatically preserves small limits, since it is defined as a right adjoint). We may therefore regard the pair $(\mathcal{E}^{\mathrm{ad}}_{/X}, \mathscr{O}_X)$ as an object of the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$. We will denote this object by $\mathrm{Re}(X)$ and refer to it as the *corporeal realization* of X.

Remark 21.4.7.2. Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site (Definition 20.6.2.1) and let $h : \mathcal{G} \to \mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ be the sheafified Yoneda embedding. Then h carries each object $X \in \mathcal{G}$ to a corporeal object of $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ (where we regard $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})$ as a fractured ∞-topos via Theorem 20.6.3.4), and the ∞-topos $\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G})^{\mathrm{ad}}_{/X}$ can be identified with $\mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}}_{/X})$ (see Proposition 20.6.3.2). According to Proposition 21.2.4.10, composition with the functor h induces an equivalence of ∞-categories

$$\mathrm{Fun}^*_{\mathrm{loc}}(\mathcal{S}\mathrm{hv}_{\tau}(\mathcal{G}), \mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}}_{/X})) \simeq \mathrm{Obj}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{S}\mathrm{hv}_{\tau^{\mathrm{ad}}}(\mathcal{G}^{\mathrm{ad}}_{/X})).$$

Under this equivalence, the structure sheaf $\mathscr{O}_X$ of Construction 21.4.7.1 corresponds to the locally left exact functor $\mathcal{G} \to \mathcal{S}\mathrm{hv}_{\tau_X}(\mathcal{G}^{\mathrm{ad}}_{/X})$, which assigns to each $T \in \mathcal{G}$ the sheafification of the presheaf $(U \in \mathcal{G}^{\mathrm{ad}}_{/X}) \mapsto \mathrm{Map}_{\mathcal{G}}(U, T)$.

Example 21.4.7.3 (Algebraic Geometry: Étale Topology)**.** Let $(\mathrm{Aff}, \mathrm{Aff}^{\text{ét}}, \tau_{\text{ét}})$ denote the classical étale site (Example 20.6.4.2), and let $X = \operatorname{Spec} R$ be an object of Aff. It follows from Remark 21.4.7.2 that the corporeal realization $\mathrm{Re}(X)$ can be identified with the étale spectrum $\mathrm{Spét}\, R$ introduced in Definition 1.2.3.3.

Example 21.4.7.4 (Spectral Algebraic Geometry: Étale Topology)**.** Let $(\mathrm{Aff}_{\mathrm{Sp}}, \mathrm{Aff}^{\text{ét}}_{\mathrm{Sp}}, \tau_{\text{ét}})$ denote the spectral étale site (Example 20.6.4.5), and let $X = \operatorname{Spec} R$ be an object of $\mathrm{Aff}_{\mathrm{Sp}}$. It follows from Remark 21.4.7.2 that the corporeal realization $\mathrm{Re}(X)$ can be identified with the étale spectrum $\mathrm{Spét}\, R$ introduced in Definition 1.4.2.5.

Our main goal in this section is to show that for any fractured ∞-topos $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$, the construction $X \mapsto \mathrm{Re}(X)$ is functorial (Remark 21.4.7.8 and Proposition 21.4.7.10), and that a morphism $f : X \to Y$ between corporeal objects of $\mathcal{E}$ belongs to $\mathcal{E}^{\mathrm{corp}}$ if and only if the induced map $\mathrm{Re}(X) \to \mathrm{Re}(Y)$ is étale (Corollary 21.4.7.11). The main step will be to show that the corporeal realization $\mathrm{Re}(X)$ can be characterized by a universal property (Theorem 21.4.7.7).

Notation 21.4.7.5. [The Global Sections Functor] Let $\chi : \infty\mathcal{T}\mathrm{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the forgetful functor and let $Q : \overline{\infty\mathcal{T}\mathrm{op}}^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{op}}$ be a Cartesian fibration classified by χ (so that the objects of $\overline{\infty\mathcal{T}\mathrm{op}}$ can be identified with pairs $(\mathcal{X}, X)$, where $\mathcal{X}$ is an ∞-topos and $X \in \mathcal{X}$ is an object). For each ∞-topos $\mathcal{X}$, we let $\mathbf{1}_{\mathcal{X}}$ denote a final object of $\mathcal{X}$. We let $\Gamma : \overline{\infty\mathcal{T}\mathrm{op}}^{\mathrm{op}} \to \mathcal{S}$ denote the functor represented by the object $(\mathcal{S}, \mathbf{1}_{\mathcal{S}}) \in \overline{\infty\mathcal{T}\mathrm{op}}$. We will denote the value of Γ on a pair $(\mathcal{X}, X)$ by $\Gamma(\mathcal{X}; X)$, given informally by the formula $\Gamma(\mathcal{X}; X) = \mathrm{Map}_{\mathcal{X}}(\mathbf{1}_{\mathcal{X}}, X)$. We will refer to Γ as the *global sections functor*.

Notation 21.4.7.6. Let $\mathcal{E}$ be an ∞-topos. For each $X \in \mathcal{E}$, evaluation at X determines a functor

$$e_X : \infty\mathcal{T}\mathrm{op}(\mathcal{E}) \to \overline{\infty\mathcal{T}\mathrm{op}}$$
$$e_X(f^* : \mathcal{E} \to \mathcal{Y}) = (\mathcal{Y}, f^* X).$$

We let $\Gamma^X : \infty\mathcal{T}\mathrm{op}(\mathcal{E})^{\mathrm{op}} \to \mathcal{S}$ denote the composition of e_X with the global sections functor Γ of Notation 21.4.7.5. If $\mathcal{E}$ is equipped with a fracture subcategory $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$, then we will abuse notation by identifying Γ^X with its restriction to the subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})^{\mathrm{op}} \subseteq \infty\mathcal{T}\mathrm{op}(\mathcal{E})^{\mathrm{op}}$.

We can now formulate the universal property enjoyed by the corporeal realization $\mathrm{Re}(X)$:

Theorem 21.4.7.7. *Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos and let $X \in \mathcal{E}^{\mathrm{corp}}$ be a corporeal object. Then the functor $\Gamma^X : \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{S}$ is represented by the* $\mathrm{Re}(X) = (\mathcal{E}^{\mathrm{ad}}_{/X}, \mathcal{O}_X) \in \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$.

We postpone the proof of Theorem 21.4.7.7 for the moment; it will be deduced from a more general result (Theorem **??**) which we will prove in §**??**.

Remark 21.4.7.8 (Functoriality)**.** Let $\mathcal{E}$ be an ∞-topos and let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fracture subcategory. We let $\mathcal{E}_{\mathrm{corp}}$ denote the full subcategory of $\mathcal{E}$ spanned by the corporeal objects (that is, the essential image of the inclusion $\mathcal{E}^{\mathrm{corp}} \hookrightarrow \mathcal{E}$). Note that we always have $\mathcal{E}^{\mathrm{corp}} \subsetneq \mathcal{E}_{\mathrm{corp}} \subsetneq \mathcal{E}$, except in the trivial case where $\mathcal{E}^{\mathrm{corp}} = \mathcal{E}$ (see Warning 20.1.3.5).

Note that we can regard the map $\Gamma^X : \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{S}$ of Notation 21.4.7.6 as a covariant functor of $X \in \mathcal{E}$. Using Theorem 21.4.7.7 (and the fact that the Yoneda embedding $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E}) \to \mathrm{Fun}(\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})^{\mathrm{op}}, \widehat{\mathcal{S}})$ is fully faithful) that we can regard the construction $X \mapsto \mathrm{Re}(X)$ as a functor $\mathrm{Re} : \mathcal{E}_{\mathrm{corp}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$. We will refer to Re as the *corporeal realization functor*.

In particular, every morphism $X \to Y$ in the ∞-category $\mathcal{E}_{\mathrm{corp}}$ induces a map

$$(\mathcal{E}^{\mathrm{ad}}_{/X}, \mathcal{O}_X) \to (\mathcal{E}^{\mathrm{ad}}_{/Y}, \mathcal{O}_Y)$$

in $\infty\mathcal{T}\mathrm{op}(\mathcal{X})$, whose underlying geometric morphism is right adjoint to the functor

$$\mathcal{E}^{\mathrm{ad}}_{/Y} \to \mathcal{E}^{\mathrm{ad}}_{/X} \qquad U \mapsto U \times_Y X.$$

Remark 21.4.7.9. Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos. Then the construction $(X \in \mathcal{E}) \mapsto \Gamma^X$ of Notation 21.4.7.6 preserves finite limits. It follows that the corporeal realization functor $\mathrm{Re} : \mathcal{E}_{\mathrm{corp}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ preserves all finite limits which exist in $\mathcal{E}_{\mathrm{corp}}$.

Proposition 21.4.7.10. *Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos. Then the corporeal realization functor $\mathrm{Re} : \mathcal{E}_{\mathrm{corp}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ is fully faithful.*

Proof. Let X and Y be corporeal objects of $\mathcal{E}$, and let $\rho_X : \mathcal{E}_{/X} \to \mathcal{E}^{\mathrm{ad}}_{/X}$ be a right adjoint to the inclusion. Using Theorem 21.4.7.7, we compute

$$\begin{aligned}
\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})}(\mathrm{Re}(X), \mathrm{Re}(Y)) &\simeq \Gamma^Y(\mathcal{E}^{\mathrm{ad}}_{/X}; \mathcal{O}_X) \\
&= \Gamma(\mathcal{X}^{\mathrm{ad}}_{/X}; \mathcal{O}_X(Y)) \\
&\simeq \Gamma(\mathcal{E}^{\mathrm{ad}}_{/X}; \rho_X(X \times Y)) \\
&\simeq \mathrm{Map}_{\mathcal{E}^{\mathrm{ad}}_{/X}}(X, \rho_X(X \times Y)) \\
&\simeq \mathrm{Map}_{\mathcal{E}_{/X}}(X, X \times Y) \\
&\simeq \mathrm{Map}_{\mathcal{E}}(X, Y).
\end{aligned}$$

□

Corollary 21.4.7.11. *Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos and let $f : X \to Y$ be a morphism between corporeal objects of $\mathcal{X}$. The following conditions are equivalent:*

(1) *The morphism f belongs to $\mathcal{E}^{\mathrm{corp}}$.*

(2) *The induced map of corporeal realizations* $\mathrm{Re}(X) \to \mathrm{Re}(Y)$ *is an étale morphism in* $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$.

Proof. The implication (1) ⇒ (2) follows from the description of the map $\mathrm{Re}(X) \to \mathrm{Re}(Y)$ given in Remark 21.4.7.8. Conversely, suppose that (2) is satisfied. Then there exists an object $U \in \mathcal{E}^{\mathrm{ad}}_{/Y}$ such that $\mathrm{Re}(X)$ is equivalent to $((\mathcal{E}^{\mathrm{ad}}_{/Y})_{/U}, \mathscr{O}_Y|_U) \simeq (\mathcal{E}^{\mathrm{ad}}_{/U}, \mathscr{O}_U) = \mathrm{Re}(U)$ as an object of $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})_{/\mathrm{Re}(Y)}$. Applying Proposition 21.4.7.10, we deduce that X and U are equivalent as objects of $\mathcal{E}_{/Y}$, so that $X \in \mathcal{E}^{\mathrm{ad}}_{/Y}$. Using Example 20.3.1.2, we deduce that f belongs to $\mathcal{E}^{\mathrm{corp}}$. □

The implication (1) ⇒ (2) of Corollary 21.4.7.11 can be generalized to the situation where X and Y are not necessarily corporeal:

Proposition 21.4.7.12. *Let* $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ *be a fractured* ∞*-topos and let* $f : X \to Y$ *be a* $\mathcal{E}^{\mathrm{corp}}$*-admissible morphism in* $\mathcal{X}$ *(Definition 20.3.1.1). If the functor* Γ^Y *of Notation 21.4.7.6 is representable by an object* $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \in \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$*, then* Γ^X *is representable by an object of* $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ *which is étale over* $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$.

Corollary 21.4.7.13. *Let* $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ *be a fractured* ∞*-topos and let* $f : X \to Y$ *be a morphism in* $\mathcal{E}$*. The following conditions are equivalent:*

(a) *The morphism* f *is* $\mathcal{E}^{\mathrm{corp}}$*-admissible.*

(b) *For every pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

in the ∞*-topos* $\mathcal{E}$*, if the functor* $\Gamma^{Y'} : \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{S}$ *is representable by an object* $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}) \in \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$*, then* $\Gamma^{X'}$ *is representable by an object of* $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$ *which is étale over* $(\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$.

Proof. The implication (a) ⇒ (b) follows from Proposition 21.4.7.12. Conversely, suppose that (b) is satisfied. Choose a corporeal object $Y' \in \mathcal{X}$ equipped with an effective epimorphism $Y' \to Y$. To prove that f is $\mathcal{E}^{\mathrm{corp}}$-admissible, it will suffice to show that the projection map $Y' \times_Y X \to Y'$ is $\mathcal{E}^{\mathrm{corp}}$-admissible. Using Corollary 21.4.7.11, we are reduced to showing that the map of corporeal realizations $\mathrm{Re}(Y' \times_Y X) \to \mathrm{Re}(Y')$ is étale, which follows immediately from assumption (b). □

Remark 21.4.7.14. Let $\mathcal{E}^{\mathrm{corp}} \subseteq \mathcal{E}$ be a fractured ∞-topos. It follows from Corollary 21.4.7.13 that the admissibility structure $\mathcal{E}^{\mathrm{ad}} \subseteq \mathcal{E}$ of Proposition 20.3.1.3 can be recovered from the subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E}) \subseteq \infty\mathcal{T}\mathrm{op}(\mathcal{E})$ of Construction 21.4.2.1. Consequently, the subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E}) \subseteq \infty\mathcal{T}\mathrm{op}(\mathcal{E})$ determines the fracture subcategory $\mathcal{E}^{\mathrm{corp}}$ up to completion (see Remark 20.3.4.7).

Proof of Proposition 21.4.7.12. Suppose that $f : X \to Y$ is an $\mathcal{E}^{\mathrm{corp}}$-admissible morphism in $\mathcal{E}$ and that the functor $\Gamma^Y : \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E}) \to \mathcal{S}$ is representable by an object $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \in \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$. In particular, we have a canonical point $\eta \in \Gamma^Y(\mathcal{Y}; \mathcal{O}_{\mathcal{Y}}) \simeq \Gamma(\mathcal{Y}; \mathcal{O}_{\mathcal{Y}}(Y))$. Let U denote the fiber product $\mathbf{1}_{\mathcal{Y}} \times_{\mathcal{O}_{\mathcal{Y}}(Y)} \mathcal{O}_{\mathcal{Y}}(X)$, so that η lifts to an element $\overline{\eta} \in \Gamma^X(\mathcal{Y}_{/U}; \mathcal{O}_{\mathcal{Y}}|_U)$. For any object $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \in \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$, evaluation on $\overline{\eta}$ determines a map θ fitting into a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})}((\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}), (\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}}|_U),) & \xrightarrow{\theta} & \Gamma^X(\mathcal{Z}; \mathcal{O}_{\mathcal{Z}}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})}((\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})) & \xrightarrow{\theta_0} & \Gamma^Y(\mathcal{Z}; \mathcal{O}_{\mathcal{Z}}). \end{array}$$

By assumption, the map θ_0 is a homotopy equivalence. Passing to vertical homotopy fibers over a point

$$(f^*, \alpha_0) \in \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})}((\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}))$$

we obtain a map

$$\Gamma(\mathcal{Z}; f^*U) \simeq \Gamma(\mathcal{Z}; f^*\mathcal{O}_{\mathcal{Y}}(X)) \times_{\Gamma(\mathcal{Z}; f^*\mathcal{O}_{\mathcal{Y}}(Y))} \Gamma(\mathcal{Z}; f^*\mathbf{1}_{\mathcal{Y}}) \to \Gamma(\mathcal{Z}; \mathcal{O}_{\mathcal{Z}}(X)) \times_{\Gamma(\mathcal{Z}; \mathcal{O}_{\mathcal{Z}}(Y))} \Gamma(\mathcal{Z}; \mathbf{1}_{\mathcal{Z}}).$$

This map is a homotopy equivalence, since the $\mathcal{E}^{\mathrm{corp}}$-admissibility of f guarantees that the diagram

$$\begin{array}{ccc} f^*\mathcal{O}_{\mathcal{Y}}(X) & \longrightarrow & f^*\mathcal{O}_{\mathcal{Y}}(Y) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{Z}}(X) & \longrightarrow & \mathcal{O}_{\mathcal{Z}}(Y) \end{array}$$

is a pullback square in $\mathcal{Z}$ (see Proposition 21.2.4.8). It follows that θ is also a homotopy equivalence. Allowing $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ to vary over all objects of $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}(\mathcal{E})$, we conclude that $\overline{\eta}$ exhibits Γ^X as the functor corepresented by the object $(\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}}|_U)$. □

Chapter 22

Scheme Theory

Part VII

Variants of Spectral Algebraic Geometry

Chapter 23

Derived Differential Topology

Chapter 24

Derived Complex Analytic Geometry

Chapter 25

Derived Algebraic Geometry

Associated to every commutative ring R, there are several candidates for a "homotopy theory" of commutative algebras over R:

(a) One can consider topological spaces (or simplicial sets) equipped with the structure of a commutative R-algebra, in which the commutative ring axioms are required to hold "on the nose."

(b) One can consider commutative differential graded algebras over R: that is, chain complexes over R which are equipped with a structure of a commutative R-algebra in which the commutative ring axioms are again required to hold "on the nose."

(c) One can consider $\mathbb{E}_\infty$-algebras over R. These can be viewed as chain complexes (or, in the connective case, as topological spaces) which are equipped with the structure of a commutative R-algebra "up to coherent homotopy."

If R contains the field $\mathbf{Q}$ of rational numbers, then (a), (b), and (c) give rise to essentially the same theory. More precisely, the category of simplicial commutative R-algebras and the category of commutative differential graded algebras over R admit model structures (Propositions HA.7.1.4.20 and HA.7.1.4.10), and the associated ∞-categories can be identified with $\mathrm{CAlg}_R^{\mathrm{cn}}$ and CAlg_R, respectively (Propositions 25.1.2.2 and HA.7.1.4.6); here CAlg_R denotes the ∞-category of $\mathbb{E}_\infty$-algebras over R and $\mathrm{CAlg}_R^{\mathrm{cn}} \subseteq \mathrm{CAlg}_R$ the full subcategory spanned by the connective $\mathbb{E}_\infty$-algebras over R.

If R does not contain the field $\mathbf{Q}$ of rational numbers, then the theory of commutative differential graded algebras over R is poorly behaved. However, one can still consider simplicial commutative algebra and $\mathbb{E}_\infty$-algebras over R, which give rise to *different* homotopy theories. The theory of spectral algebraic geometry developed earlier in this book can be loosely described as obtained from the theory of classical algebraic geometry by replacing commutative rings by $\mathbb{E}_\infty$-rings. In this section we will study a parallel (and closely related) theory which we call *derived* algebraic geometry, where we instead use simplicial commutative rings in place of ordinary commutative rings.

We begin in §25.1 with a review of the theory of simplicial commutative rings. Here we will indulge in a slight abuse of terminology: the term "simplicial commutative ring" is usually used in the literature to refer to a simplicial object of the ordinary category $\mathrm{CAlg}^{\heartsuit}$ of commutative rings, or (equivalently) to a commutative ring object in the ordinary category of simplicial sets. The collection of such objects can be organized into a simplicial model category $\mathbf{A}$. From our point of view, the real object of interest is the underlying ∞-category CAlg^{Δ}, which can be obtained as the homotopy coherent nerve of the full subcategory $\mathbf{A}^{\circ}$ of fibrant-cofibrant objects of $\mathbf{A}$. This ∞-category admits many different descriptions; for us, the most relevant of these is that it can be obtained from the category of finitely generated polynomial rings (over $\mathbf{Z}$) by freely adjoining sifted colimits. We will essentially take this universal property as our definition of CAlg^{Δ} (see Definition 25.1.1.1 and Remark 25.1.1.2); the relationship between this definition with the simplicial model category $\mathbf{A}$ can be deduced from some general results of [138] (Remark 25.1.1.3), but will not play an essential role in our exposition.

There is a close relationship between the theory of simplicial commutative rings and the theory of (connective) $\mathbb{E}_\infty$-rings. More precisely, we can associate to every simplicial commutative ring A an underlying connective $\mathbb{E}_\infty$-ring A°, which is an $\mathbb{E}_\infty$-algebra over $\mathbf{Z}$ (Construction 25.1.2.1). We therefore have forgetful functors

$$\mathrm{CAlg}^{\Delta} \to \mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}} \to \mathrm{CAlg}^{\mathrm{cn}}.$$

Heuristically, one can think of a connective $\mathbb{E}_\infty$-ring $R \in \mathrm{CAlg}^{\mathrm{cn}}$ as a space which is equipped with an addition and multiplication which satisfy the axiomatics for commutative rings "up to coherent homotopy," while a simplicial commutative ring $R \in \mathrm{CAlg}^{\Delta}$ is a space equipped with an addition and multiplication satisfying the commutative ring axioms "on the nose." In between these extremes lies the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}}$, whose objects R can be thought of as spaces equipped with an addition law which is strictly commutative and associative, and a compatible multiplication law which is commutative and associative only up to coherent homotopy. A precise consequence of this heuristic picture is that if for a simplicial commutative ring R, the space of units of the underlying $\mathbb{E}_\infty$-ring can be regarded as the 0th space of a $\mathbf{Z}$-module spectrum (Proposition 25.1.5.3).

Another important difference between the theory of $\mathbb{E}_\infty$-rings and the theory of simplicial commutative rings lies in the structure of free algebras. If A is an $\mathbb{E}_\infty$-ring and M is an A-module, then one can consider the free $\mathbb{E}_\infty$-algebra $\mathrm{Sym}^*_A(M)$ over A generated by M: as an A-module, it is given as an infinite direct sum $\bigoplus_{n \geq 0} \mathrm{Sym}^n_A()$, where each $\mathrm{Sym}^n_A(M)$ denotes the homotopy coinvariants for the action of the symmetric group Σ_n on the n-fold tensor power $M \otimes_A \cdots \otimes_A M$. In the setting of simplicial commutative rings, there is an analogous picture: given a simplicial commutative ring A and a connective A-module M, one can form a free algebra $\mathrm{LSym}^*_A(M)$ which decomposes as a direct sum $\bigoplus_{n \geqslant 0} \mathrm{LSym}^n_A(M)$ (see Construction 25.2.2.6). However, the symmetric powers $\mathrm{LSym}^n_A(M)$ which appear here *cannot* be defined as the homotopy coinvariants of symmetric group actions: instead, they are obtained as nonabelian left derived functors of the formation of symmetric powers in the setting of classical commutative algebra. In §25.2, we will analyze the derived symmetric power construction LSym^n_A as well as the analogous derived functors for exterior and divided powers (Constructions 25.2.2.1, 25.2.2.2, and 25.2.2.3) and study their behavior with respect to connectivity and finiteness properties of modules.

The theory of simplicial commutative rings was originally introduced as a tool for studying commutative algebra using techniques borrowed from algebraic topology. One of the motivating applications was to extend the theory of Kähler differentials $\Omega_{B/A}$ of a commutative ring homomorphism $A \to B$ to a homology theory (called *André-Quillen homology*), allowing the short exact sequence

$$\mathrm{Tor}^B_0(C, \Omega_{B/A}) \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

to be extended on the left. In §25.3, we study André-Quillen in the setting of *simplicial* commutative rings by assigning to each morphism $\phi : A \to B$ in CAlg^{Δ} a B-module spectrum $L^{\mathrm{alg}}_{B/A}$, which we will refer to as the *algebraic cotangent complex* of B over A (see Notation 25.3.2.1). This is generally different from the cotangent complex of the underlying morphism of $\mathbb{E}_\infty$-rings $A^\circ \to B^\circ$ considered elsewhere in this book, even when A and B are ordinary commutative rings (the algebraic cotangent complex $L^{\mathrm{alg}}_{B/A}$ computes André-Quillen homology, while the cotangent complex L_{B°/A° computes *topological* André-Quillen homology), but there is a close relationship between the two: there is an equivalence $L^{\mathrm{alg}}_{B/A} \simeq B^\circ \otimes_{B^+} L_{B^\circ/A^\circ}$ where B^+ is a certain (noncommutative) ring spectrum which acts on L_{B°/A° (see Remark 25.3.3.7). Using this description, we show that many of the pleasant features of the cotangent complexes of $\mathbb{E}_\infty$-ring can be extended to the algebraic setting; for example, a morphism of simplicial commutative rings $\phi : A \to B$ is an equivalence if and only if induces an isomorphism on π_0 and the algebraic cotangent complex $L^{\mathrm{alg}}_{B/A}$ vanishes (Corollary 25.3.6.6).

In §**??**, we study finiteness conditions on simplicial commutative rings (and on morphisms of simplicial commutative rings). If $\phi : A \to B$ is a morphism of simplicial commutative rings, we say that ϕ is *almost of finite presentation* if the functor $C \mapsto \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_A}(B, C)$ commutes with filtered colimits when restricted to n-truncated objects of CAlg^{Δ}_A, for every nonnegative integer n (where $\mathrm{CAlg}^{\Delta}_A = (\mathrm{CAlg}^{\Delta})_{A/}$ denotes the ∞-category of simplicial commutative rings equipped with a map from A). Our main result is that this condition depends only the underlying morphism $\phi^\circ : A^\circ \to B^\circ$ of $\mathbb{E}_\infty$-rings (more precisely, it is equivalent to the requirement that ϕ° be almost of finite presentation, in the sense of Definition HA.7.2.4.26; see Proposition **??**), and that a similar phenomenon occurs for weaker finiteness conditions

(Theorem **??**). Note that this is not *a priori* obvious, because the analogous statement is false for stronger finiteness conditions such as compactness: the polynomial ring $\mathbf{Z}[x]$ is a compact object of the ∞-category CAlg^{Δ} of simplicial commutative rings, but is not compact when viewed as an $\mathbb{E}_\infty$-ring or as an $\mathbb{E}_\infty$-algebra over $\mathbf{Z}$ (see Warning **??**).

In §**??**, we study sheaves of simplicial commutative rings on a topological space X and, more generally, on an ∞-topos $\mathcal{X}$ (the former can be considered as a special case of the latter, since the datum of a sheaf on a topological space X is equivalent to the datum of a sheaf on the ∞-topos $\mathcal{S}\mathrm{hv}(X)$). In §**??**, we will use the language of CAlg^{Δ}-valued sheaves to introduce the notion of *derived scheme* (Definition **??**): namely, we define a derived scheme to be a pair $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is an CAlg^{Δ}-valued sheaf on X for which the underlying pair $(X, \mathscr{O}_X^{\circ})$ is a spectral scheme (in the sense of Definition 1.1.2.8). Our main result (Corollary **??**) is that this condition is equivalent to the requirement that the pair $(X, \mathscr{O}_X)$ is locally equivalent to the Zariski spectrum $\mathrm{Spec}(A)$ of some simplicial commutative ring A (see Notation **??**). In §**??**, we will prove an analogous result for Deligne-Mumford stacks (Corollary **??**), where the role of the Zariski spectrum $\mathrm{Spec}\, A$ is instead played by a suitable notion of étale spectrum $\mathrm{Sp\acute{e}t}(A)$ (Notation **??**).

The theory of derived algebraic geometry developed in this section is closely related to the theory of spectral algebraic geometry studied throughout this book. Every derived scheme $(X, \mathscr{O}_X)$ has an underlying spectral scheme $(X, \mathscr{O}_X^{\circ})$, and every derived Deligne-Mumford stack $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ has an underlying spectral Deligne-Mumford stack $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^{\circ})$. Consequently, many basic concepts and theorems can directly imported from spectral algebraic geometry to derived algebraic geometry. However, there is one important respect in which the theory of derived algebraic geometry is substantially simpler than the theory of spectral algebraic geometry. In the setting of spectral algebraic geometry, there are (at least) two *different* smoothness conditions one can consider on a morphism $f : \mathsf{X} \to \mathsf{Y}$: differential smoothness and fiber-smoothness (see Definition 11.2.5.5). One of the difficulties in working with spectral algebraic geometry is that these two notions are distinct from one another (unless we work in characteristic zero) and neither shares all of the pleasant features of smooth morphisms from classical algebraic geometry: differentially smooth morphisms need not be flat, and fiber-smooth morphisms need not satisfy an infinitesimal lifting criterion. In §**??**, we will see that the theory of derived algebraic geometry does not share this defect: we will introduce a smoothness condition on a morphism $f : \mathsf{X} \to \mathsf{Y}$ of derived Deligne-Mumford stacks (Variant 3.4.3.3) which entails both that f is flat *and* that it satisfies an infinitesimal lifting criterion. The notion of smooth morphism will play an important role in the theory of derived algebraic stacks which we will develop in Part VIII.

Contents

25.1 Simplicial Commutative Rings

Our goal in this section is to give a short introduction to the theory of simplicial commutative rings and its relationship to the theory of $\mathbb{E}_\infty$-rings which appears elsewhere in this book. Fix a commutative ring R. In §25.1.1, we introduce the ∞-category CAlg^{Δ}_R of *simplicial commutative R-algebras*, which is obtained from the category of polynomial rings over R by formally adjoining sifted colimits (Proposition 25.1.1.5). In §25.1.2, we study the relationship of ∞-category CAlg^{Δ}_R and the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}_R$ of connective $\mathbb{E}_\infty$-algebras over R. There is a forgetful functor $\mathrm{CAlg}^{\Delta}_R \to \mathrm{CAlg}^{\mathrm{cn}}_R$, which we will denote by $A \mapsto A^\circ$; we show that this functor admits both left and right adjoints (Proposition 25.1.2.4), and that it is an equivalence when R contains the field $\mathbf{Q}$ of rational numbers (Proposition 25.1.2.2). For many purposes, it is convenient to identify a simplicial commutative R-algebra A with the underlying $\mathbb{E}_\infty$-algebra A°; for example, in §25.1.3 we define the homotopy groups of A to be the homotopy groups of A°

In §25.1.4, we study the dependence of the ∞-category CAlg^{Δ}_R on the commutative ring R. Every morphism of commutative rings $f : R \to R'$ induces a “restriction of scalars” functor $\mathrm{CAlg}^{\Delta}_{R'} \to \mathrm{CAlg}^{\Delta}_R$, and we show that this functor induces an equivalence of ∞-categories

$\mathrm{CAlg}^{\Delta}_{R'} \simeq (\mathrm{CAlg}^{\Delta}_{R})_{R'/}$ (Proposition 25.1.4.2). In particular, for any commutative ring R, we can identify $\mathrm{CAlg}^{\Delta}_{R}$ with the ∞-category $(\mathrm{CAlg}^{\Delta})_{R/}$, where $\mathrm{CAlg}^{\Delta} = \mathrm{CAlg}^{\Delta}_{\mathbf{Z}}$ denotes the ∞-category of simplicial commutative $\mathbf{Z}$-algebras.

Roughly speaking, the difference between objects of $\mathrm{CAlg}^{\Delta}_{R}$ and $\mathrm{CAlg}^{\mathrm{cn}}_{R}$ is that the multiplication on a simplicial commutative algebra is required to be "strictly" commutative, while the multiplication on an $\mathbb{E}_\infty$-algebra is only required to be commutative up to coherent homotopy. In §25.1.5, we give a precise articulation of this heuristic (Proposition 25.1.5.3) which will be applied in §25.3 to analyze the difference between the deformation theory of simplicial commutative algebras and their $\mathbb{E}_\infty$ counterparts.

25.1.1 The ∞-Category $\mathrm{CAlg}^{\Delta}_{R}$

We begin by introducing the principal objects of interest.

Definition 25.1.1.1. Let R be a commutative ring. We let Poly_R denote the category whose objects are polynomial rings $R[x_1, \ldots, x_n]$ over R, and whose morphisms are R-algebra homomorphisms. Note that the category Poly_R admits finite coproducts (given by the formation of tensor product over R).

We let $\mathrm{CAlg}^{\Delta}_{R}$ denote the full subcategory $\mathrm{Fun}^{\pi}(\mathrm{Poly}_R^{\mathrm{op}}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathrm{Poly}_R^{\mathrm{op}}, \mathcal{S})$ spanned by those functors $\mathrm{Poly}_R^{\mathrm{op}} \to \mathcal{S}$ which preserve finite products. We will refer to $\mathrm{CAlg}^{\Delta}_{R}$ as the *∞-category of simplicial commutative R-algebras*. In the special case where R is the ring $\mathbf{Z}$ of integers, we will denote $\mathrm{CAlg}^{\Delta}_{R}$ simply by CAlg^{Δ}, and refer to it as the *∞-category of simplicial commutative rings*.

Remark 25.1.1.2. Let R be a commutative ring. By virtue of Proposition HTT.5.5.8.22 , the ∞-category $\mathrm{CAlg}^{\Delta}_{R}$ can be characterized up to equivalence by the following properties:

(1) The ∞-category $\mathrm{CAlg}^{\Delta}_{R}$ is presentable.

(2) There exists a coproduct-preserving, fully faithful functor $j : \mathrm{Poly}_R \hookrightarrow \mathrm{CAlg}^{\Delta}_{R}$.

(3) The essential image of j consists of compact projective objects of $\mathrm{CAlg}^{\Delta}_{R}$ which generate $\mathrm{CAlg}^{\Delta}_{R}$ under sifted colimits.

Remark 25.1.1.3. The terminology of Definition 25.1.1.1 is motivated by the following observation, which follows immediately from Corollary HTT.5.5.9.3 :

($*$) Let $\mathbf{A}_R$ be the ordinary category of simplicial commutative R-algebras, regarded as a simplicial model category in the usual way (see [167] or Proposition HTT.5.5.9.1), and let $\mathbf{A}^{\circ}$ denote the full subcategory spanned by the fibrant-cofibrant objects. Then there is a canonical equivalence of ∞-categories $\mathrm{N}(\mathbf{A}^{\circ}) \to \mathrm{CAlg}^{\Delta}_{R}$.

Remark 25.1.1.4. Using $(*)$, we deduce that the ∞-category of discrete objects of CAlg^{Δ}_R is canonically equivalent with (the nerve of) the category $\mathrm{CAlg}^{\heartsuit}_R$ of commutative R-algebras. We will generally abuse notation and not distinguish between commutative R-algebras and the corresponding discrete objects of CAlg^{Δ}_R. In particular, we will view the polynomial algebras $R[x_1, \ldots, x_n]$ as objects of CAlg^{Δ}_R (which we can regard as compact projective generators for CAlg^{Δ}_R, by virtue of Remark 25.1.1.2).

The following result is an immediate consequence of Proposition HTT.5.5.8.15 and its proof:

Proposition 25.1.1.5. *Let R be a commutative ring and let $j : \mathrm{Poly}_R \to \mathrm{CAlg}^{\Delta}_R$ denote the Yoneda embedding. Let $\mathcal{C}$ be an ∞-category which admits small sifted colimits, and $\mathrm{Fun}_{\Sigma}(\mathrm{CAlg}^{\Delta}_R, \mathcal{C})$ the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\Delta}_R, \mathcal{C})$ spanned by those functors which preserve sifted colimits. Then:*

(1) *Composition with j induces an equivalence of ∞-categories*

$$\mathrm{Fun}_{\Sigma}(\mathrm{CAlg}^{\Delta}_R, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Poly}_R, \mathcal{C}).$$

(2) *A functor $F : \mathrm{CAlg}^{\Delta}_R \to \mathcal{C}$ belongs to $\mathrm{Fun}_{\Sigma}(\mathrm{CAlg}^{\Delta}_R, \mathcal{C})$ if and only if F is a left Kan extension of $F \circ j$ along j.*

(3) *Suppose $\mathcal{C}$ admits finite coproducts, and let $F : \mathrm{CAlg}^{\Delta}_R \to \mathcal{C}$ preserve sifted colimits. Then F preserves finite coproducts if and only if $F \circ j$ preserves finite coproducts.*

25.1.2 Comparison with $\mathbb{E}_\infty$-Algebras

We now study the relationship between simplicial commutative algebras and $\mathbb{E}_\infty$-algebras. We begin with an application of Proposition 25.1.1.5:

Construction 25.1.2.1. [The Underlying $\mathbb{E}_\infty$-Algebra] Let R be a commutative ring and let Poly_R denote the category of finitely generated polynomial rings over R, which we regard as a full subcategory of both the ∞-category CAlg^{Δ}_R of simplicial commutative R-algebras and the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}_R$ of connective $\mathbb{E}_\infty$-algebras over R. By virtue of Proposition 25.1.1.5, there is an essentially unique functor

$$\Theta : \mathrm{CAlg}^{\Delta}_R \to \mathrm{CAlg}^{\mathrm{cn}}_R$$

which commutes with small sifted colimits and restricts to the identity on Poly_R. If A is an object of CAlg^{Δ}_R, we will $\Theta(A)$ by A° and refer to it as the *underlying $\mathbb{E}_\infty$-algebra of A.*

Proposition 25.1.2.2. *Let R be a commutative ring, and let $\Theta : \mathrm{CAlg}^{\Delta}_R \to \mathrm{CAlg}^{\mathrm{cn}}_R$ be the forgetful functor of Construction 25.1.2.1. Then:*

(1) *The functor* Θ *preserves small limits and colimits.*

(2) *The functor* Θ *is conservative.*

(3) *If* R *contains the field* $\mathbf{Q}$*, then* Θ *is an equivalence of* ∞*-categories.*

Proof. We first prove (1). To prove that Θ preserves small colimits, it will suffice to show that the inclusion $\mathrm{Poly}_R \hookrightarrow \mathrm{CAlg}_R^{\mathrm{cn}}$ of Construction 25.1.2.1 preserves finite coproducts (Proposition 25.1.1.5). Since coproducts in $\mathrm{CAlg}_R^{\mathrm{cn}}$ are computed by relative tensor products over R, this follows from the fact that every polynomial algebra $R[x_1, \ldots, x_n]$ is flat as a R-module.

Consider the functor $\phi : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ defined by the composition

$$\mathrm{CAlg}_R^{\mathrm{cn}} \simeq \mathrm{CAlg}(\mathrm{Mod}_R^{\mathrm{cn}}) \to \mathrm{Mod}_R^{\mathrm{cn}} \to (\mathrm{Sp})_{\geqslant 0} \xrightarrow{\Omega^\infty} \mathcal{S}.$$

Using Corollary HA.3.2.2.5 and Corollary HA.4.2.3.3, we deduce that ϕ is conservative and preserves small limits. Let $\psi, \psi' : \mathrm{CAlg}_R^{\Delta} \to \mathcal{S}$ be the functors given by $\psi(A) = \phi(A^\circ)$ and $\psi'(A) = A(R[x])$. The functor ψ' obviously preserves small limit, and it is conservative since every object of Poly_k can be written as a coproduct of finitely many copies of $R[x]$. To complete the proofs of (1) and (2), it will suffice to show that the functors ψ and ψ' are equivalent. The functor ψ' obviously preserves small sifted colimits. Combining Proposition HA.1.4.3.9, Corollary HA.4.2.3.5, and Corollary HA.3.2.3.2, we conclude that $\psi : \mathrm{CAlg}_k^{\Delta} \to \mathcal{S}$ preserves small sifted colimits as well. By virtue of Proposition 25.1.1.5, it will suffice to show that the composite functors $\psi \circ j, \psi' \circ j : \mathrm{Poly}_R \to \mathcal{S}$ are equivalent. We now simply observe that both of these compositions can be identified with the functor which associates to each polynomial ring $R[x_1, \ldots, x_n]$ its underlying set of elements, regarded as a discrete space.

Let us now prove (3). Suppose that R is a $\mathbf{Q}$-algebra. Then, for every $n \geqslant 0$, every flat R-module M, and every $i > 0$, the homology group $\mathrm{H}_i(\Sigma_n; M^{\otimes n})$ vanishes, where Σ_n denotes the symmetric group on n letters. It follows that the symmetric power $\mathrm{Sym}_R^n(M) \in \mathrm{Mod}_R^{\mathrm{cn}}$ is discrete, so that the $\mathbb{E}_\infty$-algebra $\mathrm{Sym}_R^*(R^m)$ can be identified with the (discrete) polynomial ring $R[x_1, \ldots, x_m]$. Using Proposition HA.7.2.4.27, we conclude that the essential image of θ_0 consists of compact projective objects of $\mathrm{CAlg}_R^{\mathrm{cn}}$ which generate $\mathrm{CAlg}_R^{\mathrm{cn}}$ under colimits, so that Θ is an equivalence by Proposition HTT.5.5.8.25. □

Notation 25.1.2.3. [Tensor Products] Let R be a commutative ring, and suppose we are given a diagram

$$A_0 \leftarrow A \to A_1,$$

in the ∞-category CAlg_R^{Δ}. We will denote the pushout of this diagram by $A_0 \otimes_A A_1$. This notation is justified by the fact that the forgetful functor $B \mapsto B^\circ$ preserves small colimits

(Proposition 25.1.2.2), so that we have a canonical equivalence

$$(A_0 \otimes_A A_1)^\circ \simeq A_0^\circ \otimes_{A^\circ} A_1^\circ$$

in $\mathrm{CAlg}_R^{\mathrm{cn}}$.

Proposition 25.1.2.4. *Let R be a commutative ring. Then the forgetful functor $\Theta : \mathrm{CAlg}_R^\Delta \to \mathrm{CAlg}_R^{\mathrm{cn}}$ admits left and right adjoints*

$$\Theta^L : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CAlg}_R^\Delta \qquad \Theta^R : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CAlg}_R^\Delta .$$

Moreover, the functor Θ is both monadic and comonadic: that is, we can identify CAlg_R^Δ with the ∞-category of algebras for the monad $\Theta \circ \Theta^L$ on $\mathrm{CAlg}_R^{\mathrm{cn}}$, or with the ∞-category of coalgebras for the comonad $\Theta \circ \Theta^R$ on $\mathrm{CAlg}_R^{\mathrm{cn}}$.

Proof. Since the functor Θ preserves small limits and colimits, the existence of the functors Θ^L and Θ^R follows from Corollary HTT.5.5.2.9 . Since Θ is conservative and preserves geometric realizations of simplicial objects, the monadicity of Θ follows from the Barr-Beck theorem (Theorem HA.4.7.0.3). Similarly, the comonadicity of Θ follows from the fact that it is conservative and preserves totalizations of cosimplicial objects. □

Remark 25.1.2.5. Let R be a commutative ring, and let $\Theta^R : \mathrm{CAlg}_k^\Delta \to \mathrm{CAlg}_k^{\mathrm{cn}}$ be as in Proposition 25.1.2.4. Let A be a connective $\mathbb{E}_\infty$-algebra over R. The underlying space of the simplicial commutative k-algebra $\Theta^R(A)$ can be identified with

$$\mathrm{Map}_{\mathrm{CAlg}_k^\Delta}(R[x], \Theta^R(A)) \simeq \mathrm{Map}_{\mathrm{CAlg}_R^{\mathrm{cn}}}(R[x], A).$$

Note that this is generally *different* from the underlying space $\Omega^\infty(A) \in \mathcal{S}$, because the discrete R-algebra $R[x]$ generally does not agree with the free $\mathbb{E}_\infty$-algebra $R\{x\} \in \mathrm{CAlg}_R$ (though they do coincide whenever R is a $\mathbf{Q}$-algebra). We can think of $\mathrm{Map}_{\mathrm{CAlg}_R}(R[x], A)$ as the space of "very commutative" points A, which generally differs from the underlying space $\Omega^\infty(A) \simeq \mathrm{Map}_{\mathrm{CAlg}_R}(R\{x\}, A)$of A. The difference between these spaces can be regarded either as a measure of the failure of the polynomial $R[x]$ to be free as an $\mathbb{E}_\infty$-algebra over R, or as a measure of the failure of free $\mathbb{E}_\infty$-algebra $R\{x\}$ to be flat over R.

We can interpret the situation as follows. The affine line $\mathrm{Spec}(R[x])$ can be regarded as a commutative R-algebra in the category of R-schemes. Since the functor $\theta : \mathrm{Poly}_R \to \mathrm{CAlg}_R^{\mathrm{cn}}$ preserves finite coproducts, we can also view $R[x]$ as a commutative k-algebra object (in an appropriate sense) of the ∞-category $(\mathrm{CAlg}_R^{\mathrm{cn}})^{\mathrm{op}}$. In other words, the functor $\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ corepresented by $R[x]$ can naturally be lifted to a functor taking values in a suitable ∞-category of "commutative R-algebras in $\mathcal{S}$": this is the ∞-category CAlg_R^Δ of simplicial commutative R-algebras, and the lifting is provided by the functor Θ^R.

Proposition 25.1.2.4 allows us to identify the ∞-category CAlg_R^{Δ} with the ∞-category of coalgebras for the comonad $T = \Theta \circ \Theta^R$ on the ∞-category $\mathrm{CAlg}_R^{\mathrm{cn}}$. In other words, the ∞-category CAlg_R^{Δ} arises naturally when we attempt to correct the difference between the $\mathbb{E}_\infty$-algebras $R[x]$ and $R\{x\}$ (which measures the failure of T to agree with the identity functor on $\mathrm{CAlg}_R^{\mathrm{cn}}$). In the ∞-category CAlg_R^{Δ}, the polynomial ring $R[x]$ is both flat over R *and* free (see Remark 25.1.3.6).

25.1.3 Homotopy Groups and Truncation

Let R be a commutative ring and let A be an object of CAlg_R^{Δ}. For each integer $n \in \mathbf{Z}$, we let $\pi_n(A)$ denote the nth homotopy group of the $\mathbb{E}_\infty$-ring A°. Then $\pi_*(A)$ is a graded R-algebra which is commutative in the graded sense (that is, for homogeneous elements $x \in \pi_m(A)$ and $y \in \pi_n(A)$, we have $xy = (-1)^{mn} yx \in \pi_{m+n}(A)$). Since the functor $A \mapsto A^\circ$ is conservative, a morphism $f : A \to B$ of simplicial commutative R-algebras is an equivalence if and only if each of the induced maps $\pi_n(A) \to \pi_n(B)$ is an isomorphism.

Remark 25.1.3.1. Let R be a commutative ring, and suppose we are given a diagram

$$A_0 \leftarrow A \rightarrow A_1$$

in the ∞-category CAlg_R^{Δ}, having pushout $A_0 \otimes_A A_1$. Then Proposition HA.7.2.1.19 supplies a convergent spectral sequence

$$E_2^{s,t} = \mathrm{Tor}_s^{\pi_*(A)}(\pi_*(A_0), \pi_*(A_1))_t \Rightarrow \pi_{s+t}(A_0 \otimes_A A_1).$$

Example 25.1.3.2. Let $\phi : R \to R'$ be any homomorphism of commutative rings, and let us identify R' with the corresponding discrete object of CAlg_R^{Δ}. Then for each $n \geqslant 0$, we have a canonical equivalence

$$R' \otimes_R R[x_1, \ldots, x_n] \simeq R'[x_1, \ldots, x_n]$$

in CAlg_R^{Δ}.

Remark 25.1.3.3. Let R be a commutative ring, let A be an object of CAlg_R^{Δ}, and let $n \geqslant 0$ be an integer. The following conditions are equivalent:

(1) The $\mathbb{E}_\infty$-ring A° is n-truncated.

(2) The homotopy groups $\pi_m(A)$ vanish for $m > n$.

(3) For every object $B \in \mathrm{CAlg}_R^{\Delta}$, the mapping space $\mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(B, A)$ is n-truncated.

We will say that A is *n-truncated* if it satisfies these equivalent conditions. We let $\tau_{\leqslant n} \mathrm{CAlg}_R^{\Delta}$ denote the full subcategory of CAlg_R^{Δ} spanned by the n-truncated objects.

Remark 25.1.3.4. Let k be a commutative ring and let $n \geqslant 0$ be an integer. Then the inclusion functor $\tau_{\leqslant n} \operatorname{CAlg}_R^{\Delta} \hookrightarrow \operatorname{CAlg}_R^{\Delta}$ admits a left adjoint $\tau_{\leqslant n} : \operatorname{CAlg}_R^{\Delta} \to \tau_{\leqslant n} \operatorname{CAlg}_R^{\Delta}$, given concretely by the formula

$$(\tau_{\leqslant n} A)(P) = \tau_{\leqslant n}(A(P)).$$

for $P \in \operatorname{Poly}_R$. Note that the forgetful functor $A \mapsto A^{\circ}$ commutes with the formation of n-truncations.

Remark 25.1.3.5. For each $n \geqslant 0$, the forgetful functor

$$\Theta : \operatorname{CAlg}_R^{\Delta} \to \operatorname{CAlg}_R^{\mathrm{cn}} \qquad A \mapsto A^{\circ}$$

restricts to a functor $\Theta_{\leqslant n} : \tau_{\leqslant n} \operatorname{CAlg}_R^{\Delta} \to \tau_{\leqslant n} \operatorname{CAlg}_R^{\mathrm{cn}}$. When $n = 0$, this functor is an equivalence of categories (in this case, both sides can be identified with the ordinary category $\operatorname{CAlg}_R^{\heartsuit}$ of commutative algebras over R).

Remark 25.1.3.6. Let R be a commutative ring and let A be an object of $\operatorname{CAlg}_R^{\Delta}$. The homotopy groups $\pi_*(A)$ can be identified with the homotopy groups of the mapping space $\operatorname{Map}_{\operatorname{CAlg}_R^{\Delta}}(R[x], A)$. In particular, we have a canonical bijection $\operatorname{Hom}_{\operatorname{hCAlg}_R^{\Delta}}(R[x], A) \simeq \pi_0(A)$. More generally, evaluation separately on each variable induces a homotopy equivalence

$$\operatorname{Map}_{\operatorname{CAlg}_R^{\Delta}}(R[x_1, \ldots, x_n], A) \simeq \operatorname{Map}_{\operatorname{CAlg}_R^{\Delta}}(R[x], A)^n$$

and a bijection $\operatorname{Hom}_{\operatorname{hCAlg}_R^{\Delta}}(R[x_1, \ldots, x_n], A) \simeq \pi_0(A)^n$.

Remark 25.1.3.7. Let $f : R[x_1, \ldots, x_n] \to R[y_1, \ldots, y_m]$ be a homomorphism of polynomial rings over R, given by $x_i \mapsto f_i(y_1, \ldots, y_m)$. For every object $A \in \operatorname{CAlg}_k^{\Delta}$, composition with f induces a map of spaces

$$\operatorname{Map}_{\operatorname{CAlg}_R^{\Delta}}(R[y_1, \ldots, y_m], A) \to \operatorname{Map}_{\operatorname{CAlg}_R^{\Delta}}(R[x_1, \ldots, x_n], A).$$

Passing to homotopy groups at some point $\eta \in \operatorname{Map}_{\operatorname{CAlg}_R^{\Delta}}(R[y_1, \ldots, y_m], A)$, we get a map $\pi_*(A)^m \to \pi_*(A)^n$. For $* = 0$, this map is given by

$$(a_1, \ldots, a_m) \mapsto (f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m)).$$

For $* > 0$, it is given instead by the action of the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]$ (which we regard as a matrix taking values in $\pi_0(A)$ using the morphism η).

25.1.4 Change of Base Ring

We now study the behavior of the ∞-category CAlg^{Δ}_R as we vary the commutative ring R.

Construction 25.1.4.1 (Restriction and Extension of Scalars)**.** Let $\varphi : R \to R'$ be a homomorphism of commutative rings. Then extension of scalars along φ determines a functor $f : \mathrm{Poly}_R \to \mathrm{Poly}_{R'}$. Using Proposition 25.1.1.5, we see that there is an essentially unique functor $F : \mathrm{CAlg}^{\Delta}_k \to \mathrm{CAlg}^{\Delta}_{k'}$ which preserves sifted colimits and fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Poly}_R & \xrightarrow{f} & \mathrm{Poly}_{R'} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\ \mathrm{CAlg}^{\Delta}_R & \xrightarrow{F} & \mathrm{CAlg}^{\Delta}_{R'} . \end{array}$$

In this case, we will say that $F : \mathrm{CAlg}^{\Delta}_R \to \mathrm{CAlg}^{\Delta}_{R'}$ is the functor of *extension of scalars along* φ.

Since the functor f preserves finite coproducts, the functor F preserves small colimits and therefore admits a right adjoint $G : \mathrm{CAlg}^{\Delta}_{R'} \to \mathrm{CAlg}^{\Delta}_R$. Concretely, if we identify $\mathrm{CAlg}^{\Delta}_{R'}$ with the full subcategory of $\mathrm{Fun}(\mathrm{Poly}^{\mathrm{op}}_{R'}, \mathcal{S})$ spanned by those functors which preserve finite products, then the functor G is given by precomposition with f. In particular, we have a canonical isomorphism $\pi_*(G(A)) \simeq \pi_*(A)$. We will refer to G as the functor of *restriction of scalars along* φ.

In the situation of Construction 25.1.4.1, the canonical isomorphism $\pi_*(G(R')) \simeq R'$ shows that $G(R')$ is a discrete object of CAlg^{Δ}_R which can be identified with R' (as a commutative algebra over R).

Proposition 25.1.4.2. *Let $\varphi : R \to R'$ be a homomorphism of commutative rings. Then the restriction of scalars functor of Construction 25.1.4.1 induces an equivalence of ∞-categories*

$$\overline{G} : \mathrm{CAlg}^{\Delta}_{R'} \simeq (\mathrm{CAlg}^{\Delta}_{R'})_{R'/} \to (\mathrm{CAlg}^{\Delta}_R)_{G(R')/} \simeq (\mathrm{CAlg}^{\Delta}_R)_{R'/}.$$

Proof. Note that the equivalence $R' \simeq G(R')$ in CAlg^{Δ}_R induces a map $F(R') \to R'$ in the ∞-category $\mathrm{CAlg}^{\Delta}_{R'}$. Unwinding the definitions, we see that the functor $\overline{G}$ admits a left adjoint $\overline{F}$, given on objects by the formula $\overline{F}(A) = F(A) \otimes_{F(R')} R'$. We will show that the unit map $u : \mathrm{id}_{\mathrm{CAlg}^{\Delta}_R} \to \overline{G} \circ \overline{F}$ is an equivalence, so that the functor $\overline{F}$ is fully faithful. To complete the proof, it will then suffice to show that $\overline{G}$ is conservative, which follows immediately from the observation that the homotopy groups of $\overline{G}(B)$ can be identified with the homotopy groups of B.

Fix an object $A \in \mathrm{CAlg}^{\Delta}_R$; we wish to show that the unit map $u_A : A \to (\overline{G} \circ \overline{F})(A)$ is an equivalence in CAlg^{Δ}_R. Since the functor F preserves compact projective objects, the functor

G commutes with sifted colimits. It follows that the functor $A \mapsto u_A$ preserves sifted colimits. Since the ∞-category CAlg^{Δ}_R has compact projective generators given by the polynomial rings $R[x_1, \ldots, x_n]$, the ∞-category $(\mathrm{CAlg}^{\Delta}_R)_{R'/}$ has compact projective generators of the form $R[x_1, \ldots, x_n] \otimes_R R'$. It will therefore suffice to show that u_A is an equivalence when A has the form $A_0 \otimes_R R'$ for some $A_0 \simeq R[x_1, \ldots, x_n] \in \mathrm{CAlg}^{\Delta}_R$. In this case, a simple calculation shows that the map

$$\begin{aligned} \pi_*(A) &\to \pi_*(\overline{G} \circ \overline{F})(A) \\ &\simeq \pi_*(\overline{F}(A)) \\ &\simeq \pi_*(F(A) \otimes_{F(R')} R') \\ &\simeq \pi_*(F(A_0) \otimes_{F(R)} F(R') \otimes_{F(R')} R') \\ &\simeq \pi_*(F(A_0) \otimes_{F(R)} R') \\ &\simeq \pi_*(F(A_0)) \\ &\simeq R'[x_1, \ldots, x_n] \end{aligned}$$

is the identity. □

Corollary 25.1.4.3. *Let R be a commutative ring. Then the Construction 25.1.4.1 induces an equivalence of ∞-categories* $\mathrm{CAlg}^{\Delta}_R \simeq (\mathrm{CAlg}^{\Delta})_{R/}$.

Notation 25.1.4.4. For the remainder of this book, we will abuse notation by not distinguishing between the ∞-categories CAlg^{Δ}_R and $(\mathrm{CAlg}^{\Delta})_{R/}$: that is, we will think of objects of CAlg^{Δ}_R as simplicial commutative rings A equipped with a map $R \to A$.

More generally, if R is a simplicial commutative ring, we let CAlg^{Δ}_R denote the ∞-category $(\mathrm{CAlg}^{\Delta})_{R/}$. By virtue of Corollary 25.1.4.3, this agrees with Definition 25.1.1.1 (up to canonical equivalence) in the case where R is an ordinary commutative ring.

Note that every morphism $\varphi : R \to R'$ of simplicial commutative rings induces a pair of adjoint functors $\mathrm{CAlg}^{\Delta}_R \rightleftarrows \mathrm{CAlg}^{\Delta}_{R'}$, given by

$$(A \in \mathrm{CAlg}^{\Delta}_R) \mapsto (A \otimes_R R' \in \mathrm{CAlg}^{\Delta}_{R'}) \qquad (B \in \mathrm{CAlg}^{\Delta}_{R'}) \mapsto (B \in \mathrm{CAlg}^{\Delta}_R).$$

When R and R' are ordinary commutative rings, these constructions recover (up to canonical equivalence) the functors described in Construction 25.1.4.1.

25.1.5 Strictness of Multiplication

Let A be a connective $\mathbb{E}_\infty$-ring. Then we can think of A as consisting of an "underlying space" $\Omega^\infty A$ which is equipped with an addition and multiplication which satisfy the axioms of commutative algebra "up to coherent homotopy." Under this heuristic, the difference between objects of $\mathrm{CAlg}^{\mathrm{cn}}$ and objects of $\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}}$ is that in the latter case, we require addition to be commutative "on the nose" rather than up to coherent homotopy. Remark 25.1.2.5 expresses a similar heuristic: the difference between objects of $\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}}$ and objects of CAlg^{Δ} is that for the latter, we also require multiplication to be commutative "on the nose" rather than up to coherent homotopy. Our final goal in this section is to make this idea more precise.

Notation 25.1.5.1. The functor $\Sigma^{\infty}_{+} : \mathcal{S} \to \mathrm{Sp}^{\mathrm{cn}}$ is symmetric monoidal, where we regard $\mathcal{S}$ as equipped with the Cartesian symmetric monoidal structure and $\mathrm{Sp}^{\mathrm{cn}}$ as equipped with the symmetric monoidal structure given by the smash product. In particular, composition with Σ^{∞}_{+} induces a functor

$$\mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S}) \subseteq \mathrm{CAlg}(\mathcal{S}) \to \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{cn}}) = \mathrm{CAlg}^{\mathrm{cn}};$$

here $\mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$ denotes the full subcategory of $\mathrm{CAlg}(\mathcal{S})$ spanned by the grouplike $\mathbb{E}_{\infty}$-spaces. For any connective $\mathbb{E}_{\infty}$-ring R, we can compose with the base change functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{CAlg}^{\mathrm{cn}}_{R}$ to obtain a functor

$$\Psi : \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S}) \to \mathrm{CAlg}^{\mathrm{cn}}_{k}$$

$$X \mapsto R[X],$$

where $R[X]$ denotes the spectrum $R \otimes \Sigma^{\infty}_{+}(X)$ whose homotopy groups are the R-homology groups of X.

Proposition 25.1.5.2. *Let R be a commutative ring. Then the composite functor*

$$\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}} \xrightarrow{\Omega^{\infty}} \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S}) \xrightarrow{\Psi} \mathrm{CAlg}^{\mathrm{cn}}_{R}$$

admits an essentially unique factorization as a composition

$$\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}} \xrightarrow{\Psi^{\mathrm{alg}}} \mathrm{CAlg}^{\Delta}_{R} \xrightarrow{\Theta} \mathrm{CAlg}^{\mathrm{cn}}_{R},$$

where Ψ^{alg} commutes with sifted colimits. Here Θ denotes the forgetful functor of Proposition 25.1.2.4 and Ψ is the group algebra functor of Notation 25.1.5.1. Moreover, the functor Ψ^{alg} commutes with all small colimits.

Proof. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ spanned by those modules of the form $\mathbf{Z}^{k}$. Then $\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ is freely generated by $\mathcal{C}$ under sifted colimits (Corollary HA.7.1.4.15). Consequently, it will suffice to show that the composite functor

$$\mathcal{C} \subseteq \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}} \xrightarrow{\Omega^{\infty}} \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S}) \xrightarrow{\Psi} \mathrm{CAlg}^{\mathrm{cn}}_{k}$$

$$\mathbf{Z}^{k} \mapsto R[\mathbf{Z}^{k}]$$

admits an essentially unique factorization through the forgetful functor $\Theta : \mathrm{CAlg}_k^{\Delta} \to \mathrm{CAlg}_k^{\mathrm{cn}}$. This follows from the observation that Θ is an equivalence when restricted to discrete objects. To prove that Ψ^{alg} commutes with all small colimits, it suffices to observe that the functor

$$\mathcal{C} \to \mathrm{CAlg}_R^{\Delta} \qquad \mathbf{Z}^k \mapsto R[\mathbf{Z}^k]$$

preserves finite coproducts; see Proposition HTT.5.5.8.15 . □

Proposition 25.1.5.3. *Let R be a commutative ring, and let σ :*

$$\begin{array}{ccc} \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}} & \xrightarrow{\Psi^{\mathrm{alg}}} & \mathrm{CAlg}_R^{\Delta} \\ \downarrow \Omega^{\infty} & & \downarrow \Theta \\ \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S}) & \xrightarrow{\Psi} & \mathrm{CAlg}_R^{\mathrm{cn}} \end{array}$$

be the commutative diagram of ∞-categories supplied by Proposition 25.1.5.2. Then σ is left adjointable (Definition **??***).*

Lemma 25.1.5.4. *Let R be a commutative ring. For every object $A \in \mathrm{CAlg}_k^{\Delta}$, the canonical map*

$$\mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[x^{\pm 1}], A) \to \mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[x], A) \simeq \Omega^{\infty} A^{\circ}$$

induces a homotopy equivalence from $\mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[x^{\pm 1}], A)$ to the union of those connected components of $\Omega^{\infty} A^{\circ}$ which correspond to invertible elements of the commutative ring $\pi_0(A)$.

Proof. It follows from Notation 25.1.2.3 that the diagram

$$\begin{array}{ccc} R[x] & \longrightarrow & R[x^{\pm 1}] \\ \downarrow & & \downarrow \\ R[x^{\pm 1}] & \longrightarrow & R[x^{\pm 1}] \end{array}$$

is a pushout square in CAlg_R^{Δ}, which implies that the map $\rho : \mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[x^{\pm 1}], A) \to \mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[x], A)$ has (-1)-truncated homotopy fibers: that is, it induces a homotopy equivalence from $\mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[x^{\pm 1}], A)$ to some union of connected components of $\mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[x], A) \simeq \Omega^{\infty} A^{\circ}$. Notation 25.1.2.3 also shows that the diagram

$$\begin{array}{ccc} R[t] & \xrightarrow{t \mapsto 1} & R \\ \downarrow t \mapsto xy & & \downarrow \\ R[x, y] & \xrightarrow{y \mapsto x^{-1}} & R[x^{\pm 1}] \end{array}$$

is a pushout square in CAlg_R^{Δ}, so that a point of $\Omega^{\infty} A^{\circ}$ belongs to the essential image of ρ if and only if its homotopy class is an invertible element in $\pi_0(A)$. □

Proof of Proposition 25.1.5.3. The functor Ψ preserves small colimits by construction and the functor Ψ^{alg} preserves small colimits by virtue of Proposition 25.1.5.2. It follows from Corollary HTT.5.5.2.9 that the functors Ψ and Ψ^{alg} admit right adjoints

$$\mathrm{GL}_1 : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S}) \qquad \mathbf{G}_m : \mathrm{CAlg}_R^{\Delta} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}.$$

To complete the proof, it will suffice to show that for each object $A \in \mathrm{CAlg}_R^{\Delta}$, the canonical map

$$\rho : \Omega^{\infty}(\mathbf{G}_m(A)) \to \mathrm{GL}_1(A^{\circ})$$

is a homotopy equivalence of grouplike $\mathbb{E}_{\infty}$-spaces. Unwinding the definitions, we can identify the domain of ρ with the mapping space $\mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[x^{\pm 1}], A)$ and the codomain of ρ with the union of those connected components of $\Omega^{\infty} A^{\circ} \simeq \mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[x], A)$ which correspond to invertible elements of $\pi_0(A)$. The desired result now follows from Lemma 25.1.5.4. □

Remark 25.1.5.5. It follows from Proposition 25.1.5.3 that we have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{CAlg}_R^{\Delta} & \xrightarrow{\mathbf{G}_m} & \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}} \\ \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \Omega^{\infty}} \\ \mathrm{CAlg}_R^{\mathrm{cn}} & \xrightarrow{\mathrm{GL}_1} & \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S}). \end{array}$$

In other words, any realization of an object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$ as the underlying $\mathbb{E}_{\infty}$-algebra of a simplicial commutative ring determines a realization of the space of units $\mathrm{GL}_1(A)$ as the 0th space of a $\mathbf{Z}$-module spectrum. This can be regarded as an expression of the heuristic idea that the multiplication on a simplicial commutative ring is "strictly" commutative.

25.2 Symmetric Powers of Modules

To every $\mathbb{E}_1$-ring A, one can associate the ∞-category LMod_A of (left) A-module spectra. Conversely, from the ∞-category LMod_A together with the distinguished object $\mathbf{1} = A \in \mathrm{LMod}_A$, one can recover A: it can be identified with the endomorphism algebra of $\mathbf{1}$ (see Theorem HA.7.1.2.1).

One can think of an $\mathbb{E}_{\infty}$-ring A as consisting of an underlying $\mathbb{E}_1$-ring (which we will also denote by A) together with some additional structure (encoding the commutativity of A). By virtue of Theorem HA.7.1.2.1 , the underlying $\mathbb{E}_1$-ring is determined by the the ∞-category $\mathrm{Mod}_A \simeq \mathrm{LMod}_A$ (together with its unit object). We can therefore view the commutativity of A as an additional structure *on* the ∞-category Mod_A. In fact, we can be much more precise: according to Proposition HA.7.1.2.6 , we can recover the $\mathbb{E}_{\infty}$-structure on A from the symmetric monoidal structure on the ∞-category Mod_A (and vice versa).

Now suppose that A is a simplicial commutative ring. Then A has an underlying $\mathbb{E}_\infty$-ring (and therefore also an underlying $\mathbb{E}_1$-ring) which we will denote by A° (Construction 25.1.2.1). As an $\mathbb{E}_1$-ring, the ring spectrum A° is determined by the ∞-category Mod_{A° (together with its unit object). By analogy with the preceding discussion, it is natural to expect the fact that A° arises from a simplicial commutative ring to be reflected in some additional structure on the ∞-category Mod_{A°. For example, the $\mathbb{E}_\infty$-structure on A° determines a symmetric monoidal structure on Mod_{A°. If we are working over the field $\mathbf{Q}$, then we can recover the simplicial commutative ring A from its underlying $\mathbb{E}_\infty$-ring A° (Proposition 25.1.2.2), so there is nothing more to say. However, if we are working in positive or mixed characteristics, then the ∞-category of modules over a simplicial commutative ring has additional features that are not determined by its symmetric monoidal structure. Our goal in this section is to describe some of these features more explicitly.

We begin in §25.2.1 by describing a general paradigm. Suppose that, for every polynomial ring $A = \mathbf{Z}[x_1, \ldots, x_n]$ and every free A-module M of finite rank, we have a procedure for producing another A-module $F(M)$. In this case, the construction $(A, M) \mapsto F(M)$ admits a canonical extension to a functor defined on pairs (A, M), where R is *any* simplicial commutative ring and M is *any* connective A-module structure; this extension is characterized up to equivalence by the requirement that it commutes with sifted colimits (Corollary 25.2.1.3). We will be particularly interested in the case where $F(M) = \mathrm{CSym}^n_A(M)$ is the *classical nth symmetric power* of the module M; in this case, we will denote the extended construction by $M \mapsto \mathrm{LSym}^n_A(M)$ and refer to $\mathrm{LSym}^n_A(M)$ as the *nth derived symmetric power* of M (Construction **??**). Most of this section is devoted to the study of these derived symmetric power functors (and several variant constructions which we introduce in §25.2.2). We begin in §25.2.3 by establishing some elementary properties of derived symmetric powers: for example, they are compatible with base change in A (Proposition **??**) and carry flat modules to flat modules (Corollary 25.2.3.3).

In §25.2.4 and §25.2.5, we study the connectivity and finiteness properties of the derived symmetric power $\mathrm{LSym}^n_A(M)$. We prove an essentially connectivity estimate which asserts that if M is m-connective for $m \geqslant 2$, then $\mathrm{LSym}^n_A(M)$ is $(m+2n-2)$-connective (Proposition 25.2.4.1). The proof is based on a result of Illusie, which allows us to relate the derived symmetric powers of M to derived *divided* powers of the shift $\Sigma^{-2}(M)$ (Proposition 25.2.4.2). We also show that if M is perfect or almost perfect as an A-module, then $\mathrm{LSym}^n_A(M)$ has the same property (Proposition 25.2.5.3 and Corollary 25.2.5.2).

We refer to $\mathrm{LSym}^n_A(M)$ as the nth *derived* symmetric power of M to distinguish it from R-module given by the homotopy coinvariants for the symmetric group Σ_n on the nth tensor power $M \otimes_A \cdots \otimes_A M$. We will denote the latter by $\mathrm{Sym}^n_A(M)$ and refer to it simply as the *nth symmetric power of M*. In §25.2.6, we study the relationship between these constructions. In particular, we show that there is a canonical map $\rho : \mathrm{Sym}^n_A(M) \to \mathrm{LSym}^n_A(M)$ which is an equivalence if A is an algebra over the field $\mathbf{Q}$ of rational numbers (Proposition 25.2.6.1). Beware that it is not an equivalence in general: roughly speaking, the failure of ρ to be an equivalence can be regarded as a measure of the information that is lost in passing from the simplicial commutative ring A to its underlying $\mathbb{E}_\infty$-ring A° (note that the definition of $\mathrm{LSym}^n_A(M)$ depends on A, while $\mathrm{Sym}^n_A(M)$ depends only on A°).

25.2.1 Modules over Simplicial Commutative Rings

We begin by introducing some definitions.

Notation 25.2.1.1. Let A be a simplicial commutative ring and let A° denote its underlying $\mathbb{E}_\infty$-ring (see Construction 25.1.2.1). We let Mod_A denote the ∞-category Mod_{A°. We will refer to objects of Mod_A as *A-modules* and to Mod_A as the *∞-category of A-modules.* We will say that an A-module M is *connective* if $\pi_n M \simeq 0$ for $n < 0$ and *discrete* if $\pi_n M \simeq 0$ for $n \neq 0$; we let $\mathrm{Mod}^{\mathrm{cn}}_A$ and $\mathrm{Mod}^{\heartsuit}_A$ denote the full subcategories of Mod_A spanned by those A-modules which are connective and discrete, respectively.

Let $\mathrm{Mod}(\mathrm{Sp})$ denote the ∞-category whose objects are pairs (A, M) where A is an $\mathbb{E}_\infty$-ring and M is an A-module spectrum (Notation 2.2.1.1). We let SCRMod denote the fiber product $\mathrm{CAlg}^\Delta \times_{\mathrm{CAlg}(\mathrm{Sp})} \mathrm{Mod}(\mathrm{Sp})$ whose objects are pairs (A, M), where A is a simplicial commutative ring and M is an A-module. We let $\mathrm{SCRMod}^{\mathrm{cn}}$ denote the full subcategory of SCRMod spanned by those pairs (A, M) where M is a connective A-module.

Proposition 25.2.1.2. *Let $\mathcal{C} \subseteq \mathrm{SCRMod}^{\mathrm{cn}}$ denote the full subcategory spanned by those pairs (A, M) where A is equivalent to a polynomial ring $\mathbf{Z}[x_1, \ldots, x_m]$ for some $m \geqslant 0$ and M is equivalent to A^n for some $n \geqslant 0$. Then the objects of $\mathcal{C}$ form compact projective generators for $\mathrm{SCRMod}^{\mathrm{cn}}$: that is, the inclusion $\mathcal{C} \hookrightarrow \mathrm{SCRMod}^{\mathrm{cn}}$ extends to an equivalence of ∞-categories $\mathcal{P}_\Sigma(\mathcal{C}) = \mathrm{Fun}^\pi(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{SCRMod}^{\mathrm{cn}}$.*

Proof. Note that $\mathcal{C}$ consists of those objects of $\mathrm{SCRMod}^{\mathrm{cn}}$ which can be obtained as coproduct of finitely many copies of the objects $C = (\mathbf{Z}[x], 0)$ and $D = (\mathbf{Z}, \mathbf{Z})$. Unwinding the definitions, we see that C and D corepresent functors $\mathrm{SCRMod}^{\mathrm{cn}} \to \mathcal{S}$ given by

$$(A, M) \mapsto \Omega^\infty A^\circ \qquad (A, M) \mapsto \Omega^\infty M.$$

Since both of these functors preserve sifted colimits, the objects C and D are compact and projective, so that $\mathcal{C}$ consists of compact projective objects of $\mathrm{SCRMod}^{\mathrm{cn}}$. It follows from Proposition HTT.?? that the inclusion $f : \mathcal{C} \hookrightarrow \mathrm{SCRMod}^{\mathrm{cn}}$ extends to a fully faithful embedding $F : \mathcal{P}_\Sigma(\mathcal{C}) \to \mathrm{SCRMod}^{\mathrm{cn}}$ which commutes with sifted colimits. Since f preserves finite coproducts, the functor F preserves small colimits (Proposition **??**) and therefore admits a right adjoint G (Corollary HTT.5.5.2.9). To prove that F is an equivalence of

∞-categories, it will suffice to show that the functor G is conservative. This is clear, since the conservative functor

$$((A, M) \in \mathrm{SCRMod}^{\mathrm{cn}}) \mapsto ((\Omega^{\infty} A^{\circ}, \Omega^{\infty} M) \in \mathcal{S} \times \mathcal{S})$$

factors through G. □

Corollary 25.2.1.3. *Let $\mathcal{C} \subseteq \mathrm{SCRMod}^{\mathrm{cn}}$ be as in Proposition 25.2.1.2, let $\mathcal{E}$ be an arbitrary ∞-category which admits small sifted colimits, and let $\mathrm{Fun}_{\Sigma}(\mathrm{SCRMod}_k^{\mathrm{cn}}, \mathcal{E})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{SCRMod}^{\mathrm{cn}}, \mathcal{E})$ spanned by those functors which preserve sifted colimits. Then the restriction functor*

$$\mathrm{Fun}_{\Sigma}(\mathrm{SCRMod}^{\mathrm{cn}}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

is an equivalence of ∞-categories.

Proof. Compare Proposition HTT.5.5.8.15 with Proposition 25.2.1.2. □

25.2.2 Derived Symmetric Powers

We can state Corollary 25.2.1.3 more informally as follows: to construct a functor $F : \mathrm{SCRMod}^{\mathrm{cn}} \to \mathcal{E}$ which commutes with sifted colimits, it suffices to specify the values of F on pairs (A, M) where A is a polynomial ring and M is a finitely generated free module over A. This is very convenient, since the collection of such pairs (A, M) forms an *ordinary* category rather than an ∞-category (by virtue of the fact that the higher homotopy groups of A and M vanish). We can use this to construct many examples of such functors "by hand":

Construction 25.2.2.1 (Derived Symmetric Powers)**.** Let A be a commutative ring, let $n \geqslant 0$ be an integer, and let M be a discrete A-module. We let $\mathrm{CSym}^n_A M$ denote the *discrete* A-module given by the nth symmetric power of A of M: that is, the quotient of the n-fold tensor product $\pi_0(M \otimes_A \cdots \otimes_A M)$ by the action of the symmetric group Σ_n given by permuting the factors

$$\sigma(x_1 \otimes \cdots \otimes x_n) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

Note that if M is a free A-module of rank m with basis $\{x_1, \ldots, x_m\}$, then $\mathrm{CSym}^n_A M$ is also a free A-module of rank $\binom{n+m-1}{n}$, with basis given by the set of monomials $x_1^{d_1} \cdots x_m^{d_m}$ satisfying $d_1 + \cdots + d_m = n$. In particular, the construction $(A, M) \mapsto (A, \mathrm{CSym}^n_A M)$ determines a functor $f : \mathcal{C} \to \mathcal{C}$, where $\mathcal{C} \subseteq \mathrm{SCRMod}^{\mathrm{cn}}$ is defined as in Proposition

25.2.1.2. It follows from Corollary 25.2.1.3 that f admits an essentially unique extension $F : \mathrm{SCRMod}^{\mathrm{cn}} \to \mathrm{SCRMod}^{\mathrm{cn}}$ which commutes with sifted colimits and that the diagram

$$\begin{array}{ccc} \mathrm{SCRMod}^{\mathrm{cn}} & \xrightarrow{F} & \mathrm{SCRMod}^{\mathrm{cn}} \\ \downarrow & & \downarrow \\ \mathrm{CAlg}^{\Delta} & \xrightarrow{\mathrm{id}} & \mathrm{CAlg}^{\Delta} \end{array}$$

commutes up to canonical equivalence. We can therefore write $F(A, M) = (A, \mathrm{LSym}^n_A M)$ for some object $\mathrm{LSym}^n_A(M) \in \mathrm{Mod}^{\mathrm{cn}}_A$. We will refer to $\mathrm{LSym}^n_A(M)$ as the *derived nth symmetric power of M over A*.

Construction 25.2.2.2 (Derived Exterior Powers)**.** Let A be a polynomial ring, let $n \geqslant 0$ be an integer, and let M be a free A-module of finite rank. We let $\bigwedge^n_A(M)$ denote the nth exterior power of A of M: that is, the quotient of the n-fold tensor product $M^{\otimes n}$ by the action of the symmetric group Σ_n by signed permutations

$$\sigma(x_1 \otimes \cdots \otimes x_n) = \mathrm{sgn}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)};$$

here $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ denotes the sign of the permutation σ. Note that if M is a free A-module of rank m with basis $\{x_1, \ldots, x_m\}$, then $\bigwedge^n_A(M)$ is also a free A-module of rank $\binom{m}{n}$, with basis given the images of $x_{i_1} \otimes \cdots \otimes x_{i_n}$ where $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant m$. In particular, the construction $(A, M) \mapsto (A, \bigwedge^n_A(M))$ determines a functor $f : \mathcal{C} \to \mathcal{C}$, where $\mathcal{C} \subseteq \mathrm{SCRMod}^{\mathrm{cn}}$ is defined as in Proposition 25.2.1.2. It follows from Corollary 25.2.1.3 that f admits an essentially unique extension $F : \mathrm{SCRMod}^{\mathrm{cn}} \to \mathrm{SCRMod}^{\mathrm{cn}}$ which commutes with sifted colimits and that the diagram

$$\begin{array}{ccc} \mathrm{SCRMod}^{\mathrm{cn}} & \xrightarrow{F} & \mathrm{SCRMod}^{\mathrm{cn}} \\ \downarrow & & \downarrow \\ \mathrm{CAlg}^{\Delta} & \xrightarrow{\mathrm{id}} & \mathrm{CAlg}^{\Delta} \end{array}$$

commutes up to canonical equivalence. We can therefore write $F(A, M) = (A, \bigwedge^n_A(M))$ for some object $\bigwedge^n_A(M) \in \mathrm{Mod}^{\mathrm{cn}}_A$. We will refer to $\bigwedge^n_A(M)$ as the *derived nth exterior power of M over A*.

Construction 25.2.2.3 (Derived Divided Powers)**.** Let A be a polynomial ring, let $n \geqslant 0$ be an integer, and let M be a free A-module. We let $\Gamma^n_A(M)$ denote the nth divided power of A of M: that is, the submodule of $\pi_0(M \otimes_A \cdots \otimes_A M)$ given by taking invariants for the action of the symmetric group Σ_n given by permuting the factors. Note that if M is a free A-module of rank m, then $\Gamma^n_A(M)$ is also a free A-module of rank $\binom{n+m-1}{n}$. In particular, the construction $(A, M) \mapsto (A, \Gamma^n_A M)$ determines a functor $f : \mathcal{C} \to \mathcal{C}$, where

$\mathcal{C} \subseteq \mathrm{SCRMod}^{\mathrm{cn}}$ is defined as in Proposition 25.2.1.2. It follows from Corollary 25.2.1.3 that f admits an essentially unique extension $F : \mathrm{SCRMod}^{\mathrm{cn}} \to \mathrm{SCRMod}^{\mathrm{cn}}$ which commutes with sifted colimits and that the diagram

$$\begin{array}{ccc} \mathrm{SCRMod}^{\mathrm{cn}} & \xrightarrow{F} & \mathrm{SCRMod}^{\mathrm{cn}} \\ \downarrow & & \downarrow \\ \mathrm{CAlg}^{\Delta} & \xrightarrow{\mathrm{id}} & \mathrm{CAlg}^{\Delta} \end{array}$$

commutes up to canonical equivalence. We can therefore write $F(A, M) = (A, \Gamma^n_A(M))$ for some object $\Gamma^n_A(M) \in \mathrm{Mod}^{\mathrm{cn}}_A$. We will refer to $\Gamma^n_A(M)$ as the *derived nth divided power of M over A.*

Remark 25.2.2.4. Construction 25.2.2.3 can be regarded as dual to Construction 25.2.2.1 in the sense that if M is a free module of finite rank over a commutative ring A, then there is a canonical isomorphism

$$\Gamma^n_A(M^\vee) \simeq (\mathrm{CSym}^n_A(M))^\vee.$$

Similarly, one can contemplate a "dual" version of Construction 25.2.2.2, but this does not lead to anything new: the formation of exterior powers $M \mapsto \bigwedge^n_A(M)$ is already self-dual in the sense that there is a canonical isomorphism $\bigwedge^n_A(M^\vee) \simeq (\bigwedge^n_A(M))^\vee$ when M is a free A-module of finite rank.

Example 25.2.2.5. Let A be a simplicial commutative ring and let M be an A-module. When $n = 0$, we have $\mathrm{LSym}^n_A(M) \simeq \bigwedge^n_A(M) \simeq \Gamma^n_A(M) \simeq A$. When $n = 1$, we have $\mathrm{LSym}^n_A(M) \simeq \bigwedge^n_A(M) \simeq \Gamma^n_A(M) \simeq M$.

Construction 25.2.2.6. [Derived Symmetric Algebras] For every morphism of simplicial commutative rings $\phi : A \to B$, the underlying map of $\mathbb{E}_\infty$-rings $A^\circ \to B^\circ$ allows us to regard B° as a connective A-module. This observation determines a forgetful functor

$$G : \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\Delta}) \to \mathrm{SCRMod}^{\mathrm{cn}}$$

$$(\phi : A \to B) \mapsto (A, B^\circ).$$

The functor G preserves small limits and filtered colimits and therefore admits a left adjoint $F : \mathrm{SCRMod}^{\mathrm{cn}} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\Delta})$. Unwinding the definitions, we see that the functor F carries a pair $(A, M) \in \mathrm{SCRMod}$ to a morphism of simplicial commutative rings whose domain can be identified with A, and whose codomain we will denote by $\mathrm{LSym}^*_A(M)$ and refer to as the *derived symmetric algebra on M*. The composite functor $G \circ F : \mathrm{SCRMod}^{\mathrm{cn}} \to \mathrm{SCRMod}^{\mathrm{cn}}$ preserves sifted colimits and is therefore determined (up to equivalence) by its restriction to the full subcategory $\mathcal{C} \subseteq \mathrm{SCRMod}^{\mathrm{cn}}$ of Proposition 25.2.1.2. Note that if $A = \mathbf{Z}[x_1, \ldots, x_m]$ is a polynomial ring over $\mathbf{Z}$ and if M is a free A-module on generators $y_1, \ldots, y_n$, then $F(A, M)$ can be identified with the map of polynomial rings $\mathbf{Z}[x_1, \ldots, x_m] \to \mathbf{Z}[x_1, \ldots, x_m, y_1, \ldots, y_n]$, so that $(G \circ F)(A, M)$ can be identified with the pair $(A, \bigoplus_{k \geqslant 0} \mathrm{CSym}^k_A(M))$. Applying Corollary 25.2.1.3, we see that for *any* connective module M over *any* simplicial commutative ring A, we have a canonical equivalence of A-modules

$$\mathrm{LSym}^*_A(M) \simeq \bigoplus_{n \geqslant 0} \mathrm{LSym}^n_A(M),$$

where the right hand side is defined as in Construction 25.2.2.1.

25.2.3 Properties of Derived Functors

Let $\phi : A \to B$ be any morphism of simplicial commutative rings. Then the forgetful functor $\mathrm{Mod}_B \to \mathrm{Mod}_A$ admits a left adjoint, which we will denote by $M \mapsto B \otimes_A M$.

Proposition 25.2.3.1. *Let $q : \mathrm{SCRMod}^{\mathrm{cn}} \to \mathrm{CAlg}^{\Delta}$ denote the forgetful functor, let $n \geqslant 0$ be an integer, and let $F : \mathrm{SCRMod}^{\mathrm{cn}} \to \mathrm{SCRMod}^{\mathrm{cn}}$ be as in Construction 25.2.2.1, 25.2.2.2, or 25.2.2.3. Then F carries q-coCartesian morphisms to q-coCartesian morphisms. In other words, for every morphism $\phi : A \to B$ of simplicial commutative rings and every connective A-module M, the canonical maps*

$$B \otimes_A \mathrm{LSym}^n_A(M) \to \mathrm{LSym}^n_B(B \otimes_A M)$$

$$B \otimes_A {\bigwedge}^n_A(M) \to {\bigwedge}^n_B(B \otimes_A M)$$

$$B \otimes_A \Gamma^n_A(M) \to \Gamma^n_B(B \otimes_A M)$$

are equivalences.

Proof. We will prove that the map $\alpha_M : B \otimes_A \mathrm{LSym}^n_A M \to \mathrm{LSym}^n_B(B \otimes_A M)$; the proof in the remaining cases differs only by a change of notation. Note that the construction $M \mapsto \alpha_M$ commutes with sifted colimits. Consequently, it will suffice to prove that α_M is an equivalence in the special case where M is a free A-module of finite rank. In particular, we may assume that $M = A \otimes_{\mathbf{Z}} M_0$ where $M_0 \simeq \mathbf{Z}^m$ is a free $\mathbf{Z}$-module of finite rank. We then have a commutative diagram

$$\begin{array}{ccc} & B \otimes_A A \otimes_{\mathbf{Z}} \mathrm{LSym}^n_{\mathbf{Z}} M_0 & \\ \swarrow & & \searrow \\ B \otimes_A \mathrm{LSym}^n_A(A \otimes_{\mathbf{Z}} M_0) & \xrightarrow{\alpha_M} & \mathrm{LSym}^n_B(B \otimes_A A \otimes_{\mathbf{Z}} M_0). \end{array}$$

Consequently, it will suffice to prove that α_M is an equivalence in the special case where $A = \mathbf{Z}$ is the ring of integers. Let us therefore regard A and M as fixed, and regard the morphism α_M as a functor of $B \in \mathrm{CAlg}^{\Delta}$. Since this functor commutes with sifted colimits, it will suffice to prove that α_M is an equivalence in the special case where $B \simeq \mathbf{Z}[x_1, \ldots, x_k]$ is a polynomial ring on finitely many generators. Unwinding the definitions, we are reduced to proving that the map

$$B \otimes_{\mathbf{Z}} \mathrm{CSym}^n_{\mathbf{Z}}(\mathbf{Z}^m) \to \mathrm{CSym}^n_B(B^m)$$

is an equivalence. This follows immediately from the explicit description of CSym^n_B on free B-modules given in Construction 25.2.2.1. □

Corollary 25.2.3.2. *Let A be a simplicial commutative ring, let $n \geqslant 0$, and let M be an A-module which is (locally) free of some finite rank r. Then the A-modules $\mathrm{LSym}^n_A(M)$, $\bigwedge^n_A(M)$, and $\Gamma^n_A(M)$ are (locally) free of ranks $\binom{n+r-1}{n}$, $\binom{r}{n}$, and $\binom{n+r-1}{n}$, respectively.*

Proof. Using Proposition 25.2.3.1, we can reduce to the case where M is a free A-module of rank r. In this case, we can write $M = A \otimes_{\mathbf{Z}} \mathbf{Z}^r$. Using Proposition 25.2.3.1 again, we can reduce to the case where $A = \mathbf{Z}$ and $M = \mathbf{Z}^r$, in which case the desired results follow from explicit calculation. □

Corollary 25.2.3.3. *Let A be a simplicial commutative ring, let $n \geqslant 0$ be an integer, and let M be a flat A-module. Then the A-modules $\mathrm{LSym}^n_A(M)$, $\bigwedge^n_A(M)$, and $\Gamma^n_A(M)$ are also flat.*

Proof. Combine Corollary 25.2.3.2 with Theorem HA.7.2.2.15 . □

It follows from Corollary 25.2.3.3 that if A is a commutative ring and M is a flat A-module, then the modules $\mathrm{LSym}^n_A(M)$, $\bigwedge^n_A(M)$, and $\Gamma^n_A(M)$ are discrete. In fact, we can be more precise:

Proposition 25.2.3.4. *Let A be a commutative ring (regarded as a discrete object of CAlg^{Δ}) and let M be a flat A-module. Then:*

(1) *There is a canonical isomorphism $\alpha : \mathrm{LSym}^n_A(M) \simeq \mathrm{CSym}^n_A(M)$.*

(2) *There is a canonical isomorphism $\beta : \bigwedge^n_A(M) \simeq \bigwedge'^n_A(M)$, where $\bigwedge'^n_A(M)$ denotes the quotient of the nth tensor power $M^{\otimes n}$ by the action of the symmetric group Σ_n by signed permutations*

$$\sigma(x_1 \otimes \cdots \otimes x_n) = \mathrm{sgn}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

(3) *There is a canonical isomorphism $\gamma : \Gamma^n_A(M) \simeq \Gamma'^n_A(M)$, where $\Gamma'^n_A(M)$ denotes the submodule of $M^{\otimes n}$ spanned by the elements which invariant under the action of Σ_n.*

Proof. We will prove (1); the proofs of (2) and (3) are similar. Let $\mathcal{E} \subseteq \mathrm{SCRMod}^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{SCRMod}^{\mathrm{cn}}$ spanned by those objects of the form (A, M), where A is an ordinary commutative ring and M is a discrete A-module. Define functors

$$F, F' : \mathcal{E} \to \mathrm{SCRMod}^{\mathrm{cn}}$$

by setting $F(A, M) = (A, \mathrm{LSym}^n_A(M))$ and $F'(A, M) = (A, \mathrm{CSym}^n_A(M))$. By definition, the functors F and F' agree on the subcategory $\mathcal{C} \subseteq \mathcal{E}$ defined in Proposition 25.2.1.2, and F is a left Kan extension of its restriction $F|_{\mathcal{C}}$. It follows that there is an essentially unique natural transformation $\alpha : F \to F'$ which is the identity on objects of $\mathcal{C}$. For every object $(A, M) \in \mathcal{E}$, this natural transformation determines a map $\alpha_{A,M} : \mathrm{LSym}^n_A(M) \to \mathrm{CSym}^n_A(M)$. We claim that $\alpha_{A,M}$ is an equivalence when M is a flat A-module. To prove this, we first note that for fixed A, both sides commute with filtered colimits in M. Using Theorem HA.7.2.2.15 , we can reduce to the case where M is a free A-module of finite rank. In this case, M is obtained from a free $\mathbf{Z}$-module by extension of scalars, so the desired result follows from Proposition 25.2.3.1. □

Warning 25.2.3.5. Let A be a commutative ring and let M be a discrete A-module. If M is not flat, then the A-modules $\mathrm{LSym}^n_A(M)$, $\bigwedge^n_A(M)$, and $\Gamma^n_A(M)$ are not necessarily discrete. In particular, $\bigwedge^n_A(M)$ and $\Gamma^n_A(M)$ need not coincide with the usual nth exterior power and nth divided power module of M, in the sense of commutative algebra (however, the usual algebraic constructions can be recovered from $\bigwedge^n_A(M)$ and $\Gamma^n_A(M)$ by passing to connected components).

25.2.4 Connectivity of Derived Symmetric Powers

Our next goal is to prove the following:

Proposition 25.2.4.1. *Let A be a simplicial commutative ring, let M be a connective A-module, and let $n > 0$. If M is 1-connective, then $\mathrm{LSym}^n_A(M)$ is n-connective. If M is m-connective for $m \geqslant 2$, then $\mathrm{LSym}^n_A(M)$ is $(m + 2n - 2)$-connective.*

Our proof is based on the following result, which relates the functors defined in Constructions 25.2.2.1, 25.2.2.2, and 25.2.2.3:

Proposition 25.2.4.2 (Illusie)**.** *For every simplicial commutative ring A, every connective A-module M, and every nonnegative integer n, there are canonical equivalences*

$$\mathrm{LSym}^n_A(\Sigma M) \simeq \Sigma^n \bigwedge\nolimits^n_A(M) \qquad \bigwedge\nolimits^n_A(\Sigma M) \simeq \Sigma^n \Gamma^n_A(M)$$

in the ∞-category Mod_A.

Before giving the proof of Proposition 25.2.4.2, let us show that it implies Proposition 25.2.4.1.

Lemma 25.2.4.3. *Let A be a simplicial commutative ring and let M be an A-module which is m-connective for some $m \geqslant 0$. Then, for each $n > 0$, the derived divided power $\Gamma^n_A(M)$ is also m-connective.*

Proof. We proceed by induction on m, the case $m = 0$ being trivial. To carry out the inductive step, suppose that M is m-connective for $m > 0$, and let $N_\bullet$ denote the simplicial object of Mod_A given by the Čech nerve of the map $0 \to M$. Then each N_k is equivalent to $(\Sigma^{-1} M)^k$ and is therefore $(m-1)$-connective. It that $(\Gamma^n_A M) \simeq |\Gamma^n_A(N_\bullet)|$ is $(m-1)$-connective and that $\pi_{m-1} \Gamma^n_A(M)$ is a quotient of $\pi_{m-1}(\Gamma^n_A(N_0)) \simeq 0$, so that $\Gamma^n_A(M)$ is m-connective as desired. □

Proof of Proposition 25.2.4.1. If M is 1-connective, then $\Sigma^{-1}(M)$ is connective, so that $\bigwedge^n_A(\Sigma^{-1}(M))$ is a well-defined connective A-module. The first assertion now follows from the equivalence

$$\mathrm{LSym}^n_A(M) \simeq \Sigma^n \bigwedge\nolimits^n_A(\Sigma^{-1}(M))$$

provided by Proposition 25.2.4.2.

If M is m-connective for $m \geqslant 2$, then $\Sigma^{-2}(M)$ is connective. Proposition 25.2.4.2 then supplies an equivalence $\mathrm{LSym}^n_A(M) \simeq \Sigma^{2n} \Gamma^n_A(\Sigma^{-2}(M))$. We are therefore reduced to proving that A-module $\Gamma^n_A(\Sigma^{-2}(M))$ is $(m-2)$-connective, which follows from Lemma 25.2.4.3. □

Proof of Proposition 25.2.4.1. Let us first suppose that R is a discrete commutative ring and that we are given a short exact sequence

$$0 \to M' \xrightarrow{\rho} M \to M'' \to 0$$

between free R-modules of finite rank. Let $\mathrm{CSym}^*_R(M)$ denote the (discrete) symmetric algebra over R generated by M, and let $\mathrm{CSym}^*_R(M'')$ be defined similarly. Then We can regard $\mathrm{CSym}^*_R(M'')$ as the quotient of $\mathrm{CSym}^*_R(M)$ by a regular sequence given by a set of generators for M'. We therefore have a Koszul complex which resolves $\mathrm{CSym}^*_R(M'')$ by free $\mathrm{CSym}^*_R(M)$-modules of finite rank, given by

$$\cdots \to (\bigwedge\nolimits^1_R M') \otimes_R (\mathrm{CSym}^*_R M) \to (\bigwedge\nolimits^0_R M') \otimes_R (\mathrm{CSym}^*_R M) \to (\mathrm{LSym}^*_R M') \to 0.$$

Restricting to our attention to terms which are homogeneous of degree n, we obtain an exact sequence

$$0 \to (\bigwedge\nolimits^n_R M') \otimes_R (\mathrm{CSym}^0_R M) \xrightarrow{d_n} \cdots \to (\bigwedge\nolimits^0_R M') \otimes_R (\mathrm{CSym}^n_R M) \xrightarrow{d_0} \mathrm{CSym}^n_R M'' \to 0.$$

Applying the same construction to the dual exact sequence

$$0 \to M''^{\vee} \to M^{\vee} \to M'^{\vee} \to 0$$

and dualizing (using Remark 25.2.2.4), we obtain another exact sequence

$$0 \to \Gamma^n_R(M') \xrightarrow{d'_0} (\bigwedge^0_R M'') \otimes_R (\Gamma^n_R M) \to \cdots \xrightarrow{d'_n} (\bigwedge^n_R M'') \otimes_R (\Gamma^0_R M) \to 0.$$

Let $f_i(R, \rho)$ denote the image of the differential d_i and let $g_i(R, \rho)$ denote the image of the differential d'_i, so that we have equivalences

$$f_0(R, \rho) \simeq \mathrm{CSym}^n_R(M'') \qquad g_0(R, \rho) \simeq \Gamma^n_R(M')$$

$$f_n(R, \rho) \simeq \bigwedge^n_R(M') \qquad g_n(R, \rho) \simeq \bigwedge^n_R(M'')$$

and short exact sequence

$$0 \to f_{i+1}(R, \rho) \to (\bigwedge^i_R(M')) \otimes_R (\mathrm{CSym}^{n-i}_R(M)) \to f_i(R, \rho) \to 0$$

$$0 \to g_i(R, \rho) \to (\bigwedge^i_R(M'')) \otimes_R (\Gamma^{n-i}_R(M)) \to g_{i+1}(R, \rho) \simeq 0.$$

Let $\mathcal{E}$ denote the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{SCRMod}^{\mathrm{cn}}) \times_{\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\Delta})} \mathrm{CAlg}^{\Delta}$ whose objects are pairs $(A, \rho : M' \to M)$, where A is a simplicial commutative ring and ρ is a morphism of connective A-modules. Let $\mathcal{E}_0 \subseteq \mathcal{E}$ be the full subcategory spanned by those pairs $(A, \rho : M' \to M)$ where A is a polynomial ring $\mathbf{Z}[x_1, \ldots, x_k]$ and ρ fits into a short exact sequence

$$0 \to M' \xrightarrow{\rho} M \to M'' \to 0$$

of finitely generated free modules over A. Equivalently, we can describe $\mathcal{E}_0$ as the full subcategory of $\mathcal{E}$ generated under coproducts by the objects

$$C = (\mathbf{Z}[x], \mathrm{id} : 0 \to 0) \qquad D = (\mathbf{Z}, \mathrm{id} : \mathbf{Z} \to \mathbf{Z}) \qquad E = (\mathbf{Z}, \rho_0 : 0 \to \mathbf{Z}).$$

Note that the objects C, D, and E corepresent functors $\mathcal{E} \to \mathcal{S}$ given by

$$(A, \rho) \mapsto \Omega^{\infty} A^{\circ} \qquad (A, \rho) \mapsto \Omega^{\infty} M' \qquad (A, \rho) \mapsto \Omega^{\infty} M,$$

which preserve sifted colimits and are jointly conservative. It follows that the inclusion $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ extends to an equivalence of ∞-categories $\mathcal{P}_{\Sigma}(\mathcal{E}_0) \simeq \mathcal{E}$, so that any functor $u : \mathcal{E} \to \mathrm{SCRMod}^{\mathrm{cn}}$ admits an essentially unique extension to a functor $U : \mathcal{E} \to \mathrm{SCRMod}^{\mathrm{cn}}$ which commutes with sifted colimits. Applying this observation to the functors

$$(R, \rho : M' \to M) \mapsto (R, f_i(R, \rho)) \qquad (R, \rho : M' \to M) \mapsto (R, g_i(R, \rho)),$$

we obtain functors $\mathcal{E} \to \mathrm{SCRMod}^{\mathrm{cn}}$ which we will denote by

$$(A, \rho) \mapsto (A, F_i(A, \rho)) \qquad (A, \rho) \mapsto (A, G_i(A, \rho)).$$

We have canonical equivalences

$$F_0(A, \rho : M' \to M) \simeq \mathrm{LSym}^n_A \operatorname{cofib}(\rho) \qquad f_n(A, \rho : M' \to M) \simeq \bigwedge\nolimits^n_A M'$$

$$G_0(A, \rho : M' \to M) \simeq \Gamma^n_A M' \qquad G_n(A, \rho : M' \to M) \simeq \bigwedge\nolimits^n_A \operatorname{cofib}(\rho)$$

and fiber sequences

$$F_{i+1}(A, \rho : M' \to M) \to (\bigwedge\nolimits^i_A M') \otimes_A (\mathrm{CSym}^{n-i}_A M) \to F_i(A, \rho : M' \to M)$$

$$G_i(A, \rho : M' \to M) \to (\bigwedge\nolimits^i_A \operatorname{cofib}(\rho)) \otimes_A (\Gamma^{n-i}_A M) \to G_{i+1}(A, \rho : M' \to M).$$

In the special case where $M = 0$, the middle terms of these fiber sequences vanish, so we obtain the desired equivalences

$$\mathrm{LSym}^n_A(\Sigma M') \simeq F_0(A, \rho : M' \to 0) \simeq \cdots \simeq \Sigma^n F_n(A, \rho : M' \to 0) \simeq \Sigma^n \bigwedge\nolimits^n_A (M')$$

$$\bigwedge\nolimits^n_A (\Sigma M') \simeq G_n(A, \rho : M' \to 0) \simeq \cdots \simeq \Sigma^n G_0(A, \rho : M' \to 0) \simeq \Sigma^n \Gamma^n_A(M').$$

□

By virtue of Proposition 25.2.4.2, the derived exterior power functor and derived divided power functor of Constructions 25.2.2.2 and 25.2.2.3 are completely determined by the derived symmetric power functor of Construction 25.2.2.1. Consequently, we will mainly confine our discussion to derived symmetric powers in what follows.

25.2.5 Finiteness of Derived Symmetric Powers

We now study finiteness properties of the derived symmetric powers $\mathrm{LSym}^n_A(M)$ (note that, by virtue of Proposition 25.2.4.1, this subsumes also the study of finiteness properties of the constructions $\bigwedge^n_A(M)$ and $\Gamma^n_A(M)$).

Proposition 25.2.5.1. *Let A be a simplicial commutative ring and let M be a connective A-module which is perfect to order m for some $m \geqslant 0$. Then, for every $n \geqslant 0$, the A-modules $\mathrm{LSym}^n_A(M)$ is perfect to order n.*

Proof. Since M is perfect to order n, Corollary 2.7.2.4 implies that we can write M as the geometric realization of a simplicial object $M_\bullet$ of $\mathrm{Mod}^{\mathrm{cn}}_A$ where M_k is a free A-module of finite rank for $0 \leqslant k \leqslant m$. Since the functor LSym^n_A commutes with sifted colimits, we can write $\mathrm{LSym}^n_A(M)$ as the geometric realization of the simplicial object $\mathrm{LSym}^n_A(M_\bullet)$. It follows from Corollary 25.2.3.2 that $\mathrm{LSym}^n_A(M_k)$ is free of finite rank for $0 \leqslant k \leqslant m$, so that $\mathrm{LSym}^n_A(M) \simeq |\mathrm{LSym}^n_A(M_\bullet)|$ is perfect to order m by virtue of Corollary 2.7.2.4. □

Corollary 25.2.5.2. *Let A be a simplicial commutative ring, let $n \geqslant 0$, and let M be an A-module which is connective and almost perfect. Then the A-module $\mathrm{LSym}^n_A(M)$ is almost perfect.*

We now prove the analogous statement for perfect modules.

Proposition 25.2.5.3. *Let A be a simplicial commutative ring, let M be a connective perfect A-module, and let $n \geqslant 0$ be an integer. Then the derived symmetric power is $\mathrm{LSym}^n_A(M)$ is a perfect A-module.*

The proof of Proposition 25.2.5.3 will require an auxiliary construction.

Construction 25.2.5.4. Let R be a discrete commutative ring and suppose we are given a short exact sequence of finitely generated free R-modules

$$0 \to M' \xrightarrow{\rho} M \to M'' \to 0.$$

For each $d \geqslant 0$, let $\overline{F}^d(\rho)$ denote the $\mathrm{CSym}^*_R(M')$-submodule of $\mathrm{CSym}^*_R(M)$ generated by the symmetric powers $\mathrm{CSym}^{d'}_R(M)$ for $0 \leqslant d' \leqslant d$. Passing to homogeneous elements of some degree $n \geqslant 0$, we obtain a finite filtration

$$0 = \overline{F}^{-1,n}(\rho) \subseteq \overline{F}^{0,n}(\rho) \subseteq \overline{F}^{1,n}(\rho) \subseteq \cdots \subseteq \overline{F}^{n,n}(\rho) = \mathrm{CSym}^n_R(M)$$

whose successive quotients $\overline{F}^{d,n}(\rho)/\overline{F}^{d-1,n}(\rho)$ are canonically isomorphic to $\mathrm{CSym}^d_R(M'') \otimes_R \mathrm{CSym}^{n-d}_R(M')$. Let $\mathcal{E}_0 \subseteq \mathcal{E}$ be as in the proof of Proposition 25.2.4.2, so that the construction

$$(R, \rho) \mapsto (R, \overline{F}^{0,n}(\rho) \to \cdots \to \overline{F}^{n,n}(\rho))$$

determines a functor from $\mathcal{E}_0$ to the ∞-category

$$\mathrm{Fun}(\Delta^n, \mathrm{SCRMod}^{\mathrm{cn}}) \times_{\mathrm{Fun}(\Delta^n, \mathrm{CAlg}^{\Delta})} \mathrm{CAlg}^{\Delta}.$$

Arguing as in the proof of Proposition 25.2.4.2, we see that this functor admits an essentially unique extension to a functor

$$\mathcal{E} \to \mathrm{Fun}(\Delta^n, \mathrm{SCRMod}^{\mathrm{cn}}) \times_{\mathrm{Fun}(\Delta^n, \mathrm{CAlg}^{\Delta})} \mathrm{CAlg}^{\Delta}.$$

$$(R, \rho : M' \to M) \mapsto (R, F^{0,n}(\rho) \to \cdots \to F^{n,n}(\rho)),$$

where we have

$$F^{0,n}(\rho) \simeq \mathrm{LSym}^n_R(M') \qquad F^{n,n}(\rho) = \mathrm{LSym}^n_R(M)$$

$$\mathrm{cofib}(F^{d-1,n}(\rho) \to F^{d,n}(\rho)) \simeq \mathrm{LSym}^d_R(M'') \otimes_R \mathrm{LSym}^{n-d}_R(M').$$

Proof of Proposition 25.2.5.3. Let A be a simplicial commutative ring and let M be a connective perfect A-module; we wish to show that $\mathrm{LSym}^n_A(M)$ is also perfect. Choose an integer $m \geqslant 0$ such that M has Tor-amplitude $\leqslant m$. We proceed by induction on m. If $m = 0$, then M is a projective A-module of finite rank and the desired result follows from Corollary 25.2.3.2 (and Proposition 25.2.3.1). Let us therefore assume that $m > 0$. Our proof now proceeds also by induction on n. Since M is perfect and connective, the group $\pi_0 M$ is finitely generated as a module over $\pi_0 A$; we can therefore choose a fiber sequence $M' \xrightarrow{\rho} A^k \to M$ where M' is a connective perfect A-module having Tor-amplitude $< m$. Applying Construction 25.2.5.4, we deduce the existence of a finite sequence of maps

$$\mathrm{LSym}^n_A(M') \simeq F^{0,n}(\rho) \to F^{1,n}(\rho) \to \cdots \to F^{n,n}(\rho) \simeq \mathrm{LSym}^n_A(A^k),$$

where each of the successive quotients is given by

$$\mathrm{cofib}(F^{d-1,n}(\rho) \to F^{d,n}(\rho)) \simeq \mathrm{LSym}^d(M) \otimes_A \mathrm{LSym}^{n-d}(M').$$

Using our inductive hypotheses, we see that these cofibers are perfect for $d < n$, so that $F^{n-1,n}(\rho)$ is perfect. The fiber sequence

$$F^{n-1,n}(\rho) \to \mathrm{LSym}^n_A(A^k) \to \mathrm{LSym}^n_A(M)$$

now proves that $\mathrm{LSym}^n_A(M)$ is perfect, as desired. □

25.2.6 Comparison with Sym^*_A

Let A be an $\mathbb{E}_\infty$-ring. For any A-module M, let $\mathrm{Sym}^n_A(M)$ denote the nth symmetric power of M in the sense of Construction HA.3.1.3.9: that is, the A-module obtained by taking the (homotopy) coinvariants for the action of the symmetric group Σ_n on the n-fold tensor power $M \otimes_A \cdots \otimes_A M$. Note that if A and M are connective, then we have a canonical isomorphism

$$\pi_0(\mathrm{Sym}^n_A(M)) \simeq \mathrm{CSym}^n_{\pi_0(A)}(\pi_0(M)).$$

In particular, if A is a polynomial ring over $\mathbf{Z}$ and M is a finitely generated free module over A, then there is a canonical map $\rho : \mathrm{Sym}^n_A(M) \to \mathrm{CSym}^n_A(M)$ which exhibits $\mathrm{CSym}^n_A(M)$ as the 0-truncation of $\mathrm{Sym}^n_A(M)$. Since the functor $(A, M) \mapsto \mathrm{Sym}^n_A(M)$ commutes with sifted colimits, we can use Corollary 25.2.1.3 to extend ρ to a map $\mathrm{Sym}^n_A(M) \to \mathrm{LSym}^n_A(M)$ defined for all simplicial commutative rings A and all connective A-modules M. This map is generally *not* an equivalence. For example, when $A = M = \mathbf{Z}$, the symmetric power $\mathrm{LSym}^n_A(M)$ is discrete, but the homotopy groups of $\mathrm{Sym}^n_A(M)$ are the integral homology groups of Σ_n.

Proposition 25.2.6.1. *Let A be a simplicial commutative ring and suppose that there exists a morphism $\mathbf{Q} \to A$. Then, for every connective A-module M and every integer $n \geqslant 0$, the map $\rho_M : \mathrm{Sym}^n_A(M) \to \mathrm{LSym}^n_A(M)$ described above is an equivalence.*

Proof. The construction $M \mapsto \rho_M$ commutes with sifted colimits. Consequently, to prove that ρ_M is an equivalence for all connective A-modules M, it will suffice to prove that it is an equivalence in the special case where M is a free A-module of finite rank. In this case, we can write $M = A \otimes_{\mathbf{Q}} V$ where V is a finite-dimensional vector space over $\mathbf{Q}$. Using Proposition 25.2.3.1 (and the observation that the symmetric monoidal, colimit-preserving functor $\mathrm{Mod}_{\mathbf{Q}} \to \mathrm{Mod}_A$ commutes with the formation of symmetric powers), we can reduce to the case $A = \mathbf{Q}$. In this case, Proposition 25.2.3.4 allows us to identify $\mathrm{LSym}^n_{\mathbf{Q}} V$ with the classical symmetric power $\mathrm{CSym}^n_{\mathbf{Q}}(V) \simeq \pi_0 \mathrm{Sym}^n_{\mathbf{Q}}(V)$. We are therefore reduced to proving that the homotopy groups $\pi_i \mathrm{Sym}^n_{\mathbf{Q}}(V)$ vanish for $i > 0$. In other words, we wish to prove that the homology groups $\mathrm{H}_i(\Sigma_n; V^{\otimes n})$ vanish for $i > 0$, which follows from the fact that Σ_n is a finite group and $V^{\otimes n}$ is a vector space over the rational numbers. □

Remark 25.2.6.2. Proposition 25.2.6.1 shows that if we are working rationally, the derived symmetric powers LSym^n_A of Construction 25.2.2.1 can be recovered from the symmetric monoidal structure on the ∞-category Mod_A, and therefore depend only on the underlying $\mathbb{E}_\infty$-ring A°. Of course this is to be expected, since the forgetful functor $\mathrm{CAlg}^{\Delta}_{\mathbf{Q}} \to \mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Q}}$ is an equivalence of ∞-categories (Proposition 25.1.2.2).

25.3 The Algebraic Cotangent Complex

Let A be an $\mathbb{E}_\infty$-ring. For any A-module M, the direct sum $A \oplus M$ can be regarded as an $\mathbb{E}_\infty$-algebra over A. One can then define a *derivation* of A into M to be a section of the projection map $e : A \oplus M \to A$. Recall that sections of e are classified by A-module morphisms $L_A \to M$, where L_A is a certain A module which we refer to as the *cotangent complex* of A (see §HA.7.3.5).

In §25.3.1, we will show that if A is the underlying $\mathbb{E}_\infty$-ring of a simplicial commutative ring and the module M is connective, the direct sum $A \oplus M$ is also the underlying $\mathbb{E}_\infty$-algebra of a simplicial commutative ring (Construction 25.3.1.1). We can then consider a *different* notion of derivation from A into M: namely, a section of the projection map $e : A \oplus M \to A$ in the ∞-category of simplicial commutative rings. Such sections can also be classified by A-module morphisms $L^{\mathrm{alg}}_A \to M$, where is an A-module that we will refer to as the *algebraic cotangent complex of* A (Definition 25.3.1.6).

Our main objective in this section is to understand the relationship between the A-modules L_A and L^{alg}_A. We begin in §25.3.3 by observing that when A admits the structure of a simplicial commutative ring, the relative cotangent complex $L_{A/\mathbf{Z}}$ can be endowed with additional structure: namely, can be regarded as a left module over a certain $\mathbb{E}_1$-ring A^+ (Remark 25.3.3.5). The ring spectrum A^+ is equipped with a canonical map $\gamma : A^+ \to A$, and the algebraic cotangent complex L^{alg}_A can be obtained from $L_{A/\mathbf{Z}}$ by extending scalars along γ (Remark 25.3.3.7). Consequently, to analyze the difference between $L_{A/\mathbf{Z}}$ and L^{alg}_A, we would like understand how far γ is from being an equivalence of ring spectra. We address this question in §25.3.4 by showing that there are canonical maps of $\mathbb{E}_1$-rings $A \to A^+ \leftarrow \mathbf{Z}$ which induce, via the multiplication on A^+, an equivalence of spectra $A \otimes_S \mathbf{Z} \to A^+$ (Proposition 25.3.4.2). Beware that this is not an equivalence of *ring* spectra; the actions of A and $\mathbf{Z}$ on A^+ generally do not commute with one another (Warning **??**). Nevertheless, the equivalence $A \otimes_S \mathbf{Z} \simeq A^+$ gives us good control the difference between A and A^+. In 25.3.5, we translate this into information about the relationship between the cotangent complexes $L_{A/\mathbf{Z}}$ and L_A^{alg}.

For many applications, it is useful to consider not only the absolute algebraic cotangent complex L_A^{alg}, but also the *relative* algebraic cotangent complex $L_{B/A}^{\mathrm{alg}}$ associated to a morphism of simplicial commutative rings $\varphi : A \to B$. In §25.3.6, we introduce a canonical map of A-modules $\mathrm{cofib}(\varphi) \to L_{B/A}^{\mathrm{alg}}$ and study its connectivity properties, which will play a key role in understanding finiteness properties of simplicial commutative rings in §**??**.

Remark 25.3.0.1. The $\mathbb{E}_1$-ring A^+ was introduced by Schwede in [185], where it is denoted by DA.

25.3.1 Derivations

Let A be a commutative ring and let M be a (discrete) A-module. Then the direct sum $A \oplus M$ admits the structure of a commutative ring, with multiplication given by

$$(a, m)(a', m') = (aa', am' + a'm).$$

We will refer to $A \oplus M$ as the *trivial square-zero extension of A by M*. Using the paradigm described in §25.2.1, we can extend this construction to the setting of simplicial commutative rings:

Construction 25.3.1.1 (Trivial Square-Zero Extensions)**.** Let $\mathcal{C} \subseteq \mathrm{SCRMod}^{\mathrm{cn}}$ be the category appearing in Proposition 25.2.1.2, whose objects are pairs (A, M) where A is a polynomial ring and M is a free A-module of finite rank. The construction $(A, M) \mapsto A \oplus M$ determines a functor from $\mathcal{C}$ to the category of commutative rings, which we regard as a full subcategory of the ∞-category CAlg^{Δ} of simplicial commutative rings. Applying Corollary 25.2.1.3, we deduce that there is an essentially unique functor $F : \mathrm{SCRMod}^{\mathrm{cn}} \to \mathrm{CAlg}^{\Delta}$ which commutes with sifted colimits and is given by $(A, M) \mapsto A \oplus M$ on the subcategory $\mathcal{C}$. We will denote the value of the functor F on a pair (A, M) by $A \oplus M$, and refer to it as the *trivial square zero extension of A by M*.

Remark 25.3.1.2. The constructions

$$(A, M) \mapsto (A \oplus M)^{\circ} \qquad (A, M) \mapsto A^{\circ} \oplus M$$

determine functors from the ∞-category $\mathrm{SCRMod}^{\mathrm{cn}}$ to the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}}$ of connective $\mathbb{E}_\infty$-algebras over $\mathbf{Z}$. These functors are canonically equivalent when restricted to the full subcategory $\mathcal{C} \subseteq \mathrm{SCRMod}^{\mathrm{cn}}$ and both commute with small sifted colimits. It follows from Corollary 25.2.1.3 that they are equivalent: that is, the underlying $\mathbb{E}_\infty$-algebra of a trivial square-zero extension $A \oplus M$ in the sense of Construction 25.3.1.1 is a trivial square-zero extension of $\mathbb{E}_\infty$-algebras in the sense of Remark **??**. In particular, when A and M are discrete, then the simplicial commutative ring $A \oplus M$ of Construction 25.3.1.1 can be identified with the usual trivial square-zero extension of A by M.

Remark 25.3.1.3. In the situation of Construction 25.3.1.1, we can restrict the construction $(A, M) \mapsto A \oplus M$ to the full subcategory $\mathcal{E} \subseteq \mathrm{SCRMod}^{\mathrm{cn}}$ spanned by those pairs (A, M) where $M \simeq 0$. This restricted functor commutes with sifted colimits and agrees with the forgetful functor $(A, M) \mapsto A$ when A is a polynomial ring over $\mathbf{Z}$, and therefore agree with the forgetful functor in general: that is, whenever $M \simeq 0$, we have a canonical equivalence $A \oplus M \simeq A$.

If M is an arbitrary connective A-module, we have essentially unique maps $0 \to M \to 0$ in Mod_A, which determine maps of simplicial commutative rings $A \to A \oplus M \to A$. In other words, we can regard the trivial square-zero extension $A \oplus M$ as an object of the pointed ∞-category $(\mathrm{CAlg}^{\Delta}_A)_{/A}$.

Definition 25.3.1.4. Let A be a simplicial commutative ring and let M be a connective A-module. We let $\mathrm{Der}(A, M)$ denote the mapping space $\mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/A}}(A, A \oplus M)$. We will refer to $\mathrm{Der}(A, M)$ as the *space of derivations of A into M*.

Note that the construction $M \mapsto \mathrm{Der}(A, M)$ is functorial. In particular, given a point $\eta \in \mathrm{Der}(A, M_0)$, evaluation on η determines a map

$$\mathrm{Map}_{\mathrm{Mod}_A}(M_0, M) \to \mathrm{Der}(A, M)$$

for every connective A-module M. If this map is a homotopy equivalence for each $M \in \mathrm{Mod}^{\mathrm{cn}}_A$, then we will say that η is a *universal derivation*.

Proposition 25.3.1.5. *Let A be a simplicial commutative ring. Then there exists a connective A-module M_0 and a universal derivation $\eta \in \mathrm{Der}(A, M_0)$.*

Proof. The construction $M \mapsto \mathrm{Der}(A, M)$ determines an accessible functor

$$\mathrm{Mod}^{\mathrm{cn}}_A \to \mathcal{S}$$

which preserves small limits. We wish to show that this functor is corepresentable, which follows from Proposition HTT.5.5.2.7 . □

Construction 25.3.1.6 (The Algebraic Cotangent Complex)**.** Let A be a simplicial commutative ring. Proposition 25.3.1.5 implies that there exists a connective A-module L_A^{alg} and a universal derivation $\eta \in \mathrm{Der}(A, L_A^{\mathrm{alg}})$. It follows from general nonsense that the pair $(L_A^{\mathrm{alg}}, \eta)$ is uniquely determined up to equivalence. We will refer to L_A^{alg} as the *algebraic cotangent complex* of A.

Example 25.3.1.7. Let A be a polynomial ring $\mathbf{Z}[x_s]$ generated by a possibly infinite set of variables $\{x_s\}_{s \in S}$. Let $\Omega_{A/\mathbf{Z}}$ denote the module of Kähler differentials of A (regarded as a module over $\mathbf{Z}$, so that $\Omega_{A/\mathbf{Z}}$ is the A-module freely generated by the symbols $\{dx_s\}_{s \in S}$. The construction

$$(f \in A) \mapsto (f, \sum \frac{\partial f}{\partial x_s} dx_s)$$

determines a derivation η of A into $\Omega_{A/\mathbf{Z}}$. Moreover, for every connective A-module M, evaluation on η determines a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_A}(\Omega_{A/\mathbf{Z}}, M) \to \mathrm{Der}(A, M) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/A}}(A, A \oplus M),$$

since both sides can be identified with the space $\prod_{s \in S} \Omega^{\infty} M$. It follows that η is a universal derivation, and therefore determines an identification $L_A^{\mathrm{alg}} \simeq \Omega_{A/\mathbf{Z}}$; in particular, L_A^{alg} is a *discrete* A-module.

Example 25.3.1.8. Let A be a commutative ring. When regarded as an object of the ∞-category CAlg^{Δ}, the commutative ring A can be written as the geometric realization of a simplicial object $P_\bullet$ where each P_n is a polynomial ring over $\mathbf{Z}$ (possibly with an infinite set of generators). It follows from Remark **??** that we can identify the algebraic cotangent complex L_A^{alg} with the geometric realization of the simplicial A-module $A \otimes_{P_\bullet} L_{P_\bullet}^{\mathrm{alg}}$. It follows from Example 25.3.1.7 that this simplicial A-module is levelwise free, given by $A \otimes_{P_\bullet} \Omega_{P_\bullet/\mathbf{Z}}$. We can therefore identify L_A^{alg} with the B-module represented by the normalized chain complex

$$\cdots \to A \otimes_{P_2} \Omega_{P_2/\mathbf{Z}} \to A \otimes_{P_1} \Omega_{P_1/\mathbf{Z}} \to A \otimes_{P_0} \Omega_{P_0/\mathbf{Z}}.$$

This chain complex is often referred to as the *cotangent complex of* A; its homology is called the *André-Quillen homology of* A and has been studied by many authors (see for example [1], [132], and [167]). We refer to it instead as the *algebraic* cotangent complex to distinguish it from the complex L_A which computes the *topological* André-Quillen homology of A (which plays a larger role throughout most of this book).

25.3.2 The Relative Algebraic Cotangent Complex

The construction $A \mapsto (A, L_A^{\mathrm{alg}})$ determines a functor from the ∞-category of simplicial commutative rings to the ∞-category $\mathrm{SCRMod}^{\mathrm{cn}}$. In particular, every map of simplicial commutative rings $\phi : A \to B$ induces a morphism of B-modules

$$B \otimes_A L_A^{\mathrm{alg}} \to L_B^{\mathrm{alg}}.$$

Notation 25.3.2.1. Let $f : A \to B$ be a morphism of simplicial commutative rings. We let $L_{B/A}^{\mathrm{alg}}$ denote the cofiber of the induced map $B \otimes_A L_A^{\mathrm{alg}} \to L_B^{\mathrm{alg}}$. We will refer to $L_{B/A}^{\mathrm{alg}}$ as the *relative algebraic cotangent complex of B over A*.

Example 25.3.2.2. Let A be a simplicial commutative ring, let M be a connective A-module, and let $B = \mathrm{LSym}_A^*(M)$ be the derived symmetric algebra on M (see Construction 25.2.2.6). The universal property of B then supplies a canonical equivalence

$$L_{B/A}^{\mathrm{alg}} \simeq B \otimes_A M.$$

In the special case where $A = \mathbf{Z}$ and M is a free A-module, this recovers the description of the absolute algebraic cotangent complex L_B^{alg} given in Example 25.3.1.7.

Remark 25.3.2.3. Fix a simplicial commutative ring B. Then the construction

$$A \mapsto B \otimes_A L_A^{\mathrm{alg}}$$

determines a functor $F : \mathrm{CAlg}_{/B}^{\Delta} \to \mathrm{Mod}_B^{\mathrm{cn}}$. Unwinding the definitions, we see that this functor is left adjoint to the square-zero extension functor

$$G : \mathrm{Mod}_B^{\mathrm{cn}} \to \mathrm{CAlg}_B^{\Delta} \qquad M \mapsto B \oplus M.$$

In particular, the functor F preserves small colimits.

Remark 25.3.2.4. Let $\phi : A \to B$ be a morphism of simplicial commutative rings. For every connective B-module M, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}^{\mathrm{alg}}, M) \simeq \mathrm{Map}_{(\mathrm{CAlg}_A^{\Delta})_{/B}}(B, B \oplus M).$$

It follows that for every pushout diagram of simplicial commutative rings

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \longrightarrow & B, \end{array}$$

there is a canonical equivalence of B-modules $L_{B/A}^{\mathrm{alg}} \simeq B \otimes_{B'} L_{B'/A'}^{\mathrm{alg}}$.

Remark 25.3.2.5. For every composable pair of morphisms between simplicial commutative rings $A \to B \to C$, there is a canonical cofiber sequence

$$C \otimes_B L_{B/A}^{\mathrm{alg}} \to L_{C/A}^{\mathrm{alg}} \to L_{C/B}^{\mathrm{alg}}$$

in the stable ∞-category Mod_C.

25.3.3 Spectrum Objects of $\mathrm{CAlg}^{\Delta}_{/A}$

Let A be an $\mathbb{E}_\infty$-ring and let M be an A-module spectrum. In §HA.7.3.4 , we equipped the direct sum $A \oplus M$ with the structure of an $\mathbb{E}_\infty$-algebra (see Remark **??**). Let $\mathrm{CAlg}^{\mathrm{aug}}_A$ denote the ∞-category of augmented $\mathbb{E}_\infty$-algebras over A, whose objects are pairs (B, ϵ) where $B \in \mathrm{CAlg}_A$ and $\epsilon : B \to A$ is an augmentation. The construction $(B, \epsilon) \mapsto \mathrm{fib}(\epsilon)$ determines a functor $\rho : \mathrm{CAlg}^{\mathrm{aug}}_A \to \mathrm{Mod}_A$. This functor commutes with finite limits, and therefore induces a functor on spectrum objects

$$\overline{\rho} : \mathrm{Sp}(\mathrm{CAlg}^{\mathrm{aug}}_A) \to \mathrm{Sp}(\mathrm{Mod}_A) \simeq \mathrm{Mod}_A .$$

According to Corollary HA.7.3.4.14 , the functor $\overline{\rho}$ is an equivalence of ∞-categories; the direct sum $A \oplus M$ is then defined as the image of M under the composition

$$\mathrm{Mod}_A \xrightarrow{\overline{\rho}^{-1}} \mathrm{Sp}(\mathrm{CAlg}^{\mathrm{aug}}_A) \xrightarrow{\Omega^\infty} \mathrm{CAlg}^{\mathrm{aug}}_A .$$

Phrased differently, the functor $M \mapsto A \oplus M$ can be characterized (up to contractible choice) by the requirement that there is a canonical equivalence $M \simeq \rho(A \oplus M)$.

In §25.3.1, we gave an analogous definition in the situation where A is a simplicial commutative ring and M is a connective A-module. However, our approach was somewhat different: in essence, we gave a direct construction of the functor $(A, M) \mapsto A \oplus M$, rather than characterizing it by a universal property. It is also possible to apply the abstract formalism of §HA.7.3.4 in the setting of simplicial commutative rings. However, this yields a slightly different notion of square-zero extension.

Notation 25.3.3.1. Let A be a simplicial commutative ring and let $\mathrm{CAlg}^{\Delta}_{/A}$ denote the ∞-category whose objects are pairs (B, ϵ), where B is a simplicial commutative ring equipped with a map $\epsilon : B \to A$ (here we do not require that B is equipped with the structure of a simplicial commutative algebra over A). The construction

$$(\epsilon : B \to A) \mapsto \mathrm{fib}(\epsilon^\circ : B^\circ \to A^\circ)$$

determines a functor of ∞-categories $\psi_0 : \mathrm{CAlg}^{\Delta}_{/A} \to \mathrm{Sp}$. This functor preserves small limits, and therefore induces a functor between spectrum objects

$$\psi : \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) \to \mathrm{Sp}(\mathrm{Sp}) \simeq \mathrm{Sp} .$$

Proposition 25.3.3.2. *Let A be a simplicial commutative ring. Then:*

(1) *The functor $\psi : \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) \to \mathrm{Sp}$ admits a left adjoint φ.*

(2) *The adjunction $\mathrm{Sp} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A})$ is monadic.*

(3) *The underlying monad* $(\psi \circ \varphi) : \mathrm{Sp} \to \mathrm{Sp}$ *is a functor which preserves small colimits.*

Proof. The functor $\psi_0 : \mathrm{CAlg}^{\Delta}_{/A} \to \mathrm{Sp}$ preserves small limits and filtered colimits, so the functor $\psi : \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) \to \mathrm{Sp}$ has the same properties. Assertion (1) now follows from the adjoint functor theorem (Corollary HTT.5.5.2.9). Note that ψ is a left exact functor between stable ∞-categories and therefore right exact. It follows that ψ preserves all small colimits, which proves (3). In particular, ψ preserves geometric realizations. Since it is also conservative, assertion (2) follows from the Barr-Beck theorem (Theorem HA.4.7.0.3). □

Corollary 25.3.3.3. *Let* A *be a simplicial commutative ring. Then the functor*

$$\psi : \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) \to \mathrm{Sp}$$

factors canonically as a composition

$$\mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) \simeq \mathrm{LMod}_{A^+} \xrightarrow{u} \mathrm{Sp}$$

for some $\mathbb{E}_1$*-algebra* A^+*; here* $u : \mathrm{LMod}_{A^+} \to \mathrm{Sp}$ *denotes the forgetful functor.*

Proof. Let $\psi \circ \varphi$ be the monad of Proposition 25.3.3.2, and take A^+ to be the image of $\psi \circ \varphi$ under the equivalence of monoidal ∞-categories $\mathrm{LFun}(\mathrm{Sp}, \mathrm{Sp}) \simeq \mathrm{Sp}$. □

Remark 25.3.3.4. Let A be a simplicial commutative ring and let A° denote its underlying $\mathbb{E}_\infty$-ring. Note that the construction $B \mapsto B^\circ$ determines a forgetful functor $\mathrm{CAlg}^{\Delta}_{/A} \to (\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}})_{/A^\circ}$. Passing to spectrum objects, we obtain a functor $u : \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) \to \mathrm{Sp}((\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}})_{/A^\circ})$. Note that the map ψ of Notation 25.3.3.1 factors as a composition

$$\mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) \xrightarrow{u} \mathrm{Sp}((\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}})_{/A^\circ}) \simeq \mathrm{Mod}_{A^\circ} \xrightarrow{v} \mathrm{Sp},$$

where v is the forgetful functor. It follows that there is an essentially unique map of $\mathbb{E}_1$-rings $\alpha : A^\circ \to A^+$ for which the diagram

$$\begin{array}{ccc} \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) & \xrightarrow{u} & \mathrm{Sp}((\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}})_{/A^\circ}) \\ \downarrow{\sim} & & \downarrow{\sim} \\ \mathrm{LMod}_{A^+} & \longrightarrow & \mathrm{Mod}_{A^\circ} \end{array}$$

commutes up to homotopy, where the bottom horizontal map is given by restriction of scalars along α.

Remark 25.3.3.5. Let A be a simplicial commutative ring and let A° be the underlying $\mathbb{E}_\infty$-algebra over $\mathbf{Z}$. Then the cotangent complex $L_{A^\circ/\mathbf{Z}}$ can be described as the image of A under the composite functor

$$\mathrm{CAlg}^{\Delta}_{/A} \xrightarrow{\Sigma^\infty_+} \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) \to \mathrm{Sp}((\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}})_{/A^\circ}) \simeq \mathrm{Mod}_{A^\circ}.$$

In particular, we see that L_{A° has a canonical preimage under the functor $\mathrm{LMod}_{A^+} \to \mathrm{Mod}_{A^\circ}$ given by restriction of scalars along the map $\alpha : A^\circ \to A^+$ of Remark 25.3.3.4.

We can summarize the situation more informally as follows: if A° is the underlying $\mathbb{E}_\infty$-ring of a simplicial commutative ring A, then the relative cotangent complex $L_{A^\circ/\mathbf{Z}}$ is endowed with additional structure: it carries a left action of the $\mathbb{E}_1$-ring A^+ of Corollary 25.3.3.3 (which restricts to the tautological action of A° on $L_{A^\circ/\mathbf{Z}}$ along $\alpha : A^\circ \to A^+$).

Let us now return to the construction of square-zero extensions given in §**??**.

Construction 25.3.3.6. Fix a simplicial commutative ring A. The construction $M \mapsto A \oplus M$ then determines a functor $\theta_0 : \mathrm{Mod}^{\mathrm{cn}}_{A^\circ} \to \mathrm{CAlg}^{\Delta}_{/A}$ which commutes with small limits. Passing to spectrum objects, we obtain a functor

$$\theta : \mathrm{Mod}_{A^\circ} \simeq \mathrm{Sp}(\mathrm{Mod}_{A^\circ}) \to \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}).$$

By virtue of Remark 25.3.1.2, the composition of this functor with the forgetful map

$$\mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) \to \mathrm{Sp}((\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}})_{/A^\circ}) \simeq \mathrm{Mod}_{A^\circ}$$

is naturally equivalent to the identity functor Mod_{A°. It follows that there is an essentially unique map of $\mathbb{E}_1$-rings $\gamma : A^+ \to A^\circ$ for which the composition

$$\mathrm{Mod}_{A^\circ} \xrightarrow{\theta} \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) \simeq \mathrm{LMod}_{A^+}$$

is given by restriction of scalars along γ. Moreover, the composition $A^\circ \xrightarrow{\alpha} A^+ \xrightarrow{\gamma} A^\circ$ is homotopic to the identity.

Remark 25.3.3.7. Let A be a simplicial commutative ring. The algebraic cotangent complex L^{alg}_A is characterized by the requirement that, for any connective A-module M, we have canonical homotopy equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{Mod}_{A^\circ}}(L^{\mathrm{alg}}_A, M) &\simeq \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/A}}(A, A \oplus M) \\ &\simeq \mathrm{Map}_{\mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A})}(\Sigma^\infty_+(A), \theta(M)) \\ &\simeq \mathrm{Map}_{\mathrm{LMod}_{A^+}}(L_{A^\circ/\mathbf{Z}}, M) \end{aligned}$$

where we regard $L_{A^\circ/\mathbf{Z}}$ as a left A^+-module as in Remark 25.3.3.4, and we regard M as a left A^+-module by restricting scalars along the map $\gamma : A^+ \to A^\circ$ of Construction 25.3.3.6. It follows that we have a canonical equivalence of A-modules $L^{\mathrm{alg}}_A \simeq A^\circ \otimes_{A^+} L_{A^\circ/\mathbf{Z}}$.

More generally, if we are given a morphism $B \to A$ of simplicial commutative rings, then we can identify the relative algebraic cotangent complex $L^{\mathrm{alg}}_{A/B}$ with $A^\circ \otimes_{A^+} L_{A^\circ/B^\circ}$.

25.3.4 Calculation of the Ring Spectrum A^+

Let A be a simplicial commutative ring. Our goal in this section is to describe the $\mathbb{E}_1$-ring A^+ appearing in the statement of Corollary 25.3.3.3. This ring spectrum is essentially characterized by the requirement that the ∞-category LMod_{A^+} is equivalent to the ∞-category of spectrum objects $\mathrm{CAlg}^{\Delta}_{/A}$. Suppose we are given an object $X \in \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A})$. Then the image $\widetilde{A}$ of X under the forgetful functor $\Omega^\infty : \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) \to \mathrm{CAlg}^{\Delta}_{/A}$ as a kind of "generalized trivial square-zero extension of A." Any such extension determines a trivial square-zero extension of the underlying $\mathbb{E}_\infty$-ring A°, which is automatically of the form $A^\circ \oplus M$ for some connective A°-module M. However, to promote the $\mathbb{E}_\infty$-ring $A^\circ \oplus M$ to a simplicial commutative ring A°, we also need to make the multiplication of $A^\circ \oplus M$ *strictly commutative*. We now show that this can be used to endow M with the structure of a $\mathbf{Z}$-module spectrum, which is *a priori* unrelated to its A°-module structure.

Construction 25.3.4.1. Let A be a simplicial commutative ring and let A° denote its underlying $\mathbb{E}_\infty$-ring. Then Remark 25.1.5.5 supplies a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{CAlg}^{\Delta}_{/A} & \xrightarrow{\mathbf{G}_m} & (\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}})_{/\mathbf{G}_m(A)} \\ \downarrow{\scriptstyle\Theta} & & \downarrow \\ (\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}})_{/A^\circ} & \xrightarrow{\mathrm{GL}_1} & \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})_{/\,\mathrm{GL}_1(A^\circ)} \end{array}$$

Passing to spectrum objects and using the equivalences

$$\mathrm{CAlg}^{\Delta}_{/A} \simeq \mathrm{LMod}_{A^+} \qquad \mathrm{Sp}((\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}})_{/\mathbf{G}_m(A)}) \simeq \mathrm{Mod}_{\mathbf{Z}}$$

$$\mathrm{Sp}((\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}})_{/A^\circ}) \simeq \mathrm{Mod}_{A^\circ} \qquad \mathrm{Sp}((\mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S}))_{/\,\mathrm{GL}_1(A^\circ)}) \simeq \mathrm{Sp},$$

we obtain a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{LMod}_{A^+} & \longrightarrow & \mathrm{Mod}_{\mathbf{Z}} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{A^\circ} & \longrightarrow & \mathrm{Sp} \end{array}$$

where the bottom horizontal map and right vertical maps are the forgetful functors, and the left vertical map is given by restriction of scalars along the map $\alpha : A^\circ \to A^+$ of Remark 25.3.3.4. It follows that the upper horizontal map is given by restriction of scalars along an essentially unique morphism of $\mathbb{E}_1$-rings $\beta : \mathbf{Z} \to A^+$.

We now use Construction 25.3.4.1 to describe the structure of the ring spectrum A^+ (see [185] for another proof):

Proposition 25.3.4.2 (Schwede). *Let A be a simplicial commutative ring and let $\alpha : A^{\circ} \to A^{+}$ and $\beta : \mathbf{Z} \to A^{+}$ be the morphisms of $\mathbb{E}_1$-rings given in Remark 25.3.3.4 and Construction 25.3.4.1, respectively. Then the multiplication on A^{+} induces a homotopy equivalence of spectra*

$$A^{\circ} \otimes_S \mathbf{Z} \xrightarrow{\alpha \otimes \beta} A^{+} \otimes_S A^{+} \to A^{+}.$$

Warning 25.3.4.3. Let A be a simplicial commutative ring. In general, the $\mathbb{E}_1$-ring A^{+} is not commutative (in fact, the graded ring $\pi_* A^{+}$ need not even be graded-commutative). Consequently, the homotopy equivalence $A^{\circ} \otimes_S \mathbf{Z} \simeq A^{+}$ is generally not compatible with ring structures. More informally, the issue is that the $\mathbb{E}_1$-ring morphisms $f : \mathbf{Z} \to A^{+}$ and $f' : A^{\circ} \to A^{+}$ do not "commute" with one another. Beware also that the order of the multiplication is important: we can also consider the multiplication map

$$\mathbf{Z} \otimes_S A^{+} \xrightarrow{\beta \otimes \alpha} A^{+} \otimes_S A^{+} \to A^{+},$$

but this map is generally not a homotopy equivalence of spectra.

Proof of Proposition 25.3.4.2. We have a commutative diagram of $\mathbb{E}_1$-rings

$$\begin{array}{ccc} A^{+} & \xleftarrow{\beta} & \mathbf{Z} \\ \uparrow{\scriptstyle \alpha} & & \uparrow \\ A^{\circ} & \longleftarrow & S \end{array}$$

which induces a commutative diagram of module ∞-categories σ :

$$\begin{array}{ccc} \mathrm{LMod}_{A^{+}} & \longrightarrow & \mathrm{Mod}_{\mathbf{Z}} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{A^{\circ}} & \longrightarrow & \mathrm{Mod}_{\mathrm{Sp}} \end{array}$$

Unwinding the definitions, we see that the multiplication map $A^{\circ} \otimes_S \mathbf{Z} \to A^{+}$ is an equivalence if and only if the diagram σ is left adjointable. By construction, the diagram σ arises by applying the stabilization construction to a diagram of ∞-categories σ_0:

$$\begin{array}{ccc} \mathrm{CAlg}^{\Delta}_{/A} & \xrightarrow{\mathbf{G}_m} & (\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}})_{/\mathbf{G}_m(A)} \\ \downarrow{\scriptstyle \Theta} & & \downarrow \\ (\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Z}})_{/A^{\circ}} & \xrightarrow{\mathrm{GL}_1} & \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})_{/\mathrm{GL}_1(A^{\circ})} \end{array}$$

We are therefore reduced to showing that the diagram σ_0 is left adjointable, which follows from Proposition 25.1.5.3. $\square$

25.3.5 Comparison with the Topological Cotangent Complex

Let $\phi : A \to B$ be a morphism of simplicial commutative rings. Then there exists a "universal A-linear derivation" $B \to B \oplus L^{\mathrm{alg}}_{B/A}$ in the ∞-category $(\mathrm{CAlg}^{\Delta}_{A})_{/B}$. Passing to the underlying $\mathbb{E}_\infty$-rings, we obtain a morphism $B^\circ \to B^\circ \oplus L^{\mathrm{alg}}_{B/A}$ in the ∞-category $\mathrm{CAlg}_{A^\circ//B^\circ}$, which is classified by a morphism of B-modules

$$\rho_{B/A} : L_{B^\circ/A^\circ} \to L^{\mathrm{alg}}_{B/A}.$$

Proposition 25.3.5.1. *Let $\phi : A \to B$ be a morphism of simplicial commutative rings. Suppose that $\mathrm{fib}(\phi)$ is m-connective for some $m \geqslant -1$: that is, the underlying map $\pi_n A \to \pi_n B$ is surjective when $n = m$ and an isomorphism for $n < m$. Then the comparison map $\rho_{B/A} : L_{B^\circ/A^\circ} \to L^{\mathrm{alg}}_{B/A}$ has $(m+3)$-connective fiber. In other words, the map $\pi_n L_{B^\circ/A^\circ} \to \pi_n L^{\mathrm{alg}}_{B/A}$ is surjective when $n = m+3$ and an isomorphism for $n < m+3$.*

Proof. Under the identification $L^{\mathrm{alg}}_{B/A} \simeq B^\circ \otimes_{B^+} L_{B^\circ/A^\circ}$ supplied by Remark 25.3.3.7, the comparison map $\rho_{B/A}$ is given by

$$L_{B^\circ/A^\circ} \to B^\circ \otimes_{B^+} L_{B^\circ/A^\circ}.$$

Since $\mathrm{fib}(\phi)$ is m-connective, the relative cotangent complex L_{B°/A° is $(m+1)$-connective (this is trivial when $m = -1$ and follows from Corollary HA.7.4.3.2 if $m \geqslant 0$). It will therefore suffice to show that the map of $\mathbb{E}_1$-rings $\gamma : B^+ \to B^\circ$ appearing in Remark 25.3.3.5 has 2-connective fiber. Since γ is a left homotopy inverse to the map $\alpha : B^\circ \to B^+$ of Remark 25.3.3.4, we have $\mathrm{fib}\,\gamma) \simeq \Sigma\,\mathrm{fib}(\alpha)$. We are therefore reduced to proving that $\mathrm{fib}(\alpha)$ is 1-connective. We conclude by observing that the identification $B^+ \simeq B^\circ \otimes_S \mathbf{Z}$ induces an equivalence $\mathrm{fib}(\alpha) \simeq B^\circ \otimes_S (\tau_{\geqslant 1} S)$, where S is the sphere spectrum. □

Variant 25.3.5.2. Let A be a simplicial commutative ring. Then the universal derivation $A \to A \oplus L^{\mathrm{alg}}_{A}$ in $\mathrm{CAlg}^{\Delta}_{A}$ determines a map of $\mathbb{E}_\infty$-rings which is classified by a comparison map $\rho : L_{A^\circ} \to L^{\mathrm{alg}}_{A}$. The map ρ also has 2-connective fiber: this follows from the observation that ρ factors as a composition

$$L_{A^\circ} \xrightarrow{\rho'} L_{A^\circ/\mathbf{Z}} \xrightarrow{\rho''} L^{\mathrm{alg}}_{A/\mathbf{Z}},$$

where γ'' has 2-connective fiber by Proposition 25.3.5.1 and $\mathrm{fib}(\gamma') \simeq A \otimes_{\mathbf{Z}} L_{\mathbf{Z}}$ is 2-connective by virtue of the fact that the cofiber of the unit map $S \to \mathbf{Z}$ is 2-connective (here S denotes the sphere spectrum).

The connectivity estimate given in Proposition 25.3.5.1 is in some sense optimal: in general, the map $\pi_2 L_{B^\circ/A^\circ} \to \pi_2 L^{\mathrm{alg}}_{B/A}$ need not be injective (for example, it is not injective if $A = \mathbf{Z}$ and $B = \mathbf{Z}[x]$. However, we do have the following:

Proposition 25.3.5.3. *Let $\phi : A \to B$ be a morphism of simplicial commutative rings. Then the comparison map $\rho_{B/A} : L_{B^\circ/A^\circ} \to L^{\mathrm{alg}}_{B/A}$ is a rational equivalence. In other words, for every integer n, the map of abelian groups $\pi_n L_{B^\circ/A^\circ} \to \pi_n L^{\mathrm{alg}}_{B/A}$ becomes an isomorphism after tensoring with the rational numbers.*

Corollary 25.3.5.4. *Let $\phi : A \to B$ be a morphism of simplicial commutative rings, and suppose that the commutative ring $\pi_0 B$ is a $\mathbf{Q}$-algebra. Then the comparison map $\rho_{B/A} : L_{B^\circ/A^\circ} \to L^{\mathrm{alg}}_{B/A}$ is an equivalence of B-modules.*

Proof of Proposition 25.3.5.3. By virtue of Remark 25.3.3.7, it will suffice to show that the morphism of $\mathbb{E}_1$-rings $\gamma : B^+ \to B^\circ$ appearing in Remark 25.3.3.5 is a rational equivalence. Note that γ is left inverse to the map $\alpha : B^\circ \to B^+$ of Remark 25.3.3.4, so we are reduced to showing that α is a rational equivalence. It follows from Proposition 25.3.4.2 that as a map of spectra, α can be obtained by smashing the identity on B° with the unit map $S \to \mathbf{Z}$, where S is the sphere spectrum. We are therefore reduced to proving that the unit map $S \to \mathbf{Z}$ is a rational equivalence, which follows from the fact that the stable homotopy groups of spheres are torsion in nonzero degrees. □

25.3.6 The Hurewicz Map

Let $\phi : A \to B$ be a morphism of simplicial commutative rings. Recall that the underlying morphism of $\mathbb{E}_\infty$-rings $A^\circ \to B^\circ$ determines a map

$$\epsilon^\circ_\phi : B \otimes_A \mathrm{cofib}(\phi) \to L_{B^\circ/A^\circ}$$

(see Theorem HA.7.4.3.1). Composing with the comparison map $\gamma_{B/A}$, we obtain a map

$$\epsilon_\phi : B \otimes_A \mathrm{cofib}(\phi) \to L^{\mathrm{alg}}_{B/A}.$$

We will refer to ϵ_ϕ as the *Hurewicz map associated to ϕ.*

Proposition 25.3.6.1. *Let $\phi : A \to B$ be a morphism of simplicial commutative rings and suppose that the fiber of ϕ. Then the map*

$$\epsilon_\phi : B \otimes_A \mathrm{cofib}(\phi) \to L^{\mathrm{alg}}_{B/A}$$

is surjective on π_0. If the fiber of ϕ is connective, then the fiber of ϵ_ϕ is 2-connective. If the fiber of ϕ is m-connective for $m > 0$, then the fiber of ϵ_ϕ is $(m+3)$-connective.

Warning 25.3.6.2. The analogue of Proposition 25.3.6.1 in the setting of $\mathbb{E}_\infty$-rings is somewhat stronger: it asserts that if the fiber of ϕ is m-connective, then the fiber of ϵ°_ϕ is $(2m+2)$-connective. In the setting of simplicial commutative rings, we have a weaker statement because the derived symmetric powers $\mathrm{LSym}^n_A(M)$ are only slightly more connected than M (see Proposition 25.2.4.1).

Proof of Proposition 25.3.6.1. By construction, the map ϵ_ϕ is given by a composition

$$B \otimes_A \mathrm{cofib}(\phi) \xrightarrow{\epsilon^\circ_\phi} L_{B^\circ/A^\circ} \xrightarrow{\gamma_{B/A}} L^{\mathrm{alg}}_{B/A}.$$

If the fiber $\mathrm{fib}(\phi)$ is m-connective for $m \geqslant -1$, then the fiber of ϵ°_ϕ is $(2m+2)$-connective (Theorem HA.7.4.3.1) and the fiber of $\gamma_{B/A}$ is $(m+3)$-connective (Proposition 25.3.5.1). □

Corollary 25.3.6.3. *Let $\phi : A \to B$ be a morphism of simplicial commutative rings, so that the universal derivation $B \to B \oplus L^{\mathrm{alg}}_{B/A}$ induces a morphism of A-modules $\beta : \mathrm{cofib}(\phi) \to L^{\mathrm{alg}}_{B/A}$.*

(a) *If the fiber $\mathrm{fib}(\phi)$ is connective, then the fiber $\mathrm{fib}(\beta)$ is 1-connective.*

(b) *If the fiber $\mathrm{fib}(\phi)$ is 1-connective, then the fiber $\mathrm{fib}(\beta)$ is 3-connective.*

(c) *If the fiber $\mathrm{fib}(\phi)$ is m-connective for $m \geqslant 2$, then the fiber $\mathrm{fib}(\beta)$ is $(m+3)$-connective.*

Proof. The map β factors as a composition

$$\mathrm{cofib}(\phi) \xrightarrow{\beta_0} B \otimes_A \mathrm{cofib}(\phi) \xrightarrow{\epsilon_\phi} L^{\mathrm{alg}}_{B/A}.$$

If $\mathrm{fib}(\phi)$ is m-connective, then $\mathrm{fib}(\beta) \simeq \mathrm{fib}(\phi) \otimes_A \mathrm{cofib}(\phi)$ is $(2m+1)$-connective. It follows from Proposition 25.3.6.1 that the map ϵ_ϕ is 2-connective when $m = 0$ and $(m+3)$-connective when $m > 0$. It follows that $\mathrm{fib}(\beta)$ is 1-connective when $m = 0$, 3-connective when $m = 1$, and $(m+3)$-connective for $m > 1$. □

Corollary 25.3.6.4. *Let $\phi : A \to B$ be a morphism of simplicial commutative rings. If the fiber $\mathrm{fib}(\phi)$ is m-connective for $m \geqslant 0$, then the algebraic cotangent complex $L^{\mathrm{alg}}_{B/A}$ is $(m+1)$-connective.*

Remark 25.3.6.5. Let $\phi : A \to B$ be a morphism of simplicial commutative rings, and suppose that $\mathrm{fib}(\phi)$ is m-connective for $m \geqslant 0$: that is, the map $\pi_n A \to \pi_n B$ is surjective when $n = m$ and an isomorphism for $n < m$. It follows from Corollary 25.3.6.4 that the homotopy groups $\pi_n L^{\mathrm{alg}}_{B/A}$ vanish for $n \leqslant m$. We can restate Corollary 25.3.6.3 as follows:

(a') If $m = 0$, then the universal derivation induces a surjection $\pi_1 \mathrm{cofib}(\phi) \to \pi_1 L^{\mathrm{alg}}_{B/A}$.

(b') If $m = 1$, then the universal derivation induces an isomorphism $\pi_2 \mathrm{cofib}(\phi) \to \pi_2 L^{\mathrm{alg}}_{B/A}$ and a surjection $\pi_3 \mathrm{cofib}(\phi) \to \pi_3 L^{\mathrm{alg}}_{B/A}$.

(c') If $m \geqslant 2$, then the universal derivation induces isomorphisms

$$\pi_{m+1} \mathrm{cofib}(\phi) \to \pi_{m+1} L^{\mathrm{alg}}_{B/A} \qquad \pi_{m+2} \mathrm{cofib}(\phi) \to \pi_{m+2} L^{\mathrm{alg}}_{B/A}$$

and a surjection $\pi_{m+3} \mathrm{cofib}(\phi) \to \pi_{m+3} L^{\mathrm{alg}}_{B/A}$.

Corollary 25.3.6.6. *Let $\phi : A \to B$ be a morphism of simplicial commutative rings. Then ϕ is an equivalence if and only if it induces an isomorphism of commutative rings $\pi_0 A \to \pi_0 B$ and the algebraic cotangent complex $L^{\mathrm{alg}}_{B/A}$ vanishes.*

Proof. The "only if" direction is clear. Suppose, conversely, that ϕ induces an isomorphism of $\pi_0 A$ with $\pi_0 B$. If ϕ is not an equivalence, then $\mathrm{cofib}(\phi)$ is a nonzero connective A-module, so there exists some smallest integer $n \geqslant 0$ such that $\pi_n \mathrm{cofib}(\phi) \neq 0$. Then $\pi_n(B \otimes_A \mathrm{cofib}(\phi)) \neq 0$. It follows from Proposition 25.3.6.1 that the fiber of the map

$$\epsilon_\phi : B \otimes_A \mathrm{cofib}(\phi) \to L^{\mathrm{alg}}_{B/A}$$

is $(n+1)$-connective (and even $(n+2)$-connective if $n \geqslant 2$). In particular, it induces an isomorphism $\pi_n(B \otimes_A \mathrm{cofib}(\phi)) \to \pi_n L^{\mathrm{alg}}_{B/A}$, contradicting our assumption that $L^{\mathrm{alg}}_{B/A} \simeq 0$. □

Part VIII

Higher Algebraic Stacks

Chapter 26

Algebraic Stacks in Derived Algebraic Geometry

Chapter 27

Artin Representability

Chapter 28

Coaffine Stacks

Chapter 29

Generalized Algebraic Gerbes

Part IX

Rational and p-adic Homotopy Theory

Chapter 30

Rational Homotopy Theory

Chapter 31

p-adic Homotopy Theory

Chapter 32

Unstable Riemann-Hilbert Correspondence

Part X

Appendix

Appendix A

Coherent ∞-Topoi

Recall that a scheme $(X, \mathscr{O}_X)$ is said to be *quasi-compact* if the topological space X is quasi-compact: that is, every open covering of X admits a finite subcovering. We say that $(X, \mathscr{O}_X)$ is *quasi-separated* if the collection of quasi-compact open subsets of X is closed under pairwise intersections. Throughout this book, we study algebro-geometric objects which behave like schemes, but whose "underlying topology" is encoded by an ∞-topos rather than a topological space. Our primary goal in this appendix is to describe finiteness hypotheses on ∞-topoi which are analogous the the hypotheses of quasi-compactness and quasi-separatedness on ordinary topological spaces.

We begin in §A.1 with a discussion of the theory of *coherent* topological spaces: that is, sober topological spaces X for which the quasi-compact open subsets form a basis for the topology of X and are stable under finite intersections. If X is a coherent topological space, then the collection of quasi-compact open subsets of X forms a distributive lattice Λ. A version of the classical *Stone duality theorem* asserts that the lattice Λ determines X up to homeomorphism (Proposition A.1.5.10). This result provides an appealing algebraic approach to the theory of coherent topological spaces: we can describe such a space X by specifying a collection of well-behaved open subsets, rather than by specifying its points.

In §A.2, we introduce an ∞-categorical counterpart of the notion of coherent topological space: the notion of *coherent* ∞-topos. We say that an ∞-topos $\mathcal{X}$ is *locally coherent* if there exists a full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ of quasi-compact objects which is stable under fiber products, such that every object $X \in \mathcal{X}$ admits a covering $\{U_\alpha \rightarrow X\}$ by objects of $\mathcal{X}_0$. If this condition is satisfied, then there exists a largest subcategory with these properties, which we denote by $\mathcal{X}^{\mathrm{coh}}$ and refer to as the *∞-category of coherent objects of $\mathcal{X}$*. We will say that a locally coherent ∞-topos $\mathcal{X}$ is *coherent* if $\mathcal{X}^{\mathrm{coh}}$ contains the final object of $\mathcal{X}$. Coherent ∞-topoi exist in great abundance: in §A.3, we show that if $\mathcal{C}$ is an ∞-category which admits finite limits and is equipped with a *finitary* Grothendieck topology (that is, a Grothendieck topology which is generated by finite coverings), then the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is both coherent and locally coherent (Proposition A.3.1.3). In particular, most of the ∞-topoi that we will encounter in this book are coherent.

Most of this appendix is devoted to answering the following:

Question A.0.6.7. Let $\mathcal{X}$ be an ∞-topos which is coherent and locally coherent. To what extent can $\mathcal{X}$ be recovered from its full subcategory $\mathcal{X}^{\mathrm{coh}} \subseteq \mathcal{X}$ of coherent objects?

Before we can properly discuss Question A.0.6.7, we first need to ask something more basic:

Question A.0.6.8. Let $\mathcal{X}$ be an ∞-topos which is coherent and locally coherent. What sort of mathematical object is $\mathcal{X}^{\mathrm{coh}}$?

In §A.6, we address Question A.0.6.8 by introducing the notion of an *∞-pretopos* (Definition A.6.1.1). Roughly speaking, an ∞-pretopos is an ∞-category having the exactness properties of an ∞-topos, but which is not required to admit "infinitary" categorical constructions (such as infinite products and coproducts). If $\mathcal{X}$ is a coherent ∞-topos, then its full subcategory $\mathcal{X}^{\mathrm{coh}}$ of coherent objects is an ∞-pretopos. Conversely, if $\mathcal{C}$ is an essentially small ∞-pretopos, then $\mathcal{C}$ can be equipped with a finitary Grothendieck topology (Definition A.6.2.4) from which we can construct a coherent ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$. Unfortunately, these constructions are not quite inverse to one another (in contrast with the analogous situation in classical topos theory), due to technicalities related to failure of Whitehead's theorem in a general ∞-topos. However, we can obtain a good dictionary by imposing additional restrictions on both sides. This can be accomplished in (at least) two different ways:

(*a*) An ∞-topos $\mathcal{X}$ is said to be *hypercomplete* if every ∞-connective morphism $f : X \to Y$ in $\mathcal{X}$ is an equivalence (in other words, if Whitehead's theorem is valid for $\mathcal{X}$). In §A.6, we will introduce the more general notion of a *hypercomplete ∞-pretopos* (Definition A.6.5.3) and establish an equivalence

$$\{ \text{ Small hypercomplete } \infty\text{-pretopoi } \}$$
$$\Updownarrow$$
$$\{ \text{ Hypercomplete, coherent, locally coherent } \infty\text{-topoi } \}$$

(see Theorem A.6.6.5).

(*b*) We will say that an ∞-topos $\mathcal{X}$ is *bounded* if it can be written as a limit of n-localic ∞-topoi (Definition A.7.1.2) and that an ∞-pretopos $\mathcal{C}$ is bounded if it is essentially small and each object $C \in \mathcal{C}$ is truncated (Definition A.7.4.1). In §A.7, we establish an equivalence

$$\{ \text{ Bounded } \infty\text{-pretopoi } \}$$
$$\Updownarrow$$
$$\{ \text{ Bounded coherent } \infty\text{-topoi } \}$$

(see Theorem A.7.5.3).

Both (*a*) and (*b*) provide useful contexts in which to work. In §A.4, we will show that an ∞-topos $\mathcal{X}$ which is hypercomplete and locally coherent has enough points (Theorem A.4.0.5): this can be regarded as an ∞-categorical generalization of a classical result of Deligne on coherent Grothendieck topoi (Corollary A.4.0.6). This result is useful because it provides a general mechanism for reducing certain types of questions about arbitrary ∞-topoi to more concrete questions about the homotopy theory of spaces. We will apply this mechanism in §A.5 to show that in any ∞-topos $\mathcal{X}$, the homotopy theory of simplicial objects of $\mathcal{X}$ behaves to a large extent like the classical homotopy theory of simplicial sets (see Theorems **??**, A.5.4.1, and A.5.6.1). On the other hand, the theory of bounded ∞-pretopoi has the virtue that it admits a completely finitary description (in other words, it can be axiomatized without referring to infinite limits and colimits), and seems to provide a good language for extending concepts of first-order logic to the ∞-categorical setting. We offer some evidence for this last assertion in §A.9 by proving an ∞-categorical generalization of the "conceptual completeness" theorem of Makkai-Reyes (see [143]), asserting that a morphism of bounded ∞-pretopoi $f^* : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if $\mathcal{C}$ and $\mathcal{D}$ have the same "models" (see Theorem A.9.3.1 for a precise statement). The proof relies on a study of Pro-objects of bounded ∞-pretopoi which we carry out in §A.8 which may be of some independent interest (and will be useful for our study of profinite homotopy theory in Appendix E).

Contents

A.1 Stone Duality

Let X be a topological space. We will say that X is *coherent* if it satisfies the following conditions:

(a) The space X is *sober*: that is, every irreducible closed subset of X has a unique generic point (see Definition 1.5.3.1).

(b) The collection of quasi-compact open subsets of X forms a basis for the topology of X.

(c) The space X is quasi-compact and quasi-separated. In other words, the collection of quasi-compact open subsets of X is closed under finite intersections.

Warning A.1.0.9. Some authors do not include condition (a) in the definition of a coherent topological space. This does not matter very much: for every topological space X, one can construct a sober topological space X' having the same lattice of open sets (concretely, X' can be described as the set of irreducible closed subsets of X). Then X' is coherent if and only if X satisfies conditions (b) and (c).

Our goal in this section is to review the theory of Stone duality, which establishes an equivalence of the theory of coherent topological spaces and the theory of distributive lattices (Proposition A.1.5.10). We will also record a few facts about the relationship between Boolean algebras and distributive lattices, which are useful for working with constructible sets in algebraic geometry (see §4.3). For a much more extensive discussion of Stone duality, we refer the reader to [105].

A.1.1 Upper Semilattices

We begin by reviewing some definitions.

Definition A.1.1.1. An *upper semilattice* is a partially ordered set Λ such that every finite subset $S \subseteq \Lambda$ has a supremum $\bigvee S$.

For partially ordered set Λ to be an upper semilattice, it is necessary and sufficient that Λ has least element $\bot$ and every pair of elements $x, y \in \Lambda$ has a least upper bound. We denote this least upper bound by $x \vee y$, and refer to it as the *join* of x and y.

Remark A.1.1.2. Let Λ be an upper semilattice. Then the join operation $\vee : \Lambda \times \Lambda \to \Lambda$ endows Λ with the structure of a commutative monoid (with identity element given by $\perp$). Moreover, every element $x \in \Lambda$ is idempotent: that is, we have $x = x \vee x$. Conversely, if M is a commutative monoid in which every element is idempotent, then we can introduce a partial ordering of M by writing $x \leqslant y$ if and only if $xy = y$. This partial ordering exhibits M as an upper semilattice.

Definition A.1.1.3. Let Λ and Λ' be upper semilattices. A *distributor* from Λ to Λ' is a subset $D \subseteq \Lambda \times \Lambda'$ satisfying the following conditions:

(*i*) If $(x, x') \in D$, $y \leqslant x$, and $x' \leqslant y'$, then $(y, y') \in D$.

(*ii*) Let $S = \{y_i\}$ be a finite subset of Λ', let $y = \bigvee S$, and let $x \in \Lambda$. Then $(x, y) \in D$ if and only if we can write $x = \bigvee\{x_i\}$ for some finite collection of elements $\{x_i\} \subseteq \Lambda$ such that $(x_i, y_i) \in D$ for every index i.

(*iii*) Let $S = \{y_i\}$ be a finite subset of Λ' and let $x \in \Lambda$ be such that $(x, y_i) \in D$ for every index i. Then there exists an element $y \in \Lambda'$ such that $(x, y) \in D$, and $y \leqslant y_i$ for every index i.

We say that an upper semilattice Λ is *distributive* if the set $\{(x, y) \in \Lambda \times \Lambda : x \leqslant y\}$ is a distributor from Λ to itself.

Remark A.1.1.4. Let Λ be an upper semilattice. The set $\{(x, y) \in \Lambda \times \Lambda : x \leqslant y\}$ automatically satisfies conditions (*i*) and (*iii*) of Definition A.1.1.3. Consequently, Λ is distributive if and only if for every inequality $x \leqslant \bigvee\{y_i\}$, we can write $x = \bigvee\{x_i\}$ for some collection of elements x_i satisfying $x_i \leqslant y_i$. This is obvious if the set $\{y_i\}$ is empty. Using induction on the size of the set $\{y_i\}$, we see that Λ is distributive if and only if the following condition is satisfied:

($*$) For every inequality $x \leqslant y \vee z$ in Λ, we can write $x = y_0 \vee z_0$, where $y_0 \leqslant y$ and $z_0 \leqslant z$.

Construction A.1.1.5. Let Λ, Λ', and Λ'' be upper semilattices, and suppose we are given distributors $D \subseteq \Lambda \times \Lambda'$ and $D' \subseteq \Lambda' \times \Lambda''$. We define the *composition* D' with D to be the relation

$$D'D = \{(x, z) \in \Lambda \times \Lambda'' : (\exists y \in \Lambda')[(x, y) \in D \text{ and } (y, z) \in D']\}.$$

Then $D'D$ is a distributor from Λ to Λ''. The composition of distributors is associative. Moreover, if Λ is a distributive upper semilattice and we let id_Λ denote the distributor $\{(x, y) \in \Lambda \times \Lambda : x \leqslant y\}$, then $\mathrm{id}_P D = D$ for any distributor R from Λ' to Λ, and $D' \mathrm{id}_P = D'$ for any distributor D' from Λ to Λ'. We therefore obtain a category SLat whose objects are distributive upper semilattices, where the morphisms from Λ to Λ' are given by distributors from Λ to Λ'.

Example A.1.1.6. Let X be a topological space, and let $\Lambda(X)$ denote the collection of all quasi-compact open subsets of X. Then $\Lambda(X)$ is an upper semilattice (when regarded as a partially ordered set with respect to inclusion). The empty set $\emptyset \subseteq X$ is a least element of $\Lambda(X)$, and the join operation $\vee : \Lambda(X) \times \Lambda(X) \to \Lambda(X)$ is given by $(U, V) \mapsto U \cup V$.

A.1.2 The Spectrum of an Upper Semilattice

Example A.1.1.6 has a converse: every upper semilattice Λ is isomorphic to the partially ordered set of quasi-compact open subsets of some topological space X. In fact, there is a canonical choice for the topological space X, which we will refer to as the *spectrum* of Λ and denote by $\mathrm{Spec}(\Lambda)$.

Definition A.1.2.1. Let Λ be an upper semilattice. We say that a subset $I \subseteq \Lambda$ is an *ideal* if it is closed downwards and closed under finite joins. We say that a subset $F \subseteq \Lambda$ is a *filter* if it is closed upwards and every finite subset $S \subseteq F$ has a lower bound in F. We say that an ideal I is *prime* if $\Lambda - I$ is a filter.

Remark A.1.2.2. Any ideal $I \subseteq \Lambda$ contains the least element $\bot \in \Lambda$. Note that I is prime if and only if the following pair of conditions holds:

- (i) The empty set $\emptyset \subseteq \Lambda - I$ has a lower bound in $\Lambda - I$: that is, $I \neq \Lambda$.
- (ii) For every pair of elements $x, y \in \Lambda$ such that $x, y \notin I$, there exists $z \leqslant x, y$ such that $z \notin I$.

Construction A.1.2.3. Let Λ be a distributive upper semilattice. We let $\mathrm{Spec}(\Lambda)$ denote the collection of all prime ideals of Λ. We will refer to $\mathrm{Spec}(\Lambda)$ as the *spectrum* of Λ.

Notation A.1.2.4. Let Λ be a distributive upper semilattice. If $I \subseteq \Lambda$ is an ideal, we let $\mathrm{Spec}(\Lambda)_I$ denote the collection of those prime ideals $\mathfrak{p} \subseteq \Lambda$ such that $I \nsubseteq \mathfrak{p}$. If $x \in \Lambda$, we let $\mathrm{Spec}(\Lambda)_x = \{\mathfrak{p} \in \mathrm{Spec}(\Lambda) : x \notin \mathfrak{p}\}$.

Proposition A.1.2.5. *Let Λ be a distributive upper semilattice and let $\mathrm{Spec}(\Lambda)$ be the spectrum of Λ. Then:*

1. *There exists a topology on the set $\mathrm{Spec}(\Lambda)$, for which the open sets are those of the form $\mathrm{Spec}(\Lambda)_I$, where I ranges over the ideals of Λ.*
2. *The construction $I \mapsto \mathrm{Spec}(\Lambda)_I$ determines an isomorphism from the partially ordered set of ideals of Λ and the partially ordered set of open subsets of $\mathrm{Spec}(\Lambda)$.*
3. *For each $x \in \Lambda$, the subset $\mathrm{Spec}(\Lambda)_x \subseteq \mathrm{Spec}(\Lambda)$ is open. Moreover, the collection of sets of the form $\mathrm{Spec}(\Lambda)_x$ form a basis for the topology of $\mathrm{Spec}(\Lambda)$.*

(4) *For every finite subset* $S \subseteq \Lambda$ *having join* $\bigvee S = x$*, the open set* $\mathrm{Spec}(\Lambda)_x$ *is given by the union* $\bigcup_{y \in S} \mathrm{Spec}(\Lambda)_y$*.*

(5) *Each of the open sets* $\mathrm{Spec}(\Lambda)_x$ *is quasi-compact. Conversely, every quasi-compact open subset of* $\mathrm{Spec}(\Lambda)$ *has the form* $\mathrm{Spec}(\Lambda)_x$ *for some uniquely determined* $x \in \Lambda$*.*

(6) *The topological space* $\mathrm{Spec}(\Lambda)$ *is sober: that is, every irreducible closed subset of* $\mathrm{Spec}(\Lambda)$ *has a unique generic point.*

The proof of Proposition A.1.2.5 depends on the following basic observation:

Lemma A.1.2.6. *Let* Λ *be a distributive upper semilattice containing an element* x*. For every ideal* $I \subseteq \Lambda$ *which does not contain* x*, there exists a prime ideal* $\mathfrak{p} \subseteq \Lambda$ *which contains* I *but does not contain* x*.*

Proof. Using Zorn's lemma, we can choose an ideal $\mathfrak{p} \subseteq \Lambda$ which is maximal among those ideals which contain I and do not contain x. We will complete the proof by showing that $\mathfrak{p}$ is prime. Since $x \notin \mathfrak{p}$, it is clear that $\Lambda - \mathfrak{p}$ is nonempty. It will therefore suffice to show that every pair of elements $y, z \in \Lambda - \mathfrak{p}$ have a lower bound in $\Lambda - \mathfrak{p}$. The maximality of $\mathfrak{p}$ implies that x belongs to the ideal generated by $\mathfrak{p}$ and y. It follows that $x \leqslant y \vee y'$ for some $y' \in \mathfrak{p}$. Since Λ is distributive, we can write $x = y_0 \vee y_0'$ for some $y_0 \leqslant y$ and some $y_0' \in \mathfrak{p}$. The same argument shows that $x \leqslant z \vee z'$ for some $z' \in \mathfrak{p}$. Then $y_0 \leqslant z \vee z'$, so that $y_0 = z_0 \vee z_0'$ for some $z_0 \leqslant z$ and some $z_0' \in \mathfrak{p}$. Then z_0 is a lower bound for y and z. We claim that $z_0 \notin \mathfrak{p}$: otherwise, we deduce that $y_0 = z_0 \vee z_0' \in \mathfrak{p}$, so that $x = y_0 \vee y_0' \in \mathfrak{p}$, a contradiction. □

Proof of Proposition A.1.2.5. We first prove (1). Suppose first that we are given a finite collection of open subsets $\mathrm{Spec}(\Lambda)_{I_\alpha}$ of $\mathrm{Spec}(\Lambda)$, and let $I = \bigcap_\alpha I_\alpha$. To prove that $\bigcap_\alpha \mathrm{Spec}(\Lambda)_{I_\alpha}$ is open, it will suffice to show that $\bigcap_\alpha \mathrm{Spec}(\Lambda)_{I_\alpha} = \mathrm{Spec}(\Lambda)_I$. That is, we must show that a prime ideal $\mathfrak{p} \subseteq \Lambda$ contains I if and only if it contains some I_α. The "if" direction is obvious. For the converse, suppose that each I_α contains an element $x_\alpha \in \Lambda - \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, the finite collection of elements $\{x_\alpha\}$ have a lower bound $x \in \Lambda - \mathfrak{p}$. Since each I_α is closed downwards, we deduce that $x \in I = \bigcap_\alpha I_\alpha$.

Now suppose we are given an arbitrary collection of open subsets $\mathrm{Spec}(\Lambda)_{I_\beta}$ of $\mathrm{Spec}(\Lambda)$; we wish to show that $\bigcup_\beta \mathrm{Spec}(\Lambda)_{I_\beta}$ is open. Let I smallest ideal containing each I_β. Then a prime ideal $\mathfrak{p}$ contains I if and only if it contains each I_β; so that $\bigcup_\beta \mathrm{Spec}(\Lambda)_{I_\beta} = \mathrm{Spec}(\Lambda)_I$. This completes the proof of (1).

We now prove (2). Consider two ideals $I, J \subseteq \Lambda$; we wish to show that $I \subseteq J$ if and only if $\mathrm{Spec}(\Lambda)_I \subseteq \mathrm{Spec}(\Lambda)_J$. Let $K = I \cap J$, so that $\mathrm{Spec}(\Lambda)_K = \mathrm{Spec}(\Lambda)_I \cap \mathrm{Spec}(\Lambda)_J$ (by the argument given above). Then $K \subseteq I$. We wish to show that $K = I$ if and only if $\mathrm{Spec}(\Lambda)_K = \mathrm{Spec}(\Lambda)_I$. The "only if" direction is obvious. For the converse, we must show that if $I \neq K$, then there is a prime ideal $\mathfrak{p}$ such that $K \subseteq \mathfrak{p}$ but $I \nsubseteq \mathfrak{p}$.

To prove (3), we note that $\mathrm{Spec}(\Lambda)_x = \mathrm{Spec}(\Lambda)_I$ where I is the ideal $\{y \in \Lambda : y \leqslant x\}$; this proves that $\mathrm{Spec}(\Lambda)_x$ is open. For any ideal $J \subseteq \Lambda$, we have $\mathrm{Spec}(\Lambda)_J = \bigcup_{x \in J} \mathrm{Spec}(\Lambda)_x$, so that the open sets of the form $\mathrm{Spec}(\Lambda)_x$ form a basis for the topology of $\mathrm{Spec}(\Lambda)$. Assertion (4) follows immediately from the definition of a prime ideal.

We now prove (5). Let $x \in \Lambda$, and suppose that $\mathrm{Spec}(\Lambda)_x$ admits a covering by open sets of the form $\mathrm{Spec}(\Lambda)_{I_\alpha} \subseteq \mathrm{Spec}(\Lambda)_x$. Let J be the smallest ideal containing each I_α. It follows from the proof of (1) that $\mathrm{Spec}(\Lambda)_J = \bigcup_\alpha \mathrm{Spec}(\Lambda)_{I_\alpha} = \mathrm{Spec}(\Lambda)_x$. Invoking (2), we deduce that $J = \{y \in \Lambda : y \leqslant x\}$. In particular, $x \in J$. It follows that $x \leqslant x_1 \vee \ldots \vee x_n$ for some elements $x_i \in I_{\alpha(i)}$, from which we deduce that $\mathrm{Spec}(\Lambda)_x = \bigcup_{1 \leqslant i \leqslant n} \mathrm{Spec}(\Lambda)_{I_{\alpha(i)}}$. This proves that $\mathrm{Spec}(\Lambda)_x$ is quasi-compact. Conversely, suppose that $U \subseteq \mathrm{Spec}(\Lambda)$ is any quasi-compact open set. Then U has a finite covering by basic open sets of the form $\mathrm{Spec}(\Lambda)_{y_1}, \ldots, \mathrm{Spec}(\Lambda)_{y_n}$. It follows from (4) that $U = \mathrm{Spec}(\Lambda)_y$, where $y = y_1 \vee \ldots \vee y_n$.

We now prove (6). Suppose that $K \subseteq \mathrm{Spec}(\Lambda)$ is an irreducible closed subset. Then $K = \mathrm{Spec}(\Lambda) - \mathrm{Spec}(\Lambda)_I$ for some ideal $I \subseteq \Lambda$, which is uniquely determined by condition (2). By definition, a prime ideal $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$ is a generic point for K if K is the smallest closed subset containing $\mathfrak{p}$. According to condition (2), this is equivalent to the requirement that I be the largest ideal such that $I \subseteq \mathfrak{p}$. That is, $\mathfrak{p}$ is a generic point for K if and only if $\mathfrak{p} = I$. This proves the uniqueness of $\mathfrak{p}$. For existence, it suffices to show that I is a prime ideal. Since K is nonempty, $I \neq \Lambda$. It will therefore suffice to show that every pair of elements $x, y \in \Lambda - I$ have a lower bound in $\Lambda - I$. Since $x, y \notin I$, the open sets $\mathrm{Spec}(\Lambda)_x$ and $\mathrm{Spec}(\Lambda)_y$ have nonempty intersection with K. Because K is irreducible, we conclude that $\mathrm{Spec}(\Lambda)_x \cap \mathrm{Spec}(\Lambda)_y \cap K \neq \emptyset$. That is, there exists a prime ideal $\mathfrak{q}$ such that $x, y \notin \mathfrak{q}$ while $I \subseteq \mathfrak{q}$. Since $\mathfrak{q}$ is prime, x and y have a lower bound $z \in \Lambda - \mathfrak{q}$. Then z is a lower bound for x and y in $\Lambda - I$. □

A.1.3 Stone Duality for Upper Semilattices

It follows from Proposition A.1.2.5 that every distributive upper semilattice Λ can be recovered as the partially ordered set of quasi-compact open subsets of $\mathrm{Spec}(\Lambda)$. Our next goal is to prove a refinement of this observation: the construction $\Lambda \mapsto \mathrm{Spec}(\Lambda)$ determines a fully faithful embedding from the category SLat of distributive upper semilattices (Construction A.1.1.5) to the category of topological spaces (Proposition A.1.3.3).

Construction A.1.3.1. Let Λ and Λ' be distributive upper semilattices, and let $D \subseteq \Lambda \times \Lambda'$ be a distributor from Λ to Λ'. We define a map $\mathrm{Spec}(D) : \mathrm{Spec}(\Lambda) \to \mathrm{Spec}(\Lambda')$ by the formula

$$\mathrm{Spec}(D)(\mathfrak{p}) = \{y \in \Lambda' : (\forall x \in \Lambda)[(x, y) \in D \Rightarrow x \in \mathfrak{p}]\}.$$

We claim that, for every prime ideal $\mathfrak{p} \subseteq \Lambda$, the subset $\mathrm{Spec}(D)(\mathfrak{p})$ is a prime ideal in Λ'. It is clear that $\mathrm{Spec}(D)(\mathfrak{p})$ is closed downwards. If $\{y_1, \ldots, y_n\} \subseteq P' - \mathrm{Spec}(D)(\mathfrak{p})$ is a finite subset, then we can choose a finite subset $\{x_1, \ldots, x_n\} \subseteq P - \mathfrak{p}$ such that $(x_i, y_i) \in D$ for $1 \leqslant i \leqslant n$. Since $\mathfrak{p}$ is prime, the elements x_i have a lower bound $x \in \Lambda - \mathfrak{p}$. Then $(x, y_i) \in D$ for $1 \leqslant i \leqslant n$. Since D is a distributor, we deduce that $(x, y) \in D$ for some lower bound y for $\{y_1, \ldots, y_n\}$. Noting that $y \notin \mathrm{Spec}(D)(\mathfrak{p})$, we see that $\Lambda' - \mathrm{Spec}(D)(\mathfrak{p})$ is a filter. To show that $\mathrm{Spec}(D)(\mathfrak{p})$ is an ideal, suppose we are given a finite collection of elements $\{y'_1, \ldots, y'_m\} \subseteq \mathrm{Spec}(D)(\mathfrak{p})$. If the join $y'_1 \vee \cdots \vee y'_m$ does not belong to $\mathrm{Spec}(D)(\mathfrak{p})$, then $(x', y'_1 \vee \cdots \vee y'_m) \in D$ for some $x \in \mathfrak{p}$. We can therefore write $x' = x'_1 \vee \cdots \vee x'_m$ where $(x'_i, y'_i) \in D$ for every index i. Since each $y'_i \in \mathrm{Spec}(D)(\mathfrak{p})$, we conclude that $x'_i \in \mathfrak{p}$, so that $x' = x'_1 \vee \cdots x'_n \in \mathfrak{p}$, a contradiction.

Remark A.1.3.2. In the situation of Construction A.1.3.1, the map $\mathrm{Spec}(D) : \mathrm{Spec}(\Lambda) \to \mathrm{Spec}(\Lambda')$ is continuous. To prove this, we note that if $I \subseteq P'$ is an ideal, then $\mathrm{Spec}(D)^{-1} \mathrm{Spec}(\Lambda')_I = \mathrm{Spec}(\Lambda)_J$, where J is the ideal $\{x \in \Lambda : (\exists y \in I)[(x, y) \in D]\}$.

It follows from Remark A.1.3.2 that we can view Spec as a functor from the category SLat of distributive upper semilattices (with morphisms given by distributors) to the category $\mathcal{T}\mathrm{op}$ of topological spaces.

We can now formulate the first main result of this section.

Proposition A.1.3.3 (Duality for Distributive Upper Semilattices)**.** *The functor* Spec : SLat $\to \mathcal{T}\mathrm{op}$ *is fully faithful. Moreover, a topological space X belongs to the essential image of* Spec *if and only if it is sober and has a basis consisting of quasi-compact open sets.*

Proof. Let Λ and Λ' be distributive upper semilattices, let $f : \mathrm{Spec}(\Lambda) \to \mathrm{Spec}(\Lambda')$ be a continuous map, and let $D \subseteq \Lambda \times \Lambda'$ be a distributor. We first prove the following:

$(*)$ We have $f = \mathrm{Spec}(D)$ if and only if $D = \{(x, y) \in \Lambda \times \Lambda' : \mathrm{Spec}(\Lambda)_x \subseteq f^{-1} \mathrm{Spec}(\Lambda')_y\}$.

This shows in particular that D is uniquely determined by f, so that the functor Spec is faithful. We begin by proving the "only if" direction of $(*)$. Suppose that $f = \mathrm{Spec}(D)$. If $(x, y) \in D$, then for every prime ideal $\mathfrak{p} \subseteq \Lambda$ not containing x, we have $y \in \mathrm{Spec}(D)(\mathfrak{p}) = f(\mathfrak{p})$, so that $\mathrm{Spec}(\Lambda)_x \subseteq f^{-1} \mathrm{Spec}(\Lambda')_y$. Conversely, suppose $(x, y) \notin D$. Then $I = \{x' \in \Lambda : (x', y) \in D\}$ is an ideal of Λ which does not contain the element x. Using Lemma A.1.2.6, we can choose a prime ideal $\mathfrak{p}$ containing I and not containing x. Then $\mathfrak{p} \in \mathrm{Spec}(\Lambda)_x$ but $f(F) = \mathrm{Spec}(D)(\mathfrak{p}) \notin \mathrm{Spec}(\Lambda')_y$, so that $\mathrm{Spec}(\Lambda)_x \nsubseteq f^{-1} \mathrm{Spec}(\Lambda')_y$.

We next prove the "if" direction of $(*)$. Assume that $D = \{(x, y) \in \Lambda \times \Lambda' : \mathrm{Spec}(\Lambda)_x \subseteq f^{-1} \mathrm{Spec}(\Lambda')_y\}$, and let $\mathfrak{p} \subseteq \Lambda$ be a prime ideal. We wish to show that $f(F) = \mathrm{Spec}(D)(\mathfrak{p})$.

We have

$$\begin{aligned} y \notin f(\mathfrak{p}) &\Leftrightarrow f(F) \in \operatorname{Spec}(\Lambda')_y \\ &\Leftrightarrow F \in f^{-1} \operatorname{Spec}(\Lambda')_y \\ &\Leftrightarrow (\exists x \in \Lambda)[F \in \operatorname{Spec}(\Lambda)_x \subseteq f^{-1} \operatorname{Spec}(\Lambda')_y] \\ &\Leftrightarrow (\exists x \in \Lambda)[(x \in F) \wedge (x, y) \in D] \\ &\Leftrightarrow y \notin \operatorname{Spec}(D)(\mathfrak{p}). \end{aligned}$$

We now prove that the functor Spec is full. Let $f : \operatorname{Spec}(\Lambda) \to \operatorname{Spec}(\Lambda')$ be a continuous map, and set $D = \{(x, y) \in \Lambda \times \Lambda' : \operatorname{Spec}(\Lambda)_x \subseteq f^{-1} \operatorname{Spec}(\Lambda')_y\}$. We will show that D is a distributor, so that assertion $(*)$ immediately implies that $f = \operatorname{Spec}(D)$. Let us verify the conditions of Definition A.1.1.3:

(*i*) It is clear that if $(x, y) \in D$, $x' \leqslant x$, and $y \leqslant y'$, then $(x', y') \in D$.

(*ii*) Let $S = \{y_i\}$ be a finite subset of Λ', let $y = \bigvee S$, and let $x \in \Lambda$. Then $(x, y) \in D$ if and only if $\operatorname{Spec}(\Lambda)_x \subseteq \bigcup_i f^{-1} \operatorname{Spec}(\Lambda')_{y_i}$. In this case, $\operatorname{Spec}(\Lambda)_x$ admits a covering by quasi-compact open sets $U_{i,j}$ such that $U_{i,j} \subseteq f^{-1} \operatorname{Spec}(\Lambda')_{y_i}$. Since $\operatorname{Spec}(\Lambda)_x$ is quasi-compact, we can assume that this covering is finite. Let $U_i = \bigcup_j U_{i,j}$. Then each U_i is a quasi-compact open subset of $\operatorname{Spec}(\Lambda)$, and is therefore of the form $\operatorname{Spec}(\Lambda)_{x_i}$ for some $x_i \in \Lambda$. Since $\operatorname{Spec}(\Lambda)_x = \bigcup U_i$, we have $x = x_1 \vee \cdots \vee x_n$. Moreover, the containment $U_i \subseteq f^{-1} \operatorname{Spec}(\Lambda')_{y_i}$ implies that $(x_i, y_i) \in D$ for $1 \leqslant i \leqslant n$.

(*iii*) Let $S = \{y_i\}$ be a finite subset of Λ' and let $x \in \Lambda$ be such that $(x, y_i) \in D$ for every index i. Then $U = \bigcap \operatorname{Spec}(\Lambda')_{y_i}$ is an open subset of $\operatorname{Spec}(\Lambda')$ containing $f(\operatorname{Spec}(\Lambda)_x)$. Since f is continuous, $\operatorname{Spec}(\Lambda)_x$ is quasi-compact. We may therefore choose a finite covering of $f(\operatorname{Spec}(\Lambda)_x)$ by quasi-compact open subsets of $\operatorname{Spec}(\Lambda')$ which are contained in U. Let V be the union of these quasi-compact open sets, so that $V = \operatorname{Spec}(\Lambda)_y$ for some $y \in Y$. Then $\operatorname{Spec}(\Lambda)_x \subseteq f^{-1}V$, so that $(x, y) \in D$ and $y \leqslant y_i$ for each i.

We now describe the essential image of the functor Spec. Proposition A.1.2.5 implies that for every distributive upper semilattice Λ, the spectrum $\operatorname{Spec}(\Lambda)$ is a sober topological space having a basis of quasi-compact open sets. Conversely, suppose that X is any sober topological space having a basis of quasi-compact open sets. Let Λ be the collection of all quasi-compact open subsets of X, partially ordered by inclusion. Since the collection of quasi-compact open subsets of X is closed under finite unions, we see that Λ is an upper semilattice. We next claim that Λ is distributive. Let U, V, and W be quasi-compact open subsets of X such that $U \subseteq V \cup W$. Then $U \cap V$ and $U \cap W$ is an open covering of U. Since X has a basis of quasi-compact open sets, this covering admits a refinement $\{U_\alpha\}$ where each U_α is quasi-compact. Since U is quasi-compact, we may assume that the set of indices

α is finite. Then $U = V_1 \cup \cdots \cup V_m \cup W_1 \cup \cdots \cup W_{m'}$, where $V_i \subseteq V$, $W_i \subseteq W$, and each of the open sets V_i and W_i is quasi-compact. Let $V' = \bigcup_i V_i$ and $W' = \bigcup_i W_i$. Then V' and W' are quasi-compact open subsets of X satisfying $U = V' \cup W'$, $V' \subseteq V$, and $W' \subseteq W$.

We now define a map $\Phi : X \to \operatorname{Spec}(\Lambda)$ by the formula $\Phi(x) = \{U \in \Lambda : x \notin U\}$. To prove that ϕ is well-defined, we must show that for every point $x \in X$, the subset $\Phi(x) \subseteq \Lambda$ is a prime ideal. It is easy to see that $\Phi(x)$ is an ideal. If we are given a finite collection of elements $U_1, \ldots, U_n \in \Lambda - \Phi(x)$, then $x \in \bigcap_i U_i$. Since X has a basis of quasi-compact open sets, we can choose a quasi-compact open set $V \subseteq \bigcap_i U_i$ containing x, so that V is a lower bound for the subset $\{U_i\} \subseteq \Phi(x)$. This proves that $\Lambda - \Phi(x)$ is a filter, so that $\Phi(x)$ is prime.

For each $U \in \Lambda$, we have

$$\Phi^{-1} \operatorname{Spec}(\Lambda)_U = \{x \in X : \Phi(x) \in \operatorname{Spec}(\Lambda)_U\} = \{x \in X : x \in U\} = U.$$

Since the open sets of the form $\operatorname{Spec}(\Lambda)_U$ form a basis for the topology of $\operatorname{Spec}(\Lambda)$ (Proposition A.1.2.5), we deduce that Φ is continuous. We next show that Φ is bijective. Let $\mathfrak{p} \subseteq \Lambda$ be a prime ideal; we wish to show that there is a unique point $x \in X$ such that $F = \Phi(x)$. Let $V = \bigcup_{U \in \mathfrak{p}} U$. Note that if $\mathfrak{p} = \Phi(x)$, then V is the union of all those quasi-compact open subsets of X which do not contain the point x. It follows that x is a generic point of $X - V$. Since X is sober, we conclude that the point x is unique if it exists. To prove the existence, we will show that the closed set $K = X - V$ is irreducible. Since $\mathfrak{p}$ is prime, there exists a quasi-compact open set $W \subseteq X$ which is not contained in $\mathfrak{p}$. We claim that $W \cap K \neq \varnothing$. Assume otherwise; then $W \subseteq V = \bigcup_{U \notin F} U$. Since W is quasi-compact, it is contained in a finite union $U_1 \cup \cdots \cup U_n$ where each U_i belongs to $\mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, we conclude that $U_1 \cup \cdots \cup U_n \in \mathfrak{p}$, contradicting our assumption that $U \notin \mathfrak{p}$. To complete the proof that K is irreducible, it will suffice to show that if W and W' are open subsets of X such that $W \cap K \neq \varnothing$ and $W' \cap K \neq \varnothing$, then $W \cap W' \cap K \neq \varnothing$. Since X has a basis of quasi-compact open sets, we may assume without loss of generality that W and W' are quasi-compact. The definition of V then guarantees that W and W' belong to the filter $\Lambda - \mathfrak{p}$. It follows that W and W' have a lower bound $W'' \in \Lambda - \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, W'' is not contained in any finite union of open sets belonging to $\mathfrak{p}$. The quasi-compactness of W'' then implies that W'' is not contained in $\bigcup_{U \in \mathfrak{p}} U = V$, so that $\varnothing \neq W'' \cap K \subseteq W \cap W' \cap K$.

To complete the proof, it will suffice to show that the continuous bijection $\Phi : X \to \operatorname{Spec}(\Lambda)$ is an open map. Since X has a basis consisting of quasi-compact open sets, it will suffice to show that for every quasi-compact open set $U \subseteq X$, the set $\Phi(U) \subseteq \operatorname{Spec}(\Lambda)$ is open. In fact, we claim that $\Phi(U) = \operatorname{Spec}(\Lambda)_U$. The containment $\Phi(U) \subseteq \operatorname{Spec}(\Lambda)_U$ was established above. To verify the reverse inclusion, let $\mathfrak{p} \subseteq \Lambda$ be a prime ideal not containing U. The bijectivity of Φ implies that $\mathfrak{p} = \Phi(x)$ for some point $x \in X$. It now suffices to observe that $x \in U$ (since this is equivalent to the condition that $U \notin \mathfrak{p} = \Phi(x)$). □

A.1.4 Distributive Lattices

We now give a brief review of the theory of distributive lattices.

Definition A.1.4.1. Let Λ be a partially ordered set. We say that Λ is a *lattice* if both Λ and Λ^{op} are upper semilattices: that is, if every finite subset $S \subseteq \Lambda$ has both a greatest lower bound and a least upper bound.

Notation A.1.4.2. If Λ is a lattice and we are given a finite subset $S \subseteq \Lambda$, we will denote its greatest lower bound by $\bigwedge S$. We will refer to $\bigwedge S$ as the *meet* of S. We denote the greatest element of S by $\top$, and the join of set $S = \{x, y\}$ by $x \wedge y$.

Remark A.1.4.3. Let Λ be a lattice. Then the opposite partially ordered set Λ^{op} is also a lattice. Moreover, a subset $I \subseteq \Lambda$ is an ideal if and only if it is a filter when regarded as a subset of Λ^{op}. In particular, I is a prime ideal of Λ if and only if $\Lambda - I$ is a prime ideal of Λ^{op}.

Proposition A.1.4.4. *Let Λ be a lattice. The following conditions are equivalent:*

(1) *The lattice Λ is distributive when regarded as an upper semilattice (see Definition A.1.1.3).*

(2) *For every triple of elements $x, y, z \in \Lambda$, we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.*

(3) *The lattice Λ^{op} is distributive when regarded as an upper semilattice.*

(4) *For every triple of elements $x, y, z \in \Lambda$, we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.*

Definition A.1.4.5. We say that a lattice Λ is *distributive* if it satisfies the equivalent conditions of Proposition A.1.4.4.

Proof of Proposition A.1.4.4. We first prove that (1) ⇒ (2). Let x, y, and z be elements of a lattice Λ. Since $x \wedge y \leqslant x \wedge (y \vee z) \geqslant x \wedge z$, we automatically have

$$x \wedge (y \vee z) \geqslant (x \wedge y) \vee (x \wedge z).$$

Suppose that Λ is distributive as an upper semilattice. Then the inequality $x \wedge (y \vee z) \leqslant y \vee z$ implies that we can write $x \wedge (y \vee z) = y' \vee z'$, where $y' \leqslant y$ and $z' \leqslant z$. Then $y', z' \leqslant x$, so that $y' \leqslant x \wedge y$ and $z' \leqslant x \wedge z$. It follows that

$$x \wedge (y \vee z) = y' \vee z' \leqslant (x \wedge y) \vee (x \wedge z).$$

Conversely, suppose that (2) holds. We will prove that Λ is distributive as an upper semilattice. Suppose we have an inequality $x \leqslant y \vee z$ in Λ. Then

$$x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

where $x \wedge y \leqslant y$ and $x \wedge z \leqslant z$. This completes the proof that (1) $\Leftrightarrow$ (2), and the equivalence (3) $\Leftrightarrow$ (4) follows by the same argument.

We now prove that (2) $\Leftrightarrow$ (4). By symmetry, it will suffice to show that (2) $\Rightarrow$ (4). Let $x, y, z \in \Lambda$. If assumption (2) is satisfied, we have

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= (x \wedge (x \vee z)) \vee (y \wedge (x \vee z)) \\ &= x \vee ((y \wedge x) \vee (y \wedge z)) \\ &= (x \vee (y \wedge x)) \vee (y \wedge z) \\ &= x \vee (y \wedge z).\end{aligned}$$

□

Combining Remark A.1.4.3 with Lemma A.1.2.6, we obtain the following:

Proposition A.1.4.6. *Let Λ be a distributive lattice containing an element x and let $F \subseteq \Lambda$ be a filter which does not contain x. Then there exists a prime ideal $\mathfrak{p} \subseteq \Lambda$ which contains x such that $\mathfrak{p} \cap F = \varnothing$.*

A.1.5 Stone Duality for Distributive Lattices

We now study the restriction of the fully faithful embedding $\mathrm{Spec} : \mathrm{SLat} \hookrightarrow \mathcal{T}\mathrm{op}$ to distributive lattices.

Definition A.1.5.1. Let X be a topological space having a basis of quasi-compact open sets. We say that X is *quasi-separated* if, for every pair of quasi-compact open sets $U, V \subseteq X$, the intersection $U \cap V$ is quasi-compact.

Proposition A.1.5.2. *Let Λ be a distributive upper semilattice. The following conditions are equivalent:*

(1) *The partially ordered set Λ is a distributive lattice.*

(2) *The topological space $\mathrm{Spec}(\Lambda)$ is quasi-compact and quasi-separated.*

Proof. Suppose first that condition (2) is satisfied. Let us identify Λ with the collection of quasi-compact open subsets of $\mathrm{Spec}(\Lambda)$. For any finite collection $\{U_i\}$ of such subsets, condition (2) guarantees that $U = \bigcup U_i$ is quasi-compact, so that U is a greatest lower bound for $\{U_i\}$ in Λ. Conversely, suppose that (1) is satisfied. Let $\{U_i\}_{1 \leqslant i \leqslant n}$ be a finite collection of quasi-compact open subsets of $\mathrm{Spec}(\Lambda)$, and let U be their greatest lower bound in Λ. Then U is the largest quasi-compact open subset contained in $\bigcap U_i$. Since $\mathrm{Spec}(\Lambda)$ has a basis of quasi-compact open sets, we must have $U = \bigcap U_i$, so that $\bigcap U_i$ is quasi-compact. Taking $n = 0$, we learn that $\mathrm{Spec}(\Lambda)$ is quasi-compact; taking $n = 2$, we learn that $\mathrm{Spec}(\Lambda)$ is quasi-separated. □

Definition A.1.5.3. Let Λ and Λ' be lattices. A *lattice homomorphism* from Λ to Λ' is a map $\lambda : \Lambda \to \Lambda'$ such that, for every finite subset $S \subseteq \Lambda$, we have

$$\lambda(\bigvee S) = \bigvee \lambda(S) \qquad \lambda(\bigwedge S) = \bigwedge \lambda(S).$$

We let Lat denote the category whose objects are distributive lattices and whose morphisms are lattice homomorphisms.

Remark A.1.5.4. A map of lattices $\lambda : \Lambda \to \Lambda'$ is a lattice homomorphism if and only if λ satisfies

$$\lambda(\bot) = \bot \qquad \lambda(x \vee y) = \lambda(x) \vee \lambda(y)$$
$$\lambda(\top) = \top \qquad \lambda(x \wedge y) = \lambda(x) \wedge \lambda(y).$$

Here $\bot$ and $\top$ denote the least and greatest elements of Λ and Λ'.

Construction A.1.5.5. Let Λ and Λ' be distributive lattices, and let $\lambda : \Lambda' \to \Lambda$ be a lattice homomorphism. We let $D_\lambda \subseteq \Lambda \times \Lambda'$ denote the subset $\{(x, y) \in \Lambda \times \Lambda' : x \leqslant \lambda(y)\}$. Then D_λ is a distributor from Λ to Λ'. The construction $\lambda \mapsto D_\lambda$ determines a functor $\mathrm{Lat}^{\mathrm{op}} \to \mathrm{SLat}$.

Remark A.1.5.6. The functor $\mathrm{Lat}^{\mathrm{op}} \to \mathrm{SLat}$ of Construction A.1.5.5 is faithful. That is, we can recover a lattice homomorphism $\lambda : \Lambda' \to \Lambda$ from the underlying distributor D_λ. For each $y \in \Lambda'$, $\lambda(y)$ can be characterized as the largest element of x such that $(x, y) \in D_\lambda$.

Notation A.1.5.7. Let Λ be a distributive lattice. We let $\mathrm{Spec}(\Lambda)$ denote the spectrum of Λ, regarded as an upper semilattice (Construction A.1.2.3). If $\lambda : \Lambda \to \Lambda'$ is a lattice homomorphism, we let $\mathrm{Spec}(\lambda) : \mathrm{Spec}(\Lambda') \to \mathrm{Spec}(\Lambda)$ denote the map associated to the distributor D_λ of Construction A.1.5.5. The construction $\Lambda \mapsto \mathrm{Spec}(\Lambda)$ determines a functor $\mathrm{Lat}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}$. We will abuse notation by denoting this functor by Spec.

Remark A.1.5.8. The definition of the spectrum $\mathrm{Spec}(\Lambda)$ can be simplified a bit if we work in the setting of distributive lattices. Note than an ideal $\mathfrak{p} \subseteq \Lambda$ is prime if and only if it satisfies the following pair of conditions:

(*i*) The greatest element $\top \in \Lambda$ is not contained in $\mathfrak{p}$.

(*ii*) If $x \wedge y \in \mathfrak{p}$, then either x or y belongs to $\mathfrak{p}$.

Definition A.1.5.9. Let X and Y be coherent topological spaces. We will say that a morphism $f : X \to Y$ is *quasi-compact* if, for every quasi-compact open subset $U \subseteq Y$, the inverse image $f^{-1}(U)$ is a quasi-compact open subset of X. We let $\mathcal{T}\mathrm{op}_{\mathrm{coh}}$ denote the category whose objects are coherent topological spaces and whose morphisms are quasi-compact, continuous maps.

Proposition A.1.5.10. *The functor* Spec : $\mathrm{Lat}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}$ *induces an equivalence of categories* $\mathrm{Lat}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}_{\mathrm{coh}}$. *In other words,* Spec *is a faithful functor with the following properties:*

(1) *A topological space* X *lies in the essential image of* Spec *if and only if it is coherent.*

(2) *Let* Λ *and* Λ' *be distributive lattices. Then a continuous map* $f : \mathrm{Spec}(\Lambda) \to \mathrm{Spec}(\Lambda')$ *arises from a lattice homomorphism* $\lambda : \Lambda' \to \Lambda$ *(necessarily unique) if and only if* f *is quasi-compact.*

Proof. The faithfulness follows from Proposition A.1.3.3 and Remark A.1.5.6. Assertion (1) follows from Propositions A.1.3.3 and A.1.5.2. We now prove (2). Suppose first that $\lambda : \Lambda' \to \Lambda$ is a lattice homomorphism, let D_λ be the corresponding distributor, and $f : \mathrm{Spec}(\Lambda) \to \mathrm{Spec}(\Lambda')$ the induced map. For each $y \in \Lambda'$, we have

$$f^{-1}\,\mathrm{Spec}(\Lambda')_y = \bigcup_{\mathrm{Spec}(\Lambda)_x \subseteq f^{-1}\,\mathrm{Spec}(\Lambda')_y} \mathrm{Spec}(\Lambda)_x = \bigcup_{(x,y)\in D_\lambda} \mathrm{Spec}(\Lambda)_x = \bigcup_{x \leqslant \lambda(y)} \mathrm{Spec}(\Lambda)_x = \mathrm{Spec}(\Lambda)_{\lambda(y)},$$

so that f^{-1} carries quasi-compact open subsets of $\mathrm{Spec}(\Lambda')$ to quasi-compact open subsets of $\mathrm{Spec}(\Lambda)$. Conversely, suppose that $f : \mathrm{Spec}(\Lambda) \to \mathrm{Spec}(\Lambda')$ is a continuous map such that $f^{-1}U$ is quasi-compact whenever $U \subseteq \mathrm{Spec}(\Lambda')$ is quasi-compact. We then have $f^{-1}\,\mathrm{Spec}(\Lambda')_y = \mathrm{Spec}(\Lambda)_{\lambda(y)}$ for some map $\lambda : \Lambda' \to \Lambda$. Since the formation of inverse images commutes with unions and intersections, we conclude that λ is a lattice homomorphism. Note that $\mathrm{Spec}(\Lambda)_x \subseteq f^{-1}\,\mathrm{Spec}(\Lambda')_y$ if and only if $x \leqslant \lambda(y)$, so that the underlying distributor of the continuous map f is given by D_λ. □

Remark A.1.5.11. Let X be a coherent topological space and let $\{U_\alpha\}$ be a collection of nonempty quasi-compact open subsets of X which are closed under finite intersections. Then $\bigcap U_\alpha \neq \varnothing$. This follows from Proposition A.1.4.6.

Remark A.1.5.12. Let Λ be a distributive lattice which is given as a filtered colimit of distributive lattices Λ_α. Then the canonical map $\mathrm{Spec}(\Lambda) \simeq \varprojlim \mathrm{Spec}(P_\alpha)$ is a homeomorphism.

A.1.6 Boolean Algebras

Let Λ be a distributive lattice containing a least element $\perp$ and a greatest element $\top$. Let $x \in \Lambda$ be an element. A *complement* of x is an element $x^c \in \Lambda$ such that

$$x \wedge x^c = \perp \qquad x \vee x^c = \top.$$

We will say that x is *complemented* if there exists a complement for x.

Remark A.1.6.1. Let Λ be a distributive lattice containing an element x. If x' and x'' are complements of x, then $x' = x''$. To see this, we note that

$$x' = x' \wedge \top = x' \wedge (x \vee x'') = (x' \wedge x) \vee (x' \wedge x'') = \perp \vee (x' \wedge x'') = x' \wedge x''$$

so that $x' \leqslant x''$. The same argument shows that $x'' \leqslant x'$.

Remark A.1.6.2. Let Λ be a distributive lattice containing elements x, x^c. Then x^c is a complement of x if and only if x is a complement of x^c. In this case, we will simply say that x and x^c are *complementary*.

Remark A.1.6.3. Let $\lambda : \Lambda' \to \Lambda$ be a homomorphism of distributive lattices. Suppose that $y, y^c \in \Lambda'$ are complementary. Then $\lambda(y), \lambda(y^c) \in \Lambda$ are complementary.

Definition A.1.6.4. A *Boolean algebra* is a distributive lattice Λ for which every element $x \in \Lambda$ has a complement. We let BAlg denote the full subcategory of Lat spanned by the Boolean algebras.

Proposition A.1.6.5. *Let Λ be a distributive upper semilattice. The following conditions are equivalent:*

(1) *The partially ordered set Λ is a Boolean algebra.*

(2) *The spectrum* $\mathrm{Spec}(\Lambda)$ *is compact and Hausdorff.*

Proof. Suppose first that (1) is satisfied. Let $\mathfrak{p}$ and $\mathfrak{q}$ be distinct prime ideals of Λ. We may assume without loss of generality that there exists an element $x \in \mathfrak{p}$ which does not belong to $\mathfrak{q}$. Let x^c be a complement of x. Since $x \vee x^c = \top \notin \mathfrak{p}$, we conclude that $x^c \notin \mathfrak{p}$. Since $x \wedge x^c \in \mathfrak{q}$, we conclude that $x^c \in \mathfrak{q}$. Then $\mathrm{Spec}(\Lambda)_{x^c}$ and $\mathrm{Spec}(\Lambda)_x$ are disjoint open subsets of $\mathrm{Spec}(\Lambda)$ containing $\mathfrak{p}$ and $\mathfrak{q}$, respectively.

We now show that (2) $\Rightarrow$ (1). Let $x \in \Lambda$, so that $\mathrm{Spec}(\Lambda)_x$ is a quasi-compact open subset of $\mathrm{Spec}(\Lambda)$. Since $\mathrm{Spec}(\Lambda)$ is Hausdorff, the subset $\mathrm{Spec}(\Lambda)_x$ is also closed. Let U be the complement of $\mathrm{Spec}(\Lambda)_x$. Then U is a closed subset of $\mathrm{Spec}(\Lambda)$ and therefore compact. Since it is also an open subset of $\mathrm{Spec}(\Lambda)$, it has the form $\mathrm{Spec}(\Lambda)_y$ for some $y \in \Lambda$. We now observe that y is a complement to x. □

Corollary A.1.6.6. *Let Λ be a distributive lattice, let Λ' be a Boolean algebra, and let $D \subseteq \Lambda \times \Lambda'$ be a distributor. Then $D = D_\lambda$ for some lattice homomorphism $\lambda : \Lambda' \to \Lambda$ (necessarily unique, by Remark A.1.5.6).*

Remark A.1.6.7. Corollary A.1.6.6 implies that we can regard the category BAlg of Boolean algebras as a full subcategory of *both* the category of Lat of distributive lattices (where the morphisms are lattice homomorphisms) and the category $\mathrm{SLat}^{\mathrm{op}}$ of distributive upper semilattices (where the morphisms are distributors). Beware that the embedding $\mathrm{Lat} \hookrightarrow \mathrm{SLat}^{\mathrm{op}}$ is not full.

Proof of Corollary A.1.6.6. Using Propositions A.1.3.3 and A.1.5.10, we are reduced to proving that if $f : \mathrm{Spec}(\Lambda) \to \mathrm{Spec}(\Lambda')$ is a continuous map and $U \subseteq \mathrm{Spec}(\Lambda')$ is a quasi-compact open subset, then $f^{-1}U$ is a quasi-compact open subset of $\mathrm{Spec}(\Lambda)$. Since $\mathrm{Spec}(\Lambda')$ is Hausdorff (Proposition A.1.6.5), the subset $U \subseteq \mathrm{Spec}(\Lambda')$ is closed. The continuity of f guarantees that $f^{-1}U$ is a closed subset of $\mathrm{Spec}(\Lambda)$, hence quasi-compact (since $\mathrm{Spec}(\Lambda)$ is quasi-compact). □

Definition A.1.6.8. Let X be a topological space. We say that X is a *Stone space* if it is compact, Hausdorff, and has a basis of closed and open sets. We let $\mathcal{T}\mathrm{op}_{\mathrm{St}}$ denote the category whose objects are Stone spaces and whose morphisms are continuous maps.

Remark A.1.6.9. The category of Stone spaces admits several other descriptions, which we will discuss in §E.1 (see Proposition E.1.3.1).

Remark A.1.6.10. Let X be a Hausdorff space. Then X is automatically sober (every irreducible closed subset of X is a singleton). Moreover, an open set $U \subseteq X$ is quasi-compact if and only if it is also closed. It follows that the collection of quasi-compact open subsets of X is closed under finite intersections. It follows that a topological space X is a Stone space if and only if X is both Hausdorff and coherent.

Theorem A.1.6.11 (Stone Duality)**.** *The construction* $\Lambda \mapsto \mathrm{Spec}(\Lambda)$ *induces a fully faithful embedding* $\mathrm{Spec} : \mathrm{BAlg}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}$*, whose essential image is the full subcategory* $\mathcal{T}\mathrm{op}_{\mathrm{St}}$ *spanned by the Stone spaces.*

Proof. Combine Proposition A.1.6.5, Proposition A.1.5.10, Corollary A.1.6.6, and Remark A.1.6.10. □

A.2 Coherent ∞-Topoi

Let $\mathcal{X}$ be a Grothendieck topos. We say that $\mathcal{X}$ is *coherent* if it is equivalent to a category of the form $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathcal{C})$, where $\mathcal{C}$ is a small category which admits finite limits, and is equipped with a finitary Grothendieck topology (see Definition A.3.1.1). The theory of coherent topoi can be regarded as a generalization of the theory of coherent topological spaces described in §A.1. In this section, we will study a further generalization: the theory of *coherent ∞-topoi*.

Definition A.2.0.12. Let $\mathcal{X}$ be an ∞-topos. We will say that $\mathcal{X}$ is *quasi-compact* if every covering of $\mathcal{X}$ has a finite subcovering: that is, for every effective epimorphism $\amalg_{i \in I} U_i \to \mathbf{1}$ in $\mathcal{X}$ (where $\mathbf{1}$ is the final object of $\mathcal{X}$), there exists a finite subset $I_0 \subseteq I$ such that the map $\amalg_{i \in I} U_i \to \mathbf{1}$ is also an effective epimorphism. We say that an object $X \in \mathcal{X}$ is *quasi-compact* if the ∞-topos $\mathcal{X}_{/X}$ is quasi-compact.

Let $n \geqslant 0$ be an integer. We will define the notion of an *n-coherent ∞-topos* using induction on n. We say that an ∞-topos $\mathcal{X}$ is 0-*coherent* if it is quasi-compact. Assume that we have defined the notion of an n-coherent ∞-topos for some $n \geqslant 0$. We will say that an object $U \in \mathcal{X}$ of an ∞-topos $\mathcal{X}$ is *n-coherent* if the ∞-topos $\mathcal{X}_{/U}$ is n-coherent. We say that $\mathcal{X}$ is *locally n-coherent* if, for every object $X \in \mathcal{X}$, there exists an effective epimorphism $\amalg_i U_i \to X$, where each U_i is n-coherent. We say that $\mathcal{X}$ is *$(n+1)$-coherent* if it is locally n-coherent and the collection of n-coherent objects of $\mathcal{X}$ is closed under finite products.

Remark A.2.0.13. Let $\mathcal{X}$ be an ∞-topos. Then $\mathcal{X}$ is quasi-compact if and only if, for every collection of (-1)-truncated objects $\{U_i \in \mathcal{X}\}_{i \in I}$ such that $\tau_{\leqslant -1}(\amalg_{i \in I} U_i)$ is a final object of $\mathcal{X}$, there exists a finite subset $I_0 \subseteq I$ such that $\tau_{\leqslant -1}(\amalg_{i \in I_0} U_i)$ is a final object of $\mathcal{X}$. In particular, the condition that $\mathcal{X}$ is quasi-compact depends only on the underlying locale of (-1)-truncated objects of $\mathcal{X}$.

Remark A.2.0.14. Let $\mathcal{X}$ be an n-coherent ∞-topos for $n > 0$. The collection of $(n-1)$-coherent objects of $\mathcal{X}$ is stable under finite products. In particular, the final object of $\mathcal{X}$ is $(n-1)$-coherent, so that $\mathcal{X}$ is $(n-1)$-coherent. It follows that an n-coherent ∞-topos is also m-coherent for each $m \leqslant n$.

Remark A.2.0.15. Let $\mathcal{X}$ be a locally n-coherent ∞-topos. Then $\mathcal{X}_{/U}$ is locally n-coherent for any object $U \in \mathcal{X}$. In this case, an object $X \in \mathcal{X}$ is $(n+1)$-coherent if and only if it is n-coherent and, for every pullback diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

in $\mathcal{X}$, if U and V are n-coherent, then $U \times_X V$ is also n-coherent.

Remark A.2.0.16. Suppose that $\mathcal{X} = \prod_{1 \leqslant i \leqslant k} \mathcal{X}_i$ is a product of finitely many ∞-topoi (corresponding to a *coproduct* in the ∞-category $\infty\mathcal{T}\mathrm{op}$). Then $\mathcal{X}$ is n-coherent if and only if each $\mathcal{X}_i$ is n-coherent. It follows that if $\mathcal{Y}$ is any ∞-topos, then a finite coproduct $U = \amalg_{1 \leqslant i \leqslant k} U_i$ in $\mathcal{Y}$ is n-coherent if and only if each U_i is n-coherent.

Remark A.2.0.17. Let $\mathcal{X}$ be a locally n-coherent ∞-topos and let $X \in \mathcal{X}$ be a quasi-compact object. The assumption that $\mathcal{X}$ is locally n-coherent guarantees the existence of an effective epimorphism $\amalg_{i \in I} U_i \to X$, where each U_i is n-coherent. Since X is quasi-compact, we may assume that the index set I is finite. Then $U = \amalg_{i \in I} U_i$ is n-coherent by Remark A.2.0.16. It follows that there exists an effective epimorphism $U \to X$, where U is n-coherent.

A.2.1 Coherence of Morphisms

We now introduce a relative version of Definition A.2.0.12.

Definition A.2.1.1. Let $\mathcal{X}$ be an ∞-topos which is locally n-coherent. We will say that a morphism $f : X' \to X$ in $\mathcal{X}$ is *relatively n-coherent* if, for every n-coherent object $U \in \mathcal{X}$ and every morphism $U \to X$, the fiber product $U \times_X X'$ is also n-coherent.

Example A.2.1.2. Let $\mathcal{X}$ be a locally n-coherent ∞-topos. If $f : X' \to X$ is a morphism such that X' is n-coherent and X is $(n+1)$-coherent, then f is relatively n-coherent.

Proposition A.2.1.3. *Let $n \geqslant 0$ be an integer and $\mathcal{X}$ an ∞-topos, and let $f : X_0 \to X$ be a morphism in $\mathcal{X}$. Assume that if $n > 0$, then $\mathcal{X}$ is locally $(n-1)$-coherent and that f is relatively $(n-1)$-coherent. Then:*

(1) *The map f is relatively m-coherent for each $m < n$.*

(2) *Assume that f is an effective epimorphism and that X_0 is n-coherent. Then X is n-coherent.*

Proof. The proof proceeds by induction on n. Suppose first that $n = 0$; we must show that if f is an effective epimorphism and X_0 is quasi-compact, then X is quasi-compact. Choose an effective epimorphism $\amalg_{i \in I} X_i \to X$. Then the induced map $\amalg_{i \in I}(X_i \times_X X_0) \to X_0$ is also an effective epimorphism. Since X_0 is quasi-compact, there exists a finite subset $I_0 \subseteq I$ such that the map $\amalg_{i \in I_0}(X_i \times_X X_0) \to X_0$ is an effective epimorphism. Since f is an effective epimorphism, we conclude that the composite map $\amalg_{i \in I_0}(X_i \times_X X_0) \to X_0 \to X$ is an effective epimorphism. This map factors through $\phi : \amalg_{i \in I_0} X_i \to X$, so that ϕ is an effective epimorphism as desired.

Now suppose that $n > 0$. We begin by proving (1). Choose a morphism $U \to X$, where U is m-coherent; we must show that $U_0 = U \times_X X_0$ is m-coherent. Remark A.2.0.17 guarantees the existence of an effective epimorphism $g : V \to U$, where V is $(n-1)$-coherent. It follows from Example A.2.1.2 that g is relatively $(m-1)$-coherent. Let $V_0 = V \times_X X_0$ and $g_0 : V_0 \to U_0$ the induced map, so that g_0 is also relatively $(m-1)$-coherent. Our assumption that f is relatively $(n-1)$-coherent guarantees that V_0 is $(n-1)$-coherent, and therefore m-coherent (Remark A.2.0.14). Since g_0 is an effective epimorphism, the inductive hypothesis guarantees that U_0 is m-coherent, as desired.

We now prove (2). We will show that X satisfies the criterion for n-coherence described in Remark A.2.0.15. The inductive hypothesis guarantees that X is $(n-1)$-coherent. Choose maps $U \to X$ and $V \to X$, where U and V are $(n-1)$-coherent; we wish to show that $U \times_X V$ is $(n-1)$-coherent. Let $U_0 = U \times_X X_0$ and $V_0 = V \times_X X_0$. Since f is relatively $(n-1)$-coherent, U_0 and V_0 are $(n-1)$-coherent. Since X_0 is n-coherent, we deduce that $U_0 \times_{X_0} V_0$ is $(n-1)$-coherent. The map $f' : U_0 \times_{X_0} V_0 \to U \times_X V$ is a pullback of f and therefore relatively $(n-2)$-coherent by (1). Since f' is an effective epimorphism, the inductive hypothesis guarantees that $U \times_X V$ is $(n-1)$-coherent, as desired. □

Corollary A.2.1.4. *Let $\mathcal{X}$ be an ∞-topos and suppose we are given a full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ with the following properties:*

(*a*) *Every object $U \in \mathcal{X}_0$ is an n-coherent object of $\mathcal{X}$*

(*b*) *For every object $X \in \mathcal{X}$, there exists an effective epimorphism $\amalg U_i \to X$, where each U_i belongs to $\mathcal{X}_0$.*

Then:

(1) *A morphism $f : X' \to X$ in $\mathcal{X}$ is relatively n-coherent if and only if, for every morphism $U \to X$ where $U \in \mathcal{X}_0$, the fiber product $U' = U \times_X X'$ is n-coherent.*

(2) *An object $X \in \mathcal{X}$ is $(n+1)$-coherent if and only if it is quasi-compact and, for every pair of maps $U \to X$, $V \to X$ where $U, V \in \mathcal{X}_0$, the fiber product $U \times_X V$ is n-coherent.*

Proof. We first prove (1). The "only if" direction is obvious. For the converse, choose a map $V \to X$ where V is n-coherent; we wish to show that $V' = V \times_X X'$ is n-coherent. Condition (*b*) and the quasi-compactness of V guarantee the existence of an effective epimorphism $g : \amalg_{i \in I} U_i \to V$, where each U_i belongs to $\mathcal{X}_0$ and the index set I is finite. Let $g' : \amalg_{i \in I}(U_i \times_X X') \to V'$ be the induced map. Using our hypothesis together with Remark A.2.0.16, we see that $\amalg_{i \in I}(U_i \times_X X')$ is n-coherent. The map g is relatively $(n-1)$-coherent by Example A.2.1.2, so that g' is relatively $(n-1)$-coherent. Applying Proposition A.2.1.3, we deduce that V' is n-coherent as desired.

We now prove (2) using induction on n. The "only if" direction is again obvious. Assume therefore that X is quasi-compact and that $U \times_X V$ is n-coherent whenever $U, V \in \mathcal{X}_0$. We note that X is n-coherent: this follows from the inductive hypothesis if $n > 0$, or by assumption if $n = 0$. Using (1), we see that the map $U \to X$ is relatively n-coherent whenever $U \in \mathcal{X}_0$. Consequently, if V is an arbitrary n-coherent object of $\mathcal{X}$ and we are given a map $g : V \to X$, then $U \times_X V$ is n-coherent for each $U \in \mathcal{X}_0$. Applying (1) again, we deduce that g is relatively n-coherent. It follows that the fiber product $U \times_X V$ is n-coherent whenever U and V are n-coherent, so that X is $(n+1)$-coherent by Remark A.2.0.15. □

Corollary A.2.1.5. *Let $\mathcal{X}$ be a locally n-coherent ∞-topos, and let $f : X' \to X$ be a morphism in $\mathcal{X}$. Suppose that there exists an effective epimorphism $U \to X$ such that the induced map $f' : U' \to U$ is relatively n-coherent, where $U' = X' \times_X U$. Then f is relatively n-coherent.*

Proof. Suppose we are given a map $Y \to X$, where Y is n-coherent. We wish to prove that $Y' = X' \times_X Y$ is n-coherent. Replacing X by Y and U by $Y \times_X U$, we are reduced to proving that if X is n-coherent, then X' is also n-coherent.

Since $\mathcal{X}$ is locally n-coherent, there exists an effective epimorphism $\amalg_{i \in I} U_i \to U$, where each U_i is n-coherent. The composite map $\amalg_{i \in I} U_i \to U \to X$ is also an effective epimorphism.

Since X is quasi-compact, there exists a finite subset $I_0 \subseteq I$ such that the map $\amalg_{i \in I_0} U_i \to X$ is an effective epimorphism. The coproduct $\amalg_{i \in I_0} U_i$ is n-coherent (Remark A.2.0.16). Replacing U by $\amalg_i U_i$, we can reduce to the case where U is n-coherent. Since f' is relatively n-coherent, we deduce that U' is n-coherent. Since X is n-coherent and U is $(n-1)$-coherent, the map $U \to X$ is relatively $(n-1)$-coherent (if $n > 0$), so the induced map $U' \to X'$ is an effective epimorphism which is $(n-1)$-coherent (if $n > 0$). Proposition A.2.1.3 now implies that X' is n-coherent as desired. □

Definition A.2.1.6. Let $\mathcal{X}$ be an ∞-topos. We will say that $\mathcal{X}$ is *coherent* if it is n-coherent for every integer n. We will say that an object $U \in \mathcal{X}$ is *coherent* if the ∞-topos $\mathcal{X}_{/U}$ is coherent. We will say that $\mathcal{X}$ is *locally coherent* if, for every object $X \in \mathcal{X}$, there exists an effective epimorphism $\amalg_i U_i \to X$ where each U_i is coherent.

Example A.2.1.7. Let $\mathcal{X} = \mathcal{S}$ be the ∞-category of spaces. Then $\mathcal{X}$ is coherent and locally coherent. An object $X \in \mathcal{X}$ is n-coherent if and only if the homotopy sets $\pi_i(X, x)$ are finite for every point $x \in X$ and all $i \leqslant n$.

Remark A.2.1.8. Let $\mathcal{X}$ be an ∞-topos. The collection of coherent objects of $\mathcal{X}$ is closed under finite coproducts (Remark A.2.0.16) and under fiber products. In particular, if $\mathcal{X}$ is coherent, then the collection of coherent objects of $\mathcal{X}$ is closed under finite limits.

A.2.2 Coherence and Hypercompletion

We now show that, to some extent, the coherence of an ∞-topos $\mathcal{X}$ depends only on the hypercompletion $\mathcal{X}^{\mathrm{hyp}}$.

Lemma A.2.2.1. *Let $\mathcal{X}$ be an ∞-topos and $f^* : \mathcal{X} \to \mathcal{Y}$ a geometric morphism, which exhibits $\mathcal{Y}$ as a cotopological localization of $\mathcal{X}$ (see Definition HTT.6.5.2.17). Let $n \geqslant 0$ be an integer, and assume either that $n = 0$ or that $\mathcal{X}$ is locally $(n-1)$-coherent.*

(1) *An object $X \in \mathcal{X}$ is n-coherent if and only if $f^* X \in \mathcal{Y}$ is n-coherent.*

(2) *An object $Y \in \mathcal{Y}$ is n-coherent if and only if $f_* Y \in \mathcal{X}$ is n-coherent.*

(3) *If $n > 0$, the ∞-topos $\mathcal{Y}$ is locally $(n-1)$-coherent.*

Proof. Since f^* is a localization functor, the counit map $f^* f_* Y \to Y$ is an equivalence for each $Y \in \mathcal{Y}$. Consequently, assertion (2) follows from (1), applied to $X = f_* Y$. We prove (1) by induction on n. We first note that the inductive hypothesis implies (3). To see this, assume that $n > 0$ and let $Y \in \mathcal{Y}$, so that $Y \simeq f^* X$ for $X = f_* Y \in \mathcal{X}$. Since $\mathcal{X}$ is locally $(n-1)$-coherent, there exists an effective epimorphism $\amalg V_i \to X$ where each V_i is $(n-1)$-coherent. This induces an effective epimorphism $\amalg f^* V_i \to Y$ in $\mathcal{Y}$, and each $f^* V_i$ is $(n-1)$-coherent by the inductive hypothesis.

We now prove (1) in the case $n = 0$. Suppose that $X \in \mathcal{X}$ is quasi-compact; we wish to show that $f^*X \in \mathcal{Y}$ is quasi-compact. Choose an effective epimorphism $u : \amalg_{i \in I} U_i \to f^*X$ in $\mathcal{Y}$. For $i \in I$, let $V_i = f_* U_i \times_{f_* f^* X} X$, so that $u \simeq f^* v$ for some map $v : \amalg_{i \in I} V_i \to X$. Since f^* is a cotopological localization, the map v is an effective epimorphism. Since X is quasi-compact, there exists a finite subset $I_0 \subseteq I$ such that the induced map $v' : \amalg_{i \in I_0} V_i \to X$ is an effective epimorphism. It follows that $f^*(v') = u' : \amalg_{i \in I_0} U_i \to f^* X$ is an effective epimorphism as well.

Now suppose that f^*X is quasi-compact. We wish to prove that X is quasi-compact. Choose an effective epimorphism $v : \amalg_{i \in I} V_i \to X$, so that $u = f^* v$ is an effective epimorphism $\mathcal{Y}$. Since f^*X is quasi-compact, there exists a finite subset $I_0 \subseteq I$ such that the induced map $\amalg_{i \in I_0} f^* V_i \to f^* X$ is an effective epimorphism. Since f^* is a cotopological localization, we conclude that the map $\amalg_{i \in I_0} V_i \to X$ is an effective epimorphism.

It remains to prove (1) in the case $n > 0$. Suppose first that X is n-coherent. Using the inductive hypothesis, we deduce that f^*X is $(n-1)$-coherent; moreover, we have already seen that $\mathcal{Y}$ is locally $(n-1)$-coherent. To show that f^*X is n-coherent, it suffices to show that for every pair of maps $U \to f^*X$ and $U' \to f^*X$ where $U, U' \in \mathcal{Y}$ are $(n-1)$-coherent, the fiber product $U \times_{f^*X} U'$ is $(n-1)$-coherent (Remark A.2.0.15). Let $V = f_* U \times_{f_* f^* X} X$ and $V' = f_* U' \times_{f_* f^* X} X$. It follows from the inductive hypothesis that V and V' are $(n-1)$-coherent objects of $\mathcal{X}$, so that $V \times_X V'$ is $(n-1)$-coherent. Applying the inductive hypothesis again, we conclude that $U \times_{f^*X} U' \simeq f^*(V \times_X V')$ is $(n-1)$-coherent.

For the converse, suppose that f^*X is n-coherent. Using the inductive hypothesis, we conclude that X is $(n-1)$-coherent. To show that X is n-coherent, it suffices to show that if we are given morphisms $V \to X$, $V' \to X$ where $V, V' \in \mathcal{X}$ are $(n-1)$-coherent, then $V \times_X V'$ is $(n-1)$-coherent. By the inductive hypothesis, it suffices to show that $f^*(V \times_X V') \simeq f^* \times_{f^*X} f^* V'$ is $(n-1)$-coherent, which follows from our assumption that f^*X is n-coherent. □

Taking $\mathcal{Y}$ be the hypercompletion of $\mathcal{X}$ and allowing n to vary, we immediately obtain the following conssequence:

Proposition A.2.2.2. *Let $\mathcal{X}$ be an ∞-topos which is locally n-coherent for all $n \geqslant 0$, let $\mathcal{X}^{\mathrm{hyp}}$ be the full subcategory of $\mathcal{X}$ spanned by the hypercomplete objects, and let $L : \mathcal{X} \to \mathcal{X}^{\mathrm{hyp}}$ be a left adjoint to the inclusion. Then:*

(1) *The ∞-topos $\mathcal{X}^{\mathrm{hyp}}$ is locally n-coherent for all $n \geqslant 0$.*

(2) *An object of $\mathcal{X}^{\mathrm{hyp}}$ is coherent if and only if it is coherent when viewed as an object of $\mathcal{X}$.*

(3) *An object $X \in \mathcal{X}$ is coherent if and only if LX is coherent.*

In particular, if $\mathcal{X}$ is locally coherent, then $\mathcal{X}^{\mathrm{hyp}}$ is also locally coherent.

A.2.3 Coherence and Compactness

Let $\mathcal{X}$ be an ∞-topos. The hypothesis that an object $X \in \mathcal{X}$ be n-coherent can be regarded as a finiteness condition on X. Our next result relates this finiteness condition to the compactness properties of X.

Proposition A.2.3.1. *Let $\mathcal{X}$ be an n-coherent ∞-topos for some $n \geqslant 0$, and let $\Gamma : \mathcal{X} \to \mathcal{S}$ be the global sections functor (that is, Γ is the functor corepresented by the final object $\mathbf{1} \in \mathcal{X}$). Then the restriction of Γ to $\tau_{\leqslant n-1}\mathcal{X}$ commutes with filtered colimits.*

Corollary A.2.3.2. *Let $\mathcal{X}$ be a locally n-coherent ∞-topos. Then:*

(1) *If $U \in \mathcal{X}$ is an n-coherent object, then $\tau_{\leqslant n-1}U$ is a compact object of $\tau_{\leqslant n-1}\mathcal{X}$.*

(2) *The ∞-category $\tau_{\leqslant n-1}\mathcal{X}$ is generated, under small colimits, by objects of the form $\tau_{\leqslant n-1}U$, where $U \in \mathcal{X}$ is n-coherent.*

(3) *The ∞-category $\tau_{\leqslant n-1}\mathcal{X}$ is compactly generated.*

Proof. Assertion (1) follows immediately from Proposition A.2.3.1. Consider an arbitrary object $X \in \mathcal{X}$. Since $\mathcal{X}$ is locally coherent, we can choose a hypercovering $X_\bullet$ of X such that each X_m is a coproduct of n-coherent objects of $\mathcal{X}$. If we assume that $X \in \tau_{\leqslant n-1}\mathcal{X}$, then the map $\tau_{\leqslant n-1}|X_\bullet| \to \tau_{\leqslant n-1}X \simeq X$ is an equivalence. Consequently, X is the geometric realization of a simplicial object of $\tau_{\leqslant n-1}\mathcal{X}$, each term of which is a coproduct of objects having the form $\tau_{\leqslant n-1}U$, where U is n-coherent. This proves (2). Assertion (3) follows immediately from (1) and (2). □

Remark A.2.3.3. Let $\mathcal{X}$ be a locally $(n+1)$-coherent ∞-topos. Let $\mathcal{C}$ be the smallest full subcategory of $\tau_{\leqslant n}\mathcal{X}$ which is closed under finite colimits and contains $\tau_{\leqslant n}U$ for every $(n+1)$-coherent object $U \in \mathcal{X}$. Then $\mathcal{C}$ is the full subcategory of $\tau_{\leqslant n}\mathcal{X}$ spanned by the compact objects. To see this, we first note that every object of $\mathcal{C}$ is compact in $\tau_{\leqslant n}\mathcal{X}$. It follows from Proposition HTT.5.3.5.11 that the inclusion $\mathcal{C} \hookrightarrow \tau_{\leqslant n}\mathcal{X}$ extends to a fully faithful embedding $\phi : \mathrm{Ind}(\mathcal{C}) \to \tau_{\leqslant n}\mathcal{X}$. Proposition HTT.5.5.1.9 implies that ϕ preserves small colimits, so that ϕ is an equivalence of ∞-categories by Corollary A.2.3.2. It follows that the ∞-category of compact objects of $\tau_{\leqslant n}\mathcal{X}$ can be identified with an idempotent completion of $\mathcal{C}$. Since $\mathcal{C}$ is an $(n+1)$-category which admits finite colimits, it is already idempotent complete (Proposition HTT.**??**), so that every compact object of $\tau_{\leqslant n}\mathcal{X}$ belongs to $\mathcal{C}$.

The proof of Proposition A.2.3.1 depends on the following:

Lemma A.2.3.4. *Let $n \geqslant 0$ be an integer, and let $\mathcal{X}$ be an ∞-topos which we assume to be locally $(n-1)$-coherent if $n > 0$. Let $f : U \to X$ be a morphism in $\mathcal{X}$. If f is $(n-2)$-truncated, X is n-coherent, and U is $(n-1)$-coherent, then U is n-coherent.*

Proof. We proceed by induction on n. In the case $n = 0$, the map f is an equivalence and the n-coherence of U follows from the n-coherence of X. Assume therefore that $n > 0$, so that U is quasi-compact. We wish to show that if we are given maps $V_1 \to U$, $V_2 \to U$, where V_1 and V_2 are $(n-1)$-coherent objects of $\mathcal{X}$, then the fiber product $V_1 \times_U V_2$ is also $(n-1)$-coherent. Since U is $(n-1)$-coherent, the fiber product $V_1 \times_U V_2$ is automatically $(n-2)$-coherent. The map $V_1 \times_U V_2 \to V_1 \times_X V_2$ is a pullback of the diagonal map $U \to U \times_X U$ and therefore $(n-3)$-truncated. Since X is n-coherent, the fiber product $V_1 \times_X V_2$ is $(n-1)$-coherent, and the desired result follows from the inductive hypothesis. □

Remark A.2.3.5. Let $\mathcal{X}$ be a coherent ∞-topos and let $U \in \mathcal{X}$ be an n-truncated object. Then U is coherent if and only if it is $(n+1)$-coherent. This follows by applying Lemma A.2.3.4 in the case where X is a final object of $\mathcal{X}$.

Proof of Proposition A.2.3.1. We proceed by induction on n. In the case $n = 0$, our assumption guarantees that $\mathcal{X}$ is quasi-compact and the desired result follows immediately from the definition. Let us therefore assume that $n > 0$. Let $\mathcal{J}$ be a small filtered ∞-category, let $U, U' : \mathrm{Fun}(\mathcal{J}, \tau_{\leq n-1}\mathcal{X}) \to \mathcal{S}$ be given by

$$U(F) = \varinjlim_{J \in \mathcal{J}} \Gamma(F(J)) \qquad U'(F) = \Gamma(\varinjlim_{J \in \mathcal{J}} F(J)).$$

There is an evident natural transformation $\beta_F : U(F) \to U'(F)$; we wish to show that β_F is a homotopy equivalence. Assume for the moment that β_F is surjective on connected components. It then suffices to show that for every pair of points $\eta, \eta' \in U(F)$, the map β_F induces a homotopy equivalence of path spaces $\phi : \{\eta\} \times_{U(F)} \{\eta'\} \to \{\eta\} \times_{U'(F)} \{\eta'\}$. Since $\mathcal{J}$ is filtered, we may assume without loss of generality that η and η' are the images of points $\eta_0, \eta'_0 \in \Gamma(F(J))$. Since $\mathcal{J}$ is filtered, the map $\mathcal{J}_{J/} \to \mathcal{J}$ is left cofinal; we may therefore replace $\mathcal{J}$ by $\mathcal{J}_{J/}$ and thereby assume that J is a final object of $\mathcal{J}$. In this case, η_0 and η'_0 determine natural transformations $* \to F$, where $*$ denotes the constant functor $\mathcal{J} \to \mathcal{X}$ taking the value $\mathbf{1}$. Let $F' = * \times_F *$. Unwinding the definitions, we see that ϕ can be identified with the map $\beta_{F'}$. The desired result then follows from the inductive hypothesis, since $\mathcal{X}$ is $(n-1)$-truncated and F' takes values in $\tau_{\leq n-2}\mathcal{X}$.

It remains to prove that β_F is surjective on connected components. Choose a point $\eta \in U'(F)$, corresponding to a map $\alpha : \mathbf{1} \to \varinjlim_{J \in \mathcal{J}} F(J)$. We wish to show that α factors (up to homotopy) through $F(J)$ for some $J \in \mathcal{J}$. Note that the map $\amalg_{J \in \mathcal{J}} F(J) \to \varinjlim_{J \in \mathcal{J}} F(J)$ is an effective epimorphism. Since $\mathcal{X}$ is locally $(n-1)$-coherent, there exists a collection of $(n-1)$-coherent objects $\{U_i \in \mathcal{X}\}_{i \in I}$ such that $\amalg_{i \in I} U_i \to \mathbf{1}$ is an effective epimorphism and each of the composite maps $U_i \to \mathbf{1} \to \varinjlim_{J \in \mathcal{J}} F(J)$ factors through $F(J_i)$, for some $J_i \in \mathcal{J}$. Since $\mathcal{X}$ is quasi-compact, we can assume that the set I is finite. Let $U = \amalg_{i \in I} U_i$. Since $\mathcal{J}$ is filtered, there exists an object $J_0 \in \mathcal{J}$ and maps $J_i \to J_0$ for $i \in I$, so that the composite map $U \to \mathbf{1} \xrightarrow{\alpha} \varinjlim_{J \in \mathcal{J}} F(J)$ factors through $F(J_0)$. Since $\mathcal{J}$ is filtered, the map

$\mathcal{J}_{J_0/} \to \mathcal{J}$ is left cofinal; we may therefore replace $\mathcal{J}$ by $\mathcal{J}_{J_0/}$ and thereby reduce to the case where J_0 is an initial object of $\mathcal{J}$. Let $F_0 : \mathcal{J} \to \mathcal{X}$ be the constant functor taking the value $F(\mathcal{J}_0)$, and let $F_\bullet$ be the simplicial object of $\mathrm{Fun}(\mathcal{J}, \mathcal{X})$ given by the Čech nerve of the map $F_0 \to F$. Let $U_\bullet$ be the Čech nerve of the map $U \to \mathbf{1}$, so that we obtain a map of simplicial objects of $\mathcal{X}$ $\gamma : U_\bullet \to \varinjlim_{J \in \mathcal{J}} F_\bullet(J)$. We will prove the following assertion by induction on $m \geqslant 0$:

(∗) Let $\mathbf{\Delta}_{s,\leqslant m}$ denote the subcategory of $\mathbf{\Delta}$ whose objects are linearly ordered sets $[j]$ for $j \leqslant m$, and whose morphisms are given by injective maps $[j] \to [j']$. Let $U_\bullet^{\leqslant m}$ be the restriction of $U_\bullet$ to $\mathbf{\Delta}^{\mathrm{op}}_{s,\leqslant m}$, define $F_\bullet^{\leqslant m}$ similarly, and let $\gamma^{\leqslant m} : U_\bullet^{\leqslant m} \to \varinjlim_{J \in \mathcal{J}} F_\bullet^{\leqslant m}(J)$ be the map induced by γ. Then there exists an object $J_m \in \mathcal{J}$ such that $\gamma^{\leqslant m}$ factors through $F_\bullet^{\leqslant m}(J_m)$.

Assertion (∗) is obvious when $m = 0$, since the functor F_0 is constant. Assume that $\gamma^{\leqslant m-1}$ factors through $F_\bullet^{\leqslant m-1}(J_{m-1})$ for some $J_{m-1} \in \mathcal{J}$. Replacing $\mathcal{J}$ by $\mathcal{J}_{J_{m-1}/}$, we may assume that J_{m-1} is an initial object of $\mathcal{J}$, so that we have a canonical map $\delta_J : U_\bullet^{\leqslant m-1} \to F_\bullet^{\leqslant m-1}(J)$ for all $J \in \mathcal{J}$. Let $M(J) \in \mathcal{X}$ denote the mth matching object of $F_\bullet(J)$, for each $j \in J$, so that δ_J determines a map $\theta : U_m \to M(J)$. Using Proposition HTT.A.2.9.14 , we see that promoting δ_J to a natural transformation $U_\bullet^{\leqslant m} \to F_\bullet^{\leqslant m}(J)$ is equivalent to choosing a point of the mapping space

$$\mathrm{Map}_{\mathcal{X}_{/M(J)}}(U_m, F_m(J)) \simeq \mathrm{Map}_{\mathcal{X}_{/U_m}}(U_m, F_m(J) \times_{M(J)} U_m).$$

Consequently, to prove (∗), it suffices to show that the map

$$\varinjlim \mathrm{Map}_{\mathcal{X}_{/U_m}}(U_m, F_m(J) \times_{M(J)} U_m) \to \mathrm{Map}_{\mathcal{X}_{/U_m}}(U_m, \varinjlim F_m(J) \times_{M(J)} U_m)$$

is a homotopy equivalence. Since $\mathcal{X}$ is n-coherent and $U \in \mathcal{X}$ is $(n-1)$-coherent, the ∞-topos $\mathcal{X}_{/U_m}$ is $(n-1)$-coherent. By the inductive hypothesis, it suffices to show that the the objects $F_m(J) \times_{M(J)} U_m$ are $(n-2)$-truncated objects of $\mathcal{X}_{/U_m}$. For this, it suffices to show that the map $F_m(J) \to M(J)$ is $(n-2)$-truncated. This map is a pullback of the diagonal $F(J) \to F(J)^{\partial \Delta^m}$, and therefore $(n-m-1)$-truncated (since $F(J)$ is assumed to be $(n-1)$-truncated). This completes the proof of (∗).

Applying (∗) in the case $m = n$ and composing with the natural map $\varinjlim F_\bullet^{\leqslant n} \to F$, we deduce the existence of an object $J_n \in \mathcal{J}$ and a commutative diagram σ :

$$\begin{array}{ccc} \varinjlim U_\bullet^{\leqslant n} & \longrightarrow & F(J_n) \\ \downarrow & & \downarrow \\ \mathbf{1} & \xrightarrow{\eta} & \varinjlim_{J \in \mathcal{J}} F(J). \end{array}$$

Since the map $U \to \mathbf{1}$ is an effective epimorphism, we deduce that $\tau_{\leqslant n-1} \varinjlim U_{\bullet}^{\leqslant n} \simeq |U_{\bullet}| \simeq \mathbf{1}$. Applying the truncation functor $\tau_{\leqslant n-1}$ to the diagram σ, we conclude that η factors through $F(J_n)$. □

A.2.4 Coherence and Truncations

We now study the behavior of coherent objects under truncation.

Proposition A.2.4.1. *Let $\mathcal{X}$ be an ∞-topos and let $f : X \to Y$ be a morphism of $\mathcal{X}$, and let $n \geqslant 0$. Then:*

(a_n) *If X is n-coherent and f is n-connective, then Y is n-coherent.*

(b_n) *If Y is n-coherent and f is $(n+1)$-connective, then X is n-coherent.*

Proof. We proceed by induction on n. We begin by treating the case $n = 0$. To prove (a_0), assume that X is quasi-compact and that f is an effective epimorphism; we wish to show that Y is also quasi-compact. Fix an effective epimorphism $\amalg_{i \in I} Y_i \to Y$ and for each $i \in I$ set $X_i = Y_i \times_Y X$. Then the induced map $\amalg_{i \in I} X_i \to X$ is also an effective epimorphism. Since X is quasi-compact, we can choose a finite subset $I_0 \subseteq I$ for which the upper horizontal map in the diagram

$$\begin{array}{ccc} \amalg_{i \in I_0} X_i & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \amalg_{i \in I_0} Y_i & \longrightarrow & Y \end{array}$$

is an effective epimorphism. Since f is an effective epimorphism by assumption, it follows from the commutativity of the diagram that the lower horizontal map is also an effective epimorphism.

We now prove (b_0). Assume that f is 1-connective and that Y is quasi-compact. Suppose we are given an effective epimorphism $\rho : \amalg_{i \in I} X_i \to X$. Then the composite map $\amalg_{i \in I} X_i \to X \xrightarrow{f} Y$ is also an effective epimorphism. Using the quasi-compactness of Y, we deduce that there exists a finite subset $I_0 \subseteq I$ for which the composite map $\phi : \amalg_{i \in I_0} X_i \to X \xrightarrow{f} Y$ is also an effective epimorphism. We claim that $\rho_0 = \rho|_{\amalg_{i \in I_0} X_i}$ is also an effective epimorphism. To prove this, we note that ρ_0 factors as a composition

$$\amalg_{i \in I_0} X_i \xrightarrow{g} \amalg_{i \in I_0} X_i \times_Y X \xrightarrow{h} X$$

where g is the coproduct of maps $g_i : X_i \to X_i \times_Y X$ and h is the amalgam of maps $h_i : X_i \times_Y X \to X$ given by projection onto the second factor. Each g_i is a pullback of the diagonal map $X \to X \times_Y X$, and is therefore an effective epimorphism by virtue of our assumption that f is 1-connective. The map h is a pullback of ϕ, and is therefore an effective epimorphism. It follows that ρ_0 is also an effective epimorphism, as desired.

We now carry out the inductive step. Assume that $n > 0$ and that assertions (a_m) and (b_m) are valid for $m < n$. We begin by establishing (a_n). Suppose that X is n-coherent and that f is n-connective; we wish to show that Y is n-coherent. For each object $Y' \in \mathcal{X}_{/Y}$, we can regard the fiber product $X' = X \times_Y Y'$ as an object of $\mathcal{X}_{/X}$. Since X is n-coherent, the ∞-topos $\mathcal{X}_{/X}$ is locally $(n-1)$-coherent: in particular, we can choose an effective epimorphism $\amalg X_i \to X'$, where each X_i is an $(n-1)$-coherent object of $\mathcal{X}$. Since f is an effective epimorphism, the composite map $\amalg X_i \to X' \to Y'$ is also an effective epimorphism. Allowing Y' to vary, we deduce that the ∞-topos $\mathcal{X}_{/Y}$ is locally $(n-1)$-coherent. To complete the proof of (a_n), we must show that for every pair of $(n-1)$-coherent objects $Y_0, Y_1 \in \mathcal{X}_{/Y}$, the fiber product $Y_{01} = Y_0 \times_Y Y_1$ is also $(n-1)$-coherent. Set $X_0 = X \times_Y Y_0$ and $X_1 = X \times_Y Y_1$. Then f induces n-connective maps $X_0 \to Y_0$ and $X_1 \to Y_1$, it follows from our inductive hypothesis (b_{n-1}) that X_0 and X_1 are $(n-1)$-coherent. Using the n-coherence of X, we deduce that $X_{01} = X_0 \times_X X_1$ is $(n-1)$-coherent. The natural map $X_{01} \to Y_{01}$ is a pullback of f, and therefore n-connective. Applying (a_{n-1}), we deduce that Y_{01} is $(n-1)$-coherent, as desired.

We now prove (b_n). Assume that Y is n-coherent and that f is $(n+1)$-connective; we wish to show that X is n-coherent. Since $\mathcal{X}_{/Y}$ is locally $(n-1)$-coherent, it follows immediately from the definitions that $\mathcal{X}_{/X}$ is locally $(n-1)$-coherent. It will therefore suffice to show that if $X_0, X_1 \in \mathcal{X}_{/X}$ are $(n-1)$-coherent, then the fiber product $X_0 \times_X X_1$ are $(n-1)$-coherent. Unwinding the definitions, we have a pullback diagram

$$\begin{array}{ccc} X_0 \times_X X_1 & \xrightarrow{\delta'} & X_0 \times_Y X_1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\delta} & X \times_Y X. \end{array}$$

Our assumption that f is $(n+1)$-connective guarantees that δ is n-connective, so that δ' is also n-connective. The n-coherence of Y guarantees that $X_0 \times_Y X_1$ is $(n-1)$-coherent. Applying (b_{n-1}), we deduce that $X_0 \times_X X_1$ is also $(n-1)$-coherent, as desired. □

Corollary A.2.4.2. *Let $\mathcal{X}$ be an ∞-topos containing a morphism $f : X \to Z$. Let $n \geqslant 0$ be an integer, so that f factors as a composition $X \xrightarrow{f'} Y \xrightarrow{f''} Z$ where f' is n-connective and f'' is $(n-1)$-truncated. If X is n-coherent and Z is m-coherent for $m \geqslant n$, then Y is m-coherent.*

Proof. We proceed by induction on m. If $m = n$, the desired result follows from Proposition A.2.4.1 (together with the n-coherence of X and the n-connectivity of f'). The inductive step follows from Lemma A.2.3.4 (since f'' is $(n-1)$-truncated and therefore also $(m-2)$-truncated for $m > n$). □

Corollary A.2.4.3. *Let $\mathcal{X}$ be an ∞-topos containing an n-coherent object X. For each $m \geqslant n$, if the ∞-topos $\mathcal{X}$ is m-coherent, then the object $\tau_{\leqslant n-1}X$ is m-coherent.*

Proof. Apply Corollary A.2.4.2 in the case where Z is a final object of $\mathcal{X}$. □

Corollary A.2.4.4. *Let $\mathcal{X}$ be a coherent ∞-topos. If $X \in \mathcal{X}$ is an n-coherent object for $n \geqslant 0$, then the truncation $\tau_{\leqslant n-1}X$ is a coherent object of $\mathcal{X}$.*

A.3 ∞-Topoi of Sheaves

In §A.2, we introduced the notion of a *coherent ∞-topos* (Definition A.2.1.6). In this section, we will describe a large class of coherent ∞-topoi. More precisely, we will show that if $\mathcal{C}$ is a small ∞-category which admits finite limits and its equipped with a *finitary* Grothendieck topology (Definition A.3.1.1), then the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is coherent (Proposition A.3.1.3). It would be too optimistic to expect the converse to hold in the ∞-categorical setting: just as not every ∞-topos $\mathcal{X}$ can be obtained as an ∞-category of sheaves on a Grothendieck site, not every coherent ∞-topos $\mathcal{X}$ can be obtained as an ∞-category of sheaves on a finitary Grothendieck site. However, we will show that something slightly weaker is true: every ∞-topos $\mathcal{X}$ which is coherent, locally coherent, and hypercomplete can be realized as the ∞-category of hypercomplete sheaves on a finitary Grothendieck site (Theorem A.3.4.1). In fact, there is even a canonical choice for the site $\mathcal{C}$ (which we will study in §A.6).

A.3.1 Finitary Grothendieck Topologies

Let $\mathcal{C}$ be an ∞-category. Recall that a *sieve* on an object $C \in \mathcal{C}$ is a full subcategory $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ with the following property: for any commutative diagram

$$\begin{array}{ccc} C_0 & \longrightarrow & C_1 \\ & {\scriptstyle \phi_0}\searrow \quad \swarrow{\scriptstyle \phi_1} & \\ & C & \end{array}$$

in $\mathcal{C}$, if ϕ_1 belongs to $\mathcal{C}^{(0)}_{/C}$, then ϕ_0 also belongs to $\mathcal{C}^{(0)}_{/C}$. A *Grothendieck topology* on $\mathcal{C}$ consists of the specification, for each object $C \in \mathcal{C}$, of a special class of sieves on C, called *covering sieves*, which satisfy the following axioms:

(a) For every object $C \in \mathcal{C}$, the category $\mathcal{C}_{/C}$ is a covering sieve on C.

(b) Let $\phi : C \to D$ be a morphism in $\mathcal{C}$, $\mathcal{C}^{(0)}_{/D}$ a sieve on D, and $\phi^* \mathcal{C}^{(0)}_{/D}$ the full subcategory of $\mathcal{C}^{(0)}_{/C}$ spanned by those maps $\psi : C_0 \to C$ for which the composite map $\phi \circ \psi$ belongs to $\mathcal{C}^{(0)}_{/D}$. If $\mathcal{C}^{(0)}_{/D}$ is a covering sieve on D, then $\mathcal{C}^{(0)}_{/C}$ is a covering sieve on C.

(c) Let $D \in \mathcal{C}$ be an object, and suppose we are given sieves $\mathcal{C}^{(0)}_{/D}, \mathcal{C}^{(1)}_{/D} \subseteq \mathcal{C}_{/D}$. Suppose that $\mathcal{C}^{(0)}_{/D}$ is a covering sieve, and that for each morphism $\phi : C \to D$ belonging to $\mathcal{C}^{(0)}_{/D}$, the sieve $\phi^* \mathcal{C}^{(1)}_{/D}$ is a covering sieve on C. Then $\mathcal{C}^{(1)}_{/D}$ is a covering sieve on D.

Definition A.3.1.1. Let $\mathcal{C}$ be an ∞-category which admits pullbacks. We will say that a Grothendieck topology on $\mathcal{C}$ is *finitary* if it satisfies the following condition:

(∗) For every object $C \in \mathcal{C}$ and every covering sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$, there exists a finite collection of morphisms $\{C_i \to C\}_{1 \leqslant i \leqslant n}$ in $\mathcal{C}^{(0)}_{/C}$ which generate a covering of C (in other words, the smallest sieve $\mathcal{C}^{(1)}_{/C}$ containing each C_i is also a covering sieve on C).

Remark A.3.1.2. Let $\mathcal{C}$ be an ∞-category which admits pullbacks, and suppose that $\mathcal{C}$ is equipped with an arbitrary Grothendieck topology. Let $\mathcal{D}$ denote the same ∞-category $\mathcal{C}$, and let us say that a sieve $\mathcal{D}^{(0)}_{/D} \subseteq \mathcal{D}_{/D}$ is *covering* if it contains a finite collection of morphisms $\{D_i \to D\}$ which generate a covering sieve in $\mathcal{C}$. This collection of covering sieves determines a Grothendieck topology on $\mathcal{D}$. This Grothendieck topology is the finest finitary topology on $\mathcal{D}$ which is coarser than the original topology on $\mathcal{C}$.

Proposition A.3.1.3. *Let $\mathcal{C}$ be a small ∞-category which admits pullbacks which is equipped with a finitary Grothendieck topology. Then:*

(1) *If $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ denotes the composition of the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ with the sheafification function $\mathcal{P}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\mathcal{C})$, the functor j carries each object $C \in \mathcal{C}$ to a coherent object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$.*

(2) *The ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is locally coherent.*

(3) *If $\mathcal{C}$ has a final object, then $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is coherent.*

Proof. Since $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is generated by $j(\mathcal{C})$ under small colimits, assertion (2) follows immedaitely from (1). Since j preserves finite limits, it carries final objects of $\mathcal{C}$ to final objects of $\mathcal{S}\mathrm{hv}(\mathcal{C})$, so assertion (3) also follows from (1). We will prove the following assertions by induction on n:

(1′) The functor j carries each object $C \in \mathcal{C}$ to an n-coherent object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$.

(2′) The ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is locally n-coherent.

It is clear that (1′) implies (2′). To prove (1′), let us first assume that $n = 0$. We must show that for $C \in \mathcal{C}$, the object $j(C) \in \mathcal{S}\mathrm{hv}(\mathcal{C})$ is quasi-compact. Choose an effective epimorphism $\amalg_{i \in I} U_i \to j(C)$ in $\mathcal{S}\mathrm{hv}(\mathcal{C})$. It follows that there exists a covering $\{C_\alpha \to C\}$ in $\mathcal{C}$ such that each of the induced maps $j(C_\alpha) \to j(C)$ factors through U_i for some i. Since the topology on $\mathcal{C}$ is finitary, we may assume that this covering is finite; then we may assume that all of this indices $i \in I$ which are used belong to some finite subset $I_0 \subseteq I$, so that $\amalg_{i \in I_0} U_i \to j(C)$ is an effective epimorphism, as desired.

Now suppose that $n > 0$. Using the inductive hypothesis, we may assume that $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is locally $(n-1)$-coherent and that $j(C)$ is $(n-1)$-coherent for $C \in \mathcal{C}$. We wish to show that $j(C)$ is n-coherent. Without loss of generality, we may replace $\mathcal{C}$ by $\mathcal{C}_{/C}$ and $\mathcal{S}\mathrm{hv}(\mathcal{C})$ by $\mathcal{S}\mathrm{hv}(\mathcal{C})_{/j(C)} \simeq \mathcal{S}\mathrm{hv}(\mathcal{C}_{/C})$. We wish to show that the collection of $(n-1)$-coherent objects of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is closed under finite products. Using Corollary A.2.1.4, we are reduced to showing that $j(C') \times j(C'')$ is $(n-1)$-coherent, for every pair of objects $C', C'' \in \mathcal{C}$. This is clear, since $j(C') \times j(C'') = j(C' \times C'')$. □

A.3.2 Examples of Grothendieck Topologies

The theory of Grothendieck topologies is quite flexible and can be applied in a great deal of generality. However, for many applications, it is convenient to work not with covering *sieves*, but instead with covering *morphisms*: that is, morphisms $\phi : C \to D$ which generate a covering sieve on D. We now describe a general procedure for constructing Grothendieck topologies which are "generated" by their covering morphisms. Most of the Grothendieck topologies of interest to us in this book can be constructed using this procedure.

Proposition A.3.2.1. *Let $\mathcal{C}$ be an ∞-category and let S be a collection of morphisms in $\mathcal{C}$. Assume that:*

(a) The collection of morphisms S is contains all equivalences and is stable under composition (in particular, if $f, g : C \to D$ are homotopic morphisms in $\mathcal{C}$, then $f \in S$ if and only if $g \in S$).

(b) The ∞-category $\mathcal{C}$ admits pullbacks. Moreover, the class of morphisms S is stable under pullback: for every pullback diagram

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ D' & \longrightarrow & D \end{array}$$

such that $f \in S$, the morphism f' also belongs to S.

(c) The ∞-category $\mathcal{C}$ admits finite coproducts. Moreover, the collection of morphisms S is stable under finite coproducts: if $f_i : C_i \to D_i$ is a finite collection of morphisms in $\mathcal{C}$ which belong to S, then the induced map $\coprod_i C_i \to \coprod_i D_i$ also belongs to S.

(d) Finite coproducts in $\mathcal{C}$ are universal. That is, given a diagram $\coprod_{1 \leqslant i \leqslant n} C_i \to D \leftarrow D'$, the canonical map $\coprod_{1 \leqslant i \leqslant n}(C_i \times_D D') \to (\coprod_{1 \leqslant i \leqslant n} C_i) \times_D D'$ is an equivalence in $\mathcal{C}$.

Then there exists a Grothendieck topology on $\mathcal{C}$ which can be described as follows: a sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ on an object $C \in \mathcal{C}$ is covering if and only if it contains a finite collection of morphisms $\{C_i \to C\}_{1 \leqslant i \leqslant n}$ such that the induced map $\coprod C_i \to C$ belongs to S.

Remark A.3.2.2. The Grothendieck topologies described in Proposition A.3.2.1 are finitary.

Proof of Proposition A.3.2.1. We show that the collection of covering sieves satisfies the conditions of Definition HTT.6.2.2.1 :

(1) For every object $C \in \mathcal{C}$, the sieve $\mathcal{C}_{/C}$ covers C. This is clear, since $\mathcal{C}_{/C}$ contains the identity map $\mathrm{id}_C : C \to C$, which belongs to S by (a).

(2) If $\mathcal{C}^{(0)}_{/C}$ is a covering sieve on an object $C \in \mathcal{C}$ and $f : C' \to C$ is a morphism in $\mathcal{C}$, then the pullback sieve $f^* \mathcal{C}^{(0)}_{/C}$ covers C'. To prove this, we observe that there exists a finite collection of morphisms $C_i \to C$ belonging to $\mathcal{C}^{(0)}_{/C}$ such that the induced map $\coprod_i C_i \to C$ belongs to S. Assumption (b) guarantees that the induced map $(\coprod_i C_i) \times_C C' \to C'$ also belongs to S, and assumption (d) gives an identification $(\coprod_i C_i) \times_C C' \simeq \coprod_i (C_i \times_C C')$. It now suffices to observe that each of the morphisms $C_i \times_C C' \to C'$ belongs to the sieve $f^* \mathcal{C}^{(0)}_{/C}$.

(3) Let $\mathcal{C}^{(0)}_{/C}$ be a covering sieve on an object $C \in \mathcal{C}$, and let $\mathcal{C}^{(1)}_{/C}$ be an arbitrary sieve on C. Suppose that, for each morphism $f : C' \to C$ belonging to $\mathcal{C}^{(0)}_{/C}$, the pullback sieve $f^* \mathcal{C}^{(1)}_{/C}$ covers C'. We must show that $\mathcal{C}^{(1)}_{/C}$ covers C. Since $\mathcal{C}^{(0)}_{/C}$ is a covering sieve, there exists a finite collection of morphisms $f_i : C_i \to C$ belonging to $\mathcal{C}^{(0)}_{/C}$ such that the induced map $\coprod_i C_i \to C$ belongs to S. Each $f_i^* \mathcal{C}^{(1)}_{/C}$ is a covering sieve on C_i, so there exists a finite collection of morphisms $C_{i,j} \to C_i$ belonging to $f^* \mathcal{C}^{(1)}_{/C}$ such that the induced map $\coprod_j C_{i,j} \to C_i$ belongs to S. It follows that each of the composite maps $C_{i,j} \to C_i \to C$ belongs to the sieve $\mathcal{C}^{(1)}_{/C}$. To prove that $\mathcal{C}^{(1)}_{/C}$ is covering, it suffices to show that the map $g : \coprod_{i,j} C_{i,j} \to C$ belongs to S. To prove this, we factor g as a composition

$$\coprod_{i,j} C_{i,j} \xrightarrow{g'} \coprod_i C_i \xrightarrow{g''} C.$$

The map g'' belongs to S by assumption, and the map g' is a finite coproduct of maps belonging to S and therefore belongs to S by virtue of (c). It follows from (a) that $g \simeq g'' \circ g'$ belongs to S, as required.

□

A.3.3 Čech Descent

Recall that if $\mathcal{C}$ is an ∞-category equipped with a Grothendieck topology and $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is a functor, we say that $\mathscr{F}$ is a *sheaf* if, for every covering sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ on an object $C \in \mathcal{C}$, the functor $\mathscr{F}$ exhibits $\mathscr{F}(C)$ as a limit of the diagram $\mathscr{F}|_{(\mathcal{C}^{(0)}_{/C})^{\mathrm{op}}}$. For Grothendieck topologies which arise from the construction of Proposition A.3.2.1, we can formulate this condition in a more concrete way:

Proposition A.3.3.1. *Let $\mathcal{C}$ be an ∞-category and S a collection of morphisms in $\mathcal{C}$. Assume that $\mathcal{C}$ and S satisfy the hypotheses of Proposition A.3.2.1, together with the following additional hypothesis:*

(e) *Coproducts in the ∞-category $\mathcal{C}$ are disjoint. That is, if C and C' are objects of $\mathcal{C}$, then the fiber product $C \times_{C \amalg C'} C'$ is an initial object of $\mathcal{C}$ (see §HTT.6.1.1).*

Let $\mathcal{D}$ be an arbitrary ∞-category and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be a functor. Then $\mathscr{F}$ is a $\mathcal{D}$-valued sheaf on $\mathcal{C}$ if and only if the following conditions are satisfied:

(1) *The functor $\mathscr{F}$ preserves finite products.*

(2) *Let $f : U_0 \to X$ be a morphism in $\mathcal{C}$ which belongs to S and let $U_\bullet$ be a Čech nerve of f (see §HTT.6.1.2), regarded as an augmented simplicial object of $\mathcal{C}$. Then the composite map $\mathbf{\Delta}_+ \xrightarrow{U_\bullet} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathcal{D}$ is a limit diagram. In other words, $\mathscr{F}$ exhibits $\mathscr{F}(X)$ as a totalization of the cosimplicial object $[n] \mapsto \mathscr{F}(U_n)$.*

Proof. For every object $D \in \mathcal{D}$, let $h^D : \mathcal{D} \to \mathcal{S}$ be the functor corepresented by D. Using Proposition HTT.5.1.3.2 , we deduce that $\mathscr{F}$ is a sheaf $\mathcal{D}$-valued sheaf on $\mathcal{C}$ if and only if each composite map $h^D \circ \mathscr{F}$ is a $\mathcal{S}$-valued sheaf on $\mathcal{C}$, and that $\mathscr{F}$ satisfies conditions (1) and (2) if and only if each $h^D \circ \mathscr{F}$ satisfies the same condition. We may therefore replace $\mathscr{F}$ by $h^D \circ \mathscr{F}$ and thereby reduce to the case where $\mathcal{D} = \mathcal{S}$.

Suppose first that $\mathscr{F}$ is a sheaf; we will prove that $\mathscr{F}$ satisfies conditions (1) and (2). We begin with (1). Let $\{C_i\}_{1 \leqslant i \leqslant n}$ be a finite collection of objects in $\mathcal{C}$ and let $C = \coprod_i C_i$ be their coproduct. We wish to prove that the canonical map $\mathscr{F}(C) \to \prod_i \mathscr{F}(C_i)$ is an equivalence. The proof proceeds by induction on n. If $n = 0$, then C is an initial object of $\mathcal{C}$ so that the empty sieve is a covering of C. Since $\mathscr{F}$ is a sheaf, we deduce that $\mathscr{F}(C)$ is a final object of $\mathcal{S}$, as required. If $n = 1$, there is nothing to prove. If $n > 2$, we let $D = \coprod_{1 \leqslant i < n} C_i$, so that $C = D \amalg C_n$. The natural map $\mathscr{F}(C) \to \prod_i \mathscr{F}(C_i)$ then factors as a composition of maps

$$\mathscr{F}(C) \to \mathscr{F}(D) \times \mathscr{F}(C_n) \to (\prod_{1 \leqslant i < n} \mathscr{F}(C_i)) \times \mathscr{F}(C_n) \simeq \prod_{1 \leqslant i \leqslant n} \mathscr{F}(C_n),$$

each of which is an equivalence by the inductive hypothesis. It remains to treat the case $n = 2$. Let $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ be the sieve generated by C_1 and C_2. This sieve is evidently a covering of C, so that $\mathscr{F}(C) \simeq \varprojlim \mathscr{F}|_{(\mathcal{C}^{(0)}_{/C})^{\mathrm{op}}}$. To complete the proof, it suffices to show that the canonical map $\varprojlim \mathscr{F}|_{(\mathcal{C}^{(0)}_{/C})^{\mathrm{op}}} \to \mathscr{F}(C_1) \times \mathscr{F}(C_2)$ is an equivalence. Let $p : \Lambda^2_0 \to \mathcal{C}^{(0)}_{/C}$ be the map corresponding to the pullback diagram

$$\begin{array}{ccc} C_1 \times_C C_2 & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ C_2 & \longrightarrow & C \end{array}$$

in $\mathcal{C}$. Since $C_1 \times_C C_2$ is initial in $\mathcal{C}$, the above argument shows that $\mathscr{F}(C_1 \times_C C_2)$ is final in $\mathcal{D}$: that is, $\mathscr{F}|_{(\Lambda^2_0)^{\mathrm{op}}}$ is a right Kan extension of $\mathscr{F}|_{\{1,2\}^{\mathrm{op}}}$, so that $\varprojlim \mathscr{F}|_{(\Lambda^2_0)^{\mathrm{op}}} \simeq \mathscr{F}(C_1) \times \mathscr{F}(C_2)$ by Lemma HTT.4.3.2.7 . To complete the proof of (1), we will show that p is left cofinal. According to Theorem HTT.4.1.3.1 , it will suffice to show that for every object $(f : D \to C) \in \mathcal{C}^{(0)}_{/C}$, the ∞-category $S = \Lambda^2_0 \times_{\mathcal{C}^{(0)}_{/C}} (\mathcal{C}^{(0)}_{/C})_{f/}$ is weakly contractible. If $D \in \mathcal{C}$ is initial, then the projection map $S \to \Lambda^2_0$ is a trivial Kan fibration and the result is obvious. If D is not initial, then condition (d) guarantees that there do not exist any maps from D to an initial object of $\mathcal{C}$. Using (e), we deduce that there do not exist any maps from D into $C_1 \times_C C_2$. It follows that f factors through either the map $C_1 \to C$ or $C_2 \to C$, but not both. Without loss of generality, we may assume that f factors through $C_1 \to C$. In this case, we can identify S with the simplicial set $\{C_1\} \times_{\mathcal{C}_{/C}} \mathcal{C}_{D//C}$, which is the homotopy fiber of the composition map $q : \mathrm{Map}_{\mathcal{C}}(D, C_1) \to \mathrm{Map}_{\mathcal{C}}(D, C)$ over f. We wish to show that this homotopy fiber is contractible. By assumption, it is nonempty; it will therefore suffice to show that the morphism q is (-1)-truncated. To prove this, we need only verify that $C_1 \to C$ is a monomorphism; that is, that the diagonal map $C_1 \to C_1 \times_C C_1$ is an equivalence. Using (d), we obtain equivalences

$$C_1 \simeq C_1 \times_C C \simeq C_1 \times_C (C_1 \amalg C_2) \simeq (C_1 \times_C C_1) \amalg (C_1 \times_C C_2),$$

and the first summand maps by an equivalence to $C_1 \times_C C_1$. The second summand is trivial, by virtue of (e).

We now prove (2). Let $f : U_0 \to X$ be a morphism of S and let f be its Čech nerve, so that f generates a covering sieve $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$. We can regard $U_\bullet$ as determining a simplicial object $V : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}^{(0)}_{/X}$. Our assumption that $\mathscr{F}$ is a sheaf guarantees that $\mathscr{F}(X) \simeq \varprojlim \mathscr{F}|_{(\mathcal{C}^{(0)}_{/X})^{\mathrm{op}}}$. To prove (2), it suffices to prove that the map V is left cofinal. According to Theorem HTT.4.1.3.1 , it suffices to show that for every map $f : X' \to X$ belonging to $\mathcal{C}^{(0)}_{/X}$, the ∞-category $\mathcal{X} = \mathbf{\Delta}^{\mathrm{op}} \times_{\mathcal{C}^{(0)}_{/X}} (\mathcal{C}^{(0)}_{/X})_{f/}$ is weakly contractible. The projection map $\mathcal{X} \to \mathbf{\Delta}^{\mathrm{op}}$ is a left fibration, classified by a functor $\chi : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}$. According to Proposition HTT.3.3.4.5 , it will suffice to show that $\varinjlim(\chi)$ is contractible. Note that χ can be identified with the underlying simplicial object of the Čech nerve of the map of spaces $q : \mathrm{Map}_{\mathcal{C}_{/X}}(X', U_0) \to \Delta^0$. Since f belongs to the sieve $\mathcal{C}^{(0)}_{/X}$, the space $\mathrm{Map}_{\mathcal{C}_{/X}}(X', U_0)$ is nonempty so that q is an effective epimorphism. Since $\mathcal{S}$ is an ∞-topos, we conclude that $\varinjlim(\chi) \simeq \Delta^0$ as required.

Now suppose that $\mathscr{F}$ satisfies (1) and (2); we will show that $\mathscr{F}$ is a sheaf on $\mathcal{C}$. Choose an object $X \in \mathcal{C}$ and a covering sieve $\mathcal{C}^{(0)}_{/X}$; we wish to prove that $\mathscr{F}(X) \simeq \varprojlim \mathscr{F}|_{(\mathcal{C}^{(0)}_{/X})^{\mathrm{op}}}$. We first treat the case where $\mathcal{C}^{(0)}_{/X}$ is generated by a single morphism $f : U_0 \to X$ which belongs to S. Let $U_\bullet$ be a Čech nerve of f, so that $\mathscr{F}(X)$ can be identified with the totalization of the cosimplicial space $[n] \mapsto \mathscr{F}(U_n)$ by virtue of (2). To complete the proof, we invoke the fact (established above) that $U_\bullet$ determines a left cofinal map $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}^{(0)}_{/X}$.

Now suppose that $\mathcal{C}^{(0)}_{/X}$ is generated by a finite collection of morphisms $\{C_i \to X\}_{1 \leqslant i \leqslant n}$ such that the induced map $\coprod C_i \to X$ belongs to S. Let $C = \coprod_i C_i$ and let $\mathcal{C}^{(1)}_{/X}$ denote the sieve generated by the induced map $C \to X$. Then $\mathcal{C}^{(1)}_{/X}$ contains $\mathcal{C}^{(0)}_{/X}$ and is therefore a covering sieve; the above argument shows that $\mathscr{F}(X) \simeq \varprojlim \mathscr{F}|_{(\mathcal{C}^{(1)}_{/X})^{\mathrm{op}}}$. To complete the proof in this case, it will suffice to show that $\mathscr{F}|_{(\mathcal{C}^{(1)}_{/X})^{\mathrm{op}}}$ is a right Kan extension of $\mathscr{F}|_{(\mathcal{C}^{(0)}_{/X})^{\mathrm{op}}}$.

Fix an object $f : U \to X$ of the sieve $\mathcal{C}^{(1)}_{/X}$, and let $\mathcal{E}$ denote the full subcategory of $(\mathcal{C}^{(1)}_{/X})_{/U} \simeq \mathcal{C}_{/U}$ spanned by those objects whose image in $\mathcal{C}_{/X}$ belongs to $\mathcal{C}^{(0)}_{/X}$. We wish to prove that the canonical map $\mathscr{F}(U) \to \varprojlim \mathscr{F}|_{\mathcal{E}^{\mathrm{op}}}$ is an equivalence. By construction, the map f factors through some map $f_0 : U \to C$. Invoking (b), we have $U \simeq U \times_C C \simeq \coprod_i U \times_C C_i$, so that U can be obtained as a coproduct of objects U_i belonging to $\mathcal{C}^{(0)}_{/X}$. Let $T \subseteq \{1, \ldots, n\}$ denote the collection of indices for which U_i is not initial. We let $\mathcal{E}' \subseteq \mathcal{E}$ denote the full subcategory spanned by morphisms $U' \to U$ which factor through some U_i and such that $U' \in \mathcal{C}$ is not initial. For $i \neq j$, the fiber product $U_i \times_U U_j$ is initial (by (e)) and therefore receives no morphisms from non-initial objects of $\mathcal{C}$ (by (d)); it follows that $\mathcal{E}'$ can be decomposed as a disjoint union $\coprod_{i \in T} \mathcal{E}'_i$ where each $\mathcal{E}'_i$ denotes the full subcategory of $\mathcal{E}'$ spanned by those morphisms $U' \to U$ which factor through U_i. Since the map $U_i \to U$ is a monomorphism, each $\mathcal{E}'_i$ contains the map $U_i \to U$ as a final object, so that the inclusion $\{U_i\}_{i \in T} \to \mathcal{E}'$ is left cofinal. Condition (1) implies that $\mathscr{F}(U) \simeq \prod_{i \in T} \mathscr{F}(U_i)$, so that $\mathscr{F}(U)$ is a limit of the diagram $\mathscr{F}|_{(\mathcal{E}')^{\mathrm{op}}}$. We will prove that $\mathscr{F}|_{\mathcal{E}^{\mathrm{op}}}$ is a right Kan extension of $\mathscr{F}|_{\mathcal{E}'^{\mathrm{op}}}$, so that $\varprojlim \mathscr{F}|_{\mathcal{E}^{\mathrm{op}}} \simeq \varprojlim \mathscr{F}|_{\mathcal{E}'^{\mathrm{op}}} \simeq \mathscr{F}(U)$ by Lemma HTT.4.3.2.7. To see this, choose an object $U' \to U$ in $\mathcal{E}$; we wish to show that $\mathscr{F}(U')$ is a limit of the diagram $\mathscr{F}|_{(\mathcal{E}'_{/U'})^{\mathrm{op}}}$. Let $U'_i = U' \times_U U_i$, and let T' be the collection of indices i for which U'_i is not initial. Then $\mathcal{E}'_{/U'}$ decomposes as a disjoint union $\coprod_{i \in T'} (\mathcal{E}'_{/U'})_i$, each of which has a final object (given by the map $U'_i \to U'$). It follows that $\varprojlim \mathscr{F}|_{(\mathcal{E}'_{/U'})^{\mathrm{op}}}$ is equivalent to $\prod_{i \in T'} \mathscr{F}(U'_i)$, which is equivalent to $\mathscr{F}(U')$ by virtue of (1).

We now treat the case of a general covering sieve $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$. By definition, there exists a finite collection of morphisms $f_i : C_i \to X$ belonging to $\mathcal{C}^{(0)}_{/X}$ such that the induced map $\coprod_i C_i \to X$ belongs to S. Let $\mathcal{C}^{(1)}_{/X} \subseteq \mathcal{C}^{(0)}_{/X}$ be the sieve generated by the maps f_i. The above argument shows that $\mathscr{F}(X) \simeq \varprojlim \mathscr{F}|_{(\mathcal{C}^{(1)}_{/X})^{\mathrm{op}}}$. To prove that $\mathscr{F}(X) \simeq \varprojlim(\mathcal{C}^{(0)})_{/X})^{\mathrm{op}}$, it will suffice to show that $\mathscr{F}|_{(\mathcal{C}^{(0)}_{/X})^{\mathrm{op}}}$ is a right Kan extension of $\mathscr{F}|_{(\mathcal{C}^{(1)}_{/X})^{\mathrm{op}}}$ (Lemma HTT.4.3.2.7). Unwinding the definitions, we must show that for every $f : U \to X$ belonging to the sieve $\mathcal{C}^{(0)}_{/X}$, we have $\mathscr{F}(U) \simeq \varprojlim \mathscr{F}|_{f^* \mathcal{C}^{(1)}_{/X}}$. This is clear, since $f^* \mathcal{C}^{(1)}_{/X}$ is generated by the pullback maps $C_i \times_X U \to U$, and the induced map $\coprod_i (C_i \times_X U) \to U$ factors as a composition

$$\coprod_i (C_i \times_X U) \xrightarrow{\alpha} (\coprod_i C_i) \times_X U \xrightarrow{\beta} U,$$

where α is an equivalence by assumption (d) and the map β belongs to S by assumption (b). □

Corollary A.3.3.2. *Let $\mathcal{C}$ be an ∞-category and let S be a collection of morphisms of $\mathcal{C}$ satisfying the hypotheses of Proposition A.3.3.1. Suppose that we are given functors $\chi, \chi' : \mathcal{C}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ and a natural transformation $\rho : \chi \to \chi'$ satisfying the following conditions:*

(1) *For each object $C \in \mathcal{C}$, the ∞-category $\chi(C)$ admits finite products and totalizations of cosimplicial objects.*

(2) *For each object $C \in \mathcal{C}$, the functor $\rho(C) : \chi(C) \to \chi'(C)$ is conservative and preserves finite products and totalizations of cosimplicial objects.*

(3) *For every morphism $C \to D$ in $\mathcal{C}$, the diagram*

$$\begin{array}{ccc} \chi(C) & \longrightarrow & \chi(D) \\ \downarrow & & \downarrow \\ \chi'(C) & \longrightarrow & \chi'(D) \end{array}$$

is right adjointable.

If χ' is a $\mathcal{C}\mathrm{at}_\infty$-valued sheaf on $\mathcal{C}$, then χ is also a $\mathcal{C}\mathrm{at}_\infty$-valued sheaf on $\mathcal{C}$.

Proof. Combine Proposition A.3.3.1 with Corollary HA.5.2.2.37 . □

A.3.4 Classification of Hypercomplete Locally Coherent ∞-Topoi

We now apply the preceding results to give an "extrinsic" characterization of the class of hypercomplete locally coherent ∞-topoi.

Theorem A.3.4.1. *Let $\mathcal{X}$ be an ∞-topos. The following conditions are equivalent:*

(1) *The ∞-topos $\mathcal{X}$ is locally coherent and hypercomplete.*

(2) *There exists a small ∞-category $\mathcal{C}$ which admits fiber products, a finitary Grothendieck topology on $\mathcal{C}$, and an equivalence $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$.*

Moreover, if these conditions are satisfied, then we may assume that $\mathcal{C}$ admits finite coproducts and that the topology on $\mathcal{C}$ is subcanonical. If $\mathcal{X}$ is coherent, we may assume that $\mathcal{C}$ admits finite limits.

We will deduce Theorem A.3.4.1 from the following more precise result:

Proposition A.3.4.2. *Let $\mathcal{X}$ be an ∞-topos which is hypercomplete, and let $\mathcal{C} \subseteq \mathcal{X}$ be a full subcategory which satisfies the following conditions:*

(a) *The ∞-category $\mathcal{C}$ is essentially small.*

(b) *Each object of $\mathcal{C}$ is a coherent object of $\mathcal{X}$.*

(c) *The ∞-category $\mathcal{C}$ is closed under finite coproducts and fiber products.*

(d) *For every object $X \in \mathcal{X}$, there exists an effective epimorphism $\amalg U_\alpha \to X$ where each U_α belongs to $\mathcal{C}$.*

Then the ∞-category $\mathcal{C}$ admits a finitary Grothendieck topology which can be characterized as follows: a collection of morphisms $\{f_\alpha : U_\alpha \to X\}$ generates a covering sieve on $X \in \mathcal{C}$ if and only if the induced map $\amalg U_\alpha \to X$ is an effective epimorphism in $\mathcal{X}$. For each $X \in \mathcal{X}$, let $h_X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ denote the functor given by $h_X(Y) = \mathrm{Map}_{\mathcal{X}}(Y, X)$. Then the construction $X \mapsto h_X$ induces an equivalence of ∞-categories $\mathcal{X} \to \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$.

Proof of Theorem A.3.4.1. The implication (2) ⇒ (1) of Theorem A.3.4.1 follows immediately from Propositions A.3.1.3 and A.2.2.2. To prove the converse, suppose that $\mathcal{X}$ is locally coherent. Choose a small collection of objects $\{X_\alpha\}$ which generates $\mathcal{X}$ under small colimits. Since $\mathcal{X}$ is locally coherent, for each index α we can choose an effective epimorphism $\amalg_\beta U_{\alpha,\beta} \to X_\alpha$ where $U_{\alpha,\beta}$ is coherent. Let $\mathcal{C}$ denote an essentially small full subcategory of $\mathcal{X}$ such that each object of $\mathcal{C}$ is coherent in $\mathcal{X}$, and each $U_{\alpha,\beta}$ belongs to $\mathcal{C}$. Enlarging this collection if necessary, we may assume that it is closed under pullbacks, finite coproducts, and that it contains the a final object of $\mathcal{X}$ if $\mathcal{X}$ is coherent (see Remark **??**) Then Proposition A.3.4.2 supplies a finitary Grothendieck topology on $\mathcal{C}$ and an equivalence $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$. □

Remark A.3.4.3. In the situation of Theorem A.3.4.1, we can take $\mathcal{C}$ to be the full subcategory of $\mathcal{X}$ spanned by the coherent object of $\mathcal{X}$: see Remark A.6.6.3.

The proof of Proposition A.3.4.2 depends on the following:

Lemma A.3.4.4. *Let $\mathcal{X}$ be an ∞-topos containing a collection of objects $\{X_i\}_{i \in I}$. For every subset $J \subseteq I$, let $X_J \simeq \amalg_{i \in J} X_i$. If $C \in \mathcal{X}$ is quasi-compact, then the canonical map*

$$\varinjlim_{J \subseteq I} \operatorname{Map}_{\mathcal{X}}(C, X_J) \to \operatorname{Map}_{\mathcal{X}}(C, X_I)$$

is a homotopy equivalence, where the colimit is taken over all finite subsets $J \subseteq I$.

Proof. Let J be any subset of I and let $\phi : C \to X_J$ be a morphism in $\mathcal{X}$. Since colimits in $\mathcal{X}$ are universal, this morphism determines a decomposition $C \simeq \amalg_{i \in J} C_i$, where $C_i = C \times_{X_J} X_i$. We define the *support* of ϕ to be the subset of J consisting of those indices $i \in J$ such that C_i is not an initial object of $\mathcal{X}$.

Let $\phi : C \to X_J$ be any morphism. Since C is quasi-compact, there is a finite subset $J_0 \subseteq J$ such that the map $\amalg_{i \in J_0} C_i \to C$ is an effective epimorphism. For $i' \in J$, we have an effective epimorphism $\amalg_{i \in J_0} C_i \times_C C_{i'} \to C_{i'}$. If $i' \notin J_0$, then the left hand side is an initial object of $\mathcal{X}$ (since coproducts in $\mathcal{X}$ are disjoint), so that $C_{i'}$ is likewise initial object of $\mathcal{X}$. It follows that the support of ϕ is contained in J_0, and is therefore finite.

For each $J \subseteq I$, we can decompose the mapping space $\operatorname{Map}_{\mathcal{X}}(C, X_J)$ as a coproduct $\amalg_S \operatorname{Map}^S_{\mathcal{X}}(C, X_J)$, where S ranges over finite subsets of I and $\operatorname{Map}^S_{\mathcal{X}}(C, X_J)$ is the summand of $\operatorname{Map}^S_{\mathcal{X}}(C, X_J)$ given by maps $\phi : C \to X_J$ with support S (by convention, this summand is empty unless $S \subseteq J$). It will therefore suffice to prove that for every finite set S, the map

$$\varinjlim_{J \subseteq I} \operatorname{Map}^S_{\mathcal{X}}(C, X_J) \to \operatorname{Map}^S_{\mathcal{X}}(C, X_I)$$

is a homotopy equivalence. To prove this, we observe that $\operatorname{Map}^S_{\mathcal{X}}(C, X_J) \simeq \operatorname{Map}^S_{\mathcal{X}}(C, X_I)$ whenever $S \subseteq J$. □

Proof of Proposition A.3.4.2. The existence of desired the Grothendieck topology on $\mathcal{C}$ follows by applying Proposition A.3.2.1 to the class of effective epimorphisms between morphisms in $\mathcal{C}$. It follows from Proposition 20.4.5.1 that the functor $h : \mathcal{X} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is fully faithful and from Remark 20.4.5.2 that h admits a left exact left adjoint $F : \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{X}$. Using Proposition A.3.3.1, we see that h factors through the full subcategory $\mathcal{S}\mathrm{hv}(\mathcal{C}) \subseteq \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, and therefore through the full subcategory $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ (since $\mathcal{X}$ is assumed to be hypercomplete). It follows that the functor F factors as a composition

$$\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{L} \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}} \xrightarrow{f^*} \mathcal{X}$$

where L is a left adjoint to the inclusion and f^* is a geometric morphism of ∞-topoi. Let $f_* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ denote the right adjoint to f^* (which coincides with the functor h). We wish to show that f_* and f^* are mutually inverse equivalence of ∞-categories.

We first show that the functor f_* is conservative: that is, if $u : X \to Y$ is a morphism in $\mathcal{X}$ such that $f_*(u)$ is an equivalence, then u is an equivalence. Since $\mathcal{X}$ is hypercomplete, it will suffice to show that u is n-connective for every integer n. We proceed by induction on n. When $n = 0$, we must show that u is an effective epimorphism. Choose an object $Z \in \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ and an effective epimorphism $v : f^*Z \to Y$. Then v is adjoint to a map $v' \in \mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}}(Z, f_*Y)$. Since $f_*(u)$ is an equivalence, the map v' factors through f_*X; it follows that v factors as a composition $f^*Z \to X \xrightarrow{u} Y$ so that u is also an effective epimorphism as desired If $n > 0$, then (since u is an effective epimorphism) we are reduced to proving that the induced map $\delta : X \to X \times_Y X$ is $(n-1)$-connective. This follows from our inductive hypothesis, since $f_*(\delta)$ is also an equivalence. This completes the proof that the functor f_* is conservative.

We next prove:

$(*)$ For each $n \geqslant 0$, the functor f_* carries n-connective morphisms in $\mathcal{X}$ to n-connective morphisms in $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$.

The proof proceeds by induction on n. We begin by treating the case $n = 0$. Fix an effective epimorphism $u : X \to Y$ in $\mathcal{X}$; we wish to show that $f_*(u)$ is an effective epimorphism in $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$. Unwinding the definitions, we must show that for every object $C \in \mathcal{C}$ and every morphism $\eta : C \to Y$, there exists a covering sieve on $\{C_i \to C\}$ such that each of the composite maps $C_i \to C \to Y$ factors through g. To prove this, it suffices to choose an effective epimorphism $\amalg C_i \to C \times_Y X$, where each $C_i \in \mathcal{C}$; our assumption that u is an effective epimorphism guarantees that the composite map $\amalg C_i \to C \times_Y X \to C$ is also an effective epimorphism, so that the maps $\{C_i \to C\}$ generate a covering sieve in $\mathcal{C}$.

Now suppose $n > 0$ and that $u : X \to Y$ is an n-connective morphism in $\mathcal{X}$; we wish to show that $f_*(u)$ is an n-connective morphism in $\mathcal{S}\mathrm{hv}(\mathcal{C})$. The above argument shows that $f_*(u)$ is an effective epimorphism; it will therefore suffice to show that the diagonal map $f_*X \to f_*X \times_{f_*Y} f_*X = f_*(X \times_Y X)$ is a $(n-1)$-connective. This follows from our inductive hypothesis.

To complete the proof that f_* is an equivalence, it will suffice to show that the unit map $u_{\mathscr{F}} : \mathscr{F} \to f_*f^*\mathscr{F}$ is an equivalence for each $\mathscr{F} \in \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$. We first prove that $u_{\mathscr{F}}$ is an equivalence in the special case where $\mathscr{F}$ can be written as a coproduct $\amalg_{i \in I} h_{C_i}$ for some objects $C_i \in \mathcal{C} \subseteq \mathcal{X}$. For every subset $J \subseteq I$, let $\mathscr{F}_J \in \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ denote the coproduct $\amalg_{i \in J} h_{C_i}$ and let u_J denote the unit map $\mathscr{F}_J \to f_*f^*\mathscr{F}_J$. We wish to show that u_I is an equivalence. We first show that u_J is an equivalence when $J \subseteq I$ is finite. Write $C = \amalg_{i \in J} C_i$, so we have equivalences

$$f_*f^*\mathscr{F}_J \simeq f_*(f^* \amalg_{i \in J} h_{C_i}) \simeq f_*(\amalg_{i \in J} f^* h_{C_i}) \simeq f_*(\amalg_{i \in J} C_i) \simeq h_C$$

(where the last equivalence is a consequence of the subcanonicality of the topology on $\mathcal{C}$). Consequently, we can identify u_J with the canonical map $\amalg_{i \in J} h_{C_i} \to h_C$. Note that the fiber product

$$\amalg_{i \in J} h_{C_i} \times_{h_C} \amalg_{i \in J} h_{C_i}$$

is given by $\amalg_{i,j \in J} h_{C_i \times_C C_j}$. For $i \neq j$, the fiber product $C_i \times_C C_j$ is an initial object $\emptyset \in \mathcal{C}$. The empty sieve is a covering of $C_i \times_C C_j$, so we have an effective epimorphism from the initial object to $h_{C_i \times_C C_j}$ in the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$, so that $h_{C_i \times_C C_j}$ is also initial in $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$. It follows that $\amalg_{i \in J} h_{C_i} \times_{h_C} \amalg_{i \in J} h_{C_i}$ is equivalent to $\amalg_{i \in J} h_{C_i \times_C C_i} \simeq \amalg_{i \in J} h_{C_i}$: that is, the map u_J becomes an equivalence after pullback along u_J. To complete the proof that u_J is an equivalence, it suffices to show that u_J is an effective epimorphism. This follows from the observation that the collection of maps $\{C_i \to C\}_{i \in J}$ generates a covering sieve.

To complete the proof that u_I is an equivalence, it will suffice to show that the canonical map $\varinjlim_{J \subseteq I} u_J \to u_I$ is an equivalence in $\mathrm{Fun}(\Delta^1, \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}})$; here the colimit is taken over all finite subsets $J \subseteq I$. It is easy to see that $\mathscr{F}_I \simeq \varinjlim_{J \subseteq I} \mathscr{F}_J$ in $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$. We will complete the proof by showing that $f_* f^* \mathscr{F}_I$ is a colimit of the diagram $\{f_* f^* \mathscr{F}_J\}_{J \subseteq I}$ in the ∞-category $\mathcal{P}(\mathcal{C})$ (and therefore also in the full subcategory $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}} \subseteq \mathcal{P}(\mathcal{C})$). In other words, we claim that for each object $C \in \mathcal{C}$, the canonical map $\varinjlim_{J \subseteq I} \mathrm{Map}_{\mathcal{X}}(C, \amalg_{i \in J} C_i) \to \mathrm{Map}_{\mathcal{X}}(C, \amalg_{i \in I} C_i)$ is a homotopy equivalence. This is a special case of Lemma A.3.4.4.

We now prove that the unit map $u_{\mathscr{F}} : \mathscr{F} \to f_* f^* \mathscr{F}$ is an equivalence for an arbitrary hypercomplete sheaf $\mathscr{F} \in \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$. Since $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ is hypercomplete, it will suffice to verify the following:

($*'$) For every object $\mathscr{F} \in \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ and every $n \geqslant 0$, the unit map $u_X : \mathscr{F} \to f_* f^* \mathscr{F}$ is n-connective.

The proof of ($*'$) proceeds by induction on n. We begin with the casen $n = 0$. Fix $\mathscr{F} \in \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$; we wish to show that the unit map $u_{\mathscr{F}} : \mathscr{F} \to f_* f^* \mathscr{F}$ is an effective epimorphism. Since $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ is generated under colimits by the essential image of the Yoneda embedding $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$, we can choose an effective epimorphism $v : \mathscr{F}' \to \mathscr{F}$ in $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$, where $\mathscr{F}'$ is a coproduct of objects belonging to the essential image of j. We have a commutative diagram

$$\begin{array}{ccc} \mathscr{F}' & \longrightarrow & f_* f^* \mathscr{F}' \\ \downarrow & & \downarrow \\ \mathscr{F} & \longrightarrow & f_* f^* \mathscr{F} . \end{array}$$

It will therefore suffice to show that the composite map $\mathscr{F}' \to f_* f^* \mathscr{F}' \to f_* f^* \mathscr{F}$ is an effective epimorphism. We established above that the first map is an equivalence, and are therefore reduced to showing that the map $f_* f^*(v)$ is an effective epimorphism. This follows from ($*$) (since the pullback functor f^* preserves effective epimorphisms).

We now prove $(*')$ in the case $n > 0$. Fix an object $\mathscr{F} \in \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$; we wish to prove that $u_{\mathscr{F}}$ is n-connective. The argument above shows that $u_{\mathscr{F}}$ is an effective epimorphism; it will therefore suffice to show that the diagonal map $\beta : \mathscr{F} \to \mathscr{F} \times_{f_* f^* \mathscr{F}} \mathscr{F}$ is $(n-1)$-connective. Let $\mathscr{F}'$ be as above so that we have a pullback diagram

$$\begin{array}{ccc} \mathscr{F}' \times_{\mathscr{F}} \mathscr{F}' & \xrightarrow{\beta'} & \mathscr{F}' \times_{f_* f^* \mathscr{F}} \mathscr{F}' \\ \downarrow & & \downarrow \\ \mathscr{F} & \xrightarrow{\beta} & \mathscr{F} \times_{f_* f^* \mathscr{F}} \mathscr{F}. \end{array}$$

Since the vertical maps are effective epimorphisms, it will suffice to show that β' is $(n-1)$-connective. To prove this, it suffices to observe that the composition of β' with the equivalence $\mathscr{F}' \times_{f_* f^* \mathscr{F}} \mathscr{F}' \to (f_* f^* \mathscr{F}') \times_{f_* f^* \mathscr{F}} (f_* f^* \mathscr{F}')$ can be identified with the unit map $u_{\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}'}$, which is $(n-1)$-connective by virtue of our inductive hypothesis. □

A.4 Deligne's Completeness Theorem

Let $\mathcal{X}$ be a Grothendieck topos. We say that $\mathcal{X}$ has *enough points* if the following condition holds:

$(*)$ Let $\alpha : X \to Y$ be a morphism in $\mathcal{X}$ which is not an isomorphism. Then there exists a geometric morphism $f^* : \mathcal{X} \to \mathcal{S}\mathrm{et}$ such that $f^*(\alpha)$ is not bijective.

A useful theorem of Deligne asserts that every coherent topos has enough points. Our goal in this section is to prove an ∞-categorical version of this result. We will follow the proof of Deligne's theorem given in [142], with minor modifications.

Theorem A.4.0.5 (∞-Categorical Deligne Completeness Theorem)**.** *Let $\mathcal{X}$ be an ∞-topos which is locally coherent and hypercomplete. Then $\mathcal{X}$ has enough points. In other words, given a morphism $\alpha : X \to Y$ in $\mathcal{X}$ which is not an equivalence, there exists a geometric morphism $f^* : \mathcal{X} \to \mathcal{S}$ such that $f^*(\alpha)$ is not an equivalence.*

Note that Theorem A.4.0.5 recovers the classical version of Deligne's completeness theorem:

Corollary A.4.0.6 (Deligne)**.** *Let $\mathcal{X}$ be a coherent topos. Then $\mathcal{X}$ has enough points.*

Proof. Since $\mathcal{X}$ is a coherent topos, it can be realized as the category $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathcal{C})$ of $\mathcal{S}$et-valued sheaves on a small category $\mathcal{C}$ which admits finite limits which is equipped with a finitary Grothendieck topology. Let $\overline{\mathcal{X}}$ be the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$, so that $\mathcal{X}$ can be identified with the full subcategory of $\overline{\mathcal{X}}$ spanned by the discrete objects. Let $\alpha : X \to Y$ be a morphism in

$\mathcal{X}$ which is not an isomorphism. Then α can be regarded as a morphism in $\overline{\mathcal{X}}^{\mathrm{hyp}}$ which is not an equivalence. According to Theorem A.4.0.5, there exists a geometric morphism $\overline{f}^* : \overline{\mathcal{X}}^{\mathrm{hyp}} \to \mathcal{S}$ such that $\overline{f}^*(\alpha)$ is not an equivalence in $\mathcal{S}$. Restricting to discrete objects, we get a geometric morphism $f^* : \mathcal{X} \to \mathcal{S}\mathrm{et}$ such that $f^*(\alpha)$ is not an equivalence. □

A.4.1 Digression: Sheaves on Complete Boolean Algebras

Let Λ be a Boolean algebra (see Definition A.1.6.4). We say that Λ is *complete* if every subset $S \subseteq \Lambda$ has a least upper bound $\bigvee S$ in Λ.

Remark A.4.1.1. If Λ is a Boolean algebra, then the formation of complements $x \mapsto x^c$ determines an isomorphism of Λ with the opposite partially ordered set Λ^{op}. It follows that Λ is complete if and only if every subset $S \subseteq \Lambda$ has a greatest lower bound $\bigwedge S$ in Λ.

Remark A.4.1.2. Let Loc denote the category of locales (Definition 1.5.1.5). The construction $\mathcal{X} \mapsto \mathrm{Sub}(\mathcal{X})$ of Definition 1.5.4.1 admits a fully faithful right adjoint $\mathrm{Loc} \to \infty\mathcal{T}\mathrm{op}$ (see §HTT.6.4.2) and §HTT.6.4.5). This equivalence is given concretely by assigning to each locale Λ the full subcategory $\mathcal{S}\mathrm{hv}(\Lambda) \subseteq \mathrm{Fun}(\Lambda^{\mathrm{op}}, \mathcal{S})$ spanned by those functors $\mathscr{F}$ which satisfy the following condition: for every subset $S \subseteq \Lambda$ which is closed downwards, the canonical map $\mathscr{F}(\bigvee S) \to \varprojlim_{U \in S} \mathscr{F}(U)$ is a homotopy equivalence.

Every complete Boolean algebra Λ is a locale. The completeness of Λ immediately implies that that Λ satisfies condition (1), and the inequality $(\bigvee U_\alpha) \wedge V \geqslant \bigvee (U_\alpha \wedge V)$ follows immediately from the definitions. To verify the reverse inequality $(\bigvee U_\alpha) \wedge V \leqslant \bigvee (U_\alpha \wedge V)$, choose a complement V^c of V, so that we can write each U_α as a join $(U_\alpha \wedge V) \vee (U_\alpha \wedge V^c)$. We therefore have

$$\begin{aligned}
(\bigvee U_\alpha) \wedge V &= (\bigvee ((U_\alpha \wedge V) \vee (U_\alpha \wedge V^c)) \wedge V \\
&= (\bigvee (U_\alpha \wedge V) \vee \bigvee U(_\alpha \wedge V^c)) \wedge V \\
&\leqslant (\bigvee (U_\alpha \wedge V) \vee V^c) \wedge V \\
&= ((\bigvee (U_\alpha \wedge V)) \wedge V) \vee (V^c \wedge V) \\
&= (\bigvee (U_\alpha \wedge V)) \wedge V \\
&\leqslant \bigvee (U_\alpha \wedge V).
\end{aligned}$$

It follows that every complete Boolean algebra Λ determines a 0-localic ∞-topos $\mathcal{S}\mathrm{hv}(\Lambda) \subseteq \mathrm{Fun}(\Lambda^{\mathrm{op}}, \mathcal{S})$.

The ∞-topos of sheaves on a complete Boolean algebra Λ has many pleasant features.

Proposition A.4.1.3. *Let Λ be a complete Boolean algebra. Then the ∞-topos $\mathcal{S}\mathrm{hv}(\Lambda)$ has homotopy dimension $\leqslant 0$: that is, every 0-connective object $X \in \mathcal{S}\mathrm{hv}(\Lambda)$ admits a global section.*

Proof. Let $X \in \mathcal{S}\mathrm{hv}(\Lambda)$ be a 0-connective object which does not admit a global section. For every ordinal α, we let (α) denote the well-ordered set of ordinals $\{\beta : \beta < \alpha\}$. We will construct a compatible sequence of functors $\phi_\alpha : \mathrm{N}(\alpha) \to \mathcal{X}_{/X}$ with the following property:

- $(*)$ The composite functor $\mathrm{N}(\alpha) \xrightarrow{\phi_\alpha} \mathcal{X}_{/X} \to \mathcal{X}$ takes values in the full subcategory of $\mathcal{X}$ spanned by the (-1)-truncated objects, and determines a strictly increasing map $[\alpha] \to \Lambda$.

This leads to a contradiction for α sufficiently large (namely, for any ordinal α such that $[\alpha]$ has cardinality greater than that of Λ).

The construction of the maps ϕ_α proceeds by induction on α. If α is a limit ordinal, we let ϕ_α be the amalgamation of the functors $\{\phi_\beta\}_{\beta<\alpha}$. To complete the construction, it suffices to show that every map $\phi_\alpha : \mathrm{N}(\alpha) \to \mathcal{X}_{/X}$ can be extended to a map $\phi_{\alpha+1} : \mathrm{N}(\alpha+1) \to \mathcal{X}_{/X}$ satisfying $(*)$. The colimit of ϕ_α can be identified with a map $\psi : U \to X$ in $\mathcal{X}$, where U is (-1)-truncated. Let us identify U with an element of the Boolean algebra Λ, and ψ with a point of the space $X(U)$. Since X does not admit a global section, U is not a maximal element of Λ. Because Λ is a Boolean algebra, the object U has a complement $U' \in \Lambda$, which is not a minimal element of Λ. Since $X \in \mathcal{S}\mathrm{hv}(\Lambda)$ is 0-connective, the object U' can be written as a join $\bigvee U'_i$ where each $X(U'_i)$ is nonempty. For some index i, the element $U'_i \in \Lambda$ is nontrivial. Since $U'_i \wedge U = \emptyset$, the canonical map $X(U'_i \vee U) \to X(U'_i) \to X(U)$ is a homotopy equivalence; it follows that ψ can be lifted (up to homotopy) to a point of $X(U'_i \vee U)$. This point gives an extension $\phi_{\alpha+1}$ of ϕ_α, with $\phi_{\alpha+1}(\alpha)$ given by a map $V \to X$ where V is a (-1)-truncated object corresponding to the element $U'_i \vee U$. of Λ. □

Corollary A.4.1.4. *Let Λ be a complete Boolean algebra. Then the ∞-topos $\mathcal{S}\mathrm{hv}(\Lambda)$ is locally of homotopy dimension $\leqslant 0$.*

Proof. For each $U \in \Lambda$, let $\chi_U \in \mathcal{S}\mathrm{hv}(\Lambda)$ be the sheaf given by the formula

$$\chi_U(V) = \begin{cases} \Delta^0 & \text{if } V \leqslant U \\ \emptyset & \text{otherwise.} \end{cases}$$

The objects χ_U generate $\mathcal{S}\mathrm{hv}(\Lambda)$ under colimits. Consequently, it suffices to show that each of the ∞-topoi $\mathcal{S}\mathrm{hv}(\Lambda)_{/\chi_U}$ has homotopy dimension $\leqslant 0$. We complete the proof by observing that $\mathcal{S}\mathrm{hv}(\Lambda)_{/\chi_U}$ is equivalent to $\mathcal{S}\mathrm{hv}(\Lambda_U)$, where Λ_U denotes the complete Boolean algebra $\{V \in \Lambda : V \leqslant U\}$, and therefore has homotopy dimension $\leqslant 0$ by Proposition A.4.1.3. □

Corollary A.4.1.5. *Let Λ be a complete Boolean algebra. Then the ∞-topos $\mathcal{S}\mathrm{hv}(\Lambda)$ is hypercomplete.*

Proof. Combine Corollary A.4.1.4 with Corollary HTT.7.2.1.12 . □

The next result shows that complete Boolean algebras are in abundant supply.

Proposition A.4.1.6. *Let $\mathcal{X}$ be a 0-localic ∞-topos. Assume that $\mathcal{X}$ is not a contractible Kan complex. Then there exists a nontrivial complete Boolean algebra Λ and a geometric morphism $f^* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(\Lambda)$.*

Proof. Let $\mathcal{U}$ be the underlying locale of $\mathcal{X}$: that is, the partially ordered set of subobjects of the unit object $\mathbf{1}_{\mathcal{X}}$. Then $\mathcal{U}$ is a complete lattice: in particular, every set of elements $\{U_\alpha \in \mathcal{U}\}$ has a greatest lower bound $\wedge_\alpha U_\alpha$ and a least upper bound $\vee_\alpha U_\alpha$. In particular, $\mathcal{U}$ has a least element (which we will denote by $\varnothing$) and a greatest element (which we will denote by $\mathbf{1}$). For each element $U \in \mathcal{U}$, we let U' denote the least upper bound of the set $\{V \in \mathcal{U} : U \wedge V = \varnothing\}$. Let $\Lambda = \{U \in \mathcal{U} : U = U''\}$. We will prove:

- (a) The map $U \mapsto U''$ determines a retraction from $\mathcal{U}$ onto Λ, which commutes with finite meets and infinite joins.
- (b) As a partially ordered set, Λ is a complete Boolean algebra.

Assertion (a) implies that Λ is a left exact localization of $\mathcal{U}$, and is therefore itself a locale; moreover, the proof of Proposition HTT.6.4.5.7 gives a geometric morphism $f^* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(\Lambda)$. We begin by proving ($a$). Note that the construction $U \mapsto U'$ is order-reversing. It follows that $U \leqslant V$ implies that $U'' \leqslant V''$. Moreover, we have an evident inequality $U \leqslant U''$ which guarantees that $U''' = U'$. In particular, $U' \in \Lambda$ for each $U \in \mathcal{U}$. We next claim that the construction $U \mapsto U''$ is a left adjoint to the inclusion $\Lambda \subseteq \mathcal{U}$. In other words, we claim that for $V \in \mathcal{U}$, we have $U \leqslant V$ if and only if $U'' \leqslant V$. The "if" direction is clear (since $U \leqslant U''$), and the "only if" direction follows from the implications

$$(U \leqslant V) \Rightarrow (V' \leqslant U') \Rightarrow (U'' \leqslant V'') \Rightarrow (U'' \leqslant V),$$

since $V = V''$. It follows immediately that $U \mapsto U''$ is a retraction onto Λ which preserves infinite joins.

We now show that the construction $U \mapsto U''$ preserves finite meets (note that, since the inclusion $\Lambda \hookrightarrow \mathcal{U}$ admits a left adjoint, Λ is closed under meets in $\mathcal{U}$). The inequality $U \leqslant U''$ shows that $\mathbf{1} = \mathbf{1}''$. It therefore suffices to show that $U \mapsto U''$ preserves pairwise meets. The construction $U \mapsto U'$ is an order-reversing bijection from Λ to itself, and therefore carries finite joins in Λ to finite meets in Λ. It will therefore suffice to show that the construction $U \mapsto U'$ carries pairwise meets in $\mathcal{U}$ to pairwise joins in Λ. In other words, we must show that for $U, V \in \mathcal{U}$, the element $(U \wedge V)'$ is a join of U' and V' in Λ. It is clear that $U', V' \leqslant (U \wedge V)'$; it therefore suffices to show that if $W = W''$ is any upper bound for U' and V' in Λ, then $(U \wedge V)' \leqslant W = W''$. In other words, we must show that $(U \wedge V)' \wedge W' = \varnothing$: that is, if $X \in \mathcal{U}$ is any object such that $X \wedge W = \varnothing$ and $X \wedge (U \wedge V) = \varnothing$, then $X = \varnothing$.

We have $X \wedge U \leqslant V' \leqslant W'' = W$, so that $(X \wedge U) \leqslant X \wedge W = \varnothing$. This shows that $X \leqslant U' \leqslant W$, so that $X = X \wedge W = \varnothing$ as desired. This completes the proof of (a).

The proof of (a) shows that Λ is a locale; in particular, it is a distributive lattice. To prove (b), it suffices to show that Λ is complemented: that is, for every $U \in \Lambda$ there exists $V \in \Lambda$ such that $U \wedge V = \varnothing$ and $U \vee V = \mathbf{1}$. For this, we take $V = U'$, so that the equation $U \wedge V = \varnothing$ is obvious. To prove $U \vee V = \mathbf{1}$, it suffices to show that if U and U' are bounded by an element $W \in B$, then $W = \mathbf{1}$. In fact, the inequalities $U \leqslant W$ and $U' \leqslant W$ guarantee that $W' \leqslant U' \wedge U'' = \varnothing$, so that $W = W'' = \varnothing' = \mathbf{1}$. □

A.4.2 Surjective Geometric Morphisms

As a first step towards the proof of Theorem A.4.0.5, we discuss various formulations of the hypothesis that the hypercompletion of an ∞-topos $\mathcal{X}$ has enough points.

Proposition A.4.2.1. *Let $\mathcal{X}$ be an ∞-topos, and suppose we are given a collection of geometric morphisms $\{f_\alpha^* : \mathcal{X} \to \mathcal{X}_\alpha\}$. The following conditions are equivalent:*

(1) *A monomorphism $u : X \to Y$ in $\mathcal{X}$ is an equivalence if and only if each $f_\alpha^*(u)$ is an equivalence in $\mathcal{X}_\alpha$.*

(2) *A morphism $u : X \to Y$ in $\mathcal{X}$ is an effective epimorphism if and only if each $f_\alpha^*(u)$ is an effective epimorphism in $\mathcal{X}_\alpha$.*

(3) *For each $n \geqslant 0$, a morphism $u : X \to Y$ in $\mathcal{X}$ is n-connective if and only if each $f_\alpha^*(u)$ is n-connective.*

(4) *A morphism $u : X \to Y$ in $\mathcal{X}$ is ∞-connective if and only if each $f_\alpha^*(u)$ is ∞-connective.*

Proof. We will prove that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$. Suppose first that (1) is satisfied, and let $u : X \to Y$ be a morphism in $\mathcal{X}$. Then u factors as a composition $X \xrightarrow{u'} X' \xrightarrow{u''} Y$, where u' is an effective epimorphism and u'' is a monomorphism. If each $f_\alpha^*(u)$ is an effective epimorphism, then each $f_\alpha^*(u'')$ is an equivalence, so that (1) implies that u'' is an equivalence. It follows that $u \simeq u'$ is an effective epimorphism as desired.

Now suppose that (2) is satisfied; we prove (3) using induction on n, the case $n = -1$ being vacuous. Suppose that $u : X \to Y$ is a morphism in $\mathcal{X}$ such that each $f_\alpha^*(u)$ is n-connective. Let $v : X \to X \times_Y X$ be the diagonal map; then each $f_\alpha^*(v)$ is $(n-1)$-connective. The inductive hypothesis guarantees that v is $(n-1)$-connective, and assumption (2) guarantees that u is an effective epimorphism. It follows that u is n-connective as desired.

The implication $(3) \Rightarrow (4)$ is obvious, and the implication $(4) \Rightarrow (1)$ follows from the observation that a monomorphism $u : X \to Y$ is an equivalence if and only if it is ∞-connective. □

Definition A.4.2.2. We will say that a collection of geometric morphisms of ∞-topoi $\{f^*_\alpha : \mathcal{X} \to \mathcal{X}_\alpha\}$ is *jointly surjective* if it satisfies the equivalent conditions of Proposition A.4.2.1. We will say that a geometric morphism $f^* : \mathcal{X} \to \mathcal{Y}$ is *surjective* if the one-element collection $\{f^* : \mathcal{X} \to \mathcal{Y}\}$ is jointly surjective.

Warning A.4.2.3. Let $f : X \to Y$ be a continuous map of topological spaces, so that f determines a geometric morphism $f^* : \mathcal{S}\mathrm{hv}(Y) \to \mathcal{S}\mathrm{hv}(X)$. If f is surjective, then the geometric morphism f^* is surjective. However, the converse fails in general.

Example A.4.2.4. Let $\mathcal{X}$ be an ∞-topos, and let $f^* : \mathcal{X} \to \mathcal{X}^{\mathrm{hyp}}$ be a left adjoint to the inclusion. Then f^* is surjective.

Example A.4.2.5. Let $\mathcal{X}$ be an ∞-topos containing an object U. Then the étale geometric morphism $f^* : \mathcal{X} \to \mathcal{X}_{/U}$ is surjective if and only if the object U is 0-connective: that is, if and only if the map $U \to \mathbf{1}$ is an effective epimorphism, where $\mathbf{1}$ denotes a final object of $\mathcal{X}$. If this condition is satisfied, then we will say that $f^* : \mathcal{X} \to \mathcal{X}_{/U}$ is an *étale surjection*.

Remark A.4.2.6. Let $\mathcal{X}$ be an arbitrary ∞-topos. Since the ∞-topos $\mathcal{S}$ is hypercomplete, composition with the localization functor $\mathcal{X} \to \mathcal{X}^{\mathrm{hyp}}$ induces an equivalence between the ∞-category of points of $\mathcal{X}^{\mathrm{hyp}}$ and the ∞-category of points of $\mathcal{X}$. Note if $\mathcal{X}$ is locally foherent, then $\mathcal{X}^{\mathrm{hyp}}$ is also locally coherent (Proposition A.2.2.2). Consequently, Theorem A.4.0.5 can be reformulated as follows: if $\mathcal{X}$ is a coherent ∞-topos, then there exists a jointly surjective collection of points $\{f^*_\alpha : \mathcal{X} \to \mathcal{S}\}$.

Proposition A.4.2.7. *Let $\mathcal{X}$ be a* 0*-localic ∞-topos. Then there exists a surjective geometric morphism $f^* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(\Lambda)$, where Λ is a complete Boolean algebra.*

Proof. Let $\mathcal{U}$ be the locale of equivalence classes of (-1)-truncated objects of $\mathcal{X}$. For every proper inclusion $U \subset V$ in $\mathcal{U}$, Proposition A.4.1.6 supplies a nontrivial complete Boolean algebra $\Lambda_{U,V}$ and a left exact, join-preserving map $f_{U,V} : \mathcal{U} \to \Lambda_{U,V}$. Let Λ be the product of the Boolean algebras $\Lambda_{U,V}$, and let $f : \mathcal{U} \to \Lambda$ be the product functor; then f induces a geometric morphism $f^* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(\Lambda)$. We claim that this geometric morphism is surjective.

Let $u : X \to Y$ be a monomorphism in $\mathcal{X}$ such that $f^*(u)$ is an equivalence; we wish to prove that u is an equivalence. For each $V \in \mathcal{U}$, let $\chi_V \in \mathcal{S}\mathrm{hv}(U)$ be the sheaf given by the formula

$$\chi_V(W) = \begin{cases} \Delta^0 & \text{if } W \subseteq V \\ \varnothing & \text{otherwise.} \end{cases}$$

The ∞-category $\mathcal{S}\mathrm{hv}(\mathcal{U})$ is generated under colimits by the objects χ_V. In particular, there exists an effective epimorphism $\amalg_\alpha \chi_{V_\alpha} \to Y$. It therefore suffices to show that the induced map

$$(\amalg_\alpha \chi_{V_\alpha}) \times_Y X \to \amalg_\alpha \chi_{V_\alpha}$$

is an equivalence. This map is a coproduct of morphisms

$$u_\alpha : \chi_{V_\alpha} \times_Y X \to \chi_{V_\alpha}.$$

To complete the proof, it suffices to show that each u_α is an equivalence. We may therefore replace u by u_α, and thereby reduce to the case where Y has the form χ_V for some object $V \in \mathcal{U}$.

Since u is a monomorphism, we can identify X with χ_U for some $U \subseteq V$. We wish to show that $U = V$. Suppose otherwise, so that the geometric morphism $f^*_{U,V} : \mathcal{X} \to \mathcal{S}\mathrm{hv}(\Lambda) \to \mathcal{S}\mathrm{hv}(\Lambda_{U,V})$ is well-defined. We note that the image of χ_U in $\mathcal{S}\mathrm{hv}(\Lambda_{U,V})$ is the initial object, while the image of χ_V in $\mathcal{S}\mathrm{hv}(\Lambda_{U,V})$ is the final object. Consequently, $f^*_{U,V}(u)$ is an equivalence in $\mathcal{S}\mathrm{hv}(\Lambda_{U,V})$ between the initial and final objects, contradicting the nontriviality of $\Lambda_{U,V}$. □

A.4.3 The Diaconescu Cover

Our proof of Theorem A.4.0.5 will proceed by reducing to a more concrete statement about locales. The mechanism for this reduction is the following result:

Proposition A.4.3.1. *Let $\mathcal{C}$ be an ∞-category equipped with a Grothendieck topology. Then there exists a surjective geometric morphism $f^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{X}$, where $\mathcal{X}$ is a 0-localic ∞-topos.*

Proof. Let $\mathcal{D}$ be an ∞-category equipped with a functor $g : \mathcal{D} \to \mathcal{C}$. We will say that a sieve $\mathcal{D}^{(0)}_{/D} \subseteq \mathcal{D}_{/D}$ on an object $D \in \mathcal{D}$ is *covering* if the following condition is satisfied:

- $(*)$ For every morphism $\alpha : D' \to D$ in $\mathcal{D}$, the collection of morphisms $g(\beta) : g(D'') \to g(D')$ such that the composition $(\alpha \circ \beta) : D'' \to D$ belongs to $\mathcal{D}^{(0)}_{/D}$ generates a covering sieve on $g(D') \in \mathcal{C}$.

It is not difficult to see that this defines a Grothendieck topology on $\mathcal{D}$. Let $L_{\mathcal{C}} : \mathcal{P}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ and $L_{\mathcal{D}} : \mathcal{P}(\mathcal{D}) \to \mathcal{S}\mathrm{hv}(\mathcal{D})$ be left adjoints to the inclusions, and consider the composite functor $\overline{f}^* : \mathcal{P}(\mathcal{C}) \xrightarrow{\circ g} \mathcal{P}(\mathcal{D}) \xrightarrow{L_{\mathcal{D}}} \mathcal{S}\mathrm{hv}(\mathcal{D})$. It is clear that $\overline{f}^*$ is a geometric morphism.

We now suppose that the functor g has the following property:

- (a) For every object $D \in \mathcal{D}$ and every morphism $\beta : C \to g(D)$ in $\mathcal{D}$, there exists a morphism $\overline{\beta} : \overline{C} \to D$ in $\mathcal{D}$ such that $\beta = g(\overline{\beta})$.

We claim that $\overline{f}^*$ carries $L_{\mathcal{C}}$-equivalences to equivalences in $\mathcal{S}\mathrm{hv}(\mathcal{D})$. To prove this, it suffices to show that if we are given a collection of morphisms $\alpha_i : C_i \to C$ which generate a covering sieve on $C \in \mathcal{C}$, then the induced map $\phi : \amalg \overline{f}^* j(C_i) \to \overline{f}^* j(C)$ is an effective epimorphism in $\mathcal{S}\mathrm{hv}(\mathcal{D})$; here $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ denotes the Yoneda embedding (see Proposition HTT.6.2.3.20).

Let $e : \mathcal{D} \to \mathcal{S}\mathrm{hv}(\mathcal{D})$ be the composition of the Yoneda embedding $\mathcal{D} \to \mathcal{P}(\mathcal{D})$ with the sheafification functor $L_{\mathcal{D}}$. Then $\mathcal{S}\mathrm{hv}(\mathcal{D})$ is generated under colimits by the essential image of e. Consequently, to prove that ϕ is an effective epimorphism, it suffices to show that for every morphism $u : e(D) \to \overline{f}^* j(C)$, the induced map

$$\phi_u : \amalg(\overline{f}^* j(C_i) \times_{\overline{f}^* j(C)} e(D)) \to e(D)$$

is an effective epimorphism in $\mathcal{S}\mathrm{hv}(\mathcal{D})$. Passing to a covering of D, we may reduce to the case where u is induced by a morphism in $\mathcal{P}(\mathcal{D})$, corresponding to a map $\overline{u} : g(D) \to C$ in $\mathcal{C}$. Let $\mathcal{D}^{(0)}_{/D}$ denote the full subcategory of $\mathcal{D}_{/D}$ spanned by those morphisms $D_0 \to D$ such that the induced map $g(D_0) \to g(D) \xrightarrow{\overline{u}} C$ belongs to the sieve generated by the collection of morphisms $\{\alpha_i\}$. It is clear that $\mathcal{D}^{(0)}_{/D}$ is a sieve on D. For every morphism $D_0 \to D$ in $\mathcal{D}^{(0)}_{/D}$, the induced map $e(D_0) \to e(D)$ factors through ϕ_u. Consequently, to show that ϕ_u is an effective epimorphism, it will suffice to show that $\mathcal{D}^{(0)}_{/D}$ is a covering sieve on D: that is, that it satisfies condition $(*)$. Choose a morphism $D' \to D$ in $\mathcal{D}$. Since the collection of covering sieves in $\mathcal{C}$ forms a Grothendieck topology, there exists a collection of morphisms $\beta_j : C'_j \to g(D')$ which generate a covering sieve, each of which fits into a commutative diagram

$$\begin{array}{ccc} C'_j & \xrightarrow{\beta_j} & g(D') \\ \downarrow & & \downarrow \\ C_i & \xrightarrow{\alpha_i} & C. \end{array}$$

Condition (a) guarantees that each β_j can be lifted to a morphism $\overline{\beta}_j : \overline{C}'_j \to D'$ in $\mathcal{D}$, which belongs to the pullback of the sieve $\mathcal{D}^{(0)}_{/D}$. It follows that $\mathcal{D}^{(0)}_{/D}$ satisfies condition $(*)$ and is therefore a covering sieve on D, as required.

Since $\overline{f}^*$ carries $L_{\mathcal{C}}$-equivalences to equivalences in $\mathcal{S}\mathrm{hv}(\mathcal{D})$, it factors up to homotopy as a composition $\mathcal{P}(\mathcal{C}) \xrightarrow{L_{\mathcal{C}}} \mathcal{S}\mathrm{hv}(\mathcal{C}) \xrightarrow{f^*} \mathcal{S}\mathrm{hv}(\mathcal{D})$ where f^* is a colimit-preserving functor which (since it is equivalent to $\overline{f}^*|_{\mathcal{S}\mathrm{hv}(\mathcal{C})}$) preserves finite limits. We now make the following additional assumption:

(b) The functor g is surjective on objects.

We claim that condition (b) implies that f^* is surjective in the sense of Definition A.4.2.2. We will show that if $u : X \to Y$ is a morphism $\mathcal{S}\mathrm{hv}(X)$ such that $f^*(u)$ is an effective epimorphism in $\mathcal{S}\mathrm{hv}(\mathcal{D})$, then u is an effective epimorphism in $\mathcal{S}\mathrm{hv}(X)$. Choose an object $C \in \mathcal{C}$ and a point $\eta \in Y(C)$, and let $\mathcal{C}^{(0)}_{/C}$ be the full subcategory of $\mathcal{C}_{/C}$ spanned by those morphisms $C' \to C$ such that the image of η in $\pi_0 Y(C')$ can be lifted to $\pi_0 X(C')$; we wish to prove that $\mathcal{C}^{(0)}_{/C}$ is covering. Assumption (b) implies we can write $C = g(D)$ for some object

$D \in \mathcal{D}$. Then η determines a point $\overline{\eta} \in (f^*Y)(D)$. Let $\mathcal{D}^{(0)}_{/D} \subseteq \mathcal{D}_{/D}$ be the sieve consisting of morphisms $D' \to D$ such that the image of $\overline{\eta}$ in $\pi_0(f^*Y)(D')$ lifts to $\pi_0(f^*X)(D)$, and let $\mathcal{D}^{(1)}_{/D} \subseteq \mathcal{D}_{/D}$ be the sieve consisting of morphisms $\beta : D' \to D$ such that the image of η in $\pi_0 Y(g(D'))$ lifts to $\pi_0 X(g(D'))$. The functor g carries $\mathcal{D}^{(1)}_{/D}$ into $\mathcal{C}^{(0)}_{/C}$. Consequently, to prove that $\mathcal{C}^{(0)}_{/C}$ is a covering sieve on $C \in \mathcal{C}$, it suffices to show that $\mathcal{D}^{(1)}_{/D}$ is a covering sieve on $D \in \mathcal{D}$. Since $f^*(u)$ is an effective epimorphism, the sieve $\mathcal{D}^{(0)}_{/D}$ is covering. It therefore suffices to show that for each $\beta : D' \to D$ in $\mathcal{D}^{(0)}_{/D}$, the pullback $\beta^* \mathcal{D}^{(1)}_{/D} \subseteq \mathcal{D}_{/D'}$ is a covering sieve on D'. Replacing D by D' (and C by $g(D')$), we may assume that $\overline{\eta}$ lifts to a point $\overline{\eta}' \in (f^*X)(D)$. Note that f^*X is the sheafification of the functor $\mathcal{D} \xrightarrow{g} \mathcal{C} \xrightarrow{X} \mathcal{S}$. It follows that there exists a covering sieve $\mathcal{D}^{(2)}_{/D}$ on D such that for each morphism $D' \to D$ in $\mathcal{D}^{(2)}_{/D}$, the image of $\overline{\eta}'$ in $(f^*X)(D')$ belongs to the image of $X(g(D'))$. We clearly have a containment $\mathcal{D}^{(2)}_{/D} \subseteq \mathcal{D}^{(1)}_{/D}$, so that $\mathcal{D}^{(1)}_{/D}$ is also a covering sieve.

We now add the following further assumption:

(c) The ∞-category $\mathcal{D}$ is (the nerve of) a partially ordered set.

In this case, the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{D})$ is 0-localic, so that the geometric morphism $f^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\mathcal{D})$ satisfies the requirements of Proposition A.4.3.1.

It remains to prove that there exists a functor $g : \mathcal{D} \to \mathcal{C}$ satisfying conditions (a), (b), and (c). For this, we let A denote the partially ordered set of pairs (n, σ), where $n \geqslant 0$ and $\sigma : \Delta^n \to \mathcal{C}^{\mathrm{op}}$ is an n-simplex of $\mathcal{C}^{\mathrm{op}}$. We write $(n, \sigma) \leqslant (n', \sigma')$ if $n \leqslant n'$ and $\sigma = \sigma'|_{\Delta^{\{0,\ldots,n\}}}$. A k-simplex of the nerve $\mathrm{N}(A)$ consists of a sequence τ :

$$(n_0, \sigma_0) \leqslant \cdots \leqslant (n_k, \sigma_k).$$

Let $g(\tau)$ denote the k-simplex of $\mathcal{C}^{\mathrm{op}}$ given by the composition

$$\Delta^k \xrightarrow{\gamma} \Delta^{n_k} \xrightarrow{\sigma_k} \mathcal{C}^{\mathrm{op}},$$

where γ is given on vertices by the formula $\gamma(i) = n_i$. Then the construction $\tau \mapsto g(\tau)$ determines a map of simplicial sets $g : \mathrm{N}(A)^{\mathrm{op}} \to \mathcal{C}$. It is easy to see that this map satisfies conditions (a), (b), and (c). $\square$

Corollary A.4.3.2. *Let $\mathcal{X}$ be an ∞-topos. Then there exists a complete Boolean algebra Λ and a surjective geometric morphism $f^* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(\Lambda)$*

Proof. Using Proposition HTT.6.5.2.19 , we deduce that there exists a small ∞-category $\mathcal{C}$ equipped with a Grothendieck topology such that $\mathcal{X}$ is a cotopological localization of $\mathcal{S}\mathrm{hv}(\mathcal{C})$. Proposition A.4.3.1 gives a surjective geometric morphism $g^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{Y}$, where $\mathcal{Y}$ is 0-localic. Proposition A.4.2.7 guarantees a surjective geometric morphism $h^* : \mathcal{Y} \to \mathcal{S}\mathrm{hv}(\Lambda)$, where $\mathcal{S}\mathrm{hv}(\Lambda)$ is a complete Boolean algebra. Since $\mathcal{S}\mathrm{hv}(\Lambda)$ is hypercomplete (Corollary A.4.1.5), the functor $h^* \circ g^*$ carries ∞-connective morphisms in $\mathcal{S}\mathrm{hv}(\mathcal{C})$ to equivalences in $\mathcal{S}\mathrm{hv}(\Lambda)$, and therefore factors as a composition

$$\mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{X} \xrightarrow{f^*} \mathcal{S}\mathrm{hv}(\Lambda)$$

for some surjective geometric morphism f^*. □

A.4.4 The Proof of Deligne's Theorem

If Λ is an arbitrary Boolean algebra, we let $\mathrm{Spec}(\Lambda)$ denote the spectrum of Λ (Construction A.1.2.3), so that $\mathrm{Spec}(\Lambda)$ is a compact Hausdorff space with a basis of closed and open sets (Theorem A.1.6.11). Let $\mathcal{U}(\mathrm{Spec}(\Lambda))$ denote the collection of *all* open subsets $\mathrm{Spec}(\Lambda)$. For each $\lambda \in \Lambda$, let $U_\lambda = \{\mathfrak{p} \in \mathrm{Spec}(\Lambda) : \lambda \notin \mathfrak{p}\}$, so that the construction $\lambda \mapsto U_\lambda$ determines an injective map of partially ordered sets $\iota : \Lambda \hookrightarrow \mathcal{U}(\mathrm{Spec}(\Lambda))$, whose image is the collection of closed and open subsets of $\mathrm{Spec}(\Lambda)$. If Λ is complete, then the inclusion map ι admits a left adjoint $\phi : \mathcal{U}(\mathrm{Spec}(\Lambda)) \to \Lambda$, given by the formula $\phi(U) = \bigvee\{\lambda \in \Lambda : U_\lambda \subseteq U\}$. It is easy to see that the map ϕ preserves finite meets and arbitrary joins, and can therefore be regarded as a morphism of locales. In particular, it induces a geometric morphism of ∞-topoi $f^* : \mathcal{S}\mathrm{hv}(\mathrm{Spec}(\Lambda)) \to \mathcal{S}\mathrm{hv}(\Lambda)$.

Lemma A.4.4.1. *Let Λ be a complete Boolean algebra, and let $f^* : \mathcal{S}\mathrm{hv}(\mathrm{Spec}(\Lambda)) \to \mathcal{S}\mathrm{hv}(\Lambda)$ be the geometric morphism constructed above. Then:*

(1) *The right adjoint f_* to f^* is fully faithful. In other words, the composition $f^* f_*$ is equivalent to the identity on $\mathcal{S}\mathrm{hv}(\Lambda)$.*

(2) *For every finite collection of objects $\{X_i\}_{1 \leqslant i \leqslant n}$ and effective epimorphism $\amalg X_i \to Y$ in $\mathcal{S}\mathrm{hv}(\Lambda)$, the induced map $\amalg f_* X_i \to f_* Y$ is an effective epimorphism in $\mathcal{S}\mathrm{hv}(\mathrm{Spec}(\Lambda))$.*

Proof. Let $\iota : \Lambda \hookrightarrow \mathcal{U}(\mathrm{Spec}(\Lambda))$ be the inclusion described above. The functor $f_* : \mathcal{S}\mathrm{hv}(\Lambda) \to \mathcal{S}\mathrm{hv}(\mathrm{Spec}(\Lambda))$ is given by right Kan extension along the inclusion ι, and is fully faithful by Proposition HTT.4.3.2.15 . This proves (1). To prove (2), we note that $\mathcal{S}\mathrm{hv}(\mathrm{Spec}(\Lambda))$ is generated under colimits by objects of the form $f_* \chi_U$ for $U \in \Lambda$ (see the proof of Corollary A.4.1.4; consequently, we may assume without loss of generality that Y has the form χ_U. Each of the maps $X_i \to Y$ factors as a composition

$$X_i \xrightarrow{u_i} \chi_{U_i} \to \chi_U,$$

where u_i is an effective epimorphism. Applying Proposition A.4.1.3 (to the complete Boolean algebra $\{V \in \Lambda : V \leqslant U_i\}$), we deduce that u_i admits a section s_i. We have a commutative

diagram

$$\begin{array}{ccc} & \amalg f_* X_i & \\ {\scriptstyle \amalg s_i}\nearrow & & \searrow \\ \amalg f_* \chi_{U_i} & \xrightarrow{\psi} & f_* \chi_U . \end{array}$$

Consequently, to prove (2), it suffices to show that ψ is an effective epimorphism. For this, it suffices to observe that for each $V \in \Lambda$, the functor f_* carries χ_V to the sheaf represented by the open set $i(V) \subseteq \operatorname{Spec}(\Lambda)$, and the map $V \mapsto i(V)$ preserves finite joins. □

Lemma A.4.4.2. *Let $\mathcal{C}$ be a small ∞-category which admits finite limits and which is equipped with a finitary Grothendieck topology. Let Λ be a complete Boolean algebra, and let $g^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\Lambda)$ be a geometric morphism. Then g^* is homotopic to a composition*

$$\mathcal{S}\mathrm{hv}(\mathcal{C}) \xrightarrow{h^*} \mathcal{S}\mathrm{hv}(\operatorname{Spec}(\Lambda)) \xrightarrow{f^*} \mathcal{S}\mathrm{hv}(\Lambda),$$

where f^* is the geometric morphism of Lemma A.4.4.1.

Proof. For every ∞-topos $\mathcal{Y}$, the ∞-category of geometric morphisms from $\mathcal{S}\mathrm{hv}(\mathcal{C})$ to $\mathcal{Y}$ can be identified with the ∞-category of left-exact functors $u : \mathcal{C} \to \mathcal{Y}$ with the following property: for every every collection of morphisms $\{C_i \to C\}$ which generate a covering sieve on an object $C \in \mathcal{C}$, the induced map $\amalg u(C_i) \to u(C)$ is an effective epimorphism in $\mathcal{Y}$ (Proposition HTT.6.2.3.20). In particular, g^* is classified by a functor $u : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\Lambda)$. Let f_* denote a right adjoint to f^*, and let $u' : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\operatorname{Spec}(\Lambda))$ be the composition $f_* \circ u$. It follows from Lemma A.4.4.1 that u' determines a geometric morphism $h^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\operatorname{Spec}(\Lambda))$ such that $f^* \circ h^* \simeq g^*$. □

Lemma A.4.4.3. *Let $\mathcal{C}$ be a small ∞-category which admits finite limits and which is equipped with a finitary Grothendieck topology. Then there exists a surjective geometric morphism $h^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(X)$, where X is a Stone space.*

Proof. Corollary A.4.3.2 guarantees the existence of a surjective geometric morphism $g^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\Lambda)$, where Λ is a complete Boolean algebra. Let X be the spectrum of Λ. Lemma A.4.4.2 guarantees that g^* factors through a geometric morphism $h^* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(X)$, which is clearly surjective. □

Proof of Theorem A.4.0.5. Let $\mathcal{X}$ be an ∞-topos which is locally coherent; we wish to show that the ∞-topos $\mathcal{X}^{\mathrm{hyp}}$ has enough points. Without loss of generality, we may assume that $\mathcal{X}$ is coherent and hypercomplete (see Remark A.4.2.6). Using Theorem A.3.4.1, we can assume that $\mathcal{X} = \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ for some small ∞-category $\mathcal{C}$ which admits finite limits and is equipped with a finitary Grothendieck topology. Choose a surjective geometric morphism $h^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(X)$ as in Lemma A.4.4.3. For each point $x \in X$, let f_x^* denote the composite map $\mathcal{S}\mathrm{hv}(\mathcal{C}) \xrightarrow{h^*} \mathcal{S}\mathrm{hv}(X) \to \mathcal{S}\mathrm{hv}(\{x\}) \simeq \mathcal{S}$. It is easy to see that the collection of geometric morphisms $\{f_x^*\}_{x \in X}$ is jointly surjective, so that $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ has enough points as desired (see Remark A.4.2.6). □

A.5 Application: Homotopy Theory of Simplicial Objects

Theorem A.4.0.5 is very useful: it can be used to reduce many questions about a general ∞-topos $\mathcal{X}$ to questions about the ∞-topos $\mathcal{S}$ of spaces, which can then be attacked using methods of classical homotopy theory. In this section, we will illustrate this principle by using Theorem A.4.0.5 to study extension conditions on simplicial (and semisimplicial) objects of an arbitrary ∞-topos $\mathcal{X}$.

A.5.1 Semisimplicial Objects

We begin by reviewing some terminology.

Notation A.5.1.1. We let $\mathbf{\Delta}$ denote the category of simplices: the objects of $\mathbf{\Delta}$ are the finite linearly ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n > 0$, and the morphisms in $\mathbf{\Delta}$ are nondecreasing functions. We let $\mathbf{\Delta}_s$ denote the subcategory of $\mathbf{\Delta}$ containing all objects, where a morphism from $[m]$ to $[n]$ in $\mathbf{\Delta}_s$ is a strictly increasing function $\alpha : [m] \to [n]$.

A *simplicial object* of an ∞-category $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}$, and a *semisimplicial object* of $\mathcal{C}$ is a a functor $\mathbf{\Delta}_s^{\mathrm{op}} \to \mathcal{S}$. We will typically denote either a simplicial or semisimplicial object of $\mathcal{C}$ by $X_\bullet$, and indicate its value on an object $[n] \in \mathbf{\Delta}$ by X_n.

Remark A.5.1.2. Let $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object of an ∞-category $\mathcal{C}$. Then the composition

$$\mathbf{\Delta}_s^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}$$

is a semisimplicial object of $\mathcal{C}$, which we will refer to as the *underlying semisimpicial object of* $X_\bullet$. We will typically abuse notation by not distinguishing between $X_\bullet$ and its underlying semisimplicial object.

Notation A.5.1.3. Let $X_\bullet$ be a semisimplicial object of an ∞-category $\mathcal{C}$. We let $|X_\bullet|$ denote a colimit of the diagram $X_\bullet$, provided that such a colimit exists. In this case, we will refer to $|X_\bullet|$ as the *geometric realization* of $X_\bullet$.

Remark A.5.1.4. The inclusion $\mathbf{\Delta}_s \hookrightarrow \mathbf{\Delta}$ is right cofinal (Lemma HTT.6.5.3.7). Consequently, if $X_\bullet$ is a simplicial object of an ∞-category $\mathcal{C}$, then a geometric realization of (the underlying semsimplicial object of) $X_\bullet$ can be identified with a colimit of the diagram $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$.

Notation A.5.1.5. Let K be a simplicial set. We say that K is *nonsingular* if every nondegenerate simplex $\sigma : \Delta^n \to K$ is a monomorphism of simplicial sets. For any simplicial set K, we let $\mathbf{\Delta}_K$ denote the category of simplices of K: that is, the category whose objects are simplices $\sigma : \Delta^m \to K$, where a morphism from $\sigma : \Delta^m \to K$ to $\tau : \Delta^n \to K$ is given by a commutative diagram of simplicial sets

$$\begin{array}{ccc} \Delta^m & \longrightarrow & \Delta^n \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & K. & \end{array}$$

For any simplicial set K, we let $\mathbf{\Delta}_K^{\mathrm{nd}}$ denote the full subcategory of $\mathbf{\Delta}_K$ spanned by the nondegenerate simplices $\sigma : \Delta^m \to K$. In this case where K is nonsingular, this construction has several pleasant features:

- The category $\mathbf{\Delta}_K^{\mathrm{nd}}$ is (equivalent to) a partially ordered set.
- The inclusion $\mathbf{\Delta}_K^{\mathrm{nd}} \hookrightarrow \mathbf{\Delta}_K$ is left cofinal.
- The nerve of the category $\mathbf{\Delta}_K^{\mathrm{nd}}$ can be identified with the subdivision of the simplicial set K.

Construction A.5.1.6. Let $X_\bullet$ be a semisimplicial object of an ∞-category $\mathcal{C}$ and let K be a nonsingular simplicial set. We let $X_\bullet[K]$ denote the limit of the diagram

$$(\mathbf{\Delta}_K^{\mathrm{nd}})^{\mathrm{op}} \to \mathbf{\Delta}_s^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C},$$

provided that such a limit exists (otherwise, we will say that $X_\bullet[K]$ *does not exist*).

Example A.5.1.7. Let $K = \Delta^n$. For any semisimplicial object $X_\bullet$ of an ∞-category $\mathcal{C}$, the object $X_\bullet[K]$ can be identified with $X_n \in \mathcal{C}$.

Example A.5.1.8. Let $\mathcal{C}$ be an ∞-category which admits finite limits and let $X_\bullet$ be a semisimplicial object of $\mathcal{C}$. Then for any finite nonsingular simplicial set K, the object $X_\bullet[K] \in \mathcal{C}$ is well-defined.

Proposition A.5.1.9. *Let $\mathcal{C}$ be an ∞-category which admits fiber products and let $X_\bullet$ be a semisimplicial object of $\mathcal{C}$. Then, for any $n > 0$ and any $0 \leqslant i \leqslant n$, the object $X_\bullet[\Lambda^n_i] \in \mathcal{C}$ is well-defined.*

Proof. Let $\mathcal{J} \subseteq \mathbf{\Delta}^{\mathrm{nd}}_{\Lambda^n_i}$ be the full subcategory spanned by those simplices which contain the ith vertex. Then the inclusion $\mathcal{J} \hookrightarrow \mathbf{\Delta}^{\mathrm{nd}}_{\Lambda^n_i}$ is left cofinal. Then we can identify $\mathcal{J}$ with $\mathcal{J}_0^{\triangleleft}$, where $\mathcal{J}_0$ is the full subcategory spanned by those of positive dimension which contain the ith vertex. We are therefore reduced to showing that the diagram

$$(\mathcal{J}_0^{\mathrm{op}})^{\triangleleft} \simeq \mathcal{J}^{\mathrm{op}} \to \mathbf{\Delta}_s^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}$$

admits a limit, which is equivalent to the existence of a limit of a diagram $\mathcal{J}_0^{\mathrm{op}} \to \mathcal{C}_{/X_0}$. It now suffices to observe that because $\mathcal{C}$ admits fiber products, the ∞-category $\mathcal{C}_{/X_0}$ admits all finite limits. □

Variant A.5.1.10. Let $X_\bullet$ be a *simplicial* object of an ∞-category $\mathcal{C}$. Then for any simplicial set K, we can contemplate the existence of a limit for the diagram

$$\mathbf{\Delta}_K^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C};$$

if such a limit exists, we will denote it by $X_\bullet[K]$. In the special case where K is a nonsingular simplicial set, the left cofinality of the inclusion $\mathbf{\Delta}_K^{\mathrm{nd}} \hookrightarrow \mathbf{\Delta}_K$ implies that this definition agrees with that of Construction A.5.1.6 (and therefore depends only on the underlying semisimplicial object of $X_\bullet$).

Remark A.5.1.11. Let L be a nonsingular simplicial set and let $K \subseteq L$ be a simplicial subset. Then K is also nonsingular. Moreover, the inclusion $K \hookrightarrow L$ induces a functor $\iota : \mathbf{\Delta}_K^{\mathrm{nd}} \to \mathbf{\Delta}_L^{\mathrm{nd}}$. If $X_\bullet$ is a semisimplicial object of an ∞-category $\mathcal{C}$ for which both $X_\bullet[K]$ and $X_\bullet[L]$ exist, then ι induces a restriction map $X_\bullet[L] \to X_\bullet[K]$.

Variant A.5.1.12. If $X_\bullet$ is a simplicial object of an ∞-category $\mathcal{C}$, then one can define a $X_\bullet[L] \to X_\bullet[K]$ for *any* map of simplicial sets $K \to L$ (provided that $X_\bullet[K]$ and $X_\bullet[L]$ are defined). However, this generality will not be very important for what follows (in practice, we will generally be interested in the case where K and L are simplicial subsets of a standard simplex).

Remark A.5.1.13. Let $X_\bullet$ be a semisimplicial object of an ∞-category $\mathcal{C}$, let K be a nonsingular simplicial set, and suppose that K is a union of simplicial subsets $K_0, K_1 \subseteq K$ having intersection $K_{01} = K_0 \cap K_1$. Then we have a pushout diagram of simplicial sets

$$\begin{array}{ccc} \mathrm{N}(\mathbf{\Delta}^{\mathrm{nd}}_{K_{01}}) & \longrightarrow & \mathrm{N}(\mathbf{\Delta}^{\mathrm{nd}}_{K_0}) \\ \downarrow & & \downarrow \\ \mathrm{N}(\mathbf{\Delta}^{\mathrm{nd}}_{K_1}) & \longrightarrow & \mathrm{N}(\mathbf{\Delta}^{\mathrm{nd}}_{K}). \end{array}$$

Assume that the objects $X_\bullet[K_0], X_\bullet[K_1], X_\bullet[K_{01}] \in \mathcal{C}$ are well-defined. Using the results of §HTT.4.2.3 , we deduce that the object $X_\bullet[K]$ can be identified with the fiber product $X_\bullet[K_0] \times_{X_\bullet[K_{01}]} X_\bullet[K_1]$ (in particular, this fiber product exists in $\mathcal{C}$ if and only if $X_\bullet[K]$ is well-defined).

Remark A.5.1.14. Let $\mathcal{C}$ be an ∞-category which admits finite limits, let X be an object of $\mathcal{C}$, and let $\rho : \mathcal{C}_{/X} \to \mathcal{C}$ denote the forgetful functor. Let $U_\bullet$ be a semisimplicial object of $\mathcal{C}_{/X}$ (so that we can identify $U_\bullet$ with an *augmented* semisimplicial object of the ∞-category $\mathcal{C}$) and let $\rho U_\bullet$ denote the underlying semisimplicial object of $\mathcal{C}$. For any nonsingular simplicial set K, we have a canonical map $u : \rho(U_\bullet[K]) \to (\rho U_\bullet)[K]$ in the ∞-category $\mathcal{C}$. If K is weakly contractible, then the subdivision $\mathrm{N}(\mathbf{\Delta}_K)$ is also weakly contractible, so the comparison map u is an equivalence (Proposition HTT.4.4.2.9).

A.5.2 Kan Fibrations

We now study semisimplicial objects of an ∞-topos $\mathcal{X}$.

Definition A.5.2.1. Let $\mathcal{X}$ be an ∞-topos and let $f : X_\bullet \to Y_\bullet$ be a morphism of semisimplicial objects of $\mathcal{X}$. We will say that f is a *Kan fibration* if, for each $n > 0$ and each $0 \leqslant i \leqslant n$, the induced map

$$X_\bullet[\Delta^n] \to Y_\bullet[\Delta^n] \times_{Y_\bullet[\Lambda^n_i]} X_\bullet[\Lambda^n_i]$$

is an effective epimorphism in $\mathcal{X}$. We will say that f is a *trivial Kan fibration* if, for each $n \geqslant 0$, the natural map

$$X_\bullet[\Delta^n] \to Y_\bullet[\Delta^n] \times_{Y_\bullet[\partial\,\Delta^n]} X_\bullet[\partial\,\Delta^n]$$

is an effective epimorphism.

If $f : X_\bullet \to Y_\bullet$ is a morphism between simplicial objects of $\mathcal{X}$, we will say that f is a *Kan fibration* (*trivial Kan fibration*) if the underlying morphism of semisimplicial objects is a Kan fibration (trivial Kan fibration).

Example A.5.2.2. Let $f : X_\bullet \to Y_\bullet$ be a morphism of simplicial sets. Then f can be regarded as a morphism between simplicial objects of the ∞-topos $\mathcal{S}$ (which are levelwise discrete). The morphism f is a Kan fibration (trivial Kan fibration) in the sense of Definition A.5.2.1 if and only if it is a Kan fibration (trivial Kan fibration) in the usual sense.

Definition A.5.2.3. Let $\mathcal{X}$ be an ∞-topos and let $X_\bullet$ be a semisimplicial object of $\mathcal{X}$. We will say that $X_\bullet$ *satisfies the Kan condition* if, for each $n \geqslant 1$ and each $0 \leqslant i \leqslant n$, the map $X_n = X_\bullet[\Delta^n] \to X_\bullet[\Lambda^n_i]$ is an effective epimorphism in $\mathcal{X}$. We will say that $X_\bullet$ is a *hypercovering of* $\mathcal{X}$ if, for each $n \geqslant 0$, the canonical map $X_n = X_\bullet[\Delta^n] \to X_\bullet[\partial\,\Delta^n]$ is an effective epimorphism. More generally, if $X_\bullet$ is a semisimplicial object of the ∞-topos $\mathcal{X}_{/X}$ which is a hypercovering of $\mathcal{X}_{/X}$, then we will say that $X_\bullet$ is a *hypercovering of* X.

Remark A.5.2.4. Let $X_\bullet$ be a semisimplicial object of an ∞-topos $\mathcal{X}$. Then there is an essentially unique map $f : X_\bullet \to \mathbf{1}_\bullet$, where $\mathbf{1}_\bullet$ denotes the constant semisimplicial object whose value is a final object of $\mathcal{X}$. The morphism f is a Kan fibration if and only if $X_\bullet$ satisfies the Kan condition, and f is a trivial Kan fibration if and only if $X_\bullet$ is a hypercovering.

Example A.5.2.5. Let $\mathcal{X}$ be an ∞-topos and let $X_\bullet$ be a simplicial object of $\mathcal{X}$. Then $X_\bullet$ is a groupoid object of $\mathcal{X}$ (in the sense of Definition HTT.6.1.2.7) if and only if, for each $0 \leqslant i \leqslant n$, the natural map $X_n = X_\bullet[\Delta^n] \to X_\bullet[\Lambda^n_i]$ is an equivalence in $\mathcal{X}$. In particular, every groupoid object of $\mathcal{X}$ satisfies the Kan condition.

Remark A.5.2.6. Let $\mathcal{X}$ be an ∞-topos, let $X \in \mathcal{X}$ be an object, and let $\rho : \mathcal{X}_{/X} \to \mathcal{X}$ denote the forgetful functor. Let $U_\bullet$ be a semisimplicial object of $\mathcal{X}_{/X}$. Then $U_\bullet$ satisfies the Kan condition if and only if $\rho(U_\bullet)$ satisfies the Kan condition. This follows immediately from Remark A.5.1.14.

We now summarize some of the elementary properties of Definition A.5.2.1:

Proposition A.5.2.7. *Let $\mathcal{X}$ be an ∞-topos and suppose we are given a pullback diagram*

$$\begin{array}{ccc} X'_\bullet & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'_\bullet & \longrightarrow & Y_\bullet \end{array}$$

of semisimplicial objects of $\mathcal{X}$. If f is a Kan fibration, then f' is a Kan fibration. If f is a trivial Kan fibration, then f' is a trivial Kan fibration.

Proof. The desired result follows immediately from the fact that the class of effective epimorphisms in $\mathcal{X}$ is stable under the formation of pullback. □

Proposition A.5.2.8. *Let $\mathcal{X}$ be an ∞-topos and let $f : X_\bullet \to Y_\bullet$ be a trivial Kan fibration between semisimplicial objects of $\mathcal{X}$. For every finite nonsingular simplicial set L and every simplicial subset $K \subseteq L$, the natural map the induced map $\theta_{L,K} : X_\bullet[L] \to Y_\bullet[L] \times_{Y_\bullet[K]} X_\bullet[K]$ is an effective epimorphism in $\mathcal{X}$.*

Proof. Working by induction on the number of simplices of L which do not belong to K, we can reduce to the case where L is obtained from K by adding a single nondegenerate n-simplex whose boundary is contained in K. In this case, the map $\theta_{L,K}$ is a pullback of the map $\theta_{\Delta^n, \partial \Delta^n}$, which is an effective epimorphism by virtue of our assumption that f is a trivial Kan fibration. □

Corollary A.5.2.9. *Let $f : X_\bullet \to Y_\bullet$ be a morphism between semisimplicial objects of an ∞-topos $\mathcal{X}$. If f is a Kan fibration, then it is a trivial Kan fibration.*

Corollary A.5.2.10. *Let $X_\bullet$ be a semisimplicial object of an ∞-topos $\mathcal{X}$. If $X_\bullet$ is a hypercovering, then $X_\bullet$ satisfies the Kan condition.*

Proposition A.5.2.11. *Let $f : X_\bullet \to Y_\bullet$ and $g : Y_\bullet \to Z_\bullet$ be morphisms between semisimplicial objects of an ∞-topos $\mathcal{X}$. Then:*

(1) *If f and g are Kan fibrations, then $g \circ f$ is a Kan fibration.*

(1′) *If f is a trivial Kan fibration and $g \circ f$ is a Kan fibration, then g is a Kan fibration.*

(2) *If f and g are trivial Kan fibrations, then $g \circ f$ is a trivial Kan fibration.*

(2′) *If f and $g \circ f$ are trivial Kan fibrations, then g is a trivial Kan fibration.*

Proof. We will prove (2) and (2′); the proofs of (1) and (1′) are similar. Assume that f is a trivial Kan fibration; we wish to show that g is a trivial Kan fibration if and only if $g \circ f$ is a trivial Kan fibration. Let $n \geqslant 0$ be an integer.

Since f is a trivial Kan fibration, the natural map $X_\bullet[\partial\,\Delta^n] \to Y_\bullet[\partial\,\Delta^n]$ is an effective epimorphism in $\mathcal{X}$. Consequently, the map $Y_\bullet[\Delta^n] \to Z_\bullet[\Delta^n] \times_{Z_\bullet[\partial\,\Delta^n]} Y_\bullet[\partial\,\Delta^n]$ is an effective epimorphism if and only if the projection map $Y_\bullet[\Delta^n] \times_{Y_\bullet[\partial\,\Delta^n]} X_\bullet[\partial\,\Delta^n] \to Z_\bullet[\Delta^n] \times_{Z_\bullet[\partial\,\Delta^n]} X_\bullet[\partial\,\Delta^n]$ is an effective epimorphism. Inspecting the diagram

$$\begin{array}{ccc} X_\bullet[\Delta^n] & \longrightarrow & Y_\bullet[\Delta^n] \times_{Y_\bullet[\partial\,\Delta^n]} X_\bullet[\partial\,\Delta^n] \\ & \searrow \quad \swarrow & \\ & Z_\bullet[\Delta^n] \times_{Z_\bullet[\partial\,\Delta^n]} X_\bullet[\partial\,\Delta^n] & \end{array}$$

(in which the upper horizontal map is an effective epimorphism by virtue of our assumption on f), we deduce that the left vertical map is an effective epimorphism if and only if the right vertical map is an effective epimorphism. Allowing n to vary, we deduce that g is a trivial Kan fibration if and only if $g \circ f$ is a trivial Kan fibration. □

Corollary A.5.2.12. *Let $\mathcal{X}$ be an ∞-topos and let $f : X_\bullet \to Y_\bullet$ be a trivial Kan fibration between semisimplicial objects of $\mathcal{X}$. Then:*

(1) *The semisimplicial object $X_\bullet$ satisfies the Kan condition if and only if $Y_\bullet$ satisfies the Kan condition.*

(2) *The semisimplicial object $X_\bullet$ is a hypercovering of $\mathcal{X}$ if and only if $Y_\bullet$ is a hypercovering of $\mathcal{X}$.*

A.5.3 Connectivity of Trivial Kan Fibrations

We are now ready to state our first main result of this section:

Theorem A.5.3.1. *Let $f : X_\bullet \to X'_\bullet$ be a trivial Kan fibration between semisimplicial objects of an ∞-topos $\mathcal{X}$. Assume that one of the following conditions holds:*

(i) The morphism f can be promoted to a morphism between simplicial objects of $\mathcal{X}$.

(ii) The semisimplicial object $X'_\bullet$ satisfies the Kan condition.

Then f induces an ∞-connective map $|X_\bullet| \to |X'_\bullet|$.

Warning A.5.3.2. If $f : X_\bullet \to X'_\bullet$ does not satisfy either of the conditions (i) or (ii), then the conclusion of Theorem A.5.3.1 need not be valid. For example, suppose that $X'_\bullet$ is the semisimplicial set represented by the object $[1] \in \mathbf{\Delta}_s$ (so the simplices of $X'_\bullet$ are the *nondegenerate* simplices of Δ^1). Then the data of a map of semisimplicial sets $f : X_\bullet \to X'_\bullet$ is equivalent to the data of a bipartite graph: that is, a pair of sets V_0, V_1 (which can be defined as the inverse images of the two vertices of $X'_\bullet$) and another set E equipped with maps $V_0 \xleftarrow{d_0} E \xrightarrow{d_1} V_1$. The hypothesis that f is a trivial Kan fibration is equivalent to the requirement that the map $(d_0, d_1) : E \to V_0 \times V_1$ is surjective. If both V_0 and V_1 have more than one element, then the geometric realization $|X_\bullet|$ is *never* contractible.

Specializing Theorem A.5.3.1 to the case where $Y_\bullet$ is final we obtain the following:

Corollary A.5.3.3. *Let $X_\bullet$ be a semisimplicial object of an ∞-topos $\mathcal{X}$ which is a hypercovering. Then $|X_\bullet|$ is an ∞-connective object of $\mathcal{X}$.*

Let us now outline our strategy for proving Theorem A.5.3.1, which we will use to prove many other results in this section. The first observation is that we can reduce to proving Theorem A.5.3.1 in a single example: namely, we can assume that the ∞-topos $\mathcal{X}$ is *freely* generated by a Kan fibration (satisfying condition (i) or condition (ii)). We then observe that in this universal example, the ∞-topos $\mathcal{X}$ is locally coherent. It then follows from Deligne's theorem (Theorem A.4.0.5) that the hypercompletion $\mathcal{X}^{\mathrm{hyp}}$ has enough points, which allows us to reduce to the case $\mathcal{X} = \mathcal{S}$. In this case, we will "resolve" $X_\bullet$ and $X'_\bullet$ by semisimplicial sets, for which the desired result can be proven by combinatorial methods.

To handle case (ii) of Theorem A.5.3.1, we need the following result:

Proposition A.5.3.4. *Let $f : X_\bullet \to Y_\bullet$ be a Kan fibration between semisimplicial sets. Suppose that $Y_\bullet$ is the underlying semisimplicial set of a simplicial set $\overline{Y}_\bullet$. Then f can be lifted to a Kan fibration of simplicial sets $\overline{f} : \overline{X}_\bullet \to \overline{Y}_\bullet$.*

Taking $\overline{Y}_\bullet = \Delta^0$ in Proposition A.5.3.4, we obtain the following result of Rourke-Sanderson (see [176]):

Corollary A.5.3.5. *Let $X_\bullet$ be a semisimplicial set. If $X_\bullet$ satisfies the Kan condition, then $X_\bullet$ can be promoted to a simplicial set.*

We will prove Proposition A.5.3.4 using a slight variant of the proof of Corollary A.5.3.5 given in [113]. The key point is the following:

Lemma A.5.3.6. *Let $K_\bullet$ denote the semisimplicial subset of Δ^n given by the union of $\partial\,\Delta^n$ with the unique nondegenerate n-simplex of Δ^n (so that a k-simplex $\sigma : \Delta^k \to \Delta^n$ belongs to $X_\bullet$ if and only if σ is either an isomorphism or is not surjective). Then the inclusion $K_\bullet \hookrightarrow \Delta^n$ has the left lifting property with respect to all Kan fibrations between semisimplicial sets.*

Proof. For every nonsingular simplicial set $L_\bullet$, let $L_\bullet^{\mathrm{nd}}$ denote the semisimplicial subset of $L_\bullet$ spanned by the nondegenerate simplices. Let us say that a map of semisimplicial sets $A_\bullet \hookrightarrow B_\bullet$ is *anodyne* if it has the left lifting property with respect to all Kan fibrations. Using the small object argument (see Proposition HTT.A.1.2.5), we see that the collection of anodyne morphisms of semisimplicial sets is the smallest collection of morphisms which is closed under pushouts, retracts, transfinite composition, and contains all horn inclusions $(\Lambda^m_i)^{\mathrm{nd}} \hookrightarrow (\Delta^m)^{\hookrightarrow}$. We wish to prove that the inclusion

$$(\Delta^n)^{\mathrm{nd}} \amalg_{(\partial\,\Delta^n)^{\mathrm{nd}}} \partial\,\Delta^n \hookrightarrow \Delta^n$$

is anodyne. Our proof will proceed by induction on n.

For every semisimplicial set $A_\bullet$, let $A_\bullet^+$ denote the cone on $A_\bullet$: that is, the semisimplicial set given by

$$A_m = \begin{cases} \{\sigma \in A_m\} \amalg \{v\} & \text{if } m = 0 \\ \{\sigma \in A_m\} \amalg \{\sigma^+ | \sigma \in A_{m-1}\} & \text{if } m > 0, \end{cases}$$

whose face maps determined by the fact that they extend the face maps on $A_\bullet$ and are otherwise given by the formula

$$d_i\sigma^+ = \begin{cases} (d_i\sigma)^+ & \text{if } \sigma \in A_m \text{ and } 0 \leqslant i \leqslant m, m > 0 \\ \sigma & \text{if } \sigma \in A_m \text{ and } i = m+1 \\ v & \text{if } \sigma \in A_m \text{ and } 0 = i = m. \end{cases}$$

Let us say that a semisimplicial set $A_\bullet$ is *good* if the inclusion $A_\bullet \hookrightarrow A_\bullet^+$ is anodyne.

Our proof proceeds in several steps:

(*a*) For any monomorphism of semisimplicial sets $\iota : A_\bullet \hookrightarrow B_\bullet$, the induced map $\iota^+ : A_\bullet^+ \to B_\bullet^+$ is anodyne. Writing ι as a transfinite composition of simplex attachments, we can reduce to checking this in the case $B = (\Delta^m)^{\mathrm{nd}}$ and $A = (\partial\,\Delta^m)^{\mathrm{nd}}$, in which case a simple calculation shows that ι^+ can be identified with the horn inclusion $(\Lambda^{m+1}_{m+1})^{\mathrm{nd}} \hookrightarrow (\Delta^{m+1})^{\mathrm{nd}}$.

(b) Suppose we are given a monomorphism of semisimplicial sets $\iota : A_\bullet \hookrightarrow B_\bullet$, where $A_\bullet$ is good and there exists a retraction $r : B_\bullet^+ \to B_\bullet$ (this condition is satisfied, for example, when $B_\bullet = \Delta^n$). Then ι is a retract of the composite map $A_\bullet \to A_\bullet^+ \xrightarrow{\iota^+} B_\bullet^+$, and is therefore anodyne (by virtue of (a) and our assumption that $A_\bullet$ is good).

(c) Suppose that $\iota : A_\bullet \hookrightarrow B_\bullet$ is a monomorphism of semisimplicial sets. If ι is anodyne, then the induced map $\iota' : A_\bullet^+ \amalg_{A_\bullet} B_\bullet \to B_\bullet^+$ is also anodyne. To prove this, we can use Proposition HTT.A.1.2.5 to reduce to the case where ι is a horn inclusion $(\Lambda^m_i)^{\mathrm{nd}} \hookrightarrow (\Delta^m)^{\mathrm{nd}}$, in which case ι' can be identified with the horn inclusion $(\Lambda^{m+1}_i)^{\mathrm{nd}} \hookrightarrow (\Delta^{m+1})^{\mathrm{nd}}$.

(d) Let $\iota : A_\bullet \hookrightarrow B_\bullet$ be an anodyne morphism of simplicial sets. It follows from (c) that if $A_\bullet$ is good, then $B_\bullet$ is also good.

(e) For each $m \geqslant 0$, the simplex $B_\bullet = (\Delta^m)^{\mathrm{nd}})$ is good. To prove this, we use induction on m. When $m = 0$, the natural map $B_\bullet \hookrightarrow B_\bullet^+$ can be identified with the horn inclusion $(\Lambda^1_0)^{\mathrm{nd}} \hookrightarrow (\Delta^1)^{\mathrm{nd}}$, and is therefore anodyne. For $m > 0$, we can write $B_\bullet = A_\bullet^+$ where $A_\bullet = (\Delta^{m-1})^{\mathrm{nd}}$. It follows from our inductive hypothesis that $A_\bullet$ is good, so that the inclusion $A_\bullet \hookrightarrow B_\bullet$ is anodyne and therefore $B_\bullet$ is good by virtue of (c).

(f) Let $L_\bullet$ be a simplicial subset of $(\partial\,\Delta^n)$. We claim that the semisimplicial set $L_\bullet \amalg_{L_\bullet^{\mathrm{nd}}} (\Delta^n)^{\mathrm{nd}}$ is good. The proof proceeds by induction on the number of nondegenerate simplices of L. If $L = \varnothing$, the desired result follows from (e). Otherwise, we can write $L_\bullet$ as the union of a simplicial subset $L'_\bullet$ with a nondegenerate m-simplex of $\partial\,\Delta^n$ whose boundary belongs to $L'_\bullet$. We then have an inclusion

$$\iota : L'_\bullet \amalg_{L'^{\mathrm{nd}}_\bullet} (\Delta^n)^{\mathrm{nd}} \hookrightarrow L_\bullet \amalg_{L^{\mathrm{nd}}_\bullet} (\Delta^n)^{\mathrm{nd}}.$$

The domain of ι is good by our inductive hypothesis. Consequently, to show that the codomain of ι is good, it will suffice to show that ι is anodyne (by virtue of (d)). For this, we observe that ι is a pushout of the inclusion

$$\partial\,\Delta^m \amalg_{(\partial\,\Delta^m)^{\mathrm{nd}}} (\Delta^m)^{\mathrm{nd}} \hookrightarrow \Delta^m.$$

Since $m < n$, this map is anodyne by virtue of our (other) inductive hypothesis.

Applying (f) in the case $L_\bullet = \partial\,\Delta^n$, we deduce that the semisimplicial set $(\Delta^n)^{\mathrm{nd}} \amalg_{(\partial\,\Delta^n)^{\mathrm{nd}}} \partial\,\Delta^n$ is good. Combining this observation with (b), we conclude that the inclusion $(\Delta^n)^{\mathrm{nd}} \amalg_{(\partial\,\Delta^n)^{\mathrm{nd}}} \partial\,\Delta^n \hookrightarrow \Delta^n$ is anodyne, as desired. □

Proof of Proposition A.5.3.4. Let $\mathcal{C}$ denote the category whose objects are triples $(\overline{S}_\bullet, \overline{\alpha}, \beta)$ where $\overline{S}_\bullet$ is a simplicial set whose underlying semisimplicial set we will denote by $S_\bullet$,

$\overline{\alpha} : \overline{S}_\bullet \to \overline{Y}_\bullet$ is a map of simplicial sets whose underlying map of semisimplicial sets we denote by α, and $\beta : S_\bullet \to X_\bullet$ is a monomorphism of semisimplicial sets for which the diagram

$$\begin{array}{ccc} S_\bullet & \xrightarrow{\beta} & X_\bullet \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle f} & \\ & Y_\bullet & \end{array}$$

commutes. A morphism from $(\overline{S}_\bullet, \overline{\alpha}, \beta)$ to $(\overline{S}'_\bullet, \overline{\alpha}', \overline{\beta}')$ in the category $\mathcal{C}$ is a map $\overline{g} : \overline{S}_\bullet \to \overline{S}'_\bullet$ satisfying $\overline{\alpha} = \overline{\alpha}' \circ \overline{g}$ and $\beta = \beta' \circ g$ (where g denotes the map of semisimplicial sets underlying $\overline{g}$). Since a map of simplicial sets is determined by the underlying map of semisimplicial sets, the morphism $\overline{g}$ is unique if it exists; it follows that the category $\mathcal{C}$ is equivalent to a partially ordered set P (which we can identify with the set of isomorphism classes of objects of $\mathcal{C}$). Note that P is nonempty (since $\mathcal{C}$ contains an object $(\overline{S}_\bullet, \overline{\alpha}, \beta)$ where $\overline{S}_\bullet$ is empty) and that every directed subset of P has a least upper bound (since the category $\mathcal{C}$ admits filtered colimits). It follows from Zorn's lemma that P has a maximal element, which is represented by an object $(\overline{S}_\bullet, \overline{\alpha}, \beta) \in \mathcal{C}$. To complete the proof, it will suffice to show that $\beta : S_\bullet \hookrightarrow X_\bullet$ is an isomorphism of simplicial sets. Suppose otherwise: then there exists an n-simplex $\sigma \in X_n$ which does not belong to the image of β. Let us assume that n is chosen as small as possible, so that each face of σ belongs to the image of β.

Let $A_\bullet \subseteq \partial\,\Delta^n$ denote the semisimplicial subset consisting of the nondegenerate simplices of $\partial\,\Delta^n$, so that σ determines a map of semisimplicial sets $A_\bullet \to X_\bullet$ which factors uniquely as a composition $A_\bullet \xrightarrow{\rho} S_\bullet \xrightarrow{\beta} X_\bullet$. The map ρ extends uniquely to a map of simplicial sets $\overline{\rho}^+ : \partial\,\Delta^n \to \overline{S}_\bullet$. Let $\overline{S}'_\bullet$ denote the pushout $\overline{S}_\bullet \amalg_{\partial\,\Delta^n} \Delta^n$. The map $\overline{\alpha}$ and the n-simplex $f(\sigma) \in Y_n$ determine a map of simplicial sets $\overline{\alpha}^+ : \overline{S}'_\bullet \to \overline{Y}_\bullet$. Let $S'_\bullet$ denote the underlying semisimplicial subset of $\overline{S}'_\bullet$, which can be described as a pushout $S_\bullet \amalg_{\partial\,\Delta^n} \Delta^n$ in the category of semisimplicial sets. We claim that β can be extended to a map of semisimplicial sets $\beta' : S'_\bullet \to X_\bullet$ which carries the nondegenerate n-simplex of Δ^n to σ and fits into a commutative diagram

$$\begin{array}{ccc} S'_\bullet & \xrightarrow{\beta'} & X_\bullet \\ & {\scriptstyle\alpha'}\searrow \quad \swarrow{\scriptstyle f} & \\ & Y_\bullet. & \end{array}$$

To prove this, it suffices to solve a lifting problem of the form

$$\begin{array}{ccc} K_\bullet & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \Delta^n & \longrightarrow & Y_\bullet, \end{array}$$

where $K_\bullet$ is as in Lemma A.5.3.6; this is possibly by virtue of our assumption that f is a Kan fibration.

It follows from the maximality of $(\overline{S}_\bullet, \overline{\alpha}, \beta) \in \mathcal{C}$ that the triple $(\overline{S}'_\bullet, \overline{\alpha}', \beta')$ cannot be an object of $\mathcal{C}$: in other words, the map β' cannot be a monomorphism of semisimplicial sets. We can therefore choose a pair of k-simplices $\tau, \tau' \in S'_k$ such that $\tau \neq \tau'$ but $\beta'(\tau) = \beta'(\tau')$. Let us assume that k is chosen as small as possible, so that the 0th faces of τ and τ' are the same. Since β is injective, the simplices τ and τ' cannot both belong to S_k. We may therefore assume without loss of generality that τ is the image of a k-simplex Δ^n which does not belong to $\partial \Delta^n$. In this case, the nondegenerate n-simplex of Δ^n appears as a facet of τ, so that $\sigma \in X_n$ appears as a facet of $\beta'(\tau) = \beta'(\tau')$. It follows that τ' also cannot belong to S_k, so that we can identify τ' also with a k-simplex of Δ^n which does not belong to $\partial \Delta^n$. Then we can identify β and β' with surjective nondecreasing functions

$$\beta, \beta' : \{0 < 1 < \cdots < k-1 < k\} \to \{0 < 1 < \cdots < n-1 < n\}.$$

Since the 0th faces of β and β' agree, we have $\beta(i) = \beta'(i)$ for $i > 0$. However, the surjectivity of β and β' guarantees that $\beta(0) = 0 = \beta'(0)$. It follows that $\beta = \beta'$, contrary to our assumption. □

We now use Proposition A.5.3.4 to prove Theorem A.5.3.1 in the special case $\mathcal{X} = \mathcal{S}$:

Lemma A.5.3.7. *Let $f : X_\bullet \to X'_\bullet$ be a trivial Kan fibration between semisimplicial objects of $\mathcal{S}$ which satisfies one of the following conditions:*

(i) The morphism f can be promoted to a morphism between simplicial objects of $\mathcal{S}$.

(ii) The semisimplicial object $X'_\bullet$ satisfies the Kan condition.

Then the induced map $\theta : |X_\bullet| \to |X'_\bullet|$ is a homotopy equivalence.

Proof. Using Proposition HTT.4.2.4.4 , we may assume without loss of generality that f is obtained from a morphism $\overline{f} : \overline{X}_\bullet \to \overline{X}'_\bullet$ between semisimplicial objects of the ordinary category $\mathcal{S}\mathrm{et}_\Delta$ of simplicial sets. Moreover, in case (i), we may even assume that this is a morphism between *simplicial* objects of $\mathcal{S}\mathrm{et}_\Delta$. Without loss of generality, we may assume that $\overline{X}'_\bullet$ is Reedy fibrant and that $\overline{f}$ is a Reedy fibration: that is, for each $n \geqslant 0$, the natural maps

$$\overline{X}_\bullet[\Delta^n] \to \overline{X}_\bullet[\partial \Delta^n] \times_{\overline{X}'_\bullet[\partial \Delta^n]} \overline{X}'_\bullet[\Delta^n] \qquad \overline{X}'_\bullet[\Delta^n] \to \overline{X}'_\bullet[\partial \Delta^n]$$

are Kan fibrations of simplicial sets. It follows that for every nonsingular simplicial set K, the simplicial sets $\overline{X}_\bullet[K]$ and $\overline{X}'_\bullet[K]$ are Kan complexes that represent the objects $X_\bullet[K], X'_\bullet[K] \in \mathcal{S}$. It follows that for $n \geqslant 0$, the map of simplicial sets

$$\overline{X}_\bullet[\Delta^n] \to \overline{X}_\bullet[\partial \Delta^n] \times_{\overline{X}'_\bullet[\partial \Delta^n]} \overline{X}'_\bullet[\Delta^n]$$

is a Kan fibration which is surjective on connective components, and therefore surjective on m-simplices for all $m \geqslant 0$. In case (ii), the same argument shows that the maps $\overline{X}'_\bullet[\Delta^n] \to \overline{X}'_\bullet[\Lambda^n_i]$ are surjective on m-simplices for all $n > 0$ and $0 \leqslant i \leqslant n$.

Let $\overline{X}_{\bullet m}$ and $\overline{X}'_{\bullet m}$ denote the semisimplicial sets obtained by evaluating $\overline{X}_\bullet$ and $\overline{X}'_\bullet$ on the object $[m] \in \mathbf{\Delta}^{\mathrm{op}}$. Then we can identify θ with a homotopy colimit of the maps $\theta_m : |\overline{X}_{\bullet m}| \to |\overline{X}'_{\bullet m}|$; it will therefore suffice to show that each θ_m is a homotopy equivalence. The above argument shows that the map of semisimplicial sets $\overline{f}_m : \overline{X}_{\bullet m} \to \overline{X}'_{\bullet m}$ is a trivial Kan fibration. To show that $\overline{f}_m$ induces a homotopy equivalence of geometric realizations, it will suffice to show that $\overline{f}_m$ can be promoted to a map of simplicial sets. In case (i), this follows from our construction; in case (ii), it follows by from Corollary A.5.3.5 (applied to the semisimplicial set $\overline{X}'_{\bullet m}$) and Proposition A.5.3.4.

□

Lemma A.5.3.8. *Let $f : X_\bullet \to X'_\bullet$ be a trivial Kan fibration between semisimplicial objects of an ∞-topos $\mathcal{X}$. Then there exists a ∞-topos $\mathcal{Y}$ which is coherent and locally coherent, a trivial Kan fibration $g : Y_\bullet \to Y'_\bullet$ between semisimplicial objects of $\mathcal{Y}$, and a geometric morphism $\phi^* : \mathcal{Y} \to \mathcal{X}$ such that f is equivalent to $\phi^*(g)$. Moreover, if f satisfies condition (i) or (ii) of Theorem A.5.3.1, then we can arrange that g has the same property.*

Proof. Let $\mathcal{C}$ be an ∞-category freely generated by $\mathbf{\Delta}_s^{\mathrm{op}} \times \Delta^1$ under finite limits. More precisely, we choose a functor $u : \mathbf{\Delta}_s^{\mathrm{op}} \times \Delta^1 \to \mathcal{C}$, where $\mathcal{C}$ admits finite limits, which has the following universal property: for any ∞-category $\mathcal{D}$ which admits finite limits, composition with u induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathbf{\Delta}_s^{\mathrm{op}} \times \Delta^1, \mathcal{D})$, where $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve finite limits. The existence of u follows from Remark HTT.5.3.5.9 . Let us identify u with a morphism $Z_\bullet \to Z'_\bullet$ of the ∞-category $\mathcal{C}$. We will regard $\mathcal{C}$ as equipped with the coarsest Grothendieck topology τ for which each of the natural maps $Z_\bullet[\Delta^n] \to Z'_\bullet[\Delta^n] \times_{Z'_\bullet[\partial \Delta^n]} Z_\bullet[\partial \Delta^n]$ generates a covering sieve. It follows from Remark A.3.1.2 that the Grothendieck topology τ is finitary, so that $\mathcal{Y} = \mathcal{S}\mathrm{hv}(\mathcal{C})$ is a coherent and locally coherent ∞-topos (Proposition A.3.1.3). Let $g : Y_\bullet \to Y'_\bullet$ be the morphism of semisimplicial objects of $\mathcal{Y}$ which is obtained from u by applying the sheafified Yoneda embedding $\mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$. We can regard f as a functor $\mathbf{\Delta}_s^{\mathrm{op}} \times \Delta^1 \to \mathcal{X}$. Invoking the universal property of u, we can assume that f is given by the composition $\mathbf{\Delta}_s^{\mathrm{op}} \times \Delta^1 \xrightarrow{u} \mathcal{C} \xrightarrow{f'} \mathcal{X}$ where the functor f' preserves finite limits. Using Proposition HTT.6.2.3.20 , we see that f' induces a geometric morphism $\phi^* : \mathcal{Y} \to \mathcal{X}$ satisfying $\phi^*(g) \simeq f$. This proves the first assertion.

If f satisfies condition (i) of Theorem **??**, then we can arrange that g has the same property using a variant of the above construction where we replace $\mathbf{\Delta}_s^{\mathrm{op}} \times \Delta^1$ by $\mathbf{\Delta}^{\mathrm{op}} \times \Delta^1$. If f satisfies condition (ii) of Theorem **??**, then we can arrange that g has the same property by replacing τ by a slightly finer (but still finitary) topology, where the morphisms

$Z'_\bullet[\Delta^n] \to Z'_\bullet[\Lambda^n_i]$ are also coverings. □

Proof of Theorem A.5.3.1. Let $f : X_\bullet \to X'_\bullet$ be a trivial Kan fibration between semisimplicial objects of ∞-topos $\mathcal{X}$ which satisfies condition (i) or (ii) of Theorem A.5.3.1; we wish to prove that the induced map $|f| : |X_\bullet| \to |X'_\bullet|$ is ∞-connective. Using Lemma A.5.3.8, we can reduce to the case where the ∞-topos $\mathcal{X}$ is locally coherent. In this case, it follows from Theorem A.4.0.5 that the map $|f|$ is ∞-connective if and only if $\phi^*|f|$ is a homotopy equivalence for every geometric morphism $\phi^* : \mathcal{X} \to \mathcal{S}$. We may therefore assume without loss of generality that $\mathcal{X} = \mathcal{S}$, in which case the desired result follows from Lemma A.5.3.7. □

A.5.4 Pullbacks and Geometric Realizations

We now state our next main result:

Theorem A.5.4.1. *Let $\mathcal{X}$ be a hypercomplete ∞-topos. Suppose we are given a pullback square of semisimplicial objects σ :*

$$\begin{array}{ccc} X'_\bullet & \xrightarrow{g'} & X_\bullet \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'_\bullet & \xrightarrow{g} & Y_\bullet \end{array}$$

satisfying one of the following conditions:

(i) *The morphism f is a Kan fibration and σ can be promoted to a pullback square of simplicial objects of $\mathcal{X}$.*

(ii) *The morphisms f and g are Kan fibrations and $Y_\bullet$ satisfies the Kan condition.*

Then the diagram of geometric realizations

$$\begin{array}{ccc} |X'_\bullet| & \longrightarrow & |X_\bullet| \\ \downarrow & & \downarrow \\ |Y'_\bullet| & \longrightarrow & |Y_\bullet| \end{array}$$

is a pullback square in $\mathcal{X}$.

Warning A.5.4.2. In the statement of Theorem A.5.4.1, it is not enough to assume merely that f is a Kan fibration: for example, the geometric realization of the fiber of a Kan fibration f between semisimplicial sets need not be equivalent to the homotopy fiber of geometric realization of f (see Warning A.5.3.2).

Our proof of Theorem A.5.4.1 will follow the same basic outline as our proof of Theorem A.5.3.1: we will use Theorem A.4.0.5 to reduce to a statement about the homotopy theory of semisimplicial spaces, and then reduce further to a statement about the homotopy theory of semisimplicial sets. To carry out the second step, our basic mechanism is the following:

Lemma A.5.4.3. *Let $Y_\bullet$ be a simplicial (semisimplicial) object of $\mathcal{S}$. Then there exists a trivial Kan fibration $f : X_\bullet \to Y_\bullet$ of simplicial (semisimplicial) objects of $\mathcal{S}$, where $X_\bullet$ is levelwise discrete: that is, $X_\bullet$ is a simplicial (semisimplicial) set.*

Proof. We will prove Theorem A.5.4.3 for simplicial objects of $\mathcal{S}$; the proof in the semisimplicial case is analogous (and slightly easier). We will produce $X_\bullet$ as the union of a compatible sequence of functors $F_{\leqslant n} : \mathbf{\Delta}^{\mathrm{op}}_{\leqslant n} \to \mathcal{S}\mathrm{et} \subseteq \mathcal{S}$ and f as the limit of a compatible sequence of natural transformations $\alpha_n : F_{\leqslant n} \to Y_\bullet|_{\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n}}$. Let us assume that $n \geqslant 0$ and that we have constructed the a functor $F_{\leqslant n-1} : \mathbf{\Delta}^{\mathrm{op}}_{\leqslant n} \to \mathcal{S}\mathrm{et} \subseteq \mathcal{S}$ satisfying the following additional condition:

($*_{n-1}$) For each $m \leqslant n$, let L_m denote the mth latching object of $F_{\leqslant n-1}$: that is, the colimit $\varinjlim_{[k] \hookrightarrow [m], k<m} F_{\leqslant n-1}([k])$. For $m < n$, the natural map $L_m \to F_{\leqslant n-1}([m])$ is injective, so we can write $F_{\leqslant n-1}([m])$ as a disjoint union of L_m with an auxiliary set A_m.

It follows from ($*_{n-1}$) that the nth latching object of $F_{\leqslant n-1}$ can be identified with the disjoint union $\amalg_{[m] \hookrightarrow [n], m<n} A_m$, and is therefore discrete. Let M_n denote the nth matching object of $F_{\leqslant n-1}$. According to Corollary HTT.A.2.9.15 , to produce a functor $F_{\leqslant n}$ extending $F_{\leqslant n-1}$ and a natural transformation α_n extending α_{n-1}, it suffices to supply a commutative diagram

$$\begin{array}{ccc} & X_n & \\ \nearrow & & \searrow^{\phi} \\ L_n & \longrightarrow & Y_n \times_{Y_\bullet[\partial \Delta^n]} M_n \end{array}$$

where the bottom vertical maps is determined by $F_{\leqslant n-1}$ and α_{n-1}. This can be achieved by taking X_n to be the disjoint union of L_n with an auxiliary set A_n, and choosing the map ϕ so that $\phi|_{A_n}$ is surjective on connected components. By construction, we see that $F_{\leqslant n}$ satisfies ($*_n$) and that the maps $\{\alpha_n\}_{n \geqslant 0}$ induce a trivial Kan fibration $f : X_\bullet \to Y_\bullet$. □

Lemma A.5.4.4. *Let $\mathcal{X}$ be a hypercomplete ∞-topos and suppose we are given a pullback diagram σ :*

$$\begin{array}{ccc} X'_\bullet & \xrightarrow{g'} & X_\bullet \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'_\bullet & \xrightarrow{g} & Y_\bullet \end{array}$$

of semisimplicial objects of $\mathcal{X}$. Then there exists an equivalence $\sigma = \phi^* \widetilde{\sigma}$, where $\phi^* : \mathcal{Y} \to \mathcal{X}$ is a geometric morphism of ∞-topoi, $\widetilde{\sigma}$ is a pullback diagram of semisimplicial objects of $\mathcal{Y}$, and the ∞-topos $\mathcal{Y}$ is locally coherent and hypercomplete. Moreover, if σ satisfies condition (i) or (ii) of Theorem A.5.4.1, then we can arrange that $\widetilde{\sigma}$ has the same property.

Proof. To fix ideas, let us assume that σ satisfies condition (ii) of Theorem A.5.4.1; we will show that we can choose an equivalence $\sigma \simeq \phi^* \widetilde{\sigma}$ where $\widetilde{\sigma}$ also satisfies (ii) (the proof for the case where σ satisfies (i) is similar). We proceed as in the proof of Lemma A.5.3.8. Let $\mathcal{C}$ be an ∞-category which is freely generated by $\mathbf{\Delta}_s^{\mathrm{op}} \times \Lambda^2_2$ under finite limits, so that there is a functor $u : \mathbf{\Delta}_s^{\mathrm{op}} \times \Lambda^2_2 \to \mathcal{C}$ having the following universal property: for any ∞-category $\mathcal{D}$ which admits finite limits, composition with u induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathbf{\Delta}_s^{\mathrm{op}} \times \Lambda^2_2, \mathcal{D})$, where $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve finite limits. The existence of u follows from Remark HTT.5.3.5.9 . The map u can be identified with a diagram $\overline{Y}'_\bullet \xrightarrow{\overline{g}} \overline{Y}_\bullet \xleftarrow{\overline{f}} \overline{X}_\bullet$ of semisimplicial objects of $\mathcal{C}$ which we can complete to a pullback diagram $\overline{\sigma}$:

$$\begin{array}{ccc} \overline{X}'_\bullet & \xrightarrow{\overline{g}'} & \overline{X}_\bullet \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \overline{f}} \\ \overline{Y}'_\bullet & \xrightarrow{\overline{g}} & \overline{Y}_\bullet . \end{array}$$

Let us regard $\mathcal{C}$ as equipped with the coarsest Grothendieck topology for which the natural maps

$$\overline{X}_\bullet[\Delta^n] \to \overline{X}_\bullet[\Lambda^n_i] \times_{\overline{Y}_\bullet[\Lambda^n_i]} \overline{Y}_\bullet[\Delta^n] \quad \overline{Y}_\bullet[\Delta^n] \to \overline{Y}_\bullet[\Lambda^n_i] \quad \overline{Y}'_\bullet[\Delta^n] \to \overline{Y}'_\bullet[\Lambda^n_i] \times_{\overline{Y}_\bullet[\Lambda^n_i]} \overline{Y}_\bullet[\Delta^n]$$

are coverings. It follows from Remark A.3.1.2 that this Grothendieck topology is finitary, so Propositions A.3.1.3 and A.2.2.2 guarantee that the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ is coherent.

Let $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ denote the composition of the Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C})$ with a left adjoint to the inclusion $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}} \subseteq \mathcal{P}(\mathcal{C})$. It follows from the universal property of u that there is an essentially unique left exact functor $\rho : \mathcal{C} \to \mathcal{X}$ satisfying $\rho(\overline{\sigma}) \simeq \sigma$. Since $\mathcal{X}$ is hypercomplete and σ satisfies condition (ii) of Theorem A.5.4.1, Proposition HTT.6.2.3.20 shows that ρ admits an essentially unique factorization as a composition $\mathcal{C} \xrightarrow{j} \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}} \xrightarrow{\phi^*} \mathcal{X}$ where ϕ^* is a geometric morphism. We now complete the proof by taking $\widetilde{\sigma} = j(\overline{\sigma})$. □

Proof of Theorem A.5.4.1. Let $\mathcal{X}$ be a hypercomplete ∞-topos and suppose we are given a commutative diagram of semisimplicial objects

$$\begin{array}{ccc} X'_\bullet & \xrightarrow{g'} & X_\bullet \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'_\bullet & \xrightarrow{g} & Y_\bullet \end{array}$$

which satisfies either (i) or (ii); we wish to show that the diagram of geometric realizations

$$\begin{array}{ccc} |X'_\bullet| & \longrightarrow & |X_\bullet| \\ \downarrow & & \downarrow \\ |Y'_\bullet| & \longrightarrow & |Y_\bullet| \end{array}$$

is a pullback square in $\mathcal{X}$. Using Lemma A.5.4.4, we can reduce to the case where $\mathcal{X}$ is locally coherent. Using Theorem A.4.0.5, we can reduce further to the case where $\mathcal{X} = \mathcal{S}$.

Applying Lemma A.5.4.3, we can choose a semisimplicial set $\overline{Y}_\bullet$ and a trivial Kan fibration $\overline{Y}_\bullet \to Y_\bullet$. Applying Lemma A.5.4.3 two more times, we can choose semisimplicial sets $\overline{X}_\bullet$ and $\overline{Y}'_\bullet$ equipped with trivial Kan fibrations

$$\overline{X}_\bullet \to \overline{Y}_\bullet \times_{Y_\bullet} X_\bullet \qquad \overline{Y}'_\bullet \to \overline{Y}_\bullet \times_{Y_\bullet} Y'_\bullet.$$

Set $\overline{X}'_\bullet = \overline{X}_\bullet \times_{\overline{Y}_\bullet} \overline{Y}'_\bullet$, so that we have a pullback square $\overline{\sigma}$:

$$\begin{array}{ccc} \overline{X}'_\bullet & \xrightarrow{\overline{g}'} & \overline{X}_\bullet \\ \downarrow{\scriptstyle \overline{f}'} & & \downarrow{\scriptstyle \overline{f}} \\ \overline{Y}'_\bullet & \xrightarrow{\overline{g}} & \overline{Y}_\bullet \end{array}$$

of semisimplicial sets. Note that if σ satisfies (i), then we can arrange that $\overline{\sigma}$ is a diagram of simplicial sets.

Applying Propositions A.5.2.11 and A.5.2.7, we see that the natural map $\overline{\sigma} \to \sigma$ induces trivial Kan fibrations

$$\overline{X}'_\bullet \to X'_\bullet \qquad \overline{X}_\bullet \to X_\bullet$$
$$\overline{Y}'_\bullet \to Y'_\bullet \qquad \overline{Y}_\bullet \to Y_\bullet$$

and therefore induces an equivalence on geometric realizations (Lemma A.5.3.7). We are therefore reduced to showing that the diagram of spaces

$$\begin{array}{ccc} |\overline{X}'_\bullet| & \longrightarrow & |\overline{X}_\bullet| \\ \downarrow & & \downarrow \\ |\overline{Y}'_\bullet| & \longrightarrow & |\overline{Y}_\bullet| \end{array}$$

is a pullback square in $\mathcal{S}$. To prove this, it will suffice to show that $\overline{\sigma}$ can be promoted to a diagram of simplicial sets :such a promotion is automatically a homotopy pullback square of simplicial sets, since the model structure on $\mathcal{S}\mathrm{et}_\Delta$ is right proper (the map $\overline{f}$ is a Kan fibration, since it is the composition of a trivial Kan fibration with a pullback of f). In case (i), this promotion is automatic; in case (ii), it can be obtained by first applying Corollary A.5.3.5 to the semisimplicial set $\overline{Y}_\bullet$, and then applying Proposition A.5.3.4 to the morphisms $\overline{f}$ and $\overline{g}$. □

A.5.5 Coherence and the Kan Condition

Let $X_\bullet$ be a simplicial set. If $X_\bullet$ satisfies the Kan condition and each of the sets X_n is finite, then the homotopy groups $\pi_k(X_\bullet, x)$ are finite for all $k \geqslant 0$ and all $x \in X_0$. In other words, the geometric realization $|X_\bullet|$ is a coherent object of the ∞-topos $\mathcal{S}$. We now generalize this observation to semisimplicial objects of an arbitrary ∞-topos:

Theorem A.5.5.1. *Let $X_\bullet$ be a semisimplicial object of an ∞-topos $\mathcal{X}$. Assume that $\mathcal{X}$ is locally n-coherent for all $n \geqslant 0$, that $X_\bullet$ satisfies the Kan condition, and that each X_k is a coherent object of $\mathcal{X}$. Then the geometric realization $|X_\bullet|$ is also a coherent object of $\mathcal{X}$.*

The proof of Theorem A.5.5.1 will require some preliminaries.

Notation A.5.5.2. Let $X_\bullet$ be a semisimplicial object of an ∞-category $\mathcal{C}$. We let $X_\bullet^+$ denote the semisimplicial object of $\mathcal{C}$ obtained by composing $X_\bullet$ with the translation functor

$$s_+ : \mathbf{\Delta}_s \to \mathbf{\Delta}_s \qquad [n] \mapsto [n] \star [0] \simeq [n+1].$$

Similarly, we let $X_\bullet^-$ denote the semisimplicial object of $\mathcal{C}$ given by composing $X_\bullet$ with the functor

$$s_- : \mathbf{\Delta}_s \to \mathbf{\Delta}_s \qquad [n] \mapsto [0] \star [n] \simeq [n+1].$$

Note that the inclusion maps $[0] \star [n] \hookleftarrow [n] \hookrightarrow [n] \star [0]$ determine maps of semisimplicial objects $X_\bullet^- \to X_\bullet \leftarrow X_\bullet^+$. Similarly, the inclusions $[0] \star [n] \hookleftarrow [0] \hookrightarrow [n] \star [0]$ induce maps $|X_\bullet^-| \to X_0 \leftarrow |X_\bullet^+|$ in the ∞-category $\mathcal{C}$ (provided that the geometric realizations $|X_\bullet^-|$ and $|X_\bullet^+|$ are well-defined).

Lemma A.5.5.3. *Let $\mathcal{X}$ be an ∞-topos and let $X_\bullet$ be a semisimplicial object of $\mathcal{X}$. If $X_\bullet$ satisfies the Kan condition, then the natural maps $X_\bullet^- \to X_\bullet \leftarrow X_\bullet^+$ are Kan fibrations.*

Proof. Let $n > 0$ and $0 \leqslant i \leqslant n$. Using Remark A.5.1.13, we can identify the canonical map

$$X_\bullet^-[\Delta^n] \to X_\bullet^-[\Lambda^n_i] \times_{X_\bullet[\Lambda^n_i]} X_\bullet[\Delta^n]$$

with the restriction map $X_\bullet[\Delta^{n+1}] \to X_\bullet[\Lambda^{n+1}_{i+1}]$, which is an effective epimorphism by virtue of our assumption that $X_\bullet$ satisfies the Kan condition. The proof for $X_\bullet^+$ is similar. □

Lemma A.5.5.4. *Let $\mathcal{X}$ be an ∞-topos and let $X_\bullet$ be a semisimplicial object of $\mathcal{X}$ which satisfies the Kan condition. Then the natural maps $|X_\bullet^-| \to X_0 \leftarrow |X_\bullet^+|$ are ∞-connective.*

Proof. We will show that the natural map $\theta : |X_\bullet^-| \to X_0$ is ∞-connective; the proof for $X_\bullet^+$ is similar. Applying Lemma A.5.3.8 to the identity map $\mathrm{id} : X_\bullet \to X_\bullet$, we can reduce to the case where the ∞-topos $\mathcal{X}$ is coherent. Using Theorem A.4.0.5, we can further reduce to the case $\mathcal{X} = \mathcal{S}$. In this case, Proposition HTT.4.2.4.4 implies that we can obtain $X_\bullet$ from a semisimplicial object $\overline{X}_\bullet$ in the ordinary category $\mathcal{S}\mathrm{et}_\Delta$ of simplicial sets. Without loss of generality, we may assume that $\overline{X}_\bullet$ is Reedy fibrant. It follows that for every nonsingular simplicial set K, the simplicial set $\overline{X}_\bullet[K]$ is a Kan complex which represents the object $X_\bullet[K] \in \mathcal{S}$. Since $X_\bullet$ satisfies the Kan condition, we conclude that each of the maps $\overline{X}_\bullet[\Delta^n] \to \overline{X}_\bullet[\Lambda^n_i]$ is a Kan fibration of simplicial sets which is surjective on connective components, and is therefore surjective on m-simplices for each $m \geqslant 0$.

For $m \geqslant 0$, let $\overline{X}_{\bullet m}$ denote the semisimplicial set obtained by evaluating $\overline{X}_\bullet$ on the object $[m] \in \mathbf{\Delta}^{\mathrm{op}}$. The preceding argument shows that each $\overline{X}_{\bullet m}$ satisfies the Kan condition. The map θ can be written as a homotopy colimit of maps $\theta_m : |\overline{X}^-_{\bullet m}| \to \overline{X}_{0m}$. It will therefore suffice to show that each θ_m is a homotopy equivalence. In other words, we may replace $X_\bullet$ with $\overline{X}_{\bullet m}$ and thereby reduce to the case where $X_\bullet$ is a semisimplicial set. Using Corollary A.5.3.5, we can promote $X_\bullet$ to a simplicial set, in which case the desired result follows from the fact that the augmented simplicial set $\mathbf{\Delta}_+ \xrightarrow{[n] \mapsto [0] \star [n]} \mathbf{\Delta} \xrightarrow{X_\bullet} \mathcal{S}\mathrm{et}$ is split. □

Proof of Theorem A.5.5.1. Let $\mathcal{X}$ be an ∞-topos which is locally n-coherent for all $n \geqslant 0$, let $X_\bullet$ be a semisimplicial object of $\mathcal{X}$ which satisfies the Kan condition, and assume that each X_k is a coherent object of $\mathcal{X}$. We wish to prove that the geometric realization $|X_\bullet|$ is also a coherent object of $\mathcal{X}$. Using Proposition A.2.2.2, we can replace $\mathcal{X}$ by its hypercompletion and thereby reduce to the case where $\mathcal{X}$ is hypercomplete.

We will show that $|X_\bullet|$ is n-coherent for each $n \geqslant 0$. The proof proceeds by induction on n. In the case $n = 0$, we must show that $|X_\bullet|$ is quasi-compact. This follows from the fact that there exists an effective epimorphism $X_0 \to |X_\bullet|$, where X_0 is quasi-compact. We now carry out the inductive step. Let $X_\bullet^+$ and $X_\bullet^-$ be defined as in Notation A.5.5.2. Combining Lemma A.5.5.3 with Theorem A.5.4.1, we deduce that the diagram σ :

$$\begin{array}{ccc} |X_\bullet^- \times_{X_\bullet} X_\bullet^+| & \xrightarrow{f} & |X_\bullet^-| \\ \downarrow & & \downarrow g \\ |X_\bullet^+| & \xrightarrow{f'} & |X_\bullet| \end{array}$$

is a pullback diagram in $\mathcal{X}$. It follows from Lemma A.5.5.3 that $X_\bullet^- \times_{X_\bullet} X_\bullet^+$ satisfies the Kan condition. Applying our inductive hypothesis, we deduce that upper left corner of the diagram σ is an $(n-1)$-coherent object of $\mathcal{X}$. The upper right corner is equivalent to

X_0 (Lemma A.5.5.4) and is therefore n-coherent. Applying Example A.2.1.2, we deduce that the morphism f is relatively $(n-1)$-coherent. Since g is an effective epimorphism, it follows that f' is also relatively $(n-1)$-coherent (Corollary A.2.1.5). Applying Proposition A.2.1.3 to the morphism f' (and using the n-coherence of $|X^+_\bullet| \simeq X_0$), we deduce that $|X_\bullet|$ is n-coherent, as desired. □

A.5.6 Triviality of Kan Fibrations

We now formulate a refinement of Theorem A.5.3.1:

Theorem A.5.6.1. *Let $f : X_\bullet \to Y_\bullet$ be a morphism of semisimplicial objects of an ∞-topos $\mathcal{X}$, where $Y_\bullet$ satisfies the Kan condition. The following conditions are equivalent:*

(1) *The morphism f is a trivial Kan fibration.*

(2) *The morphism f is a Kan fibration and the induced map $|X_\bullet| \to |Y_\bullet|$ is ∞-connective.*

Remark A.5.6.2. One can show that the conclusion of Theorem A.5.6.1 remains valid if we replace the assumption that $Y_\bullet$ satisfies the Kan condition by the assumption that f can be promoted to a morphism of simplicial objects. However, this variant of Theorem A.5.6.1 requires a proof which is somewhat different from the one we give below.

Applying Theorem A.5.6.1 in the case where $Y_\bullet$ is a final object of $\mathrm{Fun}(\mathbf{\Delta}_s^{\mathrm{op}}, \mathcal{S})$, we obtain the following:

Corollary A.5.6.3. *Let $X_\bullet$ be a semisimplicial object of an ∞-topos $\mathcal{X}$. The following conditions are equivalent:*

(1) *The semisimplicial object $X_\bullet$ is a hypercovering of $\mathcal{X}$.*

(2) *The semisimplicial object $X_\bullet$ satisfies the Kan condition and the geometric realization $|X_\bullet|$ is ∞-connective.*

Combining Corollary A.5.6.3 with Remark A.5.2.6, we obtain the following:

Corollary A.5.6.4. *Let $X_\bullet$ be a semisimplicial object of an ∞-topos $\mathcal{X}$. Then $X_\bullet$ satisfies the Kan condition if and only if it is a hypercovering of its geometric realization $|X_\bullet|$.*

The proof of Theorem A.5.6.1 will again proceed by reduction to the case $\mathcal{X} = \mathcal{S}$ by means of Deligne's theorem.

Lemma A.5.6.5. *Let $f : X_\bullet \to X'_\bullet$ be a morphism of semisimplicial objects of a hypercomplete ∞-topos $\mathcal{X}$. Assume that $X'_\bullet$ satisfies the Kan condition, f is a Kan fibration, and the geometric realization $|f| : |X_\bullet| \to |X'_\bullet|$ is an equivalence. Then we can write $f = \phi^*(g)$, where $\phi^* : \mathcal{Y} \to \mathcal{X}$ is a geometric morphism of ∞-topoi, $g : Y_\bullet \to Y'_\bullet$ is a Kan fibration between semisimplicial objects of $\mathcal{Y}$, where $Y'_\bullet$ satisfies the Kan condition and $|g|$ is an equivalence.*

Proof. We begin as in Lemma A.5.3.8. Choose a functor $u : \mathbf{\Delta}_s^{\mathrm{op}} \times \Delta^1 \to \mathcal{C}$ which freely generates $\mathcal{C}$ under finite limits, and regard u as a morphism $Z_\bullet \to Z'_\bullet$ between semisimplicial objects of $\mathcal{C}$. Let us regard $\mathcal{C}$ as equipped with the coarsest Grothendieck topology for which each of the natural maps

$$Z_\bullet[\Delta^n] \to Z'_\bullet[\Delta^n] \times_{Z'_\bullet[\Lambda^n_i]} Z_\bullet[\Lambda^n_i] \qquad Z'_\bullet[\Delta^n] \to Z'_\bullet[\Lambda^n_i]$$

is a covering. Let $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ denote the composition of the Yoneda embedding with a left adjoint to the inclusion $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}} \hookrightarrow \mathcal{P}(\mathcal{C})$. The universal property of u implies that we can write f as a composition $\mathbf{\Delta}_s^{\mathrm{op}} \times \Delta^1 \xrightarrow{u} \mathcal{C} \xrightarrow{f'} \mathcal{X}$ where the functor f' preserves finite limits. Since f is a Kan fibration and $X'_\bullet$ satisfies the Kan condition, it follows from Proposition HTT.6.2.3.20 (together with the hypercompleteness of $\mathcal{X}$) that f' induces a geometric morphism $\psi^* : \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}} \to \mathcal{X}$ satisfying $f \simeq \psi^*(j(u))$.

Since the Grothendieck topology on $\mathcal{C}$ is finitary (Remark A.3.1.2), it follows from Propositions A.3.1.3 and A.2.2.2 that the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ is locally coherent and that the essential image of j consists of coherent objects of $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$. Using Theorem A.5.5.1, we deduce that the geometric realizations $D = |jZ_\bullet|$ and $D' = |jZ'_\bullet|$ are coherent objects of $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$. Choose a full subcategory $\mathcal{D} \subseteq \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ containing D and D' which satisfies hypotheses (a) through (d) of Proposition A.3.4.2. Let τ denote the (finitary) Grothendieck topology described in Proposition A.3.4.2, so that the inclusion $\iota : \mathcal{D} \hookrightarrow \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ extends to an equivalence $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}} \simeq \mathcal{S}\mathrm{hv}_\tau(\mathcal{D})^{\mathrm{hyp}}$. For each $n \geqslant -1$, we let $\delta_n : D \to D' \times_{D'^{S^n}} D^{S^n}$ denote the $(n+1)$st iterated diagonal of the canonical map $D \to D'$. Let τ' denote the coarsest Grothendieck topology on $\mathcal{D}$ which refines τ and has the property that each of the morphisms δ_n generates a covering sieve. Then τ' is also a finitary Grothendieck topology (Remark A.3.1.2) so the full subcategory $\mathcal{S}\mathrm{hv}_{\tau'}(\mathcal{D})^{\mathrm{hyp}} \subseteq \mathcal{S}\mathrm{hv}_\tau(\mathcal{D})^{\mathrm{hyp}}$ is again a locally coherent ∞-topos (Propositions A.3.1.3 and A.2.2.2). Since f induces an equivalence of geometric realizations, it follows that the functor ψ^* carries each δ_n to an equivalence in $\mathcal{X}$ and therefore factors through the localization functor $L : \mathcal{S}\mathrm{hv}_\tau(\mathcal{D})^{\mathrm{hyp}} \to \mathcal{S}\mathrm{hv}_{\tau'}(\mathcal{D})^{\mathrm{hyp}}$. By construction, L carries each δ_n to an effective epimorphism in $\mathcal{S}\mathrm{hv}_{\tau'}(\mathcal{D})^{\mathrm{hyp}}$. It follows that the induced map $LD \to LD'$ is ∞-connective and therefore an equivalence (since $\mathcal{S}\mathrm{hv}_{\tau'}(\mathcal{D})^{\mathrm{hyp}}$ is hypercomplete). We now complete the proof by setting $\mathcal{Y} = \mathcal{S}\mathrm{hv}_{\tau'}(\mathcal{D})^{\mathrm{hyp}}$ and $g = L(j(u))$. □

Proof of Theorem A.5.6.1. The implication (1) ⇒ (2) follows from Theorem A.5.3.1 and Corollary A.5.2.9. We will prove the converse. Suppose that $f : X_\bullet \to Y_\bullet$ is a Kan fibration between semisimplicial objects of $\mathcal{X}$, that $Y_\bullet$ satisfies the Kan condition, and that the induced map $|f| : |X_\bullet| \to |Y_\bullet|$ is ∞-connective; we wish to show that f is a trivial Kan fibration. Replacing $\mathcal{X}$ by its hypercompletion, we can assume that $\mathcal{X}$ is hypercomplete and that $|f|$ is an equivalence. By virtue of Lemma A.5.3.8, we may assume without loss of generality that $\mathcal{X}$ is a coherent ∞-topos. By virtue of Theorem A.4.0.5, we may further reduce to the case where $\mathcal{X} = \mathcal{S}$.

Using Lemma A.5.4.3, we can choose a trivial Kan fibration $u : Y'_\bullet \to Y_\bullet$ where $Y'_\bullet$ is a semisimplicial set. Applying Lemma A.5.4.3 again, we can choose a trivial Kan fibration $X'_\bullet \to X_\bullet \times_{Y_\bullet} Y'_\bullet$ where $X'_\bullet$ is a semisimplicial set. We then have a commutative diagram

$$\begin{array}{ccc} X'_\bullet & \xrightarrow{f'} & Y'_\bullet \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ X_\bullet & \xrightarrow{f} & Y_\bullet. \end{array}$$

Each term in this diagram satisfies the Kan condition and the vertical maps are trivial Kan fibrations. It follows from Theorem A.5.3.1 that the induced maps $|v| : |X'_\bullet| \to |X_\bullet|$ and $|u| : |Y'_\bullet| \to |Y_\bullet|$ are homotopy equivalences. Since the geometric realization of f is a homotopy equivalence, it follows that the geometric realization of f' is a homotopy equivalence. Using Corollary A.5.3.5, we can promote $Y'_\bullet$ to a simplicial set. The map f' is the composition of a trivial Kan fibration with a pullback of f, and is therefore a Kan fibration of semisimplicial sets. Applying Proposition A.5.3.4, we can promote f' to a map of simplicial sets $X'_\bullet \to Y'_\bullet$. Since this map induces a homotopy equivalence of geometric realizations, it is a trivial Kan fibration. It follows that the the composition $u \circ f' \simeq f \circ v$ is also a trivial Kan fibration. Since v is a trivial Kan fibration, Proposition A.5.2.11 implies that f is also a trivial Kan fibration. □

A.5.7 Hypercompleteness

We now describe an application of Theorem **??** to the problem of recognizing *hypercomplete* sheaves in the setting of §A.3. First, we need a bit more terminology.

Definition A.5.7.1. Let $\mathbf{\Delta}_{s,+}$ denote the category whose objects are linearly ordered sets of the form $[n] = \{0 < 1 < \ldots < n\}$ for $n \geqslant -1$, and whose morphisms are strictly increasing functions. If $\mathcal{C}$ is an ∞-category, we will refer to a functor $X_\bullet : \mathbf{\Delta}_{s,+}^{\mathrm{op}} \to \mathcal{C}$ as an *augmented semisimplicial object* of $\mathcal{C}$. If $\mathcal{C}$ admits finite limits, then for each $n \geqslant 0$ we can associate to $X_\bullet$ an nth matching object

$$M_n(X) = \varprojlim_{[m] \hookrightarrow [n]} X_m,$$

where the limit is taken over all injective maps $[m] \to [n]$ such that $m < n$.

Let S be a collection of morphisms in $\mathcal{C}$. We will say that an augmented semisimplicial object $X_\bullet : \mathbf{\Delta}_{s,+}^{\mathrm{op}} \to \mathcal{C}$ is an *S-hypercovering* if, for each $n \geqslant 0$, the canonical map $X_n \to M_n(X)$ belongs to S.

Proposition A.5.7.2. *Let $\mathcal{C}$ be an ∞-category and S a collection of morphisms in $\mathcal{C}$. Assume that $\mathcal{C}$ and S satisfy the conditions of Proposition A.3.2.1 and condition (e) of Proposition A.3.3.1. Let $\mathcal{D}$ be an arbitrary ∞-category and $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ a functor. Then $\mathscr{F}$ is a hypercomplete $\mathcal{D}$-valued sheaf on $\mathcal{C}$ if and only if the following conditions are satisfied:*

(1) *The functor $\mathscr{F}$ preserves finite products.*

(2) *Let $X_\bullet : \mathbf{\Delta}^{\mathrm{op}}_{s,+} \to \mathcal{C}$ be an S-hypercovering. Then the composite map*

$$\mathbf{\Delta}_{s,+} \xrightarrow{X_\bullet} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathcal{D}$$

is a limit diagram.

Proof of Proposition A.5.7.2. As in the proof of Proposition A.3.3.1, we may assume without loss of generality that $\mathcal{D} = \mathcal{S}$. We first prove the "only if" direction. Assume that $\mathscr{F}$ is a hypercomplete sheaf. Condition (1) follows from Proposition A.3.3.1. To prove (2), let $F : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ denote the composition of the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ with the sheafification functor $\mathcal{P}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\mathcal{C})$, and let $L : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ be a left adjoint to the inclusion. It will suffice to show that $L \circ F \circ X_\bullet$ is a colimit diagram in $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$: in other words, that $X_\bullet$ exhibits $LF(X_{-1})$ as a colimit of the diagram $\{LFX_n\}_{n \geqslant 0}$. This follows immediately from Theorem A.5.3.1, applied in the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})_{/FX_{-1}}$.

Now suppose that (1) and (2) are satisfied. Proposition A.3.3.1 guarantees that $\mathscr{F}$ is a sheaf on $\mathcal{C}$; we wish to prove that $\mathcal{C}$ is hypercomplete. Choose an ∞-connective morphism $\alpha : \mathscr{F} \to \mathscr{G}$ in $\mathcal{S}\mathrm{hv}(\mathcal{C})$, where $\mathscr{G}$ is hypercomplete (and therefore satisfies conditions (1) and (2)). We wish to show that α is an equivalence. To prove this, it will suffice to verify the following:

(∗) Let $\alpha : \mathscr{F} \to \mathscr{G}$ be an ∞-connective morphism in $\mathcal{S}\mathrm{hv}(\mathcal{C})$, where $\mathscr{F}$ and $\mathscr{G}$ both satisfy (2). Then α is an equivalence.

To prove (∗), we will show that for every object $C \in \mathcal{C}$ and each $n \geqslant 0$, the map of spaces $\alpha_C : \mathscr{F}(C) \to \mathscr{G}(C)$ is n-connective. The proof proceeds by induction on n. If $n > 0$, then the inductive hypothesis guarantees that α_C is 0-connective; it therefore suffices to show that the diagonal map $\mathscr{F}(C) \to \mathscr{F}(C) \times_{\mathscr{G}(C)} \mathscr{F}(C)$ is $(n-1)$-connective, which also follows from the inductive hypothesis. It therefore suffices to treat the case $n = 0$: that is, we must show that the map $\mathscr{F}(C) \to \mathscr{G}(C)$ is surjective on connected components. Replacing $\mathcal{C}$ by $\mathcal{C}_{/C}$, we may assume that C is a final object of $\mathcal{C}$, so that a point $\eta \in \mathscr{G}(C)$ determines a map $\mathbf{1} \to \mathscr{G}$, where $\mathbf{1}$ denotes the final object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$. Replacing $\mathscr{F}$ by $\mathscr{F} \times_{\mathscr{G}} \mathbf{1}$, we are reduced to proving the following:

(∗′) Let $\mathscr{F}$ be an ∞-connective object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ satisfying condition (2), and let $C \in \mathcal{C}$ be a final object. Then $\mathscr{F}(C)$ is nonempty.

To prove $(*')$, let $\widetilde{\mathcal{C}} \to \mathcal{C}$ be the right fibration classified by the functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$. We wish to show that $\widetilde{\mathcal{C}} \times_{\mathcal{C}} \{C\}$ is nonempty. We will construct an S-hypercovering $X_\bullet : \mathbf{\Delta}_{s,+}^{\mathrm{op}} \to \mathcal{C}$ with $X_{-1} = C$ together with a lifting $Y_\bullet : \mathbf{\Delta}_s^{\mathrm{op}} \to \widetilde{\mathcal{C}}$ of $X_\bullet|_{\mathbf{\Delta}_s^{\mathrm{op}}}$. Condition (2) and Corollary HTT.3.3.3.3 guarantee that $Y_\bullet$ extends (in an essentially unique fashion) to a map $\overline{Y}_\bullet : \mathbf{\Delta}_{s,+}^{\mathrm{op}} \to \widetilde{\mathcal{C}}$ lifting $X_\bullet$, so that $\overline{Y}_{-1}$ is the required point of $\widetilde{\mathcal{C}} \times_{\mathcal{C}} \{C\}$.

The construction of $X_\bullet$ and $Y_\bullet$ proceeds in stages. Let $\mathbf{\Delta}_{s,\leqslant m}$ denote the full subcategory of $\mathbf{\Delta}_s$ spanned by the objects $[k]$ for $k \leqslant m$, and define $\mathbf{\Delta}_{s,+,\leqslant m} \subseteq \mathbf{\Delta}_{s,+}$ similarly. We define $X_\bullet^{\leqslant m} : \mathbf{\Delta}_{s,+,\leqslant m}^{\mathrm{op}} \to \mathcal{C}$ and $Y_\bullet^{\leqslant m} : \mathbf{\Delta}_{s,\leqslant m}^{\mathrm{op}} \to \widetilde{\mathcal{C}}$ by induction on m, the case $m = -1$ being trivial. Assuming that $X_\bullet^{\leqslant m-1}$ has been defined, we can define the matching object $M_m(X) \in \mathcal{C}$. The lifting $Y_\bullet^{\leqslant m-1}$ determines a map $\partial \Delta^m \to \mathscr{F}(M_m(X))$. Since $\mathscr{F}$ is ∞-connective, there exists a collection of morphisms $\{D_i \to M_m(X)\}$ which generate a covering sieve, such that each composite map $\partial \Delta^m \to \mathscr{F}(M_m(X)) \to \mathscr{F}(D_i)$ is nullhomotopic. Without loss of generality, we may assume that the set of indices D_i is finite, and that the map $\amalg D_i \to M_m(X)$ belongs to S. Let $D = \amalg D_i$. Using condition (1), we see that the composite map $\gamma : \partial \Delta^m \to \mathscr{F}(M_m(X)) \to \mathscr{F}(D)$ is nullhomotopic. We can now define the extension $X_\bullet^{\leqslant m}$ by setting $X_m = D$, and the extension $Y_\bullet^{\leqslant m}$ using the nullhomotopy γ. □

A.6 Pretopoi in Higher Category Theory

Let $\mathcal{X}$ be a locally coherent ∞-topos (see Definition A.2.1.6). Then the structure of $\mathcal{X}$ is largely determined by the full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ spanned by the coherent objects of $\mathcal{X}$. The full subcategory $\mathcal{X}_0$ is not an ∞-topos (for example, it does not admit small colimits unless $\mathcal{X}$ is trivial), but nevertheless shares many of the defining categorical features of ∞-topoi (for example, the effectivity of groupoid objects). In this section, we will axiomatize these features by introducing the notion of a *∞-pretopos* (Definition A.6.1.1). The class of ∞-pretopoi includes all ∞-topoi $\mathcal{X}$ and is closed under passage to well-behaved full subcategories, like the full subcategory of coherent objects of a coherent ∞-topos $\mathcal{X}$ (Corollary A.6.1.7). Conversely, every ∞-pretopos $\mathcal{X}_0$ can be equipped with a finitary Grothendieck topology from which one can extract a coherent topos $\mathcal{S}\mathrm{hv}(\mathcal{X}_0)$. These processes are *approximately* inverse to one another: for a more precise statement, see Theorems **??** or **??**.

A.6.1 ∞-Pretopoi

We begin by introducing some terminology.

Definition A.6.1.1. Let $\mathcal{C}$ be an ∞-category. We will say that $\mathcal{C}$ is a *local ∞-pretopos* if it satisfies the following axioms:

(a) The ∞-category $\mathcal{C}$ admits fiber products. That is, for every pair of morphisms $f : X \to Y$ and $f' : X' \to Y$ with common codomain, there exists a fiber product

$X \times_Y X'$.

(b) The ∞-category $\mathcal{C}$ admits finite coproducts. In particular, $\mathcal{C}$ has an initial object.

(b′) The formation of finite coproducts in $\mathcal{C}$ is universal. That is, for every collection finite collection of morphisms $\{X_\alpha \to X\}$ and every map $Y \to X$, the induced map

$$\amalg_\alpha (X_\alpha \times_X Y) \to (\amalg_\alpha X_\alpha) \times_X Y$$

is an equivalence in $\mathcal{C}$.

(c) Coproducts in $\mathcal{C}$ are disjoint: that is, for every pair of objects $X, Y \in \mathcal{C}$, the fiber product $X \times_{X \amalg Y} Y$ is an initial object of $\mathcal{C}$.

(d) Every groupoid object of $\mathcal{C}$ is effective. That is, if $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ is a groupoid object of $\mathcal{C}$ (see Definition HTT.6.1.2.7), then there exists a geometric realization $|X_\bullet| \in \mathcal{C}$, and the canonical diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{d_0} & X_0 \\ \downarrow{\scriptstyle d_1} & & \downarrow \\ X_0 & \longrightarrow & |X_\bullet| \end{array}$$

is a pullback square in $\mathcal{C}$ (here $d_0, d_1 : X_1 \to X_0$ denote the face maps).

(d′) The formation of geometric realizations of groupoids in $\mathcal{C}$ is universal. In other words, given a morphism $f : X \to Y$ in $\mathcal{C}$ and a groupoid object $Y_\bullet$ of $\mathcal{C}_{/Y}$, the induced map $|X \times_Y Y_\bullet| \to X \times_Y |Y_\bullet|$ is an equivalence in $\mathcal{C}$.

We will say that $\mathcal{C}$ is an *∞-pretopos* if it is a local ∞-pretopos which admits a final object.

Remark A.6.1.2. In the situation of Definition A.6.1.1, it suffices to check condition (b′) in the case $X = \amalg X_\alpha$ and it suffices to check condition (d′) in the special case $X = |X_\bullet|$.

Remark A.6.1.3. Let $\mathcal{C}$ be a local ∞-pretopos. Then for every object $X \in \mathcal{C}$, the ∞-category $\mathcal{C}_{/X}$ is a local ∞-pretopos which admits a final object, and therefore an ∞-pretopos. Conversely, if $\mathcal{C}$ is an ∞-category which admits finite coproducts and $\mathcal{C}_{/X}$ is an ∞-pretopos for each $X \in \mathcal{C}$, then $\mathcal{C}$ is a local ∞-pretopos.

Remark A.6.1.4. Let $\mathcal{C}$ be a local ∞-pretopos and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory of $\mathcal{C}$. If $\mathcal{C}_0$ is closed under fiber products, finite coproducts, and the formation of geometric realizations of groupoid objects, then $\mathcal{C}_0$ is also a local ∞-pretopos. If, in addition, $\mathcal{C}$ is an ∞-pretopos and $\mathcal{C}_0$ contains the final object of $\mathcal{C}$, then $\mathcal{C}_0$ is also an ∞-pretopos.

Example A.6.1.5. Every ∞-topos is an ∞-pretopos (Theorem HTT.6.1.0.6).

Proposition A.6.1.6. *Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}^{\mathrm{coh}} \subseteq \mathcal{X}$ be the full subcategory spanned by the coherent objects of $\mathcal{X}$. Then $\mathcal{X}^{\mathrm{coh}}$ is a local ∞-pretopos.*

Corollary A.6.1.7. *Let $\mathcal{X}$ be a coherent ∞-topos and let $\mathcal{X}^{\mathrm{coh}} \subseteq \mathcal{X}$ be the full subcategory spanned by the coherent objects of $\mathcal{X}$. Then $\mathcal{X}^{\mathrm{coh}}$ is an ∞-pretopos.*

Proof of Proposition A.6.1.6. The ∞-topos $\mathcal{X}$ itself is a local ∞-pretopos (Example A.6.1.5). Consequently, to show that the full subcategory $\mathcal{X}^{\mathrm{coh}} \subseteq \mathcal{X}$ is a local ∞-pretopos, it will suffice to show that $\mathcal{X}^{\mathrm{coh}}$ is closed under finite coproducts, fiber products, and the geometric realization of groupoid objects (Remark A.6.1.4). The first two assertions follow from Remark A.2.1.8. To prove the third, suppose that $X_\bullet$ is a groupoid object of $\mathcal{X}^{\mathrm{coh}}$, and set $X = |X_\bullet|$ (where the geometric realization is formed in $\mathcal{X}$). Since groupoid objects of $\mathcal{X}$ are effective, we have a pullback square

$$\begin{array}{ccc} X_1 & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & X \end{array}$$

in the ∞-topos $\mathcal{X}$. Since X_0 and X_1 are coherent objects of $\mathcal{X}$, the upper horizontal morphism in this diagram is n-coherent for each integer n (Example A.2.1.2). Since the vertical maps in this diagram are effective epimorphisms, it follows that the lower horizontal map is also n-coherent for every integer n (Corollary A.2.1.5). Applying Proposition A.2.1.3 (and the coherence of X_0), we deduce that X is $(n+1)$-coherent. Allowing n to vary, we deduce that X is coherent, as desired. $\square$

A.6.2 The Effective Epimorphism Topology

Let $\mathcal{C}$ be an ∞-category which admits fiber products. For each morphism $f : U \to X$ in $\mathcal{C}$, the *Čech nerve* of f is the simplicial object $U_\bullet$ of $\mathcal{C}_{/X}$ depicted in the diagram

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} U \times_X U \rightrightarrows U,$$

so that U_n is the $(n+1)$-fold fiber product of U with itself over X. We will say that f is an *effective epimorphism* if the above diagram exhibits X as a geometric realization of $U_\bullet$ in the ∞-category $\mathcal{C}$. When $\mathcal{C}$ is a local ∞-pretopos, the class of effective epimorphisms has many good properties.

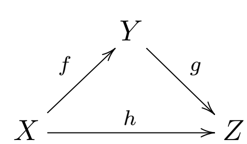
Proposition A.6.2.1. *Let $\mathcal{C}$ be a local ∞-pretopos. Then $\mathcal{C}$ admits a factorization system (see Definition HTT.5.2.8.8) (S_L, S_R), where S_L is the collection of effective epimorphisms in $\mathcal{C}$ and S_R is the collection of (-1)-truncated morphisms of $\mathcal{C}$.*

Proof. It is easy to see that S_L and S_R are closed under retracts. We next claim that every morphism $f : U \to X$ belonging to S_L is left orthogonal to every morphism $g : V \to Y$ belonging to S_R. To prove this, we can replace $\mathcal{C}$ by $\mathcal{C}_{/Y}$ and thereby reduce to the case where Y is a final object of $\mathcal{C}$. We wish to show that the canonical map $\theta : \mathrm{Map}_{\mathcal{C}}(X, V) \to \mathrm{Map}_{\mathcal{C}}(U, V)$ is a homotopy equivalence. Let $U_\bullet$ be the Čech nerve of f; our assumption that f is an effective epimorphism guarantees that we can identify X with the geometric realization $|U_\bullet|$, so that the domain of θ can be identified with the totalization of the cosimplicial space $\mathrm{Map}_{\mathcal{C}}(U_\bullet, V)$. Note that our assumption that g is (-1)-truncated guarantees that each of the spaces $\mathrm{Map}_{\mathcal{C}}(U_n, V)$ is either empty or contractible. Since there exist morphisms $[m] \to [n]$ in $\mathbf{\Delta}$ for every pair of integers $m, n \geqslant 0$, either $\mathrm{Map}_{\mathcal{C}}(U_n, V)$ is empty for all $n \geqslant 0$ (in which case θ is a map between empty spaces) or $\mathrm{Map}_{\mathcal{C}}(U_n, V)$ is contractible for all $n \geqslant 0$ (in which case θ is a morphism between contractible spaces). In either case, the map θ is automatically a homotopy equivalence.

To complete the proof that (S_L, S_R) is a factorization system on $\mathcal{C}$, it will suffice to show that every morphism $f : U \to X$ in $\mathcal{C}$ admits a factorization $U \xrightarrow{f_L} V \xrightarrow{f_R} X$ where f_L is an effective epimorphism and f_R is (-1)-truncated. To prove this, we let $U_\bullet$ denote the Čech nerve of f and $V = |U_\bullet|$. Since $\mathcal{C}$ is a local ∞-pretopos, the groupoid $U_\bullet$ is effective and therefore f_L is an effective epimorphism. It remains to show that f_R is (-1)-truncated: that is, that the diagonal map $V \simeq V \times_V V \xrightarrow{\delta} V \times_X V$ is an equivalence. Because $\mathcal{C}$ is a local ∞-pretopos, the formation of geometric realizations of groupoid objects of $\mathcal{C}$ commutes with pullbacks, so we can identify δ with the geometric realization of a map of bisimplicial objects $\delta_{\bullet\bullet} : U_\bullet \times_V U_\bullet \to U_\bullet \times_X U_\bullet$. We now observe that each δ_{mn} is the pullback of the map $\delta_{00} : U \times_V U \to U \times_X U \simeq U_1$, which is an equivalence by virtue of the fact that $U_\bullet$ is an effective groupoid object of $\mathcal{C}$. □

Corollary A.6.2.2. *Let $\mathcal{C}$ be a local ∞-pretopos.*

(a) Suppose we are given a commutative diagram



in $\mathcal{C}$ where f is an effective epimorphism. Then g is an effective epimorphism if and only if h is an effective epimorphism. In particular, the collection of effective epimorphisms is closed under composition.

(*b*) *Suppose we are given a pullback square*

$$\begin{array}{ccc} U' & \xrightarrow{g'} & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

in the ∞-category $\mathcal{C}$. If f is an effective epimorphism, then so is f'. The converse holds if g is an effective epimorphism.

(*c*) *Let $\{f_\alpha : U_\alpha \to X_\alpha\}$ be a finite collection of effective epimorphisms in $\mathcal{C}$. Then the induced map $\amalg f_\alpha : \amalg U_\alpha \to \amalg X_\alpha$ is an effective epimorphism.*

Proof. Assertions (a) and (c) follow from Propositions A.6.2.1 and HTT.5.2.8.6. Let us prove (b). Suppose we are given a pullback square

$$\begin{array}{ccc} U' & \xrightarrow{g'} & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X. \end{array}$$

in the ∞-category $\mathcal{C}$. Let $U_\bullet$ be the Čech nerve of f, so that $X' \times_X U_\bullet$ can be identified with the Čech nerve of f'. Since the formation of geometric realizations of groupoid objects in $\mathcal{C}$ commutes with pullback, we have a pullback square

$$\begin{array}{ccc} |X' \times_X U_\bullet| & \longrightarrow & |U_\bullet| \\ \downarrow & & \downarrow \\ X' & \xrightarrow{g'} & X. \end{array}$$

Consequently, if f is an effective epimorphism, then so is f'. Conversely, suppose that g and f' are effective epimorphisms. Then the preceding argument shows that g' is an effective epimorphism. It follows from (a) that $g \circ f' = f \circ g'$ is an effective epimorphism, so that f is an effective epimorphism by a second application of (a). □

Corollary A.6.2.3. *Let $\mathcal{C}$ be a local ∞-pretopos. Then $\mathcal{C}$ admits a Grothendieck topology which can be described as follows: a collection of morphisms $\{f_\alpha : U_\alpha \to X\}_{\alpha \in A}$ in $\mathcal{C}$ generate a covering sieve if and only if there exists a finite subset $A_0 \subseteq A$ such that the induced map $\amalg_{\alpha \in A_0} U_\alpha \to X$ is an effective epimorphism.*

Proof. It suffices to show that the collection of effective epimorphisms satisfies conditions (a) through (d) of Proposition A.3.2.1. Conditions (a) through (c) follow from Corollary A.6.2.2, while condition (d) follows from our assumption that $\mathcal{C}$ is a local ∞-pretopos. □

Definition A.6.2.4. Let $\mathcal{C}$ be a local ∞-pretopos. We will refer to the Grothendieck topology of Corollary A.6.2.3 as the *effective epimorphism topology*. We let $\mathcal{S}\mathrm{hv}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ spanned by those functors which are sheaves with respect to the effective epimorphism topology.

Applying Proposition A.3.3.1, we immediately obtain the following:

Proposition A.6.2.5. *Let $\mathcal{C}$ be a local ∞-pretopos and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ be a functor. Then $\mathscr{F}$ is a sheaf (with respect to the effective epimorphism topology) if and only if it satisfies the following conditions:*

(1) *For every finite collection of objects $\{X_\alpha\}_{\alpha \in A}$ in $\mathcal{C}$, the canonical map $\mathscr{F}(\amalg X_\alpha) \to \prod \mathscr{F}(X_\alpha)$ is a homotopy equivalence.*

(2) *For every effective epimorphism $f : U \to X$ in $\mathcal{C}$ having Čech nerve $U_\bullet$, the induced map $\mathscr{F}(X) \to \mathrm{Tot}\, \mathscr{F}(U_\bullet)$ is a homotopy equivalence.*

Corollary A.6.2.6. *Let $\mathcal{C}$ be a local ∞-pretopos. Then the effective epimorphism topology on $\mathcal{C}$ is subcanonical: that is, the Yoneda embedding $j : \mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ factors through $\mathcal{S}\mathrm{hv}(\mathcal{C})$.*

A.6.3 Subobjects

We now describe explicate one relationship between our theory of ∞-pretopoi (Definition A.6.1.1) and the classical theory of distributive lattices (Definition A.1.4.5).

Notation A.6.3.1. Let X be an object of an ∞-category $\mathcal{C}$. Recall that a *subobject of* X is a (-1)-truncated morphism $\iota : U \to X$ in $\mathcal{C}$. We will often abuse terminology by referring to the object $U \in \mathcal{C}$ as a *subobject of* X (in which case we implicitly assume that a (-1)-truncated morphism $U \to X$ has been specified). If U and V are subobjects of X, we write $U \subseteq V$ if there exists a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ & \searrow \quad \swarrow & \\ & X. & \end{array}$$

Note that since the map $V \to X$ is (-1)-truncated, the morphism f is unique up to a contractible space of choices (provided that f exists).

If U and V are subobjects of an object $X \in \mathcal{C}$, we say that U and V are *equivalent* if they are equivalent when viewed as objects of $\mathcal{C}_{/X}$ (in other words, if $U \subseteq V$ and $V \subseteq U$). We let $\mathrm{Sub}(X)$ denote the collection of equivalence classes of subobjects of X. We will generally abuse notation by identifying a subobject U of X with the corresponding element of $\mathrm{Sub}(X)$.

We regard $\mathrm{Sub}(X)$ as a partially ordered set with respect to the inclusion relation $\subseteq$ defined above.

Remark A.6.3.2. Let X be an object of an ∞-category $\mathcal{C}$. Then the partially ordered set $\mathrm{Sub}(X)$ can be regarded as an ∞-category, which is equivalent to the full subcategory of $\mathcal{C}_{/X}$ spanned by the (-1)-truncated objects. This equivalence determines a fully faithful embedding $\mathrm{Sub}(X) \hookrightarrow \mathcal{C}_{/X}$. If $\mathcal{C}$ is a local ∞-pretopos, then Proposition A.6.2.1 implies that this fully faithful embedding admits a left adjoint $L : \mathcal{C}_{/X} \to \mathrm{Sub}(X)$. For each object $U \in \mathcal{C}_{/X}$, the underlying map $f : U \to X$ admits an essentially unique factorization $U \xrightarrow{f'} LU \xrightarrow{f''} X$, where f' is an effective epimorphism and f'' is (-1)-truncated. Consequently, the object LU can be described explicitly as the geometric realization $|U_\bullet|$, where $U_\bullet$ is the groupoid object of $\mathcal{C}$ given by the Čech nerve of f (which is the same as the Čech nerve of f', by virtue of the fact that f'' is (-1)-truncated).

Proposition A.6.3.3. *Let $\mathcal{C}$ be a local ∞-pretopos. Then, for each $X \in \mathcal{C}$, the partially ordered set* $\mathrm{Sub}(X)$ *is a distributive lattice.*

Proof. Let us abuse notation by identifying $\mathrm{Sub}(X)$ with the full subcategory of $\mathcal{C}_{/X}$ spanned by the (-1)-truncated objects. Let $L : \mathcal{C}_{/X} \to \mathrm{Sub}(X)$ be a left adjoint to the inclusion (see Remark A.6.3.2). Note that $\mathrm{Sub}(X)$ is closed under all limits which exist in $\mathcal{C}_{/X}$. Since $\mathcal{C}$ admits pullbacks, the ∞-category $\mathcal{C}_{/X}$ admits finite products so that $\mathrm{Sub}(X)$ is a lower semilattice. Since $\mathcal{C}$ admits finite coproducts, the partially ordered set $\mathrm{Sub}(X)$ admits finite joins which are given by the formula $\bigvee_{i \in I} U_i = L(\bigwedge_{i \in I} U_i)$. This proves that the partially ordered set $\mathrm{Sub}(X)$ is a lattice. To show that this lattice is distributive, it will suffice to show that for every triple of subobjects $U, V, W \in \mathrm{Sub}(X)$, the inclusion $(U \wedge V) \vee (U \wedge W) \subseteq U \wedge (V \vee W)$ is an equality. Unwinding the definitions, we see that this is equivalent to the requirement that the composite map

$$(U \times_X V) \amalg (U \times_X W) \xrightarrow{f} U \times_X (V \amalg W) \xrightarrow{g} U \times_X L(V \amalg W)$$

is an effective epimorphism. Our assumption that $\mathcal{C}$ is a local ∞-pretopos guarantees that f is an equivalence, and the morphism g is a pullback of the natural map $V \amalg W \to L(V \amalg W)$ and is therefore an effective epimorphism by virtue of Corollary A.6.2.2. □

Remark A.6.3.4 (Functoriality)**.** Let $\mathcal{C}$ be a local ∞-pretopos and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then the construction $U \mapsto U \times_Y X$ determines a functor $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$. This functor is left exact (since it is right adjoint to the functor $f_! : \mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ given by composition with f) and therefore carries (-1)-truncated objects to (-1)-truncated objects. It therefore induces a map of partially ordered sets $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$. We claim that $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a homomorphism of distributive lattices (Definition A.1.5.3): that is, it preserves finite joins and finite meets. The preservation of finite meets follows from left exactness. For the case of joins, we must show that for every finite collection of objects $\{U_i\}_{i \in I}$ of $\mathrm{Sub}(Y)$, the inclusion $\bigvee_{i \in I} f^* U_i \subseteq f^*(\bigvee_{i \in I} U_i)$ is an equality. Unwinding the definitions, this is equivalent to the requirement that the composite map

$$\amalg_{i \in I} f^* U_i \xrightarrow{\rho} f^*(\amalg_{i \in I} U_i) \xrightarrow{\rho'} f^*(\bigvee_{i \in I} U_i)$$

is an effective epimorphism in $\mathcal{C}$. The map ρ is an equivalence (since finite coproducts in $\mathcal{C}$ are universal) and the map ρ' is a pullback of the effective epimorphism $\amalg_{i \in I} U_i \to \bigvee_{i \in I} U_i$, hence an effective epimorphism by virtue of Corollary A.6.2.2.

Proposition A.6.3.5. *Let $\mathcal{C}$ be a local ∞-pretopos and let $f : X \to Y$ be an effective epimorphism in $\mathcal{C}$. The following conditions are equivalent:*

(a) *The morphism f is an effective epimorphism.*

(b) *The induced map of distributive lattices $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is injective.*

Proof. We first show that (a) implies (b). Suppose that f is an effective epimorphism and that we are given subobjects $U, V \in \mathrm{Sub}(Y)$ satisfying $f^* U = f^* V$ in $\mathrm{Sub}(X)$; we wish to show that $U = V$. We will show that the projection maps $U \leftarrow U \times_X V \to V$ are equivalences in $\mathcal{C}$. Since both projections are (-1)-truncated morphisms, it will suffice to show that they are effective epimorphisms. By virtue of Corollary A.6.2.2, we are reduced to proving that the induced maps

$$U \times_X Y \leftarrow U \times_X V \times_X Y \to V \times_X Y$$

are effective epimorphisms. However, both of these maps are equivalences, since assumption (a) gives

$$f^* U = (f^* U) \wedge (f^* V) = f^*(U \wedge V) = (f^* U) \wedge (f^* V) = f^* V$$

in the partially ordered set $\mathrm{Sub}(X)$.

Now suppose that (b) is satisfied. It follows from Proposition A.6.2.1 that the morphism f factors as a composition $X \xrightarrow{f'} U \xrightarrow{f''} Y$ where f' is an effective epimorphism and f'' is (-1)-truncated. Then $f^* U = f^* Y$ in $\mathrm{Sub}(X)$, so condition (b) implies that $U = Y$ in $\mathrm{Sub}(Y)$: that is, the morphism f'' is an equivalence. It follows that $f \simeq f'' \circ f'$ is a composition of effective epimorphisms, and therefore an effective epimorphism as desired. □

Remark A.6.3.6. Let $\mathcal{C}$ be an ∞-pretopos containing an object X, and suppose we are given a pair of (-1)-truncated morphisms $U \to X \leftarrow V$. The following conditions are equivalent:

(i) The induced map $U \amalg V \to X$ is (-1)-truncated.

(ii) The fiber product $U \times_X V$ is an initial object of $\mathcal{C}$.

To prove this, we note that since coproducts in $\mathcal{C}$ are universal, assertion (i) is equivalent to statement that the composite map

$$\begin{aligned} U \amalg V &\simeq (U \times_X U) \amalg (V \times_X V) \\ &\to (U \times_X U) \amalg (V \times_X U) \amalg (U \times_X V) \amalg (V \times_X V) \\ &\simeq (U \amalg V) \times_X (U \times_X V) \end{aligned}$$

is an equivalence. The equivalence of this statement with (ii) follows from the disjointness of coproducts in $\mathcal{C}$.

Remark A.6.3.7. Let $f : U \to X$ be a morphism in a local ∞-pretopos $\mathcal{C}$. Using Remark A.6.3.6, we deduce that the following conditions are equivalent:

(i) The morphism f is (-1)-truncated and exhibits U as a complemented element of the distributive lattice $\mathrm{Sub}(X)$ (in other words, there exists an subobject $V \in \mathrm{Sub}(X)$ satisfying $U \wedge V = \emptyset$ and $U \vee V = X$).

(ii) There exists a morphism $g : V \to X$ such that f and g induce an equivalence $U \amalg V \xrightarrow{(f,g)} X$ (note that since coproducts in $\mathcal{C}$ are disjoint, this automatically guarantees that f and g are (-1)-truncated).

If these conditions are satisfied, then the morphism $g : V \to X$ appearing in (ii) is essentially unique (see Remark A.1.6.1).

Definition A.6.3.8. Let $\mathcal{C}$ be a local ∞-pretopos. We will say that $\mathcal{C}$ is *Boolean* if, for every object $X \in \mathcal{C}$, the distributive lattice $\mathrm{Sub}(X)$ is a Boolean algebra.

Remark A.6.3.9. Let $\mathcal{C}$ be a local ∞-pretopos. If $\mathcal{C}$ is Boolean, then every (-1)-truncated morphism $f : U \to X$ in $\mathcal{C}$ determines an essentially unique decomposition of $X \simeq U \amalg V$ in $\mathcal{C}$ (Remark A.6.3.7).

A.6.4 The Universal Property of $\mathcal{S}\mathrm{hv}(\mathcal{C})$

The collection of all ∞-pretopoi can be organized into an ∞-category.

Definition A.6.4.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be ∞-pretopoi. We let $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{C}')$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ spanned by those functors $f : \mathcal{C} \to \mathcal{C}'$ which preserve finite limits, finite coproducts, and carry effective epimorphisms in $\mathcal{C}$ to effective epimorphisms in $\mathcal{C}'$. We will refer to $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{C}')$ as the *∞-category of ∞-pretopos morphisms from $\mathcal{C}$ to $\mathcal{C}'$*.

We let $\infty\widehat{\mathcal{T}\text{op}}^{\text{pre}}$ denote the subcategory of $\widehat{\mathcal{C}\text{at}}_\infty$ whose objects are (not necessarily small) ∞-pretopoi and whose morphisms are functors which preserve finite limits, finite coproducts, and effective epimorphisms, and we let $\infty\mathcal{T}\text{op}^{\text{pre}}$ denote the full subcategory of $\infty\widehat{\mathcal{T}\text{op}}^{\text{pre}}$ spanned by those ∞-pretopoi which are essentially small.

Example A.6.4.2. Let $f^* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. Then f^* is a morphism of ∞-pretopoi: it preserves finite limits and small colimits by definition, and preserves effective epimorphisms by virtue of Remark HTT.6.2.3.6 . Consequently, we can regard the ∞-category $\infty\mathcal{T}\text{op}^{\text{op}}$ as a (non-full) subcategory of $\infty\widehat{\mathcal{T}\text{op}}^{\text{pre}}$.

Example A.6.4.3. Let $\mathcal{C}$ be a local ∞-pretopos and let $f : C \to D$ be a morphism in $\mathcal{C}$. Then the construction $(D' \in \mathcal{C}_{/D}) \mapsto (C \times_D D' \in \mathcal{C}_{/C})$ determines an ∞-pretopos morphism from $\mathcal{C}_{/D}$ to $\mathcal{C}_{/C}$.

Proposition A.6.4.4. *Let $\mathcal{C}$ be an essentially small ∞-pretopos and let $\mathcal{X}$ be an ∞-topos. Then composition with the Yoneda embedding $j : \mathcal{C} \to \mathcal{S}\text{hv}(\mathcal{C})$ induces an equivalence of ∞-categories* $\text{Fun}^*(\mathcal{S}\text{hv}(\mathcal{C}), \mathcal{X}) \to \text{Fun}^{\text{pre}}(\mathcal{C}, \mathcal{X})$*; here* $\text{Fun}^*(\mathcal{S}\text{hv}(\mathcal{C}), \mathcal{X})$ *denotes the full subcategory of* $\text{Fun}(\mathcal{S}\text{hv}(\mathcal{C}), \mathcal{X})$ *spanned by those functors which preserve small colimits and finite limits.*

Remark A.6.4.5. Proposition A.6.4.4 implies that we can regard the construction $\mathcal{C} \mapsto \mathcal{S}\text{hv}(\mathcal{C})$ as a *partially defined* left adjoint to the inclusion functor $\infty\mathcal{T}\text{op}^{\text{op}} \hookrightarrow \infty\widehat{\mathcal{T}\text{op}}^{\text{pre}}$. It is not quite a left adjoint because Proposition A.6.4.4 requires the ∞-pretopos $\mathcal{C}$ to be essentially small, but the ∞-topos $\mathcal{X}$ will almost never be essentially small.

Proof of Proposition A.6.4.4. By virtue of Proposition HTT.6.2.3.20 , composition with j induces a fully faithful embedding $\text{Fun}^*(\mathcal{S}\text{hv}(\mathcal{C}), \mathcal{X}) \to \text{Fun}(\mathcal{C}, \mathcal{X})$ whose essential image is spanned by those left exact functors $f : \mathcal{C} \to \mathcal{X}$ which satisfy the following additional condition:

$(*)$ For every collection of morphisms $\{u_\alpha : U_\alpha \to V\}_{\alpha \in A}$ in $\mathcal{C}$ which generate a covering sieve with respect to the effective epimorphism topology, the induced map $\rho : \amalg f(U_\alpha) \to f(V)$ is an effective epimorphism in the ∞-topos $\mathcal{X}$.

Note that in verifying $(*)$, we may assume without loss of generality that the set A is finite, so that the maps u_α determine an effective epimorphism $u : \amalg U_\alpha \to V$ in the ∞-pretopos $\mathcal{C}$. If f commutes with finite coproducts, then we can identify ρ with $f(u)$. Consequently, if f is a morphism of ∞-pretopoi, then f satisfies $(*)$. We now prove the converse. Suppose that f is a left exact functor which satisfies $(*)$. Applying $(*)$ in the case where A is a singleton, we deduce that f preserves effective epimorphisms. Applying $(*)$ in the case where A is empty, we deduce that f preserves initial objects. To complete the proof, it will suffice to show that f preserves pairwise coproducts. Let U and U' be objects of $\mathcal{C}$. Since coproducts in $\mathcal{C}$ are disjoint, we have a pullback square

$$\begin{array}{ccc} \emptyset & \longrightarrow & U \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U \amalg U' \end{array}$$

in the ∞-category $\mathcal{C}$, where $\emptyset$ is an initial object of $\mathcal{C}$. Note that each of the morphisms in this diagram is (-1)-truncated. Applying the left exact functor f, we obtain a pullback square

$$\begin{array}{ccc} f(\emptyset) & \longrightarrow & f(U) \\ \downarrow & & \downarrow \\ f(U') & \longrightarrow & f(U \amalg U') \end{array}$$

in the ∞-topos $\mathcal{X}$, where each morphism is (-1)-truncated. Since $f(\emptyset)$ is an initial object of $\mathcal{X}$, the induced map $f(U) \amalg f(U') \to f(U \amalg U')$ is (-1)-truncated. To show that this map is an equivalence, it suffices to show that it is an effective epimorphism, which follows by applying $(*)$ to the pair of maps $\{U \to U \amalg U', U' \to U \amalg U'\}$. □

A.6.5 Hypercomplete ∞-Pretopoi

Let $\mathcal{C}$ be a small local ∞-pretopos. Then the Yoneda embedding $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ is fully faithful. We now describe some additional conditions on $\mathcal{C}$ which guarantee that the essential image of j admits a simple description.

Definition A.6.5.1. Let $\mathcal{C}$ be a local ∞-pretopos. We will say that a semisimplicial object $X_\bullet$ of $\mathcal{C}$ *satisfies the Kan condition* if, for $n > 0$ and $0 \leqslant i \leqslant n$, the natural map $X_n = X_\bullet[\Delta^n] \to X_\bullet[\Lambda^n_i]$ is an effective epimorphism (note that $X_\bullet[\Lambda^n_i]$ is well-defined by virtue of the assumption that $\mathcal{C}$ admits fiber products: see Proposition A.5.1.9).

If $\mathcal{C}$ is an ∞-pretopos, we will say that a semisimplicial object $X_\bullet$ of $\mathcal{C}$ is a *hypercovering* if the natural map $X_n = X_\bullet[\Delta^n] \to X_\bullet[\partial\,\Delta^n]$ is an effective epimorphism for $n \geqslant 0$. More generally, if $\mathcal{C}$ is a local ∞-pretopos containing an object X, then we will say that a semisimplicial object $X_\bullet$ of $\mathcal{C}_{/X}$ is a *hypercovering of* X if it is a hypercovering of the ∞-pretopos $\mathcal{C}_{/X}$.

Remark A.6.5.2. In the special case where $\mathcal{C}$ is an ∞-topos, Definition A.6.5.1 recovers the notions of Kan condition and hypercover given in Definition A.5.2.3.

Definition A.6.5.3. Let $\mathcal{C}$ be a local ∞-pretopos. We will say that $\mathcal{C}$ is *hypercomplete* if it satisfies the following conditions:

(a) Every simplicial object $X_\bullet$ of $\mathcal{C}$ which satisfies the Kan condition, there exists an object $X \in \mathcal{C}$ such that $X_\bullet$ can be lifted to a simplicial object of $\mathcal{C}_{/X}$ which is a hypercovering of X.

(b) If $X \in \mathcal{C}$ and $X_\bullet$ is a simplicial object of $\mathcal{C}_{/X}$ which is a hypercovering of X, then X is a colimit of the diagram $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}_{/X} \to \mathcal{C}$.

Remark A.6.5.4. In the situation of Definition A.6.5.3, the converse of assertion (a) is always true: if $X_\bullet$ is a hypercovering of an object $X \in \mathcal{C}$, then the image of $X_\bullet$ under the forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$ satisfies the Kan condition (see Remark A.5.2.6).

Remark A.6.5.5. Let $\mathcal{C}$ be a local ∞-pretopos. If $\mathcal{C}$ satisfies condition (b) of Definition A.6.5.3 and $X_\bullet$ is a simplicial object of $\mathcal{C}$ which satisfies the Kan condition, then the object X appearing in condition (a) is determined uniquely up to equivalence: condition (b) supplies an equivalence $X \simeq |X_\bullet|$.

Proposition A.6.5.6. *Let $\mathcal{X}$ be an ∞-topos. Then $\mathcal{X}$ is hypercomplete as an ∞-topos (in the sense of Definition HTT.6.5.2) if and only if it is hypercomplete as an ∞-pretopos (in the sense of Definition A.6.5.3).*

Proof. If $\mathcal{X}$ satisfies condition (b) of Definition A.6.5.3, then it is hypercomplete as an ∞-topos by virtue of Theorem HTT.6.5.3.12 . Conversely, suppose that $\mathcal{X}$ is hypercomplete as an ∞-topos. If $X_\bullet$ is a semisimplicial object of $\mathcal{X}$ which is a hypercovering of an object $X \in \mathcal{X}$, then the canonical map $|X_\bullet| \to X$ is ∞-connective (Theorem A.5.3.1) and therefore an equivalence (since $\mathcal{X}$ is hypercomplete). This verifies condition (b) of Definition A.6.5.3. If $X_\bullet$ is a semisimplicial object of $\mathcal{X}$ which satisfies the Kan condition, then Corollary A.5.6.4 shows that $X_\bullet$ is a hypercovering of its geometric realization, which verifies condition (a). □

Proposition A.6.5.7. *Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}_0 \subseteq \mathcal{X}$ be a full subcategory which satisfies the following conditions:*

(a) *The full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ is closed under finite coproducts.*

(b) *The full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ is closed under fiber products.*

(c) *Every object $X \in \mathcal{X}_0$ is quasi-compact.*

(d) *For every object $X \in \mathcal{X}$, there exists an effective epimorphism $\amalg U_\alpha \to X$, where each U_α belongs to X.*

(e) *For every object $X \in \mathcal{X}$ and every simplicial object $U_\bullet$ of $\mathcal{X}_{/X}$ which is a hypercovering of X, if each U_n belongs to $\mathcal{X}_0$, then X belongs to $\mathcal{X}_0$.*

Then $\mathcal{X}_0$ is the full subcategory of $\mathcal{X}$ spanned by the coherent objects.

Proof. We first prove by induction on k that each each $X \in \mathcal{X}_0$ is a k-coherent object of $\mathcal{X}$. In the case $k = 0$, this follows from (c). Suppose that $k > 0$. It follows from (d) and our inductive hypothesis that the ∞-topos $\mathcal{X}_{/X}$ is locally k-coherent. Let U and V be $(k-1)$-coherent objects of $\mathcal{X}_{/X}$; we wish to show that $U \times_X V$ is $(k-1)$-coherent. Since $k > 0$, the objects U and V are quasi-compact. Using (a) and (d), we can choose effective epimorphisms $u : U' \to U$ and $v : V' \to V$ where $U', V' \in \mathcal{X}_0$. Then U' and V' are $(k-1)$-coherent by our inductive hypothesis, so the morphisms u and v are relatively $(k-1)$-coherent (Example A.2.1.2). It follows that the induced map $U' \times_X V' \to U \times_X V$ is an effective epimorphism which is relatively $(k-1)$-coherent. It follows from (b) that the fiber product $U' \times_X V'$ belongs to $\mathcal{X}_0$ and is therefore $(k-1)$-coherent by our inductive hypothesis, so that $U \times_X V$ is $(k-1)$-coherent by virtue of Proposition A.2.1.3. Allowing k to vary, we deduce that each object of $\mathcal{X}_0$ is coherent.

We now prove the converse. Let $X \in \mathcal{X}$ be a coherent object; we wish to prove that $X \in \mathcal{X}_0$. By virtue of (e), it will suffice to show that there exists simplicial object $U_\bullet$ of $\mathcal{X}_{/X}$ which is a hypercovering of X, such that $U_n \in \mathcal{X}_0$ for all $n \geqslant 0$. We will construct $U_\bullet$ as the union of a compatible sequence of functors $F_{\leqslant n} : \mathbf{\Delta}^{\mathrm{op}}_{\leqslant n} \to \mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_{/X}$ which have the following additional property:

$(*_n)$ For $m \leqslant n+1$, let L_m denote the mth latching object of $F_{\leqslant n}$: that is, the colimit $\varinjlim_{[k] \hookrightarrow [m], k<m} F_{\leqslant n}([k])$. Then the natural map $L_m \to F_{\leqslant n}([m])$ exhiibits L_m as a summand of $F_{\leqslant n}([m])$: that is, we have an equivalence $F_{\leqslant n}([m]) \simeq L_m \amalg A_m$ for some auxiliary object $A_m \in \mathcal{X}_0$.

Let us assume that we have already constructed the diagram $F_{\leqslant n-1}$ satisfying $(*_{n-1})$. It follows that the latching object L_n can be identified with the finite coproduct $\amalg_{[m] \hookrightarrow [n], m<n} A_m$ and therefore belongs to $\mathcal{X}_0$. Let M_n denote the matching object $\varprojlim_{[m] \hookrightarrow [n], m<n} F_{\leqslant n-1}([m])$ (where the limit is formed in the ∞-category $\mathcal{X}_{/X}$). Each $F_{\leqslant n-1}([m])$ belongs to $\mathcal{X}_0$ and is therefore coherent, so that M_n is a coherent object of $\mathcal{X}$. In particular, M_n is quasi-compact, so assumptions (a) and (d) guarantee the existence of an effective epimorphism $v : Y \to M_n$ where $Y \in \mathcal{X}_0$. Using Corollary HTT.A.2.9.15 , we see that the data of a functor $F_{\leqslant n}$ extending $F_{\leqslant n-1}$ is equivalent to the data of a commutative diagram

$$\begin{array}{ccc} & X_n & \\ \nearrow & & \searrow^{\phi} \\ L_n & \xrightarrow{\phi_0} & M_n \end{array}$$

where the map ϕ_0 prescribed by $F_{\leqslant n-1}$. We complete the construction by taking X_n to be the coproduct $L_n \amalg Y$ and ϕ to be the amalgam of ϕ_0 with v. It is easy to see that resulting functor $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{X}_{/X}$ has the desired properties. □

Our next result points to the relevance of Definition A.6.5.3:

Proposition A.6.5.8. *Let $\mathcal{C}$ be an essentially small local ∞-pretopos. If $\mathcal{C}$ is hypercomplete, then the Yoneda embedding $j : \mathcal{C} \hookrightarrow \mathcal{S}\mathrm{hv}(\mathcal{C})$ factors through the full subcategory $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ spanned by the hypercomplete objects of $\mathcal{S}\mathrm{hv}(\mathcal{C})$. Moreover, an object of $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ belongs to the essential image of j if and only if it is coherent.*

Proof. Let $X \in \mathcal{C}$ be an object; we wish to show that $j(X) \in \mathcal{S}\mathrm{hv}(\mathcal{C})$ is hypercomplete. Using Proposition A.5.7.2, we are reduced to proving that if $Y_\bullet$ is a hypercovering of an object $Y \in \mathcal{C}$, then the canonical map $\mathrm{Map}_{\mathcal{C}}(Y, X) \to \varprojlim \mathrm{Map}_{\mathcal{C}}(Y_\bullet, X)$ is a homotopy equivalence. This follows from the second assumption of Definition A.6.5.3. This proves that j factors through $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$. To complete the proof, it will suffice to show that the essential image of j satisfies conditions (a) through (e) of Proposition A.6.5.7. Conditions (b) and (d) are obvious, condition (a) follows from the proof of Proposition A.6.4.4, condition (c) from Proposition A.3.1.3. To prove (e), suppose that $\mathscr{F}_\bullet$ is a simplicial hypercovering of an object $\mathscr{F} \in \mathcal{S}\mathrm{hv}(\mathcal{X})^{\mathrm{hyp}}$ where each $\mathscr{F}_k$ belongs to the essential image of j. Since j is fully faithful, the image of $\mathscr{F}_\bullet$ in $\mathcal{S}\mathrm{hv}(\mathcal{X})^{\mathrm{hyp}}$ can be identified with $j(U_\bullet)$ for some simplicial object $U_\bullet$ of $\mathcal{C}$. Since $\mathscr{F}_\bullet$ is a hypercovering of $\mathscr{F}$, it satisfies the Kan condition (Remark A.6.5.4), so that the simplicial object $U_\bullet$ also satisfies the Kan condition. Our assumption that $\mathcal{C}$ is hypercomplete guarantees the existence of an object $X \in \mathcal{C}$ such that $U_\bullet$ can be lifted to a hypercovering of X. It follows that $j(U_\bullet)$ is a hypercovering of $j(X)$ in $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$. Since $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$ is hypercomplete, we conclude that that both $\mathscr{F}$ and $j(X)$ can be identified with the geometric realization $|j(U_\bullet)$, so that $\mathscr{F} \simeq j(X)$ belongs to the essential image of j as desired. □

Corollary A.6.5.9. *Let $\mathcal{C}$ be a local ∞-pretopos. If $\mathcal{C}$ is hypercomplete, then $\mathcal{C}$ is idempotent-complete.*

Proof. Enlarging the universe if necessary, we may assume that $\mathcal{C}$ is small. In this case, Proposition A.6.5.8 shows that the essential image of the Yoneda embedding $j : \mathcal{C} \hookrightarrow \mathcal{S}\mathrm{hv}(\mathcal{C})$ is closed under retracts. Since the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is idempotent-complete, it follows that $\mathcal{C}$ is idempotent-complete. □

A.6.6 The ∞-Category of Hypercomplete ∞-Pretopoi

We now show that the datum of a hypercomplete ∞-pretopos is essentially the same as the datum of a hypercomplete ∞-topos which is coherent and locally coherent (Theorem **??**).

Proposition A.6.6.1. *Let $\mathcal{X}$ be an ∞-topos which is locally coherent and hypercomplete, and let $\mathcal{X}^{\mathrm{coh}}$ denote the full subcategory of $\mathcal{X}$ spanned by the coherent objects. Then:*

(1) *The ∞-category $\mathcal{X}^{\mathrm{coh}}$ is a local ∞-pretopos.*

(2) *The ∞-category $\mathcal{X}^{\mathrm{coh}}$ is essentially small.*

(3) *If $X_\bullet$ is a semisimplicial object of $\mathcal{X}^{\mathrm{coh}}$ which satisfies the Kan condition, then the geometric realization $|X_\bullet|$ belongs to $\mathcal{X}^{\mathrm{coh}}$ and $X_\bullet$ is a hypercovering of $|X_\bullet|$.*

(4) *If $X \in \mathcal{X}^{\mathrm{coh}}$ is an object and $X_\bullet : \mathbf{\Delta}_s^{\mathrm{op}} \to \mathcal{X}_{/X}$ is a hypercovering of X, then the induced map $|X_\bullet| \to X$ is an equivalence.*

In particular, $\mathcal{X}^{\mathrm{coh}}$ is a hypercomplete local ∞-pretopos.

Proof. Assertion (a) follows from Proposition A.6.1.6, assertion (c) from Theorem A.5.5.1, and assertion (d) from Theorem A.5.3.1 (and the hypercompleteness of $\mathcal{X}$). We now prove (b). Since $\mathcal{X}$ is a presentable ∞-category, there exists an essentially small full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ such that every object $X \in \mathcal{X}$ admits an effective epimorphism $\amalg U_\alpha \to X$, where each U_α belongs to $\mathcal{X}_0$. Because $\mathcal{X}$ is locally coherent, we may assume without loss of generality that each object of $\mathcal{X}_0$ is coherent. Enlarging $\mathcal{X}_0$ if necessary, we may assume that $\mathcal{X}_0$ is closed under fiber products, finite coproducts (Remark A.2.1.8), and that for every simplicial object $U_\bullet$ of $\mathcal{X}_0$ which satisfies the Kan condition, the geometric realization $|U_\bullet|$ belongs to $\mathcal{X}$ (Theorem A.5.5.1). Then $\mathcal{X}_0$ satisfies the hypotheses of Proposition A.6.5.7, so that $\mathcal{X}^{\mathrm{coh}} = \mathcal{X}_0$ is essentially small. □

Warning A.6.6.2. Using a more refined argument, one can show that the ∞-category $\mathcal{X}^{\mathrm{coh}}$ is essentially small for *any* hypercomplete ∞-topos $\mathcal{X}$: the local coherence of $\mathcal{X}$ is not necessary. However, the hypothesis that $\mathcal{X}$ is hypercomplete cannot be eliminated.

Remark A.6.6.3. Let $\mathcal{X}$ be an ∞-topos which is hypercomplete and locally coherent. Combining Propositions A.6.6.1 and A.3.4.2, we see that the Yoneda embedding of $\mathcal{X}$ induces an equivalence of ∞-categories $\mathcal{X} \to \mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{coh}})^{\mathrm{hyp}}$.

Corollary A.6.6.4. *Let $\mathcal{C}$ be a local ∞-pretopos which is essentially small. The following conditions are equivalent:*

(1) *The ∞-pretopos $\mathcal{C}$ is hypercomplete (in the sense of Definition A.6.5.3).*

(2) *There exists an ∞-topos $\mathcal{X}$ which is hypercomplete and locally coherent and an equivalence of ∞-categories $\mathcal{C} \simeq \mathcal{X}^{\mathrm{coh}}$.*

(3) *The ∞-pretopos $\mathcal{C}$ satisfies the following stronger version of the requirements of Definition A.6.5.3:*

   (a') *Every semisimplicial object $X_\bullet$ of $\mathcal{C}$ which satisfies the Kan condition, there exists an object $X \in \mathcal{C}$ such that $X_\bullet$ can be lifted to a semisimplicial object of $\mathcal{C}_{/X}$ which is a hypercovering of X.*

(b') *If $X \in \mathcal{C}$ and $X_\bullet$ is a semisimplicial object of $\mathcal{C}_{/X}$ which is a hypercovering of X, then X is a colimit of the diagram $\mathbf{\Delta}_s^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}_{/X} \to \mathcal{C}$.*

Proof. The implication (1) ⇒ (2) follows from Proposition A.6.5.8, the implication (2) ⇒ (3) from Proposition A.6.6.1, and the implication (3) ⇒ (1) is trivial. □

Theorem A.6.6.5. *Let $\mathcal{E}$ denote the subcategory of $\infty\mathcal{T}\mathrm{op}$ whose objects are ∞-topoi $\mathcal{X}$ which are hypercomplete, coherent, and locally coherent and whose morphisms are functors $f^* : \mathcal{X} \to \mathcal{Y}$ which preserve small colimits, finite limits, and carry coherent objects of $\mathcal{X}$ to coherent objects of $\mathcal{Y}$. Then the construction $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{coh}}$ induces a fully faithful functor $\rho : \mathcal{E}^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}$, whose essential image is spanned by the ∞-pretopoi which are hypercomplete and essentially small.*

Proof. It follows from Proposition A.6.6.1 that if $\mathcal{X}$ is an ∞-topos which is hypercomplete, coherent, and locally coherent, then the full subcategory $\mathcal{X}^{\mathrm{coh}} \subseteq \mathcal{X}$ of coherent objects is a hypercomplete ∞-pretopos which is essentially small. This proves that the functor ρ is well-defined. We next claim that ρ is fully faithful. Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi which are hypercomplete, coherent, and locally coherent; we wish to show that the canonical map

$$\theta : \mathrm{Map}_{\mathcal{E}}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}}(\mathcal{Y}^{\mathrm{coh}}, \mathcal{X}^{\mathrm{coh}}) \simeq \mathrm{Fun}^{\mathrm{pre}}(\mathcal{Y}^{\mathrm{coh}}, \mathcal{X}^{\mathrm{coh}})^{\simeq}$$

is a homotopy equivalence. Using Remark A.6.6.3, we can identify $\mathcal{Y}$ with the hypercomplete ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{Y}^{\mathrm{coh}})^{\mathrm{hyp}}$. Since the ∞-pretopos $\mathcal{Y}^{\mathrm{coh}}$ is hypercomplete (Proposition A.6.6.1), an object of $\mathcal{S}\mathrm{hv}(\mathcal{Y}^{\mathrm{coh}})^{\mathrm{hyp}}$ is coherent if and only if it belongs to the essential image of the Yoneda embedding $j : \mathcal{Y}^{\mathrm{coh}} \to \mathcal{S}\mathrm{hv}(\mathcal{Y}^{\mathrm{coh}})^{\mathrm{hyp}}$ (Proposition A.6.5.8). Unwinding the definitions, we see the map θ fits into a pullback diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}}(\mathcal{X}, \mathcal{Y}) & \xrightarrow{\theta} & \mathrm{Fun}^{\mathrm{pre}}(\mathcal{Y}^{\mathrm{coh}}, \mathcal{X}^{\mathrm{coh}})^{\simeq} \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{Y}) & \xrightarrow{\theta'} & \mathrm{Fun}^{\mathrm{pre}}(\mathcal{Y}^{\mathrm{coh}}, \mathcal{X})^{\simeq}. \end{array}$$

It will therefore suffice to show that the map θ' is a homotopy equivalence. This follows from the observation that we can write θ' as a composition

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{Y}) \xrightarrow{\phi} \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{coh}}), \mathcal{Y}) \xrightarrow{\psi} \mathrm{Fun}^{\mathrm{pre}}(\mathcal{Y}^{\mathrm{coh}}, \mathcal{X})^{\simeq},$$

where ϕ is a homotopy equivalence by virtue of our assumption that $\mathcal{X}$ is hypercomplete and ψ is a homotopy equivalence by virtue of Proposition A.6.4.4. This shows that ρ is fully faithful. The description of the essential image of ρ follows from Corollary A.6.6.4. □

A.6.7 Truncations in ∞-Pretopoi

Let $\mathcal{C}$ be an arbitrary ∞-category. Recall that an object $C \in \mathcal{C}$ is said to be *n-truncated* if the mapping space $\mathrm{Map}_{\mathcal{C}}(C', C)$ is n-truncated for each $C' \in \mathcal{C}$. We say that a morphism $u : C \to D$ *exhibits D as an n-truncation of C* if the object D is n-truncated and, for every n-truncated object $E \in \mathcal{C}$, composition with u induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(D, E) \to \mathrm{Map}_{\mathcal{C}}(C, E)$. In this case, the morphism u (and the object $D \in \mathcal{C}$) are uniquely determined by C (up to canonical equivalence). Our next goal is to show that truncations exist in any ∞-pretopos:

Proposition A.6.7.1. *Let $\mathcal{C}$ be an ∞-pretopos and let $n \geqslant -2$ be an integer. Then:*

(a) *For every object $C \in \mathcal{C}$, there exists a morphism $u : C \to D$ which exhibits D as an n-truncation of C.*

(b) *Let $\mathcal{C}'$ be another ∞-pretopos and let $f \in \mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{C}')$ be a morphism of ∞-pretopoi (see Definition A.6.4.1). If $u : C \to D$ is a morphism in $\mathcal{C}$ which exhibits D as an n-truncation of C, then $f(u) : f(C) \to f(D)$ is a morphism in $\mathcal{C}'$ which exhibits $f(D)$ as an n-truncation of $f(C)$.*

Corollary A.6.7.2. *Let $\mathcal{C}$ be an ∞-pretopos, let $n \geqslant -2$ be an integer, and let $\mathcal{C}_{\leqslant n}$ be the full subcategory of $\mathcal{C}$ spanned by the n-truncated objects. Then the inclusion functor $\mathcal{C}_{\leqslant n} \hookrightarrow \mathcal{C}$ admits a left adjoint $\tau_{\leqslant n} : \mathcal{C} \to \mathcal{C}_{\leqslant n}$.*

Corollary A.6.7.3. *Let $\mathcal{C}$ and $\mathcal{C}'$ be ∞-pretopoi and let $f \in \mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{C}')$ be an ∞-pretopos morphism from $\mathcal{C}$ to $\mathcal{C}'$. Then the diagram of ∞-categories*

$$\begin{array}{ccc} \mathcal{C}_{\leqslant n} & \longrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle f|_{\mathcal{C}_{\leqslant n}}} & & \downarrow{\scriptstyle f} \\ \mathcal{C}'_{\leqslant n} & \longrightarrow & \mathcal{C}' \end{array}$$

is left adjointable. In particular, the diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C}_{\leqslant n} & \xleftarrow{\tau_{\leqslant n}} & \mathcal{C} \\ \downarrow{\scriptstyle f|_{\mathcal{C}_{\leqslant n}}} & & \downarrow{\scriptstyle f} \\ \mathcal{C}'_{\leqslant n} & \xleftarrow{\tau_{\leqslant n}} & \mathcal{C}' \end{array}$$

commutes up to (canonical) homotopy.

We will deduce Proposition A.6.7.1 from the following:

Proposition A.6.7.4. *Let $\mathcal{C}$ be a small local ∞-pretopos, which we regard as equipped with the effective epimorphism topology of Corollary A.6.2.3. Let $n \geqslant -2$ be an integer and suppose we are given a pair of morphisms $\mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H}$ in $\mathcal{S}\mathrm{hv}(\mathcal{C})$, where f is $(n+1)$-connective and g is n-truncated. If $\mathscr{F}$ and $\mathscr{H}$ belong to the essential image of the Yoneda embedding $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$, then so does $\mathscr{G}$.*

Proof. We proceed by induction on n. If $n = -2$, the g is an equivalence and there is nothing to prove. Let us therefore assume that $n > -2$. Let $\mathscr{F}_\bullet$, $\mathscr{G}_\bullet$, and $\mathscr{H}_\bullet$ denote the simplicial objects of $\mathcal{C}$ given by the Čech nerves of the morphisms $\mathrm{id}_{\mathscr{F}}$, f, and $g \circ f$, respectively. We then have a diagram of simplicial objects

$$\mathscr{F}_\bullet \xrightarrow{f_\bullet} \mathscr{G}_\bullet \xrightarrow{g_\bullet} \mathscr{H}_\bullet .$$

For each $k \geqslant 0$, the map f_k fits into a commutative diagram

$$\begin{array}{ccccc} & & \mathscr{G}_k & & \\ & \overset{f_k}{\nearrow} & & \overset{h}{\searrow} & \\ \mathscr{F}_k & & \xrightarrow{\mathrm{id}} & & \mathscr{F} \end{array}$$

where h is a composition of pullbacks of f, and is therefore an $(n+1)$-connective morphism in $\mathcal{S}\mathrm{hv}(\mathcal{G})$. It follows that f_k is n-connective. Note that g_k is a pullback of the diagonal map $\mathscr{G} \to \mathscr{H} \times_{\mathscr{H}^{k+1}} \mathscr{G}^{k+1}$; since g is n-truncated, we conclude that g_k is $(n-1)$-truncated. Since the functor j is fully faithful and left exact, the essential image of j is closed under finite limits; it follows that $\mathscr{F}_k$ and $\mathscr{H}_k$ belong to the image of j. Applying our inductive hypothesis, we deduce that $\mathscr{G}_k$ also belongs to the image of j. Using the fact that j is fully faithful, we can identify $\mathcal{G}_\bullet$ with the image under j of a groupoid object $X_\bullet$ of $\mathcal{C}$. Since $\mathcal{C}$ is a local ∞-pretopos, we can identify $X_\bullet$ with the Čech nerve of the induced map $u : X_0 \to |X_\bullet|$ in $\mathcal{C}$. Because the functor j is left exact, it follows that $\mathcal{G}_\bullet \simeq j(X_\bullet)$ is the Čech nerve of the induced map $j(u) : j(X_0) \to j|X_\bullet|$. The map u is an effective epimorphism in $\mathcal{C}$ and therefore generates a covering with respect to the effective epimorphism topology, so that $j(u)$ is an effective epimorphism in the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$. It follows that we can identify $j|X_\bullet|$ with the geometric realization $|\mathcal{G}_\bullet|$. Our assumption that $n > -2$ guarantees that f is an effective epimorphism so that the natural map $|\mathcal{G}_\bullet| \to \mathcal{G}$ is an equivalence, and therefore $\mathcal{G} \simeq j|X_\bullet|$ belongs to the essential image of j as desired. □

Proof of Proposition A.6.7.1. We first prove (a). Let $\mathcal{C}$ be an ∞-pretopos, let $n \geqslant -2$ be an integer, and let C be an object of $\mathcal{C}$. We wish to show that there exists a morphism $u : C \to D$ in $\mathcal{C}$ which exhibits D as an n-truncation of $\mathcal{C}$. Passing to a larger universe if necessary, we may assume that $\mathcal{C}$ is small. Let $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ be the Yoneda embedding. Since $\mathcal{C}$ has a final object, the final object $\mathbf{1} \in \mathcal{S}\mathrm{hv}(\mathcal{C})$ belongs to the essential image of

j. Applying Proposition A.6.7.4 to the diagram $j(C) \xrightarrow{u'} \tau_{\leqslant n} j(C) \to \mathbf{1}$ in $\mathcal{S}\mathrm{hv}(\mathcal{C})$, we can assume that $\tau_{\leqslant n} j(C)$ can be written as $j(D)$ for some object $D \in \mathcal{C}$. Since j is fully faithful, we can further assume that $u' = j(u)$ for some morphism $u : C \to D$ in $\mathcal{C}$. We claim that u has the desired property. Since $j(D) = \tau_{\leqslant n} j(C)$ is n-truncated, we immediately deduce that the object $D \in \mathcal{C}$ is n-truncated. To complete the proof of (a), it suffices to observe that for any n-truncated object $E \in \mathcal{C}$, the object $j(E) \in \mathcal{S}\mathrm{hv}(\mathcal{C})$ is n-truncated so that the canonical map

$$\mathrm{Map}_{\mathcal{C}}(D, E) \simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(\tau_{\leqslant n} j(C), j(E)) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(j(C), j(E)) \simeq \mathrm{Map}_{\mathcal{C}}(C, E)$$

is a homotopy equivalence.

We now prove (b). Let $u : C \to D$ be as above and let $f : \mathcal{C} \to \mathcal{C}'$ be a morphism of ∞-pretopoi. We then have a commutative diagram of ∞-pretopoi

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{C}' \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ \mathcal{S}\mathrm{hv}(\mathcal{C}) & \xrightarrow{f^*} & \mathcal{S}\mathrm{hv}(\mathcal{C}'), \end{array}$$

where f^* is a geometric moprhism of ∞-topoi. Since $j(u)$ exhibits $j(D)$ as an n-truncation of $j(C)$, it follows that $(f^* \circ j)(u) = (j' \circ f)(u)$ exhibits $(j' \circ f)(D)$ as an n-truncation of $(j' \circ f)(C)$. Since j' is left exact and fully faithful, it follows that $f(u)$ exhibits $f(D)$ as an n-truncation of $f(C)$, as desired. □

Corollary A.6.7.5. *Let $\mathcal{X}$ be a coherent ∞-topos. If X is a coherent object of $\mathcal{X}$, then every truncation $\tau_{\leqslant n} X$ is also a coherent object of $\mathcal{X}$.*

A.7 Bounded ∞-Topoi

For each $0 \leqslant n \leqslant \infty$, let $\mathcal{T}\mathrm{op}_n$ denote the $(n+1)$-category of n-topoi (see Definition HTT.6.4.5.1). For $m \leqslant n$, the construction $\mathcal{X} \mapsto \tau_{\leqslant m-1} \mathcal{X}$ determines a forgetful functor $\mathcal{T}\mathrm{op}_n \to \mathcal{T}\mathrm{op}_m$; these forgetful functors determine a tower of ∞-categories

$$\cdots \to \mathcal{T}\mathrm{op}_3 \to \mathcal{T}\mathrm{op}_2 \to \mathcal{T}\mathrm{op}_1 \to \mathcal{T}\mathrm{op}_0 \,.$$

The canonical map $\infty\mathcal{T}\mathrm{op} \to \varprojlim \{\mathcal{T}\mathrm{op}_n\}_{n \geqslant 0}$ is not an equivalence: for example, if $\mathcal{X}$ is any ∞-topos and $\mathcal{X}^{\mathrm{hyp}} \subseteq \mathcal{X}$ is the full subcategory spanned by the hypercomplete objects, then the inclusion map $\iota_* : \mathcal{X}^{\mathrm{hyp}} \hookrightarrow \mathcal{X}$ is a morphism in $\infty\mathcal{T}\mathrm{op}$ whose image in each $\mathcal{T}\mathrm{op}_n$ is an equivalence (since truncated objects of $\mathcal{X}$ are automatically hypercomplete). Our goal in this section is to describe the relationship between $\infty\mathcal{T}\mathrm{op}$ and $\varprojlim \{\mathcal{T}\mathrm{op}_n\}_{n \geqslant 0}$ in more detail: in particular, we will show that it has both a right adjoint (Theorem **??**) and a left adjoint (Theorem **??**), both of which are fully faithful.

A.7.1 Boundedness

Fix an integer $n \geqslant 0$. Recall that an ∞-topos $\mathcal{X}$ is said to be *n-localic* if it has the form $\mathcal{S}\mathrm{hv}(\mathcal{C})$, where $\mathcal{C}$ is a small $(n-1)$-category which admits finite limits and is equipped with a Grothendieck topology (see the proof of Proposition HTT.6.4.5.9). We let $\infty\mathcal{T}op^{\leqslant n}$ denote the full subcategory of $\infty\mathcal{T}\mathrm{op}$ spanned by the n-localic ∞-topoi. It follows from Proposition HTT.6.4.5.7 that the forgetful functor $\infty\mathcal{T}\mathrm{op} \rightarrow \mathcal{T}\mathrm{op}_n$ admits a fully faithful right adjoint whose essential image is the full subcategory $\infty\mathcal{T}op^{\leqslant n} \subseteq \infty\mathcal{T}\mathrm{op}$. In particular, the ∞-category $\infty\mathcal{T}op^{\leqslant n}$ is a localization of $\infty\mathcal{T}\mathrm{op}$; we let $L_n : \infty\mathcal{T}\mathrm{op} \rightarrow \infty\mathcal{T}op^{\leqslant n}$ denote a right adjoint to the inclusion.

Proposition A.7.1.1. *Let $\mathcal{X}$ be an ∞-topos. The following conditions are equivalent:*

(a) *The ∞-topos $\mathcal{X}$ can be obtained as written as a limit of a small filtered diagram $\{\mathcal{X}_\alpha\}$ in the ∞-category $\infty\mathcal{T}\mathrm{op}$, where each of the ∞-topoi $\mathcal{X}_\alpha$ is n-localic for some integer n (which might depend on α).*

(b) *The canonical map $\theta_* : \mathcal{X} \rightarrow \varprojlim L_n \mathcal{X}$ is an equivalence in $\infty\mathcal{T}\mathrm{op}$.*

Definition A.7.1.2. Let $\mathcal{X}$ be an ∞-topos. We will say that $\mathcal{X}$ is *bounded* if it satisfies the equivalent conditions of Proposition A.7.1.1. We let $\infty\mathcal{T}\mathrm{op}^{\mathrm{b}}$ denote the full subcategory of $\infty\mathcal{T}\mathrm{op}$ spanned by the bounded ∞-topoi.

Example A.7.1.3. Let $\mathcal{X}$ be an ∞-topos which is n-localic for some integer n. Then $\mathcal{X}$ is bounded.

The essential content of Proposition A.7.1.1 is contained in the following observation:

Lemma A.7.1.4. *For each $n \geqslant 0$, the functor $L_n : \infty\mathcal{T}\mathrm{op} \rightarrow \infty\mathcal{T}\mathrm{op}$ preserves small filtered limits.*

Proof. Suppose we are given a diagram of ∞-topoi

$$\mathcal{J}^{\mathrm{op}} \rightarrow \infty\mathcal{T}\mathrm{op} \qquad (\alpha \in \mathcal{J}) \mapsto \mathcal{X}_\alpha,$$

indexed by a small filtered ∞-category $\mathcal{J}$. We wish to show that the canonical map $\rho_* : L_n(\varprojlim_{\alpha \in \mathcal{J}^{\mathrm{op}}} \mathcal{X}_\alpha) \rightarrow \varprojlim_{\alpha \in \mathcal{J}^{\mathrm{op}}} L_n \mathcal{X}_\alpha$ is an equivalence of ∞-topoi. Since the domain and codomain of ρ_* are n-localic, it will suffice to show that ρ_* induces an equivalence when restricted to $(n-1)$-truncated objects. In other words, we are reduced to showing that the map $\rho'_* : \varprojlim_{\alpha \in \mathcal{J}^{\mathrm{op}}} \mathcal{X}_\alpha \rightarrow \varprojlim_{\alpha \in \mathcal{J}^{\mathrm{op}}} L_n \mathcal{X}_\alpha$ is an equivalence when restricted to $(n-1)$-truncated objects. Because $\mathcal{J}$ is filtered, the domain and codomain of ρ'_* can be identified with the corresponding limits in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$ (Theorem HTT.6.3.3.1). In particular, we see that an object of $\varprojlim_{\alpha \in \mathcal{J}^{\mathrm{op}}} \mathcal{X}_\alpha$ (or of $\varprojlim_{\alpha \in \mathcal{J}^{\mathrm{op}}} L_n \mathcal{X}_\alpha$) is $(n-1)$-truncated if and only if its image in each $\mathcal{X}_\alpha$ (or $L_n \mathcal{X}_\alpha$) is $(n-1)$-truncated. We are therefore reduced to showing that the natural map

$$\varprojlim_{\alpha \in \mathcal{J}^{\mathrm{op}}} \tau_{\leqslant n-1} \mathcal{X}_\alpha \to \varprojlim_{\alpha \in \mathcal{J}^{\mathrm{op}}} \tau_{\leqslant n-1} L_n \mathcal{X}_\alpha$$

is an equivalence of ∞-categories. This is clear, since it is given as a limit of equivalences $\tau_{\leqslant n-1} \mathcal{X}_\alpha \to \tau_{\leqslant n-1} L_n \mathcal{X}_\alpha$. □

Proof of Proposition A.7.1.1. The implication $(b) \Rightarrow (a)$ is trivial. We will prove the converse. Suppose that $\mathcal{J}$ is a small filtered ∞-category and that $\mathcal{X}$ is given as the limit of a diagram

$$\mathcal{J}^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op} \qquad (\alpha \in \mathcal{J}) \mapsto \mathcal{X}_\alpha,$$

where each of the ∞-topoi $\mathcal{X}_\alpha$ is n_α-localic for some integer $n_\alpha \geqslant 0$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\sim} & \varprojlim_{\alpha \in \mathcal{J}^{\mathrm{op}}} \mathcal{X}_\alpha \\ \downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \varprojlim \theta_{\alpha *}} \\ \varprojlim_n L_n \mathcal{X} & \longrightarrow & \varprojlim_{\alpha \in \mathcal{J}^{\mathrm{op}}} \varprojlim_n L_n \mathcal{X}_\alpha . \end{array}$$

For each $\alpha \in \mathcal{J}^{\mathrm{op}}$, the tower $\{L_n \mathcal{X}_\alpha\}_{n \geqslant 0}$ is essentially constant (with value $\mathcal{X}_\alpha$) for $n \geqslant n_\alpha$, so the canonical map $\theta_{\alpha *} : \mathcal{X}_\alpha \to \varprojlim_n L_n \mathcal{X}_\alpha$ is an equivalence. Consequently, to show that θ_* is an equivalence, it will suffice to show that the lower horizontal map in the preceding diagram is an equivalence. For this, it will suffice to show that for each $n \geqslant 0$, the natural map $\rho_* : L_n \mathcal{X} \to \varprojlim_{\alpha \in \mathcal{J}^{\mathrm{op}}} L_n \mathcal{X}_\alpha$ is an equivalence, which follows from Lemma A.7.1.4. □

Proposition A.7.1.5. *The canonical map $\infty\mathcal{T}\mathrm{op} \to \varprojlim_n \mathcal{T}\mathrm{op}_n$ admit a fully faithful right adjoint, whose essential image is the full subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{b}} \subseteq \infty\mathcal{T}\mathrm{op}$ spanned by the bounded ∞-topoi.*

Proof. Using Proposition HTT.6.4.5.7 , we can identify $\varprojlim_n \mathcal{T}\mathrm{op}_n$ with the limit of the tower of ∞-categories

$$\cdots \to \infty\mathcal{T}op^{\leqslant 3} \xrightarrow{L_2} \infty\mathcal{T}op^{\leqslant 2} \xrightarrow{L_1} \infty\mathcal{T}op^{\leqslant 1} \xrightarrow{L_0} \infty\mathcal{T}op^{\leqslant 0}.$$

We can therefore identify the objects of $\varprojlim_n \mathcal{T}\mathrm{op}_n$ with towers of ∞-topoi

$$\cdots \to \mathcal{X}_3 \xrightarrow{\phi(2)_*} \mathcal{X}_2 \xrightarrow{\phi(1)_*} \mathcal{X}_1 \xrightarrow{\phi(0)_*} \mathcal{X}_0$$

where each $\phi(n)_*$ is a geometric morphism which exhibits $\mathcal{X}_n$ as an n-localic reflection of $\mathcal{X}_{n+1}$ (so that each $\mathcal{X}_n$ is n-localic). The natural map $F : \infty\mathcal{T}\mathrm{op} \to \varprojlim_n \mathcal{T}\mathrm{op}_n$ admits a right adjoint $G : \varprojlim_n \mathcal{T}\mathrm{op}_n \to \infty\mathcal{T}\mathrm{op}$, which carries a tower $\{\mathcal{X}_n\}_{n \geqslant 0}$ as above to the limit $\varprojlim \mathcal{X}_n$ (formed in the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi). Note that for any such tower $\{\mathcal{X}_n\}_{n\geqslant 0}$, Lemma A.7.1.4 implies that the canonical map $L_m \varprojlim_n \mathcal{X}_n \to \varprojlim_n L_m \mathcal{X}_n \simeq \mathcal{X}_m$ is an equivalence. In other words, the counit $v : F \circ G \to \mathrm{id}$ is an equivalence, so that the functor G is fully faithful. To complete the proof, it suffices to observe that the full subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{b}} \subseteq \infty\mathcal{T}\mathrm{op}$ is the essential image of G: in other words, that an ∞-topos $\mathcal{X}$ is bounded if and only if the unit map $\mathcal{X} \to (G \circ F)(\mathcal{X}) = \varprojlim_n L_n \mathcal{X}$ is an equivalence. □

Example A.7.1.6. Let $\mathcal{C}$ be a small ∞-category which admits finite limits and let S be a collection of morphisms in $\mathcal{C}$ which satisfies conditions (a) through (d) of Proposition A.3.2.1 and therefore determines a Grothendieck topology on $\mathcal{C}$. Suppose further that coproducts in $\mathcal{C}$ are disjoint and that every object $C \in \mathcal{C}$ is n-truncated for some integer $n \geqslant 0$ (which might depend on $\mathcal{C}$). For each $n \geqslant 0$, let $\mathcal{C}_{\leqslant n}$ denote the full subcategory of $\mathcal{C}$ spanned by the n-truncated objects, and let $S_{\leqslant n}$ denote the collection of morphisms in $\mathcal{C}_{\leqslant n}$ which belong to S. Since coproducts in $\mathcal{C}$ are disjoint, the collection of n-truncated objects of $\mathcal{C}$ is closed under coproducts for $n \geqslant 0$, so that the collection of morphisms $S_{\leqslant n}$ also satisfies the hypotheses of Proposition A.3.2.1 and therefore determines a Grothendieck topology on $\mathcal{C}_{\leqslant n}$. The equality $\mathcal{C} = \bigcup_{n\geqslant 0} \mathcal{C}_{\leqslant n}$ induces an equivalence of ∞-categories $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \varprojlim_n \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}_{\leqslant n}, \mathcal{S})$. Using the criterion of Proposition A.3.3.1, we see that this equivalence restricts to an equivalence $\mathcal{S}\mathrm{hv}(\mathcal{C}) \simeq \varprojlim_{n\geqslant 0} \mathcal{S}\mathrm{hv}(\mathcal{C}_{\leqslant n})$ in the ∞-category $\infty\mathcal{T}\mathrm{op}$. By construction, each of the ∞-topoi $\mathcal{S}\mathrm{hv}(\mathcal{C}_{\leqslant n})$ is n-localic. It follows that the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is bounded.

A.7.2 Postnikov Completeness

Let $\mathcal{C}$ be a presentable ∞-category. For each $n \geqslant 0$, we let $\tau_{\leqslant n}\mathcal{C}$ denote the full subcategory of $\mathcal{C}$ spanned by the n-truncated objects. The inclusion functor $\tau_{\leqslant n}\mathcal{C} \hookrightarrow \mathcal{C}$ has a left adjoint, which we will denote by $\tau_{\leqslant n}$. We have a tower of ∞-categories

$$\cdots \to \tau_{\leqslant 3}\mathcal{C} \xrightarrow{\tau_{\leqslant 2}} \tau_{\leqslant 2}\mathcal{C} \xrightarrow{\tau_{\leqslant 1}} \tau_{\leqslant 1}\mathcal{C} \xrightarrow{\tau_{\leqslant 0}} \tau_{\leqslant 0}\mathcal{C}.$$

Definition A.7.2.1. Let $\mathcal{C}$ be a presentable ∞-category. We will say that $\mathcal{C}$ is *Postnikov complete* if the canonical map

$$\mathcal{C} \to \varprojlim_n \tau_{\leqslant n}\mathcal{C} \qquad C \mapsto \{\tau_{\leqslant n} C\}_{n\geqslant 0}$$

is an equivalence of ∞-categories. We let $\infty\mathcal{T}\mathrm{op}^{\mathrm{c}}$ denote the full subcategory of $\infty\mathcal{T}\mathrm{op}$ spanned by those ∞-topoi which are Postnikov complete.

Warning A.7.2.2. The terminology of Definition A.7.2.1 is slightly different from that of [138], where we said that *Postnikov towers in $\mathcal{C}$ are convergent* if $\mathcal{C}$ satisfies the requirement of Definition A.7.2.1.

Remark A.7.2.3. Let $\mathcal{X}$ be an ∞-topos. If $\mathcal{X}$ is Postnikov complete, then every object $X \in \mathcal{X}$ can be recovered as the limit of its Postnikov tower $\{\tau_{\leqslant n} X\}_{n \geqslant 0}$. It follows that every Postnikov complete ∞-topos is hypercomplete. However, the converse is false.

The failure of an ∞-topos $\mathcal{X}$ to be Postnikov complete can always be corrected by passing to a suitable modification of $\mathcal{X}$:

Theorem A.7.2.4. *Let $\mathcal{X}$ be an ∞-topos, and let $\widehat{\mathcal{X}}$ denote the limit of the tower of ∞-categories*

$$\cdots \to \tau_{\leqslant 3}\,\mathcal{X} \xrightarrow{\tau_{\leqslant 2}} \tau_{\leqslant 2}\,\mathcal{X} \xrightarrow{\tau_{\leqslant 1}} \tau_{\leqslant 1}\,\mathcal{X} \xrightarrow{\tau_{\leqslant 0}} \tau_{\leqslant 0}\,\mathcal{X} = \mathcal{X}^{\heartsuit}.$$

Then:

- (a) *The ∞-category $\widehat{\mathcal{X}}$ is an ∞-topos.*
- (b) *The canonical map $e^* : \mathcal{X} \to \widehat{\mathcal{X}}$ is a geometric morphism (that is, it preserves small colimits and finite limits).*
- (c) *For each $n \geqslant 0$, the map $e^* : \mathcal{X} \to \widehat{\mathcal{X}}$ induces an equivalence when restricted to n-truncated objects.*
- (d) *The ∞-topos $\widehat{\mathcal{X}}$ is Postnikov complete.*
- (e) *If $\mathcal{Y}$ is a Postnikov complete ∞-topos, then composition with e^* induces an equivalence of ∞-categories* $\mathrm{Fun}^*(\widehat{\mathcal{X}}, \mathcal{Y}) \to \mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$.

Definition A.7.2.5. Let $\mathcal{X}$ be an ∞-topos. We will refer to the ∞-topos $\widehat{\mathcal{X}} = \varprojlim \{\tau_{\leqslant n}\,\mathcal{X}\}_{n \geqslant 0}$ of Theorem A.7.2.4 as the *Postnikov completion* of $\mathcal{X}$.

Before giving the proof of Theorem A.7.2.4, let us describe some of its consequences.

Corollary A.7.2.6. *The inclusion functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{c}} \hookrightarrow \infty\mathcal{T}\mathrm{op}$ admits a right adjoint, which carries each ∞-topos $\mathcal{X}$ to its Postnikov completion $\widehat{\mathcal{X}}$.*

Corollary A.7.2.7. *The natural map $\rho : \infty\mathcal{T}\mathrm{op} \to \varprojlim_n \mathcal{T}\mathrm{op}_n$ admits a fully faithful left adjoint, whose essential image is the full subcategory of $\infty\mathcal{T}\mathrm{op}^{\mathrm{c}} \subseteq \infty\mathcal{T}\mathrm{op}$.*

Proof. Let $\infty\mathcal{T}\mathrm{op}^{\mathrm{b}}$ denote the full subcategory of $\infty\mathcal{T}\mathrm{op}$ spanned by the bounded ∞-pretopoi (Definition A.7.1.2). Using Proposition A.7.1.5, we deduce that there is an equivalence of ∞-categories $\varprojlim_n \mathcal{T}\mathrm{op}_n \simeq \infty\mathcal{T}\mathrm{op}^{\mathrm{b}}$ whose composition with ρ yields a functor $L : \infty\mathcal{T}\mathrm{op} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{b}}$ which is left adjoint to the inclusion $\infty\mathcal{T}\mathrm{op}^{\mathrm{b}} \hookrightarrow \infty\mathcal{T}\mathrm{op}$. Using Corollary A.7.2.6, we can regard the formation of Postnikov completions $\mathcal{X} \mapsto \widehat{\mathcal{X}}$ as a functor from the ∞-category $\infty\mathcal{T}\mathrm{op}$ to itself. To complete the proof of Corollary A.7.2.7, it will suffice to verify the following:

(i) The construction $\mathcal{X} \mapsto \widehat{\mathcal{X}}$ determines a functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{b}} \to \infty\mathcal{T}\mathrm{op}$ which is left adjoint to L.

(ii) An ∞-topos $\mathcal{Y}$ is Postnikov complete if and only if it has the form $\widehat{\mathcal{X}}$ for some $\mathcal{X} \in \infty\mathcal{T}\mathrm{op}^{\mathrm{b}}$.

To prove (i), we must establish the existence of a homotopy equivalence $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\widehat{\mathcal{X}}, \mathcal{Y}) \simeq \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, L\,\mathcal{Y})$ depending functorially on $\mathcal{X} \in \infty\mathcal{T}\mathrm{op}^{\mathrm{b}}$ and $\mathcal{Y} \in \infty\mathcal{T}\mathrm{op}$. We obtain this homotopy equivalence by inspecting the commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\widehat{\mathcal{X}}, \mathcal{Y}) & \longleftarrow & \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\widehat{\mathcal{X}}, \widehat{\mathcal{Y}}) \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha} \\
\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, L\,\mathcal{Y}) & \longrightarrow & \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\widehat{\mathcal{X}}, L\,\mathcal{Y}) & \longleftarrow & \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\widehat{\mathcal{X}}, \widehat{L\,\mathcal{Y}}) \\
\uparrow & & \uparrow & & \\
\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(L\,\mathcal{X}, L\,\mathcal{Y}) & \xrightarrow{\beta} & \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(L\widehat{\mathcal{X}}, L\,\mathcal{Y}). & &
\end{array}$$

We claim that each of the maps in this diagram is a homotopy equivalence. Here the lower vertical maps are homotopy equivalences because $L\,\mathcal{Y}$ is bounded, and the right horizontal maps are homotopy equivalences by virtue of Corollary A.7.2.6 (since $\widehat{\mathcal{X}}$ is Postnikov complete). We are therefore reduced to proving that α and β are homotopy equivalences. This follows from the observations that the geometric morphisms $\mathcal{Y} \to L\,\mathcal{Y}$ and $\widehat{\mathcal{X}} \to \mathcal{X}$ are equivalences on n-truncated objects for each $n \geqslant 0$ (and therefore induce equivalences $\widehat{\mathcal{Y}} \to \widehat{L\,\mathcal{Y}}$ and $L\widehat{\mathcal{X}} \to L\,\mathcal{X}$, respectively). This completes the proof of (i). The "if" direction of (ii) follows from assertion (d) of Theorem A.7.2.4, and the "only if" direction follows from the observation that for any Postnikov complete ∞-topos $\mathcal{X}$, we have equivalences $\mathcal{X} \leftarrow \widehat{\mathcal{X}} \to \widehat{L\,\mathcal{X}}$ in the ∞-category $\infty\mathcal{T}\mathrm{op}$. □

Corollary A.7.2.8. *The formation of Postnikov completions induces an equivalence of ∞-categories $\infty\mathcal{T}\mathrm{op}^{\mathrm{b}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{c}}$ (with homotopy inverse given by the functor $L : \infty\mathcal{T}\mathrm{op}^{\mathrm{c}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{b}}$ appearing in the proof of Corollary A.7.2.7).*

Warning A.7.2.9. Corollary A.7.2.8 asserts that the datum of a bounded ∞-topos $\mathcal{X}$ is equivalent to the datum of a Postnikov complete ∞-topos $\mathcal{Y}$. Beware that the $\mathcal{X}$ and $\mathcal{Y}$ are usually not equivalent as ∞-topoi (though they are related by a geometric morphism $\mathcal{Y} \to \mathcal{X}$ which induces an equivalence on n-truncated objects for each $n \geqslant 0$). In other words, the condition that an ∞-topos $\mathcal{Z}$ is bounded is *not* equivalent to the condition that $\mathcal{Z}$ is Postnikov complete. Both of these conditions assert that $\mathcal{Z}$ can be recovered from its truncated objects, but not in the same way:

- The ∞-topos $\mathcal{Z}$ is Postnikov complete if and only if it can be recovered as the limit of the tower of ∞-categories

$$\cdots \to \tau_{\leqslant 3}\,\mathcal{Z} \xrightarrow{\tau_{\leqslant 2}} \tau_{\leqslant 2}\,\mathcal{Z} \xrightarrow{\tau_{\leqslant 1}} \tau_{\leqslant 1}\,\mathcal{Z} \xrightarrow{\tau_{\leqslant 0}} \tau_{\leqslant 0}\,\mathcal{Z}$$

- The ∞-topos $\mathcal{Z}$ is bounded if and only if it can be recovered as the limit of its tower of geometric morphisms

$$\cdots \to \mathcal{Z}_3 \xrightarrow{\phi(2)_*} \mathcal{Z}_2 \xrightarrow{\phi(1)_*} \mathcal{Z}_1 \xrightarrow{\phi(0)_*} \mathcal{Z}_0;$$

here $\mathcal{Z}_n$ denotes the n-localic ∞-topos corresponding to the n-topos $\tau_{\leqslant n-1}\,\mathcal{Z}$.

To put the difference in another way: the hypothesis that $\mathcal{Z}$ is Postnikov complete guarantees that every object $Z \in \mathcal{Z}$ can be recovered as the limit of its Postnikov tower $\{\tau_{\leqslant n} Z\}$ *in the ∞-category* $\mathcal{Z}$ (and that conversely, every Postnikov tower arises in this way), while the hypothesis of boundedness asserts that the ∞-topos $\mathcal{Z}$ can itself be recovered as the limit of its "localic Postnikov tower" $\{\mathcal{Z}_n\}_{n \geqslant 0}$ *in the ∞-category* $\infty\mathcal{T}\mathrm{op}$.

A.7.3 The Proof of Theorem A.7.2.4

The proof of Theorem A.7.2.4 will require some auxiliary constructions.

Notation A.7.3.1. Let $f : X \to Y$ be a morphism in an ∞-topos $\mathcal{X}$ and let $n \geqslant -1$ be an integer. We let $\tau_{\leqslant n}(f)$ denote an n-truncation of X regarded as an object of $\mathcal{X}_{/Y}$, so that the map f factors canonically as a composition $X \xrightarrow{f'} \tau_{\leqslant n}(f) \xrightarrow{f''} Y$ where f' is $(n+1)$-connective and f'' is n-truncated.

Fix an ∞-topos $\mathcal{X}$. For each $n \geqslant 0$, we let $\mathcal{E}_n$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{X})$ spanned by those morphisms $f : X \to Y$ where the object Y is n-truncated and the morphism f is $(n-1)$-truncated. We have inclusions of ∞-categories

$$\mathcal{E}_0 \hookrightarrow \mathrm{Fun}(\Delta^1, \tau_{\leqslant 0}\,\mathcal{X}) \hookrightarrow \mathcal{E}_1 \hookrightarrow \mathrm{Fun}(\Delta^1, \tau_{\leqslant 1}\,\mathcal{X}) \hookrightarrow \mathcal{E}_2 \hookrightarrow \mathrm{Fun}(\Delta^1, \tau_{\leqslant 2}\,\mathcal{X}) \hookrightarrow \cdots.$$

Note that each of these inclusion functors admits a left adjoint:

- For $n \geqslant 0$, the inclusion $\mathcal{E}_n \hookrightarrow \mathrm{Fun}(\Delta^1, \tau_{\leqslant 0}\,\mathcal{X})$ admits a left adjoint which carries a morphism $f : X \to Y$ to the induced map $\tau_{\leqslant n-1}(f) \to Y$.

- For $n \geqslant 0$, the inclusion $\mathrm{Fun}(\Delta^1, \tau_{\leqslant n}\,\mathcal{X}) \hookrightarrow \mathcal{E}_{n+1}$ admits a left adjoint which carries a morphism $f : X \to Y$ to the induced map $\tau_{\leqslant n} X \to \tau_{\leqslant n} Y$.

In particular, each of the inclusion functors $\mathcal{E}_n \hookrightarrow \mathcal{E}_{n+1}$ admits a left adjoint $\rho_n : \mathcal{E}_{n+1} \to \mathcal{E}_n$. Let $\widehat{\mathcal{X}} = \varprojlim\{\tau_{\leqslant n}\mathcal{X}\}$ be the Postnikov completion of $\mathcal{X}$. The towers of ∞-categories $\{\mathrm{Fun}(\Delta^1, \tau_{\leqslant n}\mathcal{X})\}_{n \geqslant 0}$ and $\{\mathcal{E}_n\}_{n\geqslant 0}$ are both right cofinal in the tower

$$\mathcal{E}_0 \leftarrow \mathrm{Fun}(\Delta^1, \tau_{\leqslant 0}\mathcal{X}) \leftarrow \mathcal{E}_1 \leftarrow \mathrm{Fun}(\Delta^1, \tau_{\leqslant 1}\mathcal{X}) \leftarrow \mathcal{E}_2 \leftarrow \mathrm{Fun}(\Delta^1, \tau_{\leqslant 2}\mathcal{X}) \leftarrow \cdots,$$

and therefore have the same limit. This proves the following:

Lemma A.7.3.2. *Let $\mathcal{X}$ be an ∞-topos and let $\widehat{\mathcal{X}}$ be its Postnikov completion. Then the ∞-category $\mathrm{Fun}(\Delta^1, \widehat{\mathcal{X}})$ can be identified with a limit of the tower of ∞-categories*

$$\mathcal{E}_0 \xleftarrow{\rho_0} \mathcal{E}_1 \xleftarrow{\rho_1} \mathcal{E}_2 \xleftarrow{\rho_2} \mathcal{E}_3 \leftarrow \cdots$$

constructed above.

Remark A.7.3.3. Let $\mathcal{X}$ be an ∞-topos and let $X = \{X_n\}_{n\geqslant 0}$ be an object of the Postnikov completion $\widehat{\mathcal{X}}$. Then the identification $\mathrm{Fun}(\Delta^1, \widehat{\mathcal{X}}) \simeq \varprojlim\{\mathcal{E}_n\}$ of Lemma A.7.3.2 induces an identification of the ∞-category $\widehat{\mathcal{X}}_{/X}$ with the limit of a tower of ∞-categories

$$\tau_{\leqslant -1}\mathcal{X}_{/X_0} \leftarrow \tau_{\leqslant 0}\mathcal{X}_{/X_1} \leftarrow \tau_{\leqslant 1}\mathcal{X}_{/X_2} \leftarrow \tau_{\leqslant 2}\mathcal{X}_{/X_3} \leftarrow \cdots.$$

Unwinding the definitions, we see that each of the transition maps in this diagram can be identified with the composition

$$\tau_{\leqslant n}\mathcal{X}_{/X_{n+1}} \xrightarrow{\tau_{\leqslant n-1}} \tau_{\leqslant n-1}\mathcal{X}_{/X_{n+1}} \simeq \tau_{\leqslant n-1}\mathcal{X}_{/X_n},$$

where the equivalence is supplied by the fact that the map $X_{n+1} \to X_n$ is $(n+1)$-connective (see Lemma HTT.7.2.1.13). It follows that each of these transition maps preserves finite products (Lemma HTT.6.5.1.2), so that each of the induced maps $\widehat{\mathcal{X}}_{/X} \to \tau_{\leqslant n-1}\mathcal{X}_{/X_n}$ also preserves finite products.

Proposition A.7.3.4. *Let $\mathcal{X}$ be an ∞-topos. Then the Postnikov completion $\widehat{\mathcal{X}}$ is also an ∞-topos.*

Proof. We first prove that colimits in $\widehat{\mathcal{X}}$ are universal. Fix a morphism $f : X \to Y$ in $\widehat{\mathcal{X}}$, which we can identify with a compatible family of maps $f_n : X_n \to Y_n$ in $\tau_{\leqslant n}\mathcal{X}$. We wish to show that the associated pullback functor $\widehat{\mathcal{X}}_{/Y} \to \widehat{\mathcal{X}}_{/X}$ preserves small colimits. Using Lemma A.7.3.2, we can identify this pullback functor with an inverse limit of pullback functors $\tau_{\leqslant n-1}\mathcal{X}_{/Y_n} \to \tau_{\leqslant n-1}\mathcal{X}_{/X_n}$, which preserve small colimits by virtue of the fact that colimits are universal in $\mathcal{X}$.

The evaluation functor $\mathrm{Fun}(\Delta^1, \widehat{\mathcal{X}}) \to \mathrm{Fun}(\{1\}, \widehat{\mathcal{X}}) \simeq \widehat{\mathcal{X}}$ is Cartesian fibration which is classified by a functor $\chi : \widehat{\mathcal{X}}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. To complete the proof that $\mathcal{X}$ is an ∞-topos, it will suffice to show that the functor χ commutes with small limits (Theorem HTT.6.1.3.9).

Using Lemma A.7.3.2, we can identify χ with a limit of functors $\widehat{\mathcal{X}}^{\mathrm{op}} \to (\tau_{\leqslant n}\,\mathcal{X})^{\mathrm{op}} \xrightarrow{\chi_n} \widehat{\mathcal{C}\mathrm{at}}_\infty$, where each χ_n classifies the Cartesian fibration $\mathcal{E}_n \to \tau_{\leqslant n}\,\mathcal{X}$. It will therefore suffice to show that each of the functors χ_n preserves small limits, which follows from Theorem HTT.6.4.4.5 (applied to the $(n+1)$-topos $\tau_{\leqslant n+1}\,\mathcal{X}$). □

Proposition A.7.3.5. *Let $\mathcal{X}$ be an ∞-topos. Then the canonical map $e^* : \mathcal{X} \to \widehat{\mathcal{X}}$ (given on objects by the construction $X \mapsto \{\tau_{\leqslant n} X\}_{n \geqslant 0}$) preserves small colimits and finite limits.*

Proof. Each of the truncation functors $\tau_{\leqslant n} : \mathcal{X} \to \tau_{\leqslant n}\,\mathcal{X}$ preserves small colimits and final objects, so that the functor e^* preserves small colimits and final objects. To complete the proof that e^* is a geometric morphism, it will suffice to show that it preserves pullbacks. In other words, it will suffice to show that for each object $X \in \mathcal{X}$, the induced map $\mathcal{X}_{/X} \to \widehat{\mathcal{X}}_{/e^*X}$ preserves finite products. By virtue of Remark A.7.3.3, this is equivalent to the requirement that each of the composite functors $\mathcal{X}_{/X} \to \widehat{\mathcal{X}}_{/e^*X} \to \tau_{\leqslant n-1}\,\mathcal{X}_{/\tau_{\leqslant n} X}$ preserves finite products. Unwinding the definitions, we see that each of these functors can be rewritten as a composition $\mathcal{X}_{/X} \xrightarrow{\tau_{\leqslant n-1}} \tau_{\leqslant n-1}\,\mathcal{X}_{/X} \simeq \tau_{\leqslant n-1}\,\mathcal{X}_{/\tau_{\leqslant n} X}$, where the truncation functor $\tau_{\leqslant n-1}$ preserves finite products by virtue of Lemma HTT.6.5.1.2 and the equivalence is provided by Lemma HTT.7.2.1.13 . □

Remark A.7.3.6. The functor $e^* : \mathcal{X} \to \widehat{\mathcal{X}}$ of Proposition A.7.3.5 admits a right adjoint e_*, which carries an object $\{X_n\}_{n \geqslant 0}$ of $\widehat{\mathcal{X}}$ to the limit $\varprojlim X_n$ in the ∞-category $\mathcal{X}$. In particular, the composite functor $e_* e^* : \mathcal{X} \to \mathcal{X}$ associates to each object $X \in \mathcal{X}$ the limit $\varprojlim \tau_{\leqslant n} X$ of its Postnikov tower. Beware that that the natural map $X \to e_* e^* X$ need not be an equivalence. However, it is always an equivalence when the object $X \in \mathcal{X}$ is m-truncated for some integer m.

Proposition A.7.3.7. *Let $\mathcal{X}$ be an ∞-topos, let $\widehat{\mathcal{X}}$ be its Postnikov completion, and let $n \geqslant 0$ be an integer. Then the canonical map $e^* : \mathcal{X} \to \widehat{\mathcal{X}}$ is an equivalence when restricted to n-truncated objects.*

Proof. The functor e^* and its right adjoint e_* are both left exact, and therefore restrict to adjoint functors

$$\tau_{\leqslant n}\,\mathcal{X} \underset{e_{n*}}{\overset{e_n^*}{\rightleftarrows}} \tau_{\leqslant n}\widehat{\mathcal{X}}.$$

It follows from Remark A.7.3.6 that the functor e_n^* is fully faithful. It will therefore suffice to show that the functor e_n^* is essentially surjective. Let X be an n-truncated object of $\widehat{\mathcal{X}}$, given by a compatible sequence of objects $X_m \in \tau_{\leqslant m}\,\mathcal{X}$. We will show that if X is n-truncated, then the canonical map $X \to e^* X_n$ is an equivalence in $\widehat{\mathcal{X}}$. This is equivalent to the following:

(∗) For each $m \geqslant n$, the canonical map $X_m \to X_n$ is an equivalence in $\mathcal{X}$.

We prove $(*)$ by induction on m, the case $m = n$ being trivial. To carry out the inductive step, it will suffice to show that for $m > n$, the map $\rho : X_m \to X_{m-1}$ is an equivalence. Since ρ exhibits X_{m-1} as an $(m-1)$-truncation of X_m, this is equivalent to the statement that X_m is $(m-1)$-truncated: that is, that the diagonal map $\delta : X_m \to X_m^{S^m}$ is an equivalence. Note that δ is automatically (-1)-truncated (since X_m is m-truncated); consequently, it will suffice to show that δ is an effective epimorphism. Observe that the map δ factors as a composition

$$X_m \simeq \tau_{\leqslant m} X_{2m} \xrightarrow{\delta'} \tau_{\leqslant m}(X_{2m}^{S^m}) \xrightarrow{\delta''} (\tau_{\leqslant m} X_{2m})^{S^m} \simeq X_m^{S^m}.$$

Because S^m has dimension $\leqslant m$, the natural map $X_{2m}^{S^m} \to (\tau_{\leqslant m} X_{2m})^{S^m}$ is an effective epimorphism. This map factors through δ'', so that δ'' is an effective epimorphism. Consequently, to complete the proof, it will suffice to show that δ' is an effective epimorphism. In fact, the map δ' is an equivalence: it is the image under the tautological map $\widehat{\mathcal{X}} \to \tau_{\leqslant m} \mathcal{X}$ of the diagonal $X \to X^{S^m}$, which is an equivalence by virtue of our assumptions that X is n-truncated and $m > n$. □

Corollary A.7.3.8. *Let $\mathcal{X}$ be an ∞-topos and let $\widehat{\mathcal{X}} = \varprojlim\{\tau_{\leqslant n} \mathcal{X}\}$ be its Postnikov completion. Then, for each integer $n \geqslant 0$, the tautological map $F_n : \widehat{\mathcal{X}} \to \tau_{\leqslant n} \mathcal{X}$ admits a fully faithful right adjoint $G_n : \tau_{\leqslant n} \mathcal{X} \to \widehat{\mathcal{X}}$, whose essential image is the full subcategory of $\widehat{\mathcal{X}}$ spanned by the n-truncated objects.*

Proof. The existence of the right adjoint G_n follows from Corollary HTT.5.5.2.9 . Since G_n preserves finite limits, it carries n-truncated objects to n-truncated objects and therefore takes values in the full subcategory $\tau_{\leqslant n}\widehat{\mathcal{X}}$. Let $e^* : \mathcal{X} \to \widehat{\mathcal{X}}$ denote the canonical map, so that the composition $F_n \circ e^*$ can be identified with the n-truncation functor $\tau_{\leqslant n} : \mathcal{X} \to \tau_{\leqslant n} \mathcal{X}$. Passing to right adjoints, we deduce that the composite functor $\tau_{\leqslant n} \mathcal{X} \xrightarrow{G_n} \widehat{\mathcal{X}} \xrightarrow{e_*} \mathcal{X}$ is homotopic to the inclusion map. Since the functor e_* restricts to an equivalence $\tau_{\leqslant n}\widehat{\mathcal{X}} \to \tau_{\leqslant n} \mathcal{X}$ (Proposition A.7.3.7), it follows that G_n induces an equivalence $\tau_{\leqslant n} \mathcal{X} \simeq \tau_{\leqslant n}\widehat{\mathcal{X}}$. □

Proposition A.7.3.9. *Let $\mathcal{X}$ be an ∞-topos and let $\widehat{\mathcal{X}}$ denote its Postnikov completion. Then $\widehat{\mathcal{X}}$ is Postnikov complete.*

Proof. By construction, the Postnikov completion $\widehat{\mathcal{X}}$ can be identified with the limit of the tower of ∞-categories

$$\cdots \to \tau_{\leqslant 3} \mathcal{X} \to \tau_{\leqslant 2} \mathcal{X} \to \tau_{\leqslant 1} \mathcal{X} \to \tau_{\leqslant 0} \mathcal{X}.$$

Using Corollary A.7.3.8, we can identify this tower with

$$\cdots \to \tau_{\leqslant 3}\widehat{\mathcal{X}} \to \tau_{\leqslant 2}\widehat{\mathcal{X}} \to \tau_{\leqslant 1}\widehat{\mathcal{X}} \to \tau_{\leqslant 0}\widehat{\mathcal{X}}$$

(as a diagram in $(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\widehat{\mathcal{X}}/}$), so that $\widehat{\mathcal{X}}$ is Postnikov complete as desired. □

Proposition A.7.3.10. *Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi. If $\mathcal{Y}$ is Postnikov complete, then composition with the natural map $e^* : \mathcal{X} \to \widehat{\mathcal{X}}$ induces an equivalence of ∞-categories* $\mathrm{Fun}^*(\widehat{\mathcal{X}}, \mathcal{Y}) \to \mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$.

Proof. For every pair of presentable ∞-categories $\mathcal{Z}$ and $\mathcal{Z}'$, let $\mathrm{LFun}(\mathcal{Z}, \mathcal{Z}')$ denote the full subcategory of $\mathrm{Fun}(\mathcal{Z}, \mathcal{Z}')$ spanned by those functors which preserve small colimits. We first show that composition with e^* induces an equivalence of ∞-categories $\theta : \mathrm{LFun}(\widehat{\mathcal{X}}, \mathcal{Y}) \to \mathrm{LFun}(\mathcal{X}, \mathcal{Y})$. Using our assumption that $\mathcal{Y}$ is Postnikov complete, we can identify θ with the limit of a tower of functors $\theta_n : \mathrm{LFun}(\widehat{\mathcal{X}}, \tau_{\leqslant n} \mathcal{Y}) \to \mathrm{LFun}(\mathcal{X}, \tau_{\leqslant n} \mathcal{Y})$, each of which is an equivalence by virtue of Proposition A.7.3.7.

To complete the proof, it will suffice to show that a colimit-preserving functor $f^* : \widehat{\mathcal{X}} \to \mathcal{Y}$ is left exact if and only if the composite functor $f^* \circ e^* : \mathcal{X} \to \mathcal{Y}$ is left exact. The "only if" direction follows from Proposition A.7.3.5. To prove the converse, suppose that $f'^* = f^* \circ e^*$ preserves finite limits. It is then clear that the functor f^* preserves final objects. To show that f^* is left exact, it will suffice to show that for each object $X = \{X_n\}_{n \geqslant 0}$ in $\widehat{\mathcal{X}}$, the induced map $\widehat{\mathcal{X}}_{/X} \to \mathcal{Y}_{/f^* X}$ preserves finite products. Using Remark A.7.3.3 (and the Postnikov completeness of $\mathcal{Y}$), we can write the ∞-category $\mathcal{Y}_{/f^* X}$ as the limit of a tower of ∞-categories $\{\tau_{\leqslant n-1} \mathcal{Y}_{/f'^* X_n}\}$ where the transition maps preserve finite products. We are therefore reduced to showing that each of the composite maps $\widehat{\mathcal{X}}_{/X} \to \mathcal{Y}_{/f^* X} \to \tau_{\leqslant n-1} \mathcal{Y}_{/f'^* X_n}$ preserves finite products. Unwinding the definitions, we see that each of these maps can be described as a composition $\widehat{\mathcal{X}}_{/X} \xrightarrow{u} \tau_{\leqslant n-1} \mathcal{X}_{/X_n} \xrightarrow{v} \tau_{\leqslant n-1} \mathcal{Y}_{/f'^* X_n}$, where the functor v preserves finite products because f'^* is left exact, and the functor u preserves finite products by virtue of Remark A.7.3.3. □

Proof of Theorem A.7.2.4. Combine Propositions A.7.3.4, A.7.3.5, A.7.3.7, A.7.3.9, and A.7.3.10. □

A.7.4 Bounded ∞-Pretopoi

We now discuss an analogue of Definition A.7.1.2 in the setting of ∞-pretopoi.

Definition A.7.4.1. Let $\mathcal{C}$ be an ∞-pretopos. We will say that $\mathcal{C}$ is *bounded* if it is essentially small and every object $X \in \mathcal{C}$ is n-truncated for $n \gg 0$. We let $\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty}$ denote the full subcategory of $\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}$ spanned by the bounded ∞-pretopoi.

Warning A.7.4.2. The terminology of Definition A.7.4.1 is slightly abusive. Any ∞-topos $\mathcal{X}$ can be regarded as an ∞-pretopos (Example A.6.1.5), but the condition that $\mathcal{X}$ is bounded as an ∞-topos (in the sense of Definition A.7.1.2) is not equivalent to the condition that $\mathcal{X}$ is bounded as an ∞-pretopos (in the sense of Definition A.7.4.1). In practice, there is unlikely to be any confusion: an ∞-topos $\mathcal{X}$ will *never* be bounded as an ∞-pretopos except in the trivial case where $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(\varnothing)$ is an initial object of $\infty\mathcal{T}\mathrm{op}$.

The next result implies that that bounded ∞-pretopoi exist in great abundance:

Proposition A.7.4.3. *Let $\mathcal{C}$ be an ∞-pretopos and let $\mathcal{C}_{<\infty}$ denote the full subcategory of $\mathcal{C}$ spanned by those objects which are n-truncated for some $n \gg 0$. Then:*

(a) *The full subcategory $\mathcal{C}_{<\infty} \subseteq \mathcal{C}$ is closed under finite limits, finite coproducts, and geometric realizations of groupoid objects.*

(b) *The ∞-category $\mathcal{C}_{<\infty}$ is an ∞-pretopos.*

(c) *If $\mathcal{C}_{<\infty}$ is essentially small, then it is a bounded ∞-pretopos.*

Proof. We will prove (a); assertion (b) then follows from Remark A.6.1.4 and (c) is an immediate consequence of (b). Enlarging the universe if necessary, we may assume that $\mathcal{C}$ is small. Let $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ denote the Yoneda embedding, so that an object of $\mathcal{C}$ is n-truncated if and only if its image in $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is n-truncated. We may therefore replace $\mathcal{C}$ by $\mathcal{S}\mathrm{hv}(\mathcal{C})$ and thereby reduce to the case where $\mathcal{C}$ is an ∞-topos (at the cost of dropping our assumption that $\mathcal{C}$ is small). In this case, the result is clear: the collection of n-truncated objects of $\mathcal{C}$ is closed under all limits, closed under coproducts if $n \geqslant 0$, and if $X_\bullet$ is a groupoid object of $\mathcal{C}$ for which X_0 and X_1 are n-truncated, then the geometric realization $|X_\bullet|$ is $(n+1)$-truncated. $\square$

Example A.7.4.4. Let $\mathcal{X}$ be a coherent ∞-topos. We let $\mathcal{X}^{\mathrm{coh}}_{<\infty}$ denote the full subcategeory of $\mathcal{X}$ spanned by the bounded coherent objects. Note that every n-truncated object coherent object of $\mathcal{X}$ is a compact object of $\tau_{\leqslant n}\mathcal{X}$ (Corollary A.2.3.2). It follows that the ∞-category $\mathcal{X}^{\mathrm{coh}}_{<\infty}$ is essentially small. Combining this observation with Propositions A.7.4.3 and A.6.1.6, we deduce that $\mathcal{X}^{\mathrm{coh}}_{<\infty}$ is a bounded ∞-pretopos.

A.7.5 Boundedness and Coherence

Let $\mathcal{X}$ be a hypercomplete ∞-topos which is coherent and locally coherent. According to Theorem A.6.6.5, we can recover $\mathcal{X}$ from its full subcategory $\mathcal{X}^{\mathrm{coh}}$ of coherent objects, which is a hypercomplete ∞-pretopos. Our goal in this section is to prove an analogous result, where the hypercompleteness hypothesis on $\mathcal{X}$ is replaced by boundedness. We begin with a simple observation:

Proposition A.7.5.1. *Let $\mathcal{X}$ be an ∞-topos which is bounded and coherent. Then $\mathcal{X}$ is locally coherent.*

Proof. Fix an object $X \in \mathcal{X}$. We wish to show that there exists an effective epimorphism $\amalg X_i \to X$, where each $X_i \in \mathcal{X}$ is coherent. Write $\mathcal{X}$ as the limit of a filtered diagram $\{\mathcal{X}_\alpha\}$ in $\infty\mathcal{T}\mathrm{op}$, where each $\mathcal{X}_\alpha$ is n_α-localic for some integer $n_\alpha \geqslant 0$. For each index α, let $f^*_\alpha : \mathcal{X}_\alpha \to \mathcal{X}$ be the corresponding geometric morphism. Then $\mathcal{X}$ is generated under small colimits by objects which lie in the essential image of some $\mathcal{X}_\alpha$. In particular, there exists an effective epimorphism $\amalg f_\alpha^* Y_\alpha \to X$ for some objects Y_α in $\mathcal{X}_\alpha$. Since $\mathcal{X}_\alpha$ is n_α-localic, we can choose effective epimorphisms $Z_\alpha \to Y_\alpha$ in $\mathcal{X}_\alpha$, where each Z_α is $(n_\alpha - 1)$-truncated. Replacing X by $f_\alpha^* Z_\alpha$, we can reduce to the case where X is n-truncated for some integer $n \gg 0$.

Since the ∞-topos $\mathcal{X}$ is coherent, it is $(n+2)$-coherent and therefore locally $(n+1)$-coherent. We may therefore choose an effective epimorphism $\rho' : \amalg X_i' \to X$, where each X_i' is $(n+1)$-coherent. Using our assumption that X is n-truncated, we deduce that ρ factors as a composition $\amalg X_i' \to \amalg(\tau_{\leqslant n} X_i') \xrightarrow{\rho} X$. We conclude by observing that ρ is an effective epimorphism and that each of the objects $\tau_{\leqslant n} X_i'$ is a coherent object of $\mathcal{X}$ by virtue of Corollary A.2.4.4. □

Construction A.7.5.2. Let $\infty\mathcal{T}\mathrm{op}_{\mathrm{coh}}$ denote the subcategory of $\infty\mathcal{T}\mathrm{op}$ whose objects are coherent ∞-topoi and whose morphisms are functors $f^* : \mathcal{X} \to \mathcal{Y}$ which preserve small colimits, finite limits, and carry coherent objects of $\mathcal{X}$ to coherent objects of $\mathcal{Y}$. Then the construction $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{coh}}_{<\infty}$ induces a functor $\infty\mathcal{T}\mathrm{op}_{\mathrm{coh}}^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty}$.

Theorem A.7.5.3. *The forgetful functor*

$$\infty\mathcal{T}\mathrm{op}_{\mathrm{coh}}^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty} \qquad \mathcal{X} \mapsto \mathcal{X}^{\mathrm{coh}}_{<\infty}$$

of Construction A.7.5.2 admits a fully faithful left adjoint, given at the level of objects by the construction $\mathcal{C} \mapsto \mathcal{S}\mathrm{hv}(\mathcal{C})$. The essential image of this right adjoint is the full subcategory of $\infty\mathcal{T}\mathrm{op}_{\mathrm{coh}}^{\mathrm{op}}$ spanned by the bounded coherent ∞-topoi.

Proof. Let $\mathcal{C}$ be a bounded ∞-pretopos and let $\mathcal{X}$ be a coherent ∞-topos. For any geometric morphism $f^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{X}$, let $\mathcal{E}(f^*)$ denote the full subcategory of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ spanned by those coherent objects $\mathscr{F} \in \mathcal{S}\mathrm{hv}(\mathcal{C})$ for which $f^* \mathscr{F}$ is a coherent object of $\mathcal{X}$. Using Theorem A.5.5.1 and Remark A.2.1.8, we see that $\mathcal{E}(f^*)$ satisfies conditions (a), (b), (c), and (e) of Proposition A.6.5.7. If $\mathcal{E}(f^*)$ contains the essential image of the Yoneda embedding $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$, then it also satisfies condition (d) and therefore contains all coherent objects of $\mathcal{S}\mathrm{hv}(\mathcal{C})$. This proves the following:

- $(*)$ A geometric morphism $f^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{X}$ belongs to $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{coh}}}(\mathcal{X}, \mathcal{S}\mathrm{hv}(\mathcal{C}))$ if and only if $f^* \circ j$ factors through the full subcategory $\mathcal{X}^{\mathrm{coh}} \subseteq \mathcal{X}$.

It follows from $(*)$ that we have a homotopy pullback square

$$\begin{array}{ccc} \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{coh}}}(\mathcal{X}, \mathcal{S}\mathrm{hv}(\mathcal{C})) & \longrightarrow & \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty}}(\mathcal{C}, \mathcal{X}^{\mathrm{coh}}_{<\infty}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{S}\mathrm{hv}(\mathcal{C})) & \longrightarrow & \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}}(\mathcal{C}, \mathcal{X}), \end{array}$$

where the horizontal maps are given by composition with j. Proposition A.6.4.4 implies that the bottom horizontal map is a homotopy equivalence, so the upper horizontal map is a homotopy equivalence as well. It follows that the construction $\mathcal{C} \mapsto \mathcal{S}\mathrm{hv}(\mathcal{C})$ induces a left adjoint to the forgetful functor $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{coh}}_{<\infty}$ of Construction A.7.5.2.

We wish to show that this left adjoint is fully faithful. Equivalently, we wish to show that for each bounded ∞-pretopos $\mathcal{C}$, the unit map $\mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{coh}}_{<\infty}$ is an equivalence of ∞-categories. For this, we must show that that a sheaf $\mathscr{F} \in \mathcal{S}\mathrm{hv}(\mathcal{C})$ is representable by an object of $\mathcal{C}$ if and only if it is truncated and coherent. This is a special case of the following assertion:

$(*')$ Let X be an object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ and let $\mathscr{F}$ be a coherent object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ equipped with an n-truncated morphism $f : \mathscr{F} \to j(X)$. Then $\mathscr{F}$ belongs to the essential image of j.

We will prove $(*')$ using induction on n. In the special case $n = -2$, the morphism f is an equivalence and there is nothing to prove. Let us therefore assume that $n > -2$. Since $\mathscr{F}$ is coherent, it is quasi-compact. We can therefore choose an effective epimorphism $g : \amalg_{i \in I} j(Y_i) \to \mathscr{F}$ where the index set I is finite. Set $Y = \amalg_{i \in I} Y_i \in \mathcal{C}$, so that we can identify g with a map $j(Y) \to \mathscr{F}$. Let $\mathscr{G}_\bullet$ denote the Čech nerve of g and let $\mathscr{H}_\bullet$ denote the Čech nerve of the composite map $(f \circ g)$, so that we can write $\mathscr{H}_\bullet = j(U_\bullet)$ where $U_\bullet$ is the Čech nerve of the associated map $Y \to X$ in the ∞-category $\mathcal{C}$. Our assumption that f is n-truncated guarantees that each of the induced maps $\mathcal{G}_k \to \mathcal{H}_k$ is $(n-1)$-truncated. Applying our inductive hypothesis, we deduce that each $\mathcal{G}_k$ belongs to the essential image of j. Since j is fully faithful, we can write $\mathcal{G}_\bullet = j(V_\bullet)$ for some groupoid object $V_\bullet$ of $\mathcal{C}$. Because j preserves effective epimorphisms, we conclude that $\mathscr{F} \simeq |\mathcal{G}_\bullet| \simeq j(|V_\bullet|)$ belongs to the essential image of j, as desired.

To complete the proof, we must show that a coherent ∞-topos $\mathcal{X} \in \infty\mathcal{T}\mathrm{op}_{\mathrm{coh}}$ belongs to the essential image of the construction $\mathcal{C} \mapsto \mathcal{S}\mathrm{hv}(\mathcal{C})$ if and only if $\mathcal{X}$ is bounded. The "only if" direction follows immediately from Example A.7.1.6. To prove the converse, we must show that if $\mathcal{X}$ is bounded and coherent then the inclusion $\mathcal{X}^{\mathrm{coh}}_{<\infty} \hookrightarrow \mathcal{X}$ extends to a geometric morphism $f^* : \mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{coh}}_{<\infty}) \to \mathcal{X}$ which is an equivalence of ∞-topoi. The domain and codomain of f^* are both bounded ∞-topoi (for the domain, this follows from Example A.7.1.6). It will therefore suffice to show that the functor f^* restricts to an equivalence on truncated objects. Applying Proposition A.9.2.1 (after passing to a larger universe), we are reduced to proving the following:

(a) For every truncated object $\mathscr{F} \in \mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$ and every morphism $u : X \to f^* \mathscr{F}$ between truncated objects of $\mathcal{X}$, there exists a morphism $v : \mathscr{G} \to \mathscr{F}$ between truncated objects of $\mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$ and a commutative diagram

$$\begin{array}{ccc} f^*\mathscr{G} & \xrightarrow{w} & X \\ & {\scriptstyle f^*v}\searrow \quad \swarrow{\scriptstyle u} & \\ & f^*\mathscr{F} & \end{array}$$

in $\mathcal{X}$, where w is an effective epimorphism.

(*b*) For every (-1)-truncated morphism $u : \mathscr{F} \to \mathscr{F}'$ between truncated object of $\mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$, if $f^*(u)$ is an equivalence, then u is an equivalence.

We begin with the proof of (a). Choose an effective epimorphism $\amalg \mathscr{F}_i \to \mathscr{F}$, where each $\mathscr{F}_i$ is representable by an object of $\mathcal{X}^{\mathrm{coh}}_{<\infty}$. Replacing X by $X \times_{f^*\mathscr{F}} f^*\mathscr{F}_i$, we can reduce to the case where $\mathscr{F}$ is representable by an object $Y \in \mathcal{X}^{\mathrm{coh}}_{<\infty}$. Then we can regard X is an object of the bounded coherent ∞-topos $\mathcal{X}_{/Y}$. Using the proof of Proposition A.7.5.1, we can find an effective epimorphism $\amalg Y_j \to X$ where each Y_j is a truncated coherent object of $\mathcal{X}_{/Y}$, and therefore also a truncated coherent object of $\mathcal{X}$. Assertion (a) now follows from the observation that $\amalg Y_j$ belongs to the essential image of the functor f^*.

We now prove (b). Let $u : \mathscr{F} \to \mathscr{F}'$ be a (-1)-truncated morphism in $\mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$ such that f^*u is an equivalence in $\mathcal{X}$; we wish to show that u is an equivalence. Writing $\mathscr{F}'$ as a colimit of representable sheaves, we can assume without loss of generality that $\mathscr{F}'$ is representable by a truncated coherent object $X \in \mathcal{X}$. Choose an effective epimorphism $\amalg_{i \in I} \mathscr{F}_i \to \mathscr{F}$, where each $\mathscr{F}_i$ is representable by an object $X_i \in (\mathcal{X}^{\mathrm{coh}}_{<\infty})_{/X}$. Since f^*u is an equivalence, the induced map

$$\amalg_{i \in I} X_i \simeq f^*(\amalg_{i \in I} \mathscr{F}_i) \to f^*\mathscr{F} \to f^*\mathscr{F}' \simeq X$$

is an effective epimorphism in $\mathcal{X}$. Because X is quasi-compact, we can choose a finite subset $I_0 \subseteq I$ for which the induced map $\amalg_{i \in I_0} X_i \to X$ is also an effective epimorphism. Note that $\amalg_{i \in I_0} X_i$ is a truncated coherent object of $\mathcal{X}$ which represents the sheaf $\amalg_{i \in I_0} \mathscr{F}_i \in \mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$. It follows that the composite map

$$\amalg_{i \in I_0} \mathscr{F}_i \to \amalg_{i \in I} \mathscr{F}_i \to \mathscr{F} \xrightarrow{u} \mathscr{F}'$$

is an effective epimorphism, so that u is also an effective epimorphism. Since u is also (-1)-truncated, it follows that u is an equivalence. □

Corollary A.7.5.4. *Let $\mathcal{C}$ be a bounded ∞-pretopos. Then $\mathcal{C}$ is idempotent-complete.*

Proof. Using Theorem A.7.5.3, one can identify $\mathcal{C}$ with the full subcategory of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ spanned by the truncated coherent objects. In particular, there is a fully faithful embedding $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ whose essential image is closed under retracts. Since the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is idempotent-complete, it follows that $\mathcal{C}$ is idempotent-complete. □

A.8 Pro-Objects of ∞-Pretopoi

Let $\mathcal{C}$ be a bounded ∞-pretopos (Definition A.7.4.1). Let us regard $\mathcal{C}$ as equipped with the effective epimorphism topology (Definition A.6.2.4) and let $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ be the Yoneda embedding. The construction $(C \in \mathcal{C}) \mapsto \mathcal{S}\mathrm{hv}(\mathcal{C})_{/C}$ determines a functor ρ_0 from $\mathcal{C}$ to the ∞-category $\infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})}$ of ∞-topoi equipped with a geometric morphism to $\mathcal{S}\mathrm{hv}(\mathcal{C})$. Let $\mathrm{Pro}(\mathcal{C})$ denote the ∞-category of Pro-objects of $\mathcal{C}$ (see Definition A.8.1.1). Since the ∞-category $\infty\mathcal{T}\mathrm{op}$ admits filtered limits, the functor ρ_0 admits an essentially unique extension $\rho : \mathrm{Pro}(\mathcal{C}) \to \infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})}$ which commutes with filtered limits. The main result of this section can be formulated as follows:

Theorem A.8.0.5. *Let $\mathcal{C}$ be a bounded ∞-pretopos. Then the functor $\rho : \mathrm{Pro}(\mathcal{C}) \to \infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})}$ is fully faithful.*

The proof of Theorem A.8.0.5 will occupy our attention throughout this section. Roughly speaking, our strategy is to give a more direct construction of the functor ρ (realizing the value of ρ on each Pro-object of $\mathcal{C}$ with the ∞-category of sheaves on a suitable bounded ∞-pretopos) which will allow us compute the relevant mapping spaces in the ∞-category $\infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})}$.

A.8.1 ∞-Categories of Pro-Objects

We begin by reviewing the theory of Pro-objects in the setting of ∞-categories. Let $\mathcal{C}$ be an essentially small ∞-category. In §HTT.5.3.5 , we introduced an ∞-category $\mathrm{Ind}(\mathcal{C})$ of Ind-*objects of* $\mathcal{C}$, which is obtained from $\mathcal{C}$ by formally adjoining filtered direct limits. In this section, we will need the categorical dual of this construction, which associates to $\mathcal{C}$ an ∞-category $\mathrm{Pro}(\mathcal{C}) = \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ of Pro-*objects* of $\mathcal{C}$. For the discussion of shape theory in §**??**, it will be convenient work in a slightly different context.

Definition A.8.1.1. Let $\mathcal{C}$ be an accessible ∞-category which admits finite limits. A Pro-*object* of $\mathcal{C}$ is a functor $U : \mathcal{C} \to \mathcal{S}$ which is accessible and preserves finite limits. We let $\mathrm{Pro}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ spanned by the Pro-objects of $\mathcal{C}$. We will refer to $\mathrm{Pro}(\mathcal{C})$ as the ∞-*category of* Pro-*objects of* $\mathcal{C}$.

Remark A.8.1.2. Let $\mathcal{C}$ be an essentially small idempotent complete ∞-category which admits finite limits (so that $\mathcal{C}$ is accessible: see Corollary HTT.5.4.3.6). Then $\mathcal{C}^{\mathrm{op}}$ admits finite colimits, so that $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$ is the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ spanned by those functors which preserve finite limits. It follows that we have a canonical isomorphism $\mathrm{Pro}(\mathcal{C}) \simeq \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ (note that since $\mathcal{C}$ is essentially small, every functor $\mathcal{C} \to \mathcal{S}$ is automatically accessible).

Remark A.8.1.3. Let $\mathcal{C}$ be an accessible ∞-category which admits finite limits. Then the Yoneda embedding determines a fully faithful functor $j : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$. We will often abuse notation by not distinguishing between an object $C \in \mathcal{C}$ and the Pro-object $j(C) \in \mathrm{Pro}(\mathcal{C})$ corepresented by C.

Remark A.8.1.4. Let $\mathcal{C}$ be an accessible ∞-category which admits finite limits. The collection of left-exact, accessible functors from $\mathcal{C}$ to $\mathcal{S}$ is closed under small filtered colimits. It follows that the ∞-category $\mathrm{Pro}(\mathcal{C})$ admits filtered limits. Moreover, these limits are computed pointwise, so the essential image of the Yoneda embedding $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ consists of compact objects of $\mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$.

Remark A.8.1.5. Let $\mathcal{C}$ be an accessible ∞-category which admits finite limits. For every regular cardinal κ, let $\mathcal{C}^{\kappa}$ denote the full subcategory of $\mathcal{C}$ spanned by the κ-compact objects. If $U : \mathcal{C} \to \mathcal{S}$ is an accessible functor, then there exists a regular cardinal κ such that $\mathcal{C}^{\kappa}$ is closed under finite limits in $\mathcal{C}$ and U is a left Kan extension of its restriction to $\mathcal{C}^{\kappa}$. Then the restriction $U_0 = U|_{\mathcal{C}^{\kappa}}$ can be regarded as a Pro-object of $\mathcal{C}^{\kappa}$, and can therefore be written as a filtered colimit of functors represented by objects of $\mathcal{C}^{\kappa}$. It follows that U can be written as a filtered colimit of functors represented by objects of $\mathcal{C}$. Consequently, every object U of $\mathrm{Pro}(\mathcal{C})$ can be written as the limit of a filtered diagram $\{U_{\alpha}\}$, where each U_{α} belongs to the essential image of the Yoneda embedding $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$. We will generally abuse notation by identifying the diagram $\{U_{\alpha}\}$ with its limit in $\mathrm{Pro}(\mathcal{C})$. Using Remarks A.8.1.3 and A.8.1.4, we see that mapping spaces in $\mathrm{Pro}(\mathcal{C})$ can be described by the usual formula

$$\begin{aligned} \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(\{C_{\alpha}\}, \{D_{\beta}\}) &\simeq \varprojlim_{\beta} \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(\{C_{\alpha}\}, j(D_{\beta})) \\ &\simeq \varprojlim_{\beta} \varinjlim_{\alpha} \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(j(C_{\alpha}), j(D_{\beta})) \\ &\simeq \varprojlim_{\beta} \varinjlim_{\alpha} \mathrm{Map}_{\mathcal{C}}(C_{\alpha}, D_{\beta}). \end{aligned}$$

Our interest in Definition A.8.1.1 is justified by the following universal property:

Proposition A.8.1.6. *Let $\mathcal{C}$ be an accessible ∞-category which admits finite limits, let $\mathcal{D}$ be an ∞-category which admits small filtered limits, and let $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ spanned by those functors which preserve small filtered limits. Then composition with the Yoneda embedding restricts to an equivalence of ∞-categories*

$$\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

We can state Proposition A.8.1.6 more informally as follows: the ∞-category $\mathrm{Pro}(\mathcal{C})$ is obtained from $\mathcal{C}$ by freely adjoining small filtered limits.

Proof of Proposition A.8.1.6. Let $\widehat{\mathcal{S}}$ denote the ∞-category of spaces which are not necessarily small, let $\mathcal{E}$ denote the smallest full subcategory of $\mathrm{Fun}(\mathcal{C}, \widehat{\mathcal{S}})$ which contains the essential image of the Yoneda embedding and is closed under small filtered colimits, and let $\mathrm{Fun}'(\mathcal{E}^{\mathrm{op}}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{D})$ spanned by those functors which preserve small filtered limits. Using Remark HTT.5.3.5.9 , we see that composition with the Yoneda embedding induces an equivalence of ∞-categories $\mathrm{Fun}'(\mathcal{E}^{\mathrm{op}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. It will therefore suffice to show that $\mathcal{E}$ is equivalent to $\mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$ (as subcategories of $\mathrm{Fun}(\mathcal{C}, \widehat{\mathcal{S}})$. Using Remark A.8.1.4, we see that $\mathcal{E}$ is contained in the essential image of $\mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$. We will complete the proof by verifying the following:

- $(*)$ Let $F : \mathcal{C} \to \mathcal{S}$ be an accessible functor which preserves finite limits. Then F can be written as a small filtered colimit $\varinjlim F_\alpha$, where each of the functors $F_\alpha : \mathcal{C} \to \mathcal{S}$ is corepresentable by an object of $\mathcal{C}$.

To prove $(*)$, choose a regular cardinal κ such that $\mathcal{C}$ is κ-accessible and the functor $F : \mathcal{C} \to \mathcal{S}$ preserves κ-filtered colimits. Let $\mathcal{C}^\kappa$ denote the full subcategory of $\mathcal{C}$ spanned by the κ-compact objects. Enlarging κ if necessary, we may assume $\mathcal{C}^\kappa$ is closed under finite limits. Let F^κ denote the restriction of F to $\mathcal{C}^\kappa$. Since $\mathcal{C}^\kappa$ is essentially small and F^κ is left-exact, we can write F^κ as a filtered colimit of functors F^κ_α, where each F^κ_α is corepresentable by an object of $\mathcal{C}^\kappa$ (Corollary HTT.5.3.5.4). Since the Yoneda embedding $h^\kappa : (\mathcal{C}^\kappa)^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}^\kappa, \mathcal{S})$ is fully faithful, we can write $F^\kappa_\alpha = h^\kappa(C_\alpha)$ for some filtered diagram $\{C_\alpha\}$ in $(\mathcal{C}^\kappa)^{\mathrm{op}}$. Let $h : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ denote the Yoneda embedding for $\mathcal{C}$, and let $F' = \varinjlim_\alpha h(C_\alpha)$. We will complete the proof of $(*)$ by showing that $F' \simeq F$. By construction, F and F' have the same restriction to $\mathcal{C}^\kappa$. Since $\mathcal{C}$ is κ-accessible, it will suffice to show that the functors F and F' preserve small κ-filtered colimits (Proposition HTT.5.3.5.10). For the functor F, this follows by assumption. To show that F' commutes with κ-filtered colimits, it will suffice to show that each $h(C_\alpha)$ commutes with κ-filtered colimits; this follows because each C_α is κ-compact. □

Example A.8.1.7. Let $\mathcal{C}$ be a presentable ∞-category, and let $F : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ be the Yoneda embedding. It follows from Proposition HTT.5.1.3.2 that the functor j preserves small colimits. Applying Corollary HTT.5.5.2.9 , we deduce that F admits a right adjoint $G : \mathrm{Pro}(\mathcal{C}) \to \mathcal{C}$. Since G is a right adjoint, it preserves small filtered limits. Because F is fully faithful, the unit map $\mathrm{id}_{\mathcal{C}} \to G \circ F$ is an equivalence. It follows from Proposition A.8.1.6 that the functor G is uniquely determined (up to a contractible space of choices) by these conditions: that is, $G : \mathrm{Pro}(\mathcal{C}) \to \mathcal{C}$ is the unique extension of the identity functor on $\mathcal{C}$ which preserves small filtered limits. If X is an object of $\mathrm{Pro}(\mathcal{C})$ given by an inverse system $\{X_\alpha\}$ of objects of $\mathcal{C}$, then $G(X)$ is given by the limit $\varprojlim X_\alpha$.

Example A.8.1.8. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between accessible ∞-categories which admit small limits. Then the composite functor $\mathcal{C} \xrightarrow{f} \mathcal{D} \to \mathrm{Pro}(\mathcal{D})$ induces a map $\mathrm{Pro}(f) :$

$\operatorname{Pro}(\mathcal{C}) \to \operatorname{Pro}(\mathcal{D})$, which commutes with small filtered limits. If f is accessible and preserves finite limits, then composition with f induces a functor $F : \operatorname{Pro}(\mathcal{D}) \to \operatorname{Pro}(\mathcal{C})$. It is not difficult to see that the functor F (when defined) is a left adjoint to $\operatorname{Pro}(f)$.

Proposition A.8.1.9. *Let $\mathcal{C}$ be an accessible ∞-category which admits finite limits, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be an accessible full subcategory which is closed under finite limits, and let $f : \mathcal{C}_0 \to \mathcal{C}$ denote the inclusion map (so that f is accessible). Then the induced map $\operatorname{Pro}(f) : \operatorname{Pro}(\mathcal{C}_0) \to \operatorname{Pro}(\mathcal{C})$ is fully faithful. Moreover, the essential image of $\operatorname{Pro}(f)$ is spanned by those* Pro*-objects $X \in \operatorname{Pro}(\mathcal{C}) \subseteq \operatorname{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ for which X is a left Kan extension of $X|_{\mathcal{C}_0}$.*

Proof. Let $\mathcal{D}$ denote the full subcategory of $\operatorname{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ spanned by those functors which are left Kan extensions of their restrictions to $\mathcal{C}_0$. Then $\mathcal{D}^{\mathrm{op}} \subseteq \operatorname{Fun}(\mathcal{C}, \mathcal{S})$ is closed under small colimits, so that $\mathcal{D}$ is closed under small limits in $\operatorname{Fun}(\mathcal{C}, \mathcal{S})$. Using Remark A.8.1.4, we see that the intersection $\mathcal{D} \cap \operatorname{Pro}(\mathcal{C})$ is closed under small filtered limits in $\operatorname{Pro}(\mathcal{C})$. Note that the Yoneda embedding $j : \mathcal{C} \to \operatorname{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ carries $\mathcal{C}_0$ into $\mathcal{D}$. It follows that $\operatorname{Pro}(f)$ carries $\operatorname{Pro}(\mathcal{C}_0)$ into $\operatorname{Pro}(\mathcal{C}) \cap \mathcal{D}$. Let $U : \operatorname{Pro}(\mathcal{C}) \cap \mathcal{D} \to \operatorname{Pro}(\mathcal{C}_0)$ denote the functor given by composition with the inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$. Then U preserves small filtered limits (Remark A.8.1.4), and the diagram

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{\mathrm{id}} & \mathcal{C}_0 \\ \downarrow & & \downarrow \\ \operatorname{Pro}(\mathcal{C}_0) & \xrightarrow{U \circ \operatorname{Pro}(f)} & \operatorname{Pro}(\mathcal{C}_0) \end{array}$$

commutes up to homotopy. It follows that that $U \circ \operatorname{Pro}(f)$ is homotopic to the identity. Since U is fully faithful (Proposition HTT.4.3.2.15), we deduce that $\operatorname{Pro}(f)$ is an equivalence of ∞-categories. □

A.8.2 Digression: Truncated Category Objects

Let $\mathcal{C}$ be an ∞-category which admits finite limits. Recall that a *category object* of $\mathcal{C}$ is a simplicial object $C_\bullet$ of $\mathcal{C}$ satisfying the following "Segal condition": for each $n \geqslant 0$, the diagram of linearly ordered sets

$$\begin{array}{ccccccccc} & \{0,1\} & & \cdots & & \{n-1,n\} & \\ \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \\ \{0\} & & \{1\} & \cdots & \{n-1\} & & \{n\} \end{array}$$

induces an equivalence

$$C_n \to C_1 \times_{C_0} \cdots \times_{C_0} C_1.$$

Example A.8.2.1. Let $C_\bullet$ be a simplicial set. Then $C_\bullet$ is a category object of the category $\mathcal{S}\mathrm{et}$ of sets if and only if $C_\bullet$ is isomorphic to the nerve of a small category $\mathcal{E}$. Moreover, the category $\mathcal{E}$ is determined up to canonical isomorphism: the objects of $\mathcal{E}$ are the elements of the set C_0, the morphisms of $\mathcal{E}$ are the elements of the set C_1, and the composition of morphisms is determined by the map

$$C_1 \times_{C_0} C_1 \simeq C_2 \xrightarrow{\rho} C_1,$$

where ρ is induced by the inclusion of linearly ordered sets $[1] \simeq \{0, 2\} \hookrightarrow [2]$.

We now call attention to two phenomena visible in Example A.8.2.1:

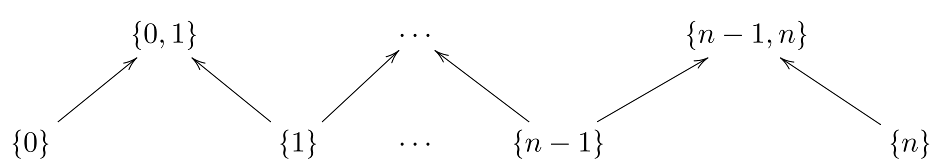
- Let $C_\bullet$ be a category object in sets, so that $C_\bullet \simeq \mathrm{N}(\mathcal{E})$ for some small category $\mathcal{E}$. To recover the category $\mathcal{E}$ (and therefore the entire simplicial set $C_\bullet$), we only need to know the sets C_0, C_1, C_2, and the maps between them.

- When reconstructing the category $\mathcal{E}$ from the simplicial set $C_\bullet$, the main step is to prove that composition of morphisms is associative. The proof of this involves studying the set C_3 and the bijection $C_3 \to C_1 \times_{C_0} C_1 \times_{C_0} C_1$. In particular, it does not make any reference to the sets C_n for $n \geqslant 4$.

Our goal in this section is to generalize these observations. We begin by introducing some terminology.

Definition A.8.2.2. Let $\mathcal{C}$ be an ∞-category which admits finite limits, and let $m \geqslant 1$ be an integer. We let $\mathbf{\Delta}_{\leqslant m}$ denote the category whose objects are the sets $[n] = \{0 < 1 < \ldots < n\}$ for $0 \leqslant n \leqslant m$, and whose morphisms are nondecreasing maps of linearly ordered sets. An *m-skeletal simplicial object* of $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}}_{\leqslant m} \to \mathcal{C}$. If C is an m-skeletal simplicial object and $n \leqslant m$, we let C_n denote the image in $\mathcal{C}$ of the object $[n] \in \mathbf{\Delta}_{\leqslant m}$.

An *m-skeletal category object* is a functor $\mathbf{\Delta}^{\mathrm{op}}_{\leqslant m} \to \mathcal{C}$ with the following property: for each $n \leqslant m$, the diagram of linearly ordered sets



induces an equivalence $C_n \to C_1 \times_{C_0} \cdots \times_{C_0} C_1$.

We let $\mathrm{CObj}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$ spanned by the category objects, and $\mathrm{CObj}_{\leqslant m}(\mathcal{C})$ the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\leqslant m}, \mathcal{C})$ spanned by the m-skeletal category objects.

We can now state our main result.

Theorem A.8.2.3. *Let $\mathcal{C}$ be an ∞-category which is equivalent to an n-category for some $n \geqslant -1$ (see Definition HTT.2.3.4.1) and admits finite limits. Then the restriction functor $\mathrm{CObj}(\mathcal{C}) \to \mathrm{CObj}_{\leqslant m}(\mathcal{C})$ is fully faithful when $m = n+1$ and an equivalence of ∞-categories when $m \geqslant n+2$.*

Corollary A.8.2.4. *Let $\mathcal{C}$ be an ∞-category which is given as a filtered colimit $\varinjlim \mathcal{C}_\alpha$, where each $\mathcal{C}_\alpha$ admits finite limits and each of the transition maps $\mathcal{C}_\alpha \to \mathcal{C}_\beta$ preserves finite limits. Suppose further that there exists an integer n such that each $\mathcal{C}_\alpha$ is equivalent to an n-category. Then the canonical map $\varinjlim \mathrm{CObj}(\mathcal{C}_\alpha) \to \mathrm{CObj}(\mathcal{C})$ is an equivalence of ∞-categories.*

Proof. Note that $\mathcal{C}$ is also equivalent to an n-category. Let K denote the $(n+1)$-skeleton of the simpicial set $\mathrm{N}(\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n+2})$. We have a commutative diagram

$$\begin{array}{ccc}
\varinjlim_\alpha \mathrm{CObj}(\mathcal{C}_\alpha) & \longrightarrow & \mathrm{CObj}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\varinjlim_\alpha \mathrm{CObj}_{\leqslant n+2}(\mathcal{C}_\alpha) & \longrightarrow & \mathrm{CObj}_{\leqslant n+2}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\varinjlim_\alpha \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n+2}, \mathcal{C}_\alpha) & \longrightarrow & \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n+2}, \mathcal{C}) \\
\downarrow & & \downarrow \\
\varinjlim_\alpha \mathrm{Fun}(K, \mathcal{C}_\alpha) & \longrightarrow & \mathrm{Fun}(K, \mathcal{C}).
\end{array}$$

The upper vertical maps are equivalences by virtue of Theorem A.8.2.3, the lower vertical maps are equivalences by virtue of our assumption that the ∞-categories $\mathcal{C}$ and $\mathcal{C}_\alpha$ are equivalent to n-categories, and the middle square is a (homotopy) pullback. We are therefore reduced to showing that the canonical map $\varinjlim_\alpha \mathrm{Fun}(K, \mathcal{C}_\alpha) \to \mathrm{Fun}(K, \mathcal{C})$ is an equivalence, which follows from the observation that K is a finite simplicial set. □

Corollary A.8.2.5. *Let $\mathcal{C}$ be an ∞-category which is given as a filtered colimit $\varinjlim \mathcal{C}_\alpha$, where each $\mathcal{C}_\alpha$ admits finite limits and each of the transition maps $\mathcal{C}_\alpha \to \mathcal{C}_\beta$ preserves finite limits. Then the induced map $\varinjlim_\alpha \mathcal{G}\mathrm{pd}(\mathcal{C}_\alpha) \to \mathcal{G}\mathrm{pd}(\mathcal{C})$ is an equivalence of ∞-categories. Here $\mathcal{G}\mathrm{pd}(\mathcal{E})$ denotes the ∞-category of groupoid objects of $\mathcal{E}$ (see Definition HTT.6.1.2.7).*

Theorem A.8.2.3 is an immediate consequence of the following more precise assertions (and Proposition HTT.4.3.2.15).

Proposition A.8.2.6. *Let $\mathcal{C}$ be an ∞-category which admits finite limits, and let $C_\bullet$ be a category object of $\mathcal{C}$. Assume that the map $C_1 \to C_0 \times C_0$ is $(n-2)$-truncated for some integer $n \geqslant 0$. Then $C_\bullet$ is a right Kan extension of its restriction to $\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n}$.*

Proposition A.8.2.7. *Let $\mathcal{C}$ be an ∞-category which admits finite limits, let $n \geqslant 1$, and let $C_\bullet$ be an n-skeletal category object of $\mathcal{C}$. Assume that the map $C_1 \to C_0 \times C_0$ is $(n-3)$-truncated. Then $C_\bullet$ can be extended to a category object $\overline{C}_\bullet$ of $\mathcal{C}$ (this extension is necessarily a right Kan extension, by virtue of Proposition A.8.2.6).*

The proofs of Propositions A.8.2.6 and A.8.2.7 will require some preliminaries. First, we need a slight modification of Construction A.5.1.6:

Notation A.8.2.8. Let $\mathcal{C}$ be an ∞-category which admits finite limits and let $C_\bullet$ be an m-skeletal simplicial object of $\mathcal{C}$. Let K be a finite nonsingular simplicial set of dimension $\leqslant m$ (meaning that every nondegenerate simplex of K has dimension $\leqslant m$), and let $\mathbf{\Delta}_K^{\mathrm{nd}}$ denote the category of nondegenerate simplices of K (see Notation A.5.1.5). We let $C_\bullet[K]$ denote a limit of the induced diagram $(\mathbf{\Delta}_K^{\mathrm{nd}})^{\mathrm{op}} \to \mathbf{\Delta}_{\leqslant m}^{\mathrm{op}} \xrightarrow{C_\bullet} \mathcal{C}$.

Lemma A.8.2.9. *Let $\mathcal{C}$ be an ∞-category which admits finite limits, let $C_\bullet$ be an m-skeletal category object of $\mathcal{C}$ for some $m \geqslant 1$. Then:*

(1) *Let $n \leqslant m+1$, let $0 < j < n$, and let $A \subseteq \Delta^n$ be the simplicial subset spanned by the edges $\{\Delta^{\{i-1,i\}}\}_{1 \leqslant i \leqslant n}$. Then the restriction map $C_\bullet[\Lambda^n_j] \to C_\bullet[A]$ is an equivalence.*

(2) *For $0 < j < n \leqslant m$, the map $C_\bullet[\Delta^n] \to C_\bullet[\Lambda^n_j]$ is an equivalence.*

Proof. We prove (1) and (2) by a simultaneous induction on n. Note that if $n \leqslant m$ and K is defined as in (1), then the composite map $C_\bullet[\Delta^n] \to C_\bullet[\Lambda^n_j] \to C_\bullet[A]$ is an equivalence by virtue of our assumption that $C_\bullet$ is an n-skeletal category object. Consequently, assertion (2) follows from (1) and the two-out-of-three property.

We now prove (1). Let S be the collection of all nondegenerate simplices σ of Δ^n which contain the vertex j together with additional vertices i and k such that $i < j < k$. Write $S = \{\sigma_1, \sigma_2, \ldots, \sigma_b\}$, where $a < a'$ whenever σ_a has dimension larger than $\sigma_{a'}$; in particular, we have $\sigma_1 = \Delta^n$. For $1 \leqslant a \leqslant b$, let τ_a be the face of σ_a obtained by the removing the vertex j, and let K_a denote the simplicial subset obtained from Δ^n by removing the simplices $\{\sigma_{a'}, \tau_{a'}\}_{a' \leqslant a}$. We have a chain of simplicial subsets

$$\Delta^{\{0,\ldots,j\}} \coprod_{\{j\}} \Delta^{\{j,j+1,\ldots,n\}} = K_b \subseteq K_{b-1} \subseteq \cdots \subseteq K_1 = \Lambda^n_j.$$

For $1 \leqslant a < b$, the inclusion $K_{a+1} \subseteq K_a$ is a pushout of an inner horn inclusion $\Lambda^{n'}_{j'} \subseteq \Delta^{n'}$ for some $0 < j' < n' < n$, so we have a pullback diagram

$$\begin{array}{ccc} C_\bullet[K_a] & \longrightarrow & C_\bullet[K_{a+1}] \\ \downarrow & & \downarrow \\ C_\bullet[\Delta^{n'}] & \longrightarrow & C_\bullet[\Lambda^{n'}_{j'}] \end{array}$$

The inductive hypothesis implies that the bottom horizontal map is an equivalence, so that $C_\bullet[K_a] \simeq C_\bullet[K_{a+1}]$ for $1 \leqslant a < b$. It follows that the restriction map

$$C_\bullet[\Lambda^n_j] \to C_\bullet[K_b] \simeq C_\bullet[\Delta^{\{0,\ldots,j\}}] \times_{C_\bullet[\{j\}]} C_\bullet[\Delta^{\{j,\ldots,n\}}]$$

is an equivalence. Let A_- be the simplicial subset of Δ^n spanned by the edges $\Delta^{\{i-1,i\}}$ for $1 \leqslant i \leqslant j$, and let A_+ be the simplicial subset of Δ^n spanned by the edges $\Delta^{\{i-1,i\}}$ for $j < i \leqslant n$. Since $C_\bullet$ is an m-coskeletal category object, the restriction maps

$$C_\bullet[\Delta^{\{0,\ldots,j\}}] \to C_\bullet[A_-] \qquad C_\bullet[\Delta^{\{j,j+1,\ldots,n\}}] \to C_\bullet[A_+]$$

are equivalences. It follows that the map

$$C_\bullet[\Lambda^n_j] \to C_\bullet[A] \simeq C_\bullet[A_-] \times_{C_\bullet[\{j\}]} C_\bullet[A_+]$$

is an equivalence. □

Proof of Proposition A.8.2.6. Let $C_\bullet$ be a category object of $\mathcal{C}$ such that the map $C_1 \to C_0 \times C_0$ is $(n-2)$-truncated. We wish to show that $C_\bullet$ is a right Kan extension of its restriction to $\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n}$. It will suffice to show that for each $m \geqslant n$, the restriction $C_\bullet|_{\mathbf{\Delta}^{\mathrm{op}}_{\leqslant m}}$ is a right Kan extension of $C_\bullet|_{\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n}}$. Using Proposition HTT.4.3.2.8 repeatedly, we are reduced to showing that $C_\bullet|_{\mathbf{\Delta}^{\mathrm{op}}_{\leqslant m}}$ is a right Kan extension of $C_\bullet|_{\mathbf{\Delta}^{\mathrm{op}}_{\leqslant m-1}}$ for $m > n$. In other words, we must show that if $m > n$, then the map $C_m \to M_m(C)$ is an equivalence, where $M_m(C)$ denotes the mth matching object of $C_\bullet$ (see Notation HTT.A.2.9.7). We will show more generally that the map $\beta_m : C_m \to M_m(C)$ is $(n-m-1)$-truncated for each $m \geqslant 1$. If $m > n$, this implies that β_m is an equivalence. We proceed by induction on m: the case $m = 1$ follows from our hypothesis that $C_1 \to C_0 \times C_0$ is $(n-2)$-truncated. Assume therefore that $m \geqslant 2$, and choose $0 < j < m$. Lemma A.8.2.9 implies that the composite map

$$C_\bullet[\Delta^m] \simeq C_m \to M_m(C) \simeq C_\bullet[\partial\,\Delta^m] \to C_\bullet[\Lambda^m_j]$$

is an equivalence. It will therefore suffice to show that the map $\gamma : C_\bullet[\partial\,\Delta^m] \to C_\bullet[\Lambda^m_j]$ is $(n-m)$-truncated. This follows from the inductive hypothesis, since γ is a pullback of the map $C_{m-1} \to M_{m-1}(C)$. □

Proof of Proposition A.8.2.7. Let $C_\bullet$ be an n-skeletal category object of $\mathcal{C}$ and assume that the map $C_1 \to C_0 \times C_0$ is $(n-3)$-truncated. Since $\mathcal{C}$ admits finite limits, there exists a simplicial object $\overline{C}_\bullet$ which is a right Kan extension of $C_\bullet$. We wish to show that $\overline{C}_\bullet$ is a category object of $\mathcal{C}$. It will suffice to show that the restriction $\overline{C}_\bullet|_{\mathbf{\Delta}^{\mathrm{op}}_{\leqslant m}}$ is an m-skeletal category object for each $m \geqslant n$. We proceed by induction on m, the case $m = n$ being trivial. Let $A \subseteq \Delta^m$ be the simplicial subset given by the union of the edges $\{\Delta^{\{i-1,i\}}\}_{1 \leqslant i \leqslant m}$; we wish to show that the map $\overline{C}_\bullet[\Delta^m] \to \overline{C}_\bullet[A]$ is an equivalence. Since $m > n \geqslant 1$, we can choose $0 < j < m$. Using the inductive hypothesis and Lemma A.8.2.9, we deduce that $\overline{C}_\bullet[\Lambda^m_j] \to \overline{C}_\bullet[A]$ is an equivalence. It will therefore suffice to show that the restriction map $\beta : \overline{C}_\bullet[\Delta^m] \to \overline{C}_\bullet[\Lambda^m_j]$ is an equivalence. Note that β is a pullback of the map $\beta' : \overline{C}_\bullet[\Delta^{m-1}] \to \overline{C}_\bullet[\partial\,\Delta^{m-1}]$. We will show that β' is an equivalence. For this, it suffices to prove the more general claim that for $1 \leqslant k < m$, the map $\beta'_k : \overline{C}_\bullet[\Delta^k] \to \overline{C}_\bullet[\partial\,\Delta^k]$ is $(n-k-2)$-truncated.

As in the proof of Proposition A.8.2.6, we proceed by induction on k, the case $k = 1$ being true by virtue of our hypothesis. Assume therefore that $k \geqslant 2$, and choose $0 < i < k$. Since $k < m$, the restriction $\overline{C}_\bullet|_{\mathbf{\Delta}_{\leqslant k}}$ is a k-skeletal category object so that Lemma A.8.2.9 implies that the composite map $\overline{C}_\bullet[\Delta^k] \xrightarrow{\beta'_k} \overline{C}_\bullet[\partial\,\Delta^k] \xrightarrow{\gamma} \overline{C}_\bullet[\Lambda^k_i]$ is an equivalence. It is therefore sufficient to show that γ is $(n-k-1)$-truncated. This follows from the inductive hypothesis, since γ is a pullback of β'_{k-1}. □

A.8.3 Filtered Colimits of ∞-Pretopoi

In §A.7, we introduced the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty}$ whose objects are bounded ∞-pretopoi and whose morphisms are functors which preserve finite limits, finite coproducts, and effective epimorphisms (Definition A.7.4.1). We now make the following observation:

Proposition A.8.3.1. *The ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty}$ admits small filtered colimits. Moreover, the inclusion $\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty} \hookrightarrow \widehat{\mathcal{C}\mathrm{at}}_\infty$ preserves small filtered colimits.*

Proof. Let $\{\mathcal{C}_\alpha\}_{\alpha \in A}$ be a diagram in the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty}$ indexed by a filtered partially ordered set A, and set $\mathcal{C} = \varinjlim_{\alpha \in A} \mathcal{C}_\alpha$. For each index $\alpha \in A$, let $f^*_\alpha : \mathcal{C}_\alpha \to \mathcal{C}$ be the canonical map. Unwinding the definitions, we must prove the following:

- (i) The ∞-category $\mathcal{C}$ is a bounded ∞-pretopos.
- (ii) For every bounded ∞-pretopos $\mathcal{E}$, a functor $g^* : \mathcal{C} \to \mathcal{E}$ is a morphism in $\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty}$ if and only if the composition $(g^* \circ f^*_\alpha) : \mathcal{C}_\alpha \to \mathcal{E}$ is a morphism in $\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty}$ for each $\alpha \in A$.

We first prove (i). By assumption, each of the ∞-categories $\mathcal{C}_\alpha$ admits finite limits and finite coproducts and each of the transition maps $\mathcal{C}_\alpha \to \mathcal{C}_\beta$ preserves finite limits and finite colimits. It follows that the ∞-category $\mathcal{C}$ admits finite limits and finite coproducts and that each of the functors f^*_α preserves finite limits and finite coproducts. In particular, each of the functors f^*_α carries n-truncated objects of $\mathcal{C}_\alpha$ to n-truncated objects of $\mathcal{C}$. Since every object $C \in \mathcal{C}$ lies in the essential image of one of the functors f^*_α, it follows immediately that C is n-truncated for some integer n. Conversely, suppose that $C \in \mathcal{C}$ is n-truncated, and write $C = f^*_\alpha C_0$ for some index α and some object $C_0 \in \mathcal{C}_\alpha$. Since the diagonal map $\delta : C \to C^{S^{n+1}}$ is an equivalence, it follows that the map $\delta_0 : C_0 \to C_0^{S^{n+1}}$ is a morphism in $\mathcal{C}_\alpha$ whose image under f_α^* is an equivalence. We may therefore arrange (after enlarging α) that the map δ_0 is an equivalence, so that the object $C_0 \in \mathcal{C}_\alpha$ is n-truncated. It follows that we have $\mathcal{C}^{\leqslant n} \simeq \varinjlim \mathcal{C}_\alpha^{\leqslant n}$, where $\mathcal{C}^{\leqslant n}$ denotes the full subcategory of $\mathcal{C}$ spanned by the n-truncated objects and $\mathcal{C}_\alpha^{\leqslant n} \subseteq \mathcal{C}_\alpha$ is defined similarly.

To complete the proof of (i), it will suffice to show that $\mathcal{C}$ is an ∞-pretopos. We first show that coproducts in $\mathcal{C}$ are universal. Suppose we are given a finite collection of morphisms $\phi_i : X_i \to X$ in $\mathcal{C}$ and a morphism $\psi : Y \to X$ in $\mathcal{C}$; we wish to prove that the induced map $\theta : \amalg(X_i \times_X Y) \to (\amalg X_i) \times_X Y$ is an equivalence. Without loss of generality, we may assume that $\psi = f_\alpha^* \psi_0$ for some morphism $\psi_0 : Y_0 \to X_0$ in $\mathcal{C}_\alpha$. Enlarging α if necessary, we may further assume that each ϕ_i is given by $f_\alpha^*(\phi_{i0})$ for some morphism $\phi_{i0} X_{i0} \to X_0$ in $\mathcal{C}_\alpha$. Then $\theta = f_\alpha^* \theta_0$, where θ_0 denotes the canonical map $\amalg(X_{i0} \times_{X_0} Y_0) \to (\amalg X_{i0}) \times_{X_0} Y_0$ in the ∞-category $\mathcal{C}_\alpha$. Since $\mathcal{C}_\alpha$ is an ∞-pretopos, the morphism θ_0 is an equivalence, so that $\theta \simeq f_\alpha^*(\theta_0)$ is an equivalence in $\mathcal{C}$.

We now claim that coproducts in $\mathcal{C}$ are disjoint. Fix objects $X, Y \in \mathcal{C}$; we wish to show that the fiber product $Z = X \times_{X \amalg Y} Y$ is an initial object of $\mathcal{C}$. Without loss of generality, we may assume that there exists an index α such that $X = f_\alpha^* X_0$ and $Y = f_\alpha^* Y_0$ for some objects $X_0, Y_0 \in \mathcal{C}_\alpha$. In this case, we have $Z = f_\alpha^*(Z_0)$, where $Z_0 = X_0 \times_{X_0 \amalg Y_0} Y_0$. Since $\mathcal{C}_\alpha$ is an ∞-pretopos, the object $Z_0 \in \mathcal{C}_\alpha$ is initial, so that $Z = f_\alpha^* Z_0$ is an initial object of $\mathcal{C}$.

We next show that for each index $\alpha \in A$, the functor $f_\alpha^* : \mathcal{C}_\alpha \to \mathcal{C}$ preserves effective epimorphisms. Fix an effective epimorphism $u : C_0 \to C$ in $\mathcal{C}_\alpha$ and let $C_\bullet$ denote the Čech nerve of u. Since f_α^* is left exact, the simplicial object $f_\alpha^* C_\bullet$ is the Čechnerve of $f_\alpha^*(u)$. To show that $f_\alpha^*(u)$ is an effective epimorphism, we must show that $f_\alpha^* C$ is a geometric realization of $f_\alpha^* C_\bullet$. In other words, we must show that for each object $D \in \mathcal{C}$, the canonical map $\rho : \operatorname{Map}_{\mathcal{C}}(f_\alpha^* C, D) \to \operatorname{Tot} \operatorname{Map}_{\mathcal{C}}(f_\alpha^* C_\bullet, D)$ is a homotopy equivalence. Enlarging α if necessary, we may assume that $D = f_\alpha^* D_0$ for some object $D_0 \in \mathcal{C}_\alpha$. For $\beta \geqslant \alpha$, let $f_{\alpha\beta}^* : \mathcal{C}_\alpha \to \mathcal{C}_\beta$ denote the transition map. Unwinding the definitions, we see that the map ρ factors as a composition

$$\varinjlim_{\beta \geqslant \alpha} \operatorname{Map}_{\mathcal{C}_\beta}(f_{\alpha\beta}^* C, f_{\alpha\beta}^* D_0) \xrightarrow{\rho'} \varinjlim_{\beta \geqslant \alpha} \operatorname{Tot} \operatorname{Map}_{\mathcal{C}_\beta}(f_{\alpha\beta}^* C_\bullet, f_{\alpha\beta}^* D_0) \xrightarrow{\rho''} \operatorname{Tot} \varinjlim_{\beta \geqslant \alpha} \operatorname{Map}_{\mathcal{C}_\beta}(f_{\alpha\beta}^* C_\bullet, f_{\alpha\beta}^* D_0).$$

The map ρ' is a filtered colimit of homotopy equivalences $\rho'_\beta : \operatorname{Map}_{\mathcal{C}_\beta}(f_{\alpha\beta}^* C, f_{\alpha\beta}^* D_0) \to \operatorname{Tot} \operatorname{Map}_{\mathcal{C}_\beta}(f_{\alpha\beta}^* C_\bullet, f_{\alpha\beta}^* D_0)$, each of which is a homotopy equivalence because the functor $f_{\alpha\beta}^*$ is left exact and preserves effective epimorphisms. Since the ∞-pretopos $\mathcal{C}_\alpha$ is bounded, there exists an integer m such that $D_0 \in \mathcal{C}_\alpha$ is m-truncated. It follows that each of the mapping spaces $\operatorname{Map}_{\mathcal{C}_\beta}(f_{\alpha\beta}^* C_\bullet, f_{\alpha\beta}^* D_0)$ is also m-truncated, so that the map ρ'' is also a homotopy equivalence. This completes the proof that ρ is a homotopy equivalence, so that f_α^* preserves effective epimorphisms.

We now claim that every groupoid object of $\mathcal{C}$ is effective. Let $C_\bullet$ be a groupoid object of $\mathcal{C}$. Choose $n \gg 0$ for which the objects $C_0, C_1 \in \mathcal{C}$ are n-truncated. Since the collection of

n-truncated objects of $\mathcal{C}$ is closed under finite limits, it follows that each C_k is n-truncated: that is, we can regard $C_\bullet$ as a groupoid object of the subcategory $\mathcal{C}^{\leqslant n} \subseteq \mathcal{C}$. Corollary A.8.2.5 implies that the canonical map $\varinjlim_\alpha \mathrm{Gpd}(\mathcal{C}_\alpha^{\leqslant n}) \to \mathcal{C}^{\leqslant n}$ is an equivalence of ∞-categories, so we can assume without loss of generality that $C_\bullet = f_\alpha^* C'_\bullet$ for some index α and some groupoid object $C'_\bullet$ in $\mathcal{C}_\alpha^{\leqslant n}$. Since $\mathcal{C}_\alpha$ is an ∞-pretopos, we can assume that $C'_\bullet$ is the Čech nerve of an effective epimorphism $u : C'_0 \to C'$ in $\mathcal{C}_\alpha$. It follows that $C_\bullet$ can be identified with the Čech nerve of the induced map $f_\alpha^*(u) : C_0 \to f_\alpha^* C'$. Because the functor f_α^* preserves effective epimorphisms, the morphism $f_\alpha^*(u)$ is an effective epimorphism, so that $f_\alpha^* C'$ is a geometric realization of the groupoid $C_\bullet$ in the ∞-category $\mathcal{C}$.

To complete the proof of (i), it will suffice to show that the formation of geometric realizations in $\mathcal{C}$ is universal. In other words, let $C_\bullet$ be a groupoid object of $\mathcal{C}$ having geometric realization $C = |C_\bullet|$, and suppose we are given a map $g : D \to C$; we wish to show that the induced map $|D \times_C C_\bullet| \to D$ is an equivalence. Since the groupoid object $C_\bullet$ is effective, it can be identified with the Čech nerve of the map $v : C_0 \to C$. It follows that $D \times_C C_\bullet$ can be identified with the Čechnerve of the induced map $v_D : D \times_C C_0 \to D$. We are therefore reduced to proving that v is an effective epimorphism. The argument of the preceding paragraph shows that we can can assume that $v = f_\alpha^*(u)$ for some $\alpha \in A$ and some effective epimorphism $u : C'_0 \to C'$ in $\mathcal{C}_\alpha$. Enlarging α if necessary, we can assume that $g = f_\alpha^*(g_0)$ for some map $g_0 : D' \to C'$ in $\mathcal{C}_\alpha$. Then $v_D = f_\alpha^* u_D$, where $u_D : D' \times_{C'} C'_0 \to D'$ is the projection onto the first factor. Since $\mathcal{C}_\alpha$ is an ∞-pretopos, the morphism u_D is an effective epimorphism in $\mathcal{C}_\alpha$ (Corollary A.6.2.2) and therefore $v_D = f_\alpha^* u_D$ is an effective epimorphism in $\mathcal{C}$. This completes the proof of (i).

We now prove (ii). The arguments above show that each of the functors $f_\alpha^* : \mathcal{C}_\alpha \to \mathcal{C}$ preserves finite limits, finite coproducts, and effective epimorphisms. This proves the "only if" direction of (ii). To prove the converse, suppose we are given an ∞-pretopos $\mathcal{E}$ and a functor $g^* : \mathcal{C} \to \mathcal{E}$ such that each composition $g^* \circ f_\alpha^*$ is a morphism of ∞-pretopoi. We claim that g^* is a morphism of ∞-pretopoi. Note that if K is a finite simplicial set and $p : K \to \mathcal{C}$ is any diagram, then we can assume that $p = f_\alpha^* \circ p_0$ for some diagram $p_0 : K \to \mathcal{C}_\alpha$. Since the functors f_α^* and $g^* \circ f_\alpha^*$ preserve K-indexed limits, it follows that the canonical map $g^* \varinjlim(p) \to \varinjlim g^*(p)$ is an equivalence in $\mathcal{E}$. Allowing p and K to vary, we deduce that the functor g^* preserves finite limits. A similar argument shows that g^* preserves finite coproducts. To show that g^* preserves effective epimorphisms, it suffices to note that our preceding arguments every effective epimorphism in $\mathcal{C}$ is equivalent to the image under f_α^* of some effective epimorphism in $\mathcal{C}_\alpha$, for some $\alpha \in A$. □

We now describe the significance of Proposition A.8.3.1 at the level of bounded coherent ∞-topoi.

Proposition A.8.3.2. *Let $\{\mathcal{C}_\alpha\}$ be a small filtered diagram of bounded ∞-pretopoi having colimit $\mathcal{C} \in \infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty}$. Then the induced map $\mathcal{S}\mathrm{hv}(\mathcal{C}) \to \varprojlim_\alpha \mathcal{S}\mathrm{hv}(\mathcal{C}_\alpha)$ is an equivalence in*

$\infty\mathcal{T}\mathrm{op}$.

Corollary A.8.3.3. *Let $\infty\mathcal{T}\mathrm{op}_{\mathrm{coh}}$ denote the ∞-category of bounded coherent ∞-topoi (see Construction A.7.5.2). Then the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{coh}}$ admits small filtered limits, and the inclusion functor $\infty\mathcal{T}\mathrm{op}_{\mathrm{coh}} \hookrightarrow \infty\mathcal{T}\mathrm{op}$ preserves small filtered limits.*

Proof. Combine Proposition A.8.3.2 with Theorem A.7.5.3. □

Proof of Proposition A.8.3.2. Since the inclusion functor $\infty\mathcal{T}\mathrm{op} \hookrightarrow \widehat{\mathcal{C}\mathrm{at}}_\infty$ preserves small filtered limits (Theorem HTT.6.3.3.1), it will suffice to show that $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is a limit of the diagram $\{\mathcal{S}\mathrm{hv}(\mathcal{C}_\alpha)\}$ in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}(\mathcal{C}) & \longrightarrow & \varprojlim \mathcal{S}\mathrm{hv}(\mathcal{C}_\alpha) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) & \longrightarrow & \varprojlim \mathrm{Fun}(\mathcal{C}_\alpha^{\mathrm{op}}, \mathcal{S}), \end{array}$$

where the vertical maps are fully faithful embeddings and the bottom horizontal map is an equivalence by virtue of our assumption that $\mathcal{C}$ is a colimit of the diagram $\{\mathcal{C}_\alpha\}$ in $\mathcal{C}\mathrm{at}_\infty$ (Proposition A.8.3.1). This immediately implies that the upper horizontal map is fully faithful. To verify essential surjectivity, we must show that if $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is a functor having the property that $\mathscr{F}|_{\mathcal{C}_\alpha^{\mathrm{op}}}$ belongs to $\mathcal{S}\mathrm{hv}(\mathcal{C}_\alpha)$ for each index α, then $\mathscr{F}$ is a sheaf on $\mathcal{C}$. This follows immediately from the criterion of Proposition A.6.2.5 (note that the proof of Proposition A.8.3.1 shows that every effective epimorphism in $\mathcal{C}$ is equivalent to the image of an effective epimorphism in $\mathcal{C}_\alpha$ for some index α). □

A.8.4 The Proof of Theorem A.8.0.5

We now have most of the ingredients needed to assemble a proof of Theorem A.8.0.5.

Definition A.8.4.1. Let $\mathcal{E}$ be an ∞-category which admits small limits. We will say that an object $X \in \mathcal{E}$ is *cocompact* if it is compact when viewed as an object of $\mathcal{E}^{\mathrm{op}}$: that is, if the functor $\mathrm{Map}_{\mathcal{E}}(\bullet, X)$ carries filtered limits in $\mathcal{E}$ to filtered colimits in $\mathcal{S}$. More generally, we will say that a morphism $X \to Y$ in $\mathcal{E}$ is *cocompact* if it exhibits X as a cocompact object of the ∞-category $\mathcal{E}_{/Y}$. We let $\mathcal{E}_{/Y}^{\mathrm{cc}}$ denote the full subcategory of $\mathcal{E}_{/Y}$ spanned by the cocompact objects.

Suppose that $\mathcal{C}$ is an essentially small ∞-category which admits finite limits. Then the ∞-category $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \simeq \mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$ is compactly generated. Moreover, if $\mathcal{C}$ is idempotent-complete, then an object of $\mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$ is compact if and only if it belongs to the essential image of the Yoneda embedding $j : \mathcal{C}^{\mathrm{op}} \to \mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$. Applying Proposition 4.4.1.2, we obtain the following:

Proposition A.8.4.2. *Let $\mathcal{C}$ be an essentially small idempotent-complete ∞-category which admits finite limits, and let $j : \mathcal{C} \to \operatorname{Pro}(\mathcal{C})$ be the Yoneda embedding. Then:*

(1) *A morphism $f : X \to Y$ in $\operatorname{Pro}(\mathcal{C})$ is cocompact if and only if there exists a pullback square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ j(X_0) & \xrightarrow{f_0} & j(Y_0) \end{array}$$

for some objects $X_0, Y_0 \in \mathcal{C}$.

(2) *For each object $Y \in \operatorname{Pro}(\mathcal{C})$, the inclusion $\operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/Y} \hookrightarrow \operatorname{Pro}(\mathcal{C})_{/Y}$ extends to an equivalence of ∞-categories $\operatorname{Pro}(\operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/Y}) \simeq \operatorname{Pro}(\mathcal{C})_{/Y}$.*

(3) *The construction $Y \mapsto \operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/Y}$ determines a functor $\operatorname{Pro}(\mathcal{C})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which commutes with filtered colimits.*

Lemma A.8.4.3. *Let $\mathcal{C}$ be a bounded ∞-pretopos. Then:*

(a) *For each object $X \in \operatorname{Pro}(\mathcal{C})$, the ∞-category $\operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X}$ is a bounded ∞-pretopos.*

(b) *For each morphism $f : X \to Y$ in $\operatorname{Pro}(\mathcal{C})$, the pullback functor $Y' \mapsto X \times_Y Y'$ induces a morphism of ∞-pretopoi $\operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/Y} \to \operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X}$.*

(c) *The construction $Y \mapsto \operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/Y}$ determines a functor $\operatorname{Pro}(\mathcal{C})^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty}$ which commutes with filtered colimits.*

Proof. Let $\chi : \operatorname{Pro}(\mathcal{C})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the functor given by $\chi(X) = \operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X}$ (see Proposition A.8.4.2) and let $j : \mathcal{C} \to \operatorname{Pro}(\mathcal{C})$ denote the Yoneda embedding. It follows from assertion (1) of Proposition A.8.4.2 that for each object $C \in \mathcal{C}$, the functor j induces an equivalence of ∞-categories $\mathcal{C}_{/C} \to \operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/j(C)}$. The ∞-category $\mathcal{C}_{/C}$ is an ∞-pretopos (Remark A.6.1.3) which is evidently bounded, so that $\operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/j(C)}$ is also a bounded ∞-pretopos. Moreover, if $f : C \to D$ is a morphism in $\mathcal{C}$, then the associated pullback functor $\operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/j(D)} \to \operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/j(C)}$ can be identified with the pullback map $\mathcal{C}_{/D} \to \mathcal{C}_{/C}$, and is therefore a morphism of ∞-pretopoi (Example A.6.4.3). It follows that the composite functor

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{j} \operatorname{Pro}(\mathcal{C})^{\mathrm{op}} \xrightarrow{\chi} \widehat{\mathcal{C}\mathrm{at}}_\infty$$

factors through the subcategory $\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty} \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$. Proposition A.8.4.2 implies that the functor χ commutes with filtered limits. Using Proposition A.8.3.1, we deduce that χ also factors through $\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty}$, which proves ($a$) and ($b$). Assertion ($c$) follows from the fact that χ preserves filtered colimits, since the inclusion $\infty\mathcal{T}\mathrm{op}^{\mathrm{pre}}_{<\infty} \hookrightarrow \widehat{\mathcal{C}\mathrm{at}}_\infty$ is conservative and preserves filtered colimits. □

Lemma A.8.4.4. *Let $\mathcal{C}$ be a bounded ∞-pretopos. For each object $X \in \mathrm{Pro}(\mathcal{C})$, let us regard the ∞-pretopos $\mathrm{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X}$ as equipped with the effective epimorphism topology (Definition A.6.2.4). Then:*

(a) *For each $X \in \mathrm{Pro}(\mathcal{C})$, the ∞-topos $\mathcal{S}\mathrm{hv}(\mathrm{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X})$ is bounded and coherent.*

(b) *The construction $X \mapsto \mathcal{S}\mathrm{hv}(\mathrm{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X})$ determines a functor $\mathrm{Pro}(\mathcal{C}) \to \infty\mathcal{T}\mathrm{op}$ which preserves filtered limits.*

Proof. Combine Lemma A.8.4.3 with Proposition A.8.3.2. □

Proof of Theorem A.8.0.5. Let $\mathcal{C}$ be a bounded ∞-pretopos and let $\overline{\rho} : \mathrm{Pro}(\mathcal{C}) \to \infty\mathcal{T}\mathrm{op}$ be the functor given on objects by $X \mapsto \mathcal{S}\mathrm{hv}(\mathrm{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X})$. Note that $\overline{\rho}$ carries the initial object $\mathbf{1} \in \mathrm{Pro}(\mathcal{C})$ to the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$, and therefore induces a functor $\rho : \mathrm{Pro}(\mathcal{C}) \simeq \mathrm{Pro}(\mathcal{C})_{/\mathbf{1}} \to \infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})}$. By construction, the composition of ρ with the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ can be identified with the composition

$$\mathcal{C} \hookrightarrow \mathcal{S}\mathrm{hv}(\mathcal{C}) \simeq \infty\mathcal{T}\mathrm{op}^{\text{ét}}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})} \subseteq \infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})},$$

and the functor ρ commutes with filtered limits by virtue of Lemma A.8.4.4. Consequently, the content of Theorem A.8.0.5 is that the functor ρ is fully faithful: that is, that it induces a homotopy equivalence $\theta_{X,Y} : \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(X, Y) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})}}(\rho(X), \rho(Y))$ for every pair of objects $X, Y \in \mathrm{Pro}(\mathcal{C})$.

Let us abuse notation by identifying $\mathcal{C}$ with its essential image under the fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$. Regard the object $X \in \mathrm{Pro}(\mathcal{C})$ as fixed. Since the functor ρ commutes with small filtered limits, the construction $Y \mapsto \theta_{X,Y}$ commutes with small filtered limits. Consequently, the collection of those objects $Y \in \mathrm{Pro}(\mathcal{C})$ for which $\theta_{X,Y}$ is a homotopy equivalence is closed under small filtered limits. We may therefore assume without loss of generality that Y is an object of $\mathcal{C}$. Let $\mathscr{F} \in \mathcal{S}\mathrm{hv}(\mathcal{C})$ denote the image of Y under the Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{S}\mathrm{hv}(\mathcal{C})$, so that we can identify $\rho(Y)$ with the overcategory $\mathcal{S}\mathrm{hv}(\mathcal{C})_{/\mathscr{F}}$. Set $\mathcal{X} = \rho(X)$, which we regard as an object of $\infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})}$ via a geometric morphism $e_* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ with left adjoint $e^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{X}$. Invoking the universal property of $\mathcal{S}\mathrm{hv}(\mathcal{C})_{/\mathscr{F}}$ (see Proposition HTT.6.3.5.5), we can identify the mapping space $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})}}(\rho(X), \rho(Y))$ with the mapping space $\mathrm{Map}_{\mathcal{X}}(\mathbf{1}_{\mathcal{X}}, e^*\mathscr{F})$, where $\mathbf{1}_{\mathcal{X}}$ denotes a final object of $\mathcal{X}$. By construction, we can identify $\mathcal{X}$ with the ∞-category $\mathcal{S}\mathrm{hv}(\mathrm{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X})$, and the objects $\mathbf{1}_{\mathcal{X}}$ and $e^*\mathscr{F}$ are the images under the Yoneda embedding $\mathrm{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X} \hookrightarrow \mathcal{S}\mathrm{hv}(\mathrm{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X}) = \mathcal{X}$ of the objects X and $X \times Y$, respectively. Under this identification, the map $\theta_{X,Y}$ corresponds to the evident homotopy equivalence $\mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(X, Y) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})_{/X}}(X, X \times Y)$. □

A.9 Conceptual Completeness

Let $\mathcal{X}$ be an ∞-topos. Recall that a *point* of $\mathcal{X}$ is a geometric morphism $\eta^* : \mathcal{X} \to \mathcal{S}$. We let $\mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{S})$ spanned by the points of $\mathcal{X}$. It is natural to ask the following:

Question A.9.0.5. To what extent can an ∞-topos $\mathcal{X}$ be recovered from its ∞-category of points $\mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$?

In §A.4, we took a first step toward answering Question A.9.0.5: according to Theorem A.4.0.5, if $\mathcal{X}$ is hypercomplete and locally coherent, then $\mathcal{X}$ has enough points (that is, the points $\eta^* : \mathcal{X} \to \mathcal{S}$ are mutually conservative). Our goal in this section is to prove the following closely related result:

Theorem A.9.0.6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be bounded coherent ∞-topoi and let $f^* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism. Then f^* is an equivalence if and only if the following conditions are satisfied:*

(a) *The functor f^* carries coherent objects of $\mathcal{X}$ to coherent objects of $\mathcal{Y}$.*

(b) *Composition with f^* induces an equivalence of ∞-categories $\mathrm{Fun}^*(\mathcal{Y}, \mathcal{S}) \to \mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$.*

Remark A.9.0.7. Theorem A.9.0.6 can be regarded as an ∞-categorical generalization of the conceptual completeness theorem of Makkai-Reyes; see [143].

Warning A.9.0.8. In the statement of Theorem A.9.0.6, one cannot replace (b) by the weaker statement that f_* induces a homotopy equivalence $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{S}, \mathcal{Y}) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{S}, \mathcal{X})$. For example, let X be a coherent topological space and let Y be the Stone space obtained by equipping X with the constructible topology (see Notation 4.3.1.3). Then the canonical map $f : Y \to X$ induces a geometric morphism $f^* : \mathcal{S}\mathrm{hv}(X) \to \mathcal{S}\mathrm{hv}(Y)$ satisfying condition (a) of Theorem A.9.0.6. The ∞-categories $\mathrm{Fun}_*(\mathcal{S}, \mathcal{S}\mathrm{hv}(X))$ and $\mathrm{Fun}_*(\mathcal{S}, \mathcal{S}\mathrm{hv}(Y))$ can be identified with X and Y respectively, where we regard X and Y as partially ordered sets with respect to the specialization ordering. The underlying map $\rho : \mathrm{Fun}^*(\mathcal{S}\mathrm{hv}(Y), \mathcal{S}) \to \mathrm{Fun}^*(\mathcal{S}\mathrm{hv}(X), \mathcal{S})$ induces a homotopy equivalence of the underlying Kan complexes (since f is bijective), but is generally not an equivalence of ∞-categories (since the specialization ordering on Y is trivial but the specialization ordering on X need not be).

A.9.1 Points and Pro-Objects

Let $\mathcal{C}$ be an bounded ∞-pretopos. In §A.8, we saw that the ∞-category $\mathrm{Pro}(\mathcal{C})$ of Pro-objects of $\mathcal{C}$ admits a "topological" incarnation: it can be identified with a full subcategory of the ∞-category $\infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})}$ of ∞-topoi with a geometric morphism to $\mathcal{S}\mathrm{hv}(\mathcal{C})$ (see Theorem A.8.0.5). We now describe a slightly different relationship between Pro-objects of $\mathcal{C}$ and the geometry of $\mathcal{S}\mathrm{hv}(\mathcal{C})$.

Proposition A.9.1.1. *Let $\mathcal{C}$ be an ∞-pretopos and let $X : \mathcal{C} \to \mathcal{S}$ be a left exact functor. The following conditions are equivalent:*

(1) *The functor X is an ∞-pretopos morphism (Definition A.6.4.1).*

(2) *For every finite collection of objects $\{U_i\}_{\in I}$ of $\mathcal{C}$ and every effective epimorphism $\amalg_{i\in I} U_i \to V$, the induced map $\amalg_{i\in I} X(U_i) \to X(V)$ is surjective on connected components.*

Proof. If (1) is satisfied, then the functor X preserves coproducts and effective epimorphisms, so condition (2) follows. Conversely, suppose that (2) is satisfied. Applying (2) in the case where I is a singleton, we deduce that X preserves effective epimorphisms. Taking $I = \varnothing$, we deduce that X preserves initial objects. Since X is left exact by assumption, it will suffice to prove X preserves pairwise coproducts. Let U and V be objects of $\mathcal{C}$ and let $i : U \to U \amalg V$ and $j : V \to U \amalg V$ denote the canonical maps. We wish to show that $X(i)$ and $X(j)$ exhibit $X(U \amalg V)$ as a coproduct of $X(U)$ with $X(V)$. Because $\mathcal{C}$ is an ∞-pretopos, coproducts in $\mathcal{C}$ are disjoint. It follows that the maps i and j are (-1)-truncated and that the fiber product $U \times_{U\amalg V} V$ is initial in $\mathcal{C}$. Using the left exactness of X, we deduce that $X(i)$ and $X(j)$ are (-1)-truncated and that the fiber product $X(U) \times_{X(U\amalg V)} X(V)$ is empty. It follows that the natural map $\rho : X(U) \amalg X(V) \to X(U \amalg V)$ is (-1)-truncated. To complete the proof, it suffices to show that ρ is surjective on connected components, which follows from (2). □

Corollary A.9.1.2. *Let $\mathcal{C}$ be an essentially small ∞-pretopos. Then composition with the Yoneda embedding $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ induces an equivalence of ∞-categories $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}(\mathcal{C}), \mathcal{S}) \to \mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$.*

Proof. Combine Propositions A.9.1.1 and HTT.6.2.3.20 . □

Definition A.9.1.3. Let $\mathcal{C}$ be an essentially small ∞-pretopos, and let us abuse notation by identifying $\mathcal{C}$ with its essential image in the ∞-category $\mathrm{Pro}(\mathcal{C})$. We will say that an object $X \in \mathrm{Pro}(\mathcal{C})$ is *prime* if it satisfies the following condition:

($*$) For every finite collection of objects $\{U_i\}_{i\in I}$ of $\mathcal{C}$ and every effective epimorphism $\amalg_{i\in I} U_i \to V$, the induced map $\amalg_{i\in I} \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(X, U_i) \to \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(X, V)$ is surjective on connected components.

Let $\mathrm{Pro}(\mathcal{C})_\circ$ denote the full subcategory of $\mathrm{Pro}(\mathcal{C})$ spanned by those objects which satisfy condition ($*$).

Remark A.9.1.4. Let $\mathcal{C}$ be an essentially small ∞-pretopos. It follows from Proposition A.9.1.1 that the equality $\mathrm{Pro}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ restricts to an equality $\mathrm{Pro}(\mathcal{C})_\circ = \mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$. Using Corollary A.9.1.2, we can identify $\mathrm{Pro}(\mathcal{C})_\circ$ with the opposite of the ∞-category $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}(\mathcal{C}), \mathcal{S})$ of points of $\mathcal{S}\mathrm{hv}(\mathcal{C})$.

Remark A.9.1.5. Let Λ be a distributive lattice (Definition A.1.4.5), which we regard as a category (where there is a unique morphism from λ to μ if $\lambda \leqslant \mu$). There is a bijective correspondence between isomorphism classes of Pro-objects of Λ and filters $F \subseteq \Lambda$ (see Definition A.1.2.1), which carries an object $X \in \mathrm{Pro}(\Lambda)$ to the filter $F_X = \{\lambda \in \Lambda : \mathrm{Hom}_{\mathrm{Pro}(\Lambda)}(X, \lambda) \neq \varnothing\}$. Let us say that the Pro-object X is *prime* if it satisfies the following analogue of condition $(*)$ of Definition A.9.1.3:

($\star$) For every finite subset $\{\lambda_i\}_{i \in I}$ of Λ, the map $\amalg_{i \in I} X(\lambda_i) \to X(\bigvee_{i \in I} \lambda_i)$ is surjective.

Unwinding the definitions, we see that a Pro-object X satisfies ($\star$) if and only if the complement of $F_X \subseteq \Lambda$ is an ideal. Consequently, the construction $X \mapsto \mathfrak{p}_X = \Lambda - F_X$ determines a bijection from the set of isomorphism classes of prime objects of $\mathrm{Pro}(\Lambda)$ to the set $\mathrm{Spec}(\Lambda)$ of prime ideals in Λ (in fact, we can be more precise: the construction $X \mapsto \mathfrak{p}_X$ determines an equivalence from the full subcategory of $\mathrm{Pro}(\Lambda)$ spanned by the prime Pro-objects to the partially ordered set of prime ideals in Λ).

Warning A.9.1.6. Let $\mathcal{C}$ be a bounded ∞-pretopos and let $X \in \mathrm{Pro}(\mathcal{C})$ be a Pro-object. Then X potentially has two different incarnations in the ∞-category $\infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})}$:

(*a*) We can consider the image of X under the fully faithful embedding $\rho : \mathrm{Pro}(\mathcal{C}) \hookrightarrow \infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})}$ of Theorem A.8.0.5: by virtue of Lemma A.8.4.4, this image can be identified with the ∞-topos $\mathcal{S}\mathrm{hv}(\mathrm{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X})$.

(*b*) If X is prime, then the image of X under the equivalence of ∞-categories $\mathrm{Pro}(\mathcal{C})_\circ \simeq \mathrm{Fun}^*(\mathcal{S}\mathrm{hv}(\mathcal{C}), \mathcal{S})^{\mathrm{op}}$ of Remark A.9.1.4 is a geometric morphism $\eta_{X*} : \mathcal{S} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$, which exhibits the ∞-topos $\mathcal{S}$ as an object of $\infty\mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(\mathcal{C})}$.

These objects are usually not the same. However, they are closely related: if X is prime, then the geometric morphism $\eta_{X*} : \mathcal{S} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ admits a factorization $\mathcal{S} \xrightarrow{\eta'_{X*}} \mathcal{S}\mathrm{hv}(\mathrm{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X}) \to \mathcal{S}\mathrm{hv}(\mathcal{C})$. Beware that if $u : X \to Y$ is a morphism in $\mathrm{Pro}(\mathcal{C})_\circ$, then the diagram of ∞-topoi

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\eta'_{X*}} & \mathcal{S}\mathrm{hv}(\mathrm{Pro}(\mathcal{C})^{\mathrm{cc}}_{/X}) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \rho(u)} \\ \mathcal{S} & \xrightarrow{\eta'_{Y*}} & \mathcal{S}\mathrm{hv}(\mathrm{Pro}(\mathcal{C})^{\mathrm{cc}}_{/Y}) \end{array}$$

commutes only up to a natural transformation $\rho(u) \circ \eta'_{X*} \to \eta'_{Y*}$, which is an equivalence if and only if u is an equivalence.

A.9.2 Detecting Equivalences of ∞-Pretopoi

Our proof Theorem A.9.0.6 rests on the following recognition criterion for equivalences of (bounded) ∞-pretopoi (which we used already in our proof of Theorem A.7.5.3):

Proposition A.9.2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be bounded ∞-pretopoi and let $f^* : \mathcal{C} \to \mathcal{D}$ be a functor which preserves coproducts, finite limits, and effective epimorphisms. Then f^* is an equivalence if and only if the following conditions are satisfied:*

(a) *For every object $C \in \mathcal{C}$ and every morphism $u : D \to f^*C$ in $\mathcal{D}$, there exists a morphism $v : C' \to C$ in $\mathcal{C}$ and a commutative diagram*

$$\begin{array}{ccc} f^*C' & \xrightarrow{w} & D \\ & {\scriptstyle f^*v} \searrow \quad \swarrow {\scriptstyle u} & \\ & f^*C & \end{array}$$

in $\mathcal{D}$, where w is an effective epimorphism.

(b) *For every (-1)-truncated morphism $u : C' \to C$ in $\mathcal{C}$, if $f^*(u)$ is an equivalence in $\mathcal{D}$, then u is an equivalence.*

Proof. The necessity of conditions (a) and (b) is obvious. For the converse, assume that (a) and (b) are satisfied; we wish to prove that the functor f^* is an equivalence of ∞-categories. We begin by showing that f^* is conservative. Let $u : C' \to C$ be a morphism in $\mathcal{C}$ such that $f^*(u)$ is an equivalence; we wish to show that u is an equivalence. Because $\mathcal{C}$ is bounded, we may assume that the morphism u is n-truncated for some integer n; we proceed by induction on n. If $n = -2$, there is nothing to prove. If $n > -2$, then the diagonal map $\delta : C' \to C' \times_C C'$ is $(n-1)$-truncated and the left exactness of f^* guarantees that $f^*(\delta)$ is an equivalence. Applying our inductive hypothesis, we deduce that δ is an equivalence: that is, the morphism u is (-1)-truncated. In this case, the desired result follows from assumption (b).

We will prove that the functor f^* is an equivalence of ∞-categories by establishing the following assertions for $n \geqslant -2$:

(1_n) For every pair of objects $C, C' \in \mathcal{C}$ where C' is n-truncated, the functor f^* induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Map}_{\mathcal{D}}(f^*C, f^*C')$.

(2_n) Every n-truncated object of $\mathcal{D}$ belongs to the essential image of f^*.

Since $\mathcal{C}$ is bounded, it follows from (1_n) (for all $n \geqslant -2$) that the functor f^* is fully faithful; since $\mathcal{D}$ is bounded, it follows from (2_n) (for all $n \geqslant -2$) that the functor f^* is essentially surjective.

We will prove (1_n) and (2_n) by a simultaneous induction on n. Let us begin with the case $n = -2$. If $C' \in \mathcal{C}$ is (-2)-truncated, then it is a final object of $\mathcal{C}$, so (since the functor f^* preserves finite limits) we conclude that f^*C' is a final object of $\mathcal{D}$. This proves (2_n). Assertion (1_n) follows from the observation that for any object $C \in \mathcal{C}$, the natural map

$\mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Map}_{\mathcal{D}}(f^*C, f^*C')$ has contractible domain and codomain and is therefore automatically a homotopy equivalence.

We now prove (1_n) for $n > -2$. Fix objects $C, C' \in \mathcal{C}$ where C' is n-truncated; we wish to prove that the natural map $\rho : \mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Map}_{\mathcal{D}}(f^*C, f^*C')$ is a homotopy equivalence. Fix $k \geqslant -1$ and suppose we are given a map $S^k \to \mathrm{Map}_{\mathcal{C}}(C, C')$; we wish to show that the induced map

$$\mathrm{fib}(\mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Map}_{\mathcal{C}}(C, C'^{S^k})) \to \mathrm{fib}(\mathrm{Map}_{\mathcal{D}}(f^*C, f^*C') \to \mathrm{Map}_{\mathcal{D}}(f^*C, f^*C'^{S^k}))$$

is surjective on connected components. Replacing $\mathcal{C}$ by $\mathcal{C}_{/C'^{S^k}}$ and $\mathcal{D}$ by $\mathcal{D}_{/f^*C'^{S^k}}$, we are reduced to proving that the map ρ is surjective on connected components. Let us therefore fix a map $\alpha : f^*C \to f^*C'$ in the ∞-category $\mathcal{D}$; we wish to show that α belongs to the essential image of ρ. Note that the pair

$$(\mathrm{id}, \alpha) : f^*C \to f^*C \times f^*C' \simeq f^*(C \times C')$$

is a pullback of the diagonal map $f^*C' \to f^*(C' \times C')$. Since C' is n-truncated, the map (id, α) exhibits f^*C as an $(n-1)$-truncated object of the ∞-pretopos $\mathcal{D}_{/f^*(C \times C')}$. Applying our inductive hypothesis (2_{n-1}) to the map $\mathcal{C}_{/C \times C'} \to \mathcal{D}_{/f^*(C \times C')}$, we deduce that there exists an object $E \in \mathcal{C}_{/C \times C'}$ and an equivalence $f^*E \simeq f^*C$ in the ∞-category $\mathcal{D}_{/f^*(C \times C')}$. Let us regard E as an object of $\mathcal{C}$ equipped with a pair of maps $\beta : E \to C$ and $\beta' : E \to C'$. Then $f^*(\beta)$ is an equivalence and therefore β is an equivalence (by virtue of the fact that f^* is conservative). We now observe that α is homotopic to the composition $f^*(\beta') \circ f^*(\beta)^{-1} \simeq f^*(\beta' \circ \beta^{-1})$ and therefore belongs to the essential image of ρ as desired. This completes the proof of (1_n).

We now verify (2_n). Let $D \in \mathcal{D}$ be an n-truncated object. Using (a), we can choose an effective epimorphism $u : D_0 \to D$, where D_0 belongs to the essential image of f^*. Since the essential image of f^* is stable under n-truncation, we may assume without loss of generality that D_0 is n-truncated. Let $D_\bullet$ denote the Čech nerve of u. Note that for $k > 0$, we have an equivalence

$$D_k \simeq D_{k-1} \times_D D_0 \simeq (D_{k-1} \times D_0) \times_{D \times D} D.$$

Since D is n-truncated, it follows that the projection map $D_k \to D_{k-1} \times D_0$ is $(n-1)$-truncated. It follows from inductive hypothesis (2_{n-1}) and induction on k that each D_k belongs to the essential image of f^*. Since each D_k is n-truncated and the functor f^* is fully faithful when restricted to n-truncated objects (by virtue of (1_n)), we can write $D_\bullet = f^*C_\bullet$ for some groupoid object $C_\bullet$ in $\mathcal{C}$. Set $C = |C_\bullet|$, and let $v : C_0 \to C$ be the canonical map. Then f^*v factors as a composition

$$f^*C_0 \simeq D_0 \xrightarrow{u} D = |f^*C_\bullet| \xrightarrow{w} f^*|C_\bullet| = f^*C.$$

where w is (-1)-truncated. Since the functor f^* preserves effective epimorphisms, the map f^*v is an effective epimorphism. It follows that $w : D \to f^*C$ is an equivalence, so that D belongs to the essential image of f^* as desired. □

A.9.3 The Proof of Theorem A.9.0.6

We now turn to the proof of Theorem A.9.0.6. It is clear that if $f^* : \mathcal{X} \to \mathcal{Y}$ is an equivalence of bounded coherent ∞-topoi, then f^* carries coherent objects of $\mathcal{X}$ to coherent objects of $\mathcal{Y}$ and that composition with f^* induces an equivalence $\mathrm{Fun}^*(\mathcal{Y}, \mathcal{S}) \to \mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$. By virtue of Theorem A.7.5.3 and Corollary A.9.1.2, the converse can be reformulated as follows:

Theorem A.9.3.1. *Let $f^* : \mathcal{C} \to \mathcal{D}$ be a morphism between bounded ∞-pretopoi, and suppose that composition with f^* induces an equivalence* $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{D}, \mathcal{S}) \to \mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$*. Then f^* is an equivalence of ∞-categories.*

Proof. We will show that the functor f^* satisfies conditions (a) and (b) of Proposition A.9.2.1. We begin with condition (b). Let $i : U \to X$ be a (-1)-truncated morphism in $\mathcal{C}$ such that $f^*(i)$ is an equivalence in $\mathcal{D}$; we wish to show that i is an equivalence. To prove this, it will suffice to show that the image of i under the Yoneda embedding $j_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{S}\mathrm{hv}(\mathcal{C})$ is an equivalence. Since $\mathcal{C}$ is bounded, the functor j takes values in the full subcategory $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}} \subseteq \mathcal{S}\mathrm{hv}(\mathcal{C})$, which has enough points by virtue of Theorem A.4.0.5. It will therefore suffice to show that for every geometric morphism $\eta^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}$, the image $\eta^*(j_{\mathcal{C}}(i))$ is a homotopy equivalence. Equivalently, we wish to show that the functor $(\eta^* \circ j_{\mathcal{C}}) \in \mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$ carries i to an equivalence. The essential surjectivity of the functor $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{D}, \mathcal{S}) \to \mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$ guarantees that $\eta^* \circ j$ factors through f^*, so the desired result follows from our assumption that $f^*(i)$ is an equivalence.

We can break condition (a) of Proposition A.9.2.1 into the following family of assertions:

(a_n) For every object $E \in \mathcal{C}$ and every n-truncated morphism $u : D \to f^*E$ in $\mathcal{D}$, there exists a morphism $v : C \to E$ in $\mathcal{C}$ and a commutative diagram

$$\begin{array}{ccc} f^*C & \xrightarrow{w} & D \\ & {\scriptstyle f^*v} \searrow \quad \swarrow {\scriptstyle u} & \\ & f^*E & \end{array}$$

in $\mathcal{D}$, where w is an effective epimorphism.

Our proof of (a_n) will proceed by induction on n. If $n = -2$, then u is an equivalence and there is nothing to prove. Let us therefore assume that $n > -2$. Replacing $\mathcal{C}$ by $\mathcal{C}_{/E}$ and $\mathcal{D}$ by $\mathcal{D}_{/f^*E}$, we may reduce to the case where E is a final object of $\mathcal{C}$, so that $D \in \mathcal{D}$ is

n-truncated. Let us say that an object $D' \in \mathcal{D}_{/D}$ is *good* if there exists an $(n-1)$-truncated map $D' \to f^* C$ for some object $C \in \mathcal{C}$. Suppose that the collection of good objects $D' \in \mathcal{D}_{/D}$ generate a covering sieve with respect to the effective epimorphism topology: that is, there exists a finite collection of good objects $\{D'_i\}_{i \in I}$ for which the induced map $\amalg_{i \in I} D'_i \to D$ is an effective epimorphism. By definition, each of the objects D'_i admits an $(n-1)$-truncated morphism $D'_i \to f^* C_i$ for some object $C_i \in \mathcal{C}_{/C}$. Applying the inductive hypothesis (a_{n-1}), we deduce the existence of effective epimorphisms $f^* C'_i \to D'_i$ for some objects $C'_i \in \mathcal{C}_{/C_i}$. We can then complete the verification of (a_n) by observing that the composite map

$$f^*(\amalg_{i \in I} C'_i) \simeq \amalg_{i \in I} f^* C'_i \to \amalg_{i \in I} D'_i \to D$$

is an effective epimorphism.

Let us now treat the case where the good objects of $\mathcal{D}_{/D}$ do *not* generate a covering sieve with respect to the effective epimorphism topology. Applying Theorem A.4.0.5 to the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{D})^{\mathrm{hyp}}$ and using the identifications $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}(\mathcal{D})^{\mathrm{hyp}}, \mathcal{S}) \simeq \mathrm{Fun}^*(\mathcal{S}\mathrm{hv}(\mathcal{D}), \mathcal{S}) \simeq \mathrm{Fun}^{\mathrm{pre}}(\mathcal{D}, \mathcal{S})$, we deduce that there exists an object $M \in \mathrm{Fun}^{\mathrm{pre}}(\mathcal{D}, \mathcal{S})$ together with a point $x \in M(D)$ which cannot be lifted to a point of $M(D')$ for any good object $D' \in \mathcal{D}_{/D}$.

In what follows, let us abuse notation by identifying $\mathcal{C}$ and $\mathcal{D}$ with their essential images in the ∞-categories $\mathrm{Pro}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{D})$. Note that precomposition with the functor f^* determines a functor $F : \mathrm{Pro}(\mathcal{D}) \to \mathrm{Pro}(\mathcal{C})$, and the functor F admits a right adjoint $G : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$ (which can be described as the essentially unique extension of f^* which commutes with filtered limits). Let us view M as an object of the ∞-category $\mathrm{Pro}(\mathcal{D})$ and x as a morphism from M to D in $\mathrm{Pro}(\mathcal{D})$. Set $X = M^{S^n} \times_{(F \circ G)(M)^{S^n}} (F \circ G)(M)$, where the fiber product is computed in the ∞-category $\mathrm{Pro}(\mathcal{D})$, and consider the composite map $X \to M^{S^n} \xrightarrow{x} D^{S^n}$. Since the object D is n-truncated, the diagonal map $\delta : D \to D^{S^n}$ is (-1)-truncated. Set $X_0 = D \times_{D^{S^n}} X$, so that projection onto the second factor gives a (-1)-truncated, cocompact morphism $\iota : X_0 \to X$ in $\mathrm{Pro}(\mathcal{D})$. We consider two cases:

(*i*) Suppose that the morphism ι is an equivalence. Let us write the object $M \in \mathrm{Pro}(\mathcal{D})_{/D}$ as the limit of a filtered diagram $\{M_\alpha\}_{\alpha \in A}$ in $\mathcal{D}_{/D}$. For each index $\alpha \in A$, set $X_\alpha = M_\alpha^{S^n} \times_{(F \circ G)(M_\alpha)^{S^n}} (F \circ G)(M_\alpha)$ and $X_{\alpha 0} = D \times_{D^{S^n}} X_\alpha$. Since $\mathrm{Pro}(\mathcal{D})^{\mathrm{cc}}_{/X}$ is the filtered colimit of the ∞-categories $\mathrm{Pro}(\mathcal{D})^{\mathrm{cc}}_{/X_\alpha}$ (Proposition A.8.4.2), there exists an index $\alpha \in A$ for which the projection map $X_{\alpha 0} \to X_\alpha$ is an equivalence. Let us write $G(M_\alpha)$ as the limit of a filtered diagram $\{C_\beta\}_{\beta \in B}$ in $\mathcal{C}$, and for each index $\beta \in B$ set

$$Y_\beta = M_\alpha^{S^n} \times_{f^* C_\beta^{S^n}} f^* C_\beta \qquad Y_{\beta 0} = D \times_{D^{S^n}} Y_\beta.$$

Using Proposition A.8.4.2 again, we deduce that $\mathrm{Pro}(\mathcal{D})^{\mathrm{cc}}_{/X_\alpha}$ can be identified with the filtered colimit of the diagram of ∞-categories $\{\mathrm{Pro}(\mathcal{D})^{\mathrm{cc}}_{/Y_\beta}\}_{\beta \in B}$, so that there exists an index $\beta \in B$ such that the projection $\iota_\beta : Y_{\beta 0} \to Y_\beta$ is an equivalence. The map

$M_\alpha \to D \times f^* C_\beta$ factors as a composition $M_\alpha \xrightarrow{e} D' \xrightarrow{(g,h)} D \times f^* C_\beta$, where the morphism e is n-connective and the morphism (g,h) is $(n-1)$-truncated. Unwinding the definitions, we have a pullback diagram

$$\begin{array}{ccc} M_\alpha^{S^n} \times_{(D \times f^* C_\beta)^{S^n}} (X \times f^* C_\beta) & \xrightarrow{\iota_\beta} & M_\alpha^{S^n} \times_{f^* C_\beta^{S^n}} f^* C_\beta \\ \downarrow & & \downarrow \\ D'^{S^n} \times_{(D \times f^* C_\beta)^{S^n}} (X \times f^* C_\beta) & \xrightarrow{\iota'} & D'^{S^n} \times_{f^* C_\beta^{S^n}} f^* C_\beta \end{array}$$

in the ∞-pretopos $\mathcal{D}$. Since e is n-connective, the vertical maps are effective epimorphisms. Since the map ι_β is an equivalence, we deduce that ι' is an equivalence. Because the morphism (g,h) is $(n-1)$-truncated, the diagonal map $D' \to D'^{S^n} \times_{(D \times f^* C_\beta)^{S^n}} (X \times f^* C_\beta)$ is an equivalence, so the composite map

$$D' \to D'^{S^n} \times_{(D \times f^* C_\beta)^{S^n}} (X \times f^* C_\beta) \xrightarrow{\iota'} D'^{S^n} \times_{f^* C_\beta^{S^n}} f^* C_\beta$$

is also an equivalence. In other words, the map $h : D' \to f^* C_\beta$ is $(n-1)$-truncated. It follows that g exhibits D' as a good object of $\mathcal{D}_{/D}$. This is a contradiction, since the morphism $x : M \to D$ factors through D'.

(ii) Suppose that the morphism ι is not an equivalence. Applying Theorem A.4.0.5 to the hypercompletion of the ∞-topos $\mathcal{S}\mathrm{hv}(\mathrm{Pro}(\mathcal{D})^{\mathrm{cc}}_{/X})$, we deduce that there exists a geometric morphism $\nu^* : \mathcal{S}\mathrm{hv}(\mathrm{Pro}(\mathcal{D})^{\mathrm{cc}}_{/X}) \to \mathcal{S}$ such that the image of X_0 under the composite map

$$\mathrm{Pro}(\mathcal{D})^{\mathrm{cc}}_{/X} \hookrightarrow \mathcal{S}\mathrm{hv}(\mathrm{Pro}(\mathcal{D})^{\mathrm{cc}}_{/X}) \xrightarrow{\nu^*} \mathcal{S}$$

is empty. Using Corollary A.9.1.2, we can identify η^* with a prime Pro-object $N \in \mathrm{Pro}(\mathrm{Pro}(\mathcal{D})^{\mathrm{cc}}_{/X}) \simeq \mathrm{Pro}(\mathcal{D})_{/X}$. By construction, the Pro-object N fits into a commutative diagram

$$\begin{array}{ccc} N & \longrightarrow & (G \circ F)(M) \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ M^{S^n} & \longrightarrow & (G \circ F)(M)^{S^n} \end{array}$$

in the ∞-category $\mathrm{Pro}(\mathcal{D})$, and the composite map $N \xrightarrow{\phi} M^{S^n} \xrightarrow{x} D^{S^n}$ does not factor through the diagonal map $D \to D^{S^n}$. Unwinding the definitions, we see that ϕ classifies a map of spaces $S^n \to \mathrm{Map}_{\mathrm{Fun}^{\mathrm{pre}}(\mathcal{D},\mathcal{S})}(M, N)$, and the commutativity of the diagram σ shows that the composite map

$$S^n \to \mathrm{Map}_{\mathrm{Fun}^{\mathrm{pre}}(\mathcal{D},\mathcal{S})}(M, N) \to \mathrm{Map}_{\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C},\mathcal{S})}(FM, FN)$$

is nullhomotopic. By assumption, precomposition with f^* induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{D}, \mathcal{S}) \to \mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$, so the map $S^n \to \mathrm{Map}_{\mathrm{Fun}^{\mathrm{pre}}(\mathcal{D},\mathcal{S})}(M, N)$ must already be nullhomotopic: that is, the map ϕ factors through the diagonal $M \to M^{S^n}$. It follows that the composite map $N \xrightarrow{\phi} M^{S^n} \xrightarrow{x} D^{S^n}$ factors through the diagonal $D \to D^{S^n}$: that is, the projection map $N \to X$ factors through X_0. This is a contradiction, since the mapping space $\mathrm{Map}_{\mathrm{Pro}(\mathcal{D})_{/X}}(N, X_0) = \emptyset$ by construction.

□

Appendix B

Grothendieck Topologies in Commutative Algebra

Let Aff denote the category of affine schemes. We will say that a collection of morphisms $\{f_\alpha : U_\alpha \to X\}$ in Aff is *jointly surjective* if the map $\coprod U_\alpha \to X$ is surjective as a map of topological spaces. There are several Grothendieck topologies on the category Aff which are useful in the study of classical algebraic geometry. For example:

- The *Zariski topology*, whose coverings are generated by collections of maps $\{\operatorname{Spec} R_\alpha \to \operatorname{Spec} R\}$ which are jointly surjective and have the property that each R_α is isomorphic to $R[x_\alpha^{-1}]$ for some element $x_\alpha \in R$ (in this case, the condition of joint surjectivity is equivalent to the condition that the elements x_α generate the unit ideal in R).
- The *étale topology*, whose coverings generated by collections of maps $\{\operatorname{Spec} R_\alpha \to \operatorname{Spec} R\}$ which are jointly surjective and have the property that each R_α is an étale R-algebra.
- The *fpqc topology*, whose coverings are generated by finite collections of maps $\{\operatorname{Spec} R_\alpha \to \operatorname{Spec} R\}$ which are jointly surjective and have the property that each R_α is flat over R.

In this appendix, we will study some Grothendieck topologies which play an analogous role in the theory of spectral algebraic geometry. For our purposes, the most important of the topologies listed above is the étale topology. We therefore begin in §B.1 by reviewing the theory of étale morphisms between $\mathbb{E}_\infty$-rings (relying heavily on the more extensive discussion given in §HA.7.5). Our main result is a structure theorem for étale morphisms (Proposition B.1.1.3), which parallels (and generalizes) the classification of étale morphisms of ordinary commutative rings (Proposition B.1.1.1).

If R is an $\mathbb{E}_\infty$-ring, we let CAlg_R denote the ∞-category of $\mathbb{E}_\infty$-algebras over R and $\mathrm{CAlg}_R^{\text{ét}} \subseteq \mathrm{CAlg}_R$ the full subcategory of étale R-algebras. This ∞-category can be equipped with a Grothendieck topology which we refer to as *the étale topology* (see Definition B.6.2.2). Most of this appendix is devoted to developing some tools for answering the following:

Question B.0.3.1. Let R be an $\mathbb{E}_\infty$-ring and let $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{C}$ be a functor with values in an ∞-category $\mathcal{C}$. How can one tell if $\mathscr{F}$ is a sheaf for the étale topology?

Remark B.0.3.2. For any $\mathbb{E}_\infty$-ring R, the ∞-category $\mathrm{CAlg}_R^{\text{ét}}$ is actually (equivalent to) an ordinary category: more precisely, it is equivalent to the ordinary category $\mathrm{CAlg}_{\pi_0 R}^{\text{ét}}$ of étale algebras over the ordinary commutative ring $\pi_0 R$ (Theorem HA.7.5.0.6). Consequently, to understand general features of the theory of étale sheaves over R, there is no loss of generality in assuming that R is discrete. However, the sheaves which arise *in practice* often depend on the $\mathbb{E}_\infty$-ring R itself, and not only on the commutative ring $\pi_0 R$. For example, the forgetful functor $\mathscr{O} : \mathrm{CAlg}_R^{\text{ét}} \to \mathrm{CAlg}$ plays the role of the structure sheaf of the étale spectrum $\mathrm{Spét}\, R$ (see Proposition 1.4.2.4), and we can recover R as the value of $\mathscr{O}$ on the initial object of $\mathrm{CAlg}_R^{\text{ét}}$.

To address Question B.0.3.1, it will be convenient to introduce some auxiliary Grothendieck topologies: the *Nisnevich topology* (which we study in §B.4) and the *finite étale topology* (see Proposition **??**). In §B.6, we will show that a functor $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ is a sheaf for the étale topology if and only if it is a sheaf for both the Nisnevich topology and the finite étale topology (Theorem B.6.4.1). This result is useful because the class of sheaves with respect to the Nisnevich and finite étale topologies admit more concrete characterizations. In §B.5, we will prove a result a Morel and Voevodsky (Theorem B.5.0.3) which characterizes Nisnevich sheaves as those functors $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ which satisfy a certain excision property (see Theorem B.5.0.3). In §B.7, we will show that a functor $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ is a sheaf for the finite étale topology if and only if it satisfies a version of Galois descent (Theorem B.7.6.1).

Our discussion of the Nisnevich topology (and of its relationship with the étale topology) will require some familiarity with the theory of commutative rings. Some of the early sections of this appendix are devoted to giving brief expository accounts of some of the requisite commutative algebra: in §B.2 we discuss the dimension theory of Noetherian rings, and in §B.3 we review the theory of Henselian rings (and the construction of the *Henselization* of a commutative ring R with respect to an ideal $I \subseteq R$).

Remark B.0.3.3. Another important Grothendieck topology on the category of (affine) schemes is the *fppf topology*, whose coverings are generated by finite collections of maps $\{\mathrm{Spec}\, R_\alpha \to \mathrm{Spec}\, R\}$ which are jointly surjective and have the property that each R_α is flat and of finite presentation over R. We will discuss analogues of the fppf topology in §**??**.

Contents

B.1 Étale Morphisms of Ring Spectra

Let $\phi : A \to B$ be a homomorphism of commutative rings. The morphism ϕ is said to be *étale* if the following conditions are satisfied:

(i) The morphism ϕ is flat: that is, it exhibits B as a flat A-module.

(ii) The multiplication map $B \otimes_A B \to B$ exhibits B as a localization $(B \otimes_A B)[e^{-1}]$ for some idempotent element $e \in B \otimes_A B$.

(iii) The commutative ring B is finitely presented as an A-algebra.

In §HA.7.5 , we studied a generalization of the theory of étale ring homomorphisms to the setting of $\mathbb{E}_\infty$-rings. Recall that if $\phi : A \to B$ is a morphism of $\mathbb{E}_\infty$-rings, then we say that ϕ is *étale* if the underlying ring homomorphism $\pi_0 A \to \pi_0 B$ is étale and ϕ induces an isomorphism of graded rings $\pi_0 B \otimes_{\pi_0 A} \pi_* A \to \pi_* B$. Note that if A is discrete, then ϕ is étale if and only if B is also discrete and ϕ is étale when regarded as a morphism of commutative rings (that is, it satisfies conditions (i), (ii) and (iii) above).

B.1.1 Structure Theory of Étale Morphisms

In the setting of classical commutative algebra, the collection of étale ring homomorphisms admits the following "extrinsic" characterization:

Proposition B.1.1.1. *Let A be a commutative ring. Then a ring homomorphism $\phi : A \to B$ is étale if and only if it exhibits B as isomorphic to an A-algebra of the form*

$$(A[y_1, \ldots, y_n]/(f_1, \ldots, f_n))[\Delta^{-1}],$$

where Δ denotes the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1 \leqslant i,j \leqslant n}$.

We would like to generalize Proposition B.1.1.1 to the setting of $\mathbb{E}_\infty$-rings. First, we need to introduce some terminology.

Notation B.1.1.2. Let A be an $\mathbb{E}_\infty$-ring, and let x be an arbitrary symbol. We let $A\{x\}$ denote the free $\mathbb{E}_\infty$-algebra over A on one generator. More precisely, we let $A\{x\}$ denote an object CAlg_A equipped with a point $\eta \in \Omega^\infty A\{x\}$ have the following universal property: for every object $B \in \mathrm{CAlg}_A$, evaluation on η induces a homotopy equivalence $\mathrm{Map}_{\mathrm{CAlg}_A}(A\{x\}, B) \to \Omega^\infty B$. We can describe $A\{x\}$ more explicitly as the symmetric algebra $\mathrm{Sym}^*_A(A)$ on A (regarded as a module over itself), so that $A\{x\}$ is equivalent to the infinite direct sum $\bigoplus_{n \geqslant 0} A_{\Sigma_n}$, where A_{Σ_n} denotes the spectrum of coinvariants with respect to the trivial action of the symmetric group Σ_n on A (see §HA.3.1.3).

More generally, given a finite collection of symbols $x_1, \ldots, x_m$, we let $A\{x_1, \ldots, x_m\}$ denote the free $\mathbb{E}_\infty$-algebra over A on m generators, which can either be described inductively by the formula $A\{x_1, \ldots, x_m\} \simeq (A\{x_1, \ldots, x_{m-1}\})\{x_m\}$, or as the symmetric algebra $\mathrm{Sym}^*_A(A^m)$. We note that if A is connective, then the commutative ring $\pi_0 A\{x_1, \ldots, x_m\}$ can be identified with the polynomial algebra $(\pi_0 A)[x_1, \ldots, x_m]$.

We can now formulate the main result of this section.

Proposition B.1.1.3. *Let R be a connective $\mathbb{E}_\infty$-ring, and let $\phi : A \to B$ be a morphism between R-algebras. The following conditions are equivalent:*

(1) *The map ϕ is étale.*

(2) *There exists a pushout diagram of R-algebras*

$$\begin{array}{ccc} R\{x_1, \ldots, x_n\} & \longrightarrow & A \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi} \\ R\{y_1, \ldots, y_n\}[\Delta^{-1}] & \longrightarrow & B, \end{array}$$

where $\phi_0(x_i) = f_i(y_1, \ldots, y_n) \in (\pi_0 R)[y_1, \ldots, y_n]$ and $\Delta \in (\pi_0 R)[y_1, \ldots, y_n]$ denotes the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1 \leqslant i,j \leqslant n}$.

B.1.2 The Proof of Proposition B.1.1.1

Proposition B.1.1.3 can be regarded as a generalization of Proposition B.1.1.1, since every étale ring homomorphism is also étale when regarded as a morphism of $\mathbb{E}_\infty$-rings. Nevertheless, we give a proof of Proposition B.1.1.1 first, since it will be needed in the proof of Proposition B.1.1.3.

Lemma B.1.2.1. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Assume that:*

(1) *The map f induces a surjection $f_0 : \pi_0 A \to \pi_0 B$.*

(2) *The commutative ring $\pi_0 B$ is finitely presented over $\pi_0 A$ (that is, the kernel of f_0 is a finitely generated ideal in $\pi_0 A$).*

(3) *The abelian group $\pi_1 L_{B/A}$ vanishes.*

Then there exists an element $a \in \pi_0 A$ such that $\pi_0 B \simeq (\pi_0 A)[a^{-1}]$.

Proof. Let I denote the kernel of f_0, and let $R = (\pi_0 A)/I^2$. It follows from Corollary HA.7.4.1.27 that, in the ∞-category CAlg_A, we have a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & \pi_0 B \\ \downarrow & & \downarrow \\ \pi_0 B & \longrightarrow & (\pi_0 B) \oplus \Sigma(I/I^2). \end{array}$$

Since $\pi_i L_{B/A} \simeq 0$ for $i \leqslant 1$, the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}_A}(B, \pi_0 B) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, \pi_0 B \oplus \Sigma(I/I^2))$$

is a homotopy equivalence, so that $\mathrm{Map}_{\mathrm{CAlg}_R}(B, R) \to \mathrm{Map}_{\mathrm{CAlg}_R}(B, \pi_0 B)$ is also a homotopy equivalence. In particular, the truncation map $B \to \pi_0 B$ lifts (in an essentially unique fashion) to a map $B \to R$. Passing to connected components, we deduce that the quotient map of commutative algebras $\phi : (\pi_0 A)/I^2 \to (\pi_0 A)/I$ admits a section (in the category of $\pi_0 A$-algebras). This implies that ϕ is an isomorphism: that is, that $I = I^2$.

Because $\pi_0 B$ is finitely presented over $\pi_0 A$, the ideal I is generated by finitely elements $y_1, \ldots, y_m$. Since $I = I^2$, we can write $y_i = \sum_j z_{i,j} y_j$ for some elements $z_{i,j} \in I$. Let Z denote the matrix $\{z_{i,j}\}_{1 \leqslant i,j \leqslant m}$. Then $\mathrm{id} - Z$ annihilates the vector $(y_1, \ldots, y_m) \in (\pi_0 A)^m$. Let $a \in \pi_0 A$ denote the determinant of $\mathrm{id} - Z$. Since the entries of Z belong to I, D is congruent to 1 modulo I and is therefore invertible in $\pi_0 B$. It follows that we have a canonical map $g : A[a^{-1}] \to B$. We claim that g induces an isomorphism $g_0 : \pi_0 A[a^{-1}] \to \pi_0 B$. The surjectivity of g is clear, and the injectivity follows from the observation that multiplication by a annihilates every element of I. □

Lemma B.1.2.2. *Let κ be a field and suppose we are given a homomorphism of polynomial rings $\psi : \kappa[x_1, \ldots, x_n] \to \kappa[y_1, \ldots, y_n]$, given by $\psi(x_i) = f_i(y_1, \ldots, y_n)$. Let $\Delta \in \kappa[y_1, \ldots, y_n]$ denote the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1 \leqslant i,j \leqslant n}$. Then $\kappa[y_1, \ldots, y_n][\Delta^{-1}]$ is flat over $\kappa[x_1, \ldots, x_n]$.*

Proof. Without loss of generality, we may suppose that κ is algebraically closed. It will suffice to show that for each maximal ideal $\mathfrak{m} \subseteq \kappa[y_1, \ldots, y_n]$ not containing Δ, the induced map of localizations $\kappa[x_1, \ldots, x_n]_{\mathfrak{n}} \to \kappa[y_1, \ldots, y_n]_{\mathfrak{m}}$ is flat, where $\mathfrak{n} = \psi^{-1}\mathfrak{m}$. Using Hilbert's Nullstellensatz, we conclude that $\mathfrak{m}$ is generated by $y_j - \lambda_j$ for some scalars $\lambda_j \in \kappa$. Making a change of coordinates if necessary, we may assume that $\mathfrak{m} = (y_1, \ldots, y_n)$. Similarly, we may assume that the polynomials f_i satisfy $f_i(0, \ldots, 0) = 0$, so that $\mathfrak{n} = (x_1, \ldots, x_n)$. In this case, we have a commutative diagram

$$\begin{array}{ccc} \kappa[x_1, \ldots, x_n]_{\mathfrak{n}} & \longrightarrow & \kappa[y_1, \ldots, y_n]_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ \kappa[[x_1, \ldots, x_n]] & \xrightarrow{\psi^\wedge} & \kappa[[y_1, \ldots, y_n]], \end{array}$$

where the vertical maps are faithfully flat (Corollary 7.3.6.9). It will therefore suffice to show that $\psi^\wedge$ is flat. In fact, we claim that $\psi^\wedge$ is an isomorphism. To prove this, we note that $\psi^\wedge$ is an inverse limit of maps of the form

$$\psi_d : \kappa[x_1, \ldots, x_n]/\mathfrak{n}^d \to \kappa[y_1, \ldots, y_n]/\mathfrak{m}^d.$$

Since ψ_d is a map between vector spaces of the same dimension over κ, we are reduced to proving that ψ_d is surjective. Using induction on d, we can reduce to the case $d = 2$, in which case the desired result follows from our assumption that $\Delta \notin \mathfrak{m}$. □

Proof of Proposition B.1.1.1. Suppose first that $B = (A[y_1, \ldots, y_n]/(f_1, \ldots, f_n))[\Delta^{-1}]$; we will prove that B is étale over A. It is clear that B is finitely presented as an A-algebra. We next prove that B is flat over A. For this, let us abuse notation by identifying Δ with the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1 \leqslant i,j \leqslant n}$ in the polynomial ring $A[y_1, \ldots, y_n]$, so that we have a pushout diagram of commutative rings

$$\begin{array}{ccc} A[x_1, \ldots, x_n] & \longrightarrow & A \\ \downarrow{\scriptstyle \psi} & & \downarrow \\ A[y_1, \ldots, y_n][\Delta^{-1}] & \longrightarrow & B \end{array}$$

where $\psi(x_i) = f_i$. It will therefore suffice to show that the map ψ is flat. Writing A as a direct limit of finitely generated subrings, we may reduce to the case A is Noetherian, so that $A[y_1, \ldots, y_n][\Delta^{-1}]$ is locally almost of finite presentation as an $\mathbb{E}_\infty$-algebra over $A[x_1, \ldots, x_n]$. Using the fiberwise flatness criterion (see Corollary 11.3.10 of [90] or Proposition **??**), we can reduce to the case where A is a field, in which case the desired result follows from Lemma B.1.2.2.

Let $B' = B \otimes_A B$. To complete the proof that B is étale over A, it will suffice to show that the multiplication map $m : B' \to B$ exhibits B as a localization $B'[e^{-1}]$ for some element $e \in B$ (since m is surjective, this will prove that the map of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} B'$ is both a closed and open immersion, so that we can choose e to be idempotent if desired). By virtue of Lemma B.1.2.1, it will suffice to show that the abelian group $\pi_1 L_{B/B'}$ vanishes. Using the exactness of the sequence $\pi_1 L_{B/B} \to \pi_1 L_{B/B'} \to \pi_0(B \otimes_{B'} L_{B'/B}$, we are reduced to proving that the group

$$\pi_0(B \otimes_{B'} L_{B'/B}) \simeq \pi_0(B \otimes_{B'} B' \otimes_B L_{B/A}) \simeq \pi_0 L_{B/A} \simeq \Omega_{B/A}$$

vanishes. This is clear, since $\Omega_{B/A}$ can be computed as the cokernel of the map $B^n \to B^n$ given by the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1 \leqslant i,j \leqslant n}$, which is invertible in B by assumption.

We now prove the converse. Suppose that $\phi : A \to B$ is an étale homomorphism of commutative rings. Since B is finitely generated as an A-algebra, we can choose a surjective ring homomorphism $\phi' : A' \to B$, where $A' = A[y_1, \ldots, y_n]$ is a polynomial algebra over A. Let $I \subseteq A'$ denote the kernel of ϕ'. We have a fiber sequence of cotangent complexes $B \otimes_{A'} L_{A'/A} \to L_{B/A} \to L_{B/A'}$. Since B is étale over A, the relative cotangent complex $L_{B/A}$ vanishes. It follows that the boundary map $\delta : \pi_1 L_{B/A'} \to \pi_0(B \otimes_{A'} L_{A'/A})$ is an isomorphism. Using Theorem HA.7.4.3.1 and Proposition HA.7.4.3.9 , we obtain isomorphisms

$$\pi_1 L_{B/A'} \simeq I/I^2 \qquad \pi_0(B \otimes_{A'} L_{A'/A}) \simeq B \otimes_{A'} \Omega_{A/A} \simeq B^n.$$

Under these isomorphisms, δ carries an element $\overline{f} \in I/I^2$ to the image in B^n of the vector of partial derivatives $[\frac{\partial f}{\partial y_j}]_{1 \leqslant j \leqslant n}$, where $f \in I$ is any representative of $\overline{f}$. It follows that we can choose a sequence of functionrs $f_1, f_2, \ldots, f_n \in I$ such that the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1 \leqslant i,j \leqslant n}$ has invertible image in B. Set $A'' = (A[y_1, \ldots, y_n]/(f_1, \ldots, f_n))[\Delta^{-1}]$, where Δ denotes the determinant of $[\frac{\partial f_i}{\partial y_j}]_{1 \leqslant i,j \leqslant n}$. Then ϕ' factors through a ring homomorphism $\phi'' : A'' \to B$. The first part of the proof shows that A'' is étale over A, and the map ϕ'' induces a surjection $\pi_0 A'' \to \pi_0 B$ by construction. We have a fiber sequence of B-modules $B \otimes_{A''} L_{A''/A} \to L_{B/A} \to L_{B/A''}$. Since B and A'' are both étale over A, the first two terms of this fiber sequence vanish. It follows in particular that $\pi_1 L_{B/A''} \simeq 0$. Applying Lemma B.1.2.1, we deduce that $B \simeq A''[a^{-1}]$ for some element $a \in A''$. Multiplying a by a power of Δ if necessary, we may assume that a is the image of some polynomial $g \in A'$. In this case, we have an isomorphism

$$B \simeq A[y_1, \ldots, y_n, y_{n+1}]/(f_1, \ldots, f_n, f_{n+1})[\Delta'^{-1}],$$

where $f_{n+1} = 1 - g y_{n+1}$ and Δ' denotes the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1 \leqslant i,j \leqslant n+1}$. □

B.1.3 The Proof of Proposition B.1.1.3

We now study the structure of étale morphisms between $\mathbb{E}_\infty$-rings.

Lemma B.1.3.1. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. The following conditions are equivalent:*

(1) *The abelian group $\pi_0 L_{B/A}$ vanishes, and $\pi_0 B$ is finitely generated as an algebra over $\pi_0 A$.*

(2) *There exist finitely many elements $x_1, \ldots, x_n \in \pi_0 B$ which generate the unit ideal, such that each of the induced maps $A \to B[x_i^{-1}]$ factors as a composition $A \xrightarrow{f'} A_i \xrightarrow{f''} B[x_i^{-1}]$ where f' is étale and f'' induces a surjection $\pi_0 A_i \to \pi_0 B[x_i^{-1}]$.*

Proof. Suppose first that (2) is satisfied. Each of the commutative rings $\pi_0 B[x_i^{-1}]$ is a quotient of an étale $\pi_0 A$-algebra, and therefore finitely generated over $\pi_0 A$. Let $B_0 \subseteq \pi_0 B$

be a finitely generated $\pi_0 A$-subalgebra containing each x_i, such that $B_0[x_i^{-1}] \to \pi_0 B[x_i^{-1}]$ is surjective for each i. Since the x_i generate the unit ideal in B, we deduce that $\pi_0 B = B_0$ is finitely generated over $\pi_0 A$.

It remains to prove that $\pi_0 L_{B/A} \simeq 0$. Since the elements x_i generate the unit ideal, it will suffice to show that $(\pi_0 L_{B/A})[x_i^{-1}] \simeq \pi_0(L_{B/A} \otimes_B B[x_i^{-1}]) \simeq \pi_0 L_{B[x_i^{-1}]/A}$ vanishes for each index i. Choose a factorization $A \xrightarrow{f'} A_i \xrightarrow{f''} B[x_i^{-1}]$ as in (2). We have a short exact sequence of abelian groups

$$\pi_0(B[x_i^{-1}] \otimes_{A'} L_{A_i/A}) \to \pi_0 L_{B[x_i^{-1}]/A} \to \pi_0 L_{B[x_i^{-1}]/A_i}.$$

Here $L_{A_i/A}$ vanishes since f' is étale (Corollary HA.7.5.4.5) and $\pi_0 L_{B[x_i^{-1}]/A_i}$ can be identified with the relative Kähler differentials $\Omega_{\pi_0 B[x_i^{-1}]/\pi_0 A_i}$ (Proposition HA.7.4.3.9), which vanishes because f'' is surjective on connected components. It follows that $\pi_0 L_{B[x_i^{-1}]/A} \simeq 0$ as desired.

Now suppose that (1) is satisfied. Let $R = \pi_0 B$. Since R is finitely generated over $\pi_0 A$, we can choose a presentation $R \simeq (\pi_0 A)[x_1, \ldots, x_n]/I$ for some ideal $I \subseteq (\pi_0 A)[x_1, \ldots, x_n]$. Then $\pi_0 L_{B/A}$ is the module of Kähler differentials of R over $\pi_0 A$ (Proposition HA.7.4.3.9). That is, $\pi_0 L_{B/A}$ is the quotient of the free R-module generated by elements $\{dx_i\}_{1 \leqslant i \leqslant n}$ by the submodule generated by elements $\{df\}_{f \in I}$. Since $\pi_0 L_{B/A} \simeq 0$, we can choose a finite collection of elements $\{f_j \in I\}_{1 \leqslant j \leqslant m}$ such that the Jacobian matrix $M = \{\frac{\partial f_j}{\partial x_i}\}$ has rank n. Let $\{a_k\}$ be the collection of determinants of n-by-n submatrices of the matrix M, so that the elements a_k generate the unit ideal in R. We will prove that each of the composite maps $A \to B[a_k^{-1}]$ factors as a composition $A \xrightarrow{f'} A_k \xrightarrow{f''} B[a_k^{-1}]$, where f' is étale and f'' is surjective on connected components. Reordering the f_j if necessary, we may suppose that $m \geqslant n$ and that a_k is the determinant of the matrix $\{\frac{\partial f_j}{\partial x_i}\}_{1 \leqslant i,j \leqslant n}$. Set

$$R' = (\pi_0 R)[x_1, \ldots, x_n, a_k^{-1}]/(f_1, \ldots, f_m),$$

so that R is an étale algebra over $\pi_0 A$ (Proposition B.1.1.1). It follows from Theorem HA.7.5.0.6 that R' can be lifted (in an essentially unique fashion) to an étale A-algebra A_k. Moreover, Corollary HA.7.5.4.6 implies that the surjective map $R' \to R[a_k^{-1}] = \pi_0 B[a_k^{-1}]$ lifts to a map $A_k \to B[a_k^{-1}]$, thereby supplying the desired factorization. □

Recall that if $\phi : A \to B$ is an étale morphism of $\mathbb{E}_\infty$-rings, then the relative cotangent complex $L_{B/A}$ vanishes (Corollary HA.7.5.4.5). We will need the following partial converse:

Lemma B.1.3.2. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings and let $1 \leqslant n \leqslant \infty$ be an integer. The following conditions are equivalent:*

(1) *The commutative ring $\pi_0 B$ is finitely presented over $\pi_0 A$, and $\pi_i L_{B/A} \simeq 0$ for $i \leqslant n$.*

(2) *The map f factors as a composition $A \xrightarrow{f'} A_i \xrightarrow{f''} B$ where f' is étale, f'' induces an isomorphism $\pi_i A' \to \pi_i B$ for $i < n$, and f'' induces a surjection $\pi_n A' \to \pi_n B$.*

Proof. Suppose first that (2) is satisfied. Then $\pi_0 B \simeq \pi_0 A'$ is étale over $\pi_0 A$, and therefore finitely presented as a $\pi_0 A$-algebra. We have a fiber sequence of B-modules $B \otimes_{A'} L_{A'/A} \to L_{B/A} \to L_{B/A'}$. Since A' is étale over A, we deduce that $L_{A'/A} \simeq 0$. Since f'' is n-connective, Corollary HA.7.4.3.2 implies that $L_{B/A} \simeq L_{B/A'}$ is $(n+1)$-connective, thereby completing the proof of (1).

Now assume that condition (1) holds. We first prove that $\pi_0 B$ is étale over $\pi_0 A$. Using Lemma B.1.3.1, we can choose a finite collection of elements $x_i \in \pi_0 B$ generating the unit ideal such that each of the induced maps $A \to B[x_i^{-1}]$ factors as a composition $A \xrightarrow{g'} A_i \xrightarrow{g''} B[x_i^{-1}]$ where g' is étale and g'' is surjective on connected components. Note that $\pi_1 L_{B[x_i^{-1}]/A_i} \simeq (\pi_1 L_{B/A})[x_i^{-1}] \simeq 0$. Using Lemma B.1.2.1, we deduce that $\pi_0 B[x_i^{-1}]$ is étale over $\pi_0 A_i$ and therefore over $\pi_0 A$, from which it follows that $\pi_0 B$ is étale over $\pi_0 A$.

Using Theorem HA.7.5.0.6 , we can choose an étale A-algebra A' and an isomorphism of $\pi_0 A$-algebras $\alpha : \pi_0 A' \simeq \pi_0 B$. Theorem HA.7.5.4.2 implies that we can lift α to a map of A-algebras $f'' : A' \to B$. To complete the proof, it will suffice to show that f'' is n-connective; this follows from Corollary HA.7.4.3.2 . □

Lemma B.1.3.3. *Let $A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings, and assume that the relative cotangent complex $L_{B/A}$ vanishes. The following conditions are equivalent:*

(1) *The commutative ring $\pi_0 B$ is finitely presented over $\pi_0 A$.*

(2) *The algebra B is of finite presentation over A.*

(3) *The algebra B is almost of finite presentation over A.*

(4) *The map $A \to B$ is étale.*

Proof. The implication (4) ⇒ (1) is obvious, the equivalences (1) ⇔ (2) ⇔ (3) follow from Theorem HA.7.4.3.18 , and the implication (1) ⇒ (4) is a special case of Lemma B.1.3.2. □

Proof of Proposition B.1.1.3. To prove that (2) implies (1), it suffices to show that the map ϕ_0 appearing in the diagram is étale. Note that the relative cotangent complex of ϕ_0 can be identified with the cofiber of the map $L_{A\{x_1,\ldots,x_n\}/A} \otimes_{A\{x_1,\ldots,x_n\}} A\{y_1,\ldots,y_n\}[\Delta^{-1}] \to L_{A\{y_1,\ldots,y_n\}[\Delta^{-1}]/A}$. This is a map of free modules of rank n, which is given on π_0 by the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1 \leqslant i,j \leqslant n}$. Since this matrix is invertible in $\pi_0 A\{y_1,\ldots,y_n\}[\Delta^{-1}]$, we deduce that the relative cotangent complex of ϕ_0 vanishes, so that ϕ_0 is étale by Lemma B.1.3.3.

We now prove that (1) ⇒ (2). Suppose that ϕ is étale. Then B is flat over A, so we have a pushout diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} \tau_{\geqslant 0} A & \longrightarrow & A \\ \downarrow & & \downarrow \\ \tau_{\geqslant 0} B & \longrightarrow & B. \end{array}$$

We may therefore replace A and B by their connective covers, and thereby reduce to the case where A and B are connective. Proposition B.1.1.1 implies that there exists an isomorphism of commutative rings $\pi_0 B \simeq (\pi_0 A)[y_1, \ldots, y_m]/(f_1, \ldots, f_m)[\overline{\Delta}^{-1}]$, where $\overline{\Delta}$ denotes the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1 \leqslant i,j \leqslant m}$ is invertible in $\pi_0 B$. Let $\{a_i \in \pi_0 A\}_{1 \leqslant i \leqslant k}$ be the nonzero coefficients appearing in the polynomials f_i. Choose a commutative diagram

$$\begin{array}{ccc} A\{x_1, \ldots, x_k\} & \xrightarrow{g_0} & A \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ A\{x_1, \ldots, x_k, y_1, \ldots, y_m\} & \xrightarrow{g_1} & B \end{array}$$

where g_0 carries each x_i to $a_i \in \pi_0 A$. For each $1 \leqslant i \leqslant m$, choose a polynomial $\overline{f}_i \in (\pi_0 A)[y_1, \ldots, y_m, x_1, \ldots, x_k]$ lying over f_i, so that we have $g_1(\overline{f}_i) = 0 \in \pi_0 B$. Let $\Delta \in (\pi_0 A)[y_1, \ldots, y_m, x_1, \ldots, x_k]$ denote the determinant of the Jacobian matrix $[\frac{\partial \overline{f}_i}{\partial y_j}]_{1 \leqslant i,j \leqslant m}$.

Using Corollary HA.7.5.4.6 , we deduce the existence of a commutative diagram

$$\begin{array}{ccccc} A\{x_1, \ldots, x_k, z_1, \ldots, z_m\} & \xrightarrow{\epsilon} & A\{x_1, \ldots, x_k\} & \longrightarrow & A \\ \downarrow{\scriptstyle h} & & & & \downarrow{\scriptstyle \phi} \\ A\{x_1, \ldots, x_k, y_1, \ldots, y_m\}[\Delta^{-1}] & & \xrightarrow{\quad g_1 \quad} & & B \end{array}$$

where $h(z_i) = \overline{f}_i$ and $\epsilon(z_i) = 0$ for $1 \leqslant i \leqslant m$. We claim that the outer square appearing in this diagram is a pushout. To see this, form a pushout diagram

$$\begin{array}{ccc} A\{x_1, \ldots, x_k, z_1, \ldots, z_m\} & \longrightarrow & A \\ \downarrow{\scriptstyle \phi_0} & & \downarrow \\ A\{x_1, \ldots, x_k, y_1, \ldots, y_m\}[\Delta^{-1}] & \longrightarrow & B' \end{array}$$

so that we have a canonical map $\psi : B' \to B$; we wish to show that ψ is an equivalence. By construction, $\psi : B' \to B$ induces an isomorphism on connected components. The first part of the proof shows that B' is étale over A, so that $L_{B'/A} \simeq 0$. Since $L_{B/A} \simeq 0$, we conclude that $L_{B/B'} \simeq 0$, so that $B \simeq B'$ by Corollary HA.7.4.3.2 . □

B.1.4 Descent for the Flat Topology

We close this section with a simple observation concerning the relationship between flat and étale morphisms of $\mathbb{E}_\infty$-rings. We will assume that the reader is familiar with the notion of a *faithfully flat* morphism of $\mathbb{E}_\infty$-rings (see Definition B.6.1.1).

Proposition B.1.4.1. *Suppose we are given morphisms of $\mathbb{E}_\infty$-rings $A \xrightarrow{f} B \xrightarrow{g} C$, where g is étale and faithfully flat. Then f is étale if and only if $g \circ f$ is étale.*

The proof of Proposition **??** requires the following:

Lemma B.1.4.2. *Let $f : A \to B$ and $g : B \to C$ be morphisms of $\mathbb{E}_\infty$-rings. Then:*

(1) *If f and g are flat, then $g \circ f$ is flat.*

(2) *If $g \circ f$ is flat and g is faithfully flat, then f is flat.*

Proof. We first prove (1). The map $\pi_0 A \to \pi_0 C$ is a composition of two flat maps between commutative rings, and therefore a flat map. It therefore suffices to show that for each integer n, the map of abelian groups $\mathrm{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 C) \to \pi_n C$ is an isomorphism. We can factor this map as a composition

$$\begin{aligned} \mathrm{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 C) &\simeq \mathrm{Tor}_0^{\pi_0 B}(\mathrm{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 B), \pi_0 C) \\ &\xrightarrow{\phi} \mathrm{Tor}_0^{\pi_0 B}(\pi_n B, \pi_0 C) \\ &\xrightarrow{\psi} \pi_n C. \end{aligned}$$

We conclude by observing that ϕ is an isomorphism because f is assumed to be flat, and the map ψ is an isomorphism because g is assumed to be flat.

We now prove (2). We first claim that f induces a flat map of commutative rings $\pi_0 A \to \pi_0 B$. To prove this, choose a monomorphism $M \to N$ of (discrete) $\pi_0 A$-modules, and let K be the kernel of the induced map $\mathrm{Tor}_0^{\pi_0 A}(M, \pi_0 B) \to \mathrm{Tor}_0^{\pi_0 A}(N, \pi_0 B)$; we wish to prove that $K = 0$. Since $\pi_0 B \to \pi_0 C$ is faithfully flat, it suffices to show that $\mathrm{Tor}_0^{\pi_0 C}(K, \pi_0 B)$ is zero. Using the flatness of $\pi_0 B \to \pi_0 C$, we can identify $\mathrm{Tor}_0^{\pi_0 C}(K, \pi_0 B)$ with the kernel of the map

$$\begin{aligned} \mathrm{Tor}_0^{\pi_0 B}(\mathrm{Tor}_0^{\pi_0 A}(M, \pi_0 B), \pi_0 C) &\simeq \mathrm{Tor}_0^{\pi_0 A}(M, \pi_0 C) \\ &\to \mathrm{Tor}_0^{\pi_0 A}(N, \pi_0 C) \\ &\simeq \mathrm{Tor}_0^{\pi_0 B}(\mathrm{Tor}_0^{\pi_0 A}(N, \pi_0 B), \pi_0 C). \end{aligned}$$

This map is a monomorhism, since $g \circ f$ is assumed to be flat.

To complete the proof that f is flat, we must show that for each integer n, the map $\mathrm{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 B) \to \pi_n B$ is an isomorphism. Since g is faithfully flat, we reduce to proving that the map ϕ above is an isomorphism. By a two-out-of-three argument, we are reduced to proving that the maps $\psi \circ \phi$ and ψ are isomorphisms. This follows from our assumption that g and $g \circ f$ are flat. □

Corollary B.1.4.3. *Suppose we are given a pushout diagram of $\mathbb{E}_\infty$-rings*

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A' \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ B & \xrightarrow{\psi'} & B', \end{array}$$

where ψ is faithfully flat. If B' is flat over A', then B is flat over A.

Proof. Since ψ is faithfully flat flat, the morphism ψ' is also faithfully flat. By virtue of Lemma B.1.4.2, it will suffice to show that the composition $\psi' \circ \phi \simeq \phi' \circ \psi$ is flat. This follows from Lemma B.1.4.2, since ψ and ϕ' are both flat.

□

Proof of Proposition B.1.4.1. The "only if" direction is obvious. For the converse, assume that $g \circ f$ is étale. Lemma B.1.4.2 implies that f is flat. It will therefore suffice to show that the commutative ring $\pi_0 B$ is étale over $\pi_0 A$. Replacing A, B, and C by their connective covers, we can reduce to the case where A, B, and C are connective. We have a fiber sequence $C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B}$. Since C is étale over both A and B, we have $L_{C/A} \simeq L_{C/B} \simeq 0$. It follows that $C \otimes_B L_{B/A} \simeq 0$. Since C is faithfully flat over B, this implies that $L_{B/A} \simeq 0$ (Remark B.6.1.2). To complete the proof that B is étale over A, it will suffice to show that $\pi_0 B$ is finitely presented as a commutative algebra over $\pi_0 A$ (Lemma B.1.3.3). We first prove that $\pi_0 B$ is finitely generated over $\pi_0 A$. Since g is étale, the commutative algebra $\pi_0 C$ is finitely presented over $\pi_0 B$. We may therefore choose a finitely generated $\pi_0 A$-subalgebra $R \subseteq \pi_0 B$ and an étale morphism $R \to R'$ such that $\pi_0 C \simeq (\pi_0 B) \otimes_R R'$. Since $\pi_0 C$ is finitely generated over $\pi_0 A$, we may assume (after enlarging R if necessary) that the map $R' \to \pi_0 C$ is surjective. Since $\pi_0 C$ is faithfully flat over $\pi_0 B$, we conclude that the inclusion $R \hookrightarrow \pi_0 B$ is surjective, so that $\pi_0 B = R$ is finitely generated over $\pi_0 A$. Choose a surjection $S \to \pi_0 B$, where S is finitely presented over $\pi_0 A$. Let I denote the kernel of ϕ; we wish to show that I is a finitely generated ideal. Using Proposition B.1.1.3, we can choose an étale morphism $S \to S'$ and an isomorphism $\pi_0 C \simeq \pi_0 B \otimes_S S'$. Replacing S by the quotient S/J for some finitely generated ideal $J \subseteq I$, we can assume that S' is faithfully flat over S. It follows that the canonical map $S'' \to \pi_0 C$ is surjection with kernel $S' \otimes_S I$. Since $\pi_0 C$ is finitely presented over $\pi_0 A$, the ideal $S' \otimes_S I$ is finitely generated as a module over S'. Because S' is faithfully flat over S, the ideal I is finitely generated as an S-module, as desired. □

B.2 Dimension Theory of Commutative Rings

In this section, we will review the dimension theory of Noetherian commutative rings (more complete accounts can be found in many standard texts on commutative algebra, such as [8] or [59]).

B.2.1 Dimension of a Local Noetherian Ring

Let A be a local Noetherian ring. Then the maximal ideal $\mathfrak{m} \subseteq A$ is finitely generated: that is, we can choose finitely many elements $a_1, \ldots, a_d \in A$ such that $\mathfrak{m} = (a_1, \ldots, a_d)$.

Definition B.2.1.1. Let A be a local Noetherian ring with maximal ideal $\mathfrak{m}$. The *dimension* of A is the smallest integer $d \geqslant 0$ for which there exists elements $a_1, \ldots, a_d \in \mathfrak{m}$ for which the ideal $(a_1, \ldots, a_d)$ contains some power of $\mathfrak{m}$.

The dimension of a local Noetherian ring admits several other characterizations:

Theorem B.2.1.2. *Let A be a local Noetherian ring with maximal ideal $\mathfrak{m}$ and let $d \geqslant 0$ be an integer. The following conditions are equivalent:*

(a) *Every chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k$ in A has length $k \leqslant d$.*

(b) *The local ring A has Krull dimension $\leqslant d$ (in the sense of Definition B.2.1.1): that is, there exist elements $a_1, \ldots, a_d \in \mathfrak{m}$ such that $(a_1, \ldots, a_d)$ contains $\mathfrak{m}^n$ for $n \gg 0$.*

(c) *There exists a constant c such that for each $n \geqslant 0$, the quotient $A/\mathfrak{m}^n$ has length $\leqslant cn^d$ as an A-module.*

Remark B.2.1.3. In the situation of Theorem B.2.1.2, there exists a polynomial $P(t) \in \mathbf{Q}[t]$ such the quotient $A/\mathfrak{m}^n$ has length $P(n)$ for all sufficiently large integers n. The polynomial $P(t)$ is called the *Hilbert-Samuel polynomial* of A, and its degree is the dimension of A.

Proof of Theorem B.2.1.2. We first show that (a) implies (b) using induction on d. If $d = 0$, then condition (a) guarantees that $\mathfrak{m}$ is the unique prime ideal of A, and therefore coincides with the nilradical of A. Since A is Noetherian, the maximal ideal $\mathfrak{m}$ is finitely generated, so we have $\mathfrak{m}^n \simeq 0$ for $n \gg 0$, which proves (b). Suppose that $d > 0$. The ring A is Noetherian and therefore contains finitely many minimal prime ideals. The assumption $d > 0$ implies that each of these minimal prime ideals is properly contained in $\mathfrak{m}$. We can therefore choose an element $x \in \mathfrak{m}$ which is not contained in any minimal prime ideal of A. For any sequence of prime ideals $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_k$ in the quotient ring $A/(x)$, their inverse images in A form a sequence of prime ideals which does not contain any minimal prime ideal of A, and can therefore be extended to a chain of prime ideals in A having length $k+1$. It follows from assumption (a) that $k < d$, so our inductive hypothesis implies that $A/(x)$ has dimension $< d$. That is, we can choose elements $\overline{y}_1, \ldots, \overline{y}_{d_1} \in \overline{\mathfrak{m}}$ such that $(\overline{y}_1, \ldots, \overline{y}_{d-1})$ contains $\overline{\mathfrak{m}}^n$ for $n \gg 0$, where $\overline{\mathfrak{m}}$ denotes the image of $\mathfrak{m}$ in $A/(x)$. Writing each $\overline{y}_i$ as the image of an element $y_i \in \mathfrak{m}$, we obtain $\mathfrak{m}^n \subseteq (x, y_1, \ldots, y_{d-1})$ so that A has dimension $\leqslant d$.

We next prove that (b) implies (c). Choose elements $x_1, \ldots, x_d \in \mathfrak{m}$ for which the ideal $(x_1, \ldots, x_d)$ contains some power of $\mathfrak{m}$. It follows that that $A/(x_1, \ldots, x_d)$ appears as a quotient of $A/\mathfrak{m}^k$ for $k \gg 0$ and is therefore an A-module of finite length. Let c be the length of $A/(x_1, \ldots, x_d)$. An easy induction shows that for every sequence of integers $n_1, \ldots, n_d \geqslant 0$, the quotient $A/(x_1^{n_1}, \ldots, x_d^{n_d})$ has length $\leqslant c \prod_{1 \leqslant i \leqslant d} n_i$. For $n \geqslant 0$, the ideal $\mathfrak{m}^n$ contains each x_i^n, so that $A/\mathfrak{m}^n$ is a quotient of $A/(x_1^n, \ldots, x_d^n)$ and therefore has length $\leqslant cd^n$.

We now show that (c) implies (a). For each A-module M of finite length, let $\ell(M)$ denote the length of M as an A-module. Suppose that there exists a chain of prime ideals $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_k$ in A having length k. We will prove that there exists a positive real number e such that $\ell(A/\mathfrak{m}^n) \geqslant en^k$ for $n \gg 0$. If (c) is satisfied, then this claim guarantees that $k \leqslant d$, so that (a) is also satisfied. Our proof proceeds by induction on k, the case $k = 0$ being trivial. Let us therefore assume that $k > 0$. Replacing A by $A/\mathfrak{p}_0$, we may assume that $\mathfrak{p}_0 = (0)$ so that A is an integral domain. Choose a nonzero element $x \in \mathfrak{p}_1$. Applying the Artin-Rees lemma to the ideal $(x) \subseteq A$, we deduce that there exists an integer r such that $\mathfrak{m}^n \cap (x) \subseteq \mathfrak{m}^{n-r}(x)$ for $n \geqslant r$. Set $\overline{A} = A/(x)$ and let $\overline{\mathfrak{m}}$ be the image of $\mathfrak{m}$ in $\overline{A}$. Applying our inductive hypothesis to $\overline{A}$, we deduce that there exists constant $\overline{e} > 0$ and $a_0 \geqslant r$ such that $\ell(\overline{A}/\mathfrak{m}^a) \geqslant \overline{e}a^{k-1}$ for $a \geqslant a_0$. For $n \geqslant a_0$, there is a unique integer $n' > 0$ such that $a_0 - r \leqslant n - rn' < a_0$. We then have

$$\begin{aligned}
\ell(A/\mathfrak{m}^n) &\geqslant \sum_{0 \leqslant i < n'} \ell((x^i) + \mathfrak{m}^n/(x^{i+1}) + \mathfrak{m}^n) \\
&= \sum_{0 \leqslant i < n'} \ell(A/(x) + \{y \in A : x^i y \in \mathfrak{m}^n\}) \\
&\geqslant \sum_{0 \leqslant i < n'} \ell(\overline{A}/\overline{\mathfrak{m}}^{n-ir}) \\
&\geqslant \sum_{0 \leqslant i < n'} \overline{e}(n - ir)^{k-1} \\
&\geqslant \int_0^{n'} \overline{e}(n - tr)^{k-1} dt \\
&= \frac{\overline{e}}{rk}(n^k - (n - rn')^k) \\
&> \frac{\overline{e}}{rk}(n^k - a_0^k).
\end{aligned}$$

It follows that for any constant $e < \frac{\overline{e}}{rk}$, we have $\ell(A/\mathfrak{m}^n) \geqslant en^k$ for $n \gg 0$. □

Remark B.2.1.4. Let A be a local Noetherian ring. It follows from characterization (c) of Theorem B.2.1.2 that the dimension of A is the same as the dimension of the completion $\widehat{A}$ of A at its maximal ideal $\mathfrak{m}$.

Definition B.2.1.5. Let A be a Noetherian ring and let $\mathfrak{p}$ be a prime ideal of A. The *height* of $\mathfrak{p}$ is the dimension of the local ring $A_{\mathfrak{p}}$. We say that A has *Krull dimension* $\leqslant d$ if each prime ideal $\mathfrak{p}$ in A has height $\leqslant d$.

Remark B.2.1.6. Let $\mathfrak{p}$ be a prime ideal in a Noetherian ring A. It follows from criterion (a) of Theorem B.2.1.2 that the height of $\mathfrak{p}$ is the largest integer h for which there exists a chain of prime ideals $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_h = \mathfrak{p}$ ending in $\mathfrak{p}$.

Remark B.2.1.7. If A is a local Noetherian ring of dimension d and $I \subsetneq A$ is an ideal, then the quotient A/I is a local Noetherian ring of dimension $\leqslant d$. More generally, if A is a Noetherian ring of Krull dimension $\leqslant d$, then any quotient ring of A has Krull dimension $\leqslant d$.

B.2.2 Flat Morphisms

We now discuss the relationship between the Krull dimensions of commutative rings A and B which are related by a ring homomorphism $\phi : A \to B$. In the case where ϕ is flat, we have the following:

Theorem B.2.2.1. *Let $\phi : A \to B$ be a flat homomorphism of commutative rings. Let $\mathfrak{p}$ be a prime ideal of B, and let $\mathfrak{q}$ be a prime ideal of A which is contained in $\phi^{-1}\mathfrak{p}$. Then we can write $\mathfrak{q} = \phi^{-1}\mathfrak{p}_0$ for some prime ideal $\mathfrak{p}_0 \subseteq \mathfrak{p} \subseteq B$.*

Proof. Replacing A by $A/\mathfrak{q}$ and B by $B/\mathfrak{q}B$, we may assume that A is an integral domain and that $\mathfrak{q}$ is the zero ideal. Replacing B by $B_{\mathfrak{p}}$, we may reduce to the case where B is a local ring with maximal ideal $\mathfrak{p}$. In this case, it will suffice to show that there exists any prime ideal $\mathfrak{p}_0 \subseteq B$ such that $\phi^{-1}\mathfrak{p}_0 = (0)$. Let K denote the fraction field of A, so that we have an injective map $A \hookrightarrow K$. Since B is flat over A, the induced map $B \to K \otimes_A B$ is also injective. It follows that $K \otimes_A B$ is nonzero, so that there there exists a maximal ideal $\mathfrak{m}$ in the ring $K \otimes_A B$. We let $\mathfrak{p}_0$ denote the inverse image of $\mathfrak{m}$ in B; it follows from the commutativity of the diagram

$$\begin{array}{ccc} |\operatorname{Spec}(K \otimes_A B)| & \longrightarrow & |\operatorname{Spec} B| \\ \downarrow & & \downarrow \\ |\operatorname{Spec} K| & \longrightarrow & |\operatorname{Spec} A| \end{array}$$

that $\phi^{-1}\mathfrak{p}_0 = (0)$, as desired. □

Corollary B.2.2.2. *Let $\phi : A \to B$ be a flat morphism of Noetherian rings, let $\mathfrak{p}$ be a prime ideal of B, and let $\mathfrak{q} = \phi^{-1}\mathfrak{p}$. If $\mathfrak{p}$ has height $\leqslant d$, then $\mathfrak{q}$ has height $\leqslant d$.*

Proof. If $\mathfrak{q}$ does not have height $\leqslant d$, then there exists a chain of prime ideals $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_{d+1} = \mathfrak{q}$ in A having length $d+1$. Applying Theorem B.2.2.1 repeatedly, we see that this chain can be lifted to a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_{d+1} = \mathfrak{p}$ in B, contradicing our asusumption that $\mathfrak{p}$ has height $\leqslant d$. □

Corollary B.2.2.3. *Let $\phi : A \to B$ be a faithfully flat morphism of commutative rings. If B has Krull dimension $\leqslant d$, then A has Krull dimension $\leqslant d$.*

Variant B.2.2.4. Let $\phi : A \rightarrow B$ be an étale homomorphism between Noetherian rings, let $\mathfrak{p}$ be a prime ideal in B, and let $\mathfrak{q} = \phi^{-1}\mathfrak{p}$. Then $\mathfrak{p}$ and $\mathfrak{q}$ have the same height.

Proof. The map ϕ induces an isomorphism between the completions of the local rings $A_{\mathfrak{q}}$ and $B_{\mathfrak{p}}$, so the desired result follows from Remark B.2.1.4. □

For later use, we record another consequence of Theorem B.2.2.1:

Corollary B.2.2.5. *Let $\phi : A \rightarrow B$ be homomorphism of commutative rings which is flat and of finite presentation (for example, an étale morphism). Then the induced map of topological spaces $f : |\operatorname{Spec} B| \rightarrow |\operatorname{Spec} A|$ is open.*

Proof. Let U be an open subset of $|\operatorname{Spec} B|$; we wish to prove that $f(U)$ is an open subset of $|\operatorname{Spec} A|$. Without loss of generality, we may assume that U is a basic open set of the form $|\operatorname{Spec} B[b^{-1}]|$. Replacing B by $B[b^{-1}]$, we are reduced to proving that the image of f is open. Since ϕ is of finite presentation, Chevalley's constructibility theorem (Theorem 4.3.3.1) implies that the image of f is a constructible subset of $|\operatorname{Spec} B|$. To show that this image is open, it will suffice to show that it is stable under generalization (Proposition 4.3.2.1), which follows immediately from Theorem B.2.2.1. □

B.2.3 Relative Dimension

We will need the following relative version of Definition B.2.1.1:

Definition B.2.3.1. Let $\phi : A \rightarrow B$ be a homomorphism of commutative rings and let $d \geqslant 0$ be an integer. We will say that ϕ is *of relative dimension* $\leqslant d$ if it exhibits B as a finitely generated A-algebra and, for every ring homomorphism $A \rightarrow \kappa$ where κ is a field, the tensor product $\operatorname{Tor}_0^A(\kappa, B)$ has Krull dimension $\leqslant d$ (in the sense of Definition B.2.1.1).

Remark B.2.3.2. In the situation of Definition B.2.4.1, the assumption that B is finitely generated as an A-algebra guarantees that any tensor product $\operatorname{Tor}_0^A(\kappa, B)$ is a finitely generated algebra over the field κ, and is therefore automatically Noetherian.

Remark B.2.3.3. In the situation of Definition B.2.3.1, it suffices to consider the case where κ is a residue field of A: see Corollary B.2.3.10 below.

Remark B.2.3.4. Let $\phi : A \rightarrow B$ and $\psi : B \rightarrow C$ be homomorphisms of commutative rings. If $\psi \circ \phi : A \rightarrow C$ has relative dimension $\leqslant d$, then ψ has relative dimension $\leqslant d$. It is clear that if C is finitely generated over A, then it is also finitely generated over B. We complete the proof by observing that for any ring homomorphism from B to a field κ, the tensor product $\operatorname{Tor}_0^B(\kappa, C)$ is a quotient of $\operatorname{Tor}_0^A(\kappa, C)$ and therefore has Krull dimension $\leqslant d$ (Remark B.2.1.7).

We now summarize some of the formal properties of Definition B.2.3.1.

Proposition B.2.3.5. *Suppose we are given a pushout diagram of commutative rings*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow{\scriptstyle\psi} & & \downarrow \\ A' & \xrightarrow{\phi'} & B'. \end{array}$$

If ϕ is of relative dimension $\leqslant d$, then so is ϕ'. The converse holds if ψ is faithfully flat.

Proof. The first assertion follows immediately from the definitions. To prove the second, suppose that ϕ' is of relative dimension d and that ψ is faithfully flat. Then B' is finitely generated over A', so we can choose finitely many elements $b_1, \ldots, b_n \in B$ whose images generate B' as an A'-algebra. Let $B_0 \subseteq B$ be the A-subalgebra generated by $b_1, \ldots, b_n$. Then the inclusion $\iota : B_0 \hookrightarrow B$ induces a surjection

$$A' \otimes_A B_0 \to A' \otimes_A B \simeq B'.$$

Since A' is faithfully flat over A, it follows that ι is surjective so that $B = B_0$ is finitely generated as an A-algebra. To complete the proof, choose a ring homomorphism $A \to \kappa$ where κ is a field and set $B_\kappa = \mathrm{Tor}_0^A(\kappa, B)$. We wish to show that B_κ has Krull dimension $\leqslant d$. Since ϕ' is faithfully flat, there exists an extension field κ' of κ for which the composite map $A \to \kappa \hookrightarrow \kappa'$ factors through A'. Our assumption that ϕ' has relative dimension $\leqslant d$ then implies that $B_{\kappa'} \simeq \kappa' \otimes_\kappa B_\kappa$ has Krull dimension $\leqslant d$. Since $B_{\kappa'}$ is faithfully flat over B_κ, it follows that B_κ also has Krull dimension $\leqslant d$ (Corollary B.2.2.3). □

Proposition B.2.3.6. *Let $\phi : A \to B$ be a homomorphism of commutative rings which has relative dimension $\leqslant d$. Suppose that A is a Noetherian ring of Krull dimension $\leqslant n$. Then B is a Noetherian ring of Krull dimension $\leqslant n + d$.*

Proof. Since B is finitely generated as an A-algebra, it is Noetherian by virtue of the Hilbert basis theorem. Fix a prime ideal $\mathfrak{p} \subseteq B$; we will show that the local ring $B_\mathfrak{p}$ has dimension $\leqslant n+d$. Set $\mathfrak{q} = \phi^{-1}\mathfrak{p} \subseteq A$. Replacing A by the localization $A_\mathfrak{q}$ and B by the tensor product $A_\mathfrak{q} \otimes_A B$, we may assume that A is a local ring with maximal ideal $\mathfrak{q}$. Since A has Krull dimension $\leqslant n$, we can choose elements $a_1, \ldots, a_n$ for which the ideal $(a_1, \ldots, a_n)$ contains $\mathfrak{q}^k$ for some integer k. Similarly, the condition that ϕ has relative dimension $\leqslant d$ guarantees that we can choose elements $\overline{b}_1, \ldots, \overline{b}_d$ in $B_\mathfrak{p}/\mathfrak{q}B_\mathfrak{p}$ such that the ideal $(\overline{b}_1, \ldots, \overline{b}_d)$ contains the image of $\mathfrak{p}^{k'}$ for some $k' > 0$. For each $1 \leqslant i \leqslant d$, choose an element $b_i \in B_\mathfrak{p}$ representing $\overline{b}_i$, and let $I \subseteq B_\mathfrak{p}$ denote the ideal $(\phi(a_1), \ldots, \phi(a_n), b_1, \ldots, b_d)$. By construction, we have

$$\mathfrak{p}^{kk'} \subseteq (\mathfrak{q}B_\mathfrak{p} + I)^k \subseteq \mathfrak{q}^k B_\mathfrak{p} + I \subseteq I$$

so that $B_\mathfrak{p}$ has dimension $\leqslant n + d$, as desired. □

Proposition B.2.3.7. *Let $\phi : A \to B$ an $\psi : B \to C$ be homomorphisms of commutative rings. If ϕ has relative dimension $\leqslant d$ and ψ has relative dimension $\leqslant d'$, then $\psi \circ \phi$ has relative dimension $\leqslant d + d'$.*

Proof. Since B is finitely generated as an A-algebra and C is finitely generated as a B-algebra, we immediately see that C is finitely generated as an A-algebra. Choose a field κ and a ring homomorphism $A \to \kappa$ and set

$$B_\kappa = \mathrm{Tor}_0^A(\kappa, B) \qquad C_\kappa = \mathrm{Tor}_0^A(\kappa, C).$$

Since ϕ has relative dimension $\leqslant d$, the ring B_κ has Krull dimension $\leqslant d$. Since ψ has relative dimension $\leqslant d'$, the map $B_\kappa \to C_\kappa$ has relative dimension $\leqslant d'$ (Proposition B.2.3.5). Applying Proposition B.2.3.6, we deduce that C_κ has Krull dimension $\leqslant d + d'$. □

Example B.2.3.8. Let A be a commutative ring. Then the polynomial ring $A[x_1, \ldots, x_d]$ has relative dimension $\leqslant d$ over A. To prove this, we may use Proposition B.2.3.7 to reduce to the case where $d = 1$. Since $A[x]$ is finitely generated as an A-algebra, it will suffice to show that for every field κ, the polynomial ring $\kappa[x]$ has Krull dimension $\leqslant 1$. This is clear, since every ideal in $\kappa[x]$ (or any localization of $\kappa[x]$) is principal.

Proposition B.2.3.9. *Let A be a finitely generated algebra over a field κ, let κ' be an extension field of κ, and set $A' = \kappa' \otimes_\kappa A$. Then A and A' have the same Krull dimension.*

Proof. Let d be the Krull dimension of A (which is finite by virtue of Example B.2.3.8 and Remark B.2.1.7). Since A' is faithfully flat over A, it follows from Corollary B.2.2.3 that the Krull dimension of A' is $\geqslant d$. We wish to prove the reverse inequality. We first treat the case where κ' is a finitely generated field extension of κ. Proceeding by induction on the number of generators, we may assume that κ' is generated (as an extension field of κ) by a single element x. There are two cases to consider:

- Suppose that x is algebraic over κ. Then κ' is a finite algebraic extension of κ. From this, it follows immediately that the inclusion $\kappa \hookrightarrow \kappa'$ has relative dimension 0, so that the inclusion $A \hookrightarrow A'$ also has relative dimension 0 (Proposition B.2.3.5) and therefore A' has Krull dimension $\leqslant d$ by virtue of Proposition B.2.3.6.

- Suppose that x is transcendental over κ, so that κ' is the fraction field of the polynomial ring $\kappa[x]$. It follows that A' is obtained from $A[x]$ by inverting all nonzero polynomials $f(x)$ whose coefficients lie in the field κ. Suppose we are given a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k$ in A'; we wish to show that $k \leqslant d$. Set $\mathfrak{q}_i = \mathfrak{p}_i \cap A[x]$. Note that the residue field of $A[x]$ at $\mathfrak{q}_k$ is not algebraic over κ, so (by virtue of Hilbert's Nullstellensatz) the prime ideal $\mathfrak{q}_k \subseteq A[x]$ is not maximal. Choose a maximal ideal $\mathfrak{m} \subseteq A[x]$ containing $\mathfrak{q}_k$, so that $\mathfrak{m}$ has height $\geqslant k + 1$. Since A has Krull dimension

$\leqslant d$, it follows from Example B.2.3.8 and Proposition B.2.3.6 that $A[x]$ has Krull dimension $\leqslant d+1$, so that $k+1 \leqslant d+1$ and therefore $k \leqslant d$ as desired.

We now treat the general case, where κ' is not assumed to be a finitely generated field extension of κ. Fix a prime ideal $\mathfrak{m} \subseteq A'$; we wish to show that $\mathfrak{p}$ has height $\leqslant d$. Without loss of generality, we may assume that the prime ideal $\mathfrak{m}$ is maximal. Since A' is a finitely generated algebra over κ', it is a Noetherian ring; in particular, the ideal $\mathfrak{m}$ is generated by finitely many elements $x_1, \ldots, x_n \in \mathfrak{m}$. We can therefore choose a subfield $\kappa'' \subseteq \kappa'$ which is a finitely generated extension field of κ such that each x_i is defined over κ'': that is, it belongs to the image of the natural map $A \otimes_\kappa \kappa'' \hookrightarrow A'$. It follows from the first part of the proof that $A \otimes_\kappa \kappa''$ has Krull dimension d. We may therefore replace κ by κ'' (and A by $A \otimes_\kappa \kappa''$) and thereby reduce to the case where each x_i belongs to the image of the inclusion $A \hookrightarrow A'$. Let $\mathfrak{n}$ be the ideal in A generated by the elements x_i, so that $A'/\mathfrak{m} = (A/\mathfrak{n}) \otimes_\kappa \kappa'$. Hilbert's Nullstellensatz implies that $A'/\mathfrak{m}$ is finite-dimensional as a vector space over κ', so that $A/\mathfrak{n}$ is finitely generated as a vector space over κ. Since $A/\mathfrak{n}$ can be identified with a subring of $A'/\mathfrak{m}$, it is an integral domain and therefore a finite extension field of κ. It follows that $\mathfrak{n}$ is a maximal ideal of A.

Our assumption that A has Krull dimension $\leqslant d$ implies that the local ring $A_\mathfrak{n}$ has dimension $\leqslant d$. We can therefore choose elements $y_1, \ldots, y_d \in \mathfrak{n}$ such that $\mathfrak{n}^k A_\mathfrak{n}$ is contained in $y_1 A_\mathfrak{n} + y_2 A_\mathfrak{n} + \cdots + y_d A_\mathfrak{n}$ for some integer $k \gg 0$. Replacing A by a localization $A[a^{-1}]$ for $a \notin \mathfrak{n}$, we may assume that $\mathfrak{n}^k \subseteq y_1 A + y_2 A + \cdots + y_d A$ Extending scalars to A', we deduce that $\mathfrak{m}^k$ is contained in the ideal $(y_1, \ldots, y_d) = y_1 A' + \cdots + y_d A'$, so that the local ring $A'_\mathfrak{m}$ has dimension $\leqslant d$ as desired. □

Corollary B.2.3.10. *Let $\phi : A \to B$ be a homomorphism of commutative rings which exhibits B as a finitely generated algebra over A. The following conditions are equivalent:*

- (a) *The map ϕ has relative dimension $\leqslant d$: that is, for every ring homomorphism $A \to \kappa$ where κ is a field, the tensor product $\mathrm{Tor}_0^A(\kappa, B)$ has Krull dimension $\leqslant d$.*

- (b) *For every residue field κ of A, the tensor product $\mathrm{Tor}_0^A(\kappa, B)$ has Krull dimension $\leqslant d$.*

Corollary B.2.3.11. *Let κ be a field, let A be a finitely generated κ-algebra, and let $d \geqslant 0$ be an integer. The following conditions are equivalent:*

- (a) *The commutative ring A has Krull dimension $\leqslant d$ (in the sense of Definition B.2.1.1).*

- (b) *The unit map $\kappa \to A$ has relative dimension $\leqslant d$ (in the sense of Definition B.2.3.1).*

Proof. The implication $(b) \Rightarrow (a)$ is trivial, and the converse follows from Proposition B.2.3.9. □

B.2.4 Quasi-Finite Morphisms

We now specialize to the study of ring homomorphism shaving relative dimension $\leqslant 0$.

Definition B.2.4.1. Let $\phi : A \to B$ be a homomorphism of commutative rings. We will say that ϕ is *quasi-finite* if it has relative dimension $\leqslant 0$: that is, if it exhibits B as a finitely generated A-algebra and, for each residue field κ of A, the tensor product $B_\kappa = \mathrm{Tor}_0^A(\kappa, B)$ has Krull dimension $\leqslant 0$.

Remark B.2.4.2. Let κ be a field and let R be a finitely generated algebra over κ. Then the following conditions are equivalent:

(a) The ring R has Krull dimension zero.

(b) The ring R has finite dimension as a vector space over κ.

(c) The Zariski spectrum $|\operatorname{Spec} R|$ is finite.

It follows that if $\phi : A \to B$ is a homomorphism of commutative rings that exhibits B as a finitely generated A-algebra, then ϕ is quasi-finite if and only if the map of Zariski spectra $|\operatorname{Spec} B| \to |\operatorname{Spec} A|$ has finite fibers (in this case, the fibers of the map $|\operatorname{Spec} B| \to |\operatorname{Spec} A|$ automatically inherit the discrete topology). Alternatively, ϕ is quasi-finite if and only if for every residue field κ of A, the tensor product $\mathrm{Tor}_0^A(\kappa, B)$ is a finite-dimensional vector space over κ.

Example B.2.4.3. Every étale homomorphism of commutative rings $\phi : A \to B$ is quasi-finite. In particular, every localization $A \to A[t^{-1}]$ is quasi-finite.

Example B.2.4.4. Let $\phi : A \to B$ be a homomorphism of commutative rings which exhibits B as a finitely generated A-module. Then ϕ is quasi-finite.

All quasi-finite ring homomorphisms can be obtained through some combination of Examples B.2.4.3 and B.2.4.4:

Theorem B.2.4.5. *[Zariski's Main Theorem, Affine Version] Let $\phi : A \to B$ be a quasi-finite map of commutative rings. Then there exists an A-subalgebra $B_0 \subseteq B$ which is finitely generated as an A-module for which the induced map $\operatorname{Spec} B \to \operatorname{Spec} B_0$ is an open immersion of schemes.*

We will deduce Theorem B.2.4.5 from the following local variant:

Theorem B.2.4.6. *Let $\phi : A \to B$ be a commutative ring homomorphism which exhibits B as a finitely generated A-algebra and let $\mathfrak{q}$ be a prime ideal of B which is an isolated point of the fiber $|\operatorname{Spec} B| \times_{|\operatorname{Spec} A|} \{\phi^{-1}(\mathfrak{q})\}$. Then there exists an A-subalgebra $B_0 \subseteq B$ with the following properties:*

(*i*) *The algebra B_0 is finitely generated as an A-module.*

(*ii*) *There exists an element $b \in B_0$ such that $b \notin \mathfrak{q}$ and the induced map $B_0[b^{-1}] \to B[b^{-1}]$ is an isomorphism.*

Proof. We use an argument of Grothendieck, borrowing some ideas from algebraic geometry. We first treat the case where A is Noetherian. Choose a surjection of A-algebras $A[x_1, \ldots, x_n] \to B$, which we can identify with a closed immersion of affine schemes $i : \operatorname{Spec} B \hookrightarrow \mathbf{A}^n_A$. Let $X \subseteq \mathbf{P}^n_A$ denote the scheme-theoretic image of i in the projective space $\mathbf{P}^n_A$, and let $f : X \to \operatorname{Spec} A$ denote the projection map. Then f admits a Stein factorization $X \xrightarrow{g} \operatorname{Spec} \overline{B} \xrightarrow{h} \operatorname{Spec} A$, where $\overline{B}$ is finitely generated as an A-module. Let $U \subseteq \operatorname{Spec} \overline{B}$ be the largest open subscheme for which the projection map $X \times_{\operatorname{Spec} \overline{B}} U \to U$ is an isomorphism, and set $V = \operatorname{Spec} B \times_{\operatorname{Spec} A} U$. Our assumption that $\mathfrak{q}$ is an isolated point of $|\operatorname{Spec} B| \times_{|\operatorname{Spec} A|} \{\phi^{-1}(\mathfrak{q})\}$ guarantees that $i(\mathfrak{q})$ is an isolated point of $f^{-1}\{\phi^{-1}\mathfrak{q}\}$, so that $\mathfrak{q}$ is contained in V (see Theorem 8.7.2.3). Consequently, there exists an element $\overline{b} \in \overline{B}$ whose image in B does not belong to $\mathfrak{q}$ such that $\operatorname{Spec} \overline{B}[\overline{b}^{-1}]$ is contained in the image of the open immersion $V \to U$. This guarantees that the natural map $\overline{B}[\overline{b}^{-1}] \to B[b^{-1}]$ is an isomorphism, where $b \in B$ denotes the image of $\overline{b}$. We now complete the proof by taking B_0 to be the image of the map $\overline{B} \to B$.

If we drop the assumption that A is Noetherian but still assume that B is finitely presented over A, then we can choose a finitely generated subalgebra $A' \subseteq A$, a finitely presented A'-algebra B', and an isomorphism of A-algebras $\operatorname{Tor}^{A'}_0(A, B') \simeq B$. Let $\mathfrak{q}'$ be the inverse image of $\mathfrak{q}$ in B'. Then $\mathfrak{q}'$ is an isolated point of the fiber $|\operatorname{Spec} B'| \times_{|\operatorname{Spec} A'|} \{\phi^{-1}(\mathfrak{q}')\}$. Applying the preceding argument, we deduce the existence of a subalgebra $B'_0 \subseteq B'$ which is finitely generated an A'-module and an element $b \in B'_0$ which is not contained in $\mathfrak{q}'$ such that $B'_0[b^{-1}] \simeq B'[b^{-1}]$. We now conclude by defining B_0 to be the A-subalgebra of B generated by the image of B'_0.

We now treat the general case. Let κ be the residue field of A at the point $\phi^{-1}(\mathfrak{q})$. Write B as a quotient $A[x_1, \ldots, x_n]/I$ for some ideal I, so that the tensor product $\operatorname{Tor}^A_0(\kappa, B)$ can be written as a quotient $\kappa[x_1, \ldots, x_n]/I_\kappa$ where I_κ is the ideal generated by the image of I. Since $\kappa[x_1, \ldots, x_n]$ is a Noetherian ring, the ideal I_κ is finitely generated. It follows that we can choose a finitely generated ideal $I' \subseteq I$ such that the quotient $B' = A[x_1, \ldots, x_n]/I'$ has the property that the natural map $\operatorname{Tor}^A_0(\kappa, B') \to \operatorname{Tor}^A_0(\kappa, B)$ is an isomorphism. Let $\mathfrak{q}'$ denote the inverse image of $\mathfrak{q}$ in B', so that $\mathfrak{q}'$ is an isolated point of the fiber $|\operatorname{Spec} B'| \times_{|\operatorname{Spec} A|} \{\phi^{-1}(\mathfrak{q}')\}$. Applying the preceding argument to $\mathfrak{q}'$, we deduce the existence of an A-algebra $B'_0 \to B'$ and an element $b' \in B'_0$ such that B'_0 is finitely generated as an A-module and $B'_0[b'^{-1}] \simeq B'[b'^{-1}]$. We conclude by defining B_0 to be the image of B'_0 in B, and b to be the image of b' in B. □

Warning B.2.4.7. The proof of Theorem B.2.4.6 given above depends on basic facts about

Stein factorizations (such as Zariski's connectedness theorem), which we will discuss in §8.7. On the other hand, much of the foundational material developed in Parts I and II depends on facts that are derived from Theorem B.2.4.6. A reader who is troubled by this circular reasoning can consult the literature for other approaches to either Theorem B.2.4.6 (for example, [172] contains a more elementary argument which avoids the use of algebraic geometry) or to developing the algebro-geometric tools needed for the proof given above (note that the proof above requires the theory of Stein factorizations only for projective morphisms).

Proof of Theorem B.2.4.5. Let $\phi : A \to B$ be a quasi-finite morphism of commutative rings. For each prime ideal $\mathfrak{q} \subseteq B$, we can use Theorem B.2.4.6 to choose an A-subalgebra $B(\mathfrak{q}) \subseteq B$ and an element $b_{\mathfrak{q}} \in B(\mathfrak{q})$ for which the induced map $B(\mathfrak{q})[b_{\mathfrak{q}}^{-1}] \to B[b_{\mathfrak{q}}^{-1}]$ is an isomorphism. The elements $b_{\mathfrak{q}}$ generate the unit ideal in B: we may therefore choose finitely many prime ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ for which the elements $b_{\mathfrak{q}_i}$ generate the unit ideal. Let $B_0 \subseteq B$ be the A-subalgebra generated by $\{B(\mathfrak{q}_i)\}_{1 \leqslant i \leqslant n}$. Then B_0 is finitely generated as an A-algebra and integral over A, hence finitely generated as an A-module. By construction, each of the maps $B_0[b_{\mathfrak{q}_i}^{-1}] \to B[b_{\mathfrak{q}_i}^{-1}]$ is surjective (and is automatically injective, since localization is exact). It follows that the induced map $\operatorname{Spec} B \to \operatorname{Spec} B_0$ is an open immersion, complementary to the vanishing locus of the ideal $(b_{\mathfrak{q}_1}, \ldots, b_{\mathfrak{q}_n}) \subseteq B_0$. □

For later reference, we record another consequence of Theorem B.2.4.6:

Corollary B.2.4.8. *Let $\phi : A \to B$ be a commutative ring homomorphism which exhibits B as a finitely generated A-algebra, let $\mathfrak{q}$ be a prime ideal of A, and let $\mathfrak{p}$ be a prime ideal of B which is maximal among those prime ideals lying over $\mathfrak{q}$. Suppose that the local ring $B_{\mathfrak{p}}/\mathfrak{q}B_{\mathfrak{p}}$ has dimension $\leqslant d$. Then there exists an element $b \in B$ which does not belong to $\mathfrak{p}$ such that $B[b^{-1}]$ is of relative dimension d over A.*

Proof. Set $R = B_{\mathfrak{p}}/\mathfrak{q}B_{\mathfrak{p}}$. Since R has dimension $\leqslant d$, we can choose elements $r_1, \ldots, r_d$ which belong to the maximal ideal of R such that $R/(r_1, \ldots, r_d)$ has a unique maximal ideal. Multiplying each r_i by a unit if necessary, we may assume that each r_i is the image of an element $b_i \in B$. Let $\phi' : A[x_1, \ldots, x_d] \to B$ be the extension of ϕ given by $\phi'(x_i) = b_i$. Then $\mathfrak{p}$ is an isolated point of its fiber under the map $|\operatorname{Spec} B| \to |\operatorname{Spec} A[x_1, \ldots, x_d]|$. Theorem B.2.4.6 implies that we can choose a map $\rho : \overline{B} \to B$ of $A[x_1, \ldots, x_d]$-algebras where $\overline{B}$ is finitely generated as an $A[x_1, \ldots, x_d]$-module and ρ induces an isomorphism $\overline{B}[b^{-1}] \simeq B[b^{-1}]$ for some $b \in \overline{B} - \rho^{-1}\mathfrak{p}$. We now conclude by observing that the canonical map $A \to B[b^{-1}]$ factors as a composition

$$A \to A[x_1, \ldots, x_d] \to \overline{B} \to \overline{B}[b^{-1}] \simeq B[b^{-1}]$$

where the first map has relative dimension $\leqslant d$ (Example B.2.3.8) the second map is finite (hence of relative dimension $\leqslant 0$ by Example B.2.4.4), and the third map is étale (hence of relative dimension $\leqslant 0$ by Example B.2.4.3), so the composite map is of relative dimension $\leqslant d$ by Proposition B.2.3.7. □

B.3 Henselian Rings

In this section, we will review some basic facts about Henselian rings which will be needed in this book. For a more detailed exposition, we refer the reader to [172].

B.3.1 Henselian Pairs

Let R be a commutative ring. Suppose that we wish to find an element $t \in R$ which is a solution to a polynomial equation $f(t) = 0$, where where $f(x) \in R[x]$ is a polynomial. One approach to this problem is to begin with an approximate solution (an element $t_0 \in R$ such that $f(t_0) \equiv 0 \mod I$ for some ideal $I \subseteq R$) and to then find better approximations $\{t_n\}_{n \geqslant 0}$ using *Newton's method*: $t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$. Provided that these expressions are well-defined (that is, that each $f'(t_n) = \frac{\partial f(x)}{\partial x}|_{x=t_n}$ is an invertible element of R), an elementary calculation gives

$$f(t_n) \equiv 0 \mod I^{n+1} \qquad t_{n+1} \equiv t_n \mod I^{n+1}.$$

If the ring R is complete with respect to I (that is, if it is equivalent to the inverse limit of the tower $\cdots \to R/I^4 \to R/I^3 \to R/I^2 \to R/I$), then the sequence $\{t_n\}_{n \geqslant 0}$ converges I-adically to an element $t \in R$ satisfying $f(t) = 0$. The following definition is intended to capture the essence of this situation:

Definition B.3.1.1. Let R be a commutative ring and let $I \subseteq R$ be an ideal. We will say that (R, I) is a *Henselian pair* if it satisfies the following condition: for every étale ring homomorphism $R \to R'$ which induces an isomorphism $R/I \to R'/IR'$, there exists an R-algebra homomorphism $R' \to R$.

We say that a commutative ring R is *Henselian* it it is a local ring and the pair $(R, \mathfrak{m})$ is Henselian, where $\mathfrak{m}$ is the maximal ideal of R.

Warning B.3.1.2. Our terminology is not completely standard; many authors do not require a Henselian ring to be local.

Remark B.3.1.3. Let R be a commutative ring and let $I \subseteq J \subseteq R$ be ideals. If the pair (R, J) is Henselian, then the pair (R, I) is also Henselian. In particular, if R is a Henselian ring, then the pair (R, I) is Henselian for any ideal $I \subsetneq R$.

Definition B.3.1.1 is motivated by the following:

Proposition B.3.1.4 (Hensel's Lemma)**.** *Let R be a commutative ring which is I-adically complete for some ideal I. Then (R, I) is a Henselian pair.*

Proof. We apply Newton's method. Let R' be an étale R-algebra. Proposition B.1.1.1 implies the existence of an isomorphism $R' \simeq R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)[\Delta^{-1}]$, where Δ denotes the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial x_j}]_{1 \leqslant i,j \leqslant n}$ (see Proposition B.1.1.3). We wish to show that every R-algebra homomorphism $\phi_0 : R' \to R/I$ can be lifted to a ring homomorphism $\phi : R' \to R$. Since R is complete with respect to I, it will suffice to construct a compatible sequence of R-algebra homomorphisms $\phi_a : R' \to R/I^{a+1}$. Assume that ϕ_a has already been constructed, and choose elements $\{y_j \in R\}_{1 \leqslant j \leqslant n}$ such that $\phi_a(x_j) \cong y_j$ modulo I^{a+1}. Since ϕ_a is a ring homomorphism, we have $f_i(\vec{y}) \in I^{a+1}$ for $1 \leqslant i \leqslant n$. Let $\Delta[\vec{x}]$ denote the determinant of the Jacobian matrix $M(\vec{x}) = [\frac{\partial f_i}{\partial x_j}]_{1 \leqslant i,j \leqslant n}$. Then $\Delta[\vec{y}]$ is invertible modulo I^{a+1} and therefore invertible (since I is contained in every maximal ideal of R, by Proposition B.3.2.2). It follows that $M(\vec{y})$ is an invertible matrix over R, so we can define $\vec{y}' = \vec{y} - M(\vec{y})^{-1} \vec{f}(\vec{y})$. A simple calculation gives shows that $f_i(\vec{y}') \in I^{2(a+1)}$, so that the assignment $x_i \mapsto y'_i$ determines a ring homomorphism $\phi_{a+1} : R' \to R/I^{a+2}$ compatible with ϕ_a. □

Corollary B.3.1.5. *Let R be a commutative ring and let $I \subseteq R$ be a nilpotent ideal. Then the pair (R, I) is Henselian.*

B.3.2 Lifting Idempotents

Let R be a commutative ring containing an ideal $I \subseteq R$. There are many equivalent ways to formulate the condition that (R, I) is a Henselian pair.

Notation B.3.2.1. If R is a commutative ring, we let $\mathrm{Idem}(R)$ denote the set of idempotent elements of R. Given a pair of commutative R-algebras A and B, we let $\mathrm{Hom}_R(A, B)$ denote the set of R-algebra homomorphisms from A to B.

Proposition B.3.2.2. *Let R be a commutative ring and let $I \subseteq R$ be an ideal. The following conditions are equivalent:*

(1) *The pair (R, I) is Henselian.*

(2) *For every étale R-algebra R', the reduction map $\theta_{R'} : \mathrm{Hom}_R(R', R) \to \mathrm{Hom}_R(R', R/I) \simeq \mathrm{Hom}_{R/I}(R'/IR', R/I)$ is bijective.*

(3) *The ideal I is contained in every maximal ideal $\mathfrak{m} \subseteq R$ and, for every R-algebra A which is a free R-module of finite rank, the reduction map $\mathrm{Idem}(A) \to \mathrm{Idem}(A/IA)$ is bijective.*

(4) *For every R-algebra A which is finitely generated as an R-module, the reduction map $\mathrm{Idem}(A) \to \mathrm{Idem}(A/IA)$ is bijective.*

Proof. We first show that (1) ⇒ (2). Assume that (R, I) is a Henselian pair. We first show that for every étale R-algebra R', the map $\theta_{R'}$ is surjective. Suppose we are given an R-algebra map $\phi_0 : R' \to R/I$. Then ϕ_0 extends to an R/I-algebra map $\overline{\phi}_0 : R'/IR' \to R/I$. Since R'/IR' is étale over R/I, the map $\overline{\phi}_0$ exhibits R/I as a direct factor of R"$/IR'$: that is, it induces an isomorphism $(R'/IR')[\overline{e}^{-1}] \simeq R/I$ for some idempotent element $\overline{e} \in R'/IR'$. Let $e \in R'$ be any element lying over $\overline{e}$. Then $R'[e^{-1}]$ is an étale R-algebra for which the unit map $R/I \to R'[e^{-1}]/IR'[e^{-1}]$ is an isomorphism. Applying our hypothesis that (R, I) is a Henselian pair, we deduce the existence of an R-algebra homomorphism $R'[e^{-1}] \to R$. Composing this with the evident map $R' \to R'[e^{-1}]$, we obtain an R-algebra map $\phi : R' \to R$ which is a preimage of ϕ_0 under the map $\theta_{R'}$.

To prove the injectivity of $\theta_{R'}$, suppose that we are given given two R-algebra maps $f, g : R' \to R$ with $\theta_{R'}(f) = \theta_{R'}(g)$. Since R' is étale over R, the multiplication map $m : R' \otimes_R R' \to R'$ induces an isomorphism $(R' \otimes_R R')[e^{-1}] \simeq R'$ for some idempotent element $e \in R' \otimes_R R'$. The maps f and g determine an R-algebra homomorphism $u : R' \otimes_R R' \to R$. Since $\theta(f) = \theta(g)$, the composite map $u' : R' \otimes_R R' \to R \to R/I$ factors through m, so that the image of $a = u(e) \in R$ under the quotient map $R \to R/I$ is an invertible element of R/I. Using the surjectivity of the map $\theta_{R[a^{-1}]} : \mathrm{Hom}_R(R[a^{-1}], R) \to \mathrm{Hom}_R(R[a^{-1}], R/I)$, we deduce that a is an invertible element of R. Then u factors through m, so that $f = g$.

We next show that (2) implies (3). Assume that the pair (R, I) satisfies (2). We first claim that I is contained in every maximal ideal $\mathfrak{m} \subseteq R$. Suppose otherwise: then the maximality of $\mathfrak{m}$ implies that $I + \mathfrak{m} = R$, so we can write $1 = a + b$ where $a \in I$ and $b \in \mathfrak{m}$. Then the image of b in R/I is invertible, so the mapping space $\mathrm{Hom}_R(R[b^{-1}], R/I)$ is nonempty. It follows from (2) that the mapping space $\mathrm{Hom}_R(R[b^{-1}], R)$ is nonempty, so that b is invertible in R. This is a contradiction, since b belongs to the maximal ideal $\mathfrak{m}$.

Now suppose that A is an R-algebra which is freely generated as an R-module by elements $a_1, \ldots a_n \in A$. The multiplication on A is then given by $a_i a_j = \sum_{1 \leqslant k \leqslant n} r^k_{i,j} a_k$ for some structure constants $r^k_{i,j} \in R$. Set $B = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ where $f_k(x_1, \ldots, x_n) = x_k - \sum_{i,j} r^k_{i,j} x_i x_j$. Unwinding the definitions, we see that for any R-algebra R', we have a canonical bijection $\mathrm{Hom}_R(B, R') \simeq \mathrm{Idem}(A \otimes_R R')$. From this description it follows that B is an étale R-algebra. Applying hypothesis (2), we deduce that the canonical map

$$\mathrm{Idem}(A) \simeq \mathrm{Hom}_R(B, R) \to \mathrm{Hom}_R(B, R/I) \simeq \mathrm{Idem}(A/IA)$$

is bijective, so that condition (3) is satisfied.

We now show that (3) implies (4). Assume (3) is satisfied and let A be an R-algebra which is generated as an R-module (not necessarily freely) by elements $a_1, \ldots, a_n \in R$. We wish to prove that the reduction map $\mathrm{Idem}(A) \to \mathrm{Idem}(A/IA)$ is bijective. We first show that it is injective. Let e and e' be idempotent elements of A with the same image in A/IA; we wish to prove that $e = e'$. Set $A' = A/(e - ee')$. Then A' is a finitely generated

R-module and the quotient A'/IA' vanishes (since the image of $e - ee'$ vanishes in A/IA). For every maximal ideal $\mathfrak{m}$ of R, assumption (3) guarantees that $I \subseteq \mathfrak{m}$, so that $A'/\mathfrak{m}A' \simeq 0$. Applying Nakayama's lemma, we deduce that the localization $A'_{\mathfrak{m}}$ vanishes. Since the maximal ideal $\mathfrak{m} \subseteq R$ was chosen arbitrarily, it follows that $A' \simeq 0$. Consequently, the idempotent $e - ee' \in A$ vanishes, so $e = ee'$. The same argument shows that $e' = ee'$, so that $e = e'$ by transitivity.

We now prove that the reduction map $\mathrm{Idem}(A) \to \mathrm{Idem}(A/IA)$ is surjective. Fix an idempotent element $\overline{e} \in A/IA$, and lift $\overline{e}$ to an element $e \in A$. The element e is not necessarily idempotent, but the difference $e - e^2$ belongs to the ideal I. Consequently, we can write

$$(e - e^2)a_i = \sum_{1 \leqslant j \leqslant n} c_{i,j} a_j$$

for some coefficients $c_{i,j} \in I \subseteq R$. Let $f(x) \in R[x]$ be the characteristic polynomial of the matrix $\left(c_{i,j}\right)$ and set $g(x) = f(x - x^2)$, so that $g(e) = f(e - e^2)$ vanishes in A. Set $A' = R[x]/(g(x))$, so that the construction $x \mapsto e$ determines an R-algebra homomorphism $A' \to A$. We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Idem}(A') & \longrightarrow & \mathrm{Idem}(A) \\
\downarrow & & \downarrow \\
\mathrm{Idem}(A'/IA') & \xrightarrow{\rho} & \mathrm{Idem}(A/IA)
\end{array}$$

where the left horizontal map is a bijection by virtue of assumption (2). Consequently, to prove that $\overline{e}$ lies in the image of the right vertical map, it will suffice to show that it lies in the image of the map $\rho : \mathrm{Idem}(A'/IA') \to \mathrm{Idem}(A/IA)$. Because the coefficients $c_{i,j}$ lies in the ideal I, we can identify A'/IA' with the quotient $(R/I)[x]/(x^n(1-x)^n)$. This is clear, since ρ factors as a composition

$$\mathrm{Idem}(A'/IA) \xrightarrow{\rho'} \mathrm{Idem}((R/I)[x]/(x - x^2)) \xrightarrow{\rho''} \mathrm{Idem}(A/IA),$$

where ρ' is bijective (since the ring homomorphism $(A'/IA' \to (R/I)[x]/(x - x^2)$ has nilpotent kernel) and the map ρ'' carries x to $\overline{e}$.

We now complete the proof by showing that (4) implies (1). Assume that condition (4) is satisfied, let R' be an étale R-algebra for which the unit map $R/I \to R'/IR'$ is an isomorphism. We wish to show that there exists an R-algebra homomorphism $\phi : R' \to R$. Since R' is étale over R, it is also quasi-finite over R. Applying Theorem B.2.4.5, we can choose an R-algebra homomorphism $\rho : A \to R'$, where A is finitely generated as an R-module and the induced map $j : \mathrm{Spec}\, R' \to \mathrm{Spec}\, A$ is an open immersion of affine schemes. Then the map $\overline{\rho} : A/IA \to R'/IR'$ also induces an open immersion of affine schemes $f : \mathrm{Spec}\, R'/IR' \to A/IA$. However, $\overline{\rho}$ is surjective (up to isomorphism, it is left inverse to the unit map $R/I \to A/IA$), so the map f is also a closed immersion. It follows that we can choose an idempotent element $\overline{e} \in A/IA$ for which $\overline{\rho}$ induces an isomorphism $(A/IA)[\overline{e}^{-1}] \to R'/IR'$. Using hypothesis (4), we can lift $\overline{e}$ to an idempotent element $e \in A$. Replacing A by $A[e^{-1}]$ and R' by $R'[\rho e^{-1}]$, we can reduce to the situation where $e = 1$: that is, the map $\overline{\rho}$ is an isomorphism.

We claim that ρ is an isomorphism. Suppose otherwise: then there exists some closed point of $x \in \operatorname{Spec} A$ which does not belong to the image of the open immersion j. Since A is finitely generated as an R-module, it follows that the image of x is a closed point of $\operatorname{Spec} R$, corresponding to a maximal ideal $\mathfrak{m} \subseteq R$. Hypothesis (4) implies that the map $\operatorname{Idem}(R/\mathfrak{m}) \to \operatorname{Idem}(R/(I + \mathfrak{m}))$ is bijective, so that $I \subseteq \mathfrak{m}$. It follows that x belongs to the closed subscheme $\operatorname{Spec} A/I \subseteq \operatorname{Spec} A$, contradicting the fact that $\overline{\rho}$ is an isomorphism.

For every maximal ideal $\mathfrak{m} \subseteq R$, the preceding argument shows that $I \subseteq \mathfrak{m}$ and therefore the natural map $R/\mathfrak{m} \to R'/\mathfrak{m}R'$ is an isomorphism. Since ρ is an isomorphism, the ring R' is finitely generated as an R-module. Applying Nakayama's lemma, we deduce that the map $R \to R'$ is surjective. Since R' is étale over R, we can write $R' = R/(e')$ for some idempotent element $e' \in R$. Then e' must belong the ideal I, and therefore to every maximal ideal $\mathfrak{m} \subseteq R$. It follows from the idempotence of e' that $e' = 0$. We conclude that the unit map $R \to R'$ is an isomorphism, so that there exists an R-algebra homomorphism $R' \to R$ as desired. □

B.3.3 Properties of Henselian Pairs

We now summarize some features of Henselian pairs which can be easily deduced from Proposition B.3.2.2.

Corollary B.3.3.1. *Let $\phi : R \to R'$ be a homomorphism of commutative rings which exhibits R' as a finitely generated R-module, and let $I \subseteq R$ be an ideal. If the pair (R, I) is Henselian, then the pair (R', IR') is Henselian.*

Proof. Use criterion (4) of Proposition B.3.2.2. □

Corollary B.3.3.2. *Let R be a Henselian commutative ring and let $I \subsetneq R$ be a proper ideal. Then the quotient R/I is also a Henselian ring.*

Corollary B.3.3.3. *Let R be a commutative ring containing ideals $I \subseteq J \subseteq R$. Suppose that the pairs (R, I) and $(R/I, J/I)$ are Henselian. Then the pair (R, J) is Henselian.*

Proof. Use criterion (2) of Proposition B.3.2.2. □

Corollary B.3.3.4. *Let $\phi : R \to R'$ be a homomorphism between local commutative rings. Suppose that R is Henselian and that ϕ exhibits R' as a finitely generated R-module. Then R' is Henselian.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of R and let $\mathfrak{m}'$ denote the maximal ideal of R'. Since R' is local, we have $R' \neq 0$. Invoking Nakayama's lemma, we deduce that $\mathfrak{m}R' \subseteq \mathfrak{m}'$. It follows from Corollary B.3.3.1 that the pair $(R', \mathfrak{m}R')$ is Henselian. By virtue of Corollary B.3.3.3, to prove that the pair $(R', \mathfrak{m}')$ is Henselian, it will suffice to show that the pair $(R'/\mathfrak{m}R', \mathfrak{m}'/\mathfrak{m}R')$ is Henselian. Note that the quotient $R'/\mathfrak{m}R'$ is a local ring which is finite-dimensional as a vector space over the field $R/\mathfrak{m}$, so that the maximal ideal of $R'/\mathfrak{m}R'$ is nilpotent. In particular, the ring $R'/\mathfrak{m}R'$ is complete with respect to its maximal ideal, so that the desired result follows from Proposition B.3.1.4. □

Corollary B.3.3.5. *Let $\phi : R \to R'$ be a homomorphism between commutative rings which exhibits R' as a finitely generated R-module. If R is isomorphic to a finite product of Henselian local rings, then R' is isomorphic to a finite product of Henselian local rings.*

Proof. Without loss of generality we may assume that R is a local ring with maximal ideal $\mathfrak{m}$. Then $A = R'/\mathfrak{m}R'$ is finite-dimensional algebra over the residue field $\kappa = R/\mathfrak{m}$, and therefore factors as a finite product $\prod A_i$ of local Artinian rings. Since R is Henselian, we can lift the factorization $A \simeq \prod_i A_i$ to a factorization $R' \simeq \prod R'_i$ (Proposition B.3.2.2). We now observe that each R'_i is a local commutative ring which is finitely generated as an R-module, and is therefore Henselian (Corollary B.3.3.4). □

Corollary B.3.3.6. *Let (R, I) be a Henselian pair and let A be a quasi-finite R-algebra with the property that A/IA is finite over R/I. Then A factors as a product $A' \times A''$, where A' is finite over R and $A''/IA'' \simeq 0$.*

Proof. Using Theorem B.2.4.5, we can factor ϕ as a composition $R \xrightarrow{\phi'} B \xrightarrow{\phi''} A$ where B is finite over R and ϕ'' induces an open immersion $j : \operatorname{Spec}(A) \to \operatorname{Spec}(B)$. Let $U \subseteq |\operatorname{Spec}(B)|$ denote the image of j and let $U_0 \subseteq |\operatorname{Spec}(B/IB)|$ denote its inverse image in $|\operatorname{Spec}(B/IB)|$. Since A/IA is finite over R/I, it is also finite over B/IB; it follows that U_0 is a closed and open subset of $|\operatorname{Spec}(B/IB)|$, and is therefore given as the vanishing locus of some idempotent element $\overline{e} \in B/IB$. Our assumption that (R, I) is a Henselian pair guarantees that we can lift e to an idempotent element $e \in B$. Let $V \subseteq |\operatorname{Spec}(B)|$ be the vanishing locus of e, so that V is a closed and open subset of $\operatorname{Spec} B$ whose intersection with $|\operatorname{Spec}(B/IB)|$ is equal to U_0. Then $V - U$ is a closed subset of $|\operatorname{Spec}(B)|$ whose image under the finite map $\operatorname{Spec}(B) \to \operatorname{Spec}(R)$ is also closed. By construction, this image does not intersect the vanishing locus of I, and is therefore empty (since (R, I) is a Henselian pair). It follows that V is contained in U, so that ϕ'' induces an isomorphism $B/(e) \simeq A/(\phi''(e))$. In particular, In particular, $A/(\phi''(e))$ is finitely generated as an R-module. We now complete the proof by taking $A' = A/(\phi''(e))$ and $A'' = A/(1 - \phi''(e))$. □

Corollary B.3.3.7. *Let (R, I) be a Henselian pair. Then the construction $A \mapsto A/IA$ induces an equivalence from the category of finite étale R-algebras to the category of finite étale R/I-modules.*

Proof. We first show that extension of scalars is fully faithful. Let A and B be finite étale R-algebras; we wish to show that the natural map

$$\mathrm{Hom}_{\mathrm{CAlg}_R^\heartsuit}(A, B) \to \mathrm{Hom}_{\mathrm{CAlg}_R^\heartsuit}(A, B/IB) \simeq \mathrm{Hom}_{\mathrm{CAlg}_{R/I}^\heartsuit}(A/IA, B/IB)$$

is bijective. This follows immediately from the fact that the pair (A, IA) is Henselian (Corollary B.3.3.1).

We now prove essential surjectivity. Let A_0 be a finite flat (R/I)-algebra. Using the structure theory of étale morphisms (see Proposition B.1.1.3), we can assume that $A_0 = A/IA$, where A is an étale R-algebra. Then A is quasi-finite over R, so we can use Corollary B.3.3.6 to split A as a product $A' \times A''$, where A' is finite (and therefore finite étale) over R and $A'/IA' \simeq A/IA \simeq A_0$. □

Corollary B.3.3.8. *Let R be a Henselian local ring with maximal ideal $\mathfrak{m}$ and let $\phi : R \to A$ be a quasi-finite ring homomorphism. Then we can write $A = A' \times A''$, where A' is finite over R and $A''/\mathfrak{m}A'' \simeq 0$.*

Proof. This is a special case of Corollary B.3.3.6, since the quasi-finite map $R/\mathfrak{m} \to A/\mathfrak{m}A$ is automatically finite. □

Corollary B.3.3.9. *Let R be a Henselian ring. Suppose we are given a faithfully flat étale map $R \to R'$. Then there exists an idempotent element $e \in R'$ such that $R'[e^{-1}]$ is local, faithfully flat over R, and finitely generated as an R-module.*

Proof. Since R' is étale over R, it is quasi-finite over R. Using Corollary B.3.3.8, we can reduce to the case where R' is finite over R. Let $\mathfrak{m}$ denote the maximal ideal of R, so that $R'/\mathfrak{m}R'$ is a nonzero étale algebra over the field $\kappa = R/\mathfrak{m}$. We can therefore choose an idempotent element $\overline{e}$ in $R'/\mathfrak{m}R'$ such that $(R'/\mathfrak{m}R')[\overline{e}^{-1}]$ is a finite separable extension of κ. Using Proposition B.3.2.2, we can lift $\overline{e}$ to an idempotent element $e \in R'$ having the desired properties. □

B.3.4 Henselization

Let R be a commutative ring and let $I \subseteq R$ be an ideal. If R is Noetherian, then the I-adic completion $\widehat{R} = \varprojlim \{R/I^n\}$ is complete with respect to the ideal $\widehat{I} = I\widehat{R}$, so that $(\widehat{R}, \widehat{I})$ is a Henselian pair by virtue of Hensel's Lemma (Proposition B.3.1.4). However, the passage from R to its completion $\widehat{R}$ is a fairly transcendental construction; in practice, it is often more convenient to work with a much more conservative enlargement of R having the same essential feature.

Proposition B.3.4.1. *Let R be a commutative ring and let $I \subseteq R$ be an ideal. Then there exists ring homomorphism $\phi : R \to A$ satisfying the following conditions:*

(a) *The pair (A, IA) is Henselian.*

(b) *For every R-algebra B, if the pair (B, IB) is Henselian, then there exists a unique R-algebra homomorphism $A \to B$.*

Moreover, the R-algebra A has the following additional property:

(c) *As an R-algebra, A can be written as a filtered colimit $\varinjlim R_\alpha$, where each R_α is an étale R-algebra for which the unit map $R/I \to R_\alpha/IR_\alpha$ is an isomorphism.*

In particular, the canonical map $R/I \to A/IA$ is an isomorphism.

Remark B.3.4.2. In the situation of Proposition B.3.4.1, the R-algebra A is uniquely determined (up to unique isomorphism) by the commutative ring R and the ideal $I \subseteq R$. We will refer to A as the *Henselization of R with respect to the ideal I*.

Proof of Proposition B.3.4.1. We define a category $\mathcal{C}$ as follows:

- The objects of $\mathcal{C}$ are étale R-algebras R' for which the natural map $R/I \to R'/IR'$ is an isomorphism.
- The morphisms of $\mathcal{C}$ are R-algebra homomorphisms.

The category $\mathcal{C}$ admits finite colimits and is therefore filtered. We let A denote the direct limit $\varinjlim_{R' \in \mathcal{C}} R'$. Then A satisfies condition (c) by construction. We will show that it also satisfies (a) and (b).

To prove (a), let B be an étale A-algebra for which the unit map $A/IA \to B/IB$ is an isomorphism. We wish to prove that there exists an A-algebra homomorphism $B \to A$. Since B is étale over A, it is finitely presented as an A-algebra. Using a direct limit argument, we can write $B = A \otimes_{R'} R''$, where R' is an object of $\mathcal{C}$ and R'' is an étale R'-algebra. Unwinding the definitions, we are reduced to proving that there exists an R'-algebra homomorphism from R'' to A. This is clear: the unit map $R'/IR' \to R''/IR''$ is an isomorphism, so we can regard $R' \to R''$ as a morphism in the category $\mathcal{C}$.

We now prove (b). Let B be an arbitrary R-algebra for which (B, IB) is a Henselian pair: we wish to prove that the map $\mathrm{Hom}_R(A, B)$ contains a single element. Using the definition of A as a direct limit, we are reduced to proving that the set $\mathrm{Hom}_R(R', B)$ has a single element for each $R' \in \mathcal{C}$. Since $B \otimes_R R'$ is étale over B and the pair (B, IB) is

Henselian, we have canonical bijections

$$\begin{aligned}\mathrm{Hom}_R(R', B) &\simeq \mathrm{Hom}_B(B \otimes_R R', B)\\ &\simeq \mathrm{Hom}_B(B \otimes_R R', B/IB)\\ &\simeq \mathrm{Hom}_R(R', B/I)\\ &\simeq \mathrm{Hom}_{R/I}(R'/IR', B/IB).\end{aligned}$$

These sets are singletons by virtue of our assumption that the unit map $R/I \to R'/IR'$ is an isomorphism. □

Remark B.3.4.3. The proof of Proposition B.3.4.1 shows that condition (b) follows from (a) and (c). Consequently, to prove that a ring homomorphism $\phi : R \to A$ exhibits A as the Henselization of R with respect to some ideal $I \subseteq R$, it suffices to show that ϕ satisfies conditions (a) and (c).

Remark B.3.4.4. Let $\phi : A \to B$ be a map which exhibits B as the Henselization of A with respect to an ideal I, and suppose we are given a pushout square

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow{\scriptstyle\psi} & & \downarrow \\ A' & \xrightarrow{\phi'} & B' \end{array}$$

where ψ exhibits A' as a finitely generated A-module. Then ϕ' exhibits B' as the Henselization of A' with respect to the ideal IA'. This follows immediately from Remark B.3.4.3 and Corollary B.3.3.1.

Corollary B.3.4.5. *Let $\phi : R \to A$ be a quasi-finite map of commutative rings and let κ denote a residue field of R at some prime ideal $\mathfrak{p} \subseteq R$. Then there exists a factorization of the canonical map $R \to \kappa$ as a composition $R \to R' \to \kappa$, where R' is étale over R, and a decomposition of $R' \otimes_R A$ as a product $A' \times A''$, where A' is a finitely generated module over R' and $\kappa \otimes_{R'} A'' \simeq 0$.*

Proof. Suppose we are given an R-algebra R' equipped with a map $\rho : R' \to \kappa$. Let us say that R' is *good* if the tensor product $R' \otimes_R A$ (formed in the ordinary category of R-modules) decomposes as a Cartesian product $A' \times A''$, where A' is a finitely generated R'-module and $\kappa \otimes_{R'} A'' \simeq 0$. Note that ρ is good if and only if there exists an idempotent element $e \in R' \otimes_R A$ such that $(R' \otimes_R A)[e^{-1}]$ is finitely generated as an R'-module and the image of e vanishes in $\kappa \otimes_R A$. From this description, it is easy to see that if R' is given as a filtered colimit of R-algebra homomorphisms R'_α and R' is good, then some R'_α is also good.

Let S denote the Henselization of the local ring $R_{\mathfrak{p}}$ with respect to its maximal ideal. It follows from Corollary B.3.3.8 that S is good. Using Proposition B.3.4.1, we can write

S as a filtered colimit $\varinjlim S_\alpha$, where each S_α is an étale $R_{\mathfrak{p}}$-algebra. It follows that some S_α is good. Note that we can write S_α as a tensor product $R_{\mathfrak{p}} \otimes_R R'$ where R' is an étale R-algebra. Then S_α can be written as a filtered colimit of localizations $R'[a^{-1}]$ where $a \notin \mathfrak{p}$. It follows that some localization $R'[a^{-1}]$ is good and étale over R, as desired. □

Proposition B.3.4.6 (Transitivity of Henselizations)**.** *Suppose we are given homomorphisms of commutative rings $\phi : A \to B$ and $\psi : B \to C$ and ideals $I, J \subseteq A$ satisfying the following conditions:*

- (a) *The map ϕ exhibits B as the Henselization of A with respect to the ideal I.*
- (b) *The map ψ exhibits C as the Henselization of B with respect to the ideal JB.*

Then the composite map $(\psi \circ \phi) : A \to C$ exhibits C as the Henselization of A with respect to the ideal $I + J$.

Proof. Let D be an A-algebra for which the pair $(D, (I + J)D)$ is Henselian. Then the pairs (D, ID) and (D, JD) are also Henselian (Remark B.3.1.3). It follows from (b) that composition with ψ induces a bijection $\mathrm{Hom}_A(C, D) \to \mathrm{Hom}_A(B, D)$, and it follows from (a) that the set $\mathrm{Hom}_A(B, D)$ is a singleton. Consequently, there is a unique A-algebra homomorphism from C to D. To complete the proof, it will suffice to show that the pair $(C, (I + J)C)$ is Henselian. Using Corollary B.3.3.3, we are reduced to proving that the pairs (C, JC) and $(C/JC, (I + J)C/JC)$ are Henselian. The first follows from (b). To prove the second, we note that (b) implies that the canonical map $B/JB \to C/JC$ is an isomorphism. We are therefore reduced to proving that the pair $(B/JB, (I + J)B/JB)$ is Henselian. This follows from Corollary B.3.3.1, since the pair (B, IB) is Henselian by virtue of assumption (a). □

B.3.5 Strictly Henselian Rings

We now restrict our attention to a special class of Henselian rings.

Definition B.3.5.1. Let R be a commutative ring. We say that R is *strictly Henselian* if R is Henselian (Definition B.3.1.1) and the residue field $R/\mathfrak{m}$ is separably closed (where $\mathfrak{m}$ denotes the maximal ideal of R).

Example B.3.5.2. Let R be a strictly Henselian ring and let $I \subsetneq R$ be an ideal. Then the quotient R/I is Henselian (Corollary B.3.3.2) with the same residue field as R, and is therefore strictly Henselian.

Proposition B.3.5.3. *Let R be a commutative ring. The following conditions are equivalent:*

- (1) *The ring R is strictly Henselian.*

(2) *For every finite collection of étale maps* $\{\phi_\alpha : R \to R_\alpha\}$ *for which the induced map* $R \to \prod_\alpha R_\alpha$ *is faithfully flat, one of the maps* ϕ_α *admits a left inverse.*

Proof. Suppose first that condition (1) is satisfied, and let $\mathfrak{m}$ denote the maximal ideal of R. Let $\{\phi_\alpha : R \to R_\alpha\}$ be as in (2). Since the map $R \to \prod_\alpha R_\alpha$ is faithfully flat, there exists an index α such that $R_\alpha/\mathfrak{m}R_\alpha$ is nonzero. Since R_α is étale over R, $R_\alpha/\mathfrak{m}R_\alpha$ is a product of separable field extensions of $\kappa = R/\mathfrak{m}$. Since κ is separably closed, we can choose a map of R-algebras $\theta : R_\alpha/\mathfrak{m} \to R/\mathfrak{m}$. The assumption that R is Henselian implies that θ lifts to a map of R-algebras $R_\alpha \to R$, which is left inverse to ϕ_α.

Now suppose that (2) is satisfied; we wish to prove that R is strictly Henselian. We first observe that R is nonzero (otherwise the map from R to an empty product is faithfully flat, contradicting (2)). For every element $x \in R$, the map $R \to R[x^{-1}] \times R[(1-x)^{-1}]$ is faithfully flat, so condition (2) implies that either x or $1-x$ is invertible in R: that is, R is a local ring.

We now claim that R is Henselian. Let R' be an étale R-algebra and choose a map of R-algebras $\theta : R' \to R/\mathfrak{m}$. We wish to prove that θ can be lifted to an R-algebra map $R' \to R$. Let $\kappa = R/\mathfrak{m}$, so that $R'/\mathfrak{m}R'$ is a product of finite separable extensions of k. We proceed by induction on the dimension n of $R'/\mathfrak{m}R'$ as a κ-vector space. Note that $n > 0$, since θ induces a surjection $R'/\mathfrak{m}R' \to \kappa$. It follows that R' is faithfully flat over R, so condition (2) implies that there is a map of R-algebras $\phi : R' \to R$. Since R' is étale over R, the kernel of the map ϕ is generated by an idempotent element $e \in R'$. If $\theta(e) = 0$, then θ factors as a composition $R' \xrightarrow{\phi} R \to R/\mathfrak{m}$ so that ϕ is the desired lifting of θ. Assume otherwise. Then $\theta(e) = 1$ (since e is idempotent and κ is a field), so that θ factors through the quotient $R'' = R'/(1-e)$ of R'. The inductive hypothesis then implies that the induced map $R'' \to R/\mathfrak{m}$ lifts to a map of R-algebras $R'' \to R$, so that the composite map $R' \to R'' \to R$ is the desired lifting of θ.

To complete the proof, we must show that the field $\kappa = R/\mathfrak{m}$ is separably closed. Assume otherwise. Then we can choose a nontrivial finite separable extension field κ' of κ. Without loss of generality, κ' is generated by a single element; we may therefore write $\kappa' = \kappa[x]/(f(x))$ for some monic polynomial f with coefficients in κ. Let $\overline{f}(x)$ be a monic polynomial with coefficients in R which lifts f (and has the same degree as f), and let $R' = R[x]/(\overline{f}(x))$. Then R' is finite as an R-module. The derivative of $\overline{f}(x)$ is invertible in $R'/\mathfrak{m}R'$, and therefore (by Nakayama's lemma) invertible in R'. It follows that R' is faithfully flat and étale over R. Using condition (2), we deduce that there is a map of R-algebras $R' \to R$. Reducing modulo $\mathfrak{m}$, we obtain a map of κ-algebras $\kappa' \to \kappa$, contradicting our assumption that κ' is a proper extension of κ. □

It will be convenient to have the following relative formulation of Proposition B.3.5.3.

Corollary B.3.5.4. *Let $\phi : R \to A$ be a map of commutative rings. The following conditions are equivalent:*

(1) *The commutative ring A is strictly Henselian.*

(2) *For every finitely presented R-algebra R' and every finite collection of étale maps $\{R' \to R'_\alpha\}$ which induce a faithfully flat map $R' \to \prod_\alpha R'_\alpha$, every R-algebra map $R' \to A$ factors through some R'_α.*

Proof. Assume that (1) is satisfied, and let $\{R' \to R'_\alpha\}$ be as in (2). For any map $R' \to A$, we obtain a finite collection of étale maps $\{\phi_\alpha : A \to R'_\alpha \otimes_R A\}$ which induce a faithfully flat map $A \to \prod_\alpha (R'_\alpha \otimes_R A)$. Proposition B.3.5.3 implies that one of the maps ϕ_α admits a left inverse, which determines a map of R'-algebras from R'_α into A.

Now suppose that (2) is satisfied. We will show that A satisfies the criterion of Proposition B.3.5.3. Choose a finite collection of étale maps $\{A \to A_\alpha\}$ which induce a faithfully flat map $A \to \prod_\alpha A_\alpha$. Using the structure theory for étale morphisms (Proposition B.1.1.3), we may assume that there exists a finitely presented R-algebra R' and étale maps $R' \to R'_\alpha$ such that $A_\alpha \simeq R'_\alpha \otimes_R A$. Replacing R' by a product of localizations if necessary, we may suppose that the map $R' \to \prod_\alpha R'_\alpha$ is faithfully flat. Condition (2) then guarantees the existence of a map of R'-algebras $R'_\alpha \to A$ for some α, which we can identify with an A-algebra map from A_α into A. □

B.4 The Nisnevich Topology

In [162], Nisnevich introduced the *completely decomposed topology* (now called the *Nisnevich topology*) associated to a Noetherian scheme X of finite Krull dimension. The Nisnevich topology on X is intermediate between the Zariski and étale topologies, sharing some of the pleasant features of each. In this section, we will describe an analogue of the Nisnevich topology in the non-Noetherian setting. We will restrict our attention to the case of affine schemes; for a generalization to (spectral) algebraic spaces, see §3.7.

B.4.1 Nisnevich Coverings

In the setting of Noetherian schemes, one can define a Nisnevich covering to be a collection of étale maps $\{p_\alpha : U_\alpha \to X\}$ having the property that, for every point $x \in X$, there exists an index α and a point $\overline{x} \in U_\alpha$ such that $p_\alpha(\overline{x}) = x$ and p_α induces an isomorphism of residue fields $\kappa_x \to \kappa_{\overline{x}}$. To handle non-Noetherian situations, we need to adopt a more complicated definition.

Definition B.4.1.1. Let R be a commutative ring. We will say that a collection of étale ring homomorphisms $\{\phi_\alpha : R \to R_\alpha\}_{\alpha \in I}$ is a *Nisnevich covering* of R if there exists a finite sequence of elements $a_1, \ldots, a_n \in R$ with the following properties:

(1) The elements $a_1, \ldots, a_n$ generate the unit ideal in R.

(2) For $1 \leqslant i \leqslant n$, there exists an index α and a ring homomorphism ψ which fits into a commutative diagram

$$\begin{array}{ccc} & R_\alpha & \\ {}^{\phi_\alpha}\nearrow & & \searrow^{\psi} \\ R & \longrightarrow & R[a_i^{-1}]/(a_1, \ldots, a_{i-1}). \end{array}$$

Remark B.4.1.2. If the commutative ring R is Noetherian, then Definition B.4.1.1 recovers the standard definition of Nisnevich covering (see Proposition B.4.3.1).

Example B.4.1.3. For any commutative ring R, the one-element family of maps $\{\mathrm{id}_R : R \to R\}$ is a Nisnevich covering of R.

Example B.4.1.4. Let R be the zero ring. Then the empty collection $\varnothing$ is a Nisnevich covering of R (take $n = 0$ in Definition B.4.1.1).

Example B.4.1.5. Let R be a commutative ring and let $a_1, \ldots, a_n \in R$ be elements which generate the unit ideal. Then the collection of maps $\{R \to R[a_i^{-1}]\}_{1 \leqslant i \leqslant n}$ is a Nisnevich covering of R. In other words, every Zariski covering of R is also a Nisnevich covering.

Remark B.4.1.6. Let R be a commutative ring. Then any Nisnevich covering $\{R \to R_\alpha\}_{\alpha \in A}$ of R also an étale covering: that is, there exists a finite subset $A_0 \subseteq A$ for which the induced map $R \to \prod_{\alpha \in A_0} R_\alpha$ is (étale and) faithfully flat.

Remark B.4.1.7. Let $\{\phi_\alpha : R \to R_\alpha\}_{\alpha \in A}$ be a Nisnevich covering of a commutative ring R. Suppose we are given a family of étale maps $\{\psi_\beta : R \to R_\beta\}_{\beta \in B}$ with the following property: for each $\alpha \in A$, there exists $\beta \in B$ and a commutative diagram

$$\begin{array}{ccc} & R_\beta & \\ {}^{\psi_\beta}\nearrow & & \searrow \\ R & \xrightarrow{\phi_\alpha} & R_\alpha. \end{array}$$

Then $\{\psi_\beta : R \to R_\beta\}_{\beta \in B}$ is also a Nisnevich covering of R.

Remark B.4.1.8. Let $\{R \to R_\alpha\}_{\alpha \in A}$ be a Nisnevich covering of a commutative ring R. Then there exists a finite subset $A_0 \subseteq A$ such that $\{R \to R_\alpha\}_{\alpha \in A_0}$ is also a Nisnevich covering of R.

Remark B.4.1.9. Let $\{R \to R_\alpha\}_{\alpha \in A}$ be a Nisnevich covering of a commutative ring R, and suppose we are given a ring homomorphism $R \to R'$. Then the collection of induced maps $\{R' \to R' \otimes_R R_\alpha\}_{\alpha \in A}$ is a Nisnevich covering of R'.

B.4.2 The Nisnevich Site

Let R be a commutative ring. We let $\mathrm{CAlg}_R^{\text{ét}}$ denote the category whose objects are étale R-algebras and whose morphisms are R-algebra homomorphisms. The collection of Nisnevich coverings determines a Grothendieck topology on $\mathrm{CAlg}_R^{\text{ét}}$:

Theorem B.4.2.1. *Let R be a commutative ring. Then there exists a Grothendieck topology on the category $(\mathrm{CAlg}_R^{ét})^{\mathrm{op}}$ which can be described as follows: if A is an étale R-algebra, then a sieve $\mathcal{C} \subseteq (\mathrm{CAlg}_R^{ét})^{\mathrm{op}}_{/A} \simeq \mathrm{CAlg}_A^{ét}$ is covering if and only if it contains a collection of morphisms $\{A \to A_\alpha\}$ which comprise a Nisnevich covering of A, in the sense of Definition B.4.1.1.*

Definition B.4.2.2. Let R be a commutative ring. We will refer to the Grothendieck topology of Theorem B.4.2.1 as the *Nisnevich topology* on the category $(\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}$. We let $\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S})$ spanned by those functors which are sheaves with respect to the Nisnevich topology.

Unwinding the definitions, we see that Theorem B.4.2.1 follows from Example B.4.1.3, Remark B.4.1.9, and the following transitivity result:

Proposition B.4.2.3. *Let R be a commutative ring, and suppose we are given a Nisnevich covering $\{R \to R_\alpha\}_{\alpha \in A}$ of R. Assume furthermore that for each $\alpha \in A$, we are given a Nisnevich covering $\{R_\alpha \to R_{\alpha,\beta}\}$. Then the family of composite maps $\{R \to R_{\alpha,\beta}\}$ is a Nisnevich covering of R.*

The proof of Proposition B.4.2.3 will require some preliminaries.

Lemma B.4.2.4. *Let R be a commutative ring and let $\{R \to R_\alpha\}_{\alpha \in A}$ be a collection of étale maps. Suppose that there exists a nilpotent ideal $I \subseteq R$ such that the family of induced maps $\{R/I \to R_\alpha/IR_\alpha\}_{\alpha \in A}$ is a Nisnevich covering of R/I. Then $\{R \to R_\alpha\}_{\alpha \in A}$ is a Nisnevich covering of R.*

Proof. Choose a sequence of elements $a_1, \ldots, a_n \in R$ which generate the unit ideal in R/I and a collection of commutative diagrams

$$\begin{array}{ccc} & R_{\alpha_i} & \\ \nearrow & & \searrow^{\phi_i} \\ R & \longrightarrow & (R/I)[a_i^{-1}]/(a_1, \ldots, a_{i-1}). \end{array}$$

Since each R_{α_i} is étale over R and I is a nilpotent ideal, we can lift ϕ_i to an R-algebra map $\psi_i : R_{\alpha_i} \to R[a_i^{-1}]/(a_1, \ldots, a_{i-1})$. Let J denote the ideal $(a_1, \ldots, a_n)$; since the a_i generate the unit ideal in R/I, we have $I + J = R$. It follows that $R \subseteq (I+J)^m \subseteq I^m + J$ for every integer n. Choosing n sufficiently large, we have $I^m = 0$ so that $R = J$; this proves that $\{R \to R_\alpha\}_{\alpha \in A}$ is a Nisnevich covering of R. □

Lemma B.4.2.5. *Let R be a commutative ring containing an element x. A collection of étale maps $\{R \to R_\alpha\}_{\alpha \in A}$ is a Nisnevich cover of R if and only if the following conditions are satisfied:*

(1) *The maps $\{R/xR \to R_\alpha/xR_\alpha\}_{\alpha \in A}$ are a Nisnevich cover of the quotient ring R/xR.*

(2) *The maps $\{R[x^{-1}] \to R_\alpha[x^{-1}]\}_{\alpha \in A}$ are a Nisnevich cover of the commutative ring $R[x^{-1}]$.*

Proof. The necessity of conditions (1) and (2) follows from Remark B.4.1.9. For the converse, suppose that conditions (1) and (2) are satisfied. Using (2), we deduce that there exists a sequence of elements $\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n \in R[x^{-1}]$ which generates the unit ideal, a sequence of indices $\alpha_1, \ldots, \alpha_n \in A$, and a sequence of commutative diagrams

$$\begin{array}{ccc} & R_{\alpha_i}[x^{-1}] & \\ \nearrow & & \searrow^{\phi_i} \\ R[x^{-1}] & \longrightarrow & R[(xa_i)^{-1}]/(\overline{a}_1, \ldots, \overline{a}_{i-1}). \end{array}$$

Multiplying each $\overline{a}_i$ by a sufficiently large power of x, we may assume that $\overline{a}_i$ is the image of an element $a_i \in R$. Let J denote the ideal $(xa_1, \ldots, xa_n) \subseteq R$. Since the $\overline{a}_i$ generate the unit ideal in $R[x^{-1}]$, the ideal J contains x^k for $k \gg 0$. It follows that x generates a nilpotent ideal in R/J. Using (1) and Lemma B.4.2.4, we deduce that the maps $\{R/J \to R_\alpha/JR_\alpha\}_{\alpha \in A}$ determine a Nisnevich covering of R/J. We may therefore choose elements $b_1, \ldots, b_m \in R$ which generate the unit ideal in R/J together with commutative diagrams

$$\begin{array}{ccc} & R_{\beta_j}/JR_{\beta_j} & \\ \nearrow & & \searrow^{\psi_j} \\ R/J & \longrightarrow & R[_j^{-1}]/(J + (b_1, \ldots, b_{j-1})). \end{array}$$

Then the sequence $xa_1, xa_2, \ldots, xa_n, b_1, \ldots, b_m$ generates the unit ideal in R; using the maps $\{\phi_i\}_{1 \leq i \leq n}$ and $\{\psi_j\}_{1 \leq j \leq m}$, we see that the family of maps $\{R \to R_\alpha\}$ determines a Nisnevich covering of R. $\square$

Proof of Proposition B.4.2.3. Let $\{R \to R_\alpha\}_{\alpha \in A}$ be a Nisnevich covering of a commutative ring R, and suppose that for each $\alpha \in A$ we are given a Nisnevich covering $\{R_\alpha \to R_{\alpha,\beta}\}$. Choose a sequence $a_1, \ldots, a_n \in R$ which generate the unit ideal and commutative diagrams

$$\begin{array}{ccc} & R_{\alpha_i} & \\ \nearrow & & \searrow \\ R & \longrightarrow & R[a_i^{-1}]/(a_1, \ldots, a_{i-1}). \end{array}$$

We prove by induction on n that the maps $\{R \to R_{\alpha,\beta}\}$ form a Nisnevich covering of R. If $n = 0$, then the unit ideal and zero ideal of R coincide, so that $R \simeq 0$ and the result is obvious (Example B.4.1.4 and Remark B.4.1.7). Otherwise, we may assume by the inductive hypothesis that the family of maps $\{R/a_1R \to R_{\alpha,\beta}/a_1R_{\alpha,\beta}\}$ is a Nisnevich covering of R/a_1R. According to Lemma B.4.2.5, it will suffice to show that the maps $\{R[a_1^{-1}] \to R_{\alpha,\beta}[a_1^{-1}]\}$ are a Nisnevich covering of $R[a_1^{-1}]$. Using Remark B.4.1.9, we see that $\{R[a_1^{-1}] \to R[a_1^{-1}] \otimes_{R_{\alpha_1}} R_{\alpha_1,\beta}\}$ is a Nisnevich covering; the desired result now follows from Remark B.4.1.7. □

B.4.3 The Noetherian Case

For every commutative ring R, we let $|\operatorname{Spec} R|$ denote the Zariski spectrum of R: this is a topological space whose points are prime ideals $\mathfrak{p} \subset R$ (Example 1.1.1.2). For each point $\mathfrak{p} \in |\operatorname{Spec} R|$, we let $\kappa(\mathfrak{p})$ denote the fraction field of the quotient $R/\mathfrak{p}$.

Proposition B.4.3.1. *Let R be a Noetherian ring, and suppose we are given a collection of étale maps $\{\phi_\alpha : R \to R_\alpha\}_{\alpha \in A}$. The following conditions are equivalent:*

(1) *The maps $\{R \to R_\alpha\}_{\alpha \in A}$ determine a Nisnevich covering of R.*

(2) *For every $\mathfrak{p} \in |\operatorname{Spec} R|$, there exists an index $\alpha \in A$ and a prime ideal $\mathfrak{q} \in |\operatorname{Spec} R_\alpha|$ such that $\mathfrak{p} = \phi_\alpha^{-1}\mathfrak{q}$ and the induced map $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ is an isomorphism.*

Proof. We first show that (1) ⇒ (2) (which does not require our assumption that R is Noetherian). Choose a sequence of elements $a_1, \ldots, a_n \in R$ which generate the unit ideal and a collection of commutative diagrams

$$\begin{array}{ccccc} & & R_{\alpha_i} & & \\ & \nearrow & & \searrow & \\ R & & \longrightarrow & & R[a_i^{-1}]/(a_1, \ldots, a_{i-1}). \end{array}$$

Let $\mathfrak{p}$ be a prime ideal of R. Since $\mathfrak{p} \neq R$, there exists an integer $i \leqslant n$ such that $a_1, \ldots, a_{i-1} \in \mathfrak{p}$ but $a_i \notin \mathfrak{p}$. Then $\mathfrak{p}$ is the inverse image of a prime ideal $\mathfrak{p}' \in R[a_i^{-1}]/(a_1, \ldots, a_{i-1})$, which also has an inverse image $\mathfrak{q} \in |\operatorname{Spec} R_{\alpha_i}|$. We have a commutative diagram of fields

$$\begin{array}{ccccc} & & \kappa(\mathfrak{q}) & & \\ & \nearrow & & \searrow & \\ \kappa(\mathfrak{p}) & & \longrightarrow & & \kappa(\mathfrak{p}'). \end{array}$$

Since the lower horizontal map is an isomorphism, we conclude that $\kappa(\mathfrak{p}) \simeq \kappa(\mathfrak{q})$.

Now suppose that (2) is satisfied; we will prove (1). Let X be the collection of all ideals $I \subseteq R$ for which the maps $\{R/I \to R_\alpha/IR_\alpha\}_{\alpha \in A}$ do not form a Nisnevich covering of R/I; we will prove that X is empty. Otherwise, X contains a maximal element I (since R is Noetherian). Replacing R by R/I, we may assume that X does not contain any nonzero ideal of R. Let J be the nilradical of R. Since R is Noetherian, J is nilpotent. If $J \neq 0$, then $J \notin X$ and assertion (1) follows from Lemma B.4.2.4. We may therefore assume that $J = 0$; that is, the ring R is reduced.

If $R = 0$ there is nothing to prove. Otherwise, since R is Noetherian, it contains finitely many associated primes $\eta_0, \eta_1, \ldots, \eta_k$. Reordering if necessary, we may assume that η_0 is not contained in η_i for $i > 0$. Choose an element $x \in R$ which belongs to $\eta_1 \cap \cdots \cap \eta_k$ but not to η_0. Then $x \neq 0$, so the principal ideal (x) does not belong to X. Using Lemma B.4.2.5, we can replace R by $R[x^{-1}]$ and thereby reduce to the case where R is a reduced ring with a unique associated prime, and therefore an integral domain.

Let κ denote the fraction field of R. Using assumption (2), we conclude that there exists an index $\alpha \in A$ and a prime ideal $\mathfrak{q} \subseteq R_\alpha$ such that $\phi_\alpha^{-1}\mathfrak{q} = (0) \subseteq R$ and $\kappa(\mathfrak{q}) \simeq \kappa$. In particular, we have an R-algebra map $R_\alpha \to \kappa(\mathfrak{q}) \to \kappa$. Since R_α is finitely presented as an R-algebra, this map factors through $R[y^{-1}] \subseteq \kappa$ for some nonzero element $y \in R$. It follows that the single map $\{R[y^{-1}] \to R_\alpha[y^{-1}]\}$ is a Nisnevich covering of $R[y^{-1}]$. Since the principal ideal (y) does not belong to X, Lemma B.4.2.5 implies that the maps $\{R \to R_\alpha\}_{\alpha \in A}$ is a Nisnevich covering of R, as desired. □

Remark B.4.3.2. Let $R = \varinjlim_\gamma R(\gamma)$ be a filtered colimit of commutative rings $R(\gamma)$, and suppose we are given a Nisnevich covering $\{R \to R_\alpha\}_{\alpha \in A}$ where the set A is finite (by virtue of Remark B.4.1.8, this assumption is harmless). Then there exists an index γ and a Nisnevich covering $\{R(\gamma) \to R(\gamma)_\alpha\}_{\alpha \in A}$ such that $R_\alpha \simeq R \otimes_{R(\gamma)} R(\gamma)_\alpha$ for each $\alpha \in A$.

Remark B.4.3.3. The theory of Nisnevich coverings (as set forth in Definition B.4.1.1) is uniquely determined by Remark B.4.3.2, Remark B.4.1.8, and Proposition B.4.3.1. That is, suppose we are given a commutative ring R and a collection of étale maps $\{R \to R_\alpha\}_{\alpha \in A}$. We can realize R as a filtered colimit of subrings which are finitely generated over $\mathbf{Z}$, and therefore Noetherian. It follows that $\{R \to R_\alpha\}_{\alpha \in A}$ is a Nisnevich covering of R if and only if there exists a finite subset $A_0 \subseteq A$, a subring $R' \subseteq R$ which is finitely generated over $\mathbf{Z}$, and a collection of étale maps $\{R' \to R'_\alpha\}_{\alpha \in A_0}$ satisfying condition (2) of Proposition B.4.3.1, such that $R_\alpha \simeq R \otimes_{R'} R'_\alpha$ for $\alpha \in A_0$.

B.4.4 Points of the Nisnevich Topology

Let R be a commutative ring and let $\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}$ denote the associated ∞-category of Nisnevich sheaves (see Definition B.4.2.2). Our next goal is to describe the ∞-category $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}, \mathcal{S})$ of points of the ∞-topos $\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}$. First, we need to introduce some definitions.

Definition B.4.4.1. Let R be a commutative ring and let A be a commutative R-algebra. We will say that A is Ind-étale if it can be written as a filtered colimit $\varinjlim R_\alpha$, where each R_α is an étale R-algebra. We will say that A is a *Henselization of* R if it is Ind-étale over R and is a Henselian local ring. We let $\mathrm{CAlg}_R^{\mathrm{Hens}}$ denote the full subcategory of $\mathrm{CAlg}_R^{\heartsuit}$ spanned by the Henselizations of R.

Warning B.4.4.2. The terminologies of Definition B.4.4.1 and Remark **??** are not quite compatible. If R is a commutative ring and $I \subseteq R$ is an ideal, then the Henselization of R with respect to I (in the sense of Remark **??**) is an Ind-étale R-algebra, but usually not a Henselian ring unless the ideal $I \subseteq R$ is maximal. On the other hand, not every Henselization A of R (in the sense of Definition B.4.4.1) arises from the construction of Proposition B.3.4.1.

Remark B.4.4.3. Let R be a commutative ring. Every étale R-algebra A is finitely presented, and is therefore compact when viewed as an object of the category $\mathrm{CAlg}_R^{\heartsuit}$ of commutative R-algebras (or even the ∞-category CAlg_R of $\mathbb{E}_\infty$-algebras over R). It follows that the inclusion $\mathrm{CAlg}_R^{\text{ét}} \hookrightarrow \mathrm{CAlg}_R$ extends to a fully faithful embedding $\mathrm{Ind}(\mathrm{CAlg}_R^{\text{ét}}) \hookrightarrow \mathrm{CAlg}_R$, whose essential image is spanned by those R-algebras which are Ind-étale in the sense of Definition B.4.4.1.

Theorem B.4.4.4. *Let R be a commutative ring, and let $\mathscr{O}$ denote the commutative ring object of $\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}$ given by $\mathscr{O}(A) = A$. Then evaluation on $\mathscr{O}$ induces an equivalence of ∞-categories $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}, \mathcal{S}) \to \mathrm{CAlg}_R^{\mathrm{Hens}}$.*

The content of Theorem B.4.4.4 lies in the following analogue of Corollary B.3.5.4 for the Nisnevich topology:

Proposition B.4.4.5. *Let R be a commutative ring. The following conditions are equivalent:*

(1) *The ring R is Henselian.*

(2) *For every Nisnevich covering $\{\phi_\alpha : R \to R_\alpha\}$ of R, one of the morphisms ϕ_α admits a left inverse.*

Proof. We first show that (1) ⇒ (2). Assume that R is a local Henselian ring with maximal ideal $\mathfrak{m}$, and let $\kappa = R/\mathfrak{m}$ denote the residue field of R. Suppose we are given a Nisnevich covering $\{R \to R_\alpha\}$. We wish to prove that one of the spaces $\mathrm{Hom}_R(R_\alpha, R)$ is nonempty. Since R is Henselian, this is equivalent to showing that one of the mapping spaces

$$\mathrm{Hom}_R(R_\alpha, \kappa) \simeq \mathrm{Hom}_\kappa(R_\alpha \otimes_R \kappa, \kappa)$$

is nonempty. This follows immediately from Proposition B.4.3.1, since the collection of maps $\{\kappa \to R_\alpha \otimes_R \kappa\}$ form a Nisnevich covering of the field κ.

Now suppose that (2) is satisfied; we will prove (1). If $R = 0$, then the empty set is a Nisnevich covering of R, contradicting assumption (2). If $x \in R$, then the pair of maps $\{R \to A[x^{-1}], A \to A[(1-x)^{-1}]\}$ determines a Nisnevich covering of R. It then follows from (2) that either x or $1-x$ is invertible in R. This proves that R is a local ring; let $\mathfrak{m}$ denote its maximal ideal. Let A be an étale R-algebra and suppose we are given a R-algebra map $\phi_0 : A \to R/\mathfrak{m}$; we wish to show that ϕ_0 can be lifted to a R-algebra map $A \to R$. Replacing A by a localization if necessary, we may assume that ϕ_0 is the only R-algebra map from A to $R/\mathfrak{m}$. In this case, it suffices to show that the map $R \to A$ admits a left inverse. Since A is finitely presented as an R-algebra, we can lift ϕ_0 to a map $\phi_1 : A \to R/I$, where I is a proper ideal generated by finitely many elements $x_1, \ldots, x_n \in R$. By construction, the collection of maps $\{R \to R[x_i^{-1}], R \to A\}$ is a Nisnevich covering of R. Note that since $x_i \in I \subseteq \mathfrak{m}$, there cannot exist an R-algebra map $R[x_i^{-1}] \to A$. Using (2), we deduce the existence of an R-algebra map $A \to R$, as desired. □

Proof of Theorem B.4.4.4. Theorem HTT.5.1.5.6 and Proposition HTT.6.1.5.2 supply an equivalence of $\mathrm{Fun}^*(\mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S}), \mathcal{S})$ with the ∞-category $\mathrm{Ind}(\mathrm{CAlg}_R^{\text{ét}})$, which we can identify with the full subcategory of CAlg_R spanned by those R-algebras which are Ind-étale over R (Remark B.4.4.3). Combining this observation with Proposition HTT.6.2.3.20 , we can identify $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}, \mathcal{S})$ with the full subcategory of CAlg_R spanned by those Ind-étale R-algebras which satisfy condition (2) of Proposition B.4.4.5. By virtue of Proposition B.4.4.5, this full subcategory coincides with $\mathrm{CAlg}_R^{\mathrm{Hens}}$. □

B.4.5 Classification of Henselizations

Let R be a commutative ring. Using Theorem B.4.4.4, we can obtain a very explicit description of the Kan complex $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}, \mathcal{S})^{\simeq}$ of points of the ∞-topos $\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}$.

Notation B.4.5.1. Let R be a commutative ring. We define a subcategory $\mathrm{Field}_R^{\mathrm{sep}} \subseteq \mathrm{CAlg}_R^{\heartsuit}$ as follows:

- The objects of $\mathrm{Field}_R^{\mathrm{sep}}$ are ring homomorphisms $\phi : R \to \kappa$ which exhibit κ as a separable algebraic extension of some residue field of R.
- The morphisms in $\mathrm{Field}_R^{\mathrm{sep}}$ are given by isomorphisms of R-algebras.

Let R be commutative ring and let A be a Henselization of R (in the sense of Definition B.4.4.1). Then A is a local ring with maximal ideal $\mathfrak{m}$. Since A is Ind-etale over R, the residue field $A/\mathfrak{m}$ is a separable algebraic extension of the residue field $\kappa(\mathfrak{p})$ at some prime ideal $\mathfrak{p} \subseteq R$. The construction $A \mapsto A/\mathfrak{m}$ determines a functor $\Phi : (\mathrm{CAlg}_R^{\mathrm{Hens}})^{\simeq} \to \mathrm{Field}_R^{\mathrm{sep}}$.

Proposition B.4.5.2. *Let R be a commutative ring. Then the functor $\Phi : (\mathrm{CAlg}_R^{\mathrm{Hens}})^{\simeq} \to \mathrm{Field}_R^{\mathrm{sep}}$ is an equivalence of categories.*

Remark B.4.5.3. Let R be an $\mathbb{E}_\infty$-ring. Proposition B.4.5.2 asserts that every separable extension κ of every residue field of R has the form $A/\mathfrak{m}$, for some Henselian ring A which is Ind-étale over R. We will refer to A as the *Henselization of R at κ*.

Proof of Proposition B.4.5.2. We first prove that Φ is fully faithful. Let A and A' be Henselizations of R, let $\mathfrak{m} \subseteq A$ and $\mathfrak{m}' \subseteq A'$ denote maximal ideals, and set $\kappa = A/\mathfrak{m}$ and $\kappa' = A'/\mathfrak{m}'$. Unwinding the definitions, we must show that every R-algebra isomorphism of κ with κ' can be lifted uniquely to an R-algebra isomorphism of A with A'. Since A is Henselian, we have

$$\mathrm{Hom}_R(R', A) \simeq \mathrm{Hom}_A(R' \otimes_R A, A) \simeq \mathrm{Hom}_A(R' \otimes_R A, \kappa) \simeq \mathrm{Hom}_\kappa(R' \otimes_R \kappa, \kappa) \simeq \mathrm{Hom}_R(R', \kappa)$$

for every étale R-algebra R'. Passing to filtered colimits, we deduce that the restriction map $\mathrm{Hom}_R(R', A) \to \mathrm{Hom}_R(R', \kappa)$ is bijective whenever R' is Ind-etale over R. In particular, every R-algebra homomorphism $\phi_0 : \kappa' \to \kappa$ induces a map $A' \to \kappa' \xrightarrow{\phi_0} \kappa$, which lifts uniquely to an R-algebra homomorphism $\phi : A' \to A$. If ϕ_0 is invertible, the same argument shows that ϕ_0^{-1} lifts to an R-algebra map $\psi : A \to A'$. The compositions $\phi \circ \psi$ and $\psi \circ \phi$ lift the identity maps from κ and κ' to themselves; by uniqueness we deduce that $\phi \circ \psi = \mathrm{id}_A$ and $\psi \circ \phi = \mathrm{id}_{A'}$. It follows that ϕ is an isomorphism of R-algebras.

We now prove that Φ is essentially surjective. Let $\kappa \in \mathrm{Field}_R^{\mathrm{sep}}$, and let $\mathcal{C}$ denote the category whose objects are étale R-algebras R' equipped with an R-algebra homomorphism $\epsilon_{R'} : R' \to \kappa$. The category $\mathcal{C}$ admits finite colimits, and is therefore filtered. We let $A = \varinjlim_{(R', \epsilon_{R'}) \in \mathcal{C}} R'$, so that A is an Ind-étale R-algebra. By construction, we have a canonical map $\epsilon : A \to \kappa$. This implies in particular that $A \neq 0$. We next claim that A is a local ring with maximal ideal $\mathfrak{m} = \ker(\epsilon)$. To prove this, choose an arbitrary element $x \notin \mathfrak{m}$; we will show that x is invertible in A. To prove this, choose a representation of x as the image of an element $x_0 \in R'$, for some étale R-algebra R' equipped with a map $\epsilon_{R'} : R' \to \kappa$. Then $\epsilon_{R'}(x_0) = \epsilon(x) \neq 0$, so that $\epsilon_{R'}$ factors through $R'[x_0^{-1}]$. It follows that the map $R' \to A$ also factors through $R'[x_0^{-1}]$, so that the image of x_0 in A is invertible.

We now claim that ϵ induces an isomorphism $A/\mathfrak{m} \to \kappa$. This map is injective by construction. To prove the surjectivity, choose an arbitrary element $y \in \kappa$; we will prove that y belongs to the image of ϵ. Since κ is a separable algebraic extension of some residue field of R, the element y satisfies a polynomial equation $f(y) = 0$ where the coefficients of f lie in R, and the discriminant Δ of f invertible in κ. Then $R' = R[Y, \Delta^{-1}]/(f)$ is an étale R-algebra equipped with a map $R' \to \kappa$ given by $Y \mapsto y$. It follows that y belongs to the image of ϵ as desired.

To complete the proof, it will suffice to show that the local ring A is Henselian. Let B be an étale A-algebra equipped with an A-algebra homomorphism $f_0 : B \to \kappa$; we wish to prove that f_0 can be lifted to a map $f : B \to A$. Using the structure theory of étale morphisms (Proposition B.1.1.1), we can write $B = A \otimes_{R'} B_0$, where R' is an étale R-algebra equipped with a map $\epsilon_{R'} : R' \to \kappa$, and B_0 is étale over R'. Then f_0 determines a map $B_0 \to \kappa$ extending $\epsilon_{R'}$. It follows that the canonical map $R' \to A$ factors through B_0; this factorization determines a map $B \to A$ having the desired properties. □

B.5 Nisnevich Excision

Let R be a commutative ring and let $\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}$ denote associated ∞-category of Nisnevich sheaves (see Definition B.4.2.2). In this section, we will prove a version of a result of Morel and Voevodsky (see [157]) which characterizes $\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}$ as the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S})$ spanned by those functors $\mathscr{F}$ which possess a certain excision property (Theorem B.5.0.3). We will follow a particularly convenient formulation given by Asok, Hoyois, and Wendt ([7]), which allows us to stay entirely within the world of *affine* schemes.

Definition B.5.0.1. Let R be a commutative ring and let $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ be a functor. We will say that $\mathscr{F}$ satisfies *Nisnevich excision* if the following conditions are satisfied:

(1) The space $\mathscr{F}(0)$ is contractible.

(2) Let $\phi : A \to A'$ be a morphism of étale R-algebras which induces an isomorphism $A/(a) \to A'/(a)$ for some element $a \in A$ (which, by abuse of notation, we will identify with its image in A'). Then the diagram of spaces

$$\begin{array}{ccc} \mathscr{F}(A) & \longrightarrow & \mathscr{F}(A') \\ \downarrow & & \downarrow \\ \mathscr{F}(A[a^{-1}]) & \longrightarrow & \mathscr{F}(A'[a^{-1}]) \end{array}$$

is a pullback square in $\mathcal{S}$.

Remark B.5.0.2. Let R be a commutative ring. The collection of all functors $\mathscr{F} : \mathrm{CAlg}_R^{\mathcal{S}} \to \mathcal{S}$ satisfying Nisnevich excision is closed under small limits and filtered colimits.

We can now state the main result of this section:

Theorem B.5.0.3 (Morel-Voevodsky, Asok-Hoyois-Wendt)**.** *Let R be a commutative ring and let $\mathscr{F} : \mathrm{CAlg}_R^{ét} \to \mathcal{S}$ be a functor. Then $\mathscr{F}$ is a sheaf with respect to the Nisnevich topology (Definition B.4.2.2) if and only if it satisfies Nisnevich excision.*

Corollary B.5.0.4. *Let R be a commutative ring. Then the full subcategory $\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}} \subseteq \mathrm{Fun}(\mathrm{CAlg}_R^{ét}, \mathcal{S})$ is closed under filtered colimits.*

Proof. Combine Theorem B.5.0.3 with Remark B.5.0.2. □

B.5.1 Non-Affine Excision

Let R be a commutative ring. We let $\mathrm{Sch}_R^{\text{ét}}$ denote the category of R-schemes which are quasi-compact, separated, and étale over R. For every object $X \in \mathrm{Sch}_R^{\text{ét}}$, we let $h_X : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ denote the functor given by $h_X(A) = \mathrm{Hom}_{\mathrm{Sch}_R^{\text{ét}}}(\operatorname{Spec} A, X)$. Note that h_X is a sheaf (of sets) with respect to the Nisnevich topology, and that h_X satisfies affine Nisnevich excision (Definition B.5.0.1). For any functor $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$, we will abuse notation by writing $\mathscr{F}(X)$ for the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S})}(h_X, \mathscr{F})$. Note that $\mathscr{F}(X)$ can be regarded as a contravariant functor of X, and that $\mathscr{F}(\operatorname{Spec} A) \simeq \mathscr{F}(A)$ when A is an étale R-algebra (the value of $\mathscr{F}$ on more general R-schemes is obtained from the original functor $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ by means of right Kan extension along the fully faithful embedding $\operatorname{Spec} : \mathrm{CAlg}_R^{\text{ét}} \to (\mathrm{Sch}_R^{\text{ét}})^{\mathrm{op}}$). The first step in the proof of Theorem B.5.0.3 is to show that this extension process behaves nicely with respect to excision:

Proposition B.5.1.1. *Let R be a commutative ring and let $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

(1) *The functor $\mathscr{F}$ satisfies Nisnevich excision (Definition B.5.0.1).*

(2) *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{j} & X \end{array}$$

in $\mathrm{Sch}_R^{\text{ét}}$, where p is affine, j is an open immersion, and p induces an isomorphism $(X - U) \times_X X' \to (X - U)$, where $(X - U)$ denotes the complement of U (endowed with the reduced scheme structure). Then the diagram of spaces

$$\begin{array}{ccc} \mathscr{F}(U') & \longleftarrow & \mathscr{F}(X') \\ \uparrow & & \uparrow \\ \mathscr{F}(U) & \longleftarrow & \mathscr{F}(X) \end{array}$$

is a pullback square.

Remark B.5.1.2. In the statement of condition (2) of Proposition B.5.1.1, the hypothesis that p is affine is not important.

The proof of Proposition B.5.1.1 will require some preliminaries.

Lemma B.5.1.3. *[7] Let A be a commutative ring containing a pair of ideals I and J, and suppose we are given an étale ring homomorphism $\phi : A \to A'$ which induces an* *isomorphism* $A/(I+J) \to A'/(I+J)A'$. *Then there exists a commutative diagram of étale ring homomorphisms*

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi'} \\ A' & \xrightarrow{\phi'} & B' \end{array}$$

where ψ *induces an isomorphism* $A/I \to B/IB$ *and* ϕ' *induces an isomorphism* $B/JB \to B'/JB'$.

Proof. Let $\overline{B}$ denote the Henselization of A with respect to the ideal I (see Proposition B.3.4.1 and Remark B.3.4.2) and let $\overline{B}'$ denote the Henselization of $\overline{B}$ with respect to the ideal $J\overline{B}$. Applying Proposition B.3.4.6, we can identify $\overline{B}$ with the Henselization of A with respect to the ideal $I+J$. Consequently, the assumption that the map $A/(I+J) \to A'/(I+J)A'$ is an isomorphism guarantees that there exists an A-algebra homomorphism $\rho : A' \to \overline{B}'$ (this follows from the construction of the Henselization given in the proof of Proposition B.3.4.1).

Since $\overline{B}'$ is the Henselization of $\overline{B}$ with respect to the ideal $J\overline{B}$, we can write $\overline{B}'$ as a filtered colimit $\varinjlim \overline{B}'_\alpha$, where each $\overline{B}'_\alpha$ is an étale $\overline{B}$-algebra for which the map $\overline{B}/J\overline{B} \to \overline{B}'_\alpha/J\overline{B}'_\alpha$ is an isomorphism. Because A' is finitely presented as an A-algebra, the map ρ factors through some map $\rho_\alpha : A' \to \overline{B}'_\alpha$.

Write $\overline{B}$ as the colimit of a diagram $\{\overline{B}_\beta\}_{\beta \in P}$ indexed by a direct partially ordered set P, where each $\overline{B}_\beta$ is an étale A-algebra for which the map $A/I \to \overline{B}_\beta/I\overline{B}_\beta$ is an isomorphism. Using the structure theory of étale morphisms (see Proposition B.1.1.3), we can write $\overline{B}'_\alpha = \overline{B} \otimes_{\overline{B}_\beta} \overline{B}'_{\alpha,\beta}$ for some $\beta \in P$, where $\overline{B}'_{\alpha,\beta}$ is étale over $\overline{B}_\beta$. Then we can write $\overline{B}'_\alpha$ as a filtered colimit $\varinjlim_{\beta' \geq \beta} \overline{B}_{\beta'} \otimes_{\overline{B}_\beta} \overline{B}'_{\alpha,\beta}$. Enlarging β if necessary, we may assume that ρ_α factors through an A-algebra map $\phi' : A' \to B'_{\alpha,\beta}$ and that the natural map $\overline{B}_\beta/J\overline{B}_\beta \to \overline{B}'_{\alpha,\beta}/J\overline{B}'_{\alpha,\beta}$ is an isomorphism. It follows that the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & \overline{B}_\beta \\ \downarrow & & \downarrow \\ A' & \xrightarrow{\phi'} & \overline{B}'_{\alpha,\beta} \end{array}$$

has the desired properties. □

Proof of Proposition B.5.1.1. Note that condition (1) of Proposition B.5.1.1 can be identified with the special case of (2) where $X = \operatorname{Spec} A$ and $U = \operatorname{Spec} A[a^{-1}]$ for some element $a \in A$. Consequently, the implication (2) ⇒ (1) is immediate from the definitions. To prove the converse, let us assume that $\mathscr{F}$ satisfies Nisnevich excision. Let $p : X' \to X$ be an affine morphism in $\mathrm{Sch}_R^{\text{ét}}$, and let U be a quasi-compact open subscheme of X. We will say that p

is *U-good* if the diagram of spaces

$$\begin{array}{ccc} \mathscr{F}(X) & \longrightarrow & \mathscr{F}(X') \\ \downarrow & & \downarrow \\ \mathscr{F}(U) & \longrightarrow & \mathscr{F}(U \times_X X') \end{array}$$

is a pullback square. To establish (2), we must show that $p : X' \to X$ is U-good whenever the induced map $X' \times_X (X - U) \to X - U$ is an isomorphism. The proof proceeds in several steps.

(a) Let $p : X' \to X$ be an affine morphism in $\mathrm{Sch}_R^{\text{ét}}$ and let $U \subseteq X$ be a quasi-compact open subscheme. Suppose that, for every map $\operatorname{Spec} A \to X$ in $\mathrm{Sch}_R^{\text{ét}}$, the induced map $p_A : X' \times_X \operatorname{Spec} A \to \operatorname{Spec} A$ is $(U \times_X \operatorname{Spec} A)$-good. Then p is U-good. To prove this, write the functor $h_X : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ as a colimit $\varinjlim \mathscr{G}_\alpha$, where each $\mathscr{G}_\alpha : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ is corepresented by an object of $\mathrm{CAlg}_R^{\text{ét}}$. Set $\mathcal{X} = \mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S})$. Unwinding the definitions, we see that p is U-good if and only if the diagram of spaces τ :

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{X}}(\varinjlim \mathscr{G}_\alpha, \mathscr{F}) & \longrightarrow & \mathrm{Map}_{\mathcal{X}}((\varinjlim \mathscr{G}_\alpha) \times_{h_X} h_U, \mathscr{F}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{X}}((\varinjlim \mathscr{G}_\alpha) \times_{h_X} h_{X'}, \mathscr{F}) & \longrightarrow & \mathrm{Map}_{\mathcal{X}}((\varinjlim \mathscr{G}_\alpha) \times_{h_X} h_{U'}, \mathscr{F}) \end{array}$$

is a pullback square, where $U' = U \times_X X'$. Because colimits in $\mathcal{X}$ are universal, we can write the preceding diagram as a limit of diagrams τ_α:

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{X}}(\mathscr{G}_\alpha, \mathscr{F}) & \longrightarrow & \mathrm{Map}_{\mathcal{X}}(\mathscr{G}_\alpha \times_{h_X} h_U, \mathscr{F}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{X}}(\mathscr{G}_\alpha \times_{h_X} h_{X'}, \mathscr{F}) & \longrightarrow & \mathrm{Map}_{\mathcal{X}}(\mathscr{G}_\alpha \times_{h_X} h_{U'}, \mathscr{F}). \end{array}$$

Choosing an object $A \in \mathrm{CAlg}_R^{\text{ét}}$ which corepresents the functor $\mathscr{G}_\alpha$, the desired result follows from our assumption that the map $p_A : X' \times_X \operatorname{Spec} A \to \operatorname{Spec} A$ is $(U \times_X \operatorname{Spec} A)$-good.

(b) Let $p : X' \to X$ be an affine morphism in $\mathrm{Sch}_R^{\text{ét}}$, let $a \in \Gamma(X; \mathcal{O}_X)$ be a global function on X, and let $U \subseteq X$ be the open subscheme on which the function a does not vanish. Suppose that p induces an isomorphism $X' \times_X (X - U) \to X - U$. Then p is U-good. To prove this, we can use (a) to reduce to the case where $X = \operatorname{Spec} A$ is affine, so that a can be regarded as an element of the ring A. Since p is affine, we can write

$X' = \operatorname{Spec} A'$. Unwinding the definitions, we wish to show that the diagram

$$\begin{array}{ccc} \mathscr{F}(A) & \longrightarrow & \mathscr{F}(A') \\ \downarrow & & \downarrow \\ \mathscr{F}(A[a^{-1}]) & \longrightarrow & \mathscr{F}(A'[a^{-1}]) \end{array}$$

is a pullback square, which follows from our assumption that $\mathscr{F}$ satisfies Nisnevich excision.

(c) Let $p : X' \to X$ be an affine morphism in $\mathrm{Sch}_R^{\text{ét}}$, let $U \subseteq V \subseteq X$ be quasi-compact open subschemes, set $U' = U \times_X X'$ and $V' = V \times_X X'$, and let $p_V : V' \to V$ be the projection map. Suppose that p_V is U-good. Then p is U-good if and only if p is V-good. To prove this, we note that there is a commutative diagram

$$\begin{array}{ccccc} \mathscr{F}(X) & \longrightarrow & \mathscr{F}(V) & \longrightarrow & \mathscr{F}(U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{F}(X') & \longrightarrow & \mathscr{F}(V') & \longrightarrow & \mathscr{F}(U'). \end{array}$$

Our assumption that p_V is U-good guarantees that the right square is a pullback, so that the left square is a pullback if and only if the outer square is a pullback.

(d) Let $\phi : A \to A'$ be a morphism of étale R-algebras. Let $U \subseteq \operatorname{Spec} A$ be the open subscheme complementary to the vanishing locus of a finitely generated ideal $I \subseteq A$. Suppose that there exists an element $a \in I$ for which ϕ induces an isomorphism $A/(a) \to A'/(a)$. It follows from (b) that the maps

$$\operatorname{Spec}(\phi) : \operatorname{Spec} A' \to \operatorname{Spec} A \qquad \operatorname{Spec}(\phi)_U : U \times_{\operatorname{Spec} A} \operatorname{Spec} A' \to U$$

are V-good, where $V = \operatorname{Spec} A[a^{-1}]$. Applying (c), we deduce that the map $\operatorname{Spec}(\phi) : \operatorname{Spec} A' \to \operatorname{Spec} A$ is U-good.

(e) Let $p : X'' \to X'$ and $q : X' \to X$ be affine morphisms in $\mathrm{Sch}_R^{\text{ét}}$ and let $U \subseteq X$ be a quasi-compact open subscheme. Suppose that p is $(U \times_X X')$-good. Then q is U-good if and only if $(q \circ p) : X'' \to X$ is U-good. To prove this, we observe that there is a commutative diagram

$$\begin{array}{ccc} \mathscr{F}(X) & \longrightarrow & \mathscr{F}(U) \\ \downarrow & & \downarrow \\ \mathscr{F}(X') & \longrightarrow & \mathscr{F}(U \times_X X') \\ \downarrow & & \downarrow \\ \mathscr{F}(X'') & \longrightarrow & \mathscr{F}(U \times_X X''), \end{array}$$

where the bottom square is a pullback by virtue of our assumption that p is $(U \times_X X')$-good. It follows that the upper square is a pullback if and only if the outer rectangle is a pullback.

(f) We now treat the general case. Let $p : X' \to X$ be an affine morphism in $\mathrm{Sch}_R^{\text{ét}}$ and let $U \subseteq X$ be a quasi-compact open subscheme for which p induces an isomorphism $X' \times_X (X - U) \to X - U$. We wish to prove that p is U-good. Using (a), we can reduce to the case where $X = \operatorname{Spec} A$ is affine, so that we can identify X' with $\operatorname{Spec} A'$ for some étale A-algebra A'. Since $U \subseteq \operatorname{Spec} A$ is quasi-compact, we can choose a finitely generated ideal $I \subseteq A$ such that U is the complement of the vanishing locus of I. Choose a set of generators $a_1, \ldots, a_n$ for the ideal I. Our assumption on p guarantees that the map $A/I \to A'/IA'$ is an isomorphism. Applying Lemma B.5.1.3 repeatedly, we deduce that there is a sequence of étale morphisms

$$A = A_0 \to A_1 \to \cdots \to A_n,$$

which induce isomorphisms $A_{i-1}/(a_i) \to A_i/(a_i)$ for $i \leqslant n$ and an A-algebra homomorphism $\rho : A' \to A_n$. It follows from (d) that each of the morphisms $\operatorname{Spec} A_{i+1} \to \operatorname{Spec} A_i$ is $(U \times_X \operatorname{Spec} A_i)$-good. Applying (e) repeatedly, we deduce that the composite map $q : \operatorname{Spec} A_n \to X$ is U-good. Set $A'_i = A' \otimes_A A_i$. Repeating the preceding argument, we deduce that the map $q' : \operatorname{Spec} A'_n \to X'$ is $(U \times_X X')$-good. We have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A'_n & \xrightarrow{q'} & X' \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \operatorname{Spec} A_n & \xrightarrow{q} & X \end{array}$$

is a pullback. The ring homomorphism ρ determines a map $s : \operatorname{Spec} A_n \to \operatorname{Spec} A'_n$ which is a section of p' Since p' is étale, the map s is the inclusion of a direct summand: in particular, it induces an isomorphism from $\operatorname{Spec} A_n$ to the open subscheme $V \subseteq \operatorname{Spec} A'_n$ complementary to the vanishing locus of an idempotent element $e \in A'_n$. Since p' and s are isomorphisms away from the inverse image of U, we have $e \subseteq IA'_n$. Applying (d), we deduce that the map s is $(U \times_X A'_n)$-good. Since the identity map $\mathrm{id} : \operatorname{Spec} A_n \to \operatorname{Spec} A_n$ is $(U \times_X \operatorname{Spec} A_n)$-good, it follows from (e) that the map p' is $(U \times_X \operatorname{Spec} A_n)$-good. Applying (e) again, we deduce that the composition $(q \circ p') : \operatorname{Spec} A'_n \to X$ is U-good. Using the equality $q \circ p' = p \circ q'$, another application of (e) shows that p is U-good, as desired.

□

B.5.2 Weak Connectivity

Our strategy for proving Theorem B.5.0.3 is to use formal arguments to reduce to the following special case:

Proposition B.5.2.1. *Let R be a Noetherian ring of finite Krull dimension, let $X \in \mathrm{Sch}_R^{\acute{e}t}$, and let $\theta : \mathscr{F} \to h_X$ be a natural transformation of functors $\mathscr{F}, h_X : \mathrm{CAlg}_R^{\acute{e}t} \to \mathcal{S}$. Assume that $\mathscr{F}$ satisfies Nisnevich excision and that θ exhibits h_X as the sheafification of $\mathscr{F}$ with respect to the Nisnevich topology. Then θ admits a section.*

Our proof of Proposition B.5.2.1 involves some auxiliary considerations. We begin by introducing a "perverse" connectivity condition on $\mathcal{S}$-valued sheaves.

Notation B.5.2.2. Let X be a Noetherian scheme. For every point $x \in X$, we let $\mathrm{ht}(x)$ denote the dimension of the local ring $\mathcal{O}_{X,x}$. Note that if $X = \operatorname{Spec} R$ is the spectrum of a commutative ring R, then $\mathrm{ht}(x)$ is the height of the prime ideal $\mathfrak{p} \subseteq R$ corresponding to x.

Definition B.5.2.3. Let R be a Noetherian ring, let $X \in \mathrm{Sch}_R^{\text{ét}}$, and suppose are given a map $\theta : \mathscr{F} \to h_X$ in $\mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S})$. For every morphism $f : U \to X$ in $\mathrm{Sch}_R^{\text{ét}}$, let $\mathscr{F}_f(U)$ denote the fiber product $\mathscr{F}(U) \times_{h_X(U)} \{f\}$. Let $n \geqslant -1$ be an integer. We will say that θ is *weakly n-connective* if if the following condition is satisfied:

- $(*)$ Let $f : U \to X$ be a morphism in $\mathrm{Sch}_R^{\text{ét}}$, let $x \in U$ be a point, and suppose we are given a map of spaces $S^k \to \mathscr{F}_f(U)$, where $-1 \leqslant k < n - \mathrm{ht}(x)$. Then there exists a map $g : U' \to U$ in $\mathrm{Sch}_R^{\text{ét}}$ and a point $x' \in U$ with $g(x') = x$ such that g induces an isomorphism of residue fields $\kappa(x) \to \kappa(x')$ and the composite map $S^k \to \mathscr{F}_f(U) \to \mathscr{F}_{f \circ g}(U')$ is nullhomotopic (when $k = -1$, this means that $\mathscr{F}_{f \circ g}(U')$ is nonempty).

Remark B.5.2.4. In the situation of Definition B.5.2.3, suppose that $\theta : \mathscr{F} \to h_X$ is weakly n-connective. For every morphism $U \to X$ in $\mathrm{Sch}_R^{\text{ét}}$, the pullback map $\mathscr{F} \times_{h_X} h_U \to h_U$ is also weakly n-connective.

We now summarize some of the features of Definition B.5.2.3 that will be used in the proof of Theorem B.5.0.3.

Lemma B.5.2.5. *Let R be a Noetherian ring, let $X \in \mathrm{Sch}_R^{\acute{e}t}$, let $\theta : \mathscr{F} \to h_X$ be a morphism in $\mathrm{Fun}(\mathrm{CAlg}_R^{\acute{e}t}, \mathcal{S})$, and let $n \geqslant 0$ be a nonnegative integer. Then θ is weakly n-connective if and only if the following conditions are satisfied:*

1. *For every map $f : U \to X$ in $\mathrm{Sch}_R^{\acute{e}t}$ and every point $x \in U$ of height $\leqslant n$, there exists a map $g : U' \to U$ in $\mathrm{Sch}_R^{\acute{e}t}$ and a point $x' \in U'$ such that $x = g(x')$, the map of residue fields $\kappa(x) \to \kappa(x')$ is an isomorphism, and $\mathscr{F}_{f \circ g}(U')$ is nonempty.*

(2) *If $n > 0$, then for every pair of maps $h_U \to \mathscr{F}$, $h_V \to \mathscr{F}$ in* $\mathrm{Fun}(\mathrm{CAlg}_R^{\acute{e}t}, \mathcal{S})$*, the induced map*

$$h_U \times_{\mathscr{F}} h_V \to h_U \times_{h_X} h_V \simeq h_{U \times_X V}$$

is $(n-1)$-connective.

Proof. Suppose first that θ is weakly n-connective. Condition (1) is obvious (take $k = -1$ in Definition B.5.2.3). To prove (2), assume that $n > 0$ and that we are given maps $\alpha : h_U \to \mathscr{F}$, $\beta : h_V \to \mathscr{F}$. We wish to show that the map $\theta' : h_U \times_{\mathscr{F}} h_V \to h_{U \times_X V}$ satisfies condition $(*)$ of Definition B.5.2.3. To this end, suppose we are given a map $f : W \to U \times_X V$ in $\mathrm{Sch}_R^{\text{ét}}$, a point $x \in W$, and a map of spaces $\eta : S^k \to \mathscr{F}'_f(W)$, where $\mathscr{F}' = h_U \times_{\mathscr{F}} h_V$ such that $-1 \leqslant k < n - 1 - \mathrm{ht}(x)$. Let f' be the induced map $W \to X$. Note that $\mathscr{F}'_f(W)$ can be identified with the space of paths joining the two points of $\mathscr{F}_{f'}(W)$ determined by α and β. Consequently, η determines a map $\eta_0 : S^{k+1} \to \mathscr{F}_{f'}(W)$. Using our assumption that θ is weakly n-connective, we can choose a map $g : W' \to W$ and a point $x' \in W'$ such that $g(x') = x$, $\kappa(x) \simeq \kappa(x')$, and the composite map

$$S^{k+1} \xrightarrow{\eta_0} \mathscr{F}_{f'}(W) \to \mathscr{F}_{f' \circ g}(W')$$

is nullhomotopic. Since $\mathscr{F}'_{f \circ g}(W')$ is homotopy equivalent to the space of paths in $\mathscr{F}_{f' \circ g}(W')$ joining the points determined by α and β, we deduce that the composite map $S^k \xrightarrow{\eta} \mathscr{F}'_f(W) \to \mathscr{F}'_{f \circ g}(W')$ is also nullhomotopic, as desired.

Now suppose that conditions (1) and (2) are satisfied. We must show that θ satisfies condition $(*)$ of Definition B.5.2.3. Choose a map $f : U \to X$ in $\mathrm{Sch}_R^{\text{ét}}$, a point $x \in U$, and a map of spaces $\eta : S^k \to \mathscr{F}_f(U)$, where $-1 \leqslant k < n - \mathrm{ht}(x)$. We wish to prove that there exists a map $g : U' \to U$ and a point $x' \in U'$ such that $g(x') = x$, $\kappa(x) \simeq \kappa(x')$, and the composite map $S^k \to \mathscr{F}_f(U) \to \mathscr{F}_{f \circ g}(U')$ is nullhomotopic. If $k = -1$, this follows from condition (1). Otherwise, we can write S^k as a homotopy pushout $* \coprod_{S^{k-1}} *$. Then η determines a pair of maps $\alpha, \beta : h_U \to \mathscr{F}$ such that $\theta \circ \alpha$ and $\theta \circ \beta$ are induced by f. Let $\mathscr{F}' = h_U \times_{\mathscr{F}} h_U$. The restriction of η to the equator of S^k gives a map $S^{k-1} \to \mathscr{F}'_\delta(U)$, where $\delta : U \to U \times_X U$ is the diagonal map. Using condition (2), we deduce the existence of a map $g : U' \to U$ and a point $x' \in U'$ such that $g(x') = x$, $\kappa(x) \simeq \kappa(x')$, and the induced map $S^{k-1} \to \mathscr{F}'_\delta(U) \to \mathscr{F}'_{\delta g}(U')$ is nullhomotopic. Unwinding the definitions, we deduce that $S^k \to \mathscr{F}_f(U) \to \mathscr{F}_{f \circ g}(U')$ is nullhomotopic, as desired. □

Lemma B.5.2.6. *Let R be a Noetherian ring, let $X \in \mathrm{Sch}_R^{\acute{e}t}$, and let $\theta : \mathscr{F} \to h_X$ be a morphism in* $\mathrm{Fun}(\mathrm{CAlg}_R^{\acute{e}t}, \mathcal{S})$*. Suppose that θ exhibits h_X as the sheafification of $\mathscr{F}$ with respect to the Nisnevich topology. Then θ is weakly n-connective for each $n \geqslant 0$.*

Proof. The proof proceeds by induction on n. We will show that θ satisfies the criteria of Lemma B.5.2.5. Condition (1) follows immediately from our assumption that θ is an effective epimorphism after Nisnevich sheafificiation. To verify (2), we may assume that $n > 0$. Choose maps $h_U \to \mathscr{F}$ and $h_V \to \mathscr{F}$. Since sheafification is left exact, the induced map $\theta' : h_U \times_{\mathscr{F}} h_V \to h_{U \times_X V}$ exhibits $h_{U \times_X V}$ as the sheafification of $h_U \times_{\mathscr{F}} h_V$ with respect to the Nisnevich topology. It follows from the inductive hypothesis that θ' is weakly $(n-1)$-connective, as desired. □

Lemma B.5.2.7. *Let R be a Noetherian ring, let $X \in \mathrm{Sch}_R^{\acute{e}t}$, and let $\theta : \mathscr{F} \to h_X$ be a weakly n-connective morphism in $\mathcal{C} = \mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S})$. Assume that $\mathscr{F}$ satisfies Nisnevich excision. Then there exists a finite set of points $x_1, \ldots, x_m \in X$ of height $> n$ and a commutative diagram*

$$\begin{array}{ccc} & \mathscr{F} & \\ \nearrow & & \searrow^{\theta} \\ h_U & \longrightarrow & h_X \end{array}$$

where $U = X - \bigcup_{1 \leqslant i \leqslant m} \overline{\{x_i\}}$.

Proof. The proof proceeds by induction on n. When $n = -1$, we take $x_1, \ldots, x_m$ to be the set of generic points of X, so that $U = \emptyset$ and the existence of the desired map $h_U \to \mathscr{F}$ follows from our assumption that $\mathscr{F}(0)$ is contractible. Assume now that $n \geqslant 0$ and that the result is known for the integer $n-1$, so that we can choose a commutative diagram

$$\begin{array}{ccc} & \mathscr{F} & \\ {}^{\phi}\nearrow & & \searrow^{\theta} \\ h_U & \longrightarrow & h_X \end{array}$$

where $U = X - \bigcup_{1 \leqslant i \leqslant m} \overline{\{x_i\}}$ where the points x_i have height $\geqslant n$. Reordering the points x_i if necessary, we may assume that $x_1, x_2, \ldots, x_k$ have height n while $x_{k+1}, \ldots, x_m$ have height $> n$. We assume that this data has been chosen so that k is as small as possible. We will complete the induction by showing that $k = 0$. Otherwise, the point x_1 has height n. Since θ is weakly n-connective, there exists a map $f : X' \to X$ and a point $x' \in X'$ such that $f(x') = x_1$, $\kappa(x) \simeq \kappa(x')$, and $\mathscr{F}_f(X')$ is nonempty. We may therefore choose a commutative diagram

$$\begin{array}{ccc} & \mathscr{F} & \\ {}^{\psi}\nearrow & & \searrow^{\theta} \\ h_{X'} & \xrightarrow{f} & h_X \end{array}$$

Replacing X' by an open subset if necessary, we may suppose that f induces an isomorphism from $f^{-1}\overline{\{x_1\}}$ to an open subset $V \subseteq \overline{\{x_1\}}$ (here we endow these closed subsets with the reduced scheme structure). Set $U' = U \times_X X'$. Our maps $\phi : h_U \to \mathscr{F}$ and $\psi : h_{X'} \to \mathscr{F}$ determine a map $\theta' : h_U \times_{\mathscr{F}} h_{X'} \to h_{U'}$. Using Lemma B.5.2.5, we see that θ' is weakly

$(n-1)$-connective. Applying the inductive hypothesis, we deduce that there exists a finite collection of points $y_1, \ldots, y_{m'} \in U'$ of height $\geqslant n$ and a commutative diagram

$$\begin{array}{ccccc} & & h_U \times_{\mathscr{F}} h_{X'} & & \\ & \nearrow & & \searrow & \\ h_W & & \longrightarrow & & h_{U'}, \end{array}$$

where W is the open subscheme $U' - \bigcup \overline{\{y_j\}}$ of U'. Replacing X' by the open subscheme $X' - \bigcup \overline{\{y_j\}}$ (which contains x', since x' is a point of height n and therefore cannot lie in the closure of any other point of height n), we may assume that $W = U'$, so that the maps $h_U \xrightarrow{\phi} \mathscr{F} \xleftarrow{\psi} h_{X'}$ induce homotopic maps from $h_{U'}$ into $\mathscr{F}$. Shrinking X' further if necessary, we may assume that X' is affine (so that the map $X' \to X$ is affine). Regard $U \cup V$ as an open subscheme of X; since $\mathscr{F}$ satisfies Nisnevich excision, the diagram

$$\begin{array}{ccc} \mathscr{F}(U \cup V) & \longrightarrow & \mathscr{F}(X') \\ \downarrow & & \downarrow \\ \mathscr{F}(U) & \longrightarrow & \mathscr{F}(U') \end{array}$$

is a pullback square in $\mathcal{S}$ (Proposition B.5.1.1). It follows that ϕ extends to a map ϕ' fitting into a commutative diagram

$$\begin{array}{ccccc} & & \mathscr{F} & & \\ & \overset{\phi'}{\nearrow} & & \overset{\theta}{\searrow} & \\ h_{U \cup V} & & \longrightarrow & & h_X, \end{array}$$

contradicting the minimality of k. □

Proof of Proposition B.5.2.1. Combine Lemmas B.5.2.6 and B.5.2.7. □

Remark B.5.2.8. Using Lemma B.5.2.7, one can show that if R is a Noetherian ring of Krull dimension $\leqslant n$, then the ∞-topos $\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}$ has homotopy dimension $\leqslant n$. We will prove a more general version of this assertion in §3.7 (see Theorem 3.7.7.1).

B.5.3 The Proof of Theorem B.5.0.3

We now explain how to deduce Theorem B.5.0.3 from Proposition B.5.2.1. Our first step is to show that it suffices to treat the case where the commutative ring R is finitely generated over $\mathbf{Z}$ (and therefore a Noetherian ring of finite Krull dimension). This requires a brief digression about functoriality.

Notation B.5.3.1. Let $f : R' \to R$ be a homomorphism of commutative rings. Then f induces determines a pushforward functor $f_* : \mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S}) \to \mathrm{Fun}(\mathrm{CAlg}_{R'}^{\text{ét}}, \mathcal{S})$, given by the formula $(f_* \mathscr{F})(A) = \mathscr{F}(R' \otimes_R A)$.

Lemma B.5.3.2. *Let R be a commutative ring and let $\mathscr{F} : \mathrm{CAlg}_R^{\acute{e}t} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

(1) *The functor $\mathscr{F}$ is a sheaf with respect to the Nisnevich topology.*

(2) *For every homomorphism of commutative rings $f : R' \to R$, the direct image $f_* \mathscr{F} \in \mathrm{Fun}(\mathrm{CAlg}_{R'}^{\acute{e}t}, \mathcal{S})$ is a sheaf for the Nisnevich topology.*

(3) *For every homomorphism of commutative rings $f : R' \to R$ where R' is finitely generated the direct image $f_* \mathscr{F} \in \mathrm{Fun}(\mathrm{CAlg}_{R'}^{\acute{e}t}, \mathcal{S})$ is a sheaf for the Nisnevich topology.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious. The implication (3) ⇒ (1) follows from the criterion of Proposition A.3.3.1, since for every étale ring homomorphism $R \to A$ which exhibits A as a Nisnevich covering of R, we can write $A = R \otimes_{R_0} A_0$ for some finitely generated subring $R_0 \subseteq R$, where A_0 is an étale R_0-algebra which is a Nisnevich covering of R_0. □

Lemma B.5.3.3. *Let R be a Noetherian ring of finite Krull dimension, let $X \in \mathrm{Sch}_R^{\acute{e}t}$, and let $\theta : \mathscr{F} \to h_X$ be a morphism in $\mathcal{C} = \mathrm{Fun}(\mathrm{CAlg}_R^{\acute{e}t}), \mathcal{S})$. Assume that $\mathscr{F}$ satisfies Nisnevich excision and that θ exhibits h_X as the Nisnevich sheafification of $\mathscr{F}$. Then the mapping space $\mathrm{Map}_{\mathcal{C}_{/h_X}}(h_X, \mathscr{F})$ is contractible.*

Proof. We will prove by induction on k that the mapping space $\mathrm{Map}_{\mathcal{C}_{/h_X}}(h_X, \mathscr{F})$ is k-connective. If $k > 0$, then it suffices to show that for every pair of maps $f, g : h_X \to \mathscr{F}$ in $\mathcal{C}_{/h_X}$, the mapping space $\mathrm{Map}_{\mathcal{C}_{/h_X}}(h_X, h_X \times_{\mathscr{F}} h_X)$ is $(k-1)$-connective, which follows from our inductive hypothesis (and the left exactness of sheafification). It will therefore suffice to treat the case $k = 0$, which follows from Proposition B.5.2.1. □

Proof of Theorem B.5.0.3. Suppose first that $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}}$ is a sheaf with respect to the Nisnevich topology: we claim that $\mathscr{F}$ satisfies Nisnevich excision. Using Example B.4.1.4, we see that the empty sieve is a covering of the zero ring $0 \in \mathrm{CAlg}_R^{\text{ét}}$, so that the space $\mathscr{F}(0)$ is contractible. To verify the second condition of Definition B.5.0.1, let us suppose we are given a morphism $\phi : A \to A'$ of étale R-algebras and an element $a \in A$ for which ϕ induces an isomorphism $A/(a) \to A'/(a)$. We wish to show that the diagram

$$\begin{array}{ccc} \mathscr{F}(A) & \longrightarrow & \mathscr{F}(A') \\ \downarrow & & \downarrow \\ \mathscr{F}(A[a^{-1}]) & \longrightarrow & \mathscr{F}(A'[a^{-1}]) \end{array}$$

is a pullback square.

Let $\mathcal{C} \subseteq \mathrm{CAlg}^{\text{ét}}_A$ be the sieve generated by the morphisms $A \to A'$ and $A \to A[a^{-1}]$. By assumption, this sieve is covering for the Nisnevich topology, so the assumption that $\mathscr{F}$ is a sheaf for the Nisnevich topology guarantees that the natural map $\mathscr{F}(A) \to \varprojlim \mathscr{F}|_{\mathcal{C}}$ is a homotopy equivalence.

Let $A^\bullet$ denote the augmented cosimplicial object of $\mathrm{CAlg}^{\text{ét}}_R$ given by the Čech nerve of ϕ, so that $A^{-1} \simeq A$ and $A^0 \simeq A'$. Localizing with respect to the element $a \in A = A^{-1}$ gives a map of augmented cosimplicial R-algebras $A^\bullet \to A^\bullet[a^{-1}]$, which we identify with a functor $\rho : \mathbf{\Delta}_+ \times \Delta^1 \to \mathrm{CAlg}^{\text{ét}}_R$. Let $\mathcal{J} \subseteq \mathbf{\Delta}_+ \times \Delta^1$ be the subcategory obtained by omitting the initial object, so that we can identify ρ with a functor $\rho' : \mathcal{J} \to \mathrm{CAlg}^{\text{ét}}_A$. Note that ρ' factors through $\mathcal{C}$. We claim that the map $\phi' : \mathcal{J} \to \mathcal{C}$ is right cofinal. We will prove this using the criterion of Theorem HTT.4.1.3.1 : it suffices to show that if B is an étale A-algebra which belongs to the sieve $\mathcal{C}$, then the ∞-category $\mathcal{J} \times_{\mathcal{C}} \mathcal{C}_{/B}$ is weakly contractible. Note that the projection map $\mathcal{J} \times_{\mathcal{C}} \mathcal{C}_{/B} \to \mathcal{J}$ is a right fibration which is classified by the functor $\psi : \mathcal{J}^{\mathrm{op}} \to \mathcal{S}$ given by the formula $\psi([n], i) = \mathrm{Map}_{\mathrm{CAlg}^{\text{ét}}_A}(\psi([n], i), B)$. By virtue of Proposition HTT.3.3.4.5 , the statement that $\mathcal{J} \times_{\mathcal{C}} \mathcal{C}_{/B}$ is weakly contractible is equivalent to the statement that the colimit $\varinjlim \psi \in \mathcal{S}$ is contractible. There are two cases to consider:

- (i) Suppose that the image of $a \in A$ is an invertible element of B. Then the canonical map $\mathrm{Map}_{\mathrm{CAlg}^{\text{ét}}_A}(A^n[a^{-1}], B) \to \mathrm{Map}_{\mathrm{CAlg}^{\text{ét}}_A}(A^n, B)$ is invertible for $n \geqslant 0$, so the functor ψ is a left Kan extension of its restriction to $(\mathbf{\Delta}_+ \times \{1\})^{\mathrm{op}} \subseteq \mathcal{J}^{\mathrm{op}}$. This subcategory has a final object $([-1], 1)$, so we can identify $\varinjlim \psi$ with the space $\psi([-1], 1) = \mathrm{Map}_{\mathrm{CAlg}^{\text{ét}}_A}(A[a^{-1}], B)$, which is contractible.
- (ii) Suppose that the image of $(*)$ in (b) is not invertible. Then the mapping space $\mathrm{Map}_{\mathrm{CAlg}^{\text{ét}}_A}(A^n[a^{-1}], B)$ is empty for each $n \geqslant 0$, so the functor ψ is a left Kan extension of its restriction to the subcategory $(\mathbf{\Delta} \times \{0\})^{\mathrm{op}} \subseteq \mathcal{J}^{\mathrm{op}}$. We can therefore identify $\varinjlim(\psi)$ with the geometric realization of the simplicial space $X_\bullet = \mathrm{Map}_{\mathrm{CAlg}^{\text{ét}}_A}(A^\bullet, B)$ Note that $X_\bullet$ is the Čech nerve of the map $\mathrm{Map}_{\mathrm{CAlg}^{\text{ét}}_A}(A', B) \to *$, and is therefore contractible (since the assumptions that $B \in \mathcal{C}$ and $\mathrm{Map}_{\mathrm{CAlg}^{\text{ét}}_A}(A[a^{-1}], B) = \varnothing$ guarantee that $\mathrm{Map}_{\mathrm{CAlg}^{\text{ét}}_A}(A', B)$ is nonempty).

Consider the maps $\mathscr{F}(A) \to \varprojlim_{J \in \mathcal{J}} \mathscr{F}(\rho(J)) \to \varprojlim_{B \in \mathcal{C}} \mathscr{F}(B)$. Since ρ is right cofinal, the second map is a homotopy equivalence. Since $\mathscr{F}$ is a Nisnevich sheaf, the composite map is a homotopy equivalence. It follows that $\mathscr{F} \circ \rho$ is a limit diagram in $\mathcal{S}$. We have a canonical isomorphism $\mathcal{J} \simeq (\mathbf{\Delta}^{\mathrm{op}} \times \Delta^1) \coprod_{\mathbf{\Delta}^{\mathrm{op}} \times \{0\}} (\mathbf{\Delta}^{\mathrm{op}}_+ \times \{0\})$. Applying the results of §HTT.4.2.3 to this decomposition, we obtain an equivalence

$$\varprojlim_{J \in \mathcal{J}} \mathscr{F}(\rho(J)) \simeq \mathscr{F}(A[a^{-1}]) \times_{\varprojlim \mathscr{F}(A^\bullet[a^{-1}])} \varprojlim \mathscr{F}(A^\bullet).$$

Consider the diagram

$$\begin{array}{ccccc} \mathscr{F}(A) & \longrightarrow & \varprojlim \mathscr{F}(A^\bullet) & \longrightarrow & \mathscr{F}(A') \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{F}(A[a^{-1}]) & \longrightarrow & \varprojlim \mathscr{F}(A^\bullet[a^{-1}]) & \longrightarrow & \mathscr{F}(A'[a^{-1}]). \end{array}$$

The above argument shows that the square on the left is a pullback. We wish to show that the outer square is a pullback. It will therefore suffice to show that the square on the right is a pullback. We can identify this square with a limit of diagrams σ_n :

$$\begin{array}{ccc} \mathscr{F}(A^n) & \longrightarrow & \mathscr{F}(A') \\ \downarrow & & \downarrow \\ \mathscr{F}(A^n[a^{-1}]) & \longrightarrow & \mathscr{F}(A'[a^{-1}]) \end{array}$$

induced by the multiplication $m : A^n \simeq A' \otimes_A \cdots \otimes_A A' \to A$. Since A' is étale over A, the map m exhibits A' as a direct factor of A^n: that is, we have a product decomposition $A^n \simeq A' \times B$ for some auxiliary factor B. Because $\mathscr{F}$ commutes with finite products (since it is Nisnevich sheaf; see Proposition A.3.3.1), we can identify σ_n with the diagram

$$\begin{array}{ccc} \mathscr{F}(A') \times \mathscr{F}(B) & \longrightarrow & \mathscr{F}(A') \\ \downarrow & & \downarrow \\ \mathscr{F}(A'[a^{-1}]) \times \mathscr{F}(B[a^{-1}]) & \longrightarrow & \mathscr{F}(A'[a^{-1}]) \end{array}$$

To show that this diagram is a pullback square, it will suffice to show that the localization map $B \to B[a^{-1}]$ is an isomorphism. This follows from our assumption that ϕ induces an isomorphism $A/(a) \to A'/(a)$ (which guarantees that m has the same property, so that $B/(a) \simeq 0$). This completes the proof that every Nisnevich sheaf satisfies Nisnevich excision.

We now prove the converse. Suppose that $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}}$ satisfies Nisnevich excision. We will prove that $\mathscr{F}$ is a sheaf for the Nisneivch topology. Using Lemma B.5.3.2, we can reduce to the case where the commutative ring R is finitely generated; in particular, we may assume that R is a Noetherian ring of finite Krull dimension. Let $\mathscr{F}'$ be the sheafification of $\mathscr{F}$ with respect to the Nisnevich topology. The first part of the proof shows that $\mathscr{F}'$ satisfies Nisnevich excision. We will show that for each $X \in \mathrm{Sch}_R^{\text{ét}}$, the map $\mathscr{F}(X) \to \mathscr{F}'(X)$ is a homotopy equivalence. Fix a point $\eta \in \mathscr{F}'(X)$, corresponding to a map $h_X \to \mathscr{F}'$. Then the homotopy fiber of $\mathscr{F}(X) \to \mathscr{F}'(X)$ over the point η can be identified with the space of sections of the induced map $\theta : \mathscr{F} \times_{\mathscr{F}'} h_X \to h_X$. Note that $\mathscr{F} \times_{\mathscr{F}'} h_X$ satisfies Nisnevich excision (Remark B.5.0.2) and that θ exhibits h_X as the Nisnevich sheafification of $\mathscr{F} \times_{\mathscr{F}'} h_X$ (since sheafification is left exact). The desired result now follows from Lemma B.5.3.3. $\square$

B.6 Topologies on Ring Spectra

In §A.3, we described a general paradigm for producing examples of Grothendieck topologies on ∞-categories. In this section, we will apply this paradigm to describe several different topologies on (the opposite of) the ∞-category CAlg of $\mathbb{E}_\infty$-rings.

B.6.1 The Flat Topology

Let $\phi : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings. Recall that ϕ is said to be *flat* if the following conditions are satisfied:

(i) The underlying ring homomorphism $\pi_0 A \to \pi_0 B$ exhibits $\pi_0 B$ as a flat A-module.

(ii) The morphism ϕ induces an isomorphism of graded rings

$$(\pi_0 B) \otimes_{\pi_0 A} (\pi_* A) \to \pi_* B.$$

We refer the reader to §HA.7.2.2 for a more extensive discussion of flat morphisms of $\mathbb{E}_\infty$-rings (as well as several reformulations of the preceding definition).

Definition B.6.1.1. Let $\phi : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings. We will say that f is *faithfully flat* if it satisfies the following conditions:

(i') The underlying map of commutative rings $\pi_0 A \to \pi_0 B$ is faithfully flat, in the sense of classical commutative algebra.

(ii) The morphism ϕ induces an isomorphism of graded rings

$$(\pi_0 B) \otimes_{\pi_0 A} (\pi_* A) \to \pi_* B.$$

Remark B.6.1.2. Let $f : A \to B$ be a faithfully flat morphism of $\mathbb{E}_\infty$-rings. A morphism $M \to N$ of A-modules is an equivalence if and only if the induced map $M \otimes_A B \to N \otimes_A B$ is an equivalence. This follows immediately from Corollary HA.7.2.1.22 .

Proposition B.6.1.3. *Let R be an $\mathbb{E}_\infty$-ring. Then there exists a Grothendieck topology on the ∞-category $\mathrm{CAlg}_R^{\mathrm{op}}$ which can be characterized as follows: if A is an $\mathbb{E}_\infty$-algebra over R, then a sieve $\mathcal{C} \subseteq (\mathrm{CAlg}_R^{\mathrm{op}})_{/A} \simeq \mathrm{CAlg}_A^{\mathrm{op}}$ is a covering if and only if it contains a finite collection of morphisms $\{A \to A_i\}_{1 \leqslant i \leqslant n}$ for which the induced map $A \to \prod_{1 \leqslant i \leqslant n} A_i$ is faithfully flat.*

Remark B.6.1.4. We will refer to the Grothendieck topology of Proposition B.6.1.3 as the *fpqc topology* on $\mathrm{CAlg}_R^{\mathrm{op}}$. We will often abuse terminology by referring to the fpqc topology on the ∞-category CAlg_R, rather than its opposite category.

Warning B.6.1.5. Let R be an $\mathbb{E}_\infty$-ring. Then the ∞-category $\mathrm{CAlg}_R^{\mathrm{op}}$ is not small. Consequently, though it makes sense to consider the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(\mathrm{CAlg}_R^{\mathrm{op}}) \subseteq \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$ of $\mathcal{S}$-valued sheaves on $\mathrm{CAlg}_R^{\mathrm{op}}$, it is not clear that $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(\mathrm{CAlg}_R^{\mathrm{op}})$ is a localization of $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$. In concrete terms, the danger is that the process of sheafification with respect to the fpqc topology might produce spaces which are not essentially small.

Proof of Proposition B.6.1.3. Let S be the collection of faithfully flat morphisms in $\mathrm{CAlg}_R^{\mathrm{op}}$. We will show that S satisfies each of the hypotheses of Proposition A.3.2.1:

(a) The collection of faithfully flat morphisms in CAlg_R contains all equivalences and is stable under composition. The first assertion is obvious. To prove the second, consider a pair of faithfully flat morphisms $A \xrightarrow{f} B \xrightarrow{g} C$; we wish to prove that $g \circ f$ is faithfully flat. The underlying map $\pi_0 A \to \pi_0 C$ is a composition of faithfully flat morphisms of commutative rings, and therefore faithfully flat. The map $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 C, \pi_i A) \to \pi_i C$ factors as a composition

$$\begin{aligned} \mathrm{Tor}_0^{\pi_0 A}(\pi_0 C, \pi_i A) &\simeq \mathrm{Tor}_0^{\pi_0 B}(\pi_0 C, \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_i A)) \\ &\xrightarrow{\alpha} \mathrm{Tor}_0^{\pi_0 B}(\pi_0 C, \pi_i B) \\ &\xrightarrow{\beta} \pi_i C. \end{aligned}$$

The map α is an isomorphism because f is faithfully flat, and the map β is an isomorphism because g is faithfully flat.

(b) It is clear that the ∞-category $\mathrm{CAlg}_R^{\mathrm{op}}$ admits pullbacks (the ∞-category CAlg_R of $\mathbb{E}_\infty$-rings is presentable and therefore admits all small limits and colimits). It therefore suffices to show that if we are given a diagram $A' \xleftarrow{g} A \xrightarrow{f} B$, where f is faithfully flat, then the induced map $A' \to B \otimes_A A'$ is faithfully flat. Since B is flat over A, Proposition HA.7.2.2.13 guarantees that the canonical maps $\gamma_i : \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_i A') \to \pi_i(B \otimes_A A')$ is an isomorphism. Taking $i = 0$, we deduce that $\pi_0(B \otimes_A A')$ is a pushout of $\pi_0 B$ and $\pi_0 A'$ over $\pi_0 A$, and therefore faithfully flat over $\pi_0 A'$. Moreover, the canonical map

$$\mathrm{Tor}_0^{\pi_0 A'}(\pi_0(B \otimes_A A'), \pi_i A') \to \pi_i(B \otimes_A A')$$

factors as a composition

$$\begin{aligned} \mathrm{Tor}_0^{\pi_0 A'}(\pi_0(B \otimes_A A'), \pi_i A') &\overset{\gamma_0^{-1}}{\simeq} \mathrm{Tor}_0^{\pi_0 A'}(\mathrm{Tor}_0^{\pi_0 A}(\pi_0 A', \pi_0 B), \pi_i A') \\ &\simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_i A') \\ &\xrightarrow{\gamma_i} \pi_i(B \otimes_A A'). \end{aligned}$$

and is therefore an isomorphism.

(c) It is clear that the category $\mathrm{CAlg}_R^{\mathrm{op}}$ admits finite coproducts, which are given by products of the corresponding $\mathbb{E}_\infty$-algebras over R. We must show that if we are given a finite collection of faithfully flat morphisms $A_i \to B_i$ and we define $A = \prod_i A_i$ and $B = \prod_i B_i$, then the induced map $A \to B$ is also faithfully flat. We have $\pi_0 A = \prod_i \pi_0 A_i$ and $\pi_0 B \simeq \prod_i \pi_0 B_i$. Since a product of faithfully flat morphisms of commutative rings is faithfully flat, we deduce that the map $\pi_0 A \to \pi_0 B$ is faithfully flat. For the other homotopy groups, we have

$$\begin{aligned}
\pi_n B &\simeq \prod_i (\pi_n B_i) \\
&\simeq \prod_i \mathrm{Tor}_0^{\pi_0 A_i}(\pi_0 B_i, \pi_n A_i) \\
&\simeq \mathrm{Tor}_0^{\prod_i \pi_0 A_i}(\prod_i \pi_0 B_i, \prod_i \pi_n A_i) \\
&\simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_n A)
\end{aligned}$$

as required.

(d) Given a finite collection of morphisms $A \to A_i$ and a morphism $A \to B$ in CAlg_R, we must show that the canonical map

$$(\prod_i A_i) \otimes_A B \to \prod_i (A_i \otimes_A B)$$

is an equivalence of $\mathbb{E}_\infty$-rings. We will show that this map is an equivalence in the ∞-category of B-modules. For this, it suffices to observe that the functor $F : \mathrm{Mod}_A \to \mathrm{Mod}_B$ given by $M \mapsto M \otimes_A B$ preserves finite limits. The functor F evidently preserves small colimits, and therefore also finite limits because the ∞-categories Mod_A and Mod_B are stable (Proposition HA.1.1.4.1).

□

Remark B.6.1.6. Let R be an $\mathbb{E}_\infty$-ring. The ∞-category $\mathrm{CAlg}_R^{\mathrm{op}}$ satisfies the additional hypothesis (e) (disjointness of coproducts) appearing in Proposition A.3.3.1. In other words, for any pair of $\mathbb{E}_\infty$-rings A and B, the relative tensor product $A \otimes_{A \times B} B$ is vanishes. To prove this, we observe that the identity element of $\pi_0(A \times B) \simeq \pi_0 A \times \pi_0 B$ can be written as a sum $e + e'$, where $e = (1, 0)$ and $e' = (0, 1)$. The image of e is trivial in $\pi_0 B$, and the image of e' is trivial in $\pi_0 A$. It follows that e and e' both have trivial image in the commutative ring $R = \pi_0(A \otimes_{A \times B} B)$, so that $1 = 0$ in R. Since every homotopy group of $A \otimes_{A \times B} B$ is a module over R, each of these groups is trivial.

Variant B.6.1.7. Let R be a *connective* $\mathbb{E}_\infty$-ring. Then Proposition B.6.1.3 has an obvious analogue for the ∞-category $\mathrm{CAlg}_R^{\mathrm{cn}}$ of *connective* $\mathbb{E}_\infty$-algebras over R, which can be proven in exactly the same way. We will refer to the resulting Grothendieck topology on $(\mathrm{CAlg}_R^{\mathrm{cn}})^{\mathrm{op}}$ also as the *fpqc topology*.

B.6.2 The (Small) Étale and Nisnevich Sites

Let R be an $\mathbb{E}_\infty$-ring. We let $\mathrm{CAlg}_R^{\text{ét}}$ denote the full subcategory of CAlg_R spanned by the étale R-algebras. We have the following analogue of Proposition B.6.1.3:

Proposition B.6.2.1. *Let R be an $\mathbb{E}_\infty$-ring. Then there exists a Grothendieck topology on the ∞-category $(\mathrm{CAlg}_R^{\acute{e}t})^{\mathrm{op}}$ which can be characterized as follows: if A is an $\mathbb{E}_\infty$-algebra over R, then a sieve $\mathcal{C} \subseteq (\mathrm{CAlg}_R^{\acute{e}t})^{\mathrm{op}}_{/A} \simeq (\mathrm{CAlg}_A^{\acute{e}t})^{\mathrm{op}}$ is a covering if and only if it contains a finite collection of morphisms $\{A \to A_i\}_{1 \leqslant i \leqslant n}$ for which the induced map $A \to \prod_{1 \leqslant i \leqslant n} A_i$ is faithfully flat.*

Proof. It will suffice to show that the collection of faithfully flat étale morphisms in $(\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}$ satisfies hypotheses (a) through (d) of Proposition A.3.2.1, which follows immediately from the proof of Proposition B.6.1.3 (together with the observation that the forgetful functor $\mathrm{CAlg}_R^{\text{ét}} \to \mathrm{CAlg}$ preserves pushouts and finite products). □

Definition B.6.2.2. Let R be an $\mathbb{E}_\infty$-ring. We will refer to the Grothendieck topology described in Proposition B.6.2.1 as the *étale topology* on $(\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}$. We let $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S})$ spanned by those functors which are sheaves for the étale topology. We will refer to $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ as the *∞-topos of étale sheaves over R.*

Remark B.6.2.3. Let R be an $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be an ∞-category, and let $\mathscr{F} : \mathrm{CAlg}_R^{\mathrm{op}} \to \mathcal{C}$ be a functor. Suppose that $\mathscr{F}$ is a $\mathcal{C}$-valued sheaf for the fpqc topology of Proposition B.6.1.3. Then, for every $\mathbb{E}_\infty$-algebra A over R, the composite map

$$\mathscr{F}_A : \mathrm{CAlg}_A^{\text{ét}} \to \mathrm{CAlg}_R \xrightarrow{\mathscr{F}} \mathcal{C}$$

is a $\mathcal{C}$-valued sheaf for the étale topology of Proposition B.6.2.1. Moreover, if $\mathscr{F}$ is hypercomplete, then each $\mathscr{F}_R$ is hypercomplete. This follows immediately from the criteria given in Propositions A.3.3.1 and A.5.7.2.

Variant B.6.2.4. Let R be an $\mathbb{E}_\infty$-ring. Then there exists a Grothendieck topology on the ∞-category $(\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}$ which can be characterized as follows: if A is an $\mathbb{E}_\infty$-algebra over R, then a sieve $\mathcal{C} \subseteq (\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}_{/A} \simeq (\mathrm{CAlg}_A^{\text{ét}})^{\mathrm{op}}$ is a covering if and only if it contains a finite collection of morphisms $\{A \to A_i\}_{1 \leqslant i \leqslant n}$ for which the the underlying ring homomorphisms $\{\pi_0 A \to \pi_0 A_i\}$ form a Nisnevich covering (Definition B.4.1.1).

Proof. By virtue of Theorem HA.7.5.0.6 we can assume without loss of generality that R is discrete, in which case the desired result was established as Theorem **??**. □

Definition B.6.2.5. Let R be an $\mathbb{E}_\infty$-ring. We will refer to the Grothendieck topology described in Variant B.6.2.4 as the *Nisnevich topology* on $(\mathrm{CAlg}^{\text{ét}}_R)^{\mathrm{op}}$. We let $\mathcal{S}\mathrm{hv}^{\mathrm{Nis}}_R$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\text{ét}}_R, \mathcal{S})$ spanned by those functors which are sheaves for the Nisnevich topology. We will refer to $\mathcal{S}\mathrm{hv}^{\mathrm{Nis}}_R$ as the *∞-topos of Nisnevich sheaves over R*.

Remark B.6.2.6. For any $\mathbb{E}_\infty$-ring R, the étale topology on $(\mathrm{CAlg}^{\text{ét}}_R)^{\mathrm{op}}$ is a refinement of the Nisnevich topology on $(\mathrm{CAlg}^{\text{ét}}_R)^{\mathrm{op}}$ (see Remark B.4.1.6). In particular, we have $\mathcal{S}\mathrm{hv}^{\text{ét}}_R \subseteq \mathcal{S}\mathrm{hv}^{\mathrm{Nis}}_R$.

Remark B.6.2.7. Let R be an $\mathbb{E}_\infty$-ring. According to Theorem HA.7.5.0.6 , the functor $A \mapsto \pi_0 A$ determines an equivalence from the ∞-category $\mathrm{CAlg}^{\text{ét}}_R$ to the ordinary category $\mathrm{CAlg}^{\text{ét}}_{\pi_0 R}$ of étale algebras over the commutative ring $\pi_0 R$. Under this equivalence, the étale and Nisnevich topologies on $\mathrm{CAlg}^{\text{ét}}_R$ correspond to the étale and Nisnevich topologies on $\mathrm{CAlg}^{\text{ét}}_{\pi_0 R}$. We therefore have equivalences of ∞-categories $\mathcal{S}\mathrm{hv}^{\text{ét}}_R \simeq \mathcal{S}\mathrm{hv}^{\text{ét}}_{\pi_0 R}$ and $\mathcal{S}\mathrm{hv}^{\mathrm{Nis}}_R \simeq \mathcal{S}\mathrm{hv}^{\mathrm{Nis}}_{\pi_0 R}$. In particular, the ∞-topoi $\mathcal{S}\mathrm{hv}^{\text{ét}}_R$ and $\mathcal{S}\mathrm{hv}^{\mathrm{Nis}}_R$ are both 1-localic.

B.6.3 The Finite Flat Topology

We now study a variant of Definition B.6.2.2, where we impose descent only for morphisms which are *finite flat*.

Proposition B.6.3.1. *Let $f : A \to A'$ be a map of $\mathbb{E}_1$-rings which exhibits A' as a flat left A-module. The following conditions are equivalent:*

(1) *The map f exhibits $\pi_0 A'$ as a finitely presented left module over $\pi_0 A$.*

(2) *The map f exhibits $\pi_0 A'$ as a finitely generated projective left module over $\pi_0 A$.*

(3) *The map f exhibits $\tau_{\geqslant 0} A'$ as a finitely generated projective left module over $\tau_{\geqslant 0} A$.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) follows from Proposition HA.7.2.2.18 . The implication (2) $\Rightarrow$ (1) is obvious. Conversely, suppose that $\pi_0 A'$ is finitely presented as a left $\pi_0 A$-module. Since $\pi_0 A'$ is flat over $\pi_0 A$, Lazard's theorem (Theorem HA.7.2.2.15) guarantees that $\pi_0 A'$ can be realized as a filtered colimit $\varinjlim M_\alpha$ of finitely generated free left modules over $\pi_0 A$. It follows that the isomorphism $\pi_0 A' \simeq \varinjlim M_\alpha$ factors through some M_α, so that $\pi_0 A'$ is a retract of M_α (as a left $\pi_0 A'$-module) and is therefore a finitely generated projective module. □

Definition B.6.3.2. We say that a morphism $f : A \to A'$ of $\mathbb{E}_\infty$-rings is *finite flat* if it satisfies the equivalent conditions of Proposition B.6.3.1. We say that f is *finite étale* if it is both finite flat and étale.

Warning B.6.3.3. The terminology of Definition B.6.3.2 is potentially misleading: in classical commutative algebra, a flat morphism $f : A \to A'$ which exhibits A' as a finitely generated A-module need not be finite flat. For example, suppose that X is a Stone space (Definition A.1.6.8) containing a point x, let A be the ring of locally constant $\mathbf{C}$-valued functions on X, and let $e : A \to \mathbf{C}$ be the map given by evaluation at x. Then e is surjective and flat, but is finite flat only if x is an isolated point of X.

Remark B.6.3.4. Let $f : A \to A'$ be an étale map of $\mathbb{E}_\infty$-rings. If f exhibits $\pi_0 A'$ as a finitely generated module over $\pi_0 A$, then it is finite étale. To prove this, it suffices to show that $\pi_0 A'$ is finitely presented as a module over $\pi_0 A$ (Proposition B.6.3.1). It follows from the structure of étale algebras (Proposition B.1.1.3) that there exists a finitely generated subring $R \subseteq \pi_0 A$ and an étale R-algebra R' such that $\pi_0 A' \simeq \pi_0 A \otimes_R R'$. Enlarging R if necessary, we can ensure that R' is a finitely generated R-module. Since R is Noetherian, the algebra R' is finitely presented as an R-module. It follows that $\pi_0 A'$ is finitely presented as a $\pi_0 A$-module.

We now summarize some of the permanence properties enjoyed by the class of finite flat morphisms:

Lemma B.6.3.5. (1) *Every equivalence $f : A \to A'$ of $\mathbb{E}_\infty$-rings is finite flat.*

(2) *The collection of finite flat morphisms in* CAlg *is closed under composition.*

(3) *Suppose we are given a pushout diagram of $\mathbb{E}_\infty$-rings*

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & B'. \end{array}$$

If f is finite flat, then g is finite flat.

(4) *Given a finite collection $f_i : A_i \to A'_i$ of finite flat morphisms, the induced map $\prod_i A_i \to \prod_i A'_i$ is finite flat.*

Proposition B.6.3.6. *Let R be an $\mathbb{E}_\infty$-ring. Then there exists a Grothendieck topology on the ∞-category $(\mathrm{CAlg}_R^{\acute{e}t})^{\mathrm{op}}$ which can be characterized as follows: if A is an $\mathbb{E}_\infty$-algebra over R, then a sieve $\mathcal{C} \subseteq (\mathrm{CAlg}_R^{\acute{e}t})^{\mathrm{op}}_{/A} \simeq (\mathrm{CAlg}_A^{\acute{e}t})^{\mathrm{op}}$ is a covering if and only if it contains a finite collection of morphisms $\{A \to A_i\}_{1 \leqslant i \leqslant n}$ for which the induced map $A \to \prod_{1 \leqslant i \leqslant n} A_i$ is finite flat and faithfully flat.*

Proof. Let S be the collection of all morphisms in $\mathrm{CAlg}_R^{\text{ét}}$ which are finite flat and faithfully flat. To prove Proposition B.6.3.6, it will suffice to show that S satisfies conditions (a) through (d) of Proposition A.3.2.1. This follows immediately from Lemma B.6.3.5 and the proof of Proposition B.6.1.3. □

We will refer to the Grothendieck topology of Proposition B.6.3.6 as the *finite étale topology* on $(\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}$.

B.6.4 A Criterion for Étale Descent

We can now state the main result of this section:

Theorem B.6.4.1. *Let R be an $\mathbb{E}_\infty$-ring and let $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ be a functor. Then $\mathscr{F}$ is a sheaf with respect to the étale topology if and only if the following conditions are satisfied:*

(1) *The functor $\mathscr{F}$ is a sheaf with respect to the Nisnevich topology.*

(2) *The functor $\mathscr{F}$ is a sheaf with respect to the finite étale topology.*

Remark B.6.4.2. Let R be an $\mathbb{E}_\infty$-ring, let $\mathcal{C} = (\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}$, and let $\mathcal{X}$ denote the full subcategory of $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ spanned by those functors $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ which are sheaves with respect to the Nisnevich and finite étale topologies. It follows from Lemma HTT.6.3.3.4 that $\mathcal{X}$ is an accessible left exact localization of $\mathcal{P}(\mathcal{C})$. Since $\mathcal{X}$ can be obtained as $S^{-1}\mathcal{P}(\mathcal{C})$ where S consists of monomorphisms, we deduce from Proposition HTT.6.2.2.17 that $\mathcal{X} = \mathcal{S}\mathrm{hv}(\mathcal{C})$, where we regard $\mathcal{C}$ as endowed with the coarsest Grothendieck topology which is finer than both the Nisnevich and finite étale topologies.

Proof of Theorem B.6.4.1. The "only if" direction is obvious. To prove the converse, set $\mathcal{C} = (\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}$, and regard $\mathcal{C}$ as endowed with the coarsest Grothendieck topology which is finer than both the Nisnevich and finite étale topologies. We wish to show that $\mathcal{X} = \mathcal{S}\mathrm{hv}(\mathcal{C})$ is contained in the ∞-category of étale sheaves on $\mathcal{C}$. Let $L : \mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S}) \to \mathcal{X}$ be a left adjoint to the inclusion, and let $j : (\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S})$ be the Yoneda embedding. Since the fpqc topology on CAlg is subcanonical (Theorem D.6.3.5), the functor j takes values in $\mathcal{X}$. Using Proposition HTT.6.2.3.20 , we are reduced to proving the following: for every collection of morphisms $\{R' \to R_\alpha\}$ which generate a covering sieve with respect to the étale topology, the induced map $\theta : \coprod_\alpha j(R_\alpha) \to j(R')$ is an effective epimorphism in $\mathcal{X}$. Without loss of generality we may replace R by R', and thereby reduce to the case where $j(R')$ is a final object of $\mathcal{X}$.

Note that the Grothendieck topology on $\mathcal{C}$ is finitary (see Definition A.3.1.1); consequently, to prove that θ is an effective epimorphism in $\mathcal{X}$, it will suffice to prove that $\eta^*(\theta)$ is an effective epimorphism in $\mathcal{S}$, for every geometric morphism $\eta^* : \mathcal{X} \to \mathcal{S}$ (Theorem A.4.0.5). The map η^* determines a geometric morphism $\eta'^* : \mathcal{S}\mathrm{hv}((\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}) \to \mathcal{S}$. According to Corollary **??**, the point η'^* is determined by a Henselian R-algebra A which is a filtered colimit $\varinjlim A_\beta$ of étale R-algebras A_β. In particular, A is local. It follows that there exists an index α such that the induced map $A \to A \otimes_R R_\alpha$ is faithfully flat. We will complete the proof by showing that $\eta^* j(R_\alpha)$ is nonempty.

According to Corollary B.3.3.9, there exists an idempotent element $e \in \pi_0(A \otimes_R R_\alpha)$ such that $A' = (A \otimes_R R_\alpha)[e^{-1}]$ is a faithfully flat finite étale A-algebra. Consequently, there exists an index β and an idempotent $e_\beta \in \pi_0(A_\beta \otimes_R R_\alpha)$ such that $A'_\beta = (A_\beta \otimes_R R_\alpha)[e_\beta^{-1}]$ is a faithfully flat finite étale A_β-algebra. It follows that the map $j(A'_\beta) \to j(A_\beta)$ is an effective epimorphism in $\mathcal{X}$. Since $\eta^* j(A_\beta)$ can be identified with $\mathrm{Map}_{\mathrm{CAlg}_R}(A_\beta, A) \neq \emptyset$, we conclude that $\eta^* j(A'_\beta)$ is nonempty. The map $R_\alpha \to A'_\beta$ induces a map of spaces $\eta^* j(A'_\beta) \to \eta^* j(R_\alpha)$, so that $\eta^* j(R_\alpha)$ is nonempty as desired. □

We also have the following companion to Theorem B.6.4.1:

Theorem B.6.4.3. *Let R be an $\mathbb{E}_\infty$-ring, let $L_{\mathrm{Nis}} : \mathrm{Fun}(\mathrm{CAlg}_R^{\acute{e}t}, \mathcal{S}) \to \mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}$ be a left adjoint to the inclusion functor (given by sheafification with respect to the Nisnevich topology), and let $\mathscr{F} : \mathrm{CAlg}_R^{\acute{e}t} \to \mathcal{S}$ be a sheaf with respect to the finite étale topology. Assume that $\mathscr{F}$ is n-truncated for some $n \geqslant 0$. Then $L_{\mathrm{Nis}} \mathscr{F}$ is a sheaf with respect to the étale topology.*

Remark B.6.4.4. Let R be an $\mathbb{E}_\infty$-ring and let

$$L_{\mathrm{Nis}}, L_{\mathrm{f\acute{e}t}}, L_{\mathrm{\acute{e}t}} : \mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{\acute{e}t}}, \mathcal{S}) \to \mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{\acute{e}t}}, \mathcal{S})$$

denote the functors given by sheafification with respect to the Nisnevich, finite étale, and étale topologies, respectively. For any functor $\mathscr{F} : \mathrm{CAlg}_R^{\mathrm{\acute{e}t}} \to \mathcal{S}$, we have a canonical map $\rho : L_{\mathrm{Nis}}(L_{\mathrm{f\acute{e}t}} \mathscr{F}) \to L_{\mathrm{\acute{e}t}} \mathscr{F}$ (since the étale topology refines both the Nisnevich and finite étale topologies), which is an $L_{\mathrm{\acute{e}t}}$-equivalence. If $\mathscr{F}$ is truncated, then Theorem B.6.4.3 implies that the domain of ρ is an étale sheaf, so that ρ is an equivalence. In other words, to sheafify a (truncated) presheaf $\mathscr{F}$ with respect to the étale topology, we can first sheafify with respect to the finite étale topology and then with respect to the Nisnevich topology.

Warning B.6.4.5. In the situation of Remark B.6.4.4, we also have a natural map $\rho' : L_{\mathrm{f\acute{e}t}}(L_{\mathrm{Nis}} \mathscr{F}) \to L_{\mathrm{\acute{e}t}} \mathscr{F}$, but the map ρ' is generally not an equivalence (the functor $L_{\mathrm{f\acute{e}t}}$ usually does not preserve the class of Nisnevich sheaves).

We will deduce Theorem B.6.4.3 from the following general observation:

Lemma B.6.4.6. *Let $f : A \to B$ be a finite morphism of commutative rings, and let*

$$L_A : \mathrm{Fun}(\mathrm{CAlg}_A^{\acute{e}t}, \mathcal{S}) \to \mathcal{S}\mathrm{hv}_A^{\mathrm{Nis}} \qquad L_B : \mathrm{Fun}(\mathrm{CAlg}_B^{\acute{e}t}, \mathcal{S}) \to \mathcal{S}\mathrm{hv}_B^{\mathrm{Nis}}$$

be left adjoint to the inclusion functors. Suppose that $\mathscr{F} : \mathrm{CAlg}_B^{\acute{e}t} \to \mathcal{S}$ is a functor which commutes with finite products and that $\mathscr{F}$ is n-truncated for some $n \gg 0$. Then the Beck-Chevalley morphism $\rho : L_A(f_* \mathscr{F}) \to f_*(L_B \mathscr{F})$ is an equivalence in the ∞-category $\mathcal{S}\mathrm{hv}_A^{\mathrm{Nis}}$.

Proof of Lemma B.6.4.6. We will show that if $\mathscr{F} : \mathrm{CAlg}_B^{\mathrm{\acute{e}t}} \to \mathcal{S}$ is *any* functor which commutes with finite products, then the Beck-Chevalley morphism $\theta : L_A(f_* \mathscr{F}) \to f_*(L_B \mathscr{F})$

is ∞-connective. In the case where $\mathscr{F}$ is n-truncated, the domain and codomain of θ will also be n-truncated, so that θ will be an equivalence as desired.

Since the Nisnevich topology is finitary, the hypercompletion of the ∞-topos $\mathcal{S}\mathrm{hv}_A^{\mathrm{Nis}}$ has enough points (Theorem A.4.0.5). It will therefore suffice to show that $\eta^*(\theta)$ is a homotopy equivalence for every geometric morphism $\eta^* : \mathcal{S}\mathrm{hv}_A^{\mathrm{Nis}} \to \mathcal{S}$. Using Theorem B.4.4.4, we can identify the point $\eta^* \in \mathrm{Fun}^*(\mathcal{S}\mathrm{hv}_A^{\mathrm{Nis}}, \mathcal{S})$ with a Henselization $\overline{A}$ of A: that is, a local Henselian ring A which can be written as the colimit of a filtered diagram $\{A_\alpha\}$ in $\mathrm{CAlg}_A^{\text{ét}}$. Using Corollary B.3.3.5, we see that $\overline{B} = \overline{A} \otimes_A B$ factors as a finite product $\prod_i \overline{B}_i$, where each $\overline{B}_i$ is a local Henselian ring. Replacing A by an étale A-algebra if necessary, we may assume that this product decomposition exists already over A: that is, we have a factorization $B \simeq \prod B_i$ with $\overline{B}_i \simeq \overline{A} \otimes_A B_i$. Unwinding the definitions, we see that $\eta^*(\theta)$ can be identified with the natural map $\varinjlim \mathscr{F}(B \otimes_A A_\alpha) \to \prod_i \varinjlim \mathscr{F}(B_i \otimes_A A_\alpha)$, which is an equivalence by virtue of our assumption that $\mathscr{F}$ commutes with finite products. □

Remark B.6.4.7. In the statement of Lemma B.6.4.6, the hypothesis that $\mathscr{F}$ is n-truncated is not essential. For example, if A is a Noetherian ring of finite Krull dimension, then the ∞-topos $\mathcal{S}\mathrm{hv}_A^{\mathrm{Nis}}$ is hypercomplete (Corollary 3.7.7.3), so our proof of Lemma B.6.4.6 immediately shows that the map θ is an equivalence. One can treat the general case using a Noetherian approximation argument. We omit the details, since the truncated version of Lemma B.6.4.6 will be sufficient for our application.

Proof of Theorem B.6.4.3. Let R be an $\mathbb{E}_\infty$-ring and let $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ be an n-truncated sheaf for the finite étale topology; we wish to show that $L_{\mathrm{Nis}}\,\mathscr{F}$ is a sheaf for the étale topology. By virtue of Theorem B.6.4.1, it will suffice to show that $L_{\mathrm{Nis}}\,\mathscr{F}$ is a sheaf with respect to the finite étale topology. Since $L_{\mathrm{Nis}}\,\mathscr{F}$ is a Nisnevich sheaf, it commutes with finite products. Using Proposition A.3.3.1, we are reduced to proving the following:

- $(*)$ Let $A \in \mathrm{CAlg}_R^{\text{ét}}$, let $u_0 : A \to A^0$ be a morphism which is finite étale and faithfully flat, and let $A^\bullet$ be the Čech nerve of u (formed in the opposite of the ∞-category $\mathrm{CAlg}_R^{\text{ét}}$). Then the canonical map $\rho : (L_{\mathrm{Nis}}\,\mathscr{F})(A) \to \mathrm{Tot}(L_{\mathrm{Nis}}\,\mathscr{F})(A^\bullet)$ is a homotopy equivalence.

Without loss of generality, we may assume that $A = R$. For each $n \geqslant 0$, let $u_n : A \to A^n$ be the canonical map, so that we have a pair of adjoint functors

$$\mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathcal{S}) \underset{u_{n*}}{\overset{u_n^*}{\rightleftarrows}} \mathrm{Fun}(\mathrm{CAlg}_{A^n}^{\text{ét}}, \mathcal{S}).$$

Unwinding the definitions, we see that ρ is obtained from the composite map

$$\begin{aligned} L_{\mathrm{Nis}}\,\mathscr{F} &\xrightarrow{\rho'} L_{\mathrm{Nis}}(\mathrm{Tot}(u_*^\bullet u^{\bullet *}\,\mathscr{F})) \\ &\xrightarrow{\rho''} \mathrm{Tot}(L_{\mathrm{Nis}}(u_*^\bullet u^{\bullet *}\,\mathscr{F})) \\ &\xrightarrow{\rho'''} \mathrm{Tot}(u_*^\bullet u^{\bullet *}(L_{\mathrm{Nis}}\,\mathscr{F})). \end{aligned}$$

by evaluating at the object $A \in \mathrm{CAlg}^{\text{ét}}_A$. The morphism ρ' is an equivalence by virtue of our assumption that $\mathscr{F}$ is a sheaf for the finite étale topology, and the map ρ''' is an equivalence by virtue of Lemma B.6.4.6. The map ρ'' fits into a commutative diagram

$$\begin{array}{ccc} L_{\mathrm{Nis}}(\varprojlim_{[m]\in\mathbf{\Delta}} u^m_* u^{m*}\mathscr{F}) & \xrightarrow{\rho''} & \varprojlim_{[m]\in\mathbf{\Delta}}(L_{\mathrm{Nis}} u^m_* u^{m*}\mathscr{F}) \\ \downarrow & & \downarrow \\ L_{\mathrm{Nis}}(\varprojlim_{[m]\in\mathbf{\Delta}_{s,\leqslant n+1}} u^m_* u^{m*}\mathscr{F}) & \longrightarrow & \varprojlim_{[m]\in\mathbf{\Delta}_{s,\leqslant n+1}}(L_{\mathrm{Nis}} u^m_* u^{m*}\mathscr{F}). \end{array}$$

We conclude by observing that the vertical maps are equivalences when $\mathscr{F}$ is n-truncated, and the lower horizontal map is an equivalence because the sheafification functor L_{Nis} is left exact. □

B.6.5 The Henselian Case

We close this section with a few observations concerning finite étale morphisms in the Henselian setting.

Notation B.6.5.1. Let R be an $\mathbb{E}_\infty$-ring. We let $\mathrm{CAlg}^{\text{fét}}_R$ denote the full subcategory of CAlg_R spanned by the finite étale R-algebras.

Proposition B.6.5.2. *Let R be a Henselian $\mathbb{E}_\infty$-ring, let $\mathfrak{m} \subseteq \pi_0 R$ be the maximal ideal, and let κ denote the residue field $\pi_0 R/\mathfrak{m}$. Then the inclusion functor $\mathrm{CAlg}^{\text{fét}}_R \hookrightarrow \mathrm{CAlg}^{\text{ét}}_R$ admits a left adjoint. Moreover, the construction $A \mapsto (\pi_0 A)/\mathfrak{m}$ determines an equivalence of ∞-categories from $\mathrm{CAlg}^{\text{fét}}_R$ to $\mathrm{CAlg}^{\text{ét}}_\kappa$.*

Proof. We may assume without loss of generality that R is discrete. Let A be an étale R-algebra, and choose a factorization $A \simeq A' \times A''$ as in Corollary B.3.4.5. We claim that the projection map $A \to A'$ exhibits A' as a $\mathrm{CAlg}^{\text{fét}}_R$-localization of A. In other words, we claim that for every finite étale A-algebra B, every R-algebra homomorphism $\phi : A \to B$ factors (necessarily uniquely) through A'. Note that ϕ induces a decomposition $B \simeq B' \times B''$ and maps $\phi' : A' \to B'$, $\phi'' : A'' \to B''$. We wish to prove that $B'' \simeq 0$. Note that $B''/\mathfrak{m}B''$ is an algebra over $A''/\mathfrak{m}A'' \simeq 0$, so that $\mathfrak{m}B'' = B''$. Since B'' is a direct factor of B, it is a finite étale R-algebra. Using Nakayama's lemma, we conclude that $B'' \simeq 0$. The final assertion is a special case of Corollary B.3.3.7.

□

Proposition B.6.5.3. *Let R be a Henselian $\mathbb{E}_\infty$-ring, let $\mathfrak{m} \subseteq \pi_0 R$ be the maximal ideal, and let $\kappa = \pi_0 R/\mathfrak{m}$ denote the residue field. Let $\mathcal{X} = \mathcal{S}\mathrm{hv}^{\text{ét}}_R$ and let $\mathcal{X}_0$ denote the full subcategory of $\mathcal{X}$ spanned by those sheaves $X : \mathrm{CAlg}^{\text{ét}}_R \to \mathcal{S}$ which are right Kan extensions of their restriction to $\mathrm{CAlg}^{\text{fét}}_R$. Then $\mathcal{X}_0$ is an accessible left exact localization of $\mathcal{X}$. Moreover, the pullback functor $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}^{\text{ét}}_{\tau_{\geqslant 0} R} \to \mathcal{S}\mathrm{hv}^{\text{ét}}_\kappa$ induces an equivalence of ∞-categories $\mathcal{X}_0 \to \mathcal{S}\mathrm{hv}^{\text{ét}}_\kappa$.*

Proof. We may assume without loss of generality that R is discrete. Let $T : \mathrm{CAlg}_R^{\text{ét}} \to \mathrm{CAlg}_R^{\text{fét}}$ be a left adjoint to the inclusion (see Proposition B.6.5.2). Then composition with T induces a left exact localization functor $L : \mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S}) \to \mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S})$, whose essential image is the collection of those functors $X : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ which are right Kan extensions of $X|_{\mathrm{CAlg}_R^{\text{fét}}}$. We will prove that $L\mathcal{X} \subseteq \mathcal{X}$, so that L induces a left exact localization on $\mathcal{X}$ whose essential image is $\mathcal{X} \cap L\,\mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S}) = \mathcal{X}_0$. To prove this, suppose that $X \in \mathcal{S}\mathrm{hv}_R^{\text{ét}}$; we wish to show that LX is a sheaf with respect to the étale topology. Choose an étale R-algebra A and a covering sieve $\mathcal{C} \subseteq \mathrm{CAlg}_A^{\text{ét}}$; we wish to show that the canonical map $LX(A) \to \varprojlim_{B \in \mathcal{C}} LX(B)$ is an equivalence. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory spanned by those étale A-algebras which are finite étale over R. Then $LX|_{\mathcal{C}}$ is a right Kan extension of $LX|_{\mathcal{C}'}$. Using Lemma HTT.4.3.2.7 , we are reduced to proving that the map $\theta : LX(A) \to \varprojlim_{B \in \mathcal{C}_0} LX(B) \simeq \varprojlim_{B \in \mathcal{C}_0} X(B)$ is an equivalence.

Write $A = A' \times A''$ where A' is a finite étale R-algebra and $A''/\mathfrak{m}A'' \simeq 0$ (Corollary B.3.4.5). Let $\mathcal{C}' \subseteq \mathrm{CAlg}_{A'}^{\text{ét}}$ be the full subcategory spanned by those étale maps $A' \to B$ which factor as a composition $A' \to \overline{B} \to B$, where the composition $A \to A' \to \overline{B}$ belongs to $\mathcal{C}$ and $\overline{B}$ is finite étale over R. Since $\mathcal{C}$ is a covering sieve, it induces a covering sieve on A'. It follows that there exists a finite collection of étale maps $A' \to B_\alpha$ such that the induced map $A' \to \prod_\alpha B_\alpha$ is faithfully flat, and each of the composite maps $A \to A' \to B_\alpha$ belongs to $\mathcal{C}$. Using Corollary B.3.4.5, we can assume that each B_α is finite étale over R, so that $A' \to B_\alpha$ belongs to $\mathcal{C}'$. It follows that $\mathcal{C}'$ is a covering sieve on A', so that the map $X(A') \to \varprojlim_{B \in \mathcal{C}'} X(B)$ is a homotopy equivalence.

Note that we can identify $\mathcal{C}_0$ with a full subcategory of $\mathcal{C}'$, so that θ factors as a composition

$$LX(A) \simeq X(A') \to \varprojlim_{B \in \mathcal{C}'} X(B) \to \varprojlim_{B \in \mathcal{C}_0} X(B).$$

To prove that this map is a homotopy equivalence, it suffices to prove that the inclusion $\mathcal{C}_0 \to \mathcal{C}'$ is right cofinal. Using Corollary HTT.4.1.3.1 , we are reduced to proving the following: for every object $B \in \mathcal{C}'$, the ∞-category $\mathcal{C}'_{/B} \times_{\mathcal{C}'} \mathcal{C}_0$ is weakly contractible. In fact, the ∞-category $\mathcal{C}'_{/B} \times_{\mathcal{C}'} \mathcal{C}_0$ is filtered: it is nonempty by construction and admits pushouts.

We now show that the pullback map $i^* : \mathcal{X}_0 \to \mathcal{S}\mathrm{hv}_k^{\text{ét}}$ is an equivalence of ∞-categories. Proposition B.6.5.2 supplies an equivalence of ∞-categories $\mathrm{CAlg}_\kappa^{\text{ét}} \simeq \mathrm{CAlg}_R^{\text{fét}}$. This equivalence determines a Grothendieck topology on $\mathrm{CAlg}_R^{\text{fét}}$, which agrees with the finite étale topology introduced in Proposition B.6.3.6. It follows immediately from Proposition HTT.4.3.2.15 that i^* is fully faithful. To prove the essential surjectivity, we must show that if $Y_0 : \mathrm{CAlg}_R^{\text{fét}} \to \mathcal{S}$ is a sheaf with respect to the finite étale topology, then $Y = Y_0 \circ T$ is a sheaf with respect to the étale topology on $\mathrm{CAlg}_R^{\text{ét}}$. Let A be an étale R-algebra and let $\mathcal{C} \subseteq \mathrm{CAlg}_A^{\text{ét}}$ be a covering sieve on A. As above, we wish to show that the canonical map $Y(A) \to \varprojlim_{B \in \mathcal{C}} Y(B)$ is a homotopy equivalence. Write $A = A' \times A''$ as in Corollary B.3.4.5

and let $\mathcal{C}_0$ and $\mathcal{C}'$ be defined as. Once again, we see that $Y|_{\mathcal{C}}$ is a right Kan extension of $Y|_{\mathcal{C}_0}$, and are therefore reduced to showing that

$$Y(A) \simeq Y_0(A') \to \varprojlim_{B \in \mathcal{C}_0} Y_0(B) \simeq \varprojlim_{B \in \mathcal{C}_0} Y(B)$$

is a homotopy equivalence, which follows from the observation that $\mathcal{C}_0$ is a covering sieve on A' with respect to the finite étale topology. □

Let R be a connective Henselian $\mathbb{E}_\infty$-ring. Let $\mathfrak{m} \subseteq \pi_0 R$ be the maximal ideal, $\kappa = (\pi_0 R)/\mathfrak{m}$ the residue field, and let $\phi : R \to \kappa$ denote the evident map, so that ϕ induces a pair of adjoint functors

$$\mathcal{S}\mathrm{hv}_R^{\text{ét}} \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathcal{S}\mathrm{hv}_\kappa^{\text{ét}} .$$

Unwinding the definitions, we see that ϕ_* induces the equivalence of ∞-categories $\mathcal{S}\mathrm{hv}_\kappa^{\text{ét}} \to \mathcal{X}_0$ appearing in the statement of Proposition B.6.5.3. It follows that the localization functor $L : \mathcal{S}\mathrm{hv}_R^{\text{ét}} \to \mathcal{X}_0$ is given by the composition $\phi_* \phi^*$. For every sheaf $X \in \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ and every *finite* étale R-algebra A, the unit map

$$X(A) \to (LX)(A) \simeq (\phi_* \phi^* X)(A) = (\phi^* X)(\pi_0 A / \mathfrak{m} \pi_0 A)$$

is a homotopy equivalence. Taking $A = R$, we obtain the following result:

Proposition B.6.5.4. *Let R be a connective Henselian $\mathbb{E}_\infty$-ring, $\mathfrak{m} \subseteq \pi_0 R$ the maximal ideal, and $\kappa = (\pi_0 R)/\mathfrak{m}$ the residue field. Let $\phi : R \to \kappa$ be the quotient map and $\phi^* : \mathcal{S}\mathrm{hv}_R^{\acute{e}t} \to \mathcal{S}\mathrm{hv}_\kappa^{\acute{e}t}$ the associated pullback functor. Let $\mathbf{1}$ denote the final object of $\mathcal{S}\mathrm{hv}_R^{\acute{e}t}$. Then the canonical map $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_R^{\acute{e}t}}(\mathbf{1}, \mathscr{F}) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_\kappa^{\acute{e}t}}(\phi^* \mathbf{1}, \phi^* \mathscr{F})$ is a homotopy equivalence for every object $\mathscr{F} \in \mathcal{S}\mathrm{hv}_R^{\acute{e}t}$.*

B.7 Galois Descent

Let R be an $\mathbb{E}_\infty$-ring. In §B.6, we proved that a presheaf $\mathscr{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ satisfies descent for étale coverings if and only if it satisfies descent for both Nisnevich and finite étale coverings. The former condition is very concrete: it is equivalent to Nisnevich excision, by virtue of Theorem B.5.0.3. Our goal in this section is to obtain a concrete interpretation of the second condition as well (Theorem B.7.6.1). The main observation is that every finite étale covering can be refined to a Galois covering (Lemma B.7.6.4).

B.7.1 Free Actions of Finite Groups

We begin by reviewing the Galois theory of commutative rings. For a much more general discussion, we refer the reader to [50] (see in particular Expose 5, Theorem 4.1).

Proposition B.7.1.1. *Let G be a finite group acting on a commutative ring R. The following conditions are equivalent:*

(1) *For every nonzero commutative ring A, the group G acts freely on the set* $\mathrm{Hom}(R, A)$ *of ring homomorphisms from R to A.*

(2) *Let $\mathfrak{p}$ be a prime ideal of R and $G_{\mathfrak{p}} \subseteq G$ the stabilizer group of $\mathfrak{p}$. Then $G_{\mathfrak{p}}$ acts faithfully on the residue field $\kappa(\mathfrak{p})$.*

Proof. Assume first that (1) is satisfied. Choose a point $\mathfrak{p} \in |\operatorname{Spec} R|$ and let $\phi : R \to \kappa(\mathfrak{p})$ be the canonical map. For any $g \in G_{\mathfrak{p}}$, the action of g on $\mathrm{Hom}(R, \kappa(\mathfrak{p}))$ is given by the inverse of the action of g on $\kappa(\mathfrak{p})$. Since the action of G is free, we conclude that any nontrivial element $g \in G_{\mathfrak{p}}$ must act nontrivially on $\kappa(\mathfrak{p})$.

Conversely, suppose that condition (2) is satisfied, and let A be a nonzero commutative ring. We wish to prove that G acts freely on $\mathrm{Hom}(R, A)$. Choose a homomorphism $\psi : A \to \kappa$ where κ is a field; composition with ψ induces a G-equivariant map $\mathrm{Hom}(R, A) \to \mathrm{Hom}(R, k)$. It will therefore suffice to show that G acts freely on $\mathrm{Hom}(R, k)$. Let $\theta : R \to \kappa$ be a ring homomorphism which is invariant under an element $g \in G$. Then $g \in G_{\mathfrak{p}}$ for $\mathfrak{p} = \ker(\theta)$, and the map θ factors as a composition

$$R \xrightarrow{\theta'} \kappa(\mathfrak{p}) \xrightarrow{\theta''} k.$$

Since θ'' injective and θ is g-invariant, we conclude that g acts trivially on the field $\kappa(\mathfrak{p})$; condition (2) then guarantees that g is the identity element of G. □

Definition B.7.1.2. We will say that an action of a finite group G on a commutative ring R is *free* if it satisfies the equivalent conditions of Proposition B.7.1.1.

B.7.2 The Invariant Subring

Let R be a commutative ring equipped with an action of a finite group G. We let R^G denote the subring of R consisting of G-invariant elements. Our first goal is to show that if the action of G is free, then $R^G \hookrightarrow R$ is finite étale. In fact, we can make a more precise statement, for which we require a bit of notation.

Notation B.7.2.1. Let R be a commutative ring acted on by a finite group G and set $R' = \prod_{g \in G} R$. We define ring homomorphisms $\phi_0, \phi_1 : R \to R'$ by the formulae

$$\phi_0(r)_g = r \qquad \phi_1(r)_g = g(r).$$

Together these maps determine a ring homomorphism $\phi : \mathrm{Tor}_0^{R^G}(R, R) \to R'$, where $R^G \subseteq R$ denotes the ring of G-invariant elements of R.

Proposition B.7.2.2. *Let G be a finite group acting on freely on a commutative ring R, and let $R^G \subseteq R$ denote the subring consisting of G-invariant elements. Then:*

(1) *The inclusion $R^G \hookrightarrow R$ is finite étale.*

(2) *Let $R' = \prod_{g \in G} R$ be as in Notation B.7.2.1. Then the map $\phi : \mathrm{Tor}_0^{R^G}(R, R) \to R'$ is an isomorphism.*

The proof of Proposition B.7.2.2 depends on the following observation from the classical Galois theory of fields:

Lemma B.7.2.3. *Let G be a group of order n which acts faithfully on a field κ, and let $\phi_0, \phi_1 : \kappa \to \kappa'$ be as in Notation B.7.2.1. Then there exists a finite sequence $x_1, \ldots, x_n \in \kappa$ such that the images $\phi_1(x_i)$ form a basis for κ', regarded as a κ-vector space via the homomorphism ϕ_0.*

Proof. It is clear that κ' has dimension n over κ. Consequently, it will suffice to show that κ' is generated as a κ-vector space by elements of the form $\phi_1(x)$. In other words, we must show that the map $\phi : \mathrm{Tor}_0^{\kappa^G}(\kappa, \kappa) \to \kappa'$ is surjective. In the ring κ', we have a unique decomposition $1 = \sum_{g \in G} e_g$, where each e_g is a nonzero idempotent element corresponding to projection of $\kappa' = \prod_{g \in G} \kappa$ onto the gth factor. The elements e_g form a basis for κ' as a κ-vector space; it will therefore suffice to show that each e_g belongs to the image of ϕ. If $h \neq g$, then since $h^{-1}g$ acts nontrivially on κ we can choose an element $x \in \kappa$ such that $h(x) \neq g(x)$. Then

$$y_h = (\phi_0(h(x)) - \phi_1(x))\phi_0(\frac{1}{h(x) - g(x)})$$

belongs to the image of ϕ; note that the gth coordinate of y_h is equal to 1, and the hth coordinate vanishes. It follows that $e_g = \prod_{h \neq g} y_h$ also belongs to the image of ϕ, as desired. □

Proof of Proposition B.7.2.2. Consider the following weaker version of assertion (1):

(1′) The inclusion $R^G \hookrightarrow R$ is faithfully flat.

Since the diagonal inclusion $\phi_0 : R \to R'$ is finite étale, assertions (1′) and (2) imply (1) by faithfully flat descent. Assertions (1′) and (2) are local on $|\operatorname{Spec} R^G|$; we may therefore replace R^G by its localization at some prime and thereby reduce to the case where R^G is a local ring with a unique maximal ideal $\mathfrak{m}$.

Note that R integral over R^G: every element $x \in R$ is a solution to the polynomial equation $\prod_{g \in G}(X - g(x)) = 0$, whose coefficients are G-invariant. We may therefore write

R as a union of subalgebras R_α which are finitely generated as R^G-modules. It follows from Nakayama's lemma that each quotient $R_\alpha/\mathfrak{m}R_\alpha$ is nonzero, so that the direct limit $R/\mathfrak{m}R \simeq \varinjlim R_\alpha/\mathfrak{m}R_\alpha$ is nonzero. We conclude that $\mathfrak{m}$ is contained in a maximal ideal of R.

Choose a maximal ideal $\mathfrak{n} \subset R$ containing $\mathfrak{m}$. For each $g \in G$, let $\mathfrak{n}^g$ denote the inverse image of $\mathfrak{n}$ under the action of g on R. We next claim that R is a semi-local ring, whose maximal ideals are precisely those of the form $\mathfrak{n}^g$ for $g \in G$. To see this, it suffices to show that if $x \in R$ is an element which does not belong to any of the ideals $\mathfrak{n}^g$, then x is invertible in R. Note that for each $g \in G$, our assumption $x \notin \mathfrak{n}^g$ is equivalent to $g(x) \notin \mathfrak{n}$. Since $\mathfrak{n}$ is a prime ideal, we conclude that $y = \prod_{g \in G} g(x) \notin \mathfrak{n}$. Then y is a G-invariant element of R which does not belong to $\mathfrak{m}$, so that y is invertible in R^G and therefore x is invertible in R.

Let $H \subseteq G$ be the stabilizer of the ideal $\mathfrak{n} \subset R$ and let n be the order of H. Our assumption that G acts freely on R guarantees that H acts freely on the residue field $R/\mathfrak{n}$. Using Lemma B.7.2.3, we deduce the existence of elements $\overline{x}_1, \ldots, \overline{x}_n \in R/\mathfrak{n}$ whose images under the map ϕ_1 of Notation B.7.2.1 form a basis for $\prod_{h \in H}(R/\mathfrak{n})$ as a vector space over $R/\mathfrak{n}$. Choose elements $x_i \in \bigcap_{g \in G - H} \mathfrak{n}^g \subseteq R$ which reduce to the elements $\overline{x}_i$ modulo $\mathfrak{n}$, and let $g_1, \ldots, g_m \in G$ be a set of representatives for the set of right cosets G/H. For each maximal ideal $\mathfrak{n}'$ of R, the images $\phi_1(g_i(x_j))$ form a basis for the vector space $\prod_{g \in G} R/\mathfrak{n}'$. Using Nakayama's lemma, we conclude that the elements $\phi_1(g_i(x_j))$ form a basis of R' (viewed as an R-module via ϕ_0).

Let G act on R' via the formula $g(\{r_{g'}\}_{g' \in G}) = \{g(r_{g'g^{-1}})\}_{g' \in G}$. The map $\{r_{i,j}\} \mapsto \sum_{i,j} \phi_0(r_{i,j})\phi_1(g_i(x_j))$ determines a G-equivariant isomorphism $\bigoplus_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n} R \simeq R'$. Passing to fixed points, we see that R is freely generated by the elements $g_i(x_j)$ as an R^G-module, which immediately implies both (1′) and (2). □

B.7.3 Descent

Let R be a commutative ring equipped with a free action of a finite group G. Proposition B.7.2.2 guarantees that R is faithfully flat over R^G. If $R^\bullet$ denotes the cosimplicial commutative ring obtained as the Čech nerve of the inclusion $R^G \hookrightarrow R$, then we have a canonical isomorphism $R^k \simeq \prod_{g_1, \ldots, g_k \in G} R$. Using faithfully flat descent, we obtain the following result:

Proposition B.7.3.1. *Let R be a commutative ring equipped with a free action of a finite group G. Then the construction $M \mapsto M \otimes_{R^G} R$ determines an equivalence of categories from the category of (discrete) R^G-modules to the category of (discrete) R-modules equipped with a compatible action of G. Moreover, if M is any R^G-module, the augmented cochain complex*

$$M \to R^0 \otimes_{R^G} M \to R^1 \otimes_{R^G} M \to \cdots$$

associated to the cosimplicial abelian group $R^\bullet \otimes_{R^G} M$ is acyclic.

Remark B.7.3.2. Let R be a commutative ring equipped with a free action of a finite group G, and let N be an R-module equipped with a compatible action of the group G. Unwinding the definitions, we see that the cochain complex appearing in Proposition B.7.3.1 is the standard complex for computing the group cohomology $\mathrm{H}^*(G; N)$. Proposition B.7.3.1 gives

$$\mathrm{H}^n(G; N) \simeq \begin{cases} 0 & \text{if } n > 0 \\ M & \text{if } n = 0. \end{cases}$$

where $M \simeq N^G$ is an R^G-module (unique up to canonical isomorphism) such that $M \otimes_{R^G} R \simeq N$.

B.7.4 Digression: Group Actions on Spectra

Let G be a discrete group. Then G is a monoid object in the category of sets, and therefore determines a simplicial set BG (see §HA.4.1.2). The simplicial set BG is a Kan complex with a unique vertex, and can therefore be viewed as an object of the ∞-category of pointed spaces $\mathcal{S}_*$. We refer to BG as the *classifying space* of G.

Remark B.7.4.1. The classifying space BG is determined up to equivalence by the requirements that BG is 1-connective, 1-truncated, and the fundamental group of BG (taken with respect to its base point) is isomorphic to G (see, for example, Proposition HTT.7.2.2.12).

Definition B.7.4.2. Let $\mathcal{C}$ be an ∞-category. A *G-equivariant object* of $\mathcal{C}$ is a map of simplicial sets $BG \to \mathcal{C}$.

Remark B.7.4.3. Evaluation at the base point of BG determines a forgetful functor $\theta : \mathrm{Fun}(BG, \mathcal{C}) \to \mathcal{C}$. We will generally abuse notation by identifying a G-equivariant object C of $\mathcal{C}$ with its image $\theta(C) \in \mathcal{C}$. In this situation, we will also say that the group G *acts on* the object $\theta(C) \in \mathcal{C}$ (the action itself is given by the object $C \in \mathrm{Fun}(BG, \mathcal{C})$).

Definition B.7.4.4. Let $\mathcal{C}$ be an ∞-category and let $F : BG \to \mathcal{C}$ be a functor which determines an action of G on the object $F(*) = X \in \mathcal{C}$. We let X^G denote a limit of the diagram F (if such a limit exists).

Remark B.7.4.5. The notation of Definition B.7.4.4 is abusive: the object X^G depends on the G-equivariant object $F : BG \to \mathcal{C}$, and not only on the underlying object $X \in \mathcal{C}$.

Let $\mathcal{C}$ be an ∞-category which admits small limits, let G be a discrete group, and suppose we are given a G-equivariant object $BG \to \mathcal{C}$, corresponding to an action of G on an object $X \in \mathcal{C}$. We can regard BG as a simplicial object in the category of sets: in particular, BG determines a simplicial space $S_\bullet$ with $S_n \simeq G^n$. According to Example HTT.A.2.9.31 , we can identify BG with the colimit of the diagram $S_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}$. It follows that X^G can be identified with the limit of a cosimplicial object of $X^\bullet \in \mathcal{C}$, where each X^n is a limit of the induced diagram $S_n \to BG \to \mathcal{C}$. Since S_n is discrete, we obtain $X^n = \prod_{g_1,\ldots,g_n \in G} X$.

Now suppose that $\mathcal{C}$ is the ∞-category of spectra. Applying Variant HA.1.2.4.9 , we deduce the existence of a spectral sequence of abelian groups $\{E_r^{p,q}, d_r\}_{r \geqslant 1}$ with

$$E_1^{p,q} \simeq \prod_{g_1,\ldots,g_p \in G} \pi_{-q} X.$$

Note that each $\pi_{-q}X$ is an abelian group acted on by G; unwinding the definitions, we see that the differential d_1 is the standard differential in the cochain complex

$$\pi_{-q}X \to \prod_{g \in G} \pi_{-q}X \to \prod_{g,g' \in G} \pi_{-q}X \to \cdots$$

which computes the cohomology of the group G with coefficients in $\pi_{-q}X$. We therefore obtain a canonical isomorphism $E_2^{p,q} \simeq \mathrm{H}^p(G; \pi_{-q}X)$.

In good cases, the spectral sequence described above will converge to the homotopy groups $\pi_{-p-q}X^G$. For example, Corollary HA.1.2.4.12 yields the following result:

Proposition B.7.4.6. *Let G be a discrete group, and let $BG \to \mathrm{Sp}$ be a G-equivariant object of the ∞-category of spectra whose underlying spectrum is X. Assume that for every integer n and for each $k > 0$, the cohomology group $\mathrm{H}^k(G; \pi_n X)$ vanishes. Then for every integer n, the map $\pi_n(X^G) \to \pi_n X$ is injective and its image is the group of G-invariant elements of $\pi_n X$.*

B.7.5 Galois Extensions of $\mathbb{E}_\infty$-Rings

We now study actions of finite groups on $\mathbb{E}_\infty$-rings.

Definition B.7.5.1. Let G be a finite group acting on an $\mathbb{E}_\infty$-ring R. We will say that the action is *free* if the induced action of G on the commutative ring $\pi_0 R$ is free, in the sense of Definition B.7.1.2.

Let $f : R \to R'$ be a map of $\mathbb{E}_\infty$-rings and G a finite group. We will say that f is a *Galois extension* (with Galois group G) if there exists a free action of G on R' such that f factors as a composition $R \simeq R'^G \to R'$.

Warning B.7.5.2. Let $f : R \to R'$ be a Galois extension $\mathbb{E}_\infty$-rings. The Galois group G is not uniquely determined by f. For example, if $R' \simeq R^n$, then any group G of order n can appear as a Galois group for f.

Warning B.7.5.3. The notion of Galois extension introduced in Definition B.7.5.1 is very restrictive. For a much more general analogue of Galois theory in this context, we refer the reader to [175].

Remark B.7.5.4. Using Theorem HA.7.5.0.6 , we see that a map $f : R \to R'$ of $\mathbb{E}_\infty$-rings is a Galois extension if and only if the induced map of commutative rings $\pi_0 R \to \pi_0 R'$ is a Galois extension.

Proposition B.7.5.5. *Let R be an $\mathbb{E}_\infty$-ring equipped with a free action of a finite group G. Then:*

(1) *For every integer n, the map $\pi_n R^G \to \pi_n R$ is injective, and its image is the subgroup of G-invariant elements of $\pi_n R$.*

(2) *The map $R^G \to R$ is finite étale.*

(3) *The canonical map $R \otimes_{R^G} R \to \prod_{g \in G} R$ is an equivalence.*

Proof. Combine Propositions B.7.2.2, B.7.3.1, and B.7.4.6. □

B.7.6 Étale Descent and Galois Descent

Let R be an $\mathbb{E}_\infty$-ring and let $\mathscr{F} : \mathrm{CAlg}^{\text{ét}}_R \to \mathcal{S}$ be a functor. We will say that $\mathscr{F}$ *satisfies Galois descent* if, for every object $A \in \mathrm{CAlg}^{\text{ét}}_R$ equipped with a free action of a finite group G, the canonical map $\mathscr{F}(A^G) \to \mathscr{F}(A)^G$ is a homotopy equivalence.

We can now state the main result of this section.

Theorem B.7.6.1. *Let R be an $\mathbb{E}_\infty$-ring and let $\mathscr{F} : \mathrm{CAlg}^{\acute{e}t}_R \to \mathcal{S}$ be a functor. Then $\mathscr{F}$ is a sheaf with respect to the finite étale topology if and only if the following conditions are satisfied:*

(1) *The functor $\mathscr{F}$ satisfies Galois descent.*

(2) *The functor $\mathscr{F}$ commutes with finite products.*

Corollary B.7.6.2. *Let R be an $\mathbb{E}_\infty$-ring and let $\mathscr{F} : \mathrm{CAlg}^{\acute{e}t}_R \to \mathcal{S}$ be a functor. Then $\mathscr{F}$ is a sheaf with respect to the étale topology if and only if the following conditions are satisfied:*

(1) *The functor $\mathscr{F}$ satisfies Galois descent.*

(2) *The functor $\mathscr{F}$ is a sheaf with respect to the Nisnevich topology.*

Proof. Combine Theorem B.7.6.1 and Theorem B.6.4.1 (note that if $\mathscr{F} : \mathrm{CAlg}^{\text{ét}}_R \to \mathcal{S}$ is a sheaf for the Nisnevich topology, then $\mathscr{F}$ commutes with finite products). □

In order to prove Theorem B.7.6.1, we need to show that there is a sufficiently large supply of Galois extensions in the setting of $\mathbb{E}_\infty$-rings.

Lemma B.7.6.3. *Let $f : R \to R'$ be a finite étale map of $\mathbb{E}_\infty$-rings, and suppose that $\pi_0 R'$ is a projective $\pi_0 R$-module of rank n. Then there exists a map $R \to A$ which is finite étale and faithfully flat such that $R' \otimes_R A \simeq A^n$.*

Proof. We proceed by induction on n. If $n = 0$, we can take $A = R$. Assume $n > 0$. Then f is faithfully flat. Replacing R by R' and R' by $R' \otimes_R R'$, we can assume that f admits a left inverse $g : R' \to R$. Since f is étale, the map g determines a decomposition $R' \simeq R \times R''$. Then R'' is finite étale of rank $(n-1)$ over R. By the inductive hypothesis, we can choose a faithfully flat finite étale map $R \to A$ such that $R'' \otimes_R A \simeq A^{n-1}$. It follows that $R' \otimes_R A \simeq A^n$ as desired. □

Lemma B.7.6.4. *Let R be an $\mathbb{E}_\infty$-ring and let $f : R \to R'$ be a faithfully flat finite étale morphism. Then there exists a map $g : R' \to R''$ such that the composite map $R \to R''$ is a Galois extension.*

Proof. For each $i \geqslant 0$, there exists a largest open subset U_i of $|\operatorname{Spec} \pi_0 R|$ over which the localization of the module $\pi_0 R'$ has rank i. Then $|\operatorname{Spec} \pi_0 R|$ is the disjoint union of the open sets U_i, so each U_i is also closed and therefore has the form $|\operatorname{Spec}(\pi_0 R)[e_n^{-1}]|$ where e_i is some idempotent element of $\pi_0 R$. Since R' is faithfully flat over R, we have $e_0 = 0$. Let n be the least common multiple of the set $\{i : e_i \neq 0\}$. Replacing R' by $\prod_i R'[e_i^{-1}]^{\frac{n}{i}}$, we can assume that $\pi_0 R'$ has constant rank n over $\pi_0 R$.

Let $R'^{\otimes n}$ be the n-fold tensor power of R' over R. For every pair of integers $1 \leqslant i < j \leqslant n$, the multiplication on the ith and jth tensor factors induces a map of étale R-algebras $f_{i,j} : R'^{\otimes n} \to R'^{\otimes (n-1)}$. Since $f_{i,j}$ admits a section, it induces an equivalence $R'^{\otimes(n-1)} \simeq R'^{\otimes n}[\epsilon_{i,j}^{-1}]$ for some idempotent elements $\epsilon_{i,j} \in \pi_0 R'^{\otimes n}$. Let $R'' = R'^{\otimes n}[\prod_{i,j} \epsilon_{i,j}^{-1}]$. Then R'' carries an action of the symmetric group Σ_n (as one can see, for example, by reducing to the discrete case using Theorem HA.7.5.0.6) in the ∞-category CAlg_R. To complete the proof, it suffices to show that Σ_n acts freely on $\pi_0 R''$ and that the induced map $R \to R''^{\Sigma_n}$ is an equivalence. This assertion is local on R; we may therefore invoke Lemma B.7.6.3 to reduce to the case where $R' = R^n$. In this case, $R'' \simeq \prod_{\sigma \in \Sigma_n} R$ and the desired result is obvious. □

Proof of Theorem B.7.6.1. Let G be a finite group acting on an object $A \in \mathrm{CAlg}_R^{\text{ét}}$. Write BG as the colimit of simplicial space $S_\bullet$ (with $S_n \simeq G^n$) and let $A^\bullet$ be the cosimplicial $\mathbb{E}_\infty$-ring given by $A^n \simeq \varinjlim_{S_n} A \simeq \prod_{g_1,\ldots,g_n \in G} A$, so that $A^G \simeq \varprojlim A^\bullet$. Using Proposition B.7.5.5, we see that $A^\bullet$ can be identified with the Čech nerve of the map $A^G \to A$. Similarly, we see that $\mathscr{F}(A)^G$ can be identified with the limit of the cosimplicial space $[n] \mapsto \prod_{g_1,\ldots,g_n \in G} \mathscr{F}(A)$. If $\mathscr{F}$ commutes with finite products, then this cosimplicial object is given by $[n] \mapsto \mathscr{F}(A^n)$. We have proven:

- $(*)$ If $\mathscr{F}$ commutes with finite products, then $\mathscr{F}$ satisfies Galois descent if and only if, for every Galois extension $A^G \to A$ with Čech nerve $A^\bullet$ in $\mathrm{CAlg}_R^{\text{ét}}$, the induced map

$\mathscr{F}(A^G) \to \varprojlim \mathscr{F}(A^\bullet)$ is a homotopy equivalence.

If $\mathscr{F}$ is a sheaf with respect to the finite étale topology, then Proposition A.3.3.1 implies that $\mathscr{F}$ commutes with finite products and satisfies the criterion of $(*)$, and therefore satisfies Galois descent.

Conversely, suppose that $\mathscr{F}$ commutes with finite products and satisfies Galois descent. We wish to prove that $\mathscr{F}$ is a sheaf with respect to the finite étale topology. We proceed as in the proof of Proposition A.3.3.1. Let $A \in \mathrm{CAlg}_R^{\text{ét}}$ and let $\mathcal{C}^{(0)} \subseteq \mathrm{CAlg}_A^{\text{ét}}$ be a sieve on A; we wish to prove that the map $\mathscr{F}(A) \to \varprojlim \mathscr{F}|_{\mathcal{C}^{(0)}}$ is an equivalence. Suppose first that $\mathcal{C}^{(0)}$ is generated by a Galois extension $f : A \to A^0$. Let $A^\bullet$ be the Čech nerve of f. It follows from $(*)$ that the canonical map $\mathscr{F}(A) \to \varprojlim \mathscr{F}(A^\bullet)$. The desired result now follows follows from the observation that $A^\bullet$ is given by a right cofinal map $\mathbf{\Delta} \to \mathcal{C}^{(0)}$.

Suppose now that $\mathcal{C}^{(0)}$ is generated by a single map $f : A \to A'$ which is finite étale and faithfully flat. Using Lemma B.7.6.4, we see that there exists a map $A' \to A''$ such that A'' is a Galois extension of A. Let $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(0)}$ be the sieve generated by A'', so that $\mathscr{F}(A) \simeq \varprojlim \mathscr{F}|_{\mathcal{C}^{(1)}}$ by the above argument. By virtue of Lemma HTT.4.3.2.7 , it will suffice to show that $\mathscr{F}|_{\mathcal{C}^{(0)}}$ is a right Kan extension of $\mathscr{F}|_{\mathcal{C}^{(1)}}$. Choose a map $A' \to B$ and let $\mathcal{C}^{(2)}$ be the pullback of the sieve $\mathcal{C}^{(1)}$ to $\mathrm{CAlg}_B^{\text{ét}}$; we wish to prove that $\mathscr{F}(B) \simeq \varprojlim \mathscr{F}|_{\mathcal{C}^{(2)}}$. This follows from the above argument, since $\mathcal{C}^{(2)}$ is generated by the Galois extension $B \to B \otimes_A A''$.

Now suppose that $\mathcal{C}^{(0)}$ is generated by a finite collection of finite étale morphisms $\{A \to A_i\}_{1 \leqslant i \leqslant n}$ such that the induced map $A \to \prod_i A_i$ is faithfully flat. Let $A' = \prod_i A_i$ and let $\mathcal{C}^{(1)}$ denote the sieve generated by the the map $A \to A'$. The above argument shows that $\mathscr{F}(A) \simeq \varprojlim \mathscr{F}|_{\mathcal{C}^{(1)}}$. To prove that $\mathscr{F}(A) \simeq \varprojlim \mathscr{F}|_{\mathcal{C}^{(0)}}$, it will suffice to show that $\mathscr{F}|_{\mathcal{C}^{(1)}}$ is a right Kan extension of $\mathscr{F}|_{\mathcal{C}^{(0)}}$. Fix an object $g : A \to B$ in $\mathcal{C}^{(1)}$, and let $\mathcal{C}^{(2)}$ be the pullback of $\mathcal{C}^{(0)}$ to $\mathrm{CAlg}_B^{\text{ét}}$; we wish to prove that $\mathscr{F}(B) \simeq \varprojlim \mathscr{F}|_{\mathcal{C}^{(2)}}$. By construction, g factors through some map $g_0 : \prod_i A_i \to B$, so that g_0 determines a decomposition $B \simeq \prod_i B_i$. Let $T \subseteq \{1, \ldots, n\}$ denote the collection of indices i such that $B_i \neq 0$. Let $\mathcal{C}' \subseteq \mathcal{C}^{(2)}$ be the full subcategory spanned by those morphisms $B \to B'$ which factor through some B_i, where $B' \neq 0$. Note that in this case B_i is uniquely determined and the index i belongs to T; it follows that $\mathcal{C}'$ decomposes as a disjoint union of full subcategories $\coprod_{i \in T} \mathcal{C}'_i$. Each of the categories $\mathcal{C}'_i$ contains the projection $B \to B_i$ as an initial object, so the inclusion $\{B_i\}_{i \in T} \hookrightarrow \mathcal{C}'$ is right cofinal. We therefore obtain homotopy equivalences

$$\mathscr{F}(B) \simeq \mathscr{F}(\prod_{i \in T} B_i) \simeq \prod_{i \in T} \mathscr{F}(B_i) \simeq \varprojlim \mathscr{F}|_{\mathcal{C}'}.$$

By virtue of Lemma HTT.4.3.2.7 , we are reduced to proving that $\mathscr{F}|_{\mathcal{C}^{(2)}}$ is a right Kan extension of $\mathscr{F}|_{\mathcal{C}'}$. To see this, choose an object $B \to B'$ in $\mathcal{C}^{(2)}$; we wish to show that $\mathscr{F}(B') \simeq \varprojlim \mathscr{F}|_{\mathcal{C}'_{B'/}}$. Let $B'_i = B_i \otimes_B B'$ for $1 \leqslant i \leqslant n$, and let T' be the collection of indices

i for which $B'_i \neq 0$. Then $\mathcal{C}'_{B'/}$ decomposes as a disjoint union $\coprod_{i \in T'} (\mathcal{C}'_{B'/})_i$, each of which has a final object (given by the map $B' \to B'_i$). We therefore have homotopy equivalences

$$\varprojlim \mathscr{F}|_{\mathcal{C}'_{/B'}} \simeq \prod_{i \in T'} \mathscr{F}(B'_i) \simeq \mathscr{F}(\prod_{i \in T'} B'_i) \simeq \mathscr{F}(B'),$$

as desired.

We now treat the case of a general covering sieve $\mathcal{C}^{(0)} \subseteq \mathrm{CAlg}^{\text{ét}}_A$. By definition, there exists a finite collection of finite étale maps maps $f_i : A \to A_i$ which generate a covering sieve $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(0)}$. The above argument shows that $\mathscr{F}(A) \simeq \varprojlim \mathscr{F}|_{\mathcal{C}^{(1)}}$. To complete the proof, it will suffice (by Lemma HTT.4.3.2.7) to show that $\mathscr{F}|_{\mathcal{C}^{(0)}}$ is a right Kan extension of $\mathscr{F}|_{\mathcal{C}^{(1)}}$. Unwinding the definitions, we must show that for every map $g : A \to B$ in $\mathcal{C}^{(0)}$, we have $\mathscr{F}(A) \simeq \varprojlim \mathscr{F}|_{g^* \mathcal{C}^{(1)}}$. This is clear, since $g^* \mathcal{C}^{(1)}$ is a covering sieve on B generated by finitely many morphisms $B \to A_i \otimes_A B$. □

Appendix C

Prestable ∞-Categories

In §HA.1.1.1 , we introduced the notion of a *stable ∞-category* (Definition HA.1.1.1.9). The theory of stable ∞-categories can be regarded as an ∞-categorical analogue of the classical theory of abelian categories. To explicitly connect these theories, one can consider the data of a stable ∞-category $\mathcal{C}$ equipped with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$: from this data, one can extract an abelian category $\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0}$ which appears as a full subcategory of $\mathcal{C}$. In good cases, one can reconstruct the entire ∞-category $\mathcal{C}$ (together with its t-structure) from the subcategory $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$: for example, if the t-structure on $\mathcal{C}$ is right complete, then we can identify $\mathcal{C}$ with the ∞-category $\mathrm{Sp}(\mathcal{C}_{\geqslant 0})$ of spectrum objects of $\mathcal{C}_{\geqslant 0}$. Our goal in this appendix is to develop a theory of *prestable* ∞-categories: that is, ∞-categories which behave like the "connective parts" of t-structures on stable ∞-categories.

We begin in §C.1 with an axiomatic approach: we define a prestable ∞-category $\mathcal{C}$ to be an ∞-category which satisfies a short list of conditions, analogous to (but slightly weaker than) the conditions defining a stable ∞-category (Definition C.1.2.1). Our first main result asserts the equivalence of this "intrinsic" definition with an "extrinsic" one: an ∞-category $\mathcal{C}$ is prestable if and only if it appears as a full subcategory of a stable ∞-category $\mathcal{D}$ which is closed under finite colimits and extensions (Corollary C.1.2.3). Under mild additional assumptions, one can arrange that $\mathcal{C}$ is the connective part of a (uniquely determined) t-structure on $\mathcal{D}$ (Proposition C.1.2.9). We will be primarily interested in the case where the stable ∞-category $\mathcal{D}$ is presentable and its t-structure is compatible with filtered colimits: in this case, we will say that $\mathcal{C}$ is a *Grothendieck prestable ∞-category* (Definition C.1.4.2). The collection of Grothendieck prestable ∞-categories can itself be organized into an ∞-category Groth_{∞}, which we will study in §C.3.

For any connective $\mathbb{E}_1$-ring A, the stable ∞-category RMod_A of right A-modules admits a t-structure which is compatible with filtered colimits, and the full subcategory $\mathrm{RMod}_A^{\mathrm{cn}} \subseteq \mathrm{RMod}_A$ of *connective* right A-modules is an example of a Grothendieck prestable ∞-category. The central result of this appendix is the following converse: every Grothendieck prestable

∞-category $\mathcal{C}$ can be obtained as a left exact localization of $\mathrm{RMod}_A^{\mathrm{cn}}$, for some connective $\mathbb{E}_1$-ring A (Theorem C.2.4.1). This can be regarded as a generalization of the classical Gabriel-Popescu theorem, which asserts that every Grothendieck abelian category arises as an exact localization of the abelian category $\mathrm{RMod}_A^{\heartsuit}$ for some associative ring A (see [75]). Like the classical Gabriel-Popescu theorem, Theorem C.2.4.1 is very useful: it supplies a mechanism for reducing questions about arbitrary Grothendieck prestable ∞-categories to questions about structured ring spectra. In §C.4, we will use this mechanism to construct a well-behaved tensor product on the ∞-category Groth_{∞}, which refines (and generalizes) the tensor product on presentable stable ∞-categories studied in §**??**.

Recall that an abelian category $\mathcal{A}$ is said to be *Grothendieck* if it admits exact filtered colimits and a small generator. The theory of Grothendieck prestable ∞-categories is closely related to the theory of Grothendieck abelian categories: every Grothendieck prestable ∞-category $\mathcal{C}$ determines a Grothendieck abelian category $\mathcal{C}^{\heartsuit}$ (which can be defined as the full subcategory of $\mathcal{C}$ spanned by the discrete objects), and every Grothendieck abelian category $\mathcal{A}$ arises in this way (though not uniquely). In §**??**, we will carry out a detailed study of this relationship and introduce a family of intermediate objects (which we refer to as *Grothendieck abelian n-categories* for $1 \leqslant n < \infty$) which interpolate between classical homological algebra and our theory of Grothendieck prestable ∞-categories.

Except in trivial cases, a Grothendieck prestable ∞-category $\mathcal{C}$ is never small: the definition requires that $\mathcal{C}$ admits arbitrary colimits (such as infinite direct sums). However, we can often select out an essentially small full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ by imposing some additional finiteness conditions on the objects of $\mathcal{C}$. In good cases, one might hope to recover the ∞-category $\mathcal{C}$ (up to equivalence) from the full subcategory $\mathcal{C}_0$. In §C.6, we will consider several variations on this theme, and give a detailed description of the mechanism relating $\mathcal{C}$ with $\mathcal{C}_0$.

Contents

C.1 Prestability

Recall that an ∞-category $\mathcal{C}$ is *stable* if it is pointed, admits finite colimits, and the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence of ∞-categories (Corollary HA.1.4.2.27). The collection of stable ∞-categories includes many, but not all, of the ∞-categories which are relevant to the study of homological algebra. For example, if R is a commutative ring, then the collection of chain complexes of R-modules can be organized into a stable ∞-category Mod_R whose homotopy category is the classical derived category of R (see §HA.1.3.2). The property of stability is not inherited by subcategories: for example, the full subcategory $\mathrm{Mod}_R^{\mathrm{cn}} \subseteq \mathrm{Mod}_R$ of *connective R-modules* (that is, chain complexes whose homology vanishes in negative degrees) is not stable. In this section, we will study a larger class of ∞-categories which we call *prestable ∞-categories*, which includes $\mathrm{Mod}_R^{\mathrm{cn}}$ and variants thereof.

C.1.1 The Spanier-Whitehead Construction

Recall that the Spanier-Whitehead category $\mathcal{SW}$ can be defined as follows (see Definition 0.2.3.1):

- The objects of $\mathcal{SW}$ are pairs (X, m), where X is a pointed finite space and m is an integer.
- Given a pair of objects $(X, m), (Y, n) \in \mathcal{SW}$, the set of morphisms from (X, m) to (Y, n) is given by the direct limit $\varinjlim_k [\Sigma^{m+k} X, \Sigma^{n+k} Y]$, where $[U, V]$ denotes the set of homotopy classes of pointed maps from U to V.

The Spanier-Whitehead category $\mathcal{SW}$ arises naturally as the homotopy category of the stable ∞-category $\mathrm{Sp}^{\mathrm{fin}}$ of finite spectra, which is obtained from the ∞-category $\mathcal{S}_*^{\mathrm{fin}}$ of pointed finite spaces by "inverting" the suspension functor. This is a special case of a more general construction.

Construction C.1.1.1 (Spanier-Whitehead)**.** Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits and let $\Sigma : \mathcal{C} \to \mathcal{C}$ denote the suspension functor (given by $\Sigma X = 0 \amalg_X 0 = \mathrm{cofib}(X \to 0)$). We let $\mathrm{SW}(\mathcal{C})$ denote the ∞-category given by the colimit of the sequence

$$\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \cdots$$

We will refer to $\mathrm{SW}(\mathcal{C})$ as the *Spanier-Whitehead ∞-category of $\mathcal{C}$*.

Remark C.1.1.2. In more concrete terms, we can identify the objects of $\mathrm{SW}(\mathcal{C})$ with pairs (C, m), where C is an object of the ∞-category $\mathcal{C}$ and m is an integer; morphism spaces in $\mathrm{SW}(\mathcal{C})$ are computed by the formula

$$\mathrm{Map}_{\mathrm{SW}(\mathcal{C})}((C, m), (D, n)) \simeq \varinjlim_k \mathrm{Map}_{\mathcal{C}}(\Sigma^{m+k} C, \Sigma^{n+k} D)$$

where the colimit is taken over $k \geqslant \max\{-m, -n\}$. Roughly speaking, we can think of (C, m) as an m-fold suspension of the object $C \in \mathcal{C}$ (which exists in the original ∞-category $\mathcal{C}$ for $m \geqslant 0$, but has otherwise been formally adjoined by passing from $\mathcal{C}$ to $\mathrm{SW}(\mathcal{C})$).

Remark C.1.1.3. The passage from an ∞-category $\mathcal{C}$ to its homotopy category $\mathrm{h}\mathcal{C}$ commutes with filtered colimits. Consequently, if $\mathcal{C}$ is a pointed ∞-category which admits finite colimits, then the homotopy category of $\mathrm{SW}(\mathcal{C})$ can be identified with the colimit of the diagram

$$\mathrm{h}\mathcal{C} \xrightarrow{\Sigma} \mathrm{h}\mathcal{C} \xrightarrow{\Sigma} \mathrm{h}\mathcal{C} \xrightarrow{\Sigma} \cdots .$$

In particular, it depends only on the homotopy category $\mathrm{h}\mathcal{C}$ and the suspension functor $\Sigma : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{C}$.

Example C.1.1.4. Let $\mathcal{C}$ be the ∞-category of pointed finite spaces. Then the homotopy category $\mathrm{hSW}(\mathcal{C})$ is the Spanier-Whitehead category $\mathcal{SW}$ of Definition 0.2.3.1.

Remark C.1.1.5. Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. Then the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ preserves finite colimits. Consequently, we can regard $\mathrm{SW}(\mathcal{C})$ as the direct limit of the diagram

$$\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \cdots$$

both in the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories, and in the subcategory $\mathcal{C}\mathrm{at}_\infty^{\mathrm{rex}}$ whose objects are ∞-categories which admits finite colimits and whose morphisms are functors which preserve finite colimits. In particular, each of the natural maps $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$ preserves finite colimits.

Remark C.1.1.6. The construction $\mathcal{C} \mapsto \mathrm{Ind}(\mathcal{C})$ determines a functor $\mathrm{Ind} : \mathcal{C}\mathrm{at}_\infty^{\mathrm{rex}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$, where $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is the ∞-category of presentable stable ∞-categories (where the morphisms are given by functors which preserve small colimits). Moreover, the functor Ind commutes with colimits. Consequently, if $\mathcal{C}$ is a (small) ∞-category which admits finite colimits, then we can identify $\mathrm{Ind}(\mathrm{SW}(\mathcal{C}))$ with the direct limit of the sequence

$$\mathrm{Ind}(\mathcal{C}) \xrightarrow{\Sigma} \mathrm{Ind}(\mathcal{C}) \xrightarrow{\Sigma} \mathrm{Ind}(\mathcal{C}) \xrightarrow{\Sigma} \cdots$$

in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Using Corollary HTT.5.5.3.4 and Theorem HTT.5.5.3.18 , we can identify this colimit with the limit of the tower of right adjoint functors

$$\mathrm{Ind}(\mathcal{C}) \xleftarrow{\Omega} \mathrm{Ind}(\mathcal{C}) \xleftarrow{\Omega} \mathrm{Ind}(\mathcal{C}) \xleftarrow{\Omega} \cdots ;$$

that is, with the ∞-category $\mathrm{Sp}(\mathrm{Ind}(\mathcal{C}))$ of spectrum objects of $\mathrm{Ind}(\mathcal{C})$. Using an elaboration of this observation, we obtain a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C}\mathrm{at}_\infty^{\mathrm{rex}} & \xrightarrow{\mathrm{SW}} & \mathcal{C}\mathrm{at}_\infty^{\mathrm{rex}} \\ \downarrow{\scriptstyle \mathrm{Ind}} & & \downarrow{\scriptstyle \mathrm{Ind}} \\ \mathcal{P}\mathrm{r}^{\mathrm{L}} & \xrightarrow{\mathrm{Sp}} & \mathcal{P}\mathrm{r}^{\mathrm{L}} . \end{array}$$

The construction $\mathcal{C} \mapsto \mathrm{SW}(\mathcal{C})$ can be characterized by a universal property:

Proposition C.1.1.7. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. Then:*

- (a) *The ∞-category $\mathrm{SW}(\mathcal{C})$ is stable.*
- (b) *For any stable ∞-category $\mathcal{D}$, composition with the canonical map $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$ induces an equivalence $\mathrm{Fun}^{\mathrm{rex}}(\mathrm{SW}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{D})$; here $\mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{D})$ is the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve finite colimits, and $\mathrm{Fun}^{\mathrm{rex}}(\mathrm{SW}(\mathcal{C}), \mathcal{D})$ is defined similarly.*

Proof. It follows immediately from Remark C.1.1.2 that the Spanier-Whitehead ∞-category $\mathrm{SW}(\mathcal{C})$ is also pointed (in fact, each of the natural maps $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$ carries zero objects of $\mathcal{C}$ to zero objects of $\mathrm{SW}(\mathcal{C})$). Unwinding the definitions, we see that the suspension functor on $\mathrm{SW}(\mathcal{C})$ is induced by the solid arrows in the diagram

$$\begin{array}{ccccccc} \mathcal{C} & \xrightarrow{\Sigma} & \mathcal{C} & \xrightarrow{\Sigma} & \mathcal{C} & \xrightarrow{\Sigma} & \cdots \\ \downarrow{\scriptstyle\Sigma} & \nearrow^{\mathrm{id}} & \downarrow{\scriptstyle\Sigma} & \nearrow & \downarrow{\scriptstyle\Sigma} & \nearrow & \downarrow \\ \mathcal{C} & \xrightarrow{\Sigma} & \mathcal{C} & \xrightarrow{\Sigma} & \mathcal{C} & \longrightarrow & \cdots ; \end{array}$$

it follows by inspection that the dotted arrows provide a homotopy inverse to the suspension functor. Assertion (a) now follows from Proposition HA.1.4.2.11 .

We now prove (b). By virtue of Remark C.1.1.5, it will suffice to show that precomposition with the suspension functor on $\mathcal{C}$ induces an equivalence θ from $\mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{D})$ to itself. Since right exact functors commute with suspension, the functor θ is also given by postcomposition with the suspension functor on $\mathcal{D}$, and is therefore an equivalence (since $\mathcal{D}$ is assumed to be stable). □

C.1.2 Prestability

We now introduce our main objects of study in this section.

Definition C.1.2.1. Let $\mathcal{C}$ be an ∞-category. We will say that $\mathcal{C}$ is *prestable* if the following conditions are satisfied:

- (a) The ∞-category $\mathcal{C}$ is pointed and admits finite colimits.
- (b) The suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is fully faithful.
- (c) For every morphism $f : Y \to \Sigma Z$ in $\mathcal{C}$, there exists a pullback square σ :

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & \Sigma Z \end{array}$$

in $\mathcal{C}$. Moreover, σ is also a pushout square.

If $\mathcal{C}$ is an ∞-category which satisfies conditions (a) of Definition C.1.2.1, then conditions (b) and (c) can be reformulated in terms of the Spanier-Whitehead ∞-category:

Proposition C.1.2.2. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. Then the following conditions are equivalent:*

- (b) *The suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is fully faithful.*
- (b') *The canonical map $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$ is fully faithful.*
- (b'') *There exists a fully faithful embedding $\rho : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is a stable ∞-category and ρ preserves finite colimits.*

If these conditions are satisfied, then the following further conditions are equivalent:

- (c) *For every morphism $f : Y \to \Sigma Z$ in $\mathcal{C}$, there exists a pullback square σ :*

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & \Sigma Z \end{array}$$

  in $\mathcal{C}$. Moreover, σ is also a pushout square.

- (c') *For every fully faithful embedding $\rho : \mathcal{C} \to \mathcal{D}$ which preserves finite colimits, if $\mathcal{D}$ is stable, then the essential image of ρ is closed under extensions in $\mathcal{D}$.*
- (c'') *The essential image of the canonical map $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$ is closed under extensions.*
- (c''') *There exists a fully faithful embedding $\rho : \mathcal{C} \to \mathcal{D}$ which preserves finite colimits, the ∞-category $\mathcal{D}$ is stable, and the essential image of ρ is closed under extensions in $\mathcal{D}$.*

Proof. The implication $(b) \Rightarrow (b')$ follows immediately from the definition of $\mathrm{SW}(\mathcal{C})$ and the implication $(b') \Rightarrow (b'')$ is a tautology. To show that $(b'') \Rightarrow (b)$, suppose that $\rho : \mathcal{C} \to \mathcal{D}$ is a fully faithful embedding which commutes with finite colimits and that $\mathcal{D}$ is stable. We then have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\rho} & \mathcal{D} \\ \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \Sigma} \\ \mathcal{C} & \xrightarrow{\rho} & \mathcal{D} \end{array}$$

where the horizontal maps and the right vertical map are fully faithful; it follows that the left vertical map is also fully faithful.

Assume now that (b), (b'), and (b'') are satisfied. We first show that (c) implies (c'). Let $\rho : \mathcal{C} \to \mathcal{D}$ be as above, and suppose we are given a fiber sequence

$$\rho(C') \to D \to \rho(C'')$$

in the stable ∞-category $\mathcal{D}$; we wish to show that D belongs to the essential image of ρ. Note that we can identify D with the fiber of a map $f : \rho(C'') \to \Sigma\rho(C') \simeq \rho(\Sigma C')$. Using our assumption that ρ is fully faithful, we can assume that $f = \rho(f_0)$ for some map $f_0 : C'' \to \Sigma C'$ in the ∞-category $\mathcal{C}$. It follows from assumption (c) that we can complete f_0 to a fiber sequence $C \to C'' \xrightarrow{f_0} \Sigma C'$ which is also a cofiber sequence. Since the functor ρ preserves finite colimits, we obtain a cofiber sequence $\rho(C) \to \rho(C'') \xrightarrow{f} \rho(\Sigma C)$ in the ∞-category $\mathcal{D}$. Because $\mathcal{D}$ is stable, we obtain $D \simeq \mathrm{fib}(f) \simeq \rho(C)$ so that D belongs to the essential image of ρ, as desired.

The implications $(c') \Rightarrow (c'') \Rightarrow (c''')$ are immediate. We will complete the proof by showing that $(c''') \Rightarrow (c)$. Assume that we have a functor $\rho : \mathcal{C} \to \mathcal{D}$ as above, and let $f : Y \to \Sigma Z$ be a morphism in $\mathcal{C}$. Set $D = \mathrm{fib}(\rho(f))$, so that we have a fiber sequence $\rho(Z) \to D \xrightarrow{g} \rho(Y)$ in the ∞-category $\mathcal{D}$. Since the essential image of ρ is closed under extensions, we can write $D = \rho(X)$ for some $X \in \mathcal{C}$. Because ρ is fully faithful and preserves finite colimits, the cofiber sequence $D \xrightarrow{g} \rho(Y) \xrightarrow{\rho(f)} \rho(\Sigma Z)$ can be lifted to a cofiber sequence $X \xrightarrow{g_0} Y \xrightarrow{f} \Sigma Z$ in the ∞-category $\mathcal{C}$. Note that this cofiber sequence is also a fiber sequence, since its essential image under the fully faithful embedding ρ is a fiber sequence. □

Corollary C.1.2.3. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

- (i) *The ∞-category $\mathcal{C}$ is prestable.*
- (ii) *There exists a fully faithful embedding $\rho : \mathcal{C} \hookrightarrow \mathcal{D}$, where $\mathcal{D}$ is a stable ∞-category and the essential image of ρ is closed under finite colimits and extensions.*

Corollary C.1.2.4. *Let $\mathcal{C}$ be a prestable ∞-category and suppose we are given a cofiber sequence $C' \to C \to C''$ in $\mathcal{C}$. If any two of the objects $C, C', C'' \in \mathcal{C}$ vanishes, then so does the third.*

Corollary C.1.2.5. *Let $\mathcal{C}$ be a prestable ∞-category. If $u : C \to C'$ is a morphism in $\mathcal{C}$ satisfying* $\mathrm{cofib}(u) \simeq 0$*, then u is an equivalence.*

Corollary C.1.2.6. *Let $\mathcal{C}$ be a prestable ∞-category. Then every pushout square in $\mathcal{C}$ is also a pullback square.*

Proof. Choose a fully faithful embedding $\rho : \mathcal{C} \hookrightarrow \mathcal{D}$ which preserves finite colimits, where $\mathcal{D}$ is stable. If $\sigma \in \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ is a pushout square in $\mathcal{C}$, then $\rho(\sigma)$ is a pushout square in $\mathcal{D}$. Since $\mathcal{D}$ is stable, it follows that $\rho(\sigma)$ is also a pullback square in $\mathcal{D}$ (Proposition HA.1.1.3.4). Because ρ is fully faithful, we conclude that σ is a pullback square in $\mathcal{C}$. □

Example C.1.2.7. Any stable ∞-category is prestable.

Example C.1.2.8. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$. Then the full subcategory $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$ is prestable.

The prestable ∞-categories which arise from Example C.1.2.8 can be characterized as follows:

Proposition C.1.2.9. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

- (a) *The ∞-category $\mathcal{C}$ is prestable and admits finite limits.*
- (b) *The ∞-category $\mathcal{C}$ is pointed and admits finite colimits, the canonical map $\rho : \mathcal{C} \to \mathrm{SW}(\mathcal{C})$ is fully faithful. Moreover, the stable ∞-category $\mathrm{SW}(\mathcal{C})$ admits a t-structure $(\mathrm{SW}(\mathcal{C})_{\geqslant 0}, \mathrm{SW}(\mathcal{C})_{\leqslant 0})$ where $\mathrm{SW}(\mathcal{C})_{\geqslant 0}$ is the essential image of ρ.*
- (c) *There exists a stable ∞-category $\mathcal{D}$ equipped with a t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ and an equivalence of ∞-categories $\mathcal{C} \simeq \mathcal{D}_{\geqslant 0})$.*

Proof. We first show that $(a) \Rightarrow (b)$. Assume that $\mathcal{C}$ is prestable and admits finite limits, and let $\mathrm{SW}(\mathcal{C})_{\geqslant 0}$ denote the full subcategory of $\mathrm{SW}(\mathcal{C})$ spanned by the essential image of the canonical map $\rho : \mathcal{C} \hookrightarrow \mathrm{SW}(\mathcal{C})$ (which is fully faithful by virtue of Proposition C.1.2.2). Let $\mathrm{SW}(\mathcal{C})_{\leqslant 0}$ denote the full subcategory of $\mathrm{SW}(\mathcal{C})$ spanned by those objects of the form $\Omega^n \rho(C)$, where C is an n-truncated object of $\mathcal{C}$. We will prove (b) by showing that the pair $(\mathrm{SW}(\mathcal{C})_{\geqslant 0}, \mathrm{SW}(\mathcal{C})_{\leqslant 0})$ is a t-structure on the stable ∞-category $\mathrm{SW}(\mathcal{C})$: that is, that it satisfies the conditions of Definition HA.1.2.1.1 :

(1) For each object $X \in \mathrm{SW}(\mathcal{C})_{\geqslant 0}$ and $Y \in \mathrm{SW}(\mathcal{C})_{\leqslant 0}$, every map $u : X \to \Omega Y$ is nullhomotopic. Write $X = \rho(C)$ and $Y = \Omega^n \rho(D)$ for some objects $C, D \in \mathcal{C}$ where D is n-truncated. Since ρ is fully faithful, the mapping space

$$\mathrm{Map}_{\mathrm{SW}(\mathcal{C})}(X, \Omega Y) \simeq \mathrm{Map}_{\mathrm{SW}(\mathcal{C})}(\rho(C), \Omega^{n+1} \rho(D)) \simeq \Omega^{n+1} \mathrm{Map}_{\mathcal{C}}(X, Y)$$

is contractible.

(2) We have $\Sigma\, \mathrm{SW}(\mathcal{C})_{\geqslant 0} \subseteq \mathrm{SW}(\mathcal{C})_{\geqslant 0}$ and $\Omega\, \mathrm{SW}(\mathcal{C})_{\leqslant 0} \subseteq \mathrm{SW}(\mathcal{C})_{\leqslant 0}$. The first inclusion follows from the observation that $\Sigma \rho(\mathcal{C}) = \rho(\Sigma\, \mathcal{C}) \subseteq \rho(\mathcal{C})$, and the second follows from the observation that every n-truncated object of $\mathcal{C}$ is also $(n+1)$-truncated.

(3) For every object $X \in \mathrm{SW}(\mathcal{C})$, there exists a fiber sequence $X' \to X \to X''$ where $X' \in \mathrm{SW}(\mathcal{C})_{\geqslant 0}$ and $X'' \in \mathrm{SW}(\mathcal{C})_{\leqslant -1}$. Write $X = \Omega^n \rho(C)$ for some object $C \in \mathcal{C}$, set $C' = \Sigma^n \Omega^n C$, and let C'' denote the cofiber of the canonical map $C' \to C$. Since the functor ρ preserves cofiber sequences, we obtain a cofiber sequence

$$\Omega^n \rho(C') \to \Omega^n \rho(C) \to \Omega^n \rho(C'').$$

The first term in this sequence can be rewritten as

$$\Omega^n \rho(\Sigma^n \Omega^n C) \simeq \Omega^n \Sigma^n \rho(\Omega^n C) \simeq \rho(\Omega^n C)$$

and therefore belongs to $\mathrm{SW}(\mathcal{C})_{\geqslant 0}$. It will therefore suffice to show that $\Omega^n \rho(C'')$ belongs to $\mathrm{SW}(\mathcal{C})_{\leqslant 0}$. We will prove this by verifying that the object $C'' \in \mathcal{C}$ is $(n-1)$-truncated. Fix an object $D \in \mathcal{C}$; we wish to show that the mapping space $\mathrm{Map}_{\mathcal{C}}(D, C'')$ is $(n-1)$-truncated. In other words, we wish to show that the homotopy groups of $\mathrm{Map}_{\mathcal{C}}(D, C'')$ vanish in degrees $\geqslant n$, for any choice of base point. Note that $\mathrm{Map}_{\mathcal{C}}(D, C'') \simeq \mathrm{Map}_{\mathrm{SW}(\mathcal{C})}(\rho(D), \rho(C''))$ is an infinite loop space; we may therefore assume without loss of generality that our base point is chosen to be the zero map $D \to C''$. In other words, it will suffice to show that for $m \geqslant n$, any map $u : \Sigma^m D \to C''$ is nullhomotopic. To prove this, set $Y = \Omega^n(\rho(\Sigma^m D) \times_{\rho(C'')} \rho(C)) \in \mathrm{SW}(\mathcal{D})$, so that we have a fiber sequence $\rho(\Omega^n C) \to Y \to \rho(\Sigma^{m-n} D)$. Because the essential image of ρ is closed under extensions, we can write $Y = \rho(E)$ for some object $E \in \mathcal{C}$. We then have a commutative diagram of cofiber sequences

$$\begin{array}{ccccc} \Sigma^n \Omega^n C & \longrightarrow & \Sigma^n E & \longrightarrow & \Sigma^m D \\ \downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ \Sigma^n \Omega^n C & \longrightarrow & C & \longrightarrow & C''. \end{array}$$

It follows from the adjointness of Σ^n and Ω^n that the map v factors through $\Sigma^n \Omega^n C$. It follows that u factors through C, and therefore (since $\Sigma^m D$ is an n-fold suspension) through the composite map $\Sigma^n \Omega^n C \to C \to C''$ which is nullhomotopic.

The implication $(b) \Rightarrow (c)$ is trivial. We complete the proof by showing that $(c) \Rightarrow (a)$. Let $\mathcal{D}$ be a stable ∞-category equipped with a t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$; we wish to show that $\mathcal{D}_{\geqslant 0}$ is a prestable ∞-category which admits finite limits. The prestability of $\mathcal{D}_{\geqslant 0}$ follows from Example C.1.2.8, and the existence of finite limits follows from the fact that $\mathcal{D}_{\geqslant 0}$ is a colocalization of the stable ∞-category $\mathcal{D}$. □

Remark C.1.2.10. Let $\mathcal{C}$ be a prestable ∞-category which admits finite limits. It follows from Proposition C.1.2.9 that there exists a stable ∞-category $\mathcal{D}$ with a t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ and an equivalence $\mathcal{C} \simeq \mathcal{D}_{\geqslant 0}$. The stable ∞-category $\mathcal{D}$ is not uniquely determined. However, there are two canonical choices for $\mathcal{D}$:

(a) One can take $\mathcal{D}$ to be the Spanier-Whitehead ∞-category $\mathrm{SW}(\mathcal{C})$, as in the proof of Proposition C.1.2.9: that is, we can take $\mathcal{D}$ to be the colimit of the sequence

$$\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \to \cdots .$$

In this case, the t-structure on $\mathcal{D}$ is right bounded: that is, we have $\mathcal{D} = \bigcup \mathcal{D}_{\geqslant -n}$.

(*b*) One can take $\mathcal{D}$ to be the ∞-category $\mathrm{Sp}(\mathcal{C})$ of spectrum objects of $\mathcal{C}$, defined as the homotopy limit of the tower of ∞-categories

$$\cdots \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$$

which we can identify with

$$\cdots \to \mathrm{SW}(\mathcal{C})_{\geqslant -2} \xrightarrow{\tau_{\geqslant -1}} \mathrm{SW}(\mathcal{C})_{\geqslant -1} \xrightarrow{\tau_{\geqslant 0}} \mathrm{SW}(\mathcal{C})_{\geqslant 0}.$$

In other words, we can identify $\mathcal{D}$ with the right completion of the ∞-category $\mathrm{SW}(\mathcal{C})$ with respect to its t-structure; in particular, the t-structure on the ∞-category $\mathcal{D}$ is right complete.

An arbitrary stable ∞-category $\mathcal{D}$ with a t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ and an equivalence $\mathcal{C} \simeq \mathcal{D}_{\geqslant 0}$ lies somewhere between these two extremes: more precisely, one has functors

$$\varinjlim(\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \to \cdots) \simeq \bigcup_{n \geqslant 0} \mathcal{D}_{\geqslant -n} \subseteq \mathcal{D} \to \varprojlim_{n \geqslant 0} \mathcal{D}_{\geqslant -n} \simeq \mathrm{Sp}(\mathcal{C}).$$

Notation C.1.2.11. Let $\mathcal{C}$ be a prestable ∞-category. We let $\mathcal{C}^{\heartsuit}$ denote the full subcategory of $\mathcal{C}$ spanned by the discrete objects. We will refer to $\mathcal{C}^{\heartsuit}$ as the *heart* of $\mathcal{C}$. Note that if $\mathcal{C}$ admits finite limits, then Proposition C.1.2.9 implies that we can identify $\mathcal{C}^{\heartsuit}$ with the heart of a t-structure on the stable ∞-category $\mathrm{SW}(\mathcal{C})$. In particular, $\mathcal{C}^{\heartsuit}$ is an abelian category.

Definition C.1.2.12. Let $\mathcal{C}$ be a prestable ∞-category which admits finite limits. We will say that that an object $X \in \mathcal{C}$ is *∞-connective* if $\tau_{\leqslant n} X \simeq 0$ for every integer n. We will say that $\mathcal{C}$ is *separated* if every ∞-connective object of $\mathcal{C}$ is a zero object. We say that $\mathcal{C}$ is *complete* it is a homotopy limit of the tower of ∞-categories

$$\cdots \to \tau_{\leqslant 2}\,\mathcal{C} \xrightarrow{\tau_{\leqslant 1}} \tau_{\leqslant 1}\,\mathcal{C} \xrightarrow{\tau_{\leqslant 0}} \tau_{\leqslant 0}\,\mathcal{C} = \mathcal{C}^{\heartsuit}.$$

In other words, $\mathcal{C}$ is complete if it is Postnikov complete (in the sense of Definition A.7.2.1).

Remark C.1.2.13. If a prestable ∞-category $\mathcal{C}$ is complete, then it is separated.

Remark C.1.2.14. Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ is separated in the sense of Definition C.1.2.12 (when regarded as a prestable ∞-category) if and only if $\mathcal{C} \simeq *$.

C.1.3 The Prestable Dold-Kan Correspondence

Let $\mathcal{C}$ be an ∞-category which admits finite colimits and let $X_\bullet \in \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{C})$ be a simplicial object of $\mathcal{C}$. For each $n \geqslant 0$, we define the *n-skeleton* $\mathrm{sk}_n(X_\bullet)$ to be the colimit $\varinjlim_{[m] \in \boldsymbol{\Delta}^{\mathrm{op}}_{\leqslant n}} X_m$ (note that this colimit exists in $\mathcal{C}$, since it can be rewritten as a colimit over the subcategory $\mathbf{\Delta}^{\rm op}_{s,\leqslant n} \subseteq \mathbf{\Delta}^{\rm op}_{\leqslant n}$ having finite nerve; see Lemma HA.1.2.4.17). Note that$\mathbf{\Delta}_{\leqslant n}$ is contained in $\mathbf{\Delta}_{\leqslant n'}$ for $n \leqslant n'$, so that the skeleta of $X_\bullet$ can be arranged in a diagram

$$\mathrm{sk}_0(X_\bullet) \to \mathrm{sk}_1(X_\bullet) \to \mathrm{sk}_2(X_\bullet) \to \cdots .$$

The construction $X_\bullet \mapsto \{\mathrm{sk}_n(X_\bullet)\}_{n \geqslant 0}$ determines a functor

$$\mathrm{sk}_* : \mathrm{Fun}(\mathbf{\Delta}^{\rm op}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geqslant 0}), \mathcal{C})$$

from simplicial objects of $\mathcal{C}$ to filtered objects of $\mathcal{C}$ (here $\mathbf{Z}_{\geqslant 0}$ denotes the set of nonnegative integers, endowed with its usual ordering). In §HA.1.2.4 , we proved that this functor is an equivalence when $\mathcal{C}$ is a stable ∞-category (see Theorem HA.1.2.4.1 and its proof). We now establish the following refinement:

Theorem C.1.3.1 (The Prestable Dold-Kan Correspondence)**.** *Let $\mathcal{C}$ be a prestable ∞-category. Then the functor*

$$\mathrm{sk}_* : \mathrm{Fun}(\mathbf{\Delta}^{\rm op}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geqslant 0}), \mathcal{C})$$

is fully faithful. Moreover, the essential image of sk_* *is spanned by those diagrams*

$$X(0) \xrightarrow{f(1)} X(1) \xrightarrow{f(2)} X(2) \to \cdots$$

which possess the following property:

- $(*)$ *For each $n > 0$, the cofiber* $\mathrm{cofib}(f(n))$ *belongs to the essential image of the iterated suspension functor $\Sigma^n : \mathcal{C} \to \mathcal{C}$.*

Proof. Let $\mathrm{SW}(\mathcal{C})$ denote the Spanier-Whitehead category of $\mathcal{C}$ and let $\rho : \mathcal{C} \to \mathrm{SW}(\mathcal{C})$ be the canonical map. Since ρ commutes with finite colimits, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}(\mathbf{\Delta}^{\rm op}, \mathcal{C}) & \xrightarrow{\mathrm{sk}_*} & \mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geqslant 0}), \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathbf{\Delta}^{\rm op}, \mathrm{SW}(\mathcal{C})) & \xrightarrow{\mathrm{sk}_*} & \mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geqslant 0}), \mathrm{SW}(\mathcal{C})), \end{array}$$

where the vertical maps are given by composition with ρ (and are therefore fully faithful). Since the bottom horizontal map is an equivalence of ∞-categories (Theorem HA.1.2.4.1), it follows immediately that the functor $\mathrm{sk}_* : \mathrm{Fun}(\mathbf{\Delta}^{\rm op}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geqslant 0}), \mathcal{C})$ is fully faithful.

For each $n \geqslant 0$, consider the following hypotheses on a simplicial object $X_\bullet$ of $\mathrm{SW}(\mathcal{C})$:

- (a_n) The object X_n belongs to the essential image of the functor $\rho : \mathcal{C} \to \mathrm{SW}(\mathcal{C})$.
- (b_n) The cofiber of the map $\mathrm{sk}_{n-1}(X_\bullet) \to \mathrm{sk}_n(X_\bullet)$ belongs to the essential image of the functor $(\rho \circ \Sigma^n) : \mathcal{C} \to \mathrm{SW}(\mathcal{C})$ (here we adopt the convention that $\mathrm{sk}_{-1}(X_\bullet) \simeq 0$).

To complete the proof, it will suffice to show that a simplicial object $X_\bullet$ of $\mathrm{SW}(\mathcal{C})$ satisfies condition (a_n) for all $n \geqslant 0$ if and only if it satisfies condition (b_n) for all $n \geqslant 0$. We will prove this by showing that if $X_\bullet$ satisfies condition (a_m) for $0 \leqslant m < n$, then conditions (a_n) and (b_n) are equivalent.

Let us henceforth regard $n \geqslant 0$ as fixed. Let $X_{\leqslant n}$ denote the restriction of X to the full subcategory $\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n} \subseteq \mathbf{\Delta}^{\mathrm{op}}$ and define $X_{\leqslant n-1}$ similarly. Let $\overline{X}', \overline{X} : \mathbf{\Delta}^{\mathrm{op}}_{+,\leqslant n} \to \mathrm{SW}(\mathcal{C})$ be left Kan extensions of $X_{\leqslant n-1}$ and $X_{\leqslant n}$, respectively, so that $\overline{X}'([-1]) \simeq \mathrm{sk}_{n-1} X$, $\overline{X}([-1]) \simeq \mathrm{sk}_n(X)$, and $\overline{X}'([n])$ can be identified with the nth latching object of X. The canonical identification $X_{\leqslant n-1} \simeq X_{\leqslant n}|_{\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n-1}}$ induces a natural transformation of functors $\overline{X}' \to \overline{X}$. Let us denote the cofiber of this natural transformation by $\overline{X}''$, so that we have a cofiber sequence $\overline{X}' \to \overline{X} \to \overline{X}''$ in the ∞-category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{+,\leqslant n}, \mathrm{SW}(\mathcal{C}))$. Note that both $\overline{X}'$ and $\overline{X}$ are left Kan extensions of their restrictions to $\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n}$, so that $\overline{X}''$ has the same property. Set $Z = \mathrm{cofib}(\mathrm{sk}_{n-1}(X_\bullet)\, \mathrm{sk}_n(X_\bullet)) \simeq \overline{X}''([-1])$. Since the objects $\overline{X}''([m])$ vanish for $0 \leqslant m < n$, Corollary HA.1.2.4.18 supplies an equivalence $Z \simeq \Sigma^n \overline{X}''([n])$. In other words, we have a cofiber sequence

$$\overline{X}'([n]) \to X_n \to \Sigma^{-n} Z.$$

Our hypothesis that $X_\bullet$ satisfies (a_m) for $m < n$ guarantees that $\overline{X}'([n])$ belongs to the essential image of ρ. Since the essential image of ρ is closed under cofibers and extensions, it follows that X_n belongs to the essential image of ρ if and only if $\Sigma^{-n} Z$ belongs to the essential image of ρ. We conclude that $(a_n) \Leftrightarrow (b_n)$, as desired. □

C.1.4 Grothendieck Prestable ∞-Categories

Let $\mathcal{A}$ be an abelian category. Recall that $\mathcal{A}$ is said to be *Grothendieck* if it is presentable and filtered colimits in $\mathcal{A}$ are exact. We now discuss an analogous condition in the setting of prestable ∞-categories.

Proposition C.1.4.1. *Let $\mathcal{C}$ be a presentable ∞-category. The following conditions are equivalent:*

- *(a) The ∞-category $\mathcal{C}$ is prestable and filtered colimits in $\mathcal{C}$ are left exact (see Definition HTT.7.3.4.2).*
- *(b) The ∞-category $\mathcal{C}$ is prestable and the functor $\Omega : \mathcal{C} \to \mathcal{C}$ commutes with filtered colimits.*
- *(c) The ∞-category $\mathcal{C}$ is prestable and the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ commutes with filtered colimits.*

(*d*) *There exists a presentable stable ∞-category $\mathcal{D}$, a t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ on $\mathcal{D}$ which is compatible with filtered colimits, and an equivalence $\mathcal{C} \simeq \mathcal{D}_{\geqslant 0}$.*

Definition C.1.4.2. Let $\mathcal{C}$ be a prestable ∞-category. We will say that $\mathcal{C}$ is *Grothendieck* if it presentable and satisfies the equivalent conditions of Proposition C.1.4.1.

Example C.1.4.3. Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ is automatically a prestable ∞-category. It is a Grothendieck prestable ∞-category if and only if it is presentable (in this case, the functor $\Omega : \mathcal{C} \to \mathcal{C}$ automatically commutes with filtered colimits since it is an equivalence of ∞-categories).

Example C.1.4.4. Let $\mathcal{C}$ be a prestable ∞-category. If $\mathcal{C}$ is compactly generated (Definition HTT.5.5.7.1), then $\mathcal{C}$ is a Grothendieck prestable ∞-category (note that filtered colimits are left exact in any compactly generated ∞-category).

Example C.1.4.5. Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{D}(\mathcal{A})$ be the derived ∞-category of $\mathcal{A}$ (see §HA.1.3.5). Then $\mathcal{D}(\mathcal{A})$ admits a t-structure $(\mathcal{D}(\mathcal{A})_{\geqslant 0}, \mathcal{D}(\mathcal{A})_{\leqslant 0})$ which is right complete and compatible with filtered colimits (Proposition HA.1.3.5.21). It follows that $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ is a Grothendieck prestable ∞-category.

Remark C.1.4.6. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Then the heart $\mathcal{C}^{\heartsuit}$ (see Notation C.1.2.11) is a Grothendieck abelian category (Remark HA.1.3.5.23). It follows from Example C.1.4.5 that every Grothendieck abelian category arises in this way. However, the correspondence is many-to-one. For example, if R is a commutative ring, then the Grothendieck abelian category $\mathrm{Mod}_R^{\heartsuit}$ can be identified with the heart of $\mathrm{Mod}_{\overline{R}}^{\mathrm{cn}}$ for *any* connective $\mathbb{E}_\infty$-ring $\overline{R}$ satisfying $\pi_0 \overline{R} \simeq R$. For a more detailed discussion about the relationship between a Grothendieck prestable ∞-category and its heart $\mathcal{C}^{\heartsuit}$, we refer the reader to §C.5.

Proof of Proposition C.1.4.1. The implication $(a) \Rightarrow (b)$ is trivial, the implication $(b) \to (c)$ follows from the observation that $\mathrm{Sp}(\mathcal{C})$ can be realized as the homotopy limit of the tower

$$\cdots \xrightarrow{\Omega} \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C},$$

and the implication $(c) \Rightarrow (d)$ follows by taking $\mathcal{D} = \mathrm{Sp}(\mathcal{C})$ (see Remark C.1.2.10). We will complete the proof by showing that (d) implies (a). Suppose that $\mathcal{E}$ is a presentable stable ∞-category equipped with a t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ which is compatible with filtered colimits. Then $\mathcal{D}_{\geqslant 0}$ is prestable (Proposition C.1.2.9); we wish to show that filtered colimits in $\mathcal{D}_{\geqslant 0}$ are left exact. Let K be a finite simplicial set and let $F_{\geqslant 0} : \mathrm{Fun}(K, \mathcal{D}_{\geqslant 0}) \to \mathcal{D}_{\geqslant 0}$ be a right adjoint to the diagonal map; we wish to show that the functor $F_{\geqslant 0}$ commutes with filtered colimits. Unwinding the definitions, we can write $F_{\geqslant 0}$ as a composition

$$\mathrm{Fun}(K, \mathcal{D}_{\geqslant 0}) \hookrightarrow \mathrm{Fun}(K, \mathcal{D}) \xrightarrow{F} \mathcal{D} \xrightarrow{\tau_{\geqslant 0}} \mathcal{D}_{\geqslant 0},$$

where $F : \mathrm{Fun}(K, \mathcal{D}) \to \mathcal{D}$ is a right adjoint to the diagonal map. Since $\mathcal{D}$ is stable, the functor F preserves filtered colimits (in fact, it preserves all small colimits). It therefore suffices to observe that the truncation functor $\tau_{\geqslant 0}$ preserves filtered colimits, by virtue of our assumption that the t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ is compatible with filtered colimits. □

C.1.5 Additive ∞-Categories

Let $\mathcal{A}$ be a category. Recall that $\mathcal{A}$ is said to be *additive* if it satisfies the following three conditions:

(i) The category $\mathcal{A}$ is pointed: that is, there is an object $0 \in \mathcal{A}$ which is both initial and final.

Using (i), we can associate to every pair of objects $X, Y \in \mathcal{A}$ a *zero morphism* $0 \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$, given by the composition $X \to 0 \to Y$.

(ii) The category $\mathcal{A}$ admits finite products and finite coproducts. Moreover, for every pair of objects $X, Y \in \mathcal{A}$, the canonical map

$$\begin{bmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{bmatrix} : X \amalg Y \to X \times Y$$

is an isomorphism.

Using (ii), we can endow each $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ with the structure of a commutative monoid, where the sum of morphisms $f, g : X \to Y$ is given by the composition

$$(f + g) : X \to X \times X \xrightarrow{(f,g)} Y \times Y \simeq Y \amalg Y \to Y.$$

(iii) For every pair of objects $X, Y \in \mathcal{A}$, the addition law defined above endows the set $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ with the structure of an abelian group.

In this section, we will study the following ∞-categorical generalization of the notion of additive category:

Definition C.1.5.1. Let $\mathcal{C}$ be an ∞-category. We will say that $\mathcal{C}$ is *additive* if it satisfies the following conditions:

(a) The ∞-category $\mathcal{C}$ admits finite products.

(b) The ∞-category $\mathcal{C}$ admits finite coproducts.

(c) The homotopy category $\mathrm{h}\mathcal{C}$ is an additive category.

Remark C.1.5.2. Let $\mathcal{C}$ be an additive ∞-category. Then $\mathcal{C}$ is pointed. Moreover, for every pair of objects $X, Y \in \mathcal{C}$, the additivity of $\mathrm{h}\mathcal{C}$ implies that the canonical map

$$\begin{bmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{bmatrix} : X \amalg Y \to X \times Y$$

is an equivalence. We will henceforth denote both the coproduct $X \amalg Y$ and the product $X \times Y$ by $X \oplus Y$, which we refer to as the *direct sum* of X and Y.

Remark C.1.5.3. Let $\mathcal{C}$ be an ∞-category. If $\mathcal{C}$ satisfies condition (a) of Definition C.1.5.1, then we can regard $\mathcal{C}$ as equipped with the Cartesian symmetric monoidal structure introduced in §HA.2.4.1 . Let $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C})$ denote the ∞-category of commutative monoid objects of $\mathcal{C}$ (see §HA.2.4.2). If $\mathcal{C}$ satisfies also condition (b) of Definition C.1.5.1, then it can also be equipped with the coCartesian symmetric monoidal structure introduced in §**??**. If $\mathcal{C}$ satisfies condition (c) of Definition C.1.5.1, then the Cartesian and coCartesian symmetric monoidal structures on $\mathcal{C}$ are equivalent. In this case, Propositions HA.2.4.2.5 and HA.2.4.3.8 imply that the vertical maps in the diagram

$$\begin{array}{ccc} & \mathrm{CAlg}(\mathcal{C}) & \\ \swarrow & & \searrow \\ \mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C}) & \longrightarrow & \mathcal{C} \end{array}$$

are equivalences of ∞-categories, so that the forgetful functor $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C}) \to \mathcal{C}$ is also an equivalence of ∞-categories. In other words, every object of $\mathcal{C}$ admits the structure of a commutative monoid (with respect to the direct sum $\oplus$) in an essentially unique way.

Example C.1.5.4. Let $\mathcal{C}$ be an additive ∞-category, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory which is closed under finite coproducts. Then $\mathcal{C}_0$ is also an additive ∞-category.

Example C.1.5.5. Any stable ∞-category is additive (see Lemma HA.1.1.2.9).

Example C.1.5.6. Any prestable ∞-category is additive (this follows from Examples C.1.5.4 and C.1.5.5, together with Corollary C.1.2.3).

We now establish a converse to Example C.1.5.6: every (small) additive ∞-category $\mathcal{C}$ admits a canonical embedding into a (Grothendieck) prestable ∞-category, whose essential image is closed under finite coproducts.

Proposition C.1.5.7. *Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts, let $\mathcal{P}_{\Sigma}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ be the full subcategory spanned by those functors which preserve finite products, and let $j : \mathcal{C} \to \mathcal{P}_{\Sigma}(\mathcal{C})$ be the Yoneda embedding. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is additive.*

(2) *The ∞-category $\mathcal{P}_\Sigma(\mathcal{C})$ is prestable.*

Lemma C.1.5.8. *Let $\mathcal{C}$ be a small additive ∞-category. Then the ∞-category $\mathcal{P}_\Sigma(\mathcal{C})$ is also additive.*

Proof. Since $\mathcal{P}_\Sigma(\mathcal{C})$ is presentable, it admits finite limits and colimits. Let $j : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})$ be the Yoneda embedding, so that j preserves finite products (Proposition HTT.5.1.3.2) and finite coproducts (Proposition HTT.5.5.8.10). In particular, if 0 is a zero object of $\mathcal{C}$, then $j(0)$ is a zero object of $\mathcal{P}_\Sigma(\mathcal{C})$. Consequently, for every pair of objects $X, Y \in \mathcal{P}_\Sigma(\mathcal{C})$, we obtain a canonical map $\theta_{X,Y} : X \amalg Y \to X \times Y$. We wish to prove that $\theta_{X,Y}$ is an equivalence for all $X, Y \in \mathcal{P}_\Sigma(\mathcal{C})$. Since $\mathcal{P}_\Sigma(\mathcal{C})$ is closed under sifted colimits in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ and the formation of products in $\mathcal{S}$ commutes with small colimits in each variable, it follows that the construction $(X, Y) \mapsto X \times Y$ preserves sifted colimits separately in each variable. Consequently, the collection of those pairs (X, Y) for which $\theta_{X,Y}$ is an equivalence is closed under sifted colimits. We may therefore assume without loss of generality that X and Y belong to the essential image of j, in which case the desired result follows from the additivity of $\mathcal{C}$.

Arguing as in Remark C.1.5.3, we see that the forgetful functor $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{P}_\Sigma(\mathcal{C})) \to \mathcal{P}_\Sigma(\mathcal{C})$ is an equivalence of ∞-categories, so that we can regard each object $Y \in \mathcal{P}_\Sigma(\mathcal{C})$ as a commutative monoid object of $\mathcal{P}_\Sigma(\mathcal{C})$. In particular, if $X \in \mathcal{P}_\Sigma(\mathcal{C})$ is another object, then the mapping space $\mathrm{Map}_{\mathcal{P}_\Sigma}(X, Y)$ can be regarded as a commutative monoid object of the ∞-category $\mathcal{S}$, so that $\pi_0 \mathrm{Map}_{\mathcal{P}_\Sigma}(X, Y)$ inherits the structure of a commutative monoid. To complete the proof, it will suffice to show that each of the commutative monoids $\pi_0 \mathrm{Map}_{\mathcal{P}_\Sigma}(X, Y)$ is an abelian group: in other words, each of the mapping spaces $\mathrm{Map}_{\mathcal{P}_\Sigma}(X, Y)$ is *grouplike* commutative monoid object of $\mathcal{S}$, (see Definition **??**). Since the full subcategory of $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{S})$ spanned by the grouplike commutative monoids is closed under limits, the collection of those objects $X \in \mathcal{P}_\Sigma(\mathcal{C})$ for which $\pi_0 \mathrm{Map}_{\mathcal{P}_\Sigma}(X, Y)$ is an abelian group is closed under small colimits. We may therefore assume without loss of generality that X belongs to the essential image of j, so that X is a compact projective object of $\mathcal{P}_\Sigma$. Write Y as the geometric realization of a simplicial object $Y_\bullet$, where Y_0 is a coproduct of objects belonging to the essential image of j. Since X is projective, the map

$$\pi_0 \mathrm{Map}_{\mathcal{P}_\Sigma}(X, Y_0) \to \pi_0 \mathrm{Map}_{\mathcal{P}_\Sigma}(X, Y)$$

is surjective. We may therefore replace Y by Y_0 and thereby reduce to the case where Y has the form $\coprod_{\alpha \in A} j(Y_\alpha)$ for some objects $Y_\alpha \in \mathcal{C}$. Since X is compact, the commutative monoid $\pi_0 \mathrm{Map}_{\mathcal{P}_\Sigma}(X, Y)$ can be written as a filtered colimit of commutative monoids of the form $\pi_0 \mathrm{Map}_{\mathcal{P}_\Sigma}(X, \coprod_{\alpha \in A_0} j(Y_\alpha))$, where A_0 ranges over all finite subsets of A. We may therefore replace Y by $j(\coprod_{\alpha \in A_0} Y_\alpha)$ and thereby reduce to the case where Y also belongs to the essential image of j. In this case, the existence of inverses in the commutative monoid $\mathrm{Map}_{\mathcal{P}_\Sigma}(X, Y)$ follows from our assumption that $\mathcal{C}$ is additive. □

Proof of Proposition C.1.5.7. Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts. Then the Yoneda embedding $j : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})$ is fully faithful and preserves finite coproducts (Proposition HTT.5.5.8.10). Using Examples C.1.5.4 and C.1.5.6, we deduce that if $\mathcal{P}_\Sigma(\mathcal{C})$ is prestable, then $\mathcal{C}$ is additive. For the converse, assume that $\mathcal{C}$ is additive. Using Remark HA.5.2.6.26 , we can identify the ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ of connective spectra with the full subcategory $\mathrm{Mon}^{\mathrm{gp}}_{\mathrm{Comm}}(\mathcal{S}) \subseteq \mathrm{Mon}_{\mathrm{Comm}}(\mathcal{S})$ spanned by the grouplike commutative monoid objects of $\mathcal{S}$. This identification furnishes an equivalence $\mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}^{\mathrm{cn}}) \simeq \mathrm{Mon}^{\mathrm{gp}}_{\mathrm{Comm}}(\mathcal{P}_\Sigma(\mathcal{C}))$, where $\mathrm{Mon}^{\mathrm{gp}}_{\mathrm{Comm}}(\mathcal{P}_\Sigma(\mathcal{C}))$ denotes the full subcategory of $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{P}_\Sigma(\mathcal{C}))$ spanned by those commutative monoid objects X having the property that for each object $C \in \mathcal{C}$, the object $X(C) \simeq \mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{C})}(j(C), X) \in \mathrm{Mon}_{\mathrm{Comm}}(\mathcal{S})$ is grouplike. It follows from Lemma C.1.5.8 that $\mathcal{P}_\Sigma(\mathcal{C})$ is additive, so that the forgetful functor

$$\mathrm{Mon}^{\mathrm{gp}}_{\mathrm{Comm}}(\mathcal{P}_\Sigma(\mathcal{C})) = \mathrm{Mon}_{\mathrm{Comm}}(\mathcal{P}_\Sigma(\mathcal{C})) \to \mathcal{P}_\Sigma(\mathcal{C})$$

is an equivalence of ∞-categories. We therefore obtain an equivalence $\mathcal{P}_\Sigma(\mathcal{C}) \simeq \mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}^{\mathrm{cn}})$. In particular, there is a fully faithful embedding $\mathcal{P}_\Sigma(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ whose essential image is closed under finite colimits and extensions, so that $\mathcal{P}_\Sigma(\mathcal{C})$ is prestable by virtue of Corollary C.1.2.3. □

Remark C.1.5.9. Let $\mathcal{C}$ be a small additive ∞-category and let $\mathcal{E} = \mathcal{P}_\Sigma(\mathcal{C})$. It follows from Proposition C.1.5.7 that the ∞-category $\mathcal{E}$ is prestable. The proof of Proposition C.1.5.7 yields an equivalence $\mathcal{E} \simeq \mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}^{\mathrm{cn}})$, which yields equivalences

$$\mathrm{Sp}(\mathcal{E}) \simeq \mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \qquad \mathcal{E}^{\heartsuit} \simeq \mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}^{\heartsuit}).$$

In particular, $\mathcal{E}$ is a Grothendieck prestable ∞-category and the canonical map $\mathcal{E} \to \varprojlim \tau_{\leqslant n} \mathcal{E}$ is an equivalence (that is, $\mathcal{E}$ is *complete* in the sense of Definition C.1.2.12).

Remark C.1.5.10. Let $\mathcal{E}$ be an ∞-category which is projectively generated (see Definition HTT.5.5.8.23). Then the following conditions are equivalent:

(a) The ∞-category $\mathcal{E}$ is additive.

(b) The ∞-category $\mathcal{E}$ is Grothendieck prestable.

(c) There exist a small additive ∞-category $\mathcal{C}$ and an equivalence $\mathcal{P}_\Sigma(\mathcal{C}) \simeq \mathcal{E}$.

The implication $(c) \Rightarrow (b)$ follows from Remark C.1.5.9, the implication $(b) \Rightarrow (a)$ follows from Example **??**. To show that $(a) \Rightarrow (c)$, take $\mathcal{C} \subseteq \mathcal{E}$ to be the full subcategory spanned by the compact projective objects. The assumption that $\mathcal{E}$ is projective generated guarantees the existence of an equivalence $\mathcal{P}_\Sigma(\mathcal{C}) \simeq \mathcal{E}$ (Proposition HTT.5.5.8.25). Since $\mathcal{C}$ is a full subcategory of $\mathcal{E}$ which is closed under finite coproducts, the additivity of $\mathcal{E}$ guarantees the additivity of $\mathcal{C}$ (Example **??**).

Example C.1.5.11. Let $\mathcal{C}$ be an additive ∞-category which is generated under finite coproducts by a single object $C \in \mathcal{C}$. Let $j : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})$ be the Yoneda embedding, and let us abuse notation by identifying $\mathcal{P}_\Sigma(\mathcal{C})$ with a full subcategory of its stabilization $\mathrm{Sp}(\mathcal{P}_\Sigma(\mathcal{C}))$. Then $j(C)$ is a compact object of the stable $\mathrm{Sp}(\mathcal{P}_\Sigma(\mathcal{C}))$ whose desuspensions generate $\mathrm{Sp}(\mathcal{P}_\Sigma(\mathcal{C}))$ under small colimits. Applying Theorem **??** (and its proof), we deduce that there exists an $\mathbb{E}_1$-ring A and an equivalence of stable ∞-categories $\rho : \mathrm{RMod}_A \to \mathrm{Sp}(\mathcal{P}_\Sigma(\mathcal{C}))$ carrying A to $j(C)$. For each $n \geqslant 0$, we have a canonical homotopy equivalence

$$\Omega^{\infty-n} A \simeq \mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{C})}(j(C), \Sigma^n j(C)).$$

Since $j(C)$ is a projective object of $\mathcal{P}_\Sigma(\mathcal{C})$, it follows that $\Omega^{\infty-n}A$ is n-connective for each n: that is, the spectrum A is connective. Taking $n = 0$ and using the fact that j is fully faithful, we obtain a homotopy equivalence $\Omega^\infty A \simeq \mathrm{Map}_{\mathcal{C}}(C, C)$: more informally, we can regard A as the "endomorphism ring" of the object $C \in \mathcal{C}$. Since $\mathrm{RMod}_A^{\mathrm{cn}}$ is the smallest full subcategory of RMod_A which contains A and is closed under small colimits, the functor ρ restricts to an equivalence of Grothendieck prestable ∞-categories $\mathrm{RMod}_A^{\mathrm{cn}} \simeq \mathcal{P}_\Sigma(\mathcal{C})$.

C.2 The Gabriel-Popescu Theorem

Let $\mathcal{A}$ be a Grothendieck abelian category. Recall that an object $C \in \mathcal{A}$ is a *generator* of $\mathcal{A}$ if every object of $\mathcal{A}$ can be written as a quotient of a direct sum $\bigoplus C$ of copies of C.

Theorem C.2.0.12 (Gabriel-Popescu)**.** *Let $\mathcal{A}$ be a Grothendieck abelian category, let $C \in \mathcal{A}$ be a generator, let $A = \mathrm{Hom}_{\mathcal{A}}(C, C)$ be its endomorphism ring, and let $\mathrm{RMod}_A^\heartsuit$ denote the abelian category of right A-modules. Then the construction $D \mapsto \mathrm{Hom}_{\mathcal{A}}(C, D)$ determines a fully faithful embedding $G : \mathcal{A} \to \mathrm{RMod}_A^\heartsuit$. Moreover, the left adjoint of G (given by the construction $M \mapsto M \otimes_A C$) is an exact functor from $\mathrm{RMod}_A^\heartsuit$ to $\mathcal{A}$.*

Since every Grothendieck abelian category has a generator, Theorem C.2.0.12 implies that every Grothendieck abelian category can be realized as a left exact localization of the category of modules over a (possibly noncommutative) ring. This has many pleasant consequences: for example, it can be used to show that there is a well-behaved tensor product on Grothendieck abelian categories (see Theorem C.5.4.16).

Our goal in this section is to establish an analogue of Theorem C.2.0.12 for Grothendieck prestable ∞-categories (Theorem **??**), and to study some of its applications. For example, we will see that Theorem C.2.0.12 can be recovered from our result as a special case (Theorem C.2.2.1), and that every Grothendieck prestable ∞-category $\mathcal{C}$ can be obtained as a left exact localization of the ∞-category of $\mathrm{RMod}_A^{\mathrm{cn}}$, where A is a connective $\mathbb{E}_1$-ring (Theorem C.2.4.1).

C.2.1 The Gabriel-Popescu Theorem for Prestable ∞-Categories

We begin by introducing some terminology.

Definition C.2.1.1. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. A *generating subcategory* for $\mathcal{C}$ is a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ with the following property: for every object $X \in \mathcal{C}$, there exists a collection of maps $\rho_\alpha : C_\alpha \to X$, where each C_α belongs to $\mathcal{C}_0$ and the induced map $\bigoplus_\alpha \pi_0 C_\alpha \to \pi_0 X$ is an epimorphism in the abelian category $\mathcal{C}^\heartsuit$.

When $\mathcal{C}_0$ is the full subcategory of $\mathcal{C}$ spanned by a single object C, then we say that C is a *generator* of $\mathcal{C}$ if $\mathcal{C}_0$ is a generating subcategory of $\mathcal{C}$.

Warning C.2.1.2. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. If $\mathcal{C}_0 \subseteq \mathcal{C}$ is a full subcategory which generates $\mathcal{C}$ under small colimits, then $\mathcal{C}_0$ is a generating subcategory of $\mathcal{C}$ in the sense of Definition C.2.1.1. However, the converse is not true in general. For example, if $\mathcal{C}$ is stable, then *any* full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is a generating subcategory. However, we will show below that the converse does hold when $\mathcal{C}$ is separated (in the sense of Definition C.1.2.12); see Corollary C.2.1.7.

Remark C.2.1.3. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Then $\mathcal{C}$ is presentable, so there exists an essentially full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ which generates $\mathcal{C}$ under small colimits. In particular, there exists an essentially small generating subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$.

Remark C.2.1.4. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. If $\mathcal{C}_0 \subseteq \mathcal{C}$ is a generating subcategory of $\mathcal{C}$ spanned by a small collection of objects $\{C_\alpha\}$, then the single object $C = \bigoplus_\alpha C_\alpha$ is a generator of $\mathcal{C}$. Combining this observation with Remark C.2.1.3, we see that every Grothendieck prestable ∞-category $\mathcal{C}$ admits a generator $C \in \mathcal{C}$.

Warning C.2.1.5. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $C \in \mathcal{C}$ be a generator. Then the truncation $\pi_0 C$ is a generator of the Grothendieck abelian category $\mathcal{C}^\heartsuit$. However, the converse is false in general. For example, $\mathbf{Z}$ is a generator for the Grothendieck abelian category $\mathrm{Sp}^\heartsuit$ of abelian groups, but is *not* a generator for the Grothendieck prestable ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ (any map from $\mathbf{Z}$ to the sphere spectrum is nullhomotopic).

We can now formulate our main result:

Theorem C.2.1.6 (∞-Categorical Gabriel-Popescu Theorem)**.** *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be an essentially small generating subcategory. Assume that $\mathcal{C}$ is separated and that $\mathcal{C}_0$ is closed under finite coproducts in $\mathcal{C}$. Then:*

(1) *The inclusion functor $\mathcal{C}_0 \to \mathcal{C}$ extends to a left exact functor $F : \mathcal{P}_\Sigma(\mathcal{C}_0) \to \mathcal{C}$ which commutes with small colimits.*

(2) *The functor F admits a fully faithful right adjoint $G : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C}_0)$.*

We will give the proof of Theorem C.2.1.6 in §**??**. For the moment, let us enumerate some of its consequences.

Corollary C.2.1.7. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be an essentially small generating subcategory. If $\mathcal{C}$ is separated, then it is generated under small colimits by $\mathcal{C}_0$.*

Corollary C.2.1.8. *Let $\mathcal{C}$ be a separated Grothendieck prestable ∞-category and let $C \in \mathcal{C}$ be a generator. Then there exists a connective $\mathbb{E}_1$-ring A and a pair of adjoint functors $\mathrm{RMod}_A^{\mathrm{cn}} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}$, where $G(C) \simeq A$, the functor F is left exact, and the functor G is fully faithful.*

Proof. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ denote the full subcategory spanned by the finite direct sums $C \oplus C \oplus \cdots \oplus C$. Example C.1.5.11 supplies a connective $\mathbb{E}_1$-ring A and an equivalence of ∞-categories $\mathcal{P}_\Sigma(\mathcal{C}_0) \simeq \mathrm{RMod}_A^{\mathrm{cn}}$. Applying Theorem C.2.1.6, we obtain the desired adjunction $\mathrm{RMod}_A^{\mathrm{cn}} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}$. □

Remark C.2.1.9. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Then we can regard $\mathcal{C}$ as tensored over the ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ of connective spectra (see Corollary C.4.1.2). It follows that for every object $C \in \mathcal{C}$, we can associate a connective spectrum $\mathrm{End}_{\mathcal{C}}(C)$ which is *universal* among connective spectra E equipped with a map $E \otimes C \to C$. In this case, we can regard $\mathrm{End}_{\mathcal{C}}(C)$ as a connective $\mathbb{E}_1$-ring and C as an $\mathrm{End}_{\mathcal{C}}(C)$-module object of $\mathcal{C}$. Suppose that $\mathcal{C}$ is separated and C is a generator of $\mathcal{C}$, and consider an adjunction $\mathrm{RMod}_A^{\mathrm{cn}} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}$ as in the statement of Corollary C.2.1.8. Then G is fully faithful and $G(C) \simeq A$, so we obtain a canonical equivalence

$$\mathrm{End}_{\mathcal{C}}(C) \simeq \mathrm{End}_{\mathrm{RMod}_A^{\mathrm{cn}}}(A) \simeq A.$$

In other words, we can identify A with the (connective) endomorphism ring $\mathrm{End}_{\mathcal{C}}(C)$. Under this identification, the functor F is given by $(M \in \mathrm{RMod}_A^{\mathrm{cn}}) \mapsto (M \otimes_A C \in \mathcal{C})$.

Corollary C.2.1.10. *Let $\mathcal{C}$ be presentable stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ which is left separated, right complete, and compatible with filtered colimits. Let $\mathcal{C}_0 \subseteq \mathcal{C}_{\geq 0}$ be a generating subcategory which is closed under finite direct sums. Then the functor $G : \mathcal{C} \to \mathrm{Fun}^\pi(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Sp})$ given by $G(C)(C_0) = \underline{\mathrm{Map}}_{\mathcal{C}}(C_0, C)$ is fully faithful. Moreover, G admits a t-exact left adjoint $F : \mathrm{Fun}^\pi(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Sp}) \to \mathcal{C}$.*

Proof. Applying Theorem C.2.1.6 to the inclusion $\mathcal{C}_0 \subseteq \mathcal{C}_{\geqslant 0}$ and identifying $\mathcal{P}_\Sigma(\mathcal{C}_0)$ with the ∞-category $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Sp}^{\mathrm{cn}})$ as in the proof of Proposition C.1.5.7, we obtain adjoint functors $\mathrm{Fun}^\pi(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Sp}^{\mathrm{cn}}) \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{C}_{\geqslant 0}$ where f is left exact and g is fully faithful. Passing to stabilizations, we obtain the desired adjunction $\mathrm{Fun}^\pi(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Sp}) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}$. □

Corollary C.2.1.11. *Let $\mathcal{C}$ be presentable stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ which is left separated, right complete, and compatible with filtered colimits. Let C be a generator for the Grothendieck prestable ∞-category $\mathcal{C}_{\geqslant 0}$, and let $A \in \mathrm{Alg}^{\mathrm{cn}}$ denote the connective cover of the endomorphism ring $\underline{\mathrm{Map}}_{\mathcal{C}}(C, C)$. Then the construction $(D \in \mathcal{C}) \mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ induces a fully faithful embedding $G : \mathcal{C} \to \mathrm{RMod}_A$. Moreover, G admits a t-exact left adjoint F, given by $F(M) = M \otimes_A C$.*

C.2.2 The Gabriel-Popescu Theorem for Abelian Categories

We now show that the ∞-categorical Gabriel-Popescu theorem (Theorem C.2.1.6) implies the classical Gabriel-Popescu theorem (Theorem C.2.0.12). In fact, we can deduce a slightly stronger "many-object" version, due to Kuhn (see [126]).

Theorem C.2.2.1 (Kuhn)**.** *Let $\mathcal{A}$ be a Grothendieck abelian category, let $\mathcal{A}_0 \subseteq \mathcal{A}$ be an essentially small full subcategory which is closed under finite direct sums, and let $\mathrm{Fun}^\pi(\mathcal{A}_0^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{A}_0^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ spanned by those functors which preserve finite products. Suppose that, for every object $X \in \mathcal{A}$, there exists an epimorphism $\bigoplus C_\alpha \to X$, where each C_α belongs to $\mathcal{A}_0$. Then:*

- (a) *The construction $X \mapsto \mathrm{Hom}_{\mathcal{A}}(\bullet, X)$ determines a fully faithful embedding $g : \mathcal{A} \to \mathrm{Fun}^\pi(\mathcal{A}_0^{\mathrm{op}}, \mathcal{S}\mathrm{et})$.*
- (b) *The functor g admits an exact left adjoint $f : \mathrm{Fun}^\pi(\mathcal{A}_0^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{A}$.*

Example C.2.2.2 (The Case of a Single Generator)**.** Let $\mathcal{A}$ be a Grothendieck abelian category and let $C \in \mathcal{A}$ be a generator. Then the full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ spanned by the objects $\{C^n\}_{n \geqslant 0}$ satisfies the hypotheses of Theorem C.2.2.1, and we can identify $\mathrm{Fun}^\pi(\mathcal{A}_0^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ with the abelian category of (discrete) right modules over the endomorphism ring $A = \mathrm{Hom}_{\mathcal{A}}(C, C)$. Applying Theorem C.2.2.1, we obtain an adjunction $\mathrm{RMod}_A^\heartsuit \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{A}$, where the functor f is exact and the functor $g = \mathrm{Hom}_{\mathcal{A}}(C, \bullet)$ is fully faithful: this proves Theorem C.2.0.12.

Proof of Theorem C.2.2.1. Let $\mathcal{D}(\mathcal{A})$ denote the derived ∞-category of $\mathcal{A}$ (see §HA.1.3.5). We regard $\mathcal{D}(\mathcal{A})$ as endowed with the t-structure $(\mathcal{D}(\mathcal{A})_{\geqslant 0}, \mathcal{D}(\mathcal{A})_{\leqslant 0})$ of Proposition HA.1.3.5.21 , so that $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ is a separated Grothendieck prestable ∞-category (Example C.1.4.5). Let us abuse notation by identifying $\mathcal{A}$ with the heart of $\mathcal{D}(\mathcal{A})$.

Let X be an arbitrary object of $\mathcal{D}(\mathcal{A})$, which we can represent by a chain complex

$$\cdots \to I_2 \xrightarrow{d_2} I_1 \xrightarrow{d_1} I_0 \xrightarrow{d_0} I_{-1} \xrightarrow{d_{-1}} I_{-2} \to \cdots .$$

Our assumption on $\mathcal{A}_0$ guarantees that we can choose an epimorphism $\rho : \bigoplus C_\alpha \to \ker(d_0)$ in $\mathcal{A}$, where each C_α belongs to $\mathcal{A}_0$. Then we can regard ρ as a morphism from $\bigoplus C_\alpha$ to X in the ∞-category $\mathcal{D}(\mathcal{A})$ which induces an epimorphism on π_0. Allowing X to vary over all objects of $\mathcal{D}(\mathcal{A})_{\geqslant 0}$, we conclude that $\mathcal{A}_0 \subseteq \mathcal{A} \simeq \mathcal{D}(\mathcal{A})^\heartsuit \subseteq \mathcal{D}(\mathcal{A})_{\geqslant 0}$ is a generating subcategory of $\mathcal{D}(\mathcal{A})_{\geqslant 0}$. Applying the ∞-categorical Gabriel-Popescu theorem (Theorem C.2.1.6), we obtain an adjunction $\mathcal{P}_\Sigma(\mathcal{A}_0) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}(\mathcal{A})_{\geqslant 0}$ where F is left exact and G is fully faithful. Restricting to discrete objects, we obtain an adjunction $\tau_{\leqslant 0} \mathcal{P}_\Sigma(\mathcal{A}_0) \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{A}$, where f is exact and g is fully faithful. It now suffices to observe that we have a canonical equivalence $\tau_{\leqslant 0} \mathcal{P}_\Sigma(\mathcal{A}_0) = \mathrm{Fun}^\pi(\mathcal{A}_0^{\mathrm{op}}, \mathcal{S}\mathrm{et})$, which carries g to the functor $X \mapsto \mathrm{Hom}_{\mathcal{A}}(\bullet, X)$. □

Corollary C.2.2.3. *Let $\mathcal{A}$ be a category. The following conditions are equivalent:*

(*a*) *The category $\mathcal{A}$ is a Grothendieck abelian category.*

(*b*) *There exists a Grothendieck abelian category $\mathcal{B}$ such that $\mathcal{A}$ is an accessible left exact localization of $\mathcal{B}$ (that is, there exists an accessible left exact functor $L : \mathcal{B} \to \mathcal{A}$ with a fully faithful right adjoint).*

(*c*) *There exists a (possibly noncommutative) ring A such that $\mathcal{A}$ is an accessible left exact localization of the abelian category $\mathcal{M}$ of right A-modules.*

Proof. The implication $(c) \Rightarrow (b)$ is obvious, and the implication $(a) \to (c)$ follows from Theorem C.2.0.12 (since every Grothendieck abelian category admits a generator). We complete the proof by showing that $(b) \Rightarrow (a)$. Let us identify $\mathcal{A}$ with a full subcategory of $\mathcal{B}$ via the left adjoint of the localization functor $L : \mathcal{B} \to \mathcal{A}$. Note that $\mathcal{A}$ is closed under small limits in $\mathcal{B}$, and in particular it is closed under finite products. Since $\mathcal{B}$ is an additive category, it follows that $\mathcal{A}$ is an additive category. Because $\mathcal{B}$ is a presentable category and the localization functor L is accessible, it follows that $\mathcal{A}$ is presentable: in particular, it admits kernels and cokernels. To complete the proof that $\mathcal{A}$ is abelian, it will suffice to show that for every morphism $f : C \to D$ in $\mathcal{A}$, the canonical map

$$\theta : \mathrm{coker}_{\mathcal{A}}(\ker_{\mathcal{A}}(f) \to C) \to \ker_{\mathcal{A}}(D \to \mathrm{coker}_{\mathcal{A}}(f))$$

is an isomorphism; here the subscript indicates that the relevant kernels and cokernels are computed in the category $\mathcal{A}$. Note that limits in the category $\mathcal{A}$ can be computed in $\mathcal{B}$ (that is, the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ preserves limits) and that colimits in $\mathcal{A}$ are computed by forming colimits in $\mathcal{B}$ and then applying the localization functor L. We may therefore identify θ with the canonical map

$$L\,\mathrm{coker}_{\mathcal{B}}(\ker_{\mathcal{B}}(f) \to C) \to \ker_{\mathcal{B}}(D \to L\,\mathrm{coker}_{\mathcal{B}}(f)).$$

Since the functor L is left exact, the morphism θ is equivalent to the image under the functor L of the map

$$\mathrm{coker}_{\mathcal{B}}(\ker_{\mathcal{B}}(f) \to C) \to \ker_{\mathcal{B}}(D \to \mathrm{coker}_{\mathcal{B}}(f)),$$

which is an isomorphism since $\mathcal{B}$ is an abelian category.

We now complete the proof that $(b) \Rightarrow (a)$ by showing that the abelian category $\mathcal{A}$ is Grothendieck. Let $\{f_\alpha : C_\alpha \to D_\alpha\}$ be a filtered diagram of monomorphisms in $\mathcal{A}$ having colimit $f : C \to D$ in the category $\mathcal{B}$. Since filtered colimits in $\mathcal{B}$ are left exact, f is a monomorphism. Because the functor L is left exact, the induced map $Lf : LC \to LD$ (which we can identify with the colimit of $\{f_\alpha\}$ in the category $\mathcal{A}$) is also a monomorphism. This proves that filtered colimits in $\mathcal{A}$ are left exact. □

C.2.3 Localizations of Prestable ∞-Categories

Let $\mathcal{A}$ be a category. Then $\mathcal{A}$ is a Grothendieck abelian category if and only if it is equivalent to a left-exact localization of the abelian category of (right) modules over some associative ring A (Corollary C.2.2.3). In §C.2.4, we will prove an analogous result in the setting of Grothendieck prestable ∞-categories (Theorem C.2.4.1). The proof will require some general facts about localizations of prestable ∞-categories, which we review in this section.

Proposition C.2.3.1. *Let $\mathcal{C}$ be a prestable ∞-category which admits finite limits and let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory. Suppose that the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{D}$ which is left exact. Then $\mathcal{D}$ is also a prestable ∞-category which admits finite limits. If $\mathcal{C}$ is a Grothendieck prestable ∞-category and L is accessible, then $\mathcal{D}$ is also a Grothendieck prestable ∞-category.*

Remark C.2.3.2. Let $L : \mathcal{C} \to \mathcal{D}$ be as in Proposition C.2.3.1. We will say that an object $C \in \mathcal{C}$ is *L-acyclic* if LC is a zero object of $\mathcal{D}$. For any morphism $\alpha : C' \to C$ in $\mathcal{C}$, the following conditions are equivalent:

(i) The morphism $L\alpha$ is an equivalence in $\mathcal{D}$.

(ii) The object $\mathrm{cofib}(\alpha) \in \mathcal{C}$ is L-acyclic.

The implication $(i) \Rightarrow (ii)$ follows from the observation $L(\mathrm{cofib}(\alpha)) = \mathrm{cofib}(L\alpha)$. To prove the converse, note that we have a cofiber sequence $LC' \xrightarrow{L\alpha} LC \to L\,\mathrm{cofib}(\alpha)$ which is also a fiber sequence by virtue of Corollary C.1.2.6, so the vanishing of $L\,\mathrm{cofib}(\alpha)$ guarantees that $L\alpha$ is an equivalence.

Proof of Proposition C.2.3.1. Since $\mathcal{D}$ is a localization of $\mathcal{C}$, it is closed under all limits which exist in $\mathcal{C}$; in particular, $\mathcal{D}$ admits finite limits. Moreover, since $\mathcal{C}$ admits finite colimits, the ∞-category $\mathcal{D}$ also admits finite colimits (which are given by forming the relevant colimits in $\mathcal{C}$ and then applying the functor L). Any final object of $\mathcal{D}$ is also final in $\mathcal{C}$ and therefore a zero object, so that $\mathcal{D}$ is pointed. Consider the adjunctions

$$\mathcal{C} \underset{\Omega_{\mathcal{C}}}{\overset{\Sigma_{\mathcal{C}}}{\rightleftarrows}} \mathcal{C} \qquad \mathcal{D} \underset{\Omega_{\mathcal{D}}}{\overset{\Sigma_{\mathcal{D}}}{\rightleftarrows}} \mathcal{D}.$$

For any object $D \in \mathcal{D}$, the unit map $u : D \to \Omega_{\mathcal{D}}\Sigma_{\mathcal{D}} D$ is given by the composition

$$D \xrightarrow{u'} \Omega_{\mathcal{C}}\Sigma_{\mathcal{C}} D \xrightarrow{u''} \Omega_{\mathcal{C}} L \Sigma_{\mathcal{C}} D \simeq \Omega_{\mathcal{D}}\Sigma_{\mathcal{D}} D.$$

The assumption that C is prestable guarantees that u' is an equivalence and the left exactness of L implies that u'' is an equivalence. It follows that u is an equivalence for every object $D \in \mathcal{D}$: that is, the suspension functor $\Sigma_{\mathcal{D}}$ is fully faithful.

To complete the proof that $\mathcal{D}$ is prestable, suppose that we are given a pullback diagram σ :

$$\begin{array}{ccc} X & \longrightarrow & X'' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma_{\mathcal{D}} X' \end{array}$$

in the ∞-category $\mathcal{D}$; we wish to show that σ is also a pushout square in $\mathcal{D}$. To prove this, we note that (by virtue of the left exactness of L) the diagram σ can be obtained by applying L to the pullback square

$$\begin{array}{ccc} X & \longrightarrow & \Sigma_{\mathcal{C}} X' \times_{\Sigma_{\mathcal{D}} X'} X'' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma_{\mathcal{C}} X', \end{array}$$

which is also a pushout square in $\mathcal{C}$ by virtue of our assumption that $\mathcal{C}$ is prestable. Since the functor L preserves finite colimits, it follows that σ is a pushout square in $\mathcal{D}$.

Now suppose that $\mathcal{C}$ is a Grothendieck prestable ∞-category and that the localization functor L is accessible. Then the ∞-category $\mathcal{D}$ is also presentable (Theorem HTT.5.5.1.1). To show that $\mathcal{D}$ is a Grothendieck prestable ∞-category, it will suffice to show that filtered colimits in $\mathcal{D}$ are left exact. Let $\mathcal{J}$ be a small filtered ∞-category and let

$$F_{\mathcal{C}} : \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C} \qquad F_{\mathcal{D}} : \mathrm{Fun}(\mathcal{J}, \mathcal{D}) \to \mathcal{D}$$

be left adjoint to the diagonal maps. Then the functor $F_{\mathcal{C}}$ is left exact, and we wish to show that $F_{\mathcal{D}}$ is also left exact. This follows from the observation that $F_{\mathcal{D}}$ can be written as a composition $\mathrm{Fun}(\mathcal{J}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \xrightarrow{F_{\mathcal{C}}} \mathcal{C} \xrightarrow{L} \mathcal{D}$, since the functor L is left exact. □

We now classify the left exact localizations appearing in Proposition C.2.3.1.

Definition C.2.3.3. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. We will say that a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is *localizing* if it satisfies the following conditions:

(i) The ∞-category $\mathcal{C}_0$ is accessible and closed under small coproducts in $\mathcal{C}$.

(ii) Given a cofiber sequence $C' \to C \to C''$ in $\mathcal{C}$, if any two of the objects C, C', C'' belong to $\mathcal{C}_0$, then so does the third.

(iii) Given a cofiber sequence $C' \to C \to C''$ in $\mathcal{C}$ where $C \in \mathcal{C}_0$ and $C'' \in \mathcal{C}^{\heartsuit}$, we have $C' \in \mathcal{C}_0$.

Remark C.2.3.4. Let $\alpha : C' \to C$ be a morphism in a Grothendieck prestable ∞-category $\mathcal{C}$. Then the assertion $\mathrm{cofib}(\alpha) \in \mathcal{C}^{\heartsuit}$ is equivalent to the requirement that α exhibit C' as a (-1)-truncated object of $\mathcal{C}_{/C}$: that is, that it exhibits C' as a *subobject* of C. Consequently, assertion (iii) of Definition C.2.3.3 asserts that the full subcategory $\mathcal{C}_0$ is closed under the formation of subobjects.

Remark C.2.3.5. Condition (ii) of Definition C.2.3.3 guarantees that $\mathcal{C}_0$ is closed under the formation of cofibers in $\mathcal{C}$. Combined with condition (i), this implies that $\mathcal{C}_0$ is closed under all small colimits.

Example C.2.3.6. Let $\mathcal{C}$ be a presentable stable ∞-category. Then a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is localizing if and only if it is accessible and closed under small colimits and desuspensions.

Example C.2.3.7. Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves small colimits and finite limits. If $\mathcal{D}_0 \subseteq \mathcal{D}$ is a localizing subcateegory, then the inverse image $F^{-1}(\mathcal{D}_0) \subseteq \mathcal{C}$ is also a localizing subcategory.

In particular, if we let $\mathcal{C}_0 \subseteq \mathcal{C}$ denote the full subcategory spanned by those objects C for which $FC \simeq 0$, then $\mathcal{C}_0$ is a localizing subcategory of $\mathcal{C}$.

Proposition C.2.3.8. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. The following conditions are equivalent:*

(a) *The full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is localizing (see Definition C.2.3.3).*

(b) *There exists an accessible left exact localization $L : \mathcal{C} \to \mathcal{D}$ such that $\mathcal{C}_0$ is the full subcategory of $\mathcal{C}$ spanned by the L-acyclic objects.*

Proof of Proposition C.2.3.8. The implication $(b) \Rightarrow (a)$ is a special case of Example C.2.3.7. For the converse, suppose that (a) is satisfied. Let S be the collection of all morphisms $u : C \to D$ in $\mathcal{C}$ such that $\mathrm{cofib}(u) \in \mathcal{C}_0$. We first claim that S is strongly saturated (in the sense of Definition HTT.5.5.4.5). That is, it satisfies the following conditions:

(1) Given a pushout square

$$\begin{array}{ccc} C & \xrightarrow{u} & D \\ \downarrow & & \downarrow \\ C' & \xrightarrow{u'} & D' \end{array}$$

in $\mathcal{C}$, if u belongs to S then u' also belongs to S. This is clear, since $\mathrm{cofib}(u) \simeq \mathrm{cofib}(u')$.

(2) The full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by S is closed under small colimits. This follows from Remark C.2.3.5, since the construction $u \mapsto \mathrm{cofib}(u)$ preserves small colimits.

(3) Given a pair of composable morphisms $C \xrightarrow{f} D \xrightarrow{g} E$, if any two of the morphisms f, g, and $g \circ f$ belong to S, then so does the third. This follows by applying assumption (ii) of Definition C.2.3.3 to the cofiber sequence $\mathrm{cofib}(f) \to \mathrm{cofib}(g \circ f) \to \mathrm{cofib}(g)$.

Since $\mathcal{C}_0$ is an accessible category, it follows from Proposition HTT.5.4.6.6 that the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by S is accessible. It follows that S is of small generation (Lemma HTT.5.5.4.14). Let $\mathcal{D} = S^{-1}\mathcal{C}$ be the full subcategory of $\mathcal{C}$ spanned by the S-local objects; applying Proposition HTT.5.5.4.15 we deduce that the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ admits an accessible left adjoint $L : \mathcal{C} \to \mathcal{D}$. Note that an object $C \in \mathcal{C}$ satisfies $LC \simeq 0$ if and only if the canonical map $v : 0 \to C$ belongs to S: that is, if and only if $\mathrm{cofib}(v) \simeq C$ belongs to $\mathcal{C}_0$. To complete the proof of (a), it will suffice to show that the functor L is left exact. According to Proposition HTT.6.2.1.1 , it will suffice to show that for every pullback square σ :

$$\begin{array}{ccc} C & \xrightarrow{u} & D \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ C' & \xrightarrow{u'} & D' \end{array}$$

in $\mathcal{C}$, if $u' \in S$, then $u \in S$. The condition that σ is a pullback square in $\mathcal{C}$ guarantees that it exhibits $\Sigma^\infty C$ as the connective cover of the fiber product $\Sigma^\infty D \times_{\Sigma^\infty D'} \Sigma^\infty C'$ in the stable ∞-category $\mathrm{Sp}(\mathcal{C})$. In particular, the total fiber of the diagram $\Sigma^\infty(\sigma)$ belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant -2}$, so the total cofiber of $\Sigma^\infty(\sigma)$ belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. In other words, the cofiber of the map $\mathrm{cofib}(u) \to \mathrm{cofib}(u')$ is a discrete object of $\mathcal{C}$, so that $\mathrm{cofib}(u') \in \mathcal{C}_0$ implies $\mathrm{cofib}(u) \in \mathcal{C}_0$ by virtue of assumption (iii) of Definition C.2.3.3. □

Notation C.2.3.9. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a localizing subcategory. We let $\mathcal{C}/\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by the S-local objects, where S is the collection of those morphisms in $\mathcal{C}$ whose cofiber belongs to $\mathcal{C}_0$. It follows from Remark C.2.3.2 that any functor $L : \mathcal{C} \to \mathcal{D}$ as in part (b) of Proposition C.2.3.8 is left adjoint to an equivalence of ∞-categories $\mathcal{D} \simeq \mathcal{C}/\mathcal{C}_0 \subseteq \mathcal{C}$.

The quotient category $\mathcal{C}/\mathcal{C}_0$ is characterized by the following universal property:

Proposition C.2.3.10. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a localizing subcategory, let $L : \mathcal{C} \to \mathcal{C}/\mathcal{C}_0$ be a left adjoint to the inclusion functor, and let $\mathcal{D}$ be any Grothendieck prestable ∞-category. Then composition with L induces a fully faithful embedding of ∞-categories $F : \mathrm{LFun}(\mathcal{C}/\mathcal{C}_0, \mathcal{D}) \to \mathrm{LFun}(\mathcal{C}, \mathcal{D})$, where $\mathrm{LFun}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve small colimits, and $\mathrm{LFun}(\mathcal{C}/\mathcal{C}_0, \mathcal{D})$ is defined similarly; the essential image of F is spanned by those colimit-preserving functors $\mathcal{C} \to \mathcal{D}$ which annihilate each object of $\mathcal{C}_0$.*

Proof. Let S be the collection of morphisms u in $\mathcal{C}$ for which $\mathrm{cofib}(u) \in \mathcal{C}_0$. By virtue of Proposition HTT.**??** , it will suffice to show that a colimit-preserving functor $F : \mathcal{C} \to \mathcal{D}$ carries each morphism of S to an equivalence in $\mathcal{D}$ if and only if F annihilates each object of $\mathcal{C}_0$. The "only if" direction is obvious, and the "if" direction follows from Corollary C.1.2.5. □

Remark C.2.3.11. In the situation of Proposition C.2.3.10, a colimit-preserving functor $F : \mathcal{C}/\mathcal{C}_0 \to \mathcal{D}$ is left exact if and only if the composite functor $\mathcal{C} \xrightarrow{L} \mathcal{C}/\mathcal{C}_0 \xrightarrow{F} \mathcal{D}$ is left exact. The "if" direction is obvious (since we can identify F with the restriction of $F \circ L$ to the quotient $\mathcal{C}/\mathcal{C}_0$), and the converse follows from the left exactness of L (Proposition C.2.3.8).

C.2.4 Classification of Grothendieck Prestable ∞-Categories

We now extend the results of §C.2.3 to establish the following characterization for the class of Grothendieck prestable ∞-categories:

Theorem C.2.4.1. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is prestable and Grothendieck.*

(2) *There exists a connective $\mathbb{E}_1$-ring A for which the ∞-category $\mathcal{C}$ is an accessible left exact localization of $\mathrm{RMod}_A^{\mathrm{cn}}$.*

The implication (2) ⇒ (1) of Theorem C.2.4.1 is an immediate consequence of Proposition C.2.3.1. If $\mathcal{C}$ is a *separated* Grothendieck prestable ∞-category, then assertion (2) follows from Corollary C.2.1.8: in this case, we can take A to be the (connective) endomorphism ring of any generator $C \in \mathcal{C}$. However, this strategy will not work in general: for example, if

$\mathcal{C}$ is stable, then the zero object $0 \in \mathcal{C}$ is a generator (see Warning C.2.1.2). Our proof will instead proceed in two steps: first, we write $\mathcal{C}$ as an (accessible) left-exact localization of a separated Grothendieck prestable ∞-category $\mathcal{C}'$. We will then deduce Theorem C.2.4.1 by applying Corollary C.2.1.8 to the ∞-category $\mathcal{C}'$.

We begin with some general remarks about t-structures and Ind-completions.

Proposition C.2.4.2. *Let $\mathcal{C}$ be a small stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$. Then the inclusion functor $\mathrm{Ind}(\mathcal{C}_{\geqslant 0}) \hookrightarrow \mathrm{Fun}^{\pi}(\mathcal{C}_{\geqslant 0}^{\mathrm{op}}, \mathcal{S})$ admits an exact left adjoint $L : \mathrm{Fun}^{\pi}(\mathcal{C}_{\geqslant 0}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Ind}(\mathcal{C}_{\geqslant 0})$.*

Lemma C.2.4.3. *Let $\mathcal{C}$ be a small stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$. Then $\mathrm{Ind}(\mathcal{C})$ inherits a t-structure $(\mathrm{Ind}(\mathcal{C})_{\geqslant 0}, \mathrm{Ind}(\mathcal{C})_{\leqslant 0})$, where $\mathrm{Ind}(\mathcal{C})_{\geqslant 0}$ is the essential image of the fully faithful functor $\mathrm{Ind}(\mathcal{C}_{\geqslant 0}) \to \mathrm{Ind}(\mathcal{C})$, and $\mathrm{Ind}(\mathcal{C})_{\leqslant 0}$ is defined similarly. If the t-structure on $\mathcal{C}$ is right bounded, then the t-structure on $\mathrm{Ind}(\mathcal{C})$ is right complete.*

Proof. The first assertion is straightforward. To prove the second, let us assume that the t-structure on $\mathcal{C}$ is right bounded. It is clear from the construction that $\mathrm{Ind}(\mathcal{C})_{\leqslant 0}$ is closed under filtered colimits in $\mathrm{Ind}(\mathcal{C})$. To prove that $\mathrm{Ind}(\mathcal{C})$ is right complete, it will suffice to show that the intersection $\bigcap_n \mathrm{Ind}(\mathcal{C})_{\leqslant -n}$ consists only of zero objects of $\mathrm{Ind}(\mathcal{C})$ (Proposition HA.1.2.1.19). To this end, let us suppose that $X \in \bigcap_n \mathrm{Ind}(\mathcal{C})_{\leqslant -n}$. Then $\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(Y, X)$ is contractible for any $Y \in \bigcup_n \mathrm{Ind}(\mathcal{C})_{\geqslant -n}$, and therefore for any Y belonging to the essential image of the Yoneda embedding $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$. Since $\mathrm{Ind}(\mathcal{C})$ is generated under filtered colimits by the essential image of j, we conclude that $\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(Y, X)$ is contractible for all Y and therefore X is a final object of $\mathrm{Ind}(\mathcal{C})$. □

Lemma C.2.4.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories equipped with t-structures $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ and $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which is exact and t-exact. Then the induced map $F|_{\mathcal{C}_{\geqslant 0}} : \mathcal{C}_{\geqslant 0} \to \mathcal{D}_{\geqslant 0}$ is left exact.*

Proof. Suppose we are given maps $X_0 \to X \leftarrow X_1$ in the ∞-category $\mathcal{C}_{\geqslant 0}$. Let $X_{01} = X_0 \times_X X_1$ denote the fiber product of X_0 with X_1 over X in the ∞-category $\mathcal{C}$, so that the truncation $\tau_{\geqslant 0} X_{01}$ is the fiber product of X_0 with X_1 over X in the smaller ∞-category $\mathcal{C}_{\geqslant 0}$. To show that $F|_{\mathcal{C}_{\geqslant 0}}$ is left exact, we wish to show that the diagram

$$\begin{array}{ccc} F(\tau_{\geqslant 0} X_{01}) & \longrightarrow & F(X_0) \\ \downarrow & & \downarrow \\ F(X_1) & \longrightarrow & F(X) \end{array}$$

is a pullback diagram in $\mathcal{D}_{\geqslant 0}$: in other words, that it induces an equivalence

$$\rho : F(\tau_{\geqslant 0} X_{01}) \to \tau_{\geqslant 0}(F(X_0) \times_{F(X)} F(X_1))$$

in the ∞-category $\mathcal{D}$. Using the exactness of F, we can rewrite the codomain of ρ as $\tau_{\geqslant 0}F(X_{01})$, so that the assertion that ρ is an equivalence follows because F is t-exact. $\square$

Proof of Proposition C.2.4.2. Without loss of generality, we can replace $\mathcal{C}$ by the union $\bigcup_n \mathcal{C}_{\geqslant -n}$ and thereby reduce to the case where the t-structure on $\mathcal{C}$ is right bounded, so that the pair of ∞-categories $(\mathrm{Ind}(\mathcal{C}_{\geqslant 0}), \mathrm{Ind}(\mathcal{C}_{\leqslant 0}))$ determines a right complete t-structure on $\mathrm{Ind}(\mathcal{C})$ (Lemma C.2.4.3). In this case, we can identify $\mathrm{Ind}(\mathcal{C})$ with the ∞-category of spectrum objects $\mathrm{Sp}(\mathrm{Ind}(\mathcal{C}_{\geqslant 0}))$, so that the inclusion $\mathrm{Ind}(\mathcal{C}_{\geqslant 0}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}_{\geqslant 0}, \mathcal{S}) \hookrightarrow \mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}_{\geqslant 0}, \mathcal{S})$ induces a functor

$$G : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Sp}(\mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}_{\geqslant 0}, \mathcal{S})) \simeq \mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}_{\geqslant 0}, \mathrm{Sp}).$$

Set $\mathcal{E} = \mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}_{\geqslant 0}, \mathrm{Sp})$, and regard $\mathcal{E}$ as a stable ∞-category equipped with a t-structure (setting $\mathcal{E}_{\geqslant 0} = \mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}_{\geqslant 0}, \mathrm{Sp}_{\geqslant 0})$ and $\mathcal{E}_{\leqslant 0} = \mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}_{\geqslant 0}, \mathrm{Sp}_{\leqslant 0})$). The functor G preserves small limits and filtered colimits and therefore admits a left adjoint $F : \mathcal{E} \to \mathrm{Ind}(\mathcal{C})$; since G is left t-exact, the functor F is right t-exact. Note that the ∞-category $\mathcal{C}^{\mathrm{op}}_{\geqslant 0}$ is additive (since it is closed under finite products in the stable ∞-category $\mathcal{C}^{\mathrm{op}}$), so that we can identify $\mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}_{\geqslant 0}, \mathcal{S})$ with the full subcategory

$$\mathcal{E}_{\geqslant 0} = \mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}_{\geqslant 0}, \mathrm{Sp}_{\geqslant 0}) \subseteq \mathcal{E}$$

spanned by the connective objects. Under this identification, we see that L is given by the restriction of the functor $F : \mathcal{E} \to \mathrm{Ind}(\mathcal{C})$ to the full subcategory $\mathcal{E}_{\geqslant 0} \subseteq \mathcal{E}$. It will therefore suffice to show that F is left t-exact.

Let E be an object of $\mathcal{E}_{\leqslant 0}$; we wish to show that $F(E) \in \mathrm{Ind}(\mathcal{C})_{\leqslant 0}$. Writing E as the colimit of its truncations $\tau_{\geqslant -n}E$, we can reduce to the case where E belongs to $\mathcal{E}_{\geqslant -n}$ for some integer n. In this case, E can be written as as successive extension of shifts of objects belonging to the heart $\mathcal{E}^{\heartsuit}$. We may therefore assume without loss of generality that E belongs to the heart $\mathcal{E}^{\heartsuit}$: that is, that it is a discrete object of the ∞-category

$$\mathcal{E}_{\geqslant 0} \simeq \mathrm{Fun}^{\pi}(\mathcal{C}^{\mathrm{op}}_{\geqslant 0}, \mathcal{S}) = \mathcal{P}_{\Sigma}(\mathcal{C}_{\geqslant 0}).$$

Let $j : \mathcal{C}_{\geqslant 0} \to \mathcal{P}_{\Sigma}(\mathcal{C}_{\geqslant 0}) \simeq \mathcal{E}_{\geqslant 0}$ be the Yoneda embedding and let $J : \mathcal{C}_{\geqslant 0} \to \mathcal{E}^{\heartsuit}$ be the functor given by $J(C) = \pi_0 j(C)$. More concretely, we can identify $\mathcal{E}^{\heartsuit}$ with the abelian category of additive functors from the homotopy category $\mathrm{h}\mathcal{C}^{\mathrm{op}}_{\geqslant 0}$ to the category of abelian groups; for each object $C \in \mathcal{C}_{\geqslant 0}$, we can identify $J(C)$ with the functor

$$(C' \in \mathcal{C}^{\mathrm{op}}_{\geqslant 0}) \mapsto \mathrm{Ext}^0_{\mathcal{C}}(C', C)$$

represented by C.

Since $\mathcal{E}_{\geqslant 0}$ is generated (freely) under sifted colimits by the essential image of j, the abelian category $\mathcal{E}^{\heartsuit}$ is generated under colimits by the essential image of J. In particular, every object E of the abelian category $\mathcal{E}^{\heartsuit}$ can be written as the cokernel of a map

$$u : \bigoplus J(C_i) \to \bigoplus J(D_j)$$

whose domain and codomain are (possibly infinite) direct sums of objects belonging to the essential image of the functor J. We can therefore write E as a filtered colimit of objects $\{E_\alpha\}$, where each E_α is the cokernel (in the abelian category $\mathcal{E}^{\heartsuit}$) of a map $u_\alpha : \bigoplus J(C_i) \to \bigoplus J(D_j)$ whose domain and codomain are *finite* direct sums of objects belonging to the essential image of J. Since J commutes with finite direct sums, and the functor $F : \mathcal{E} \to \mathrm{Ind}(\mathcal{C})$ commutes with filtered colimits, it will suffice to show that $F(E) \in \mathrm{Ind}(\mathcal{C})_{\leqslant 0}$ in the special case where $E \in \mathcal{E}^{\heartsuit}$ is given as the cokernel of a single map $u : J(C) \to J(D)$ in $\mathcal{E}^{\heartsuit}$; here C and D are objects of $\mathcal{C}_{\geqslant 0}$. By virtue of Yoneda's lemma, we can identify u with a map from C to D in the ∞-category $\mathcal{C}_{\geqslant 0}$ (which is well-defined up to homotopy).

Let $\mathrm{im}(u)$ denote the image of the map u (formed in the abelian category $\mathcal{E}^{\heartsuit}$), so that we have an exact sequence $0 \to \mathrm{im}(u) \to J(D) \to E \to 0$ in the abelian category $\mathcal{E}^{\heartsuit}$ and therefore a fiber sequence

$$F(\mathrm{im}(u)) \to F(J(D)) \to F(E)$$

in the stable ∞-category $\mathrm{Ind}(\mathcal{C})$. Let X denote the fiber product $\mathrm{im}(u) \times_{J(D)} j(D)$, formed in the ∞-category $\mathcal{E}_{\geqslant 0}$. Since the functor F is exact, we have a pushout square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(j(D)) \\ \downarrow & & \downarrow \\ F(\mathrm{im}(u)) & \longrightarrow & F(J(D)) \end{array}$$

in the stable ∞-category $\mathrm{Ind}(\mathcal{C})$; we may therefore identify $F(E)$ with the cofiber of the map $F(X) \to F(j(D))$. Let us identify objects of $\mathcal{E}_{\geqslant 0}$ with product-preserving functors $\mathcal{C}_{\geqslant 0}^{\mathrm{op}} \to \mathcal{S}$, so that $j(D)$ is identified with the functor represented by D. Unwinding the definitions, we see that X can be identified with the subfunctor of $j(D)$ which assigns to each object $C' \in \mathcal{C}_{\geqslant 0}$ the subspace $X(C') \subseteq j(D)(C') = \mathrm{Map}_{\mathcal{C}}(C', D)$ spanned by those maps $C' \to D$ which factor through C (the factorization itself is not specified). In other words, X is the *image* of the canonical map $v : j(C) \to j(D)$, formed in the ∞-topos $\mathrm{Fun}(\mathcal{C}_{\geqslant 0}^{\mathrm{op}}, \mathcal{S})$. It follows that X can be computed as the geometric realization of the simplicial object $X_\bullet$ of $\mathcal{E}_{\geqslant 0}$ given by the Čech nerve of v. Let $C_\bullet$ be the simplicial object of $\mathcal{C}$ given by the Čech nerve of the map $u : C \to D$. Since the functor $j : \mathcal{C}_{\geqslant 0} \to \mathcal{E}_{\geqslant 0}$ preserves finite limits, we can write $X_\bullet = j(\tau_{\geqslant 0} C_\bullet)$.

Let K be the cokernel of the map $\pi_0 C \to \pi_0 D$ determined by u (formed in the abelian category $\mathcal{C}^{\heartsuit}$ and let D' denote the fiber of the natural map $D \to K$. Unwinding the definitions, we see that $\tau_{\geqslant 0} C_\bullet$ can be identified with the Čech nerve of the natural map $C \to D'$. Let us abuse notation by identifying $\mathcal{C}$ with its essential image in $\mathrm{Ind}(\mathcal{C})$, so that the composite functor $F \circ j : \mathcal{C}_{\geqslant 0} \to \mathrm{Ind}(\mathcal{C})$ is simply the identity. Then $FX_\bullet = Fj(\tau_{\geqslant 0} C_\bullet)$ is the Čech nerve of the map $C \to D'$ in $\mathrm{Ind}(\mathcal{C})$. We therefore have $FX \simeq |FX_\bullet| \simeq FD'$, so that the cofiber of the map $FX \to D$ can be identified with K and therefore belongs to $\mathcal{C}^\heartsuit \subseteq \mathcal{C}_{\leqslant 0}$, as desired. □

Proof of Theorem C.2.4.1. It follows immediately from Proposition C.2.3.1 that if A is a connective $\mathbb{E}_1$-ring, then every accessible left exact localization of $\mathrm{RMod}_A^{\mathrm{cn}}$ is a Grothendieck prestable ∞-category. We wish to prove that every Grothendieck prestable ∞-category $\mathcal{C}$ arises in this way. Without loss of generality, we may assume that there exists a presentable stable ∞-category $\mathcal{E}$ equipped with a t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ which is compatible with filtered colimits such that $\mathcal{C} = \mathcal{E}_{\geqslant 0}$. Since $\mathcal{E}$ is presentable, there exists a regular cardinal κ for which the inclusion $\mathcal{E}^\kappa \hookrightarrow \mathcal{E}$ extends to an equivalence of ∞-categories $\mathrm{Ind}_\kappa(\mathcal{E}^\kappa) \simeq \mathcal{E}$; here $\mathcal{E}^\kappa$ denotes the full subcategory of $\mathcal{E}$ spanned by the κ-compact objects and $\mathrm{Ind}_\kappa(\mathcal{E}^\kappa)$ is the full subcategory of $\mathrm{Fun}((\mathcal{E}^\kappa)^{\mathrm{op}}, \mathcal{S})$ spanned by those functors which preserve κ-small colimits (see §HTT.5.3.5). Enlarging κ if necessary, we may assume that the truncation functors $\tau_{\geqslant 0}$ and $\tau_{\leqslant 0}$ on $\mathcal{E}$ carry κ-compact objects to κ-compact objects. Then the t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ on $\mathcal{E}$ induces a t-structure $(\mathcal{E}^\kappa_{\geqslant 0}, \mathcal{E}^\kappa_{\leqslant 0})$ on the stable ∞-category $\mathcal{E}^\kappa$, where $\mathcal{E}^\kappa_{\geqslant 0} = \mathcal{E}^\kappa \cap \mathcal{E}_{\geqslant 0}$ and $\mathcal{E}^\kappa_{\leqslant 0} = \mathcal{E}^\kappa \cap \mathcal{E}_{\leqslant 0}$. The inclusion $\mathcal{E}^\kappa \hookrightarrow \mathcal{E}$ admits an essentially unique extension to a functor $F : \mathrm{Ind}(\mathcal{E}^\kappa) \to \mathcal{E}$. Since the t-structure on $\mathcal{E}$ is compatible with filtered colimits, the functor F is t-exact. Moreover, the restriction $F|_{\mathrm{Ind}(\mathcal{E}^\kappa_{\geqslant 0})}$ admits a right adjoint which we can identify with the inclusion $\mathcal{E}_{\geqslant 0} \simeq \mathrm{Ind}_\kappa(\mathcal{E}^\kappa_{\geqslant 0}) \subseteq \mathrm{Ind}(\mathcal{E}_{\geqslant 0})$, and is therefore fully faithful. Consequently, $\mathcal{C} = \mathcal{E}_{\geqslant 0}$ is an accessible left exact localization of the Grothendieck prestable ∞-category $\mathrm{Ind}(\mathcal{E}^\kappa_{\geqslant 0})$. It follows from Proposition C.2.4.2 that $\mathrm{Ind}(\mathcal{E}^\kappa_{\geqslant 0})$ is an accessible left exact localization of the ∞-category $\mathcal{C}' = \mathrm{Fun}^\pi((\mathcal{E}^\kappa_{\geqslant 0})^{\mathrm{op}}), \mathcal{S})$. We may therefore replace $\mathcal{C}$ by $\mathcal{C}'$ and thereby reduce to the case where the Grothendieck prestable ∞-category $\mathcal{C}$ is complete. Replacing $\mathcal{E}$ by $\mathrm{Sp}(\mathcal{C})$, we may assume that the t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ is both left and right complete. It follows from Remark C.2.1.4 that there exists an object $C \in \mathcal{E}_{\geqslant 0}$ which is a generator for $\mathcal{E}$. Let A be the connective cover of the endomorphism ring $\mathrm{End}_{\mathcal{E}}(C)$. Applying Theorem **??**, we deduce that $\mathcal{C} \simeq \mathcal{E}_{\geqslant 0}$ is an accessible left exact localization of the ∞-category $\mathrm{RMod}_A^{\mathrm{cn}}$ of connective right A-module spectra. □

C.2.5 Proof of the Gabriel-Popescu Theorem

Let $\mathcal{C}$ be a separated Grothendieck prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a generating subcategory. Theorem C.2.1.6 has two parts:

(1) The inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ extends to a functor $F : \mathcal{P}_\Sigma(\mathcal{C}_0) \to \mathcal{C}$ which preserves small colimits and finite limits.

(2) The functor F admits a fully faithful right adjoint $G : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C}_0)$.

For later applications, it will be convenient to have slightly stronger versions of both (1) and (2), where we loosen the requirement that $\mathcal{C}_0 \subseteq \mathcal{C}$ is a generating subcategory.

Notation C.2.5.1. Let $\mathcal{C}$ be an ∞-category and let $k \geqslant 0$ be an integer. Recall that we say an object $X \in \mathcal{C}$ is *k-truncated* if, for every object $Y \in \mathcal{C}$ and every point $\eta \in \mathrm{Map}_{\mathcal{C}}(Y, X)$, the homotopy groups $\pi_i(\mathrm{Map}_{\mathcal{C}}(Y, X), \eta)$ vanish for $i > k$. We let $\tau_{\leqslant k}\mathcal{C}$ denote the full subcategory of $\mathcal{C}$ spanned by the k-truncated objects.

In what follows, it will be convenient to extend the preceding definition to the case $k = \infty$. We therefore adopt the following convention: when $k = \infty$, every object of $\mathcal{C}$ is k-truncated, and $\tau_{\leqslant k}\mathcal{C} = \mathcal{C}$.

Proposition C.2.5.2. *Let $\mathcal{C}$ be a separated Grothendieck prestable ∞-category, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be an essentially small full subcategory which is closed under finite coproducts, and let $0 \leqslant k \leqslant \infty$. Assume that the following condition is satisfied:*

$(*_k)$ *Every object $C \in \mathcal{C}_0$ is k-truncated. Moreover, for every k-truncated object $X \in \mathcal{C}$, there exists a collection of objects $\{C_i\}_{i \in I}$ in $\mathcal{C}_0$ and a map $\amalg_{i \in I} C_i \to X$ which induces an epimorphism $\pi_0(\amalg_{i \in I} C_i) \to \pi_0 X$ in the abelian category $\mathcal{C}^\heartsuit$.*

Then the inclusion map $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ extends to a left exact functor $F : \mathcal{P}_\Sigma(\mathcal{C}_0) \to \mathcal{C}$ which preserves small colimits.

Proposition C.2.5.3. *Let $\mathcal{C}$ be a separated Grothendieck prestable ∞-category, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be an essentially small full subcategory which is closed under finite coproducts and satisfies condition $(*_k)$ of Proposition C.2.5.2, for some $0 \leqslant k \leqslant \infty$. Then the functor $F : \mathcal{P}_\Sigma(\mathcal{C}_0) \to \mathcal{C}$ admits a right adjoint $G : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C}_0)$ which restricts to a fully faithful embedding $\tau_{\leqslant k}\mathcal{C} \to \tau_{\leqslant k}\mathcal{P}_\Sigma(\mathcal{C}_0)$.*

Remark C.2.5.4. When $k = \infty$, condition $(*_k)$ of Proposition C.2.5.2 asserts that $\mathcal{C}_0 \subseteq \mathcal{C}$ is generating subcategory, in the sense of Definition C.2.1.1. Consequently, Theorem C.2.1.6 can be regarded as the special case of Propositions C.2.5.2 and C.2.5.3 where we take $k = \infty$. When $k = 0$, Propositions C.2.5.2 and C.2.5.3 are essentially equivalent to Theorem C.2.2.1. In general, we can regard Propositions C.2.5.2 and C.2.5.3 as "interpolating" between the classical and ∞-categorical versions of the Gabriel-Popescu theorem.

Proof of Proposition C.2.5.2. Let $\mathcal{C}$ be a separated Grothendieck prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be an essentially small generating subcategory which is closed under finite direct sums and satisfies condition $(*_k)$ for $0 \leqslant k \leqslant \infty$. The inclusion $\iota : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ preserves finite coproducts, so ι admits an essentially unique extension to a functor $F : \mathcal{P}_\Sigma(\mathcal{C}_0) \to \mathcal{C}$ which commutes with small colimits (Proposition HTT.5.5.8.15). We wish to show that F is left exact. By virtue of Proposition C.3.2.1, it will suffice to show that for every discrete object $M \in \mathcal{P}_\Sigma(\mathcal{C}_0)$, the image $F(M) \in \mathcal{C}$ is discrete. Since $\mathcal{C}$ is separated, it will suffice to prove the following assertion for each $n > 0$:

(a_n) If M is a discrete object of $\mathcal{P}_\Sigma(\mathcal{C})$, then $\pi_n FM \simeq 0$.

Our proof of (a_n) will proceed by induction on n. Let us therefore assume that $n > 0$, and that (a_m) is satisfied for all $m < n$. Note that if M is an $(n-2)$-truncated object of $\mathcal{C}$, then M can be written as a successive extension of objects $\Sigma^i(\pi_i M)$ for $k \leqslant n-2$. The functor F preserves suspensions and cofiber sequences, so that FM can be written as a successive extension of objects of the form $\Sigma^i F(\pi_i M)$ for $0 \leqslant i \leqslant n-2$. Using assertions (a_m) for $m < n$, we obtain the following:

(a') Let M be an $(n-2)$-truncated object of $\mathcal{C}$. Then $\pi_{n-1}(FM) \simeq 0$.

We now turn to the proof of (a_n). Let $h : \mathcal{C}_0 \to \mathcal{P}_\Sigma(\mathcal{C}_0)$ be the Yoneda embedding. Then $\mathcal{P}_\Sigma(\mathcal{C}_0)$ is generated under small colimits by the essential image of h. Let M be a discrete object of $\mathcal{P}_\Sigma(\mathcal{C}_0)$, and choose a collection of objects $\{C_i\}_{i \in I}$ of $\mathcal{C}_0$ and a map $\rho : \amalg_{i \in I} h(C_i) \to M$ which induces an epimorphism on π_0. For every finite subset $I_0 \subseteq I$, let $M_{I_0} \subseteq M$ denote the image of $\amalg_{i \in I_0} \pi_0(h(C_i))$ in M. Then we can write M as a filtered colimit $\varinjlim M_{I_0}$. The functor F commutes with filtered colimits, so we have $\pi_n FM \simeq \varinjlim \pi_n FM_{I_0}$. It will therefore suffice to show that $\pi_n FM_{I_0} \simeq 0$ for each finite subset $I_0 \subseteq I$. Replacing M by M_{I_0} and setting $C = \amalg_{i \in I_0} C_i$, we can reduce to the case where there exists an object $C \in \mathcal{C}_0$ and a map $\rho : h(C) \to M$ which induces an epimorphism on π_0. Set $Q = \mathrm{fib}(\rho)$, so that we have a cofiber sequence $Q \to h(C) \to M$ in the ∞-category $\mathcal{P}_\Sigma(\mathcal{C}_0)$. Applying the functor F, we obtain a cofiber sequence $F(Q) \to C \to F(M)$ in the ∞-category $\mathcal{C}$, giving rise to a long exact sequence

$$\pi_n FQ \xrightarrow{\alpha} \pi_n C \to \pi_n FM \to \pi_{n-1} FQ \xrightarrow{\beta} \pi_{n-1} C$$

in the abelian category $\mathcal{C}^\heartsuit$. Assertion (a_n) is now reduced to the following pair of assertions:

(b) The map $\alpha : \pi_n FQ \to \pi_n C$ is an epimorphism in the abelian category $\mathcal{C}^\heartsuit$.

(c) The map $\beta : \pi_{n-1} FQ \to \pi_{n-1} C$ is a monomorphism in the abelian category $\mathcal{C}^\heartsuit$.

We first prove (b). Note that α is obtained by applying π_0 to a morphism $u : \Omega^n(FQ) \to \Omega^n(C)$ in $\mathcal{C}$. Using $(*_k)$, we deduce that $\Omega^n(C)$ is k-truncated, so that $\pi_0 \Omega^n(C)$ is generated by the images of maps $\pi_0(f)$, where $f : D \to \Omega^n C$ is a morphism in $\mathcal{C}$ whose domain D is contained in $\mathcal{C}_0$. It will therefore suffice to show that any such f factors through u. Note that f determines a map $h(f) : h(D) \to \Omega^n h(C)$ in $\mathcal{P}_\Sigma(\mathcal{C}_0)$. Since M is discrete, the map $\Omega^n Q \to \Omega^n h(C)$ is an equivalence; it follows that $h(f)$ factors as a composition $h(D) \to \Omega^n Q \to \Omega^n h(C)$. Applying the functor F, we obtain a map

$$D \simeq F(h(D)) \to F(\Omega^n Q) \to \Omega^n FQ$$

whose composition with u is homotopic to f.

The proof of (c) is a bit more involved. Set $Q' = \Omega^{n-1}Q$, so that we have a cofiber sequence $\Sigma^{n-1}Q' \to Q \to \tau_{\leqslant n-2}Q$ in the ∞-category $\mathcal{C}$. Applying the functor F and passing to homotopy, we obtain an exact sequence

$$\pi_0 FQ' \xrightarrow{\gamma} \pi_{n-1}FQ \to \pi_{n-1}F(\tau_{\leqslant n-2}Q)$$

in the abelian category $\mathcal{C}^{\heartsuit}$, where the third term vanishes by virtue of (a'). It follows that γ is an epimorphism in $\mathcal{C}^{\heartsuit}$. Choose a collection of objects $\{D_j\}_{j \in J}$ and a map $g : \amalg_{j \in J} h(D_j) \to Q'$ in $\mathcal{P}_{\Sigma}(\mathcal{C}_0)$ which is an epimorphism in π_0. Let γ_J denote the epimorphism given by the composition

$$\pi_0 \amalg_{j \in J} D_j \to \pi_0 FQ' \xrightarrow{\gamma} \pi_{n-1}FQ.$$

To show that β is a monomorphism, it will suffice to show that $\ker(\beta \circ \gamma_J)$ and $\ker(\gamma_J)$ coincide (as subobjects of $\pi_0 \amalg_{j \in J} D_j$). For every finite subset $J_0 \subseteq J$, let γ_{J_0} denote the restriction of γ_J to the finite coproduct $\pi_0(\amalg_{j \in J_0} F(D_j)$. Since filtered colimits in $\mathcal{C}$ are left exact, we can identify $\ker(\beta \circ \gamma_J)$ and $\ker(\gamma_J)$ with the filtered colimits $\varinjlim \ker(\beta \circ \gamma_{J_0})$ and $\varinjlim \ker(\gamma_{J_0})$. It will therefore suffice to show that for each finite subset $J_0 \subseteq J$, the kernels $\ker(\beta \circ \gamma_{J_0})$ and $\ker(\gamma_{J_0})$ coincide as subobjects of $\pi_0 D$, where $D = \amalg_{j \in J_0} D_j$. The restriction of g to $\amalg_{j \in J_0} h(D_j)$ is classified by an element $\xi \in \pi_0 Q'(D) = \pi_{n-1}Q(D)$. Let ξ' denote the image of ξ in $\pi_{n-1}h(C)(D)$, so that ξ' classifies a map $\overline{g} : D \to \Omega^{n-1}C$ in $\mathcal{C}$. The exactness of the sequence

$$\pi_0 \operatorname{fib}(\overline{g}) \xrightarrow{\delta} \pi_0 D \xrightarrow{\beta \circ \gamma_{J_0}} \pi_{n-1}C$$

shows that $\ker(\beta \circ \gamma_{J_0}) = \operatorname{im}(\delta)$. It will therefore suffice to show that $\operatorname{im}(\delta) \subseteq \ker(\gamma_J)$. Assumption $(*_k)$ ensures that C and D are k-truncated, so that $\operatorname{fib}(\overline{g})$ is k-truncated. Using $(*_k)$ again, we deduce that $\operatorname{im}(\delta)$ can be written as the union of the images of maps $\pi_0(f) : \pi_0 E \to \pi_0 D$, where $f : E \to D$ is a morphism in $\mathcal{C}_0$ which factors through $\operatorname{fib}(\overline{g})$. For any such map f, the composition $\overline{g} \circ f$ is nullhomotopic: that is, the image of ξ' in $\pi_{n-1}h(C)(E)$ vanishes. Using the exactness of the sequence $\pi_n M(E) \to \pi_{n-1}Q(E) \to \pi_{n-1}h(C)(E)$ (and the vanishing of $\pi_n M(E)$), we conclude that the image of ξ in $\pi_{n-1}Q(E) \simeq \pi_0 Q'(E)$ vanishes: that is, the composition $h(E) \xrightarrow{h(f)} h(D) \to \amalg_{j \in J} h(D_j) \xrightarrow{g} Q'$ is nullhomotopic. Applying F and passing to homotopy, we conclude that $\operatorname{im}(\pi_0(f)) \subseteq \ker(\gamma_{J_0})$, as desired. □

Proof of Proposition C.2.5.3. Let $F : \mathcal{P}_{\Sigma}(\mathcal{C}_0) \to \mathcal{C}$ be as in Proposition C.2.5.2. The functor F admits a right adjoint $G : \mathcal{C} \to \mathcal{P}_{\Sigma}(\mathcal{C}_0) \subseteq \operatorname{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{S})$, given by the restricted Yoneda embedding $G(X)(C) = \mathrm{Map}_{\mathcal{C}}(C, X)$. Proposition C.2.5.2 implies that F is left exact, so it carries k-truncated objects to k-truncated objects. It follows that F and G restrict to adjoint functors

$$\tau_{\leqslant k}\,\mathcal{P}_{\Sigma}(\mathcal{C}_0) \underset{G_k}{\overset{F_k}{\rightleftarrows}} \tau_{\leqslant k}\,\mathcal{C}\,.$$

We wish to show that G_k is fully faithful. Equivalently, we wish to show that for each k-truncated object $X \in \mathcal{C}$, the counit map $v : (F \circ G)(X) \to X$ is an equivalence in $\mathcal{C}$. Since $\mathcal{C}$ is separated, it will suffice to show that the map $\pi_n(v) : \pi_n(F \circ G)(X) \to \pi_n X$ is an isomorphism in $\mathcal{C}^{\heartsuit}$ for each $n \geqslant 0$. Replacing X by $\Omega^n X$ (and using the left exactness of the functor F), we can assume without loss of generality that $n = 0$. Note that if $C \in \mathcal{C}_0$, then any morphism $C \to X$ factors through v. Since X is assumed to be k-truncated, assertion $(*_k)$ immediately implies that $\pi_0(v)$ is an epimorphism in $\mathcal{C}^{\heartsuit}$.

We now show that $\pi_0(v)$ is a monomorphism. Let $h : \mathcal{C}_0 \to \mathcal{P}_{\Sigma}(\mathcal{C}_0)$ denote the Yoneda embedding. Choose a collection of objects $\{C_i\}_{i \in I}$ of $\mathcal{C}_0$ and a morphism $f_I : \amalg_{i \in I} h(C_i) \to G(X)$ in $\mathcal{P}_{\Sigma}(\mathcal{C}_0)$ which induces an epimorphism on π_0. Then the induced map $\amalg_{i \in I} \pi_0 C_i \xrightarrow{\pi_0 F(f_I)} \pi_0(F \circ G)(X)$ is also an epimorphism. It will therefore suffice to show that $\ker(\pi_0(F(f_I)))$ and $\ker(\pi_0(v \circ F(f_I)))$ coincide (as subobjects of $\amalg_{i \in I} \pi_0 C_i$). For each finite subset $I_0 \subseteq I$, let f_{I_0} denote the restriction of f_I to $\amalg_{i \in I_0} h(C_i)$. Since filtered colimits in $\mathcal{C}$ are exact, we obtain isomorphisms

$$\ker(\pi_0(F(f_I))) \simeq \varinjlim_{I_0} \ker(\pi_0(F(f_{I_0}))) \qquad \ker(\pi_0(F(v \circ f_I))) \simeq \varinjlim_{I_0} \ker(\pi_0(F(v \circ f_{I_0}))).$$

It will therefore suffice to show that for each finite subset $I_0 \subseteq I$, the kernels $\ker(\pi_0(F(f_{I_0})))$ and $\ker(\pi_0(v \circ F(f_{I_0})))$ coincide (as subobjects of $\amalg_{i \in I_0} \pi_0 C_i$). Set $C = \amalg_{i \in I_0} C_i$ and $f_0 = f_{I_0}$, so that f_0 can be identified with $G(\rho)$, where $\rho : C \to X$ is the morphism in $\mathcal{C}$ given by the composition $v \circ F(f_0)$. We have a fiber sequence $\mathrm{fib}(\rho) \to C \xrightarrow{\rho} X$ in $\mathcal{C}$, hence a short exact sequence $\pi_0 \,\mathrm{fib}(F(f_0)) \xrightarrow{\alpha} \pi_0 C \xrightarrow{\pi_0 \rho} \pi_0 X$ in the abelian category $\mathcal{C}^{\heartsuit}$. It will therefore suffice to show that $\mathrm{im}(\alpha) \subseteq \ker(F(f_0))$ (as subobjects of $\pi_0 C$). Assumption $(*_k)$ guarantees that C is k-truncated, and X is k-truncated by assumption. It follows that $\mathrm{fib}(\rho)$ is k-truncated. Applying $(*_k)$, we are reduced to proving that for every object $D \in \mathcal{C}_0$ and every morphism $\phi : D \to C$ which factors through $\mathrm{fib}(\rho)$, we have $\mathrm{im}(\pi_0 \phi) \subseteq \ker(F(f_0))$. This is clear: if $\rho \circ \phi$ is nullhomotopic, then the induced map

$$D \simeq (F \circ G)(D) \xrightarrow{(F \circ G)(\phi)} (F \circ G)(C) \xrightarrow{(F \circ G)(\rho)} (F \circ G)(X)$$

is also nullhomotopic, and therefore vanishes after applying π_0. □

C.3 The ∞-Category of Grothendieck Prestable ∞-Categories

In §C.1, we introduced the notion of a Grothendieck prestable ∞-category (Definition C.1.4.2). The collection of Grothendieck prestable ∞-categories can itself be organized into a (very large) ∞-category Groth_∞:

Definition C.3.0.5. Let $\widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the ∞-category of (not necessarily small) ∞-categories, and let $\mathcal{P}\mathrm{r}^{\mathrm{L}} \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the subcategory whose objects are presentable ∞-categories and whose morphisms are functors which preserve small colimits (see Definition HTT.5.5.3.1). We let Groth_∞ denote the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ whose objects are Grothendieck prestable ∞-categories (see Definition C.1.4.2).

C.3.1 Comparison with Stable ∞-Categories

The ∞-category Groth_∞ contains the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ of presentable stable ∞-categories as a full subcategory. Note that the inclusion functor $\mathcal{P}\mathrm{r}^{\mathrm{St}} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}}$ admits a left adjoint, given by the construction $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ (see Proposition HA.4.8.2.18). When restricted to Groth_∞, this left adjoint is not far from being an equivalence of ∞-categories.

Proposition C.3.1.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories. Then the canonical map*

$$\theta : \mathrm{LFun}(\mathcal{C}, \mathcal{D}) \to \mathrm{LFun}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D}))$$

is a fully faithful embedding, whose essential image consists of those functors $\mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ which preserve small colimits and are right t-exact (with respect to the t-structure of Remark C.1.2.10).

Proof. Writing $\mathrm{Sp}(\mathcal{C})$ as the colimit of the diagram

$$\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \to \cdots$$

in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, we can identify $\mathrm{LFun}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D})$ with the homotopy limit of the tower

$$\cdots \to \mathrm{LFun}(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \to \mathrm{LFun}(\mathcal{C}, \mathrm{Sp}(\mathcal{D})).$$

where the transition maps are given by precomposition with the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$. Since colimit-preserving functors commute with suspension, these transition maps can also be described as the functors given by postcomposition with the suspension functor on $\mathrm{Sp}(\mathcal{D})$, and are therefore equivalences. We can therefore identify θ with the map

$$\theta' : \mathrm{LFun}(\mathcal{C}, \mathcal{D}) \to \mathrm{LFun}(\mathcal{C}, \mathrm{Sp}(\mathcal{D}))$$

obtained by composing with the functor $\Sigma^\infty_{\mathcal{D}} : \mathcal{D} \to \mathrm{Sp}(\mathcal{D})$. It follows that θ is fully faithful, and that its essential image consists of those colimit-preserving functors $f : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ such that $f \circ \Sigma^\infty_{\mathcal{C}}$ factors through the full subcategory $\mathrm{Sp}(\mathcal{D})_{\geqslant 0} \subseteq \mathrm{Sp}(\mathcal{D})$: that is, those functors which are right t-exact. □

Example C.3.1.2. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. It follows from Proposition C.3.1.1 that evaluation on the sphere spectrum $S \in \mathrm{Sp}^{\mathrm{cn}}$ induces an equivalence of ∞-categories $\mathrm{LFun}(\mathrm{Sp}^{\mathrm{cn}}, \mathcal{C}) \to \mathcal{C}$.

Remark C.3.1.3. Let $\mathcal{C}$ be a presentable stable ∞-category. Let us say that a full subcategory $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$ is a *core* if it is closed under small colimits and extensions. We let $\mathrm{Core}(\mathcal{C})$ denote the collection of all cores of $\mathcal{C}$, which we regard as a partially ordered set with respect to inclusions.

If $f : \mathcal{C} \to \mathcal{D}$ is a functor between presentable stable ∞-categories which preserves small colimits, then the construction

$$(\mathcal{D}_{\geqslant 0} \subseteq \mathcal{D}) \mapsto (f^{-1}\,\mathcal{D}_{\geqslant 0} \subseteq \mathcal{C})$$

carries cores of $\mathcal{D}$ to cores of $\mathcal{C}$. We can therefore view the construction $\mathcal{C} \mapsto \mathrm{Core}(\mathcal{C})$ as a contravariant functor from the homotopy category $\mathrm{h}\mathcal{P}\mathrm{r}^{\mathrm{St}}$ of presentable stable ∞-categories to the ordinary category of (large) partially ordered sets. Consequently, the construction $\mathcal{C} \mapsto \mathrm{Core}(\mathcal{C})$ can be regarded as a functor from the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ to the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$. This functor classifies a Cartesian fibration $q : \mathsf{Groth}^+_\infty \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$.

We will refer to Groth^+_∞ as the *∞-category of cored stable ∞-categories*. Unwinding the definitions, we see that the objects of Groth^+_∞ are pairs $(\mathcal{C}, \mathcal{C}_{\geqslant 0})$, where $\mathcal{C}$ is a presentable stable ∞-category and $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$ is a core; a morphism from $(\mathcal{C}, \mathcal{C}_{\geqslant 0})$ to $(\mathcal{D}, \mathcal{D}_{\geqslant 0})$ is given by a colimit preserving functor $f : \mathcal{C} \to \mathcal{D}$ satisfying $f(\mathcal{C}_{\geqslant 0}) \subseteq \mathcal{D}_{\geqslant 0}$.

If $\mathcal{C}$ is a Grothendieck prestable ∞-category, then the essential image of the functor $\Sigma^\infty : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ determines a core $\mathrm{Sp}(\mathcal{C})_{\geqslant 0} \subseteq \mathrm{Sp}(\mathcal{C})$. Using Proposition C.3.1.1, we obtain the following:

Corollary C.3.1.4. *The construction $\mathcal{C} \mapsto (\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{C})_{\geqslant 0})$ determines a fully faithful embedding from the ∞-category Groth_∞ of Grothendieck prestable ∞-categories to the ∞-category Groth^+_∞ of cored stable ∞-categories.*

Remark C.3.1.5. Let $\mathcal{C}$ be a presentable stable ∞-category and let $\mathcal{C}_{\geqslant 0} \in \mathrm{Core}(\mathcal{C})$ be a core. According to Proposition HA.1.4.4.11 , the subcategory $\mathcal{C}_{\geqslant 0}$ can be extended to an accessible t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ on $\mathcal{C}$ if and only if $\mathcal{C}_{\geqslant 0}$ is generated (as a full subcategory of $\mathcal{C}$ closed under colimits and extensions) by a small collection of objects of $\mathcal{C}$. In this case, $(\mathcal{C}, \mathcal{C}_{\geqslant 0})$ belongs to the essential image of the fully faithful embedding of Corollary C.3.1.4 if and only if the t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ is compatible with filtered colimits and right complete. Assuming that $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ is compatible with filtered colimits, the right completeness is equivalent to the statement that the intersection $\bigcap_{n \geqslant 0} \mathcal{C}_{\leqslant -n}$ contains only zero objects of $\mathcal{C}$ (Proposition HA.**??**): in other words, that the objects $\{\Sigma^{-n} C : C \in \mathcal{C}_{\geqslant 0}\}$ generate the ∞-category $\mathcal{C}$ under small colimits. However, the hypothesis that the t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ be compatible with filtered colimits is difficult to formulate directly in terms of $\mathcal{C}_{\geqslant 0}$, unless $\mathcal{C}_{\geqslant 0}$ is generated by compact objects (a situation we will study in §C.6; see Proposition C.6.3.1).

Remark C.3.1.6. Let $f : \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor between Grothendieck prestable ∞-categories. Then the inverse image construction

$$f^{-1} : \mathrm{Core}(\mathcal{D}) \to \mathrm{Core}(\mathcal{C})$$

of Remark C.3.1.3 admits a left adjoint $f_+ : \mathrm{Core}(\mathcal{C}) \to \mathrm{Core}(\mathcal{D})$: for every core $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$, we can take $f_+(\mathcal{C}_{\geqslant 0})$ to be the smallest full subcategory of $\mathcal{D}$ which is closed under colimits and extensions and contains $f(C)$ for each object $C \in \mathcal{C}_{\geqslant 0}$. It follows that the Cartesian fibration $q : \mathsf{Groth}_{\infty}^{+} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$ of Corollary C.3.1.4 is also a coCartesian fibration.

Remark C.3.1.7. Let $\mathcal{C}$ be a presentable stable ∞-category. Then the collection of cores of $\mathcal{C}$ is closed under intersections. It follows that $\mathrm{Core}(\mathcal{C})$ is a complete lattice: when viewed as a category, it admits all limits and colimits. Combining this observation with Proposition HTT.4.3.1.5 and Corollary HTT.**??**, we deduce that the ∞-category $\mathsf{Groth}_{\infty}^{+}$ admits small limits and colimits, which are preserved by the forgetful functor $q : \mathsf{Groth}_{\infty}^{+} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$.

We regard the introduction of the ∞-category $\mathsf{Groth}_{\infty}^{+}$ as a technical device: we are not really interested in objects of $\mathsf{Groth}_{\infty}^{+}$ unless they arise from Grothendieck prestable ∞-categories by means of Corollary C.3.1.4. However, for certain constructions it is more convenient to work in the ∞-category $\mathsf{Groth}_{\infty}^{+}$ than in Groth_{∞} because the former admits arbitrary small limits and colimits (Remark C.3.1.7) but the latter does not.

Remark C.3.1.8. Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories and let $f : \mathcal{C} \to \mathcal{D}$ be a functor which preserves small colimits, so that f admits a right adjoint g (Corollary HTT.5.5.2.9). Then f is an equivalence if and only if the following conditions are satisfied:

(a) The induced map $F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ is an equivalence.

(b) The functor g does not annihilate any nonzero objects of $\mathcal{D}$.

The necessity of conditions (a) and (b) is obvious. Conversely, if condition (a) is satisfied, then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow{\scriptstyle \Sigma^\infty_{\mathcal{C}}} & & \downarrow{\scriptstyle \Sigma^\infty_{\mathcal{D}}} \\ \mathrm{Sp}(\mathcal{C}) & \xrightarrow{F} & \mathrm{Sp}(\mathcal{D}), \end{array}$$

where the bottom horizontal map is an equivalence and the vertical maps are fully faithful. It follows immediately that f is fully faithful. Let G denote a homotopy inverse to F. In order to show that f is an equivalence, it will suffice to show that for each object $D \in \mathrm{Sp}(\mathcal{D})_{\geqslant 0}$, we have $GD \in \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Since the functor G is an equivalence, we can choose a fiber sequence $D' \to D \to D''$ in the stable ∞-category $\mathrm{Sp}(\mathcal{D})$ where $GD' \in \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ and $GD'' \in \mathrm{Sp}(\mathcal{C})_{\leqslant -1}$. Then $D' \simeq (F \circ G)(D')$ belongs to $\mathrm{Sp}(\mathcal{D})_{\geqslant 0}$, so that $D'' \simeq \mathrm{cofib}(D' \to D)$ also belongs to $\mathrm{Sp}(\mathcal{D})_{\geqslant 0}$. The condition $GD'' \in \mathrm{Sp}(\mathcal{C})_{\leqslant -1}$ guarantees that the mapping space

$$\mathrm{Map}_{\mathcal{C}}(C, g(\Omega^\infty_{\mathcal{D}} D'')) \simeq \mathrm{Map}_{\mathcal{C}}(C, \Omega^\infty_{\mathcal{C}} GD'') \simeq \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty_{\mathcal{C}} C, GD'')$$

vanishes for every object $C \in \mathcal{C}$. Applying assumption (b), we deduce that $\Omega^\infty_{\mathcal{D}} D''$ vanishes, so that $D'' \simeq \Sigma^\infty_{\mathcal{D}} \Omega^\infty_{\mathcal{D}} D'' \simeq 0$. Thus $GD \simeq GD' \in \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ as desired.

C.3.2 Left Exact Functors

We now restrict our attention to functors between Grothendieck prestable ∞-categories which preserve finite limits.

Proposition C.3.2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories and let $f : \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor. Then the following conditions are equivalent:*

(1) *The functor f is left exact.*

(2) *The functor f carries discrete objects of $\mathcal{C}$ to discrete objects of $\mathcal{D}$.*

(3) *The induced map $F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ is left t-exact.*

Lemma C.3.2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be prestable ∞-categories which admit finite limits and let $f : \mathcal{C} \to \mathcal{D}$ be a right exact functor. The following conditions are equivalent:*

(1) *The functor f is left exact.*

(2) *The induced functor $\mathrm{SW}(f) : \mathrm{SW}(\mathcal{C}) \to \mathrm{SW}(\mathcal{D})$ is t-exact.*

Proof. Suppose first that (1) is satisfied. Let $X \in \mathrm{SW}(\mathcal{C})_{\leqslant n}$; we wish to show that $\mathrm{SW}(f)(X)$ belongs to $\mathrm{SW}(\mathcal{D})_{\leqslant n}$. Replacing X by a suspension if necessary, we may assume that X is the image of an object $X_0 \in \mathcal{C}$ under the identification $\mathcal{C} \simeq \mathrm{SW}(\mathcal{C})_{\geqslant 0}$. In this case, the object X_0 is n-truncated. Since f is left exact, the object $f(X_0) \in \mathcal{D}$ is n-truncated, so that $\mathrm{SW}(f)(X)$ belongs to $\mathrm{SW}(\mathcal{D})_{\leqslant n}$.

Now suppose that (2) is satisfied; we wish to show that the functor f preserves limits indexed by an arbitrary finite simplicial set K. Let $q_0 : K \to \mathcal{C}$ be an arbitrary diagram, so that q_0 can be identified with a diagram $q : K \to \mathrm{SW}(\mathcal{C})_{\geqslant 0}$. Extend q to a limit diagram $\overline{q} : K^{\triangleleft} \to \mathrm{SW}(\mathcal{C})$ in the ∞-category $\mathrm{SW}(\mathcal{C})$. Then $\tau_{\geqslant 0}\overline{q}$ is a limit diagram in the ∞-category $\mathrm{SW}(\mathcal{C})_{\geqslant 0} \simeq \mathcal{C}$. We wish to show that the image of $\tau_{\geqslant 0}\overline{q}$ under the functor $\mathrm{SW}(f) : \mathrm{SW}(\mathcal{C}) \to \mathrm{SW}(\mathcal{D})$ is a limit diagram in $\mathrm{SW}(\mathcal{D})_{\geqslant 0} \simeq \mathcal{D}$. Since $\mathrm{SW}(f)$ is t-exact, we can identify this image with $\tau_{\geqslant 0}\,\mathrm{SW}(f)(\overline{q})$. This is a limit diagram because the functors

$$\mathrm{SW}(f) : \mathrm{SW}(\mathcal{C}) \to \mathrm{SW}(\mathcal{D}) \qquad \tau_{\geqslant 0} : \mathrm{SW}(\mathcal{D}) \to \mathrm{SW}(\mathcal{D})_{\geqslant 0}$$

both preserve finite limits. □

Proof of Proposition C.3.2.1. The implication (1) ⇒ (2) is immediate. We next prove that (2) implies (3). Suppose that (2) is satisfied, and let $C \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$; we wish to prove that $F(C) \in \mathrm{Sp}(\mathcal{D})_{\leqslant 0}$. Since $\mathrm{Sp}(\mathcal{C})$ is right complete, we can write $C = \varinjlim \tau_{\geqslant -n} C$. Because F commutes with filtered colimits, we have $F(C) = \varinjlim F(\tau_{\geqslant -n} C)$. Since $\mathrm{Sp}(\mathcal{D})_{\leqslant 0}$ is closed under filtered colimits, we are reduced to proving that each $F(\tau_{\geqslant -n} C)$ belongs to $\mathrm{Sp}(\mathcal{D})_{\leqslant 0}$. The proof proceeds by induction on n, the case $n < 0$ being trivial. To carry out the inductive step, we note that the exactness of F supplies a fiber sequence

$$F(\tau_{\geqslant 1-n} C) \to F(\tau_{\geqslant -n} C) \to F(\Sigma^{-n} \pi_n C).$$

The inductive hypothesis implies that $F(\tau_{\geqslant 1-n} C)$ belongs to $\mathrm{Sp}(\mathcal{D})_{\leqslant 0}$. Since $\mathrm{Sp}(\mathcal{D})_{\leqslant 0}$ is closed under extensions, we are reduced to proving that the object $F(\Sigma^{-n}\pi_n C) \simeq \Sigma^{-n} F(\pi_n C)$ belongs to $\mathrm{Sp}(\mathcal{D})_{\leqslant 0}$. In fact, it belongs to $\mathrm{Sp}(\mathcal{D})_{\leqslant -n}$: since $\pi_n C$ belongs to the heart $\mathrm{Sp}(\mathcal{C})^{\heartsuit}$, we can write $\pi_n C \simeq \Sigma^{\infty}_{\mathcal{C}} X$ for some discrete object $X \in \mathcal{C}$, so that $F(\pi_n C) \simeq \Sigma^{\infty}_{\mathcal{D}} F(X) \in \mathrm{Sp}(\mathcal{D})^{\heartsuit}$ by virtue of assumption (2).

We now complete the proof by showing that (3) implies (1). Let us identify $\mathrm{SW}(\mathcal{C})$ and $\mathrm{SW}(\mathcal{D})$ with the full subcategories

$$\bigcup_{n \geqslant 0} \mathrm{Sp}(\mathcal{C})_{\geqslant -n} \subseteq \mathrm{Sp}(\mathcal{C}) \qquad \mathrm{SW}(\mathcal{D}) \subseteq \bigcup_{n \geqslant 0} \mathrm{Sp}(\mathcal{D})_{\geqslant -n} \subseteq \mathrm{Sp}(\mathcal{D}).$$

It follows from (3) that the functor F_0 is t-exact, so that f is left exact by virtue of Lemma C.3.2.2. □

Notation C.3.2.3. Let $\mathsf{Groth}^{\mathrm{lex}}_{\infty}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$ whose objects are Grothendieck prestable ∞-categories and whose morphisms are functors $F : \mathcal{C} \to \mathcal{D}$ which preserve small colimits and finite limits.

Proposition C.3.2.4. *The ∞-category* $\mathsf{Groth}_\infty^{\mathrm{lex}}$ *admits small limits. Moreover, the forgetful functors*

$$\mathsf{Groth}_\infty^{\mathrm{lex}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}} \qquad \mathsf{Groth}_\infty^{\mathrm{lex}} \to \mathsf{Groth}_\infty^{+}$$

preserve small limits.

Proof. Let $\{\mathcal{C}_\alpha\}$ be a small diagram of Grothendieck prestable ∞-categories where the transition maps preserve small colimits and finite limits, and let $\mathcal{C}$ denote a limit of $\{\mathcal{C}_\alpha\}$ in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$. It follows from Proposition **??** that the ∞-category $\mathcal{C}$ is presentable and that for any presentable ∞-category $\mathcal{D}$, a functor $f : \mathcal{D} \to \mathcal{C}$ preserves small colimits if and only if each of the maps $\mathcal{D} \to \mathcal{C} \to \mathcal{C}_\alpha$ preserves small colimits. Applying Proposition C.3.2.4, we see that $\mathcal{C}$ is a prestable ∞-category and that a functor $f : \mathcal{D} \to \mathcal{C}$ preserves finite limits if and only if each of the functors $\mathcal{D} \to \mathcal{C} \to \mathcal{C}_\alpha$ preserve small limits. To show that $\mathcal{C}$ is a limit of the diagram $\{\mathcal{C}_\alpha\}$ in Groth_∞, it will suffice to show that $\mathcal{C}$ is Grothendieck: that is, that the functor $\Omega_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ commutes with filtered colimits (see Proposition C.1.4.1). This is equivalent to the statement that each of the composite maps $\mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \to \mathcal{C}_\alpha$ commutes with filtered colimits. This is clear, since we can rewrite this composition as $\mathcal{C} \to \mathcal{C}_\alpha \xrightarrow{\Omega_{\mathcal{C}_\alpha}} \mathcal{C}_\alpha$, where $\Omega_{\mathcal{C}_\alpha}$ commutes with filtered colimits by virtue of our assumption that $\mathcal{C}_\alpha$ is a Grothendieck prestable ∞-category. This completes the proof that the ∞-category $\mathsf{Groth}_\infty^{\mathrm{lex}}$ admits small colimits which are preserved by the inclusion $\mathsf{Groth}_\infty^{\mathrm{lex}} \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$.

We now complete the proof by showing that the functor $\mathsf{Groth}_\infty^{\mathrm{lex}} \to \mathsf{Groth}_\infty^{+}$ of Corollary C.3.1.4 preserves small limits. For any pointed presentable ∞-category $\mathcal{E}$, we can identify the ∞-category $\mathrm{Sp}(\mathcal{E})$ of spectrum objects of $\mathcal{E}$ with the homotopy limit of the diagram

$$\to \mathcal{E} \xrightarrow{\Omega} \mathcal{E} \xrightarrow{\Omega} \mathcal{E}$$

in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$. Moreover, this identification is functor with respect to *left exact* colimit-preserving functors. Consequently, if $\mathcal{C} \simeq \varprojlim \mathcal{C}_\alpha$ is as above, then we can identify $\mathrm{Sp}(\mathcal{C})$ with the limit of the diagram $\{\mathrm{Sp}(\mathcal{C}_\alpha)\}$ in the ∞-category of presentable stable ∞-categories. Each of the functors $\mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C}_\alpha)$ is t-exact, so that an object of $\mathrm{Sp}(\mathcal{C})$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ if and only if its image in each $\mathrm{Sp}(\mathcal{C}_\alpha)$ belongs to $\mathrm{Sp}(\mathcal{C}_\alpha)_{\geqslant 0}$; this proves that the object $(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{C})_{\geqslant 0})$ is a limit of the diagram $\{(\mathrm{Sp}(\mathcal{C}_\alpha), \mathrm{Sp}(\mathcal{C}_\alpha)_{\geqslant 0})$ in $\mathsf{Groth}_\infty^{+}$. □

Corollary C.3.2.5. *The construction* $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ *determines a functor* $\mathsf{Groth}_\infty^{\mathrm{lex}} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$ *which preserves small limits.*

C.3.3 Filtered Colimits of Grothendieck Prestable ∞-Categories

Recall that the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable stable ∞-categories admits small limits (Proposition HTT.5.5.3.13) and small colimits (Theorem HTT.5.5.3.18). Our first goal is to show that the full subcategory $\mathsf{Groth}_\infty \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}}$ of Definition C.3.0.5 enjoys the following closure property:

Theorem C.3.3.1. *The full subcategory* $\mathsf{Groth}_\infty \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}}$ *is closed under small filtered colimits. Consequently, the* ∞*-category* Groth_∞ *admits filtered colimits, which are preserved by the inclusion* $\mathsf{Groth}_\infty \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$.

To prove Theorem C.3.3.1, it will be convenient to first study *lax* (co)limits of Grothendieck prestable ∞-categories. Here, we do not need any requirements on the indexing ∞-category:

Proposition C.3.3.2. *Let* S *be a small simplicial set and let* $p : \mathcal{C} \to S$ *be a map of simplicial sets having the following properties:*

- (i) *The map* p *is both a Cartesian fibration and a coCartesian fibration.*
- (ii) *For each* $s \in S$*, the fiber* $\mathcal{C}_s = \mathcal{C} \times_S \{s\}$ *is a Grothendieck prestable* ∞*-category.*

Let $\mathcal{E} = \mathrm{Fun}_S(S, \mathcal{C}) = \mathrm{Fun}(S, \mathcal{C}) \times_{\mathrm{Fun}(S,S)} \{\mathrm{id}_S\}$ *denote the* ∞*-category of sections of* p*. Then* $\mathcal{E}$ *is a Grothendieck prestable* ∞*-category.*

Proof. Proposition HTT.5.5.3.17 implies that $\mathcal{E}$ is presentable. For each $s \in S$, let $e_s : \mathcal{E} \to \mathcal{C}_s$ denote the functor given by evaluation at s. Using Proposition HTT.5.1.2.2 , we see that the functors e_s preserve small limits and colimits. We next show that $\mathcal{E}$ is prestable: that is, it satisfies conditions (a) through (c) of Definition C.1.2.1:

- (a) We have already seen that $\mathcal{E}$ admits small colimits (and therefore finite colimits). Choose a morphism $g : \varnothing \to \mathbf{1}$ in $\mathcal{E}$, where $\varnothing \in \mathcal{E}$ is an initial object and $\mathbf{1} \in \mathcal{E}$ is a final object. For each vertex $s \in S$, the functor e_s carries g to a morphism from an initial object to a final object in the pointed ∞-category $\mathcal{C}_s$, so that $e_s(g)$ is an equivalence. It follows that g is an equivalence, so that $\mathcal{E}$ is pointed.
- (b) Fix an object $E \in \mathcal{E}$, and let $u : E \to \Omega_{\mathcal{E}}\Sigma_{\mathcal{E}} E$ be the unit map. For each $s \in S$, the evaluation functor $e_s : \mathcal{E} \to \mathcal{C}_s$ preserves finite limits and colimits and therefore carries u to the unit map $e_s(E) \to \Omega_{\mathcal{C}_s}\Sigma_{\mathcal{C}_s} e_s(E)$, which is an equivalence since $\mathcal{C}_s$ is prestable. It follows that u is an equivalence for each $E \in \mathcal{E}$, so that the suspension functor $\Sigma_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ is fully faithful.
- (c) Suppose we are given a pullback square σ :

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & \Sigma_{\mathcal{E}} Z \end{array}$$

  in $\mathcal{E}$; we wish to show that σ is also a pushout square. Since the functors $\{e_s\}_{s \in S}$ preserve finite limits and commute with suspension each $e_s(\sigma)$ is a pullback square in

$\mathcal{C}_s$ whose lower right-hand corner is a suspension (and whose lower left-hand corner is a zero object). The prestability of $\mathcal{C}_s$ guarantees that each $e_s(\sigma)$ is a pushout square in $\mathcal{C}_s$. Since the functors $\{e_s\}_{s \in S}$ preserve pushout squares and are mutually conservative, it follows that σ is a pushout square in $\mathcal{E}$.

To complete the proof, it will suffice to show that $\mathcal{E}$ is a *Grothendieck* prestable ∞-category: that is, that the functor $\Omega_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ commutes with filtered colimits (see Proposition C.1.4.1). Since the evaluation functors $\{e_s\}_{s \in S}$ are mutually conservative and commute with filtered colimits, this is equivalent to the requirement that each composition $e_s \circ \Omega_{\mathcal{E}}$ commutes with filtered colimits. Because e_s is left exact, we have an equivalence $e_s \circ \Omega_{\mathcal{E}} \simeq \Omega_{\mathcal{C}_s} \circ e_s$; we are therefore reduced to showing that $\Omega_{\mathcal{C}_s}$ commutes with filtered colimits, which follows from our assumption that $\mathcal{C}_s$ is a Grothendieck prestable ∞-category. □

We will deduce Theorem C.3.3.1 from Proposition C.3.3.2, together with the following:

Proposition C.3.3.3. *Let A be a small filtered partially ordered set, let $p : \mathcal{C} \to \mathrm{N}(A)$ be a map of simplicial sets satisfying conditions (i) and (ii) of Proposition C.3.3.2. Let $\mathcal{E} = \mathrm{Fun}_{\mathrm{N}(A)}(\mathrm{N}(A), \mathcal{C})$ be the ∞-category of sections of $\mathcal{C}$, and let $\mathcal{E}^{\mathrm{cart}} \subseteq \mathcal{E}$ denote the full subcategory spanned by those sections $s : \mathrm{N}(A) \to \mathcal{C}$ which carry each edge of $\mathrm{N}(A)$ to a p-Cartesian morphism in $\mathcal{C}$. Then the inclusion $\mathcal{E}^{\mathrm{cart}} \hookrightarrow \mathcal{E}$ admits a left exact left adjoint $L : \mathcal{E} \to \mathcal{E}^{\mathrm{cart}}$.*

Proof of Theorem C.3.3.1 from Proposition C.3.3.3. Let $\mathcal{J}$ be a small filtered ∞-category and suppose we are given a diagram $f : \mathcal{J} \to \mathsf{Groth}_{\infty} \subseteq \mathcal{P}r^{\mathrm{L}}$. We wish to show that the colimit of f (formed in the ∞-category $\mathcal{P}r^{\mathrm{L}}$) is a Grothendieck prestable ∞-category. Using Proposition HTT.5.3.1.18 , we can choose a small filtered partially ordered set A and a left cofinal map $g : \mathrm{N}(A) \to \mathcal{J}$. Replacing f by $f \circ g$, we can reduce to the case where $\mathcal{J} = \mathrm{N}(A)$. Let $p : \mathcal{C} \to \mathrm{N}(A)$ be a presentable fibration classified by f (see Proposition HTT.5.5.3.3). Then p satisfies conditions (i) and (ii) of Proposition C.3.3.2, so that $\mathcal{E} = \mathrm{Fun}_{\mathrm{N}(A)}(\mathrm{N}(A), \mathcal{C})$ is a Grothendieck prestable ∞-category. Using Theorem HTT.5.5.3.18 and Proposition HTT.**??** , we can identify the colimit of f (formed in the ∞-category $\mathcal{P}r^{\mathrm{L}}$) with the full subcategory $\mathcal{E}^{\mathrm{cart}} \subseteq \mathcal{E}$ spanned by the Cartesian sections of p. It follows from Proposition C.3.3.3 that the inclusion $\mathcal{E}^{\mathrm{cart}} \subseteq \mathcal{E}$ admits a left exact left adjoint $L : \mathcal{E} \to \mathcal{E}^{\mathrm{cart}}$ (which is automatically accessible: see Proposition HTT.5.4.7.7). Applying Proposition C.2.3.1, we deduce that $\mathcal{E}^{\mathrm{cart}}$ is a Grothendieck prestable ∞-category. □

Proof of Proposition C.3.3.3. We proceed as in the proof of Proposition HTT.6.3.3.3 . For every cofinal subset $B \subseteq A$, let $\mathcal{E}_B$ denote the full subcategory of $\mathcal{E} = \mathrm{Fun}_{\mathrm{N}(A)}(\mathrm{N}(A), \mathcal{C})$ spanned by those sections s such that s is a p-right Kan extension of $s|_{\mathrm{N}(B)}$. By virtue of Lemma C.4.3.1, it will suffice to prove the following:

(a) For every cofinal subset $B \subseteq A$, the full subcategory $\mathcal{E}_B$ is an accessible left exact localization of $\mathcal{E}$.

(b) Let P be the set of all cofinal subsets of A. Then $\mathcal{E}^{\text{cart}} = \bigcap_{B \in P} \mathcal{E}_B$.

To prove (a), consider the restriction functor $\phi : \mathcal{E} \to \text{Fun}_{\text{N}(A)}(\text{N}(B), \mathcal{C})$. Using Proposition HTT.4.3.2.15, we deduce that ϕ restricts to a trivial Kan fibration $\phi_0 : \mathcal{E}_B \to \text{Fun}_{\text{N}(A)}(\text{N}(B), \mathcal{C})$. Any choice of section of ϕ_0 determines a fully faithful embedding $\text{Fun}_{\text{N}(A)}(\text{N}(B), \mathcal{C}) \to \mathcal{E}$ which is right adjoint to ϕ, having essential image $\mathcal{E}_B$. This proves that $\mathcal{E}_B$ is a localization of $\mathcal{E}$. Since $\text{Fun}_{\text{N}(A)}(\text{N}(B), \mathcal{C}) \simeq \text{Fun}_{\text{N}(B)}(\text{N}(B), \mathcal{C} \times_{\text{N}(A)} \text{N}(B))$ is a Grothendieck prestable ∞-category (Proposition C.3.3.2), it follows that $\mathcal{E}_B$ is also a Grothendieck prestable ∞-category (and, in particular, accessible). The left exactness of the localization $\mathcal{E} \to \mathcal{E}_B$ follows from the left exactness of ϕ (which is immediate from the fact that limits are computed levelwise; see Proposition HTT.5.1.2.2).

We now prove (b). Suppose first that $s : \text{N}(A) \to \mathcal{C}$ is a section of p which carries each edge of $\text{N}(A)$ to a p-Cartesian edge of $\mathcal{C}$. We claim that s belongs to $\mathcal{E}_B$ for every cofinal subset $B \subseteq A$. To prove this, it suffices to show that for each element $\alpha \in A$, the section s exhibits $s(\alpha)$ as a p-limit of the diagram $s|_{\text{N}(B_\alpha)}$, where $B_\alpha = \{\beta \in B : \beta \geqslant \alpha\}$. By assumption, s carries each edge of $\text{N}(A)$ to a p-Cartesian morphism in $\mathcal{C}$. Using Propositions HTT.4.3.1.9 , HTT.4.3.1.10 , and Corollary HTT.4.4.4.10 , we are reduced to proving that the simplicial set $\text{N}(B_\alpha)$ is weakly contractible. This is clear, since the cofinality of B in A guarantees that $\text{N}(B_\alpha)$ is filtered.

To complete the proof, it will suffice to show that every object $s \in \bigcap_{B \in P} \mathcal{C}_B$ belongs to $\mathcal{E}^{\text{cart}}$. Fix a pair of elements $\alpha \leqslant \beta$ in A; we wish to show that the map $s(\alpha) \to s(\beta)$ is p-Cartesian. To prove this, set $B = \{\gamma \in A : \gamma \geqslant \beta\}$. Since A is filtered, B is cofinal in A. Our assumption on s guarantees that s is a p-right Kan extension of $s|_{\text{N}(B)}$, so that $s(\alpha)$ is a p-limit of the diagram $s|_{\text{N}(B)}$. Since β is an initial object of $\text{N}(B)$, we conclude that the map $s(\alpha) \to s(\beta)$ is p-Cartesian, as desired. □

We now restrict our attention to the colimits of filtered diagrams in the subcategory $\mathsf{Groth}_\infty^{\text{lex}} \subseteq \mathsf{Groth}_\infty$ of Notation C.3.2.3.

Lemma C.3.3.4. *Let $\mathcal{J}$ be a small filtered ∞-category and suppose we are given a diagram $f : \mathcal{J} \to \mathsf{Groth}_\infty^{\text{lex}}$, which we will denote by $\{\mathcal{C}_\alpha\}_{\alpha \in \mathcal{J}}$. Let $\mathcal{C}$ denote the colimit $\varinjlim_{\alpha \in \mathcal{J}} \mathcal{C}_\alpha$ (formed in the ∞-category Groth_∞). For each $\alpha \in \mathcal{J}$, the tautological map $\rho_\alpha : \mathcal{C}_\alpha \to \mathcal{C}$ is left exact.*

Proof. Arguing as in the proof of Theorem C.3.3.1, we can use Proposition HTT.5.3.1.18 to reduce to the case where $\mathcal{J} = \text{N}(A)$ for some filtered partially ordered set A. Replacing A by the cofinal subset $\{\beta \in A : \beta \geqslant \alpha\}$, we can assume that α is a least element of A. Let $p : \mathcal{C} \to \text{N}(A)$ be a presentable fibration classified by f, let $\mathcal{E} = \text{Fun}_{\text{N}(A)}(\text{N}(A), \mathcal{C})$ be the

∞-category of sections of p, and let $\mathcal{E}^{\mathrm{cart}} \subseteq \mathcal{E}$ be as in Proposition C.3.3.3. Then we can identify the right adjoint to ρ_α with the functor $e_\alpha : \mathcal{E}^{\mathrm{cart}} \to \mathcal{C}_\alpha$ given by evaluation at α. It follows that ρ_α can be computed as the composition

$$\mathcal{C}_\alpha \xrightarrow{\overline{\rho}_\alpha} \mathcal{E} \xrightarrow{L} \mathcal{E}^{\mathrm{cart}},$$

where $\overline{\rho}_\alpha$ is given by p-left Kan extension along the inclusion $\{\alpha\} \hookrightarrow A$ and L is a left adjoint to the inclusion $\mathcal{E}^{\mathrm{cart}} \to \mathcal{E}$. Since the functor L is left exact (Proposition C.3.3.3), it will suffice to show that $\overline{\rho}_\alpha$ is left exact. Because the evaluation functors $\{e_\beta : \mathcal{E} \to \mathcal{C}_\beta\}_{\beta \in A}$ preserve small limits and are mutually conservative, this is equivalent to the assertion that each composition $e_\beta \circ \overline{\rho}_\alpha$ is left exact. Using the fact that α is a least element of A, we see that $e_\beta \circ \overline{\rho}_\alpha$ can be identified with the transition functor $\mathcal{C}_\alpha \to \mathcal{C}_\beta$ determined by the diagram $f : \mathrm{N}(A) \to \mathsf{Groth}_\infty^{\mathrm{lex}}$, and is therefore left exact by assumption. □

Proposition C.3.3.5. *The ∞-category $\mathsf{Groth}_\infty^{\mathrm{lex}}$ admits small filtered colimits, which are preserved by the inclusion functors $\mathsf{Groth}_\infty^{\mathrm{lex}} \hookrightarrow \mathsf{Groth}_\infty \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$.*

Proof. By virtue of Theorem C.3.3.1, it will suffice to prove the following:

- $(*)$ Let $\mathcal{J}$ be an essentially small filtered ∞-category, let $\{\mathcal{C}_\alpha\}_{\alpha \in \mathcal{J}}$ be a diagram in the ∞-category $\mathsf{Groth}_\infty^{\mathrm{lex}}$ indexed by $\mathcal{J}$, let $\mathcal{C} \simeq \varinjlim_\alpha \mathcal{C}_\alpha$ be its colimit in the ∞-category Groth_∞, and for each $\alpha \in \mathcal{J}$ let $f_\alpha : \mathcal{C}_\alpha \to \mathcal{C}$ be the tautological map. Then a morphism $h : \mathcal{C} \to \mathcal{D}$ in Groth_∞ is left exact if and only if, for each $\alpha \in \mathcal{J}$, the composition $(h \circ f_\alpha) : \mathcal{C}_\alpha \to \mathcal{D}$ is left exact.

The "only if" direction of $(*)$ follows immediately from Lemma C.3.3.4. For each $\alpha \in \mathcal{J}$, let $g_\alpha : \mathcal{C} \to \mathcal{C}_\alpha$ denote a right adjoint to f_α. Using Lemma HTT.6.3.3.7 , we see that the identity functor $\mathrm{id}_{\mathcal{C}}$ can be written as a filtered colimit $\varinjlim_{\alpha \in \mathcal{J}} f_\alpha \circ g_\alpha$. It follows that any functor $h : \mathcal{C} \to \mathcal{D}$ which preserves small colimits can be written as a filtered colimit $\varinjlim_{\alpha \in \mathcal{J}} (h \circ f_\alpha \circ g_\alpha)$. The functors g_α are automatically left exact (since they are right adjoints). Consequently, if each $h \circ f_\alpha : \mathcal{C}_\alpha \to \mathcal{D}$ is left exact, then the composition $h \circ f_\alpha \circ g_\alpha$ is left exact. If $\mathcal{D}$ is a Grothendieck prestable ∞-category, then the formation of filtered colimits in $\mathcal{D}$ commutes with finite limits, so that $h \simeq \varinjlim_{\alpha \in \mathcal{J}} (h \circ f_\alpha \circ g_\alpha)$ is also left exact. □

C.3.4 Compact Functors

We now study a different class of functors between Grothendieck prestable ∞-categories, which we will refer to as *compact* functors.

Proposition C.3.4.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories, let $f : \mathcal{C} \to \mathcal{D}$ be a functor which preserves small colimits, and let $F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ be the right t-exact functor determined by f (see Proposition C.3.1.1). Then the functors f and F admit right adjoints $g : \mathcal{D} \to \mathcal{C}$ and $G : \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{C})$, and the following conditions are equivalent:*

(*a*) *The functor g commutes with small filtered colimits.*

(*b*) *The functor G commutes with small colimits.*

Definition C.3.4.2. Let $\mathcal{C}$ and $\mathcal{D}$ be a Grothendieck prestable ∞-categories. We will say that a colimit-preserving functor $f : \mathcal{C} \to \mathcal{D}$ is *compact* if it satisfies the equivalent conditions of Proposition C.3.4.1. We let $\mathsf{Groth}^{\mathrm{c}}_{\infty}$ denote the subcategory of Groth_{∞} whose objects are Grothendieck prestable ∞-categories and whose morphisms are compact functors.

Remark C.3.4.3. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism between Grothendieck prestable ∞-categories which preserves small colimits. If f is compact, then it carries compact objects of $\mathcal{C}$ to compact objects of $\mathcal{D}$. The converse holds if $\mathcal{C}$ is compactly generated (Proposition HTT.5.5.7.2).

Example C.3.4.4. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. According to Example C.3.1.2, evaluation on the sphere spectrum $S \in \mathrm{Sp}^{\mathrm{cn}}$ induces an equivalence of ∞-categories $\mathrm{LFun}(\mathrm{Sp}^{\mathrm{cn}}, \mathcal{C}) \to \mathcal{C}$. If $f : \mathrm{Sp}^{\mathrm{cn}} \to \mathcal{C}$ is a colimit-preserving functor, then the following conditions are equivalent:

(*a*) The functor f is compact (in the sense of Definition C.3.4.2).

(*b*) For every connective finite spectrum X, the object $f(X) \in \mathcal{C}$ is compact.

(*c*) The object $f(S) \in \mathcal{C}$ is compact.

The equivalence of (*a*) and (*b*) follows from Remark C.3.4.3, and the equivalence of (*b*) and (*c*) follows because every connective finite spectrum X can be build from the sphere spectrum S using finite colimits.

Proof of Proposition C.3.4.1. The existence of the functors g and G follows from the adjoint functor theorem. Unwinding the definitions, we see that the functor g can be identified with the composition

$$\mathcal{D} \xrightarrow{\Sigma^{\infty}_{\mathcal{D}}} \mathrm{Sp}(\mathcal{D}) \xrightarrow{G} \mathrm{Sp}(\mathcal{C}) \xrightarrow{\Omega^{\infty}_{\mathcal{C}}} \mathcal{C}\,.$$

Here the functor $\Sigma^{\infty}_{\mathcal{D}}$ commutes with small colimits (since it is a left adjoint) and the functor $\Omega^{\infty}_{\mathcal{C}}$ commutes with small filtered colimits (by virtue of our assumption that $\mathcal{C}$ is Grothendieck; see Proposition C.1.4.1). It follows immediately that (*b*) implies (*a*). Conversely, suppose that (*a*) is satisfied. The functor G is a right adjoint and is therefore left exact. Since the domain and codomain of G are stable ∞-categories, the functor G is also right exact. Consequently, to show that G preserves all small colimits, it will suffice to show that it preserves small filtered colimits.

For each integer n, let $G_n : \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{C})$ denote the functor given on objects by the formula $G_n(X) = G(\tau_{\geqslant -n} X)$. Since G is exact, for each $X \in \mathrm{Sp}(\mathcal{D})$ we have a canonical fiber sequence $G_n(X) \to G(X) \to G(\tau_{\leqslant -n-1}X)$. Since F is right t-exact, the functor G is left t-exact, so $G(\tau_{\leqslant -n-1}X)$ belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant -n-1}$. Using the right completeness of the t-structure on $\mathrm{Sp}(\mathcal{C})$, we deduce that the canonical map $\varinjlim_{n \geqslant 0} G_n(X) \to G(X)$ is an equivalence.

For each integer m, let $G_{n,m} : \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{C})$ denote the functor given by $\tau_{\geqslant -m} G(\tau_{\geqslant -n} X)$. Using the right completeness of $\mathrm{Sp}(\mathcal{C})$, we see that G_n can be written as a colimit of the diagram $\varinjlim_{m \geqslant 0} G_{n,m}$. We can therefore write the functor G as a direct limit

$$\varinjlim_{n \geqslant 0} \varinjlim_{m \geqslant 0} G_{n,m} \simeq \varinjlim_{n \geqslant 0} G_{n,n}.$$

It will therefore suffice to show that each of the functors $G_{n,n}$ commutes with filtered colimits. Unwinding the definitions, we have

$$G_{n,n} \simeq \Sigma^{-n}_{\mathrm{Sp}(\mathcal{C})} \circ G_{0,0} \circ \Sigma^{n}_{\mathrm{Sp}(\mathcal{D})}.$$

It will therefore suffice to show that the functor

$$\begin{aligned} G_{0,0} &= \tau_{\geqslant 0} \circ G \circ \tau_{\geqslant 0} \\ &\simeq \Sigma^{\infty}_{\mathcal{C}} \circ \Omega^{\infty}_{\mathcal{C}} \circ G \circ \Sigma^{\infty}_{\mathcal{D}} \circ \Omega^{\infty}_{\mathcal{D}} \\ &\simeq \Sigma^{\infty}_{\mathcal{C}} \circ g \circ \Omega^{\infty}_{\mathcal{D}} \end{aligned}$$

commutes with small filtered colimits. This follows from assumption (a), since the functor $\Sigma^{\infty}_{\mathcal{C}}$ preserves all small colimits and the functor $\Omega^{\infty}_{\mathcal{D}}$ preserves small filtered colimits (by virtue of our assumption that $\mathcal{D}$ is Grothendieck; see Proposition C.1.4.1). □

C.3.5 Colimits in $\mathsf{Groth}^{\mathrm{c}}_{\infty}$

In §C.3.3, we proved that the ∞-category Groth_{∞} of Grothendieck prestable ∞-categories admits filtered colimits (Theorem C.3.3.1). If we restrict our attention to diagrams where the transition maps are given by compact functors, then the analogous assertion holds even for non-filtered diagrams:

Proposition C.3.5.1. *The ∞-category* $\mathsf{Groth}^{\mathrm{c}}_{\infty}$ *admits small colimits.*

Proof. Let $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ denote the ∞-category whose objects are presentable ∞-categories and whose morphisms are accessible functors which preserve small limits (see Definition HTT.5.5.3.1). According to Corollary HTT.5.5.3.4 , there is a canonical equivalence of ∞-categories $\mathcal{P}\mathrm{r}^{\mathrm{R}} \simeq (\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\mathrm{op}}$ which is the identity on objects and replaces each morphism in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ with its left adjoint. Let $\mathcal{M} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{R}}$ denote the subcategory given by the inverse image of $\mathsf{Groth}^{\mathrm{c}}_{\infty}$ under this equivalence. More concretely, $\mathcal{E}$ is the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$ whose objects are Grothendieck prestable ∞-categories and whose morphisms are functors which preserve small limits and small filtered colimits. We will complete the proof by showing that the ∞-category $\mathcal{E}$ admits small limits.

Suppose we are given a diagram of Grothendieck prestable ∞-categories $\{\mathcal{C}_\alpha\}$, where each of the transition maps $g_{\alpha,\beta} : \mathcal{C}_\alpha \to \mathcal{C}_\beta$ preserves small limits and small filtered colimits. Let $\mathcal{C}$ denote the limit of the diagram $\{\mathcal{C}_\alpha\}$ in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$. It follows from Theorem HTT.5.5.3.18 that $\mathcal{C}$ is also the limit of the diagram $\{\mathcal{C}_\alpha\}$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{R}}$; in particular, $\mathcal{C}$ is presentable. Note that the ∞-category of spectrum objects $\mathrm{Sp}(\mathcal{C})$ can be identified with the limit $\varprojlim \mathrm{Sp}(\mathcal{C}_\alpha)$. Let $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ denote the full subcategory of $\mathrm{Sp}(\mathcal{C})$ given by the limit of the diagram of ∞-categories $\{\mathrm{Sp}(\mathcal{C}_\alpha)_{\leqslant 0}\}$. It follows from Theorem HTT.5.5.3.18 that $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ is a presentable ∞-category. Since the inclusion $\mathrm{Sp}(\mathcal{C})_{\leqslant 0} \hookrightarrow \mathrm{Sp}(\mathcal{C})$ preserves small limits and small filtered colimits, it admits a left adjoint $L : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ (Corollary HTT.5.5.2.9). Since $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ is closed under extensions in $\mathrm{Sp}(\mathcal{C})$, Proposition HA.1.2.1.16 implies that it extends to a t-structure $(\mathrm{Sp}(\mathcal{C})_{\geqslant 0}, \mathrm{Sp}(\mathcal{C})_{\leqslant 0})$ on $\mathrm{Sp}(\mathcal{C})$.

Since each $\mathcal{C}_\alpha$ is a Grothendieck prestable ∞-category, each of the loop functors $\Omega_{\mathcal{C}_\alpha} : \mathcal{C}_\alpha \to \mathcal{C}_\alpha$ preserves small filtered colimits. It follows that the loop functor $\Omega_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ preserves small filtered colimits, so that the functor $\Omega^\infty_{\mathcal{C}} : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ preserves small filtered colimits and therefore restricts to a functor $U : \mathrm{Sp}(\mathcal{C})_{\geqslant 0} \to \mathcal{C}$ which preserves small filtered colimits. We claim that U preserves all small limits. To prove this, let K be a small simplicial set and let $q : K \to \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ be a diagram having a limit $\overline{q} : K^{\triangleleft} \to \mathrm{Sp}(\mathcal{C})$ in $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$, so that $\tau_{\geqslant 0}\overline{q}$ is a limit of the diagram $\tau_{\geqslant 0} q \simeq q$ in the ∞-category $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. We wish to show that $U(\tau_{\geqslant 0}\overline{q})$ is a limit diagram in $\mathcal{C}$. Note that we have an evident fiber sequence of diagrams

$$U(\tau_{\geqslant 0}\overline{q}) \to \Omega^\infty_{\mathcal{C}}(\overline{q}) \to \Omega^\infty_{\mathcal{C}}(\tau_{\leqslant -1}\overline{q}).$$

It will therefore suffice to show that $\Omega^\infty_{\mathcal{C}}(\overline{q})$ and $\Omega^\infty_{\mathcal{C}}(\tau_{\leqslant -1}\overline{q})$ are limit diagrams in $\mathcal{C}$. The first assertion is clear (since the functor $\Omega^\infty_{\mathcal{C}}$ preserves limits) and the second follows from the observation that $\Omega^\infty_{\mathcal{C}}(\tau_{\leqslant -1}\overline{q})$ is equivalent to the constant diagram taking the value $0 \in \mathcal{C}$.

Since the t-structure $(\mathrm{Sp}(\mathcal{C})_{\geqslant 0}, \mathrm{Sp}(\mathcal{C})_{\leqslant 0})$ is compatible with filtered colimits, the ∞-category $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ is prestable and Grothendieck. We claim that the map $U : \mathrm{Sp}(\mathcal{C})_{\geqslant 0} \to \mathcal{C}$ exhibits $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ as a limit of the diagram $\{\mathcal{C}_\alpha\}$ in $\mathcal{E}$. For any presentable ∞-categories $\mathcal{D}$ and $\mathcal{D}'$, let $\mathrm{Fun}'(\mathcal{D}, \mathcal{D}')$ denote the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{D}')$ spanned by those functors which preserve small limits and small filtered colimits. We wish to show that if $\mathcal{D}$ is a Grothendieck prestable ∞-category, then the composite map

$$\theta : \mathrm{Fun}'(\mathcal{D}, \mathrm{Sp}(\mathcal{C})_{\geqslant 0}) \xrightarrow{U\circ} \mathrm{Fun}'(\mathcal{D}, \mathcal{C}) \to \varprojlim_{\alpha} \mathrm{Fun}'(\mathcal{D}, \mathcal{C}_\alpha)$$

is an equivalence of ∞-categories. Let $\mathrm{Fun}''(\mathrm{Sp}(\mathcal{D}), \mathrm{Sp}(\mathcal{C}))$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Sp}(\mathcal{D}), \mathrm{Sp}(\mathcal{C}))$ spanned by those functors which preserve small limits, small filtered colimits, and are left t-exact, and define $\mathrm{Fun}''(\mathrm{Sp}(\mathcal{D}), \mathrm{Sp}(\mathcal{C}_\alpha))$ similarly. Using Propositions **??** and C.3.4.1, we can identify θ with the natural map $\mathrm{Fun}''(\mathrm{Sp}(\mathcal{D}), \mathrm{Sp}(\mathcal{C})) \to$

$\varprojlim \mathrm{Fun}''(\mathrm{Sp}(\mathcal{D}), \mathrm{Sp}(\mathcal{C}_\alpha))$, which is an equivalence by virtue of our construction of $\mathcal{C}$ and the t-structure on $\mathrm{Sp}(\mathcal{C})$. □

Remark C.3.5.2. Let $\{\mathcal{C}_\alpha\}$ and $\mathcal{C}$ be as in the proof of Proposition C.3.5.1 and let X be an object of $\mathrm{Sp}(\mathcal{C})$. Then the following conditions are equivalent:

- The object X belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant -1}$.
- For every index α, the image of X under the forgetful functor $G_\alpha : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C}_\alpha)$ belongs to $\mathrm{Sp}(\mathcal{C}_\alpha)_{\leqslant -1}$.
- For each index α and each object $C \in \mathcal{C}_\alpha$, the mapping space $\mathrm{Map}_{\mathrm{Sp}(\mathcal{C}_\alpha)}(\Sigma^\infty_{\mathcal{C}_\alpha} C, G_\alpha(X))$ is contractible.
- For each index α and each object $C \in \mathcal{C}_\alpha$, the mapping space $\mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty_{\mathcal{C}} f_\alpha(C), X)$ is contractible, where f_α denotes the tautological map from $\mathcal{C}_\alpha$ into the direct limit $\mathcal{C} \simeq \varinjlim \mathcal{C}_\alpha$.
- For each object $C \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty_{\mathcal{C}} C, X)$ is contractible (note that any object $C \in \mathcal{C}$ can be obtained as a colimit of objects belonging to the essential images of the functors f_α).

It follows that $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ can be identified with the smallest full subcategory of $\mathrm{Sp}(\mathcal{C})$ which contains the essential image of the functor $\Sigma^\infty_{\mathcal{C}} : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ and is closed under small colimits and extensions (see Proposition HA.1.4.4.11). In other words, the composite functor

$$\mathsf{Groth}^{\mathrm{c}}_\infty \hookrightarrow \mathsf{Groth}_\infty \hookrightarrow \mathsf{Groth}^{+}_\infty$$

$$\mathcal{C} \mapsto (\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{C})_{\geqslant 0})$$

preserves small colimits.

The inclusion functor $\mathsf{Groth}^{\mathrm{c}}_\infty \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$ does not preserve small colimits in general. In other words, the functor $U : \mathrm{Sp}(\mathcal{C})_{\geqslant 0} \to \mathcal{C}$ appearing in the proof of Proposition C.3.5.1 is generally not an equivalence of ∞-categories (because $\mathcal{C}$ is generally not a Grothendieck prestable ∞-category). However, Remark C.3.5.2 and Corollary C.3.1.4 immediately imply the following:

Proposition C.3.5.3. *The inclusion functor* $\mathsf{Groth}^{\mathrm{c}}_\infty \hookrightarrow \mathsf{Groth}_\infty$ *preserves small colimits.*

Remark C.3.5.4. Let $\{\mathcal{C}_\alpha\}$ be a small diagram in the ∞-category $\mathsf{Groth}^{\mathrm{c}}_\infty$, and suppose that the transition functors $f_{\alpha,\beta} : \mathcal{C}_\alpha \to \mathcal{C}_\beta$ admit right adjoints $g_{\alpha,\beta} : \mathcal{C}_\beta \to \mathcal{C}_\alpha$ which preserve *all* small colimits (in other words, the associated exact functors $G_{\alpha,\beta} : \mathrm{Sp}(\mathcal{C}_\beta) \to \mathrm{Sp}(\mathcal{C}_\alpha)$ are t-exact). In this case, the map $U : \mathrm{Sp}(\mathcal{C})_{\geqslant 0} \to \mathcal{C}$ can be identified with an inverse limit of equivalences

$$\mathrm{Sp}(\mathcal{C}_\alpha)_{\geqslant 0} \hookrightarrow \mathrm{Sp}(\mathcal{C}_\alpha) \xrightarrow{\Omega^\infty_{\mathcal{C}_\alpha}} \mathcal{C}_\alpha$$

and is therefore itself an equivalence. The proof of Proposition C.3.5.1 then shows that the colimit of the diagram $\{\mathcal{C}_\alpha\}$ in the ∞-category $\mathsf{Groth}^{\mathrm{c}}_\infty$ is also a colimit in the larger ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of all presentable ∞-categories. In this case, we can also describe $\mathcal{C}$ as the *limit* of the diagram in the ∞-category $\mathsf{Groth}^{\mathrm{lex}}_\infty$ determined by functors $g_{\alpha,\beta}$.

C.3.6 Separated and Complete Grothendieck Prestable ∞-Categories

We close this section with a few remarks about separated and complete Grothendieck prestable ∞-categories (see Definition C.1.2.12).

Proposition C.3.6.1. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category, and let $\mathcal{C}^{\mathrm{sep}} \subseteq \mathcal{C}$ be the full subcategory spanned by those objects C with the following property: for every ∞-connective object $D \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(D, C)$ is contractible. Then:*

- (a) *The inclusion functor $\mathcal{C}^{\mathrm{sep}} \hookrightarrow \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{C}^{\mathrm{sep}}$ which is left exact.*
- (b) *The ∞-category $\mathcal{C}^{\mathrm{sep}}$ is a separated Grothendieck prestable ∞-category.*
- (c) *For any separated Grothendieck prestable ∞-category $\mathcal{D}$, composition with L induces an equivalence of ∞-categories $\mathrm{LFun}(\mathcal{C}^{\mathrm{sep}}, \mathcal{D}) \to \mathrm{LFun}(\mathcal{C}, \mathcal{D})$.*

Proof. We first prove (b). Without loss of generality, we may assume that $\mathcal{C} = \mathcal{E}_{\geqslant 0}$ for some presentable stable ∞-category $\mathcal{E}$ equipped with a t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ which is compatible with filtered colimits. Then the ∞-category of ∞-connective objects of $\mathcal{C}$ can be identified with the intersection $\mathcal{E}_{\geqslant\infty} = \bigcap_{n \geqslant 0} \mathcal{E}_{\geqslant n}$. The subcategory $\mathcal{E}_{\geqslant\infty}$ is presentable and closed under colimits and extensions in $\mathcal{E}$. It follows from Proposition HA.1.4.4.11 that $\mathcal{E}$ admits an accessible t-structure $(\mathcal{E}_{\geqslant\infty}, \mathcal{E}')$. Since $\mathcal{E}_{\geqslant\infty}$ is closed under desuspensions, the subcategory $\mathcal{E}' \subseteq \mathcal{E}$ is closed under suspensions and is therefore a stable subcategory of $\mathcal{E}$. Unwinding the definitions, we see that $\mathcal{E}'$ can be identified with the full subcategory of $\mathcal{E}$ spanned by those objects C for which the mapping space $\mathrm{Map}_{\mathcal{E}}(D, C)$ is contractible for every ∞-connective object $D \in \mathcal{C}$. In particular we have

$$\mathcal{E}_{\leqslant 0} \subseteq \mathcal{E}' \qquad \mathcal{C}^{\mathrm{sep}} = \mathcal{E}_{\geqslant 0} \cap \mathcal{E}' .$$

It follows that the pair of subategories $(\mathcal{C}^{\mathrm{sep}}, \mathcal{E}_{\leqslant 0})$ determines a t-structure on $\mathcal{E}'$. Since $\mathcal{E}_{\leqslant 0}$ is closed under the formation of filtered colimits in $\mathcal{E}$, it is also closed under the formation of filtered colimits in $\mathcal{E}' \subseteq \mathcal{E}$: that is, the t-structure $(\mathcal{C}^{\mathrm{sep}}, \mathcal{E}_{\leqslant 0})$ is compatible with filtered colimits. It follows that $\mathcal{C}^{\mathrm{sep}}$ is a Grothendieck prestable ∞-category. Moreover, if $C \in \mathcal{C}^{\mathrm{sep}}$

is ∞-connective as an object of $\mathcal{C}^{\mathrm{sep}}$, then it is ∞-connective as an object of $\mathcal{C}$. It then follows from the definition of $\mathcal{C}^{\mathrm{sep}}$ that the identity map $\mathrm{id} : C \to C$ is nullhomotopic so that $C \simeq 0$. This proves that $\mathcal{C}^{\mathrm{sep}}$ is separated.

By construction, the inclusion $\mathcal{E}' \hookrightarrow \mathcal{E}$ admits a left adjoint $\overline{L} : \mathcal{E} \to \mathcal{E}'$. Since $\mathcal{E}_{\leqslant 0}$ is contained in $\mathcal{E}'$, the functor $\overline{L}$ is equivalent to the identity on $\mathcal{E}_{\leqslant 0}$ and is therefore left t-exact. If $E \in \mathcal{E}_{\geqslant 0}$, then $\overline{L}E$ is an extension of an object of $\mathcal{E}_{\geqslant \infty}$ by E, and therefore also belong to $\mathcal{E}_{\geqslant 0}$: this proves that L is also left t-exact. It follows that $\overline{L}$ restricts to a left exact localization functor $L : \mathcal{C} \to \mathcal{C}_0$, which proves (a).

We now prove (c). Let $\mathcal{D}$ be a Grothendieck prestable ∞-category. It follows from (a) that the canonical map $\mathrm{LFun}(\mathcal{C}^{\mathrm{sep}}, \mathcal{D}) \to \mathrm{LFun}(\mathcal{C}, \mathcal{D})$ is a fully faithful embedding whose essential image consists of those colimit-preserving functors $F : \mathcal{C} \to \mathcal{D}$ for which the right adjoint $G : \mathcal{D} \to \mathcal{C}$ factors through $\mathcal{C}^{\mathrm{sep}}$. This is equivalent to the condition that the mapping space $\mathrm{Map}_{\mathcal{C}}(C, GD) \simeq \mathrm{Map}_{\mathcal{D}}(FC, D)$ is contractible whenever $C \in \mathcal{C}$ is ∞-connective and $D \in \mathcal{D}$ is arbitrary. The assumption that C is ∞-connective guarantees that it belongs to the essential image of the iterated suspension functor $\Sigma^n_{\mathcal{C}}$ for every $n \geqslant 0$, so that FC belongs to the essential image of the iterated suspension functor $\Sigma^n_{\mathcal{D}}$ for each $n \geqslant 0$. If $\mathcal{D}$ is separated, then $FC \simeq 0$ and the contractibility of $\mathrm{Map}_{\mathcal{D}}(FC, D)$ is automatic. □

In the situation of Proposition C.3.6.1, we will refer to $\mathcal{C}^{\mathrm{sep}}$ as the *separated quotient* of $\mathcal{C}$.

Corollary C.3.6.2. *Let* $\mathsf{Groth}^{\mathrm{sep}}_{\infty}$ *denote the full subcategory of* Groth_{∞} *spanned by the separated Grothendieck prestable* ∞*-categories. Then the inclusion functor* $\mathsf{Groth}^{\mathrm{sep}}_{\infty} \hookrightarrow \mathsf{Groth}_{\infty}$ *admits a left adjoint, given on objects by the construction* $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{sep}}$*.*

Proposition C.3.6.3. *Let* $\mathcal{C}$ *be a Grothendieck prestable* ∞*-category and let* $\widehat{\mathcal{C}}$ *denote the homotopy limit of the tower of* ∞*-categories*

$$\cdots \to \tau_{\leqslant 2}\, \mathcal{C} \to \tau_{\leqslant 1}\, \mathcal{C} \to \tau_{\leqslant 0}\, \mathcal{C}\,.$$

Then:

- (a) *The natural map* $f : \mathcal{C} \to \widehat{\mathcal{C}}$ *preserves small colimits and finite limits.*
- (b) *The* ∞*-category* $\widehat{\mathcal{C}}$ *is a complete Grothendieck prestable* ∞*-category.*
- (c) *For every complete Grothendieck prestable* ∞*-category* $\mathcal{D}$*, composition with* f *induces an equivalence of* ∞*-categories* $\mathrm{LFun}(\widehat{\mathcal{C}}, \mathcal{D}) \to \mathrm{LFun}(\mathcal{C}, \mathcal{D})$*.*

In the situation of Proposition C.3.6.3, we will refer to $\widehat{\mathcal{C}}$ as the *completion* of the prestable ∞-category $\mathcal{C}$.

Proof of Proposition **??**. Without loss of generality, we may assume that $\mathcal{C} = \mathcal{E}_{\geqslant 0}$ for some presentable stable ∞-category $\mathcal{E}$ equipped with a t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ which is compatible with filtered colimits. Let $\widehat{\mathcal{E}}$ denote the left completion of $\mathcal{E}$ with respect to the t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$: that is, the homotopy inverse limit of the tower

$$\cdots \to \mathcal{E}_{\leqslant 2} \xrightarrow{\tau_{\leqslant 1}} \mathcal{E}_{\leqslant 1} \xrightarrow{\tau_{\leqslant 0}} \mathcal{E}_{\leqslant 0}.$$

It follows from Proposition HA.1.2.1.17 that $\widehat{\mathcal{E}}$ is a stable ∞-category which inherits a t-structure $(\widehat{\mathcal{E}}_{\geqslant 0}, \widehat{\mathcal{E}}_{\leqslant 0})$ and a t-exact functor $F : \mathcal{E} \to \widehat{\mathcal{E}}$ which induces equivalences $\mathcal{E}_{\leqslant n} \simeq \widehat{\mathcal{E}}_{\leqslant n}$ for every integer n. Note that the ∞-category $\mathcal{E}$ is presentable (Proposition **??**). Since the t-structure on $\mathcal{E}$ is compatible with filtered colimits, each of the functors in the diagram

$$\cdots \to \mathcal{E}_{\leqslant 2} \xrightarrow{\tau_{\leqslant 1}} \mathcal{E}_{\leqslant 1} \xrightarrow{\tau_{\leqslant 0}} \mathcal{E}_{\leqslant 0}.$$

commutes with filtered colimits. It follows that the projection map $\widehat{\mathcal{E}} \to \mathcal{E}_{\leqslant 0}$ commutes with filtered colimits, so that the t-structure on $\widehat{\mathcal{E}}$ is also compatible with filtered colimits.

Unwinding the definitions, we can identify $\widehat{\mathcal{C}}$ with $\widehat{\mathcal{E}}_{\geqslant 0}$ so that $\widehat{\mathcal{C}}$ is a Grothendieck prestable ∞-category. Since the t-structure on $\widehat{\mathcal{E}}$ is left complete, the prestable ∞-category $\widehat{\mathcal{C}}$ is complete; this proves (b), and assertion (a) follows from the t-exactness of F. We will complete the proof by establishing (c). Let $\mathcal{D}$ be a complete Grothendieck prestable ∞-category, so that $\mathcal{D}$ can be identified with the homotopy limit of the tower

$$\cdots \to \tau_{\leqslant 2}\mathcal{D} \xrightarrow{\tau_{\leqslant 1}} \tau_{\leqslant 1}\mathcal{D} \xrightarrow{\tau_{\leqslant 0}} \tau_{\leqslant 0}\mathcal{D} \simeq \mathcal{D}^{\heartsuit}.$$

in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. It follows that the canonical map $\theta : \mathrm{LFun}(\widehat{\mathcal{C}}, \mathcal{D}) \to \mathrm{LFun}(\mathcal{C}, \mathcal{D})$ can be identified with the homotopy limit of a tower of maps

$$\theta_n : \mathrm{LFun}(\widehat{\mathcal{C}}, \tau_{\leqslant n}\mathcal{D}) \to \mathrm{LFun}(\mathcal{C}, \tau_{\leqslant n}\mathcal{D}).$$

Let $g : \widehat{\mathcal{C}} \to \mathcal{C}$ be a right adjoint to f. Passing to right adjoints we can identify θ_n with the functor $\mathrm{RFun}(\tau_{\leqslant n}\mathcal{D}, \widehat{\mathcal{C}})^{\mathrm{op}} \to \mathrm{RFun}(\tau_{\leqslant n}\mathcal{D}, \mathcal{C})^{\mathrm{op}}$ given by composition with g; here $\mathrm{RFun}(\tau_{\leqslant n}\mathcal{D}, \widehat{\mathcal{C}})$ denotes the full subcategory of $\mathrm{Fun}(\tau_{\leqslant n}\mathcal{D}, \widehat{\mathcal{C}})$ spanned by those accessible functors which preserve small limits and $\mathrm{RFun}(\tau_{\leqslant n}\mathcal{D}, \mathcal{C})$ is defined similarly. Since any left exact functor carries n-truncated objects to n-truncated objects, it will suffice to show that the functor g restricts to an equivalence of ∞-categories $g_{\leqslant n} : \tau_{\leqslant n}\widehat{\mathcal{C}} \to \tau_{\leqslant n}\mathcal{C}$. This is clear: the left exactness of f guarantees that $g_{\leqslant n}$ is right adjoint to the functor $f|_{\tau_{\leqslant n}\mathcal{C}} : \tau_{\leqslant n}\mathcal{C} \to \tau_{\leqslant}\, n\widehat{\mathcal{C}}$, which is an equivalence by virtue of Proposition HA.1.2.1.17. □

Corollary C.3.6.4. *Let* $\mathsf{Groth}_{\infty}^{\mathrm{comp}}$ *denote the full subcategory of* Groth_{∞} *spanned by the separated Grothendieck prestable ∞-categories. Then the inclusion functor* $\mathsf{Groth}_{\infty}^{\mathrm{comp}} \hookrightarrow \mathsf{Groth}_{\infty}$ *admits a left adjoint, given at the level of objects by the construction* $\mathcal{C} \mapsto \widehat{\mathcal{C}}$.

C.4 Tensor Products of Prestable ∞-Categories

Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category whose objects are presentable ∞-categories and whose morphisms are functors which preserve small colimits (see Definition HTT.5.5.3.1). In §HA.4.8.1 , we constructed a symmetric monoidal structure on the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$: given a pair of presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the tensor product $\mathcal{C} \otimes \mathcal{D} \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$ is universal among ∞-categories $\mathcal{E}$ which admit small colimits and are equipped with a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserve small colimits separately in each variable. Our goal in this section is to show that the symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ induces a symmetric monoidal structure on the full subcategory $\mathsf{Groth}_\infty \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}}$ spanned by the Grothendieck prestable ∞-categories (Theorem **??**).

C.4.1 Additive ∞-Categories

Recall that an ∞-category $\mathcal{C}$ is *additive* if $\mathcal{C}$ admits finite products and coproducts and the homotopy category $\mathrm{h}\mathcal{C}$ is additive (see Definition C.1.5.1). Our starting point is the following characterization of presentable additive ∞-categories:

Theorem C.4.1.1. *Let $\mathcal{C}$ be a presentable ∞-category. The following conditions are equivalent:*

(a) *The ∞-category $\mathcal{C}$ is additive.*

(b) *The functor $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}^{\mathrm{cn}}$ induces an equivalence of ∞-categories $\mathcal{C} \simeq \mathcal{S} \otimes \mathcal{C} \to \mathrm{Sp}^{\mathrm{cn}} \otimes \mathcal{C}$. Here the tensor products are formed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories.*

Corollary C.4.1.2. *The functor $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}^{\mathrm{cn}}$ exhibits $\mathrm{Sp}^{\mathrm{cn}}$ as an idempotent object of the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (see Definition HA.4.8.2.1). In other words, the functor Σ^∞_+ induces equivalences*

$$\mathrm{Sp}^{\mathrm{cn}} \simeq \mathrm{Sp}^{\mathrm{cn}} \otimes \mathcal{S} \to \mathrm{Sp}^{\mathrm{cn}} \otimes \mathrm{Sp}^{\mathrm{cn}} \leftarrow \mathcal{S} \otimes \mathrm{Sp}^{\mathrm{cn}} \simeq \mathrm{Sp}^{\mathrm{cn}} .$$

Proof. This is an immediate consequence of Theorem C.4.1.1, since the ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ is additive. □

Corollary C.4.1.3. *The ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ of connective spectra admits an essentially unique symmetric monoidal structure for which the unit object is the sphere spectrum $S \in \mathrm{Sp}^{\mathrm{cn}}$. Moreover, the forgetful functor*

$$G : \mathrm{Mod}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$$

is fully faithful, and its essential image consists of the additive presentable ∞-categories.

Proof. The first two assertions follow from Corollary C.4.1.2, Proposition HA.4.8.2.9 , and Proposition HA.4.8.2.10 . Let $F : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathrm{Mod}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ be right adjoint to G; then a presentable ∞-category $\mathcal{C}$ belongs to the essential image of G if and only if the unit map

$$\mathcal{C} \to (G \circ F)(\mathcal{C}) \simeq \mathrm{Sp}^{\mathrm{cn}} \otimes \mathcal{C}$$

is an equivalence. By virtue of Theorem C.4.1.1, this is equivalent to the requirement that $\mathcal{C}$ is additive. □

Corollary C.4.1.4. *Let $\mathcal{P}\mathrm{r}^{\mathrm{Add}}$ denote the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ whose objects are additive presentable ∞-categories. Then:*

(*a*) *The inclusion functor $\mathcal{P}\mathrm{r}^{\mathrm{Add}} \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$ admits a left adjoint $L : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{Add}}$, given on the level of objects by $\mathcal{C} \mapsto \mathrm{Sp}^{\mathrm{cn}} \otimes \mathcal{C}$.*

(*b*) *The localization functor L is compatible with the symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (in the sense of Definition HA.2.2.1.6). Consequently, there is an essentially unique symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{Add}}$ for which the functor $L : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{Add}}$ is symmetric monoidal.*

(*c*) *The inclusion functor $\mathcal{P}\mathrm{r}^{\mathrm{Add}} \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$ preserves tensor products.*

Remark C.4.1.5. The inclusion functor $\mathcal{P}\mathrm{r}^{\mathrm{Add}} \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$ is lax symmetric monoidal (since it is right adjoint to the symmetric monoidal functor L) and preserves tensor products, but it is not a symmetric monoidal functor because it fails to preserve unit objects. The unit object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is the ∞-category $\mathcal{S}$ of spaces, while the unit object of $\mathcal{P}\mathrm{r}^{\mathrm{Add}} \simeq \mathrm{Mod}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ is the ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ of connective spectra.

Proof of Corollary C.4.1.4. Assertions (*a*) and (*b*) follow from Corollary C.4.1.3 and Proposition HA.4.8.2.7 . To prove (*c*), it suffices to observe that if $\mathcal{C}$ and $\mathcal{D}$ are additive presentable ∞-categories, then the tensor product $\mathcal{C} \otimes \mathcal{D}$ (formed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$) is again additive (since it admits the structure of a $\mathrm{Sp}^{\mathrm{cn}}$-module). □

The proof of Theorem C.4.1.1 will require some preliminaries. First, we prove an analogue of Theorem C.4.1.1 for $\mathbb{E}_\infty$-spaces which need not be grouplike.

Definition C.4.1.6. Let $\mathcal{C}$ be an ∞-category. We will say that $\mathcal{C}$ is *semiadditive* if it satisfies the following conditions:

(*a*) The ∞-category $\mathcal{C}$ admits finite products and finite coproducts.

(*b*) The ∞-category $\mathcal{C}$ admits a zero object 0. In particular, to every pair of objects X and Y we can associate a *zero morphism* (well-defined up to homotopy) given by the composition $X \to 0 \to Y$.

(c) For every pair of objects X and Y, the canonical map $X \amalg Y \to X \times Y$ represented by the matrix $\begin{bmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{bmatrix}$ is an equivalence in $\mathcal{C}$.

Remark C.4.1.7. Let $\mathcal{C}$ be an ∞-category which admits finite sums and finite products. Then $\mathcal{C}$ is semiadditive if and only if the homotopy category of $\mathcal{C}$ is semiadditive.

Remark C.4.1.8. Let $\mathcal{C}$ be an ∞-category which admits finite sums and finite products. Then we can regard $\mathcal{C}$ as endowed with the *Cartesian* symmetric monoidal structure (given by the formation of products) or with the *coCartesian* symmetric monoidal structure (given by the formation of coproducts). The ∞-category $\mathcal{C}$ is semiadditive if and only if these two symmetric monoidal structures on $\mathcal{C}$ are the same.

Let us regard the ∞-category $\mathcal{S}$ as endowed with the symmetric monoidal structure given by the Cartesian product and let $\mathrm{CAlg}(\mathcal{S})$ denote the ∞-category of commutative algebra objects of $\mathcal{S}$: that is, the ∞-category of $\mathbb{E}_\infty$-spaces. We then have the following analogue of Theorem C.4.1.1:

Proposition C.4.1.9. *Let $\mathcal{C}$ be a presentable ∞-category. The following conditions are equivalent:*

(a) *The ∞-category $\mathcal{C}$ is semiadditive (Definition C.4.1.6).*

(b) *The free functor $\mathrm{Sym}^*_{\mathcal{S}} : \mathcal{S} \to \mathrm{CAlg}(\mathcal{S})$ induces an equivalence of ∞-categories*

$$\theta : \mathcal{C} \simeq \mathcal{S} \otimes \mathcal{C} \to \mathrm{CAlg}(\mathcal{S}) \otimes \mathcal{C}.$$

Here the tensor products are formed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories.

Proof. Let us regard $\mathcal{C}$ as endowed with the Cartesian symmetric monoidal structure. We then have

$$\begin{aligned} \mathrm{CAlg}(\mathcal{S}) \otimes \mathcal{C} &\simeq \mathrm{Fun}'(\mathcal{C}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{S})) \\ &\simeq \mathrm{CAlg}(\mathrm{Fun}'(\mathcal{C}^{\mathrm{op}}, \mathcal{S})) \\ &\simeq \mathrm{CAlg}(\mathcal{C}), \end{aligned}$$

where $\mathrm{Fun}'(\mathcal{C}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{S}))$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ spanned by those functors which preserve small limits and $\mathrm{Fun}'(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is defined similarly. Unwinding the definitions, we see that θ can be identified with the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$. Note that the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ (given by the formation of Cartesian products) is always coCartesian (Proposition HA.3.2.4.7). If the functor θ is an equivalence of ∞-categories, then the Cartesian symmetric monoidal structure on $\mathcal{C}$ is also coCartesian so that $\mathcal{C}$ is semiadditive by virtue of Remark C.4.1.8. Conversely, if the Cartesian symmetric monoidal structure on $\mathcal{C}$ is coCartesian, then the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of ∞-categories by virtue of Proposition HA.2.4.3.9 . □

Construction C.4.1.10. Let $I = \{x, y\}$ be a set with two elements and let M be the free $\mathbb{E}_\infty$-space generated by I (concretely, M can be identified with a product of two copies of the nerve of the groupoid $\mathcal{F}\mathrm{in}^{\simeq}$ of finite sets). Then $\pi_0 M$ can be identified with the commutative monoid freely generated by x and y. The construction

$$x \mapsto x \qquad y \mapsto x + y$$

determines a map of sets $I \to \pi_0 M$ which we can lift to a morphism of $\mathbb{E}_\infty$-spaces $\sigma : M \to M$, which we will refer to as the *shearing map.*

For any $\mathbb{E}_\infty$-space Z, composition with σ induces a map of spaces

$$Z^2 \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{S})}(M, Z) \xrightarrow{\circ\sigma} \mathrm{Map}_{\mathrm{CAlg}(\mathcal{S})}(M, Z) \simeq Z^2$$

which induces the homomorphism of commutative monoids

$$\pi_0 Z^2 \to \pi_0 Z^2 \qquad (a, b) \mapsto (a, a + b).$$

It follows that the commutative monoid $\pi_0 Z$ is a group if and only if Z is local with respect to the single map $\{\sigma\}$.

According to Remark HA.5.2.6.26 , we can identify the ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ of connective spectra with the full subcategory of $\mathrm{CAlg}(\mathcal{S})$ spanned by those $\mathbb{E}_\infty$-spaces Z for which $\pi_0 Z$ is a group. In other words, the ∞-category of connective spectra can be obtained as the localization of $\mathrm{CAlg}(\mathcal{S})$ with respect to the single morphism σ. In particular, there exists a functor $L : \mathrm{CAlg}(\mathcal{S}) \to \mathrm{Sp}^{\mathrm{cn}}$ with the following universal property: for any ∞-category $\mathcal{C}$ which admits small colimits, composition with L induces an equivalence $\mathrm{LFun}(\mathrm{Sp}^{\mathrm{cn}}, \mathcal{C}) \to \mathrm{LFun}(\mathrm{CAlg}(\mathcal{S}), \mathcal{C})$ whose essential image is spanned by those functors $F \in \mathrm{LFun}(\mathrm{CAlg}(\mathcal{S}), \mathcal{C})$ for which $F(\sigma)$ is an equivalence in $\mathcal{C}$.

Proof of Theorem C.4.1.1. Let $\mathcal{C}$ be a presentable ∞-category. Using Proposition HA.4.8.1.17 , we can identify the tensor product $\mathrm{Sp}^{\mathrm{cn}} \otimes \mathcal{C}$ with the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}^{\mathrm{cn}})$ spanned by those functors which are accessible and preserve small limits. In particular, the ∞-category $\mathrm{Sp}^{\mathrm{cn}} \otimes \mathcal{C}$ admits a fully faithful embedding into the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ whose essential image is closed under finite products. Since $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is a stable ∞-category, it is additive. It follows that $\mathrm{Sp}^{\mathrm{cn}} \otimes \mathcal{C}$ is additive. If condition (b) is satisfied, then $\mathcal{C}$ is equivalent to $\mathrm{Sp}^{\mathrm{cn}} \otimes \mathcal{C}$ and is therefore also additive.

We now prove the converse. Suppose that $\mathcal{C}$ is additive; we wish to show that the canonical map $F : \mathcal{C} \to \mathrm{Sp}^{\mathrm{cn}} \otimes \mathcal{C}$ is an equivalence of ∞-categories. Unwinding the definitions, we see that F is left adjoint to the functor $G : \mathrm{LFun}(\mathrm{Sp}^{\mathrm{cn}}, \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{C}$ given by evaluation at the sphere spectrum S. Let $L : \mathrm{CAlg}(\mathcal{S}) \to \mathrm{Sp}^{\mathrm{cn}}$ be the localization functor appearing in Construction C.4.1.10. Then the functor G factors as a composition

$$\mathrm{LFun}(\mathrm{Sp}^{\mathrm{cn}}, \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{G'} \mathrm{LFun}(\mathrm{CAlg}(\mathcal{S}), \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{G''} \mathcal{C},$$

where the functor G'' is given by evaluation on the free $\mathbb{E}_\infty$-space on a single generator and is therefore an equivalence of ∞-categories by virtue of Proposition C.4.1.9 (since any additive ∞-category is also semiadditive). It will therefore suffice to show that the functor G' is an equivalence of ∞-categories. For this, it will suffice to show that for every colimit-preserving functor $H : \mathrm{CAlg}(\mathcal{S}) \to \mathcal{C}^{\mathrm{op}}$, the map $H(\sigma)$ is an equivalence in $\mathcal{C}$, where $\sigma : \mathrm{Sym}^*_{\mathcal{S}}(I) \to \mathrm{Sym}^*_{\mathcal{S}}(I)$ is the shearing map appearing in Construction C.4.1.10. Note that if $G''(H)$ is an object $C \in \mathcal{C}$, then $H(\sigma)$ can be identified with the map from $C \oplus C$ to itself represented by the matrix $\begin{bmatrix} \mathrm{id}_C & 0 \\ \mathrm{id}_C & \mathrm{id}_C \end{bmatrix}$, which is an equivalence by virtue of our assumption that the homotopy category $\mathrm{h}\mathcal{C}$ is an additive category. □

C.4.2 Tensor Products

Every prestable ∞-category is additive, so we can regard the ∞-category Groth_∞ of Grothendieck prestable ∞-categories as a full subcategory of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{Add}}$ of presentable additive ∞-categories. The main result of this section can be formulated as follows:

Theorem C.4.2.1. *Let $\mathsf{Groth}_\infty \subseteq \mathcal{P}\mathrm{r}^{\mathrm{Add}}$ denote the ∞-category of Grothendieck prestable ∞-categories (see Definition C.3.0.5). Then Groth_∞ contains the unit object of $\mathcal{P}\mathrm{r}^{\mathrm{Add}}$ and is closed under tensor products. Consequently, Groth_∞ inherits the structure of a symmetric monoidal ∞-category for which the inclusion $\mathsf{Groth}_\infty \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{Add}}$ is symmetric monoidal.*

Remark C.4.2.2. Let $\mathcal{C}$ and $\mathcal{D}$ be presentable stable ∞-categories equipped with t-structures $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ and $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ which are compatible with filtered colimits. Then $\mathcal{C}_{\geqslant 0}$ and $\mathcal{D}_{\geqslant 0}$ are Grothendieck prestable ∞-categories. Choose a functor $H : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which exhibits $\mathcal{E}$ as a tensor product of $\mathcal{C}$ with $\mathcal{D}$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Let $\mathcal{E}_{\geqslant 0}$ be the smallest full subcategory of $\mathcal{E}$ which is closed under small colimits and contains the essential image of the restriction $h = H|_{\mathcal{C}_{\geqslant 0} \times \mathcal{D}_{\geqslant 0}}$. Since the formation of stabilizations commutes with tensor products, we can identify $\mathcal{E}_{\geqslant 0}$ with the smallest full subcategory of

$$\mathcal{E} \simeq \mathrm{Sp}(\mathcal{C}_{\geqslant 0}) \otimes \mathrm{Sp}(\mathcal{D}_{\geqslant 0}) \simeq \mathrm{Sp}(\mathcal{C}_{\geqslant 0} \otimes \mathcal{D}_{\geqslant 0})$$

which contains the essential image of the functor

$$\Sigma^\infty : \mathcal{C}_{\geqslant 0} \otimes \mathcal{D}_{\geqslant 0} \to \mathrm{Sp}(\mathcal{C}_{\geqslant 0} \otimes \mathcal{D}_{\geqslant 0}).$$

The content of Theorem C.4.2.1 is that the tensor product $\mathcal{C}_{\geqslant 0} \otimes \mathcal{D}_{\geqslant 0}$ is also a Grothendieck prestable ∞-category. Unwinding the definitions, this translates into the following assertions:

(*a*) The functor $\Sigma^\infty : \mathcal{C}_{\geqslant 0} \otimes \mathcal{D}_{\geqslant 0} \to \mathrm{Sp}(\mathcal{C}_{\geqslant 0} \otimes \mathcal{D}_{\geqslant 0})$ is fully faithful. In other words, the functor $h : \mathcal{C}_{\geqslant 0} \times \mathcal{D}_{\geqslant 0} \to \mathcal{E}_{\geqslant 0}$ exhibits $\mathcal{E}_{\geqslant 0}$ as a tensor product of $\mathcal{C}_{\geqslant 0}$ with $\mathcal{D}_{\geqslant 0}$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

(b) The full subcategory $\mathcal{E}_{\geqslant 0}$ is closed under extensions in $\mathcal{E}$, and therefore determines a t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ on $\mathcal{E}$.

(c) The full subcategory $\mathcal{E}_{\leqslant 0} \subseteq \mathcal{E}$ is closed under filtered colimits.

Remark C.4.2.3. Let $\mathcal{C}$ and $\mathcal{D}$ be presentable stable ∞-categories, and let

$$m : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$$

be a functor which exhibits $\mathcal{C} \otimes \mathcal{D}$ as a tensor product of $\mathcal{C}$ with $\mathcal{D}$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Let $\mathrm{Core}(\mathcal{C})$, $\mathrm{Core}(\mathcal{D})$, and $\mathrm{Core}(\mathcal{C} \otimes \mathcal{D})$ be the partially ordered sets described in Remark C.3.1.3. Then m induces a map of partially ordered sets

$$m_! : \mathrm{Core}(\mathcal{C}) \times \mathrm{Core}(\mathcal{D}) \to \mathrm{Core}(\mathcal{C} \otimes \mathcal{D}),$$

which we can describe explicitly as follows: given cores $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$ and $\mathcal{D}_{\geqslant 0} \subseteq \mathcal{D}$, we define $m_!(\mathcal{C}_{\geqslant 0}, \mathcal{D}_{\geqslant 0})$ to be the smallest full subcategory of $\mathcal{C} \otimes \mathcal{D}$ which is closed under colimits and extensions and contains the objects $m(C, D)$ for each $C \in \mathcal{C}_{\geqslant 0}$, $D \in \mathcal{D}_{\geqslant 0}$. It is not difficult to see that this construction exhibits the construction $\mathcal{C} \mapsto \mathrm{Core}(\mathcal{C})$ as a (covariant) lax symmetric monoidal functor from the homotopy category $\mathrm{h}\mathcal{P}\mathrm{r}^{\mathrm{St}}$ to the category of partially ordered sets. Equivalently, the construction $\mathcal{C} \mapsto \mathrm{Core}(\mathcal{C})$ can be regarded as a lax symmetric monoidal functor from the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ to the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$. It follows that the coCartesian fibration $q : \mathsf{Groth}_{\infty}^{+} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$ of Remark C.3.1.6 can be regarded as a symmetric monoidal functor. In more concrete terms, the symmetric monoidal structure on $\mathsf{Groth}_{\infty}^{+}$ is given by the construction

$$(\mathcal{C}, \mathcal{C}_{\geqslant 0}) \otimes (\mathcal{D}, \mathcal{D}_{\geqslant 0}) = (\mathcal{C} \otimes \mathcal{D}, m_!(\mathcal{C}_{\geqslant 0}, \mathcal{D}_{\geqslant 0}))$$

described above.

Now suppose that $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck prestable ∞-categories. Theorem C.4.2.1 asserts that the tensor product $\mathcal{C} \otimes \mathcal{D}$ (formed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$) is again a Grothendieck prestable ∞-category. Since the stabilization functor $\mathrm{Sp}(\bullet) : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$ is symmetric monoidal, we can identify the ∞-category $\mathrm{Sp}(\mathcal{C} \otimes \mathcal{D})$ with the tensor product $\mathrm{Sp}(\mathcal{C}) \otimes \mathrm{Sp}(\mathcal{D})$. Moreover, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{D} & \xrightarrow{m} & \mathcal{C} \otimes \mathcal{D} \\ \downarrow{\scriptstyle \Sigma^{\infty}_{\mathcal{C}} \times \Sigma^{\infty}_{\mathcal{D}}} & & \downarrow{\scriptstyle \Sigma^{\infty}_{\mathcal{C} \otimes \mathcal{D}}} \\ \mathrm{Sp}(\mathcal{C}) \times \mathrm{Sp}(\mathcal{D}) & \xrightarrow{m'} & \mathrm{Sp}(\mathcal{C}) \otimes \mathrm{Sp}(\mathcal{D}) \end{array}$$

where the vertical maps are fully faithful embeddings. Since $\mathcal{C} \otimes \mathcal{D}$ is generated under small colimits by the essential image of m, the functor $\Sigma^{\infty}_{\mathcal{C} \otimes \mathcal{D}}$ identifies $\mathcal{C} \otimes \mathcal{D}$ with the smallest full subcategory of $\mathrm{Sp}(\mathcal{C}) \otimes \mathrm{Sp}(\mathcal{D})$ which contains the image of functor $m'|_{\mathrm{Sp}(\mathcal{C})_{\geqslant 0} \times \mathrm{Sp}(\mathcal{D})_{\geqslant 0}}$ and is closed under small colimits. One consequence of Theorem C.4.2.1 is that this full subcategory is *also* closed under extensions: that is, it can be identified with the core $m'_!(\mathrm{Sp}(\mathcal{C})_{\geqslant 0}, \mathrm{Sp}(\mathcal{D})_{\geqslant 0})$ defined above. It follows that we can regard the construction $\mathcal{C} \mapsto (\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{C})_{\geqslant 0})$ of Corollary C.3.1.4 as a *symmetric monoidal* functor from the ∞-category Groth_∞ to the ∞-category Groth_∞^+.

C.4.3 The Proof of Theorem C.4.2.1

The proof of Theorem C.4.2.1 will require the following generalization of Lemma HTT.6.3.3.4:

Lemma C.4.3.1. *Let $\mathcal{X}$ be a presentable ∞-category equipped with a small collection of full subcategories $\{\mathcal{X}_\alpha \subseteq \mathcal{X}\}_{\alpha \in A}$. Assume the following:*

(1) *Each of the ∞-categories $\mathcal{X}_\alpha$ is presentable.*

(2) *Each of the inclusion functors $\mathcal{X}_\alpha \hookrightarrow \mathcal{X}$ admits a left exact left adjoint $L_\alpha : \mathcal{X} \to \mathcal{X}_\alpha$.*

(3) *Filtered colimits in the ∞-category $\mathcal{X}$ are left exact.*

Then the intersection $\bigcap \mathcal{X}_\alpha$ is presentable, and the inclusion functor $\bigcap \mathcal{X}_\alpha \hookrightarrow \mathcal{X}$ admits a left exact left adjoint L.

Proof. Without loss of generality, we may assume that A is an initial segment of the ordinals: that is, $A = \{\alpha : \alpha < \beta\}$, for some ordinal β. Choose a regular cardinal κ such that each of the functors L_α commutes with κ-filtered colimits (when regarded as a functor from the ∞-category $\mathcal{X}$ to itself). Let $\beta\kappa$ denote the ordinal product of α with κ, and let $[\beta\kappa]$ denote the linearly ordered set of ordinals $\leqslant \beta\kappa$. We will construct a map $\mathrm{N}[\beta\kappa] \to \mathrm{Fun}(\mathcal{X}, \mathcal{X})$, which we view as a transfinite sequence of functors $\{F_\gamma : \mathcal{X} \to \mathcal{X}\}_{\gamma \leqslant \beta\kappa}$. Let $F_0 = \mathrm{id}_{\mathcal{X}}$ and $F_\gamma = \varinjlim_{\delta < \gamma} F_\delta$ when γ is a nonzero limit ordinal. To complete the construction, it will suffice to define $F_{\gamma+1}$ assuming that F_γ has already been defined. Writing $\gamma = \beta\delta + \alpha$ for $\alpha < \beta$, we set $F_{\gamma+1} = L_\alpha \circ F_\gamma$.

Since each L_α is an accessible left exact functor and filtered colimits in $\mathcal{X}$ are left exact (see Example **??**), it follows by induction on γ that each of the functors F_γ is left exact and accessible. Set $L = F_{\beta\kappa}$. We claim that L is a localization functor with essential image $\bigcap_{\alpha \in A} \mathcal{X}_\alpha$. To prove this, it suffices to verify the following:

(a) For each $X \in \mathcal{X}$ and each $Y \in \bigcap_{\alpha \in A} \mathcal{X}_\alpha$, the canonical map $X \to L(X)$ induces a homotopy equivalence $\mathrm{Map}_{\mathcal{X}}(L(X), Y) \to \mathrm{Map}_{\mathcal{X}}(X, Y)$. More generally, we claim that each of the maps

$$\mathrm{Map}_{\mathcal{X}}(F_\gamma(X), Y) \to \mathrm{Map}_{\mathcal{X}}(F_0(X), Y) = \mathrm{Map}_{\mathcal{X}}(X, Y)$$

is a homotopy equivalence. The proof proceeds by induction on γ, the case $\gamma = 0$ being trivial. If γ is a nonzero limit ordinal, we invoke the fact that $F_\gamma \simeq \varinjlim_{\delta<\gamma} F_\delta$. To handle the case of successor ordinals, it suffices to show that $\mathrm{Map}_{\mathcal{X}}(F_{\gamma+1}(X), Y) \to \mathrm{Map}_{\mathcal{X}}(F_\gamma(X), Y)$ is a homotopy equivalence for each ordinal γ. Writing $\gamma = \beta\delta + \alpha$, we have $F_{\gamma+1}(X) = L_\alpha F_\gamma(X)$. The desired result now follows from our assumption that Y belongs to the subcategory $\mathcal{X}_\alpha \subseteq \mathcal{X}$.

(b) For each $X \in \mathcal{X}$, we must show that the object $L(X)$ belongs to $\bigcap \mathcal{X}_\alpha$. Fix $\alpha \in A$; we wish to show that $L(X) \in \mathcal{X}_\alpha$. Note that the collection of ordinals of the form $\beta\delta + \alpha + 1$, where $\delta < \kappa$ is cofinal in the set $\{\gamma : \gamma < \beta\kappa\}$. It follows that

$$L(X) \simeq \varinjlim_{\delta<\kappa} F_{\beta\delta+\alpha+1}(X) = \varinjlim_{\delta<\kappa} L_\alpha F_{\beta\delta+\alpha}(X)$$

is a κ-filtered colimit of objects belonging to $\mathcal{X}_\alpha$. Since the functor L_α commutes with κ-filtered colimits, we conclude that $L(X) \in \mathcal{X}_\alpha$ as desired.

□

Proof of Theorem C.4.2.1. It is clear that the unit object $\mathrm{Sp}^{\mathrm{cn}} \in \mathcal{P}\mathrm{r}^{\mathrm{Add}}$ is a Grothendieck prestable ∞-category. It will therefore suffice to show that if $\mathcal{C}$ and $\mathcal{C}'$ are Grothendieck prestable ∞-categories, then the tensor product $\mathcal{C} \otimes \mathcal{C}'$ is also a Grothendieck prestable ∞-category. Using Theorem C.2.4.1, we may assume without loss of generality that there exist inclusions

$$\mathcal{C} \subseteq \mathcal{D} \qquad \mathcal{C}' \subseteq \mathcal{D}'$$

which admit left exact left adjoints

$$L : \mathcal{D} \to \mathcal{C} \qquad L' : \mathcal{D}' \to \mathcal{C}',$$

where $\mathcal{D} = \mathrm{RMod}_A^{\mathrm{cn}}$ and $\mathcal{D}' = \mathrm{RMod}_{A'}^{\mathrm{cn}}$ for some connective $\mathbb{E}_1$-rings A and A'.

We first note that L induces a functor $F : \mathcal{D} \otimes \mathcal{D}' \to \mathcal{C} \otimes \mathcal{D}'$. Using Corollary C.4.1.3, we can view $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{D}'$ as ∞-categories tensored over $\mathrm{Sp}^{\mathrm{cn}}$, the functor L with a $\mathrm{Sp}^{\mathrm{cn}}$-linear functor, and F with the induced map

$$\mathcal{D} \otimes_{\mathrm{Sp}^{\mathrm{cn}}} \mathcal{D}' \to \mathcal{C} \otimes_{\mathrm{Sp}^{\mathrm{cn}}} \mathcal{D}'.$$

Applying Theorem HA.4.8.4.6 , we can identify F with the map $\mathrm{RMod}_A(\mathcal{D}) \to \mathrm{RMod}_A(\mathcal{C})$ determined by L, which is left adjoint to the inclusion $\mathrm{RMod}_A(\mathcal{C}) \hookrightarrow \mathrm{RMod}_A(\mathcal{D})$. It follows that the functor F admits a fully faithful right adjoint $G : \mathcal{C} \otimes \mathcal{D}' \to \mathcal{D} \otimes \mathcal{D}'$. Let $\mathcal{E} \subseteq \mathcal{D} \otimes \mathcal{D}'$ denote the essential image of G and let $\overline{L} = (G \circ F) : \mathcal{D} \otimes \mathcal{D}' \to \mathcal{E}$ be a left adjoint to the inclusion of $\mathcal{E}$ into $\mathcal{D} \otimes \mathcal{D}'$. Note that we can identify $\overline{L}$ with the upper horizontal map in the commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{RMod}_A(\mathcal{D}) & \longrightarrow & \mathrm{RMod}_A(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow{L} & \mathcal{C}. \end{array}$$

Since the vertical maps are conservative and preserve finite limits (Corollary HA.4.2.3.3) and the functor L is left exact, the functor $\overline{L}$ is also left exact.

Let $F' : \mathcal{D} \otimes \mathcal{D}' \to \mathcal{D} \otimes \mathcal{C}'$ be the functor determined by L'. Repeating the above reasoning, we deduce that F' admits a fully faithful right adjoint $G' : \mathcal{D} \otimes \mathcal{C}' \to \mathcal{D} \otimes \mathcal{D}'$ whose essential image is a full subcategory $\mathcal{E}'$ of $\mathcal{D} \otimes \mathcal{D}'$, and that the inclusion $\mathcal{E}' \hookrightarrow \mathcal{D} \otimes \mathcal{D}'$ admits a left exact left adjoint $\overline{L}' : \mathcal{D} \otimes \mathcal{D}' \to \mathcal{E}'$.

For every pair of presentable ∞-categories $\mathcal{X}$ and $\mathcal{Y}$, let $\mathrm{Fun}'(\mathcal{X}^{\mathrm{op}} \times \mathcal{Y}^{\mathrm{op}}, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}} \times \mathcal{Y}^{\mathrm{op}}, \mathcal{S})$ spanned by those functors which preserve small limits separately in each variable; it then follows from Proposition HTT.5.5.2.2 that we can identify $\mathrm{Fun}'(\mathcal{X}^{\mathrm{op}} \times \mathcal{Y}^{\mathrm{op}}, \mathcal{S})$ with the tensor product $\mathcal{X} \otimes \mathcal{Y}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Since L and L' are localization functors, composition with the functor $L \times L'$ induces a fully faithful embedding

$$\mathrm{Fun}'(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}'^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}'(\mathcal{D}^{\mathrm{op}} \times \mathcal{D}'^{\mathrm{op}}, \mathcal{S})$$

whose essential image consists of those functors $H : \mathcal{D}^{\mathrm{op}} \times \mathcal{D}'^{\mathrm{op}} \to \mathcal{S}$ which carry each $(L \times L')$-equivalence in $\mathcal{D} \times \mathcal{D}'$ to a homotopy equivalence in $\mathcal{S}$. Since every $(L \times L')$-equivalence can be written as a composition of an $(L \times \mathrm{id})$-equivalence with an $(\mathrm{id} \times L')$-equivalence, it follows that $\mathrm{Fun}'(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}'^{\mathrm{op}}, \mathcal{S})$ can be identified with the fiber product

$$\mathrm{Fun}'(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}'^{\mathrm{op}}, \mathcal{S}) \times_{\mathrm{Fun}'(\mathcal{D}^{\mathrm{op}} \times \mathcal{D}'^{\mathrm{op}}, \mathcal{S})} \mathrm{Fun}'(\mathcal{D}^{\mathrm{op}} \times \mathcal{C}'^{\mathrm{op}}, \mathcal{S}).$$

In other words, the tensor product $\mathcal{C} \otimes \mathcal{C}'$ is equivalent to the intersection of the full subcategories $\mathcal{E}, \mathcal{E}' \subseteq \mathcal{D} \otimes \mathcal{D}'$. It follows from Theorem HA.4.8.5.16 that we can identify $\mathcal{D} \otimes \mathcal{D}'$ with the ∞-category $\mathrm{RMod}^{\mathrm{cn}}_{A \otimes A'}$, and it follows from Lemma C.4.3.1 that the intersection $\mathcal{E} \cap \mathcal{E}'$ is an accessible left exact localization of $\mathcal{D} \otimes \mathcal{D}'$. Applying Theorem C.2.4.1, we conclude that $\mathcal{C} \otimes \mathcal{C}' \simeq \mathcal{E} \cap \mathcal{E}'$ is a Grothendieck prestable ∞-category, as desired. □

Remark C.4.3.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories. Suppose that F admits a fully faithful right adjoint G. Then, for any presentable ∞-category $\mathcal{E}$, the induced map $(F \otimes \mathrm{id}_{\mathcal{E}}) : \mathcal{C} \otimes \mathcal{E} \to \mathcal{D} \otimes \mathcal{E}$ admits a fully faithful right adjoint. This is clear, since a right adjoint to $(F \otimes \mathrm{id}_{\mathcal{E}})$ is given by the composition

$$\mathcal{D} \otimes \mathcal{E} \simeq \mathrm{Fun}'(\mathcal{E}^{\mathrm{op}}, \mathcal{D}) \xrightarrow{G \circ} \mathrm{Fun}'(\mathcal{E}^{\mathrm{op}}, \mathcal{C}) \simeq \mathcal{C} \otimes \mathcal{E};$$

here $\mathrm{Fun}'(\mathcal{E}^{\mathrm{op}}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{D})$ spanned by those functors which preserve small limits and $\mathrm{Fun}'(\mathcal{E}^{\mathrm{op}}, \mathcal{C})$ is defined similarly.

C.4.4 Left Exact and Compact Functors

We now study properties of functors between prestable ∞-categories which are preserved by the tensor product of Theorem C.4.2.1.

Proposition C.4.4.1. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between Grothendieck prestable ∞-categories which preserves small colimits. If f is left exact, then for any Grothendieck prestable ∞-category $\mathcal{E}$ the induced map $(f \otimes \mathrm{id}_{\mathcal{E}}) : \mathcal{C} \otimes \mathcal{E} \to \mathcal{D} \otimes \mathcal{E}$ is also left exact.*

Proof. Using Theorem C.2.4.1, we can assume that there is a connective $\mathbb{E}_1$-ring A and an inclusion $\mathcal{E} \hookrightarrow \mathrm{RMod}_A^{\mathrm{cn}}$ which admits a left exact left adjoint $L : \mathrm{RMod}_A^{\mathrm{cn}} \to \mathcal{E}$. The proof of Theorem C.4.2.1 shows that L induces left exact functors

$$L_{\mathcal{C}} : \mathrm{RMod}_A(\mathcal{C}) \simeq \mathcal{C} \otimes \mathrm{RMod}_A^{\mathrm{cn}} \to \mathcal{C} \otimes \mathcal{E}$$

$$L_{\mathcal{D}} : \mathrm{RMod}_A(\mathcal{D}) \simeq \mathcal{D} \otimes \mathrm{RMod}_A^{\mathrm{cn}} \to \mathcal{D} \otimes \mathcal{E}$$

which admit fully faithful right adjoints $G_{\mathcal{C}}$ and $G_{\mathcal{D}}$. We then have

$$\begin{aligned} f \otimes \mathrm{id}_{\mathcal{E}} &\simeq (f \otimes \mathrm{id}_{\mathcal{E}}) \circ L_{\mathcal{C}} \circ G_{\mathcal{E}} \\ &\simeq L_{\mathcal{D}} \circ (f \otimes \mathrm{id}_{\mathrm{RMod}_A^{\mathrm{cn}}}) \\ &\simeq G_{\mathcal{E}}. \end{aligned}$$

The functor $L_{\mathcal{D}}$ is left exact, the functor $G_{\mathcal{E}}$ preserves all small colimits, and $f \otimes \mathrm{id}_{\mathrm{RMod}_A^{\mathrm{cn}}}$ can be identified with the functor $\mathrm{RMod}_A(\mathcal{C}) \to \mathrm{RMod}_A(\mathcal{D})$ induced by f (by virtue of Theorem HA.4.8.4.6). This functor is left exact by virtue of our assumption that f is left exact. □

Corollary C.4.4.2. *The symmetric monoidal structure on the ∞-category Groth_∞ restricts to a symmetric monoidal structure on the subcategory $\mathsf{Groth}_\infty^{\mathrm{lex}}$ of Notation C.3.2.3.*

Proposition C.4.4.3. *Let $f : \mathcal{C} \to \mathcal{D}$ be a compact functor between Grothendieck presentable ∞-categories. Then, for any Grothendieck prestable ∞-category $\mathcal{E}$, the induced map $(f \otimes \mathrm{id}_{\mathcal{E}}) : \mathcal{C} \otimes \mathcal{E} \to \mathcal{D} \otimes \mathcal{E}$ is compact.*

Proof. Let $F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ be the functor obtained from f by applying the stabilization construction $\mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$. Using Proposition C.3.4.1, we see that F admits a right adjoint $G : \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{C})$ which commutes with small colimits. Then F and G induce adjoint functors

$$\mathrm{Sp}(\mathcal{C}) \otimes \mathrm{Sp}(\mathcal{E}) \underset{(G \otimes \mathrm{id})}{\overset{(F \otimes \mathrm{id})}{\rightleftarrows}} \mathrm{Sp}(\mathcal{D}) \otimes \mathrm{Sp}(\mathcal{E})$$

which commute with small colimits. We now observe that the functor $F \otimes \mathrm{id}$ can be identified with the image of the map $f \otimes \mathrm{id}_{\mathcal{E}} : \mathcal{C} \otimes \mathcal{E} \to \mathcal{D} \otimes \mathcal{E}$ under the stabilization functor $\mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$, so that $f \otimes \mathrm{id}_{\mathcal{E}}$ is compact (see Proposition C.3.4.1). □

Corollary C.4.4.4. *The symmetric monoidal structure on the* ∞*-category* Groth_∞ *restricts to a symmetric monoidal structure on the subcategory* $\mathsf{Groth}^{\mathrm{c}}_\infty \subseteq \mathsf{Groth}_\infty$ *of Definition C.3.4.2.*

Remark C.4.4.5. In the situation of Proposition C.4.4.3, suppose that the functor f admits a right adjoint g which preserves *all* small colimits. Then the adjunction between f and g determines a pair of adjoint functors

$$\mathcal{C} \otimes \mathcal{E} \underset{(g \otimes \mathrm{id}_{\mathcal{E}}),}{\overset{(f \otimes \mathrm{id}_{\mathcal{E}})}{\rightleftarrows}} \mathcal{D} \otimes \mathcal{E}$$

so that the functor $f \otimes \mathrm{id}_{\mathcal{E}}$ also admits a right adjoint which preserves small colimits.

C.4.5 Tensor Products and Colimits

The symmetric monoidal structure on the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is closed: that is, for every presentable ∞-category $\mathcal{C}$, the tensor product functor $\mathcal{D} \mapsto \mathcal{C} \otimes \mathcal{D}$ admits a right adjoint, given by the construction $\mathcal{E} \mapsto \mathrm{LFun}(\mathcal{C}, \mathcal{E})$. It follows that the tensor product on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ preserves small colimits separately in each variable. Our next goal is to prove an analogous assertion for the ∞-category $\mathsf{Groth}^{\mathrm{c}}_\infty$:

Proposition C.4.5.1. *The tensor product*

$$\otimes : \mathsf{Groth}^{\mathrm{c}}_\infty \times \mathsf{Groth}^{\mathrm{c}}_\infty \to \mathsf{Groth}^{\mathrm{c}}_\infty$$

of Corollary C.4.4.4 preserves small colimits separately in each variable.

Warning C.4.5.2. Proposition C.4.5.1 does not follow from the corresponding assertion concerning the tensor product on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, since the inclusion functor $\mathsf{Groth}^{\mathrm{c}}_\infty \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$ does not preserve small colimits.

Proposition C.4.5.1 is an immediate consequence of Remarks C.3.5.2, C.4.2.3, and the following stronger result:

Proposition C.4.5.3. *The tensor product*

$$\otimes : \mathsf{Groth}^{+}_\infty \times \mathsf{Groth}^{+}_\infty \to \mathsf{Groth}^{+}_\infty$$

of Remark C.4.2.3 preserves small colimits separately in each variable.

Proof. Let $q : \mathsf{Groth}^{+}_\infty \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$ be the forgetful functor given by $(\mathcal{C}, \mathcal{C}_{\geqslant 0}) \mapsto \mathcal{C}$. Note that a small diagram $p : K^{\triangleright} \to \mathsf{Groth}^{+}_\infty$ is a colimit diagram if and only $q \circ p$ is a colimit diagram in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ and p is a q-colimit diagram (Proposition HTT.4.3.1.5). Since the tensor product on the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ preserves small colimits separately in each variable, it will suffice to show that for each object $(\mathcal{C}, \mathcal{C}_{\geqslant 0})$, the formation of tensor product with $(\mathcal{C}, \mathcal{C}_{\geqslant 0})$ determines a functor $T : \mathsf{Groth}^{+}_\infty \to \mathsf{Groth}^{+}_\infty$ which preserves q-colimit diagrams. Using Propositions HTT.4.3.1.9 and HTT.4.3.1.10 , we are reduced to proving the following more concrete assertions:

(a) For every presentable stable ∞-category $\mathcal{D}$, the functor T induces a map of partially ordered sets $\mathrm{Core}(\mathcal{D}) \to \mathrm{Core}(\mathcal{C} \otimes \mathcal{D})$ which preserves colimits (in other words, the formation of suprema).

(b) The functor T carries q-coCartesian morphisms in Groth_∞^+ to q-coCartesian morphisms in Groth_∞^+.

We first prove (a). Let $m : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a functor which exhibits $\mathcal{E}$ as a tensor product of $\mathcal{C}$ with $\mathcal{D}$, and let $m_! : \mathrm{Core}(\mathcal{C}) \times \mathrm{Core}(\mathcal{D}) \to \mathrm{Core}(\mathcal{E})$ be as in Remark C.4.2.3. Let $\{\mathcal{D}^\alpha_{\geqslant 0}\}$ be a collection of cores of $\mathcal{D}$ having supremum $\mathcal{D}_{\geqslant 0} \in \mathrm{Core}(\mathcal{D})$. We wish to show that $m_!(\mathcal{C}_{\geqslant 0}, \mathcal{D}_{\geqslant 0})$ is a supremum of the set $\{m_!(\mathcal{C}_{\geqslant 0}, \mathcal{D}^\alpha_{\geqslant 0})\}$ in the partially ordered set $\mathrm{Core}(\mathcal{E})$. To prove this, suppose that $\mathcal{E}_{\geqslant 0} \subseteq \mathcal{E}$ is any core which contains each $m_!(\mathcal{C}_{\geqslant 0}, \mathcal{D}^\alpha_{\geqslant 0})$. Let $\mathcal{D}' \subseteq \mathcal{D}$ be the full subcategory spanned by those objects $D \in \mathcal{D}$ which satisfy $m(C, D) \in \mathcal{E}_{\geqslant 0}$ for each $C \in \mathcal{C}_{\geqslant 0}$. Since the functor m preserves small colimits separately in each variable, it follows that $\mathcal{D}' \subseteq \mathcal{D}$ is a core. By construction, it contains each $\mathcal{D}^\alpha_{\geqslant 0}$ and therefore contains the supremum $\mathcal{D}_{\geqslant 0}$, which proves that $m_!(\mathcal{C}_{\geqslant 0}, \mathcal{D}_{\geqslant 0}) \subseteq \mathcal{E}_{\geqslant 0}$ as desired.

Let us now prove (b). Suppose we are given a colimit preserving functor between presentable stable ∞-categories $f : \mathcal{D} \to \mathcal{D}'$, and form a commutative diagram

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{D} & \xrightarrow{m} & \mathcal{E} \\ \downarrow{\scriptstyle \mathrm{id} \times f} & & \downarrow{\scriptstyle g} \\ \mathcal{C} \times \mathcal{D}' & \xrightarrow{m'} & \mathcal{E}' \end{array}$$

where the horizontal maps exhibit $\mathcal{E}$ and $\mathcal{E}'$ as tensor products of $\mathcal{D}$ and $\mathcal{D}'$ with $\mathcal{C}$, respectively. Define

$$m_! : \mathrm{Core}(\mathcal{C}) \times \mathrm{Core}(\mathcal{D}) \to \mathrm{Core}(\mathcal{E}) \qquad m'_! : \mathrm{Core}(\mathcal{C}) \times \mathrm{Core}(\mathcal{D}') \to \mathrm{Core}(\mathcal{E}')$$

as in Remark C.4.2.3. Unwinding the definitions, we wish to show for every core $\mathcal{D}_{\geqslant 0} \subseteq \mathcal{D}$, if $\mathcal{D}'_{\geqslant 0} \subseteq \mathcal{D}'$ is the smallest core containing $f(\mathcal{D}_{\geqslant 0})$, then $m'_!(\mathcal{C}_{\geqslant 0}, \mathcal{D}'_{\geqslant 0})$ is the smallest core containing $g(m_!(\mathcal{C}_{\geqslant 0}, \mathcal{D}_{\geqslant 0}))$. Let $\mathcal{E}'_{\geqslant 0} \in \mathrm{Core}(\mathcal{E}')$ be any core satisfying $m(\mathcal{C}_{\geqslant 0}, \mathcal{D}_{\geqslant 0}) \subseteq g^{-1} \mathcal{E}'_{\geqslant 0}$, and let $\mathcal{D}''$ be the full subcategory of $\mathcal{D}'$ spanned by those objects D' which satisfy $m'(\mathcal{C}_{\geqslant 0} \times \{D'\}) \subseteq \mathcal{E}'_{\geqslant 0}$. Then $\mathcal{D}''$ is a core containing $f(\mathcal{D}_{\geqslant 0})$, so we have $\mathcal{D}'_{\geqslant 0} \subseteq \mathcal{D}''$ and therefore $m'_!(\mathcal{C}_{\geqslant 0}, \mathcal{D}'_{\geqslant 0}) \subseteq \mathcal{E}'_{\geqslant 0}$ as desired. □

C.4.6 Completed Tensor Products

We now consider the interaction between the tensor product on Grothendieck prestable ∞-categories and the separatedness and completeness conditions introduced in Definition C.1.2.12.

Proposition C.4.6.1. *Let* $\mathsf{Groth}_\infty^{\mathrm{comp}} \subseteq \mathsf{Groth}_\infty^{\mathrm{sep}} \subseteq \mathsf{Groth}_\infty$ *denote the full subcategories of* Groth_∞ *spanned by the complete and separated Grothendieck prestable ∞-categories, and let*

$$L : \mathsf{Groth}_\infty \to \mathsf{Groth}_\infty^{\mathrm{sep}} \qquad L' : \mathsf{Groth}_\infty \to \mathsf{Groth}_\infty^{\mathrm{comp}}$$

denote left adjoints to the inclusion maps. Suppose that $F : \mathcal{C} \to \mathcal{D}$ *is a functor between Grothendieck prestable ∞-categories which preserves small colimits, and let* $\mathcal{E}$ *be an arbitrary Grothendieck prestable ∞-categories. Then:*

- *If* LF *is an equivalence in the ∞-category* $\mathsf{Groth}_\infty^{\mathrm{sep}}$*, then* $L(F \otimes \mathrm{id}_{\mathcal{E}})$ *is also an equivalence in the ∞-category* $\mathsf{Groth}_\infty^{\mathrm{sep}}$*.*
- *If* $L'F$ *is an equivalence in the ∞-category* $\mathsf{Groth}_\infty^{\mathrm{comp}}$*, then* $L'(F \otimes \mathrm{id}_{\mathcal{E}})$ *is also an equivalence in the ∞-category* $\mathsf{Groth}_\infty^{\mathrm{comp}}$*.*

In other words, the localization functors L *and* L' *are compatible with the symmetric monoidal structure on* Groth_∞*, in the sense of Definition HA.2.2.1.6.*

Proof. We will prove the assertion for the localization functor L; the proof for L' is similar. Assume that LF is an equivalence. We wish to prove that $L(F \otimes \mathrm{id}_{\mathcal{E}})$ is also an equivalence. In other words, we wish to prove that for every separated Grothendieck presentable ∞-category $\mathcal{X}$, composition with the functor $F \otimes \mathrm{id}_{\mathcal{E}}$ induces an equivalence of ∞-categories

$$\theta : \mathrm{LFun}(\mathcal{D} \otimes \mathcal{E}, \mathcal{X}) \to \mathrm{LFun}(\mathcal{C} \otimes \mathcal{E}, \mathcal{X}).$$

This is clear, because θ can be identified with the natural map

$$\mathrm{LFun}(\mathcal{E}, \mathrm{LFun}(\mathcal{D}, \mathcal{X})) \to \mathrm{LFun}(\mathcal{E}, \mathrm{LFun}(\mathcal{C}, \mathcal{X}))$$

obtained by composition with the equivalence $\mathrm{LFun}(\mathcal{D}, \mathcal{X}) \xrightarrow{\circ F} \mathrm{LFun}(\mathcal{C}, \mathcal{X})$. □

Corollary C.4.6.2. *The ∞-categories* $\mathsf{Groth}_\infty^{\mathrm{sep}}$ *and* $\mathsf{Groth}_\infty^{\mathrm{comp}}$ *admit essentially unique symmetric monoidal structures for which the localization functors*

$$L : \mathsf{Groth}_\infty \to \mathsf{Groth}_\infty^{\mathrm{sep}} \qquad L' : \mathsf{Groth}_\infty \to \mathsf{Groth}_\infty^{\mathrm{comp}}$$

are symmetric monoidal.

Warning C.4.6.3. If we equip the ∞-categories $\mathsf{Groth}_\infty^{\mathrm{sep}}$ and $\mathsf{Groth}_\infty^{\mathrm{comp}}$ with the symmetric monoidal structures described in Corollary C.4.6.2, then the inclusion functors

$$\mathsf{Groth}_\infty^{\mathrm{comp}} \hookrightarrow \mathsf{Groth}_\infty^{\mathrm{sep}} \hookrightarrow \mathsf{Groth}_\infty$$

are lax symmetric monoidal, but not symmetric monoidal. We will denote the tensor product on $\mathsf{Groth}_\infty^{\mathrm{comp}}$ by

$$\widehat{\otimes} : \mathsf{Groth}_\infty^{\mathrm{comp}} \times \mathsf{Groth}_\infty^{\mathrm{comp}} \to \mathsf{Groth}_\infty^{\mathrm{comp}}$$

and refer to it as the *completed tensor product* on $\mathsf{Groth}_\infty^{\mathrm{comp}}$. Unwinding the definitions, we see that the completed tensor product is given on objects by the formula $\mathcal{C} \widehat{\otimes} \mathcal{D} = \widehat{\mathcal{C} \otimes \mathcal{D}}$, where $\mathcal{C} \mapsto \widehat{\mathcal{C}}$ denotes the completion functor of Proposition C.3.6.3 (that is, the localization functor L' of Corollary C.4.6.2).

C.5 Grothendieck Abelian Categories

In §C.1, we introduced the notion of a *Grothendieck prestable ∞-category* (Definition C.1.4.2). In this section, we will investigate the relationship between the theory of Grothendieck prestable ∞-categories and the more classical theory of Grothendieck abelian categories. Recall that if $\mathcal{C}$ is a Grothendieck prestable ∞-category, then the full subcategory $\mathcal{C}^{\heartsuit} \subseteq \mathcal{C}$ spanned by the discrete objects is a Grothendieck abelian category (Remark C.1.4.6). We now consider the converse:

Question C.5.0.4. Given a Grothendieck abelian category $\mathcal{A}$, is there a canonical way to construct a Grothendieck prestable ∞-category $\mathcal{C}$ and an equivalence $\mathcal{A} \simeq \mathcal{C}^{\heartsuit}$?

We have already given an affirmative answer to Question C.5.0.4: if $\mathcal{A}$ is a Grothendieck prestable ∞-category, then the derived ∞-category $\mathcal{D}(\mathcal{A})$ can be endowed with the t-structure $(\mathcal{D}(\mathcal{A})_{\geqslant 0}, \mathcal{D}(\mathcal{A})_{\leqslant 0})$ of Proposition HA.1.3.5.21 , and the ∞-category $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ is a Grothendieck prestable ∞-category whose heart is equivalent to $\mathcal{A}$ (Example C.1.4.5). However, there are (at least) two other ways to answer Question C.5.0.4:

- If $\mathcal{A}$ is a Grothendieck abelian category, then t-structure on $\mathcal{D}(\mathcal{A})$ need not be left complete. Consequently, we can consider the left completion of $\mathcal{D}(\mathcal{A})$, which we will denote by $\widehat{\mathcal{D}}(\mathcal{A})$ and refer to as the *completed derived ∞-category of* $\mathcal{A}$. This completion inherits a t-structure $(\widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0}, \widehat{\mathcal{D}}(\mathcal{A})_{\leqslant 0})$, and $\widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ is another Grothendieck prestable ∞-category whose heart can be identified with $\mathcal{A}$.

- Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathrm{Ch}(\mathcal{A})$ be the (differential graded) category of chain complexes with values in $\mathcal{A}$. Then $\mathrm{Ch}(\mathcal{A})$ admits a model structure, where the cofibrations are monomorphisms of chain complexes and the weak equivalences are quasi-isomorphisms of chain complexes (Proposition HA.1.3.5.3). The ∞-category $\mathcal{D}(\mathcal{A})$ is defined as the differential graded nerve of the full subcategory $\mathrm{Ch}(\mathcal{A})^{\circ} \subseteq \mathrm{Ch}(\mathcal{A})$ spanned by the fibrant objects (Definition HA.1.3.5.8). If $A_\bullet$ is a fibrant object of $\mathrm{Ch}(\mathcal{A})$, then each A_n is an injective object of $\mathcal{A}$: that is, we have $\mathrm{Ch}(\mathcal{A})^{\circ}) \subseteq \mathrm{Ch}(\mathcal{A}^{\mathrm{inj}})$, where $\mathcal{A}^{\mathrm{inj}} \subseteq \mathcal{A}$ denotes the full subcategory spanned by the injective objects of $\mathcal{A}$. In general, this inclusion is strict: an unbounded chain complex of injective objects of $\mathcal{A}$ need not be a fibrant object of $\mathrm{Ch}(\mathcal{A})$. We let $\check{\mathcal{D}}(\mathcal{A})$ denote the differential graded nerve of the category $\mathrm{Ch}(\mathcal{A}^{\mathrm{inj}})$ of *all* chain complexes of injective objects of $\mathcal{A}$. Then $\check{\mathcal{D}}(\mathcal{A})$ admits a t-structure $(\check{\mathcal{D}}(\mathcal{A})_{\geqslant 0}, \check{\mathcal{D}}(\mathcal{A})_{\leqslant 0})$, where $\check{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ is a Grothendieck prestable ∞-category whose heart is $\mathcal{A}$. We will refer to $\check{\mathcal{D}}(\mathcal{A})$ as the *unseparated derived ∞-category of $\mathcal{A}$*.

If $\mathcal{A}$ is a Grothendieck abelian, then the ∞-categories described above are related by functors

$$\check{\mathcal{D}}(\mathcal{A})_{\geqslant 0} \to \mathcal{D}(\mathcal{A})_{\geqslant 0} \to \widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$$

which preserve small colimits and finite limits. Each can be characterized by a universal property:

- The unseparated derived ∞-category $\check{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ is universal among Grothendieck prestable ∞-categories $\mathcal{E}$ equipped with a functor $\mathcal{A} \to \mathcal{E}^{\heartsuit}$ which preserves small colimits and finite limits (Corollary C.5.8.9).
- The derived ∞-category $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ is universal among *separated* Grothendieck prestable ∞-categories $\mathcal{E}$ equipped with a functor $\mathcal{A} \to \mathcal{E}^{\heartsuit}$ which preserves small colimits and finite limits (Theorem C.5.4.9).
- The completed derived ∞-category $\widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ is universal among *complete* Grothendieck prestable ∞-categories $\mathcal{E}$ equipped with a functor $\mathcal{A} \to \mathcal{E}^{\heartsuit}$ which preserves small colimits and finite limits (Corollary C.5.9.5).

The relationship between Grothendieck prestable ∞-categories and Grothendieck abelian categories is analogous to the relationship between ∞-topoi and ordinary (Grothendieck) topoi. Another of our goals in this section is to pursue this analogy by introducing "linear" versions of some useful concepts from the study of higher topoi, as summarized in the following table:

Nonlinear Concept	Linear Concept
∞-topos	Grothendieck prestable ∞-category
Hypercomplete ∞-topos	Separated Grothendieck prestable ∞-category
Postnikov-complete ∞-topos	Complete Grothendieck prestable ∞-category
Bounded ∞-topos	Anticomplete Grothendieck prestable ∞-category
$(n+1)$-localic ∞-topos	Anticomplete, n-complicial Grothendieck prestable ∞-category
$(n+1)$-topos	Grothendieck abelian n-category

cc

C.5.1 Localizing Subcategories of Abelian Categories

In §C.2, we introduced the notion of a *localizing subcategory* of a Grothendieck prestable ∞-category $\mathcal{C}$ (Definition C.2.3.3). We now review the analogous localization theory of Grothendieck abelian categories. We begin by establishing a 1-categorical analogue of Proposition C.2.3.1.

Proposition C.5.1.1. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{B} \subseteq \mathcal{A}$ be a full subcategory. Suppose that the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ admits a left adjoint $L : \mathcal{A} \to \mathcal{B}$ which is left exact. Then $\mathcal{B}$ is an abelian category and L is exact (when viewed as a functor from $\mathcal{A}$ to $\mathcal{B}$).*

Proof. Since $\mathcal{B}$ is a localization of $\mathcal{A}$, it is closed under all limits which exist in $\mathcal{A}$. In particular, it contains the zero object of $\mathcal{A}$ and is closed under finite direct sums, and is therefore an additive category. Moreover, every morphism $f : X \to Y$ in $\mathcal{B}$ has a kernel (which can be computed in the abelian category $\mathcal{A}$) and a cokernel (which can be computed by first taking the cokernel in the abelian category $\mathcal{A}$ and then applying the localization functor L). To complete the proof that $\mathcal{B}$ is abelian, it will suffice to show that for every f as above, the canonical map $\alpha : L\,\mathrm{coker}(\ker(f) \to X) \to \ker(Y \to L\,\mathrm{coker}(f))$ is an isomorphism (where the kernels and cokernals are formed in the abelian category $\mathcal{A}$). Using the left exactness of L, we see that α is obtained by applying the functor L to the natural map $\alpha_0 : \mathrm{coker}(\ker(f) \to X) \to \ker(Y \to \mathrm{coker}(f))$, which is an isomorphism by virtue of our assumption that $\mathcal{A}$ is an abelian category. The functor $L : \mathcal{A} \to \mathcal{B}$ is left exact by assumption and right exact by virtue of the fact that it is left adjoint to the inclusion, and is therefore exact. □

In the situation of Proposition C.5.1.1, the category $\mathcal{B}$ can be obtained (up to equivalence) from $\mathcal{A}$ by formally inverting the L-equivalences: that is, the morphisms $f : X \to Y$ in the category $\mathcal{A}$ such that Lf is an isomorphism. Since $L : \mathcal{A} \to \mathcal{B}$ is an exact functor, the morphism Lf is an isomorphism if and only if $\mathrm{coker}(Lf) = L\,\mathrm{coker}(f)$ and $\ker(Lf) = L\ker(f)$ both vanish: that is, if and only if the functor L annihilates $\ker(f)$ and $\mathrm{coker}(f)$. Consequently, the category $\mathcal{B}$ can be recovered from $\mathcal{A}$ together with the full subcategory $\mathcal{A}_0 = \{X \in \mathcal{A} : LX \simeq 0\}$. One can then ask: which full subcategories $\mathcal{A}_0 \subseteq \mathcal{A}$ can arise in this way? When $\mathcal{A}$ is a Grothendieck abelian category, this is answered by the following:

Definition C.5.1.2. Let $\mathcal{A}$ be a Grothendieck abelian category. We will say that a full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ is *localizing* if it satisfies the following conditions:

(*a*) For every exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$, the object X belongs to $\mathcal{A}_0$ if and only if both X' and X'' belong to $\mathcal{A}_0$. In other words, the full subcategory $\mathcal{A}_0$ is closed under the formation of subobjects, quotient objects, and extensions.

(*b*) The full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ is closed under the formation of (small) coproducts.

Remark C.5.1.3. If $\mathcal{A}$ is an arbitrary abelian category, then a full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ satisfying condition (a) of Definition C.5.1.2 is often called a *Serre subcategory* of $\mathcal{A}$.

Definition C.5.1.4. Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a localizing subcategory. We will say that a morphism $f : X \to Y$ in $\mathcal{A}$ is a $\mathcal{A}_0$*-equivalence* if the kernel and cokernel of f belong to $\mathcal{A}_0$. We will say that an object $Z \in \mathcal{A}$ is $\mathcal{A}_0$*-local* if, for every $\mathcal{A}_0$-equivalence $f : X \to Y$, the induced map $\mathrm{Hom}_{\mathcal{A}}(Y, Z) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$ is bijective. We let $\mathcal{A}/\mathcal{A}_0$ denote the full subcategory of $\mathcal{A}$ spanned by the $\mathcal{A}_0$-local objects.

Remark C.5.1.5. Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a localizing subcategory. Every morphism $f : X \to Y$ in $\mathcal{A}$ admits an essentially unique factorization $X \xrightarrow{f'} W \xrightarrow{f''} Y$, where f' is an epimorphism and f'' is a monomorphism. Note that f is a $\mathcal{A}_0$-equivalence if and only if f' and f'' are $\mathcal{A}_0$-equivalences. Consequently, an object $Z \in \mathcal{A}$ is $\mathcal{A}_0$-local if only if the induced map $\mathrm{Hom}_{\mathcal{A}}(Y, Z) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$ is bijective for any $\mathcal{A}_0$-equivalence $X \to Y$ which is either an epimorphism or a monomorphism. This is equivalent to the following pair of conditions:

- (a) For every exact sequence $0 \to A \to X \xrightarrow{f} Y \to 0$ in $\mathcal{A}$ where $A \in \mathcal{A}_0$, the induced map $\mathrm{Hom}_{\mathcal{A}}(Y, Z) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$ is surjective (note that it is automatically injective, since f is an epimorphism).
- (b) For every exact sequence $0 \to X \xrightarrow{f} Y \to A \to 0$ in $\mathcal{A}$ where $A \in \mathcal{A}_0$, the induced map $\mathrm{Hom}_{\mathcal{A}}(Y, Z) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$ is bijective.

Note that condition (a) is equivalent to the requirement that $\mathrm{Hom}_{\mathcal{A}}(A, Z) \simeq 0$ for each $A \in \mathcal{A}_0$. Moreover, if this condition is satisfied, then the map appearing in (b) is automatically injective. Unwinding the definitions, we see that the surjectivity is equivalent to the requirement that every exact sequence $0 \to Z \to \widetilde{Z} \to A \to 0$ in $\mathcal{A}$ splits provided that $A \in \mathcal{A}_0$: that is, to the condition that $\mathrm{Ext}^1_{\mathcal{A}}(A, Z) \simeq 0$ for $A \in \mathcal{A}_0$. Consequently, an object $Z \in \mathcal{A}$ is $\mathcal{A}_0$-local if and only if we have $\mathrm{Hom}_{\mathcal{A}}(A, Z) \simeq \mathrm{Ext}^1_{\mathcal{A}}(A, Z) \simeq 0$ for all $A \in \mathcal{A}_0$: that is, if and only if every extension of A by Z admits a *unique* splitting.

The relevance of Definition C.5.1.2 is explained by the next result:

Proposition C.5.1.6. *Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a full subcategory. The following conditions are equivalent:*

- (1) *The full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ is localizing.*
- (2) *There exists a full subcategory $\mathcal{B} \subseteq \mathcal{A}$ for which the inclusion functor admits a left exact left adjoint $L : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}_0 = \{A \in \mathcal{A} : LA \simeq 0\}$.*

Moreover, if these conditions are satisfied, then $\mathcal{B}$ is a Grothendieck abelian category.

Remark C.5.1.7. In the situation of Proposition C.5.1.6, let $f : A \to A'$ be a morphism in $\mathcal{A}$. Then Lf is an isomorphism in $\mathcal{B}$ if and only if the objects $\ker(Lf) \simeq L\ker(f)$ and $\mathrm{coker}(Lf) \simeq L\,\mathrm{coker}(f)$ vanish: that is, if and only if f is a $\mathcal{A}_0$-equivalence, in the sense of Definition **??**. It follows that the essential image of the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ can be identified with the full subcategory $\mathcal{A}/\mathcal{A}_0 \subseteq \mathcal{A}$ spanned by the $\mathcal{A}_0$-local objects.

Proof of Proposition C.5.1.6. Suppose first that (2) is satisfied. Then $L : \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories (Proposition C.5.1.1), so the full subcategory $\mathcal{A}_0 = \{A \in \mathcal{A} : LA \simeq 0\}$ is closed under the formation of subobjects, quotient objects, and extensions. Since L preserves small colimits, the full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ is closed under small colimits and therefore under small coproducts. This proves that (2) ⇒ (1).

Now suppose that (1) is satisfied. Then the full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ is closed under quotients and direct sums, and is therefore closed under all small colimits. Let $E \in \mathcal{A}$ be a generator for the Grothendieck abelian category $\mathcal{A}$, so that every object $X \in \mathcal{A}$ can be written as a filtered colimit of subobjects X_α, each of which can be described as a quotient of E^n for $n \geqslant 0$. Note that if $X \in \mathcal{A}_0$, then each X_α belongs to $\mathcal{A}_0$. It follows that $\mathcal{A}_0$ is generated under filtered colimits by the (essentially small) subcategory of $\mathcal{A}_0$ spanned by objects which appear as quotients of some E^n. Every such quotient is small as an object of $\mathcal{A}$ and therefore also as an object of $\mathcal{A}_0$, so that the category $\mathcal{A}_0$ is accessible. Let S denote the collection of all $\mathcal{A}_0$-equivalences in $\mathcal{A}$; it follows from Proposition HTT.5.4.6.6 that S spans an accessible subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{A})$.

Let $\mathcal{D}(\mathcal{A})$ denote the derived ∞-category of $\mathcal{A}$ (see §**??**), let $\mathcal{D}_{\mathcal{A}_0}(\mathcal{A})$ denote the full subcategory of $\mathcal{D}(\mathcal{A})$ spanned by those chain complexes in $\mathcal{A}$ whose homologies lie in the subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$. Using the assumption that $\mathcal{A}_0$ is a Serre subcategory of $\mathcal{A}$, we immediately deduce that $\mathcal{D}_{\mathcal{A}_0}(\mathcal{A})$ is a stable subcategory of $\mathcal{D}(\mathcal{A})$. Since $\mathcal{A}_0$ is closed under direct sums, the full subcategory $\mathcal{D}_{\mathcal{A}_0}(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})$ is closed under coproducts and therefore under all small colimits. Set $\mathcal{D}_{\mathcal{A}_0}(\mathcal{A})_{\geqslant 0} = \mathcal{D}(\mathcal{A})_{\geqslant 0} \cap \mathcal{D}_{\mathcal{A}_0}(\mathcal{A})$. We claim that $\mathcal{D}_{\mathcal{A}_0}(\mathcal{A})_{\geqslant 0}$ is a localizing subcategory of the Grothendieck prestable ∞-category $\mathcal{D}(\mathcal{A})_{\geqslant 0}$: that is, that it satisfies conditions (i), (ii), and (iii) of Definition C.2.3.3. Since $\mathcal{D}_{\mathcal{A}_0}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ are both closed under small colimits in $\mathcal{D}(\mathcal{A})$, it is clear that $\mathcal{D}_{\mathcal{A}_0}(\mathcal{A})_{\geqslant 0}$ is closed under colimits in $\mathcal{D}(\mathcal{A})$. Using Proposition HTT.5.4.6.6 , we see that the accessibility of the subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ implies the accessibility of the subcategory $\mathcal{D}_{\mathcal{A}_0}(\mathcal{A})_{\geqslant 0} \subseteq \mathcal{D}(\mathcal{A})$.

Let $\mathcal{C}$ denote the quotient $\mathcal{D}(\mathcal{A})_{\geqslant 0}/\mathcal{D}_{\mathcal{A}_0}(\mathcal{A})_{\geqslant 0}$ (see Notation C.2.3.9). Then $\mathcal{C}$ is a Grothendieck prestable ∞-category, so the full subcategory $\mathcal{C}^\heartsuit \subseteq \mathcal{C}$ of discrete objects is a Grothendieck abelian category (Remark C.1.4.6). Since the localization functor $\mathcal{D}(\mathcal{A})_{\geqslant 0} \to \mathcal{D}(\mathcal{A})_{\geqslant 0}/\mathcal{D}_{\mathcal{A}_0}(\mathcal{A})_{\geqslant 0}$ and its right adjoint are left exact, they determine adjoint functors $\mathcal{D}(\mathcal{A})^\heartsuit \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}^\heartsuit$ which exhibit $\mathcal{C}^\heartsuit$ as an exact localization of the Grothendieck abelian category $\mathcal{D}(\mathcal{A})^\heartsuit \simeq \mathcal{A}$. By construction, an object $A \in \mathcal{A}$ is annihilated by F if and only if it belongs to the full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$. We conclude by taking $\mathcal{B} \subseteq \mathcal{A}$ to be the essential image of the functor G. □

Remark C.5.1.8. Let $\mathcal{A}$ be a Grothendieck abelian category, let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a localizing subcategory, and let $\mathcal{B}$ be an arbitrary category which admits small colimits. Let $\mathrm{LFun}(\mathcal{A}, \mathcal{B})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ spanned by those functors which preserve small colimits and define $\mathrm{LFun}(\mathcal{A} / \mathcal{A}_0, \mathcal{B})$ similarly. It follows from Proposition HTT.5.5.4.20 that composition with the localization functor $L : \mathcal{A} \to \mathcal{A} / \mathcal{A}_0$ induces a fully faithful embedding $\mathrm{LFun}(\mathcal{A} / \mathcal{A}_0, \mathcal{B}) \hookrightarrow \mathrm{LFun}(\mathcal{A}, \mathcal{B})$, whose essential image is spanned by those functors which carry $\mathcal{A}_0$-equivalences in $\mathcal{A}$ to isomorphisms in $\mathcal{B}$. In particular, if $F : \mathcal{A} \to \mathcal{B}$ is a colimit-preserving functor which carries $\mathcal{A}_0$-equivalences to isomorphisms, then F admits an essentially unique factorization through the quotient $\mathcal{A} / \mathcal{A}_0$. Note that if $\mathcal{B}$ is an abelian category and F is exact, then the condition that F preserves $\mathcal{A}_0$-equivalences is equivalent to the requirement that F annihilates the objects of $\mathcal{A}_0$. In this case, the induced map $\mathcal{A} / \mathcal{A}_0 \to \mathcal{B}$ is also an exact functor.

C.5.2 Comparison of Localizing Subcategories

Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{C}^{\heartsuit}$ denote the full subcategory of $\mathcal{C}$ spanned by the discrete objects. Then $\mathcal{C}^{\heartsuit}$ is a Grothendieck abelian category. We study the relationship between localizing subcategories of $\mathcal{C}$ (in the sense of Definition C.2.3.3) and localizing subcategories of $\mathcal{C}^{\heartsuit}$ (in the sense of Definition C.5.1.2).

Proposition C.5.2.1. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a localizing subcategory (in the sense of Definition C.2.3.3). Then:*

(*a*) *The ∞-category $\mathcal{C}_0$ is Grothendieck prestable.*

(*b*) *An object of $\mathcal{C}_0$ is discrete if and only if it is discrete when viewed as an object of $\mathcal{C}$: that is, we have $\mathcal{C}_0^{\heartsuit} = \mathcal{C}_0 \cap \mathcal{C}^{\heartsuit}$.*

(*c*) *The full subcategory $\mathcal{C}_0^{\heartsuit} \subseteq \mathcal{C}^{\heartsuit}$ is a localizing subcategory of the Grothendieck abelian category $\mathcal{C}^{\heartsuit}$ (in the sense of Definition C.5.1.2).*

(*d*) *There is a canonical equivelence of Grothendieck abelian categories $(\mathcal{C} / \mathcal{C}_0)^{\heartsuit} \simeq \mathcal{C}^{\heartsuit} / \mathcal{C}_0^{\heartsuit}$.*

Proof. Assertion (*a*) follows immediately from the definitions. To prove (*b*), suppose that X is a discrete object of $\mathcal{C}_0$. Let $\pi_0 X$ denote the 0-truncation of X in the Grothendieck prestable ∞-category $\mathcal{C}$, so that we have a fiber sequence $X' \to X \to \pi_0 X$. Using condition (*iii*) of Proposition C.2.3.3, we deduce that $X' \in \mathcal{C}_0$. Since $\pi_0 X' \simeq 0$, we can write $X' = \Sigma Y$ for some object $Y \in \mathcal{C}$. We then have a fiber sequence $Y \to 0 \to X'$ in $\mathcal{C}$, so that condition (*ii*) of Definition C.2.3.3 gives $Y \in \mathcal{C}_0$. Since X is discrete, we deduce that

$\pi_1 \operatorname{Map}_{\mathcal{C}}(Y, X) \simeq \pi_0 \operatorname{Map}_{\mathcal{C}}(X', X) \simeq \pi_0 \operatorname{Map}_{\mathcal{C}}(X', X') \simeq 0$. In particular, the identity map $\mathrm{id} : X' \to X'$ is nullhomotopic, so that $X' \simeq 0$ and therefore $X \simeq \pi_0 X$ is a discrete object of $\mathcal{C}$.

To prove (c) and (d), we note that the functors appearing in the adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C} / \mathcal{C}_0$ are left exact, and therefore induce an adjunction between Grothendieck abelian categories $\mathcal{C}^{\heartsuit} \underset{G^{\heartsuit}}{\overset{F^{\heartsuit}}{\rightleftarrows}} (\mathcal{C} / \mathcal{C}_0)^{\heartsuit}$. Unwinding the definitions, we see that an object $C \in \mathcal{C}^{\heartsuit}$ is annihilated by $F^{\heartsuit}$ if and only if it belongs to $\mathcal{C}_0 \cap \mathcal{C}^{\heartsuit} = \mathcal{C}_0^{\heartsuit}$. It follows from Proposition C.5.1.6 that $\mathcal{C}_0^{\heartsuit}$ is a localizing subcategory of $\mathcal{C}^{\heartsuit}$, and that the quotient $\mathcal{C}^{\heartsuit} / \mathcal{C}_0^{\heartsuit}$ can be identified with the essential image of the fully faithful embedding $G^{\heartsuit}$. □

Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. It follows from Proposition C.5.2.1 that the construction $\mathcal{C}_0 \mapsto \mathcal{C}_0^{\heartsuit}$ determines a map

$$\{\text{Localizing subcategories of } \mathcal{C}\} \to \{\text{Localizing subcategories of } \mathcal{C}^{\heartsuit}\}.$$

We will see in a moment that this map is surjective: that is, given a localizing subcategory $\mathcal{A}_0 \subseteq \mathcal{C}^{\heartsuit}$, we can always find a localizing subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ satisfying $\mathcal{C}_0^{\heartsuit} = \mathcal{A}_0$. The localizing subcategory $\mathcal{C}_0$ is generally not unique. However, there are (at least) two natural choices for the subcategory $\mathcal{C}_0$: one which is as large as possible (Proposition C.5.2.7), and one which is as small as possible (Proposition C.5.2.8).

Proposition C.5.2.2. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a localizing subcategory. For each object $C \in \mathcal{C}_0$, we have $\pi_n C \in \mathcal{C}_0^{\heartsuit}$ for $n \geqslant 0$.*

Proof. Since the localization functor $L : \mathcal{C} \to \mathcal{C} / \mathcal{C}_0$ is left exact, we have $L(\pi_n C) \simeq \pi_n(LC) \simeq 0$ for $n \geqslant 0$. □

The converse of Proposition C.5.2.2 is not necessary true: in general, we cannot test whether or not an object $C \in \mathcal{C}$ belongs to $\mathcal{C}_0$ by studying the homotopy groups $\pi_n C \in \mathcal{C}^{\heartsuit}$.

Definition C.5.2.3. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a localizing subcategory. We will say that $\mathcal{C}_0$ is *separating* if, for every object $C \in \mathcal{C}$ which satisfies $\pi_n C \in \mathcal{C}_0^{\heartsuit}$ for $n \geqslant 0$, we have $C \in \mathcal{C}_0$.

Remark C.5.2.4. Unwinding the definitions, we see that a localizing subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is separating (in the sense of Definition C.5.2.3) if and only if the Grothendieck prestable ∞-category $\mathcal{C} / \mathcal{C}_0$ is separated (in the sense of Definition C.1.2.12).

Remark C.5.2.5. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Suppose we are given localizing subcategories $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C}$, where $\mathcal{C}_1$ is separating. Then $\mathcal{C}_0 \subseteq \mathcal{C}_1$ if and only if $\mathcal{C}_0^{\heartsuit} \subseteq \mathcal{C}_1^{\heartsuit}$. The "only if" direction is obvious, and the "if" direction follows from Proposition C.5.2.2.

Remark C.5.2.6. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a separated localizing subcategory, and let $\mathcal{A}_0 = \mathcal{C}_0^{\heartsuit}$. It follows from Remark C.5.2.5 that $\mathcal{C}_0$ is the largest localizing subcategory of $\mathcal{C}$ whose heart is $\mathcal{A}_0$.

Proposition C.5.2.7. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Then the construction $\mathcal{C}_0 \mapsto \mathcal{C}_0^{\heartsuit}$ induces a bijection*

$$\theta : \{\textit{Separating localizing subcategories of } \mathcal{C}\} \to \{\textit{Localizing subcategories of } \mathcal{C}^{\heartsuit}\}.$$

Proof. The injectivity of θ follows from Remark C.5.2.6. To prove surjectivity, let $\mathcal{A}_0 \subseteq \mathcal{C}^{\heartsuit}$ be a localizing subcategory of $\mathcal{C}^{\heartsuit}$ and define $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory of $\mathcal{C}$ spanned by those objects C satisfying $\pi_n C \in \mathcal{A}_0$ for $n \geqslant 0$. We must show that $\mathcal{C}_0$ is a localizing subcategory of $\mathcal{C}$: that is, that it satisfies the requirements of Definition C.2.3.3:

- (i) The full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is accessible and closed under small coproducts. The accessibility of $\mathcal{C}_0$ follows from the accessibility of $\mathcal{A}_0$ (established in the proof of Proposition C.5.1.6) using Proposition HTT.5.4.6.6 . Since $\mathcal{A}_0 \subseteq \mathcal{C}^{\heartsuit}$ is closed under the formation of coproducts and each of the functors $\pi_n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ commutes with coproducts, we conclude that $\mathcal{C}_0 \subseteq \mathcal{C}$ is closed under coproducts.

- (ii) Suppose we are given a cofiber sequence $X' \to X \to X''$ in $\mathcal{C}$. We must show that if any two of the objects X, X', and X'' belongs to $\mathcal{C}_0$, then so does the third. For simplicity, let us assume that X' and X'' belong to $\mathcal{C}_0$; the proofs in the other two cases differ only by minor changes in notation. For each $n \geqslant 0$, we have an exact sequence $\pi_n X' \to \pi_n X \to \pi_n X''$ in the abelian category $\mathcal{C}^{\heartsuit}$. It follows that we can write $\pi_n X$ as an extension of a subobject of $\pi_n X''$ by a quotient object of $\pi_n X'$. Since X' and X'' belong to $\mathcal{C}_0$, the objects $\pi_n X'$ and $\pi_n X''$ belong to $\mathcal{A}_0$. Since $\mathcal{A}_0$ is a localizing subcategory of $\mathcal{C}^{\heartsuit}$, it follows that any subobject of $\pi_n X''$ and any quotient object of $\pi_n X'$ belong to $\mathcal{A}_0$, so that $\pi_n X \in \mathcal{A}_0$ by virtue of the fact that $\mathcal{A}_0$ is closed under extensions. Allowing n to vary, we deduce that $X \in \mathcal{C}_0$.

- (iii) Suppose we are given a cofiber sequence $X' \to X \to X''$ in $\mathcal{C}$ where $X \in \mathcal{C}_0$ and $X'' \in \mathcal{C}^{\heartsuit}$; we wish to show that $X' \in \mathcal{C}_0$. The canonical map $\pi_n X' \to \pi_n X$ is a monomorphism for $n \geqslant 0$ (and an isomorphism for $n > 0$). Our assumption that $X \in \mathcal{C}_0$ guarantees that each $\pi_n X$ belongs to $\mathcal{A}_0$, so that $\pi_n X' \in \mathcal{A}_0$ by virtue of the fact that $\mathcal{A}_0$ is closed under the formation of subobjects. Allowing n to vary, we deduce that $X' \in \mathcal{C}_0$ as desired.

□

We now consider localizing subcategories which are, in some sense, as far as possible from being separating.

Proposition C.5.2.8. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category, let $\mathrm{Sp}(\mathcal{C})$ denote the ∞-category of spectrum objects of $\mathcal{C}$, and regard $\mathrm{Sp}(\mathcal{C})$ as equipped with the t-structure of Remark C.1.2.10. Let $\mathcal{A}_0$ be a localizing subcategory of the Grothendieck abelian category $\mathrm{Sp}(\mathcal{C})^{\heartsuit} \simeq \mathcal{C}^{\heartsuit}$, and let $\mathcal{E} \subseteq \mathrm{Sp}(\mathcal{C})$ denote the full subcategory spanned by those objects X with the property that $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(A, X)$ vanishes for each object $A \in \mathcal{A}_0$ and every integer n. Then:*

- (a) *The ∞-category $\mathcal{E}$ is presentable, and the inclusion functor $\mathcal{E} \hookrightarrow \mathrm{Sp}(\mathcal{C})$ admits a left adjoint $L : \mathrm{Sp}(\mathcal{C}) \to \mathcal{E}$.*
- (b) *There is a unique t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ on $\mathcal{E}$ for which $\mathcal{E}_{\leqslant 0} = \mathrm{Sp}(\mathcal{C})_{\leqslant 0} \cap \mathcal{E}$.*
- (c) *The functor L is t-exact.*
- (d) *The t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ is right complete and compatible with filtered colimits.*
- (e) *The full subcategory $\mathcal{E}_{\geqslant 0} \subseteq \mathcal{E}$ is a Grothendieck prestable ∞-category.*
- (f) *Let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by those objects which are annihilated by the functor $\mathcal{C} \simeq \mathrm{Sp}(\mathcal{C})_{\geqslant 0} \xrightarrow{L} \mathcal{E}_{\geqslant 0}$. Then $\mathcal{C}_0$ is a localizing subcategory of $\mathcal{C}$ satisfying $\mathcal{C}_0^{\heartsuit} = \mathcal{A}_0$.*
- (g) *The full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is the smallest localizing subcategory of $\mathcal{C}$ which contains $\mathcal{A}_0$.*

Proof. Choose a small collection of objects $\{A_\alpha\}$ of $\mathcal{A}_0$ which generate $\mathcal{A}_0$ under filtered colimits. Unwinding the definitions, we see that an object $X \in \mathrm{Sp}(\mathcal{C})$ belongs to $\mathcal{E}$ if and only if it is S-local, where S denotes the collection of all morphisms of the form $0 \to \Sigma^n A_\alpha$. Assertion (a) now follows from Proposition HTT.5.5.4.15 .

Choose a small collection of objects $\{C_\beta\}$ of $\mathcal{C}$ which generate $\mathcal{C}$ under small colimits. Let $\mathcal{E}_{\geqslant 0}$ denote the smallest full subcategory of $\mathcal{E}$ which is closed under small colimits and extensions and contains every object of the form $L(\Sigma^\infty C_\beta)$. Applying Proposition HA.1.4.4.11 , we see that $\mathcal{E}_{\geqslant 0}$ can be extended to an accessible t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ on $\mathcal{E}$. Unwinding the definitions, we see that an object $X \in \mathcal{E}$ belongs to $\mathcal{E}_{\leqslant 0}$ if and only if each of the mapping spaces

$$\mathrm{Map}_{\mathcal{E}}(L\Sigma^\infty C_\beta, \Omega X) \simeq \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty C_\beta, \Omega X)$$

is discrete. It follows that $\mathcal{E}_{\leqslant 0} = \mathcal{E} \cap \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$, which proves (b).

Let $\mathcal{D}$ denote the full subcategory of $\mathrm{Sp}(\mathcal{C})$ spanned by those objects X such that $LX \simeq 0$, and let $\overline{\mathcal{D}}$ be the full subcategory of $\mathrm{Sp}(\mathcal{C})$ spanned by those objects X such that $\pi_n X \in \mathcal{A}_0$ for every integer n. We next prove:

- $(*)$ There are inclusions $\overline{\mathcal{D}} \cap \mathrm{Sp}(\mathcal{C})_{\leqslant 0} \subseteq \mathcal{D} \subseteq \overline{\mathcal{D}}$.

Note that $\mathcal{D}$ is generated under small colimits by the collection of objects $\{\Sigma^n A_\alpha\}$. Since each of these objects belongs to $\overline{\mathcal{D}}$ and $\overline{\mathcal{D}}$ is closed under small colimits, it follows that $\mathcal{D} \subseteq \overline{\mathcal{D}}$. To prove the other inclusion, we note that the t-structure on $\mathrm{Sp}(\mathcal{C})$ is right complete, so that every object $X \in \mathrm{Sp}(\mathcal{C})$ can be obtained as a colimit $\varinjlim \tau_{\geqslant -n} X$. If X belongs to $\overline{\mathcal{D}} \cap \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$, then each truncation $\tau_{\geqslant -n} X$ can be written as a finite extension of desuspensions of objects of $\mathcal{A}_0$, and is therefore annihilated by L. Since L preserves small colimits, it follows that $LX \simeq 0$.

Let X be any object of $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ and form a fiber sequence $X' \xrightarrow{u} X \to LX$. Then X' is annihilated by the functor L, so that $(*)$ implies that the homotopy groups $\pi_n X'$ belong to $\mathcal{A}_0$ for every integer n. Our assumption that X belong to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ guarantees that we can factor the map u as a composition $X' \xrightarrow{u'} X'' \xrightarrow{u''} X$, where $X'' \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$, the maps $\pi_n X' \to \pi_n X''$ are isomorphisms for $n < 0$, the map $\pi_0 X' \to \pi_0 X''$ is an epimorphism, and the map $\pi_0 X'' \to \pi_0 X$ is a monomorphism. It follows that the objects $\pi_n X''$ belong to $\mathcal{A}_0$ for all $n \in \mathbf{Z}$. Applying $(*)$ again, we see that $X'' \in \mathcal{D}$. By construction, X' is a final object of $\mathcal{D} \times_{\mathrm{Sp}(\mathcal{C})} \mathrm{Sp}(\mathcal{C})_{/X}$, so that X' is a retract of X'' in the ∞-category $\mathrm{Sp}(\mathcal{C})_{/X}$. It follows $X' \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ and that the map $\pi_0(u) : \pi_0 X' \to \pi_0 X$ is a monomorphism in $\mathrm{Sp}(\mathcal{C})^{\heartsuit}$, from which we conclude that $LX \simeq \mathrm{cofib}(u)$ also belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. This proves that the functor $L : \mathrm{Sp}(\mathcal{C}) \to \mathcal{E}$ is left t-exact. The right t-exactness of L follows immediately from the fact that L is left adjoint to the left t-exact inclusion functor $\mathcal{E} \hookrightarrow \mathrm{Sp}(\mathcal{C})$. This completes the proof of (c).

We now prove (d). Let $\{X_\alpha\}$ be a small filtered diagram in the ∞-category $\mathcal{E}_{\leqslant 0}$, and let $X = \varinjlim X_\alpha$ be its colimit in the ∞-category $\mathrm{Sp}(\mathcal{C})$. Then X belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ (since the t-structure on $\mathrm{Sp}(\mathcal{C})$ is compatible with filtered colimits). It follows from (c) that $LX \in \mathcal{E}_{\leqslant 0}$. Note that LX can be identified with the colimit of the diagram $\{X_\alpha\}$ in the ∞-category $\mathcal{E}$, so that the t-structure on $\mathcal{E}$ is compatible with filtered colimits. The right completeness of $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ now follows from Proposition **??**, since the intersection $\bigcap_{n \geqslant 0} \mathcal{E}_{\leqslant -n} \subseteq \bigcap_{n \geqslant 0} \mathrm{Sp}(\mathcal{C})_{\leqslant -n}$ contains only zero objects of $\mathrm{Sp}(\mathcal{C})$.

Assertion (e) follows from (a), (b), and (d) (see Proposition C.1.4.1). We now prove (f). It follows from (c) that L restricts to a functor $L_{\geqslant 0} : \mathrm{Sp}(\mathcal{C})_{\geqslant 0} \to \mathcal{E}_{\geqslant 0}$. Unwinding the definitions, we see that $L_{\geqslant 0}$ is left adjoint to the composite functor

$$\mathcal{E}_{\geqslant 0} \hookrightarrow \mathcal{E} \hookrightarrow \mathrm{Sp}(\mathcal{C}) \xrightarrow{\tau_{\geqslant 0}} \mathrm{Sp}(\mathcal{C})_{\geqslant 0}.$$

We claim that this composite functor is fully faithful. To prove this, choose an arbitrary object $X \in \mathcal{E}$, and consider the fiber sequence

$$\tau_{\geqslant 0} X \to X \to \tau_{\leqslant -1} X$$

where the truncations are formed with respect to the t-structure on $\mathrm{Sp}(\mathcal{C})$. Applying the functor L (and invoking our assumption that $X \in \mathcal{E}$), we obtain a fiber sequence

$$L(\tau_{\geqslant 0} X) \xrightarrow{u_X} X \to L(\tau_{\leqslant -1} X)$$

in the ∞-category $\mathcal{E}$. It follows from (c) that the map u_X exhibits $L(\tau_{\geqslant 0}X)$ as the connective cover of X with respect to the t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$. In particular, if X belongs to $\mathcal{E}_{\geqslant 0}$, then the map u is an equivalence. Allowing X to vary, we conclude that the functor $\tau_{\geqslant 0} : \mathcal{E}_{\geqslant 0} \to \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ is fully faithful. That is, $L_{\geqslant 0} : \mathrm{Sp}(\mathcal{C})_{\geqslant 0} \to \mathcal{E}_{\geqslant 0}$ exhibits $\mathcal{E}_{\geqslant 0}$ as an accessible left exact localization of $\mathcal{C} \simeq \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$, so that $\mathcal{C}_0$ is a localizing subcategory of $\mathcal{C}$ (Proposition C.2.3.8). By construction, an object $C \in \mathcal{C}$ belongs to $\mathcal{C}_0$ if and only if $\Sigma^{\infty} C$ belongs to $\mathcal{D} \subseteq \mathrm{Sp}(\mathcal{C})$, so the equality $\mathcal{C}_0^{\heartsuit} = \mathcal{A}_0$ is a special case of $(*)$.

We now prove (g). Let $\mathcal{C}_1$ be any localizing subcategory of $\mathcal{C}$ which contains $\mathcal{A}_0$. The inclusion $\mathcal{C}/\mathcal{C}_1 \hookrightarrow \mathcal{C}$ extends to a fully faithful embedding $g : \mathrm{Sp}(\mathcal{C}/\mathcal{C}_1) \hookrightarrow \mathrm{Sp}(\mathcal{C})$. To show that $\mathcal{C}_0 \subseteq \mathcal{C}_1$, it will suffice to show that the essential image of g is contained in $\mathrm{Sp}(\mathcal{C}/\mathcal{C}_0) \simeq \mathcal{E} \subseteq \mathrm{Sp}(\mathcal{C})$. This is equivalent to the requirement that the left adjoint of g annihilates the subcategory $\mathcal{A}_0 \subseteq \mathrm{Sp}(\mathcal{C})^{\heartsuit}$, which follows from our assumption that $\mathcal{C}_1$ contains $\mathcal{A}_0$. □

C.5.3 Complicial Prestable ∞-Categories

We now study a class of Grothendieck prestable ∞-categories $\mathcal{C}$ which are "controlled" by their n-truncated objects, for some $n \geqslant 0$.

Definition C.5.3.1. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $n \geqslant 0$ be an integer. We will say that $\mathcal{C}$ is *n-complicial* if, for every object $X \in \mathcal{C}$, there exists a morphism $f : \overline{X} \to X$ where $\overline{X}$ is n-truncated and the induced map $\pi_0 \overline{X} \to \pi_0 X$ is an epimorphism in the abelian category $\mathcal{C}^{\heartsuit}$.

The terminology of Definition C.5.3.1 is motivated by the following observation:

Proposition C.5.3.2. *Let $\mathcal{A}$ be a Grothendieck abelian category, let $\mathcal{D}(\mathcal{A})$ denote the (unbounded) derived ∞-category of $\mathcal{A}$ (see §HA.**??**), and regard $\mathcal{D}(\mathcal{A})$ as equipped with the t-structure $(\mathcal{D}(\mathcal{A})_{\geqslant 0}, \mathcal{D}(\mathcal{A})_{\leqslant 0})$ of Proposition HA.1.3.5.21 . Then $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ is a 0-complicial Grothendieck prestable ∞-category.*

Proof. We have already seen that $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ is a Grothendieck prestable ∞-category (Example C.1.4.5). We claim that it is 0-complicial. To prove this, choose an object $X_\bullet \in \mathcal{D}(\mathcal{A})_{\geqslant 0}$, which we can identify with a chain complex

$$\cdots X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{d_{-1}} X_{-2} \to \cdots$$

in the abelian category $\mathcal{A}$. Let $\overline{X} = \ker(d_0) \in \mathcal{A}$. Then there is an evident map $\overline{X} \to X$ in $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ (where we abuse notation by identifying $\overline{X}$ with its image under the equivalence $\mathcal{A} \simeq \mathcal{D}(\mathcal{A})^{\heartsuit}$) which induces an epimorphism on homology in degree zero. □

Our first goal is to show that n-complicial Grothendieck prestable ∞-categories exist in abundance.

Proposition C.5.3.3. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a localizing subcategory. If $\mathcal{C}$ is n-complicial, then $\mathcal{C}/\mathcal{C}_0$ is n-complicial.*

Proof. Let us regard $\mathcal{C}/\mathcal{C}_0$ as a full subcategory of $\mathcal{C}$, and let $L : \mathcal{C} \to \mathcal{C}/\mathcal{C}_0$ be a left adjoint to the inclusion. For any object $X \in \mathcal{C}/\mathcal{C}_0$, we can choose a morphism $\alpha : \overline{X} \to X$, where $\overline{X}$ is an n-truncated object of $\mathcal{C}$ and the induced map $\pi_0\overline{X} \to \pi_0 X$ is an epimorphism in the abelian category $\mathcal{C}^{\heartsuit}$. Since X belongs to $\mathcal{C}/\mathcal{C}_0$, the morphism α factors as a composition $\overline{X} \to L\overline{X} \xrightarrow{\alpha'} X$. Because L is left exact, the object $L\overline{X} \in \mathcal{C}/\mathcal{C}_0$ is also n-truncated. The induced map $\pi_0 L\overline{X} \to \pi_0 X$ is the image of $\pi_0(\alpha)$ under the exact functor $L^{\heartsuit} : \mathcal{C}^{\heartsuit} \to (\mathcal{C}/\mathcal{C}_0)^{\heartsuit}$, and is therefore an epimorphism the abelian category $(\mathcal{C}/\mathcal{C}_0)^{\heartsuit}$. □

Proposition C.5.3.4. *Let $\mathcal{A}$ be an essentially small additive ∞-category. Suppose that, for every pair of objects $X, Y \in \mathcal{A}$, the mapping space $\mathrm{Map}_{\mathcal{A}}(X,Y)$ is n-truncated. Then $\mathcal{P}_{\Sigma}(\mathcal{A}) = \mathrm{Fun}^{\pi}(\mathcal{A}^{\mathrm{op}}, \mathcal{S})$ is an n-complicial Grothendieck prestable ∞-category.*

Proof. It follows from Remark C.1.5.10 that $\mathrm{Fun}^{\pi}(\mathcal{A}^{\mathrm{op}}, \mathcal{S})$ is Grothendieck prestable ∞-category. Let $j : \mathcal{A} \to \mathrm{Fun}^{\pi}(\mathcal{A}^{\mathrm{op}}, \mathcal{S})$ be the Yoneda embedding. Since $\mathrm{Fun}^{\pi}(\mathcal{A}^{\mathrm{op}}, \mathcal{S})$ is generated under small colimits by the essential image of j, for every object $X \in \mathrm{Fun}^{\pi}(\mathcal{A}^{\mathrm{op}}, \mathcal{S})$ we can choose a map $\overline{X} \to X$ which is an epimorphism on π_0, where $\overline{X}$ is a small coproduct of objects belonging to the essential image of j. Our assumption that the mapping spaces in $\mathcal{A}$ are n-truncated guarantees that j takes values in $\tau_{\leqslant n} \mathrm{Fun}^{\pi}(\mathcal{A}^{\mathrm{op}}, \mathcal{S})$, so that $\overline{X}$ is n-truncated. □

Example C.5.3.5. Let A be an $\mathbb{E}_1$-ring which is connective and n-truncated. Then the Grothendieck prestable ∞-category $\mathrm{LMod}_A^{\mathrm{cn}}$ is n-complicial.

Any Grothendieck prestable ∞-category $\mathcal{C}$ which both separated and n-complicial can be obtained by combining Propositions C.5.3.4 and C.5.3.3:

Proposition C.5.3.6. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $n \geqslant 0$. The following conditions are equivalent:*

- (a) *The Grothendieck prestable ∞-category $\mathcal{C}$ is separated and n-complicial.*
- (b) *There exists an essentially small additive $(n+1)$-category $\mathcal{A}$, a separating localizing subcategory $\mathcal{E} \subseteq \mathcal{P}_{\Sigma}(\mathcal{A})$, and an equivalence $\mathcal{C} \simeq \mathcal{P}_{\Sigma}(\mathcal{A})/\mathcal{E}$.*

Proof. The implication $(b) \Rightarrow (a)$ follows immediately from Propositions C.5.3.3 and C.5.3.4. For the converse, we note that if $\mathcal{C}$ is n-complicial then we can choose an an essentially small generating subcategory $\mathcal{A} \subseteq \mathcal{C}$ (in the sense of Definition C.2.1.1) which consists of n-truncated objects. Enlarging $\mathcal{A}$ if necessary, we may assume that $\mathcal{A}$ is closed under finite direct sums. If $\mathcal{C}$ is separated, then Theorem C.2.1.6 implies that the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$

extends to a left exact localization functor $\mathcal{P}_\Sigma(\mathcal{A}) \to \mathcal{C}$, so that $\mathcal{C}$ can be identified with $\mathcal{P}_\Sigma(\mathcal{A})/\mathcal{E}$ for some localizing subcategory $\mathcal{E} \subseteq \mathcal{P}_\Sigma(\mathcal{A})$ (Proposition C.2.3.8); since $\mathcal{C}$ is separated, the localizing subcategory $\mathcal{E}$ is separating. □

Notation C.5.3.7. For Grothendieck prestable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we let $\mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which are left exact and preserve small colimits. Note that if $F \in \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$, then F induces a functor $F_{\leqslant n} : \tau_{\leqslant n}\,\mathcal{C} \to \tau_{\leqslant n}\,\mathcal{D}$, for each $n \geqslant 0$. We will say that F is an *n-equivalence* if the functor $F_{\leqslant n}$ is an equivalence of ∞-categories.

Proposition C.5.3.8. *Let $\mathcal{C}$ be a separated Grothendieck prestable ∞-category and let $n \geqslant 0$. Then there exists a separated, n-complicial Grothendieck prestable ∞-category $\mathcal{C}'$ and an n-equivalence $\lambda \in \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}', \mathcal{C})$.*

Proof. Choose an essentially small full subcategory $\mathcal{A} \subseteq \tau_{\leqslant n}\,\mathcal{C}$ with the property that, for every object $C \in \tau_{\leqslant n}\,\mathcal{C}$, there exists a map $\bigoplus_{i \in I} A_i \to C$ which induces an epimorphism $\bigoplus_{i \in I} \pi_0 A_i \to \pi_0 C$ in $\mathcal{C}^{\heartsuit}$, where each A_i belongs to $\mathcal{A}$ (for example, we choose a regular cardinal κ such that $\tau_{\leqslant n}\,\mathcal{C}$ is κ-accessible, and take $\mathcal{A}$ to be the full subcategory of $\tau_{\leqslant n}\,\mathcal{C}$ spanned by the κ-compact objects). Enlarging $\mathcal{A}$ if necessary, we may assume that $\mathcal{A}$ is closed under finite direct sums. Set $\overline{\mathcal{C}} = \mathcal{P}_\Sigma(\mathcal{A}) = \mathrm{Fun}^{\pi}(\mathcal{A}^{\mathrm{op}}, \mathcal{S})$. Using Proposition C.2.5.2, we see that the inclusion functor $\mathcal{A} \hookrightarrow \tau_{\leqslant n}\,\mathcal{C} \hookrightarrow \mathcal{C}$ extends to a left exact functor $F : \overline{\mathcal{C}} \to \mathcal{C}$ (Proposition C.2.5.2). Let $\overline{\mathcal{C}}_0$ be the full subcategory of $\overline{\mathcal{C}}$ spanned by those objects which are annihilated by F. Then $\overline{\mathcal{C}}_0$ is a localizing subcategory of $\overline{\mathcal{C}}$ (see Example C.2.3.7). Set $\mathcal{C}' = \overline{\mathcal{C}}/\overline{\mathcal{C}}_0$, so that the functor F factors (up to homotopy) as a composition

$$\overline{\mathcal{C}} \to \overline{\mathcal{C}}/\overline{\mathcal{C}}_0 = \mathcal{C}' \xrightarrow{\lambda} \mathcal{C},$$

where λ is a functor preserving small colimits and finite limits (Remark C.2.3.11). We claim that λ has the desired properties. To see this, we first note that $\overline{\mathcal{C}}$ is n-complicial (Proposition C.5.3.4), so the quotient $\mathcal{C}' \simeq \overline{\mathcal{C}}/\overline{\mathcal{C}}_0$ is also n-complicial (Proposition C.5.3.3). The functor λ is conservative by construction. Since λ is left exact, the separatedness of $\mathcal{C}$ implies the separatedness of $\mathcal{C}'$.

We now complete the proof by showing that λ restricts to an equivalence of ∞-categories $\lambda_n : \tau_{\leqslant n}\,\mathcal{C}' \to \tau_{\leqslant n}\,\mathcal{C}$. Let $G : \mathcal{C} \to \overline{\mathcal{C}}$ be a right adjoint to F. By construction, the functor G factors through the full subcategory $\overline{\mathcal{C}}/\overline{\mathcal{C}}_0$; moreover, when regarded as a functor from $\mathcal{C}$ to $\overline{\mathcal{C}}/\overline{\mathcal{C}}_0 = \mathcal{C}'$, the functor G is right adjoint to λ. It follows that λ and G determine adjoint functors $\tau_{\leqslant n}\,\mathcal{C}' \underset{G_n}{\overset{\lambda_n}{\rightleftarrows}} \tau_{\leqslant n}\,\mathcal{C}$. Proposition C.2.5.3 implies that the functor G_n is fully faithful. Since the functor λ is conservative, it follows that λ_n and G_n are mutually inverse equivalences of ∞-categories. □

Our next goal is to show that for any separated Grothendieck prestable ∞-category $\mathcal{C}$, the functor $\lambda : \mathcal{C}' \to \mathcal{C}$ of Proposition C.5.3.8 is unique (up to canonical equivalence). We will deduce this from the following universal property of n-complicial Grothendieck prestable ∞-categories:

Proposition C.5.3.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories and let $n \geqslant 0$ be an integer. Suppose that $\mathcal{C}$ is n-complicial and that $\mathcal{D}$ is separated. Then the restriction functor $\phi : \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\tau_{\leqslant n}\,\mathcal{C}, \tau_{\leqslant n}\,\mathcal{D})$ is a fully faithful embedding, whose essential image is spanned by those functors $f : \tau_{\leqslant n}\,\mathcal{C} \to \tau_{\leqslant n}\,\mathcal{D}$ which are left exact and commute with small colimits.*

Proof. Consider first the special case where $\mathcal{C} = \mathcal{P}_{\Sigma}(\mathcal{A})$, where $\mathcal{A}$ is an essentially small additive $(n+1)$-category. Using Proposition HTT.5.5.8.15 , we see that composition with the Yoneda embedding $j : \mathcal{A} \to \mathcal{C}$ induces equivalences of ∞-categories

$$\mathrm{LFun}(\mathcal{C}, \mathcal{D}) \xrightarrow{\mu} \mathrm{Fun}^{\pi}(\mathcal{A}, \mathcal{D}) \qquad \mathrm{LFun}(\tau_{\leqslant n}\,\mathcal{C}, \tau_{\leqslant n}\,\mathcal{D}) \simeq \mathrm{LFun}(\mathcal{C}, \tau_{\leqslant n}\,\mathcal{D}) \to \mathrm{Fun}^{\pi}(\mathcal{A}, \tau_{\leqslant n}\,\mathcal{D}).$$

Note that the functor j takes values in the full subcategory $\tau_{\leqslant n}\,\mathcal{C}$. Consequently, the functor μ carries $\mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{LFun}(\mathcal{C}, \mathcal{D})$ to the full subcategory $\mathrm{Fun}^{\pi}(\mathcal{A}, \tau_{\leqslant n}\,\mathcal{D})$. These identifications fit into a commutative diagram

$$\begin{array}{ccc} \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}) & \xrightarrow{\phi} & \mathrm{LFun}(\tau_{\leqslant n}\,\mathcal{C}, \tau_{\leqslant n}\,\mathcal{D}) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \sim} \\ \mathrm{Fun}^{\pi}(\mathcal{A}, \tau_{\leqslant n}\,\mathcal{D}) & \xrightarrow{\mathrm{id}} & \mathrm{Fun}^{\pi}(\mathcal{A}, \tau_{\leqslant n}\,\mathcal{D}) \end{array}$$

To complete the proof, it will suffice to show that if $F : \mathcal{C} \to \mathcal{D}$ is a colimit-preserving functor such that $F(\mathcal{A}) \subseteq \tau_{\leqslant n}\,\mathcal{D}$ and the composite functor

$$\tau_{\leqslant n}\,\mathcal{C} \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\tau_{\leqslant n}} \tau_{\leqslant n}\,\mathcal{D}$$

is left exact, then F is left exact. For this, it will suffice to show that for each discrete object $C \in \mathcal{C}^{\heartsuit}$, the image $F(C) \in \mathcal{D}$ is also discrete (Proposition C.3.2.1). Since $(\tau_{\leqslant n} F) : \tau_{\leqslant n}\,\mathcal{C} \to \tau_{\leqslant n}\,\mathcal{D}$ is left exact, it is automatic that $\tau_{\leqslant n} F(C)$ is a discrete object of $\mathcal{D}$. We will complete the proof by showing that $F(C)$ is n-truncated. This is a special case of the following more general assertion:

$(*)$ For each n-truncated object $Z \in \mathcal{C}$, the object $F(Z) \in \mathcal{D}$ is also n-truncated.

Since $\mathcal{D}$ is separated, $(*)$ is equivalent to the assertion that for each n-truncated object $C \in \mathcal{C}$, the homotopy groups $\pi_m F(Z) \in \mathcal{D}^{\heartsuit}$ vanish for $m > n$. We prove this by induction on m. Let Z be an n-truncated object of $\mathcal{C}$, and choose a map $Y \to Z$, where Y is a coproduct of objects belonging to $\mathcal{A}$ and the induced map $\pi_0 Y \twoheadrightarrow \pi_0 Z$ is an epimorphism in the abelian category $\mathcal{C}^{\heartsuit}$. We then have a cofiber sequence $X \to Y \to Z$, where X and Y are also n-truncated. The cofiber sequence $FX \to FY \to FZ$ yields a long exact sequence of homotopy groups

$$\pi_m FY \to \pi_m FZ \to \pi_{m-1} FX \xrightarrow{\beta} \pi_{m-1} FY.$$

The functor F commutes with coproducts and carries $\mathcal{A}$ into $\tau_{\leqslant n}\mathcal{D}$, so that FY is n-truncated. It follows that $\pi_m FY$ vanishes for $m > n$. Consequently, to show that $\pi_m FZ$ vanishes, it will suffice to show that the map β is a monomorphism in the abelian category $\mathcal{D}^{\heartsuit}$. For $m > n+1$, our inductive hypothesis implies that $\pi_{m-1} FX$ vanishes, and there is nothing to prove. When $m = n+1$, we note that the n-truncatedness of Z implies that the morphism $X \to Y$ is $(n-1)$-truncated. Since the functor $\tau_{\leqslant n} F$ is left exact, the map $\tau_{\leqslant n} FX \to \tau_{\leqslant n} FY$ is also $(n-1)$-truncated, which guarantees that β is injective as desired. This completes the proof of Proposition C.5.3.9 in the special case $\mathcal{C} = \mathcal{P}_{\Sigma}(\mathcal{C})$.

We now treat the general case. Let $\mathcal{C}$ be any n-complicial Grothendieck prestable ∞-category, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory consisting of objects which are n-connective for all n. Since $\mathcal{D}$ is separated, any colimit-preserving functor $\mathcal{C} \to \mathcal{D}$ must annihilate $\mathcal{C}_0$. We may therefore replace $\mathcal{C}$ by $\mathcal{C} / \mathcal{C}_0$ and thereby reduce to the case where $\mathcal{C}$ is also separated. Applying Proposition C.5.3.6, we obtain an equivalence $\mathcal{C} \simeq \mathcal{P}_{\Sigma}(\mathcal{A}) / \mathcal{E}$ for some essentially small additive $(n+1)$-category $\mathcal{A}$ and some separating localizing subcategory $\mathcal{E} \subseteq \mathcal{P}_{\Sigma}(\mathcal{A})$. This equivalence determines a commutative diagram

$$\begin{array}{ccc} \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}) & \xrightarrow{\phi} & \mathrm{LFun}(\tau_{\leqslant n}\mathcal{C}, \tau_{\leqslant n}\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{LFun}^{\mathrm{lex}}(\mathcal{P}_{\Sigma}(\mathcal{A}), \mathcal{D}) & \longrightarrow & \mathrm{LFun}(\tau_{\leqslant n}\mathcal{P}_{\Sigma}(\mathcal{A}), \tau_{\leqslant n}\mathcal{D}) \end{array}$$

where the vertical maps are fully faithful. It follows from the first part of the proof that the functor ϕ must also be fully faithful. To complete the proof, it will suffice (by virtue of Proposition C.2.3.10 and Remark C.2.3.11) to show that if $F : \mathcal{P}_{\Sigma}(\mathcal{A}) \to \mathcal{D}$ is functor which is left exact and preserves small colimits for which the composite functor $\mathcal{P}_{\Sigma}(\mathcal{A}) \xrightarrow{F} \mathcal{D} \xrightarrow{\tau_{\leqslant m}} \mathcal{D}$ factors through $\mathcal{C}$, then the functor F annihilates $\mathcal{E}$. This is clear: since $\mathcal{D}$ is separated and F is left exact, the functor F annihilates $\mathcal{E}$ if and only if it annihilates the abelian category $\mathcal{E}^{\heartsuit}$. $\square$

C.5.4 Grothendieck Abelian n-Categories

Let $\mathcal{A}$ be a category. According to Remark C.1.4.6, $\mathcal{A}$ is a Grothendieck abelian category if and only if there exists a Grothendieck prestable ∞-category $\mathcal{C}$ and an equivalence $\mathcal{A} \simeq \mathcal{C}^{\heartsuit}$. We now consider a mild generalization:

Definition C.5.4.1. Let $\mathcal{A}$ be an ∞-category and let $n \geqslant 0$. We will say that $\mathcal{A}$ is a *Grothendieck abelian* $(n+1)$*-category* if there exists an equivalence $\mathcal{A} \simeq \tau_{\leqslant n} \mathcal{C}$, where $\mathcal{C}$ is a Grothendieck prestable ∞-category. We let Groth_n denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ whose objects are Grothendieck abelian $(n+1)$-categories and whose morphisms are functors which preserve small colimits. We let $\mathsf{Groth}_n^{\mathrm{lex}}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ whose objects are Grothendieck abelian $(n+1)$-categories and whose morphisms are functors which preserve small colimits and finite limits. In the special case $n = 0$, we will denote Groth_n and $\mathsf{Groth}_n^{\mathrm{lex}}$ by $\mathsf{Groth}_{\mathrm{ab}}$ and $\mathsf{Groth}_{\mathrm{ab}}^{\mathrm{lex}}$, respectively.

Remark C.5.4.2. Every Grothendieck abelian n-category is a presentable ∞-category. Consequently, for each $n \geqslant 0$, we can regard Groth_n as a full subcategory of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories.

Remark C.5.4.3. For each $n \geqslant 0$, the ∞-category Groth_n is actually an $(n+2)$-category: that is, for every pair of objects $\mathcal{A}, \mathcal{B} \in \mathsf{Groth}_n$, the mapping space $\mathrm{Map}_{\mathsf{Groth}_n}(\mathcal{A}, \mathcal{B})$ is $(n+1)$-truncated. (it is the underlying Kan complex of the ∞-category $\mathrm{LFun}(\mathcal{A}, \mathcal{B})$ of colimit-preserving functors from $\mathcal{A}$ to $\mathcal{B}$).

Example C.5.4.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor between Grothendieck prestable ∞-categories. Tensoring with $\tau_{\leqslant n} \mathcal{S}$, we obtain a colimit-preserving functor

$$F_{\leqslant n} : \tau_{\leqslant n} \mathcal{C} \simeq \mathcal{C} \otimes \tau_{\leqslant n} \mathcal{S} \xrightarrow{F \otimes \mathrm{id}} \mathcal{D} \otimes \tau_{\leqslant n} \mathcal{S} \simeq \tau_{\leqslant n} \mathcal{D},$$

given concretely by the formula $F_{\leqslant n}(C) = \tau_{\leqslant n} F(C)$. This construction determines a functor $\mathsf{Groth}_\infty \to \mathsf{Groth}_n$, given on objects by $\mathcal{C} \mapsto \tau_{\leqslant n} \mathcal{C}$.

Note that if F is left exact, then we can identify the functor $F_{\leqslant n} : \tau_{\leqslant n} \mathcal{C} \to \tau_{\leqslant n} \mathcal{D}$ with the restriction $F|_{\tau_{\leqslant n} \mathcal{C}}$. It follows that $F_{\leqslant n}$ is also left exact, so that the construction $\mathcal{C} \mapsto \tau_{\leqslant n} \mathcal{C}$ determines a functor $\mathsf{Groth}_\infty^{\mathrm{lex}} \to \mathsf{Groth}_n^{\mathrm{lex}}$.

Every Grothendieck abelian category $\mathcal{A}$ admits a *canonical* realization as the heart of a Grothendieck prestable ∞-category $\mathcal{C}$: we can take $\mathcal{C}$ to be the ∞-category $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ of connective objects of the derived ∞-category $\mathcal{D}(\mathcal{A})$. Using the results of §C.5.3, we can provide an analogous realization for Grothendieck abelian n-categories:

Proposition C.5.4.5. *Let $\mathsf{Groth}_\infty^{\mathrm{lex,sep}}$ denote the subcategory of Groth_∞ whose objects are separated Grothendieck prestable ∞-categories and whose morphisms are left exact functors which preserve small colimits. Let $n \geqslant 0$, and let $L : \mathsf{Groth}_\infty^{\mathrm{lex,sep}} \to \mathsf{Groth}_n^{\mathrm{lex}}$ be the functor $\mathcal{C} \mapsto \tau_{\leqslant n} \mathcal{C}$ of Example C.5.4.4 (restricted to separated prestable ∞-categories). Then L admits a fully faithful left adjoint $\mathsf{Groth}_n^{\mathrm{lex}} \hookrightarrow \mathsf{Groth}_\infty^{\mathrm{lex,sep}}$, whose essential image is spanned by those Grothendieck prestable ∞-categories $\mathcal{C}$ which are separated and n-complicial.*

Proof. Let $\mathcal{E} \subseteq \mathsf{Groth}_{\infty}^{\mathrm{lex,sep}}$ denote the full subcategory spanned by those separated Grothendieck prestable ∞-categories which are n-complicial. We first claim that $\mathcal{E}$ is a colocalization of $\mathsf{Groth}_{\infty}^{\mathrm{lex,sep}}$. In other words, for every object $\mathcal{C} \in \mathsf{Groth}_{\infty}^{\mathrm{lex,sep}}$, we claim that there exists a morphism $\lambda : \mathcal{C}' \to \mathcal{C}$ in $\mathsf{Groth}_{\infty}^{\mathrm{lex,sep}}$ where $\mathcal{C}'$ is n-complicial and, for every n-complicial separated Grothendieck prestable ∞-category $\mathcal{C}''$, composition with λ induces a homotopy equivalence

$$\mathrm{Map}_{\mathsf{Groth}_{\infty}^{\mathrm{lex,sep}}}(\mathcal{C}'', \mathcal{C}') \to \mathrm{Map}_{\mathsf{Groth}_{\infty}^{\mathrm{lex,sep}}}(\mathcal{C}'', \mathcal{C}).$$

By virtue of Proposition C.5.3.9, to guarantee the latter property it suffices to arrange that λ is an n-equivalence, which is always possible by virtue of Proposition C.5.3.8.

Note that the functor L carries n-equivalences in $\mathsf{Groth}_{\infty}^{\mathrm{lex,sep}}$ to equivalences in $\mathsf{Groth}_{n}^{\mathrm{lex}}$, and therefore factors as a composition

$$\mathsf{Groth}_{\infty}^{\mathrm{lex,sep}} \xrightarrow{L'} \mathcal{E} \xrightarrow{L''} \mathsf{Groth}_{n}^{\mathrm{lex}}$$

where L' is right adjoint to the inclusion $\mathcal{E} \hookrightarrow \mathsf{Groth}_{\infty}^{\mathrm{lex,sep}}$, and L'' is the restriction $L|_{\mathcal{E}}$. To complete the proof, it will suffice to show that the functor L'' is an equivalence of ∞-categories. The assertion that L'' is fully faithful follows immediately from Proposition C.5.3.9. We will complete the proof by showing that L'' is essentially surjective. Let $\mathcal{A}$ be a Grothendieck abelian $(n+1)$-category, so that $\mathcal{A} \simeq \tau_{\leqslant n}\mathcal{C}$ for some Grothendieck prestable ∞-category $\mathcal{C}$. We wish to show that we can arrange that $\mathcal{C}$ is separated and n-complicial. To prove this, we first replace $\mathcal{C}$ by its separated quotient (see Proposition C.3.6.1) to arrange that $\mathcal{C}$ is separated, and then apply Proposition C.5.3.8. □

Remark C.5.4.6. Let $\mathcal{C}$ be a separated Grothendieck prestable ∞-category. The proof of Proposition C.5.4.5 shows that the functor $\lambda : \mathcal{C}' \to \mathcal{C}$ appearing in Proposition C.5.3.8 is determined uniquely (up to equivalence) by $\mathcal{C}$.

Warning C.5.4.7. In the statement of Proposition C.5.4.5, the left exactness assumption is essential. The construction $\mathcal{C} \mapsto \tau_{\leqslant n}\mathcal{C}$ determines a functor $\mathsf{Groth}_{\infty}^{\mathrm{sep}} \to \mathsf{Groth}_{n}$ (see Example C.5.4.4) which does *not* have a fully faithful right adjoint: for example, if $\mathcal{C}$ is any separated Grothendieck prestable ∞-category, then the suspension functor $\Sigma^n : \mathcal{C} \to \mathcal{C}$ is a morphism in $\mathsf{Groth}_{\infty}^{\mathrm{sep}}$ whose image in Groth_{n} is nullhomotopic.

Arguing as in Example C.5.4.4, we see that the construction $\mathcal{C} \mapsto \tau_{\leqslant m}\mathcal{C}$ determines a forgetful functor $\mathsf{Groth}_{n} \to \mathsf{Groth}_{m}$ for all $m \leqslant n$. We therefore obtain a tower of ∞-categories

$$\mathsf{Groth}_{\infty} \to \cdots \to \mathsf{Groth}_{2} \to \mathsf{Groth}_{1} \to \mathsf{Groth}_{0} = \mathsf{Groth}_{\mathrm{ab}},$$

which determines a functor $\mathsf{Groth}_{\infty} \to \varprojlim \mathsf{Groth}_{n}$.

Theorem C.5.4.8. *The functor $\theta : \mathsf{Groth}_{\infty} \to \varprojlim \mathsf{Groth}_{n}$ described above restricts to an equivalence of ∞-categories $\theta^{\mathrm{comp}} : \mathsf{Groth}_{\infty}^{\mathrm{comp}} \to \varprojlim \mathsf{Groth}_{n}$.*

Proof. We first show that θ^{comp} is fully faithful. Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories. Unwinding the definitions, we have canonical homotopy equivalences

$$\begin{aligned} \mathrm{Map}_{\varprojlim \mathsf{Groth}_n}(\theta(\mathcal{C}), \theta(\mathcal{D})) &\simeq \varprojlim \mathrm{Map}_{\mathsf{Groth}_n}(\tau_{\leqslant n}\,\mathcal{C}, \tau_{\leqslant n}\,\mathcal{D}) \\ &\simeq \varprojlim \mathrm{LFun}(\tau_{\leqslant n}\,\mathcal{C}, \tau_{\leqslant n}\,\mathcal{D})^{\simeq} \\ &\simeq \varprojlim \mathrm{LFun}(\mathcal{C}, \tau_{\leqslant n}\,\mathcal{D})^{\simeq} \\ &\simeq \mathrm{LFun}(\mathcal{C}, \widehat{\mathcal{D}})^{\simeq} \end{aligned}$$

Using these equivalences, we can identify the natural map $\mathrm{Map}_{\mathsf{Groth}_\infty}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}_{\varprojlim \mathsf{Groth}_n}(\theta(\mathcal{C}), \theta(\mathcal{D}))$ with the map $\mathrm{LFun}(\mathcal{C}, \mathcal{D})^{\simeq} \to \mathrm{LFun}(\mathcal{D}, \widehat{\mathcal{D}})^{\simeq}$ given by composition with the functor $\mathcal{D} \to \widehat{\mathcal{D}}$. In particular, this map is an equivalence whenever $\mathcal{D}$ is complete.

We now prove essential surjectivity. Suppose we are given an object of the inverse limit $\varprojlim \mathsf{Groth}_n$, which we will identify with a sequence of objects $\mathcal{C}_n \in \mathsf{Groth}_n$ together with equivalences $\alpha_n : \mathcal{C}_n \simeq \tau_{\leqslant n}\,\mathcal{C}_{n+1}$. For each $n \geqslant 0$, Proposition C.5.4.5 guarantees the existence of a separated n-complicial Grothendieck prestable ∞-category $\mathcal{D}_n$ and an equivalence $\mathcal{C}_n \simeq \tau_{\leqslant n}\,\mathcal{D}_n$. Applying Proposition C.5.4.5, we see that each of the functors α_n admits an essentially unique extension to a functor $\beta_n : \mathcal{D}_n \to \mathcal{D}_{n+1}$ which preserves small colimits and finite limits. Let $\mathcal{D}$ denote the colimit $\varinjlim \mathcal{D}_n$, formed in the ∞-category Groth_∞ (see Theorem C.3.3.1). The underlying ∞-category of $\mathcal{D}$ can be described as the inverse limit of the tower

$$\cdots \to \mathcal{D}_3 \xrightarrow{\gamma_2} \mathcal{D}_2 \xrightarrow{\gamma_1} \mathcal{D}_1 \xrightarrow{\gamma_0} \mathcal{D}_0,$$

where γ_n denotes a right adjoint to β_n. In particular, for each $m \geqslant 0$, we can identify $\tau_{\leqslant m}\,\mathcal{D}$ with the limit of the tower

$$\cdots \to \tau_{\leqslant m}\,\mathcal{D}_3 \xrightarrow{\gamma_2} \tau_{\leqslant m}\,\mathcal{D}_2 \xrightarrow{\gamma_1} \tau_{\leqslant m}\,\mathcal{D}_1 \xrightarrow{\gamma_0} \tau_{\leqslant m}\,\mathcal{D}_0\,.$$

By construction, this tower is eventually equivalent to the constant tower with value $\mathcal{C}_n$. We therefore have canonical equivalences $\tau_{\leqslant m}\,\mathcal{D} \simeq \mathcal{C}_m$ which are easily seen to be compatible with the maps α_n, so that $\mathcal{D}$ is a preimage of the object $\{\mathcal{C}_n\} \in \varprojlim \mathsf{Groth}_n$ under the functor θ. It follows that the completion $\widehat{\mathcal{D}}$ is a preimage of the object $\{\mathcal{C}_n\}$ under the functor θ^{comp}. □

It follows from Proposition C.5.3.2 that in the special case $n = 0$, the fully faithful embedding $\mathsf{Groth}^{\mathrm{lex}}_{\mathrm{ab}} = \mathsf{Groth}^{\mathrm{lex}}_0 \hookrightarrow \mathsf{Groth}^{\mathrm{lex,sep}}_\infty$ of Proposition C.5.4.5 carries a Grothendieck abelian category $\mathcal{A}$ to the Grothendieck prestable ∞-category $\mathcal{D}(\mathcal{A})_{\geqslant 0}$. We therefore have the following:

Theorem C.5.4.9. *Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{C}$ be a separated Grothendieck prestable ∞-category. Then restriction to the heart induces a fully faithful embedding*

$$\mathrm{LFun}^{\mathrm{lex}}(\mathcal{D}(\mathcal{A})_{\geqslant 0}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{C}^{\heartsuit}),$$

whose essential image is spanned by those exact functors $\mathcal{A} \to \mathcal{C}^{\heartsuit}$ which preserve small colimits.

Remark C.5.4.10. It follows from Proposition C.5.4.5 that we can regard the construction $\mathcal{A} \mapsto \mathcal{D}(\mathcal{A})_{\geqslant 0}$ as a fully faithful embedding from the ∞-category $\mathsf{Groth}_{\mathrm{ab}}^{\mathrm{lex}}$ to the ∞-category $\mathsf{Groth}_{\infty}^{\mathrm{lex,sep}} \subseteq \mathsf{Groth}_{\infty}$: that is, the formation of derived ∞-categories is functorial (at least with respect to exact functors which preserve small colimits).

Remark C.5.4.11. Let $\mathcal{A}$ be a Grothendieck abelian category. Using Propositions C.5.3.2 and C.5.4.5, we see that the derived ∞-category $\mathcal{D}(\mathcal{A})$ (together with its t-structure) is determined uniquely up to equivalence by the following properties:

(a) The ∞-category $\mathcal{D}(\mathcal{A})$ is presentable and stable.

(b) The t-structure $(\mathcal{D}(\mathcal{A})_{\geqslant 0}, \mathcal{D}(\mathcal{A})_{\leqslant 0})$ is right complete, left separated, and compatible with filtered colimits.

(c) The Grothendieck prestable ∞-category $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ is 0-complicial.

(d) The heart $\mathcal{D}(\mathcal{A})^{\heartsuit}$ is equivalent to $\mathcal{A}$.

In order to study Grothendieck abelian $(n+1)$-categories, it will be convenient to work with a larger class of objects (in which colimit and tensor product constructions will be *a priori* well-defined).

Notation C.5.4.12. For each $n \geqslant 0$, let $\mathcal{P}r_n^{Add}$ denote the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ spanned by those presentable ∞-categories $\mathcal{A}$ which satisfy satisfy the following conditions:

(a) The ∞-category $\mathcal{A}$ is additive.

(b) The ∞-category $\mathcal{A}$ is (equivalent to) an $(n+1)$-category: that is, the mapping spaces $\mathrm{Map}_{\mathcal{A}}(X, Y)$ are n-truncated, for every pair of objects $X, Y \in \mathcal{A}$.

Note that we can regard Groth_n as a full subcategory of $\mathcal{P}r_n^{Add}$.

Proposition C.5.4.13. *Let $n \geqslant 0$ be an integer. Then:*

(a) *The functor*

$$\mathcal{S} \to \tau_{\leqslant n}\,\mathrm{Sp}^{\mathrm{cn}} \qquad X \mapsto \tau_{\leqslant n}\Sigma^{\infty}_{+} X$$

exhibits $\tau_{\leqslant n}\,\mathrm{Sp}^{\mathrm{cn}}$ as an idempotent object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories (see Definition **??***).*

(*b*) *The forgetful functor* $\pi : \mathrm{Mod}_{\tau_{\leqslant n}\,\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ *is a fully faithful embedding, whose essential image is the full subcategory* $\mathcal{P}r_n^{Add} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}}$.

In particular, any Grothendieck abelian $(n+1)$*-category* $\mathcal{A}$ *admits an essentially unique action of the* ∞*-category* $\tau_{\leqslant n}\,\mathrm{Sp}^{\mathrm{cn}}$.

Proof. It follows from Proposition HA.4.8.2.15 that the functor $\tau_{\leqslant n} : \mathcal{S} \to \tau_{\leqslant n}\,\mathcal{S}$ exhibits $\tau_{\leqslant n}\,\mathcal{S}$ as an idempotent object of the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. In particular, the forgetful functor $\mathrm{Mod}_{\tau_{\leqslant n}\,\mathcal{S}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ is fully faithful, and Proposition HA.4.8.2.15 also implies that its essential image consists of those presentable ∞-categories which are equivalent to $(n+1)$-categories. Consequently, for any presentable ∞-category $\mathcal{C}$, the tensor product $(\tau_{\leqslant n}\,\mathcal{S}) \otimes \mathcal{C}$ is universal among presentable ∞-categories $\mathcal{E}$ which are equivalent to $(n+1)$-categories and receive a functor $\mathcal{C} \to \mathcal{E}$. We can therefore identify $(\tau_{\leqslant n}\,\mathcal{S}) \otimes \mathcal{C}$ with the ∞-category $\tau_{\leqslant n}\,\mathcal{C}$ of n-truncated objects of $\mathcal{C}$. In particular, we can identify the tensor product $(\tau_{\leqslant n}\,\mathcal{S}) \otimes \mathrm{Sp}^{\mathrm{cn}}$ with the ∞-category $\tau_{\leqslant n}\,\mathrm{Sp}^{\mathrm{cn}}$ of n-truncated connective spectra. Since the suspension functor $\Sigma^{\infty}_{+} : \mathcal{S} \to \mathrm{Sp}^{\mathrm{cn}}$ and the truncation functor $\tau_{\leqslant n} : \mathcal{S} \to \tau_{\leqslant n}\,\mathcal{S}\,\mathrm{Sp}^{\mathrm{cn}}$ and $\tau_{\leqslant n}\,\mathcal{S}$ as idempotent objects of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (Proposition HA.4.8.2.15 and Corollary C.4.1.2), their tensor product

$$\mathcal{S} \simeq \mathcal{S} \otimes \mathcal{S} \xrightarrow{\tau_{\leqslant n} \otimes \Sigma^{\infty}_{+}} (\tau_{\leqslant n}\,\mathcal{S}) \otimes \mathrm{Sp}^{\mathrm{cn}} \simeq \tau_{\leqslant n}\,\mathrm{Sp}^{\mathrm{cn}}$$

$$X \mapsto \tau_{\leqslant n}(\Sigma^{\infty}_{+} X)$$

exhibits $\tau_{\leqslant n}\,\mathrm{Sp}^{\mathrm{cn}}$ as an idempotent object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. This proves (a).

It follows from (a) that there exists an essentially unique commutative algebra structure on the object $\tau_{\leqslant n}\,\mathrm{Sp}^{\mathrm{cn}} \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$ for which the functor $\tau_{\leqslant n}\Sigma^{\infty}_{+} : \mathcal{S} \to \tau_{\leqslant n}\,\mathrm{Sp}^{\mathrm{cn}}$ is the unit map (see Proposition HA.4.8.2.9). It follows from the uniqueness that the multiplication on $\tau_{\leqslant n}\,\mathrm{Sp}^{\mathrm{cn}}$ corresponds to the symmetric monoidal structure given by $(X, Y) \mapsto \tau_{\leqslant n}(X \otimes Y)$; in particular, when $n = 0$, it is given by the classical tensor product of abelian groups. Using Proposition HA.4.8.2.10 , we see that the forgetful functor $\mathrm{Mod}_{\tau_{\leqslant n}\,\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ is a fully faithful embedding, whose essential image consists of those presentable ∞-categories $\mathcal{A}$ for which the canonical map

$$e : \mathcal{A} \to (\tau_{\leqslant n}\,\mathrm{Sp}^{\mathrm{cn}}) \otimes \mathcal{A} \simeq (\tau_{\leqslant n}\,\mathcal{S}) \otimes \mathrm{Sp}^{\mathrm{cn}} \otimes \mathcal{A}$$

is an equivalence. If $\mathcal{A}$ satisfies this condition, then it can be regarded as a module over both $\mathrm{Sp}^{\mathrm{cn}}$ and $\tau_{\leqslant n}\,\mathcal{S}$, and is therefore both additive and equivalent to an $(n+1)$-category (Corollary C.4.1.3 and Proposition HA.4.8.2.15). Conversely, if $\mathcal{A}$ is additive and equivalent to an $(n+1)$-category, then e factors as a composition of equivalences

$$\mathcal{A} \xrightarrow{\sim} (\tau_{\leqslant n}\,\mathcal{S}) \otimes \mathcal{A} \xrightarrow{\sim} (\tau_{\leqslant n}\,\mathcal{S}) \otimes (\mathrm{Sp}^{\mathrm{cn}} \otimes \mathcal{A})$$

and is therefore an equivalence, which proves (b). $\square$

Remark C.5.4.14. It follows from Proposition C.5.4.13 that the $\mathcal{P}r_n^{Add} \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$ admits a left adjoint L, given by the construction $\mathcal{C} \mapsto (\tau_{\leqslant n} \mathrm{Sp}^{\mathrm{cn}}) \otimes \mathcal{C}$. Moreover, the localization functor L is compatible with the symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (see Definition HA.2.2.1.6). Consequently, there is an essentially unique symmetric monoidal structure on the ∞-category $\mathcal{P}r_n^{Add}$ with respect to which the localization functor L is symmetric monoidal. The tensor product on $\mathcal{P}r_n^{Add}$ agrees with the tensor product on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, but the unit object is different (the unit object of $\mathcal{P}r_0^{Add}$ is given $\tau_{\leqslant n} \mathrm{Sp}^{\mathrm{cn}}$), rather than the ∞-category $\mathcal{S}$ of spaces).

Corollary C.5.4.15. *Let $n \geqslant 0$ be an integer. Then the inclusion functor $\iota : \mathcal{P}r_n^{Add} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{Add}}$ admits a left adjoint, which assigns to each presentable additive ∞-category $\mathcal{C}$ the full subcategory $\tau_{\leqslant n} \mathcal{C} \subseteq \mathcal{C}$ spanned by the n-truncated objects.*

Proof. Using Proposition C.5.4.13 and Corollary C.4.1.3, we can identify ι with the forgetful functor $\mathrm{Mod}_{\tau_{\leqslant n} \mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \rightarrow \mathrm{Mod}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. This functor has a left adjoint given by the construction

$$\mathcal{C} \mapsto (\tau_{\leqslant n} \mathrm{Sp}^{\mathrm{cn}}) \otimes_{\mathrm{Sp}^{\mathrm{cn}}} \mathcal{C} \simeq (\tau_{\leqslant n} \mathcal{S}) \otimes \mathcal{C} \simeq \tau_{\leqslant n} \mathcal{C} .$$

□

In §C.4, we showed that the collection of Grothendieck prestable ∞-categories is closed under tensor products (Theorem C.4.2.1). We now prove an analogue for Grothendieck abelian categories:

Theorem C.5.4.16. *Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck abelian $(n+1)$-categories for some $n \geqslant 0$. Then the tensor product $\mathcal{A} \otimes \mathcal{B}$ (formed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$) is also a Grothendieck abelian $(n+1)$-category.*

Remark C.5.4.17. In the case where $n = 0$ and the abelian categories $\mathcal{A}$ and $\mathcal{B}$ are compactly generated, Theorem C.5.4.16 was proven by Schäppi in [179]; in this case, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is again compactly generated.

Proof of Theorem C.5.4.16. Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck abelian $(n+1)$-categories; we wish to show that the tensor product $\mathcal{A} \otimes \mathcal{B}$ is also a Grothendieck abelian $(n+1)$-category. Choose Grothendieck prestable ∞-categories $\mathcal{C}$ and $\mathcal{D}$ satisfying $\mathcal{A} \simeq \tau_{\leqslant n} \mathcal{C}$ and $\mathcal{B} \simeq \tau_{\leqslant n} \mathcal{D}$. It follows from Theorem C.4.2.1 that the tensor product $\mathcal{C} \otimes \mathcal{D}$ is again a Grothendieck prestable ∞-category. In the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, we have equivalences

$$\begin{aligned} \mathcal{A} \otimes \mathcal{B} &\simeq (\tau_{\leqslant n} \mathcal{S} \otimes \mathcal{C}) \otimes (\tau_{\leqslant n} \mathcal{S} \otimes \mathcal{D}) \\ &\simeq (\tau_{\leqslant n} \mathcal{S} \otimes \tau_{\leqslant n} \mathcal{S}) \otimes (\mathcal{C} \otimes \mathcal{D}) \\ &\simeq \tau_{\leqslant n} \mathcal{S} \otimes (\mathcal{C} \otimes \mathcal{D}) \\ &\simeq \tau_{\leqslant n} (\mathcal{C} \otimes \mathcal{D}), \end{aligned}$$

so that $\mathcal{A} \otimes \mathcal{B}$ is also a Grothendieck abelian $(n+1)$-category. □

Remark C.5.4.18. Theorem C.4.2.1 can be regarded as an ∞-categorical analogue of Theorem C.5.4.16, and the preceding argument shows that it immediately implies Theorem C.5.4.16. When $n = 0$, one can also prove Theorem C.5.4.16 directly (without appeal to higher category theory) by mimicking our earlier proof of Theorem C.4.2.1, using the classical Gabriel-Popescu theorem in place of its ∞-categorical analogue.

Since the ∞-category Groth_n contains the unit object $\tau_{\leqslant n}\,\mathrm{Sp}^{\mathrm{cn}}$ of $\mathcal{P}r_n^{Add}$, Theorem C.5.4.16 implies the following:

Corollary C.5.4.19. *For each $n \geqslant 0$, the symmetric monoidal structure on the ∞-category $\mathcal{P}r_n^{Add} \simeq \mathrm{Mod}_{\tau_{\leqslant n}\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ restricts to a symmetric monoidal structure on the full subcategory $\mathsf{Groth}_n \subseteq \mathcal{P}r_n^{Add}$. In other words, there is an essentially unique symmetric monoidal structure on the ∞-category Groth_n for which the inclusion functor $\mathsf{Groth}_n \hookrightarrow \mathcal{P}r_n^{Add}$ is symmetric monoidal.*

Remark C.5.4.20. Since the localization functor

$$\mathcal{P}\mathrm{r}^{\mathrm{Add}} \simeq \mathrm{Mod}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \xrightarrow{\tau_{\leqslant}\mathrm{Sp}^{\mathrm{cn}} \otimes_{\mathrm{Sp}^{\mathrm{cn}}}} \mathrm{Mod}_{\tau_{\leqslant n}\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \simeq \mathcal{P}r_n^{Add}$$

of Remark C.5.4.14 is symmetric monoidal, it follows that the functor

$$\mathsf{Groth}_\infty \to \mathsf{Groth}_n \qquad \mathcal{C} \mapsto \tau_{\leqslant n}\,\mathcal{C}$$

of Warning C.5.4.7 is also symmetric monoidal.

We next prove an analogue of Proposition C.3.2.4:

Proposition C.5.4.21. *Let $n \geqslant 0$ be an integer. Then the ∞-category $\mathsf{Groth}_n^{\mathrm{lex}}$ admits small limits, which are preserved by the inclusion functors*

$$\mathsf{Groth}_n^{\mathrm{lex}} \hookrightarrow \mathsf{Groth}_n \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}} \hookrightarrow \widehat{\mathcal{C}\mathrm{at}}_\infty.$$

Proof. Let $\{\mathcal{C}_\alpha\}$ be a small diagram in $\mathsf{Groth}_n^{\mathrm{lex}}$. Using Proposition C.5.4.5, we can lift $\{\mathcal{C}_\alpha\}$ to a diagram $\{\overline{\mathcal{C}}_\alpha\}$ in $\mathsf{Groth}_\infty^{\mathrm{lex}}$, where each $\overline{\mathcal{C}}_\alpha$ is separated and n-complicial. Set $\overline{\mathcal{C}} = \varprojlim\{\overline{\mathcal{C}}_\alpha\}$ (formed in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$). It follows from Proposition C.3.2.4 that $\overline{\mathcal{C}}$ is a Grothendieck prestable ∞-category, which is also a limit for the diagram $\{\overline{\mathcal{C}}\}_\alpha$ in the ∞-categories Groth_∞ and $\mathsf{Groth}_\infty^{\mathrm{lex}}$. It follows that $\mathcal{C} = \tau_{\leqslant n}\overline{\mathcal{C}}$ is a Grothendieck abelian $(n+1)$-category. Moreover, since the transition functors $\overline{\mathcal{C}}_\alpha \to \overline{\mathcal{C}}_\beta$ are left exact, an object of $\overline{\mathcal{C}}$ is n-truncated if and only if its image in each $\overline{\mathcal{C}}_\alpha$ is n-truncated. We can therefore identify $\mathcal{C}$ with a limit of the diagram $\{\mathcal{C}_\alpha\}$ in $\widehat{\mathcal{C}\mathrm{at}}_\infty$. For any presentable ∞-category $\mathcal{D}$, a functor $\mathcal{D} \to \mathcal{C}$ preserves small colimits and finite limits if and only if each of the composite maps $\mathcal{D} \to \mathcal{C} \to \mathcal{C}_\alpha$ preserves small colimits and finite limits (this is a formal consequence of our assumption that the transition maps $\mathcal{C}_\alpha \to \mathcal{C}_\beta$ preserve small colimits and finite limits). It follows that $\mathcal{C}$ is also a limit of the diagram $\{\mathcal{C}_\alpha\}$ in the ∞-categories $\mathsf{Groth}_n^{\mathrm{lex}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (hence also in the full subcategory $\mathsf{Groth}_n \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}}$). □

C.5.5 Anticomplete Prestable ∞-Categories

Recall that a Grothendieck prestable ∞-category $\mathcal{C}$ is said to be *complete* if it is Postnikov-complete: that is, if it is equivalent to the limit of the tower of ∞-categories

$$\cdots \to \tau_{\leqslant 3}\,\mathcal{C} \xrightarrow{\tau_{\leqslant 2}} \tau_{\leqslant 2}\,\mathcal{C} \xrightarrow{\tau_{\leqslant 1}} \tau_{\leqslant 1}\,\mathcal{C} \xrightarrow{\tau_{\leqslant 0}} \tau_{\leqslant 0}\,\mathcal{C} = \mathcal{C}^{\heartsuit}.$$

We now study Grothendieck prestable ∞-categories $\mathcal{C}$ which are, in some sense, as far as possible from being complete. First, we need to introduce some terminology.

Proposition C.5.5.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories, let $f : \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor, and let $n \geqslant 0$ be an integer. The following conditions are equivalent:*

- (a) *For every discrete object $C \in \mathcal{C}$, the object $f(C) \in \mathcal{D}$ is n-truncated.*
- (b) *For every k-truncated object $C \in \mathcal{C}$, the object $f(C) \in \mathcal{D}$ is $(n+k)$-truncated.*

Proof. The implication $(b) \Rightarrow (a)$ is obvious. To prove the converse, we note that a k-truncated object $C \in \mathcal{C}$ can be written as a successive extension of the objects $\Sigma^i(\pi_i C)$ for $0 \leqslant i \leqslant k$. If (a) is satisfied, then $f(C)$ can be written as a successive extension of the objects $\Sigma^i f(\pi_i C) \in \tau_{\leqslant n+i}\,\mathcal{D} \subseteq \tau_{\leqslant n+k}\,\mathcal{D}$, so that $f(C)$ belongs to $\tau_{\leqslant n+k}\,\mathcal{D}$. □

Definition C.5.5.2. Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories and let $f : \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor. We will say that *f has amplitude $\leqslant n$* if the equivalent conditions of Proposition C.5.5.1 are satisfied. We will say that *f has bounded amplitude* if f has amplitude $\leqslant n$ for some $n \geqslant 0$. We let $\mathrm{LFun}^b(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{LFun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which are of bounded amplitude.

Example C.5.5.3. Let $f : \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor between Grothendieck prestable ∞-categories. Then f has amplitude $\leqslant 0$ if and only if f is left exact (Proposition C.3.2.1).

Definition C.5.5.4. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. We will say that $\mathcal{C}$ is *anticomplete* if it satisfies the following condition:

- $(*)$ Let $\mathcal{D}$ be a Grothendieck prestable ∞-category and let $\widehat{\mathcal{D}}$ be its completion. Then the canonical map
$$\mathrm{LFun}^b(\mathcal{C}, \mathcal{D}) \to \mathrm{LFun}^b(\mathcal{C}, \widehat{\mathcal{D}})$$
is an equivalence of ∞-categories.

We now consider some examples of anticomplete Grothendieck prestable ∞-categories.

Proposition C.5.5.5. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

(a) *The ∞-category $\mathcal{C}$ is a compactly generated prestable ∞-category, the functor $\Omega : \mathcal{C} \to \mathcal{C}$ preserves compact objects, and every compact object of $\mathcal{C}$ is n-truncated for $n \gg 0$.*

(b) *There exist an essentially small stable ∞-category $\mathcal{E}$ equipped with a bounded t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ and an equivalence $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{E}_{\geqslant 0})$.*

Moreover, if $\mathcal{C}$ satisfies these conditions, then $\mathcal{C}$ is anticomplete.

Remark C.5.5.6. For a refinement of Proposition C.5.5.5, we refer the reader to Theorem C.6.7.1.

Proof of Proposition C.5.5.5. We first show that (b) implies (a). Let $\mathcal{E}$ be an essentially small stable ∞-category equipped with a bounded t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$. Then $\mathcal{C} = \mathrm{Ind}(\mathcal{E}_{\geqslant 0})$ is a prestable ∞-category (see the proof of Proposition C.2.4.2) which is compactly generated, and therefore a Grothendieck prestable ∞-category. Let $j : \mathcal{E}_{\geqslant 0} \to \mathcal{C}$ be the Yoneda embedding, so that an object $C \in \mathcal{C}$ is compact if and only if C is a direct summand of $j(E)$ for some $E \in \mathcal{E}_{\geqslant 0}$. Since the t-structure on $\mathcal{E}$ is bounded, the object $E \in \mathcal{E}_{\geqslant 0}$ is n-truncated for $n \gg 0$, so that $j(E) \in \mathcal{C}$ is also n-truncated and therefore C is n-truncated. Finally, we note that ΩC is a direct summand of $j(\tau_{\geqslant 0}\Omega E)$ and is therefore also a compact object of $\mathcal{C}$.

We now prove the converse. Suppose that $\mathcal{C}$ is a compactly generated prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory spanned by the compact objects. Then $\mathcal{C}_0$ is an essentially small prestable ∞-category (Proposition C.6.1.1). Let $\mathcal{E} = \mathrm{SW}(\mathcal{C}_0)$ denote the Spanier-Whitehead ∞-category of $\mathcal{C}_0$ (Construction C.1.1.1). If $\Omega : \mathcal{C} \to \mathcal{C}$ preserves compact objects, then $\mathcal{C}_0$ is closed under finite limits in $\mathcal{C}$. Applying Proposition C.1.2.9, we deduce that $\mathcal{E}$ admits a t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$, where $\mathcal{E}_{\geqslant 0} \simeq \mathcal{C}_0$. This t-structure is always right-bounded, and is left-bounded if and only if every compact object $C \in \mathcal{C}$ is n-truncated for some $n \gg 0$. In this case, we have $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}_0) \simeq \mathrm{Ind}(\mathcal{E}_{\geqslant 0})$, so that $\mathcal{C}$ satisfies condition (b).

We now complete the proof by showing that if $\mathcal{C} = \mathrm{Ind}(\mathcal{E}_{\geqslant 0})$ as above, then $\mathcal{C}$ is anticomplete. Let $\mathcal{D}$ be a Grothendieck prestable ∞-category and let $\widehat{\mathcal{D}}$ denote its completion; we wish to show that the canonical map

$$\rho : \mathrm{LFun}^b(\mathrm{Ind}(\mathcal{E}_{\geqslant 0}), \mathcal{D}) \to \mathrm{LFun}^b(\mathrm{Ind}(\mathcal{E}_{\geqslant 0}), \widehat{\mathcal{D}})$$

is an equivalence of ∞-categories. The proof of Proposition C.2.4.2 supplies a t-exact equivalence $\mathrm{Sp}(\mathrm{Ind}(\mathcal{E}_{\geqslant 0})) \simeq \mathrm{Ind}(\mathcal{E})$. Using Proposition C.3.1.1, we can identify ρ with the canonical map

$$\rho' : \mathrm{Fun}'(\mathrm{Ind}(\mathcal{E}), \mathrm{Sp}(\mathcal{D})) \to \mathrm{Fun}'(\mathrm{Ind}(\mathcal{E}), \mathrm{Sp}(\widehat{\mathcal{D}})).$$

Here $\mathrm{Fun}'(\mathrm{Ind}(\mathcal{E}), \mathrm{Sp}(\mathcal{D}))$ denotes the full subcategory of $\mathrm{Fun}(\mathrm{Ind}(\mathcal{E}), \mathrm{Sp}(\mathcal{D}))$ spanned by those functors which preserve small colimits, are right t-exact, and carry $\mathrm{Ind}(\mathcal{E})_{\leqslant 0}$ into

$\mathrm{Sp}(\mathcal{D})_{\leqslant n}$ for some $n \gg 0$, and the ∞-category $\mathrm{Fun}'(\mathrm{Ind}(\mathcal{E}), \mathrm{Sp}(\widehat{\mathcal{D}}))$ is defined similarly. The map ρ' fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}'(\mathrm{Ind}(\mathcal{E}), \mathrm{Sp}(\mathcal{D})) & \xrightarrow{\rho'} & \mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}), \mathrm{Sp}(\widehat{\mathcal{D}})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}''(\mathcal{E}, \mathrm{Sp}(\mathcal{D})) & \xrightarrow{\rho''} & \mathrm{Fun}''(\mathcal{E}, \mathrm{Sp}(\widehat{\mathcal{D}})) \end{array}$$

where $\mathrm{Fun}''(\mathcal{E}, \mathrm{Sp}(\mathcal{D}))$ denote the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathrm{Sp}(\mathcal{D}))$ spanned by those functors which are exact, right t-exact, and carries $\mathcal{E}_{\leqslant 0}$ into $\mathrm{Sp}(\mathcal{D})_{\leqslant n}$ for some $n \gg 0$, the ∞-category $\mathrm{Fun}''(\mathcal{E}, \mathrm{Sp}(\widehat{\mathcal{D}}))$ is defined similarly, and the vertical maps are equivalences of ∞-categories. We now observe that ρ'' is an equivalence of ∞-categories, since the t-structure on $\mathcal{E}$ is left bounded and the map $\mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\widehat{\mathcal{D}})$ induces an equivalence $\mathrm{Sp}(\mathcal{D})_{\leqslant n} \to \mathrm{Sp}(\widehat{\mathcal{D}})_{\leqslant n}$ for every integer n. □

Proposition C.5.5.7. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category with heart $\mathcal{A} = \mathcal{C}^{\heartsuit}$, let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a localizing subcategory of $\mathcal{A}$, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the smallest localizing subcategory of $\mathcal{C}$ which contains $\mathcal{A}_0$ (see Proposition C.5.2.8). If $\mathcal{C}$ is anticomplete, then $\mathcal{C} / \mathcal{C}_0$ is also anticomplete.*

Proof. Let $\mathcal{D}$ be a Grothendieck prestable ∞-category with completion $\widehat{\mathcal{D}}$. We wish to show that the canonical map $\rho : \mathrm{LFun}^b(\mathcal{C} / \mathcal{C}_0, \mathcal{D}) \to \mathrm{LFun}^b(\mathcal{C} / \mathcal{C}_0, \widehat{\mathcal{D}})$ is an equivalence of ∞-categories. Note that the functor ρ fits into a commutative diagram of ∞-categories σ :

$$\begin{array}{ccc} \mathrm{LFun}^b(\mathcal{C} / \mathcal{C}_0, \mathcal{D}) & \xrightarrow{\rho} & \mathrm{LFun}^b(\mathcal{C} / \mathcal{C}_0, \widehat{\mathcal{D}}) \\ \downarrow & & \downarrow \\ \mathrm{LFun}^b(\mathcal{C}, \mathcal{D}) & \xrightarrow{\overline{\rho}} & \mathrm{LFun}^b(\mathcal{C}, \widehat{\mathcal{D}}), \end{array}$$

where the map $\overline{\rho}$ is an equivalence of ∞-categories (by virtue of our assumption that $\mathcal{C}$ is anticomplete) and the vertical maps are fully faithful embeddings. It will therefore suffice to show that the diagram σ is a pullback square. Using Proposition C.2.3.10 and Remark C.2.3.11, we are reduced to proving the following:

(∗) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves small colimits and has bounded amplitude. If the composite functor $\mathcal{C} \xrightarrow{F} \mathcal{D} \to \widehat{\mathcal{D}}$ annihilates $\mathcal{C}_0$, then F annihilates $\mathcal{C}_0$.

To prove this, let $\mathcal{C}_1$ denote the full subcategory of $\mathcal{C}$ spanned by those objects $C \in \mathcal{C}$ satisfying $F(C) \simeq 0$. Note that if $X \in \mathcal{A}_0$ and F has amplitude $\leqslant n$, then $F(X) \in \tau_{\leqslant n} \mathcal{D}$ is annihilated by the completion functor $\mathcal{D} \to \widehat{\mathcal{D}}$, so that $F(X) \simeq 0$ (since the completion functor $\mathcal{D} \to \widehat{\mathcal{D}}$ is an equivalence on discrete objects). It follows that $\mathcal{A}_0 \subseteq \mathcal{C}_1$. Since $\mathcal{C}_1$ is a localizing subcategory of $\mathcal{C}$ (Example C.2.3.7), we have $\mathcal{C}_0 \subseteq \mathcal{C}_1$, so that F annihilates $\mathcal{C}_0$ as desired. □

Lemma C.5.5.8. *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be Grothendieck prestable ∞-categories, and suppose we are given functors $f \in \mathrm{LFun}^b(\mathcal{C}, \mathcal{D})$ and $g \in \mathrm{LFun}^{\mathrm{lex}}(\mathcal{D}, \mathcal{E})$. If g induces a conservative functor $\mathcal{D}^{\heartsuit} \to \mathcal{E}^{\heartsuit}$ and $(g \circ f)$ has amplitude $\leqslant n$, then f has amplitude $\leqslant n$.*

Proof. Note that our assumption that f has bounded amplitude guarantees that the conclusion of Lemma C.5.5.8 is satisfied for $n \gg 0$. We now proceed by descending induction on n. To carry out the inductive step, let us suppose that f has amplitude $\leqslant n+1$ and that $g \circ f$ has amplitude $\leqslant n$; we wish to show that f has amplitude $\leqslant n$. Let $C \in \mathcal{C}$ be a discrete object; then $f(C) \in \mathcal{D}$ is $(n+1)$-truncated and we wish to show that it is n-truncated. Because g is left exact, we have $g(\pi_{n+1} f(C)) \simeq \pi_{n+1}(g \circ f)(C) \simeq 0$. Since $g|_{\mathcal{D}^{\heartsuit}}$ is conservative, it follows that $\pi_{n+1} f(C) \simeq 0$, so that $f(C)$ is n-truncated as desired. □

Proposition C.5.5.9. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Then there exists an anticomplete Grothendieck prestable ∞-category $\check{\mathcal{C}}$ and a functor $\lambda \in \mathrm{LFun}^{\mathrm{lex}}(\check{\mathcal{C}}, \mathcal{C})$ which induces an equivalence of completions.*

Proof. Let $\widehat{\mathcal{C}}$ be the completion of $\mathcal{C}$. For each $n \geqslant 0$, the ∞-category $\tau_{\leqslant n}\widehat{\mathcal{C}}$ is presentable. We can therefore choose an essentially small dense subcategory $\mathcal{K}_n \subseteq \tau_{\leqslant n}\widehat{\mathcal{C}}$ (see Definition 20.4.1.1). Let us abuse notation by identifying $\widehat{\mathcal{C}}$ with its essential image in $\mathrm{Sp}(\widehat{\mathcal{C}})$. Let $\mathcal{E}$ be the smallest stable subcategory of $\mathrm{Sp}(\widehat{\mathcal{C}})$ which contains each $\mathcal{K}_n$ and is closed under truncation (so that the t-structure on $\mathrm{Sp}(\widehat{\mathcal{C}})$ induces a t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ on $\mathcal{E}$). Note that the t-structure on $\mathcal{E}$ is bounded. The inclusion $\mathcal{E} \hookrightarrow \mathrm{Sp}(\widehat{\mathcal{C}})$ extends to a t-exact functor $\mathrm{Ind}(\mathcal{E}) \to \mathrm{Sp}(\widehat{\mathcal{C}})$ which preserves small colimits, and therefore determines a functor of Grothendieck prestable ∞-categories $F : \mathrm{Ind}(\mathcal{E}_{\geqslant 0}) \to \widehat{\mathcal{C}}$ which preserves small colimits and finite limits. Note that the heart of $\mathrm{Ind}(\mathcal{E}_{\geqslant 0})$ can be identified with $\mathrm{Ind}(\mathcal{E}^{\heartsuit})$, so that F induces an exact functor of abelian categories $F^{\heartsuit} : \mathrm{Ind}(\mathcal{E}^{\heartsuit}) \to \widehat{\mathcal{C}}^{\heartsuit}$. Let $\mathcal{A}_0 \subseteq \mathrm{Ind}(\mathcal{E}^{\heartsuit})$ be the full subcategory spanned by those objects which are annihilated by $F^{\heartsuit}$, and let $\mathcal{A}_0^+ \subseteq \mathrm{Ind}(\mathcal{E}_{\geqslant 0})$ be the smallest localizing subcategory of $\mathrm{Ind}(\mathcal{E}_{\geqslant 0})$ which contains $\mathcal{A}_0$ (Proposition C.5.2.8). We define $\check{\mathcal{C}}$ to be the quotient $\mathrm{Ind}(\mathcal{E}_{\geqslant 0})/\mathcal{A}_0^+$. It follows from Propositions C.5.5.5 and C.5.5.7 that the $\check{\mathcal{C}}$ is an anticomplete Grothendieck prestable ∞-category. The functor F annihilates $\mathcal{A}_0$ and therefore also annihilates $\mathcal{A}_0^+$ (Example C.2.3.7). It follows from Proposition C.2.3.10 and Remark C.2.3.11 that F factors as a composition

$$\mathrm{Ind}(\mathcal{E}_{\geqslant 0}) \to \check{\mathcal{C}} \xrightarrow{\overline{F}} \widehat{\mathcal{C}},$$

where $\overline{F}$ preserves small colimits and finite limits. Since $\check{\mathcal{C}}$ is anticomplete, the functor $\overline{F}$ factors as a composition $\mathcal{C}' \xrightarrow{\lambda} \mathcal{C} \to \widehat{\mathcal{C}}$ where λ preserves small colimits and has bounded amplitude. Applying Lemma C.5.5.8, we see that λ is left exact.. We will complete the proof by showing that λ induces an equivalence of completions. To prove this, it will suffice to show that the functor $\overline{F}$ induces an equivalence $\tau_{\leqslant n}\check{\mathcal{C}} \to \tau_{\leqslant n}\widehat{\mathcal{C}}$, for each $n \geqslant 0$.

Let $G : \widehat{\mathcal{C}} \to \mathrm{Ind}(\mathcal{E}_{\geqslant 0})$ be a right adjoint to F. By construction, G factors through $\mathcal{C}' = \mathrm{Ind}(\mathcal{E}_{\geqslant 0})/\mathcal{A}_0^+$ (which we regard as a full subcategory of $\mathrm{Ind}(\mathcal{E}_{\geqslant 0})$. Moreover, when regarded as a functor from $\widehat{\mathcal{C}}$ to $\check{\mathcal{C}}$, the functor G is right adjoint to $\overline{F}$. By construction, the intersection $\mathcal{E}_{\geqslant 0} \cap \tau_{\leqslant n}\widehat{\mathcal{C}}$ contains $\mathcal{K}_n$, and is therefore dense in $\tau_{\leqslant n}\widehat{\mathcal{C}}$ (Remark 20.4.1.10). It follows that the functor G is fully faithful when restricted to $\tau_{\leqslant n}\widehat{\mathcal{C}}$. To complete the proof, it will suffice to show that $\overline{F}|_{\tau_{\leqslant n}\check{\mathcal{C}}}$ is conservative. Let $\alpha : X \to Y$ be a morphism of n-truncated objects of $\check{\mathcal{C}}$ such that $\overline{F}(\alpha)$ is an equivalence. Let Z denote the cofiber of α, formed in the ∞-category $\mathrm{Ind}(\mathcal{E}_{\geqslant 0})$. Since $\overline{F}(\alpha)$ is an equivalence, we have $F(Z) \simeq 0$. The functor F is left exact, so we have $F(\pi_i Z) \simeq \pi_i F(Z) \simeq 0$ for all $i \geqslant 0$. In other words, the homotopy groups $\pi_i Z$ belong to $\mathcal{A}_0$. Since X and Y are n-truncated, the object Z is $(n+1)$-truncated, and can therefore be written as a *finite* extension of objects of the form $\Sigma^i(\pi_i Z)$. It follows that $Z \in \mathcal{A}_0^+$, so that α is an equivalence as desired. □

Corollary C.5.5.10. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. The following conditions are equivalent:*

- (a) *The Grothendieck prestable ∞-category $\mathcal{C}$ is anticomplete.*
- (b) *For every Grothendieck prestable ∞-category $\mathcal{D}$, the canonical map* $\mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}) \to \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}, \widehat{\mathcal{D}})$ *is an equivalence of ∞-categories.*

Proof. The implication $(a) \Rightarrow (b)$ follows from Lemma C.5.5.8. Conversely, suppose that (b) is satisfied. Choose a map $\lambda : \check{\mathcal{C}} \to \mathcal{C}$ satisfying the hypotheses of Proposition C.5.5.9. We then have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}, \check{\mathcal{C}}) & \longrightarrow & \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{LFun}^{\mathrm{lex}}(\check{\mathcal{C}}, \check{\mathcal{C}}) & \longrightarrow & \mathrm{LFun}^{\mathrm{lex}}(\check{\mathcal{C}}, \mathcal{C}), \end{array}$$

where the horizontal maps are given by postcomposition with λ and the vertical maps are given by precomposition with λ. Since $\mathcal{C}$ satisfies (b), the upper horizontal map is an equivalence of ∞-categories. We can therefore choose a functor $\mu \in \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}, \check{\mathcal{C}})$ for which $\lambda \circ \mu$ is homotopic to the identity $\mathrm{id}_{\mathcal{C}}$. It follows that $\lambda \circ \mu \circ \lambda$ is homotopic to λ. Since $\check{\mathcal{C}}$ is anticomplete, the implication $(a) \Rightarrow (b)$ shows that the lower horizontal map is also an equivalence of ∞-categories, so that $\mu \circ \lambda$ is also homotopic to the identity $\mathrm{id}_{\check{\mathcal{C}}}$. It follows that λ is an equivalence of ∞-categories, so that $\mathcal{C} \simeq \check{\mathcal{C}}$ is anticomplete. □

In the situation of Proposition C.5.5.9, the map $\lambda : \check{\mathcal{C}} \to \mathcal{C}$ depends functorially on $\mathcal{C}$. More precisely, we have the following:

Corollary C.5.5.11. *Let* $\mathsf{Groth}_\infty^{\mathrm{ch,lex}}$ *denote the full subcategory of* $\mathsf{Groth}_\infty^{\mathrm{lex}}$ *spanned by the anticomplete Grothendieck prestable* ∞*-categories. Then* $\mathsf{Groth}_\infty^{\mathrm{ch,lex}}$ *is a colocalization of* $\mathsf{Groth}_\infty^{\mathrm{lex}}$*: that is, the inclusion functor* $\mathsf{Groth}_\infty^{\mathrm{ch,lex}} \hookrightarrow \mathsf{Groth}_\infty^{\mathrm{lex}}$ *admits a right adjoint.*

Proof. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\lambda : \check{\mathcal{C}} \to \mathcal{C}$ be as in the statement of Proposition C.5.5.9. We claim that λ exhibits $\check{\mathcal{C}}$ as a $\mathsf{Groth}_\infty^{\mathrm{ch,lex}}$-colocalization of $\mathcal{C}$. To prove this, it will suffice to show that for any anticomplete Grothendieck prestable ∞-category $\mathcal{D}$, the upper horizontal map in the diagram

$$\begin{array}{ccc} \mathrm{LFun}^{\mathrm{lex}}(\mathcal{D}, \check{\mathcal{C}}) & \longrightarrow & \mathrm{LFun}^{\mathrm{lex}}(\mathcal{D}, \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{LFun}^{\mathrm{lex}}(\mathcal{D}, \widehat{\mathcal{C}}) & \longrightarrow & \mathrm{LFun}^{\mathrm{lex}}(\mathcal{D}, \widehat{\mathcal{C}}) \end{array}$$

is an equivalence of ∞-categories. This is clear: the vertical maps are equivalences because $\mathcal{D}$ is anticomplete, and the lower horizontal map is an equivalence because the functor $\widehat{\lambda} : \widehat{\mathcal{C}}' \to \widehat{\mathcal{C}}$ is an equivalence. □

We close by noting the following consequence of the proof of Proposition C.5.5.9:

Proposition C.5.5.12. *Let* $\mathcal{C}$ *be an anticomplete Grothendieck prestable* ∞*-category. Then:*

- (a) *There is an equivalence* $\mathcal{C} \simeq \mathcal{D} / \mathcal{D}_0$*, where* $\mathcal{D}$ *is an anticomplete Grothendieck prestable* ∞*-category satisfying the hypotheses of Proposition C.5.5.5 and* $\mathcal{D}_0$ *is the smallest localizing subcategory of* $\mathcal{D}$ *which contains some localizing subcategory of* $\mathcal{D}^{\heartsuit}$*.*
- (b) *There exists an essentially small generating subcategory* $\mathcal{C}_0 \subseteq \mathcal{C}$ *having that every object* $C \in \mathcal{C}_0$ *is* n*-truncated for some* $n \gg 0$ *(where* n *might depend on* C*).*

Proof. The proof of Proposition C.5.5.9 shows that there exists a left exact, colimit-preserving functor $\lambda : \mathcal{C}' \to \mathcal{C}$, where $\mathcal{C}'$ has the form described in (a) and λ induces an equivalence after passing to completions. Since $\mathcal{C}$ and $\mathcal{C}'$ are both anticomplete, it follows that λ is an equivalence (see Proposition C.5.9.2), which proves (a). To prove (b), we note that the essential image of the composite functor

$$\mathcal{D}^c \hookrightarrow \mathcal{D} \to \mathcal{D} / \mathcal{D}_0 \simeq \mathcal{C}$$

is a generating subcategory of $\mathcal{C}$, where $\mathcal{D}^c$ denotes the full subcategory of $\mathcal{D}$ spanned by the compact objects. □

Using the theory of anticomplete Grothendieck prestable ∞-categories, we can eliminate the separatedness restrictions from several of the results of §C.5.3.

Definition C.5.5.13. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $n \geqslant 0$ be an integer. We will say that $\mathcal{C}$ is *weakly n-complicial* if, for every truncated object $X \in \mathcal{C}$, there exists a morphism $f : \overline{X} \to X$ where $\overline{X}$ is n-truncated and the induced map $\pi_0 \overline{X} \to \pi_0 X$ is an epimorphism in the abelian category $\mathcal{C}^{\heartsuit}$.

Remark C.5.5.14. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. The hypothesis that $\mathcal{C}$ is weakly n-complicial depends only the full subcategory $\bigcup_{n \geqslant 0} \tau_{\leqslant n} \mathcal{C}$ of truncated objects of $\mathcal{C}$. Consequently, $\mathcal{C}$ is weakly n-complicial if and only if the completion $\widehat{\mathcal{C}}$ is weakly n-complicial. Note that if we were to replace "weakly n-complicial" by "n-complicial", then the analogous statement would be false: if $\mathcal{C}$ is a Grothendieck prestable ∞-category which is n-complicial and separated but not complete, then $\widehat{\mathcal{C}}$ cannot be n-complicial (see Proposition C.5.4.5).

Proposition C.5.5.15. *Let R be a connective $\mathbb{E}_1$-ring. The following conditions are equivalent:*

(1) *The $\mathbb{E}_1$-ring R is n-truncated.*

(2) *The Grothendieck prestable ∞-category $\mathrm{LMod}_R^{\mathrm{cn}}$ is n-complicial.*

(3) *The Grothendieck prestable ∞-category $\mathrm{LMod}_R^{\mathrm{cn}}$ is weakly n-complicial.*

Proof. For any R-module M, we can choose a free R-module P and a morphism $f : P \to M$ which induces a surjection $\pi_0 P \to \pi_0 M$. If R is n-truncated, then P is also n-truncated. This proves that (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is obvious. We will complete the proof by showing that (3) $\Rightarrow$ (1). Assume that $\mathrm{LMod}_R^{\mathrm{cn}}$ is weakly n-complicial. Then, for every integer $m \geqslant 0$, we can choose a map of connective R-modules $f : N \to \tau_{\leqslant m} R$ which is surjective on π_0, where N is n-truncated. Then the tautological map $R \to \tau_{\leqslant m} R$ factors through f, and therefore factors as a composition $R \to \tau_{\leqslant n} R \to \tau_{\leqslant m} R$. It follows that the homotopy groups $\pi_i R$ must vanish for $n < i \leqslant m$. Since m is arbitrary, we conclude that R is n-truncated. □

Proposition C.5.5.16. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Then:*

(i) *If $\mathcal{C}$ is n-complicial, then it is weakly n-complicial.*

(ii) *If $\mathcal{C}$ is anticomplete and weakly n-complicial, then $\mathcal{C}$ is n-complicial.*

Proof. Assertion (i) follows immediately from the definitions. Let us prove (ii). Assume that $\mathcal{C}$ is anticomplete and weakly n-complicial, and let X be an object of $\mathcal{C}$. Since $\mathcal{C}$ is anticomplete, Proposition C.5.5.12 implies that there exists a map $\bigoplus_{i \in I} X_i \to X$ which induces an epimorphism $\bigoplus_{i \in I} \pi_0 X_i \to \pi_0 X$ in the abelian category $\mathcal{C}^{\heartsuit}$, where each X_i is a truncated object of $\mathcal{C}$. Since $\mathcal{C}$ is weakly n-complicial, we can choose maps $\overline{X}_i \to X_i$ which induce epimorphisms $\pi_0 \overline{X}_i \to \pi_0 X_i$, where each $\overline{X}_i$ is n-truncated. Then $\overline{X} = \bigoplus_{i \in I} \overline{X}_i$ is an n-truncated object of $\mathcal{C}$ equipped with a map $\overline{X} \to X$ which is an epimorphism on π_0. □

Corollary C.5.5.17. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $n \geqslant 0$. The following conditions are equivalent:*

(*i*) *The Grothendieck prestable ∞-category $\mathcal{C}$ is weakly n-complicial.*

(*ii*) *There exists an n-complicial Grothendieck prestable ∞-category $\mathcal{C}'$ and a functor $\lambda \in \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}', \mathcal{C})$ which induces an equivalence of completions $\widehat{\lambda} : \widehat{\mathcal{C}}' \xrightarrow{\sim} \widehat{\mathcal{C}}$.*

Proof. Let $\lambda : \mathcal{C}' \to \mathcal{C}$ be as in (*ii*). If $\mathcal{C}'$ is n-complicial, then it is weakly n-complicial, so that $\mathcal{C}$ is also weakly n-complicial (Remark C.5.5.14). Conversely, assume that $\mathcal{C}$ is weakly n-complicial and let $\lambda : \mathcal{C}' \to \mathcal{C}$ be as in Proposition C.5.5.9. Then $\mathcal{C}'$ is weakly n-complicial (Remark C.5.5.14) and anticomplete, hence n-complicial (Proposition C.5.5.16). □

Proposition C.5.5.18. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $n \geqslant 0$. Then there exists an anticomplete, n-complicial Grothendieck prestable ∞-category $\mathcal{C}'$ and an n-equivalence $\lambda \in \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}', \mathcal{C})$.*

Proof. Let $\widehat{\mathcal{C}}$ be the completion of $\mathcal{C}$. Then $\widehat{\mathcal{C}}$ is separated. Applying Proposition C.5.3.8, we can choose an n-complicial Grothendieck prestable ∞-category $\mathcal{C}'$ and an n-equivalence $\mu \in \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}', \mathcal{C})$. Using Proposition C.5.5.9, we can choose an anticomplete Grothendieck prestable ∞-category $\mathcal{C}''$ and a functor $\nu \in \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}'', \mathcal{C}')$ which induces an equivalence $\widehat{\mathcal{C}}'' \to \widehat{\mathcal{C}}'$. Since $\mathcal{C}'$ is n-complicial, it is weakly n-complicial. Applying Remark C.5.5.14, we see that $\mathcal{C}''$ is also weakly n-complicial. Since $\mathcal{C}''$ is anticomplete, it is n-complicial (Proposition C.5.5.16). Replacing $\mathcal{C}'$ by $\mathcal{C}''$ (and μ by the composite functor $\mu \circ \nu$), we may reduce to the case where $\mathcal{C}'$ is anticomplete. It follows that μ factors as a composition $\mathcal{C}' \xrightarrow{\lambda} \mathcal{C} \to \widehat{\mathcal{C}}$. Since μ is an n-equivalence, the map λ is also an n-equivalence. □

Proposition C.5.5.19. *Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories and let $n \geqslant 0$ be an integer. Suppose that $\mathcal{C}$ is anticomplete and n-complicial. Then the restriction functor $\phi : \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\tau_{\leqslant n}\mathcal{C}, \tau_{\leqslant n}\mathcal{D})$ is a fully faithful embedding, whose essential image is spanned by those functors $f : \tau_{\leqslant n}\mathcal{C} \to \tau_{\leqslant n}\mathcal{D}$ which are left exact and commute with small colimits.*

Proof. Since $\mathcal{C}$ is anticomplete, we can replace $\mathcal{D}$ by its completion $\widehat{\mathcal{D}}$ and thereby reduce where $\mathcal{D}$ is complete. In particular, $\mathcal{D}$ is separated, so the desired result follows from Proposition C.5.3.9. □

Proposition C.5.5.20. *Let $n \geqslant 0$, let $\mathsf{Groth}_n^{\mathrm{lex}}$ be as in Definition C.5.4.1, and let $L : \mathsf{Groth}_\infty^{\mathrm{lex}} \to \mathsf{Groth}_n^{\mathrm{lex}}$ be the functor given by $L(\mathcal{C}) = \tau_{\leqslant n}\mathcal{C}$. Then L admits a fully faithful left adjoint $\mathsf{Groth}_n^{\mathrm{lex}} \hookrightarrow \mathsf{Groth}_\infty^{\mathrm{lex}}$, whose essential image is spanned by those Grothendieck prestable ∞-categories $\mathcal{C}$ which are anticomplete and n-complicial.*

Proof. Let $\mathcal{E} \subseteq \mathsf{Groth}_{\infty}^{\mathrm{lex}}$ denote the full subcategory spanned by those Grothendieck prestable ∞-categories which are anticomplete and n-complicial. We first claim that $\mathcal{E}$ is a colocalization of $\mathsf{Groth}_{\infty}^{\mathrm{lex}}$. In other words, for every object $\mathcal{C} \in \mathsf{Groth}_{\infty}^{\mathrm{lex}}$, we claim that there exists a morphism $\lambda : \mathcal{C}' \to \mathcal{C}$ in $\mathsf{Groth}_{\infty}^{\mathrm{lex}}$ where $\mathcal{C}'$ is anticomplete and n-complicial and, for every anticomplete n-complicial Grothendieck prestable ∞-category $\mathcal{D}$, composition with λ induces a homotopy equivalence

$$\mathrm{Map}_{\mathsf{Groth}_{\infty}^{\mathrm{lex}}}(\mathcal{D}, \mathcal{C}') \to \mathrm{Map}_{\mathsf{Groth}_{\infty}^{\mathrm{lex}}}(\mathcal{D}, \mathcal{C}).$$

By virtue of Proposition C.5.5.19, to guarantee the latter property it suffices to arrange that λ is an n-equivalence, which is always possible by virtue of Proposition C.5.5.18.

Note that the functor L carries n-equivalences in $\mathsf{Groth}_{\infty}^{\mathrm{lex}}$ to equivalences in $\mathsf{Groth}_{n}^{\mathrm{lex}}$, and therefore factors as a composition

$$\mathsf{Groth}_{\infty}^{\mathrm{lex}} \xrightarrow{L'} \mathcal{E} \xrightarrow{L''} \mathsf{Groth}_{n}^{\mathrm{lex}}$$

where L' is right adjoint to the inclusion $\mathcal{E} \hookrightarrow \mathsf{Groth}_{\infty}^{\mathrm{lex}}$, and L'' is the restriction $L|_{\mathcal{E}}$. To complete the proof, it will suffice to show that the functor L'' is an equivalence of ∞-categories. The assertion that L'' is fully faithful follows immediately from Proposition C.5.5.19, and essential surjectivity follows from Proposition C.5.5.18. □

Remark C.5.5.21. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $n \geqslant 0$. The proof of Proposition C.5.5.20 shows that the functor $\lambda : \mathcal{C}' \to \mathcal{C}$ appearing in Proposition C.5.5.18 is determined uniquely (up to equivalence) by $\mathcal{C}$.

C.5.6 Digression: Injective Objects of Grothendieck Abelian Categories

Let $\mathcal{A}$ be an abelian category. Recall that an object $Q \in \mathcal{A}$ is said to be *injective* if the functor $X \mapsto \mathrm{Hom}_{\mathcal{A}}(X, Q)$ is exact. In this section, we review some standard facts about injective objects in Grothendieck abelian categories which will be needed elsewhere in this book. For a more detailed exposition, we refer the reader to [74]. We begin by recalling a theorem of Grothendieck (for a proof, see [87] or Corollary HA.1.3.5.7):

Proposition C.5.6.1 (Grothendieck)**.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Then $\mathcal{A}$ has enough injectives. In other words, for every object $X \in \mathcal{A}$, there exists a monomorphism $X \hookrightarrow Q$, where Q is injective.*

In the situation of Proposition C.5.6.1, the injective object Q is not uniquely determined. However, there is a choice of Q which is, in some sense, as small as possible: the *injective hull* of M.

Notation C.5.6.2. Let $\mathcal{A}$ be a Grothendieck abelian category containing an object X. We let $\mathrm{Sub}(X)$ denote the partially ordered set of isomorphism classes of subobjects of X. In what follows, we will abuse notation by identifying elements of $\mathrm{Sub}(X)$ with monomorphisms $i : Y \hookrightarrow X$ in the category $\mathcal{A}$ (or simply with the domains of those elements). We will also write $Y \subseteq X$ to indicate that Y is an element of $\mathrm{Sub}(X)$ (or that Y is the domain of a monomorphism $Y \hookrightarrow X$, which we will often not specify explicitly).

Definition C.5.6.3. Let $\mathcal{A}$ be a Grothendieck abelian category, let X be an object of $\mathcal{A}$, and let $X_0 \subseteq X$ be a subobject of X. We say that X is an *essential extension* of X_0 if, for every nonzero subobject $X' \subseteq X$, the intersection $X' \times_X X_0 \in \mathrm{Sub}(X')$ is nonzero.

Lemma C.5.6.4. *Let $\mathcal{A}$ be a Grothendieck abelian category, let Q be an object of $\mathcal{A}$, and let $X \subseteq Q$ be a subobject of Q. Then:*

(a) *Let $S \subseteq \mathrm{Sub}(Q)$ be the collection of subobjects of Q which contain X and are essential extensions of X. Then there exists a maximal element $E \in S$.*

(b) *If Q is injective, then E is also injective.*

Proof. To prove (a), it will suffice to show that S satisfies the hypotheses of Zorn's lemma. It is clear that S is nonempty (since $X \in S$). If $\{Y_\alpha\}$ is a filtered subset of S, then $Y = \varinjlim_\alpha Y_\alpha$ can be identified with a subobject of Q containing X. Any subobject $Y' \subseteq Y$ can be identified with the colimit $\varinjlim_\alpha (Y' \times_Y Y_\alpha)$. If Y' is nonzero, then some intersection $Y' \times_Y Y_\alpha$ must also be nonzero. The assumption $Y_\alpha \in S$ then implies that $Y' \times_Y X \simeq (Y' \times_Y Y_\alpha) \times_{Y_\alpha} X$ is nonzero. It follows that Y is an essential extension of X and is therefore an upper bound for $\{Y_\alpha\}$ in the partially ordered set S.

We now prove (b). Assume that Q is injective; we wish to show that E is injective. Let M be an object of $\mathcal{A}$, let M_0 be a subobject of M, and let $f_0 : M_0 \to E$ be a morphism; we wish to show that f_0 can be extended to a morphism $f : M \to E$. To prove this, we let T be the collection of all pairs (M_1, f_1), where M_1 is a subobject of M containing M_0 and $f_1 : M_1 \to E$ is an extension of f_0. The set T satisfies the hypotheses of Zorn's Lemma, and therefore contains a maximal element (M_1, f_1). Using our assumption that Q is injective, we can extend f_1 to a map $g : M \to Q$. If g factors through E, then we can complete the proof by setting $f = g$. Otherwise, g induces a map $\overline{g} : E \amalg_{M_1} M \to Q$ whose image is strictly larger than E. The maximality of E then implies that $\mathrm{im}(\overline{g})$ cannot be an essential extension of X: that is, there exists a subobject $K \subseteq \mathrm{im}(\overline{g})$ such that $K \times_Q X \simeq 0$. We then must also have $K \times_Q E \simeq 0$ (since E is an essential extension of X), so that we can identify $K \oplus E$ with a subobject of $\mathrm{im}(\overline{g})$. It follows that $M_2 = M \times_{\mathrm{im}(g)} (K \oplus E)$ is strictly larger that M_1. Let $f_2 : M_2 \to E$ be the map given by composing the projections

$$M_2 = M \times_{\mathrm{im}(g)} (K \oplus E) \to K \oplus E \qquad K \oplus E \to E.$$

Then $(M_2, f_2) \in T$ is a proper extension of (M_1, f_1), contrary to our maximality assumption. □

Definition C.5.6.5. Let $\mathcal{A}$ be a Grothendieck abelian category and let $f : X \to Q$ be a morphism in $\mathcal{A}$. We will say that f *exhibits Q as an injective hull of X* if Q is injective, f is a monomorphism, and f exhibits Q as an essential extension of X.

In any Grothendieck abelian category, injective hulls exist and are well-defined up to isomorphism:

Proposition C.5.6.6. *Let $\mathcal{A}$ be a Grothendieck abelian category and let X be an object of $\mathcal{A}$. Then:*

(1) *There exists a morphism $f : X \hookrightarrow Q$ which exhibits Q as an injective hull of X.*

(2) *Let $f : X \hookrightarrow Q$ be a monomorphism, where Q is injective. Then, for every essential extension $g : X \hookrightarrow Y$, there exists a monomorphism $h : Y \hookrightarrow Q$ extending f. In other words, Q contains every essential extension of X.*

(3) *Let $f : X \hookrightarrow Q$ be a monomorphism, where Q is injective. If $g : X \hookrightarrow Q'$ exhibits Q' as an injective hull of X, then f factors as a composition $X \xrightarrow{g} Q' \hookrightarrow Q' \oplus Q'' \simeq Q$ for some injective object $Q'' \in \mathcal{A}$. In other words, Q is isomorphic (in the category $\mathcal{A}_{X/}$) to the direct sum of Q' with an auxiliary (injective) object.*

(4) *Let $f : X \hookrightarrow Q$ and $g : X \hookrightarrow Q'$ be morphisms which exhibit Q and Q' as injective hulls of X. Then there exists an isomorphism $h : Q' \to Q$ such that $f = h \circ g$.*

Warning C.5.6.7. In part (4) of Proposition C.5.6.6, the isomorphism $h : Q' \to Q$ need not be unique. In other words, injective hulls are unique up to isomorphism but *not* up to unique (or canonical) isomorphism.

Proof of Proposition C.5.6.6. Assertion (1) follows from Lemma C.5.6.4. To prove (2), suppose we are given morphisms $f : X \to Q$ and $g : X \to Y$. If Q is injective and g is a monomorphism, then we can write $f = h \circ g$ for some morphism $h : Y \to Q$. Then $X \times_Y \ker(h) \simeq \ker(f) \simeq 0$. If Y is an essential extension of X, then it follows that h is a monomorphism.

We now prove (3). Suppose we are given a monomorphism $f : X \hookrightarrow Q$ and a morphism $g : X \hookrightarrow Q'$, where Q is injective and g exhibits Q' as an injective hull of X. It follows from (2) that there exists a monomorphism $h : Q' \hookrightarrow Q$ such that $f = h \circ g$. Using the injectivity of Q', we see that the monomorphism h splits: that is, we can write Q as a direct sum $Q' \oplus Q''$. Since Q'' is a retract of Q, it is also an injective object of $\mathcal{C}$.

Assertion (4) is an immediate consequence of (3). □

Proposition C.5.6.8 (Krause [124])**.** *Let $\mathcal{A}$ be a Grothendieck abelian category and let Q_* be a chain complex of injective objects of $\mathcal{A}$. Then Q_* splits as a direct sum $Q'_* \oplus Q''_*$, where:*

- (a) *The complex Q'_* is homotopically minimal: that is, for every integer n, the object Q'_n is an injective hull of $\ker(d'_n : Q'_n \to Q'_{n-1})$; here d'_* denote the differential on the chain complex Q'_*.*
- (b) *The chain complex Q''_* is contractible: that is, there is a chain homotopy $h : Q''_* \to Q''_{*+1}$ from 0 to $\mathrm{id}_{Q''_*}$.*

Proof. For each integer n, let $d_n : Q_n \to Q_{n-1}$ denote the differential on Q_*. Choose a subobject $E_n \subseteq Q_n$ which is maximal among essential extensions of $\ker(d_n)$. Since Q_n is injective, Lemma C.5.6.4 implies that E_n is also injective. Consequently, the inclusion $E_n \hookrightarrow Q_n$ is a split monomorphism; we can therefore choose another subobject $A_n \subseteq Q_n$ such that $Q_n \simeq A_n \oplus E_n$.

The differential

$$d_{n+1} : Q_{n+1} \to \ker(d_n) \subseteq E_n \subseteq Q_n$$

is injective when restricted to A_{n+1}, and therefore induces an isomorphism $\phi_{n+1} : A_{n+1} \to B_n$ for some subobject $B_n \subseteq E_n$. Since A_{n+1} is a direct summand of Q_{n+1}, it is an injective object of $\mathcal{A}$. It follows that B_n is also injective, so the inclusion $B_n \hookrightarrow E_n$ splits. We can therefore choose another subobject $Q'_n \subseteq E_n$ such that $E_n \simeq Q'_n \oplus E_n$.

Let us write the composition

$$Q'_n \hookrightarrow Q_n \xrightarrow{d_n} \ker(d_{n-1}) \subseteq E_{n-1} \simeq Q'_{n-1} \oplus B_{n-1}$$

as (d'_n, ψ_n) for some pair of maps

$$d'_n : Q'_n \to Q'_{n-1} \qquad \psi_n : Q'_n \to B_{n-1}.$$

Set $Q''_n = A_n \oplus B_n$, and let $d''_n : Q''_n \to Q''_{n-1}$ denote the map given by the composition

$$Q''_n = A_n \oplus B_n \to A_n \xrightarrow{\phi_n} B_{n-1} \to A_{n-1} \oplus B_{n-1} = Q''_{n-1}.$$

Then (Q'_*, d'_*) and (Q''_*, d''_*) are chain complexes satisfying conditions (a) and (b), and the maps

$$Q'_n \oplus Q''_n \simeq Q'_n \oplus A_n \oplus B_n \xrightarrow{\mathrm{id} - \rho_n} Q'_n \oplus A_n \oplus B_n \simeq Q_n$$

determine an isomorphism of chain complexes $(Q_*, d_*) \simeq (Q'_*, d'_*) \oplus (Q''_*, d''_*)$, where ρ_n is given by the composition

$$Q'_n \oplus A_n \oplus B_n \to Q'_n \xrightarrow{\psi_n} B_{n-1} \xrightarrow{\phi_n^{-1}} A_n \to Q'_n \oplus A_n \oplus B_n.$$

□

Definition C.5.6.9. Let $\mathcal{A}$ be a Grothendieck abelian category and let X be an object of $\mathcal{A}$.

(a) We say that X is *indecomposable* if it is nonzero and cannot be written as a direct sum $X_0 \oplus X_1$, where X_0 and X_1 are nonzero subobjects of X.

(b) We say that X is *coirreducible* if it is nonzero and, for every pair of nonzero subobjects $X_0, X_1 \subseteq X$, the intersection $X_0 \times_X X_1$ is also nonzero.

Proposition C.5.6.10. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then:*

(a) *Every coirreducible object $X \in \mathcal{A}$ is indecomposable.*

(b) *Let Q be an injective object of $\mathcal{A}$. If Q is indecomposable, then Q is an injective hull of each nonzero subobject $X \subseteq Q$.*

(c) *Every indecomposable injective object $Q \in \mathcal{A}$ is coirreducible.*

(d) *Let $f : X \hookrightarrow Y$ be a monomorphism in $\mathcal{A}$. If Y is coirreducible and X is nonzero, then X is coirreducible.*

(e) *Let $f : X \hookrightarrow Y$ be a monomorphism in $\mathcal{A}$ which exhibits Y as an essential extension of X. If X is coirreducible, then Y is coirreducible.*

(f) *Let $f : X \hookrightarrow Q$ be a morphism in $\mathcal{C}$ which exhibits Q as an injective hull of X. Then X is coirreducible if and only if Q is indecomposable.*

Proof. Assertion (a) follows immediately from the definitions. To prove (b), suppose that Q is an indecomposable injective object of $\mathcal{A}$ and let $X \subseteq Q$ be a nonzero subobject. It follows from Proposition C.5.6.6 that Q splits as a direct sum $Q' \oplus Q''$, where Q' is an injective hull of X. The indecomposability of Q implies that $Q'' \simeq 0$, so that Q is an injective hull of X.

Assertion (c) is a reformulation of (b). Assertion (d) follows immediately from the definitions. To prove (e), suppose that Y is an essential extension of X and that we are given nonzero subobjects $Y_0, Y_1 \subseteq Y$. Then the intersections $X_0 = Y_0 \times_Y X$ and $X_1 = Y_1 \times_Y X$ are also nonzero. If X is coirreducible, then $X_0 \times_X X_1$ is nonzero, so that $Y_0 \times_Y Y_1$ is also nonzero.

We now prove (f). Let $f : X \hookrightarrow Q$ exhibit Q as an injective hull of X. It follows from (d) and (e) that X is coirreducible if and only if Q is coirreducible. Since Q is injective, it follows from (a) and (c) that Q is coirreducible if and only if it is indecomposable. □

Corollary C.5.6.11. *Let R be a Noetherian commutative ring and let $Q \in \mathrm{Mod}_R^{\heartsuit}$. The following conditions are equivalent:*

(1) *The R-module Q is indecomposable and injective.*

(2) *The R-module Q is an injective hull some residue field κ of R.*

Proof. The implication (2) $\Rightarrow$ (1) follows immediately from Proposition C.5.6.10, since every residue field κ of R is coirreducible as an R-module. For the converse, suppose that (1) is satisfied. Since Q is a nonzero module over a Noetherian ring R, it has an associated prime: that is, there exists a monomorphism $\iota : R/\mathfrak{p} \hookrightarrow Q$ for some prime ideal $\mathfrak{p}$ of R. Let κ be the fraction field of $R/\mathfrak{p}$. The injectivity of Q guarantees that ι extends to a map $\bar{\iota} : \kappa \to Q$. Since $\ker(\bar{\iota}) \cap (R/\mathfrak{p}) \simeq 0$, we have $\ker(\bar{\iota}) \simeq 0$: that is, the map ι is injective. Assertion (b) of Proposition C.5.6.10 guarantees that $\bar{\iota}$ exhibits Q as an injective hull of κ. □

Recall that an abelian category $\mathcal{A}$ is said to be *locally Noetherian* if it is a Grothendieck abelian category and every object $X \in \mathcal{A}$ can be written as a union of its Noetherian subobjects (see Definition C.6.8.5).

Proposition C.5.6.12. *Let $\mathcal{A}$ be a locally Noetherian abelian category and let Q be an object of $\mathcal{A}$. The following conditions are equivalent:*

(a) *The object $Q \in \mathcal{A}$ is injective.*

(b) *For every Noetherian object $X \in \mathcal{A}$ and every subobject $X' \subseteq X$, the restriction map $\mathrm{Hom}_{\mathcal{A}}(X, Q) \to \mathrm{Hom}_{\mathcal{A}}(X', Q)$ is surjective.*

Proof. The implication $(a) \Rightarrow (b)$ is immediate. Conversely, suppose that (b) is satisfied. To show that Q is injective, it will suffice to show that for *every* object $X \in \mathcal{A}$ and every subobject $X' \subseteq X$, the restriction map $\mathrm{Hom}_{\mathcal{A}}(X, Q) \to \mathrm{Hom}_{\mathcal{A}}(X', Q)$ is surjective. To prove this, fix a map $f' : X' \to Q$, and let P be the partially ordered set of equivalence classes of pairs (X_0, f_0), where X_0 is a subobject of X containing X' and $f_0 : X_0 \to Q$ is a morphism satisfying $f_0|_{X'} = f'$. It is easy to see that P satisfies the hypotheses of Zorn's Lemma and therefore admits a maximal element (X_0, f_0). Replacing X' by X_0 and f' by f_0, we may reduce to the case where the morphism f' cannot be extended to any subobject of X properly containing X'. If Y is a Noetherian subobject of X and we set $Y' = X' \times_X Y$, then assumption (b) implies that $f'|_{Y'}$ can be extended to a map $Y \to Q$, so that f' can be extended to a map $Y \amalg_{Y'} X' \to Q$. The maximality of f' guarantees that $Y' \simeq Y$: that is, the Noetherian subobject Y is contained in X'. Allowing Y to vary, we deduce that X' contains every Noetherian subobject $Y \subseteq X$, and therefore coincides with X by virtue of our assumption that $\mathcal{A}$ is locally Noetherian. □

Corollary C.5.6.13. *Let $\mathcal{A}$ be a locally Noetherian abelian category. Then the collection of injective objects of $\mathcal{A}$ is closed under filtered colimits.*

Proof. Let $\{Q_\alpha\}$ be a filtered diagram of injective objects of $\mathcal{A}$ having colimit Q; we wish to show that Q is injective. By virtue of Proposition C.5.6.12, it will suffice to show that for every monomorphism $f : X' \hookrightarrow X$ where X is Noetherian, the induced map $\rho : \mathrm{Hom}_{\mathcal{A}}(X, Q) \to \mathrm{Hom}_{\mathcal{A}}(X', Q)$ is surjective. Since X and X' are compact objects of $\mathcal{A}$ (see Proposition C.6.8.7), we can write ρ as a filtered colimit of maps $\rho_\alpha : \mathrm{Hom}_{\mathcal{A}}(X, Q_\alpha) \to \mathrm{Hom}_{\mathcal{A}}(X', Q_\alpha)$, each of which is surjective by virtue of our assumption that Q_α is injective. □

Lemma C.5.6.14. *Let $\mathcal{A}$ be a locally Noetherian abelian category and let Q be a nonzero injective object of $\mathcal{A}$. Then there exists subobject $X \subseteq Q$ which is Noetherian and coirreducible.*

Proof. Since Q is nonzero and $\mathcal{A}$ is locally Noetherian, there exists a Noetherian object $Y \in \mathcal{A}$ and a nonzero map $f : Y \to Q$. Let $S \subseteq \mathrm{Sub}(Y)$ be the collection of all subobjects $Y' \subseteq Y$ for which there exists a nonzero map $g : Y \to Q$ which annihilates Y'. The existence of f shows that S is nonempty. Since Y is Noetherian, we can choose a maximal element $Y' \in S$. Let $g : Y \to Q$ be a nonzero map which annihilates Y', and set $X = \mathrm{im}(g)$. Then X is nonzero and Noetherian (Proposition C.6.8.2). We will complete the proof by showing that X is coirreducible. Suppose otherwise: then there exist nonzero subobjects $X_0, X_1 \subseteq X$ such that $X_0 \times_X X_1$ is nonzero. It follows that we can identify $X_0 \oplus X_1$ with a subobject of Q. Let $q : X_0 \oplus X_1 \to X_0$ be the projection map onto the first factor. It follows from the injectivity of Q that we can extend q to a map $\overline{q} : Q \to Q$. The map $(\overline{q} \circ g) : Y \to Q$ is nonzero (since its image contains X_0), and the kernel $\ker(\overline{q} \circ g)$ is strictly larger than Y' (since it contains the fiber product $Y \times_X X_1$). This contradicts the maximality of Y'. □

Lemma C.5.6.15. *Let $\mathcal{A}$ be a locally Noetherian abelian category and let Q be a nonzero injective object of $\mathcal{A}$. Then Q splits as a direct sum $Q' \oplus Q''$, where Q' is indecomposable.*

Proof. Applying Lemma C.5.6.14, we can choose a subobject $X \subseteq Q$ which is coirreducible. Using Proposition C.5.6.6, we can decompose Q as a direct sum $Q' \oplus Q''$, where Q' is an injective hull of X. It follows from Proposition C.5.6.10 that Q' is indecomposable. □

Proposition C.5.6.16. *Let $\mathcal{A}$ be a locally Noetherian abelian category and let Q be an object of $\mathcal{A}$. The following conditions are equivalent:*

(1) *The object Q is injective.*

(2) *There exists an isomorphism $Q \simeq \bigoplus_\alpha Q_\alpha$, where each Q_α is an injective object of $\mathcal{A}$.*

(3) *There exists an isomorphism $Q \simeq \bigoplus_\alpha Q_\alpha$, where each Q_α is an indecomposable injective object of $\mathcal{A}$.*

Proof. The implication (3) ⇒ (2) is obvious and the implication (2) ⇒ (1) follows from Corollary C.5.6.13. We will complete the proof by showing that (1) implies (3). Suppose that Q is injective. Let S denote the collection of all subsets $\mathcal{J} \subseteq \mathrm{Sub}(Q)$ with the following properties:

(*i*) Each element of $\mathcal{J}$ is an indecomposable injective object of $\mathcal{A}$.

(*ii*) The canonical map $\bigoplus_{Q_0 \in \mathcal{J}} Q_0 \to Q$ is a monomorphism.

Note that S is nonempty (since it contains the empty set). Moreover, since the collection of monomorphisms in $\mathcal{A}$ is closed under filtered colimits, the set S is closed under directed unions. Applying Zorn's Lemma, we deduce that S contains a maximal element $\mathcal{J}$. Set $Q' = \bigoplus_{Q_0 \in \mathcal{J}} Q_0$, so that the canonical map $Q' \to Q$ is a monomorphism. Since Q is injective, it follows that Q splits as a direct sum $Q' \oplus Q''$ for some auxiliary subobject $Q'' \subseteq Q$. If Q'' is nonzero, then Lemma C.5.6.15 implies that there exists an indecomposable direct summand $Q''_0 \subseteq Q''$. In this case, we would have $\mathcal{J} \cup \{Q''_0\} \in S$, contradicting the maximality of $\mathcal{J}$. It follows that $Q'' \simeq 0$, so that $Q \simeq \bigoplus_{Q_0 \in \mathcal{J}} Q_0$ satisfies condition (3). □

C.5.7 Injective Objects of Stable ∞-Categories

Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathrm{Sp}(\mathcal{C})$ denote the ∞-category of spectrum objects of $\mathcal{C}$. We will regard $\mathrm{Sp}(\mathcal{C})$ as equipped with the t-structure $(\mathrm{Sp}(\mathcal{C})_{\geqslant 0}, \mathrm{Sp}(\mathcal{C})_{\leqslant 0})$ of Remark C.1.2.10. In this section, we will introduce the notion of *injective object* of the ∞-category $\mathrm{Sp}(\mathcal{C})$ (Definition C.5.7.2) and describe its relationship with the theory of injective objects of the abelian category $\mathcal{C}^{\heartsuit}$.

Proposition C.5.7.1. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let Q be an object of $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. The following conditions are equivalent:*

(1) *The object Q is projective when viewed as an object of $\mathrm{Sp}(\mathcal{C})^{\mathrm{op}}_{\leqslant 0}$: in other words, for every cosimplicial object $X^\bullet$ of $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$, the map*

$$|\operatorname{Map}_{\mathrm{Sp}(\mathcal{C})}(X^\bullet, Q)| \to \operatorname{Map}_{\mathrm{Sp}(\mathcal{C})}(\operatorname{Tot} X^\bullet, Q)$$

is a homotopy equivalence.

(2) *For every $X \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$, the abelian group $\operatorname{Ext}^1_{\mathcal{C}}(X, Q)$ vanishes.*

(3) *For every object $X \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ and every integer $n > 0$, the abelian group $\operatorname{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X, Q)$ vanishes.*

(4) *For every object $X \in \mathcal{C}^{\heartsuit}$ and every integer $i > 0$, the abelian group $\operatorname{Ext}^i_{\mathrm{Sp}(\mathcal{C})}(X, Q)$ vanishes.*

(5) *Given a fiber sequence*

$$X' \to X \to X'',$$

where $X', X, X'' \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$, the induced map $\operatorname{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(X', Q) \to \operatorname{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(X, Q)$ is surjective.

Proof. Apply Proposition HA.?? to the ∞-category $\mathrm{Sp}(\mathcal{C})^{\mathrm{op}}$. □

Definition C.5.7.2. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $Q \in \mathrm{Sp}(\mathcal{C})$. We will say that Q is *injective* if it belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ and satisfies the equivalent conditions of Proposition C.5.7.1.

Proposition C.5.7.3. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let Q be an object of $\mathrm{Sp}(\mathcal{C})$. The following conditions are equivalent:*

- (a) *The object Q is injective.*
- (b) *For every object $X \in \mathrm{Sp}(\mathcal{C})$, the canonical map $\mathrm{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(X, Q) \to \mathrm{Hom}_{\mathcal{C}^\heartsuit}(\pi_0 X, \pi_0 Q)$ is an isomorphism.*

Moreover, if these conditions are satisfied, then $\pi_0 Q$ is an injective object of the abelian category $\mathcal{C}^\heartsuit$.

Proof. Suppose first that (a) is satisfied, and let X be an object of $\mathrm{Sp}(\mathcal{C})$; we wish to show that the canonical map $\rho : \mathrm{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(X, Q) \to \mathrm{Hom}_{\mathcal{C}^\heartsuit}(\pi_0 X, \pi_0 Q)$ is an isomorphism. Using the fact that Q belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$, we can identify $\mathrm{Hom}_{\mathcal{C}^\heartsuit}(\pi_0 X, \pi_0 Q)$ with $\mathrm{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(\tau_{\geqslant 0} X, Q)$. Using this identification, we can fit ρ into a short exact sequence of abelian groups

$$\mathrm{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(\tau_{\leqslant -1} X, Q) \to \mathrm{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(X, Q) \xrightarrow{\rho} \mathrm{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(\tau_{\geqslant 0} X, Q) \to \mathrm{Ext}^1_{\mathrm{Sp}(\mathcal{C})}(\tau_{\leqslant -1} X, Q).$$

If Q is injective, then the first and last terms of this short exact sequence vanish (Proposition C.5.7.1), so that ρ is an isomorphism.

Now suppose that (b) is satisfied. Applying (b) in the case $X = \tau_{\geqslant 1} Q$, we deduce that the group $\mathrm{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(\tau_{\geqslant 1} Q, Q)$ vanishes. In particular, the canonical map $\tau_{\geqslant 1} Q \to Q$ is nullhomotopic, so that Q belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. Applying (b) in the case $X = \Sigma^{-n} C$ for $C \in \mathcal{C}^\heartsuit$, we deduce that $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(C, Q)$ vanishes for $n \neq 0$. Using criterion (3) of Proposition C.5.7.1, we deduce that Q is injective.

Now suppose that Q satisfies (a) and (b). For any short exact sequence $0 \to X' \to X \to X'' \to 0$ in the abelian category $\mathcal{C}^\heartsuit$, we have a short exact sequence of abelian groups

$$\mathrm{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(X, Q) \xrightarrow{\theta} \mathrm{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(X', \pi_0 Q) \to \mathrm{Ext}^1_{\mathrm{Sp}(\mathcal{C})}(X'', Q)$$

where the third term vanishes by virtue of the injectivity of Q. It follows that the map θ is surjective, so that condition (b) shows that $\mathrm{Hom}_{\mathcal{C}^\heartsuit}(X, \pi_0 Q) \to \mathrm{Hom}_{\mathcal{C}^\heartsuit}(X', \pi_0 Q)$ is surjective. It follows that $\pi_0 Q$ is an injective object of the abelian category $\mathcal{C}^\heartsuit$. □

If R is a connective $\mathbb{E}_1$-ring, then Corollary HA.7.2.2.19 asserts that the homotopy category of the ∞-category of projective left R-modules is equivalent to the ordinary category of projective modules over $\pi_0 R$. Our next goal is to prove an analogous result for injective modules:

Theorem C.5.7.4. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathrm{Sp}(\mathcal{C})^{\mathrm{inj}}$ denote the full subcategory of $\mathrm{Sp}(\mathcal{C})$ spanned by the injective objects. Let $\mathcal{A} = \mathcal{C}^{\heartsuit}$ and let $\mathcal{A}^{\mathrm{inj}}$ denote the full subcategory of $\mathcal{A}$ spanned by the injective objects. Then the construction Then the construction $Q \mapsto \pi_0 Q$ determines an equivalence of categories $\theta : \mathrm{hSp}(\mathcal{C})^{\mathrm{inj}} \to \mathcal{A}^{\mathrm{inj}}$.*

Warning C.5.7.5. In the situation of Theorem C.5.7.4, the domain of the equivalence $\theta : \mathrm{hSp}(\mathcal{C})^{\mathrm{inj}} \simeq \mathcal{A}^{\mathrm{inj}}$ is the *homotopy* category of $\mathrm{Sp}(\mathcal{C})^{\mathrm{inj}}$. In general, the mapping spaces in $\mathrm{Sp}(\mathcal{C})^{\mathrm{inj}}$ are not discrete (and contain information which is not seen by the abelian category $\mathcal{A} = \mathcal{C}^{\heartsuit}$).

The proof of Theorem C.5.7.4 will require some preliminaries.

Lemma C.5.7.6. *Let α be an ordinal and let $(\alpha) = \{\beta : \beta < \alpha\}$ be the collection of ordinals smaller than α. Let $F : \mathrm{N}(\alpha) \to \mathcal{S}$ be a functor with the following property: for every ordinal $\beta < \alpha$, the map $F(\beta) \to \varprojlim_{\gamma < \beta} F(\gamma)$ has connected homotopy fibers. Then $\varprojlim_{\beta < \alpha} F(\beta)$ is connected.*

Proof. Using Proposition HTT.4.2.4.4 , we may assume without loss of generality that F arises from a diagram $X : (\alpha) \to \mathcal{S}\mathrm{et}_{\Delta}$ which is fibrant with respect to the injective model structure. Then $\varprojlim_{\beta < \alpha} F(\beta)$ is represented by the Kan complex $\varprojlim_{\beta < \alpha} X(\beta)$ (Theorem HTT.4.2.4.1). The assumption that X is fibrant is equivalent to the requirement that each of the maps $\theta_\beta : X(\beta) \to \varprojlim_{\gamma < \beta} X(\gamma)$ is a Kan fibration, and we are given that each of the maps θ_β has connected homotopy fibers. Let x and y be vertices of $\varprojlim_{\beta < \alpha} X(\beta)$, and let x_β and y_β denote the images of x and y in $X(\beta)$ for $\beta < \alpha$. To show that x and y belong to the same path component of $\varprojlim_{\beta < \alpha} X(\beta)$, we must construct a compatible system of edges $\{e_\beta : x_\alpha \to y_\beta\}$ in $X(\beta)$. The construction proceeds by induction on β. Assume that the edges $\{e_\gamma\}_{\gamma < \beta}$ have been constructed, thereby determining an edge $e' : x' \to y'$ in $\varprojlim_{\gamma < \beta} X(\gamma)$. Since θ_β is a Kan fibration, we can choose an edge $\overline{e}' : x_\beta \to \overline{y}'$ in $X(\beta)$ lying over e'. Since the fiber $X(\beta)_{y'}$ is path connected, we can choose a path $\overline{e}'' : \overline{y}' \to y_\beta$ in $X(\beta)_{y'}$. Using the fact that θ_β is a Kan fibration again, we conclude that there exists a 2-simplex σ:

$$\begin{array}{ccccc} & & \overline{y}' & & \\ & \overset{\overline{e}'}{\nearrow} & & \overset{\overline{e}''}{\searrow} & \\ x_\beta & & \xrightarrow{e_\beta} & & y_\beta \end{array}$$

lying over the degenerate 2-simplex associated to e', which produces the desired edge e_β lying over e'. $\square$

Lemma C.5.7.7. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Then there exists a small collection of objects $\{X_i \in \mathcal{C}^{\heartsuit}\}_{i \in I}$ such that an object $Q \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ is injective if and only if the abelian groups $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X_i, Q)$ vanish for all $i \in I$ and $n > 0$.*

Proof. Choose a set of objects $\{Y_j\}_{j \in J}$ which generates the abelian category $\mathrm{h}\mathcal{C}^{\heartsuit}$ under small colimits, and let $\{X_i\}_{i \in I}$ be a collection of representatives for all isomorphism classes of quotients of the objects $\{Y_j\}_{j \in J}$. Since $\mathcal{C}^{\heartsuit}$ is a Grothendieck abelian category, the collection I is small (see the proof of Proposition HA.1.3.5.3). We claim that this collection of objects has the desired property.

Let Q be an object of $\mathcal{C}_{\leqslant 0}$ such that $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X_i, Q) \simeq 0$ for $i \in I$ and $n > 0$; we wish to prove that Q is injective. To prove this, consider an arbitrary object $Z \in \mathcal{C}^{\heartsuit}$ and integer $n > 0$; we will show that $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(Z, Q) \simeq 0$. To this end, we first construct a transfinite sequence of subobjects $Z_\alpha \subseteq Z$ as follows. Assume that α is an ordinal that the subobjects $\{Z_\beta \subseteq Z\}_{\beta < \alpha}$ have been constructed. If the induced map $\phi : \varinjlim Z_\beta \to Z$ is not an isomorphism, then there exists an index $j \in J$ and a map $Y_j \to Z$ which does not factor through ϕ. We define Z_α to be the image of the map $Y_j \oplus (\varinjlim_{\beta < \alpha} Z_\beta) \to Z$.

The proof of Proposition HA.1.3.5.3 shows that the collection of isomorphism classes of subobjects of Z is small, so this process must eventually stop: that is, we have $\varinjlim_{\beta<\alpha} Z_\beta \simeq Z$ for some ordinal α. Then $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(Z, Q) \simeq \pi_0 \varprojlim_{\beta < \alpha} \mathrm{Map}_{\mathcal{C}}(\Sigma^{-n} Z_\beta, Q)$. To prove that this group vanishes, it suffices (by Lemma C.5.7.6) to show that for each $\beta < \alpha$, the map

$$\theta : \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma^{-n} Z_\beta, Q) \to \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(\varinjlim_{\gamma<\beta} \Sigma^{-n} Z_\gamma, Q)$$

has connected homotopy fibers. By construction, the map $\varinjlim_{\gamma<\beta} Z_\gamma \to Z_\beta$ is a monomorphism in $\mathcal{C}^{\heartsuit}$ whose cokernel is given by X_i for some $i \in I$. We then have a fiber sequence

$$\Sigma^{n-1} X_i \to \varinjlim_{\gamma<\beta} \Sigma^{-n} Z_\gamma \to \Sigma^{-n} Z_\beta$$

so that θ is a pullback of the map $\theta' : * \to \mathrm{Map}_{\mathcal{C}}(\Sigma^{-n-1} X_i, Q)$. It therefore suffices to show that θ' has connected homotopy fibers: that is, that the homotopy group $\pi_1 \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma^{-n-1} X_i, Q) \simeq \mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X_i, Q)$ vanishes, which follows from our assumption on I. □

Proposition C.5.7.8. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let Q_0 be an injective object of the abelian category $\mathcal{C}^{\heartsuit}$. Then there exists a map $\phi : Q_0 \to Q$ in $\mathcal{C}_{\leqslant 0}$, where Q is an injective object of $\mathcal{C}$ and ϕ induces an equivalence $Q_0 \simeq \pi_0 Q$.*

Proof. Let $\{X_i\}_{i \in I}$ be as in Lemma C.5.7.7. Since the ∞-category $\mathrm{Sp}(\mathcal{C})$ is presentable, we can choose a regular cardinal κ such that each of the objects X_i is κ-compact. We will extend the object Q_0 to a transfinite sequence of objects $\{Q_\alpha \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}\}_{\alpha<\kappa}$ with the following properties:

(*a*) If $\lambda < \kappa$ is a nonzero limit ordinal, then $Q_\lambda \simeq \varinjlim_{\alpha<\lambda} Q_\lambda$.

(*b*) The map $Q_0 \to Q_\alpha$ induces an equivalence $Q_0 \to \pi_0 Q_\alpha$ for each $\alpha < \kappa$.

(c) Let $\alpha < \kappa$ and let $\eta \in \mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X_i, Q_\alpha)$ for some $i \in I$ and some $n > 0$. Then the image of η vanishes in $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X_i, Q_{\alpha+1})$.

Assuming that such a construction is possible, let $Q = \varinjlim_{\alpha<\kappa} Q_\alpha$. Then the natural map $Q_0 \to Q$ induces an equivalence $Q_0 \simeq \pi_0 Q$ by virtue of (b). We claim that Q is injective. To prove this, consider an arbitrary class $\eta \in \mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X_i, Q)$ for $i \in I$ and $n > 0$. Since X_i is κ-compact, η is the image of a class $\eta_0 \in \mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X_i, Q_\alpha)$ for some $\alpha < \kappa$. The image of η_0 in $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X_i, Q_{\alpha+1})$ vanishes by (c), so that $\eta = 0$ as desired.

It remains to construct the sequence Q_α. We proceed by induction on α, the case where α is a limit ordinal being prescribed by condition (a). Let us therefore suppose that Q_α has been defined. Let S be the set of all triples (n, i, η), where $\eta \in \mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X_i, Q_\alpha)$. Choose a well-ordering of the set S having order type β. For each $\gamma < \beta$, let $\eta_\gamma \in \mathrm{Ext}^{n_\gamma}_{\mathcal{C}}(X_{i_\gamma}, Q_\alpha)$ denote the corresponding class. We will construct a transfinite sequence of objects $\{P_\gamma\}_{\gamma \leqslant \beta+1}$ of $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ with the following properties:

(a′) We have $P_0 = Q_\alpha$. If $\lambda \leqslant \beta + 1$ is a limit ordinal, then $P_\lambda \simeq \varinjlim_{\gamma<\lambda} P_\gamma$.

(b′) For each $\gamma \leqslant \beta + 1$, the map $Q_0 \to P_\gamma$ induces an equivalence $Q_0 \simeq \pi_0 P_\gamma$.

(c′) For each $\gamma \leqslant \beta$, the image of η_γ in $\mathrm{Ext}^{n_\gamma}(X_{i_\gamma}, P_{\gamma+1})$ vanishes.

Assuming that this construction is possible, we can complete the proof by setting $Q_{\alpha+1} = P_{\beta+1}$.

The construction of the objects P_γ proceeds by induction on γ. When γ is a limit ordinal, the definition of P_γ is determined by (a′). Let us therefore assume that P_γ has been constructed. Let $n = n_\gamma$ and $i = i_\gamma$, and let η denote the image of η_γ in $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X_i, P_\gamma)$. Then η determines a map $\phi : \Sigma^{-n} X_i \to P_\gamma$. If $n > 1$, we define $Q_{\gamma+1}$ to be the cofiber of ϕ; it is then clear that $P_{\gamma+1}$ has the desired properties. Let us therefore assume that $n = 1$, so that ϕ induces a map $\psi : X_i \to \pi_{-1} P_\gamma$ in the abelian category $\mathcal{C}^\heartsuit$. Let K denote the fiber of ψ, so that the composite map

$$\Sigma^{-1} K \to \Sigma^{-1} X_i \to \pi_{-1} P_\gamma \to \tau_{\leqslant -1} P_\gamma$$

is nullhomotopic. It follows that the map $\Sigma^{-1} K \to \Sigma^{-1} X_i \xrightarrow{\phi} P_\gamma$ factors through some map $\xi : \Sigma^{-1} K \to \pi_0 P_\gamma \simeq Q_0$. The map ξ determines an extension

$$0 \to Q_0 \to E \to K \to 0$$

in the abelian category $\mathcal{C}^\heartsuit$. Since Q_0 is an injective object of $\mathcal{C}^\heartsuit$, this extension splits so that $\xi = 0$. Let X_i/K denote the cofiber of the map $K \to X_i$, so that ϕ factors as a composition

$$\Sigma^{-1} X_i \to \Sigma^{-1}(X_i/K) \xrightarrow{\phi'} P_\gamma.$$

We now define $P_{\gamma+1}$ to be the cofiber of ϕ'. Then $P_{\gamma+1}$ satisfies condition (c') by construction. To verify (b'), we observe that there is an exact sequence

$$0 \to \pi_0 Q_\gamma \to \pi_0 Q_{\gamma+1} \to X_i/K \xrightarrow{\psi'} \pi_{-1} Q_\gamma,$$

where ψ' is the map induced by ψ. Since K is the fiber of ψ, the map ψ' is injective so that $\pi_0 Q_{\gamma+1} \simeq \pi_0 Q_\gamma \simeq Q_0$ as desired. □

Proof of Theorem C.5.7.4. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and set $\mathcal{A} = \mathcal{C}^\heartsuit$. It follows from Proposition C.5.7.3 that the functor $\pi_0 : \mathrm{Sp}(\mathcal{C})^{\mathrm{inj}} \to \mathcal{A}^{\mathrm{inj}}$ is fully faithful, and from Proposition C.5.7.8 that it is essentially surjective. □

Example C.5.7.9 (Injective Hulls)**.** Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. We will say that a morphism $f : X \to Q$ in $\mathrm{Sp}(\mathcal{C})$ *exhibits Q as an injective hull of X* if it satisfies the following conditions:

(i) The object $Q \in \mathrm{Sp}(\mathcal{C})$ is injective.

(ii) The induced map $\pi_0(f) : \pi_0 X \to \pi_0 Q$ is a monomorphism in the abelian category $\mathcal{C}^\heartsuit$.

(iii) The map $\pi_0(f)$ exhibits $\pi_0 Q$ as an essential extension of $\pi_0 X$ (see Definition C.5.6.3).

Note that for any object $X \in \mathrm{Sp}(\mathcal{C})$, we can choose a morphism $f : X \to Q$ which exhibits Q as an injective hull of X. To prove this, we first observe that there exists a morphism $f_0 : \pi_0 X \to Q_0$ in the abelian category $\mathcal{C}^\heartsuit$ which exhibits Q_0 as an injective hull of $\pi_0 X$ (Proposition C.5.6.6). Using Proposition C.5.7.8, we can assume that $Q_0 = \pi_0 Q$ for some injective object $Q \in \mathrm{Sp}(\mathcal{C})$, and Proposition C.5.7.3 implies that we can lift f_0 to a morphism $f : X \to Q$ satisfying conditions (i), (ii), and (iii).

Note that if $f : X \to Q$ and $g : X \to Q'$ are morphisms in $\mathrm{Sp}(\mathcal{C})$ which exhibit Q and Q' are injective hulls of X, then Theorem C.5.6.6 guarantees that there exists an isomorphism $h_0 : \pi_0 Q' \simeq \pi_0 Q$ such that $(\pi_0 f) = h_0 \circ (\pi_0 g)$ in the abelian category $\mathcal{C}^\heartsuit$. Using Proposition C.5.7.3, we see that h_0 can be lifted to an equivalence $h : Q' \to Q$ such that $f \simeq h \circ g$. In other words, any two injective hulls of X are equivalent (though the equivalence is not uniquely determined).

Example C.5.7.10. [Brown-Comenetz Duality] Let Sp denote the ∞-category of spectra, endowed with the t-structure described in Proposition **??**. Then the heart $\mathrm{Sp}^\heartsuit$ is equivalent to the category of abelian groups. An abelian group A is injective if and only if it *divisible*: that is, if and only if the multiplication map $A \xrightarrow{n} A$ is surjective for every integer n. Theorem C.5.7.4 implies that for every injective abelian group A, there is an injective spectrum I_A with $\pi_0 I_A \simeq A$. Moreover, the spectrum I_A is determined uniquely (up to a connected space of choices). In particular, there exists an injective spectrum I with $\pi_0 I \simeq \mathbf{Q}/\mathbf{Z}$, which is unique up to equivalence. The spectrum I is called the *Brown-Comenetz dual of the sphere spectrum.* It is characterized up to homotopy equivalence by the following universal property in the stable homotopy category hSp: for every spectrum X, there is a canonical isomorphism $\mathrm{Ext}^n_{\mathrm{Sp}}(X, I) \simeq \mathrm{Hom}(\pi_n X, \mathbf{Q}/\mathbf{Z})$, where the latter Hom is computed in the category of abelian groups. In particular, the homotopy groups of I are dual to the stable homotopy groups of spheres.

Proposition C.5.7.11. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $n \geqslant 0$ be an integer. The following conditions are equivalent:*

(a) *The ∞-category $\mathcal{C}$ is weakly n-complicial.*

(b) *Every injective object of $\mathrm{Sp}(\mathcal{C})$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant -n}$.*

Proof. Suppose first that (a) is satisfied, and let $Q \in \mathrm{Sp}(\mathcal{C})$ be an injective object. Then $Q \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. It follows that for each $m \geqslant 0$, the object $\Omega^{\infty - m} Q$ is an m-truncated object of $\mathcal{C}$. Assumption (a) then implies that there exists a map $\rho : X \to \Omega^{\infty-m} Q$ which induces an epimorphism

$$\pi_0 X \to \pi_0(\Omega^{\infty-m} Q) \simeq \pi_{-m} Q$$

in the abelian category $\mathcal{C}^{\heartsuit}$, where X is n-truncated. If $m > n$, then the injectivity of Q implies that $\mathrm{Ext}^m_{\mathrm{Sp}(\mathcal{C})}(\Sigma^{\infty} X, Q) \simeq 0$, so that ρ is nullhomotopic and therefore $\pi_{-m} Q \simeq 0$. Since the t-structure on $\mathrm{Sp}(\mathcal{C})$ is right separated, we conclude that $Q \in \mathrm{Sp}(\mathcal{C})_{\geqslant -n}$.

We now prove the converse. Suppose that (b) is satisfied, and let X be an m-truncated object of $\mathcal{C}$. We wish to show that there exists an n-truncated object $\overline{X} \in \mathcal{C}$ and a morphism $\overline{X} \to X$ which induces an epimorphism on π_0. The proof proceeds by induction on m. If $m \leqslant n$, we can take $\overline{X} = X$. We will therefore assume that $m > n$. Choose a morphism $f : \Sigma^{\infty-m} X \to Q$ in $\mathrm{Sp}(\mathcal{C})$ which exhibits Q as an injective hull of $\Sigma^{\infty-m} X$ (see Example C.5.7.9). Then f induces a map $g : X \to \Omega^{\infty-m} Q$ in the ∞-category $\mathcal{C}$. Note that g is a morphism between m-truncated objects of $\mathcal{C}$ and induces a monomophism on π_m, so the fiber $\mathrm{fib}(g)$ is $(m-1)$-truncated. Applying our inductive hypothesis, we deduce that there exists an n-truncated object $\overline{X} \in \mathcal{C}$ and a morphism $h : \overline{X} \to \mathrm{fib}(g)$ which is an epimorphism on π_0. We will complete the proof by showing that the composite map $\overline{X} \xrightarrow{h} \mathrm{fib}(g) \to X$ is also an epimorphism on π_0. This follows from the exactness of the sequence $\pi_0 \mathrm{fib}(g) \to \pi_0 X \to \pi_{-m} Q$, since the third term vanishes by virtue of assumption (b). □

C.5.8 Chain Complexes of Injectives

Let $\mathcal{A}$ be a Grothendieck abelian category. It follows from Proposition C.5.5.20 that there exists an essentially unique Grothendieck prestable ∞-category $\mathcal{C}$ which is 0-complicial, anticomplete, and satisfies $\mathcal{C}^{\heartsuit} = \mathcal{A}$. In this section, we will describe an explicit construction of $\mathcal{C}$: its stabilization $\mathrm{Sp}(\mathcal{C})$ can be obtained as the differential graded nerve of the category of chain complexes of injective objects of $\mathcal{A}$.

Remark C.5.8.1. For further discussion of chain complexes of injective objects of Grothendieck abelian categories, we refer the reader to the work of Krause (see [124] and [125]) and Neeman ([161]); the results of this section are essentially a translation of [125]).

Definition C.5.8.2. Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{A}^{\mathrm{inj}}$ denote the full subcategory of $\mathcal{A}$ spanned by the injective objects. We let $\mathrm{Ch}(\mathcal{A}^{\mathrm{inj}})$ denote the category of chain complexes with values in $\mathcal{A}^{\mathrm{inj}}$, which we regard as a differential graded category. We let $\check{\mathcal{D}}(\mathcal{A})$ denote the differential graded nerve $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}^{\mathrm{inj}}))$ (see Construction HA.1.3.1.6). We will refer to $\check{\mathcal{D}}(\mathcal{A})$ as the *unseparated derived ∞-category of* $\mathcal{A}$.

Remark C.5.8.3. In the situation of Definition C.5.8.2, the derived ∞-category $\mathcal{D}(\mathcal{A})$ is defined as the differential graded nerve of the full subcategory $\mathrm{Ch}(\mathcal{A})^{\circ} \subseteq \mathrm{Ch}(\mathcal{A}^{\mathrm{inj}})$ spanned by those chain complexes which are fibrant with respect to the model structure of Proposition HA.1.3.5.3 . Consequently, we can regard $\mathcal{D}(\mathcal{A})$ as a full subcategory of $\check{\mathcal{D}}(\mathcal{A})$.

Notation C.5.8.4. Let $\mathcal{A}$ is a Grothendieck abelian category. We let $\check{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ denote the full subcategory of $\check{\mathcal{D}}(\mathcal{A})$ spanned by those chain complexes $Q_{\bullet}$ satisfying $\mathrm{H}_n(Q_{\bullet}) \simeq 0$ for $n < 0$. We set $\check{\mathcal{D}}(\mathcal{A})_{\leqslant 0} = \mathcal{D}(\mathcal{A})_{\leqslant 0}$ be the subcategory of $\check{\mathcal{D}}(\mathcal{A})$ spanned by those chain complexes $Q_{\bullet}$ which are fibrant (with respect to the model structure of Proposition HA.1.3.5.3) and satisfy $\mathrm{H}_n(Q_{\bullet}) \simeq 0$ for $n > 0$.

Proposition C.5.8.5. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then $\check{\mathcal{D}}(\mathcal{A})$ is a stable ∞-category. Moreover, the pair of full subcategories $(\check{\mathcal{D}}(\mathcal{A})_{\geqslant 0}, \check{\mathcal{D}}(\mathcal{A})_{\leqslant 0})$ is a t-structure on $\check{\mathcal{D}}(\mathcal{A})$.*

Proof. The first assertion follows from Proposition HA.1.3.2.10 , and the second from Proposition HA.1.3.5.18 . □

To compute morphisms in the ∞-category $\check{\mathcal{D}}(\mathcal{A})$, the following observation is useful.

Lemma C.5.8.6. *Let $\mathcal{A}$ be a Grothendieck abelian category, let $i : M'_* \to M_*$ be a quasi-isomorphism in $\mathrm{Ch}_{\leqslant 0}(\mathcal{A})$, and let Q_* be a complex of injective objects of $\mathcal{A}$. Then composition with i induces a quasi-isomorphism of chain complexes of abelian groups*

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M'_*, Q_*).$$

Proof. Since Q_* is levelwise injective, we have a short exact sequence of chain complexes

$$0 \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*/M'_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M'_*, Q_*) \to 0.$$

It will therefore suffice to show that the chain complex of abelian groups $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*/M'_*, Q_*)$ is acyclic. In other words, we must show that for every integer n, every chain map $M_*/M'_* \to Q_*[n]$ is nullhomotopic. To prove this, we are free to replace Q_* by the chain complex

$$\cdots \to 0 \to Q_{1-n} \to Q_{-n} \to \cdots$$

In this case, Q_* is fibrant with respect to the model structure of Proposition HA.1.3.5.3 (see Proposition HA.1.3.5.6), so the desired result follows from Proposition HA.1.3.5.11 (since the quotient M_*/M'_* is acyclic). □

Example C.5.8.7. Let $\mathcal{A}$ be a Grothendieck abelian category and let X be an object of $\mathcal{A}$, and let $X^{\heartsuit}$ denote the image of X under the equivalence $\mathcal{A} \simeq \mathcal{D}(\mathcal{A})^{\heartsuit} \simeq \check{\mathcal{D}}(\mathcal{A})^{\heartsuit}$. As a chain complex, we can identify $X^{\heartsuit}$ with any choice of injective resolution of X. Applying Lemma C.5.8.6, we see that for any object $Q_\bullet \in \check{\mathcal{D}}(\mathcal{A})$, we can identify the groups $\mathrm{Ext}^*_{\check{\mathcal{D}}(\mathcal{A})}(Y^{\heartsuit}, Q_\bullet)$ with the cohomology groups of the chain complex

$$\cdots \to \mathrm{Hom}_{\mathcal{A}}(X, Q_2) \to \mathrm{Hom}_{\mathcal{A}}(X, Q_1) \to \mathrm{Hom}_{\mathcal{A}}(X, Q_0) \to \mathrm{Hom}_{\mathcal{A}}(X, Q_{-1}) \to \mathrm{Hom}_{\mathcal{A}}(X, Q_{-2}) \to \cdots .$$

The main result of this section can be stated as follows:

Theorem C.5.8.8. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then:*

(a) The unseparated derived ∞-category $\check{\mathcal{D}}(\mathcal{A})$ is presentable and stable.

(b) The t-structure $(\check{\mathcal{D}}(\mathcal{A})_{\geq 0}, \check{\mathcal{D}}(\mathcal{A})_{\leq 0})$ is right complete and compatible with filtered colimits. Equivalently, $\check{\mathcal{D}}(\mathcal{A})_{\geq 0}$ is a Grothendieck prestable ∞-category and $\check{\mathcal{D}}(\mathcal{A})$ is equivalent to the stabilization $\mathrm{Sp}(\check{\mathcal{D}}(\mathcal{A})_{\geq 0})$.

(c) The Grothendieck prestable ∞-category $\check{\mathcal{D}}(\mathcal{A})_{\geq 0}$ is anticomplete and 0-complicial.

(d) The construction $Q_\bullet \to \mathrm{H}_0(Q_\bullet)$ induces an equivalence of categories $\check{\mathcal{D}}(\mathcal{A})^{\heartsuit} \to \mathcal{A}$.

It follows from Proposition C.5.5.20 that the ∞-category $\check{\mathcal{D}}(\mathcal{A})$ and its t-structure are characterized uniquely (up to canonical equivalence) by the assertions of Theorem C.5.8.8. Moreover, Proposition C.5.5.20 supplies the following analogue of Theorem C.5.4.9:

Corollary C.5.8.9. *Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{C}$ be any Grothendieck prestable ∞-category. Then restriction to the heart induces a fully faithful embedding*

$$\mathrm{LFun}^{\mathrm{lex}}(\check{\mathcal{D}}(\mathcal{A})_{\geq 0}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{C}^{\heartsuit}),$$

whose essential image is spanned by those exact functors $\mathcal{A} \to \mathcal{C}^{\heartsuit}$ which preserve small colimits.

Remark C.5.8.10. It follows from Proposition C.5.5.20 (or from the universal property of Corollary C.5.8.9) that the construction $\mathcal{A} \mapsto \breve{\mathcal{D}}(\mathcal{A})$ is functorial: if $f : \mathcal{A} \to \mathcal{A}'$ is an exact functor between Grothendieck abelian categories which preserves small colimits, then f induces a functor $F : \breve{\mathcal{D}}(\mathcal{A}) \to \breve{\mathcal{D}}(\mathcal{A}')$. The functor F is determined (up to equivalence) by the requirements that it is t-exact, preserves small colimits, and that the composite functor

$$\mathcal{A} \simeq \breve{\mathcal{D}}(\mathcal{A})^{\heartsuit} \xrightarrow{F} \breve{\mathcal{D}}(\mathcal{A}')^{\heartsuit} \simeq \mathcal{A}'$$

is equivalent to the original functor f.

Corollary C.5.8.11. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{A} = \mathcal{C}^{\heartsuit}$. Then the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{C}$ admits an essentially unique extension to a functor $\lambda : \breve{\mathcal{D}}(\mathcal{A})_{\geqslant 0} \to \mathcal{C}$ which preserves small colimits and finite limits. The following conditions are equivalent:*

- (a) *The functor λ is an equivalence of ∞-categories.*
- (b) *The Grothendieck prestable ∞-category $\mathcal{C}$ is anticomplete and 0-complicial.*
- (c) *The Grothendieck prestable ∞-category $\mathcal{C}$ is anticomplete and weakly 0-complicial.*
- (d) *The Grothendieck prestable ∞-category $\mathcal{C}$ is anticomplete and every injective object of $\mathrm{Sp}(\mathcal{C})$ belongs to the heart of $\mathrm{Sp}(\mathcal{C})$.*

Proof. The existence (and essential uniqueness) of λ follows from Corollary **??**. The implication $(a) \Rightarrow (b)$ follows from Theorem C.5.8.8, and the converse follows from Proposition C.5.5.20. The equivalence $(b) \Leftrightarrow (c)$ follows from Proposition C.5.5.16, and the equivalence $(c) \Leftrightarrow (d)$ from Proposition C.5.7.11. □

We now turn to the proof of Theorem C.5.8.8, following [125]. We first consider the following special case:

Proposition C.5.8.12. *Let $\mathcal{A}_0$ be a small abelian category. Assume that every object $X \in \mathcal{A}_0$ admits a finite resolution*

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to X \to 0,$$

where each P_i is a projective object of $\mathcal{A}_0$. Set $\mathcal{A} = \mathrm{Ind}(\mathcal{A}_0)$. Then:

- (i) *An object $X \in \mathcal{D}(\mathcal{A})$ is compact if and only if the homotopy groups $\pi_n X$ are compact objects of $\mathcal{D}(\mathcal{A})$ for every $n \in \mathbf{Z}$, and vanish for all but finitely many n.*
- (ii) *The derived ∞-category $\mathcal{D}(\mathcal{A})$ is compactly generated.*
- (iii) *The Grothendieck prestable ∞-category $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ is anticomplete.*

(*iv*) *The ∞-categories $\check{\mathcal{D}}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ are the same. That is, every chain complex Q_* of injective objects of $\mathcal{A}$ is a fibrant object of* $\mathrm{Ch}(\mathcal{A})$ *(with respect to the model structure of Proposition HA.1.3.5.3).*

Proof. Let us abuse notation by identifying $\mathcal{A}$ with the full subcategory $\mathcal{D}(\mathcal{A})^{\heartsuit} \subseteq \mathcal{D}(\mathcal{A})$ and $\mathcal{A}_0$ with its essential image in $\mathcal{A} = \mathrm{Ind}(\mathcal{A}_0)$. Let P be a projective object of $\mathcal{A}_0$ and let Ab denote the category of abelian groups, so that the construction $Y \mapsto \mathrm{Hom}_{\mathcal{A}}(P, Y)$ determines an exact functor $e : \mathcal{A} \to \mathrm{Ab}$. The induced map $\mathrm{Ch}(e) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathrm{Ab})$ carries quasi-isomorphisms to quasi-isomorphisms. Let $\{Q^j_*\}_{j \in J}$ be any collection of objects of $\mathcal{D}(\mathcal{A})$, which we will identify with fibrant objects of the category $\mathrm{Ch}(\mathcal{A})$. Choose a trivial cofibration $u : \bigoplus_{j \in J} Q^j_* \to Q_*$, where Q_* is another fibrant object of $\mathrm{Ch}(\mathcal{A})$. Then u is a quasi-isomorphism of chain complexes and the chain complex Q_* represents the coproduct of the objects Q^j_* in the ∞-category $\mathcal{D}(\mathcal{A})$. Using Example C.5.8.7, we see that composition with u induces isomorphisms of abelian groups

$$\begin{aligned}
\bigoplus_{j \in J} \pi_0 \mathrm{Map}_{\mathcal{D}(\mathcal{A})}(P, Q^j_*) &\simeq \bigoplus_{j \in J} \mathrm{H}_0(\mathrm{Ch}(e)(Q^j_*)) \\
&\simeq \mathrm{H}_0(\bigoplus_{j \in J} \mathrm{Ch}(e)(Q^j_*)) \\
&\simeq \mathrm{H}_0(\mathrm{Ch}(e)(\bigoplus_{j \in J} Q^j_*)) \\
&\xrightarrow{u} \mathrm{H}_0(\mathrm{Ch}(e)(Q_*)) \\
&\simeq \pi_0 \mathrm{Map}_{\mathcal{D}(\mathcal{A})}(P, Q_*).
\end{aligned}$$

Applying Proposition **??**, we deduce that P is a compact object of $\mathcal{D}(\mathcal{A})$.

By assumption, every object $X \in \mathcal{A}_0$ admits a finite resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to X \to 0,$$

where each $P_i \in \mathcal{A}_0$ is projective and therefore compact as an object of $\mathcal{D}(\mathcal{A})$. It follows that X is also compact when viewed as an object of $\mathcal{D}(\mathcal{A})$. Every compact object of $\mathcal{A} = \mathrm{Ind}(\mathcal{A}_0)$ is a retract of an object belonging to $\mathcal{A}_0$, and is therefore also a compact object of $\mathcal{D}(\mathcal{A})$.

Let $\mathcal{C} \subseteq \mathcal{D}(\mathcal{A})$ denote the full subcategory spanned by those objects X such that $\pi_n X$ is a compact object of $\mathcal{D}(\mathcal{A})$ for every integer n which vanishes for all but finitely many n. Note that $\mathcal{C}$ is a stable subcategory of $\mathcal{D}(\mathcal{A})$. Each $X \in \mathcal{C}$ can be written as a finite extension of the objects $\Sigma^n(\pi_n X)$, and is therefore also compact when viewed as an object of $\mathcal{D}(\mathcal{A})$. It follows that the inclusion $\mathcal{C} \hookrightarrow \mathcal{D}(\mathcal{A})$ extends to a fully faithful embedding $f : \mathrm{Ind}(\mathcal{C}) \hookrightarrow \mathcal{D}(\mathcal{A})$. Applying Corollary HTT.5.5.2.9 , we see that f admits a right adjoint $g : \mathcal{D}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{C})$. Note that if Q_* is fibrant object of $\mathrm{Ch}(\mathcal{A})$ such that $g(Q_*)$ vanishes as an object of $\mathcal{C}$, then for every projective object $P \in \mathcal{A}_0$ we have

$$\mathrm{H}_n(\mathrm{Ch}(e)(Q_*)) \simeq \mathrm{Map}_{\mathcal{D}(\mathcal{A})}(\Sigma^n P, Q_*) \simeq 0$$

where e is defined as above. It follows that the chain complex Q_* is acyclic, and therefore vanishes when viewed as an object of $\mathcal{D}(\mathcal{A})$. Applying this argument to $Q_* = \mathrm{fib}(v)$ for some morphism v in $\mathcal{D}(\mathcal{A})$, we deduce that v is an equivalence if and only if $g(v)$ is an equivalence: that is, the functor g is conservative, so that f and g are mutually inverse equivalence of ∞-categories. This proves (ii), and shows that an object of $\mathcal{D}(\mathcal{A})$ is compact if and only if it is a retract of an object of $\mathcal{C}$. Assertion (i) now follows from the observation that $\mathcal{C}$ is closed under the formation of retracts. Assertion (iii) follows from (i) and (ii) together with Proposition C.5.5.5.

We will deduce (iv) from the following assertion:

$(*)$ Let Q_* be an acyclic chain complex of injective objects of $\mathcal{A}$ and let X be any object of $\mathcal{A}$. Then the chain complex of abelian groups

$$\cdots \to \mathrm{Hom}_{\mathcal{A}}(X, Q_2) \to \mathrm{Hom}_{\mathcal{A}}(X, Q_1) \to \mathrm{Hom}_{\mathcal{A}}(X, Q_0) \to \mathrm{Hom}_{\mathcal{A}}(X, Q_{-1}) \to \mathrm{Hom}_{\mathcal{A}}(X, Q_{-2}) \to \cdots$$

is also acyclic.

Let us assume $(*)$ for the moment, and see how it leads to a proof of $(*)$. Suppose that Q'_* is any chain complex of injective objects of $\mathcal{A}$. Choose a trivial cofibration $f_* : Q'_* \to Q_*$ in $\mathrm{Ch}(\mathcal{A})$, where Q_* is fibrant. Then each of the maps $f_k : Q'_k \to Q_k$ is a monomorphism whose domain Q'_k is injective. It follows that each f_k is a split monomorphism. Set $Q''_* = Q_*/Q'_*$. Each of the projection maps $Q_k \to Q''_k$ admits a section s_k which exhibits Q''_k as a direct summand of Q_k, so that Q''_k is an injective object of $\mathcal{A}$ (Proposition HA.1.3.5.6). Since f_* is a quasi-isomorphism, the chain complex Q''_* is acyclic. For each integer k, let Z_k denote the kernel of the differential $Q''_k \to Q''_{k-1}$. Applying $(*)$ in the case $X = Z_k$, we deduce that the sequence

$$\mathrm{Hom}_{\mathcal{A}}(Z_k, Q''_{k+1}) \to \mathrm{Hom}_{\mathcal{A}}(Z_k, Q''_k) \to \mathrm{Hom}_{\mathcal{A}}(Z_k, Q''_{k-1})$$

is exact. Consequently, the inclusion $Z_k \hookrightarrow Q''_k$ lifts to a map Q''_{k+1}: in other words, the chain complex Q''_* is split exact. It follows that there exists a contracting homotopy $h : Q''_* \to Q''_{*+1}$. Let $\overline{h}$ denote the composite map $Q_* \to Q''_* \xrightarrow{h} Q''_{*+1} \xrightarrow{s} Q_*$. Then $\overline{h}$ determines a chain homotopy from the identity id_{Q_*} to a retraction $r : Q_* \to Q'_*$. In particular, we see that Q'_* is a retract of Q_*, and is therefore a fibrant object of $\mathrm{Ch}(\mathcal{A})$.

It remains to prove $(*)$. Fix an acyclic chain complex Q_* consisting of injective objects of $\mathcal{A}$. For each object $X \in \mathcal{A}$, let $T_*(X)$ denote the chain complex of abelian groups given by $T_k(X) = \mathrm{Hom}_{\mathcal{A}}(X, Q_k)$. Let us say an object $X \in \mathcal{A}$ is *good* if the chain complex $T_*(X)$ is acyclic. We now proceed in several steps.

(a) Every projective object $P \in \mathcal{A}$ is good (since the acyclicity of Q_* immediately implies the acyclicity of the chain complex $T_*(P) = \mathrm{Hom}_{\mathcal{A}}(P, Q_*)$ when P is projective).

(*b*) Suppose we are given a short exact sequence $0 \to X' \to X \to X'' \to 0$. Then the diagram of chain complexes

$$0 \to T_*(X'') \to T_*(X) \xrightarrow{\rho} T_*(X') \to 0$$

is also exact (this follows from the injectivity of the objects Q_k). In particular, we see that if any two of the objects X, X', and X'' is good, then so is the third. Moreover, the map ρ is surjective: that is, it is a fibration with respect to the projective model structure on the category Ch(Ab) of chain complexes of abelian groups.

(*c*) Let X be an object of $\mathcal{A}$ which admits a finite projective resolution

$$0 \to P_n \to \cdots \to P_0 \to X \to 0.$$

Then X is good. This follows from (*a*) and (*b*), using induction on n.

(*d*) Every object of $\mathcal{A}_0$ is good. This follows from (*c*) together with our assumption that every object of $\mathcal{A}_0$ admits a finite projective resolution (note that any projective object of $\mathcal{A}_0$ is also injective as an object of $\mathcal{A} = \mathrm{Ind}(\mathcal{A}_0)$).

(*e*) Let I be a well-ordered set and let $\{X_i\}_{i \in I}$ be a diagram in $\mathcal{A}$ with the property that, for each $i \in I$, the map $\varinjlim_{j<i} X_j \to X_i$ is a monomorphism in $\mathcal{A}$. If each X_i is good, then the colimit $\varinjlim X_i$ is good (this follows from (*a*) together with Corollary HTT.A.2.9.25).

(*f*) Let X be an object of $\mathcal{A}_0$ and let X' be a subobject of X (in the abelian category $\mathcal{A}$). We claim that X' is good. To prove this, choose an epimorphism $\bigoplus_{i \in I} Y_i \to X'$, where each Y_i belongs to $\mathcal{A}_0$. We proceed by (transfinite) induction on the cardinality of the set I. If I is finite, then $X' \in \mathcal{A}_0$ and the desired result follows from (*d*). Otherwise, we can choose a well-ordering of the set I for which each initial segment $I_{\leqslant j} = \{i \in I : i \leqslant j\}$ has cardinality smaller than I. For each $j \in I$, let X'_j be the subobject of X' given by the image of the composite map $\bigoplus_{i \in I_j} Y_i \to \bigoplus_{i \in I} Y_i \to X'$. It follows from our inductive hypothesis that each X'_j is good, so that (*e*) guarantees that $X' \simeq \varinjlim_{j \in I} X'_j$ is also good.

(*g*) Let X'' be an object of $\mathcal{A}$ and suppose there exists an epimorphism $u : X \to X''$, where $X \in \mathcal{A}_0$. Then X'' is good. This follows by applying (*b*) to the short exact sequence $0 \to X' \to X \xrightarrow{u} X'' \to 0$, since X is good by virtue of (*d*) and $X' \simeq \ker(u)$ is good by virtue of (*f*).

Now suppose that X is an arbitrary object of $\mathcal{A}$; we will show that X is good. Choose an epimorphism $\bigoplus_{i \in I} Y_i \to X$ where each Y_i belongs to $\mathcal{A}_0$. We proceed by induction on the cardinality of I. If I is finite, then the desired result follows from (g). Otherwise, we can choose a well-ordering of the set I for which each initial segment $I_{\leqslant j} = \{i \in I : i \leqslant j\}$ has cardinality smaller than I. For each $j \in J$, let X_j denote the image of the composite map $\bigoplus_{i \in I_j} Y_i \to \bigoplus_{i \in I} Y_i \to X$. Our inductive hypothesis guarantees that each X_j is good, so that $X \simeq \varinjlim_{j \in I} X_j$ is also good (by virtue of (e)). □

We will prove the general case Theorem C.5.8.8 by reducing to the situation of Proposition C.5.8.12, using the following observation:

Proposition C.5.8.13. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then there exists an essentially small abelian category $\mathcal{B}_c$ in which every object admits a projective resolution of length* 2*, such that $\mathcal{A}$ is a left exact localization of the Grothendieck abelian category* $\mathrm{Ind}(\mathcal{B}_c)$.

Proof. Choose an essentially small subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ which contains a set of generators for $\mathcal{A}$ and is closed under finite limits. Let $\mathcal{B} = \mathrm{Fun}^{\pi}(\mathcal{A}_0^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ be the full subcategory of $\mathrm{Fun}(\mathcal{A}_0^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ spanned by the product-preserving functors. It follows from the Gabriel-Popescu theorem (Theorem C.2.2.1) that the construction $X \mapsto \mathrm{Hom}_{\mathcal{A}}(*, X)$ induces a fully faithful embedding $G : \mathcal{A} \to \mathcal{B}$ and that the left adjoint to G is left exact (in other words, G exhibits $\mathcal{A}$ as a left-exact localization of $\mathcal{B}$). The objects $\{G(A)\}_{A \in \mathcal{A}_0}$ form compact projective generators for the abelian category $\mathcal{B}$. In particular, the category $\mathcal{B}$ is compactly generated: that is, we have an equivalence $\mathcal{B} \simeq \mathrm{Ind}(\mathcal{B}_c)$, where $\mathcal{B}_c$ denotes the full subcategory of $\mathcal{B}$ spanned by the compact objects. To complete the proof, it will suffice to establish the following:

- (a) The full subcategory $\mathcal{B}_c \subseteq \mathcal{B}$ is abelian.
- (b) Every object of $\mathcal{B}_c$ admits a projective resolution of length 2.

Unwinding the definitions, we see that an object $X \in \mathcal{B}$ is compact if and only if it admits a resolution

$$G(A') \xrightarrow{e} G(A) \to X \to 0$$

for some objects $A', A \in \mathcal{A}_0$. In this case, the fact that G is fully faithful guarantees that we can write $e = G(e_0)$ for some map $e_0 : A' \to A$ in the category $\mathcal{A}_0$. In this case, the left exactness of G supplies an isomorphism $\ker(e) \simeq G(\ker(e_0))$; the resulting exact sequence

$$0 \to G(\ker(e_0)) \to G(A') \to G(A) \to X \to 0$$

is a projective resolution of X having length 2, which proves (b).

Note that the full subcategory $\mathcal{B}_c \subseteq \mathcal{B}$ of compact objects is closed under finite colimits; in particular, it is closed under finite direct sums and the formation of cokernels. To prove (a), it will suffice to show that $\mathcal{B}_c$ is closed under kernels. Suppose that we are given a morphism $f : X \to Y$ in $\mathcal{B}_c$; we wish to show that the kernel $\ker(f)$ also belongs to $\mathcal{B}_c$. Choosing projective resolutions as above, we can arrange that f fits into a commutative diagram

$$\begin{array}{ccccccc} G(A') & \longrightarrow & G(A) & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow f & & \\ G(A') & \longrightarrow & G(A) & \longrightarrow & Y. & \longrightarrow & 0. \end{array}$$

A simple diagram chase then yields an exact sequence

$$G(A') \times_{G(B)} G(B') \to G(A) \times_{G(B)} G(B') \to \ker(f) \to 0.$$

Using the fact that G is fully faithful and left exact, we can rewrite this sequence as $G(A' \times_B B') \to G(A \times_B B') \to \ker(f) \to 0$, so that $\ker(f)$ is a compact object of $\mathcal{B}$ as desired. □

Remark C.5.8.14. Let $\mathcal{A}$ be a Grothendieck abelian category and let Q_* be chain complex of injective objects of $\mathcal{A}$. Assume that Q_* is homotopically minimal (in the sense of Proposition C.5.6.8). Then the following conditions are equivalent:

(1) The objects Q_n vanish for $n > 0$.

(2) The object Q_* belongs to $\check{\mathcal{D}}(\mathcal{A})_{\leq 0}$.

(3) For every object $X \in \mathcal{A} \simeq \check{\mathcal{D}}(\mathcal{A})^{\heartsuit}$, the groups $\mathrm{Ext}^n_{\check{\mathcal{D}}(\mathcal{A})}(X, Q_*)$ vanish for $n < 0$.

The implications (1) ⇒ (2) ⇒ (3) are evident (and do not require the assumption that Q_* is homotopically minimal). Assume that (3) is satisfied and let $n > 0$; we wish to show that $Q_n \simeq 0$. Suppose otherwise. Then the homotopical minimality of Q_* guarantees that $Z = \ker(d_n : Q_n \to Q_{n-1})$ is nonzero. Using Example C.5.8.7, we see that the inclusion $\iota : Z \hookrightarrow Q_n$ represents a class $\eta \in \mathrm{Ext}^{-n}_{\check{\mathcal{D}}(\mathcal{A})}(Z, Q_*)$. Assumption (3) guarantees that $\eta = 0$, so that ι factors as a composition $Z \xrightarrow{\bar{\iota}} Q_{n+1} \xrightarrow{d_{n+1}} Q_n$. Then d_{n+1} is injective when restricted to $\mathrm{im}(\bar{\iota})$, contradicting the homotopical minimality of Q_*.

Proposition C.5.8.15. *Let $\mathcal{A}$ be a Grothendieck abelian category, let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a localizing subcategory, and let $\mathcal{A}/\mathcal{A}_0$ be the full subcategory of $\mathcal{A}$ spanned by the $\mathcal{A}_0$-local objects (see Definition* **??***). Then:*

(a) *Every injective object of $\mathcal{A}/\mathcal{A}_0$ is also injective when regarded as an object of $\mathcal{A}$. Consequently, we can regard $\check{\mathcal{D}}(\mathcal{A}/\mathcal{A}_0)$ as a full subcategory of $\check{\mathcal{D}}(\mathcal{A})$.*

(b) *Let Q_* be an object of $\check{\mathcal{D}}(\mathcal{A})$. Then Q_* belongs to the essential image of the inclusion $\check{\mathcal{D}}(\mathcal{A}/\mathcal{A}_0) \hookrightarrow \check{\mathcal{D}}(\mathcal{A})$ if and only if, for every object $Y \in \mathcal{A}_0$, the groups $\mathrm{Ext}^n_{\check{\mathcal{D}}(\mathcal{A})}(X, Q_*)$ vanish for every integer n (here we abuse notation by identifying X with its image under the equivalence $\mathcal{A} \simeq \check{\mathcal{D}}(\mathcal{A})^{\heartsuit}$).*

(c) *We have* $\check{\mathcal{D}}(\mathcal{A}/\mathcal{A}_0)_{\leqslant 0} = \check{\mathcal{D}}(\mathcal{A}/\mathcal{A}_0) \cap \check{\mathcal{D}}(\mathcal{A})_{\leqslant 0}$.

Proof. Assertion (a) follows from the fact that the inclusion $\mathcal{A}/\mathcal{A}_0 \hookrightarrow \mathcal{A}$ admits an exact left adjoint (Proposition C.5.1.6). The inclusion

$$\check{\mathcal{D}}(\mathcal{A}/\mathcal{A}_0)_{\leqslant 0} \subseteq \check{\mathcal{D}}(\mathcal{A}/\mathcal{A}_0) \cap \check{\mathcal{D}}(\mathcal{A})_{\leqslant 0}$$

follows immediately from the definitions, and the reverse inclusion follows from Proposition C.5.6.8 and Remark C.5.8.14.

The "only if" direction of (b) follows immediately from Example C.5.8.7. To prove the converse, suppose that Q_* is a chain complex of injective objects of $\mathcal{A}$ and that the groups $\mathrm{Ext}^n_{\check{\mathcal{D}}(\mathcal{A})}(X, Q_*)$ vanish for all n and all $X \in \mathcal{A}_0$. We wish to prove that Q_* belongs to the essential image of the inclusion $\check{\mathcal{D}}(\mathcal{A}/\mathcal{A}_0) \hookrightarrow \check{\mathcal{D}}(\mathcal{A})$. Using Proposition C.5.6.8, we can reduce to the case where Q_* is homotopically minimal. We will prove in this case that $Q_* \in \check{\mathcal{D}}(\mathcal{A}/\mathcal{A}_0)$: that is, each of the objects Q_n is $\mathcal{A}_0$-local. By virtue of Remark C.5.1.5, this is equivalent to the requirement that the groups $\mathrm{Ext}^i_{\mathcal{A}}(X, Q_n)$ vanish for all $X \in \mathcal{A}_0$ and $i = 0, 1$. For $i = 1$, the vanishing is automatic (since Q_n is an injective object of $\mathcal{A}$). To handle the case $i = 0$, suppose (for a contradiction) that there exists some nonzero map $f : X \to Q_n$ in the abelian category $\mathcal{A}$. Replacing X by $\mathrm{im}(f)$, we may suppose that f is a monomorphism. Since Q_* is homotopically minimal, there exists a nonzero subobject $X' \subseteq X$ for which the composite map

$$X' \hookrightarrow X \xrightarrow{f} Q_n \xrightarrow{d_n} Q_{n-1}$$

vanishes. Using Example C.5.8.7, we see that this composite map represents an element $\eta \in \mathrm{Ext}^{-n}_{\check{\mathcal{D}}(\mathcal{A})}(X', Q_*)$. By assumption, η must vanish, so that $f|_{X'}$ factors as a composition $X' \xrightarrow{g} Q_{n+1} \xrightarrow{d_{n+1}} Q_n$. Invoking our assumption that Q_* is homotopically minimal, we deduce that the fiber product $X' \times_{Q_{n+1}} \ker(d_{n+1})$ is nonzero, contradicting our assumption that f is a monomorphism. □

Proof of Theorem C.5.8.8. Let $\mathcal{A}$ be a Grothendieck abelian category. Applying Proposition C.5.8.13, we can write $\mathcal{A}$ as a quotient $\mathcal{B}/\mathcal{B}_0$ where $\mathcal{B} \simeq \mathrm{Ind}(\mathcal{B}_c)$ is a compactly generated Grothendieck abelian category, the full subcategory $\mathcal{B}_c \subseteq \mathcal{B}$ of compact objects is abelian, every object of $\mathcal{B}_c$ admits a projective resolution of length 2, and $\mathcal{B}_0 \subseteq \mathcal{B}$ is a localizing subcategory. Applying Propositions C.5.8.12 and C.5.3.2, we see that $\check{\mathcal{D}}(\mathcal{B})$ is a presentable stable ∞-category, the t-structure $(\check{\mathcal{D}}(\mathcal{B})_{\geqslant 0}, \check{\mathcal{D}}(\mathcal{B})_{\leqslant 0})$ is right complete and compatible with filtered colimits, and the Grothendieck prestable ∞-category $\check{\mathcal{D}}(\mathcal{B})_{\geqslant 0}$ is anticomplete and 0-complicial. Combining Propositions C.5.8.15 and C.5.2.8, we deduce that $\check{\mathcal{D}}(\mathcal{A})$ is a presentable stable ∞-category, that the t-structure $(\check{\mathcal{D}}(\mathcal{A})_{\geqslant 0}, \check{\mathcal{D}}(\mathcal{A})_{\leqslant 0})$ is right complete and compatible with filtered colimits, and that the Grothendieck prestable ∞-category $\check{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$

can be identified with the quotient $\breve{\mathcal{D}}(\mathcal{B})_{\geqslant 0}/\mathcal{C}_0$, where $\mathcal{C}_0 \subseteq \breve{\mathcal{D}}(\mathcal{B})_{\geqslant 0}$ is the smallest localizing subcategory which contains $\mathcal{A}_0$. Applying Proposition C.5.5.7, we deduce that $\breve{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ is also anticomplete. Proposition C.5.3.3 guarantees that $\breve{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ is 0-complicial. □

C.5.9 Completed Derived ∞-Categories

We now observe that the theory of anticomplete Grothendieck prestable ∞-categories is closely related to the theory of complete Grothendieck prestable ∞-categories.

Notation C.5.9.1. Let $\mathsf{Groth}_{\infty}^{\mathrm{lex,comp}}$ denote the subcategory of Groth_{∞} whose objects are complete Grothendieck prestable ∞-categories and whose morphisms are left exact functors which preserve small colimits.

Proposition C.5.9.2. *Let $\mathsf{Groth}_{\infty}^{\mathrm{lex,comp}}$ denote the full subcategory of $\mathsf{Groth}_{\infty}^{\mathrm{lex}}$ spanned by the complete Grothendieck prestable ∞-categories. Then the construction $\mathcal{C} \mapsto \widehat{\mathcal{C}}$ determines an equivalence of ∞-categories $\mathsf{Groth}_{\infty}^{\mathrm{ch,lex}} \to \mathsf{Groth}_{\infty}^{\mathrm{lex,comp}}$.*

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories. It follows immediately from the definitions that if $\mathcal{C}$ is anticomplete, then the canonical map

$$\mathrm{Map}_{\mathsf{Groth}_{\infty}^{\mathrm{lex}}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}_{\mathsf{Groth}_{\infty}^{\mathrm{lex}}}(\widehat{\mathcal{C}}, \widehat{\mathcal{D}})$$

is a homotopy equivalence. This proves that the completion functor $\mathsf{Groth}_{\infty}^{\mathrm{ch,lex}} \to \mathsf{Groth}_{\infty}^{\mathrm{lex,comp}}$ is fully faithful. Essential surjectivity follows from Proposition C.5.5.9. □

Let $\mathcal{C}$ be an anticomplete Grothendieck prestable ∞-category and let $\widehat{\mathcal{C}}$ be its completion. Using Remark C.5.5.14 and Proposition C.5.5.16, we deduce that $\mathcal{C}$ is n-complicial if and only if $\widehat{\mathcal{C}}$ is weakly n-complicial. Combining this observation with Propositions C.5.9.2 and C.5.5.20, we obtain the following variant of Proposition C.5.4.5:

Proposition C.5.9.3. *Let n be a nonnegative integer and let $L : \mathsf{Groth}_{\infty}^{\mathrm{lex,comp}} \to \mathsf{Groth}_{n}^{\mathrm{lex}}$ be the functor given by $L(\mathcal{C}) = \tau_{\leqslant n}\mathcal{C}$ (see Example C.5.4.4). Then L admits a fully faithful left adjoint, whose essential image is spanned by those Grothendieck prestable ∞-categories $\mathcal{C}$ which are complete and weakly n-complicial.*

We now specialize Proposition C.5.9.3 to the case $n = 0$.

Definition C.5.9.4. Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{D}(\mathcal{A})$ be the derived ∞-category of $\mathcal{A}$. We let $\widehat{\mathcal{D}}(\mathcal{A})$ denote the left completion of $\mathcal{D}(\mathcal{A})$ with respect to its t-structure (see §HA.1.2.1): that is, the limit of the tower

$$\cdots \mathcal{D}(\mathcal{A})_{\leqslant 3} \xrightarrow{\tau_{\leqslant 2}} \mathcal{D}(\mathcal{A})_{\leqslant 2} \xrightarrow{\tau_{\leqslant 1}} \mathcal{D}(\mathcal{A})_{\leqslant 1} \xrightarrow{\tau_{\leqslant 0}} \mathcal{D}(\mathcal{A})_{\leqslant 0}.$$

We will refer to $\widehat{\mathcal{D}}(\mathcal{A})$ as the *completed derived ∞-category* of $\mathcal{A}$. It is equipped with a t-structure $(\widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0}, \widehat{\mathcal{D}}(\mathcal{A})_{\leqslant 0})$, where $\widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ is the completion of the Grothendieck prestable ∞-category $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ (in the sense of Proposition C.3.6.3) and $\widehat{\mathcal{D}}(\mathcal{A})_{\leqslant 0}$ is equivalent to $\mathcal{D}(\mathcal{A})_{\leqslant 0}$.

If $\mathcal{A}$ is a Grothendieck abelian category, then we can identify $\widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ with the image of $\mathcal{A}$ under the functor $\mathsf{Groth}^{\mathrm{lex}}_{\mathrm{ab}} = \mathsf{Groth}^{\mathrm{lex}}_{0} \to \mathsf{Groth}^{\mathrm{lex,comp}}_{\infty}$ of Proposition C.5.9.3. We therefore obtain the following analogue of Theorem C.5.4.9:

Corollary C.5.9.5. *Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{C}$ be a Grothendieck prestable ∞-category. If $\mathcal{C}$ is complete, then the restriction functor $\mathrm{LFun}^{\mathrm{lex}}(\widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{C}^{\heartsuit})$ is a fully faithful embedding, whose essential image consists of those functors $F : \mathcal{A} \to \mathcal{C}^{\heartsuit}$ which are left exact and preserve small colimits (here we abuse notation by identifying $\mathcal{A}$ with the heart $\widehat{\mathcal{D}}(\mathcal{A})^{\heartsuit}_{\geqslant 0}$).*

Corollary C.5.9.6. *The construction $\mathcal{A} \mapsto \widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ determines a fully faithful embedding $\mathsf{Groth}^{\mathrm{lex}}_{\mathrm{ab}} \hookrightarrow \mathsf{Groth}^{\mathrm{lex,comp}}_{\infty}$, which is left adjoint to the heart functor*

$$\mathsf{Groth}^{\mathrm{lex,comp}}_{\infty} \to \mathsf{Groth}^{\mathrm{lex}}_{\mathrm{ab}} \qquad \mathcal{C} \mapsto \mathcal{C}^{\heartsuit}.$$

We also have the following variant of Corollary C.5.8.11:

Corollary C.5.9.7. *Let $\mathcal{C}$ be a complete Grothendieck prestable ∞-category and let $\mathcal{A} = \mathcal{C}^{\heartsuit}$. Then the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{C}$ admits an essentially unique extension to a functor $\lambda : \widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0} \to \mathcal{C}$ which preserves small colimits and finite limits. The following conditions are equivalent:*

(a) The functor λ is an equivalence of ∞-categories.

(b) The Grothendieck prestable ∞-category $\mathcal{C}$ is weakly 0-complicial.

(c) Every injective object of $\mathrm{Sp}(\mathcal{C})$ belongs to the heart of $\mathrm{Sp}(\mathcal{C})$.

Proof. The existence (and essential uniqueness) of λ follows from Corollary C.5.9.5. The equivalence $(a) \Leftrightarrow (b)$ follows from Proposition C.5.9.3 and the equivalence $(b) \Leftrightarrow (c)$ follows from Proposition C.5.7.11. □

C.6 Finiteness Conditions on Prestable ∞-Categories

Let $\mathcal{C}$ be a presentable ∞-category. Recall that $\mathcal{C}$ is said to be *compactly generated* if every object $C \in \mathcal{C}$ can be written as the colimit of a filtered diagram $\varinjlim \{C_\alpha\}$, where each C_α is a compact object of $\mathcal{C}$. In this case, the ∞-category $\mathcal{C}$ is equivalent to $\mathrm{Ind}(\mathcal{C}_0)$, where $\mathcal{C}_0 \subseteq \mathcal{C}$ is the full subcategory spanned by the compact objects. Moreover, the construction $\mathcal{C}_0 \mapsto \mathrm{Ind}(\mathcal{C}_0)$ establishes an equivalence between the following data:

(*a*) Essentially small ∞-categories $\mathcal{C}_0$ which are idempotent complete and admit finite colimits.

(*b*) Compactly generated ∞-categories $\mathcal{C}$.

In this section, we will consider several variants of the equivalence between (*a*) and (*b*) which arise in the study of prestable ∞-categories. We begin in §C.6.1 by showing that in the situation above, the ∞-category $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ is prestable if and only if $\mathcal{C}_0$ is prestable (Proposition C.6.1.1). Consequently, the equivalence between (*a*) and (*b*) persists when we restrict our attention to prestable ∞-categories (Corollary C.6.1.5). Moreover, the theory of compactly generated ∞-categories simplifies (to some extent) when we work in the prestable setting: in §C.6.3, we supply a simple criterion which can be used to verify that a Grothendieck prestable ∞-category $\mathcal{C}$ is compactly generated (Corollary C.6.3.3).

Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. If $C \in \mathcal{C}$ is a compact object, then $\pi_0 C$ is a compact object of the Grothendieck abelian category $\mathcal{C}^{\heartsuit}$. However, the converse is false: for example, a compact object of the abelian category $\mathcal{C}^{\heartsuit}$ need not be compact when viewed as an object of $\mathcal{C}$. For a prototypical example, we can take $\mathcal{C} = \mathrm{Mod}_R^{\mathrm{cn}}$ to be the ∞-category of connective modules over a commutative ring R. If $M \in \mathrm{Mod}_R^{\heartsuit}$ is a discrete R-module, then M is compact as an object of $\mathrm{Mod}_R^{\heartsuit}$ if and only if it is finitely presented, but is compact as an object of $\mathrm{Mod}_R^{\mathrm{cn}}$ if and only if it is perfect (that is, if and only if it admits a finite resolution by finitely generated projective R-modules). In general, the second condition is much stronger than the first. However, if R is a Noetherian ring (or, more generally, a coherent ring), then the difference is smaller: in this case, every finitely presented R-module M admits a resolution

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M$$

where each P_n is a projective R-module of finite rank, but we might not be able to arrange that $P_n \simeq 0$ for $n \gg 0$. In this case, we cannot conclude that M is perfect, but we can conclude that it is *almost compact*: that is, that it is compact when viewed as an object of $\tau_{\leqslant n} \mathrm{Mod}_R^{\mathrm{cn}}$, for each $n \geqslant 0$. In §C.6.5, we generalize this observation by introducing the notion of a *coherent* Grothendieck prestable ∞-category: that is, a Grothendieck prestable ∞-category $\mathcal{C}$ which has "enough" almost compact objects (Definition C.6.5.1). We then study two situations in which a coherent Grothendieck stable ∞-category $\mathcal{C}$ can be recovered from its almost compact objects:

- In §C.6.6, we show that a Grothendieck prestable ∞-category $\mathcal{C}$ which is separated and coherent can be functorially recovered from the full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ spanned by its almost compact objects (Theorem C.6.6.14).

- In §C.6.7, we show that Grothendieck prestable ∞-category $\mathcal{C}$ which is anticomplete and coherent is also compactly generated, so that $\mathcal{C}$ can be recovered from the full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ of compact objects (Corollary C.6.7.3); moreover, an object $C \in \mathcal{C}$ is compact if and only if it is truncated and almost compact.

In many cases of interest, one has even stronger finiteness conditions. Recall that a Grothendieck abelian category $\mathcal{A}$ is said to be *locally Noetherian* if every object $X \in \mathcal{A}$ can be written as a (filtered) colimit of its Noetherian subobjects (Definition C.6.8.5). In §C.6.9, we will consider an analogous condition in the setting of Grothendieck prestable ∞-categories (Definition C.6.9.1) and study some of its consequences.

Remark C.6.0.8. The theory of coherent Grothendieck prestable ∞-categories presented here can be regarded as a "linear" version of the theory of coherent ∞-topoi presented in Appendix A. In particular, Theorem C.6.6.14 and Corollary C.6.7.3 are directly analogous to Theorems A.6.6.5 and A.7.5.3, respectively.

C.6.1 Prestability and Ind-Completion

Our starting point is the following result:

Proposition C.6.1.1. *Let $\mathcal{C}$ be a small idempotent-complete ∞-category which admits finite colimits. Then $\mathcal{C}$ is prestable if and only if $\operatorname{Ind}(\mathcal{C})$ is prestable.*

Remark C.6.1.2. Every compactly generated prestable ∞-category $\mathcal{C}$ is a Grothendieck prestable ∞-category: filtered colimits in $\mathcal{C}$ are left exact by virtue of Remark HA.**??**.

Lemma C.6.1.3. *Let $\mathcal{C}$ be a compact generated prestable ∞-category. Then an object $C \in \mathcal{C}$ is compact if and only if $\Sigma^{\infty} C$ is a compact object of $\operatorname{Sp}(\mathcal{C})$.*

Proof. The "only if" direction follows from the fact that the functor $\Sigma^{\infty} : \mathcal{C} \to \operatorname{Sp}(\mathcal{C})$ is fully faithful and preserves filtered colimits. The converse follows from the fact that $\Omega^{\infty} : \operatorname{Sp}(\mathcal{C}) \to \mathcal{C}$ preserves filtered colimits (Remark C.6.1.2 and Proposition C.1.4.1). ☐

Proof of Proposition C.6.1.1. Assume first that $\operatorname{Ind}(\mathcal{C})$ is prestable. Let $j : \mathcal{C} \to \operatorname{Ind}(\mathcal{C})$ be the Yoneda embedding. Since $\mathcal{C}$ is idempotent-complete, the functor j induces an equivalence from $\mathcal{C}$ to the full subcategory of $\operatorname{Ind}(\mathcal{C})$ spanned by the compact objects. Since j preserves finite colimits, the diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{j} & \operatorname{Ind}(\mathcal{C}) \\ \downarrow{\scriptstyle \Sigma_{\mathcal{C}}} & & \downarrow{\scriptstyle \Sigma_{\operatorname{Ind}(\mathcal{C})}} \\ \mathcal{C} & \xrightarrow{j} & \operatorname{Ind}(\mathcal{C}) \end{array}$$

commutes up to homotopy. Since $\Sigma_{\operatorname{Ind}(\mathcal{C})}$ and j are fully faithful, it follows that $\Sigma_{\mathcal{C}}$ is also fully faithful. To complete the proof that $\mathcal{C}$ is prestable, it will suffice to show that for every cofiber sequence $X \to X'' \to \Sigma X'$ in $\mathrm{Ind}(\mathcal{C})$, if both X'' and X' are compact, then X is compact. This follows immediately from Lemma C.6.1.3.

Now suppose that $\mathcal{C}$ is prestable. Then there exists a fully faithful embedding $f : \mathcal{C} \hookrightarrow \mathcal{D}$, where $\mathcal{D}$ is stable and the essential image of f is closed under colimits and extensions (Corollary C.1.2.3). Without loss of generality, we may assume that $\mathcal{D}$ is idempotent complete. The functor f induces a fully faithful embedding $\mathrm{Ind}(f) : \mathrm{Ind}(\mathcal{C}) \hookrightarrow \mathrm{Ind}(\mathcal{D})$ which preserves small colimits, and the functor $\mathrm{Ind}(\mathcal{D})$ is stable by virtue of Proposition HA.1.1.3.6 . To show that $\mathrm{Ind}(\mathcal{C})$ is prestable, it will suffice to show that the essential image of $\mathrm{Ind}(f)$ is closed under extensions (Corollary C.1.2.3). Let us abuse notation by identifying $\mathcal{D}$ with the full subcategory of $\mathrm{Ind}(\mathcal{D})$ spanned by the compact objects. Suppose we are given a fiber sequence $X' \to X \to X''$ in $\mathrm{Ind}(\mathcal{D})$, where the objects X' and X'' are the images of diagrams $\{C'_\alpha\}_{\alpha \in A}$ and $\{C''_\beta\}_{\beta \in B}$ in the ∞-category $\mathcal{C}$ indexed by filtered partially ordered sets A and B. We wish to show that X has the same property. Writing X as a filtered colimit of the fiber products $X \times_{X''} f(C''_\beta)$, we can assume that X" belongs to the essential image of $\mathcal{C}$. In particular, X'' is a compact object of $\mathrm{Ind}(\mathcal{D})$, so that the natural map $X'' \to \Sigma X'$ factors through some C'_α. In this case, we can write X as a filtered colimit $\varinjlim_{\gamma \geqslant \alpha} X_\gamma$, where X_γ denotes the fiber of the induced map $X'' \to \Sigma X'_\gamma$. We are therefore reduced to showing that each X_γ belongs to the essential image of $\mathcal{C}$. Note X_γ is a compact object of $\mathrm{Ind}(\mathcal{D})$ (since it is an extension of compact objects), and can therefore be identified with an object of $\mathcal{D}$. Since the essential image of f is closed under extensions, it follows that X_γ belongs to the essential image of $\mathcal{C}$ as desired. □

Notation C.6.1.4. We let $\mathcal{C}\mathrm{at}_\infty^{\mathrm{PSt}}$ denote the subcategory of $\mathcal{C}\mathrm{at}_\infty$ whose objects are small prestable ∞-categories and whose morphisms are right exact functors. We will refer to $\mathcal{C}\mathrm{at}_\infty^{\mathrm{PSt}}$ as *the ∞-category of prestable ∞-categories.*

Let $\mathcal{C}\mathrm{at}_\infty^{\star}$ denote the subcategory of $\mathcal{C}\mathrm{at}_\infty$ whose objects are small idempotent-complete ∞-categories which admit finite colimits and whose morphisms are functors which preserve finite colimits. According to Lemma HA.5.3.2.9 , the construction $\mathcal{C} \mapsto \mathrm{Ind}(\mathcal{C})$ induces an equivalence from $\mathcal{C}\mathrm{at}_\infty^{\star}$ to the subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ whose objects are compactly generated ∞-categories and whose morphisms are functors which preserve small colimits and compact objects. Combining that observation with Proposition C.6.1.1 and Remark C.3.4.3, we obtain the following:

Corollary C.6.1.5. *Let $\mathcal{C}\mathrm{at}_\infty^{\mathrm{PSt},\star}$ denote the full subcategory of $\mathcal{C}\mathrm{at}_\infty^{\mathrm{PSt}}$ whose objects are small idempotent-complete prestable ∞-categories. Then the construction $\mathcal{C} \mapsto \mathrm{Ind}(\mathcal{C})$ determines a fully faithful embedding $\mathrm{Ind} : \mathcal{C}\mathrm{at}_\infty^{\mathrm{PSt},\star} \hookrightarrow \mathsf{Groth}_\infty^{\mathrm{c}}$, whose essential image is spanned by the compactly generated prestable ∞-categories.*

Warning C.6.1.6. To obtain the equivalence of Corollary C.6.1.5, it is essential that our definition of a prestable ∞-category $\mathcal{C}$ does not require that $\mathcal{C}$ admits finite limits. For example, if R is an arbitrary connective $\mathbb{E}_\infty$-ring, then the ∞-category $\mathrm{Mod}_R^{\mathrm{cn}}$ of connective R-modules is a compactly generated prestable ∞-category, whose full subcategory of compact objects is given by the intersection $\mathrm{Mod}_R^{\mathrm{perf,cn}} = \mathrm{Mod}_R^{\mathrm{cn}} \cap \mathrm{Mod}_R^{\mathrm{perf}}$ spanned by the connective perfect R-modules. This ∞-category almost never admits finite limits (according to Proposition C.1.2.9, it admits finite limits if and only if the stable ∞-category $\mathrm{Mod}_R^{\mathrm{perf}} \simeq \mathrm{SW}(\mathrm{Mod}_R^{\mathrm{perf,cn}})$ admits a t-structure $(\mathrm{Mod}_R^{\mathrm{perf,cn}}, (\mathrm{Mod}_R^{\mathrm{perf}})_{\leqslant 0})$: in other words, if and only if the collection of perfect R-modules is stable under the formation of truncations).

C.6.2 Tensor Products of Compactly Generated Prestable ∞-Categories

We let $\mathsf{Groth}_\infty^{\mathrm{cg}}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ whose objects are compactly generated prestable ∞-categories and whose morphisms are compact functors $f : \mathcal{C} \to \mathcal{D}$. We regard $\mathsf{Groth}_\infty^{\mathrm{cg}}$ as a full subcategory of the ∞-category $\mathsf{Groth}_\infty^{\mathrm{c}}$ introduced in Definition C.3.4.2 (namely, the essential image of the functor Ind of Corollary C.6.1.5).

Proposition C.6.2.1. *The ∞-category $\mathsf{Groth}_\infty^{\mathrm{cg}}$ admits small colimits and the inclusion functors*

$$\mathsf{Groth}_\infty^{\mathrm{cg}} \hookrightarrow \mathsf{Groth}_\infty^{\mathrm{c}} \qquad \mathsf{Groth}_\infty^{\mathrm{cg}} \hookrightarrow \mathsf{Groth}_\infty$$

preserve small colimits.

Proof. By virtue of Proposition C.3.5.1, it will suffice to show that if $\{\mathcal{C}_\alpha\}$ is a diagram in $\mathsf{Groth}_\infty^{\mathrm{c}}$ having colimit $\mathcal{C}$ and each $\mathcal{C}_\alpha$ is compactly generated, then $\mathcal{C}$ is compactly generated. For each index α, let $f_\alpha : \mathcal{C}_\alpha \to \mathcal{C}$ be the tautological map and $G_\alpha : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C}_\alpha)$ the stabilization of the right adjoint to f_α. Each of the functors f_α is a morphism in $\mathsf{Groth}_\infty^{\mathrm{c}}$ and therefore carries compact objects of $\mathcal{C}_\alpha$ to compact objects of $\mathcal{C}$. Moreover, an object $X \in \mathrm{Sp}(\mathcal{C})$ belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ if and only if the groups

$$\mathrm{Ext}^*_{\mathrm{Sp}(\mathcal{C}_\alpha)}(\Sigma^\infty_{\mathcal{C}_\alpha} C, G_\alpha X) \simeq \mathrm{Ext}^*_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty_{\mathcal{C}} f_\alpha(C), X)$$

vanish for each $* < 0$ and every compact object $C \in \mathcal{C}_\alpha$. It follows that the collection of objects $\{\Sigma^\infty_{\mathcal{C}} f_\alpha C\}$ where $C \in \mathcal{C}_\alpha$ is compact satisfy the hypotheses of Proposition C.6.3.1, so that $\mathcal{C} \simeq \mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ is a compactly generated prestable ∞-category. □

Proposition C.6.2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be compactly generated prestable ∞-categories. Then the tensor product $\mathcal{C} \otimes \mathcal{D}$ (formed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$) is a compactly generated prestable ∞-category.*

Proof. The prestability of $\mathcal{C} \otimes \mathcal{D}$ follows from Theorem C.4.2.1 and the existence of compact generators follows because the functor $\mathrm{Ind} : \mathcal{C}\mathrm{at}_\infty^{\mathrm{rex}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ is symmetric monoidal, where $\mathcal{C}\mathrm{at}_\infty^{\mathrm{rex}}$ denotes the ∞-category whose objects are small ∞-categories which admit finite colimits and whose morphisms are functors which preserve finite colimits. □

Since the unit object $\mathrm{Sp}^{\mathrm{cn}} \in \mathsf{Groth}_\infty^{\mathrm{c}}$ is compactly generated, Propositions C.6.2.2, **??**, and C.6.2.1 yield the following:

Corollary C.6.2.3. *The symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ induces a symmetric monoidal structure on the ∞-category $\mathsf{Groth}_\infty^{\mathrm{cg}}$ for which the inclusion $\mathsf{Groth}_\infty^{\mathrm{cg}} \hookrightarrow \mathsf{Groth}_\infty^{\mathrm{c}}$ is symmetric monoidal functor. Moreover, the tensor product $\otimes : \mathsf{Groth}_\infty^{\mathrm{cg}} \times \mathsf{Groth}_\infty^{\mathrm{cg}} \to \mathsf{Groth}_\infty^{\mathrm{cg}}$ preserves small colimits separately in each variable.*

C.6.3 Digression: A Criterion for Compact Generation

We close this section by establishing a result which is useful for constructing examples of compactly generated prestable ∞-categories.

Proposition C.6.3.1. *Let $\mathcal{C}$ be a presentable stable ∞-category, and let $\{C_\alpha\}_{\alpha \in A}$ be a collection of compact objects of $\mathcal{C}$. Suppose that for every nonzero object $D \in \mathcal{C}$, there exists $\alpha \in A$ for which the graded abelian group $\mathrm{Ext}^*_{\mathcal{C}}(C_\alpha, D)$ is nonzero. Then:*

- (a) *Let $\mathcal{C}_{\leqslant 0}$ be the full subcategory of $\mathcal{C}$ spanned by those objects D for which the groups $\mathrm{Ext}^*_{\mathcal{C}}(C_\alpha, D)$ vanish for all $\alpha \in A$ and $* < 0$. Then $\mathcal{C}_{\leqslant 0}$ can be extended to a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ on $\mathcal{C}$.*
- (b) *The t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ is right complete and compatible with filtered colimits.*
- (c) *Let $\mathcal{E} \subseteq \mathcal{C}$ be the smallest full subcategory which contains the objects C_α and is closed under finite colimits and extensions. Then the inclusion $\mathcal{E} \hookrightarrow \mathcal{C}$ extends to an equivalence of ∞-categories $\mathrm{Ind}(\mathcal{E}) \simeq \mathcal{C}_{\geqslant 0}$.*
- (d) *The ∞-category $\mathcal{C}_{\geqslant 0}$ is compactly generated and prestable.*
- (e) *Let D be an object of $\mathcal{C}_{\geqslant 0}$. Then D is compact as an object of $\mathcal{C}_{\geqslant 0}$ if and only if it is compact as an object of $\mathcal{C}$.*

Proof. The existence of the t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ follows immediately from Proposition HA.1.4.4.11 . Since each C_α is a compact object of $\mathcal{C}$, it follows immediately that the full subcategory $\mathcal{C}_{\leqslant 0} \subseteq \mathcal{C}$ is closed under filtered colimits. In particular, it is closed under arbitrary coproducts. Any object $D \in \bigcap_{n \geqslant 0} \mathcal{C}_{\leqslant -n}$ satisfies $\mathrm{Ext}^*_{\mathcal{C}}(C_\alpha, D) \simeq 0$ for all $\alpha \in A$ and is therefore a zero object of $\mathcal{C}$. The right completeness of $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ is therefore a consequence of Proposition HA.1.2.1.19 . This proves (a) and (b).

Since the collection of compact objects of $\mathcal{C}$ is closed under finite colimits and extensions, every object of $\mathcal{E}$ is compact when viewed as an object of $\mathcal{C}$. Note that the full subcategory $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$ is also closed under finite colimits and extensions and therefore contains $\mathcal{E}$. In particular, every object of $\mathcal{E}$ is compact when viewed as an object of $\mathcal{C}_{\geqslant 0}$, so the inclusion $\mathcal{E} \hookrightarrow \mathcal{C}_{\geqslant 0}$ extends to a fully faithful embedding $\theta : \mathrm{Ind}(\mathcal{E}) \to \mathcal{C}_{\geqslant 0}$ which commutes with small colimits. Let $\mathcal{C}' \subseteq \mathcal{C}_{\geqslant 0}$ be the essential image of θ. To prove (c), we must show that $\mathcal{C}' = \mathcal{C}_{\geqslant 0}$. The proof of Proposition HA.1.4.4.11 shows that $\mathcal{C}_{\geqslant 0}$ is the smallest full subcategory of $\mathcal{C}$ which contains the objects C_α and is closed under small colimits and extensions. Consequently, to establish the inclusion $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}'$, it will suffice to show that the ∞-category $\mathcal{C}'$ is closed under extensions in $\mathcal{C}$. Suppose we are given a fiber sequence

$$C' \to C \to C''$$

where $C', C'' \in \mathcal{C}'$; we wish to prove that $C \in \mathcal{C}'$. Write C'' as a filtered colimit $\varinjlim C''_\beta$, where each C''_β belongs to $\mathcal{E}$. It follows that C can be written as a filtered colimit of objects of the form $C \times_{C''} C''_\beta$. Since $\mathcal{C}'$ is closed under filtered colimits in $\mathcal{C}$, it will suffice to show that each $C \times_{C''} C''_\beta$ belongs to $\mathcal{C}'$. Replacing C'' by C''_β, we may reduce to the case where $C''_\beta \in \mathcal{E}$. In this case, we can realize C as the fiber of a map $\eta : C'' \to \Sigma C'$. Write C' as a filtered colimit $\varinjlim C'_\gamma$, where each C'_γ belongs to $\mathcal{E}$. Since C'' is a compact object of $\mathcal{C}$, it follows that η factors through $\Sigma C'_{\gamma_0}$ for some index γ_0. We may therefore write C as a filtered colimit of objects of the form $\mathrm{fib}(C'' \to \Sigma C'_\gamma)$, which are extensions of objects of $\mathcal{E}$ and therefore belong to $\mathcal{E}$. This proves (c).

Assertion (d) is an immediate consequence of (b) and (c). To prove (e), it suffices to observe that any compact object of $\mathcal{C}_{\geqslant 0} \simeq \mathrm{Ind}(\mathcal{E})$ is a retract of an object of $\mathcal{E}$, and is therefore compact when viewed as an object of $\mathcal{C}$. □

Remark C.6.3.2. Let $\mathcal{D}$ be a compactly generated prestable ∞-category, and choose a collection of compact objects $\{D_\alpha\}_{\alpha \in A}$ which generate $\mathcal{D}$ under small colimits. Set $\mathcal{C} = \mathrm{Sp}(\mathcal{D})$. Then the objects $\{\Sigma^\infty D_\alpha \in \mathcal{C}\}$ satisfy the hypotheses of Proposition C.6.3.1, and the t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ of Proposition C.6.3.1 agrees with the one determined by the prestability of $\mathcal{D}$ (that is, $\mathcal{C}_{\geqslant 0}$ can be identified with the essential image of the functor $\Sigma^\infty : \mathcal{C} \to \mathrm{Sp}(\mathcal{C}) = \mathcal{D}$). Consequently, *every* compactly generated prestable ∞-category $\mathcal{D}$ can be obtained from the construction of Proposition C.6.3.1.

Corollary C.6.3.3. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. The following conditions are equivalent:*

(a) *The ∞-category $\mathcal{C}$ is compactly generated.*

(b) *For every nonzero object $C \in \mathcal{C}$, there exists a morphism $\alpha : C_0 \to C$, where C_0 is compact and α is not nullhomotopic.*

Proof. Suppose first that (a) is satisfied, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory spanned by the compact objects. Then the composition of the Yoneda embedding $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ with the restriction functor $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{S})$ is fully faithful. Consequently, if $C \in \mathcal{C}$ is nonzero, then $j(C)|_{\mathcal{C}_0^{\mathrm{op}}}$ cannot be a final object of $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{S})$: that is, there exists a compact object $C_0 \in \mathcal{C}_0$ such that the mapping space $\mathrm{Map}_{\mathcal{C}}(C_0, C)$ is not contractible. It follows that for some integer n, the homotopy group $\pi_n \mathrm{Map}_{\mathcal{C}}(C_0, C)$ is nontrivial. Replacing C_0 by the suspension $\Sigma^n C_0$, it follows that there exists a map $C_0 \to C$ which is not nullhomotopic.

Now suppose that (b) is satisfied. If X is a nonzero object of $\mathrm{Sp}(\mathcal{C})$, then $\Omega^{\infty-n} X$ is a nonzero object of $\mathcal{C}$ for some $n \gg 0$, so condition (b) guarantees that there is compact object $C_0 \in \mathcal{C}$ such that $\pi_0 \mathrm{Map}_{\mathcal{C}}(C_0, \Omega^{\infty-n} X) \simeq \mathrm{Ext}^n_{\mathcal{C}}(\Sigma^\infty C_0, X) \neq 0$. Consequently, the collection of objects $\{\Sigma^\infty C_0\}_{C_0 \in \mathcal{C}_0}$ satisfies the hypotheses of Proposition C.6.3.1. We conclude that there exists a t-structure $(\mathrm{Sp}(\mathcal{C})_{\geqslant 0}, \mathrm{Sp}(\mathcal{C})_{\leqslant 0})$ on $\mathrm{Sp}(\mathcal{C})$, where $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ is a compactly generated prestable ∞-category and $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ is the full subcategory spanned by those objects X for which the groups

$$\mathrm{Ext}^{-n}_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty C_0, X) \simeq \pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^n C, \Omega^\infty X)$$

vanish for $C_0 \in \mathcal{C}_0$ and $n > 0$. Applying (b) again, we deduce that $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ is spanned by those objects $X \in \mathrm{Sp}(\mathcal{C})$ satisfying $\Omega^{\infty+1} X \simeq 0$, so that $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ is the essential image of the fully faithful embedding $\Sigma^\infty : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$. □

C.6.4 Almost Compact Objects

Let $\mathcal{C}$ be an ∞-category which admits filtered colimits. Recall that an object $C \in \mathcal{C}$ is said to be *compact* if the functor $D \mapsto \mathrm{Map}_{\mathcal{C}}(C, D)$ commutes with filtered colimits. In practice, it is often useful to consider a slightly weaker condition (see Definition HA.7.2.4.8):

Definition C.6.4.1. Let $\mathcal{C}$ be a presentable ∞-category. We say that an object $C \in \mathcal{C}$ is *almost compact* if, for every integer $n \geqslant 0$, the construction $D \mapsto \mathrm{Map}_{\mathcal{C}}(C, D)$ determines a functor $\tau_{\leqslant n} \mathcal{C} \to \mathcal{S}$ which commutes with filtered colimits.

Remark C.6.4.2. Let $\mathcal{C}$ be a presentable ∞-category. An object $C \in \mathcal{C}$ is almost compact if and only if, for every $n \geqslant 0$, the truncation $\tau_{\leqslant n} C$ is compact when viewed as an object of $\tau_{\leqslant n} \mathcal{C}$.

Example C.6.4.3. Let $\mathcal{C}$ be a presentable ∞-category. Every compact object of $\mathcal{C}$ is almost compact.

Proposition C.6.4.4. *Let $\mathcal{C}$ be a presentable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory spanned by those objects which are almost compact. Then $\mathcal{C}_0$ is closed under finite colimits and under geometric realizations of simplicial objects.*

Proof. Closure under finite colimits follows immediately from Remark C.6.4.2 and the closure of compact objects of $\tau_{\leqslant n} \mathcal{C}$ under finite colimits (Corollary HTT.5.3.4.15). Similarly, to show that $\mathcal{C}_0$ is closed under geometric realizations of simplicial objects, it will suffice to show that for every simplicial object $X_\bullet$ of $\tau_{\leq n}\mathcal{C}$, if each X_k is a compact object of $\tau_{\leq n}\mathcal{C}$, then the geometric realization $|X_\bullet|$ (formed in the ∞-category $\tau_{\leq n}\mathcal{C}$) is also compact. This follows from the fact that $\tau_{\leq n}\mathcal{C}$ is equivalent to an $(n+1)$-category, so that the geometric realization $|X_\bullet|$ can be identified with the finite colimit $\varinjlim_{[k]\in\mathbf{\Delta}^{\mathrm{op}}_{s,\leq n+1}} X_k$. □

Proposition C.6.4.5. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{C}^{\mathrm{ac}} \subseteq \mathcal{C}$ be the full subcategory spanned by the almost compact objects. The following conditions are equivalent:*

- (a) *The loop functor $\Omega : \mathcal{C} \to \mathcal{C}$ carries $\mathcal{C}^{\mathrm{ac}}$ to itself.*
- (b) *For every morphism $f : X \to Y$ in $\mathcal{C}^{\mathrm{ac}}$, the fiber $\mathrm{fib}(f)$ belongs to $\mathcal{C}^{\mathrm{ac}}$.*
- (c) *The full subcategory $\mathcal{C}^{\mathrm{ac}} \subseteq \mathcal{C}$ is closed under finite limits.*
- (d) *For every object $X \in \mathcal{C}^{\mathrm{ac}}$, the object $\pi_0 X$ also belongs to $\mathcal{C}^{\mathrm{ac}}$.*

Proof. We first prove that (a) implies (b). Let $f : X \to Y$ be a morphism in $\mathcal{C}$, where X and Y are almost compact. Applying Proposition C.6.4.4, we deduce that the cofiber $\mathrm{cofib}(f)$ is almost compact. From assumption (a) (and the prestability of $\mathcal{C}$) we deduce that $\mathrm{fib}(f) \simeq \Omega\,\mathrm{cofib}(f)$ is almost compact.

We next show that (b) implies (c). By virtue of Corollary HTT.4.4.2.4 , it will suffice to show that $\mathcal{C}^{\mathrm{ac}}$ contains the final object of $\mathcal{C}$ (which is clear) and is closed under fiber products. This follows from (b): if $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ are morphisms in $\mathcal{C}^{\mathrm{ac}}$, then we can identify the fiber product $X_0 \times_X X_1$ with the fiber of the induced map $X_0 \oplus X_1 \xrightarrow{f_0 - f_1} X$.

The implication $(c) \Rightarrow (a)$ is obvious. We complete the proof by showing that (a) and (d) are equivalent. Suppose that (a) is satisfied and that $X \in \mathcal{C}^{\mathrm{ac}}$. Then $\Omega X \in \mathcal{C}^{\mathrm{ac}}$. Since the collection of almost compact objects of $\mathcal{C}$ is closed under finite colimits, it follows that $\pi_0 X \simeq \mathrm{cofib}(\Sigma\Omega X \to X)$ also belongs to $\mathcal{C}^{\mathrm{ac}}$.

Conversely, suppose that (d) is satisfied. For every object $X \in \mathcal{C}^{\mathrm{ac}}$, the cofiber sequence $X \to \pi_0 X \to \Sigma^2 \Omega X$ shows that $\Sigma^2\Omega X$ is almost compact. Since the double suspension functor Σ^2 induces a fully faithful embedding $\tau_{\leq n}\mathcal{C} \to \tau_{\leq n+2}\mathcal{C}$ which commutes with filtered colimits for each $n \geq 0$, it follows that $\Omega X \in \mathcal{C}$ is also almost compact. □

Corollary C.6.4.6. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category which satisfies the equivalent conditions of Proposition C.6.4.5, and let $C \in \mathcal{C}$ be an object. The following conditions are equivalent:*

- (i) *The object C is almost compact.*
- (ii) *For each $n \geq 0$, the truncation $\tau_{\leq n} C$ is almost compact.*

(*iii*) *For each $n \geqslant 0$, the object $\pi_n C \in \mathcal{C}^{\heartsuit}$ is almost compact (when viewed as an object of $\mathcal{C}$).*

Proof. The implication $(i) \Rightarrow (ii)$ follows from the existence of a cofiber sequence $\Sigma^{n+1}\Omega^{n+1}C \to C \to \tau_{\leqslant n} C$, and the implication $(ii) \Rightarrow (iii)$ from the existence of an equivalence $\pi_n C \simeq \Omega^n(\tau_{\leqslant n} C)$. We will complete the proof by showing that $(iii) \Rightarrow (i)$. Assume that (iii) is satisfied; we wish to show that C is almost compact. Equivalently, we wish to show that $\tau_{\leqslant m} C$ is compact when viewed as an object of $\tau_{\leqslant m}\mathcal{C}$, for each $m \geqslant 0$. In fact, we will a stronger assertion: for each $m \geqslant 0$, the truncation $\tau_{\leqslant m} C$ is almost compact (when viewed as an object of $\mathcal{C}$). The proof proceeds by induction on m, the case $m = -1$ being trivial. To carry out the inductive step, we observe that there exists a fiber sequence

$$\tau_{\leqslant m} C \to \tau_{\leqslant m-1} C \to \Sigma^{m+1}(\pi_m C).$$

The object $\tau_{\leqslant m-1} C$ is almost compact by our inductive hypothesis, and $\Sigma^{m+1}(\pi_m C)$ is almost compact by virtue of assumption (iii). Invoking assumption (b) of Proposition C.6.4.5, we conclude that $\tau_{\leqslant m} C$ is almost compact. □

C.6.5 Coherent Grothendieck Prestable ∞-Categories

We now turn our attention to Grothendieck prestable ∞-categories which have "enough" almost compact objects.

Definition C.6.5.1. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory spanned by the almost compact objects. We will say that $\mathcal{C}$ is *coherent* if it satisfies the following conditions:

(a) The full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is closed under finite limits.

(b) The full subcategory $\mathcal{C}_0$ is a generating subcategory of $\mathcal{C}$, in the sense of Definition C.2.1.1. That is, for every object $X \in \mathcal{C}$, there exists a morphism $\bigoplus C_\alpha \to X$ which induces an epimorphism on π_0, where each C_α is almost compact.

We will say that $\mathcal{C}$ is *weakly coherent* if it satisfies condition (a) and the following weaker version of (b):

(b') For every truncated object $X \in \mathcal{C}$, there exists a morphism $\bigoplus C_\alpha \to X$ which induces an epimorphism on π_0, where each C_α is almost compact.

Warning C.6.5.2. Let $\mathcal{C}$ be a compactly generated prestable ∞-category. Then $\mathcal{C}$ automatically satisfies condition (b) of Definition C.6.5.1, but need not satisfy condition (a).

Example C.6.5.3. Let R be a connective $\mathbb{E}_1$-ring and let $\mathcal{C} = \mathrm{LMod}_R^{\mathrm{cn}}$ denote the ∞-category of connective left R-modules. Using Proposition HA.7.2.4.18 , we see that the following assertions are equivalent:

(i) The $\mathbb{E}_1$-ring R is left coherent.

(ii) The Grothendieck prestable ∞-category $\mathcal{C}$ is coherent.

(iii) The Grothendieck prestable ∞-category $\mathcal{C}$ is weakly coherent.

We now discuss some criteria which can be used to recognize weakly coherent Grothendieck prestable ∞-categories.

Proposition C.6.5.4. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is weakly coherent.*

(2) *For each $n \geqslant 0$, the ∞-category $\tau_{\leqslant n}\mathcal{C}$ is compactly generated and the collection of compact objects of $\tau_{\leqslant n}\mathcal{C}$ is closed under finite limits.*

Proof. Suppose first that (1) is satisfied. Chose $n \geqslant 0$, and let $\mathcal{E} \subseteq \tau_{\leqslant n}\mathcal{C}$ be the full subcategory spanned by those objects which are almost compact (when viewed as objects of $\mathcal{C}$). Since the collection of almost compact objects of $\mathcal{C}$ is closed under finite colimits (Proposition C.6.4.4) and under the truncation functor $\tau_{\leqslant n}$ (Corollary C.6.4.6), it follows that $\mathcal{E}$ is closed under finite colimits in $\tau_{\leqslant n}\mathcal{C}$. Applying Propositions HTT.5.3.5.11 and HTT.5.5.1.9 , we see that the inclusion $\mathcal{E} \hookrightarrow \tau_{\leqslant n}\mathcal{C}$ extends to a fully faithful embedding $F : \mathrm{Ind}(\mathcal{E}) \to \tau_{\leqslant n}\mathcal{C}$ which commutes with small colimits.

Let $G : \tau_{\leqslant n}\mathcal{C} \to \mathrm{Ind}(\mathcal{E})$ be a right adjoint to F. Note that if $X \in \tau_{\leqslant n}\mathcal{C}$ satisfies $GX \simeq 0$, then the mapping space $\mathrm{Map}_{\mathcal{C}}(C, X) \simeq \mathrm{Map}_{\mathcal{C}}(\tau_{\leqslant n}C, X)$ is contractible for all almost compact objects $C \in \mathcal{C}$ (Corollary C.6.4.6). If $\mathcal{C}$ is weakly coherent, this implies that $\pi_0 X \simeq 0$. Applying the same argument to $\Omega^i X$, we deduce that $\pi_i X \simeq 0$ for $0 \leqslant i \leqslant n$ and therefore $X \simeq 0$.

Now let X be any object of $\tau_{\leqslant n}\mathcal{C}$ and let $v : (F \circ G)(X) \to X$ be the counit map. Since F is fully faithful, the map $G(v)$ is an equivalence in $\mathrm{Ind}(\mathcal{E})$. The functor G is left exact (since it is a right adjoint), so we have $G(\mathrm{fib}(v)) = \mathrm{fib}\, G(v) \simeq 0$. It follows from the preceding argument that $\mathrm{fib}(v) = 0$. For every almost compact object $C \in \mathcal{C}$, every map $C \to X$ factors through v. Our assumption that $\mathcal{C}$ is weakly coherent ensures that v induces an epimorphism on π_0, so that the fiber sequence $\mathrm{fib}(v) \to (F \circ G)(X) \to X$ is also a cofiber sequence. The vanishing of $\mathrm{fib}(v)$ now shows that v is an equivalence. Allowing X to vary, we conclude that F is an equivalence of ∞-categories so that the ∞-category $\tau_{\leqslant n}\mathcal{C}$ is compactly generated.

It follows immediately from the definitions that each object of $\mathcal{E}$ is compact when viewed as an object of $\tau_{\leqslant n}\mathcal{C}$. Conversely, since F is an equivalence, every compact object of $\tau_{\leqslant n}\mathcal{C}$

is a retract of an object of $\mathcal{E}$. Since $\mathcal{E}$ is evidently closed under retracts, it follows that an object of $\tau_{\leqslant n}\mathcal{C}$ is compact if and only if it belongs to $\mathcal{E}$. Since the collections of n-truncated objects and almost compact objects of $\mathcal{C}$ are both closed under finite limits, it follows that the collection of compact objects of $\tau_{\leqslant n}\mathcal{C}$ is closed under finite limits. Allowing n to vary, we deduce that (2) is satisfied.

We now prove the converse. Assume that $\mathcal{C}$ satisfies (2); we will prove that $\mathcal{C}$ is weakly coherent. We first show that the collection of almost compact objects of $\mathcal{C}$ is closed under finite limits. By virtue of Proposition C.6.4.5, it will suffice to show that if $X \in \mathcal{C}$ is almost compact, then $\Omega X \in \mathcal{C}$ is also almost compact. For this, we must prove that for each $n \geqslant 0$, the object $\tau_{\leqslant n}(\Omega X) \simeq \Omega(\tau_{\leqslant n+1} X)$ is a compact object of $\tau_{\leqslant n}\mathcal{C}$. This is clear: our hypothesis that X is almost compact guarantees that $\tau_{\leqslant n+1} X$ is a compact object of $\tau_{\leqslant n+1}\mathcal{C}$, so that assumption (2) shows that $\Omega(\tau_{\leqslant n+1} X)$ is compact when viewed as an object of $\tau_{\leqslant n+1}\mathcal{C}$ (and therefore also when viewed as an object of the smaller ∞-category $\tau_{\leqslant n}\mathcal{C}$).

We now claim that if $n \geqslant 0$ and C is a compact object of $\tau_{\leqslant n}\mathcal{C}$, then C is almost compact when viewed as an object of $\mathcal{C}$. In other words, the object C is compact when viewed as an object of $\tau_{\leqslant m}\mathcal{C}$ for any $m \geqslant n$. To prove this, we invoke assumption (2) to write C as the colimit of a filtered diagram $\{C_\alpha\}$, where each C_α is a compact object of $\tau_{\leqslant m}\mathcal{C}$. Since truncation in $\mathcal{C}$ commutes with filtered colimits, we have $C \simeq \varinjlim_\alpha \tau_{\leqslant n} C_\alpha$. Using the compactness of C in $\tau_{\leqslant n}\mathcal{C}$, we deduce that C is a retract of $\tau_{\leqslant n} C_\alpha$. We are therefore reduced to proving that $\tau_{\leqslant n} C_\alpha$ is a compact object of $\tau_{\leqslant m}\mathcal{C}$, which follows by inspecting the cofiber sequence

$$\Sigma^{n+1}\Omega^{n+1} C_\alpha \to C_\alpha \to \tau_{\leqslant n} C_\alpha.$$

If X is any n-truncated object of $\mathcal{C}$, then assumption (2) guarantees that we can write X as the colimit of a filtered diagram $\{C_\alpha\}$, where each C_α is a compact object of $\tau_{\leqslant n}\mathcal{C}$ (and therefore an almost compact object of $\mathcal{C}$). Since the map $\bigoplus C_\alpha \to X$ is an epimorphism on π_0, we see that $\mathcal{C}$ satisfies condition (b') of Definition C.6.5.1 and is therefore weakly coherent, as desired. □

Corollary C.6.5.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between Grothendieck prestable ∞-categories which induces an equivalence $\tau_{\leqslant n}\mathcal{C} \to \tau_{\leqslant n}\mathcal{D}$ for all $n \geqslant 0$. Then $\mathcal{C}$ is weakly coherent if and only if $\mathcal{D}$ is weakly coherent.*

Proposition C.6.5.6. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Then $\mathcal{C}$ is weakly coherent if and only if the following conditions are satisfied:*

(i) The collection of almost compact objects of $\mathcal{C}$ is closed under finite limits.

(ii) Every object $D \in \mathcal{C}^\heartsuit$ can be written as a filtered colimit $\varinjlim D_\alpha$, where each D_α is an almost compact object of $\mathcal{C}$ which belongs to $\mathcal{C}^\heartsuit$.

Proof. The necessity of (i) is obvious, and the necessity of (ii) follows from the proof of Proposition C.6.5.4. Conversely, suppose that (i) and (ii) are satisfied. To prove that $\mathcal{C}$ is weakly coherent, we must show that for every n-truncated object $X \in \mathcal{C}$, there exists a map $u : \coprod C_i \to X$ which is an epimorphism on π_0, where each C_i is an almost compact object of $\mathcal{C}$. We proceed by induction on n. In the case $n = 0$, the desired result follows immediately from (ii). Let us therefore assume that $n > 0$. Set $\overline{X} = \tau_{\leqslant n-1} X$, so that our inductive hypothesis guarantees the existence of a map $\overline{u} : \bigoplus \overline{C}_i \to \overline{X}$ which is an epimorphism on π_0, where each $\overline{C}_i$ is an almost compact object of $\mathcal{C}$. Using (ii), we can write $\pi_n X$ as a filtered colimit $\varinjlim D_\alpha$, where each D_α is an almost compact object of $\mathcal{C}$ which belongs to $\mathcal{C}^\heartsuit$. We then have a fiber sequence $X \to \overline{X} \xrightarrow{v} \varinjlim \Sigma^{n+1} D_\alpha$ in the ∞-category $\mathcal{C}$. For each index i, the assumption that $\overline{C}_i$ is almost compact guarantees that the composite map

$$\overline{C}_i \to \bigoplus \overline{C}_i \xrightarrow{\overline{u}} \overline{X} \xrightarrow{v} \varinjlim \Sigma^{n+1} D_\alpha$$

factors through some $\Sigma^{n+1} D_{\alpha(i)}$ for some index $\alpha(i)$. Choose such a factorization and set $C_i = \operatorname{fib}(\overline{C}_i) \to \Sigma^{n+1} D_{\alpha(i)}$. It follows from (i) that C_i is an almost compact object of $\mathcal{C}$. We now complete the proof by observing that $\overline{u}$ fits into a commutative diagram

$$\begin{array}{ccc} \bigoplus C_i & \xrightarrow{u} & X \\ \downarrow & & \downarrow \\ \bigoplus \overline{C}_i & \xrightarrow{\overline{u}} & \overline{X}, \end{array}$$

where u is also an epimorphism on π_0 (since the vertical maps induce isomorphisms on π_0). □

In the setting of weakly n-complicial Grothendieck prestable ∞-categories, the criterion of Proposition C.6.5.4 can be sharpened.

Proposition C.6.5.7. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category which is weakly n-complicial for some $n \geqslant 0$. The following conditions are equivalent:*

(1) *The Grothendieck prestable ∞-category $\mathcal{C}$ is weakly coherent.*

(2) *The ∞-category $\tau_{\leqslant n} \mathcal{C}$ is compactly generated and the collection of compact objects of $\tau_{\leqslant n} \mathcal{C}$ is closed under finite limits.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition C.6.5.4 (and does not require our assumption that $\mathcal{C}$ is weakly n-complicial). Assume that (2) is satisfied, and let $\mathcal{A}$ be the full subcategory of $\mathcal{C}^\heartsuit$ spanned by those objects which are compact when viewed as objects of $\tau_{\leqslant n} \mathcal{C}$. Assumption (2) implies that $\mathcal{A}$ is closed under the formation of kernels and cokernels in $\mathcal{C}^\heartsuit$ and is therefore an abelian category. Let $\mathcal{E} \subseteq \operatorname{Sp}(\mathcal{C})$ be the full subcategory spanned by those objects X having the following properties:

- The object X is truncated: that is, $X \in \mathrm{Sp}(\mathcal{C})_{\leqslant m}$ for $m \gg 0$.
- For every integer m, the object $\pi_m X \in \mathrm{Sp}(\mathcal{C})^{\heartsuit} \simeq \mathcal{C}^{\heartsuit}$ belongs to $\mathcal{A}$.
- The object X belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant -k}$ for $k \gg 0$ (since the t-structure on $\mathrm{Sp}(\mathcal{C})$ is right complete, this is equivalent to the requirement that $\pi_{-k} X$ vanishes for $k \gg 0$).

Then $\mathcal{E}$ is an essentially small stable subcategory of $\mathrm{Sp}(\mathcal{C})$ which is closed under truncations, so that the t-structure on $\mathrm{Sp}(\mathcal{C})$ induces a bounded t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ on $\mathcal{E}$. Set $\mathcal{C}' = \mathrm{Ind}(\mathcal{E}_{\geqslant 0})$. Then the inclusion functor $\iota : \mathcal{E}_{\geqslant 0} \hookrightarrow \mathrm{Sp}(\mathcal{C})_{\geqslant 0} \simeq \mathcal{C}$ extends to a functor $\lambda \in \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}', \mathcal{C})$. We next prove:

(*i*) For every cofiber sequence $C' \to C \to C''$ in $\mathcal{C}$, if C' and C'' are compact objects of $\tau_{\leqslant n} \mathcal{C}$, then C is also a compact object of $\tau_{\leqslant n} \mathcal{C}$.

To prove (*i*), we invoke assumption (2) to write C as the colimit of a filtered diagram $\{C_\alpha\}$, where each C_α is a compact object of $\tau_{\leqslant n} \mathcal{C}$. Using the compactness of C'', we deduce that there exists an index α for which the composite map $C_\alpha \to C \to C''$ induces an epimorphism on π_0. Set $C'_\alpha = C' \times_C C_\alpha \simeq \mathrm{fib}(C_\alpha \to C'')$. Since the collection of compact objects of $\tau_{\leqslant n} \mathcal{C}$ is closed under finite limits, C'_α is compact in $\tau_{\leqslant n} \mathcal{C}$. We now observe that the pullback diagram

$$\begin{array}{ccc} C'_\alpha & \longrightarrow & C_\alpha \\ \downarrow & & \downarrow \\ C' & \longrightarrow & C \end{array}$$

is also a pushout square (in both $\mathcal{C}$ and $\tau_{\leqslant n} \mathcal{C}$), so that C is also a compact object of $\tau_{\leqslant n} \mathcal{C}$. We now show:

(*ii*) An object $C \in \tau_{\leqslant n} \mathcal{C}$ is compact (when viewed as an object of $\tau_{\leqslant n} C$) if and only if $\pi_i C$ in $\mathcal{A}$ for $0 \leqslant i \leqslant n$.

The "if" direction follows immediately from (*i*) (since C can be written as a successive extension of objects of the form $\Sigma^i(\pi_i C)$) To prove the converse, we observe that $\pi_i C$ fits into a cofiber sequence

$$\Sigma(\Omega^{i+1} C) \to \Omega^i C \to \pi_i C$$

in the ∞-category $\tau_{\leqslant n} \mathcal{C}$.

Combining (*ii*) with our assumption that $\tau_{\leqslant n} \mathcal{C}$ is compactly generated, we deduce:

(*iii*) The functor λ induces an equivalence $\tau_{\leqslant n} \mathcal{C}' \to \tau_{\leqslant n} \mathcal{C}$.

We now prove:

(*iv*) The ∞-category $\mathcal{C}'$ is n-complicial.

Fix an object $X \in \mathcal{C}'$; we wish to prove that there exists a map $\overline{X} \to X$ in $\mathcal{C}'$ which is an epimorphism on π_0, where $\overline{X}$ is n-truncated. Write X as a filtered colimit $\varinjlim\{X_\alpha\}$, where each X_α is an object of $\mathcal{E}_{\geqslant 0}$. We can then view each X_α as a truncated object of ∞-category $\mathcal{C}$. Since $\mathcal{C}$ is weakly n-complicial, we can choose morphisms $\overline{X}_\alpha \to X_\alpha$ in $\mathcal{C}$ which induce epimorphisms on π_0, where each $\overline{X}_\alpha$ is an n-truncated object of $\mathcal{C}$. Using our assumption that $\tau_{\leqslant n}\mathcal{C}$ is compactly generated, we can write each $\overline{X}_\alpha$ as the colimit of a filtered diagram $\{\overline{X}_{\alpha,\beta}\}$ where each $\overline{X}_{\alpha,\beta}$ is a compact object of $\tau_{\leqslant n}\mathcal{C}$. Then we can view each $\overline{X}_{\alpha,\beta}$ as an n-truncated object of $\mathcal{E}_{\geqslant 0}$. Taking $\overline{X} = \bigoplus_{\alpha,\beta}\overline{X}_{\alpha,\beta}$, we obtain a proof of (iv).

Combining (iii), (iv), and Proposition C.5.9.3, we deduce that the functor $\lambda : \mathcal{C}' \to \mathcal{C}$ induces an equivalence on completions: that is, it restricts to an equivalence $\tau_{\leqslant m}\mathcal{C}' \to \tau_{\leqslant m}\mathcal{C}$ for all $m \leqslant 0$. Identifying $\tau_{\leqslant m}\mathcal{C}'$ with $\mathrm{Ind}(\mathcal{E}_{\geqslant 0} \cap \mathcal{E}_{\leqslant m})$, we see that an object $X \in \tau_{\leqslant m}\mathcal{C}$ is compact if and only if $\pi_i X \in \mathcal{A}$ for $0 \leqslant i \leqslant m$. Allowing m to vary, we conclude that an object $X \in \mathcal{C}$ is almost compact if and only if $\pi_i X \in \mathcal{A}$ for all $i \geqslant 0$. From this description, we immediately deduce that $\mathcal{C}$ satisfies the equivalent conditions of Proposition C.6.4.5. To complete the proof that $\mathcal{C}$ is weakly coherent, it will suffice to show that for each truncated object $X \in \mathcal{C}$, there exists a map $\bigoplus_\alpha C_\alpha \to X$ which is an epimorphism on π_0, where each C_α is almost compact. Using our assumption that $\mathcal{C}$ is weakly n-complicial, we can reduce to the case where X belongs to $\tau_{\leqslant n}\mathcal{C}$, in which case the desired result follows immediately from (iii). □

Corollary C.6.5.8. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category which is n-complicial for some $n \geqslant 0$. The following conditions are equivalent:*

(1) *The Grothendieck prestable ∞-category $\mathcal{C}$ is coherent.*

(2) *The ∞-category $\tau_{\leqslant n}\mathcal{C}$ is compactly generated and the collection of compact objects of $\tau_{\leqslant n}\mathcal{C}$ is closed under finite limits.*

Proof. The implication (1) ⇒ (2) follows immediately from Proposition C.6.5.7. Conversely, suppose that condition (2) is satisfied and let X be an object of $\mathcal{C}$; we wish to show that there exists a map $\bigoplus C_\alpha \to X$ which is an epimorphism on π_0, where each C_α is almost compact. Using our assumption that $\mathcal{C}$ is n-complicial, we can reduce to the case where X is n-truncated, in which case the desired result follows from Proposition C.6.5.7. □

Let $\mathcal{A}$ be a Grothendieck abelian category. Then the following conditions are equivalent:

- The category $\mathcal{A}$ is compactly generated and the collection of compact objects of $\mathcal{A}$ is closed under finite limits.
- There is an equivalence $\mathcal{A} \simeq \mathrm{Ind}(\mathcal{A}_0)$ for some essentially small abelian category $\mathcal{A}_0$.

Specializing Proposition **??** to the case $n = 0$, we obtain the following:

Corollary C.6.5.9. *Let $\mathcal{A}_0$ be an essentially small abelian category and set $\mathcal{A} = \mathrm{Ind}(\mathcal{A}_0)$. Then the Grothendieck prestable ∞-categories $\check{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ and $\mathcal{D}(\mathcal{A})_{\geqslant 0}$ are coherent, and the Grothendieck prestable ∞-category $\widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ is weakly coherent.*

C.6.6 Separated Coherent Grothendieck Prestable ∞-Categories

Let $\mathcal{C}$ be a coherent Grothendieck prestable ∞-category. Our next goal is to articulate a precise sense in which $\mathcal{C}$ is "controlled" by its almost compact objects. We begin with an elementary observation.

Proposition C.6.6.1. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category which is separated and coherent and let $\mathcal{C}^{\mathrm{ac}} \subseteq \mathcal{C}$ be the full subcategory spanned by the almost compact objects. Then $\mathcal{C}^{\mathrm{ac}}$ is essentially small.*

Warning C.6.6.2. The conclusion of Proposition C.6.6.1 is false if $\mathcal{C}$ is not separated: if X is a nonzero object of $\mathcal{C}$ whose truncations $\tau_{\leqslant n} X$ vanish for all n, then taking a coproduct of any number of copies of X yields an almost compact object of $\mathcal{C}$.

We will deduce Proposition C.6.6.1 from the following more precise assertion:

Lemma C.6.6.3. *Let $\mathcal{C}$ be a coherent separated Grothendieck prestable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a generating subcategory which is closed under finite colimits and extensions and consists of almost compact objects of $\mathcal{C}$. Then every almost compact object $X \in \mathcal{C}$ can be obtained as the geometric realization of a simplicial object $X_\bullet$, where each X_n belongs to $\mathcal{C}_0$.*

Proof. Let X be an object of $\mathcal{C}$ which is almost compact. We will construct a sequence of cofiber sequences
$$X(n) \to X \to \Sigma^n Y(n),$$
where each $X(n)$ belongs to $\mathcal{C}_0$. The construction proceeds by induction. In the case $n = 0$, we take $X(n) = 0$ and $Y(n) = X$. To carry out the inductive step, let us assume that we have constructed a cofiber sequence $X(n) \xrightarrow{u} X \to \Sigma^n Y(n)$ with $X(n) \in \mathcal{C}_0$. Then X and $X(n)$ are both almost compact, so that $Y(n) \simeq \Omega^n \operatorname{cofib}(u)$ is also almost compact. Using our assumption that $\mathcal{C}_0$ is a generating subcategory of $\mathcal{C}$, we can choose a map $\bigoplus_{i \in I} C_i \to Y(n)$ which induces an epimorphism $\bigoplus \pi_0 C_i \to \pi_0 Y(n)$, where each C_i belongs to $\mathcal{C}_0$. Since $Y(n)$ is almost compact, the truncation $\pi_0 Y(n)$ is a compact object of the abelian category $\mathcal{C}^{\heartsuit}$, so we can choose a finite subset $I_0 \subseteq I$ for which the induced map $v : \bigoplus_{i \in I_0} C_i \to Y(n)$ also induces an epimorphism on π_0. Set $C = \bigoplus_{i \in I_0} C_i$, so that C is an object of $\mathcal{C}_0$. We now define $X(n+1) = X \times_{\Sigma^n Y(n)} \Sigma^n C$, so that we have $\operatorname{cofib}(X(n+1) \to X) \simeq \Sigma^{n+1}(\operatorname{fib}(v))$. By construction, we have a cofiber sequence
$$X(n) \to X(n+1) \to \Sigma^n C$$

so that $X(n+1)$ also belongs to $\mathcal{C}_0$. Note that each of the maps $X(n) \to X$ induces an isomorphism $\pi_i X(n) \to \pi_i X$ for $i \leqslant n-2$. Invoking our assumption that $\mathcal{C}$ is separated, we see that X can be identified with the direct limit $\varinjlim X(n)$.

By construction, the cofiber of each map $X(n) \to X(n+1)$ is an n-fold suspension of an object of $\mathcal{C}_0$. Applying Theorem C.1.3.1, we deduce the existence of an equivalence $\{X(n+1)\}_{n \geqslant 0} \simeq \{\mathrm{sk}_n(E_\bullet)\}_{n \geqslant 0}$ for some simplicial object $X_\bullet$ of $\mathcal{C}_0$, so that $X \simeq |X_\bullet|$. □

Remark C.6.6.4. In the situation of Lemma C.6.6.3, the closure properties of $\mathcal{C}_0$ can be weakened: all we really need is that $\mathcal{C}_0$ is closed under finite direct sums.

Proof of Proposition C.6.6.1. Since $\mathcal{C}$ is presentable, there exists a small collection of objects $\{C_\alpha\}$ which span a generating subcategory of $\mathcal{C}$ (in the sense of Definition C.2.1.1). Using our assumption that $\mathcal{C}$ is coherent, we can arrange that each C_α is almost compact. Let $\mathcal{C}_0$ denote the smallest full subcategory of $\mathcal{C}$ which contains each C_α and is closed under finite colimits, geometric realizations, and extensions. Then $\mathcal{C}_0$ is essentially small, and Lemma C.6.6.3 shows that every almost compact object of $\mathcal{C}$ belongs to $\mathcal{C}_0$. □

Theorem C.6.6.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be separated Grothendieck prestable ∞-categories and let $\mathcal{C}^{\mathrm{ac}} \subseteq \mathcal{C}$ denote the full subcategory spanned by the almost compact objects. If $\mathcal{C}$ is coherent, then the restriction functor $\mathrm{LFun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{ac}}, \mathcal{D})$ is fully faithful, and its essential image is spanned by those functors $\mathcal{C}^{\mathrm{ac}} \to \mathcal{D}$ which preserve finite colimits.*

Proof. Since $\mathcal{C}^{\mathrm{ac}}$ is closed under finite limits and colimits in $\mathcal{C}$, the inclusion functor $\mathcal{C}^{\mathrm{ac}} \hookrightarrow \mathcal{C}$ extends to a functor $F : \mathrm{Ind}(\mathcal{C}^{\mathrm{ac}}) \to \mathcal{C}$ which preserves small colimits and finite limits. Let $G : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\mathrm{ac}})$ be a right adjoint to F. Let us identify $\mathrm{Ind}(\mathcal{C}^{\mathrm{ac}})$ with the full subcategory of $\mathrm{Fun}((\mathcal{C}^{\mathrm{ac}})^{\mathrm{op}}, \mathcal{S})$ spanned by the left exact functors, so that G is given by the formula $G(C)(D) = \mathrm{Map}_{\mathcal{C}}(D, C)$. If $\mathcal{C}$ is coherent, then $\mathcal{C}^{\mathrm{ac}} \subseteq \mathcal{C}$ is a generating subcategory of $\mathcal{C}$, so that Theorem C.2.1.6 implies that G is fully faithful.

It follows from Proposition C.6.6.1 that $\mathcal{C}^{\mathrm{ac}}$ is an essentially small prestable ∞-category so that $\mathrm{Ind}(\mathcal{C}^{\mathrm{ac}})$ is a Grothendieck prestable ∞-category (Proposition C.6.1.1). Applying Proposition C.2.3.8, we see that G restricts to an equivalence $\mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\mathrm{ac}})/\mathcal{E}$, where $\mathcal{E} \subseteq \mathrm{Ind}(\mathcal{C}^{\mathrm{ac}})$ is the localizing subcategory spanned by those objects of $\mathrm{Ind}(\mathcal{C}^{\mathrm{ac}})$ which are annihilated by F.

It follows from Proposition C.6.5.7 that the functor F induces an equivalence $\tau_{\leqslant n} \mathrm{Ind}(\mathcal{C}^{\mathrm{ac}}) \to \tau_{\leqslant n} \mathcal{C}$ for each $n \geqslant 0$. Consequently, if $X \in \mathrm{Ind}(\mathcal{C}^{\mathrm{ac}})$ is annihilated by F, then we must have $\tau_{\leqslant n} X \simeq 0$ for all $n \geqslant 0$: that is, X belongs to the essential image of the suspension functor $\Sigma^m : \mathrm{Ind}(\mathcal{C}^{\mathrm{ac}}) \to \mathrm{Ind}(\mathcal{C}^{\mathrm{ac}})$ for each $m \geqslant 0$. It follows that any colimit-preserving functor $H : \mathrm{Ind}(\mathcal{C}^{\mathrm{ac}}) \to \mathcal{D}$ must carry X into the essential image of the functor $\Sigma^m : \mathcal{D} \to \mathcal{D}$ for each $m \geqslant 0$. Since $\mathcal{D}$ is separated, we must have $HX \simeq 0$. That is, any colimit-preserving functor $\mathrm{Ind}(\mathcal{C}^{\mathrm{ac}}) \to \mathcal{D}$ automatically annihilates the localizing subcategory $\mathcal{E} \subseteq \mathrm{Ind}(\mathcal{C}^{\mathrm{ac}})$.

Applying Proposition C.2.3.10, we deduce that composition with F induces an equivalence of ∞-categories $\mathrm{LFun}(\mathcal{C}, \mathcal{D}) \to \mathrm{LFun}(\mathrm{Ind}(\mathcal{C}^{\mathrm{ac}}), \mathcal{D})$. It now suffices to observe that composition with the Yoneda embedding $\mathcal{C}^{\mathrm{ac}} \hookrightarrow \mathrm{Ind}(\mathcal{C}^{\mathrm{ac}})$ determines a fully faithful embedding $\mathrm{LFun}(\mathrm{Ind}(\mathcal{C}^{\mathrm{ac}}), \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{ac}}, \mathcal{D})$, whose essential image is spanned by those functors which preserve finite colimits (Propositions HTT.5.3.5.11 and HTT.5.5.1.9). □

Remark C.6.6.6. In the situation of Theorem C.6.6.5, let $f : \mathcal{C} \to \mathcal{D}$ be a functor which preserves small colimits. Then the following conditions are equivalent:

(i) The functor f is left exact.

(ii) The composite functor $\mathrm{Ind}(\mathcal{C}^{\mathrm{ac}}) \to \mathcal{C} \xrightarrow{f} \mathcal{D}$ is left exact.

(iii) The restriction $f|_{\mathcal{C}^{\mathrm{ac}}}$ is left exact.

The equivalence of (i) and (ii) follows from Remark C.2.3.11, and the equivalence of (ii) and (iii) from the fact that filtered colimits in $\mathcal{D}$ are left exact.

Remark C.6.6.7. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category which is coherent and separated, and let $\mathcal{C}^{\mathrm{ac}} \subseteq \mathcal{C}$ denote the full subcategory spanned by the almost compact objects. Then $\mathcal{C}^{\mathrm{ac}}$ has the following properties:

(a) The ∞-category $\mathcal{C}^{\mathrm{ac}}$ is prestable (since it is a full subcategory of the prestable ∞-category $\mathcal{C}$ which is closed under finite colimits and extensions).

(b) The ∞-category $\mathcal{C}^{\mathrm{ac}}$ admits finite limits (since $\mathcal{C}$ is assumed to be coherent).

(c) The ∞-category $\mathcal{C}^{\mathrm{ac}}$ admits geometric realizations of simplicial objects (since the collection of almost compact objects of $\mathcal{C}$ is closed under geometric realizations, by virtue of Proposition C.6.4.4).

(d) The ∞-category $\mathcal{C}^{\mathrm{ac}}$ is essentially small (Proposition C.6.6.1).

(e) The ∞-category $\mathcal{C}^{\mathrm{ac}}$ is separated (since $\mathcal{C}$ is separated).

Our next goal is to show that every ∞-category $\mathcal{E}$ satisfying conditions (a) through (e) of Remark C.6.6.7 has the form $\mathcal{C}^{\mathrm{ac}}$, where $\mathcal{C}$ is a Grothendieck prestable ∞-category which is separated and coherent (Theorem C.6.6.14). The main ingredient we will need is the following:

Proposition C.6.6.8. *Let $\mathcal{E}$ be an essentially small separated prestable ∞-category which admits finite limits and geometric realizations. Let X be an object of $\mathrm{Ind}(\mathcal{E})$, and let us view X as a left exact functor $\mathcal{E}^{\mathrm{op}} \to \mathcal{S}$. The following conditions are equivalent:*

(a) *For every simplicial object* $E_\bullet$ *of* $\mathcal{E}$*, the canonical map* $X(|E_\bullet|) \to \operatorname{Tot} X(E_\bullet)$ *is a homotopy equivalence.*

(b) *For every simplicial object* $E_\bullet$ *of* $\mathcal{E}$ *which satisfies* $|E_\bullet| \simeq 0$*, the totalization* $\operatorname{Tot} X(E_\bullet)$ *is contractible.*

(c) *For every diagram* $E(0) \to E(1) \to E(2) \to \cdots$ *in* $\mathcal{E}$*, where each* $E(n)$ *belongs to the essential image of* $\Sigma^n : \mathcal{E} \to \mathcal{E}$*, the limit* $\varprojlim X(E(n))$ *is contractible.*

(d) *The object* X *belongs to the separated quotient* $\operatorname{Ind}(\mathcal{E})/\operatorname{Ind}(\mathcal{E})_{\geqslant\infty}$*; here* $\operatorname{Ind}(\mathcal{E})_{\geqslant\infty}$ *denotes the localizing subcategory of* $\operatorname{Ind}(\mathcal{E})$ *spanned by those objects* Y *for which* $\tau_{\leqslant n} Y \simeq 0$ *for all* $n \geqslant 0$*.*

Proof. The implication $(a) \Rightarrow (b)$ is obvious. Conversely, suppose that $E_\bullet$ is an arbitrary simplicial object of $\mathcal{E}$. Let $E''_\bullet$ denote the constant simplicial object of $\mathcal{E}$ with the value $|E_\bullet|$, and form a fiber sequence $E'_\bullet \to E_\bullet \to E''_\bullet$. It is easy to see that each of the maps $E_n \to E''_n = |E_\bullet|$ induces an epimorphism on π_0, so that the fiber sequence $E'_\bullet \to E_\bullet \to E''_\bullet$ is also a cofiber sequence. Passing to geometric realizations, we obtain a cofiber sequence $|E'_\bullet| \to |E_\bullet| \xrightarrow{u} |E''_\bullet|$ where u is an equivalence, so that $|E'_\bullet| \simeq 0$. Since the functor X is left exact, applying the functor X yields a fiber sequence of cosimplicial (pointed) spaces

$$X(E''_\bullet) \to X(E_\bullet) \to X(E'_\bullet)$$

and therefore a fiber sequence of totalizations

$$\operatorname{Tot} X(E''_\bullet) \xrightarrow{v} \operatorname{Tot} X(E_\bullet) \to \operatorname{Tot} X(E'_\bullet).$$

Note that the cosimplicial space $X(E''_\bullet)$ is constant with value $X(|E_\bullet|)$ and that v can be dentified with the tautological map $X(|E_\bullet|) \to \operatorname{Tot} X(E_\bullet)$. If assumption (b) is satisfied, then $\operatorname{Tot} X(E'_\bullet)$ is contractible and the map v is a homotopy equivalence; this proves $(b) \Rightarrow (a)$.

Now let $E_\bullet$ be a simplicial object of $\mathcal{E}$, and consider its diagram of partial skeleta

$$\operatorname{sk}_0(E_\bullet) \to \operatorname{sk}_1(E_\bullet) \to \operatorname{sk}_2(E_\bullet) \to \cdots$$

Since $\tau_{\leqslant n} \mathcal{E}$ is equivalent to an $(n+1)$-category, the canonical map $\tau_{\leqslant n} \operatorname{sk}_{n+1}(E_\bullet) \to \tau_{\leqslant n}|E_\bullet|$ is an equivalence for every n. Using the separatedness of $\mathcal{E}$, we see that $|E_\bullet| \simeq 0$ if and only if $\operatorname{sk}_n(E_\bullet)$ belongs to the essential image of the functor $\Sigma^n : \mathcal{E} \to \mathcal{E}$ for every $n \geqslant 0$. Conversely, given any diagram

$$E(0) \to E(1) \to E(2) \to \cdots$$

in $\mathcal{E}$ where each $E(n)$ belongs to the essential image of $\Sigma^n : \mathcal{E} \to \mathcal{E}$, each cofiber $\operatorname{cofib}(E(n-1) \to E(n))$ is an extension of $\Sigma E(n-1)$ by $E(n)$, and therefore also belongs to the essential image of $\Sigma^n : \mathcal{E} \to \mathcal{E}$. It then follows from Theorem C.1.3.1 that we have an equivalence $\{E(n)\}_{n \geqslant 0} \simeq \{\mathrm{sk}_n(E_\bullet)\}_{n \geqslant 0}$ for an essentially unique simplicial object $E_\bullet$ of $\mathcal{E}$. The left exactness of X then gives equivalences $X(E(n)) = X(\mathrm{sk}_n(E_\bullet)) \simeq \mathrm{Tot}^n X(E_\bullet)$, so that $\varprojlim \{X(E(n))\}_{n \geqslant 0} \simeq \varprojlim \{\mathrm{Tot}^n X(E_\bullet)\}_{n \geqslant 0} \simeq \mathrm{Tot}\, X(E_\bullet)$. This shows that conditions (*b*) and (*c*) are equivalent.

We now show that (*d*) implies (*c*). Let $j : \mathcal{E} \to \mathrm{Ind}(\mathcal{E})$ denote the Yoneda embedding. If $\{E(n)\}_{n \geqslant 0}$ is a diagram in $\mathcal{E}$ which each $E(n)$ is an n-fold suspension, then $\{j(E(n))\}_{n \geqslant 0}$ is a diagram in $\mathrm{Ind}(\mathcal{E})$ where each $j(E(n))$ is also an n-fold suspension. It follows that the direct limit $\varinjlim j(E(n))$ belongs to the full subcategory $\mathrm{Ind}(\mathcal{E})_{\geqslant \infty}$. Assumption (*d*) then implies that the space

$$\varprojlim X(E(n)) \simeq \varprojlim \mathrm{Map}_{\mathrm{Ind}(\mathcal{E})}(j(E(n)), X) \simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{E})}(\varinjlim j(E(n)), X)$$

is contractible.

We now complete the proof by showing that (*c*) implies (*d*). Suppose that X satisfies (*c*); we wish to show that X belongs to the quotient $\mathrm{Ind}(\mathcal{E}) / \mathrm{Ind}(\mathcal{E})_{\geqslant \infty}$. In other words, we wish to show that if we are given any cofiber sequence $Y' \to Y \to Y''$ with $Y'' \in \mathrm{Ind}(\mathcal{E})_{\geqslant \infty}$, then the induced map $\theta : \mathrm{Map}_{\mathrm{Ind}(\mathcal{E})}(Y, X) \to \mathrm{Map}_{\mathrm{Ind}(\mathcal{E})}(Y', X)$ is a homotopy equivalence. Since $\pi_0 Y'' \simeq 0$, we also have a cofiber sequence $\Omega Y'' \to Y' \to Y$, so that θ fits into a fiber sequence

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{E})}(Y, X) \xrightarrow{\theta} \mathrm{Map}_{\mathrm{Ind}(\mathcal{E})}(Y', X) \to \mathrm{Map}_{\mathrm{Ind}(\mathcal{E})}(\Omega Y'', X).$$

It will therefore suffice to prove the contractibility of the space $\mathrm{Map}_{\mathrm{Ind}(\mathcal{E})}(\Omega Y'', X)$.

Let $\mathcal{C}_0$ denote the full subcategory of $\mathrm{Ind}(\mathcal{E})$ spanned by those objects of the form $\varinjlim j(E(n))$, where $\{E(n)\}_{n \geqslant 0}$ is a diagram in $\mathcal{E}$ for which each $E(n)$ is an n-fold suspension. Let $\mathcal{C}$ denote the smallest full subcategory of $\mathrm{Ind}(\mathcal{E})$ which contains $\mathcal{C}_0$ and is closed under small colimits and extensions. Note that the collection of those objects $Z \in \mathrm{Ind}(\mathcal{E})$ for which $\mathrm{Map}_{\mathrm{Ind}(\mathcal{E})}(Z, X)$ is contractible is closed under small colimits and extensions. Consequently, if (*c*) is satisfied, then $\mathrm{Map}_{\mathrm{Ind}(\mathcal{E})}(Z, X)$ is contractible for each $Z \in \mathcal{C}$. Similarly, $\mathrm{Ind}(\mathcal{E})_{\geqslant \infty}$ contains $\mathcal{C}_0$ and is closed under small colimits and extensions, so we have $\mathcal{C} \subseteq \mathrm{Ind}(\mathcal{E})_{\geqslant \infty}$. We will complete the proof by showing that $\mathcal{C} = \mathrm{Ind}(\mathcal{E})_{\geqslant \infty}$. To this end, we first prove the following:

- $(*)$ Let Z be a nonzero object of $\mathrm{Ind}(\mathcal{E})_{\geqslant \infty}$. Then there exists a morphism $e : C \to Z$ which is not nullhomotopic, where $C \in \mathcal{C}_0$.

To prove $(*)$, we first note that our assumption that Z is nonzero guarantees that there exists an object $E(0) \in \mathcal{E}$ and a morphism $e_0 : j(E(0)) \to Z$ which is not nullhomotopic. To prove $(*)$, it will suffice to show that we can extend e_0 to a compatible sequence of maps $\{e_n : j(E(n)) \to Z\}_{n \geqslant 0}$, where each $E(n) \in \mathcal{E}$ is an n-fold suspension. The construction proceeds by induction. Let us therefore suppose that $n > 0$ and that $e_n : j(E(n)) \to Z$ has been constructed. Let us represent Z by a diagram $\{Z_\alpha\}_{\alpha \in A}$ in $\mathcal{E}$, indexed by a filtered partially ordered set A. It follows that there exists some index α such that e_n is represented by a map $\overline{e}_n : E(n-1) \to Z_\alpha$ in the ∞-category $\mathcal{E}$. Since $\tau_{\leqslant n} Z$ vanishes, there exists $\beta \geqslant \alpha$ for which the map $\tau_{\leqslant n} Z_\alpha \to \tau_{\leqslant n} Z_\beta$ is nullhomotopic. It follows that the composite map

$$E(n) \xrightarrow{\overline{e}_n} Z_\alpha \to Z_\beta$$

factors through $E(n+1) = \tau_{\geqslant n+1} Z_\beta$, which provides the desired extension of e_n.

We now use $(*)$ to show that $\mathcal{C} = \mathrm{Ind}(\mathcal{E})_{\geqslant \infty}$. It follows from Proposition HA.1.4.4.11 that $\mathcal{C}$ is a presentable ∞-category. Applying Corollary HTT.5.5.2.9 , we deduce that the inclusion $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{E})$ admits a right adjoint G. Let Z be an object of $\mathrm{Ind}(\mathcal{E})_{\geqslant \infty}$ and let $v : G(Z) \to Z$ be the counit map; we wish to show that v is an equivalence. Suppose otherwise: then $\mathrm{cofib}(v)$ is a nonzero object of $\mathrm{Ind}(\mathcal{E})_{\geqslant \infty}$, so assertion $(*)$ implies that there exists an object $C \in \mathcal{C}_0$ and a morphism $e : C \to \mathrm{cofib}(v)$ which is not nullhomotopic. Let Z_C denote the fiber product $Z \times_{\mathrm{cofib}(v)} C$, so that we have a diagram of cofiber sequences

$$\begin{array}{ccccc} G(Z) & \longrightarrow & Z_C & \longrightarrow & C \\ \downarrow & \swarrow & \downarrow & & \downarrow e \\ G(Z) & \xrightarrow{v} & Z & \longrightarrow & \mathrm{cofib}(v). \end{array}$$

Since $\mathcal{C}$ is closed under extensions, the object Z_C belongs to $\mathcal{C}$. The universal property of $G(Z)$ then guarantees the existence of a dotted arrow as indicated in the diagram. This shows that e is nullhomotopic, contrary to our assumption. $\square$

Proposition C.6.6.9. *Let $\mathcal{E}$ be an essentially small separated prestable ∞-category which admits finite limits and geometric realizations. Let $\mathcal{C}$ be the full subcategory of* $\mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S})$ *spanned by those functors which preserve finite limits and totalizations of cosimplicial objects. Then $\mathcal{C}$ is a separated coherent Grothendieck prestable ∞-category, and the Yoneda embedding $j : \mathcal{E} \hookrightarrow \mathcal{C}$ induces an equivalence from $\mathcal{E}$ to the full subcategory $\mathcal{C}^{\mathrm{ac}} \subseteq \mathcal{C}$ spanned by the almost compact objects.*

Proof. The ∞-category $\mathrm{Ind}(\mathcal{E})$ is Grothendieck prestable by virtue of Proposition C.6.1.1. Using Proposition C.6.6.8, we see that $\mathcal{C}$ can be identified with the separated quotient $\mathrm{Ind}(\mathcal{E}) / \mathrm{Ind}(\mathcal{E})_{\geqslant \infty}$ of $\mathrm{Ind}(\mathcal{E})$, so that $\mathcal{C}$ is a separated Grothendieck prestable ∞-category. For each $n \geqslant 0$, we have an equivalence

$$\tau_{\leqslant n} \mathcal{C} \simeq \tau_{\leqslant n}(\mathrm{Ind}(\mathcal{E}) / \mathrm{Ind}(\mathcal{E})_{\geqslant \infty}) = \tau_{\leqslant n} \mathrm{Ind}(\mathcal{E}) \simeq \mathrm{Ind}(\tau_{\leqslant n} \mathcal{E}).$$

It follows immediately that the Yoneda embedding $j : \mathcal{E} \to \mathcal{C}$ carries each object of $\mathcal{E}$ to an almost compact object of $\mathcal{C}$. Using the criterion of Proposition C.6.5.4, we see that $\mathcal{C}$

is weakly coherent. Since $\mathcal{C}$ is a localization of $\mathrm{Ind}(\mathcal{E})$, the essential image of $j : \mathcal{E} \to \mathcal{C}$ is a generating subcategory of $\mathcal{C}$ (in the sense of Definition C.2.1.1) which consists of almost compact objects, so that $\mathcal{C}$ is coherent. To complete the proof, it will suffice to show that every almost compact object of $\mathcal{C}$ belongs to the essential image of j. To see this, we first observe that the essential image of j is closed under finite limits and colimits, and is therefore also closed under extensions. Applying Lemma C.6.6.3, we see that every almost compact object $C \in \mathcal{C}$ can be written as a geometric realization $|C_\bullet|$, where each C_k belongs to the essential image of j. By construction, the functor j preserves geometric realizations, so that every almost compact object of $\mathcal{C}$ belongs to the essential image of j. □

Proposition C.6.6.9 shows that every ∞-category $\mathcal{E}$ which satisfies the conditions of Remark C.6.6.7 has the form $\mathcal{C}^{\mathrm{ac}}$, for some separated coherent Grothendieck prestable ∞-category $\mathcal{C}$. It follows from Theorem C.6.6.5 that the ∞-category $\mathcal{C}$ is uniquely determined up to equivalence. In fact, the ∞-category $\mathcal{C}$ depends functorially on $\mathcal{E}$. To articulate this more precisely, it will be convenient to introduce a variant of Definition C.3.4.2.

Definition C.6.6.10. Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves small colimits. Let $G : \mathcal{D} \to \mathcal{C}$ be a right adjoint to F (which exists by virtue of Corollary HTT.5.5.2.9). We will say that F is *almost compact* if, for each $n \geqslant 0$, the functor

$$G|_{\tau_{\leqslant n}\mathcal{D}} : \tau_{\leqslant n}\mathcal{D} \to \tau_{\leqslant n}\mathcal{C}$$

commutes with filtered colimits.

Example C.6.6.11. Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories. Then every functor $F : \mathcal{C} \to \mathcal{D}$ which is compact (in the sense of Definition C.3.4.2) is almost compact (in the sense of Definition C.6.6.10).

Example C.6.6.12. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $f : \mathrm{Sp}^{\mathrm{cn}} \to \mathcal{C}$ be a colimit-preserving functor. Then f is almost compact (in the sense of Definition C.6.6.10) if and only if the object $f(S) \in \mathcal{C}$ is almost compact (in the sense of Definition C.6.4.1).

Proposition C.6.6.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck prestable ∞-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves small colimits. Then:*

(1) *If F is almost compact, then F carries almost compact objects of $\mathcal{C}$ to almost compact objects of $\mathcal{D}$.*

(2) *If $\mathcal{C}$ is weakly coherent and F carries almost compact objects of $\mathcal{C}$ to almost compact objects of $\mathcal{D}$, then F is almost compact.*

Proof. Assume that F is almost compact and that $C \in \mathcal{C}$ is almost compact. For each $n \geqslant 0$, the functor $\tau_{\leqslant n}\,\mathcal{D} \to \mathcal{S}$ given by $D \mapsto \mathrm{Map}_{\mathcal{D}}(FC, D)$ can be expressed as a composition of functors

$$\tau_{\leqslant n}\,\mathcal{D} \xrightarrow{G} \tau_{\leqslant n}\,\mathcal{C} \xrightarrow{\mathrm{Map}_{\mathcal{C}}(C,\bullet)} \mathcal{S}$$

which commute with filtered colimits. Allowing n to vary, we deduce that FC is a compact object of $\mathcal{D}$, which proves (1).

Now suppose that $\mathcal{C}$ is weakly coherent. Then for each $n \geqslant 0$, the ∞-category $\tau_{\leqslant n}\,\mathcal{C}$ is compactly generated (Proposition C.6.5.7). It follows that the functor $G|_{\tau_{\leqslant n}\,\mathcal{D}}$ commutes with filtered colimits if and only if, for each compact object C of $\tau_{\leqslant n}\,\mathcal{C}$, the composite functor

$$\tau_{\leqslant n}\,\mathcal{D} \xrightarrow{G} \tau_{\leqslant n}\,\mathcal{C} \xrightarrow{\mathrm{Map}_{\mathcal{C}}(C,\bullet)} \mathcal{S}$$

commutes with filtered colimits. Note that in this case, C is almost compact when viewed as an object of $\mathcal{C}$. Consequently, if the functor F preserves almost compact objects, then FC is an almost compact object of $\mathcal{D}$, so that the construction $D \mapsto \mathrm{Map}_{\mathcal{C}}(C, GD) \simeq \mathrm{Map}_{\mathcal{D}}(FC, D)$ commutes with filtered colimits when restricted to $\tau_{\leqslant n}\,\mathcal{D}$, as desired. □

We can now formulate our main result.

Theorem C.6.6.14. *Let $\mathsf{Groth}_{\infty}^{\mathrm{coh}}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$ whose objects are coherent separated Grothendieck prestable ∞-categories and whose morphisms are almost compact functors. For each object $\mathcal{C} \in \mathsf{Groth}_{\infty}^{\mathrm{coh}}$, let $\mathcal{C}^{\mathrm{ac}}$ denote the full subcategory of $\mathcal{C}$ spanned by the almost compact objects. Then the construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{ac}}$ determines a fully faithful embedding $\rho : \mathsf{Groth}_{\infty}^{\mathrm{coh}} \to \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{PSt}}$, whose essential image is spanned by those prestable ∞-categories which are separated and admit finite limits and geometric realizations.*

Proof. It follows from Proposition C.6.6.13 that the functor ρ is well-defined and from Theorem C.6.6.5 (together with Proposition C.6.6.13) that ρ is fully faithful. Essential surjectivity follows from Proposition C.6.6.9. □

We conclude this section with another observation about the relationship between separatedness and coherence.

Proposition C.6.6.15. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let $\mathcal{C}_{\geqslant\infty} \subseteq \mathcal{C}$ be the localizing subcategory spanned by those objects $X \in \mathcal{C}$ satisfying $\pi_n X \simeq 0$ for $n \geqslant 0$. Then $\mathcal{C}$ is coherent if and only if the quotient $\mathcal{C}\,/\,\mathcal{C}_{\geqslant\infty}$ is coherent.*

Proof. Let us regard $\mathcal{C}\,/\,\mathcal{C}_{\geqslant\infty}$ as a full subcategory of $\mathcal{C}$, and let $L : \mathcal{C} \to \mathcal{C}\,/\,\mathcal{C}_{\geqslant\infty}$ denote a left adjoint to the inclusion. Note that $\mathcal{C}\,/\,\mathcal{C}_{\geqslant\infty}$ contains every truncated object of $\mathcal{C}$ (so that L is equivalent to the identity when restricted to $\tau_{\leqslant n}\,\mathcal{C}$, for any $n \geqslant 0$). From this, we deduce:

(*i*) An object of $\mathcal{C}/\mathcal{C}_{\geqslant\infty}$ is almost compact if and only if it is almost compact when regarded as an object of $\mathcal{C}$.

(*ii*) An object $X \in \mathcal{C}$ is almost compact if and only if $LX \in \mathcal{C}/\mathcal{C}_{\geqslant\infty}$ is almost compact.

Suppose first that $\mathcal{C}$ is coherent. Then the collection of almost compact objects of $\mathcal{C}$ is closed under finite limits. Applying (*i*), we deduce that the collection of almost compact objects of $\mathcal{C}/\mathcal{C}_{\geqslant\infty}$ is closed under finite limits. For every object $X \in \mathcal{C}/\mathcal{C}_{\geqslant\infty}$, our assumption that $\mathcal{C}$ is coherent guarantees that we can find a morphism $u : \bigoplus C_\alpha \to X$ in $\mathcal{C}$, where each C_α is an almost compact object of $\mathcal{C}$ and u induces an epimorphism on π_0. Then $L(u)$ is a morphism in $\mathcal{C}/\mathcal{C}_{\geqslant\infty}$ which is also an epimorphism on π_0, whose domain $L(\bigoplus C_\alpha)$ can be identified with a coproduct in the ∞-category $\mathcal{C}/\mathcal{C}_{\geqslant\infty}$ of objects LC_α, which are almost compact by virtue of (*ii*). Allowing X to vary, we conclude that $\mathcal{C}/\mathcal{C}_{\geqslant\infty}$ is coherent.

We now prove the converse. Suppose that $\mathcal{C}/\mathcal{C}_{\geqslant\infty}$ is coherent. Then the collection of almost compact objects of $\mathcal{C}/\mathcal{C}_{\geqslant\infty}$ is closed under finite limits. Using (*ii*) and the left exactness of L, we deduce that the collection of almost compact objects of $\mathcal{C}$ is closed under finite limits. For every object $X \in \mathcal{C}$, our assumption that $\mathcal{C}/\mathcal{C}_{\geqslant\infty}$ is coherent guarantees that we can find a map $u : \bigoplus C_\alpha \to LX$ which is surjective on π_0, where each C_α is an almost compact object of $\mathcal{C}/\mathcal{C}_{\geqslant\infty}$. For each index α, set $\overline{C}_\alpha = C_\alpha \times_{LX} X$. Since L is left exact, each of the projection maps $\overline{C}_\alpha \to C_\alpha$ induces an equivalence $L\overline{C}_\alpha \simeq C_\alpha$. Using (*ii*), we see that each $\overline{C}_\alpha$ is an almost compact object of $\mathcal{C}$. Amalgamating the projection maps $\overline{C}_\alpha \to X$, we obtain a morphism $\overline{u} : \bigoplus \overline{C}_\alpha \to X$. Since $\overline{u}$ and u are equivalent after applying the functor L, our assumption that u induces an epimorphism on π_0 guarantees that $\overline{u}$ also induces an epimorphism on π_0. Allowing X to vary, we conclude that $\mathcal{C}$ is coherent, as desired. □

C.6.7 Anticomplete Coherent Grothendieck Prestable ∞-Categories

In the setting of anticomplete Grothendieck prestable ∞-categories, coherence takes a particularly simple form.

Theorem C.6.7.1. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. The following conditions are equivalent:*

(*a*) *The ∞-category $\mathcal{C}$ is a compactly generated, the functor $\Omega : \mathcal{C} \to \mathcal{C}$ preserves compact objects, and every compact object of $\mathcal{C}$ is truncated.*

(*b*) *There exist an essentially small stable ∞-category $\mathcal{E}$ equipped with a bounded t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ and an equivalence $\mathcal{C} \simeq \operatorname{Ind}(\mathcal{E}_{\geqslant 0})$.*

(*c*) *The Grothendieck prestable ∞-category $\mathcal{C}$ is anticomplete and coherent.*

(*d*) *The Grothendieck prestable ∞-category $\mathcal{C}$ is anticomplete and weakly coherent.*

Proof. The equivalence $(a) \Leftrightarrow (b)$ follows from Proposition C.5.5.5. We next show that $(b) \Rightarrow (c)$. Assume that there is an equivalence $f : \mathrm{Ind}(\mathcal{E}_{\geqslant 0}) \simeq \mathcal{C}$ for some essentially small stable ∞-category $\mathcal{E}$ equipped with a bounded t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$. Proposition C.5.5.5 implies that $\mathcal{C}$ is anticomplete. Let us abuse notation by identifying $\mathcal{E}_{\geqslant 0}$ with its essential image under the fully faithful embedding

$$\mathcal{E}_{\geqslant 0} \hookrightarrow \mathrm{Ind}(\mathcal{E}_{\geqslant 0}) \xrightarrow{f} \mathcal{C}.$$

For each $n \geqslant 0$, the functor f induces an equivalence $\mathrm{Ind}(\mathcal{E}_{\geqslant 0} \cap \mathcal{E}_{\leqslant n}) \simeq \tau_{\leqslant n}\mathcal{C}$. It follows that an object $X \in \mathcal{C}$ is almost compact if and only if each truncation $\tau_{\leqslant n}X$ belongs to $\mathcal{E}_{\geqslant 0}$. From this description, it follows immediately that the functor $\Omega : \mathcal{C} \to \mathcal{C}$ carries almost compact objects to almost compact objects. Moreover, the collection of almost compact objects of $\mathcal{C}$ contains $\mathcal{E}_{\geqslant 0}$ and is therefore a generating subcategory of $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{E}_{\geqslant 0})$. This proves that $\mathcal{C}$ is coherent, and Proposition C.5.5.5 implies that $\mathcal{C}$ is anticomplete.

The implication $(c) \Rightarrow (d)$ is obvious. We will complete the proof by showing that (d) implies (b). Assume that $\mathcal{C}$ is anticomplete and weakly coherent. Let $\mathcal{E}_{\geqslant 0}$ be the full subcategory of $\mathcal{C}$ spanned by those objects which are truncated and almost compact. Then $\mathcal{E}_{\geqslant 0}$ is closed under finite colimits in $\mathcal{C}$ and is therefore a prestable ∞-category. Since $\mathcal{C}$ is weakly coherent, the ∞-category $\mathcal{E}_{\geqslant 0}$ is also closed under finite limits in $\mathcal{C}$. Let $\mathcal{E} = \mathrm{SW}(\mathcal{E}_{\geqslant 0})$ be the Spanier-Whitehead ∞-category of $\mathcal{E}_{\geqslant 0}$ and let us abuse notation by identifying $\mathcal{E}_{\geqslant 0}$ with its image in $\mathcal{E}$, so that $\mathcal{E}$ admits a right-bounded t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$ (see Proposition C.1.2.9). By construction, every object of $\mathcal{E}_{\geqslant 0}$ is truncated, so this t-structure is also left-bounded. Set $\mathcal{C}' = \mathrm{Ind}(\mathcal{E}_{\geqslant 0})$, so that the inclusion $\mathcal{E}_{\geqslant 0} \hookrightarrow \mathcal{C}$ extends to functor $\lambda : \mathcal{C}' \to \mathcal{C}$ which preserves small colimits and finite limits. Using Proposition C.6.5.4, we see that λ induces an equivalence

$$\tau_{\leqslant n}\mathcal{C}' \simeq \mathrm{Ind}(\mathcal{E}_{\geqslant 0} \cap \mathcal{E}_{\leqslant n}) \to \tau_{\leqslant n}\mathcal{C}$$

for each $n \geqslant 0$. The ∞-category $\mathcal{C}'$ is anticomplete (by virtue of the implication $(b) \Rightarrow (c)$) and the ∞-category $\mathcal{C}$ is anticomplete by assumption. Applying Proposition C.5.9.2, we deduce that λ is an equivalence of ∞-categories. □

Corollary C.6.7.2. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. The following conditions are equivalent:*

(*i*) *The Grothendieck prestable ∞-category $\mathcal{C}$ is weakly coherent.*

(*ii*) *There exists a coherent Grothendieck prestable ∞-category $\mathcal{C}'$ and a functor $\lambda \in \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}', \mathcal{C})$ which induces an equivalence of completions $\widehat{\mathcal{C}}' \simeq \widehat{\mathcal{C}}$.*

Proof. The implication $(ii) \Rightarrow (i)$ follows immediately from Corollary C.6.5.5. Conversely, suppose that (i) is satisfied. Using Proposition C.5.5.9, we can choose a map $\lambda \in \mathrm{LFun}^{\mathrm{lex}}(\mathcal{C}', \mathcal{C})$ which induces an equivalence of completions $\widehat{\mathcal{C}}' \to \widehat{\mathcal{C}}$, where $\mathcal{C}'$ is anticomplete. Since $\mathcal{C}$ is weakly coherent, the Grothendieck prestable ∞-category $\mathcal{C}'$ is also weakly coherent (Corollary C.6.5.5) and therefore coherent (Theorem C.6.7.1). □

Combining Theorem C.6.7.1 with Corollary C.6.1.5, we obtain the following:

Corollary C.6.7.3. *Let $\mathcal{C}\mathrm{at}_\infty^{\mathrm{PSt},b}$ denote the full subcategory of $\mathcal{C}\mathrm{at}_\infty^{\mathrm{PSt}}$ spanned by those small prestable ∞-categories $\mathcal{C}$ satisfying the following conditions:*

(i) *The ∞-category $\mathcal{C}$ admits finite limits.*

(ii) *Every object of $\mathcal{C}$ is truncated.*

Then the construction $\mathcal{C} \mapsto \mathrm{Ind}(\mathcal{C})$ determines a fully faithful embedding $\mathrm{Ind} : \mathcal{C}\mathrm{at}_\infty^{\mathrm{PSt},b} \hookrightarrow \mathsf{Groth}_\infty^{\mathrm{c}}$, whose essential image is spanned by those Grothendieck prestable ∞-categories which are coherent and anticomplete.

Remark C.6.7.4. An ∞-category $\mathcal{C}$ which satisfies conditions (i) and (ii) of Corollary C.6.7.3 is automatically idempotent complete (since $\mathcal{C} = \bigcup_{n \geqslant 0} \tau_{\leqslant n} \mathcal{C}$, and each $\tau_{\leqslant n} \mathcal{C}$ is an $(n+1)$-category which admits finite limits).

Remark C.6.7.5. Let $\mathcal{C}$ and $\mathcal{D}$ be prestable ∞-categories satisfying conditions (i) and (ii) of Corollary C.6.7.3. Then a functor $f : \mathcal{C} \to \mathcal{D}$ is left exact if and only if the induced map $\mathrm{Ind}(f) : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ is left exact (this follows from the left exactness of filtered colimits in $\mathcal{D}$). Restricting to left exact functors on both sides, the equivalence of Corollary C.6.7.3 can be regarded a "linear analogue" of Theorem A.7.5.3.

C.6.8 Locally Noetherian Abelian Categories

We now give a brief review of the classical theory of locally Noetherian abelian categories. For a more complete exposition, we refer the reader to [74].

Definition C.6.8.1. Let $\mathcal{A}$ be an abelian category. For every object $X \in \mathcal{A}$, we let $\mathrm{Sub}(X)$ denote the partially ordered set of (isomorphism classes of) subobjects of X. We say that an object $X \in \mathcal{A}$ is *Noetherian* if the partially ordered set $\mathrm{Sub}(X)$ satisfies the ascending chain condition: that is, if every increasing sequence

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$$

in $\mathrm{Sub}(X)$ eventually stabilizes.

Proposition C.6.8.2. *Let $\mathcal{A}$ be an abelian category containing an exact sequence $0 \to X' \to X \to X'' \to 0$. Then X is Noetherian if and only if X' and X'' are Noetherian. In other words, the collection of Noetherian objects of $\mathcal{A}$ is closed under the formation of subobjects, quotient objects, and extensions.*

Proof. It is easy to see that if X is Noetherian, then X' and X'' must be Noetherian (note that we can identify $\mathrm{Sub}(X')$ and $\mathrm{Sub}(X'')$ with partially ordered subsets of $\mathrm{Sub}(X)$). To prove the converse, assume that X' and X'' are Noetherian, and suppose we are given an ascending sequence $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ of subobjects of X. For each $n \geqslant 0$, let X'_n and X''_n denote the kernel and image of the composite map $X_n \hookrightarrow X \to X''$, respectively. Then we can identify $\{X'_n\}_{n \geqslant 0}$ as an ascending chain of subobjects of X', and $\{X''_n\}_{n \geqslant 0}$ as an ascending chain of subobjects of X''. Using our assumption that X' and X'' are Noetherian, we deduce that there is an integer n_0 for which the inclusions $X'_{n_0} \hookrightarrow X'_n$ and $X''_{n_0} \hookrightarrow X''_n$ are isomorphisms for $n \geqslant n_0$. Applying the Snake Lemma to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X'_{n_0} & \longrightarrow & X_{n_0} & \longrightarrow & X''_{n_0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X'_n & \longrightarrow & X_n & \longrightarrow & X''_n & \longrightarrow & 0, \end{array}$$

we deduce that the chain $\{X_n\}_{n \geqslant 0}$ is constant for $n \geqslant n_0$. □

Corollary C.6.8.3. *Let $\mathcal{A}$ be an abelian category containing an exact sequence $X' \to X \to X''$. If X' and X'' are Noetherian, then X is Noetherian.*

Proposition C.6.8.4. *Let $\mathcal{A}$ be a Grothendieck abelian category containing an object X. The following conditions are equivalent:*

(1) *The object X is the colimit of its Noetherian subobjects.*

(2) *The object X can be written as the colimit of a small filtered diagram $\{X_\alpha\}$, where each X_α is a Noetherian object of $\mathcal{A}$.*

(3) *There exists an epimorphism $\bigoplus_\alpha X_\alpha \to X$, where each X_α is a Noetherian object of $\mathcal{A}$.*

Proof. It follows from Proposition C.6.8.2 that the Noetherian subobjects of X form a filtered partially ordered set, which shows that (1) ⇒ (2). The implication (2) ⇒ (3) is immediate. To show that (3) implies (1), we note that the existence of an epimorphism $\bigoplus_{\alpha \in A} X_\alpha \to X$ implies that X can be identified with the colimit of the subobjects given by the images of the maps $\bigoplus_{\alpha \in A_0} X_\alpha \to X$, where A_0 ranges over the finite subsets of A; each of these images is a Noetherian subobject of X by virtue of Proposition C.6.8.2. □

Definition C.6.8.5. Let $\mathcal{A}$ be an abelian category. We will say that $\mathcal{A}$ is *locally Noetherian* if it is a Grothendieck abelian category and every object $X \in \mathcal{A}$ satisfies the equivalent conditions of Proposition C.6.8.4.

Example C.6.8.6. Let R be an associative ring. Then the abelian category $\mathrm{LMod}_R^{\heartsuit}$ of (discrete) left R-modules is locally Noetherian if and only if the ring R is left Noetherian (that is, if and only if every left ideal $I \subseteq R$ is finitely generated).

Proposition C.6.8.7. *Let $\mathcal{A}$ be a locally Noetherian abelian category and let $X \in \mathcal{A}$ be a Noetherian object. Then X is compact: that is, the functor $Y \mapsto \mathrm{Hom}_{\mathcal{A}}(X, Y)$ commutes with filtered colimits.*

Proof. Suppose we are given a diagram $\{Y_\alpha\}_{\alpha \in A}$ in the category $\mathcal{A}$ which is indexed by a filtered partially ordered set A. We wish to show that the canonical map $\theta : \varinjlim_{\alpha \in A} \mathrm{Hom}_{\mathcal{A}}(X, Y_\alpha) \to \mathrm{Hom}_{\mathcal{A}}(X, Y)$ is an isomorphism. We first show that θ is injective. Suppose that we are given an element $f \in \ker(\theta)$, which we can represent by a morphism $f_\alpha : X \to Y_\alpha$ for some $\alpha \in A$. For each $\beta \geqslant \alpha$, let $K_\beta \subseteq X$ denote the kernel of the composite map $X \xrightarrow{f_\alpha} Y_\alpha \to Y_\beta$. Since $\mathcal{A}$ is a Grothendieck abelian category, the vanishing of the map $\theta(f) : X \to Y$ guarantees that X is the direct limit of the subobjects $\{K_\beta\}_{\beta \geqslant \alpha}$. Our assumption that X is Noetherian guarantees that $X = K_\beta$ for some $\beta \geqslant \alpha$, so that the composite map $X \xrightarrow{f_\alpha} Y_\alpha \to Y_\beta$ vanishes and therefore $f = 0$, as desired.

We now verify the surjectivity of the map θ. Fix a map $g : X \to Y$. For each $\alpha \in A$, let X_α denote the fiber product $X \times_Y Y_\alpha$. Let $q_\alpha : X_\alpha \to X$ denote the projection onto the first factor. To prove that g belongs to the image of θ, it will suffice to show that q_α admits a section for some $\alpha \in A$. Since $\mathcal{A}$ is a Grothendieck abelian category, the canonical map $\varinjlim X_\alpha \to X$ is an isomorphism. In particular, X is the direct limit of the the diagram of subobjects $\{\mathrm{im}\, q_\alpha\}_{\alpha \in A}$. Since X is Noetherian, it follows that there exists an index $\alpha \in A$ such that q_α is an epimorphism. Write X_α as a filtered colimit of Noetherian subobjects $\{X_{\alpha,i}\}_{i \in I}$. Then X is the colimit of the diagram of subobjects $\{\mathrm{im}\, q_\alpha|_{X_{\alpha,i}}\}_{i \in I}$, so there exists an index $i \in I$ such that $q_\alpha|_{X_{\alpha,i}}$ is an epimorphism. Let $K = \ker(q_\alpha|_{X_{\alpha,i}}$. Then K is Noetherian (Proposition C.6.8.2), and the composite map $K \hookrightarrow X_\alpha \to \varprojlim_{\beta \geqslant \alpha} X_\beta \simeq X$ vanishes. It follows from the first part of the proof that there exists $\beta \geqslant \alpha$ for which the composite map $K \hookrightarrow X_\alpha \to X_\beta$ vanishes. It follows that the map q_β admits a section (given by composing the induced map $X_{\alpha,i}/K \to X_\beta$ with the inverse of the canonical isomorphism $X_{\alpha,i}/K \xrightarrow{\sim} X$). □

Corollary C.6.8.8. *Let $\mathcal{A}$ be a locally Noetherian abelian category, and let $\mathcal{A}_0 \subseteq \mathcal{A}$ be the full subcategory spanned by the Noetherian objects. Then the inclusion $\mathcal{A}_0 \hookrightarrow \mathcal{A}$ extends to an equivalence of ∞-categories $F : \mathrm{Ind}(\mathcal{A}_0) \to \mathcal{A}$.*

Proof. There exists an essentially unique functor $F : \mathrm{Ind}(\mathcal{A}_0) \to \mathcal{A}$ which extends the inclusion map and commutes with filtered colimits. It follows from Proposition C.6.8.7 that F is fully faithful and from Definition C.6.8.5 that F is essentially surjective. □

Corollary C.6.8.9. *Let $\mathcal{A}$ be a locally Noetherian abelian category. Then $\mathcal{A}$ is compactly generated, and an object $X \in \mathcal{A}$ is compact if and only if it is Noetherian.*

Remark C.6.8.10. Suppose that $\mathcal{A}_0$ is a small abelian category in which every object is Noetherian. Then $\mathrm{Ind}(\mathcal{A}_0)$ is a locally Noetherian abelian category, and an object of $\mathrm{Ind}(\mathcal{A}_0)$ is Noetherian if and only if it belongs to the essential image of the fully faithful embedding $j : \mathcal{A}_0 \hookrightarrow \mathrm{Ind}(\mathcal{A}_0)$. To prove this, we first note that $\mathrm{Ind}(\mathcal{A}_0)$ compactly generated. It follows that filtered colimits in $\mathrm{Ind}(\mathcal{A}_0)$ are left exact and therefore $\mathrm{Ind}(\mathcal{A}_0)$ is a Grothendieck abelian category. We next claim that if $M \in \mathrm{Ind}(\mathcal{A}_0)$ belongs to the essential image of j, then every subobject of M belongs to the essential image of j. To prove this, write $M = j(N)$ for some $N \in \mathcal{A}_0$. Let M' be a subobject of M, and write $M' \simeq \varinjlim_{\alpha \in A} j(N_\alpha)$ where the colimit is taken over some filtered partially ordered set A. Then $M' \simeq \varinjlim_{\alpha \in A} j(\mathrm{im}(N_\alpha \to N))$. Since N is Noetherian, we deduce that there exists $\alpha \in A$ such that $M' \simeq \mathrm{im}(N_\alpha \to N)$, so that M' belongs to the image of j.

The above argument shows that for each object $N \in \mathcal{A}_0$, the partially ordered set of isomorphism classes of subobjects of N is isomorphic to the partially ordered set of isomorphism classes of subobjects of $j(N)$. Since $N \in \mathcal{A}_0$ is Noetherian, we conclude that $j(N) \in \mathrm{Ind}(\mathcal{A}_0)$ is Noetherian. Now suppose that $M \simeq \varinjlim_{\alpha \in A} j(N_\alpha)$ is an arbitrary object of $\mathrm{Ind}(\mathcal{A}_0)$. Then M is equivalent to the filtered colimit of subobjects $\varinjlim_{\alpha \in A} \mathrm{im}(j(N_\alpha) \to M)$. Each of these subobjects is a quotient of $j(N_\alpha)$, and therefore Noetherian. This proves that the abelian category $\mathrm{Ind}(\mathcal{A}_0)$ is locally Noetherian. If the object M above is Noetherian, then there exists an index $\alpha \in A$ such that $M \simeq \mathrm{im}(j(N_\alpha) \to M)$. Then M is a quotient of $j(N_\alpha)$. Since $\ker(j(N_\alpha) \to M)$ is a subobject of $j(N_\alpha)$, it belongs to the essential image of j. It follows that every Noetherian object $M \in \mathrm{Ind}(\mathcal{A}_0)$ belongs to the essential image of j.

The collection of locally Noetherian abelian categories is closed under passage to left exact localization:

Proposition C.6.8.11. *Let $\mathcal{A}$ be a locally Noetherian abelian category and let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a localizing subcategory. Then $\mathcal{A}_0$ and $\mathcal{A}/\mathcal{A}_0$ are locally Noetherian.*

Proof. It follows immediately from the definitions that $\mathcal{A}_0$ is locally Noetherian. Let $L : \mathcal{A} \to \mathcal{A}/\mathcal{A}_0$ be a left adjoint to the inclusion. For each object $X \in \mathcal{A}$, let $\mathrm{Sub}_{\mathcal{A}}(X)$ denote the set of isomorphism classes of subobjects of X, and for $Y \in \mathcal{A}/\mathcal{A}_0$ define $\mathrm{Sub}_{\mathcal{A}/\mathcal{A}_0}(Y)$ similarly. For each object $X \in \mathcal{A}$, the construction $(Y_0 \in \mathrm{Sub}_{\mathcal{A}/\mathcal{A}_0}(LX)) \mapsto (Y_0 \times_{LX} X \in \mathrm{Sub}_{\mathcal{A}}(X))$ determines an order-preserving map $\rho : \mathrm{Sub}_{\mathcal{A}/\mathcal{A}_0}(LX) \to \mathrm{Sub}_{\mathcal{A}}(X)$. It follows from the left exactness of L that this map is injective (it has a left inverse, given by the construction

$(X_0 \in \mathrm{Sub}_{\mathcal{A}}(X)) \mapsto (LX_0 \in \mathrm{Sub}_{\mathcal{A}/\mathcal{A}_0}(LX)))$. Consequently, if X is a Noetherian object of $\mathcal{A}$, then LX is a Noetherian object of $\mathcal{A}/\mathcal{A}_0$. Every object $X \in \mathcal{A}$ can be written as a union of its Noetherian subobjects, so that $LX \in \mathcal{A}/\mathcal{A}_0$ can be written as a filtered colimit of subobjects which are Noetherian in $\mathcal{A}/\mathcal{A}_0$. Since the functor L is essentially surjective, it follows that $\mathcal{A}/\mathcal{A}_0$ is locally Noetherian as desired. □

C.6.9 Locally Noetherian Prestable ∞-Categories

We now introduce an ∞-categorical analogue of Definition C.6.8.5.

Definition C.6.9.1. Let $\mathcal{C}$ be prestable ∞-category. We will say that $\mathcal{C}$ is *locally Noetherian* if it is a weakly coherent Grothendieck prestable ∞-category (Definition C.6.5.1) and the abelian category $\mathcal{C}^{\heartsuit}$ is locally Noetherian.

Remark C.6.9.2. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. The condition that $\mathcal{C}$ is locally Noetherian depends only on the completion of $\mathcal{C}$. In particular, if $F : \mathcal{C} \to \mathcal{D}$ is a functor of Grothendieck prestable ∞-categories which induces an equivalence $\tau_{\leqslant n}\mathcal{C} \simeq \tau_{\leqslant n}\mathcal{D}$ for $n \geqslant 0$, then $\mathcal{C}$ is locally Noetherian if and only if $\mathcal{D}$ is locally Noetherian (see Corollary C.6.5.5).

Proposition C.6.9.3. *Let $\mathcal{C}$ be a locally Noetherian prestable ∞-category and let X be an object of $\mathcal{C}$. Then X is almost compact if and only if $\pi_n X$ is a Noetherian object of $\mathcal{C}^{\heartsuit}$ for each $n \geqslant 0$.*

Proof. Combine Corollaries C.6.4.6 and C.6.8.9. □

Example C.6.9.4. Let R be a connective $\mathbb{E}_1$-ring. Then R is left Noetherian if and only if the Grothendieck prestable ∞-category $\mathrm{LMod}_R^{\mathrm{cn}}$ is locally Noetherian (this follows from Examples C.6.5.3 and C.6.8.6.

Example C.6.9.5. Let $\mathcal{A}$ be a locally Noetherian abelian category. Then the Grothendieck prestable ∞-categories $\check{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$, $\mathcal{D}(\mathcal{A})_{\geqslant 0}$, and $\widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ are locally Noetherian (this follows immediately from Corollary C.6.5.9).

Warning C.6.9.6. It follows immediately from the definition that if $\mathcal{C}$ is a locally Noetherian prestable ∞-category, then the heart $\mathcal{C}^{\heartsuit}$ is a locally Noetherian abelian category. However, the converse is not true in general. For a counterexample, take $\mathcal{C} = \mathrm{Mod}_R^{\heartsuit}$, where R is a connective $\mathbb{E}_\infty$-ring for which $\pi_0 R$ is Noetherian but some homotopy group $\pi_n R$ is not finitely generated as a module over $\pi_0 R$).

Proposition C.6.9.7. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. The following conditions are equivalent:*

(a) *The prestable ∞-category $\mathcal{C}$ is locally Noetherian.*

(b) *For every integer $n \geqslant 0$, the ∞-category $\tau_{\leqslant n}\mathcal{C}$ is compactly generated. Moreover, if $C \in \tau_{\leqslant n}\mathcal{C}$ is a compact object, then $\pi_n C$ is a Noetherian object of $\mathcal{C}^{\heartsuit}$.*

Proof. The implication $(a) \Rightarrow (b)$ follows from Proposition C.6.9.3. Conversely, suppose that (b) is satisfied. We will prove the following:

$(*)$ For $n \geqslant 0$, an object $C \in \tau_{\leqslant n}\mathcal{C}$ is compact if and only if $\pi_m C$ is a Noetherian object of $\mathcal{C}^{\heartsuit}$ for $0 \leqslant m \leqslant n$.

Assume $(*)$ for the moment. Since the collection of Noetherian objects of $\mathcal{C}^{\heartsuit}$ is closed under finite direct sums (Proposition C.6.8.2), it follows that the collection of compact objects of $\tau_{\leqslant n}\mathcal{C}$ is also closed under finite direct sums, for each $n \geqslant 0$. Moreover, if $f : C \to D$ is a morphism between compact objects of $\tau_{\leqslant n}\mathcal{C}$, then we have exact sequences $\pi_{m+1} D \to \pi_m \operatorname{fib}(f) \to \pi_m C$ in the abelian category $\mathcal{C}^{\heartsuit}$ where the outer terms are Noetherian (by virtue of $(*)$). It follows from Corollary C.6.8.3 that $\pi_m \operatorname{fib}(f)$ is Noetherian for $0 \leqslant m \leqslant n$, so that $\operatorname{fib}(f)$ is also a compact object of $\tau_{\leqslant n}\mathcal{C}$ (by virtue of $(*)$). It follows that the collection of compact objects of $\tau_{\leqslant n}\mathcal{C}$ is closed under finite limits, so that $\mathcal{C}$ is weakly coherent (Proposition C.6.5.4). Applying (b) in the case $n = 0$, we see that $\mathcal{C}^{\heartsuit}$ is a compactly generated abelian category in which every compact object is Noetherian, so that $\mathcal{C}^{\heartsuit}$ is locally Noetherian and therefore $\mathcal{C}$ is locally Noetherian.

It remains to prove $(*)$. The "only if" direction follows immediately from (b) (note that if C is a compact object of $\tau_{\leqslant n}\mathcal{C}$, then $\tau_{\leqslant m} C$ is a compact object of $\tau_{\leqslant m}\mathcal{C}$ for $0 \leqslant m \leqslant n$). To prove the converse, let us consider the following more refined statement:

$(*_k)$ Let $0 \leqslant k \leqslant n$ and let C be an object of $\tau_{\leqslant n}\mathcal{C}$ such that $\pi_m C$ is Noetherian for $0 \leqslant m \leqslant k$. Then there exists a compact object $D \in \tau_{\leqslant n}\mathcal{C}$ and a map $f : D \to C$ such that the cofiber $\operatorname{cofib}(f)$ (formed in the ∞-category $\mathcal{C}$) satisfies $\pi_m \operatorname{cofib}(f) \simeq 0$ unless $k < m \leqslant n$.

Note that the "if" direction of (b) is an immediate consequence of $(*_n)$. Let us regard n as fixed; we will prove assertion $(*_k)$ using induction on k. The case $k = -1$ is trivial (in this case, we can take $C' = 0$). Let us therefore assume that $k \geqslant 0$ and that C is an n-truncated object of $\mathcal{C}$ such that $\pi_m C$ is Noetherian for $0 \leqslant m \leqslant k$. Applying assumption (b), we can write C as the colimit of a filtered diagram $\{C_\alpha\}$, where each C_α is a compact object of $\tau_{\leqslant n}\mathcal{C}$. It follows from our assumption that $\pi_0 C$ is Noetherian that we can choose some index α for which the map $u : C_\alpha \to C$ induces an epimorphism on π_0. For every integer m, we have a short exact sequence $\pi_{m+1} C \to \pi_m \operatorname{fib}(u) \to \pi_m C_\alpha$. Using assumption (b) and Proposition C.6.8.2, we see that $\pi_m \operatorname{fib}(u)$ is a Noetherian object of $\mathcal{C}^{\heartsuit}$ for $m < k$. Applying our inductive hypothesis, we deduce that there exists a compact object $E \in \tau_{\leqslant n}\mathcal{C}$ and a morphism $g : E \to \operatorname{fib}(u)$ such that $\pi_m \operatorname{cofib}(g)$ vanishes unless $k \leqslant m \leqslant n$. Let D' denote the cofiber of the composite map $E \xrightarrow{g} \operatorname{fib}(u) \to C_\alpha$, so that we have a cofiber sequence $\operatorname{cofib}(g) \to D' \xrightarrow{f'} C$. It follows that D' is an n-truncated object of $\mathcal{C}$. Moreover, it is a compact object of $\tau_{\leqslant n} \mathcal{C}$ (since both E and C_α are compact objects of $\tau_{\leqslant n} \mathcal{C}$).

Let K denote the kernel of the map $(\pi_n f') : \pi_n D' \to \pi_n C$, and let D be the cofiber of the canonical map

$$\Sigma^n K = \Sigma^n \Omega^n \operatorname{fib}(f') \to \operatorname{fib}(f') \to D'.$$

Since D' is a compact object of $\tau_{\leqslant n} \mathcal{C}$, assumption (b) implies that $\pi_n D'$ is a Noetherian object of $\mathcal{C}^{\heartsuit}$, so the subobject $K \subseteq \pi_n D'$ is also Noetherian. In particular, K is a compact object of $\mathcal{C}^{\heartsuit}$ (Proposition C.6.8.7), so that $\Sigma^n K$ is a compact object of $\tau_{\leqslant n} \mathcal{C}$. It follows that $D = \operatorname{cofib}(K \to D')$ is also a compact object of $\tau_{\leqslant n} \mathcal{C}$. The map f' admits an essentially unique factorization $D' \to D \xrightarrow{f} C$. Unwinding the definitions, we have isomorphisms $\pi_m \operatorname{cofib}(f) \xleftarrow{\sim} \pi_m \operatorname{cofib}(f') \simeq \pi_{m-1} \operatorname{cofib}(g)$ for $m \neq n+1$ and $\pi_{n+1} \operatorname{cofib}(f) \simeq 0$. It follows that $\pi_m \operatorname{cofib}(f)$ vanishes unless $k < m \leqslant n$, so that f satisfies the requirements of $(*_k)$. $\square$

Proposition C.6.9.8. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Then $\mathcal{C}$ is locally Noetherian if and only if it satisfies the following conditions:*

(i) The abelian category $\mathcal{C}^{\heartsuit}$ is locally Noetherian.

(ii) Each Noetherian object $C \in \mathcal{C}^{\heartsuit}$ is almost compact when viewed as an object of $\mathcal{C}$.

Proof. The necessity of condition (i) is immediate, and the necessity of (ii) follows from Proposition C.6.9.3. Conversely, suppose that conditions (i) and (ii) are satisfied; we wish to show that $\mathcal{C}$ is locally Noetherian. We will show that $\mathcal{C}$ satisfies condition (b) of Proposition C.6.9.7. Fix an integer $n \geqslant 0$, and let $\mathcal{E} \subseteq \tau_{\leqslant n} \mathcal{C}$ be the full subcategory spanned by those objects $C \in \tau_{\leqslant n} \mathcal{C}$ for which the objects $\pi_m C \in \mathcal{C}^{\heartsuit}$ are Noetherian for $0 \leqslant m \leqslant n$. It follows from (ii) that each object of $\mathcal{E}$ is an almost compact object of $\mathcal{C}$, and in particular a compact object of $\tau_{\leqslant n} \mathcal{C}$. It follows that the inclusion $\mathcal{E} \hookrightarrow \tau_{\leqslant n} \mathcal{C}$ extends to a fully faithful embedding $f : \operatorname{Ind}(\mathcal{E}) \to \tau_{\leqslant n} \mathcal{C}$ which commutes with filtered colimits. To complete the proof, it will suffice to show that f is essentially surjective. When $n = 0$, this follows from Corollary C.6.8.8. We now proceed by induction on n. Assume therefore that $n > 0$ and that we are given an object $C \in \tau_{\leqslant n} \mathcal{C}$; we wish to show that C belongs to the essential image of f. Note that we have a fiber sequence

$$C \to \tau_{\leqslant n-1} C \xrightarrow{\eta} \Sigma^{n+1}(\pi_n C)$$

in the prestable ∞-category $\mathcal{C}$. Using our inductive hypothesis, we see that $\tau_{\leqslant n-1} C$ can be written as the colimit of a filtered diagram $\{D_\alpha\}$, where each D_α is an $(n-1)$-truncated object of $\mathcal{E}$. For each index α, let η_α denote the composite map $D_\alpha \to \tau_{\leqslant n-1} \to \Sigma^{n+1}(\pi_n C)$, so that $C \simeq \varinjlim \mathrm{fib}(\eta_\alpha)$. Since the essential image of f is closed under filtered colimits, it will suffice to show that each fiber $\mathrm{fib}(\eta_\alpha)$ belongs to the essential image of f. Using assumption (i) (and Corollary C.6.8.8), we can write $\pi_n C$ as a union of Noetherian subobjects. Since D_α is an almost compact object of $\mathcal{C}$, the map $\eta_\alpha : D_\alpha \to \Sigma^{n+1}(\pi_n C)$ factors through $\Sigma^{n+1} E$ for some Noetherian subobject $E \subseteq \pi_n C$. For every Noetherian subobject $E' \subseteq \pi_n C$ containing E, let $\eta_{E'}$ denote the composite map $D_\alpha \to \Sigma^{n+1}(E) \to \Sigma^{n+1}(E')$, so that we can identify $\mathrm{fib}(\eta_\alpha)$ with the colimit $\varinjlim_{E \subseteq E' \subseteq \pi_n C} \mathrm{fib}(\eta_{E'})$. We now complete the proof by observing that each $\mathrm{fib}(\eta_{E'})$ belongs to $\mathcal{E}$. □

Proposition C.6.9.9. *Let $\mathcal{C}$ be a locally Noetherian prestable ∞-category, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a localizing subcategory, and let $L : \mathcal{C} \to \mathcal{C} / \mathcal{C}_0$ be a left adjoint to the inclusion. Then:*

(1) *The functor L is almost compact (Definition C.6.6.10).*

(2) *The quotient $\mathcal{C} / \mathcal{C}_0$ is locally Noetherian.*

Proof. Let $\mathcal{A}_0 = \mathcal{C}_0 \cap \mathcal{C}^\heartsuit$, and let $\mathcal{C}_1$ be the smallest localizing subcategory of $\mathcal{C}$ which contains $\mathcal{A}_0$. Then L factors as a composition $\mathcal{C} \xrightarrow{L'} \mathcal{C} / \mathcal{C}_1 \xrightarrow{L''} \mathcal{C} / \mathcal{C}_0$, where L' is a left adjoint to the inclusion $\iota : \mathcal{C} / \mathcal{C}_1 \hookrightarrow \mathcal{C}$ and the functor L'' induces an equivalence on completions. By virtue of Proposition C.5.2.8, an object $X \in \mathcal{C}$ is $\mathcal{C}_1$-local if and only if $\mathrm{Ext}^*_{\mathrm{Sp}(\mathcal{C})}(A, X)$ vanishes for each object $A \in \mathcal{A}_0$. Since $\mathcal{C}$ is locally Noetherian, every object $A \in \mathcal{A}_0$ can be written as a union of Noetherian subobjects. Consequently, an object $X \in \mathcal{C}$ is $\mathcal{C}_1$-local if and only if $\mathrm{Ext}^*_{\mathrm{Sp}(\mathcal{C})}(A, X)$ vanishes for every Noetherian object $A \in \mathcal{A}_0$. Fix an integer $n \geqslant 0$. Since every Noetherian object of $\mathcal{C}^\heartsuit$ is compact when viewed as an object of $\tau_{\leqslant n} \mathcal{C}$ (Proposition C.6.9.8), it follows that the collection of n-truncated $\mathcal{C}_1$-local objects of $\mathcal{C}$ is closed under filtered colimits. Allowing n to vary, we deduce that the functor L' is almost compact. Since L'' induces an equivalence on completions, it follows that the functor $L \simeq L'' \circ L'$ is also almost compact.

We now claim that $\mathcal{C} / \mathcal{C}_0$ is locally Noetherian. It follows from Proposition C.6.8.11 that the abelian category $(\mathcal{C} / \mathcal{C}_0)^\heartsuit \simeq \mathcal{C}^\heartsuit / \mathcal{A}_0$ is locally Noetherian. Moreover, using (1) (or the proof of Proposition C.6.8.11), we see that the functor L carries Noetherian objects of $\mathcal{C}^\heartsuit$ to Noetherian objects of $(\mathcal{C} / \mathcal{C}_0)^\heartsuit$. Every object $X \in \mathcal{C}^\heartsuit$ can be written as a filtered union of its Noetherian subobjects $\{X_\alpha\}$, so that $LX \in (\mathcal{C} / \mathcal{C}_0)^\heartsuit$ is a filtered union of Noetherian subobjects $\{LX_\alpha\}$. If LX is a Noetherian object of $(\mathcal{C} / \mathcal{C}_0)^\heartsuit$, we must have $LX \simeq LX_\alpha$ for some α. In this case, X_α is an almost compact object of $\mathcal{C}$ (Proposition C.6.9.8), so that (1) implies that $LX \simeq LX_\alpha$ is an almost compact object of $\mathcal{C} / \mathcal{C}_0$. In particular, every Noetherian object of $(\mathcal{C} / \mathcal{C}_0)^\heartsuit$ is an almost compact object of $\mathcal{C} / \mathcal{C}_0$, so that $\mathcal{C} / \mathcal{C}_0$ is locally Noetherian by virtue of Proposition C.6.9.8. □

C.6.10 Injective Objects in the Locally Noetherian Setting

Let $\mathcal{A}$ be a locally Noetherian abelian category. It follows from Proposition C.5.6.12 that an object $Q \in \mathcal{A}$ is injective if and only if the group $\mathrm{Ext}^1_{\mathcal{A}}(X, Q)$ vanishes for every Noetherian object $X \in \mathcal{A}$. We now establish an ∞-categorical analogue:

Proposition C.6.10.1. *Let $\mathcal{C}$ be a locally Noetherian prestable ∞-category and let $Q \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. The following conditions are equivalent:*

- (a) *The object Q is injective: that is, for each object $X \in \mathcal{C}^{\heartsuit}$, the groups $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X, Q)$ vanish for $n > 0$ (see Definition HA.**??**).*
- (b) *For each Noetherian object $X \in \mathcal{C}^{\heartsuit}$, the groups $\mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X, Q)$ vanish for $n > 0$.*

Proof. The implication $(a) \Rightarrow (b)$ is immediate. Conversely, suppose that (b) is satisfied. Choose an object $X \in \mathcal{C}^{\heartsuit}$ and an element $\eta \in \mathrm{Ext}^n_{\mathrm{Sp}(\mathcal{C})}(X, Q)$ for $n > 0$, which we can identify with a morphism $f : X \to \Sigma^n Q$. We wish to show that f is nullhomotopic. Define a transfinite sequence of subobjects $\{X_\alpha \subseteq X\}$ as follows:

- If $\alpha = 0$, we set $X_\alpha = 0$.
- If α is a nonzero limit ordinal, we set $X_\alpha = \varinjlim_{\beta < \alpha} X_\beta$.
- If $\alpha = \beta + 1$ is a successor ordinal, we take X_α to be the preimage in X of a nonzero Noetherian subobject $Y_\alpha \subseteq X/X_\beta$, provided that such an object exists: otherwise, we take $Y_\alpha = 0$ and set $X_\alpha = X_\beta$.

For each ordinal α, let K_α denote the homotopy fiber product $\{f\} \times_{\mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(X_\alpha, \Sigma^n Q)} \{0\}$ (in other words, the classifying space for nullhomotopies of $f|_{X_\alpha}$). Note that K_0 is contractible, that $K_\alpha \simeq \varprojlim_{\beta < \alpha} K_\beta$ when α is a nonzero limit ordinal, and that we have canonical fiber sequences $K_\alpha \xrightarrow{\rho_\beta} K_\beta \to \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(Y_\alpha, \Sigma^n Q)$ when $\alpha = \beta + 1$ is a successor ordinal. Since each Y_α is Noetherian, assumption (b) guarantees that the mapping spaces $\mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(Y_\alpha, \Sigma^n Q)$ are connected. It follows that each of the maps ρ_β is surjective on connected components, so that each of the spaces K_α is nonempty. Because X has a bounded number of isomorphism classes of subobjects, we must have $X_\alpha \simeq X$ for $\alpha \gg 0$, so the fact that K_α is nonempty guarantees that the map $f : X \to \Sigma^n Q$ is nullhomotopic as desired. □

Proposition C.6.10.2. *Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Then $\mathcal{C}$ is locally Noetherian if and only if it satisfied the following conditions:*

- (i) *The abelian category $\mathcal{C}^{\heartsuit}$ is locally Noetherian.*
- (ii) *The collection of injective objects of $\mathrm{Sp}(\mathcal{C})$ is closed under small filtered colimits.*

Proof. The necessity of condition (i) is obvious, and the necessity of (ii) follows from Proposition C.6.10.1. Conversely, suppose that $\mathcal{C}$ satisfies conditions (i) and (ii); we wish to show that $\mathcal{C}$ is locally Noetherian. By virtue of Proposition C.6.9.8, it will suffice to show that every Noetherian object $X \in \mathcal{C}^{\heartsuit}$ is compact when viewed as an object of $\mathrm{Sp}(\mathcal{C})_{\leqslant n}$, for every integer $n \geqslant 0$. Our proof proceeds by induction on n; when $n = 0$, the desired result follows from Proposition C.6.8.7. To carry out the inductive step, it will suffice to show that the construction $(Y \in \mathrm{Sp}(\mathcal{C})_{\leqslant n}) \mapsto (\mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(X, Y) \in \mathcal{S})$ commutes with colimits indexed by $\mathrm{N}(A)$ for every filtered partially ordered set A. Set $\mathcal{C}' = \mathrm{Fun}(\mathrm{N}(A), \mathcal{C})$ and let $\vec{Y}$ be an object of $\mathrm{Sp}(\mathcal{C}')_{\leqslant n} \simeq \mathrm{Fun}(\mathrm{N}(A), \mathrm{Sp}(\mathcal{C})_{\leqslant n})$. Choose a morphism $f : \vec{Y} \to \Sigma^n \vec{Q}$ in $\mathrm{Sp}(\mathcal{C}')$ which exhibits $\vec{Q}$ as an injective hull of $\vec{Y}$ (Example C.5.7.9), so that $\vec{Y}' = \mathrm{fib}(f)$ belongs to $\mathrm{Sp}(\mathcal{C}')_{\leqslant n-1}$. We then have a commutative diagram of fiber sequences

$$\begin{array}{ccccc} \varinjlim \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(X, \vec{Y}'(\alpha)) & \longrightarrow & \varinjlim \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(X, \vec{Y}(\alpha)) & \longrightarrow & \varinjlim \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(X, \Sigma^n \vec{Q}(\alpha)) \\ \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho''} \\ \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(X, \varinjlim \vec{Y}'(\alpha)) & \longrightarrow & \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(X, \varinjlim \vec{Y}(\alpha)) & \longrightarrow & \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(X, \Sigma^n \varinjlim \vec{Q}(\alpha)). \end{array}$$

Note that for each $\alpha \in A$, evaluation at α determines a functor $e_\alpha : \mathrm{Sp}(\mathcal{C}') \to \mathrm{Sp}(\mathcal{C})$ with a t-exact left adjoint (given by left Kan extension along the inclusion $\{\alpha\} \hookrightarrow A$). It follows that e_α carries injective objects of $\mathrm{Sp}(\mathcal{C}')$ to injective objects of $\mathrm{Sp}(\mathcal{C})$. In particular, the objects $\vec{Q}(\alpha)$ are injective for each $\alpha \in A$, so the mapping spaces $\mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(X, \Sigma^n \vec{Q}(\alpha))$ are n-connective Assumption (ii) guarantees that $\varinjlim \vec{Q}(\alpha)$ is also an injective object of $\mathrm{Sp}(\mathcal{C})$, so that the codomain of ρ'' is also n-connective. It follows that the map

$$\pi_m(\rho'') : \pi_m \varinjlim \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(X, \Sigma^n \vec{Q}(\alpha)) \to \pi_m \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(X, \Sigma^n \varinjlim \vec{Q}(\alpha))$$

is an isomorphism for every integer m: if $m \neq n$, then both sides vanish; if $m = n$, then we invoke the fact that X is compact as an object of the abelian category $\mathcal{C}^{\heartsuit}$, by virtue of Proposition C.6.8.7. It follows that ρ'' is a homotopy equivalence between connected spaces. Consequently, to show that ρ is a homotopy equivalence, it will suffice to show that ρ' is a homotopy equivalence, which follows from our inductive hypothesis. □

Corollary C.6.10.3. *Let $\mathcal{C}$ be a locally Noetherian prestable ∞-category. Then the collection of injective objects of* $\mathrm{Sp}(\mathcal{C})$ *is closed under (possibly infinite) coproducts.*

Definition C.6.10.4. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let Q be a nonzero object of $\mathrm{Sp}(\mathcal{C})$. We will say that Q is *indecomposable* if it cannot be written as a direct sum $Q' \oplus Q''$, where Q' and Q'' are nonzero objects of $\mathrm{Sp}(\mathcal{C})$.

Remark C.6.10.5. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category and let Q be an injective object of $\mathrm{Sp}(\mathcal{C})$. Then Q is indecomposable if and only if $\pi_0 Q$ is an indecomposable injective object of the abelian category $\mathcal{C}^{\heartsuit}$:this is an immediate consequence of Theorem C.5.7.4.

We have the following analogue of Proposition C.5.6.16:

Proposition C.6.10.6. *Let $\mathcal{C}$ be a locally Noetherian prestable ∞-category and let Q be an object of $\mathrm{Sp}(\mathcal{C})$. The following conditions are equivalent:*

(1) *The object Q is injective.*

(2) *There exists an equivalence $Q \simeq \bigoplus_\alpha Q_\alpha$, where each Q_α is an injective object of $\mathrm{Sp}(\mathcal{C})$.*

(3) *There exists an equivalence $Q \simeq \bigoplus_\alpha Q_\alpha$, where each Q_α is an indecomposable injective object of $\mathrm{Sp}(\mathcal{C})$.*

Proof. The implication (3) ⇒ (2) is obvious and the implication (2) ⇒ (1) follows from Corollary C.6.10.3. We will show that (1) ⇒ (3). Let Q be an injective object of $\mathrm{Sp}(\mathcal{C})$. Then $\pi_0 Q$ is an injective object of the abelian category $\mathcal{C}^\heartsuit$. Using Proposition C.5.6.16, we can choose an equivalence $u_0 : \amalg_\alpha Q_{0\alpha} \simeq \pi_0 Q$, where each $Q_{0\alpha}$ is an indecomposable injective object of $\mathcal{C}^\heartsuit$. Using Proposition C.5.7.8, we can lift each $Q_{0\alpha}$ to an injective object $Q_\alpha \in \mathrm{Sp}(\mathcal{C})$ (which is indecomposable by virtue of Remark C.6.10.5). Applying Proposition C.5.7.3 (and the injectivity of Q), we can lift u_0 to a morphism $u : \bigoplus Q_\alpha \to Q$ in $\mathrm{Sp}(\mathcal{C})$. It follows from Corollary C.6.10.3 that the domain of u is injective. Since u_0 is an isomorphism, Theorem C.5.7.4 guarantees that u is an equivalence. □

Example C.6.10.7. Let R be a Noetherian $\mathbb{E}_\infty$-ring. Using Corollary C.5.6.11 and Theorem C.5.7.4, we see that the following conditions on an R-module Q are equivalent:

(1) The R-module Q is indecomposable and injective.

(2) The R-module Q is an injective hull (in the sense of Example C.5.7.9) of some residue field κ of R.

Appendix D

Descent for Modules and Linear ∞-Categories

Let $f : U \to X$ be a map of schemes, and let $p, q : U \times_X U \to U$ denote the two projection maps. A *descent datum* is a pair $(\mathscr{F}, \alpha)$, where $\mathscr{F}$ is a quasi-coherent sheaf on U and $\alpha : p^* \mathscr{F} \to q^* \mathscr{F}$ is an isomorphism of quasi-coherent sheaves on $U \times_X U$ which satisfies a suitable cocycle condition. There is a functor from the category of quasi-coherent sheaves on X to the category of descent data, given by $\mathscr{F}_0 \mapsto (f^* \mathscr{F}_0, \alpha)$, where α denotes the evident isomorphism

$$p^*(f^* \mathscr{F}_0) \simeq (f \circ p)^* \mathscr{F}_0 = (f \circ q)^* \mathscr{F}_0 \simeq q^*(f^* \mathscr{F}_0).$$

The classical theory of *faithfully flat descent* guarantees that this functor is an equivalence of categories whenever the map f is faithfully flat and quasi-compact. This is the basis for an important technique in algebraic geometry: one can often reduce questions about X (or about quasi-coherent sheaves on X) to questions about U, which may be easier to answer.

In the special case where the schemes $X = \operatorname{Spec} A$ and $U = \operatorname{Spec} B$ are affine, the theory of faithfully flat descent can be phrased entirely in the language of commutative algebra: it asserts that if B is faithfully flat over A, then the category of (discrete) A-modules can be identified with the category whose objects are pairs (M_B, α), where M_B is a (discrete) B-module and $\alpha : B \otimes_A M_B \to M_B \otimes_A B$ is an isomorphism of $(B \otimes_A B)$-modules satisfying a cocycle condition. Our goal in this appendix is to discuss some analogous statements in a higher-categorical setting, which differs in three important respects:

(a) In place of the ordinary category $\operatorname{Mod}_A^{\heartsuit}$ of discrete A-modules, we study the ∞-category Mod_A of *A-module spectra*. This necessitates working with a more elaborate notion of descent datum: to recover an A-module M from the tensor product $M_B = B \otimes_A M$, one needs more than just the equivalence $\alpha : B \otimes_A M_B \simeq M_B \otimes_A B$. Instead, we should ask that M_B be promoted to an object of the totalization of the cosimplicial ∞-category

$$\mathrm{Mod}_B \rightrightarrows \mathrm{Mod}_{B \otimes_A B} \rightrightarrows \mathrm{Mod}_{B \otimes_A B \otimes_A B} \rightrightarrows \cdots .$$

More informally, this amounts to the requirement that M_B is equipped with an equivalence $\alpha : B \otimes_A M_B \simeq M_B \otimes_A B$ satisfying a *cocycle condition up to coherent homotopy*.

(*b*) In order for the theory developed in this appendix to be useful for the study of spectral algebraic geometry, it is important that we do not require that A and B are ordinary commutative rings: the theory of faithfully flat descent makes sense more generally for modules over $\mathbb{E}_\infty$-rings (in fact, many of the results of this appendix make sense more generally for $\mathbb{E}_2$-rings).

(*c*) If A is an ordinary commutative ring, then one can study A-modules in any additive category $\mathcal{A}$ (the usual theory of discrete A-modules is obtained by taking $\mathcal{A}$ to be the category of abelian groups). More generally, if A is a (connective) $\mathbb{E}_\infty$-ring, one can study A-modules in any additive ∞-category $\mathcal{C}$. Much of the basic apparatus of descent theory makes sense in this more general setting (under some mild assumptions on $\mathcal{C}$).

We begin our study of descent in §D.1 by introducing the notion of an *additive A-linear ∞-category*, where A is a connective $\mathbb{E}_\infty$-ring (Definition D.1.2.1). Roughly speaking, an additive A-linear ∞-category is a presentable ∞-category $\mathcal{C}$ in which each mapping space $\mathrm{Map}_{\mathcal{C}}(C, D)$ can be regarded as (the 0th space of) a connective A-module spectrum. We will be primarily interested in the situations where $\mathcal{C}$ is a stable ∞-category (in which case we can drop the connectivity assumption on A; see Variant D.1.5.1), but the cases where $\mathcal{C}$ is a prestable ∞-category or an ordinary abelian category are also of considerable interest.

Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be an additive A-linear ∞-category. Then every object $X \in \mathcal{C}$ can be regarded as an A-module in a canonical way. If $\phi : A \to B$ is a morphism of connective $\mathbb{E}_\infty$-rings, then one can consider *B-modules* in $\mathcal{C}$: that is, objects $X \in \mathcal{C}$ which are equipped with an action of B which is compatible with the tautological action of A. The collection of such B-modules forms an ∞-category that we will denote $\mathrm{LMod}_B(\mathcal{C})$. In §D.2, we will see that the ∞-category $\mathrm{LMod}_B(\mathcal{C})$ is naturally an additive B-linear ∞-category, which we can think of as obtained from $\mathcal{C}$ by *extension of scalars* along the morphism ϕ (to emphasize this perspective, we will often denote $\mathrm{LMod}_B(\mathcal{C})$ by $B \otimes_A \mathcal{C}$).

To every additive A-linear ∞-category $\mathcal{C}$ and every morphism $\phi : A \to B$ as above, one can form a cosimplicial ∞-category

$$\mathrm{LMod}_B(\mathcal{C}) \rightrightarrows \mathrm{LMod}_{B \otimes_A B}(\mathcal{C}) \rightrightarrows \mathrm{LMod}_{B \otimes_A B \otimes_A B}(\mathcal{C}) \rightrightarrows \cdots$$

whose inverse limit can be thought of as B-modules in $\mathcal{C}$ equipped with descent data. The fundamental question of this appendix can be formulated as follows: under what conditions is this ∞-category equivalent to $\mathcal{C}$? In §D.3, we supply a partial answer to this question by introducing the notion of a *universal descent morphism* of $\mathbb{E}_\infty$-rings. Roughly speaking, a morphism $\phi : A \to B$ is a universal descent morphism if every *stable* A-linear ∞-category $\mathcal{C}$ is equivalent to the totalization of the cosimplicial ∞-category above. This condition can be reformulated in more concrete terms and verified for a large class of morphisms of $\mathbb{E}_\infty$-rings (including, for example, all morphisms which are faithfully flat and étale).

In §D.4, we study the problem of descent in the setting of a *prestable* A-linear ∞-category $\mathcal{C}$ which is not necessarily stable. In this case, one still has a descent theorem under a the slightly more restrictive hypothesis that $\phi : A \to B$ is a faithfully flat universal descent morphism (see Theorem D.4.1.6); this result applies in particular if ϕ is faithfully flat and étale. In this case, there is a close relationship between properties of the ∞-category $\mathcal{C}$ and properties of the ∞-category $\mathrm{LMod}_B(\mathcal{C})$, which we will explore in detail in §D.5.

Unfortunately, we do not know if an arbitrary faithfully flat morphism of $\mathbb{E}_\infty$-rings $\phi : A \to B$ is an universal descent morphism (this is true under some mild assumptions regarding the cardinality of A and B, however: see Proposition D.3.3.1). Nevertheless, we will show in §D.6 that many of the descent theorems established in §D.3 for universal descent morphisms are valid for arbitrary faithfully flat morphisms under some mild additional restrictions (see Theorem D.6.3.1).

Contents

D.1 Ring Actions on ∞-Categories

Let $\mathcal{A}$ be an additive category. Then $\mathcal{A}$ is enriched over the category of abelian groups: for every pair of objects $X, Y \in \mathcal{A}$, the set $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ of morphisms from X to Y admits the structure of an abelian group, and for every triple of objects $X, Y, Z \in \mathcal{A}$ the composition map $\mathrm{Hom}_{\mathcal{A}}(Y, Z) \times \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$ is bilinear and is therefore classified by a group homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(Y, Z) \otimes \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Y)$$

(here the tensor product is formed in the category of abelian groups).

Many additive categories $\mathcal{A}$ which arise in practice come equipped with additional structure. If R is a commutative ring, then the following data are equivalent:

(a) An enrichment of $\mathcal{A}$ over the category of R-modules: that is, an R-module structure on each of sets $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ for which the composition maps $\mathrm{Hom}_{\mathcal{A}}(Y, Z) \times \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$ are R-bilinear (and therefore classifies by R-module homomorphisms $\mathrm{Hom}_{\mathcal{A}}(Y, Z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Y)$).

(b) An action of R on each object $X \in \mathcal{A}$ via a ring homomorphism $\phi_X : R \to \mathrm{Hom}_{\mathcal{A}}(X, X)$, which depends functorially on X in the following sense: for every morphism $f : X \to Y$ in $\mathcal{A}$ and every element $a \in R$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \phi_X(a)} & & \downarrow{\scriptstyle \phi_Y(a)} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

(c) An action of R on the identity functor $\mathrm{id}_{\mathcal{A}}$: that is, a ring homomorphism $\phi : R \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{A},\mathcal{A})}(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}})$.

(d) A monoidal functor $\Phi : \mathrm{Mod}_R^{\mathrm{ff}} \to \mathrm{Fun}(\mathcal{A}, \mathcal{A})$ which commutes with finite direct sums; here $\mathrm{Mod}_R^{\mathrm{ff}}$ denotes the category of free R-modules of finite rank and we regard $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ as a monoidal category with respect to composition.

We will refer to any of these equivalent data as an *action of R on the additive category $\mathcal{A}$*.

In this section, we will introduce an ∞-categorical generalization of the preceding definition, where we allow $\mathcal{A}$ to be an additive ∞-category and R to be a (connective) ring spectrum. In this book, we will be primarily interested in the case where R is an $\mathbb{E}_\infty$-ring. However, the definition below makes sense whenever R is an $\mathbb{E}_2$-ring, and working in this greater generality has some advantages.

D.1.1 Ring Actions on Additive ∞-Categories

Let R be a connective $\mathbb{E}_2$-ring. We let $\mathrm{LMod}_R^{\mathrm{ff}}$ denote the full subcategory of LMod_R spanned by those left R-modules which are free of finite rank over R. The full subcategory $\mathrm{LMod}_R^{\mathrm{ff}}$ contains the unit object $R \in \mathrm{LMod}_R$ and is closed under the tensor product $\otimes_R$, and therefore inherits the structure of a monoidal ∞-category.

Definition D.1.1.1. Let $\mathcal{A}$ be an additive ∞-category and let R be a connective $\mathbb{E}_2$-ring. An *action of R on $\mathcal{A}$* is a monoidal functor $\mathrm{LMod}_R^{\mathrm{ff}} \to \mathrm{Fun}(\mathcal{A}, \mathcal{A})$ which commutes with finite direct sums.

Remark D.1.1.2. Let $\mathcal{A}$ be an additive ∞-category, let R be a connective $\mathbb{E}_2$-ring, and let $\phi : \mathrm{LMod}_R^{\mathrm{ff}} \to \mathrm{Fun}(\mathcal{A}, \mathcal{A})$. be an action of R on $\mathcal{A}$. Then ϕ carries each object of $\mathrm{LMod}_R^{\mathrm{ff}}$ to a direct sum of finitely many copies of the identity functor $\mathrm{id}_{\mathcal{A}}$. Consequently, if $\mathrm{Fun}'(\mathcal{A}, \mathcal{A})$ is any full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ which is closed under finite direct sums, then ϕ automatically factors through $\mathrm{Fun}'(\mathcal{A}, \mathcal{A})$. For example, we can take $\mathrm{Fun}'(\mathcal{A}, \mathcal{A})$ to be the full subcategory $\mathrm{Fun}^{\pi}(\mathcal{A}, \mathcal{A}) \subseteq \mathrm{Fun}(\mathcal{A}, \mathcal{A})$ spanned by those functors which preserve finite products.

Remark D.1.1.3. Let $\mathcal{A}$ be an additive ∞-category. Giving an action of a connective $\mathbb{E}_2$-ring on $\mathcal{A}$ is equivalent to exhibiting $\mathcal{A}$ as an ∞-category which is left-tensored over the monoidal ∞-category $\mathrm{LMod}_R^{\mathrm{ff}}$ via an action $a : \mathrm{LMod}_R^{\mathrm{ff}} \times \mathcal{A} \to \mathcal{A}$ which preserves finite direct sums in the first variable. By virtue of Remark **??**, it follows automatically that a preserves finite direct sums in the second variable as well.

Remark D.1.1.4. Let $\mathcal{A}$ be a small additive ∞-category. Then the ∞-category $\mathrm{Ind}(\mathcal{A})$ is also additive. Let us abuse notation by identifying $\mathcal{A}$ with its essential image under the Yoneda embedding $j : \mathcal{A} \to \mathrm{Ind}(\mathcal{A})$, and let $\mathrm{Fun}'(\mathrm{Ind}(\mathcal{A}), \mathrm{Ind}(\mathcal{A}))$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Ind}(\mathcal{A}), \mathrm{Ind}(\mathcal{A}))$ spanned by those functors which preserve small filtered colimits and carry $\mathcal{A}$ into itself. Then the restriction functor $\mathrm{Fun}'(\mathrm{Ind}(\mathcal{A}), \mathrm{Ind}(\mathcal{A})) \to \mathrm{Fun}(\mathcal{A}, \mathcal{A})$ is an equivalence of ∞-categories.

Let R be a connective $\mathbb{E}_2$-ring. Invoking Remark D.1.1.2, we see that giving an action of R on $\mathrm{Ind}(\mathcal{A})$ is equivalent to giving an action of R on $\mathcal{A}$ itself. Using a similar argument (using sifted colimits in place of filtered colimits), we see that giving an action of R on $\mathcal{A}$ is equivalent to giving an action of R on the presentable additive ∞-category $\mathcal{P}_{\Sigma}(\mathcal{A}) = \mathrm{Fun}^{\pi}(\mathcal{A}^{\mathrm{op}}, \mathcal{S})$.

In what follows, we will generally restrict our attention to the study of actions on *presentable* additive ∞-categories $\mathcal{A}$ (by virtue of Remark D.1.1.4, this does not really involve any loss of generality).

Remark D.1.1.5. Let R be a connective $\mathbb{E}_2$-ring. Then the objects $\{R^n \in \mathrm{LMod}_R^{\mathrm{cn}}\}_{n \geqslant 0}$ form a set of compact projective generators for $\mathrm{LMod}_R^{\mathrm{cn}}$ which is closed under the formation fo finite coproducts (Corollary HA.7.1.4.15). It follows that we can identify $\mathrm{LMod}_R^{\mathrm{cn}}$ with the ∞-category $\mathcal{P}_\Sigma(\mathrm{LMod}_R^{\mathrm{ff}})$ obtained by freely adjoining sifted colimits to $\mathrm{LMod}_R^{\mathrm{ff}}$.

Suppose that $\mathcal{C}$ is a presentable ∞-category equipped with a monoidal structure for which the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable, let $\mathrm{Fun}^{\otimes}(\mathrm{LMod}_R^{\mathrm{cn}}, \mathcal{C})$ denote the ∞-category of monoidal functors from $\mathrm{LMod}_R^{\mathrm{cn}}$ to $\mathcal{C}$, and define $\mathrm{Fun}^{\otimes}(\mathrm{LMod}_R^{\mathrm{ff}}, \mathcal{C})$ similarly. It follows from the formalism of §HA.4.8.1 that the restriction functor $\mathrm{Fun}^{\otimes}(\mathrm{LMod}_R^{\mathrm{cn}}, \mathcal{C}) \to \mathrm{Fun}^{\otimes}(\mathrm{LMod}_R^{\mathrm{ff}}, \mathcal{C})$ induces an equivalence from the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{LMod}_R^{\mathrm{cn}}, \mathcal{C})$ spanned by those monoidal functors which preserve small colimits to the full subategory of $\mathrm{Fun}^{\otimes}(\mathrm{LMod}_R^{\mathrm{ff}}, \mathcal{C})$ spanned by those functors which preserve finite coproducts.

D.1.2 R-Linear ∞-Categories

Let R be a connective $\mathbb{E}_2$-ring. Applying Remark D.1.1.5 in the special case where $\mathcal{C} = \mathrm{LFun}(\mathcal{A}, \mathcal{A})$ for some presentable additive ∞-category $\mathcal{A}$, we see that the following data are equivalent:

- (i) Actions of R on $\mathcal{A}$: that is, monoidal functors $\mathrm{LMod}_R^{\mathrm{ff}} \to \mathrm{Fun}(\mathcal{A}, \mathcal{A})$ which preserve finite coproducts.
- (ii) Monoidal functors $\mathrm{LMod}_R^{\mathrm{ff}} \to \mathrm{LFun}(\mathcal{A}, \mathcal{A})$ which preserve finite coproducts (see Remark D.1.1.2).
- (iii) Monoidal functors $\mathrm{LMod}_R^{\mathrm{cn}} \to \mathrm{LFun}(\mathcal{A}, \mathcal{A})$ which preserve small colimits.
- (iv) Left actions of $\mathrm{LMod}_R^{\mathrm{cn}}$ (regarded as an algebra object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$) on $\mathcal{A}$ (regarded as an object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$).

This motivates the following definition:

Definition D.1.2.1. Let R be a connective $\mathbb{E}_2$-ring, and regard the monoidal ∞-category $\mathrm{LMod}_R^{\mathrm{cn}}$ as an algebra object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories. We let $\mathrm{LinCat}_R^{\mathrm{Add}}$ denote the ∞-category $\mathrm{LMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. We will refer to the objects of $\mathrm{LinCat}_R^{\mathrm{Add}}$ as *additive R-linear ∞-categories*, and to $\mathrm{LinCat}_R^{\mathrm{Add}}$ itself as the *∞-category of R-linear ∞-categories*.

Remark D.1.2.2. Let R be a connective $\mathbb{E}_2$-ring and let $\mathcal{A} \in \mathrm{LinCat}_R^{\mathrm{Add}}$ be a presentable ∞-category equipped with an action of $\mathrm{LMod}_R^{\mathrm{cn}}$. Then $\mathcal{A}$ can also be regarded as a module over the ∞-category $\mathrm{LMod}_S^{\mathrm{cn}} = \mathrm{Sp}^{\mathrm{cn}}$, and is therefore automatically additive (Corollary C.4.1.3). Conversely, if $\mathcal{A}$ is a presentable additive ∞-category, then the preceding discussion shows that promoting $\mathcal{A}$ to an object of $\mathrm{LinCat}_R^{\mathrm{Add}}$ is equivalent to giving an action of R on $\mathcal{A}$ (in the sense of Definition D.1.1.1).

Remark D.1.2.3. If R is a discrete commutative ring, then the theory of additive R-linear ∞-categories is closely related to the theory of *differential graded categories* over R (see §HA.1.3.1).

Notation D.1.2.4. Let R be an $\mathbb{E}_2$-ring and let $\mathcal{A}$ be an additive R-linear ∞-category. The action of R on $\mathcal{A}$ determines a functor $a : \mathrm{LMod}_R^{\mathrm{cn}} \times \mathcal{A} \to \mathcal{A}$. Given a connective left R-module M and an object $X \in \mathcal{A}$, we will denote the image of (M, X) under this functor by $M \otimes_R X$.

Warning D.1.2.5. Let R be a connective $\mathbb{E}_2$-ring. We have defined an additive R-linear ∞-category to be a presentable ∞-category $\mathcal{A}$ equipped with a left action of the presentable monoidal ∞-category $\mathrm{LMod}_R^{\mathrm{cn}}$. There are several obvious variants:

(a) We can consider right actions of $\mathrm{LMod}_R^{\mathrm{cn}}$ on $\mathcal{A}$.

(b) We can consider left actions of $\mathrm{RMod}_R^{\mathrm{cn}}$ on $\mathcal{A}$.

(c) We can consider right actions of $\mathrm{RMod}_R^{\mathrm{cn}}$ on $\mathcal{A}$.

Using the fact that the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ belongs to the identity component of the orthogonal group $\mathrm{O}(2)$ (which acts on the $\mathbb{E}_2$-operad), we see that datum (c) is (noncanonically) equivalent to the datum of an action of R on $\mathcal{A}$, and the data (a) and (b) are (noncanonically) equivalent to one another (both are equivalent to giving an action of the $\mathbb{E}_2$-ring R^{rev} on $\mathcal{A}$, where R^{rev} denotes the reverse of R in the sense of Construction HA.5.2.5.18). Note that this somewhat technical point is relevant only when R is not fully commutative: if R is a connective $\mathbb{E}_\infty$-ring, then the data (a), (b), and (c) are all *canonically* equivalent to supplying an action of R on $\mathcal{A}$, and we can identify the $\mathrm{LinCat}_R^{\mathrm{Add}}$ with the (symmetrically defined) ∞-category $\mathrm{Mod}_{\mathrm{Mod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

Warning D.1.2.6. The definition of additive R-linear ∞-category $\mathcal{A}$ supplied by Definition D.1.2.1 is perhaps unnecessarily restrictive, because we require $\mathcal{A}$ to be a presentable ∞-category. A more liberal definition might encompass *any* additive ∞-category $\mathcal{A}$ which equipped with an action of R, in the sense of Definition D.1.1.1. Our choice of terminology is motivated by the desire to avoid awkward language, since most of the actions we wish to study in this book are on *presentable* ∞-categories.

D.1.3 The Center of an Additive ∞-Category

Let $\mathcal{A}$ be an additive category. Then the endomorphism ring $E = \mathrm{Hom}_{\mathrm{Fun}(\mathcal{A},\mathcal{A})}(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}})$ is a commutative ring. For any commutative ring R, equipping $\mathcal{A}$ with the structure of an R-linear category is equivalent to choosing a ring homomorphism $\phi : R \to E$. In the ∞-categorical setting, there is a similar picture with one important difference: the endomorphism ring E is an $\mathbb{E}_2$-ring, rather than an $\mathbb{E}_\infty$-ring.

Notation D.1.3.1. For every connective $\mathbb{E}_1$-ring R, we can regard $\mathrm{LMod}_R^{\mathrm{cn}}$ as a presentable additive ∞-category with a distinguished object (given by R, regarded as a left module over itself). The construction $R \mapsto (\mathrm{LMod}_R^{\mathrm{cn}}, R)$ determines a functor

$$\Theta_* : \mathrm{Alg}(\mathrm{Sp}^{\mathrm{cn}}) \to \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{P}\mathrm{r}^{\mathrm{Add}}) \simeq \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Mod}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})).$$

According to Theorem HA.4.8.5.11 , the functor Θ_* admits a right adjoint G which carries a presentable ∞-category $\mathcal{C}$ with a distinguished object $X \in \mathcal{C}$ to its endomorphism algebra $\mathrm{End}(X) \in \mathrm{Alg}(\mathrm{Sp}^{\mathrm{cn}})$, a connective $\mathbb{E}_1$-ring satisfying $\Omega^\infty \mathrm{End}(X) \simeq \mathrm{Map}_{\mathcal{C}}(X, X)$.

According to Theorem HA.4.8.5.16 , the functor Θ_* is symmetric monoidal. It follows that the functor G is lax symmetric monoidal. Suppose that $\mathcal{C}$ is an associative algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{Add}}$: that is, a monoidal presentable ∞-category for which tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Then we can regard $(\mathcal{C}, \mathbf{1})$ as an algebra object of $\mathrm{Alg}_{\mathbb{E}_0}(\mathcal{P}\mathrm{r}^{\mathrm{Add}})$, where $\mathbf{1}$ denotes the unit object of $\mathcal{C}$. It follows that $\mathrm{End}(\mathbf{1}) = G(\mathcal{C}, \mathbf{1})$ can be regarded as an object of $\mathrm{Alg}(\mathrm{Alg}(\mathrm{Sp}^{\mathrm{cn}})) \simeq \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}^{\mathrm{cn}})$. In other words, we can regard $\mathrm{End}(\mathbf{1})$ as an $\mathbb{E}_2$-ring.

Remark D.1.3.2. Let $\mathcal{C}$ be a monoidal ∞-category with unit object $\mathbf{1}$. Assume that $\mathcal{C}$ is additive and presentable and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C}$ preserves small colimits separately in each variable. For any connective $\mathbb{E}_2$-ring R, we have a canonical homotopy equivalence $\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}}(R, \mathrm{End}(\mathbf{1})) \simeq \mathrm{Map}_{\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})}(\mathrm{LMod}_R^{\mathrm{cn}}, \mathcal{C})$.

Construction D.1.3.3 (The Connective Center)**.** Let $\mathcal{A}$ be a presentable additive ∞-category. Then the ∞-category $\mathrm{LFun}(\mathcal{A}, \mathcal{A})$ of colimit-preserving functors from $\mathcal{A}$ to itself is also an additive presentable ∞-category, and the composition map

$$\circ : \mathrm{LFun}(\mathcal{A}, \mathcal{A}) \times \mathrm{LFun}(\mathcal{A}, \mathcal{A}) \to \mathrm{LFun}(\mathcal{A}, \mathcal{A})$$

preserves small colimits separately in each variable and determines a monoidal structure on $\mathrm{LFun}(\mathcal{A}, \mathcal{A})$ (in fact, it is *strictly* associative: it exhibits $\mathrm{LFun}(\mathcal{A}, \mathcal{A})$ as a simplicial monoid). We let $\mathfrak{Z}^{\mathrm{cn}}(\mathcal{A})$ denote the connective $\mathbb{E}_2$-ring given by $\mathrm{End}(\mathrm{id}_{\mathcal{A}})$ (see Notation D.1.3.1). Then $\mathfrak{Z}^{\mathrm{cn}}(\mathcal{A})$ is a connective $\mathbb{E}_2$-ring which we will refer to as the *connective center* of $\mathcal{A}$.

Remark D.1.3.4. Let $\mathcal{A}$ be a presentable additive ∞-category. For every connective $\mathbb{E}_2$-ring R, Remark D.1.3.2 supplies a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}^{\mathrm{cn}})}(R, \mathfrak{Z}^{\mathrm{cn}}(\mathcal{A})) \simeq \mathrm{Map}_{\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})}(\mathrm{LMod}_R^{\mathrm{cn}}, \mathrm{LFun}(\mathcal{A}, \mathcal{A})).$$

In other words, giving an action of R on $\mathcal{A}$ is equivalent to giving a morphism of $\mathbb{E}_2$-rings $R \to \mathfrak{Z}^{\mathrm{cn}}(A)$.

Remark D.1.3.5. Let $\mathcal{A}$ be a presentable additive ∞-category. Then we have a canonical homotopy equivalence $\Omega^{\infty} \mathfrak{Z}^{\mathrm{cn}}(\mathcal{A}) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{A},\mathcal{A})}(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}})$. In particular, for every nonnegative integer n, we have a canonical isomorphism $\pi_n \mathfrak{Z}^{\mathrm{cn}}(\mathcal{A}) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{A},\mathcal{A})}(\mathrm{id}_{\mathcal{A}}, \Omega^n_{\mathcal{A}})$.

Example D.1.3.6. Let $\mathcal{A}$ be a presentable additive ∞-category, and suppose that $\mathcal{A}$ is equivalent to an m-category for some $m \geqslant 1$. Then the functor $\Omega^n_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ vanishes for $n \geqslant m$. It follows that the homotopy groups $\pi_n \mathfrak{Z}^{\mathrm{cn}}(\mathcal{A})$ vanish for $n \geqslant m$.

In particular, if $\mathcal{A}$ is (equivalent to) an ordinary category, then the connective center $\mathfrak{Z}^{\mathrm{cn}}(\mathcal{A})$ is discrete. It follows that for every connective $\mathbb{E}_2$-ring R, giving an action of R on $\mathcal{A}$ is equivalent to giving an action of the commutative ring $\pi_0 R$ on $\mathcal{A}$.

Example D.1.3.7 (Linearity over $\mathbf{F}_p$)**.** Let p be a prime number and let $\mathbf{F}_p$ denote the finite field $\mathbf{Z}/p\mathbf{Z}$ with p elements. A theorem of Hopkins and Mahowald asserts that, when regarded as an $\mathbb{E}_2$-ring, the field $\mathbf{F}_p$ is obtained from the sphere spectrum by attaching a single 1-cell to "kill p": more precisely, if $A = \mathrm{Sym}^*_{\mathbb{E}_2}(S)$ denotes the free $\mathbb{E}_2$-ring on a single generator x of degree zero, then there is a pushout diagram of $\mathbb{E}_2$-rings

$$\begin{array}{ccc} A & \xrightarrow{x \mapsto 0} & S \\ \downarrow{\scriptstyle x \mapsto p} & & \downarrow \\ S & \longrightarrow & \mathbf{F}_p. \end{array}$$

It follows that if $\mathcal{A}$ is any presentable additive ∞-category, then the mapping space $\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}^{\mathrm{cn}}}(\mathbf{F}_p, \mathfrak{Z}^{\mathrm{cn}}(A))$ can be identified with the homotopy fiber product

$$\{0\} \times_{\mathrm{Map}_{\mathrm{Fun}(\mathcal{A},\mathcal{A})}(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}})} \{p\}.$$

In particular, a presentable additive ∞-category $\mathcal{A}$ admits an action of $\mathbf{F}_p$ if and only if the map $p : \mathrm{id}_{\mathcal{A}} \to \mathrm{id}_{\mathcal{A}}$ is nullhomotopic.

Example D.1.3.8 (Linearity over $\mathbf{Q}$)**.** Let $\mathcal{A}$ be a presentable additive ∞-category. Suppose that $n > 0$ and that every object $X \in \mathcal{A}$, the map $n\,\mathrm{id}_X : X \to X$ is an equivalence. It follows that multiplication by n induces an equivalence from the identity functor $\mathrm{id}_{\mathcal{A}}$ to itself, so that n is invertible in the commutative ring $\pi_0 \mathfrak{Z}^{\mathrm{cn}}(\mathcal{A})$. If this condition is satisfied for *every* positive integer n, then $\pi_0 \mathfrak{Z}^{\mathrm{cn}}(\mathcal{A})$ is an algebra over the field $\mathbf{Q}$ of rational numbers and therefore the mapping space $\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}^{\mathrm{cn}})}(\mathbf{Q}, \mathfrak{Z}^{\mathrm{cn}}(\mathcal{A}))$ is contractible. It follows that $\mathcal{A}$ admits an essentially unique action of the field $\mathbf{Q}$ of rational numbers. Conversely, if $\mathcal{A}$ is an additive $\mathbf{Q}$-linear ∞-category, then the multiplication map $n\,\mathrm{id}_X : X \to X$ is an equivalence for each $X \in \mathcal{A}$ and every positive integer n.

Example D.1.3.9. Let $\mathcal{A}$ be a Grothendieck abelian category, let $\mathcal{D}(\mathcal{A})$ denote the derived ∞-category of $\mathcal{A}$, and let $\mathcal{C} = \mathcal{D}(\mathcal{A})_{\geqslant 0}$ be the associated Grothendieck prestable ∞-category.

It follows from Theorem C.5.4.9 that restriction to the heart induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{A},\mathcal{A})}(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}}),$$

hence an equivalence of connective $\mathbb{E}_2$-rings $\mathfrak{Z}^{\mathrm{cn}}(\mathcal{C}) \to \mathfrak{Z}^{\mathrm{cn}}(\mathcal{A})$ (Remark D.1.3.5). In particular, we deduce the following:

- The connective center $\mathfrak{Z}^{\mathrm{cn}}(\mathcal{C})$ is discrete (see Example D.1.3.6).
- For every connective $\mathbb{E}_2$-ring R, giving an action of R on $\mathcal{C}$ is equivalent to giving an action of R on the abelian category $\mathcal{A}$ (and any such action factors through the discrete commutative ring $\pi_0 R$).

The same reasoning applies if we take $\mathcal{C}$ to be the complete Grothendieck prestable ∞-category $\widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0}$ (see Corollary C.5.9.5), or either of the stable ∞-categories $\mathcal{D}(\mathcal{A})$ and $\widehat{\mathcal{D}}(\mathcal{A})$.

D.1.4 Special Classes of R-Linear ∞-Categories

In this book, we will generally be interested in studying an additive R-linear ∞-categories which satisfy some additional conditions.

Definition D.1.4.1. Let R be a connective $\mathbb{E}_2$-ring and let $\mathcal{C} \in \mathrm{LinCat}_R^{\mathrm{Add}}$ be an additive R-linear ∞-category.

(*a*) We say that $\mathcal{C}$ is a *stable R-linear ∞-category* if the underlying ∞-category of $\mathcal{C}$ is stable.

(*b*) We say that $\mathcal{C}$ is a *prestable R-linear ∞-category* if the underlying ∞-category of $\mathcal{C}$ is a Grothendieck prestable ∞-category.

(*c*) We say that $\mathcal{C}$ is an *abelian R-linear ∞-category* if the underlying ∞-category of $\mathcal{C}$ is (equivalent to) a Grothendieck abelian category.

We let $\mathrm{LinCat}_R^{\mathrm{St}}$ denote the full subcategory of $\mathrm{LinCat}_R^{\mathrm{Add}}$ spanned by the stable R-linear ∞-categories, $\mathrm{LinCat}_R^{\mathrm{PSt}} \subseteq \mathrm{LinCat}_R^{\mathrm{Add}}$ the full subcategory spanned by the prestable R-linear ∞-categories, and $\mathrm{LinCat}_R^{\mathrm{Ab}} \subseteq \mathrm{LinCat}_R^{\mathrm{Add}}$ the full subcategory spanned by the abelian R-linear ∞-categories.

Warning D.1.4.2. As with Definition D.1.2.1, the terminology of Definition D.1.4.1 is potentially misleading. Throughout this book, if we say that a stable ∞-category $\mathcal{C}$ is R-linear, we are implicitly asserting that $\mathcal{C}$ is a *presentable* stable ∞-category. Likewise, if we say that a prestable ∞-category (or abelian category) $\mathcal{C}$ is R-linear, we are implicitly asserting that $\mathcal{C}$ is a Grothendieck prestable ∞-category (or Grothendieck abelian category).

Of course, it is possible to adopt more liberal definitions which could be applied to more general stable ∞-categories, prestable ∞-categories, or abelian categories. However, the conventions of Definition D.1.4.1 are better suited to our undertakings in this book.

Remark D.1.4.3. Every presentable stable ∞-category $\mathcal{C}$ is a Grothendieck prestable ∞-category. Consequently, for any connective $\mathbb{E}_2$-ring R, we can regard $\mathrm{LinCat}_R^{\mathrm{St}}$ as a full subcategory of $\mathrm{LinCat}_R^{\mathrm{PSt}}$.

Example D.1.4.4. Let S denote the sphere spectrum, which we regard as a connective $\mathbb{E}_2$-ring. Then the forgetful functor $\mathrm{LinCat}_S^{\mathrm{Add}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ determines an equivalence of ∞-categories $\mathrm{LinCat}_S^{\mathrm{Add}} \simeq \mathcal{P}\mathrm{r}^{\mathrm{Add}}$, which restricts to give equivalences

$$\mathrm{LinCat}_S^{\mathrm{St}} \simeq \mathcal{P}\mathrm{r}^{\mathrm{St}} \qquad \mathrm{LinCat}_S^{\mathrm{PSt}} \simeq \mathsf{Groth}_\infty \qquad \mathrm{LinCat}_S^{\mathrm{Ab}} \simeq \mathsf{Groth}_{\mathrm{ab}}.$$

Remark D.1.4.5. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category. Then $\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Sp} \otimes \mathcal{C}$ is a presentable stable ∞-category and $\mathcal{C}^\heartsuit \simeq \mathrm{Ab} \otimes \mathcal{C}$ is a Grothendieck abelian category. If $\mathcal{C}$ is equipped with a left action of $\mathrm{LMod}_R^{\mathrm{cn}}$ for some connective $\mathbb{E}_2$-ring R, then $\mathrm{Sp} \otimes \mathcal{C}$ and $\mathrm{Ab} \otimes \mathcal{C}$ inherit left actions of $\mathrm{LMod}_R^{\mathrm{cn}}$. We therefore obtain forgetful functors

$$\begin{array}{ccccc} \mathrm{LinCat}_R^{\mathrm{St}} & \leftarrow & \mathrm{LinCat}_R^{\mathrm{PSt}} & \rightarrow & \mathrm{LinCat}_R^{\mathrm{Ab}} \\ \mathrm{Sp}(\mathcal{C}) & \leftarrow\!\shortmid & \mathcal{C} & \mapsto & \mathcal{C}^\heartsuit. \end{array}$$

In other words, if $\mathcal{C}$ is a prestable R-linear ∞-category, then $\mathrm{Sp}(\mathcal{C})$ is a stable R-linear ∞-category and $\mathcal{C}^\heartsuit$ is an abelian R-linear ∞-category.

Remark D.1.4.6. Let $\mathcal{A}$ be a Grothendieck abelian category and let R be a connective $\mathbb{E}_2$-ring. Then any colimit-preserving (monoidal) functor from the ∞-category $\mathrm{LMod}_R^{\mathrm{cn}}$ to the category $\mathrm{LFun}(\mathcal{A}, \mathcal{A})$ automatically factors through the ∞-category $\tau_{\leqslant 0} \mathrm{LMod}_R^{\mathrm{cn}} \simeq \mathrm{LMod}_R^\heartsuit$ of discrete left R-modules, which depends only on the underlying commutative ring $\pi_0 R$. It follows that the natural map $R \to \pi_0 R$ induces an equivalence of ∞-categories $\mathrm{LinCat}_{\pi_0 R}^{\mathrm{Ab}} \to \mathrm{LinCat}_R^{\mathrm{Ab}}$.

D.1.5 The Stable Case

Let R be a connective $\mathbb{E}_2$-ring. The inclusion functor $\mathrm{LMod}_R^{\mathrm{cn}} \hookrightarrow \mathrm{LMod}_R$ can be regarded as a morphism between algebra objects of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, and therefore induces a forgetful functor

$$\theta : \mathrm{LMod}_{\mathrm{LMod}_R}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{LMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \simeq \mathrm{LinCat}_R^{\mathrm{Add}}.$$

This functor admits a left adjoint, given by the construction

$$\begin{aligned} \mathcal{A} \mapsto\ & \mathrm{LMod}_R \otimes_{\mathrm{LMod}_R^{\mathrm{cn}}} \mathcal{A} \\ \simeq\ & \mathrm{Sp}(\mathrm{LMod}_R^{\mathrm{cn}}) \otimes_{\mathrm{LMod}_R^{\mathrm{cn}}} \mathcal{A} \\ \simeq\ & (\mathrm{Sp} \otimes \mathrm{LMod}_R^{\mathrm{cn}}) \otimes_{\mathrm{LMod}_R^{\mathrm{cn}}} \mathcal{A} \\ \simeq\ & \mathrm{Sp} \otimes \mathcal{A}; \end{aligned}$$

here the unit for the adjunction is provided by the canonical map

$$\mathcal{A} \simeq \mathcal{S} \otimes \mathcal{A} \xrightarrow{\Sigma^{\infty}_{+}} \mathrm{Sp} \otimes \mathcal{A}.$$

Since the functor $\Sigma^{\infty}_{+} : \mathcal{S} \to \mathrm{Sp}$ exhibits the ∞-category of spectra as an idempotent object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (see Proposition HA.4.8.2.18), it follows that the functor θ is fully faithful. Moreover, the essential image of θ is spanned by those additive R-linear ∞-categories $\mathcal{A}$ for which the unit map $\mathcal{A} \to \mathrm{Sp} \otimes \mathcal{A}$ is an equivalence of ∞-categories: by virtue of Proposition **??**, this is the full subcategory $\mathrm{LinCat}^{\mathrm{St}}_{R} \subseteq \mathrm{LinCat}^{\mathrm{Add}}_{R}$ of *stable* R-linear ∞-categories. This motivates the following variant of Definition D.1.4.1:

Variant D.1.5.1. Let R be an $\mathbb{E}_2$-ring (not necessarily connective). We let $\mathrm{LinCat}^{\mathrm{St}}_{R}$ denote the ∞-category $\mathrm{LMod}_{\mathrm{LMod}_R}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. We will refer to the objects of $\mathrm{LinCat}^{\mathrm{St}}_{R}$ as *stable R-linear ∞-categories*, and to $\mathrm{LinCat}^{\mathrm{St}}_{R}$ as the *∞-category of stable R-linear ∞-categories*.

Warning D.1.5.2. If R is a connective $\mathbb{E}_2$-ring, then the definition of a stable R-linear ∞-category $\mathcal{C}$ given in Definition D.1.4.1 is equivalent (but not identical) to the definition given in Variant D.1.5.1: in the first case, we require that $\mathcal{C}$ is equipped with a left action of the monoidal ∞-category $\mathrm{LMod}^{\mathrm{cn}}_{R}$ of connective R-modules, and in the second we require that it is equipped with a left action of the monoidal ∞-category LMod_R of *all* R-modules. It follows from the argument supplied above that if $\mathcal{C}$ is stable, then any action of $\mathrm{LMod}^{\mathrm{cn}}_{R}$ on $\mathcal{C}$ admits an essentially unique extension to an action of the larger ∞-category LMod_R.

In what follows, we will abuse terminology by not distinguishing between the notions of stable R-linear ∞-category given in Definition D.1.4.1 and Variant D.1.5.1. The advantage of the former is that it makes sense without the assumption that $\mathcal{C}$ is stable, and the advantage of the latter is that it makes sense without the assumption that R is connective.

Remark D.1.5.3. Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ and $\mathcal{C}'$ be stable R-linear ∞-categories. We will refer to the morphisms from $\mathcal{C}$ to $\mathcal{C}'$ in $\mathrm{LinCat}^{\mathrm{St}}_{R}$ as *R-linear functors* from $\mathcal{C}$ to $\mathcal{C}'$. Every R-linear functor $F : \mathcal{C} \to \mathcal{C}'$ determines a colimit-preserving functor between the underlying (presentable) ∞-categories of $\mathcal{C}$ and $\mathcal{C}'$, which therefore admits a right adjoint G (Corollary HTT.5.5.2.9). If the functor G commutes with small colimits, then it inherits the structure of an R-linear functor: this is a special case of Remark D.7.4.4.

In the stable setting, we have the following variant of Construction D.1.3.3:

Construction D.1.5.4 (The Center)**.** For every $\mathbb{E}_1$-ring R, we can regard LMod_R as a presentable stable ∞-category with a distinguished object (given by R, regarded as a left module over itself). The construction $R \mapsto (\mathrm{LMod}_R, R)$ determines a functor $\Theta_* : \mathrm{Alg}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}))$. According to Theorem HA.4.8.5.11 , the functor Θ_* admits a right adjoint G which carries a presentable stable ∞-category $\mathcal{C}$ with a distinguished object $X \in \mathcal{C}$

to its endomorphism algebra $\underline{\mathrm{End}}(X) \in \mathrm{Alg}(\mathrm{Sp})$, an $\mathbb{E}_1$-ring satisfying $\Omega^{\infty-n}\,\mathrm{End}(X) \simeq \mathrm{Map}_{\mathcal{C}}(X, \Sigma^n X)$ for each $n \in \mathbf{Z}$.

According to Theorem HA.4.8.5.16 , the functor Θ_* is symmetric monoidal. It follows that the functor G is lax symmetric monoidal. Suppose that $\mathcal{C}$ is an associative algebra object of $\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$: that is, a monoidal presentable stable ∞-category for which tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Then we can regard $(\mathcal{C}, \mathbf{1})$ as an algebra object of $\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}))$, where $\mathbf{1}$ denotes the unit object of $\mathcal{C}$. It follows that $\underline{\mathrm{End}}(\mathbf{1}) = G(\mathcal{C}, \mathbf{1})$ can be regarded as an object of $\mathrm{Alg}(\mathrm{Alg}(\mathrm{Sp})) \simeq \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp})$. In other words, we can regard $\underline{\mathrm{End}}(\mathbf{1})$ as an $\mathbb{E}_2$-ring.

Suppose that $\mathcal{C}$ is an arbitrary presentable stable ∞-category. Then the ∞-category $\mathrm{LFun}(\mathcal{C}, \mathcal{C})$ of colimit-preserving functors from $\mathcal{A}$ to itself is also a presentable stable ∞-category ∞-category, and the composition map

$$\circ : \mathrm{LFun}(\mathcal{C}, \mathcal{C}) \times \mathrm{LFun}(\mathcal{C}, \mathcal{C}) \to \mathrm{LFun}(\mathcal{C}, \mathcal{C})$$

preserves small colimits separately in each variable and determines a monoidal structure on $\mathrm{LFun}(\mathcal{C}, \mathcal{C})$. We let $\mathfrak{Z}(\mathcal{C})$ denote the $\mathbb{E}_2$-ring given by $\underline{\mathrm{End}}(\mathrm{id}_{\mathcal{C}}) = G(\mathrm{LFun}(\mathcal{C}, \mathcal{C}), \mathrm{id}_{\mathcal{C}})$. Then $\mathfrak{Z}(\mathcal{C})$ is an $\mathbb{E}_2$-ring which we will refer to as the *center* of $\mathcal{C}$.

Remark D.1.5.5. Let $\mathcal{C}$ be a presentable stable ∞-category. For every $\mathbb{E}_2$-ring R, there is a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp})}(R, \mathfrak{Z}(\mathcal{C})) \simeq \mathrm{Map}_{\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})}(\mathrm{LMod}_R, \mathrm{LFun}(\mathcal{C}, \mathcal{C})).$$

In other words, promoting $\mathcal{C}$ to a stable R-linear ∞-category (in the sense of Variant D.1.5.1) is equivalent to providing a morphism of $\mathbb{E}_2$-rings $R \to \mathfrak{Z}(\mathcal{C})$. Using Remark D.1.3.4 and Warning D.1.5.2, we deduce that the connective center $\mathfrak{Z}^{\mathrm{cn}}(\mathcal{C})$ of Construction D.1.3.3 can be identified with the connective cover of the center $\mathfrak{Z}(\mathcal{C})$.

Remark D.1.5.6. Let $\mathcal{C}$ be a presentable stable ∞-category. For every integer n, we have a canonical homotopy equivalence $\Omega^{\infty-n}\,\mathfrak{Z}(\mathcal{C}) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$, and therefore a canonical isomorphism of abelian groups $\pi_n\,\mathfrak{Z}(\mathcal{C}) \simeq \mathrm{Ext}^{-n}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$.

Remark D.1.5.7. If $\mathcal{C}$ is a presentable stable ∞-category, then the $\mathbb{E}_2$-ring $\mathfrak{Z}(\mathcal{C})$ can be regarded as an incarnation of the *topological Hochschild cohomology* of the ∞-category $\mathcal{C}$.

D.1.6 Limits and Colimits

We close this section with a few remarks about limits and colimits of R-linear ∞-categories. Note that the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ admits small limits and colimits (see §HTT.5.5.3). Consequently, for any connective $\mathbb{E}_2$-ring R, the ∞-category $\mathrm{LinCat}_R^{\mathrm{Add}} = \mathrm{LMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ of additive R-linear ∞-categories admits small limits and colimits. However, when we restrict our attention to additive R-linear ∞-categories which satisfy additional requirements, then the matter becomes more delicate.

Remark D.1.6.1 (Limits of Prestable R-Linear ∞-Categories)**.** Let R be a connective $\mathbb{E}_2$-ring, and let $\mathrm{LinCat}_R^{\mathrm{PSt,lex}}$ denote the subcategory of $\mathrm{LinCat}_R^{\mathrm{PSt}}$ whose objects are prestable R-linear ∞-categories and whose morphisms are R-linear functors which preserve small colimits and finite limits. We have an evident forgetful functor

$$q : \mathrm{LinCat}_R^{\mathrm{PSt,lex}} \to \mathrm{LinCat}_S^{\mathrm{PSt,lex}} \simeq \mathsf{Groth}_\infty^{\mathrm{lex}}.$$

It follows from Proposition C.3.2.4 that for every small diagram $p : K \to \mathrm{LinCat}_R^{\mathrm{PSt,lex}}$, the composite map $(q \circ p) : K \to \mathsf{Groth}_\infty^{\mathrm{lex}}$ can be extended to a limit diagram which is preserved by the inclusions $\mathsf{Groth}_\infty^{\mathrm{lex}} \hookrightarrow \mathsf{Groth}_\infty \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$. Combining this observation with Corollary HA.4.2.3.3 , we deduce that the diagram p admits a limit in $\mathrm{LinCat}_R^{\mathrm{PSt,lex}}$ which is preserved by the inclusion functors $\mathrm{LinCat}_R^{\mathrm{PSt,lex}} \hookrightarrow \mathrm{LinCat}_R^{\mathrm{PSt}} \hookrightarrow \mathrm{LinCat}_R^{\mathrm{Add}}$.

Remark D.1.6.2 (Filtered Colimits of Prestable R-Linear ∞-Categories)**.** Let R be a connective $\mathbb{E}_2$-ring. It follows from Theorem C.3.3.1 that the full subcategory $\mathrm{LinCat}_R^{\mathrm{PSt}} \subseteq \mathrm{LinCat}_R^{\mathrm{Add}}$ is closed under small filtered colimits. In particular, $\mathrm{LinCat}_R^{\mathrm{PSt}}$ admits small filtered colimits, which are preserved by the forgetful functors

$$\mathrm{LinCat}_R^{\mathrm{PSt}} \hookrightarrow \mathrm{LinCat}_R^{\mathrm{Add}} \to \mathcal{P}\mathrm{r}^{\mathrm{Add}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}.$$

Moreover, the subcategory $\mathrm{LinCat}_R^{\mathrm{PSt,lex}} \subseteq \mathrm{LinCat}_R^{\mathrm{PSt}}$ also admits small filtered colimits, which are preserved by the inclusion $\mathrm{LinCat}_R^{\mathrm{PSt,lex}} \hookrightarrow \mathrm{LinCat}_R^{\mathrm{PSt}}$ (Proposition C.3.3.5).

Remark D.1.6.3 (Colimits of Prestable R-Linear ∞-Categories along Compact Functors)**.** Let R be a connective $\mathbb{E}_2$-ring and let $\mathrm{LinCat}_R^{\mathrm{PSt},c}$ denote the subcategory of $\mathrm{LinCat}_R^{\mathrm{PSt}}$ whose objects are prestable R-linear ∞-categories and whose morphisms are compact R-linear functors (see Definition C.3.4.2). We have an evident forgetful functor

$$q : \mathrm{LinCat}_R^{\mathrm{PSt},c} \to \mathrm{LinCat}_S^{\mathrm{PSt},c} \simeq \mathsf{Groth}_\infty^{\mathrm{c}}.$$

It follows from Proposition C.3.5.1 that for every small diagram $p : K \to \mathrm{LinCat}_R^{\mathrm{PSt},c}$, the composite map $(q \circ p) : K \to \mathsf{Groth}_\infty^{\mathrm{c}}$ can be extended to a colimit diagram which is preserved by the inclusion $\mathsf{Groth}_\infty^{\mathrm{c}} \hookrightarrow \mathsf{Groth}_\infty$. Moreover, this colimit is also preserved by the formation of tensor product with any Grothendieck prestable ∞-category $\mathcal{C}$ (Proposition C.4.5.1). Combining this observation with Corollary HA.4.2.3.5 , we deduce that the diagram p admits a colimit in $\mathrm{LinCat}_R^{\mathrm{PSt},c}$ which is preserved by the inclusion $\mathrm{LinCat}_R^{\mathrm{PSt},c} \hookrightarrow \mathrm{LinCat}_R^{\mathrm{PSt}}$.

Remark D.1.6.4 (Limits and Colimits of Stable R-Linear ∞-Categories)**.** Let R be an arbitrary $\mathbb{E}_2$-ring. Then the ∞-category $\mathrm{LinCat}_R^{\mathrm{St}} = \mathrm{LMod}_{\mathrm{LMod}_R}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ of stable R-linear

∞-categories admits small limits and colimits. In the special case where R is connective, we can identify $\mathrm{LinCat}_R^{\mathrm{St}}$ with a full subcategory of $\mathrm{LinCat}_R^{\mathrm{PSt,lex}}$ (since every right exact functor between stable ∞-categories is also left exact) and the inclusion $\mathrm{LinCat}_R^{\mathrm{St}} \hookrightarrow \mathrm{LinCat}_R^{\mathrm{PSt,lex}}$ preserves small limits.

Remark D.1.6.5 (Limits of Abelian R-Linear ∞-Categories)**.** Let R be a connective $\mathbb{E}_2$-ring, and let $\mathrm{LinCat}_R^{\mathrm{Ab,lex}}$ denote the subcategory of $\mathrm{LinCat}_R^{\mathrm{Ab}}$ whose objects are abelian R-linear ∞-categories and whose morphisms are R-linear functors which preserve small colimits and finite limits. We have an evident forgetful functor $q : \mathrm{LinCat}_R^{\mathrm{Ab,lex}} \to \mathsf{Groth}_{\mathrm{ab}}^{\mathrm{lex}}$. It follows from Proposition C.5.4.21 that for $p : K \to \mathrm{LinCat}_R^{\mathrm{Ab,lex}}$, the composite map $(q \circ p) : K \to \mathsf{Groth}_{\mathrm{ab}}^{\mathrm{lex}}$ can be extended to a limit diagram which is preserved by the inclusions $\mathsf{Groth}_{\mathrm{ab}}^{\mathrm{lex}} \hookrightarrow \mathsf{Groth}_{\mathrm{ab}} \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$. Combining this observation with Corollary HA.4.2.3.3 , we deduce that the diagram p admits a limit in $\mathrm{LinCat}_R^{\mathrm{Ab,lex}}$ which is preserved by the inclusion functors $\mathrm{LinCat}_R^{\mathrm{Ab,lex}} \hookrightarrow \mathrm{LinCat}_R^{\mathrm{Ab}} \hookrightarrow \mathrm{LinCat}_R^{\mathrm{Add}}$.

Remark D.1.6.6 (Filtered Colimits of Abelian R-Linear ∞-Categories)**.** Let R be a connective $\mathbb{E}_2$-ring. Then the ∞-category $\mathrm{LinCat}_R^{\mathrm{Ab}}$ is closed under filtered colimits in $\mathrm{LinCat}_R^{\mathrm{Add}}$: this can be deduced from a suitable 1-categorical analogue of Theorem C.3.3.1. In particular, $\mathrm{LinCat}_R^{\mathrm{Ab}}$ admits small filtered colimits, which are preserved by the forgetful functors

$$\mathrm{LinCat}_R^{\mathrm{Ab}} \hookrightarrow \mathrm{LinCat}_R^{\mathrm{Add}} \to \mathcal{P}\mathrm{r}^{\mathrm{Add}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}.$$

Moreover, the subcategory $\mathrm{LinCat}_R^{\mathrm{Ab,lex}} \subseteq \mathrm{LinCat}_R^{\mathrm{Ab}}$ also admits small filtered colimits, which are preserved by the inclusion $\mathrm{LinCat}_R^{\mathrm{Ab,lex}} \hookrightarrow \mathrm{LinCat}_R^{\mathrm{Ab}}$.

Remark D.1.6.7. Let R be a connective $\mathbb{E}_2$-ring. Then the functors

$$\begin{array}{ccccc} \mathrm{LinCat}_R^{\mathrm{St}} & \leftarrow & \mathrm{LinCat}_R^{\mathrm{PSt}} & \to & \mathrm{LinCat}_R^{\mathrm{Ab}} \\ \mathrm{Sp}(\mathcal{C}) & \mapsfrom & \mathcal{C} & \mapsto & \mathcal{C}^{\heartsuit} \end{array}$$

preserve small filtered colimits (since they are given by tensoring with the objects $\mathrm{Sp}, \tau_{\leqslant 0}\,\mathrm{Sp}^{\mathrm{cn}}$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{Add}}$).

Remark D.1.6.8. Let R be a connective $\mathbb{E}_2$-ring. Then the functors

$$\begin{array}{ccccc} \mathrm{LinCat}_R^{\mathrm{St}} & \leftarrow & \mathrm{LinCat}_R^{\mathrm{PSt,lex}} & \to & \mathrm{LinCat}_R^{\mathrm{Ab,lex}} \\ \mathrm{Sp}(\mathcal{C}) & \mapsfrom & \mathcal{C} & \mapsto & \mathcal{C}^{\heartsuit} \end{array}$$

preserve small limits: this follows from Proposition C.5.5.20 and Corollary C.3.2.5.

Warning D.1.6.9. Let R be a connective $\mathbb{E}_2$-ring. The stabilization construction $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ determines a functor $\mathrm{LinCat}_R^{\mathrm{PSt},c} \to \mathrm{LinCat}_R^{\mathrm{St}}$ which preserves small colimits. However, the construction

$$\mathrm{LinCat}_R^{\mathrm{PSt},c} \to \mathrm{LinCat}_R^{\mathrm{Ab}} \qquad \mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$$

does *not* preserve small colimits in general. However, it does preserve all colimits in $\mathrm{LinCat}_R^{\mathrm{PSt},c}$ which remain colimits in the larger ∞-category $\mathrm{LinCat}_R^{\mathrm{Add}}$: for example, colimits of diagrams which satisfy the hypothesis described in Remark C.3.5.4.

D.2 Tensor Products and Extension of Scalars

Let R be a connective $\mathbb{E}_2$-ring. In §D.1, we introduced the the ∞-category $\mathrm{LinCat}_R^{\mathrm{Add}}$ of *additive R-linear ∞-categories* (Definition D.1.2.1), as well as the full subcategories

$$\mathrm{LinCat}_R^{\mathrm{St}}, \mathrm{LinCat}_R^{\mathrm{PSt}}, \mathrm{LinCat}_R^{\mathrm{Ab}} \subseteq \mathrm{LinCat}_R^{\mathrm{Add}}$$

of stable, prestable, and abelian R-linear ∞-categories. Note that these ∞-categories can be regarded as *contravariant* functors of R: if $\phi : R \to R'$ is a morphism of connective $\mathbb{E}_2$-rings, then extension of scalars along ϕ determines a monoidal functor $\mathrm{LMod}_R^{\mathrm{cn}} \to \mathrm{LMod}_{R'}^{\mathrm{cn}}$ and therefore a forgetful functor

$$\mathrm{LinCat}_{R'}^{\mathrm{Add}} = \mathrm{LMod}_{\mathrm{LMod}_{R'}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{LMod}_{\mathrm{LMod}_{R}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) = \mathrm{LinCat}_{R}^{\mathrm{Add}},$$

which we will refer to as *restriction of scalars along ϕ*. In other words, every additive R'-linear ∞-category $\mathcal{A}$ can be regarded as an additive R-linear ∞-category. Note that the underlying ∞-category $\mathcal{A}$ does not change, so if $\mathcal{A}$ is stable, prestable, or abelian as an R'-linear ∞-category, then it is also stable, prestable or abelian as an R-linear ∞-category. In other words, restriction of scalars induces forgetful functors

$$\mathrm{LinCat}_{R'}^{\mathrm{St}} \to \mathrm{LinCat}_{R}^{\mathrm{St}} \qquad \mathrm{LinCat}_{R'}^{\mathrm{PSt}} \to \mathrm{LinCat}_{R}^{\mathrm{PSt}} \qquad \mathrm{LinCat}_{R'}^{\mathrm{Ab}} \to \mathrm{LinCat}_{R}^{\mathrm{Ab}}.$$

In this section, we will study left adjoints to these forgetful functors, given by *extension of scalars* along the morphism $\phi : R \to R'$.

D.2.1 The Bar Construction

We begin with a general discussion of relative tensor products in the setting of presentable ∞-categories.

Construction D.2.1.1. Let $\mathcal{C}$ be a monoidal ∞-category, let $\mathcal{M}$ be an ∞-category which is right-tensored over $\mathcal{C}$, and let $\mathcal{N}$ be an ∞-category which is left-tensored over $\mathcal{C}$. Assume that $\mathcal{C}$, $\mathcal{M}$, and $\mathcal{N}$ are presentable, and that the tensor product functors

$$\mathcal{M} \times \mathcal{C} \to \mathcal{M} \qquad \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{C} \times \mathcal{N} \to \mathcal{N}$$

preserve small colimits separately in each variable. Then we can regard $\mathcal{C}$ as an algebra object in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories, and we can regard $\mathcal{M}$ and $\mathcal{N}$

as right and left modules over $\mathcal{C}$, respectively. We let $\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$ denote the relative tensor product of $\mathcal{M}$ with $\mathcal{N}$ over $\mathcal{C}$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$: that is, the geometric realization of the two-sided bar construction $\mathrm{Bar}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})_{\bullet}$ of Construction HA.4.4.2.7 .

Let us now specialize to the case where $\mathcal{C} = \mathrm{LMod}_R^{\mathrm{cn}}$ for some connective $\mathbb{E}_2$-ring R. In this case, the left action of $\mathcal{C}$ on $\mathcal{N}$ exhibits $\mathcal{N}$ as an additive R-linear ∞-category, and the right action of $\mathcal{C}$ on $\mathcal{M}$ exhibits $\mathcal{M}$ as an additive R^{rev}-linear ∞-category, where R^{rev} denotes the reverse of R (see Warning D.1.2.5). In this case, we will denote the tensor product $\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$ by $\mathcal{M} \otimes_R \mathcal{N}$, and refer to it as the *tensor product of $\mathcal{M}$ with $\mathcal{N}$ over R.*

Remark D.2.1.2 (Tensor Products of Ordinary Categories)**.** Let R be an $\mathbb{E}_2$-ring and let $\mathcal{M} \in \mathrm{RMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ and $\mathcal{N} \in \mathrm{LMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ be as Construction D.2.1.1. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are (equivalent to the nerves of) ordinary categories. Then the right and left actions of $\mathrm{LMod}_R^{\mathrm{cn}}$ on $\mathcal{M}$ and $\mathcal{N}$ factor through the monoidal functor $\pi_0 : \mathrm{LMod}_R^{\mathrm{cn}} \to \mathrm{LMod}_R^{\heartsuit}$. Moreover, the induced map of two-sided bar constructions $\mathrm{Bar}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{M}, \mathcal{N})_{\bullet} \to \mathrm{Bar}_{\mathrm{LMod}_R^{\heartsuit}}(\mathcal{M}, \mathcal{N})_{\bullet}$ induces an equivalence in each degree, and therefore an equivalence

$$\mathcal{M} \otimes_R \mathcal{N} = |\mathrm{Bar}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{M}, \mathcal{N})_{\bullet}| \simeq |\mathrm{Bar}_{\mathrm{LMod}_R^{\heartsuit}}(\mathcal{M}, \mathcal{N})| = \mathcal{M} \otimes_{\mathrm{LMod}_R^{\heartsuit}} \mathcal{N}.$$

Remark D.2.1.3. For every presentable ∞-category $\mathcal{C}$, let $\mathcal{C}^{\heartsuit}$ denote the full subcategory of $\mathcal{C}$ spanned by the discrete objects. The construction $\mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$ determines a functor from the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories to the ∞-category $\mathrm{Mod}_{\mathcal{S}\mathrm{et}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ of presentable 1-categories. This functor is symmetric monoidal and commutes with small colimits, and therefore commutes with the formation of tensor products. Consequently, if R is a connective $\mathbb{E}_2$-ring and we are given objects $\mathcal{M} \in \mathrm{RMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ and $\mathcal{N} \in \mathrm{LMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ as in Construction D.2.1.1, then we have canonical equivalences

$$\begin{aligned} (\mathcal{M} \otimes_R \mathcal{N})^{\heartsuit} &= (\mathcal{M} \otimes_{\mathrm{Mod}_R^{\mathrm{cn}}} \mathcal{N})^{\heartsuit} \\ &\simeq \mathcal{M}^{\heartsuit} \otimes_{\mathrm{Mod}_R^{\heartsuit}} \mathcal{N}^{\heartsuit} \\ &\simeq \mathcal{M}^{\heartsuit} \otimes_R \mathcal{N}^{\heartsuit} \end{aligned}$$

where the final equivalence is provided by Remark D.2.1.2.

Remark D.2.1.4 (Tensor Products of Stable Categories)**.** Let R be a connective $\mathbb{E}_2$-ring and let $\mathcal{M} \in \mathrm{RMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ and $\mathcal{N} \in \mathrm{LMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ be as Construction D.2.1.1. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are stable. Then the right and left actions of $\mathrm{LMod}_R^{\mathrm{cn}}$ on $\mathcal{M}$ and $\mathcal{N}$ factor through the monoidal inclusion functor $\mathrm{LMod}_R^{\mathrm{cn}} \hookrightarrow \mathrm{LMod}_R$. As in Remark D.2.1.2, the induced map of two-sided bar constructions $\mathrm{Bar}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{M}, \mathcal{N})_{\bullet} \to \mathrm{Bar}_{\mathrm{LMod}_R}(\mathcal{M}, \mathcal{N})_{\bullet}$ induces an equivalence in each degree, and therefore an equivalence

$$\mathcal{M} \otimes_R \mathcal{N} = |\mathrm{Bar}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{M}, \mathcal{N})_{\bullet}| \simeq |\mathrm{Bar}_{\mathrm{LMod}_R}(\mathcal{M}, \mathcal{N})| = \mathcal{M} \otimes_{\mathrm{LMod}_R} \mathcal{N}.$$

Variant D.2.1.5. Let R be an arbitrary $\mathbb{E}_2$-ring, and let $\mathcal{N}$ be a stable R-linear ∞-category in the sense of Variant D.1.5.1: that is, a presentable stable ∞-category equipped with a left action of the monoidal ∞-category LMod_R. Similarly, let $\mathcal{M}$ be a presentable stable ∞-category equipped with a right action of the monoidal ∞-category LMod_R. We let $\mathcal{M} \otimes_R \mathcal{N}$ denote the relative tensor product $\mathcal{M} \otimes_{\mathrm{LMod}_R} \mathcal{N} = |\mathrm{Bar}_{\mathrm{LMod}_R}(\mathcal{M}, \mathcal{N})_\bullet|$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (or, equivalently, in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ of presentable stable ∞-categories). It follows from Remark D.2.1.4 that when R is connective, this construction is canonically equivalent to the relative tensor product given in Construction D.2.1.1.

Remark D.2.1.6. The construction $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ determines a symmetric monoidal functor from the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories to the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}} \simeq \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ of presentable stable ∞-categories. Consequently, if R is a connective $\mathbb{E}_2$-ring and we are given objects $\mathcal{M} \in \mathrm{RMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ and $\mathcal{N} \in \mathrm{LMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ as in Construction D.2.1.1, then we have canonical equivalences

$$\begin{aligned}
\mathrm{Sp}(\mathcal{M} \otimes_R \mathcal{N}) &= \mathrm{Sp}(\mathcal{M} \otimes_{\mathrm{Mod}_R^{\mathrm{cn}}} \mathcal{N}) \\
&\simeq \mathrm{Sp}(\mathcal{M}) \otimes_{\mathrm{Mod}_R} \mathrm{Sp}(\mathcal{N}) \\
&\simeq \mathrm{Sp}(\mathcal{M}) \otimes_R \mathrm{Sp}(\mathcal{N})
\end{aligned}$$

where the final equivalence is provided by Remark D.2.1.4.

D.2.2 Closure Properties of $\otimes_R$

We now study the closure properties under tensor product of the special classes of R-linear ∞-categories introduced in Definition D.1.4.1.

Proposition D.2.2.1. *Let R be a connective $\mathbb{E}_2$-ring, and suppose we are given ∞-categories*

$$\mathcal{M} \in \mathrm{RMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \qquad \mathcal{N} \in \mathrm{LMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}).$$

Then:

- (a) *The relative tensor product $\mathcal{M} \otimes_R \mathcal{N}$ is an additive presentable ∞-category.*
- (b) *If either $\mathcal{M}$ or $\mathcal{N}$ is stable, then the relative tensor product $\mathcal{M} \otimes_R \mathcal{N}$ is stable.*
- (c) *If $\mathcal{M}$ and $\mathcal{N}$ are Grothendieck prestable ∞-categories, then the tensor product $\mathcal{M} \otimes_R \mathcal{N}$ is also a Grothendieck prestable ∞-category.*
- (d) *If $\mathcal{M}$ and $\mathcal{N}$ are (equivalent to) Grothendieck abelian categories, then the relative tensor product $\mathcal{M} \otimes_R \mathcal{N}$ is (equivalent to) a Grothendieck abelian category.*

Remark D.2.2.2. In the special case where R is the sphere spectrum, assertion (c) fo Proposition D.2.2.1 reduces to Theorem C.4.2.1, and assertion (d) reduces to Theorem C.5.4.16.

Proof of Proposition D.2.2.1. Since $\mathcal{M}$ is additive, it can be regarded as a module over the commutative algebra object $\mathrm{Sp}^{\mathrm{cn}} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ (Corollary C.4.1.3), so that $\mathcal{M} \otimes_R \mathcal{N}$ is also a module over $\mathrm{Sp}^{\mathrm{cn}}$ and therefore additive (Corollary C.4.1.3). This proves (a). Similarly, if $\mathcal{M}$ or $\mathcal{N}$ is stable, then it can be regarded as a module over $\mathrm{Sp} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ by virtue of Proposition HA.4.8.2.18 , so that $\mathcal{M} \otimes_R \mathcal{N}$ is also a module over Sp and therefore a stable ∞-category. This proves (b).

We now prove (c). Let us identify $\mathcal{M} \otimes_R \mathcal{N}$ with the geometric realization of a simplicial object $\mathcal{E}_\bullet$ of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ given by the two-sided bar construction $\mathrm{Bar}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{M}, \mathcal{N})_\bullet$ of Construction **??**. By virtue of Lemma HTT.6.5.3.7 , this is equivalent to the colimit of the underlying semisimplicial object

$$([n] \in \mathbf{\Delta}_s^{\mathrm{op}}) \mapsto \mathcal{E}_n = \mathcal{M} \otimes (\mathrm{LMod}_R^{\mathrm{cn}})^{\otimes n} \otimes \mathcal{N}.$$

Note that each $\mathcal{E}_n$ is a Grothendieck prestable ∞-category by virtue of Theorem C.4.2.1. To show that the colimit (formed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$) is again a Grothendieck prestable ∞-category, it will suffice to show that the semisimplicial object $\mathcal{E}_\bullet$ satisfies the hypotheses of Remark C.3.5.4: that is, for every injective map $\alpha : [m] \hookrightarrow [n]$, the associated functor $f_\alpha : \mathcal{E}_n \to \mathcal{E}_m$ admits a right adjoint which preserves small colimits. Factoring α as a composition, we may reduce to the case $m = n - 1$, so that f_α is one of the face maps of the simplicial object $\mathcal{E}_\bullet$. Unwinding the definitions, we see that f_α is obtained from one of the maps

$$a : \mathcal{M} \otimes \mathrm{LMod}_R^{\mathrm{cn}} \to \mathcal{M} \qquad m : \mathrm{LMod}_R^{\mathrm{cn}} \otimes \mathrm{LMod}_R^{\mathrm{cn}} \to \mathrm{LMod}_R^{\mathrm{cn}} \qquad a' : \mathrm{LMod}_R^{\mathrm{cn}} \otimes \mathcal{N} \to \mathcal{N}$$

by tensoring with some auxiliary object of Groth_∞; here the functor a is given by the right action of $\mathrm{LMod}_R^{\mathrm{cn}}$ on $\mathcal{M}$, the functor m is given by the monoidal structure on $\mathrm{LMod}_R^{\mathrm{cn}}$, and the functor a' is given by the left action of $\mathrm{LMod}_R^{\mathrm{cn}}$ on $\mathcal{N}$. By virtue of Remark C.4.4.5, it will suffice to show that the functors a, m, and a' admit right adjoints which preserve small colimits. We will prove this for the functor a; the proof in the other cases differs by a slight change in notation. We first note that a admits a right homotopy inverse f, given by the natural map $\mathcal{M} \simeq \mathcal{M} \otimes \mathrm{Sp}^{\mathrm{cn}} \to \mathcal{M} \otimes \mathrm{LMod}_R^{\mathrm{cn}}$. Let $g : \mathcal{M} \otimes \mathrm{LMod}_R \to \mathcal{M}$ be a right adjoint to f, so that g is a left homotopy inverse to the right adjoint of a. Consequently, to show that the right adjoint of a preserves small colimits, it will suffice to show that g is conservative and preserves small colimits. Using Theorem HA.4.8.4.6 , we can identify g with the forgetful functor $\mathrm{RMod}_R(\mathcal{M}) \to \mathcal{M}$, so that the desired result is a special case of Corollary HA.4.2.3.5 .

We now prove (d). Let us abuse notation by identifying $\mathcal{M}$ and $\mathcal{N}$ with their homotopy categories (which are Grothendieck abelian categories). Define

$$\mathcal{M}_+ = \mathcal{D}(\mathcal{M})_{\geqslant 0} \qquad \mathcal{N}_+ = \mathcal{D}(\mathcal{N})_{\geqslant 0},$$

so that $\mathcal{M}_+$ and $\mathcal{N}_+$ are Grothendieck prestable ∞-categories whose hearts can be identified with $\mathcal{M}$ and $\mathcal{N}$, respectively. It follows from Example D.1.3.9 that the action of R on $\mathcal{N}$ admits an essentially unique lift to an action of determines an action of R on $\mathcal{N}_+$; similarly, the action of R^{rev} on $\mathcal{M}$ admits an essentially unique lift to an action of R^{rev} on $\mathcal{M}_+$. Applying Remark D.2.1.3, we compute

$$\begin{aligned}\mathcal{M} \otimes_R \mathcal{N} &\simeq \mathcal{M}_+^{\heartsuit} \otimes_R \mathcal{N}_+^{\heartsuit} \\ &\simeq (\mathcal{M}_+ \otimes_R \mathcal{N}_+)^{\heartsuit}.\end{aligned}$$

It follows from part (c) that the tensor product $\mathcal{M}_+ \otimes_R \mathcal{N}_+$ is a Grothendieck prestable ∞-category, so that its heart $(\mathcal{M}_+ \otimes_R \mathcal{N}_+)^{\heartsuit}$ is a Grothendieck abelian category. □

D.2.3 The Commutative Case

Let us now specialize to the study of additive ∞-categories equipped with actions of $\mathbb{E}_\infty$-rings.

Remark D.2.3.1. Let R be a connective $\mathbb{E}_\infty$-ring. Then $\mathrm{Mod}_R^{\mathrm{cn}}$ is a symmetric monoidal ∞-category which we can regard as a *commutative* algebra object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. In this case, the ∞-category

$$\mathrm{LinCat}_R^{\mathrm{Add}} = \mathrm{LMod}_{\mathrm{LMod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \simeq \mathrm{Mod}_{\mathrm{Mod}_R^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$$

of additive R-linear ∞-categories inherits the structure of a symmetric monoidal ∞-category, and the symmetric monoidal structure on $\mathrm{LinCat}_R^{\mathrm{Add}}$ is given (at the level of the underlying ∞-categories) by the relative tensor product $\otimes_R$ of Construction D.2.1.1 (see Theorem HA.4.5.2.1).

Note that the full subcategory $\mathrm{LinCat}_R^{\mathrm{PSt}} \subseteq \mathrm{LinCat}_R^{\mathrm{Add}}$ of prestable R-linear ∞-categories contains the unit object $\mathrm{Mod}_R^{\mathrm{cn}} \in \mathrm{LinCat}_R^{\mathrm{Add}}$ and is closed under tensor products (by virtue of Proposition D.2.2.1). It follows that $\mathrm{LinCat}_R^{\mathrm{PSt}}$ inherits the structure of a symmetric monoidal ∞-category.

Variant D.2.3.2. Let R be a connective $\mathbb{E}_\infty$-ring. Then the abelian category $\mathrm{Mod}_R^{\heartsuit}$ of discrete R-modules can be regarded as a commutative algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Consequently, the ∞-category $\mathrm{Mod}_{\mathrm{Mod}_R^{\heartsuit}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ inherits the structure of a symmetric monoidal ∞-category. Note that we can identify $\mathrm{Mod}_{\mathrm{Mod}_R^{\heartsuit}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ with the full subcategory of $\mathrm{LinCat}_R^{\mathrm{Add}}$ spanned by the additive R-linear ∞-categories which are equivalent to their homotopy categories. It follows from Remark D.2.1.2 that under this identification, the tensor product on $\mathrm{Mod}_{\mathrm{Mod}_R^{\heartsuit}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ corresponds to the relative tensor product $(\mathcal{M}, \mathcal{N}) \mapsto \mathcal{M} \otimes_R \mathcal{N}$ of Construction D.2.1.1. Let us identify ∞-category $\mathrm{Mod}_R^{\mathrm{Ab}}$ of abelian R-linear ∞-categories with a full subcategory of $\mathrm{Mod}_{\mathrm{Mod}_R^{\heartsuit}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. This subcategory contains the unit object $\mathrm{Mod}_R^{\heartsuit}$ and is closed under tensor products (by virtue of Proposition D.2.2.1), and therefore inherits the structure of a symmetric monoidal ∞-category.

Variant D.2.3.3. Let R be an arbitrary $\mathbb{E}_\infty$-ring. Then Mod_R is a symmetric monoidal ∞-category which we can regard as commutative algebra object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. In this case, the ∞-category

$$\mathrm{LinCat}_R^{\mathrm{St}} = \mathrm{LMod}_{\mathrm{LMod}_R}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \simeq \mathrm{Mod}_{\mathrm{Mod}_R}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$$

of stable R-linear ∞-categories (as defined in Variant D.1.5.1) inherits a symmetric monoidal structure whose tensor product is given (at the level of the underlying stable ∞-categories) by the the construction $(\mathcal{M}, \mathcal{N}) \mapsto \mathcal{M} \otimes_R \mathcal{N}$ of Variant D.2.1.5 (which agrees with Construction D.2.1.1 whenever R is connective).

Remark D.2.3.4. Let R be a connective $\mathbb{E}_\infty$-ring. Then the constructions

$$\begin{array}{ccccc} \mathrm{LinCat}_R^{\mathrm{St}} & \leftarrow & \mathrm{LinCat}_R^{\mathrm{PSt}} & \rightarrow & \mathrm{LinCat}_R^{\mathrm{Ab}} \\ \mathrm{Sp}(\mathcal{C}) & \leftarrow\!\shortmid & \mathcal{C} & \mapsto & \mathcal{C}^{\heartsuit} \end{array}$$

are symmetric monoidal (with respect to the symmetric monoidal structures described in Remark D.2.3.1, Variant D.2.3.2, and Variant D.2.3.3). Note that on all three ∞-categories, the tensor product functor $\otimes_R$ is given by Construction D.2.1.1. However, the unit objects are different: the unit object of $\mathrm{LinCat}_R^{\mathrm{St}}$ is the stable ∞-category Mod_R, the unit object of $\mathrm{LinCat}_R^{\mathrm{PSt}}$ is the prestable ∞-category $\mathrm{Mod}_R^{\mathrm{cn}}$, and the unit object of $\mathrm{LinCat}_R^{\mathrm{Ab}}$ is the abelian category $\mathrm{Mod}_R^{\heartsuit}$.

Remark D.2.3.5. If κ is a field, the tensor product

$$\otimes_\kappa : \mathrm{LinCat}_\kappa^{\mathrm{Ab}} \times \mathrm{LinCat}_\kappa^{\mathrm{Ab}} \rightarrow \mathrm{LinCat}_\kappa^{\mathrm{Ab}}$$

is closely related to Deligne's tensor product for (small) κ-linear abelian categories (see [48]). The restriction to the setting of small abelian categories is technically inconvenient, since Deligne's tensor product is only well-defined (at least as an abelian category) under somewhat restrictive assumptions. We refer the reader to [65] for further discussion.

D.2.4 Extension of Scalars

We now return to the setting of $\mathbb{E}_2$-rings.

Construction D.2.4.1 (Extension of Scalars)**.** Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_2$-rings. For every additive R-linear ∞-category $\mathcal{C}$, we let $R' \otimes_R \mathcal{C}$ denote the additive R'-linear ∞-category given by the relative tensor product

$$\mathrm{Mod}^{\mathrm{cn}}_{R'} \otimes_R \mathcal{C} = \mathrm{Mod}^{\mathrm{cn}}_{R'} \otimes_{\mathrm{Mod}^{\mathrm{cn}}_R} \mathcal{C} = |\,\mathrm{Bar}_{\mathrm{Mod}^{\mathrm{cn}}_R}(\mathrm{Mod}^{\mathrm{cn}}_{R'}, \mathcal{C})_\bullet|.$$

We will say that $R' \otimes_R \mathcal{C}$ is obtained from $\mathcal{C}$ by *extension of scalars* along the morphism ϕ.

Remark D.2.4.2. If $\phi : R \to R'$ is a morphism of connective $\mathbb{E}_2$-rings, then extension of scalars along ϕ determines a functor $\mathrm{LinCat}^{\mathrm{Add}}_R \to \mathrm{LinCat}^{\mathrm{Add}}_{R'}$ which is left adjoint to the forgetful functor $\mathrm{LinCat}^{\mathrm{Add}}_{R'} \to \mathrm{LinCat}^{\mathrm{Add}}_R$ given by restriction of scalars along ϕ.

Variant D.2.4.3. Let $\phi : R \to R'$ be an arbitrary morphism of $\mathbb{E}_2$-rings. If $\mathcal{C}$ is a stable R-linear ∞-category, we let $R' \otimes_R \mathcal{C}$ denote the relative tensor product

$$\mathrm{LMod}_{R'} \otimes_R \mathcal{C} = \mathrm{LMod}_{R'} \otimes_{\mathrm{LMod}_R} \mathcal{C} = |\,\mathrm{Bar}_{\mathrm{LMod}_R}(\mathrm{LMod}_{R'}, \mathcal{C})_\bullet|.$$

Then $R' \otimes_R \mathcal{C}$ is a stable R'-linear ∞-category which we will say is obtained from $\mathcal{C}$ by extension of scalars along ϕ.

If R and R' are connective, then the assumption that $\mathcal{C}$ is stable guarantees that the inclusion maps

$$\mathrm{LMod}^{\mathrm{cn}}_R \hookrightarrow \mathrm{LMod}_R \qquad \mathrm{LMod}^{\mathrm{cn}}_{R'} \hookrightarrow \mathrm{LMod}_{R'}$$

induce a levelwise equivalence of two-sided bar constructions $\mathrm{Bar}_{\mathrm{LMod}^{\mathrm{cn}}_R}(\mathrm{LMod}^{\mathrm{cn}}_{R'}, \mathcal{C})_\bullet \to \mathrm{Bar}_{\mathrm{LMod}_R}(\mathrm{LMod}_{R'}, \mathcal{C})_\bullet$. Consequently, as an additive R'-linear ∞-category, the extension of scalars $R' \otimes_R \mathcal{C}$ agrees with the one given by Construction D.2.4.1.

Remark D.2.4.4. Let R be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be an additive R-linear ∞-category. Let A be a connective $\mathbb{E}_1$-algebra over R (that is, an algebra object of the monoidal ∞-category $\mathrm{LMod}^{\mathrm{cn}}_R$) and let $\mathrm{LMod}_A(\mathcal{C})$ denote the ∞-category of A-module objects of $\mathcal{C}$. Then Theorem HA.4.8.4.6 supplies an equivalence of ∞-categories $\mathrm{LMod}_A(\mathcal{C}) \simeq \mathrm{LMod}^{\mathrm{cn}}_A \otimes_{\mathrm{LMod}^{\mathrm{cn}}_R} \mathcal{C}$. In particular, if $\phi : R \to R'$ is a morphism of connective $\mathbb{E}_2$-rings, then the extension of scalars $R' \otimes_R \mathcal{C}$ can be identified with the ∞-category $\mathrm{LMod}_{R'}(\mathcal{C})$.

If the ∞-category $\mathcal{C}$ is stable, then the same reasoning applies without any connectivity assumptions on R and R'.

Proposition D.2.4.5. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_2$-rings. Then:*

(a) *If $\mathcal{C}$ is a stable R-linear ∞-category, then the extension of scalars $R' \otimes_R \mathcal{C}$ is a stable R-linear ∞-category.*

(b) *If $\mathcal{C}$ is a prestable R-linear ∞-category, then the extension of scalars $R' \otimes_R \mathcal{C}$ is a prestable R'-linear ∞-category.*

(c) *If $\mathcal{C}$ is an abelian R-linear ∞-category, then the extension of scalars $R' \otimes_R \mathcal{C}$ is an abelian R'-linear ∞-category.*

Proof. Assertions (a) and (b) follow immediately from Proposition D.2.2.1. To prove (c), we observe that $\mathcal{C}$ can be regarded as an Ab-module object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (Proposition C.5.4.13), so that $R' \otimes_R \mathcal{C}$ inherits the structure of an Ab-module and is therefore equivalent to its homotopy category. We therefore have equivalences

$$\begin{aligned} R' \otimes_R \mathcal{C} &\simeq (R' \otimes_R \mathcal{C})^{\heartsuit} \\ &\simeq (\mathrm{Mod}^{\mathrm{cn}}_{R'} \otimes_{\mathrm{Mod}^{\mathrm{cn}}_R} \mathcal{C})^{\heartsuit} \\ &\simeq \mathrm{Mod}^{\heartsuit}_{R'} \otimes_{\mathrm{Mod}^{\heartsuit}_R} \mathcal{C}^{\heartsuit} \\ &\simeq \mathrm{Mod}^{\heartsuit}_{R'} \otimes_R \mathcal{C} \end{aligned}$$

so that the desired result again follows from Proposition D.2.2.1. □

Remark D.2.4.6. One can deduce Proposition D.2.4.5 directly (without use of Proposition D.2.2.1) from the description of the extension of scalars functor supplied by Remark D.2.4.4.

Remark D.2.4.7. Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_2$-rings and let $\mathcal{C}$ be a prestable R-linear ∞-category. Then the stabilization $\mathrm{Sp}(\mathcal{C})$ inherits a t-structure $(\mathrm{Sp}(\mathcal{C})_{\geqslant 0}, \mathrm{Sp}(\mathcal{C})_{\leqslant 0})$, where $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ is the essential image of the functor $\Sigma^{\infty} : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$. Similarly, the ∞-category

$$\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C})) = R' \otimes_R \mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Sp}(R' \otimes_R \mathcal{C})$$

inherits a t-structure, where $\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ is the essential image of the functor

$$\Sigma^{\infty} : (R' \otimes_R \mathcal{C}) \to \mathrm{Sp}(R' \otimes_R \mathcal{C}).$$

We claim that this t-structure can be described explicitly as follows:

(a) An object $X \in \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))$ belongs to $\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ if and only if its image in $\mathrm{Sp}(\mathcal{C})$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$.

(b) An object $X \in \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))$ belongs to $\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0}$ if and only if its image in $\mathrm{Sp}(\mathcal{C})$ belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$.

To prove the "only if" directions of (a) and (b), we observe that the forgetful functor $G : \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C})) \to \mathrm{Sp}(\mathcal{C})$ is right adjoint to the stabilization of the natural map $\mathcal{C} \simeq R \otimes_R \mathcal{C} \to R' \otimes_R \mathcal{C}$, and therefore t-exact by the proof of Proposition D.2.2.1. The "if" directions of (a) and (b) then follow from the additional observation that the functor G is conservative.

D.3 Universal Descent Morphisms

Let $f : A \to B$ be a faithfully flat map of commutative rings. A classical theorem of Grothendieck asserts that the category of A-modules is equivalent to the category $\mathcal{C}$ whose objects are pairs (M, η), where M is a B-module and η is a "descent datum" for M: that is, an automorphism of $B \otimes_A M$ which is compatible with the evident involution of $B \otimes_A B$ and satisfies a suitable cocycle condition. Our goal in this section is to prove an analogue of Grothendieck's result in the ∞-categorical setting. More precisely, we will introduce the notion of a *universal descent morphism* of ring spectra (Definition D.3.1.1) and show that, if $f : A \to B$ is a universal descent morphism, then the ∞-category Mod_A can be identified with a suitably defined ∞-category of "B-modules equipped with descent data" (Corollary D.3.4.2). Moreover, an analogous statement holds for any arbitrary stable A-linear ∞-category (Theorem D.3.4.1).

The collection of universal descent morphisms is fairly robust: for example, it includes all faithfully flat morphisms $f : A \to B$ for which $\pi_0 B$ admits a countable (or even mildly uncountable) presentation as a $\pi_0 A$-module (Proposition D.3.3.1).

Remark D.3.0.8. Most of the results of this section (including the definition of universal descent morphism) were explained to us by Akhil Mathew. We refer the reader to [**?**] for further discussion.

D.3.1 Propagating Modules

Definition D.3.1.1. Let A be an $\mathbb{E}_2$-ring, let M be a left A-module spectrum, and let $\mathcal{C}$ denote the smallest stable subcategory of LMod_A which contains all A-modules of the form $N \otimes_A M$ and is closed under retracts. We will say that M is a *propagating module* for A if $\mathcal{C} = \mathrm{LMod}_A$. We will say that a morphism $f : A \to B$ of $\mathbb{E}_2$-rings is a *universal descent morphism* if it exhibits B as a propagating module for A.

Warning D.3.1.2. In the situation of Definition D.3.1.1, the monoidal structure on LMod_A is generally not symmetric. Consequently, the notion of propagating module is *a priori* asymmetric as well: it might be more appropriate to refer to an object $M \in \mathrm{LMod}_A$ as a *left propagating* module if it satisfies the requirements of Definition D.3.1.1, and a *right propagating module* if the ∞-category LMod_A is generated (under finite colimits and retracts) by objects of the form $M \otimes_A N$. However, we will see in a moment that an $\mathbb{E}_1$-algebra $B \in \mathrm{Alg}_A$ is left propagating as an A-module if and only if it is right propagating (Remark D.3.2.2). In practice, we will primarily be interested in universal descent morphisms between $\mathbb{E}_\infty$-rings, in which case the potential asymmetry disappears.

Remark D.3.1.3. Let A be an $\mathbb{E}_2$-ring and let $M \in \mathrm{LMod}_A$. Then the full subcategory $\mathcal{C} \subseteq \mathrm{LMod}_A$ appearing in Definition D.3.1.1 is closed under the operation $\bullet \mapsto N \otimes_A \bullet$, for each object $N \in \mathrm{LMod}_A$. It follows that M is a propagating module for A if and only if $A \in \mathcal{C}$.

Remark D.3.1.4. Let $\{A_i\}_{1 \leqslant i \leqslant n}$ be a finite collection of $\mathbb{E}_2$-rings with product A, and let M be a left A-module which we can write as a product $\prod_{1 \leqslant i \leqslant n} M_i$, where each M_i is a left A_i-module (see Lemma D.3.5.5). Then M is a propagating A-module if and only if each M_i is a propagating A_i-module.

Remark D.3.1.5. Let $\phi : A \to A'$ be a morphism of $\mathbb{E}_2$-rings, let $M \in \mathrm{LMod}_A$, and set $M' = A' \otimes_A M \in \mathrm{LMod}_{A'}$. Let $\mathcal{C} \subseteq \mathrm{LMod}_A$ be the subcategory appearing in Definition D.3.1.1, and let $\mathcal{C}' \subseteq \mathrm{LMod}_{A'}$ be defined in an analogous way. Then the extension of scalars functor

$$\mathrm{LMod}_A \to \mathrm{LMod}_{A'} \qquad N \mapsto A' \otimes_A N$$

carries $\mathcal{C}$ into $\mathcal{C}'$. In particular, if M is a propagating module for A, then $A' = A' \otimes_A A \in \mathcal{C}'$, so that M' is a propagating module for A' (Remark D.3.1.3).

Proposition D.3.1.6. (1) *Every equivalence of $\mathbb{E}_2$-rings is a universal descent morphism.*

(2) *Let $f : A \to B$ and $g : B \to C$ be universal descent morphisms in $\mathrm{Alg}^{(2)}$. Then $g \circ f : A \to C$ is a universal descent morphism.*

(3) *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B' \end{array}$$

be a pushout diagram of $\mathbb{E}_\infty$-rings. If f is a universal descent morphism, then so is f'.

(4) *The collection of universal descent morphisms is closed under finite products.*

Proof. Assertion (1) is obvious, assertion (3) follows immediately from Remark D.3.1.3, and assertion (4) follows immediately from Remark D.3.1.4. To prove (2), let $\mathcal{C} \subseteq \mathrm{LMod}_A$ denote the smallest stable subcategory of LMod_A which contains the essential image of the forgetful functor $\mathrm{LMod}_C \to \mathrm{LMod}_A$ and is closed under retracts. By virtue of Variant D.3.2.3, it will suffice to show that $\mathcal{C} = \mathrm{LMod}_A$. Since f is a universal descent morphism, we are reduced to proving that $\mathcal{C}$ contains the essential image of the forgetful functor $\mathrm{LMod}_B \to \mathrm{LMod}_A$. Let $\mathcal{D} \subseteq \mathrm{LMod}_B$ be the inverse image of $\mathcal{C}$; we wish to show that $\mathcal{D} = \mathrm{LMod}_B$. Using our assumption that g is a universal descent morphism, we are reduced to showing that $\mathcal{D}$ contains the essential image of the forgetful functor $\mathrm{LMod}_C \to \mathrm{LMod}_B$, which follows immediately from our construction. □

Remark D.3.1.7. Let A be an $\mathbb{E}_\infty$-ring, and let S be the collection of all universal descent morphisms in the ∞-category $\mathrm{CAlg}_A^{\mathrm{op}}$. Then Proposition D.3.1.6 implies that S satisfies the hypotheses of Proposition A.3.2.1, and therefore determines a Grothendieck topology on the (large) ∞-category $\mathrm{CAlg}_A^{\mathrm{op}}$. We will refer to this Grothendieck topology as the *universal descent topology*.

D.3.2 The Adams Tower

Let A be an $\mathbb{E}_2$-ring and let $B \in \mathrm{Alg}_A$ be an $\mathbb{E}_1$-algebra over A. Then we can associate to B a cosimplicial object $B^\bullet$ of LMod_A, given levelwise by the formula

$$[n] \mapsto B^n = B \otimes_A \cdots \otimes_A B.$$

We let $\mathrm{Tot}^\bullet(B/A)$ denote the Tot-tower of this cosimplicial object: that is, the sequence of morphisms

$$\cdots \to \mathrm{Tot}^2(B/A) \to \mathrm{Tot}^1(B/A) \to \mathrm{Tot}^0(B/A) \simeq B$$

where $\mathrm{Tot}^n(B/A) \simeq \varprojlim_{[m] \in \mathbf{\Delta}_{\leqslant n}} B^m$. We will abuse notation by identifying $\mathrm{Tot}^\bullet(B/A)$ with the corresponding Pro-object of LMod_A.

Proposition D.3.2.1. *Let A be an $\mathbb{E}_2$-ring and let $B \in \mathrm{Alg}_A$ be an $\mathbb{E}_1$-algebra over A. The following conditions are equivalent:*

(1) *The image of B in LMod_A is a propagating module for A.*

(2) *The unit map $f : A \to B$ induces an equivalence*

$$A \simeq \{\mathrm{Tot}^\bullet(A/A)\} \to \{\mathrm{Tot}^\bullet(B/A)\}$$

in the ∞-category $\mathrm{Pro}(\mathrm{LMod}_A)$.

Proof. We first prove that (1) $\Rightarrow$ (2). Let $\mathcal{C}$ denote the full subcategory of LMod_A spanned by those objects M for which the canonical map $\theta_M : \{M \otimes_A \mathrm{Tot}^\bullet(A/A)\} \to \{M \otimes_A \mathrm{Tot}^\bullet(B/A)\}$ is an equivalence in $\mathrm{Pro}(\mathrm{LMod}_A)$. Since the construction $M \mapsto \theta_M$ is an exact functor, the ∞-category $\mathcal{C}$ is a stable subcategory of LMod_A which is closed under retracts. It follows immediately from the definitions that if $M \in \mathcal{C}$, then $N \otimes_A M \in \mathcal{C}$. Consequently, to prove (2), it will suffice to show $\mathcal{C}$ contains a propagating module for A. By virtue of (1), it will suffice to show that $B \in \mathcal{C}$. This is clear, since $\{B \otimes \mathrm{Tot}^\bullet(B/A)\}$ can be identified with the Tot-tower associated to the split cosimplicial object $B^{\bullet+1}$.

Now suppose that (2) is satisfied. Let $\mathcal{D}$ denote the smallest stable subcategory of LMod_A which contains all objects of the form $N \otimes_A B$ and is closed under retracts. Then $B^\bullet$ is a cosimplicial object of $\mathcal{D}$. Since each term in the tower $\mathrm{Tot}^\bullet(B/A)$ can be written as a finite limit of objects of the form B^n, it follows that $\{\mathrm{Tot}^\bullet(B/A)\}$ can be identified with a

Pro-object of $\mathcal{D}$. Assumption (2) implies that A is equivalent to a retract of $\mathrm{Tot}^n(B/A)$ for some integer n, so that $A \in \mathcal{D}$. Invoking Remark D.3.1.3, we see that $\mathcal{D} = \mathrm{LMod}_A$, so that condition (1) is satisfied. □

Remark D.3.2.2. It follows from Proposition D.3.2.1 that if $B \in \mathrm{Alg}_A$ is an $\mathbb{E}_1$-algebra over A, then B is left propagating if and only if it is right propagating, in the sense of Warning D.3.1.2.

Variant D.3.2.3. Let $f : A \to B$ be a morphism of $\mathbb{E}_2$-rings. The following conditions are equivalent:

(1) The map f is a universal descent morphism.

(2) Let $\mathcal{C} \subseteq \mathrm{LMod}_A$ be the smallest stable subcategory which contains the essential image of the forgetful functor $\mathrm{LMod}_B \to \mathrm{LMod}_A$ and is closed under retracts. Then $\mathcal{C} = \mathrm{LMod}_A$.

Proof. Let $\mathcal{C}$ be as in (2). Then $\mathcal{C}$ contains all objects of the form $M \otimes_A B$, so that the implication (1) ⇒ (2) follows immediately from the definitions. Conversely, suppose that (2) is satisfied. Let $\mathcal{D} \subseteq \mathrm{LMod}_A$ be the full subcategory spanned by those objects M for which the canonical map

$$\theta_M : \{M \otimes_A \mathrm{Tot}^\bullet(A/A)\} \to \{M \otimes_A \mathrm{Tot}^\bullet(B/A)\}$$

is an equivalence in $\mathrm{Pro}(\mathrm{LMod}_A)$. Then $\mathcal{D}$ is a stable subcategory of LMod_A which is closed under retracts, and we wish to show that $A \in \mathcal{D}$. By virtue of (2), it will suffice to show that $\mathcal{D}$ contains the essential image of the forgetful functor $\mathrm{LMod}_B \to \mathrm{LMod}_A$. This follows from the observation that for $M \in \mathrm{LMod}_B$, the simplicial object $M \otimes_A B^\bullet \simeq M \otimes_B B^{\bullet+1}$ is a split cosimplicial object with limit M. □

D.3.3 Faithfully Flat Morphisms

Our next result supplies a large class of universal descent morphisms.

Proposition D.3.3.1. *Let $\phi : A \to B$ be a faithfully flat morphism of $\mathbb{E}_2$-rings. Suppose that, as a left module over $\pi_0 A$, the commutative ring $\pi_0 B$ admits a presentation using fewer than $\aleph_\omega$ generators and relations (here $\aleph_\omega$ denotes the first infinite singular cardinal). Then ϕ is a universal descent morphism.*

Remark D.3.3.2. The technical cardinality assumption appearing in Proposition D.3.3.1 is automatically satisfied if the commutative rings $\pi_0 A$ and $\pi_0 B$ have cardinality $< \aleph_\omega$. Assuming the generalized continuum hypothesis, this condition is satisfied for all commutative rings which arise in ordinary mathematical practice.

Remark D.3.3.3. There exist many examples of universal descent morphisms which are not faithfully flat. For example, the canonical map from the real K-theory spectrum KO to the complex K-theory spectrum KU is a universal descent morphism.

Remark D.3.3.4. We do not know if every faithfully flat morphism of ring spectra is a universal descent morphism. However, many of the descent theorems in this section can be proven for arbitrary faithfully flat morphisms as well; see §D.6 for more details.

Corollary D.3.3.5. *Every faithfully flat étale morphism of $\mathbb{E}_2$-rings is a universal descent morphism.*

The proof of Proposition D.3.3.1 is based on a theorem of Mitchell on vanishing of derived functors of the inverse limit for filtered diagrams of small cofinality. We formulate a version of this result as follows:

Lemma D.3.3.6. *Let n be a nonnegative integer, let J be a filtered partially ordered set of cardinality $\leqslant \aleph_n$, and let $\{X_j\}_{j \in J}$ be a diagram of spaces indexed by J^{op}. If each of the spaces X_j is m-connective for some integer m, then the inverse limit $\varprojlim_{j \in J} X_j$ is $(m-n)$-connective.*

Proof. We proceed by induction on n. Note that if J has cardinality $< \aleph_n$, the desired result follows from the inductive hypothesis (note that if J is finite, there is nothing to prove). We may therefore assume that the cardinality of J is exactly $\aleph_n$.

We now proceed by induction on m. If $m < n$ then there is nothing to prove. If $m > n$, then to show that $X = \varprojlim_{j \in J} X_j$ is $(m-n)$-connective, it will suffice to show that it is nonempty and that the fiber product $\{x\} \times_X \{y\}$ is $(m-n-1)$-connective for every pair of points $x, y \in X$. This follows from the inductive hypothesis, applied to the diagram $\{\{x\} \times_{X_j} \{y\}\}_{j \in J}$. It will therefore suffice to treat the case $m = n$: that is, we must show that if each X_j is n-connective, then the limit X is nonempty.

Since the cardinality of J is $\aleph_n$, we can choose a bijection $\mu : \aleph_n \to J$. We define a sequence of subsets $J_\alpha \subseteq J$ for $\alpha < \aleph_n$, each having cardinality $< \aleph_n$, using transfinite induction on α. Assuming that J_β has been constructed for $\beta < \alpha$, we take J_α to be any filtered subset of J having cardinality $< \aleph_n$ which contains $\mu(\alpha)$ and the union $\bigcup_{\beta < \alpha} J_\beta$. Then we can write $\mathrm{N}(J)$ as a filtered colimit of simplicial sets $\varinjlim_{\alpha < \aleph_n} \mathrm{N}(J_\alpha)$. It follows that we can write X as the limit of a diagram $\{Y_\alpha\}_{\alpha < \aleph_n}$, where $Y_\alpha = \varprojlim_{j \in J_\alpha} X_j$ (see §HTT.4.2.3).

Without loss of generality, we can assume that the diagram $\{Y_\alpha\}_{\alpha < \aleph_n}$ is given by a map $\mathrm{N}(\aleph_n)^{\mathrm{op}} \to \mathcal{S}$ which is obtained as the nerve of a functor of ordinary categories $\rho : \aleph_n^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta$ (Proposition HTT.4.2.4.4). Moreover, we may assume that ρ is fibrant with respect to the injective model structure on $\mathrm{Fun}(\aleph_n^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta)$. In this case, the limit Y of the diagram $\{Y_\alpha\}_{\alpha < \aleph_n}$ in the ∞-category $\mathcal{S}$ coincides with its limit in the ordinary category of simplicial sets (Theorem HTT.4.2.4.1). We wish to prove that Y is nonempty. Since the diagram $\{Y_\alpha\}$ is indexed by a well-ordered set, it will suffice to show that for each $\alpha < \aleph_n$, the canonical map $\theta : Y_\alpha \to \varprojlim_{\beta<\alpha} Y_\beta$ is surjective on vertices. Because ρ is a fibrant diagram, the map θ is a Kan fibration. It will therefore suffice to show that the spaces

$$Y_\alpha = \varprojlim_{j \in J_\alpha} X_j \qquad \text{and} \qquad \varprojlim_{\beta<\alpha} Y_\beta \simeq \varprojlim_{j \in \bigcup_{\beta<\alpha} J_\beta} X_j$$

are connected. This follows from our inductive hypothesis, since the filtered partially ordered sets J_α and $\bigcup_{\beta<\alpha} J_\beta$ have cardinality $< \aleph_n$. □

Lemma D.3.3.7. *Let A be a connective $\mathbb{E}_1$-ring, let M be a flat left A-module, and let N be a connective left A-module. Assume that $\pi_0 M$ is an $\aleph_n$-compact object of the category of discrete $\pi_0 A$-modules. Then $\mathrm{Ext}^m_A(M, N) \simeq 0$ for $m > n$.*

Proof. Let us identify N with the limit of its Postnikov tower

$$\cdots \to \tau_{\leqslant 2} N \to \tau_{\leqslant 1} N \to \tau_{\leqslant 0} N,$$

so that we have a Milnor exact sequence

$$\lim{}^1 \{\mathrm{Ext}^{m-1}_A(M, \tau_{\leqslant k} N)\} \to \mathrm{Ext}^m_A(M, N) \to \lim{}^0 \{\mathrm{Ext}^m_A(M, \tau_{\leqslant k} N)\}.$$

It will therefore suffice to show that the abelian groups $\lim^1\{\mathrm{Ext}^{m-1}_A(M, \tau_{\leqslant k} N)\}$ and $\lim^0\{\mathrm{Ext}^m_A(M, \tau_{\leqslant k} N)\}$ are trivial for $m > n$. To prove this, we will show that the maps $\mathrm{Ext}^{m-1}_A(M, \tau_{\leqslant k} N) \to \mathrm{Ext}^{m-1}_A(M, \tau_{\leqslant k-1} N)$ are surjective for $k \geqslant 1$, and that the groups $\mathrm{Ext}^m(M, \tau_{\leqslant k} N)$ vanish for all k. Using the exact sequences

$$\mathrm{Ext}^{m-1}_A(M, \tau_{\leqslant k} N) \to \mathrm{Ext}^{m-1}_A(M, \tau_{\leqslant k-1} N) \to \mathrm{Ext}^{m+k}_A(M, \pi_k N)$$

$$\mathrm{Ext}^{m+k}_A(M, \pi_k N) \to \mathrm{Ext}^m_A(M, \tau_{\leqslant k} N) \to \mathrm{Ext}^m_A(M, \tau_{\leqslant k-1} N),$$

we are reduced to proving that the groups $\mathrm{Ext}^{m+k}_A(M, \pi_k N)$ vanish. Replacing m by $m+k$ and N by $\pi_k N$, we can further reduce to the case where N is discrete. In this case, we have a canonical isomorphism $\mathrm{Ext}^m_A(M, N) \simeq \mathrm{Ext}^m_{\pi_0 A}(\pi_0 A \otimes_A M, N)$. We may therefore replace A by $\pi_0 A$ (and M by $\pi_0 A \otimes_A M$) and thereby reduce to the case where A is discrete. Since M is flat over A, it follows that M is also discrete.

Since M is flat over A, it can be written as the colimit of a diagram $\{M_\alpha\}_{\alpha \in P}$ indexed by a filtered partially ordered set P, where each M_α is a free left A-module of finite rank (Theorem HA.7.2.2.15). For each $\aleph_n$-small filtered subset $P' \subseteq P$, let $M_{P'}$ denote the colimit $\varinjlim_{\alpha \in P'} M_\alpha$. Then M can be written as a filtered colimit of the diagram $\{M_{P'}\}$, where P' ranges over all $\aleph_n$-small filtered subsets of P. Since M is $\aleph_n$-compact, the identity map $\mathrm{id}_M : \varinjlim_{P'} M_{P'} \to M$ factors through some $M_{P'}$, so that M is a retract of $M_{P'}$. We may therefore replace M by $M_{P'}$ and P by P', and thereby reduce to the case where P is $\aleph_n$-small. We have a canonical isomorphism

$$\mathrm{Ext}^m_A(M,N) \simeq \pi_0 \mathrm{Map}_{\mathrm{Mod}_A}(M, \Sigma^m N) \simeq \pi_0 \varprojlim_{\alpha \in P} \mathrm{Map}_{\mathrm{Mod}_A}(M_\alpha, \Sigma^m N)$$

To show that this group vanishes, it will suffice (by virtue of Lemma D.3.3.6) to show that the mapping spaces $\mathrm{Map}_{\mathrm{Mod}_A}(M_\alpha, \Sigma^m N)$ are n-connective for each $\alpha \in P$. This is clear, since M_α is a free left A-module of finite rank and $\Sigma^m N$ is n-connective. □

Proof of Proposition D.3.3.1. Since B is flat over A, we can identify B with the image of its connective cover $\tau_{\geqslant 0} B$ under the base change functor $\mathrm{LMod}_{\tau_{\geqslant 0} A} \to \mathrm{LMod}_A$. By virtue of Remark D.3.1.3, to prove that B is propagating as a left A-module, it will suffice to show that $\tau_{\geqslant 0} B$ is propagating as a left $\tau_{\geqslant 0} A$-module. We may therefore replace ϕ by the induced map $\tau_{\geqslant 0} A \to \tau_{\geqslant 0} B$ and thereby reduce to the case where A is connective.

Let $\mathcal{C}$ denote the smallest stable subcategory of LMod_A which contains all objects of the form $M \otimes_A B$ and is closed under retracts. By virtue of Remark D.3.1.3, it will suffice to show that A belongs to $\mathcal{C}$. Let K be the fiber of the map $\phi : A \to B$, and let $\rho : K \to A$ be the canonical map. For each integer $m \geqslant 0$, let $\rho(m) : K^{\otimes m} \to A^{\otimes m} \simeq A$ be the the mth tensor power of ρ, formed in the monoidal ∞-category LMod_A. Then $\rho(m+1)$ is given by the composition $K^{\otimes m+1} \xrightarrow{\mathrm{id}_{K^{\otimes m}}} K^{\otimes m} \xrightarrow{\rho(m)} A$, so we have a fiber sequence

$$K^{\otimes m} \otimes_A B \to \mathrm{cofib}(\rho(m+1)) \to \mathrm{cofib}(\rho(m)).$$

It follows by induction on m that each $\mathrm{cofib}(\rho(m))$ belongs to $\mathcal{C}$. Consequently, to prove that $A \in \mathcal{C}$, it will suffice to show that A is a retract of $\mathrm{cofib}(\rho(m))$ for some $m \geqslant 0$. This condition holds whenever the homotopy class of $\rho(m)$ vanishes (when regarded as an element of $\mathrm{Ext}^0_A(K^{\otimes m}, A) \simeq \mathrm{Ext}^m_A((\Sigma K)^{\otimes m}, A)$. Our assumptions imply that $\Sigma K \simeq \mathrm{cofib}(\phi)$ is a flat A-module, and that $\pi_0 \mathrm{cofib}(K)$ admits a presentation using fewer than $\aleph_n$ generators and relations for some integer $n \geqslant 0$. It follows that $(\Sigma K)^{\otimes m}$ has the same properties for each $m > 0$, so that $\mathrm{Ext}^m_A((\Sigma K)^{\otimes m}, A)$ vanishes for $m > n$ by virtue of Lemma D.3.3.7. □

D.3.4 Comonadicity

Suppose that we are given a pair of adjoint functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ between ∞-categories. Let $U = F \circ G : \mathcal{D} \to \mathcal{D}$. Then the functor U admits the structure of a comonad: that is, it is an associative coalgebra object of the monoidal ∞-category $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$ (see §HA.4.7). Equivalently, we can regard U as an associative algebra in the monoidal ∞-category $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})$, which has a left action on the ∞-category $\mathcal{D}^{\mathrm{op}}$. We can regard $\mathrm{LMod}_U(\mathcal{D}^{\mathrm{op}})$ as (the opposite of) the ∞-category of *U-comodule objects* of $\mathcal{D}$: that is, objects

$D \in \mathcal{D}$ equipped with a map $\alpha : D \to U(D)$ together with additional coherence data. The functor F factors canonically as a composition $\mathcal{C} \xrightarrow{F'} \mathrm{LMod}_U(\mathcal{D}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{F''} \mathcal{D}$. More informally: for every object $C \in \mathcal{C}$, the image $F(C) \in \mathcal{D}$ is naturally equipped with the structure of a U-comodule (given by the unit map $F(C) \to F(GF(C)) \simeq (F \circ G)(F(C)) = U(F(C))$). We may therefore think of a U-comodule structure on an object $D \in \mathcal{D}$ as a kind of descent data for D: it is a structure whose presence indicates that the object D might lift to an object of C.

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *comonadic* if F admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$, and the functor $F' : \mathcal{C} \to \mathrm{LMod}_U(\mathcal{D}^{\mathrm{op}})^{\mathrm{op}}$ described above is an equivalence of ∞-categories. In other words, F is comonadic if $\mathcal{C}$ can be identified with the ∞-category of objects of $\mathcal{D}$ equipped with descent data.

Theorem D.3.4.1. *Let $f : A \to B$ be a universal descent morphism of $\mathbb{E}_2$-rings, and let $\mathcal{C}$ be a stable A-linear ∞-category. Then the extension-of-scalars functor $F : \mathcal{C} \to \mathrm{LMod}_B(\mathcal{C})$ (given by tensoring with B) is comonadic.*

Corollary D.3.4.2. *Let $f : A \to B$ be a universal descent morphism of $\mathbb{E}_2$-rings. Then the base-change functor $\mathrm{LMod}_A \to \mathrm{LMod}_B$ is comonadic.*

Proof. Apply Theorem **??** in the case $\mathcal{C} = \mathrm{LMod}_A$. □

Proof of Theorem D.3.4.1. According to Theorem HA.4.7.3.5 , it will suffice to verify that F satisfies the following conditions:

(1) The functor F is conservative.

(2) For every F-split cosimplicial object $C^\bullet$ of $\mathcal{C}$, the canonical map $F(\varprojlim C^\bullet) \to \varprojlim F(C^\bullet)$ is an equivalence in $\mathrm{LMod}_B(\mathcal{C})$.

We first prove (1). Since F is an exact functor between stable ∞-categories, it will suffice to show that if $C \in \mathcal{C}$ is an object for which $F(C) \simeq 0$, then $C \simeq 0$. To prove this, let $\mathcal{E}$ denote the full subcategory of LMod_A spanned by those A-modules M for which the tensor product $M \otimes_A C$ vanishes. Then $\mathcal{E}$ is a stable subcategory of LMod_A which is closed under retracts. We wish to prove that $A \in \mathcal{E}$. Since f is a universal descent morphism, it will suffice to show that $\mathcal{E}$ contains the essential image of the forgetful functor $\mathrm{LMod}_B \to \mathrm{LMod}_A$ (see Variant D.3.2.3). This is clear: if $M \in \mathrm{LMod}_B(\mathcal{C})$, then our assumption that $F(C) \simeq 0$ implies that

$$M \otimes_A C \simeq M \otimes_B F(C) \simeq 0.$$

We now prove (2). Let $C^\bullet$ be an F-split cosimplicial object of $\mathcal{C}$. For each object $M \in \mathrm{LMod}_A$, let us regard $M \otimes_A C^\bullet$ as another cosimplicial object of $\mathcal{C}$, so that we have a canonical map $\theta_M : F(\varprojlim(M \otimes_A C^\bullet)) \to \varprojlim(M \otimes_A C^\bullet)$ in $\mathrm{LMod}_B(\mathcal{C})$. Let $\mathcal{E}'$ denote the full subcategory of LMod_A spanned by those objects M for which θ_M is an equivalence. Then $\mathcal{E}'$ is a stable subcategory of LMod_A which is closed under retracts. We wish to prove that $A \in \mathcal{E}'$. Since f is a universal descent morphism, it will suffice to show that $\mathcal{E}$ contains the essential image of the forgetful functor $\mathrm{LMod}_B \to \mathrm{LMod}_A$ (see Variant D.3.2.3). To prove this, we observe that for $M \in \mathrm{LMod}_B$, we can identify $M \otimes_A C^\bullet$ with $M \otimes_B F(C^\bullet)$. Since $F(C^\bullet)$ is a split cosimplicial object of $\mathrm{LMod}_B(\mathcal{C})$, it follows that $M \otimes_A C^\bullet$ is a split cosimplicial object of $\mathcal{C}$, and therefore θ_M is an equivalence as desired. □

Theorem D.3.4.1 admits a converse:

Proposition D.3.4.3. *Let $f : A \to B$ be a morphism of $\mathbb{E}_2$-rings. Suppose that for every A-linear ∞-category $\mathcal{C}$, the base change functor $F_{\mathcal{C}} : \mathcal{C} \to \mathrm{LMod}_B(\mathcal{C})$ is comonadic. Then f is a universal descent morphism.*

Proof. Let $B^\bullet$ be the cosimplicial object of Alg_A appearing in the discussion preceeding Proposition D.3.2.1. We first prove the following:

- $(*)$ Let $\mathcal{C}$ be a locally small stable ∞-category which is left-tensored over LMod_A. Suppose that $\mathcal{C}$ admits small colimits and that the action $\otimes_A : \mathrm{LMod}_A \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Then, for each object $C \in \mathcal{C}$, the canonical map $A \to \mathrm{Tot}(B^\bullet)$ exhibits C as a totalization of the cosimplicial object $B^\bullet \otimes_A C$ in $\mathcal{C}$.

Note that if $\mathcal{C}$ presentable (and therefore an A-linear ∞-category), then assertion $(*)$ follows immediately from the comonadicity of the base change functor $\mathcal{C} \to \mathrm{LMod}_B(\mathcal{C})$. To treat the general case, fix a regular cardinal κ and let LMod_A^κ denote the full subcategory of LMod_A spanned by the κ-compact objects. Note that LMod_A^κ contains the unit object of LMod_A and is closed under tensor products, and therefore inherits the structure of a monoidal ∞-category. Fix objects $C, C' \in \mathcal{C}$; we wish to show that the canonical map

$$\mathrm{Map}_{\mathcal{C}}(C', C) \to \varprojlim \mathrm{Map}_{\mathcal{C}}(C', B^\bullet \otimes_A C)$$

is a homotopy equivalence. Choose an essentially small stable subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ containing C and C' which is closed under κ-small colimits. Then we can regard $\mathcal{C}_0$ as an ∞-category tensored over LMod_A^κ, and the tensor product functor $\otimes_A : \mathrm{LMod}_A^\kappa \times \mathcal{C}_0 \to \mathcal{C}_0$ preserves κ-small colimits separately in each variable. Using Lemma HA.5.3.2.11 , we see that $\mathrm{Ind}_\kappa(\mathcal{C}_0)$ inherits the structure of an ∞-category left-tensored over $\mathrm{Ind}_\kappa(\mathrm{LMod}_A^\kappa) \simeq \mathrm{LMod}_A$. Let $j : \mathcal{C}_0 \to \mathrm{Ind}_\kappa(\mathcal{C}_0)$ denote the Yoneda embedding. Since $\mathrm{Ind}_\kappa(\mathcal{C}_0)$ is presentable, the base change functor $\mathrm{Ind}_\kappa(\mathcal{C}_0) \to \mathrm{LMod}_B(\mathrm{Ind}_\kappa(\mathcal{C}_0))$ is comonadic, so that $j(C) \simeq \varprojlim B^\bullet \otimes_A j(C)$ and therefore the canonical map

$$\begin{aligned} \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C}_0)}(j(C'), j(C)) &\to \varprojlim \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(j(C'), B^\bullet \otimes_A j(C)) \\ &\simeq \varprojlim \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(j(C'), j(B^\bullet \otimes_A C)). \end{aligned}$$

is a homotopy equivalence. Assertion $(*)$ now follows because the Yoneda embedding j is fully faithful.

For each object $M \in \mathrm{LMod}_A$, the construction $N \mapsto N \otimes_A M$ determines a functor $r_M : \mathrm{LMod}_A \to \mathrm{LMod}_A$ which preserves small colimits and therefore admits a right adjoint. Let us indicate this right adjoint by the notation $K \mapsto {}^M K$, which we regard as a functor from $\mathrm{LMod}_A^{\mathrm{op}}$ to itself (so that $\mathrm{Map}_{\mathrm{LMod}_A}(N \otimes_A M, K) \simeq \mathrm{Map}_{\mathrm{LMod}_A}(N, {}^M K)$). This construction exhibits $\mathrm{LMod}_A^{\mathrm{op}}$ as an ∞-category which is left-tensored over LMod_A. Moreover, the action map

$$\mathrm{LMod}_A \times \mathrm{LMod}_A^{\mathrm{op}} \to \mathrm{LMod}_A^{\mathrm{op}} \qquad (M, K) \mapsto {}^M K$$

preserves small colimits separately in each variable. Applying $(*)$, we deduce that for each object $N \in \mathrm{LMod}_A$, the canonical map $|{}^{B^\bullet} N| \to N$ is an equivalence in LMod_A. Applying Ω^∞, we obtain a homotopy equivalence

$$\begin{aligned}
\varinjlim_{n \geqslant 0} \mathrm{Map}_{\mathrm{LMod}_A}(\mathrm{Tot}^n(B/A), N) &\simeq \varinjlim_{n \geqslant 0} \Omega^\infty({}^{\mathrm{Tot}^n(B/A)} N) \\
&\simeq \Omega^\infty \varinjlim_{n \geqslant 0} ({}^{\mathrm{Tot}^n(B/A)} N) \\
&\simeq \Omega^\infty |{}^{B^\bullet} N| \\
&\to \Omega^\infty N \\
&\simeq \mathrm{Map}_{\mathrm{LMod}_A}(A, N),
\end{aligned}$$

where $\{\mathrm{Tot}^n(B/A)\}_{n \geqslant 0}$ is the Tot-tower defined in the discussion preceding Proposition D.3.2.1. It follows that the Tot-tower is equivalent to A as an object of $\mathrm{Pro}(\mathrm{LMod}_A)$, so that Proposition D.3.2.1 implies that B is a propagating object of LMod_A and therefore $f : A \to B$ is a universal descent morphism. □

D.3.5 Effective Descent for Objects

If we restrict our attention to $\mathbb{E}_\infty$-rings, then we can reformulate Theorem D.3.4.1 using the language of sheaves. For this, we need to introduce a bit of notation.

Notation D.3.5.1. Fix a $\mathbb{E}_\infty$-ring A and a stable A-linear ∞-category $\mathcal{C}$. We let $\mathrm{Mod}(\mathcal{C})$ denote the fiber product $\mathrm{CAlg}(\mathrm{Mod}_A) \times_{\mathrm{Alg}(\mathrm{Mod}_A)} \mathrm{LMod}(\mathcal{C})$ whose objects are pairs (B, M), where $B \in \mathrm{CAlg}(\mathrm{Mod}_A) \simeq \mathrm{CAlg}_A$ is an $\mathbb{E}_\infty$-algebra over A and M is a left B-module object of $\mathcal{C}$. We will denote the fiber of $\mathrm{Mod}(\mathcal{C})$ over an object $B \in \mathrm{CAlg}_A$ by $\mathrm{Mod}_B(\mathcal{C})$.

In the situation of Notation D.3.5.1, the coCartesian fibration $q : \mathrm{Mod}(\mathcal{C}) \to \mathrm{CAlg}(\mathrm{Mod}_A)$ is classified by a functor $\mathrm{CAlg}_A \to \widehat{\mathcal{C}\mathrm{at}}_\infty$: in other words, the construction $B \mapsto \mathrm{Mod}_B(\mathcal{C})$ can be regarded as a functor of B.

Theorem D.3.5.2. *Let A be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a stable A-linear ∞-category. Then the construction $B \mapsto \mathrm{Mod}_B(\mathcal{C})$ determines a $\mathcal{P}\mathrm{r}^{\mathrm{L}}$-valued sheaf with respect to the universal descent topology on the ∞-category $\mathrm{CAlg}_A^{\mathrm{op}}$ (see Remark D.3.1.7).*

Remark D.3.5.3. Let A be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a stable A-linear ∞-category. It follows from Theorem D.3.5.2 and Corollary D.3.3.5 that the construction $B \mapsto \mathrm{LMod}_B(\mathcal{C})$ is a sheaf with respect to the étale topology on the ∞-category $\mathrm{CAlg}_A^{\text{ét}}$. In fact, this is true more generally when A is an $\mathbb{E}_2$-ring (the proof in this case requires only slight modification).

Remark D.3.5.4. It follows from Proposition D.3.4.3 that the universal descent topology is the *finest* Grothendieck topology for which Theorem D.3.5.2 holds.

Lemma D.3.5.5. *Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. Then the construction $A \mapsto \mathrm{LMod}_A(\mathcal{C})$ commutes with finite products (when regarded as a functor $\mathrm{Alg}_R \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$).*

Proof. Let $\{A_i\}_{1 \leqslant i \leqslant n}$ be a finite collection of $\mathbb{E}_1$-algebras over R and let $A = \prod_{1 \leqslant i \leqslant n} A_i$. We wish to show that the canonical functor

$$\theta : \mathrm{LMod}_A(\mathcal{C}) \to \prod_{1 \leqslant i \leqslant n} \mathrm{LMod}_{A_i}(\mathcal{C})$$

is an equivalence of ∞-categories. By virtue of Proposition HA.5.2.2.36 , it will suffice to verify the following assertions:

(*a*) The functor θ is conservative. That is, if $\alpha : M \to N$ is a morphism in $\mathrm{LMod}_A(\mathcal{C})$ is such that each of the induced maps $\alpha_i : A_i \otimes_A M \to A_i \otimes_A M$ is an equivalence, then α is an equivalence. It suffices to show that the image of α is an equivalence in the ∞-category $\mathcal{C}$. This is clear, since α is equivalent to the product of the morphisms α_i in the ∞-category $\mathcal{C}$.

(*b*) Suppose we are given objects $M_i \in \mathrm{LMod}_{A_i}(\mathcal{C})$, and let $M \simeq \prod_{1 \leqslant i \leqslant n} M_i$ (regarded as an A-module). Then the canonical map $\phi : A_i \otimes_A M \to M_i$ is an equivalence for $1 \leqslant i \leqslant n$. To prove this, we see that the domain of ϕ is given by the product $\prod_{1 \leqslant j \leqslant n} (A_i \otimes_A A_j) \otimes_{A_j} M_j$. To prove that ϕ is an equivalence, it suffices to show that $A_i \otimes_A A_j \simeq 0$ for $i \neq j$, and that the canonical map $A_i \to A_i \otimes_A A_i$ is an equivalence. Since each A_j is flat as a left A-module, we have $\pi_*(A_i \otimes_A A_j) \simeq (\pi_* A_i) \otimes_{\pi_* A} (\pi_* A_j)$, so the desired result follows from a simple algebraic calculation.

□

Lemma D.3.5.6. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category, let $\mathcal{M}$ be an ∞-category left-tensored over $\mathcal{C}$, and suppose we are given a pushout diagram of commutative algebra objects of $\mathcal{C}$:*

$$\begin{array}{ccc} A & \longleftarrow & B \\ \uparrow & & \uparrow \\ A' & \longleftarrow & B' \end{array}$$

Then the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{LMod}_A(\mathcal{M}) & \longrightarrow & \mathrm{LMod}_B(\mathcal{M}) \\ \downarrow & & \downarrow \\ \mathrm{LMod}_{A'}(\mathcal{M}) & \longrightarrow & \mathrm{LMod}_{B'}(\mathcal{M}) \end{array}$$

is right adjointable.

Proof. This follows immediately from Proposition HA.4.6.2.17. □

Lemma D.3.5.7. *Let $f : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings, let $B^\bullet$ be the Čech nerve of f (formed in the ∞-category $\mathrm{CAlg}^{\mathrm{op}}$), and let $\mathcal{C}$ be a stable A-linear ∞-category. The following conditions are equivalent:*

(1) *The base change functor $\mathcal{C} \to \mathrm{Mod}_B(\mathcal{C})$ is comonadic.*

(2) *The canonical map $\mathcal{C} \to \varprojlim \mathrm{Mod}_{B^\bullet}(\mathcal{C})$ is an equivalence of ∞-categories.*

Proof. It follows from Lemma D.3.5.6 that the augmented cosimplicial ∞-category $\mathcal{C}^\bullet$ satisfies the (dual) Beck-Chevalley condition: that is, for every morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$, the diagram

$$\begin{array}{ccc} \mathcal{C}^m & \xrightarrow{d^0} & \mathcal{C}^{m+1} \\ \downarrow & & \downarrow \\ \mathcal{C}^n & \xrightarrow{d^0} & \mathcal{C}^{n+1} \end{array}$$

is right adjointable. The desired result now follows from Corollary HA.4.7.5.3. □

Lemma D.3.5.8. *Let $f : A \to B$ be a universal descent morphism of $\mathbb{E}_\infty$-rings and let $B^\bullet$ be the Čech nerve of f (formed in the ∞-category $\mathrm{CAlg}^{\mathrm{op}}$). For any A-linear ∞-category $\mathcal{C}$, the canonical map $\mathcal{C} \to \varprojlim \mathrm{Mod}_{B^\bullet}(\mathcal{C})$ is an equivalence of ∞-categories.*

Proof. Combine Lemma D.3.5.7 with Theorem D.3.4.1. □

Proof of Theorem D.3.5.2. Combine Proposition A.3.3.1, Lemma D.3.5.5, and Lemma D.3.5.8. □

D.3.6 Effective Descent for ∞-Categories

Our next goal is to formulate and prove a categorification of Theorem D.3.5.2.

Remark D.3.6.1. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category whose objects are presentable ∞-categories and whose morphisms are colimit preserving functors. We let $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ denote the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ spanned by the stable ∞-categories. We can identify the ∞-category Sp of spectra with a commutative algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ with the ∞-category $\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ (see Proposition HA.4.8.2.18). According to Theorem HA.4.8.5.16 , the construction $A \mapsto \mathrm{LMod}_A(\mathrm{Sp})$ determines a symmetric monoidal functor $\mathrm{Alg}(\mathrm{Sp}) \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$. Passing to algebra objects (and using Theorem HA.5.1.2.2), we obtain a functor

$$\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}) \simeq \mathrm{Alg}(\mathrm{Alg}(\mathrm{Sp})) \to \mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{St}}) \to \mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}).$$

We let $\mathrm{LinCat}^{\mathrm{St}}$ denote the fiber product $\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}) \times_{\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})} \mathrm{LMod}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. We will refer to $\mathrm{LinCat}^{\mathrm{St}}$ as the *∞-category of stable linear ∞-categories.*

There is an evident categorical fibration $\theta : \mathrm{LinCat}^{\mathrm{St}} \to \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp})$. By construction, the fiber of θ over an $\mathbb{E}_2$-ring A can be identified with the ∞-category $\mathrm{LinCat}^{\mathrm{St}}_A$ of stable A-linear ∞-categories introduced in Variant D.1.5.1. We may therefore think of $\mathrm{LinCat}^{\mathrm{St}}$ as an ∞-category whose objects are pairs $(A, \mathcal{C})$, where A is an $\mathbb{E}_2$-ring and $\mathcal{C}$ is a stable A-linear ∞-category.

The forgetful functor $\theta : \mathrm{LinCat}^{\mathrm{St}} \to \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp})$ is both a Cartesian fibration and a coCartesian fibration. In particular, as a coCartesian fibration, θ is classified by a functor $\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}) \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. This functor assigns to each $\mathbb{E}_2$-ring A the ∞-category $\mathrm{LinCat}^{\mathrm{St}}_A$ of stable A-linear ∞-categories, and to each morphism $\phi : A \to B$ of $\mathbb{E}_2$-rings the extension of scalars functor

$$\mathrm{LinCat}^{\mathrm{St}}_A \to \mathrm{LinCat}^{\mathrm{St}}_B \qquad \mathcal{C} \mapsto B \otimes_A \mathcal{C} \simeq \mathrm{LMod}_B(\mathcal{C})$$

described in Variant D.2.4.3.

Theorem D.3.6.2. *The construction $A \mapsto \mathrm{LinCat}^{\mathrm{St}}_A$ of Remark D.3.6.1 determines a functor $\mathrm{CAlg} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which is a sheaf with respect to the universal descent topology of Remark D.3.1.7.*

The proof of Theorem D.3.6.2 will require some preliminaries.

Lemma D.3.6.3. *Let $\{A_i\}_{1 \leqslant i \leqslant n}$ be a finite collection of $\mathbb{E}_2$-rings having product A. Then the canonical map $\phi : \mathrm{LinCat}^{\mathrm{St}}_A \to \prod_{1 \leqslant i \leqslant n} \mathrm{LinCat}^{\mathrm{St}}_{A_i}$ is an equivalence of ∞-categories.*

Proof. We will prove that ϕ satisfies the hypotheses of Proposition HA.5.2.2.36 :

(*a*) Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathrm{LinCat}^{\mathrm{St}}_A$ whose image in each $\mathrm{LinCat}^{\mathrm{St}}_{A_i}$ is an equivalence. We wish to show that F is an equivalence. Using Lemma D.3.5.5, we deduce that $\mathcal{C}$ and $\mathcal{D}$ can be identified with the products $\prod_i \mathrm{LMod}_{A_i}(\mathcal{C})$ and $\prod_i \mathrm{LMod}_{A_i}(\mathcal{D})$, respectively. Since the induced map $\mathrm{LMod}_{A_i}(\mathcal{C}) \to \mathrm{LMod}_{A_i}(\mathcal{D})$ is an equivalence for each index i, we conclude that F induces an equivalence of ∞-categories $f : \mathcal{C} \to \mathcal{D}$.

(b) Suppose we are given a finite collection of objects $(A_i, \mathcal{C}_i)$ in $\mathrm{LinCat}^{\mathrm{St}}$. For each index i, let $\mathcal{D}_i$ denote the A-linear ∞-category obtained from $\mathcal{C}_i$ by restriction of scalars and set $\mathcal{D} = \prod_i \mathcal{D}_i$. We wish to prove that for each index i, the canonical map $\mathrm{LMod}_{A_i} \otimes_{\mathrm{LMod}_A} \mathcal{D} \to \mathcal{C}_i$ is an equivalence. We have

$$\mathrm{LMod}_{A_i}(\mathcal{D}) \simeq \mathrm{LMod}_{A_i}(\prod_j \mathcal{D}_j) \simeq \prod_j \mathrm{LMod}_{A_i}(\mathcal{D}_j) \simeq \prod_j \mathrm{LMod}_{A_i \otimes_A A_j}(\mathcal{C}_j).$$

For $i \neq j$, the tensor product $A_i \otimes_A A_j$ is trivial, so that $\mathrm{LMod}_{A_i \otimes_A A_j}(\mathcal{C}_j)$ is a contractible Kan complex. For $i = j$, the tensor product $A_i \otimes_A A_j$ is equivalent to A_i, so that the forgetful functor $\mathrm{LMod}_{A_i \otimes_A A_j}(\mathcal{C}_j) \to \mathcal{C}_j$ is an equivalence. Passing to the product over i, we obtain the desired result.

$\square$

Proof of Theorem D.3.6.2. It follows from Lemma D.3.6.3 that the construction $A \mapsto \mathrm{LinCat}^{\mathrm{St}}_A$ preserves finite products. By virtue of Proposition A.3.3.1, it will suffice to show that if $f : A \to A^0$ is a universal descent morphism in CAlg with Čech nerve $A^\bullet$, then the canonical map $\mathrm{LinCat}^{\mathrm{St}}_A \to \varprojlim \mathrm{LinCat}^{\mathrm{St}}_{A^\bullet}$ is an equivalence of ∞-categories. We proceed by showing that this functor satisfies the conditions of Proposition HA.5.2.2.36:

(a) Fix an morphism $F : \mathcal{C} \to \mathcal{D}$ in $\mathrm{LinCat}^{\mathrm{St}}_A$ whose image in $\mathrm{LinCat}^{\mathrm{St}}_{A^0}$ is an equivalence. It follows that F induces an equivalence of cosimplicial ∞-categories $\mathrm{LMod}_{A^\bullet}(\mathcal{C}) \to \mathrm{LMod}_{A^\bullet}(\mathcal{D})$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & & \downarrow \\ \varprojlim \mathrm{LMod}_{A^\bullet}(\mathcal{C}) & \longrightarrow & \varprojlim \mathrm{LMod}_{A^\bullet}(\mathcal{D}), \end{array}$$

where the vertical maps are equivalences of ∞-categories by Theorem D.3.5.2. It follows that F is an equivalence of ∞-categories.

(b) Let $\theta : \mathrm{LinCat}^{\mathrm{St}} \to \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp})$ be the forgetful functor. Suppose we are given a diagram $X^\bullet : \mathbf{\Delta} \to \mathrm{LinCat}^{\mathrm{St}}$ lying over the cosimplicial $\mathbb{E}_\infty$-ring $A^\bullet$, and write $X^\bullet = (A^\bullet, \mathcal{C}^\bullet)$. Then $X^\bullet$ can be extended to a θ-limit diagram $\overline{X}^\bullet$ with $\overline{X}^{-1} = (A, \mathcal{C})$. We must show that if $X^\bullet$ carries every morphism in $\mathbf{\Delta}$ to a θ-coCartesian morphism in $\mathrm{LinCat}^{\mathrm{St}}$, then $\overline{X}$ has the same property. This follows from the calculation

$$\begin{aligned} \mathrm{LMod}_{A^0}(\mathcal{C}) &\simeq \mathrm{LMod}_{A^0}(\varprojlim \mathcal{C}^\bullet) \\ &\simeq \varprojlim \mathrm{LMod}_{A^0}(\mathcal{C}^\bullet) \\ &\simeq \varprojlim \mathcal{C}^{\bullet+1} \\ &\simeq \mathcal{C}^0 . \end{aligned}$$

□

D.4 Étale Descent for Prestable ∞-Categories

In §D.3, we showed that the construction $A \mapsto \mathrm{LinCat}^{\mathrm{St}}_A$ determines a functor $\mathrm{CAlg} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which is a sheaf with respect to the universal descent topology of Remark D.3.1.7 (Theorem D.3.6.2). In particular, it is a sheaf with respect to the étale topology (since every faithfully flat étale map of $\mathbb{E}_\infty$-rings is a universal descent morphism; see Corollary D.3.3.5). Our goal in this section is to prove an analogous result for *prestable* A-linear ∞-categories (Theorem D.4.1.2).

D.4.1 Formulation of the Theorem

It follows from Theorem HA.4.8.5.16 that the construction $A \mapsto \mathrm{LMod}_A(\mathrm{Sp}^{\mathrm{cn}})$ determines a symmetric monoidal functor from the ∞-category $\mathrm{Alg}^{\mathrm{cn}} = \mathrm{Alg}(\mathrm{Sp}^{\mathrm{cn}})$ of connective $\mathbb{E}_1$-rings to the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{Add}} \simeq \mathrm{Mod}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ of presentable additive ∞-categories. Passing to algebra objects (and using Theorem HA.5.1.2.2), we obtain a functor $\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}^{\mathrm{cn}}) \simeq \mathrm{Alg}(\mathrm{Alg}^{\mathrm{cn}}) \to \mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{Add}}) \to \mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

Construction D.4.1.1. We let $\mathrm{LinCat}^{\mathrm{PSt}}$ denote the ∞-category

$$\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}^{\mathrm{cn}}) \times_{\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})} \mathrm{LMod}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \times_{\mathcal{P}\mathrm{r}^{\mathrm{L}}} \mathsf{Groth}_\infty$$

whose objects are pairs $(A, \mathcal{C})$, where A is a connective $\mathbb{E}_2$-ring and $\mathcal{C}$ is a prestable A-linear ∞-category. There is an evident Cartesian fibration $\theta : \mathrm{LinCat}^{\mathrm{PSt}} \to \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}^{\mathrm{cn}})$ whose fiber over a connective $\mathbb{E}_2$-ring A can be identified with the ∞-category $\mathrm{LinCat}^{\mathrm{PSt}}_A$ of prestable A-linear ∞-categories introduced in Definition D.1.4.1. It follows from Proposition D.2.4.5 (and Corollary HTT.5.2.2.5) that θ is also a coCartesian fibration: it associates to every morphism $\phi : A \to B$ of connective $\mathbb{E}_2$-rings the functor

$$\mathrm{LinCat}^{\mathrm{PSt}}_A \to \mathrm{LinCat}^{\mathrm{PSt}}_B \qquad \mathcal{C} \mapsto B \otimes_A \mathcal{C} \simeq \mathrm{LMod}_B(\mathcal{C})$$

given by extension of scalars along ϕ (see Construction D.2.4.1). As a coCartesian fibration, the map θ is classified by a functor $\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp}^{\mathrm{cn}}) \to \widehat{\mathcal{C}\mathrm{at}}_\infty$, given at the level of objects by $A \mapsto \mathrm{LinCat}_A^{\mathrm{PSt}}$.

We can now state our main result of this section:

Theorem D.4.1.2 (Étale Descent for Prestable ∞-categories)**.** *The construction $R \mapsto \mathrm{LinCat}_R^{\mathrm{PSt}}$ determines a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which is a sheaf with respect to the étale topology.*

Remark D.4.1.3. In the statement of Theorem D.4.1.2, the restriction to to $\mathbb{E}_\infty$-rings is not essential: for any connective $\mathbb{E}_2$-ring A, the construction $R \mapsto \mathrm{LinCat}_R^{\mathrm{PSt}}$ determines a $\widehat{\mathcal{C}\mathrm{at}}_\infty$-valued sheaf on the ∞-category $\mathrm{Alg}_R^{\text{ét}}$ of étale R-algebras. This can be established using minor modifications of the proof of Theorem D.4.1.2 we present below.

We will deduce Theorem D.4.1.2 from a somewhat stronger statement (Theorem D.4.1.6). To formulate it, we need a variant of Remark D.3.1.7.

Construction D.4.1.4 (Flat Universal Descent Topology)**.** Let S be the collection of morphisms $f : A \to B$ between connective $\mathbb{E}_\infty$-rings which are flat universal descent morphisms (see Definition D.3.1.1). It follows from Propositions D.3.1.6 and B.6.1.3 that S satisfies the hypotheses of Proposition A.3.2.1, and therefore determines a Grothendieck topology on the (large) ∞-category $(\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}}$. We will refer to this Grothendieck topology as the *flat universal descent topology.*

Remark D.4.1.5. Let $f : A \to B$ be a flat universal descent morphism of connective $\mathbb{E}_\infty$-rings. Then f is faithfully flat: the assumption that f is a universal descent morphism guarantees that the extension of scalars functor $M \mapsto B \otimes_A M$ cannot annihilate any nonzero A-modules. It follows that the fpqc topology is a refinement of the flat universal descent topology. We do not know if these topologies are the same.

Every faithfully flat étale morphism of $\mathbb{E}_\infty$-rings $f : A \to B$ is also universal descent morphism (Corollary D.3.3.5). Consequently, the flat universal descent topology is a refinement of the étale topology. Theorem D.4.1.2 is therefore a consequence of the following:

Theorem D.4.1.6. *The construction $R \mapsto \mathrm{LinCat}_R^{\mathrm{PSt}}$ determines a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which is a sheaf with respect to the flat universal descent topology of Construction D.4.1.4.*

D.4.2 Comparison of Stable and Prestable ∞-Categories

We would like to deduce Theorem D.4.1.6 from the analogous result for the construction $R \mapsto \mathrm{LinCat}_R^{\mathrm{St}}$ (which satisfies descent for the universal descent topology, by virtue of Theorem D.3.6.2). For this, we need to compare a prestable ∞-category $\mathcal{C}$ with its stabilization $\mathrm{Sp}(\mathcal{C})$.

Lemma D.4.2.1. *Let R be a connective $\mathbb{E}_2$-ring. Then the diagram of ∞-categories*

$$\begin{array}{ccc} \mathrm{LinCat}_R^{\mathrm{PSt}} & \xrightarrow{\mathrm{Sp}(\bullet)} & \mathrm{LinCat}_R^{\mathrm{St}} \\ \downarrow & & \downarrow \\ \mathsf{Groth}_\infty & \xrightarrow{\mathrm{Sp}(\bullet)} & \mathcal{P}\mathrm{r}^{\mathrm{St}} \end{array}$$

is a pullback square.

Remark D.4.2.2. More informally, Lemma **??** asserts that the data of a prestable R-linear ∞-category $\mathcal{C}$ is equivalent to the data of a stable R-linear ∞-category $\mathcal{D}$ (given by the stabilization $\mathrm{Sp}(\mathcal{C})$) together with a t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ which is right complete and compatible with filtered colimits. In particular, if $\mathcal{C}$ is a Grothendieck prestable ∞-categories, then giving an action of R on $\mathcal{C}$ is equivalent to giving an action of R on the associated stable ∞-category $\mathrm{Sp}(\mathcal{C})$.

Proof of Lemma D.4.2.1. We first show that natural map

$$\theta : \mathrm{LinCat}_R^{\mathrm{PSt}} \to \mathrm{LinCat}_R^{\mathrm{St}} \times_{\mathcal{P}\mathrm{r}^{\mathrm{St}}} \mathsf{Groth}_\infty$$

is fully faithful. Let $\mathcal{C}$ and $\mathcal{D}$ be prestable R-linear ∞-categories. We wish to show that the diagram $\sigma_{\mathcal{C},\mathcal{D}}$:

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{LinCat}_R^{\mathrm{PSt}}}(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathrm{Map}_{\mathrm{LinCat}_R^{\mathrm{St}}}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D})) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{St}}}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D})) & \longrightarrow & \mathrm{Map}_{\mathsf{Groth}_\infty}(\mathcal{C}, \mathcal{D}) \end{array}$$

is a pullback square in $\mathcal{S}$. Let us regard the prestable R-linear ∞-category $\mathcal{D}$ as fixed, and write

$$\mathcal{C} \simeq \mathrm{LMod}_R^{\mathrm{cn}} \otimes_{\mathrm{LMod}_R^{\mathrm{cn}}} \mathcal{C} \simeq |\, \mathcal{C}_\bullet \,|$$

where $\mathcal{C}_\bullet$ is the simplicial prestable R-linear ∞-category given by the two-sided bar construction $[n] \mapsto (\mathrm{LMod}_R^{\mathrm{cn}})^{\otimes n+1} \otimes \mathcal{C}$. Then $\sigma_{\mathcal{C},\mathcal{D}}$ is the totalization of the cosimplicial diagram $\sigma_{\mathcal{C}_\bullet,\mathcal{D}}$. It will therefore suffice to show that $\sigma_{\mathcal{C},\mathcal{D}}$ is a homotopy equivalence in the special case where $\mathcal{C} \simeq \mathrm{LMod}_R^{\mathrm{cn}} \otimes \mathcal{C}'$ is freely generated by a Grothendieck prestable ∞-category $\mathcal{C}'$. Unwinding the definitions, we wish to show that the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathsf{Groth}_\infty}(\mathcal{C}', \mathcal{D}) & \longrightarrow & \mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{St}}}(\mathrm{Sp}(\mathcal{C}'), \mathrm{Sp}(\mathcal{D})) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathsf{Groth}_\infty}(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{St}}}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D})) \end{array}$$

is a homotopy pullback square. By virtue of Proposition C.3.1.1, this is equivalent to the following:

(∗) Let $F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ be a functor which preserves small colimits. Then F is right t-exact if and only if the composite map $\mathrm{Sp}(\mathcal{C}') \to \mathrm{Sp}(\mathcal{C}) \xrightarrow{F} \mathrm{Sp}(\mathcal{D})$ is right t-exact.

Since $\mathrm{LMod}_R^{\mathrm{cn}}$ is generated under small colimits by its unit object, the ∞-category $\mathcal{C} \simeq \mathrm{LMod}_R^{\mathrm{cn}} \otimes \mathcal{C}'$ is generated under small colimits by the essential image of the natural map $\mathcal{C}' \to \mathcal{C}$, from which assertion (∗) follows immediately. This completes the proof that θ is fully faithful.

We now prove that θ is essentially surjective. Let $\mathcal{C}$ be a Grothendieck prestable ∞-category, and suppose that $\mathrm{Sp}(\mathcal{C})$ has the structure of a stable R-linear ∞-category, encoded by a colimit-preserving monoidal functor

$$\phi : \mathrm{LMod}_R \to \mathrm{LFun}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{C})).$$

Using Proposition C.3.1.1, we can identify $\mathrm{LFun}(\mathcal{C}, \mathcal{D})$ with the full subcategory of $\mathrm{LFun}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D}))$ spanned by the right t-exact functors. Let $\mathcal{E}$ be denote the fiber product

$$\mathrm{LMod}_R \times_{\mathrm{LFun}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{C}))} \mathrm{LFun}(\mathcal{C}, \mathcal{C}),$$

which we can identify with a monoidal full subcategory of LMod_R. Then $\mathcal{E}$ contains the unit object of LMod_R and is closed under small colimits, and therefore contains all connective R-modules. It follows that ϕ restricts to a colimit-preserving monoidal functor $\mathrm{LMod}_R^{\mathrm{cn}} \hookrightarrow \mathcal{E} \to \mathrm{LFun}(\mathcal{C}, \mathcal{C})$, so we can regard $\mathcal{C}$ as a prestable R-linear ∞-category. □

D.4.3 Sheaves of ∞-Categories

We now collection some formal categorical observations which are relevant to the proof of Theorem D.4.1.6.

Lemma D.4.3.1. *The ∞-category $\mathcal{C}\mathrm{at}_\infty$ is generated (under small colimits) by the objects $\Delta^0, \Delta^1 \in \mathcal{C}\mathrm{at}_\infty$.*

Proof. Let $\mathcal{C}$ be the smallest full subcategory of $\mathcal{C}\mathrm{at}_\infty$ which contains Δ^0 and Δ^1 and is closed under small colimits. Let $\overline{\mathcal{C}} \subseteq \mathcal{S}\mathrm{et}_\Delta$ be the full subcategory spanned by those simplicial sets K which are categorically equivalent to ∞-categories belonging to $\mathcal{C}$. We wish to prove that $\mathcal{C} = \mathcal{C}\mathrm{at}_\infty$, or equivalently that $\overline{\mathcal{C}} = \mathcal{S}\mathrm{et}_\Delta$. Since $\overline{\mathcal{C}}$ is stable under filtered colimits in $\mathcal{S}\mathrm{et}_\Delta$, it will suffice to show that $\overline{\mathcal{C}}$ contains every finite simplicial set K. We proceed by induction on dimension n of K and the number of nondegenerate n-simplices of K. If $K = \varnothing$, then there is nothing to prove. Otherwise, we have a homotopy pushout diagram

$$\begin{array}{ccc} \partial\,\Delta^n & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow \\ K_0 & \longrightarrow & K \end{array}$$

where $\partial \Delta^n$ and K_0 belong to $\overline{\mathcal{C}}$ by the inductive hypothesis. Since $\mathcal{C}$ is stable under pushouts, $\overline{\mathcal{C}}$ is stable under homotopy pushouts; consequently, to show that $K \in \overline{\mathcal{C}}$, it will suffice to show that $\Delta^n \in \overline{\mathcal{C}}$. If $n > 2$, then the inclusion $\Lambda^n_1 \subseteq \Delta^n$ is a categorical equivalence. It therefore suffices to show that $\Lambda^n_1 \in \overline{\mathcal{C}}$, which follows from the inductive hypothesis. We are therefore reduced to the case $n \leqslant 1$: that is, we must show that $\Delta^0, \Delta^1 \in \overline{\mathcal{C}}$, which is clear. □

Lemma D.4.3.2. *Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology, and let $\alpha : \mathscr{F} \to \mathscr{F}'$ be a natural transformation of functors $\mathscr{F}, \mathscr{F}' : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$. For every object $C \in \mathcal{C}$ and every point $\eta \in \mathscr{F}'(C)$, we define a functor $\mathscr{F}_\eta : \mathcal{C}_{/C} \to \mathcal{S}$ by taking the fiber of the induced transformation $\mathscr{F}|_{\mathcal{C}_{/C}} \to \mathscr{F}'|_{\mathcal{C}_{/C}}$ (over the point determined by η).*

Assume that $\mathscr{F}'$ is a sheaf on $\mathcal{C}$. Then the following conditions are equivalent:

1. *The functor $\mathscr{F}$ is a sheaf on $\mathcal{C}$.*
2. *For every object $C \in \mathcal{C}$ and every point $\eta \in \mathscr{F}'(C)$, the functor $\mathscr{F}_\eta$ is a sheaf on $\mathcal{C}_{/C}$ (with respect to the induced Grothendieck topology).*

Moreover, if $\mathscr{F}'$ is hypercomplete, then $\mathscr{F}$ is hypercomplete if and only if each $\mathscr{F}_\eta$ is hypercomplete.

Proof. The implication (1) ⇒ (2) is obvious, since the full subcategory $\mathcal{S}\mathrm{hv}(\mathcal{C}_{/C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}_{/C}, \mathcal{S})$ is closed under small limits. Suppose that (2) is satisfied. Fix an object $C \in \mathcal{C}$ and a covering sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$; we wish to prove that the canonical map $\theta : \mathscr{F}(C) \to \varprojlim \mathscr{F}|_{\mathcal{C}^{(0)}_{/C}}$ is a homotopy equivalence. We have a commutative diagram

$$\begin{array}{ccc} \mathscr{F}(C) & \xrightarrow{\theta} & \varprojlim \mathscr{F}|_{\mathcal{C}^{(0)}_{/C}} \\ \downarrow & & \downarrow \\ \mathscr{F}'(C) & \xrightarrow{\theta'} & \varprojlim \mathscr{F}'|_{\mathcal{C}^{(0)}_{/C}}, \end{array}$$

where the map θ' is a homotopy equivalence. Consequently, to show that θ is a homotopy equivalence, it will suffice to show that θ induces a homotopy equivalence after passing to the homotopy fiber over any point $\eta \in \mathscr{F}'(C)$; this is precisely the content of assumption (2).

Now suppose that $\mathscr{F}'$ is hypercomplete. Since the collection of hypercomplete sheaves on $\mathcal{C}_{/C}$ is closed under limits, it is easy to see that $\mathscr{F}$ is hypercomplete only if each $\mathscr{F}_\eta$ is hypercomplete. Conversely, suppose that each $\mathscr{F}_\eta$ is hypercomplete; we wish to prove that $\mathscr{F}$ is hypercomplete. Choose an ∞-connective morphism $\beta : \mathscr{F} \to \mathscr{G}$, where $\mathscr{G}$ is hypercomplete; we wish to prove that β induces an equivalence $\beta_C : \mathscr{F}(C) \to \mathscr{G}(C)$ for each $C \in \mathcal{C}$. Since $\mathscr{F}'$ is hypercomplete, the map α factors through β; it will therefore suffice to show that β_C induces a homotopy equivalence after passing to the homotopy fiber over every point $\eta \in \mathscr{F}'(C)$. For this, it suffices to show that the induced map $\beta_\eta : \mathscr{F}_\eta \to \mathscr{G}_\eta$ is an equivalence. This is clear, since β_η is ∞-connective and both $\mathscr{F}_\eta$ and $\mathscr{G}_\eta$ are hypercomplete objects of $\mathcal{S}\mathrm{hv}(\mathcal{C}_{/C})$. □

Corollary D.4.3.3. *Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology and let $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ be a $\mathcal{C}\mathrm{at}_\infty$-valued presheaf on $\mathcal{C}$. Then X is a sheaf if and only if it satisfies the following conditions:*

(a) *The $\mathcal{S}$-valued functor $C \mapsto X(C)^{\simeq}$ is a $\mathcal{S}$-valued sheaf on $\mathcal{C}$.*

(b) *For every object $C \in \mathcal{C}$ and every pair of objects $x, y \in X(C)$, the construction*

$$(D \in \mathcal{C}_{/C}) \mapsto \mathrm{Map}_{X(D)}(x_D, y_D)$$

determines a $\mathcal{S}$-valued sheaf on $\mathcal{C}_{/C}$ (with respect to the induced Grothendieck topology); here x_D and y_D denote the images of x and y under the functor $X(C) \to X(D)$.

Proof. For every ∞-category $\mathcal{J}$, let $X^{\mathcal{J}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ be the functor given by

$$C \mapsto \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{J}, X(C)) \simeq \mathrm{Fun}(\mathcal{J}, X(C))^{\simeq}.$$

Then X is a $\mathcal{C}\mathrm{at}_\infty$-valued sheaf on $\mathcal{C}$ if and only if each $X^{\mathcal{J}}$ is a $\mathcal{S}$-valued sheaf on $\mathcal{C}$. If this condition is satisfied, then condition (a) follows by taking $\mathcal{J} = \Delta^0$ and condition (b) follows by applying Lemma D.4.3.2 to the map $X^{\Delta^1} \to X^{\partial \Delta^1}$. Conversely, suppose that (a) and (b) are satisfied, and let $\mathcal{E} \subseteq \mathcal{C}\mathrm{at}_\infty$ be the full subcategory spanned by those ∞-categories $\mathcal{J}$ for which $X^{\mathcal{J}}$ is a sheaf. Since the construction $\mathcal{J} \mapsto X^{\mathcal{J}}$ carries colimits in $\mathcal{C}\mathrm{at}_\infty$ to limits in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, we deduce that $\mathcal{E}$ is closed under small colimits in $\mathcal{C}\mathrm{at}_\infty$. Condition ($a$) implies that $\mathcal{E}$ contains Δ^0, and therefore also contains the coproduct $\Delta^0 \amalg \Delta^0 \simeq \partial \Delta^1$. Applying Lemma D.4.3.2 to the map $X^{\Delta^1} \to X^{\partial \Delta^1}$, we see that condition (b) implies that $\Delta^1 \in \mathcal{J}$. It follows from Lemma D.4.3.1 that $\mathcal{E} = \mathcal{C}\mathrm{at}_\infty$, so that X is a $\mathcal{C}\mathrm{at}_\infty$-valued sheaf on $\mathcal{C}$ as desired. □

D.4.4 Faithful Flatness

We now collect some facts about faithfully flat morphisms between $\mathbb{E}_1$-ring spectra. Here, we must be careful to distinguish between left and right flatness.

Definition D.4.4.1. Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_1$-rings. We will say that B is *left faithfully flat* over A if the following conditions are satisfied:

(a) The morphism ϕ exhibits B as a flat left A-module, in the sense of Definition HA.7.2.2.10 .

(b) For every nonzero right A-module M, the tensor product $M \otimes_A B$ is also nonzero.

We say that B is *right faithfully flat* if the following dual conditions are satisfied:

(a') The morphism ϕ exhibits B as a flat right A-module, in the sense of Definition HA.7.2.2.10.

(b') For every nonzero left A-module M, the tensor product $B \otimes_A M$ is also nonzero.

Remark D.4.4.2. Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_k$-rings for $2 \leqslant k \leqslant \infty$. Then ϕ is right faithfully flat if and only if it is left faithfully flat. In this case, we will simply say that ϕ is *faithfully flat*.

Lemma D.4.4.3. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_1$-rings. The following conditions are equivalent:*

(1) *The map f is left faithfully flat.*

(2) *The cofiber* $\mathrm{cofib}(f)$ *is flat when regarded as a left A-module.*

Proof. Suppose first that (2) is satisfied. We have a fiber sequence of left A-modules $A \to B \to \mathrm{cofib}(f)$. Since A and $\mathrm{cofib}(f)$ are flat as left modules over A, B is also flat as a left A-module. Applying Theorem HA.7.2.2.15, we can write $\mathrm{cofib}(f)$ as a filtered colimit $\varinjlim N_\alpha$, where each N_α is free of finite rank as a left A-module. Then, as a left A-module, B can be realized as a filtered colimit of left modules $B_\alpha = B \times_{\mathrm{cofib}(f)} N_\alpha$, each of which fits into a fiber sequence $A \to B_\alpha \to N_\alpha$. Since A is connective and N_α is free, this fiber sequence splits. Consequently, for any right A-module M, the canonical map $M \simeq M \otimes_A A \to M \otimes_A N_\alpha$ admits a left homotopy inverse and therefore induces an injective map $\pi_* M \to \pi_*(M \otimes_A N_\alpha)$. Passing to the colimit over α, we deduce that the map $\pi_* M \to \pi_*(M \otimes_A B)$ is injective. Consequently, if $M \otimes_A B$ vanishes, then so does M.

Now suppose that (1) is satisfied. Since A and B are connective, $\mathrm{cofib}(f)$ is connective. We wish to show that $\mathrm{cofib}(f)$ is flat (as a left A-module). By virtue of Theorem HA.7.2.2.15, it will suffice to show that for every discrete right A-module M, the relative tensor product $M \otimes_A \mathrm{cofib}(f)$ is discrete. We have a fiber sequence of spectra $M \to M \otimes_A B \to M \otimes_A \mathrm{cofib}(f)$. Since B is flat as a left A-module, the spectrum $M \otimes_A B$ is discrete. Consequently, to prove that $M \otimes_A \mathrm{cofib}(f)$ is discrete, it suffices to show that the map $\theta : \pi_0 M \to \pi_0(M \otimes_A B) \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 M, \pi_0 B)$ is a monomorphism. Let $K \subseteq \pi_0 M$ denote the kernel of θ. Since $\pi_0 B$ is flat over $\pi_0 A$, we can identify $\mathrm{Tor}_0^{\pi_0 A}(K, \pi_0 B)$ with a submodule of $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 M, \pi_0 B)$. This submodule is generated by $\theta(K) = 0$ as a module over $\pi_0 B$, and therefore vanishes. It follows that $K \otimes_A B \simeq 0$, contradicting our assumption that ϕ is left faithfully flat. □

Proposition D.4.4.4. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_2$-rings, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let X be an object of* $\mathrm{Sp}(\mathcal{C})$.

(a) *If ϕ is faithfully flat, then X belongs to* $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$ *if and only if* $R' \otimes_R X \in \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))$ *belongs to* $\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0} \subseteq \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))$.

(b) *If ϕ is a flat universal descent morphism, then X belongs to* $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ *if and only if* $R' \otimes_R X \in \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))$ *belongs to* $\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0} \subseteq \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))$.

Proof. We first prove (a). It follows immediately from the definitions (making no assumptions on ϕ) that if X belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$, then $R' \otimes_R X$ belongs to $\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$. Conversely, suppose that $R' \otimes_R X$ belongs to $\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 0}$ and that ϕ is faithfully flat. We wish to prove that X belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant 0}$. Since the t-structure on $\mathrm{Sp}(\mathcal{C})$ is right complete, this is equivalent to the assertion that $\pi_n X \simeq 0$ for $n < 0$. To prove this, it will suffice to show that the canonical map $\rho : \pi_n X \to \pi_n(R' \otimes_R X)$ is a monomorphism in the abelian category $\mathcal{C}^{\heartsuit}$. By virtue of Lemma D.4.4.3, the cofiber $\mathrm{cofib}(\phi)$ is flat as a right R-module and can therefore (by Proposition HA.7.2.2.15) be realized as the colimit of a filtered diagram $\{M_\alpha\}$ of *free* right R-modules. For each index α, set $R'_\alpha = R' \times_{\mathrm{cofib}(\phi)} M_\alpha$. Since the t-structure on $\mathrm{Sp}(\mathcal{C})$ is compatible with filtered colimits, the map ρ is a filtered colimit of morphisms

$$\rho_\alpha : \pi_n X \to \pi_n(R'_\alpha \otimes_R X).$$

The abelian category $\mathcal{C}^{\heartsuit}$ is Grothendieck, so it will suffice to show that each ρ_α is a monomorphism in $\mathcal{C}^{\heartsuit}$. In fact, ρ_α is a split monomorphism, since the fiber sequence of right R-modules $R \to R'_\alpha \to M_\alpha$ splits (because M_α is free and R is connective).

We now prove (b). If R' is flat over R, then (using Proposition HA.7.2.2.15) we can write R' as the colimit of a filtered diagram $\{N_\beta\}$ of finitely generated free right R-modules. If $X \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$, then $R' \otimes_R X \simeq \varinjlim_\beta N_\beta \otimes_R X$ also belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$, since the t-structure on $\mathrm{Sp}(\mathcal{C})$ is compatible with filtered colimits and each $N_\beta \otimes_R X$ is a direct sum of finitely many copies of X.

Conversely, suppose that $R' \otimes_R X \in \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0}$. Choose a fiber sequence $X' \to X \to X''$ in the ∞-category $\mathrm{Sp}(\mathcal{C})$, where $X' \in \mathrm{Sp}(\mathcal{C})_{\geqslant 1}$ and $X'' \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. If R' is flat over R, then the preceding arguments show that

$$R' \otimes_R X' \in \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\geqslant 1} \qquad R' \otimes_R X'' \in \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0}.$$

Consequently, the assumption that $R' \otimes_R X$ belongs to $\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0}$ guarantees that $R' \otimes_R X' \simeq 0$. If ϕ is a universal descent morphism, it follows from Theorem D.3.4.1 that $X' \simeq 0$, so that $X \simeq X'' \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ as desired. $\square$

Corollary D.4.4.5. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ and $\mathcal{D}$ be prestable R-linear ∞-categories and let $F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ be an R-linear functor. Suppose that there exists a faithfully flat map $R \to R'$ of connective $\mathbb{E}_2$-rings for which the induced functor $R' \otimes_R \mathrm{Sp}(\mathcal{C}) \to R' \otimes_R \mathrm{Sp}(\mathcal{D})$ is right t-exact. Then F is right t-exact.*

D.4.5 The Proof of Theorem D.4.1.6

We wish to show that the construction $R \mapsto \mathrm{LinCat}_R^{\mathrm{PSt}}$ determines a $\widehat{\mathcal{C}\mathrm{at}}_\infty$-valued sheaf with respect to the flat universal descent topology on $(\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}}$. We will prove this by verifying conditions (a) and (b) of Corollary D.4.3.3. We begin by verifying condition (b). Let A be a connective $\mathbb{E}_\infty$-ring and suppose that we are given prestable A-linear ∞-categories $\mathcal{C}$ and $\mathcal{D}$. We wish to show that the functor $\mathscr{F} : \mathrm{CAlg}_A^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, given by the formula $\mathscr{F}(B) = \mathrm{Map}_{\mathrm{LinCat}_B^{\mathrm{PSt}}}(B \otimes_A \mathcal{C}, B \otimes_A \mathcal{D})$, is a sheaf with respect to the flat universal descent topology. Define $\mathscr{F}' : \mathrm{CAlg}_A^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ by the formula

$$\mathscr{F}'(B) = \mathrm{Map}_{\mathrm{LinCat}_B}(B \otimes_A \mathrm{Sp}(\mathcal{C}), B \otimes_A \mathrm{Sp}(\mathcal{D})).$$

We have an evident natural transformation $\alpha : \mathscr{F} \to \mathscr{F}'$. Using Lemma D.4.2.1, we see that α identifies each $\mathscr{F}(B)$ with the summand of $\mathscr{F}'(B)$ consisting of those B-linear functors $B \otimes_A \mathrm{Sp}(\mathcal{C}) \to B \otimes_A \mathrm{Sp}(\mathcal{D})$ which are right t-exact. Since $\mathscr{F}'$ is a sheaf for the universal descent topology (Theorem D.3.5.2) and the condition of right t-exactness is local with respect to the fpqc topology (Corollary D.4.4.5), it follows that $\mathscr{F}$ is a sheaf with respect to the flat universal descent topology.

We now prove (a). Define $\widehat{\mathcal{S}}$-valued functors $\mathscr{G}, \mathscr{G}' : \mathscr{G}_R^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ by the formulae

$$\mathscr{G}(A) = (\mathrm{LinCat}_A^{\mathrm{PSt}})^{\simeq} \qquad \mathscr{G}'(A) = (\mathrm{LinCat}_A^{\mathrm{St}})^{\simeq}.$$

The construction $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ determines a map $\beta : \mathscr{G} \to \mathscr{G}'$ and the functor $\mathscr{G}'$ is a sheaf with respect to the universal descent topology (Theorem D.3.5.2). To conclude that $\mathscr{G}$ is a sheaf with respect to the flat universal descent topology, it will suffice to show that the morphism β satisfies the hypotheses of Lemma D.4.3.2. Fix a connective $\mathbb{E}_\infty$-ring A and a stable A-linear ∞-category $\mathcal{C}$, and let $\mathscr{H} : \mathscr{G}_A^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be the functor given by $\mathscr{H}(B) = \mathscr{G}(B) \times_{\mathscr{G}'(B)} \{\mathcal{C}\}$. Using Lemma D.4.2.1, we see that $\mathscr{H}(B)$ can be identified with the discrete (but large) collection of t-structures on the stable ∞-category $B \otimes_A \mathcal{C}$ which are right complete and compatible with filtered colimits. It follows from (b) that this $\mathscr{H}$ is a separated set-valued presheaf on $\mathscr{G}_A^{\mathrm{cn}}$ and we wish to prove that it is actually a sheaf. Let us therefore fix a connective A-algebra B and a finite collection of morphisms $\{B \to B_i\}_{i \in I}$ for which the map $B \to \prod B_i$ is a flat universal descent morphism, and suppose we are given a collection of elements $\eta_i \in \mathscr{H}(B_i)$ which are compatible (in the sense that η_i and η_j have the same image in $\mathscr{H}(B_i \amalg_B B_j)$ for every pair of elements $i, j \in I$); we wish to show that the element $\{\eta_i\}_{i \in I} \in \prod \mathscr{H}(B_i)$ can be lifted to an element $\eta \in \mathscr{H}(B)$.

Set $B^0 = \prod_{i \in I} B_i$ and let $B^\bullet$ denote the cosimplicial object of $\mathrm{CAlg}_B^{\mathrm{cn}}$ given by the Čech nerve of the map $B \to B^0$. Then we can regard $\mathrm{LMod}_{B^\bullet}(\mathcal{C})$ as a cosimplicial object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, and the elements of η_i determine a lifting of $\mathrm{LMod}_{B^\bullet}(\mathcal{C})$ to the ∞-category Groth_∞, which we will denote by $\mathrm{LMod}_{B^\bullet}(\mathcal{C})_{\geqslant 0}$. Note that for every injective map $[m] \hookrightarrow [n]$

the transition maps $B^m \to B^n$ is flat, so the associated functor $\mathrm{LMod}_{B^m}(\mathcal{C}) \to \mathrm{LMod}_{B^n}(\mathcal{C})$ is left t-exact. It follows that the totalization $\mathcal{E} = \mathrm{Tot}(\mathrm{LMod}_{B^\bullet}(\mathcal{C})_{\geqslant 0})$ is again a Grothendieck prestable ∞-category (see Proposition C.3.2.4). Moreover, we have a canonical equivalence

$$\begin{aligned} \mathrm{Sp}(\mathcal{E}) &\simeq \mathrm{Tot}\,\mathrm{Sp}(\mathrm{LMod}_{B^\bullet}(\mathcal{C}))_{\geqslant 0}) \\ &\simeq \mathrm{Tot}\,\mathrm{LMod}_{B^\bullet}(\mathcal{C}) \\ &\simeq \mathrm{LMod}_B(\mathcal{C}), \end{aligned}$$

where the last equivalence is a consequence of Theorem **??**. In other words, the ∞-category $\mathrm{LMod}_B(\mathcal{C})$ admits a t-structure $(\mathrm{LMod}_B(\mathcal{C})_{\geqslant 0}, \mathrm{LMod}_B(\mathcal{C})_{\leqslant 0})$ which is right t-exact and compatible with filtered colimits which is characterized by the following property:

- $(*)$ An object $X \in \mathrm{LMod}_B(\mathcal{C})$ belongs to $\mathrm{LMod}_B(\mathcal{C})_{\geqslant 0}$ if and only if each of the tensor products $B^n \otimes_B X$ belongs to $\mathrm{LMod}_{B^n}(\mathcal{C})_{\geqslant 0}$ (moreover, it suffices to check this condition in the special case $n = 0$).

This t-structure determines an element $\eta \in \mathscr{H}(B)$. To complete the proof, we wish to show that η is a preimage of the collection $\{\eta_i\} \in \prod_{i \in I} \mathscr{H}(B_i)$. By virtue of Remark D.2.4.7, this can be restated as follows:

- $(*')$ An object $Y \in \mathrm{LMod}_{B^0}(\mathcal{C})$ belongs to $\mathrm{LMod}_{B^0}(\mathcal{C})_{\geqslant 0}$ if and only if its image under the forgetful functor $\mathrm{LMod}_{B^0}(\mathcal{C}) \to \mathrm{LMod}_B(\mathcal{C})$ belongs to $\mathrm{LMod}_B(\mathcal{C})_{\geqslant 0}$.

By virtue of $(*)$ together with the left adjointability of the diagram of forgetful functors

$$\begin{array}{ccc} \mathrm{LMod}_{B^1}(\mathcal{C}) & \longrightarrow & \mathrm{LMod}_{B^0}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{LMod}_{B^0}(\mathcal{C}) & \longrightarrow & \mathrm{LMod}_B(\mathcal{C}), \end{array}$$

we can restate $(*')$ as follows:

- $(*'')$ An object $Y \in \mathrm{LMod}_{B^0}(\mathcal{C})$ belongs to $\mathrm{LMod}_{B^0}(\mathcal{C})_{\geqslant 0}$ if and only if its image under the composite functor $\mathrm{LMod}_{B^0}(\mathcal{C}) \xrightarrow{\phi} \mathrm{LMod}_{B^1}(\mathcal{C}) \xrightarrow{\psi} \mathrm{LMod}_{B^0}(\mathcal{C})$ belongs to $\mathrm{LMod}_{B^0}(\mathcal{C})_{\geqslant 0}$. (Here ϕ and ψ are induced by the two B^0-algebra structures on B^1.)

Assertion $(*'')$ now follows from Proposition D.4.4.4 and Remark D.2.4.7. This completes the proof of Theorem D.4.1.6.

D.5 Local Properties of R-Linear ∞-Categories

Let P be some property of Grothendieck prestable ∞-categories. In this section, we will study the following closely related questions:

Question D.5.0.1. Under what conditions is the property P stable under base change? That is, if we are given a morphism $\phi : R \to R'$ of connective $\mathbb{E}_\infty$-rings and a prestable R-linear ∞-category $\mathcal{C}$ which has the property P, then when can we conclude that the prestable R'-linear ∞-category $R' \otimes_R \mathcal{C}$ also has the property P?

Question D.5.0.2. Under what conditions can the property P be tested locally? That is, if we are given a morphism $\phi : R \to R'$ of connective $\mathbb{E}_\infty$-rings which is a “covering” in some sense (for example, a faithfully flat morphism) and a prestable R-linear ∞-category $\mathcal{C}$ for which $R' \otimes_R \mathcal{C}$ has the property P, then when can we conclude that $\mathcal{C}$ also has the property P?

Our goal in this section is to provide partial answers to Questions D.5.0.1 and D.5.0.2 in a variety of circumstances. Our main results can be summarized as follows:

- The properties of being stable, separated, and complete are compatible with arbitrary base change and can be tested locally with respect to the flat universal descent topology (Propositions D.5.1.1, D.5.1.2, and D.5.1.3).

- The property that an R-linear functor $F : \mathcal{C} \to \mathcal{D}$ is compact (left exact) is stable under arbitrary base change and can be tested locally with respect to the (flat) universal descent topology (Propositions D.5.2.1 and D.5.2.2).

- The property of being compactly generated is stable under arbitrary base change and can be tested locally with respect to the étale topology (Theorem D.5.3.1).

- The property of being anticomplete is stable under fiber-smooth base change (Theorem D.5.4.1) and can be tested locally with respect to the étale topology (Theorem D.5.4.9).

- The property of being weakly coherent is stable under base base change along quasi-finite morphisms which are almost of finite presentation (Theorem D.5.5.1) and can be tested locally with respect to the étale topology (Corollary D.5.5.11).

- The property of being locally Noetherian is stable under base change along morphisms which are almost of finite presentation and can be tested locally with respect to the flat topology (Propositions D.5.6.1 and D.5.6.4).

- For each $n \geqslant 0$, the properties of being n-complicial and weakly n-complicial are stable under flat base change. Moreover, the property of being weakly n-complicial can be tested locally with respect to the flat universal descent topology (Proposition D.5.7.1).

D.5.1 Stable, Separated, and Complete Prestable ∞-Categories

We with some easy cases of Questions D.5.0.1 and D.5.0.2.

Proposition D.5.1.1. *Let R be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a prestable R-linear ∞-category. Then:*

- (a) *Assume that $\mathcal{C}$ is stable (see Definition C.1.2.12). Then for any connective $\mathbb{E}_1$-algebra R' over R, the Grothendieck prestable ∞-category $\mathrm{LMod}_{R'}(\mathcal{C})$ is also stable.*
- (b) *If there exists a flat universal descent morphism $R \to R'$ of connective $\mathbb{E}_2$-rings (for example, a morphism which is étale and faithfully flat) such that $R' \otimes_R \mathcal{C}$ is stable, then $\mathcal{C}$ is stable.*

Proof. Assertion (a) is trivial. To prove (b), suppose that $R' \otimes_R \mathcal{C}$ is stable. We wish to prove that the functor $\Sigma^\infty : \mathcal{C} \simeq \mathrm{Sp}(\mathcal{C})_{\geqslant 0} \subseteq \mathrm{Sp}(\mathcal{C})$ is an equivalence of ∞-categories. Equivalently, we wish to show that every object $C \in \mathrm{Sp}(\mathcal{C})_{\leqslant 0}$ is zero. Since R' is flat over R, the tensor product $R' \otimes_R C \in \mathrm{LMod}_{R'}(\mathcal{C})$ belongs to $\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\leqslant 0}$, and therefore vanishes (by virtue of the stability of $R' \otimes_R \mathcal{C}$). Since $R \to R'$ is a universal descent morphism, it follows that $C \simeq 0$ as desired. □

Proposition D.5.1.2. *Let R be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a prestable R-linear ∞-category. Then:*

- (a) *Assume that $\mathcal{C}$ is separated (see Definition C.1.2.12). Then for any connective $\mathbb{E}_1$-algebra R' over R, the Grothendieck prestable ∞-category $\mathrm{LMod}_{R'}(\mathcal{C})$ is also separated.*
- (b) *If there exists a universal descent morphism $R \to R'$ of connective $\mathbb{E}_2$-rings (for example, a morphism which is étale and faithfully flat) such that $R' \otimes_R \mathcal{C}$ is separated, then $\mathcal{C}$ is separated.*

Proof. We first prove (a). Assume that $\mathcal{C}$ is separated and let X be an object of the ∞-category $\bigcap_{n \geqslant 0} \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\geqslant n}$. Then the image of X under the forgetful functor $\theta : \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C})) \to \mathrm{Sp}(\mathcal{C})$ belongs to $\bigcap \mathrm{Sp}(\mathcal{C})_{\geqslant n}$ (see Remark D.2.4.7). The assumption that $\mathcal{C}$ is separated then gives $\theta(X) \simeq 0$. Since the functor θ is conservative, we conclude that $X \simeq 0$. Allowing X to vary, we deduce that $\mathrm{LMod}_{R'}(\mathcal{C})$ is separated.

We now prove (b). Assume that $R' \otimes_R \mathcal{C}$ is separated and let $X \in \bigcap_{n \geqslant 0} \mathrm{Sp}(\mathcal{C})_{\geqslant n}$. Then the tensor product $R' \otimes_R X$ belongs to $\bigcap_{n \geqslant 0} \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\geqslant n}$. It follows that $R' \otimes_R X \simeq 0$. If $R \to R'$ is a universal descent morphism, this implies that $X \simeq 0$. □

Proposition D.5.1.3. *Let R be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a prestable R-linear ∞-category. Then:*

(a) *Assume that $\mathcal{C}$ is complete (see Definition C.1.2.12). Then for any connective $\mathbb{E}_1$-algebra R' over R, the Grothendieck prestable ∞-category $\mathrm{LMod}_{R'}(\mathcal{C})$ is also complete.*

(b) *If there exists a flat universal descent morphism $R \to R'$ of connective $\mathbb{E}_2$-rings (for example, a morphism which is étale and faithfully flat) such that $R' \otimes_R \mathcal{C}$ is complete, then $\mathcal{C}$ is complete.*

Proof. If $\mathcal{C}$ is complete, then the canonical map $\mathrm{Sp}(\mathcal{C}) \to \varprojlim_{n \geq 0} \mathrm{Sp}(\mathcal{C})_{\leq n}$ is an equivalence of ∞-categories left-tensored over $\mathrm{LMod}_R^{\mathrm{cn}}$. It follows that, for every algebra object $R' \in \mathrm{LMod}_R^{\mathrm{cn}}$, the induced functor

$$\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C})) \to \varprojlim \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C})_{\leq n}) \simeq \varprojlim \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\leq n}$$

is also an equivalence, so that $R' \otimes_R \mathcal{C}$ is also complete. This proves (a).

We now prove (b). Assume that $\phi : R \to R'$ is a universal descent morphism of connective $\mathbb{E}_2$-rings and that $R' \otimes_R \mathcal{C}$ is complete. We then have a pair of adjoint functors

$$\mathrm{Sp}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C})).$$

Since ϕ is a universal descent morphism, this adjunction is comonadic: that is, it exhibits $\mathrm{Sp}(\mathcal{C})$ as the ∞-category of coalgebras over the comonad

$$U = F \circ G : \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C})) \to \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C})),$$

given at the level of objects by $X \mapsto R' \otimes_R X$. If the map ϕ is flat, then an object $X \in \mathrm{Sp}(\mathcal{C})$ belongs to $\mathrm{Sp}(\mathcal{C})_{\leq n}$ if and only if $R' \otimes_R X$ belongs to $\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))_{\leq n}$ (Proposition D.4.4.4). Note that the comonad U commutes with truncations. It follows that the canonical map $\mathrm{Sp}(\mathcal{C}) \to \varprojlim_{n \geq 0} \mathrm{Sp}(\mathcal{C})_{\leq n}$ can be identified with the opposite of the composition of equivalences

$$\begin{aligned}
\mathrm{Sp}(\mathcal{C})^{\mathrm{op}} &\simeq \mathrm{RMod}_U(\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))^{\mathrm{op}}) \\
&\simeq \mathrm{RMod}_U(\varprojlim \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))^{\mathrm{op}}_{\leq n}) \\
&\simeq \varprojlim \mathrm{RMod}_U(\mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{C}))^{\mathrm{op}}_{\leq n}) \\
&\simeq \varprojlim \mathrm{Sp}(\mathcal{C})^{\mathrm{op}}_{\leq n},
\end{aligned}$$

and is therefore an equivalence of ∞-categories. □

D.5.2 Left Exact and Compact Functors

We now consider a slight variant of Questions D.5.0.1 and D.5.0.2. Suppose we are given a connective $\mathbb{E}_2$-ring R and an R-linear functor $f : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are prestable

R-linear ∞-categories. What is the relationship between properties of the functor f and properties of the induced map $\mathrm{LMod}_{R'}(\mathcal{C}) \to \mathrm{LMod}_{R'}(\mathcal{D})$, where R' is an $\mathbb{E}_1$-algebra over R?

Proposition D.5.2.1. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ and $\mathcal{D}$ be prestable R-linear ∞-categories, and let $f : \mathcal{C} \to \mathcal{D}$ be an R-linear functor. Then:*

- (a) *If the functor f is left exact, then for any connective $\mathbb{E}_1$-algebra R' over R, the induced functor $f_{R'} : \mathrm{LMod}_{R'}(\mathcal{C}) \to \mathrm{LMod}_{R'}(\mathcal{C})$ is also left exact.*
- (b) *If there exists a flat universal descent morphism $R \to R'$ of connective $\mathbb{E}_2$-rings (for example, a morphism which is étale and faithfully flat) for which the map $f_{R'}$ is left exact, then f is left exact.*

Proof. Assertion (a) follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{LMod}_{R'}(\mathcal{C}) & \xrightarrow{f_{R'}} & \mathrm{LMod}_{R'}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D}, \end{array}$$

since the vertical maps (given by the forgetful functors) are conservative and preserve finite limits (Corollary HA.4.2.3.3). To prove (b), we note that there is also a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow{\scriptstyle R'\otimes_R} & & \downarrow{\scriptstyle R'\otimes_R} \\ \mathrm{LMod}_{R'}(\mathcal{C}) & \xrightarrow{f_{R'}} & \mathrm{LMod}_{R'}(\mathcal{D}). \end{array}$$

If $R \to R'$ is a flat universal descent morphism, then Proposition D.4.4.4 (together with Proposition C.3.2.1) guarantee that the vertical maps in this diagram are conservative and left exact. Consequently, the left exactness of f follows from the left exactness of $f_{R'}$. □

Proposition D.5.2.2. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ and $\mathcal{D}$ be prestable R-linear ∞-categories, and let $f : \mathcal{C} \to \mathcal{D}$ be an R-linear functor. Then:*

- (a) *Suppose that f is compact. Then, for any connective $\mathbb{E}_1$-algebra R' over R, the induced functor $f_{R'} : \mathrm{LMod}_{R'}(\mathcal{C}) \to \mathrm{LMod}_{R'}(\mathcal{D})$ is compact.*
- (b) *If there exists a universal descent morphism $R \to R'$ of connective $\mathbb{E}_2$-rings (for example, a morphism which is étale and faithfully flat) for which the induced map $f_{R'} : R' \otimes_R \mathcal{C} \to R' \otimes_R \mathcal{D}$ is compact, then f is compact.*

Proof. To prove (a), let g and $g_{R'}$ denote right adjoints to f and $f_{R'}$, so that we have a commutative diagram σ :

$$\begin{array}{ccc} \mathrm{LMod}_{R'}(\mathcal{D}) & \xrightarrow{g_{R'}} & \mathrm{LMod}_{R'}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow{g} & \mathcal{C}\,. \end{array}$$

where the vertical maps are conservative and preserve small colimits (Corollary HA.4.2.3.5). It follows immediately that if g commutes with filtered colimits, then so does $g_{R'}$.

We now prove (b). Passing to stabilizations in the diagram σ, we obtain a commutative diagram of stable ∞-categories τ :

$$\begin{array}{ccc} \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{D})) & \xrightarrow{G_{R'}} & \mathrm{LMod}_{R'}(\mathrm{Sp}(\mathcal{D})) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(\mathcal{D}) & \xrightarrow{G} & \mathrm{Sp}(\mathcal{C}). \end{array}$$

For every left R-module M, let $G_M : \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{C})$ denote the functor given on objects by $G_M(X) = G(M \otimes_R X)$. By virtue of Proposition C.3.2.4, to show that the functor g commutes with filtered colimits it will suffice to show that G_R commutes with all colimits. In fact, we will prove that the functor G_M commutes with colimits for *all* left R-modules M.

Let $\mathcal{E} \subseteq \mathrm{LMod}_R$ denote the full subcategory spanned by those left R-modules M for which the functor G_M commutes with small colimits. We wish to show that $\mathcal{E} = \mathrm{LMod}_R$. Since the construction $M \mapsto G_M$ commutes with finite limits, the ∞-category $\mathcal{E}$ is a stable subcategory of LMod_R which is closed under retracts. Since $R \to R'$ is a universal descent morphism, it will suffice to show that $\mathcal{E}$ contains the essential image of the forgetful functor $\mathrm{LMod}_{R'} \to \mathrm{LMod}_R$ (see Variant D.3.2.3). In other words, we are reduced to proving that the functor G_M commutes with small colimits in the special case where M admits the structure of a left R'-module. In this case, the commutativity of the diagram τ allows us to identify G_M with the composition of colimit-preserving functors

$$\mathcal{D} \xrightarrow{R' \otimes_R} \mathrm{LMod}_{R'}(\mathcal{D}) \xrightarrow{M \otimes_{R'}} \mathrm{LMod}_{R'}(\mathcal{D}) \xrightarrow{G_{R'}} \mathrm{LMod}_{R'}(\mathcal{C}) \to \mathcal{C}\,.$$

□

Remark D.5.2.3. Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $C \in \mathcal{C}$ be an object. Then C determines an R-linear functor $\lambda : \mathrm{LMod}_R^{\mathrm{cn}} \to \mathcal{C}$, given by $\lambda(M) = M \otimes_R C$, and the object $C \in \mathcal{C}$ is compact if and only if the functor λ is compact (in the sense of Definition C.3.5.1). Applying Proposition D.5.2.2, we deduce:

- (a) If C is compact, then for any $\mathbb{E}_1$-algebra A over R, the tensor product $A \otimes_R C$ is a compact object of $\mathrm{LMod}_A(\mathcal{C})$.

(b) If there exists a universal descent morphism of $\mathbb{E}_2$-rings $R \to R'$ (for example, a morphism that is étale and faithfully flat) and $R' \otimes_R C$ is a compact object of $\mathrm{LMod}_{R'}(\mathcal{C})$, then C is compact.

D.5.3 Compactly Generated Prestable ∞-Categories

Recall that a stable ∞-category $\mathcal{C}$ is said to be *compactly generated* if is equivalent to an ∞-category of the form $\mathrm{Ind}(\mathcal{C}_0)$, where $\mathcal{C}_0$ is an essentially small stable ∞-category. In the case where $\mathcal{C}$ is equipped with an action of a commutative ring R, a theorem of Toën asserts that the hypothesis that $\mathcal{C}$ is compactly generated can be tested locally with respect to the étale topology on R ([211]). This was generalized to the setting of $\mathbb{E}_\infty$-rings by Antieau and Gepner ([2]). Our goal in this section is to prove the following analogous result in the setting of prestable ∞-categories:

Theorem D.5.3.1. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable R-linear ∞-category. Then:*

(a) *If $\mathcal{C}$ is compactly generated, then $\mathrm{LMod}_{R'}(\mathcal{C})$ is compactly generated for any connective $\mathbb{E}_1$-algebra R' over R.*

(b) *If there exists a faithfully flat étale morphism $R \to R'$ such that $R' \otimes_R \mathcal{C}$ is compactly generated, then $\mathcal{C}$ is compactly generated.*

Remark D.5.3.2. Theorem D.5.3.1 is valid also when R is a connective $\mathbb{E}_2$-ring; the proof we give in this section requires only minor modifications.

Part (a) of Theorem D.5.3.1 is a special case of the following:

Lemma D.5.3.3. *Let R be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a Grothendieck prestable ∞-category equipped with a right action of $\mathrm{LMod}_R^{\mathrm{cn}}$, and let $\mathcal{D}$ be a Grothendieck prestable ∞-category equipped with a left action of $\mathrm{LMod}_R^{\mathrm{cn}}$. Assume that $\mathcal{C}$ and $\mathcal{D}$ are compactly generated. Then:*

(a) *The tensor product $\mathcal{C} \otimes_R \mathcal{D} = \mathcal{C} \otimes_{\mathrm{LMod}_R^{\mathrm{cn}}} \mathcal{D}$ is compactly generated.*

(b) *Let $\mathcal{E} \subseteq \mathcal{C} \otimes_R \mathcal{D}$ be the smallest full subcategory which is closed under cofibers and contains every object of the form $C \otimes_R D$, where C is a compact object of $\mathcal{C}$ and D is a compact object of $\mathcal{D}$. Then every compact object of $\mathcal{C} \otimes_R \mathcal{D}$ is a retract of an object of $\mathcal{E}$.*

Proof. Let $\mathcal{E}$ be as in (b). It is easy to see that $\mathcal{E}$ is closed under finite coproducts and is therefore closed under finite colimits in $\mathcal{C} \otimes_R \mathcal{D}$. As in the proof of Proposition D.2.2.1, we can identify $\mathcal{C} \otimes_R \mathcal{D}$ with the geometric realization of a semisimplicial object ($[n] \in$

$\mathbf{\Delta}_s^{op}) \mapsto \mathrm{Bar}\,\mathrm{LMod}_R^{cn}(\mathcal{C},\mathcal{D})_n$ of the ∞-category Groth_∞^c of Definition C.3.4.2. It follows that the natural map $\mathcal{C}\otimes\mathcal{D}\to\mathcal{C}\otimes_R\mathcal{D}$ preserves compact objects. Since the collection of compact objects of $\mathcal{C}\otimes_R\mathcal{D}$ is closed under finite colimits, each object of $\mathcal{E}$ is compact in $\mathcal{C}\otimes_R\mathcal{D}$. It follows that the inclusion $\mathcal{E}\hookrightarrow\mathcal{C}\otimes_R\mathcal{D}$ extends to a fully faithful embedding $f:\mathrm{Ind}(\mathcal{E})\to\mathcal{C}\otimes_R\mathcal{D}$ which commutes with small colimits. Let g be a right adjoint to f. Since we can identify $\mathcal{C}\otimes_R\mathcal{D}$ with the totalization of $\mathrm{Bar}_{\mathrm{LMod}_R^{cn}}(\mathcal{C},\mathcal{D})_\bullet$ in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$, the functor g is conservative. It follows that f is an equivalence of ∞-categories, which proves (a) and (b). □

Let $\mathcal{C}$ be an R-linear prestable ∞-category, for some connective $\mathbb{E}_2$-ring R. If $\mathcal{C}$ is compactly generated, then there is a close relationship between compact objects of $\mathcal{C}$ and compact objects of the localization $\mathcal{C}[a^{-1}]=R[a^{-1}]\otimes_R\mathcal{C}$ (where a is an element of $\pi_0 R$):

Proposition D.5.3.4 (Thomason's Trick)**.** *Let R be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a compactly generated prestable R-linear ∞-category. Let a be an element of $\pi_0 R$ and let D be a compact object of $\mathcal{C}[a^{-1}]$. Then there exists a compact object $C\in\mathcal{C}$ and an equivalence $R[a^{-1}]\otimes_R C\simeq D\oplus\Sigma D$ in the ∞-category $\mathcal{C}[a^{-1}]$.*

Proof. For each object $C\in\mathcal{C}$, let $C[a^{-1}]$ denote the tensor product $R[a^{-1}]\otimes_R C\in\mathcal{C}[a^{-1}]$. Note that, as an object of $\mathcal{C}$, we can identify $C[a^{-1}]$ with the colimit of the filtered diagram

$$C\xrightarrow{a}C\xrightarrow{a}C\xrightarrow{a}C\xrightarrow{a}\cdots.$$

Consequently, for objects $C,C'\in\mathcal{C}$, if C is compact then we have a canonical isomorphism of abelian groups

$$\pi_0\,\mathrm{Map}_{\mathcal{C}[a^{-1}]}(C[a^{-1}],C'[a^{-1}])\simeq\pi_0\,\mathrm{Map}_{\mathcal{C}}(C,C'[a^{-1}])\simeq(\pi_0\,\mathrm{Map}_{\mathcal{C}}(C,C'))[a^{-1}].$$

Let $\mathcal{E}$ denote the full subcategory $R[a^{-1}]\otimes_R\mathcal{C}$ spanned by those objects of the form $R[a^{-1}]\otimes_R C$, where $C\in\mathcal{C}$ is compact. The above argument shows that for any morphism $f:C[a^{-1}]\to C'[a^{-1}]$ between objects of $\mathcal{E}$, there exists an integer $n\gg 0$ such that $a^n f$ is induced by a map $f_0:C\to C'$ in $\mathcal{C}$. We then have $\mathrm{cofib}(f)\simeq\mathrm{cofib}(a^n f)\simeq\mathrm{cofib}(f_0)[a^{-1}]$. In particular, the ∞-category $\mathcal{E}$ is closed under the formation of cofibers. A similar argument (replacing C' by $\Sigma C'$ and fibers in place of cofibers) shows that $\mathcal{E}$ is closed under extensions.

Let X be any nonzero object of $\mathcal{C}[a^{-1}]$. Then the image of X in $\mathcal{C}$ is nonzero, so our assumption that $\mathcal{C}$ is compactly generated guarantees that there is a nonzero morphism $f_0:C\to X$ in $\mathcal{C}$, where $C\in\mathcal{C}$ is compact (Corollary C.6.3.3). The map f_0 induces a nonzero map $f:C[a^{-1}]\to X$ in $\mathcal{C}[a^{-1}]$, whose domain lies in the full subcategory $\mathcal{E}$. Arguing as in Proposition C.6.3.1, we see that $\mathcal{E}$ is a set of compact generators for $\mathcal{C}[a^{-1}]$. In particular, every compact object $D\in\mathcal{C}[a^{-1}]$ can be obtained as a direct summand of an object $E\in\mathcal{E}$. Write $E\simeq D\oplus D'$, and let $q:E\to E$ be the direct sum of the zero map $0:D\to D$ with the identity map $\mathrm{id}_{D'} : D' \to D$. Then $D \oplus \Sigma D \simeq \mathrm{cofib}(q)$ belongs to $\mathcal{E}$, and can therefore be written as $C[a^{-1}]$ for some compact object $C \in \mathcal{C}$. □

Proof of Part (*b*) *of Theorem D.5.3.1.* Let R be a connective $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $\mathrm{CAlg}_R^{\text{ét}}$ denote the ∞-category of étale R-algebras. We define a functor $\chi : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ by the formula

$$\chi(R') = \begin{cases} \Delta^0 & \text{if } R' \otimes_R \mathcal{C} \text{ is compactly generated} \\ \varnothing & \text{otherwise.} \end{cases}$$

To prove Theorem **??**, it will suffice to show that the functor χ is a sheaf with respect to the étale topology. By virtue of Theorem B.6.4.1, it will suffice to show that χ is a sheaf with respect to both the finite étale topology and the Nisnevich topology.

We begin by showing that χ is a sheaf with respect to the Nisnevich topology. By virtue of Theorem B.5.0.3, it will suffice to show that χ satisfies affine Nisnevich excision. Note that if $R' \simeq 0$, then the ∞-category $R' \otimes_R \mathcal{C}$ is a contractible Kan complex and therefore compactly generated, so that $\chi(R') \simeq \Delta^0$. To complete the proof, it will suffice to show that if $\phi : A \to A'$ is a morphism of étale R-algebras which induces an isomorphism $(\pi_0 A)/(a) \to (\pi_0 A')/(a)$ for some element $a \in \pi_0 A$, then the diagram

$$\begin{array}{ccc} \chi(A) & \longrightarrow & \chi(A') \\ \downarrow & & \downarrow \\ \chi(A[a^{-1}]) & \longrightarrow & \chi(A'[a^{-1}]) \end{array}$$

is a pullback square. In other words, we must show that if $A' \otimes_R \mathcal{C}$ and $A[a^{-1}] \otimes_R \mathcal{C}$ are compactly generated, then $A \otimes_R \mathcal{C}$ is compactly generated. To prove this, we can assume without loss of generality that $R = A$. We will show that $\mathcal{C}$ is compactly generated by verifying that it satisfies the requirements of Corollary C.6.3.3. Let $C \in \mathcal{C}$ be a nonzero object; we wish to show that there is a compact object $C_0 \in \mathcal{C}$ and a nonzero map $f : C_0 \to C$. There are two cases to consider:

(i) Suppose that the object $C[a^{-1}] = A[a^{-1}] \otimes_A C$ vanishes. Let $\mathcal{D} \subseteq \mathcal{C}$ be the full subcategory spanned by those objects $D \in \mathcal{C}$ satisfying $D[a^{-1}] \simeq 0$, and define $\mathcal{D}' \subseteq (A' \otimes_A \mathcal{C})$ similarly. Using Theorem D.3.5.2 and Proposition D.4.4.4, we see that the construction $B \mapsto (B \otimes_A \mathcal{C})$ is a $\widehat{\mathcal{C}\mathrm{at}}_\infty$-valued sheaf with respect to the étale topology. In particular, it satisfies Nisnevich excision (Theorem B.5.0.3), so the diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{A[a^{-1}]\otimes_A} & A[a^{-1}] \otimes_A \mathcal{C} \\ \downarrow{\scriptstyle A'\otimes_A} & & \downarrow{\scriptstyle A'[a^{-1}]\otimes_{A[a^{-1}]}} \\ A' \otimes_A \mathcal{C} & \xrightarrow{A'[a^{-1}]\otimes_{A'}} & A'[a^{-1}] \otimes_A \mathcal{C} \end{array}$$

is a pullback square. Passing to homotopy fibers in the horizontal direction, we conclude that the functor $D \mapsto A' \otimes_A D$ induces an equivalence of ∞-categories $\mathcal{D} \to \mathcal{D}'$. By assumption, the ∞-category $A' \otimes_A \mathcal{C}$ is compactly generated. It follows from Proposition 7.1.1.12 that $\mathcal{D}'$ is compactly generated, so that $\mathcal{D}$ is also compactly generated. Our assumption that $C[a^{-1}]$ vanishes guarantees that C belongs to the full subcategory $\mathcal{D} \subseteq \mathcal{C}$. Since $\mathcal{D}$ is compactly generated, Corollary C.6.3.3 guarantees that there exists a nonzero map $C_0 \to C$, where C_0 is a compact object of $\mathcal{D}$. Note that the tensor product $A' \otimes_A C_0 \in \mathcal{D}'$ is a compact object of $A' \otimes_A \mathcal{C}$ (by virtue of Proposition 7.1.1.12) and the tensor product $A[a^{-1}] \otimes_A C_0$ is a compact object of $A[a^{-1}] \otimes_A \mathcal{C}$ (since it vanishes). Applying Remark D.5.2.3, we deduce that C_0 is a compact object of $\mathcal{C}$, as desired.

(ii) Suppose that the object $C[a^{-1}] = A[a^{-1}] \otimes_A C$ does not vanish. Since $A[a^{-1}] \otimes_A \mathcal{C}$ is a compactly generated prestable ∞-category, Corollary C.6.3.3 guarantees that there exists a compact object $D_0 \in A[a^{-1}] \otimes_A \mathcal{C}$ and a nonzero map $f : D_0 \to C[a^{-1}]$. Let D'_0 denote the tensor product $A'[a^{-1}] \otimes_{A[a^{-1}]} D_0$, which we regard as a compact object of the ∞-category $A'[a^{-1}] \otimes_A \mathcal{C}$. Replacing D_0 by $D_0 \oplus \Sigma D_0$ if necessary, we can assume that D'_0 has the form $D'[a^{-1}]$ for some compact object $D' \in (A' \otimes_A \mathcal{C})$ (Proposition D.5.3.4). Since the diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{A[a^{-1}]\otimes_A} & A[a^{-1}] \otimes_A \mathcal{C} \\ \downarrow{\scriptstyle A'\otimes_A} & & \downarrow{\scriptstyle A'[a^{-1}]\otimes_{A[a^{-1}]}} \\ A' \otimes_A \mathcal{C} & \xrightarrow{A'[a^{-1}]\otimes_{A'}} & A'[a^{-1}] \otimes_A \mathcal{C} \end{array}$$

is a pullback square, we can choose an object $D \in \mathcal{C}$ satisfying $D_0 \simeq D[a^{-1}]$ and $D' \simeq A' \otimes_A D$. It follows from Remark D.5.2.3 that the object D is compact. As in the proof of Proposition 7.1.1.12, we have a canonical isomorphism

$$(\pi_0 \operatorname{Map}_{\mathcal{C}}(D, C))[a^{-1}] \simeq \pi_0 \operatorname{Map}_{\mathcal{C}[a^{-1}]}(D_0, C[a^{-1}]) \neq 0.$$

In particular, the group $\pi_0 \operatorname{Map}_{\mathcal{C}}(D, C)$ must be nonzero, so there exists a nonzero map $f : D \to C$.

We now argue that χ is a sheaf with respect to the finite étale topology. Since χ is a sheaf for the Nisnevich topology, it commutes with finite products. It will therefore suffice to show that if $\phi : A \to B$ is a finite étale faithfully flat morphism in $\mathrm{CAlg}^{\text{ét}}_R$ for which $B \otimes_R \mathcal{C}$ is compactly generated, then $A \otimes_R \mathcal{C}$ is also compactly generated. As before, we may assume without loss of generality that $R = A$.

Let $F : \mathcal{C} \to \mathrm{LMod}_B(\mathcal{C})$ denote the functor given by $F(C) = B \otimes_A C$. Note that since B is finitely generated and projective as an A-module, the composite functor $\mathcal{C} \xrightarrow{F} \mathrm{LMod}_B(\mathcal{C}) \to \mathcal{C}$

is a retract of a product of finitely many copies of the identity functor, and therefore preserves small limits. It follows that the functor f also preserves small limits. Applying Corollary HTT.5.5.2.9 , we deduce that F admits a left adjoint $F^L : \mathrm{LMod}_B(\mathcal{C}) \to \mathcal{C}$.

If C is any nonzero object of $\mathcal{C}$, then $f(C)$ is a nonzero object of $\mathrm{LMod}_B(\mathcal{C})$. Since $\mathrm{LMod}_B(\mathcal{C})$ is compactly generated, Corollary C.6.3.3 implies that there exists a compact object $D \in \mathrm{LMod}_B(\mathcal{C})$ and a nonzero map $\alpha : D \to F(C)$. Then α determines a nonzero map $F^L(D) \to C$, and the object $F^L(D) \in \mathcal{C}$ is compact by virtue of Proposition HTT.5.5.7.2 (since the functor F commutes with filtered colimits). Applying Corollary C.6.3.3, we deduce that $\mathcal{C}$ is compactly generated. □

D.5.4 Anticomplete Prestable ∞-Categories

We now study descent questions for the class of anticomplete prestable R-linear ∞-categories. Our starting point is the following result:

Theorem D.5.4.1. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings which satisfies the following conditions:*

- (i) *The morphism ϕ is flat.*
- (ii) *The $\mathbb{E}_\infty$-ring R' has finite* Tor*-amplitude when regarded as a module over $R' \otimes_R R'$.*

Let $\mathcal{C}$ be a prestable R-linear ∞-category. If $\mathcal{C}$ is anticomplete, then $R' \otimes_R \mathcal{C}$ is anticomplete.

Remark D.5.4.2. Hypotheses (i) and (ii) of Theorem D.5.4.1 are satisfied when the morphism $\phi : R \to R'$ is étale. More generally, they are satisfies if ϕ is *fiber-smooth*, in the sense of Definition 11.2.3.1 (see Proposition 11.3.3.1).

Remark D.5.4.3. Theorem D.5.4.1 is valid more generally in the setting of $\mathbb{E}_2$-rings; we leave the requisite modifications to the reader.

Warning D.5.4.4. In the setting of Theorem D.5.4.1, hypothesis (ii) is essential. For example, suppose that $\phi : R \to R'$ is a morphism of Noetherian rings of finite Krull dimension. If R is regular, then the prestable ∞-category $\mathrm{Mod}_R^{\mathrm{cn}}$ is anticomplete. However, the tensor product $R' \otimes_R \mathrm{Mod}_R^{\mathrm{cn}} \simeq \mathrm{Mod}_{R'}^{\mathrm{cn}}$ is anticomplete only if R' is also regular.

Before giving the proof of Theorem D.5.4.1, we need a few observations about the class of anticomplete prestable R-linear ∞-categories.

Lemma D.5.4.5. *Let $\mathcal{C}$ be a compactly generated monoidal prestable ∞-category (where the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable), and suppose that the ∞-category $\mathrm{Sp}(\mathcal{C})$ is locally rigid (see Definition D.7.4.1). Let $\mathrm{LMod}_{\mathcal{C}}^{\mathrm{lex}}(\mathsf{Groth}_\infty)$*

be the subcategory of $\mathrm{LMod}_{\mathcal{C}}(\mathsf{Groth}_\infty)$ *whose objects are* $\mathcal{C}$*-linear Grothendieck prestable* ∞*-categories* $\mathcal{M}$ *and whose morphisms are* $\mathcal{C}$*-linear functors which preserve small colimits and finite limits, and let* $\mathrm{LMod}^{\mathrm{ch}}_{\mathcal{C}}(\mathsf{Groth}_\infty)$ *denote the full subcategory of* $\mathrm{LMod}^{\mathrm{lex}}_{\mathcal{C}}(\mathsf{Groth}_\infty)$ *spanned by the anticomplete prestable* $\mathcal{C}$*-linear* ∞*-categories. Then:*

(1) *The* ∞*-category* $\mathrm{LMod}^{\mathrm{ch}}_{\mathcal{C}}(\mathsf{Groth}_\infty)$ *is a colocalization of* $\mathrm{LMod}^{\mathrm{lex}}_{\mathcal{C}}(\mathsf{Groth}_\infty)$*: that is, the inclusion functor* $\mathrm{LMod}^{\mathrm{ch}}_{\mathcal{C}}(\mathsf{Groth}_\infty) \hookrightarrow \mathrm{LMod}^{\mathrm{lex}}_{\mathcal{C}}(\mathsf{Groth}_\infty)$ *admits a right adjoint* $\mathcal{M} \mapsto \breve{\mathcal{M}}$*.*

(2) *For every* $\mathcal{C}$*-linear Grothendieck prestable* ∞*-category* $\mathcal{M}$*, the canonical map* $\breve{\mathcal{M}} \to \mathcal{M}$ *induces an equivalence of completions.*

Remark D.5.4.6. Let $\mathcal{C}$ be as in Lemma D.5.4.5 and let $\mathcal{M}$ be a $\mathcal{C}$-linear Grothendieck prestable ∞-category. Then Lemma D.5.4.5 implies that there exists a $\mathcal{C}$-linear functor $\lambda : \breve{\mathcal{M}} \to \mathcal{M}$ which exhibits $\breve{\mathcal{M}}$ as a $\mathrm{LMod}^{\mathrm{ch}}_{\mathcal{C}}(\mathsf{Groth}_\infty)$-colocalization of $\mathcal{M}$. In particular, the Grothendieck prestable ∞-category $\breve{\mathcal{M}}$ is anticomplete. Moreover, part (2) of Lemma D.5.4.5 guarantees that λ induces an equivalence on completions, so that the image of λ under the forgetful functor $\mathrm{LMod}_{\mathcal{C}}(\mathsf{Groth}_\infty) \to \mathsf{Groth}_\infty$ can be identified with the functor appearing in Proposition C.5.5.9. We can summarize the situation more informally as follows:

(i) If $\mathcal{M}$ is a Grothendieck prestable ∞-category equipped which is equipped with a left action of $\mathcal{C}$ and $\lambda : \breve{\mathcal{M}} \to \mathcal{M}$ is as in Proposition C.5.5.9, then the ∞-category $\breve{\mathcal{M}}$ inherits a left action of $\mathcal{C}$.

(ii) The ∞-category $\breve{\mathcal{M}}$ of (i) is universal among anticomplete prestable $\mathcal{C}$-linear ∞-categories equipped with a left exact $\mathcal{C}$-linear functor to $\mathcal{M}$.

Applying Lemma D.5.4.5 in the case $\mathcal{C} = \mathrm{LMod}^{\mathrm{cn}}_R$, where R is a connective $\mathbb{E}_2$-ring, we obtain the following:

Corollary D.5.4.7. *Let* R *be a connective* $\mathbb{E}_2$*-ring and let* $\mathcal{M}$ *be a prestable* R*-linear* ∞*-category. The following conditions are equivalent:*

(a) *The prestable* ∞*-category* $\mathcal{M}$ *is anticomplete.*

(b) *For every prestable* R*-linear* ∞*-category* $\mathcal{N}$ *with completion* $\widehat{\mathcal{N}}$*, the canonical map* $\theta : \mathrm{Map}_{\mathrm{LinCat}^{\mathrm{PSt,lex}}_R}(\mathcal{M}, \mathcal{N}) \to \mathrm{Map}_{\mathrm{LinCat}^{\mathrm{PSt,lex}}_R}(\mathcal{M}, \widehat{\mathcal{N}})$ *is a homotopy equivalence.*

Proof of Lemma D.5.4.5. In what follows, we will assume that the reader is familiar with the formalism of ∞-operads. Let us regard the ∞-category Groth_∞ of Grothendieck prestable ∞-categories as equipped with symmetric monoidal structure described in §C.4. This symmetric monoidal structure is encoded by a coCartesian fibration $q : \mathsf{Groth}^{\otimes}_\infty \to \mathcal{F}\mathrm{in}_*$. Let $\mathcal{LM}^{\otimes}$ denote the ∞-operad of Definition HA.4.2.1.7 (so that $\mathcal{LM}^{\otimes}$-algebras are given by pairs (A, M), where A is an associative algebra and M is a left A-module) and set $\mathcal{O}^{\otimes} = \mathsf{Groth}^{\otimes}_\infty \times_{\mathcal{F}\mathrm{in}_*} \mathcal{LM}^{\otimes}$. The ∞-operad $\mathcal{O}^{\otimes}$ can be described more concretely as follows:

- Objects of $\mathcal{O}$ are given by pairs $(\mathcal{E}, \sigma)$, where $\mathcal{E}$ is a Grothendieck prestable ∞-category and $\sigma \in \{\mathfrak{a}, \mathfrak{m}\}$ is a formal symbol.

- Given a finite collection of objects $\{(\sigma_i, \mathcal{E}_i)\}_{i \in I}$ $(\sigma, \mathcal{E})$ of $\mathcal{O}$, a morphism $\phi \in \mathrm{Mul}_{\mathcal{O}}(\{(\sigma_i, \mathcal{E}_i)\}_{i \in I}, (\sigma, \mathcal{E}))$ consists of the following data:

  - A functor $\prod_{i \in I} \mathcal{E}_i \to \mathcal{E}$ which preserves small colimits separately in each variable.
  - A linear ordering $\leqslant$ on the set I.

  This data is required to satisfy the following condition:

  - If $\sigma = \mathfrak{a}$, then $\sigma_i = \mathfrak{a}$ for all $i \in I$. If $\sigma = \mathfrak{m}$, then there is a unique element $i \in I$ such that $\sigma_i = \mathfrak{m}$, and i is a maximal element of $(I, \leqslant)$.

Consider the subcategory $\mathcal{O}_1^{\otimes} \subseteq \mathcal{O}^{\otimes}$ which defines an ∞-operad admitting the following description:

- An object $(\sigma, \mathcal{E})$ of $\mathcal{O}$ belongs to $\mathcal{O}_1$ if either $\sigma = \mathfrak{m}$ or $\sigma = \mathfrak{a}$ and $\mathcal{E}$ is compactly generated.

- Let $\phi \in \mathrm{Mul}_{\mathcal{O}}(\{(\sigma_i, \mathcal{E}_i)\}_{i \in I}, (\sigma, \mathcal{E}))$ classify a functor $f : \prod_{i \in I} \mathcal{E}_i \to \mathcal{E}$ which preserves small colimits separately in each variables. Then ϕ belongs to $\mathrm{Mul}_{\mathcal{O}_1}(\{(\sigma_i, \mathcal{E}_i)\}_{i \in I}, (\sigma, \mathcal{E}))$ if one of the following conditions is satisfied:

  - We have $\sigma = \mathfrak{a}$ and $f(\{E_i\})$ is a compact object of $\mathcal{E}$ whenever each E_i is a compact object of $\mathcal{E}_i$.
  - We have $\sigma = \mathfrak{m}$, so that there is a unique element $i \in I$ such that $\sigma_i = \mathfrak{m}$. Moreover, for every collection of compact objects $\{E_j \in \mathcal{E}_j\}_{j \in I - \{i\}}$, the functor

    $$\mathcal{E}_i \simeq \mathcal{E}_i \times \prod_{j \neq i} \{E_j\} \hookrightarrow \prod_{j \in I} \mathcal{E}_j \xrightarrow{f} \mathcal{E}$$

    has bounded amplitude, in the sense of Definition C.5.5.2.

Unwinding the definitions, we can identify objects of $\mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O})$ with pairs $(\mathcal{C}, \mathcal{M})$, where $\mathcal{C}$ is a monoidal Grothendieck prestable ∞-category, $\mathcal{M}$ is a Grothendieck prestable ∞-category which is left-tensored over $\mathcal{C}$, and the action maps

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{C} \times \mathcal{M} \to \mathcal{M}$$

preserve small colimits separately in each variable. Under this identification, $\mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}_1)$ corresponds to the subcategory of $\mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O})$ which can be described as follows:

- An object $(\mathcal{C}, \mathcal{M}) \in \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O})$ belongs to $\mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}_1)$ if and only if $\mathcal{C}$ is compactly generated, the collection of compact objects of $\mathcal{C}$ contains the unit object and is stable under tensor products, and for each compact object $C \in \mathcal{C}$ the functor $M \mapsto C \otimes M$ has bounded amplitude.

- A morphism from $(\mathcal{C}, \mathcal{M})$ to $(\mathcal{C}', \mathcal{M}')$ in $\mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O})$ is a morphism of $\mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}_1)$ if and only if its domain and codomain belong to $\mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}_1)$, the underlying monoidal functor $\mathcal{C} \to \mathcal{C}'$ preserves compact objects, and the underlying functor $\mathcal{M} \to \mathcal{M}'$ has bounded amplitude.

Now suppose that we are given an object $(\mathcal{C}, \mathcal{M}) \in \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O})$ having the property that $\mathcal{C}$ is compactly generated and $\mathrm{Sp}(\mathcal{C})$ is locally rigid. If C is a compact object of $\mathcal{C}$, then $\Sigma^\infty C$ is a compact object of $\mathrm{Sp}(\mathcal{C})$. It follows that $\Sigma^\infty C$ is a dualizable object of $\mathrm{Sp}(\mathcal{C})$, whose dual $D = (\Sigma^\infty C)^\vee$ is also compact. We therefore have $D \in \mathrm{Sp}(\mathcal{C})_{\geqslant -n}$ for some $n \gg 0$. Let $\mathcal{M}$ be any left $\mathcal{C}$-module in the ∞-category Groth_∞. For $M \in \mathrm{Sp}(\mathcal{M})_{\leqslant k}$ and $N \in \mathrm{Sp}(\mathcal{M})_{\geqslant n+k+1}$, the mapping space

$$\mathrm{Map}_{\mathrm{Sp}(\mathcal{M})}(N, (\Sigma^\infty C) \otimes M) \simeq \mathrm{Map}_{\mathrm{Sp}(\mathcal{M})}(D \otimes N, M)$$

is contractible. Allowing N to vary, we deduce that tensor product with $\Sigma^\infty C$ carries $\mathrm{Sp}(\mathcal{M})_{\leqslant k}$ into $\mathrm{Sp}(\mathcal{M})_{\leqslant n+k}$. It follows that tensor product with C induces a functor $\mathcal{M} \to \mathcal{M}$ of bounded amplitude. Allowing C to vary, we conclude that $(\mathcal{C}, \mathcal{M})$ belongs to the subcategory $\mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}_1) \subseteq \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}_0)$. This proves the following:

$(*)$ Suppose we are given a monoidal Grothendieck prestable ∞-category $\mathcal{C}$ (where the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable), which we regard as an associative algebra object of the ∞-category Groth_∞. If $\mathcal{C}$ is compactly generated and $\mathrm{Sp}(\mathcal{C})$ is locally rigid, then the inclusion

$$\{\mathcal{C}\} \times_{\mathrm{Alg}(\mathsf{Groth}_\infty)} \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}_1) \hookrightarrow \{\mathcal{C}\} \times_{\mathrm{Alg}(\mathsf{Groth}_\infty)} \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}) \simeq \mathrm{LMod}_{\mathcal{C}}(\mathsf{Groth}_\infty)$$

is an equivalence onto the subcategory of $\mathrm{LMod}_{\mathcal{C}}(\mathsf{Groth}_\infty)$ whose morphisms are $\mathcal{C}$-linear functors of bounded amplitude.

Consider the full subcategory $\mathcal{O}_0^\otimes \subseteq \mathcal{O}_1^\otimes$ which defines an ∞-operad admitting the following description:

- An object $(\sigma, \mathcal{E})$ of $\mathcal{O}_1$ belongs to $\mathcal{O}_0$ if either $\sigma = \mathfrak{a}$ or $\sigma = \mathfrak{m}$ and $\mathcal{E}$ is anticomplete.

Then we can identify $\mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}_0)$ with the full subcategory of $\mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}_1)$ spanned by those pairs $(\mathcal{C}, \mathcal{M})$ where $\mathcal{M}$ is anticomplete.

Fix an object $X \in \mathcal{O}_1^\otimes$, given by a sequence $\{(\sigma_i, \mathcal{E}_i)\}_{1 \leqslant i \leqslant n}$. For each $1 \leqslant i \leqslant n$ satisfying $\sigma_i = \mathfrak{m}$, choose a functor $\lambda_i : \mathcal{E}'_i \to \mathcal{E}_i$ satisfying the requirements of Proposition C.5.5.9. For

$1 \leqslant i \leqslant n$ satisfying $\sigma_i = \mathfrak{a}$, set $\mathcal{E}'_i = \mathcal{E}_i$ and let $\lambda_i : \mathcal{E}'_i \to \mathcal{E}_i$ be the identity map. Define $X' = \{(\sigma_i, \mathcal{E}'_i)\}_{1 \leqslant i \leqslant n} \in \mathcal{O}_0^{\otimes}$, so that $\{\lambda_i\}_{1 \leqslant i \leqslant n}$ determines a map $\lambda : X' \to X$ in $\mathcal{O}_1^{\otimes}$. We will prove the following:

$(*')$ For each $X \in \mathcal{O}_1^{\otimes}$ as above, the map $\lambda : X' \to X$ exhibits X' as a $\mathcal{O}_0^{\otimes}$-colocalization of X.

Assume $(*')$ for the moment. Allowing X to vary over all objects of $\mathcal{O}_1^{\otimes}$, we deduce that the inclusion functor $\mathcal{O}_0^{\otimes} \hookrightarrow \mathcal{O}_1^{\otimes}$ admits a right adjoint G, given by $X \mapsto X'$. Then G is a functor of ∞-operads, which induces a functor $\mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}_1) \to \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}_0)$ which is right adjoint to the inclusion, given on objects by the construction $(\mathcal{C}, \mathcal{M}) \mapsto (\mathcal{C}, \breve{\mathcal{M}})$, where the counit map determines a $\mathcal{C}$-linear functor $\breve{\mathcal{M}} \to \mathcal{M}$ which is an equivalence on completions. Combining this observation with $(*)$, we obtain the following:

$(*'')$ Let $\mathcal{C} \in \mathrm{Alg}(\mathsf{Groth}_\infty)$ be as in $(*)$, let $\mathrm{LMod}'_{\mathcal{C}}(\mathsf{Groth}_\infty)$ be the subcategory of $\mathrm{LMod}_{\mathcal{C}}(\mathsf{Groth}_\infty)$ whose morphisms are $\mathcal{C}$-linear functors of bounded amplitude, and let $\mathrm{LMod}''_{\mathcal{C}}(\mathsf{Groth}_\infty) \subseteq \mathrm{LMod}'_{\mathcal{C}}(\mathsf{Groth}_\infty)$ be the full subcategory spanned by those $\mathcal{C}$-modules which are anticomplete. Then the inclusion $\mathrm{LMod}''_{\mathcal{C}}(\mathsf{Groth}_\infty) \hookrightarrow \mathrm{LMod}'_{\mathcal{C}}(\mathsf{Groth}_\infty)$ admits a right adjoint $\mathrm{LMod}'_{\mathcal{C}}(\mathsf{Groth}_\infty) \to \mathrm{LMod}''_{\mathcal{C}}(\mathsf{Groth}_\infty)$, which we will denote by $\mathcal{M} \to \breve{\mathcal{M}}$. Moreover, the counit map $\breve{\mathcal{M}} \to \mathcal{M}$ induces an equivalence of completions.

Note that in the situation of $(*'')$, a $\mathcal{C}$-linear functor $f : \mathcal{M} \to \mathcal{N}$ is left exact if and only if the composite map $\breve{\mathcal{M}} \to \mathcal{M} \xrightarrow{f} \mathcal{N}$ is left exact (Proposition C.3.2.1). It follows that the construction $\mathcal{M} \to \breve{\mathcal{M}}$ restricts to a functor $\mathrm{LMod}^{\mathrm{lex}}_{\mathcal{C}}(\mathsf{Groth}_\infty) \to \mathrm{LMod}^{\mathrm{ch}}_{\mathcal{C}}(\mathsf{Groth}_\infty)$ which is right adjoint to the inclusion, which completes the proof of Lemma D.5.4.5.

It remains to prove $(*')$. Let $\lambda : X' \to X$ be as in $(*')$, and let Y be any object of $\mathcal{O}_0^{\otimes}$; we wish to show that composition with λ induces a homotopy equivalence $\mathrm{Map}_{\mathcal{O}_1^{\otimes}}(Y, X) \to \mathrm{Map}_{\mathcal{O}_1^{\otimes}}(Y, X)$. Using the fact that $\mathcal{O}_1^{\otimes}$ is an ∞-operad, we can reduce to the case where $X = (\sigma, \mathcal{E}) \in \mathcal{O}_1$. If $\sigma = \mathfrak{a}$, then λ is an equivalence and there is nothing to prove. Let us therefore assume that $\sigma = \mathfrak{m}$, and let $\lambda : \breve{\mathcal{E}} \to \mathcal{E}$ be as in Proposition C.5.5.9. Unwinding the definitions, we wish to prove the following:

$(*''')$ Suppose we are given Grothendieck prestable ∞-categories $\mathcal{D}_1, \ldots, \mathcal{D}_k$ and $\mathcal{D}$, where $\mathcal{D}_i$ is compactly generated for $1 \leqslant i \leqslant k$ and $\mathcal{D}$ is anticomplete. Let $\mathrm{Fun}^{\rho}(\mathcal{D}_1 \times \cdots \times \mathcal{D}_k \times \mathcal{D}, \mathcal{E})$ be the full subcategory of $\mathrm{Fun}(\mathcal{D}_1 \times \cdots \times \mathcal{D}_k \times \mathcal{D}, \mathcal{E})$ spanned by those functors f which preserve small colimits separately in each variable and have the property that, for every sequence of compact objects $\{D_i \in \mathcal{D}_i\}_{1 \leqslant i \leqslant k}$, the induced map $f(D_1, \ldots, D_k, \bullet) : \mathcal{D} \to \mathcal{E}$ has bounded amplitude. Define $\mathrm{Fun}^{\rho}(\mathcal{D}_1 \times \cdots \times \mathcal{D}_k \times \mathcal{D}, \breve{\mathcal{E}})$ similarly. Then composition with λ induces a homotopy equivalence

$$\mathrm{Fun}^{\rho}(\mathcal{D}_1 \times \cdots \times \mathcal{D}_k \times \mathcal{D}, \breve{\mathcal{E}})^{\simeq} \to \mathrm{Fun}^{\rho}(\mathcal{D}_1 \times \cdots \times \mathcal{D}_k \times \mathcal{D}, \mathcal{E})^{\simeq}.$$

In fact, in the situation of $(*''')$, we will show that λ induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\rho}(\mathcal{D}_1 \times \cdots \times \mathcal{D}_k \times \mathcal{D}, \breve{\mathcal{E}}) \to \mathrm{Fun}^{\rho}(\mathcal{D}_1 \times \cdots \times \mathcal{D}_k \times \mathcal{D}, \mathcal{E}).$$

To prove this, we let $\mathcal{D}_i^c$ denote the full subcategory of $\mathcal{D}_i$ spanned by the compact objects for $1 \leqslant i \leqslant k$, let $\mathrm{LFun}^b(\mathcal{D}, \mathcal{E})$ be the full subcategory of $\mathrm{LFun}(\mathcal{D}, \mathcal{E})$ spanned by the functors of bounded amplitude, and define $\mathrm{LFun}^b(\mathcal{D}, \breve{\mathcal{E}})$ similarly. Restriction to compact objects then yields a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}^{\rho}(\mathcal{D}_1 \times \cdots \times \mathcal{D}_k \times \mathcal{D}, \breve{\mathcal{E}}) & \longrightarrow & \mathrm{Fun}^{\rho}(\mathcal{D}_1 \times \cdots \times \mathcal{D}_k \times \mathcal{D}, \mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{\mathrm{rex}}(\mathcal{D}_1^c \times \cdots \times \mathcal{D}_k^c, \mathrm{LFun}^b(\mathcal{D}, \breve{\mathcal{E}})) & \longrightarrow & \mathrm{Fun}^{\mathrm{rex}}(\mathcal{D}_1^c \times \cdots \times \mathcal{D}_k^c, \mathrm{LFun}^b(\mathcal{D}, \mathcal{E})) \end{array}$$

where $\mathrm{Fun}^{\mathrm{rex}}(\mathcal{D}_1^c \times \cdots \times \mathcal{D}_k^c, \mathrm{LFun}^b(\mathcal{D}, \mathcal{E}))$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{D}_1^c \times \cdots \times \mathcal{D}_k^c, \mathrm{LFun}^b(\mathcal{D}, \mathcal{E}))$ spanned by those functors which preserve finite colimits separately in each variable, and $\mathrm{Fun}^{\mathrm{rex}}(\mathcal{D}_1^c \times \cdots \times \mathcal{D}_k^c, \mathrm{LFun}^b(\mathcal{D}, \breve{\mathcal{E}}))$ is defined similarly. Since each $\mathcal{D}_i$ is compactly generated, the vertical maps are equivalences of ∞-categories. We now complete the proof of $(*''')$ by observing that the bottom horizontal map is also an equivalence, since composition with λ induces an equivalence $\mathrm{LFun}(\mathcal{D}, \breve{\mathcal{E}}) \to \mathrm{LFun}(\mathcal{D}, \mathcal{E})$ by virtue of our assumption that $\mathcal{D}$ is anticomplete. □

Lemma D.5.4.8. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable R-linear ∞-category. Suppose that $M \in \mathrm{Mod}_R^{\mathrm{cn}}$ has* Tor*-amplitude $\leqslant m$ and that $C \in \mathcal{C}$ is n-truncated. Then the tensor product $M \otimes_R C$ is $(m+n)$-truncated.*

Proof. We proceed by induction on m. If $m = 0$, then M is a flat R-module. Applying Theorem HA.7.2.2.15 , we can write M as a filtered colimit $\varinjlim_\alpha M_\alpha$, where each M_α is a free R-module of finite rank. Each of the tensor products $M_\alpha \otimes_R C$ is a direct sum of finitely many copies of C, and is therefore n-truncated. Since $\mathcal{C}$ is a Grothendieck prestable ∞-category, it follows that the $M \otimes_R C \simeq \varinjlim (M_\alpha \otimes_R C)$ is also n-truncated.

We now carry out the inductive step. Assume that $m > 0$, and choose a free R-module F and a map $\alpha : F \to M$ which is surjective on π_0. Then we have a cofiber sequence $\mathrm{fib}(\alpha) \to F \to M$ in the ∞-category $\mathrm{Mod}_R^{\mathrm{cn}}$, where $\mathrm{fib}(\alpha)$ and F have Tor-amplitude $\leqslant m-1$. Tensoring with the object C, we obtain a cofiber sequence

$$\mathrm{fib}(\alpha) \otimes_R C \to F \otimes_R C \to M \otimes_R C$$

in the ∞-category $\mathcal{C}$. Our inductive hypothesis implies that the first two terms of this sequence are $(m+n-1)$-truncated, so the third term must be $(m+n)$-truncated. □

Proof of Theorem D.5.4.1. Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings for which R' is flat over R and of finite Tor-amplitude over $R' \otimes_R R'$, and let $\mathcal{C}$ be a anticomplete prestable R-linear ∞-category. We wish to show that $\mathcal{C}' = R' \otimes_R \mathcal{C}$ is also anticomplete. We will prove this by verifying that $R' \otimes_R \mathcal{C}$ satisfies condition (b) of Corollary D.5.4.7. Let $\mathcal{D}$ be a prestable R'-linear ∞-category and let $\widehat{\mathcal{D}}$ be its completion; we wish to show that the canonical map

$$\theta : \mathrm{Map}_{\mathrm{LinCat}_{R'}^{\mathrm{PSt,lex}}}(\mathcal{C}', \mathcal{D}) \to \mathrm{Map}_{\mathrm{LinCat}_{R'}^{\mathrm{PSt,lex}}}(\mathcal{C}', \widehat{\mathcal{D}})$$

is a homotopy equivalence.

Let $\lambda : \mathcal{C} \to \mathcal{C}'$ denote the R-linear functor given by $\lambda(C) = R' \otimes_R C$, so that precomposition with λ induces a homotopy equivalence $\rho : \mathrm{Map}_{\mathrm{LinCat}_{R'}^{\mathrm{PSt}}}(\mathcal{C}', \mathcal{D}) \simeq \mathrm{Map}_{\mathrm{LinCat}_{R}^{\mathrm{PSt}}}(\mathcal{C}, \mathcal{D})$. Since R' is flat over R, the functor λ is left exact, so that ρ restricts to a fully faithful embedding $\rho^{\mathrm{lex}} : \mathrm{Map}_{\mathrm{LinCat}_{R'}^{\mathrm{PSt,lex}}}(\mathcal{C}', \mathcal{D}) \simeq \mathrm{Map}_{\mathrm{LinCat}_{R}^{\mathrm{PSt,lex}}}(\mathcal{C}, \mathcal{D})$. Similarly, we have a fully faithful embedding $\widehat{\rho}^{\mathrm{lex}} : \mathrm{Map}_{\mathrm{LinCat}_{R'}^{\mathrm{PSt,lex}}}(\mathcal{C}', \widehat{\mathcal{D}}) \simeq \mathrm{Map}_{\mathrm{LinCat}_{R}^{\mathrm{PSt,lex}}}(\mathcal{C}, \widehat{\mathcal{D}})$. These maps fit into a commutative diagram σ :

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{LinCat}_{R'}^{\mathrm{PSt,lex}}}(\mathcal{C}', \mathcal{D}) & \xrightarrow{\theta} & \mathrm{Map}_{\mathrm{LinCat}_{R'}^{\mathrm{PSt,lex}}}(\mathcal{C}', \widehat{\mathcal{D}}) \\ \downarrow{\scriptstyle \rho^{\mathrm{lex}}} & & \downarrow{\scriptstyle \widehat{\rho}^{\mathrm{lex}}} \\ \mathrm{Map}_{\mathrm{LinCat}_{R}^{\mathrm{PSt,lex}}}(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathrm{Map}_{\mathrm{LinCat}_{R}^{\mathrm{PSt,lex}}}(\mathcal{C}, \widehat{\mathcal{D}}), \end{array}$$

where the bottom horizontal map is a homotopy equivalence by virtue of our assumption that $\mathcal{C}$ is anticomplete (Corollary D.5.4.7). To complete the proof, it will suffice to show that σ is a pullback square. Unwinding the definitions, this is equivalent to the following assertion:

$(*)$ Let $F : \mathcal{C}' \to \mathcal{D}$ be an R'-linear functor. If both of the composite functors

$$\mathcal{C} \xrightarrow{\lambda} \mathcal{C}' \xrightarrow{F} \mathcal{D} \qquad \mathcal{C}' \xrightarrow{F} \mathcal{D} \to \widehat{\mathcal{D}}$$

are left exact, then F is left exact.

To show that the functor $F : \mathcal{C}' \to \mathcal{D}$ appearing in $(*)$ is left exact, it is enough to show that for each discrete object $C \in \mathcal{C}'$, the image $F(C) \in \mathcal{D}$ is also discrete (Proposition C.3.2.1). Set $A = R' \otimes_R R'$ so that we have two natural maps $\iota_0, \iota_1 : R' \to A$. Extension of scalars along ι_0 determines an exact functor $\mu : \mathrm{LMod}_{R'}(\mathcal{C}) \to \mathrm{LMod}_A(\mathcal{C})$ and restriction of scalars along ι_1 determines an exact functor $\nu : \mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_{R'}(\mathcal{C})$, and the composition $\nu \circ \mu$ is homotopic to the composition $\mathrm{LMod}_{R'}(\mathcal{C}) \to \mathcal{C} \xrightarrow{\lambda} \mathrm{LMod}_{R'}(\mathcal{C})$. Consequently, if $F \circ \lambda$ is left exact, then $F \circ \nu \circ \mu$ is left exact. It follows that $(F \circ \nu \circ \mu)(C) = F(A \otimes_R C)$ is a discrete A-module object of $\mathcal{D}$. Applying Lemma D.5.4.8 (and our assumption that R

has finite Tor-amplitude as an A-module), we conclude that $F(C) \simeq R \otimes_A F(A \otimes_R C)$ is a truncated object of $\mathcal{D}$. Since the completion map $\mathcal{D} \to \widehat{\mathcal{D}}$ is an equivalence when restricted to truncated objects and the image of $F(C)$ in $\widehat{\mathcal{D}}$ belongs to the heart of $\widehat{\mathcal{D}}$, it follows that $F(C) \in \mathcal{D}^{\heartsuit}$, as desired. □

We now prove a converse to Theorem D.5.4.1:

Theorem D.5.4.9. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings which satisfies the following conditions:*

(i) The morphism ϕ is flat.

(ii) The $\mathbb{E}_\infty$-ring R' has finite Tor*-amplitude when regarded as a module over $R' \otimes_R R'$.*

(iii) The map ϕ is a universal descent morphism.

Let $\mathcal{C}$ be a prestable R-linear ∞-category. If $R' \otimes_R \mathcal{C}$ is anticomplete, then $\mathcal{C}$ is anticomplete.

Remark D.5.4.10. Hypotheses (i), (ii) and (iii) of Theorem D.5.4.9 are satisfied if ϕ is étale and faithfully flat, or more generally if ϕ is fiber-smooth and faithfully flat (Proposition 11.3.3.1 and Remark 11.2.3.3).

Proof of Theorem D.5.4.9. We will prove that $\mathcal{C}$ is anticomplete by showing that it satisfies criterion (b) of Corollary D.5.4.7. Let $\mathcal{D}$ be a prestable R-linear ∞-category and let $\widehat{\mathcal{D}}$ denote its completion; we wish to show that the canonical map

$$\theta : \mathrm{Map}_{\mathrm{LinCat}_R^{\mathrm{PSt,lex}}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}_{\mathrm{LinCat}_R^{\mathrm{PSt,lex}}}(\mathcal{C}, \widehat{\mathcal{D}})$$

is a homotopy equivalence. Let $R^\bullet$ denote the Čech nerve of the morphism ϕ (formed in the ∞-category $\mathrm{CAlg}^{\mathrm{op}}$). Using hypotheses (i) and (iii), Theorem D.4.1.6, and Proposition D.5.2.1, we conclude that θ can be obtained as the totalization of a map of cosimplicial spaces

$$\theta^\bullet : \mathrm{Map}_{\mathrm{LinCat}_{R^\bullet}^{\mathrm{PSt,lex}}}(\mathrm{LMod}_{R^\bullet}(\mathcal{C}), \mathrm{LMod}_{R^\bullet}(\mathcal{D})) \to \mathrm{Map}_{\mathrm{LinCat}_R^{\mathrm{PSt,lex}}}(\mathrm{LMod}_{R^\bullet}(\mathcal{C}), \mathrm{LMod}_{R^\bullet}(\widehat{\mathcal{D}}))$$

We will complete the proof by showing that each θ^k is a homotopy equivalence. By virtue of Corollary D.5.4.7, it will suffice to show that the Grothendieck prestable ∞-category $\mathrm{LMod}_{R^k}(\mathcal{C})$ is anticomplete. This follows from Theorem D.5.4.1, since R^k is flat over R and of finite Tor-amplitude over $R^k \otimes_R R^k$. □

D.5.5 Weakly Coherent Prestable ∞-Categories

We now consider descent for the property of weak coherence (see Definition C.6.5.1).

Theorem D.5.5.1. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings which is almost of finite presentation, and suppose that the underlying ring homomorphism $\pi_0 R \to \pi_0 R'$ is quasi-finite (Definition B.2.4.1). Let $\mathcal{C}$ be an R-linear prestable ∞-category. If $\mathcal{C}$ is weakly coherent, then $R' \otimes_R \mathcal{C}$ is weakly coherent.*

The proof of Theorem D.5.5.1 will require a number of preliminaries. We begin by studying the special case where $R' \simeq R[t^{-1}]$ is a localization of R.

Lemma D.5.5.2. *Let R be a connective $\mathbb{E}_\infty$-ring and let t be an element of $\pi_0 R$. Let $\mathcal{C}$ be an additive R-linear ∞-category which is compactly generated and let $\mathcal{C}[t^{-1}]$ denote the ∞-category $R[t^{-1}] \otimes_R \mathcal{C} \simeq \mathrm{LMod}_{R[t^{-1}]}(\mathcal{C})$. Then:*

- *(a) The ∞-category $\mathcal{C}[t^{-1}]$ is compactly generated.*
- *(b) An object $C \in \mathcal{C}[t^{-1}]$ is compact if and only if it is a direct summand of an object of the form $C_0[t^{-1}]$, for some compact object $C_0 \in \tau_{\leqslant n} \mathcal{C}$.*

Proof. Let $\mathcal{E} \subseteq \mathcal{C}[t^{-1}]$ be the full subcategory spanned by objects of the form $C_0[t^{-1}]$, where C_0 is a compact object of $\mathcal{C}$. By virtue of Lemma D.5.3.3, it will suffice to show that $\mathcal{E}$ is closed under the formation of cofibers. To prove this, let $f : C_0[t^{-1}] \to D_0[t^{-1}]$ be a morphism in $\mathcal{E}$. The compactness of C_0 guarantees that, for $k \gg 0$, the map $t^k f : C_0[t^{-1}] \to D_0[t^{-1}]$ is obtained by localizing a map $f_0 : C_0 \to D_0$ in $\mathcal{C}$, so that $\mathrm{cofib}(f) \simeq \mathrm{cofib}(t^k f) \simeq \mathrm{cofib}(f_0)[t^{-1}]$. □

Remark D.5.5.3. In the situation of Lemma D.5.5.2, suppose that $\mathcal{C}$ is an n-category for some $n < \infty$. Then every compact object of $\mathcal{C}[t^{-1}]$ has the form $C_0[t^{-1}]$, where C_0 is a compact object of $\mathcal{C}$; see Proposition HTT.**??**.

Lemma D.5.5.4. *Let R be a connective $\mathbb{E}_\infty$-ring and let t be an element of $\pi_0 R$. Let $\mathcal{C}$ be a weakly coherent prestable R-linear ∞-category. Then $\mathcal{C}[t^{-1}]$ is also weakly coherent.*

Proof. For each integer n, the ∞-category $\tau_{\leqslant n} \mathcal{C}$ is compactly generated. It follows from Lemma D.5.5.2 that the ∞-category $\tau_{\leqslant n} \mathcal{C}[t^{-1}]$ is also compactly generated. Moreover, Remark D.5.5.3 shows that an object of $\tau_{\leqslant n} \mathcal{C}[t^{-1}]$ is compact if and only if it has the form $C_0[t^{-1}]$, where C_0 is a compact object of $\tau_{\leqslant n} \mathcal{C}$. Arguing as in the proof of Lemma D.5.5.2, we see that every morphism $f : C \to D$ between compact objects of $\tau_{\leqslant n} \mathcal{C}[t^{-1}]$ is obtained from a morphism $f_0 : C_0 \to D_0$ between compact objects of $\tau_{\leqslant n} \mathcal{C}$ by applying the localization functor $M \mapsto M[t^{-1}]$. Our assumption that $\mathcal{C}$ is weakly coherent guarantees that $\mathrm{fib}(f_0)$ is a compact object of $\tau_{\leqslant n} \mathcal{C}$, so that $\mathrm{fib}(f) \simeq \mathrm{fib}(f_0)[t^{-1}]$ is a compact object of $\tau_{\leqslant n} \mathcal{C}[t^{-1}]$. It follows that the collection of compact objects of $\tau_{\leqslant n} \mathcal{C}[t^{-1}]$ is closed under finite limits. Applying Proposition C.6.5.4, we deduce that $\mathcal{C}[t^{-1}]$ is weakly coherent. □

Lemma D.5.5.5. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable R-linear ∞-category which satisfies the following conditions:*

(a) *For every integer n, the ∞-category $\tau_{\leqslant n}\mathcal{C}$ is compactly generated.*

(b) *There exists a collection of elements $t_1, t_2, \ldots, t_k \in \pi_0 R$ which generate the unit ideal such that each $\mathcal{C}[t_i^{-1}]$ is weakly coherent.*

Then $\mathcal{C}$ is weakly coherent.

Proof. By virtue of Proposition C.6.5.4, it will suffice to show that for each $n \geqslant 0$, the collection of compact objects of $\tau_{\leqslant n}\mathcal{C}$ is closed under finite limits. Fix a morphism $f : C \to D$ between compact objects of $\tau_{\leqslant n}\mathcal{C}$; we wish to show that $\mathrm{fib}(f)$ is compact. By virtue of Remark D.5.2.3, it will suffice to show that the localization $\mathrm{fib}(f)[t_i^{-1}]$ is a compact object of $\tau_{\leqslant n}\mathcal{C}[t_i^{-1}]$ for $1 \leqslant i \leqslant k$, which follows from the weak coherence of $\mathcal{C}[t_i^{-1}]$. □

We now consider a special case of Theorem D.5.5.1 which lies at the opposite extreme.

Lemma D.5.5.6. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable R-linear ∞-category. Let M be a connective R-module and let $C \in \mathcal{C}$. If M is almost perfect and C is almost compact, then the object $M \otimes_R C$ is almost compact.*

Proof. Using Proposition HA.**??** , we can write M as the geometric realization of a simplicial object $P_\bullet$ of Mod_R where each P_m is a free R-module of finite rank. Then $M \otimes_R C$ can be identified with the geometric realization $|P_\bullet \otimes_R C|$, which is almost compact by virtue of Proposition C.6.4.4. □

Lemma D.5.5.7. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings which exhibits R' as an almost perfect module over R. Then:*

(a) *If $C \in \mathrm{LMod}_{R'}(\mathcal{C})$ is an object whose image under the forgetful functor $\mathrm{LMod}_{R'}(\mathcal{C}) \to \mathcal{C}$ is an almost compact object of $\mathcal{C}$, then C is an almost compact as an object of $\mathrm{LMod}_{R'}(\mathcal{C})$.*

(b) *Suppose that $\tau_{\leqslant n}\mathcal{C}$ is compactly generated ∞-category for each $n \geqslant 0$. If $C \in \mathrm{LMod}_{R'}(\mathcal{C})$ is almost compact, then the image of C under the forgetful functor $\mathrm{LMod}_{R'}(\mathcal{C}) \to \mathcal{C}$ is almost compact.*

Proof. To prove (a), we note that $C \simeq R' \otimes_{R'} C$ can be written as the geometric realization of two-sided bar construction $\mathrm{Bar}_{R'}(R', C)_\bullet$. By virtue of Proposition C.6.4.4, it will suffice to show that each $\mathrm{Bar}_{R'}(R', C)_k$ is an almost compact object of $\mathrm{LMod}_{R'}(\mathcal{C})$. The extension of scalars functor $\mathcal{C} \to \mathrm{LMod}_{R'}(\mathcal{C})$ is compact, and therefore sends almost compact objects to almost compact objects. It will therefore suffice to show that each iterated tensor product

$R' \otimes_R R' \otimes_R \cdots \otimes_R C$ is an almost compact object of $\mathcal{C}$, which follows from Lemma D.5.5.6 provided that C is almost compact when viewed as an object of $\mathcal{C}$.

We will deduce (b) from the following more precise assertions:

(b_n) Suppose that $\tau_{\leqslant n}\mathcal{C}$ is compactly generated ∞-category for some $n \geqslant 0$. If C is a compact object of $\tau_{\leqslant n}\operatorname{LMod}_{R'}(\mathcal{C})$, then the image of C in $\tau_{\leqslant n}\mathcal{C}$ is compact.

To prove (b_n), let $\mathcal{E}$ be the full subcategory of $\tau_{\leqslant n}\operatorname{LMod}_{R'}(\mathcal{C})$ spanned by those objects C whose image in $\tau_{\leqslant n}\mathcal{C}$ is compact; we wish to show that $\mathcal{E}$ contains all compact objects of $\tau_{\leqslant n}\operatorname{LMod}_{R'}(\mathcal{C})$. Since $\mathcal{E}$ is evidently closed under retracts and cofibers (formed in the ∞-category $\tau_{\leqslant n}\operatorname{LMod}_{R'}(\mathcal{C})$), it will suffice to show that $\mathcal{E}$ contains every object of the form $\tau_{\leqslant n}(R' \otimes_R C_0)$, where C_0 is a compact object of $\tau_{\leqslant n}\mathcal{C}$ (see Lemma D.5.3.3). This is clear, since $\tau_{\leqslant n}(R' \otimes_R C_0)$ can be built from C_0 using finite colimits. □

Lemma D.5.5.8. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings which exhibits R' as an almost perfect module over R, and let $\mathcal{C}$ be a prestable R-linear ∞-category. If $\mathcal{C}$ is weakly coherent, then $\operatorname{LMod}_{R'}(\mathcal{C})$ is weakly coherent.*

Proof. We show that $\operatorname{LMod}_{R'}(\mathcal{C})$ satisfies the requirements of Definition C.6.5.1. It follows from Lemma D.5.5.7 that an object $C \in \operatorname{LMod}_{R'}(\mathcal{C})$ is almost compact if and only if its image in $\mathcal{C}$ is almost compact. Using the weak coherence of $\mathcal{C}$, we see that the collection of almost compact objects of $\operatorname{LMod}_{R'}(\mathcal{C})$ is closed under finite limits. If X is a truncated object of $\operatorname{LMod}_{R'}(\mathcal{C})$, then the weak coherence of $\mathcal{C}$ implies that there exists a morphism $\bigoplus C_\alpha \to X$ in the ∞-category $\mathcal{C}$ which is an epimorphism on π_0, where each C_α is an almost compact object of $\mathcal{C}$. Then the induced map $\bigoplus(R' \otimes_R C_\alpha) \to X$ is a morphism in the ∞-category $\operatorname{LMod}_{R'}(\mathcal{C})$ which induces an epimorphism on π_0, where each $R' \otimes_R C_\alpha$ is an almost compact object of $\operatorname{LMod}_{R'}(\mathcal{C})$. □

We will need the following weak converse:

Lemma D.5.5.9. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings and let $\mathcal{C}$ be a prestable R-linear ∞-category. Assume that:*

(i) For each $n \geqslant 0$, the ∞-category $\tau_{\leqslant n}\mathcal{C}$ is compactly generated.

(ii) The morphism ϕ exhibits R' as an almost perfect R-module.

(iii) The morphism ϕ induces an isomorphism of commutative rings $\pi_0 R \to \pi_0 R'$.

(iv) The Grothendieck prestable ∞-category $\operatorname{LMod}_{R'}(\mathcal{C})$ is weakly coherent.

Then $\mathcal{C}$ is weakly coherent.

Proof. Let X be an almost compact object of $\mathcal{C}$. Then $R' \otimes_R X$ is an almost compact object of $\mathrm{LMod}_{R'}(\mathcal{C})$. Using assumption (iv), we deduce that $\pi_0(R' \otimes_R X)$ is an almost compact object of $\mathrm{LMod}_{R'}(\mathcal{C})$. Using (i), (ii), and Lemma D.5.5.7, we see that $\pi_0(R' \otimes_R X)$ is almost compact when viewed as an object of $\mathcal{C}$. Using (iii), we see that the canonical map $\pi_0(R' \otimes_R X) \to \pi_0 X$ is an isomorphism in $\mathcal{C}^{\heartsuit}$, so that $\pi_0 X$ is an almost compact object of $\mathcal{C}$. Invoking Proposition C.6.4.5, we deduce that the collection of almost compact objects of $\mathcal{C}$ is closed under finite limits. To complete the proof, it will suffice (by virtue of Proposition C.6.5.6) to show that every object $C \in \mathcal{C}^{\heartsuit}$ can be written as a filtered colimit $\varinjlim C_\alpha$, where each $C_\alpha \in \mathcal{C}^{\heartsuit}$ is almost compact when viewed as an object of $\mathcal{C}$. Assumption (iii) implies that C admits an essentially unique R'-module structure, so that the weak coherence of $\mathrm{LMod}_{R'}(\mathcal{C})$ implies that we can choose such an equivalence $C \simeq \varinjlim C_\alpha$ where each C_α is almost compact when viewed as an object of $\mathrm{LMod}_{R'}(\mathcal{C})$. It follows from (i), (ii), and Lemma D.5.5.7 that each C_α is also almost compact when viewed as an object of $\mathcal{C}$. □

Proof of Theorem D.5.5.1. Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings and let $\mathcal{C}$ be a weakly coherent prestable R-linear ∞-category. Assume that ϕ is almost of finite presentation and that the underlying homomorphism of commutative rings $R \to R'$ is quasi-finite. We wish to show that $\mathrm{LMod}_{R'}(\mathcal{C})$ is also weakly coherent.

Since $\pi_0 R'$ is finitely generated as an algebra over $\pi_0 R$, we can extend ϕ to a map $R\{x_1, \ldots, x_n\} \to R'$ which induces a surjection

$$\pi_0(R\{x_1, \ldots, x_n\}) \simeq (\pi_0 R)[x_1, \ldots, x_n] \xrightarrow{\rho} \pi_0 R';$$

here $R\{x_1, \ldots, x_n\}$ denotes the free $\mathbb{E}_\infty$-algebra over R on n generators. Since $\pi_0 R'$ is finitely presented over $\pi_0 R$, the kernel of ρ is generated by finitely many polynomials $\{g_i(x_1, \ldots, x_n)\}_{1 \leqslant i \leqslant m}$. These polynomials classify a morphism $\vec{g} : R\{y_1, \ldots, y_m\} \to R\{x_1, \ldots, x_n\}$ of $\mathbb{E}_\infty$-algebras over R. Let $\overline{R} = R\{x_1, \ldots, x_n\} \otimes_{R\{y_1, \ldots, y_m\}} R$ denote the $\mathbb{E}_\infty$-algebra over R obtained from $R\{x_1, \ldots, x_n\}$ by "killing" the polynomials $g_i(x_1, \ldots, x_m)$, so that ϕ factors as a composition

$$R \xrightarrow{\phi'} \overline{R} \xrightarrow{\phi''} R'.$$

By construction, ϕ' is almost of finite presentation and ϕ'' induces an isomorphism on π_0. Since ϕ is almost of finite presentation, it follows that ϕ'' is also locally almost of finite presentation. Using Corollary 5.2.2.2, we see that ϕ'' exhibits R' as an almost perfect $\overline{R}$-module. Consequently, in order to show that $R' \otimes_R \mathcal{C}$ is weakly coherent, it will suffice to show that $\overline{R} \otimes_R \mathcal{C}$ is weakly coherent (Lemma D.5.5.8). We may therefore replace R' by $\overline{R}$

and thereby reduce to the case where the morphism ϕ fits into a pushout diagram σ :

$$\begin{array}{ccc} R\{y_1,\ldots,y_m\} & \longrightarrow & R \\ \downarrow{\scriptstyle \vec{g}} & & \downarrow{\scriptstyle \phi} \\ R\{x_1,\ldots,x_n\} & \longrightarrow & R'; \end{array}$$

here the upper horizontal map is given by $y_i \to 0$.

Write R as the colimit of a diagram $\{R_\alpha\}_{\alpha \in A}$ indexed by a filtered partially ordered set A, where each R_α is a compact object of $\mathrm{CAlg}^{\mathrm{cn}}$. For α sufficiently large, we can arrange that all of the coefficients of the polynomials $g_i(x_1,\ldots,x_n)$ can be lifted to elements of $\pi_0 R_\alpha$, so that σ can be lifted to a pushout square σ_α :

$$\begin{array}{ccc} R_\alpha\{y_1,\ldots,y_m\} & \longrightarrow & R_\alpha \\ \downarrow & & \downarrow{\scriptstyle \phi_\alpha} \\ R_\alpha\{x_1,\ldots,x_n\} & \longrightarrow & R'_\alpha. \end{array}$$

Enlarging α further, we can assume that the ring homomorphism $\pi_0 R_\alpha \to \pi_0 R'_\alpha$ is quasi-finite. In this case, we have a canonical equivalence $R' \otimes_R \mathcal{C} \simeq R'_\alpha \otimes_{R_\alpha} \mathcal{C}$. We may therefore replace ϕ by ϕ_α and thereby reduce to the case where R is Noetherian (so that R' is also Noetherian).

The assumption that R' is Noetherian guarantees that the commutative ring $\pi_0 R'$ is almost perfect when viewed as an R'-module (Proposition HA.7.2.4.17). By virtue of Lemma D.5.5.9, to show that $R' \otimes_R \mathcal{C}$ is weakly coherent, it will suffice to show that $(\pi_0 R') \otimes_R \mathcal{C}$ is weakly coherent. We may therefore replace R' by $\pi_0 R'$ and thereby reduce to the case where R' is discrete.

Applying the affine version of Zariski's main theorem (in the form of Theorem B.2.4.5), we see that the map $\pi_0 R \to R'$ factors as a composition $\pi_0 R \to B \xrightarrow{\psi} R'$ where B is a commutative ring which is finitely generated as a module over $\pi_0 R$ and the map ψ induces an open immersion of affine schemes $j : \operatorname{Spec} R' \to \operatorname{Spec} B$. Because R is Noetherian, Proposition HA.7.2.4.17 implies that B is almost perfect when viewed as an R-module, so that Lemma D.5.5.8 guarantees that $B \otimes_R \mathcal{C}$ is weakly coherent. The complement of the image of j is a closed subset of $\operatorname{Spec} B$, which we can identify with the vanishing locus of a finitely generated ideal $(b_1,\ldots,b_k)$. Applying Lemma D.5.5.4, we deduce that each of the ∞-categories $B[b_i^{-1}] \otimes_R \mathcal{C} \simeq R'[b_i^{-1}] \otimes_R \mathcal{C}$ is weakly coherent. Since the elements b_i generate the unit ideal in R', Lemma D.5.5.5 guarantees that $R' \otimes_R \mathcal{C}$ is also weakly coherent, as desired. □

Let us now list some consequences of Theorem D.5.5.1.

Corollary D.5.5.10. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings which is almost of finite presentation, and suppose that the underlying ring homomorphism $\pi_0 R \to \pi_0 R'$ is quasi-finite (Definition B.2.4.1). Let $\mathcal{C}$ be an R-linear prestable ∞-category. If $\mathcal{C}$ is coherent, then $R' \otimes_R \mathcal{C}$ is coherent.*

Proof. It follows from Theorem D.5.5.1 that $R' \otimes_R \mathcal{C}$ is weakly coherent. Let X be an object of $R' \otimes_R \mathcal{C} \simeq \mathrm{LMod}_{R'}(\mathcal{C})$. Since $\mathcal{C}$ is coherent, we can choose a morphism $u : \bigoplus C_\alpha \to X$ in $\mathcal{C}$, where each C_α is an almost compact object of $\mathcal{C}$ and $\pi_0(u)$ is an epimorphism in the abelian category $\mathcal{C}^\heartsuit$. Extending scalars, we see that u classifies a morphism $u_{R'} : \bigoplus (R' \otimes_R C_\alpha) \to X$ in $\mathrm{LMod}_{R'}(\mathcal{C})$ The map $u_{R'}$ is also an epimorphism on π_0 (since this can be checked after applying the forgetful functor $\mathrm{LMod}_{R'}(\mathcal{C}) \to \mathcal{C}$), and each $R' \otimes_R C_\alpha$ is an almost compact object $\mathrm{LMod}_{R'}(\mathcal{C})$ (since the restriction-of-scalars functor $\mathrm{LMod}_{R'}(\mathcal{C}) \to \mathcal{C}$ commutes with filtered colimits). Allowing X to vary, we deduce that $R' \otimes_R \mathcal{C}$ is coherent. □

We also have the following stronger version of Lemma D.5.5.5:

Corollary D.5.5.11. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings which is étale and faithfully flat, and let $\mathcal{C}$ be a prestable R-linear ∞-category. If $R' \otimes_R \mathcal{C}$ is weakly coherent, then $\mathcal{C}$ is weakly coherent.*

Proof. Applying Lemma D.5.4.5, we can choose a anticomplete prestable R-linear ∞-category $\breve{\mathcal{C}}$ and a left exact R-linear functor $\lambda : \breve{\mathcal{C}} \to \mathcal{C}$ which induces an equivalence of completions. Then λ induces an R'-linear functor $\lambda_{R'} : R' \otimes_R \breve{\mathcal{C}} \to R' \otimes_R \mathcal{C}$. Note that the functor $\lambda_{R'}$ is also left exact and induces an equivalence of completions. Applying Corollary C.6.5.5, we see that $R' \otimes_R \breve{\mathcal{C}}$ is weakly coherent. Using Theorem D.5.4.1, we see that $R' \otimes_R \breve{\mathcal{C}}$ is anticomplete. Applying Theorem C.6.7.1, we see that $R' \otimes_R \breve{\mathcal{C}}$ is compactly generated, that every compact object of $R' \otimes_R \breve{\mathcal{C}}$ is truncated, and that the collection of compact objects of $R' \otimes_R \breve{\mathcal{C}}$ is closed under finite limits. Theorem D.5.3.1 then guarantees that $\breve{\mathcal{C}}$ is compactly generated, and Remark D.5.2.3 implies that an object $C \in \breve{\mathcal{C}}$ is compact if and only if the object $R' \otimes_R C \in \mathrm{LMod}_{R'}(\breve{\mathcal{C}})$ is compact. Since the extension-of-scalars functor $\breve{\mathcal{C}} \to \mathrm{LMod}_{R'}(\breve{\mathcal{C}})$ is left exact and conservative, it follows the collection of compact objects of $\breve{\mathcal{C}}$ is closed under finite limits and that every compact object of $\breve{\mathcal{C}}$ is truncated. Using Theorem C.6.7.1 again, we conclude that $\breve{\mathcal{C}}$ is weakly coherent, so that $\mathcal{C}$ is also weakly coherent (Corollary C.6.5.5). □

Remark D.5.5.12. We do not know if the analogue of Corollary D.5.5.11 holds for coherent prestable ∞-categories.

Corollary D.5.5.13. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings which is étale and faithfully flat. If R' is coherent, then R is coherent.*

Proof. Combine Corollary D.5.5.11 with Example C.6.5.3. □

D.5.6 Locally Noetherian Prestable ∞-Categories

Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings which is almost of finite presentation. It follows from Proposition HA.7.2.4.31 that if R is Noetherian, then R' is Noetherian. We establish a relative version of this assertion:

Proposition D.5.6.1. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings and let $\mathcal{C}$ be a prestable R-linear ∞-category. If $\mathcal{C}$ is locally Noetherian and ϕ is almost of finite presentation, then $R' \otimes_R \mathcal{C}$ is also locally Noetherian.*

Our starting point is the following variant of the Hilbert basis theorem.

Lemma D.5.6.2. *Let $\mathcal{A}$ be a Grothendieck abelian category containing an object X, let $\mathcal{B} = \mathrm{LMod}_{\mathbf{Z}[t]}(\mathcal{A})$ denote the category of $\mathbf{Z}[t]$-modules in $\mathcal{A}$, and set $X[t] = \mathbf{Z}[t] \otimes_{\mathbf{Z}} X \in \mathcal{B}$. If X is a Noetherian object of $\mathcal{A}$, then $X[t]$ is a Noetherian object of $\mathcal{B}$.*

Proof. For each $n \geqslant 0$, let $\mathbf{Z}[t]_{\leqslant n}$ denote the subset of the polynomial ring $\mathbf{Z}[t]$ consisting of polynomials having degree $\leqslant n$, and let $e_n : \mathbf{Z}[t]_{\leqslant n} \to \mathbf{Z}$ be the map given by $e_n(c_0 + c_1 t + \cdots + c_n t^n) = a_n$. We let $X[t]_{\leqslant n}$ denote the tensor product $\mathbf{Z}[t]_{\leqslant n} \otimes_{\mathbf{Z}} X$, which we will identify with a subobject of $X[t]$ (in the abelian category $\mathcal{A}$).

Let Y be a subobject of $X[t]$ in the abelian category $\mathcal{B}$. For each $n \geqslant 0$, let $Y_{\leqslant n}$ denote the fiber product $Y \times_{X[t]} X[t]_{\leqslant n}$ (formed in the abelian category $\mathcal{A}$). Then e_n induces a map $e_n^Y : Y_{\leqslant n} \to X$ whose kernel can be identified with $Y_{\leqslant n-1}$. Let $S_n(Y)$ denote the image of e_n^Y, which we regard as a subobject of X.

Now suppose we are given an nondecreasing sequence

$$Y(0) \subseteq Y(1) \subseteq Y(2) \subseteq Y(3) \subseteq \cdots$$

of subobjects of $X[t]$ in the abelian category $\mathcal{B}$. Then $\{S_n(Y(n))\}_{n \geqslant 0}$ is a nondecreasing sequence of subobject of X in the abelian category $\mathcal{A}$. Since X is Noetherian, this sequence must eventually stabilize: that is, we can choose an integer $n_0 \gg 0$ such that $S_n(Y(n)) = S_{n_0}(Y(n_0))$ for $n \geqslant n_0$. For each $k < n_0$, the sequence of subobjects $\{S_k(Y(n))\}_{n \geqslant 0}$ must also stabilize. We can therefore choose $n_1 \gg n_0$ such that $S_k(Y(n)) = S_k(Y(n_1))$ for $n \geqslant n_1$ and $k < n_0$. We will complete the proof by showing that $Y(n) = Y(n_1)$ for $n \geqslant n_1$. To prove this, it will suffice to show that we have $Y(n)_{\leqslant k} = Y(n_1)_{\leqslant k}$ for every integer k. This follows by induction on k, by inspecting the diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y(n_1)_{\leqslant k-1} & \longrightarrow & Y(n_1)_{\leqslant k} & \longrightarrow & S_k(Y(n_1)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y(n)_{\leqslant k-1} & \longrightarrow & Y(n)_{\leqslant k} & \longrightarrow & S_k(Y(n)) & \longrightarrow & 0. \end{array}$$

□

Lemma D.5.6.3. *Let R be a commutative ring and let $\mathcal{C}$ be a prestable R-linear ∞-category. If $\mathcal{C}$ is locally Noetherian, then $R[t_1, \ldots, t_m] \otimes_R \mathcal{C}$ is also locally Noetherian for each $m \geqslant 0$.*

Proof. Working by induction on m, we can reduce to the case where $m = 0$. We first note that if $\mathcal{C}$ is locally Noetherian, then $\tau_{\leqslant n} \mathcal{C}$ is compactly generated for each $n \geqslant 0$, so that $\tau_{\leqslant n}(R[t] \otimes_R \mathcal{C})$ is also compactly generated (Lemma D.5.3.3). Let $\mathcal{E} \subseteq \tau_{\leqslant n}(R[t] \otimes_R \mathcal{C})$ be the full subcategory spanned by those objects X for which the homotopy groups $\pi_k X$ are Noetherian objects of the abelian category $\mathrm{LMod}_{R[t]}(\mathcal{C}^{\heartsuit})$ for $0 \leqslant k \leqslant m$. For each compact object $C \in \tau_{\leqslant n} \mathcal{C}$, the objects $\pi_k C \in \mathcal{C}^{\heartsuit}$ are Noetherian for $0 \leqslant k \leqslant n$, so that Lemma D.5.6.2 guarantees that $R[t] \otimes_R C$ belongs to $\mathcal{E}$. The ∞-category $\mathcal{E}$ is clearly closed under retracts, and Corollary C.6.8.3 shows that $\mathcal{E}$ is closed under the formation of cofibers. Applying Lemma D.5.3.3, we see that every compact object of $\tau_{\leqslant n}(R[t] \otimes_R \mathcal{C})$ belongs to $\mathcal{E}$. Applying Proposition C.6.9.7, we deduce that $R[t] \otimes_R \mathcal{C}$ is locally Noetherian. □

Proof of Proposition D.5.6.1. Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings which is almost of finite presentation and let $\mathcal{C}$ be locally Noetherian R-linear prestable ∞-category; we wish to show that $R' \otimes_R \mathcal{C}$ is also locally Noetherian. Since $\pi_0 R'$ is finitely generated as an algebra over $\pi_0 R$, we can factor ϕ as a composition $R \xrightarrow{\phi'} R\{t_1, \ldots, t_n\} \xrightarrow{\phi''} R'$, where $R\{t_1, \ldots, t_n\}$ denotes the free $\mathbb{E}_\infty$-algebra over R on generators $t_1, \ldots, t_n$ and the morphism ϕ'' is surjective on π_0.

Let S denote the sphere spectrum. The canonical map $S \to \pi_0 S \simeq \mathbf{Z}$ exhibits $\mathbf{Z}$ as an almost perfect S-module (Proposition HA.7.2.4.17). Applying Lemma D.5.5.8, we deduce that $\mathbf{Z} \otimes_S \mathcal{C}$ is weakly coherent. Since $(\mathbf{Z} \otimes_S \mathcal{C})^{\heartsuit} \simeq \mathcal{C}^{\heartsuit}$ is a locally Noetherian abelian category, it follows that $\mathbf{Z} \otimes_S \mathcal{C}$ is a locally Noetherian Grothendieck prestable ∞-category. Applying Lemma D.5.6.3, we deduce that $\mathbf{Z}[t_1, \ldots, t_n] \otimes_S \mathcal{C}$ is also locally Noetherian. Note that $S\{t_1, \ldots, t_n\}$ is a Noetherian $\mathbb{E}_\infty$-ring (Proposition HA.7.2.4.31) so that the canonical map $S\{t_1, \ldots, t_n\} \to \pi_0 S\{t_1, \ldots, t_n\} \simeq \mathbf{Z}[t_1, \ldots, t_n]$ exhibits $\mathbf{Z}[t_1, \ldots, t_n]$ as an almost perfect module over $S\{t_1, \ldots, t_n\}$. Applying Lemma D.5.5.9, we conclude that $S\{t_1, \ldots, t_n\} \otimes_S \mathcal{C}$ is weakly coherent. Using the equivalence of hearts $(S\{t_1, \ldots, t_n\} \otimes_S \mathcal{C})^{\heartsuit} \simeq (\mathbf{Z}[t_1, \ldots, t_n] \otimes_S \mathcal{C})^{\heartsuit}$, we conclude that $S\{t_1, \ldots, t_n\} \otimes_S \mathcal{C} \simeq R\{t_1, \ldots, t_n\} \otimes_R \mathcal{C}$ is locally Noetherian. We may therefore replace ϕ by ϕ'' and thereby reduce to the case where the map $\phi : R \to R'$ is surjective on π_0. In this case, the weak coherence of $R' \otimes_R \mathcal{C}$ follows from Theorem D.5.5.1.

To complete the proof, it will suffice to show that every compact object X of $(R' \otimes_R \mathcal{C})^{\heartsuit}$ is also Noetherian. Note that since $R' \otimes_R \mathcal{C}$ is weakly coherent, the object X is almost compact as an object of $R' \otimes_R \mathcal{C}$, and therefore also as an object of $\mathcal{C}$ (Lemma D.5.5.7). Our assumption that $\mathcal{C}$ is locally Noetherian then guarantees that X is Noetherian when viewed as an object of the abelian category $\mathcal{C}^{\heartsuit}$, and therefore also as an object of the abelian category $(R' \otimes_R \mathcal{C})^{\heartsuit}$. □

We now consider the problem of descent.

Proposition D.5.6.4. *Let $\phi : R \to R'$ be morphism of connective $\mathbb{E}_\infty$-rings and let $\mathcal{C}$ be a prestable R-linear ∞-category. If ϕ is faithfully flat and $R' \otimes_R \mathcal{C}$ is locally Noetherian, then $\mathcal{C}$ is locally Noetherian.*

We first establish the analogue of Proposition D.5.6.4 for abelian categories.

Lemma D.5.6.5. *Let $\phi : R \to R'$ be a homomorphism of commutative rings and let $\mathcal{A}$ be an R-linear abelian category. If ϕ is faithfully flat and $R' \otimes_R \mathcal{A}$ is locally Noetherian, then $\mathcal{A}$ is locally Noetherian.*

Proof. Let X be an object of $\mathcal{A}$ and set $X' = R' \otimes_R X$. Since $R' \otimes_R \mathcal{A} \simeq \mathrm{LMod}_{R'}(\mathcal{A})$ is locally Noetherian, we can write X' as a union of subobjects $\{X'_\alpha\}$ which are Noetherian when regarded as objects of $\mathrm{LMod}_{R'}(\mathcal{A})$. For each index α, set $X_\alpha = X \times_{X'} X'_\alpha$. Since filtered colimits in $\mathcal{A}$ are left exact, we have $X \simeq \varinjlim X_\alpha$. We will show that each X_α is a Noetherian object of $\mathcal{A}$, so that X can be written as a union of Noetherian subobjects. To prove this, we first observe that each tensor product $R' \times_R X_\alpha$ can be identified with a subobject of $X' = R' \otimes_R X$ (since R' is flat over R) which is contained in X'_α. The faithful flatness of R' over R implies that the construction

$$(Y \hookrightarrow X_\alpha) \mapsto (R' \otimes_R Y \hookrightarrow X'_\alpha)$$

induces an injection from the set of isomorphism classes of subobjects of X_α in the abelian category $\mathcal{A}$ to the set of isomorphism classes of subobjects of X'_α in the abelian category $\mathrm{LMod}_{R'}(\mathcal{A})$. Since the latter set satisfies the ascending chain condition (by virtue of our assumption that X'_α is a Noetherian object of $\mathrm{LMod}_{R'}(\mathcal{A})$), the former set must also satisfy the ascending chain condition. □

Proof of Proposition D.5.6.4. Let $\phi : R \to R'$ be a faithfully flat morphism of connective $\mathbb{E}_\infty$-rings, let $\mathcal{C}$ be a prestable R-linear ∞-category, and assume that $\mathcal{C}' = R' \otimes_R \mathcal{C}$ is locally Noetherian. We wish to show that $\mathcal{C}$ is locally Noetherian. It follows from Lemma D.5.6.5 that the abelian category $\mathcal{C}^\heartsuit$ is locally Noetherian. It will therefore suffice to show that for every Noetherian object $C \in \mathcal{C}^\heartsuit$ is compact when viewed as an object of $\tau_{\leqslant n} \mathcal{C}$ for each $n \geqslant 0$ (Proposition C.6.9.8).

Fix a filtered diagram $\{D_\alpha\}$ of n-truncated objects of $\mathcal{C}$ having colimit D; we wish to show that the canonical map $\rho : \varinjlim \mathrm{Map}_{\mathcal{C}}(C, D_\alpha) \to \mathrm{Map}_{\mathcal{C}}(C, D)$ is a homotopy equivalence. We will prove that the homotopy fibers of ρ are m-truncated for every integer $m \geqslant -2$. Note that this is trivial when $m = n$ (since the domain and codomain of ρ are both n-truncated). We will handle the general case using descending induction on m. For each index α, set $D'_\alpha = R' \otimes_R D_\alpha$, and set $D' = \varinjlim D'_\alpha \simeq R' \otimes_R D$. Let D'_α / D_α denote the cofiber of the canonical map $D_\alpha \to D'_\alpha$, and define D'/D similarly. Note that our hypothesis that R' is faithfully flat over R guarantees that each D'_α/D_α is also n-truncated. We have a commutative diagram of fiber sequences

$$\begin{array}{ccccc} \varinjlim \mathrm{Map}_{\mathcal{C}}(C, D_\alpha) & \longrightarrow & \varinjlim \mathrm{Map}_{\mathcal{C}}(C, D'_\alpha) & \longrightarrow & \varinjlim \mathrm{Map}_{\mathcal{C}}(C, D'_\alpha/D_\alpha) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho''} \\ \mathrm{Map}_{\mathcal{C}}(C, D) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(C, D') & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(C, D'/D). \end{array}$$

It follows from our inductive hypothesis that the map ρ'' has $(m+1)$-truncated homotopy fibers. Consequently, to show that the homotopy fibers of ρ are m-truncated, it will suffice to show that ρ' is a homotopy equivalence. Set $C' = R' \otimes_R C$, so that ρ' can be identified with the canonical map $\varinjlim \mathrm{Map}_{\mathcal{C}'}(C', D'_\alpha) \to \mathrm{Map}_{\mathcal{C}'}(C', D')$. To show that this map is a homotopy equivalence, it will suffice to show that C' is compact when viewed as an object of $\tau_{\leqslant n} \mathcal{C}'$. Using our assumption that $\mathcal{C}'$ is locally Noetherian (together with Proposition C.6.9.3), we are reduced to showing that C' is a Noetherian object of the abelian category $\mathcal{C}'^{\heartsuit}$. By virtue of Corollary C.6.8.9, this is equivalent to the assertion that C' is a compact object of $\mathcal{C}'^{\heartsuit}$. Since the forgetful functor $\mathcal{C}'^{\heartsuit} \to \mathcal{C}^{\heartsuit}$ commutes with filtered colimits, this follows from the fact that C is a compact object of $\mathcal{C}^{\heartsuit}$ (Proposition C.6.8.7). □

D.5.7 Complicial Prestable ∞-Categories

We now consider descent properties for the class of n-complicial Grothendieck prestable ∞-categories.

Proposition D.5.7.1. *Let R be a connective $\mathbb{E}_2$-ring, let R' be an $\mathbb{E}_1$-algebra over R which is flat when regarded as a right R-module, and let $\mathcal{C}$ be a prestable R-linear ∞-category. Then:*

- *(a) If $\mathcal{C}$ is n-complicial for some $n \geqslant 0$, then $\mathrm{LMod}_{R'}(\mathcal{C})$ is also n-complicial.*
- *(b) If $\mathcal{C}$ is weakly n-complicial for some $n \geqslant 0$, then $\mathrm{LMod}_{R'}(\mathcal{C})$ is also weakly n-complicial.*
- *(c) If R' is faithfully flat as a right R-module and $\mathrm{LMod}_{R'}(\mathcal{C})$ is weakly n-complicial for some $n \geqslant 0$, then $\mathcal{C}$ is also weakly n-complicial.*

Warning D.5.7.2. Proposition **??** is not true if we drop the flatness hypothesis on ϕ.

Proof of Proposition D.5.7.1. We first prove (b). Suppose that $\mathcal{C}$ is weakly n-complicial; we wish to show that $\mathrm{LMod}_{R'}(\mathcal{C})$ is also weakly n-complicial. Fix a truncated object $X \in \mathrm{LMod}_{R'}(\mathcal{C})$. Let us abuse notation by identifying X with its image under the forgetful functor $\mathrm{LMod}_{R'}(\mathcal{C}) \to \mathcal{C}$. Since $\mathcal{C}$ is weakly n-complicial, we can choose a morphism $\alpha : \overline{X} \to X$ in $\mathcal{C}$, where $\overline{X}$ is n-truncated and the induced map $\pi_0 \overline{X} \to \pi_0 X$ is an epimorphism in $\mathcal{C}^{\heartsuit}$. Extending scalars, we obtain a morphism $\alpha' : R' \otimes_R \overline{X} \to X$ in $\mathrm{LMod}_{R'}(\mathcal{C})$. Note that if we regard α' as a morphism of $\mathcal{C}$, then α factors through α'. It follows that α' induces an epimorphism $\pi_0(R' \otimes_R \overline{X}) \to \pi_0 X$ in the abelian category $\mathcal{C}^{\heartsuit}$, hence also in the abelian category $\mathrm{LMod}_{R'}(\mathcal{C})^{\heartsuit}$. We conclude by observing that our flatness assumption on R' guarantees that as a right R-module, R' can be written as a filtered colimit of free R-modules of finite rank. Since the collection of n-truncated objects of $\mathcal{C}$ is closed under filtered colimits, it follows that $R' \otimes_R \overline{X}$ is n-truncated. Allowing X to vary, we conclude that $\mathrm{LMod}_{R'}(\mathcal{C})$ is weakly n-complicial, as desired.

Assertion (a) follows from exactly the same argument, where we now drop the assumption that X is truncated. We will complete the proof by establishing (c). Assume that R' is faithfully flat over R and that $\mathrm{LMod}_{R'}(\mathcal{C})$ is weakly n-complicial; we will show that $\mathcal{C}$ is weakly n-complicial. By virtue of Proposition C.5.7.11, it will suffice to show that every injective object $Q \in \mathrm{Sp}(\mathcal{C})$ belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant -n}$. The injectivity of Q guarantees that Q belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. Using the flatness of R' as a right R-module, we deduce that $R' \otimes_R Q$ belongs to $\mathrm{Sp}(\mathrm{LMod}_{R'}(\mathcal{C}))_{\leqslant 0}$. Since $\mathrm{LMod}_{R'}(\mathcal{C})^{\heartsuit}$ is a Grothendieck abelian category, we can choose a monomorphism $\phi : \pi_0(R' \otimes_R Q) \hookrightarrow Q'_0$ in $\mathrm{LMod}_{R'}(\mathcal{C})^{\heartsuit}$, where $Q'_0 \in \mathrm{LMod}_{R'}(\mathcal{C})^{\heartsuit}$ is injective. Using Propositon HA.**??** , we can lift Q'_0 to an injective object $Q' \in \mathrm{Sp}(\mathrm{LMod}_{R'}(\mathcal{C}))$. The injectivity of Q' then guarantees that we can lift ϕ to a map $\overline{\phi} : R' \otimes_R Q \to Q'$. Let ψ denote the composite map

$$Q \to R' \otimes_R Q \xrightarrow{\overline{\phi}} Q'.$$

Since ϕ is a monomorphism by construction and R' is faithfully flat over R, the morphism ψ induces a monomorphism $\pi_0 Q \to \pi_0 Q'$ in the abelian category $\mathcal{C}^{\heartsuit}$. In other words, the cofiber $\mathrm{cofib}(\psi)$ belongs to $\mathrm{Sp}(\mathcal{C})_{\leqslant 0}$. Using the exactness of the sequence

$$\mathrm{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(Q', Q) \to \mathrm{Ext}^0_{\mathrm{Sp}(\mathcal{C})}(Q, Q) \to \mathrm{Ext}^1_{\mathrm{Sp}(\mathcal{C})}(\mathrm{cofib}(\psi), Q),$$

and the injectivity of Q, we deduce that ψ admits a left homotopy inverse: that is, Q is a direct summand of Q' (when we regard Q' as an object of $\mathcal{C}$). Since $\mathrm{LMod}_{R'}(\mathcal{C})$ is weakly n-complicial, Q' belongs to $\mathrm{Sp}(\mathrm{LMod}_{R'}(\mathcal{C}))_{\geqslant -n}$ (Proposition C.5.7.11), so that Q belongs to $\mathrm{Sp}(\mathcal{C})_{\geqslant -n}$ as desired. □

We do not know if it is possible to replace “weakly n-complicial” by “n-complicial” in part (c) of Proposition D.5.7.1. However, we do have the following weaker result:

Corollary D.5.7.3. *Let $\phi : R \to R'$ be a flat universal descent morphism of connective $\mathbb{E}_\infty$-rings which exhibits R' as a module of finite* Tor*-amplitude over $R' \otimes_R R'$. Let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $n \geqslant 0$. Then:*

(1) *If $R' \otimes_R \mathcal{C}$ is anticomplete and n-complicial, then $\mathcal{C}$ is anticomplete and n-complicial.*

(2) *If $R' \otimes_R \mathcal{C}$ is separated and n-complicial, then $\mathcal{C}$ is separated and n-complicial.*

Proof. Assume first that $R' \otimes_R \mathcal{C}$ is anticomplete and n-complicial. It follows from Theorem D.5.4.9 that $\mathcal{C}$ is anticomplete and from Proposition D.5.7.1 that $\mathcal{C}$ is weakly n-complicial. Applying Proposition C.5.5.16, we deduce that $\mathcal{C}$ is n-complicial.

Now suppose that $R' \otimes_R \mathcal{C}$ is separated and n-complicial. Using Propositions D.5.7.1 and D.5.1.2, we deduce that $\mathcal{C}$ is separated and weakly n-complicial. Using Lemma D.5.4.5, we can choose a left exact R-linear functor $\lambda : \breve{\mathcal{C}} \to \mathcal{C}$ where $\breve{\mathcal{C}}$ is anticomplete and λ induces an equivalence of completions. Applying Remark C.5.5.14, we deduce that $\breve{\mathcal{C}}$ is also weakly n-complicial. Proposition C.5.5.16 then implies that $\breve{\mathcal{C}}$ is n-complicial. Let $\breve{\mathcal{C}}^{\mathrm{sep}}$ denote the separated quotient of $\breve{\mathcal{C}}$ (see Proposition C.3.6.1. Then $\breve{\mathcal{C}}^{\mathrm{sep}}$ inherits the structure of an R-linear prestable ∞-category, and Proposition C.5.3.3 implies that $\breve{\mathcal{C}}^{\mathrm{sep}}$ is n-complicial. The functor λ factors as a composition $\breve{\mathcal{C}} \to \breve{\mathcal{C}}^{\mathrm{sep}} \xrightarrow{\mu} \mathcal{C}$, where μ is left exact and induces an equivalence of completions. It follows that the induced map $\mu_{R'} : R' \otimes_R \breve{\mathcal{C}}^{\mathrm{sep}} \to R' \otimes_R \mathcal{C}$ is also left exact and induces an equivalence on completions. It follows from Proposition D.5.1.2 that $R' \otimes_R \breve{\mathcal{C}}^{\mathrm{sep}}$ is separated and from Proposition D.5.7.1 that it is n-complicial. Applying Proposition C.5.3.9, we deduce that $\mu_{R'}$ is an equivalence. Applying Theorem D.4.1.6, we conclude that μ is also an equivalence, so that $\mathcal{C} \simeq \breve{\mathcal{C}}^{\mathrm{sep}}$ is n-complicial as desired. □

We now specialize to the case $n = 0$.

Corollary D.5.7.4. *Let $\phi : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings, let $\mathcal{C}$ be a prestable R-linear ∞-category, and let $\mathcal{A} = \mathcal{C}^\heartsuit$ be its heart. Then:*

(a) *Assume that the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ extends to an equivalence $\breve{\mathcal{D}}(\mathcal{A})_{\geqslant 0} \simeq \mathcal{C}$ and the morphism ϕ is flat and exhibits R' as a module of finite* Tor*-amplitude over $R' \otimes_R R'$. Then the inclusion $\mathrm{LMod}_{R'}(\mathcal{A}) \hookrightarrow \mathrm{LMod}_{R'}(\mathcal{C})$ extends to an equivalence $\breve{\mathcal{D}}(\mathrm{LMod}_{R'}(\mathcal{A})) \to \mathrm{LMod}_{R'}(\mathcal{C})$.*

(a′) *If the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ extends to an equivalence $\mathcal{D}(\mathcal{A})_{\geqslant 0} \simeq \mathcal{C}$ and the morphism ϕ is flat, then the inclusion $\mathrm{LMod}_{R'}(\mathcal{A}) \hookrightarrow \mathrm{LMod}_{R'}(\mathcal{C})$ extends to an equivalence $\mathcal{D}(\mathrm{LMod}_{R'}(\mathcal{A})) \to \mathrm{LMod}_{R'}(\mathcal{C})$.*

(a″) *If the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ extends to an equivalence $\widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0} \simeq \mathcal{C}$ and the morphism ϕ is flat, then the inclusion $\mathrm{LMod}_{R'}(\mathcal{A}) \hookrightarrow \mathrm{LMod}_{R'}(\mathcal{C})$ extends to an equivalence $\widehat{\mathcal{D}}(\mathrm{LMod}_{R'}(\mathcal{A})) \to \mathrm{LMod}_{R'}(\mathcal{C})$.*

(b) *If the inclusion $\mathrm{LMod}_{R'}(\mathcal{A}) \hookrightarrow \mathrm{LMod}_{R'}(\mathcal{C})$ extends to an equivalence $\breve{\mathcal{D}}(\mathrm{LMod}_{R'}(\mathcal{A})) \to \mathrm{LMod}_{R'}(\mathcal{C})$ and ϕ is flat universal descent morphism which exhibits R' as a module of finite* Tor*-amplitude over $R' \otimes_R R'$, then the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ extends to an equivalence $\breve{\mathcal{D}}(\mathcal{A})_{\geqslant 0} \simeq \mathcal{C}$.*

(b′) *If the inclusion $\mathrm{LMod}_{R'}(\mathcal{A}) \hookrightarrow \mathrm{LMod}_{R'}(\mathcal{C})$ extends to an equivalence $\mathcal{D}(\mathrm{LMod}_{R'}(\mathcal{A})) \to \mathrm{LMod}_{R'}(\mathcal{C})$ and ϕ is flat universal descent morphism which exhibits R' as a module of* *finite* Tor*-amplitude over* $R' \otimes_R R'$*, then the inclusion* $\mathcal{A} \hookrightarrow \mathcal{C}$ *extends to an equivalence* $\mathcal{D}(\mathcal{A})_{\geqslant 0} \simeq \mathcal{C}$*.*

(b'') *If the inclusion* $\mathrm{LMod}_{R'}(\mathcal{A}) \hookrightarrow \mathrm{LMod}_{R'}(\mathcal{C})$ *extends to an equivalence* $\widehat{\mathcal{D}}(\mathrm{LMod}_{R'}(\mathcal{A})) \to \mathrm{LMod}_{R'}(\mathcal{C})$ *and* ϕ *is flat universal descent morphism, then the inclusion* $\mathcal{A} \hookrightarrow \mathcal{C}$ *extends to an equivalence* $\widehat{\mathcal{D}}(\mathcal{A})_{\geqslant 0} \simeq \mathcal{C}$*.*

Proof. Assertion (a) follows from Corollary C.5.8.11, Proposition D.5.7.1, and Theorem D.5.4.1. Assertion (a') follows from Remark C.5.4.11, Proposition D.5.7.1, and Proposition D.5.1.2. Assertion (a'') follows from Corollary C.5.9.7, Proposition D.5.7.1, and Proposition D.5.1.3. Assertion (b) follows from Corollaries C.5.8.11 and D.5.7.3. Assertion (b') follows from Corollary D.5.7.3 and Remark C.5.4.11. Assertion (b'') follows from Corollary C.5.9.7, Proposition D.5.7.1, and Proposition D.5.1.3. □

D.6 Descent for the Flat Topology

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, and let $\widehat{A} = \varprojlim A/\mathfrak{m}^n$ denote its completion. Then the natural map $A \to \widehat{A}$ is a faithfully flat map of commutative rings. It follows from the theory of faithfully flat descent that the category of A-modules is equivalent to the category of $\widehat{A}$-modules equipped with descent data. Using this fact, one can reduce many questions about A (and the category of A-modules) to questions about $\widehat{A}$ (and the category of $\widehat{A}$-modules).

In §D.3, we introduced the notion of an universal descent morphism of $\mathbb{E}_\infty$-rings (Definition D.3.1.1) and proved that if $f : A \to B$ is a universal descent morphism, then the base change functor $\mathrm{Mod}_A \to \mathrm{Mod}_B$ is comonadic (that is, we can identify A-module spectra with objects of Mod_B equipped with suitable descent data). This result applies in particular when $f : A \to B$ is faithfully flat and $\pi_0 B$ is countably presented over $\pi_0 A$ (Proposition D.3.3.1). However, this countable presentation assumption is typically not satisfied for $B = \widehat{A}$ (if A is a local Noetherian ring as above), and we do not know if it is true that all faithfully flat morphisms are universal descent morphisms. Consequently, the results of §D.3 are inadequate for many of the applications of the theory of descent. Our goal in this section is to remedy the situation by proving an analogue of Theorem D.3.5.2 for the fpqc topology.

D.6.1 Flat Descent for Stable ∞-Categories

We begin with a few general remarks.

Remark D.6.1.1. Let CAlg denote the ∞-category of $\mathbb{E}_\infty$-rings. In §A.3, we introduced the *fpqc topology* on the ∞-category $\mathrm{CAlg}^{\mathrm{op}}$. If A is an $\mathbb{E}_\infty$-ring, then a sieve on A is covering with respect to the fpqc topology if and only if it contains a finite collection of maps $\{\phi_\alpha : A \to A_\alpha\}$ which induces a faithfully flat morphism $A \to \prod_\alpha A_\alpha$.

For every $\mathbb{E}_\infty$-ring A, the fpqc topology on $\mathrm{CAlg}^{\mathrm{op}}$ determines a Grothendieck topology on the ∞-category $\mathrm{CAlg}_A^{\mathrm{op}}$ of $\mathbb{E}_\infty$-algebras over A. If A is connective, we also obtain a Grothendieck topology on the ∞-category $(\mathrm{CAlg}_A^{\mathrm{cn}})^{\mathrm{op}}$ of connective $\mathbb{E}_\infty$-algebras over A. We will refer to both of these topologies as the *fpqc topology*.

Definition D.6.1.2. Let A be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a stable A-linear ∞-category. We will say that $\mathcal{C}$ *satisfies flat descent* if the functor

$$\chi : \mathrm{CAlg}_A \to \widehat{\mathcal{C}\mathrm{at}}_\infty \qquad B \mapsto B \otimes_A \mathcal{C}$$

is a $\widehat{\mathcal{C}\mathrm{at}}_\infty$-valued sheaf with respect to the fpqc topology on $\mathrm{CAlg}_A^{\mathrm{op}}$. We will say that $\mathcal{C}$ *satisfies flat hyperdescent* if the functor χ is a hypercomplete sheaf with respect to the fpqc topology.

In the situation of Definition D.6.1.2, if the $\mathbb{E}_\infty$-ring A is connective, then it suffices to restrict our attention to $\mathbb{E}_\infty$-algebras over A which are also connective:

Proposition D.6.1.3. *Let A be a connective $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be a stable A-linear ∞-category, and let $\chi : \mathrm{CAlg}_A \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the functor given by $\chi(B) = B \otimes_A \mathcal{C}$. Then:*

1. *The functor χ is a sheaf for the fpqc topology on the ∞-category $\mathrm{CAlg}_A^{\mathrm{op}}$ if the restriction $\chi|_{\mathrm{CAlg}_A^{\mathrm{cn}}}$ is a sheaf with respect to the fpqc topology on $(\mathrm{CAlg}_A^{\mathrm{cn}})^{\mathrm{op}}$.*
2. *The functor χ is a hypercomplete sheaf for the fpqc topology on the ∞-category $\mathrm{CAlg}_A^{\mathrm{op}}$ if the restriction $\chi|_{\mathrm{CAlg}_A^{\mathrm{cn}}}$ is a hypercomplete sheaf with respect to the fpqc topology on $(\mathrm{CAlg}_A^{\mathrm{cn}})^{\mathrm{op}}$.*

Before giving the proof of Proposition D.6.1.3, let us introduce a bit of useful terminology:

Definition D.6.1.4. Let $R^\bullet$ be an augmented cosemisimplicial object of CAlg. We will say that $R^\bullet$ is a *flat hypercovering* if it determines an S-hypercovering in the ∞-category $\mathrm{CAlg}^{\mathrm{op}}$ in the sense of Definition A.5.7.1, where S is the collection of faithfully flat morphisms in CAlg. In other words, $R^\bullet$ is a flat hypercovering if each of the maps $L_n(R^\bullet) \to R^n$ is faithfully flat, where $L_n(R^\bullet)$ denotes the nth latching object of the cosemisimplicial $\mathbb{E}_\infty$-ring $R^\bullet$. In this case, we will also say that the underlying semisimplicial $\mathbb{E}_\infty$-ring of $R^\bullet$ is a *flat hypercovering* of R^{-1}.

Proof of Proposition D.6.1.3. We will prove (1); the proof of (2) is similar. The "only if" direction is obvious. Conversely, suppose that $\chi|_{\mathrm{CAlg}_A^{\mathrm{cn}}}$ is a sheaf with respect to the flat topology on $\mathrm{CAlg}_A^{\mathrm{cn}}$. We wish to show that χ is a sheaf with respect to the fpqc topology. Using Proposition A.3.3.1 and Lemma D.3.5.5, we are reduced to proving the following:

$(*)$ Let $f : B \to B^0$ be a faithfully flat morphism of A-algebras, and let $B^\bullet : \mathbf{\Delta}_+ \to \mathrm{CAlg}_A$ be the Čech nerve of f (regarded as a morphism in the ∞-category $\mathrm{CAlg}_A^{\mathrm{op}}$). Then $\chi(B^\bullet)$ is a limit diagram in $\widehat{\mathcal{C}\mathrm{at}}_\infty$.

According to Proposition HA.5.2.2.36 , it will suffice to verify the following:

(a) The extension of scalars functor $\phi : \mathrm{LMod}_B(\mathcal{C}) \to \mathrm{LMod}_{B^0}(\mathcal{C})$ is conservative. To prove this, we let $\tau_{\geqslant 0}B$ and $\tau_{\geqslant 0}B^0$ be the connective covers of B and B^0, respectively. Since f is flat, the canonical map $B \otimes_{\tau_{\geqslant 0}B} \tau_{\geqslant 0}B^0 \to B^0$ is an equivalence. It follows from Lemma D.3.5.6 that ϕ fits into a homotopy commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{LMod}_B(\mathcal{C}) & \xrightarrow{\phi} & \mathrm{LMod}_B(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{LMod}_{\tau_{\geqslant 0}B}(\mathcal{C}) & \xrightarrow{\phi_0} & \mathrm{LMod}_{\tau_{\geqslant 0}B^0}(\mathcal{C}). \end{array}$$

Here the vertical maps are the evident forgetful functors (and therefore conservative). Consequently, to show that ϕ is conservative, it suffices to show that ϕ_0 is conservative, which follows from our assumption that $\chi|_{\mathrm{CAlg}_A^{\mathrm{cn}}}$ is a sheaf with respect to the fpqc topology.

(b) Let $M^\bullet$ be a cosimplicial object of $\mathcal{C}$ which is a module over the underlying cosimplicial algebra of $B^\bullet$ such that each of the maps $B^p \otimes_{B^q} M^q \to M^p$ is an equivalence. Let $M = \varprojlim M^\bullet$, regarded as a B-module object of $\mathcal{C}$. Then we must show that the canonical map $B^p \otimes_B M \to M^p$ is an equivalence for each $p \geqslant 0$. To prove this, we note that since f is flat, the map $\tau_{\geqslant 0}B^p \otimes_{\tau_{\geqslant 0}B^q} B^q \to B^p$ is an equivalence for every morphism $[p] \to [q]$ in $\mathbf{\Delta}_+$. Let us regard $M^\bullet$ as a cosimplicial module over the underlying cosimplicial algebra of $\tau_{\geqslant 0}B^\bullet$. Using Lemma D.3.5.6, we conclude that each of the maps $\tau_{\geqslant 0}B^p \otimes_{\tau_{\geqslant 0}B^q} M^q \to M^p$ is an equivalence. Combining our assumption that $\chi|_{\mathrm{CAlg}_A^{\mathrm{cn}}}$ is a sheaf with respect to the fpqc topology and Proposition HA.5.2.2.36 , we conclude that each of the maps $\tau_{\geqslant 0}B^p \otimes_{\tau_{\geqslant 0}B} M \to M^p$ is an equivalence for $p \geqslant 0$. The desired result now follows from Lemma D.3.5.6.

□

D.6.2 Flat Descent for Prestable ∞-Categories

The criterion of Proposition D.6.1.3 has the virtue of making sense at the level of *prestable* ∞-categories:

Definition D.6.2.1. Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable A-linear ∞-category. We will say that $\mathcal{C}$ *satisfies flat descent* if the functor

$$\chi : \mathrm{CAlg}_A^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty \qquad B \mapsto B \otimes_A \mathcal{C}$$

is a $\widehat{\mathcal{C}\mathrm{at}}_\infty$-valued sheaf with respect to the fpqc topology on $(\mathrm{CAlg}^{\mathrm{cn}}_A)^{\mathrm{op}}$. We will say that $\mathcal{C}$ *satisfies flat hyperdescent* if the functor χ is a hypercomplete sheaf with respect to the fpqc topology.

Remark D.6.2.2. Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be an stable A-linear ∞-category. It follows from Proposition D.6.1.3 that $\mathcal{C}$ satisfies flat (hyper)descent when regarded as a stable A-linear ∞-category (in the sense of Definition D.6.1.2) if and only if it satisfies flat (hyper)descent when regarded as an additive A-linear ∞-category (in the sense of Definition D.6.2.1).

Proposition D.6.2.3. *Let A be connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable A-linear ∞-category. Then:*

(1) *If $\mathcal{C}$ satisfies flat descent, then the stable A-linear ∞-category* $\mathrm{Sp}(\mathcal{C})$ *satisfies flat descent.*

(2) *If $\mathcal{C}$ satisfies flat hyperdescent, then the stable A-linear ∞-category* $\mathrm{Sp}(\mathcal{C})$ *satisfies flat hyperdescent.*

Proof. We will prove (2); the proof of (1) is similar. Assume that $\mathcal{C}$ satisfies flat hyperdescent; we wish to show that $\mathrm{Sp}(\mathcal{C})$ has the same property. By virtue of Proposition D.6.1.3, it will suffice to show that the construction

$$\mathrm{CAlg}^{\mathrm{cn}}_A \to \widehat{\mathcal{C}\mathrm{at}}_\infty \qquad B \mapsto B \otimes_A \mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Sp}(\mathrm{LMod}_B(\mathcal{C}))$$

is a hypercomplete sheaf with respect to the fpqc topology. We will show that this functor satisfies the hypotheses of Proposition A.5.7.2. It is easy to see that the construction $B \mapsto \mathrm{Sp}(\mathrm{LMod}_B(\mathcal{C}))$ commutes with finite products. Let B be a connective A-algebra and let $B^\bullet$ be a semicosimplicial object of $\mathrm{CAlg}^{\mathrm{cn}}_B$ which is a flat hypercovering of B; we wish to show that the canonical map $\theta : \mathrm{Sp}(\mathrm{LMod}_B(\mathcal{C})) \to \varprojlim \mathrm{Sp}(\mathrm{LMod}_{B^\bullet}(\mathcal{C}))$ is an equivalence of ∞-categories. The functor θ factors as a composition

$$\mathrm{Sp}(\mathrm{LMod}_B(\mathcal{C})) \xrightarrow{\theta'} \mathrm{Sp}(\varprojlim \mathrm{LMod}_{B^\bullet}(\mathcal{C})) \xrightarrow{\theta''} \varprojlim \mathrm{Sp}(\mathrm{LMod}_{B^\bullet}(\mathcal{C})),$$

where θ' is an equivalence of ∞-categories by virtue of our assumption that the prestable B-linear ∞-category $\mathcal{C}$ satisfies flat hyperdescent. It will therefore suffice to show that the functor θ'' is also an equivalence of ∞-categories. By virtue of Remark C.3.2.5, it will suffice to show that the cosimplicial ∞-category $\mathrm{LMod}_{B^\bullet}(\mathcal{C})$ is given by a diagram in $\mathsf{Groth}^{\mathrm{lex}}_\infty$: that is, that for each morphism $[m] \to [n]$ in $\mathbf{\Delta}_+$, the associated functor $\mathrm{LMod}_{B^m}(\mathcal{C}) \to \mathrm{LMod}_{B^n}(\mathcal{C})$ (given by extension of scalars) is left exact. Since the forgetful functor $\mathrm{LMod}_{B^n}(\mathcal{C}) \to \mathcal{C}$ is both conservative and left exact, we are reduced to proving the left exactness of the composite functor

$$\mathrm{LMod}_{B^m}(\mathcal{C}) \to \mathrm{LMod}_{B^n}(\mathcal{C}) \to \mathcal{C} \qquad C \mapsto B^n \otimes_{B^m} C.$$

To prove this, let $\mathcal{E} \subseteq \mathrm{RMod}^{\mathrm{cn}}_{B^m}$ be the full subcategory spanned by those connective right B^m-modules M for which the functor

$$\mathrm{LMod}_{B^m}(\mathcal{C}) \to \mathcal{C} \qquad C \mapsto M \otimes_{B^m} C$$

is left exact. Since filtered colimits in $\mathcal{C}$ are left exact, the ∞-category $\mathcal{E}$ is closed under filtered colimits. We wish to show that $B^n \in \mathcal{E}$. Our assumption that $B^\bullet$ is a flat hypercovering of B guarantees that B^n is flat as a module over B^m and can therefore be obtained as a filtered colimit of free B^m-modules of finite rank (Theorem HA.7.2.2.15). It will therefore suffice to show that $\mathcal{E}$ contains all free B^m-modules of finite rank. Since $\mathcal{E}$ is closed under the formation of direct sums, we are reduced to proving that $\mathcal{E}$ contains the module B^m itself, which is equivalent to the left exactness of the forgetful functor $\mathrm{LMod}_{B^m}(\mathcal{C}) \to \mathcal{C}$. □

D.6.3 The Descent Theorem

We can now state the main result of this section:

Theorem D.6.3.1. *Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable A-linear ∞-category. If $\mathcal{C}$ is complete (see Definition C.1.2.12), then $\mathcal{C}$ satisfies flat hyperdescent.*

We will give the proof Theorem D.6.3.1 later in this section. For the moment, let us describe some of its consequences.

Corollary D.6.3.2. *Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{A}$ be an abelian A-linear ∞-category. Then $\mathcal{A}$ satisfies flat hyperdescent.*

Proof. We will show that $\mathcal{C}$ satisfies the hypotheses of Proposition A.5.7.2. Since the functor $B \mapsto \mathrm{LMod}_B(\mathcal{A})$ commutes with finite products, it will suffice to show that for every flat hypercovering $B^\bullet$ of an object $B \in \mathrm{CAlg}^{\mathrm{cn}}_A$, the canonical map $\theta : \mathrm{LMod}_B(\mathcal{A}) \to \varprojlim \mathrm{LMod}_{B^\bullet}(\mathcal{A})$ is an equivalence of ∞-categories.

Let $\mathcal{C}$ be a complete prestable A-linear ∞-category satisfying $\mathcal{C}^\heartsuit \simeq \mathcal{A}$ (the existence of $\mathcal{C}$ follows from Example D.1.3.9). It follows from Theorem D.6.3.1 that $\mathcal{C}$ satisfies flat hyperdescent, so that the canonical map $\mathrm{LMod}_B(\mathcal{C}) \to \varprojlim \mathrm{LMod}_{B^\bullet}(\mathcal{C})$ is an equivalence. Here it suffices to take the limit of the underlying cosemisimplicial ∞-category of $\mathrm{LMod}_{B^\bullet}(\mathcal{C})$, in which the transition functors are left exact. Passing to hearts, we deduce that the map θ is an equivalence, as desired. □

Corollary D.6.3.3. *Let A be an $\mathbb{E}_\infty$-ring. Then the stable A-linear ∞-category Mod_A satisfies flat hyperdescent.*

Proof. Without loss of generality, we may assume that A is the sphere spectrum. Then A is connective. By virtue of Proposition D.6.2.3, it will suffice to show that the prestable ∞-category $\mathrm{Mod}^{\mathrm{cn}}_A$ satisfies flat hyperdescent, which follows immediately from Theorem D.6.3.1. □

Corollary D.6.3.4. *Let $R^{\bullet} : \mathbf{\Delta}_{s,+} \to \mathrm{CAlg}$ be a flat hypercovering of an $\mathbb{E}_{\infty}$-ring $R = R^{-1}$, let M be an R-module spectrum, and let $M^{\bullet}$ be the cosemisimplicial $R^{\bullet}|_{\mathbf{\Delta}_s}$-module spectrum given levelwise by the formula $M^n = M \otimes_R R^n$. Then the canonical map $M \to \varprojlim M^{\bullet}$ is an equivalence.*

Proof. Combine Proposition A.5.7.2, Proposition HA.5.2.2.36 , and Corollary D.6.3.3. □

Theorem D.6.3.5. *The identity functor* CAlg → CAlg *is a hypercomplete* CAlg*-valued sheaf on* $\mathrm{CAlg}^{\mathrm{op}}$ *(with respect to the fpqc topology).*

Remark D.6.3.6. Theorem D.6.3.5 implies that the fpqc topology on the ∞-category $\mathrm{CAlg}^{\mathrm{op}}$ is *subcanonical*.

Proof of Theorem D.6.3.5. We will show that the identity functor id : CAlg → CAlg satisfies the hypotheses of Proposition A.5.7.2. Since the functor id clearly preserves finite products, it will suffice to show that every hypercovering $R^{\bullet} : \mathbf{\Delta}_{s,+} \to \mathrm{CAlg}$ is a limit diagram in CAlg. This is an immediate consequence of Corollary D.6.3.4. □

D.6.4 Digression: Faithfully Flat Monads

We now introduce an auxiliary notion which will be useful in the proof of Theorem D.6.3.1.

Definition D.6.4.1. Let $\mathcal{C}$ be a prestable ∞-category which admits finite limits, let $T : \mathcal{C} \to \mathcal{C}$ be a monad, and let $T_{\mathrm{red}} : \mathcal{C} \to \mathcal{C}$ be the cofiber of the unit map $\mathrm{id} \to T$ (formed in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$). We will say that T is *faithfully flat* if the functor T is right exact and the functor T_{red} is left exact.

Remark D.6.4.2. In the situation of Definition D.6.4.1, the right exactness of T guarantees the right exactness of T_{red}, and the left exactness of T_{red} guarantees the left exactness of T.

Remark D.6.4.3. Let $\mathcal{C}$ be a prestable ∞-category which admits finite limits and let $T : \mathcal{C} \to \mathcal{C}$ be a faithfully flat monad. For each object $C \in \mathcal{C}$ and each integer $n \geqslant 0$, let $\pi_n C \in \mathcal{C}^{\heartsuit}$ denote the nth homotopy object of $\Sigma^{\infty}(C)$ taken with respect to the natural t-structure on the Spanier-Whitehead ∞-category $\mathrm{SW}(\mathcal{C})$ of Construction C.1.1.1. The cofiber sequence $C \to TC \to T_{\mathrm{red}} C$ determines a long exact sequence

$$\cdots \to \pi_{n+1} T_{\mathrm{red}} C \xrightarrow{\delta_n} \pi_n C \to \pi_n TC \to \pi_n T_{\mathrm{red}} C \xrightarrow{\delta_{n-1}} \pi_{n-1} C,$$

depending functorially on the object C. We therefore have a commutative diagram

$$\begin{array}{ccc} \pi_{n+1} T_{\mathrm{red}} C & \xrightarrow{\delta_n} & \pi_n C \\ \downarrow & & \downarrow \\ \pi_{n+1} T_{\mathrm{red}}(\tau_{\leqslant n} C) & \longrightarrow & \pi_n(\tau_{\leqslant n} C), \end{array}$$

where the lower right corner vanishes (by virtue of our assumption that the functor T_{red} is left exact) and the right horizontal map is an isomorphism. It follows that each boundary map δ_n vanishes: that is, we have short exact sequences $0 \to \pi_n C \to \pi_n TC \to \pi_n T_{\mathrm{red}} C \to 0$ for each $n \geqslant 0$.

Remark D.6.4.4. Let A be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a prestable A-linear ∞-category. For every connective $\mathbb{E}_1$-algebra B over A, we have a pair of adjoint functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{LMod}_B(\mathcal{C})$$

which determines a monad $T \simeq G \circ F$ on $\mathcal{C}$, given on objects by the formula $M \mapsto B \otimes_A M$. Since the functor G is conservative and preserves small colimits, Theorem HA.4.7.3.5 implies that $\mathrm{LMod}_B(\mathcal{C})$ can be identified with the ∞-category $\mathrm{LMod}_T(\mathcal{C})$ of T-modules in $\mathcal{C}$.

Lemma D.6.4.5. *Let A be a connective $\mathbb{E}_2$-ring, let $\mathcal{C}$ prestable A-linear ∞-category, and let $B \in \mathrm{Alg}_A$ be an $\mathbb{E}_1$-algebra over A which is right faithfully flat over A (Definition D.4.4.1). Then the monad T of Remark D.6.4.4 is faithfully flat.*

Proof. It is clear that the monad T is right t-exact. Let B/A denote the cofiber of the map of A-modules $A \to B$. For each object $C \in \mathcal{C}$, the cofiber of the unit map $C \to TC$ can be identified with $(B/A) \otimes_A C$. We wish to prove that the functor $C \mapsto (B/A) \otimes_A C$ is left exact. We now argue as in the proof of Proposition D.6.2.3. Let $\mathcal{E} \subseteq \mathrm{RMod}_A^{\mathrm{cn}}$ be the full subcategory spanned by those connective right A-modules M for which the construction $C \mapsto M \otimes_A C$ is left exact. It is easy to see that $\mathcal{E}$ contains all free A-modules of finite rank. Since filtered colimits in $\mathcal{C}$ are left exact, the ∞-category $\mathcal{E}$ is closed under filtered colimits in $\mathrm{RMod}_A^{\mathrm{cn}}$, and therefore contains all flat right A-modules (Theorem HA.7.2.2.15). It now suffices to observe that the right faithful flatness of B over A guarantees that B/A is flat as a right A-module (Lemma D.4.4.3). □

Proposition D.6.4.6. *Let $\mathcal{C}$ be a prestable ∞-category, let T be a faithfully flat monad on $\mathcal{C}$, and let $F : \mathcal{C} \to \mathrm{LMod}_T(\mathcal{C})$ be a left adjoint to the forgetful functor $G : \mathrm{LMod}_T(\mathcal{C}) \to \mathcal{C}$. If $\mathcal{C}$ is complete (in the sense of Definition C.1.2.12), then the functor F is comonadic.*

Proof. According to Theorem HA.4.7.3.5 , it will suffice to show that the functor F is conservative and preserves totalizations of F-split cosimplicial objects of $\mathcal{C}$. For each $n \geqslant 0$, define $\pi_n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ as in Remark D.6.4.3. We first show that F is conservative. Let $\alpha : X \to Y$ be a morphism in $\mathcal{C}$ such that $F(\alpha)$ is an equivalence. Then $T \operatorname{cofib}(\alpha) \simeq GF(\operatorname{cofib}(\alpha)) \simeq G \operatorname{cofib}(F(\alpha)) \simeq 0$. Applying Remark D.6.4.3, we see that the natural map $\pi_n \operatorname{cofib}(\alpha) \to \pi_n T \operatorname{cofib}(\alpha)$ is a monomorphism (in the abelian category $\mathcal{C}^{\heartsuit}$) for each $n \geqslant 0$, so that each $\pi_n \operatorname{cofib}(\alpha)$ vanishes. Since $\mathcal{C}$ is left complete, we conclude that $\operatorname{cofib}(\alpha) \simeq 0$ and therefore α is an equivalence.

Let $X^\bullet$ be an F-split cosimplicial object of $\mathcal{C}$; we wish to show that $X^\bullet$ admits a totalization in $\mathcal{C}$ which is preserved by the functor T. Since the monad T is faithfully flat, it induces an exact functor $T^\heartsuit$ from the abelian category $\mathcal{C}^\heartsuit$ to itself. Note that for every object $C \in \mathcal{C}^\heartsuit$, there exists a monomorphism $C \hookrightarrow T^\heartsuit C$; in particular, the functor $T^\heartsuit$ does not annihilate any nonzero objects of $\mathcal{C}^\heartsuit$. For every object $Y \in \mathcal{C}$, we have canonical isomorphisms $\pi_n TY \simeq T^\heartsuit \pi_n Y$. Since $X^\bullet$ is F-split, it follows that $T^\heartsuit \pi_n X^\bullet$ is a split cosimplicial object of $\mathcal{C}^\heartsuit$ for each $n \geqslant 0$. Let

$$A(n)^0 \xrightarrow{d(n)} A(n)^1 \longrightarrow A(n)^2 \longrightarrow \cdots$$

be the unnormalized chain complex (in $\mathcal{C}^\heartsuit$) associated to the cosimplicial object $\pi_n X^\bullet$. It follows that $T^\heartsuit(A(n)^*)$ is split exact: in particular, we have an exact sequence

$$0 \to K \to T^\heartsuit A(n)^0 \to T^\heartsuit A(n)^1 \to \cdots .$$

Since the functor $T^\heartsuit$ is exact, we can write $K = \ker(T^\heartsuit d(n)) \simeq T^\heartsuit \ker d(n)$. Since $T^\heartsuit$ is exact and does not annihilate any nonzero objects of $\mathcal{C}^\heartsuit$, we deduce the exactness of the sequence

$$0 \to \ker d(n) \to A(n)^0 \to A(n)^1 \to \cdots .$$

Applying Corollary **??** (in the stable ∞-category $\mathrm{Sp}(\mathcal{C})$), we deduce that $X^\bullet$ admits a totalization $X \in \mathcal{C}$ and that the natural map $X \to X^0$ induces isomorphisms $\pi_n X \simeq \ker d(n) \subseteq \pi_n X^0$ for every integer $n \geqslant 0$. Using the exactness of $T^\heartsuit$ and the identification $T^\heartsuit \pi_n X \simeq \pi_n TX$, we see that the natural map $\alpha : TX \to \varprojlim TX^\bullet$ determines an exact sequence

$$0 \to \pi_n TX \to T^\heartsuit A(n)^0 \to T^\heartsuit A(n)^1 \to \cdots$$

so that α is an equivalence (again by virtue of Corollary HA.1.2.4.12). □

D.6.5 The Proof of Theorem D.6.3.1

We now turn to the proof of Theorem D.6.3.1. We begin by giving a concrete criterion for flat descent, which follows easily from the Barr-Beck theorem:

Proposition D.6.5.1. *Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable A-linear ∞-category. Then $\mathcal{C}$ satisfies flat descent if and only if, for every faithfully flat map of connective A-algebras $B \to B^0$, the base-change functor $\mathrm{LMod}_B(\mathcal{C}) \to \mathrm{LMod}_{B^0}(\mathcal{C})$ is comonadic.*

Proof. Using Proposition A.3.3.1 and Lemma D.3.5.5, we see that $\mathcal{C}$ satisfies flat descent if and only if, for every faithfully flat morphism of connective A-algebras $f : B \to B^0$, the following condition is satisfied:

($*'$) Let $B^\bullet : \mathbf{\Delta}_+ \to \mathrm{CAlg}_A$ be the Čech nerve of f (regarded as a morphism in $\mathrm{CAlg}_A^{\mathrm{op}}$, and let $\mathcal{C}^\bullet$ be the augmented cosimplicial ∞-category given by the formula $\mathcal{C}^\bullet = \mathrm{LMod}_{B^\bullet}(\mathcal{C})$. Then $\mathcal{C}^\bullet$ is a limit diagram in $\widehat{\mathcal{C}\mathrm{at}}_\infty$.

The desired result now follows from Lemma D.3.5.8. □

Proof of Theorem D.6.3.1. Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a prestable A-linear ∞-category; we wish to show that $\mathcal{C}$ satisfies flat hyperdescent. We first prove that $\mathcal{C}$ satisfies flat descent. By virtue of Proposition D.6.5.1, it will suffice to show that for every faithfully flat morphism between connective A-algebras $B \to B^0$, the base-change functor $F : \mathrm{LMod}_B(\mathcal{C}) \to \mathrm{LMod}_{B^0}(\mathcal{C})$ is comonadic. It follows from Remark D.6.4.4 that we can identify $\mathrm{LMod}_{B^0}(\mathcal{C})$ with the ∞-category $\mathrm{LMod}_T(\mathrm{LMod}_B(\mathcal{C}))$, where T denotes the monad given by $C \mapsto B^0 \otimes_B C$. The faithful flatness of B^0 over B guarantees that the monad T is faithfully flat (Lemma D.6.4.5), so that the comonadicity of F is a consequence of Proposition D.6.4.6.

For each integer $n \geqslant 0$, let $\chi_n : \mathrm{CAlg}_A^{\mathrm{cn}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the functor given by

$$\chi_n(B) = (\tau_{\leqslant n}\,\mathcal{S}) \otimes \mathrm{Mod}_B^{\mathrm{cn}} \otimes_{\mathrm{Mod}_A^{\mathrm{cn}}} \mathcal{C} \simeq \tau_{\leqslant n}\,\mathrm{LMod}_B(\mathcal{C}),$$

which assigns to each connective A-algebra B the ∞-category of n-truncated B-module objects of $\mathcal{C}$. We claim that χ_n is an $\mathcal{P}\mathrm{r}^{\mathrm{L}}$-valued sheaf with respect to the fpqc topology. To prove this, it will suffice to show that for every faithfully flat map between connective A-algebras $B \to B^0$ as above having Čech nerve $B^\bullet$, the upper horizontal map in the commutative diagram of ∞-categories

$$\begin{array}{ccc} \tau_{\leqslant n}\,\mathrm{LMod}_B(\mathcal{C}) & \longrightarrow & \varprojlim \tau_{\leqslant n}\,\mathrm{LMod}_{B^\bullet}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{LMod}_B(\mathcal{C}) & \longrightarrow & \varprojlim \mathrm{LMod}_{B^\bullet}(\mathcal{C}) \end{array}$$

is an equivalence of ∞-categories (beware that the vertical maps do not preserve small colimits). It follows from the first part of the proof that the lower horizontal map preserves small colimits, and the vertical maps are fully faithful embeddings. Unwinding the definitions, we are reduced to proving that an object $C \in \mathrm{LMod}_B(\mathcal{C})$ is n-truncated whenever $B^0 \otimes_B C$ is n-truncated. This is clear: the prestable ∞-category $\mathcal{C}$ is complete and therefore also separated, so that an object $C \in \mathrm{LMod}_B(\mathcal{C})$ is n-truncated if and only if the homotopy objects $\pi_m C$ vanish for $m > n$, and we have monomorphisms $\pi_m C \to \pi_m(B^0 \otimes_B C)$ in the abelian category $\mathcal{C}^\heartsuit$ by virtue of Remark D.6.4.3.

Note that for each $n \geqslant 0$, the functor χ_n takes values in the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ spanned by the presentable $(n+1)$-categories. Since this subcategory is itself an $(n+2)$-category, the functor χ_n is automatically $(n+1)$-truncated and in particular hypercomplete.

The completeness of $\mathcal{C}$ implies that each of the prestable ∞-categories $\mathrm{LMod}_B(\mathcal{C})$ is also complete (Proposition D.5.1.3), so that the functor $B \mapsto \mathrm{LMod}_B(\mathcal{C})$ is given by the inverse limit $\varprojlim_{n \geqslant 0} \chi_n(B)$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and is therefore also hypercomplete. □

D.6.6 Effective Descent for Complete Prestable ∞-Categories

We now discuss a categorification of Theorem D.6.3.1, where our attention is focused not on objects of A-linear ∞-categories, but on the A-linear ∞-categories themselves.

Notation D.6.6.1. For every connective $\mathbb{E}_2$-ring A, let $\mathrm{LinCat}_A^{\mathrm{comp}}$ denote the full subcategory of $\mathrm{LinCat}_A^{\mathrm{PSt}}$ spanned by those prestable A-linear ∞-categories which are complete (in the sense of Definition C.1.2.12). It follows from Proposition D.5.1.3 that for any morphism $\phi : A \to B$ of connective $\mathbb{E}_2$-rings, the extension of scalars functor $\mathcal{C} \mapsto B \otimes_A \mathcal{C}$ carries $\mathrm{LinCat}_A^{\mathrm{comp}}$ into $\mathrm{LinCat}_B^{\mathrm{comp}}$. Consequently, we can regard the construction $A \mapsto \mathrm{LinCat}_A^{\mathrm{comp}}$ as a functor from the ∞-category $\mathrm{Alg}_{\mathbb{E}_2}^{\mathrm{cn}}$ of connective $\mathbb{E}_2$-rings to the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$.

Proposition D.6.6.2. *The functor*

$$\mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty \qquad A \mapsto \mathrm{LinCat}_A^{\mathrm{comp}}$$

is a $\widehat{\mathcal{C}\mathrm{at}}_\infty$-valued sheaf with respect to the fpqc topology on $(\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}}$.

Remark D.6.6.3. We will see in a moment that the sheaf $A \mapsto \mathrm{LinCat}_A^{\mathrm{comp}}$ is hypercomplete (Theorem D.6.8.1).

Proof of Proposition D.6.6.2. We will show that the construction $A \mapsto \mathrm{LinCat}_A^{\mathrm{comp}}$ satisfies the hypotheses of Proposition A.3.3.1. Note that since the construction $A \mapsto \mathrm{LinCat}_A^{\mathrm{comp}}$ satisfies descent for the étale topology (Remark D.6.8.2), it commutes with finite products. It will therefore suffice to verify condition (2) of Proposition A.3.3.1. Suppose we are given a faithfully flat map of $\mathbb{E}_\infty$-rings $f : A \to A^0$ having Čech nerve $A^\bullet$. We wish to show that the induced map $\mathrm{LinCat}_A^{\mathrm{comp}} \to \varprojlim \mathrm{LinCat}_{A^\bullet}^{\mathrm{comp}}$ is an equivalence of ∞-categories. We proceed by showing that this functor satisfies the conditions of Proposition HA.5.2.2.36 :

(a) Fix an morphism $F : \mathcal{C} \to \mathcal{D}$ in the ∞-category $\mathrm{LinCat}_A^{\mathrm{comp}}$ whose image in $\mathrm{LinCat}_{A^0}^{\mathrm{comp}}$ is an equivalence. It follows that F induces an equivalence of cosimplicial ∞-categories $\mathrm{LMod}_{A^\bullet}(\mathcal{C}) \to \mathrm{LMod}_{A^\bullet}(\mathcal{D})$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & & \downarrow \\ \varprojlim \mathrm{LMod}_{A^\bullet}(\mathcal{C}) & \longrightarrow & \varprojlim \mathrm{LMod}_{A^\bullet}(\mathcal{D}) \end{array}$$

where the vertical maps are equivalences of ∞-categories (since $\mathcal{C}$ and $\mathcal{D}$ satisfy flat descent by virtue of Theorem D.6.3.1). It follows that F is an equivalence of ∞-categories.

(b) Let $q : \mathcal{E} \to \mathrm{CAlg}^{\mathrm{cn}}$ be a coCartesian fibration classified by the functor $A \mapsto \mathrm{LinCat}_A^{\mathrm{comp}}$: more informally, $\mathcal{E}$ is the ∞-category whose objects are pairs $(A, \mathcal{C})$ where A is a connective $\mathbb{E}_\infty$-ring and $\mathcal{C}$ is a complete prestable A-linear ∞-category. Suppose we are given a diagram $X^\bullet : \mathbf{\Delta} \to \mathcal{E}$ which carries each morphism in $\mathbf{\Delta}$ to a q-coCartesian morphism of $\mathcal{E}$ and satisfies $q(X^\bullet) = A^\bullet$. We must show that $X^\bullet$ can be extended to a q-limit diagram $\overline{X}^\bullet : \mathbf{\Delta}_+ \to \mathcal{E}$ which also carries each morphism of $\mathbf{\Delta}_+$ to a q-coCartesian morphism in $\mathcal{E}$. Write $X^\bullet = (A^\bullet, \mathcal{C}^\bullet)$. To prove the existence of the q-limit diagram $\overline{X}^\bullet$, we must show that the cosimplicial object $\mathcal{C}^\bullet$ admits a totalization $\mathcal{C}$ in the ∞-category $\mathrm{LinCat}_A^{\mathrm{PSt}}$ of prestable A-linear ∞-categories, and that $\mathcal{C}$ is complete. Note that the completeness of $\mathcal{C}$ is automatic, since the ∞-category of complete prestable ∞-categories is a localization of Groth_∞ and is therefore closed under small limits (Proposition C.3.6.3). To prove that $\mathcal{C}^\bullet$ admits a totalization, it will suffice to show that the underlying cosemisimplicial object $([n] \in \mathbf{\Delta}_s) \mapsto \mathcal{C}^n$ admits a totalization in the ∞-category Groth_∞. This is a special case of Proposition **??**, since the flatness of A^0 over A guarantees that the transition functors $\mathcal{C}^m \to \mathcal{C}^n \simeq \mathrm{LMod}_{A^n}(\mathcal{C}^m)$ associated to injective maps $[m] \hookrightarrow [n]$ are left exact.

To complete the proof, we must show that the diagram $\overline{X}^\bullet$ carries each morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}_+$ to a q-coCartesian morphism in $\mathcal{E}$. If $m \neq -1$, then this follows from the analogous assumption on $X^\bullet$. If $m = -1$, then we can factor α as a composition $[-1] \to [0] \to [n]$ and thereby reduce to the case $n = 0$. In this case, the desired result follows from the calculation

$$\mathrm{LMod}_{A^0}(\mathcal{C}) \simeq \mathrm{LMod}_{A^0}(\varprojlim \mathcal{C}^\bullet) \simeq \varprojlim \mathrm{LMod}_{A^0}(\mathcal{C}^\bullet) \simeq \varprojlim \mathcal{C}^{\bullet+1} \simeq \mathcal{C}^0 .$$

□

D.6.7 Digression on Hypercompleteness

Our next goal is to show that the sheaf $A \mapsto \mathrm{LinCat}_A^{\mathrm{comp}}$ of Proposition D.6.6.2 is hypercomplete. We begin with some general remarks about hypercomplete objects of ∞-topoi.

Lemma D.6.7.1. *Let $\mathcal{X}$ be an ∞-topos containing an object X. The following conditions are equivalent:*

(1) *The object $X \in \mathcal{X}$ is hypercomplete.*

(2) *For every ∞-connective morphism $E \to E'$ in $\mathcal{X}$, the map $\mathrm{Map}_{\mathcal{X}}(E', X) \to \mathrm{Map}_{\mathcal{X}}(E, X)$ is surjective on connected components.*

Proof. The implication (1) ⇒ (2) is obvious. Suppose that (2) is satisfied. We wish to prove that for every ∞-connective morphism $\alpha : E \to E'$, the map $\theta_\alpha : \mathrm{Map}_{\mathcal{X}}(E', X) \to \mathrm{Map}_{\mathcal{X}}(E, X)$ is a homotopy equivalence. We will prove that θ_α is n-connective using induction on n, the case $n = -1$ being vacuous. Since θ_α is surjective on connected components (by (2)), it will suffice to show that the diagonal map

$$\mathrm{Map}_{\mathcal{X}}(E', X) \to \mathrm{Map}_{\mathcal{X}}(E', X) \times_{\mathrm{Map}_{\mathcal{X}}(E,X)} \mathrm{Map}_{\mathcal{X}}(E', X) \simeq \mathrm{Map}_{\mathcal{X}}(E' \amalg_E E', X)$$

is $(n-1)$-connected. This follows from the inductive hypothesis, since the codiagonal $E' \amalg_E E' \to E'$ is also ∞-connective. □

Lemma D.6.7.2. *Let $\mathcal{X}$ be an ∞-topos. Then the collection of hypercomplete objects of $\mathcal{X}$ is closed under small coproducts.*

Proof. Suppose we are given a collection of hypercomplete objects $\{X_\alpha\}_{\alpha \in A}$ having coproduct $X \in \mathcal{X}$. We wish to prove that X is hypercomplete. According to Lemma D.6.7.1, it will suffice to show that if $\phi : E \to E'$ is an ∞-connective morphism in $\mathcal{X}$, then the induced map $\mathrm{Map}_{\mathcal{X}}(E', X) \to \mathrm{Map}_{\mathcal{X}}(E, X)$ is surjective on connected components. Fix a morphism $f : E \to X$. For each index α, let E_α denote the fiber product $X_\alpha \times_X E$, so that the induced map $E_\alpha \to E'$ admits a factorization

$$E_\alpha \xrightarrow{g_\alpha} E'_\alpha \xrightarrow{h_\alpha} E'$$

where g_α is an effective epimorphism and h_α is a monomorphism. Let $E'' = \amalg_\alpha E'_\alpha$, so that the maps h_α induce a map $\psi : E'' \to E'$. We claim that ψ is an equivalence. Since ϕ factors through ψ, we deduce that ψ is an effective epimorphism. It will therefore suffice to show that the diagonal map

$$\amalg_\alpha E'_\alpha E'' \to E'' \times_{E'} E'' \simeq \amalg_{\alpha,\beta} E'_\alpha \times_{E'} E'_\beta$$

is an equivalence. Because each h_α is a monomorphism, each of the diagonal maps $E'_\alpha \to E'_\alpha \times_{E'} E'_\alpha$ is an equivalence; we are therefore reduced to proving that $E'_\alpha \times_{E'} E'_\beta \simeq \varnothing$ for $\alpha \neq \beta$. This follows from the existence of an effective epimorphism

$$\varnothing \simeq E_\alpha \times_E E_\beta \to E_\alpha \times_{E'} E_\beta \to E'_\alpha \times_{E'} E'_\beta.$$

This completes the proof that ψ is an equivalence, so that we can identify ϕ with the coproduct of morphisms $\phi_\alpha : E_\alpha \to E'_\alpha$. To prove that f factors through ϕ, it suffices to show that each restriction $f|_{E_\alpha}$ factors through ϕ_α. This follows from our assumption that X_α is hypercomplete, since ϕ_α is a pullback of ϕ and therefore ∞-connective. □

Lemma D.6.7.3. *Let $\mathcal{X}$ be an ∞-topos, and let $f : U \to X$ be an effective epimorphism in $\mathcal{X}$. Assume that U is hypercomplete. Then the following conditions are equivalent:*

(1) *The object X is hypercomplete.*

(2) *The fiber product $U \times_X U$ is hypercomplete.*

Proof. The implication (1) ⇒ (2) is obvious, since the full subcategory $\mathcal{X}^{\mathrm{hyp}} \subseteq \mathcal{X}$ spanned by the hypercomplete objects is closed under small limits. We will prove that (2) ⇒ (1). Let $L : \mathcal{X} \to \mathcal{X}^{\mathrm{hyp}}$ be a left adjoint to the inclusion, so that L is left exact (see §HTT.6.5.2). Let $U_\bullet$ be a Čech nerve of the map $U \to X$, so that $LU_\bullet$ is a Čech nerve of the induced map $LU \to LX$. Using assumption (2) and our assumption that U is hypercomplete, we deduce that U_0 and U_1 are both hypercomplete, so that $U_n \simeq U_1 \times_{U_0} \cdots \times_{U_0} U_1$ is hypercomplete for all $n \geqslant 0$. It follows that the canonical map $U_\bullet \to LU_\bullet$ is an equivalence. We therefore obtain equivalences $X \simeq |U_\bullet| \simeq |LU_\bullet| \simeq LX$, so that X is also hypercomplete (the last equivalence here results from the observation that $LU \to LX$ is an effective epimorphism, since it can be identified with the composition of f with the ∞-connective map $X \to LX$). □

Proposition D.6.7.4. *Let $\mathcal{C}$ be an ∞-category which admits finite limits and is equipped with a Grothendieck topology. Assume that every object $C \in \mathcal{C}$ represents a functor $e_C : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ which is a hypercomplete sheaf on $\mathcal{C}$. Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ be a sheaf on $\mathcal{C}$. The following conditions are equivalent:*

(1) *The sheaf $\mathscr{F}$ is hypercomplete.*

(2) *For every pair of objects $C, C' \in \mathcal{C}$ and maps $\eta : e_C \to \mathscr{F}$, $\eta' : e_{C'} \to \mathscr{F}$, the fiber product $e_C \times_{\mathscr{F}} e_{C'}$ is hypercomplete.*

(3) *For every object $C \in \mathcal{C}$ and every pair of maps $\eta, \eta' : e_C \to \mathscr{F}$, the equalizer of the diagram*

$$e_C \underset{\eta'}{\overset{\eta}{\rightrightarrows}} \mathscr{F}$$

is hypercomplete.

Proof. The implication (1) ⇒ (3) is clear, since the collection of hypercomplete objects of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is stable under small limits. The implication (3) ⇒ (2) follows from the observation that $e_C \times_{\mathscr{F}} e_{C'}$ can be identified with the equalizer of the pair of maps

$$\mathscr{F} \leftarrow e_C \leftarrow e_{C \times C'} \to e_{C'} \to \mathscr{F} .$$

We will prove that (2) ⇒ (1). Let $\mathscr{F}' = \amalg_{\eta \in \mathscr{F}(C)} e_C$, so we have an effective epimorphism $\mathscr{F}' \to \mathscr{F}$. Lemma D.6.7.2 implies that $\mathscr{F}'$ is hypercomplete. By virtue of Lemma D.6.7.3, it will suffice to prove that the fiber product $\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}'$ is hypercomplete. This fiber product can be identified with the coproduct

$$\amalg_{\eta \in \mathscr{F}(C), \eta' \in \mathscr{F}(C')} e_C \times_{\mathscr{F}} e_{C'}$$

which is hypercomplete by virtue of assumption (2) and Lemma D.6.7.2. □

D.6.8 Effective Hyperdescent for Complete Prestable ∞-Categories

Our final goal in this section is to establish a slightly stronger form of Proposition D.6.6.2:

Theorem D.6.8.1. *The functor*

$$\mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty \qquad A \mapsto \mathrm{LinCat}_A^{\mathrm{comp}}$$

is a hypercomplete $\widehat{\mathcal{C}\mathrm{at}}_\infty$*-valued sheaf with respect to the fpqc topology on* $(\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}}$.

Remark D.6.8.2. Theorem D.6.8.1 can be regarded as an analogue for the fpqc topology of some of the results of §D.4: it follows from Theorem D.4.1.2 and Proposition D.5.1.3 that the construction $A \mapsto \mathrm{LinCat}_A^{\mathrm{comp}}$ satisfies descent for the étale topology.

Remark D.6.8.3. In the statement of Theorem D.6.8.1, the restriction to the setting of *complete* prestable ∞-categories is needed only to guarantee that the prestable ∞-categories $\mathcal{C}$ in question satisfy flat hyperdescent. Several variants are possible: for example, if we let $\mathrm{LinCat}_A^{\flat}$ denote the full subcategory of $\mathrm{LinCat}_A^{\mathrm{PSt}}$ spanned by those prestable ∞-categories which satisfy flat (hyper)descent, then the same argument shows that the construction $A \mapsto \mathrm{LinCat}_A^{\flat}$ is a (hypercomplete) sheaf with respect to the fpqc topology.

Corollary D.6.8.4. *The functor*

$$\mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty \qquad A \mapsto \mathrm{LinCat}_A^{\mathrm{Ab}}$$

is a hypercomplete $\widehat{\mathcal{C}\mathrm{at}}_\infty$*-valued sheaf with respect to the fpqc topology on* $(\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}}$.

Remark D.6.8.5. The sheaf $A \mapsto \mathrm{LinCat}_A^{\mathrm{Ab}}$ is automatically hypercomplete, since it is 2-truncated.

Proof of Corollary D.6.8.4. We will show that the construction $A \mapsto \mathrm{LinCat}_A^{\mathrm{Ab}}$ satisfies hypothesis (2) of Proposition A.3.3.1 (hypothesis (1) is easy and left to the reader). Suppose we are given a faithfully flat map of $\mathbb{E}_\infty$-rings $f : A \to A^0$ having Čech nerve $A^\bullet$. We wish to show that the induced map $\theta : \mathrm{LinCat}_A^{\mathrm{Ab}} \to \varprojlim \mathrm{LinCat}_{A^\bullet}^{\mathrm{Ab}}$ is an equivalence of ∞-categories. We first claim that θ is fully faithful. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian A-linear ∞-categories. Unwinding the definitions, we wish to prove that the canonical map

$$\begin{aligned} \mathrm{Map}_{\mathrm{LinCat}_A^{\mathrm{Ab}}}(\mathcal{C}, \mathcal{D}) &\to \varprojlim \mathrm{Map}_{\mathrm{LinCat}_{A^\bullet}^{\mathrm{Ab}}}(A^\bullet \otimes_A \mathcal{C}, A^\bullet \otimes_A \mathcal{D}) \\ &\simeq \varprojlim \mathrm{Map}_{\mathrm{LinCat}_A^{\mathrm{Ab}}}(\mathcal{C}, A^\bullet \otimes_A \mathcal{D}) \end{aligned}$$

is a homotopy equivalence. To prove this, it suffice to show that the canonical map $\mathcal{D} \to \varprojlim A^\bullet \otimes_A \mathcal{D}$ is an equivalence, which follows from Corollary D.6.3.2 and Proposition C.5.4.21.

To show that θ is essentially surjective, it will suffice to show that θ induces a homotopy equivalence $\theta^{\simeq} : (\mathrm{LinCat}_A^{\mathrm{Ab}})^{\simeq} \to \varprojlim(\mathrm{LinCat}_{A^\bullet}^{\mathrm{Ab}})^{\simeq}$. Let $X^\bullet$ denote the augmented cosimplicial space given by $X^n = (\mathrm{LinCat}_{A^n}^{\mathrm{Ab}})^{\simeq}$ (where, by convention, we set $A^{-1} = A$). For each $n \geqslant -1$, let Y^n denote the full subcategory of $(\mathrm{LinCat}_{A^n}^{\mathrm{comp}})^{\simeq}$ spanned by the complicial prestable A^n-linear ∞-categories. According to Proposition D.5.7.1, the condition that a complete prestable ∞-category be complicial is stable under base change along flat morphisms and can be tested locally for the fpqc topology. Consequently, we can regard the construction $[n] \mapsto Y^n$ as a functor $\mathbf{\Delta}_{s,+} \to \widehat{\mathcal{S}}$. For each $n \geqslant -1$, the construction

$$(\mathcal{C} \in \mathrm{LinCat}_{A^n}^{\mathrm{comp}}) \mapsto (\mathcal{C}^{\heartsuit} \in \mathrm{LinCat}_{A^n}^{\mathrm{Ab}})$$

determines a map of spaces $Y^n \to X^n$, which is a homotopy equivalence by virtue of Example D.1.3.9 and Corollary C.5.9.5. This map depends functorially on n, and therefore exhibits $Y^\bullet$ as the underlying augmented cosemisimplicial space of the augmented cosimplicial space $X^\bullet$. By virtue of Lemma **??**, to show that $X^\bullet$ is a limit diagram in $\widehat{\mathcal{S}}$, it will suffice to show that $Y^\bullet$ is a limit diagram in $\widehat{\mathcal{S}}$. This follows immediately from Theorem D.6.8.1 and Proposition D.5.7.1. □

The proof of Theorem D.6.8.1 will require a brief digression. Suppose that $\mathcal{C}$ is a small ∞-category, and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ be a presheaf of ∞-categories on $\mathcal{C}$, classified by a Cartesian fibration $p : \widetilde{\mathcal{C}} \to \mathcal{C}$. According to Proposition HTT.3.3.3.1 , we can identify $\varprojlim \mathscr{F}$ with the full subcategory of $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \widetilde{\mathcal{C}})$ spanned by the Cartesian sections of p. Let $X, Y \in \varprojlim \mathscr{F} \subseteq \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \widetilde{\mathcal{C}})$ so that the pair (X, Y) determines a functor $\mathcal{C} \to \mathcal{C}' = \mathrm{Fun}(\partial\,\Delta^1, \widetilde{\mathcal{C}}) \times_{\mathrm{Fun}(\partial\,\Delta^1, \mathcal{C})} \mathcal{C}$. We let $\mathcal{D}$ denote the fiber product $\mathrm{Fun}(\Delta^1, \widetilde{\mathcal{C}}) \times_{\mathcal{C}'} \mathcal{C}$. The projection $\mathcal{D} \to \mathcal{C}$ is a right fibration, whose fiber over an object $C \in \mathcal{C}$ can be identified with the Kan complex $\mathrm{Hom}_{\widetilde{\mathcal{C}}_C}(X(C), Y(C))$ (see §HTT.1.2.2). This right fibration is classified by a functor $\underline{\mathrm{Hom}}_{\mathscr{F}}(X, Y) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, given informally by the formula $\underline{\mathrm{Hom}}_{\mathscr{F}}(X, Y)(C) = \mathrm{Map}_{\mathscr{F}(C)}(X(C), Y(C))$. Let $\mathcal{D}_0$ be the full subcategory of $\mathcal{D}$ whose fiber over an object $C \in \mathcal{C}$ is given by the full subcategory of $\mathrm{Hom}_{\widetilde{\mathcal{C}}_C}(X(C), Y(C))$ spanned by the equivalences in $\widetilde{\mathcal{C}}_C$. The projection $\mathcal{D}_0 \to \mathcal{C}$ is also a right fibration, classified by a functor $\underline{\mathrm{Hom}}^{\simeq}_{\mathscr{F}}(X, Y) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$.

Proposition D.6.8.6. *Let $\mathcal{C}$ be an ∞-category which admits finite limits and is equipped with a Grothendieck topology. Assume that for every object $C \in \mathcal{C}$, the functor $e_C : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ represented by C is a hypercomplete sheaf on $\mathcal{C}$. Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ be a $\mathcal{C}\mathrm{at}_\infty$-valued sheaf on $\mathcal{C}$. The following conditions are equivalent:*

(1) *The sheaf $\mathscr{F}$ is hypercomplete.*

(2) *For every object $C \in \mathcal{C}$ and every pair of objects $X, Y \in \mathscr{F}(C) \simeq \varprojlim \mathscr{F}|_{\mathcal{C}^{\mathrm{op}}_{/C}}$, the functor $\underline{\mathrm{Hom}}_{\mathscr{F}}(X, Y) : (\mathcal{C}_{/C})^{\mathrm{op}} \to \mathcal{S}$ is a hypercomplete sheaf on $\mathcal{C}_{/C}$.*

Proof. Suppose first that (1) is satisfied; we will prove (2). Replacing $\mathcal{C}$ by $\mathcal{C}_{/C}$, we may suppose that $X, Y \in \varprojlim \mathscr{F}$. For every simplicial set K, let $\mathscr{F}^K$ denote the composition

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{Fun}(K,\bullet)} \mathcal{C}\mathrm{at}_\infty .$$

and let $*$ denote the constant functor $\mathscr{F}^{\varnothing} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ taking the value Δ^0. Then the pair (X, Y) determines a natural transformation $* \to \mathscr{F}^{\partial \Delta^0}$, and $\underline{\mathrm{Hom}}_{\mathscr{F}}(X, Y)$ can be identified with the fiber product $* \times_{\mathscr{F}^{\partial \Delta^1}} \mathscr{F}^{\Delta^1}$. Since $\mathscr{F}$ is hypercomplete, we deduce that $\mathscr{F}^{\Delta^1}$, $\mathscr{F}^{\partial \Delta^1}$, and $* \simeq \mathscr{F}^{\varnothing}$ are hypercomplete. It follows that $\underline{\mathrm{Hom}}_{\mathscr{F}}(X, Y)$ is a hypercomplete $\mathcal{C}\mathrm{at}_\infty$-valued sheaf on $\mathcal{C}$, and therefore a hypercomplete $\mathcal{S}$-valued sheaf on $\mathcal{C}$ (since the inclusion $\mathcal{S} \subseteq \mathcal{C}\mathrm{at}_\infty$ preserves small limits).

Now assume (2). Fix an object $C \in \mathcal{C}$, and let $f : x \to y$ be a morphism in $\mathscr{F}(C)$. Since $\mathscr{F}$ is a sheaf, we deduce that f is an equivalence if and only if there exists a covering sieve $\{C_\alpha \to C\}$ on C such that the image of f under each of the induced functors $\mathscr{F}(C) \to \mathscr{F}(C_\alpha)$ is an equivalence. Combining this observation with (2) and Lemma D.4.3.2, we deduce:

(∗) For every object $C \in \mathcal{C}$ and every pair of objects $X, Y \in \mathscr{F}(C) \simeq \varprojlim \mathscr{F}|_{\mathcal{C}^{\mathrm{op}}_{/C}}$, the functor $\underline{\mathrm{Hom}}^{\simeq}_{\mathscr{F}}(X, Y) : (\mathcal{C}_{/C})^{\mathrm{op}} \to \mathcal{S}$ is a hypercomplete sheaf on $\mathcal{C}_{/C}$.

For every ∞-category $\mathcal{D}$, let $\chi_{\mathcal{D}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ be the functor corepresented by $\mathcal{D}$. Let $\mathcal{C}\mathrm{at}'_\infty$ denote the full subcategory of $\mathcal{C}\mathrm{at}_\infty$ spanned by those ∞-categories $\mathcal{D}$ for which the composite functor

$$\mathscr{F}_{\mathcal{D}} : \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathcal{C}\mathrm{at}_\infty \xrightarrow{\chi_{\mathcal{D}}} \mathcal{S}$$

is a hypercomplete sheaf on $\mathcal{C}$. We wish to prove that $\mathcal{C}\mathrm{at}'_\infty = \mathcal{C}\mathrm{at}_\infty$. Since the collection of hypercomplete sheaves is stable under small limits in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ and the construction $\mathcal{D} \mapsto \mathscr{F}_{\mathcal{D}}$ carries colimits to limits, we conclude that $\mathcal{C}\mathrm{at}'_\infty \subseteq \mathcal{C}\mathrm{at}_\infty$ is stable under small colimits. By virtue of Lemma D.4.3.1, it will suffice to show that $\Delta^0, \Delta^1 \in \mathcal{C}\mathrm{at}'_\infty$. The inclusion $\Delta^0 \in \mathcal{C}\mathrm{at}'_\infty$ follows from (∗) together with Proposition D.6.7.4. It follows that $\partial \Delta^1 \in \mathcal{C}\mathrm{at}'_\infty$, so that $\mathscr{F}_{\partial \Delta^1}$ is a hypercomplete sheaf on $\mathcal{C}$. Applying Lemma D.4.3.2 to the restriction map $\mathscr{F}_{\Delta^1} \to \mathscr{F}_{\partial \Delta^1}$, we deduce that $\mathscr{F}_{\Delta^1}$ is hypercomplete so that $\Delta^1 \in \mathcal{C}\mathrm{at}'_\infty$ as desired (the hypotheses of Lemma D.4.3.2 are satisfied by virtue of assumption (2)). □

Proof of Theorem D.6.8.1. Proposition D.6.6.2 shows that the functor $A \mapsto \mathrm{LinCat}^{\mathrm{comp}}_A$ is a sheaf with respect to the fpqc topology. We wish to show show that this sheaf is hypercomplete. Note that every object of $\mathrm{CAlg}^{\mathrm{cn}}$ corepresents a hypercomplete sheaf on $(\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}}$ (this follows from Theorem D.6.3.5). We will complete the proof by showing that the construction $A \mapsto \mathrm{LinCat}^{\mathrm{comp}}_A$ satisfies the criterion of Proposition D.6.8.6. For this, we must show that for every connective $\mathbb{E}_\infty$-ring A and every pair of objects $\mathcal{C}, \mathcal{D} \in \mathrm{LinCat}^{\mathrm{comp}}_A$, the functor

$$\mathrm{CAlg}^{\mathrm{cn}}_A \to \widehat{\mathcal{S}} \qquad B \mapsto \mathrm{Map}_{\mathrm{LinCat}^{\mathrm{comp}}_B}(B \otimes_A \mathcal{C}, B \otimes_A \mathcal{D})$$

is a hypercomplete sheaf with respect to the fpqc topology on $(\mathrm{CAlg}_A^{\mathrm{cn}})^{\mathrm{op}}$. This is clear, since the prestable A-linear ∞-category $\mathcal{D}$ satisfies flat hyperdescent (Theorem D.6.3.1). $\square$

D.7 Duality for Stable ∞-Categories

Let R be an $\mathbb{E}_\infty$-ring and let $\mathrm{LinCat}_R^{\mathrm{St}}$ denote the ∞-category of stable R-linear ∞-categories (see Variant D.1.5.1). Then $\mathrm{LinCat}_R^{\mathrm{St}}$ is a symmetric monoidal ∞-category with respect to the R-linear tensor product $\otimes_R$ (see Variant D.2.3.3). In particular, one can consider dualizable objects of $\mathrm{LinCat}_R^{\mathrm{St}}$, in the sense of §HA.4.6.1 . Our goal in this section is to prove the following result:

Theorem D.7.0.7. *Let R be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. Then $\mathcal{C}$ is dualizable (as an object of the symmetric monoidal ∞-category $\mathrm{LinCat}_R^{\mathrm{St}}$) if and only if it is compactly assembled (see Definition 21.1.2.1). In particular, if $\mathcal{C}$ is compactly generated, then it is dualizable.*

Our proof of Theorem D.7.0.7 has three essentially disjoint steps. We begin by showing that a compactly generated stable ∞-category $\mathcal{C}$ is always dualizable when viewed as an object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ (Proposition D.7.2.3). We then apply this result to give a precise characterization of the dualizable objects of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ (Proposition D.7.3.1). We then complete the proof by showing that an object of $\mathrm{LinCat}_R^{\mathrm{St}}$ is dualizable if and only if its image under the forgetful functor $\mathrm{LinCat}_R^{\mathrm{St}} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$ is dualizable (Corollary D.7.7.6).

D.7.1 Digression: Mapping Spectra

We begin by introducing some terminology which is useful for studying R-linear ∞-categories in general.

Notation D.7.1.1. Let $\mathcal{C}$ be an ∞-category which is left tensored over a monoidal ∞-category $\mathcal{E}$. Given a pair of objects $C, D \in \mathcal{C}$, we let $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ denote an object of $\mathcal{E}$ which classifies morphisms from C to D (if such an object exists). That is, $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ is an object of $\mathcal{E}$ equipped with a map $\alpha : \underline{\mathrm{Map}}_{\mathcal{C}}(C, D) \otimes C \to D$ with the following universal property: for every object $E \in \mathcal{E}$, composition with α induces a homotopy equivalence $\mathrm{Map}_{\mathcal{E}}(E, \underline{\mathrm{Map}}_{\mathcal{C}}(C, D)) \to \mathrm{Map}_{\mathcal{C}}(E \otimes C, D)$. Note that if such a pair $(\underline{\mathrm{Map}}_{\mathcal{C}}(C, D), \alpha)$ exists, then it is well-defined up to a contractible space of choices. Moreover $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ is contravariantly functorial in C, and covariantly functorial in D.

Example D.7.1.2. Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. For every pair of objects $C, D \in \mathcal{C}$, the construction $(M \in \mathrm{LMod}_R) \mapsto \mathrm{Map}_{\mathcal{C}}(M \otimes_R C, D)$ determines a functor $\mathrm{LMod}_R^{\mathrm{op}} \to \mathcal{S}$ which preserves small limits, and is therefore representable by an object of LMod_R which we will denote by $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$.

Example D.7.1.3. Let $\mathcal{C}$ be a presentable stable ∞-category. Then there is an (essentially unique) action of the ∞-category Sp of spectra on $\mathcal{C}$ for which the underlying map $\mathrm{Sp} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. It follows from Example D.7.1.2 (in the special case $R = S$) that for every pair of objects $C, D \in \mathcal{C}$, the mapping object $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D) \in \mathrm{Sp}$ is well-defined.

Warning D.7.1.4. Notation D.7.1.1 is somewhat abusive, because the object $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ depends not only on the pair $C, D \in \mathcal{C}$ but also on the left action of an auxiliary ∞-category $\mathcal{E}$ on $\mathcal{C}$. This poses some danger of confusion. For example, let R be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a stable R-linear ∞-category. Then $\mathcal{C}$ is left tensored over both the monoidal ∞-category LMod_R of left R-modules and the full subcategory $\mathrm{LMod}_R^{\mathrm{cn}} \subseteq \mathrm{LMod}_R$ of connective left R-modules. Notation D.7.1.1 then assigns two different meanings to the expression $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$: we can consider a classifying object for morphisms from C to D in either LMod_R or $\mathrm{LMod}_R^{\mathrm{cn}}$. We will follow the convention of Example D.7.1.2: unless otherwise specified, the expression $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ indicates an object of LMod_R which classifies morphisms from C to D. The analogous classifying object in $\mathrm{LMod}_R^{\mathrm{cn}}$ can be identified with the connective cover $\tau_{\geq 0}\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$.

Remark D.7.1.5. Let $\phi : R' \to R$ be a morphism of $\mathbb{E}_2$-rings and let $\mathcal{C}$ be an R-linear stable ∞-category. Then we can also regard $\mathcal{C}$ as an R'-linear stable ∞-category (by neglect of structure). For every pair of objects $C, D \in \mathcal{C}$, the classifying object $\underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ does not depend on whether we view $\mathcal{C}$ as an R-linear stable ∞-category or an R'-linear stable ∞-category. More precisely, if we let $\underline{\mathrm{Map}}^R_{\mathcal{C}}(C, D) \in \mathrm{LMod}_R$ and $\underline{\mathrm{Map}}^{R'}_{\mathcal{C}}(C, D) \in \mathrm{LMod}_{R'}$ denote the relevant classifying objects, then $\underline{\mathrm{Map}}^{R'}_{\mathcal{C}}(C, D)$ can be identified with the image of $\underline{\mathrm{Map}}^R_{\mathcal{C}}(C, D)$ under the restriction-of-scalars functor $\mathrm{LMod}_R \to \mathrm{LMod}_{R'}$.

D.7.2 Duality for Compactly Generated ∞-Categories

Let $\mathcal{C}$ be a compactly generated stable ∞-category, so that there exists an equivalence $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}_c)$ where $\mathcal{C}_c$ denotes the full subcategory of $\mathcal{C}$ spanned by the compact objects. Our goal in this section is to show that $\mathcal{C}$ is a dualizable object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, whose dual can be identified with the compactly generated ∞-category $\mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}})$. This is a consequence of a more precise statement (Proposition D.7.2.3) which we prove below.

Construction D.7.2.1. Let $\mathcal{C}$ be a compactly generated stable ∞-category and let $\mathcal{C}_c$ denote the full subcategory of $\mathcal{C}$ spanned by the compact objects. Then the construction $(C, D) \mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(C, D)$ of Example D.7.1.3 determines a functor $\underline{\mathrm{Map}}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$, which we can identify with a map $\rho : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Sp})$. Let ρ_c denote the restriction of ρ to the full subcategory $\mathcal{C}_c^{\mathrm{op}} \subseteq \mathcal{C}^{\mathrm{op}}$. Note that the functor ρ preserves small limits, so that ρ_c is left exact and therefore (by virtue of the fact that the domain and codomain of ρ_c are stable) also right exact. Using Propositions HTT.5.3.5.10 and HTT.5.5.1.9, we deduce that this functor admits an essentially unique extension $\widehat{\rho}_c : \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{Sp})$ which preserves small colimits. We will identify this extension with a bifunctor

$$\Phi_{\mathcal{C}} : \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \times \mathcal{C} \to \mathrm{Sp}.$$

Lemma D.7.2.2. *Let $\mathcal{C}$ be a compactly generated stable ∞-category and let $\Phi_{\mathcal{C}} : \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \times \mathcal{C} \to \mathrm{Sp}$ be defined as in Construction D.7.2.1. Then the functor $\Phi_{\mathcal{C}}$ preserves small colimits separately in each variable.*

Proof. By construction, the functor $\Phi_{\mathcal{C}}$ preserves small colimits in the first variable. It will therefore suffice to show that for each object $X \in \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}})$, the functor $C \mapsto \Phi_{\mathcal{C}}(X, C) \in \mathrm{Sp}$ preserves small colimits. The collection of those objects $X \in \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}})$ which satisfy this condition is evidently closed under small colimits; we may therefore assume that X is the image of some object $D \in \mathcal{C}_c^{\mathrm{op}}$. We are therefore reduced to showing that the functor $C \mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(D, C)$ preserves small colimits. This functor evidently preserves small limits, and is therefore exact. We are therefore reduced to showing that the functor $C \mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(D, C)$ preserves filtered colimits: that is, that for every integer n, the construction $C \mapsto \Omega^{\infty+n}\underline{\mathrm{Map}}_{\mathcal{C}}(D, C)$ determines a functor $\mathcal{C} \to \mathcal{S}$ which preserves filtered colimits. This is clear, since this functor is corepresentable by the compact object $\Sigma^n D \in \mathcal{C}$. □

Let $\mathcal{C}$ be a compactly generated stable ∞-category and let $\mathcal{C}_c$ denote the full subcategory of $\mathcal{C}$ spanned by the compact objects. Using Lemma D.7.2.2, we see that the functor $\Phi_{\mathcal{C}}$ of Construction D.7.2.1 induces a colimit-preserving functor $e_{\mathcal{C}} : \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \otimes \mathcal{C} \to \mathrm{Sp}$, where the tensor product is taken in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ of presentable stable ∞-categories.

Proposition D.7.2.3. *Let $\mathcal{C}$ be a compactly generated stable ∞-category. Then the functor $e_{\mathcal{C}} : \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \otimes \mathcal{C} \to \mathrm{Sp}$ constructed above is a duality datum in the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ (see Definition HA.4.6.1.1). In particular, $\mathcal{C}$ is a dualizable object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, and its dual can be identified with $\mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}})$.*

Proof. According to Lemma HA.4.6.1.6, it will suffice to show that for every pair of presentable stable ∞-categories $\mathcal{D}$ and $\mathcal{E}$, the composite map

$$\theta : \mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}}(\mathcal{D}, \mathcal{C} \otimes \mathcal{E}) \to \mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}}(\mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \otimes \mathcal{D}, \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \otimes \mathcal{C} \otimes \mathcal{E}) \to \mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}}(\mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \otimes \mathcal{D}, \mathcal{E})$$

is a homotopy equivalence. Proposition HTT.5.5.1.9 and the definition of the tensor product on $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, we can identify $\mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}}(\mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \otimes \mathcal{D}, \mathcal{E})$ with the subcategory of $\mathrm{Fun}(\mathcal{C}_c^{\mathrm{op}} \times \mathcal{D}, \mathcal{E})$ whose objects are functors which are exact in the first argument and colimit-preserving in the second, and whose morphisms are equivalences. Under this identification, θ corresponds to the map

$$\mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}}(\mathcal{D}, \mathcal{C} \otimes \mathcal{E}) \to \mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}}(\mathcal{D}, \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_c^{\mathrm{op}}, \mathcal{E}))$$

given by composition with the equivalence

$$\mathcal{C} \otimes \mathcal{E} \simeq \mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_c^{\mathrm{op}}, \mathcal{E})$$

of Proposition HA.4.8.1.17 . □

D.7.3 Duality for Compactly Assembled ∞-Categories

We now extend Proposition D.7.2.3 to the setting of compactly assembled stable ∞-categories.

Proposition D.7.3.1. *Let $\mathcal{C}$ be a presentable stable ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is dualizable as an object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$.*

(2) *Let $F : \overline{\mathcal{D}} \to \mathcal{D}$ be a functor between presentable stable ∞-categories which admits a fully faithful right adjoint G. Then every colimit-preserving functor $U : \mathcal{C} \to \mathcal{D}$ is equivalent to $F \circ \overline{U}$ for some colimit-preserving functor $\overline{U} : \mathcal{C} \to \overline{\mathcal{D}}$.*

(3) *Let $F : \overline{\mathcal{C}} \to \mathcal{C}$ be a functor between presentable stable ∞-categories which admits a fully faithful right adjoint G. Then F admits a colimit-preserving section $G' : \mathcal{C} \to \overline{\mathcal{C}}$.*

(4) *The ∞-category $\mathcal{C}$ is a retract (in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ of presentable stable ∞-categories) of a compactly generated stable ∞-category.*

(5) *The ∞-category $\mathcal{C}$ is compactly assembled (in the sense of Definition 21.1.2.1).*

Proof. We first show that (1) ⇒ (2). Let $F : \overline{\mathcal{D}} \to \mathcal{D}$ be as in (2). Then F exhibits $\mathcal{D}$ as a quotient $\overline{\mathcal{D}}/\overline{\mathcal{D}}_0$, where $\overline{\mathcal{D}}_0$ is the localizing subcategory of $\overline{\mathcal{D}}$ spanned by objects which are annihilated by F. In particular, for any presentable stable ∞-category $\mathcal{E}$, composition with F induces a fully faithful embedding

$$\mathrm{LFun}(\mathcal{D}, \mathcal{C} \otimes \mathcal{E}) \to \mathrm{LFun}(\overline{\mathcal{D}}, \mathcal{C} \otimes \mathcal{E})$$

whose essential image is spanned by those functors which annihilate $\overline{\mathcal{D}}_0$. Let $\mathcal{C}^\vee$ denote a dual of $\mathcal{C}$, and let $\mathcal{A}$ denote the smallest localizing subcategory of $\mathcal{C}^\vee \otimes \overline{\mathcal{D}}$ which contains the essential image of $\mathcal{C}^\vee \otimes \overline{\mathcal{D}}_0$. Then, for any presentable stable ∞-category $\mathcal{E}$, the natural map $\mathrm{LFun}(\mathcal{C}^\vee \otimes \mathcal{D}, \mathcal{E}) \to \mathrm{LFun}(\mathcal{C}^\vee \otimes \overline{\mathcal{D}}, \mathcal{E})$ is a fully faithful embedding, whose essential image consists of those colimit-preserving functors $\mathcal{C}^\vee \otimes \overline{\mathcal{D}} \to \mathcal{E}$ which annihilate $\mathcal{A}$. It follows that the functor F exhibits $\mathcal{C}^\vee \otimes \mathcal{D}$ as the quotient of $\mathcal{C}^\vee \otimes \overline{\mathcal{D}}$ by the localizing subcategory $\mathcal{A}$. In particular, the functor $\mathcal{C}^\vee \otimes \overline{\mathcal{D}} \to \mathcal{C}^\vee \otimes \mathcal{D}$ is essentially surjective, which proves (2).

The implication (2) ⇒ (3) is trivial. We now prove (3) ⇒ (4). Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be an essentially small dense full subcategory. Enlarging $\mathcal{C}_0$ if necessary, we may assume that $\mathcal{C}_0$ is a stable subcategory of $\mathcal{C}$. Then the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ extends to a colimit-preserving functor $F : \mathrm{Ind}(\mathcal{C}_0) \to \mathcal{C}$, and the density of $\mathcal{C}_0$ in $\mathcal{C}$ guarantees that F has a fully faithful right adjoint G. Invoking assumption (3), we deduce that F admits a section $G' : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}_0)$ which preserves small colimits. Then F and G' exhibit $\mathcal{C}$ as a retract of $\mathrm{Ind}(\mathcal{C}_0)$.

Note that since the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ is idempotent complete, the collection of dualizable objects of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ is closed under retracts. Consequently, the implication (4) $\Rightarrow$ (1) follows from Proposition D.7.2.3. The implication (4) $\Rightarrow$ (5) is trivial. We now complete the proof by showing that (5) $\Rightarrow$ (1). We proceed as in the proof of Theorem 21.1.2.10. Let $G : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ be the Ind-extension of the identity functor $\mathrm{id}_{\mathcal{C}}$, and let $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ be a left adjoint to G (whose existence is guaranteed by our assumption that $\mathcal{C}$ is compactly assembled; see Theorem 21.1.2.10). If $\mathcal{C}_0 \subseteq \mathcal{C}$ is an essentially small full subcategory, we will abuse notation by identifying $\mathrm{Ind}(\mathcal{C}_0)$ with its essential image in $\mathrm{Ind}(\mathcal{C})$. Since $\mathcal{C}$ is accessible, there exists a small collection of objects $\{X_\alpha\}$ which generates $\mathcal{C}$ under small filtered colimits. Choose an essentially small stable subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ for which each $F(X_\alpha)$ belongs to the subcategory $\mathrm{Ind}(\mathcal{C}_0) \subseteq \mathrm{Ind}(\mathcal{C})$. Since the functor F commutes with small filtered colimits, it follows that F factors through a functor $F_0 : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}_0)$. Let $G_0 = G|_{\mathrm{Ind}(\mathcal{C}_0)}$. Then $G_0 \circ F_0$ is equivalent to the identity functor $\mathrm{id}_{\mathcal{C}}$, so that F_0 and G_0 exhibit $\mathcal{C}$ as a retract of $\mathrm{Ind}(\mathcal{C}_0)$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. □

D.7.4 Locally Rigid Monoidal ∞-Categories

In order to deduce Theorem D.7.0.7 from Proposition D.7.2.3, we will need to compare the condition of dualizability in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ with the *a priori* unrelated condition of dualizability in the ∞-category $\mathrm{LinCat}_R^{\mathrm{St}} = \mathrm{Mod}_{\mathrm{Mod}_R}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$, where R is an $\mathbb{E}_\infty$-ring. The comparison rests on some features of the monoidal ∞-category Mod_R which we now axiomatize.

Definition D.7.4.1. Let $\mathcal{C}$ be a monoidal stable ∞-category. We will say that $\mathcal{C}$ is *locally rigid* if it satisfies the following conditions:

(1) The ∞-category $\mathcal{C}$ is compactly generated.

(2) The tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable.

(3) The unit object $\mathbf{1} \in \mathcal{C}$ is compact.

(4) Every compact object of $\mathcal{C}$ admits a left dual and a right dual.

Example D.7.4.2. Let R be an $\mathbb{E}_2$-ring. Then LMod_R is a locally rigid stable monoidal ∞-category. That is, every perfect R-module is both left and right dualizable as an object of LMod_R. This is clear, since the collection of left and right dualizable objects of LMod_R is a stable subcategory which contains the unit object $R \in \mathrm{LMod}_R$.

Remark D.7.4.3. Let $\mathcal{C}$ be a locally rigid stable monoidal ∞-category and suppose that $X \in \mathcal{C}$ is an object which admits a right dual $X^\vee$. Then for any object $Y \in \mathcal{C}$, we have a canonical homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, Y \otimes X^\vee)$. Since the operation $Y \mapsto Y \otimes X^\vee$ commutes with filtered colimits and the unit object $\mathbf{1}$ is compact, we deduce that the functor $Y \mapsto \mathrm{Map}_{\mathcal{C}}(X, Y)$ commutes with filtered colimits: that is, X Is a compact object of $\mathcal{C}$. The same argument shows that every left dualizable object of $\mathcal{C}$ is compact. Consequently, the following conditions on an object $X \in \mathcal{C}$ are equivalent:

(a) The object $X \in \mathcal{C}$ is compact.

(b) The object X admits a left dual.

(c) The object X admits a right dual.

Remark D.7.4.4. Let $\mathcal{C}$ be a locally rigid monoidal ∞-category, let $\mathcal{M}$ and $\mathcal{N}$ be presentable ∞-categories which are left-tensored over $\mathcal{C}$, and suppose that the action maps

$$\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M} \qquad \mathcal{C} \times \mathcal{N} \to \mathcal{N}$$

preserves small colimits separately in each variable. Let $F : \mathcal{M} \to \mathcal{N}$ be a $\mathcal{C}$-linear functor which commutes with small colimits and let $G : \mathcal{N} \to \mathcal{M}$ be a right adjoint to F. For every pair of objects $C \in \mathcal{C}$ and $N \in \mathcal{N}$, the counit map $(F \circ G)(N) \to N$ induces a map

$$F(C \otimes G(N)) \to C \otimes (F \circ G)(N) \to C \otimes N,$$

which is adjoint to a morphism $\theta_{C,N} : C \otimes G(N) \to G(C \otimes N)$ in $\mathcal{C}$.

Suppose that the functor G preserves small colimits. We claim that each of the maps $\theta_{C,N}$ is an equivalence. Note that the construction $C \mapsto \theta_{C,N}$ commutes with filtered colimits. We can therefore reduce to the case where C is compact, and therefore right dualizable. We claim that for each object $M \in \mathcal{M}$, composition with $\theta_{C,N}$ induces a homotopy equivalence $\mathrm{Map}_{\mathcal{M}}(M, C \otimes G(N)) \to \mathrm{Map}_{\mathcal{M}}(M, G(C \otimes N))$. Unwinding the definitions, we see that this map is given by a composition of homotopy equivalences

$$\begin{aligned} \mathrm{Map}_{\mathcal{M}}(M, C \otimes G(N)) &\simeq \mathrm{Map}_{\mathcal{M}}(C^\vee \otimes M, G(N)) \\ &\simeq \mathrm{Map}_{\mathcal{N}}(F(C^\vee \otimes M), N) \\ &\simeq \mathrm{Map}_{\mathcal{N}}(C^\vee \otimes F(M), N) \\ &\simeq \mathrm{Map}_{\mathcal{N}}(F(M), C \otimes N) \\ &\simeq \mathrm{Map}_{\mathcal{M}}(M, G(C \otimes N)). \end{aligned}$$

Applying Remark **??**, we see that G can be regarded as a $\mathcal{C}$-linear functor from $\mathcal{N}$ to $\mathcal{M}$.

Example D.7.4.5. Let $\mathcal{C}$ and $\mathcal{D}$ be locally rigid stable monoidal ∞-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor which preserves small colimits. Then F carries (left or right) dualizable objects of $\mathcal{C}$ to (left or right) dualizable objects of $\mathcal{D}$, and therefore carries compact objects of $\mathcal{C}$ to compact objects of $\mathcal{D}$ (Remark D.7.4.3). Applying Proposition HTT.5.5.7.2 , we deduce that the functor F admits a right adjoint G which commutes with filtered colimits. Using Remark D.1.5.3, we conclude that G has the structure of a $\mathcal{C}$-linear functor.

D.7.5 Frobenius Algebras in $\mathcal{P}\mathrm{r}^{\mathrm{St}}$

Let $\mathcal{C}$ be a stable monoidal ∞-category. If $\mathcal{C}$ is presentable and the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable, then we can identify $\mathcal{C}$ with an associative algebra object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ of presentable stable ∞-categories. When $\mathcal{C}$ is locally rigid (Definition D.7.4.1), we can say more:

Proposition D.7.5.1. *Let $\mathcal{C}$ be a locally rigid monoidal stable ∞-category, and let $\lambda : \mathcal{C} \to \mathrm{Sp}$ denote the functor given by $\lambda(C) = \underline{\mathrm{Map}}_{\mathcal{C}}(\mathbf{1}, C)$. Then $(\mathcal{C}, \lambda)$ is a Frobenius algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. In other words, the composite map $u : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \xrightarrow{\lambda} \mathrm{Sp}$ is a duality datum in the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$.*

Proof. The map u classifies a bifunctor $\beta : \mathcal{C} \times \mathcal{C} \to \mathrm{Sp}$, given by the formula $\beta(C, D) = \underline{\mathrm{Map}}_{\mathcal{C}}(\mathbf{1}, C \otimes D)$. Let $\Phi_{\mathcal{C}}$ be as in Construction D.7.2.1. Using Proposition D.7.2.3, we deduce that there exists an essentially unique functor $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}})$ which is equipped with an equivalence $\beta \simeq \Phi_{\mathcal{C}} \circ (F \times \mathrm{id}_{\mathcal{C}})$. We wish to show that F is an equivalence of ∞-categories. Note that for every compact object $C \in \mathcal{C}$, the functor $\beta(C, \bullet)$ carries $D \in \mathcal{C}$ to the spectrum

$$\beta(C, D) = \underline{\mathrm{Map}}_{\mathcal{C}}(\mathbf{1}, C \otimes D) \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(C^{\vee}, D),$$

so that $\beta(C, \bullet)$ is representable by the object $C^{\vee} \in \mathcal{C}_c$. Let us abuse notation by identifying $\mathcal{C}_c^{\mathrm{op}}$ with its essential image in $\mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}})$. Unwinding the definitions, we see that F carries $\mathcal{C}_c$ into $\mathcal{C}_c^{\mathrm{op}}$, and is given on $\mathcal{C}_c$ by the formula $C \mapsto C^{\vee}$. If $C, C' \in \mathcal{C}$ are compact, then the canonical map

$$\mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Map}_{\mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}})}(F(C), F(C')) \simeq \mathrm{Map}_{\mathcal{C}_c}(C'^{\vee}, C^{\vee})$$

is a homotopy equivalence. It follows that $F|_{\mathcal{C}_c}$ is fully faithful. Since F commutes with filtered colimits and carries compact objects of $\mathcal{C}$ to compact objects of $\mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}})$, we conclude that F is fully faithful. Since every object of $\mathcal{C}_c$ admits a left dual, the essential image of F contains $\mathcal{C}_c^{\mathrm{op}}$ and is closed under filtered colimits. It follows that F is essentially surjective and is therefore an equivalence of ∞-categories. □

Remark D.7.5.2. [Serre Automorphism] Let $\mathcal{C}$ be a locally rigid stable monoidal ∞-category, let $\lambda : \mathcal{C} \to \mathrm{Sp}$ be defined as in Proposition D.7.5.1, and regard $(\mathcal{C}, \lambda)$ as a Frobenius algebra object of the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. Let $\sigma : \mathcal{C} \to \mathcal{C}$ denote the Serre automorphism of the Frobenius algebra object $(\mathcal{C}, \lambda)$ (see Remark HA.4.6.5.4). Then σ is a monoidal equivalence from $\mathcal{C}$ to itself. Moreover, if $m, m' : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ classify the tensor product on $\mathcal{C}$ and its opposite, then we have a commutative diagram of ∞-categories

$$\begin{array}{ccccc} \mathcal{C}\otimes\mathcal{C} & & \xrightarrow{\mathrm{id}\otimes\sigma} & & \mathcal{C}\otimes\mathcal{C} \\ \downarrow{\scriptstyle m} & & & & \downarrow{\scriptstyle m'} \\ \mathcal{C} & & & & \mathcal{C} \\ & \searrow^{\lambda} & & \swarrow^{\lambda} & \\ & & \mathrm{Sp}. & & \end{array}$$

In other words, for every pair of objects $C, D \in \mathcal{C}$, we have a canonical equivalence of spectra

$$\underline{\mathrm{Map}}_{\mathcal{C}}(\mathbf{1}, C \otimes D) \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(\mathbf{1}, D \otimes \sigma(C)).$$

If C is a compact object of $\mathcal{C}$, we can rewrite the left hand side of this equivalence as

$$\underline{\mathrm{Map}}_{\mathcal{C}}(\mathbf{1}, C \otimes D) \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(C^{\vee}, D) \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(\mathbf{1}, D \otimes (C^{\vee})^{\vee}).$$

We can informally summarize the situation as follows: if $\mathcal{C}$ is a locally rigid stable monoidal ∞-category, then the double duality functor $C \mapsto (C^{\vee})^{\vee}$ extends to a monoidal equivalence of $\mathcal{C}$ with itself.

Example D.7.5.3. Let R be an $\mathbb{E}_2$-ring. Then the ∞-category LMod_R is locally rigid (Example D.7.4.2). According to Remark D.7.5.2, double duality determines a monoidal equivalence $\sigma : \mathrm{LMod}_R \to \mathrm{LMod}_R$. Using Propositon HA.7.1.2.6 , we see that σ arises from an equivalence from R to itself in the ∞-category $\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp})$ of $\mathbb{E}_2$-rings. One can show that this equivalence is induced by the homotopy from the identity map from the ∞-operad $\mathbb{E}_2^{\otimes}$ to itself which is determined by the action of the circle group SO(2) on the $\mathbb{E}_2^{\otimes}$ (see Example HA.5.4.2.18).

D.7.6 A Recognition Criterion

Let $\mathcal{M}$ be a presentable stable ∞-category. If there exists a compact generator $M \in \mathcal{M}$, then we can identify $\mathcal{M}$ as the ∞-category RMod_A of right A-modules, where $A = \mathrm{End}_{\mathcal{M}}(M) = \underline{\mathrm{Map}}_{\mathcal{M}}(M, M)$ is the $\mathbb{E}_1$-ring of endomorphisms of M (see Theorem HA.7.1.2.1). We now establish a mild generalization of this statement:

Proposition D.7.6.1. *Let $\mathcal{C}$ be a locally rigid stable monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is left-tensored over $\mathcal{C}$. Assume that $\mathcal{M}$ is presentable and that the tensor product $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ preserves small colimits separately in each variable. Let $A \in \mathrm{Alg}(\mathcal{C})$ be an algebra object of $\mathcal{C}$, and let $M \in \mathrm{LMod}_A(\mathcal{M})$, so that tensor product with M determines a $\mathcal{C}$-linear functor $\lambda : \mathrm{RMod}_A(\mathcal{C}) \to \mathcal{M}$ (see Theorem HA.4.8.4.1). Then λ is an equivalence if and only if the following conditions are satisfied:*

(1) *The object $M \in \mathcal{M}$ is compact.*

(2) *The object $A \in \mathcal{C}$ classifies endomorphisms of M in $\mathcal{C}$. That is, for every object $C \in \mathcal{C}$, the action of A on M induces a homotopy equivalence*

$$\mathrm{Map}_{\mathcal{C}}(C, A) \to \mathrm{Map}_{\mathcal{M}}(C \otimes M, M).$$

(3) *The ∞-category $\mathcal{M}$ is generated under small colimits by objects of the form $C \otimes M$, where $C \in \mathcal{C}$.*

Proof. To prove that conditions (1), (2), and (3) are necessary, it suffices to show that they are satisfied when $\mathcal{M} = \mathrm{RMod}_A(\mathcal{C})$ and $M = A$ (regarded as a right module over itself). In this case, M corepresents the composite functor $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C} \xrightarrow{\phi} \mathcal{S}$, where $\phi : \mathcal{C} \to \mathcal{S}$ is the functor corepresented by the unit object $\mathbf{1}$. Since $\mathcal{C}$ is locally rigid, $\mathbf{1}$ is a compact object of $\mathcal{C}$, and condition (1) follows. Condition (2) follows from Proposition HA.4.2.4.2 , and condition (3) follows because every right A-module N can be written as $N \otimes_A A$, which is computed as the geometric realization of the simplicial object with entries of the form $N \otimes A^{\otimes k}$ for $k > 0$.

Conversely, suppose that conditions (1), (2), and (3) are satisfied. We first claim that the functor λ is fully faithful. To prove this, it will suffice to show that for every pair of objects $N, N' \in \mathrm{RMod}_A(\mathcal{C})$, the functor λ induces a homotopy equivalence

$$\theta_{N,N'} : \mathrm{Map}_{\mathrm{RMod}_A(\mathcal{C})}(N, N') \to \mathrm{Map}_{\mathcal{M}}(N \otimes_A M, N' \otimes_A M).$$

Let us first regard N' as fixed. The collection of those $N \in \mathrm{RMod}_A(\mathcal{C})$ for which $\theta_{N,N'}$ is an equivalence is closed under small colimits in $\mathrm{RMod}_A(\mathcal{C})$. Since $\mathrm{RMod}_A(\mathcal{C})$ is generated under small colimits by objects of the form $C \otimes A$, we may assume that $N = C \otimes A$ for some $C \in \mathcal{C}$. Since $\mathcal{C}$ is compactly generated, we may further reduce to the case where C is a compact object of $\mathcal{C}$. Then C admits a left dual ${}^{\vee}C$. Replacing N' by ${}^{\vee}C \otimes N'$, we may reduce to the case where $C = \mathbf{1}$, so that $N = A$. In this case, we can identify θ with the canonical map $\phi_{N'} : \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, N') \to \mathrm{Map}_{\mathcal{M}}(M, N' \otimes_A M)$.

Let $\mathcal{E} \subseteq \mathrm{RMod}_A(\mathcal{C})$ denote the full subcategory spanned by those objects N' satisfying the following condition: for every integer d, the map $\phi_{\Sigma^d N'}$ is a homotopy equivalence. Since the construction $N' \mapsto \phi_{N'}$ preserves finite limits, $\mathcal{E}$ is a stable subcategory of

$\mathrm{RMod}_A(\mathcal{C})$, and therefore closed under finite colimits in $\mathrm{RMod}_A(\mathcal{C})$. Using condition (1), we deduce that $\mathcal{E}$ is closed under filtered colimits, and therefore closed under small colimits in $\mathrm{RMod}_A(\mathcal{C})$. Consequently, to show that $\mathcal{E} = \mathrm{RMod}_A(\mathcal{C})$, it will suffice to show that $\mathcal{E}$ contains every right A-module of the form $C' \otimes A$, where C' is a compact object of $\mathcal{C}$. In this case, C' is right dualizable, so we can identify $\phi_{N'}$ with the canonical map $\mathrm{Map}_{\mathcal{C}}(C'^{\vee}, A) \to \mathrm{Map}_{\mathcal{M}}(C'^{\vee} \otimes M, M)$. This map is a homotopy equivalence by virtue of assumption (2). This completes the proof that λ is fully faithful.

Since λ is a fully faithful functor which preserves small colimits, the essential image of λ is closed under small colimits in $\mathcal{M}$. Since this essential image contains every object of the form $C \otimes M$, where $C \in \mathcal{C}$, assumption (3) guarantees that λ is essentially surjective. □

Combining Proposition D.7.6.1 with the results of §HA.4.7.1 , we obtain the following:

Corollary D.7.6.2. *Let $\mathcal{C}$ be a locally rigid stable monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is left-tensored over $\mathcal{C}$. Let $M \in \mathcal{M}$ be an object. Then the following conditions are equivalent:*

1. *There exists an algebra object $A \in \mathrm{Alg}(\mathcal{C})$ and a $\mathcal{C}$-linear equivalence of ∞-categories $\lambda : \mathrm{RMod}_A(\mathcal{C}) \to \mathcal{M}$ with $\lambda(A) \simeq M$.*
2. *The ∞-category $\mathcal{M}$ is presentable, the tensor product $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ preserves small colimits separately in each variable, M is a compact object of $\mathcal{M}$, and $\mathcal{M}$ is generated under small colimits by objects of the form $C \otimes M$, where $C \in \mathcal{C}$.*

Corollary D.7.6.3. *Let R be an $\mathbb{E}_2$-ring, let $\mathcal{M}$ be an R-linear ∞-category, and let $M \in \mathcal{M}$ be an object. The following conditions are equivalent:*

1. *There exists an R-algebra $A \in \mathrm{Alg}_R$ and an R-linear equivalence of ∞-categories $\lambda : \mathrm{RMod}_A \simeq \mathcal{M}$ with $\lambda(A) \simeq M$.*
2. *The object $M \in \mathcal{M}$ is compact, and the suspensions $\{\Sigma^k M\}_{k \in \mathbf{Z}}$ generate $\mathcal{M}$ under small colimits.*

Corollary D.7.6.4. *Let $\mathcal{C}$ be a locally rigid stable monoidal ∞-category, and let $\mathcal{M} \in \mathrm{LMod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$. Suppose that there exists a compact object $M \in \mathcal{M}$ such that $\mathcal{M}$ is generated under small colimits by objects of the form $C \otimes M$, where $C \in \mathcal{C}$. Then:*

1. *There exists an algebra object $A \in \mathrm{Alg}(\mathcal{C})$ and a $\mathcal{C}$-linear equivalence $\mathcal{M} \simeq \mathrm{RMod}_A(\mathcal{C})$.*
2. *As a left $\mathcal{C}$-module object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$, $\mathcal{M}$ admits a left dual ${}^{\vee}\mathcal{M} \in \mathrm{RMod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$.*

Proof. Assertion (1) follows immediately from Corollary D.7.6.2, and (2) follows from (1) and Remark HA.4.8.4.8 . □

D.7.7 Smoothness of Locally Rigid Stable ∞-Categories

We now turn to the proof of Theorem D.7.0.7. The main point is to establish the following:

Proposition D.7.7.1. *Let $\mathcal{C}$ be a locally rigid stable monoidal ∞-category. Then $\mathcal{C}$ is smooth when viewed as an algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$: that is, it is dualizable as module over $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}$ (see Definition HA.4.6.4.13).*

Lemma D.7.7.2. *Let $\mathcal{C}$ and $\mathcal{C}'$ be locally rigid stable monoidal ∞-categories, which we regard as algebra objects of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. Then the tensor product $\mathcal{C} \otimes \mathcal{C}' \in \mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$ is also locally rigid.*

Proof. For every pair of objects $C \in \mathcal{C}$, $C' \in \mathcal{C}'$, we let $C \boxtimes C'$ denote the image of (C, C') under the tautological map $\mathcal{C} \times \mathcal{C}' \to \mathcal{C} \otimes \mathcal{C}'$. If C and C' are compact, then $C \boxtimes C'$ is a compact object of $\mathcal{C} \otimes \mathcal{C}'$. Note that the unit object of $\mathcal{C} \otimes \mathcal{C}'$ can be written as $\mathbf{1}_{\mathcal{C}} \boxtimes \mathbf{1}_{\mathcal{C}'}$, where $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{C}'}$ denote the unit objects of $\mathcal{C}$ and $\mathcal{C}'$, respectively. It follows that the unit object of $\mathcal{C} \otimes \mathcal{C}'$ is compact. We also note that if $C \in \mathcal{C}$ and $C' \in \mathcal{C}'$ are both left dualizable (right dualizable), then $C \boxtimes C'$ is left dualizable (right dualizable). Consequently, the collection of left (right) dualizable objects of $\mathcal{C} \boxtimes \mathcal{C}'$ forms a stable subcategory which contains all objects of the form $C \boxtimes C'$, where C and C' are compact. It follows that every compact object of $\mathcal{C} \otimes \mathcal{C}'$ is (left and right) dualizable. □

Proof of Proposition D.7.7.1. Let $\mathcal{C}$ be a locally rigid stable monoidal ∞-category, and regard $\mathcal{C}$ as an associative algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. Let $\mathcal{C}^{\mathrm{rev}}$ denote the opposite algebra (so that $\mathcal{C}^{\mathrm{rev}}$ agrees with $\mathcal{C}$ as an ∞-category, but is equipped with the opposite tensor product), and let $\mathcal{C}^{e} \in \mathrm{LMod}_{\mathcal{C} \otimes \mathcal{C}^{\mathrm{rev}}}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$ denote the evaluation module of $\mathcal{C}$ (see Construction HA.4.6.3.7). We wish to show that $\mathcal{C}^{e}$ is left dualizable. Note that $\mathcal{C} \otimes \mathcal{C}^{\mathrm{rev}}$ is locally rigid (Lemma D.7.7.2). The desired result now follows immediately from Corollary D.7.6.4, since the unit object $\mathbf{1} \in \mathcal{C}$ is compact, and $\mathcal{C}$ is generated under small colimits by objects of the form $C \otimes \mathbf{1} \otimes D$ where $C, D \in \mathcal{C}$. □

Corollary D.7.7.3. *Let $\mathcal{C}$ be a locally rigid stable monoidal ∞-category, and let $\mathcal{M} \in \mathrm{LMod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{M}$ admits a right dual $\mathcal{M}^{\vee}$ (that is, $\mathcal{M}$ is dualizable as an object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$).*

(2) *The ∞-category $\mathcal{M}$ admits a left dual ${}^{\vee}\mathcal{M}$ (that is, $\mathcal{M}$ is dualizable as a $\mathcal{C}$-module).*

Proof. It follows from Propositions D.7.5.1 and D.7.7.1 that $\mathcal{C}$ is smooth and proper when regarded as an algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{St}}$. The equivalence of (1) and (2) now follows from Propositions HA.4.6.4.4 and HA.4.6.4.12. □

Theorem D.7.0.7 is a special case of the following more general result:

Corollary D.7.7.4. *Let $\mathcal{C}$ be a locally rigid stable monoidal ∞-category, and let $\mathcal{M} \in \mathrm{LMod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$. If $\mathcal{M}$ is compactly assembled, then $\mathcal{M}$ admits a left dual ${}^{\vee}\mathcal{M} \in \mathrm{RMod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$.*

Proof. Combine Corollary D.7.7.3 with Proposition D.7.3.1. □

Warning D.7.7.5. Let $\mathcal{C}$ be a locally rigid stable monoidal ∞-category, and let $\mathcal{M} \in \mathrm{LMod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$ be an ∞-category left-tensored over $\mathcal{C}$ which satisfies the equivalent conditions of Corollary D.7.7.3. Then $\mathcal{M}$ admits left and right duals ${}^{\vee}\mathcal{M}, \mathcal{M}^{\vee} \in \mathrm{RMod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^{\mathrm{St}})$. These duals are canonically equivalent to one another as ∞-categories, but are generally *not* equivalent as ∞-categories right-tensored over $\mathcal{C}$. Instead, the actions of $\mathcal{C}$ on ${}^{\vee}\mathcal{M}$ and $\mathcal{M}^{\vee}$ differ by composition with the Serre automorphism of $\mathcal{C}$ described in Remark D.7.5.2 (see Remark HA.4.6.5.13). However, this automorphism is trivial whenever the monoidal structure on $\mathcal{C}$ is symmetric (Remark HA.4.6.5.15): for example, if $\mathcal{C} = \mathrm{Mod}_R$ where R is an $\mathbb{E}_{\infty}$-ring.

Remark D.7.7.6. Let R be an $\mathbb{E}_{\infty}$-ring and let $\mathcal{C}$ be a compactly generated stable R-linear ∞-category. Then:

(1) The ∞-category $\mathcal{C}$ is dualizable as an object of $\mathrm{LinCat}_R^{\mathrm{St}}$ (this follows from Corollary D.7.7.4).

(2) Let $e : \mathcal{C}^{\vee} \otimes_R \mathcal{C} \to \mathrm{Mod}_R$ be a duality datum in LinCat_R, and let $\lambda : \mathrm{Mod}_R \to \mathrm{Sp}$ denote the forgetful functor. Then the composite map

$$\mathcal{C}^{\vee} \otimes \mathcal{C} \to \mathcal{C}^{\vee} \otimes_R \mathcal{C} \xrightarrow{e} \mathrm{Mod}_R \xrightarrow{\lambda} \mathrm{Sp}$$

is a duality datum in $\mathcal{P}\mathrm{r}^{\mathrm{St}}$ (combine (1) with Proposition D.7.5.1, Proposition D.7.6.1, and Corollary HA.4.6.5.14).

(3) Let $\mathcal{C}_c$ denote the full subcategory of $\mathcal{C}$ spanned by the compact objects. Then $\mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}})$ admits the structure of a stable R-linear ∞-category. Moreover, the functor $e_{\mathcal{C}} : \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \otimes \mathcal{C} \to \mathrm{Sp}$ is homotopic to a composition

$$\mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \otimes \mathcal{C} \to \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \otimes_R \mathcal{C} \xrightarrow{\overline{e}_{\mathcal{C}}} \mathrm{Mod}_R \xrightarrow{\lambda} \mathrm{Sp},$$

where $\overline{e}_{\mathcal{C}}$ exhibits $\mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}})$ as the dual of $\mathcal{C}$ in the ∞-category $\mathrm{LinCat}_R^{\mathrm{St}}$ (combine (2) with Proposition D.7.2.3).

Corollary D.7.7.7. *Let R be an $\mathbb{E}_{\infty}$-ring and let $\mathcal{C}$ be a compactly assembled stable R-linear ∞-category. Then $\mathcal{C}$ satisfies flat hyperdescent (see Definition D.6.1.2).*

Proof. Using Lemma D.3.5.5 and Proposition A.5.7.2, we are reduced to proving the following:

(∗) Let $A^{\bullet} : \mathbf{\Delta}_s \rightarrow \mathrm{CAlg}_{R/}$ be a flat hypercovering of an R-algebra A. Then the canonical map $\theta : \mathrm{Mod}_A \otimes_{\mathrm{Mod}_R} \mathcal{C} \rightarrow \varprojlim(\mathrm{Mod}_{A^{\bullet}} \otimes_{\mathrm{Mod}_R} \mathcal{C})$ is an equivalence of R-linear ∞-categories.

Let $\mathcal{D}$ be another R-linear ∞-category; we wish to show that θ induces a homotopy equivalence

$$\phi : \mathrm{Map}_{\mathrm{LinCat}_R}(\mathcal{D}, \mathrm{Mod}_A \otimes_{\mathrm{Mod}_R} \mathcal{C}) \rightarrow \varprojlim \mathrm{Map}_{\mathrm{LinCat}_R}(\mathcal{D}, \mathrm{Mod}_{A^{\bullet}} \otimes_{\mathrm{Mod}_R} \mathcal{C}).$$

Corollary D.7.7.4 implies that $\mathcal{C}$ is a dualizable object of LinCat_R. Let us denote its dual by $\mathcal{C}^{\vee}$. Then ϕ can be identified with the canonical map

$$\mathrm{Map}_{\mathrm{LinCat}_R}(\mathcal{D} \otimes_{\mathrm{Mod}_R} \mathcal{C}^{\vee}, \mathrm{Mod}_A) \rightarrow \varprojlim \mathrm{Map}_{\mathrm{LinCat}_R}(\mathcal{D} \otimes_{\mathrm{Mod}_R} \mathcal{C}^{\vee}, \mathrm{Mod}_{A^{\bullet}}).$$

We are therefore reduced to proving that $\mathrm{Mod}_A \simeq \varprojlim \mathrm{Mod}_{A^{\bullet}}$, which follows from Corollary D.6.3.3. □

Appendix E

Profinite Homotopy Theory

Let X be a simply connected space. Using rational homotopy theory (see §**??**), one can associate to X another simply connected space $X_{\mathbf{Q}}$ called the *rationalization of* X, whose homotopy $\pi_n X_{\mathbf{Q}} \simeq \mathbf{Q} \otimes_{\mathbf{Z}} (\pi_n X)$. The space $X_{\mathbf{Q}}$ captures all "rational information" about the homotopy type of the space X, while completely ignoring torsion. In this appendix, we give an overview of the subject of *profinite* homotopy theory which takes the opposite approach: emphasizing the role of "torsion information" (while completely neglecting rational information).

Definition E.0.7.8. Let X be a space. We will say that X is π*-finite* if it satisfies the following conditions:

(1) The space X is n-truncated for some integer n.

(2) The set $\pi_0 X$ is finite.

(3) For each vertex $x \in X$ and each integer $m \geqslant 1$, the group $\pi_m(X, x)$ is finite.

Let $\mathcal{S}_\pi$ denote the full subcategory of $\mathcal{S}$ spanned by the π-finite spaces.

Remark E.0.7.9. A space X is π-finite if and only if it is a truncated coherent object of the ∞-topos $\mathcal{S}$ (see Example A.2.1.7). In particular, the ∞-category $\mathcal{S}_\pi$ is a bounded ∞-pretopos (Example A.7.4.4).

Remark E.0.7.10. The full subcategory $\mathcal{S}_\pi \subseteq \mathcal{S}$ is closed under finite coproducts, finite limits, and retracts. In particular, $\mathcal{S}_\pi$ is an idempotent-complete ∞-category which admits finite limits. Note also that $\mathcal{S}_\pi$ is essentially small (and therefore accessible).

Definition E.0.7.11. A *profinite space* is a Pro-object of the ∞-category $\mathcal{S}_\pi$ (see Definition A.8.1.1): that is, a functor $U : \mathcal{S}_\pi \to \mathcal{S}$ which preserves finite limits. We let $\mathcal{S}_\pi^\wedge = \mathrm{Pro}(\mathcal{S}_\pi)$ denote the full subcategory of $\mathrm{Fun}(\mathcal{S}_\pi, \mathcal{S})^{\mathrm{op}}$ spanned by the profinite spaces. We will refer to $\mathcal{S}_\pi^\wedge$ as the ∞*-category of profinite spaces.*

Example E.0.7.12. Let X be a space. We let $X_{\pi}^{\wedge} : \mathcal{S}_{\pi} \to \mathcal{S}$ denote the functor given by $T \mapsto \mathrm{Map}_{\mathcal{S}}(X, T)$. Then $X_{\pi}^{\wedge}$ is a profinite space, which we will refer to as the *profinite completion* of X.

Under the dictionary of Remark **??**, the profinite completion $X_{\pi}^{\wedge}$ can be identified with the diagram $(\mathcal{S}_{/X} \times_{\mathcal{S}} \mathcal{S}_{\pi}) \to \mathcal{S}_{\pi}$: that is, the filtered system $\{T_{\alpha}\}$ indexed by all maps $X \to T_{\alpha}$, where T_{α} is π-finite.

Our main goal in this section is to address the following (closely related) questions:

(a) To what extent does the homotopy theory of profinite spaces approximate the classical homotopy theory of spaces?

(b) To what extent does the profinite completion of a space X approximate the space X itself?

Let us now outline the contents of this section. We begin in §E.1 by considering a simplified variant of the ∞-category $\mathcal{S}_{\pi}^{\wedge}$, where we consider Pro-objects not of the ∞-category $\mathcal{S}_{\pi}$ of π-finite spaces, but instead of the ordinary category $\mathcal{S}\mathrm{et}^{\mathrm{fin}}$ of finite sets. The category $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$ is well-understood: it is equivalent to the category of *Stone spaces*: that is, topological spaces which are compact, Hausdorff, and totally disconnected (Theorem E.1.4.1). In §E.2, we study an analogous fully faithful embedding from the ∞-category $\mathcal{S}_{\pi}^{\wedge}$ of profinite spaces to the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi (Theorem E.2.4.1). We say that an ∞-topos $\mathcal{X}$ is *profinite* if it belongs to the essential image of the embedding $\mathcal{S}_{\pi}^{\wedge} \hookrightarrow \infty\mathcal{T}\mathrm{op}$. The class of profinite ∞-topoi admits several different descriptions, whose equivalence will be established in §E.3 (see in particular Proposition E.3.1.1, Theorem E.3.2.8, and Theorem E.3.4.1).

The category $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$ of profinite sets can be regarded as a full subcategory of the ∞-category $\mathcal{S}_{\pi}^{\wedge}$ of profinite spaces: namely, the full subcategory spanned by those profinite spaces which are 0-*truncated*. In §E.4, we study the general notion of an n-truncated map of profinite spaces for $n \geqslant 0$, together with the dual notion of an n-connective map. Our main result (Theorem E.4.1.2) asserts that just as in classical homotopy theory, any map of profinite spaces $f : X \to Z$ admits an essentially unique factorization $X \xrightarrow{f'} Y \xrightarrow{f''} Z$ where f' is $(n+1)$-connective f'' is n-truncated.

Let X be a profinite space, represented by a filtered diagram $\{X_{\alpha}\}$ of π-finite spaces. Then we can form the inverse limit $\varprojlim_{\alpha} X_{\alpha}$ in the ∞-category of spaces. This inverse limit depends functorially on X: we will denote it by $\mathrm{Mat}(X)$, and refer to it as the *materialization* of X. If we choose a base point $x \in \mathrm{Mat}(X)$, then the fiber product $\Omega X = \{x\} \times_X \{x\}$ is a group object in the ∞-category of profinite spaces. In §E.5, we will prove a converse to this assertion: every group object G in the ∞-category of profinite spaces arises as the loop space of a (connected) profinite space, which we will denote by BG and refer to as the *classifying* *space* of G (Theorem E.5.0.4). This can be regarded as homotopy-theoretic analogue of the classical fact that every group object in the category of profinite sets can be written as a filtered inverse limit of finite groups (Proposition E.5.1.3).

One of the most important structural features of the classical homotopy theory of spaces is that colimits are universal in the ∞-category of spaces: that is, the formation of colimits in $\mathcal{S}$ commutes with pullback (Lemma HTT.6.1.3.14). The analogous statement for profinite spaces is not true in complete generality (Warning E.6.0.9). Nevertheless, in §E.6, we will see that it is approximately true: arbitrary colimits are preserved by pullback along maps of π-finite spaces (Corollary E.6.0.8), and many colimits (such as pushouts and geometric realizations) are preserved by pullback along arbitrary maps of profinite spaces (Theorem E.6.3.1).

In §E.7, we will study finiteness conditions on profinite spaces. For a simply connected profinite space X, we show that the following conditions are equivalent (see Theorem E.7.0.5):

(a) For every prime number p and every integer $n \geqslant 2$, the map $\pi_n X \xrightarrow{p} \pi_n X$ has finite cokernel.

(b) For every prime number p and every integer $n \geqslant 2$, the cohomology group $\mathrm{H}^n(X; \mathbf{F}_p)$ is finite.

If these conditions are satisfied, we say that the profinite space X has *finite type*. In §E.8, we show that the materialization functor $X \mapsto \mathrm{Mat}(X)$ is fully faithful when restricted to profinite spaces of finite type (Theorem E.8.2.1). An important special case occurs when X is the profinite completion $Y_\pi^\wedge$ of a simply connected space Y with finitely generated homotopy groups. In this case, X is of finite type (so that passage from X to $\mathrm{Mat}(X)$ involves no loss of information), and the canonical map $Y \to \mathrm{Mat}(X)$ exhibits the homotopy groups of X as the profinite completion of the homotopy groups of Y (Proposition E.8.2.4).

If X is a simply connected space, then we can think of its rationalization $X_{\mathbf{Q}}$ as an invariant which records rational data about X (for example, its homology and cohomology with rational coefficients) while discarding integral and torsion information, while its profinite completion $X_\pi^\wedge$ records "p-adic" data for each prime number p, but discards all information about how these data are related for different values of p. In good cases, one can recover X by "attaching" its rationalization $X_{\mathbf{Q}}$ to its profinite completion $X_\pi^\wedge$ by means of a map of rational spaces $X_{\mathbf{Q}} \to \mathrm{Mat}(X_\pi^\wedge)_{\mathbf{Q}}$. In §E.9 we will describe this reconstruction procedure in detail, following the work of Sullivan in [196].

Remark E.0.7.13. In [219], Quick develops another approach to the homotopy theory of profinite spaces, using the formalism of simplicial profinite sets. For a discussion of the relationship of Definition E.0.7.11 with Quick's theory, we refer the reader to the work of Barnea-Harpaz-Horel ([13]); see also Theorem E.1.5.3.

Contents

E.1 Profinite Sets and Stone Spaces

Recall that a topological space X is a *Stone space* if it is compact, Hausdorff, and has a basis consisting of closed and open sets (Definition A.1.6.8). The category $\mathcal{T}\mathrm{op}_{\mathrm{St}}$ of Stone spaces admits many different descriptions:

(*a*) According to the Stone duality theorem (Theorem A.1.6.11), a topological space X is a Stone space if and only if it is homeomorphic to the spectrum $\mathrm{Spec}(\Lambda)$ of a Boolean algebra Λ. Moreover, the construction $\Lambda \mapsto \mathrm{Spec}(\Lambda)$ determines a (contravariant) equivalence from the category of Boolean algebras to the category $\mathcal{T}\mathrm{op}_{\mathrm{St}}$ of Stone spaces.

(b) For any filtered inverse system of finite sets $\{S_\alpha\}$, the inverse limit $\varprojlim S_\alpha$ is a Stone space (when endowed with the inverse limit topology). This construction determines an equivalence from the category of profinite sets to the category of Stone spaces (Theorem E.1.4.1).

(c) Let p be a prime number. We say that a commutative ring B is an $\mathbf{F}_p$-*Boolean algebra* if $p = 0$ in B and every element $x \in B$ satisfies the equation $x^p = x$. For any $\mathbf{F}_p$-Boolean algebra B, the Zariski spectrum $|\operatorname{Spec} B|$ is a Stone space. Moreover, the construction $B \mapsto |\operatorname{Spec} B|$ induces a (contravariant) equivalence from the category of $\mathbf{F}_p$-Boolean algebras to the category of Stone spaces (Theorem **??**). When $p = 2$, this reduces to the equivalence described in (a) (see Proposition **??**).

Our goal in this section is to give a detailed explanation of (b) (we discuss (a) in §A.1 and (c) in §**??**). Let $\mathcal{S}\mathrm{et}^{\mathrm{fin}}$ denote the category of finite sets. We will refer to Pro-objects of $\mathcal{S}\mathrm{et}^{\mathrm{fin}}$ as *profinite sets*. In this section, we will review some of the basic facts about the category $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$ of profinite sets and construct the equivalence of categories $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}}) \simeq \mathcal{T}\mathrm{op}_{\mathrm{St}}$. We will then discuss the relationship between the ordinary category $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$ of profinite sets and the ∞-category $\mathcal{S}_\pi^\wedge = \mathrm{Pro}(\mathcal{S}_\pi)$ of profinite spaces.

E.1.1 Finite Sets and Inverse Limits

Our starting point is the following fundamental fact about profinite sets:

Proposition E.1.1.1. *Let A be a filtered partially ordered set, and suppose we are given a functor $X : A^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. If the set $X(\alpha)$ is finite for each $\alpha \in A$, then the inverse limit $\varprojlim_{\alpha \in A} X(\alpha)$ is nonempty.*

Proof. Let S denote the collection of all subfunctors $X_0 \subseteq X$ such that the set $X_0(\alpha)$ is nonempty for each $\alpha \in A$. We regard S as a linearly ordered set with respect to inclusions. Note that any linearly ordered subset of S has an infimum in S, since the intersection of any chain of nonempty finite subsets of a finite set is again nonempty. It follows from Zorn's lemma that S has a minimal element $X_0 \subseteq X$. We will show that for each $\alpha \in A$, the set $X_0(\alpha)$ has a single element, so that $\varprojlim_{\alpha \in A} X_0(\alpha)$ consists of a single element. The desired result will then follow from the existence of a map $\varprojlim_{\alpha \in A} X_0(\alpha) \to \varprojlim_{\alpha \in A} X(\alpha)$.

Let $\alpha \in A$ and choose elements $x, y \in X_0(\alpha)$; we will prove that $x = y$. For $\beta \geqslant \alpha$, let $\phi_\beta : X_0(\beta) \to X_0(\alpha)$ be the corresponding map of finite sets, and define subfunctors $X_x, X_y \subseteq X_0$ by the formulae

$$X_x(\beta) = \begin{cases} \phi_\beta^{-1}(X_0(\alpha) - \{x\}) & \text{if } \beta \geqslant \alpha \\ X_0(\beta) & \text{otherwise.} \end{cases}$$

$$X_y(\beta) = \begin{cases} \phi_\beta^{-1}(X_0(\alpha) - \{y\}) & \text{if } \beta \geqslant \alpha \\ X_0(\beta) & \text{otherwise.} \end{cases}$$

Since X_0 was chosen to be a minimal element of S, we must have $X_x, X_y \notin S$. It follows that there exist elements $\beta, \gamma \in A$ such that the sets $X_x(\beta)$ and $X_y(\gamma)$ are empty. Since A is filtered, we may assume without loss of generality that $\beta = \gamma$. Note also that we must have $\beta \geqslant \alpha$, since otherwise $X_x(\beta) = X_0(\beta) \neq \varnothing$. Since $X_x(\beta) = \varnothing$, the map ϕ_β must be the constant map taking the value $x \in X_0(\alpha)$. The same argument shows that ϕ_β takes the constant value y. Since $X_0(\beta) \neq \varnothing$, this proves that $x = y$ as desired. □

E.1.2 The Topology of a Profinite Set

Let $\mathcal{S}\text{et}$ denote the category of sets. We will abuse notation by identifying $\mathcal{S}\text{et}$ with the full subcategory of $\mathcal{T}\text{op}$ spanned by those topological spaces which are endowed with the discrete topology. Since the category $\mathcal{T}\text{op}$ admits filtered inverse limits, the inclusion $\mathcal{S}\text{et} \subseteq \mathcal{T}\text{op}$ extends to a functor $\psi : \text{Pro}(\mathcal{S}\text{et}^{\text{fin}}) \to \mathcal{T}\text{op}$ which preserves filtered inverse limits (moreover, this extension is unique up to unique isomorphism).

Proposition E.1.2.1. *The functor* $\psi : \text{Pro}(\mathcal{S}\text{et}^{\text{fin}}) \to \mathcal{T}\text{op}$ *described above is fully faithful.*

Proof. Let us identify $\text{Pro}(\mathcal{S}\text{et}^{\text{fin}})$ with the category of left exact functors $F : \mathcal{S}\text{et}^{\text{fin}} \to \mathcal{S}\text{et}$. The functor ψ admits a left adjoint ϕ, which carries a topological space X to the left exact functor $\phi(X)$ given by the formula

$$\phi(X)(J) = \text{Hom}_{\mathcal{T}\text{op}}(X, J).$$

To prove that ψ is fully faithful, it will suffice to show that if S is a profinite set and $X = \psi(S)$, then the adjoint map $\phi(X) \to S$ is an isomorphism of profinite sets.

Choose a filtered partially ordered set A and an isomorphism of profinite sets $S \simeq \varprojlim_{\alpha \in A} S_\alpha$ in $\text{Pro}(\mathcal{S}\text{et}^{\text{fin}})$, where each S_α is a finite set. Then $X = \psi(S)$ can be identified with the inverse limit of the diagram $\{S_\alpha\}$ in the category $\mathcal{T}\text{op}$ of topological spaces. Unwinding the definitions, we must show that for every finite set T, the natural map

$$\theta : \varinjlim \text{Hom}_{\mathcal{S}\text{et}}(S_\alpha, T) \to \text{Hom}_{\mathcal{T}\text{op}}(X, T)$$

is a bijection. We first show that θ is injective. Suppose we are given a pair of maps $f_0, f_1 : S_\alpha \to T$ such that the composite maps $X \xrightarrow{\phi_\alpha} S_\alpha \to T$ coincide. We wish to show that there exists $\beta \geqslant \alpha$ such that the composite maps $S_\beta \to S_\alpha \to T$ coincide. Let $S' = \{s \in S_\alpha : f_0(s) \neq f_1(s)\}$. For each $s \in S'$, the inverse image $\phi_\alpha^{-1}\{s\} \subseteq X$ is empty. Using Proposition E.1.1.1, we deduce that the inverse image of $\{s\}$ in S_{β_s} is empty for some $\beta_s \geqslant \alpha$. Since S' is finite, we may choose $\beta \in A$ such that $\beta \geqslant \beta_s$ for all $s \in S'$. Then the inverse image of S' in S_β is empty, so that β has the desired property.

We now show that θ is surjective. Suppose we are given a continuous map $f : X \to T$. We wish to show that f factors through $\phi_\alpha : X \to S_\alpha$ for some index $\alpha \in A$. If T is empty, then X is empty and so (by Proposition E.1.1.1) the set S_α is empty for some $\alpha \in A$, and therefore f factors through S_α. Let us therefore assume that T is nonempty. Fix $t \in T$ and let $X_t = f^{-1}\{t\}$. Note that X_t is both open and closed in X. Since X is compact, X_t is also compact. By construction, the topological space X has a basis consisting of sets of the form $\phi_\alpha^{-1}\{s\}$, where $s \in S_\alpha$. In particular, for every point $x \in X_t$, we can choose a $\alpha_x \in A$ and a point $s_x \in S_{\alpha_x}$ such that $x \in \phi_{\alpha_x}^{-1}\{s_x\} \subseteq X_t$. The sets $U_x = \phi_{\alpha_x}^{-1}\{s_x\}$ form an open covering of X_t. Since X_t is compact, there exist finitely many points $x_1, \ldots, x_n \in X_t$ such that $X_t = \bigcup_{1 \leqslant i \leqslant n} U_{x_i}$. Since A is filtered, we can choose an index $\alpha_t \in A$ such that $\alpha_t \geqslant \alpha_{x_i}$ for $1 \leqslant i \leqslant n$. Because T is finite, we may further choose α such that $\alpha \geqslant \alpha_t$ for all $t \in T$. Let $S_t = \{s \in S_\alpha : \varnothing \neq \phi_\alpha^{-1}\{s\} \subseteq X_t\}$. Then

$$X_t \subseteq \bigcup_{1 \leqslant i \leqslant n} U_{x_i} \subseteq \phi_\alpha^{-1} S_t \subseteq X_t.$$

Note that the subsets $S_t \subseteq S_\alpha$ are disjoint. Since T is nonempty, there exists a map of finite sets $f' : S_\alpha \to T$ such that $S_t \subseteq f'^{-1}\{t\}$ for each $t \in T$. Then $f = f' \circ \phi_\alpha$ as desired. □

E.1.3 Stone Spaces

Our next goal is to describe the essential image of the fully faithful embedding $\psi : \mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}}) \to \mathcal{T}\mathrm{op}$ appearing in Proposition E.1.2.1. First, let us review a bit of classical point-set topology.

Proposition E.1.3.1. *Let X be a compact Hausdorff space. The following conditions are equivalent:*

- (a) *There exists a basis for the topology of X consisting of sets which are both closed and open.*
- (b) *Every connected subset of X is a singleton.*

Proof. Suppose first that (a) is satisfied, and let $S \subseteq X$ be connected. Then S is nonempty; we wish to show that it contains only a single element. Suppose otherwise, and choose distinct points $x, y \in S$. Since X is Hausdorff, there exists an open set $U \subseteq X$ containing x but not y. Using condition (a), we can assume that the set U is also closed. Then $U \cap S$ and $(X - U) \cap S$ is a decomposition of S into nonempty closed and open subsets, contradicting the connectedness of S.

To prove the converse, we need the following fact:

- $(*)$ Let $x, y \in X$. Assume that, for every closed and open subset $U \subseteq X$, if x belongs to U then y also belongs to U. Then there is a connected subset of X containing both x and y.

To prove $(*)$, consider the collection S of all closed subsets $Y \subseteq X$ which contain both x and y, having the property that any closed and open subset $U \subseteq Y$ containing x also contains y. Then S is nonempty (since $X \in S$). We claim that every linearly ordered subset of S has a lower bound in S. Suppose we are given such a linearly ordered set $\{Y_\alpha\}$, and let $Y = \bigcap Y_\alpha$. Then Y contains the points x and y. If $Y \notin S$, then we can decompose Y as the disjoint union of (closed and open) subsets $Y_-, Y_+ \subseteq Y$, with $x \in Y_-$ and $y \in Y_+$. Let us regard Y_- and Y_+ as compact subsets of X. Since X is Hausdorff, we can choose disjoint open sets $U_-, U_+ \subseteq X$ with $Y_- \subseteq U_-$ and $Y_+ \subseteq U_+$. The intersection

$$(X - U_-) \cap (X - U_+) \cap \bigcap_\alpha Y_\alpha$$

is empty. Since X is compact, we conclude that there exists an index α such that $(X - U_-) \cap (X - U_+) \cap Y_\alpha = \varnothing$. Then $Y_\alpha \cap U_-$ and $Y_\alpha \cap U_+$ are disjoint closed and open subsets of Y_α containing x and y respectively, contradicting our assumption that $Y_\alpha \in S$. This completes the proof that $Y \in S$, so that S satisfies the hypotheses of Zorn's lemma. We may therefore choose a minimal element $Z \in S$.

To complete the proof of $(*)$, it will suffice to show that Z is connected. Assume otherwise: then there exists a decomposition of Z into closed and open nonempty subsets $Z', Z'' \subseteq Z$. Since $Z \in S$, we have either $x, y \in Z'$ or $x, y \in Z''$; let us suppose that $x, y \in Z'$. The minimality of Z implies that $Z' \notin S$, so that Z' can be further decomposed into closed and open subsets $Z'_-, Z'_+ \subseteq Z'$ containing x and y, respectively. Then Z'_- and $Z'_+ \cup Z''$ are closed and open subsets of Z containing x and y, respectively, contradicting our assumption that $Z \in S$. This completes the proof of $(*)$.

Now suppose that (b) is satisfied; we wish to prove (a). It follows from condition $(*)$ that for every pair of distinct points $x, y \in X$, there exists a closed and open subset $V_{x,y}$ which contains y but does not contain x. Let $U \subseteq X$ be an open set; we wish to show that U contains a closed and open neighborhood of each point $x \in U$. Then $X - U$ is covered by the open sets $\{V_{x,y}\}_{y \in X-U}$. Since $X - U$ is compact, we can choose a finite subset $\{y_1, \ldots, y_n\} \subseteq X - U$ such that $X - U \subseteq \bigcup_{1 \leqslant i \leqslant n} V_{x,y_i}$. It follows that $X - \bigcup_{1 \leqslant i \leqslant n} V_{x,y_i}$ is a closed and open subset of X which contains x and is contained in U. □

Definition E.1.3.2. Let X be a topological space. We say that X is a *Stone space* if it is compact, Hausdorff, and satisfies the equivalent conditions of Proposition E.1.3.1. We let $\mathcal{T}\mathrm{op}$ denote the category of topological spaces, and $\mathcal{T}\mathrm{op}_{\mathrm{St}}$ the full subcategory of $\mathcal{T}\mathrm{op}$ spanned by the Stone spaces.

Remark E.1.3.3. Let X be a compact Hausdorff space. The collection of closed and open subsets of X is closed under finite intersections. Consequently, to show that the X is a Stone space, it suffices to verify that the collection of closed and open sets forms a subbasis for the topology of X.

Remark E.1.3.4. Let X be a Stone space. Then every closed subset $Y \subseteq X$ is also a Stone space (with the induced topology).

E.1.4 The Stone Duality Theorem

We now establish a refinement of Proposition E.1.2.1:

Theorem E.1.4.1 (Stone Duality)**.** *The functor $\psi : \mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}}) \to \mathcal{T}\mathrm{op}$ of Proposition E.1.2.1 induces an equivalence of categories $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}}) \simeq \mathcal{T}\mathrm{op}_{\mathrm{St}}$.*

Lemma E.1.4.2. *The category $\mathcal{T}\mathrm{op}_{\mathrm{St}}$ of Stone spaces is closed under the formation of projective limits (in the larger category $\mathcal{T}\mathrm{op}$ of topological spaces).*

Proof. Suppose we are given an arbitrary diagram $\{X_\alpha\}$ of Stone spaces; we wish to show that $\varprojlim X_\alpha$ is also a Stone space. Note that $\varprojlim_\alpha X_\alpha$ can be identified with a closed subspace of the product $\prod_\alpha X_\alpha$. It will therefore suffice to show that $\prod_\alpha X_\alpha$ is a Stone space (Remark E.1.3.4). This product is obviously Hausdorff, compact by virtue of Tychanoff's theorem, and has a subbasis consisting of inverse images of open subsets of the spaces X_α. Since each X_α has a basis of closed and open sets, we conclude that $\prod_\alpha X_\alpha$ has a subbasis consisting of closed and open sets, and is therefore a Stone space by Remark E.1.3.3. □

Proof of Theorem E.1.4.1. Using Proposition E.1.2.1 and Lemma E.1.4.2, we see that the functor ψ is a fully faithful embedding $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}}) \to \mathcal{T}\mathrm{op}_{\mathrm{St}}$. To complete the proof, it will suffice to show that this map is essentially surjective.

Fix a Stone space X, and let $\mathcal{C}$ be the category whose objects are pairs (T, f), where T is a finite set (which we regard as a discrete topological space) and $f : X \to T$ is a surjection. Let $Y = \varprojlim_{(T,f)\in\mathcal{C}} T$, so that Y is a Stone space belonging to the essential image of ψ. To show that X belongs to the essential image of ψ, it will suffice to show that the canonical map $f : X \to Y$ is a homeomorphism.

Note that Y has a basis of open sets consisting of inverse images of points under the maps $Y \to T$, where $(T, f) \in \mathcal{C}$. It follows that the image of f is dense in Y. Since X is compact and Y is Hausdorff, the map f is closed and therefore surjective. To complete the proof that f is a homeomorphism, it will suffice to show that f is injective. To this end, suppose we are given two distinct points $x, y \in X$. Since X is a Stone space, we can choose a continuous map $u : X \to \{0, 1\}$ such that $u(x) = 0$ and $u(y) = 1$. By construction, this map factors through Y, so that $u(x) \neq u(y)$. □

Corollary E.1.4.3. *Let $f : \{X_\alpha\} \to \{Y_\beta\}$ be a map in the category $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$. If f induces a bijection of sets $\varprojlim X_\alpha \to \varprojlim Y_\beta$, then f is an isomorphism in $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$.*

Proof. According to Theorem E.1.4.1, it will suffice to show that F is a homeomorphism of Stone spaces. This is clear, since it is continuous bijection between compact Hausdorff spaces. □

Remark E.1.4.4. Since the functor $\psi : \mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}}) \to \mathcal{T}\mathrm{op}$ admits a left adjoint (see the proof of Proposition E.1.2.1), it follows from Theorem E.1.4.1 that the inclusion functor $\mathcal{T}\mathrm{op}_{\mathrm{St}} \hookrightarrow \mathcal{T}\mathrm{op}$ also admits a left adjoint. Unwinding the definitions, we see that this left adjoint carries a topological space X to $\varprojlim_E X/E$, where the inverse limit is taken over all equivalence relations E on X for which X/E is finite and the map $X \to X/E$ is continuous. In the special case where X has the discrete topology, we will denote this inverse limit by βX, and refer to it as the *Stone-Čech compactification* of X.

Warning E.1.4.5. The Stone-Čech compactification of an arbitrary topological space X can be defined as an initial object in the category of compact Hausdorff spaces Y equipped with a map $X \to Y$. If the topology on X is discrete, then the Stone-Čech compactification of X is a Stone space, and is therefore homeomorphic to the space βX of Remark E.1.4.4. However, this need not be true if X is not discrete.

E.1.5 Profinite Kan Complexes

Our goal for the remainder of this section is to describe a concrete model for the ∞-category $\mathcal{S}_{\pi}^{\wedge}$ of profinite spaces. First, we need to introduce a bit of notation.

Definition E.1.5.1. Let $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_{\Delta}$ denote the category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}}))$ of simplicial objects of $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$. We will refer to the objects of $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_{\Delta}$ as *simplicial profinite sets.*

If X is a simplicial set with only finitely many vertices in each degree, then we can regard X as simplicial profinite set (by means of the full faithful embedding $\mathcal{S}\mathrm{et}^{\mathrm{fin}} \hookrightarrow \mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$). We will regard $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_{\Delta}$ as a simplicial category, with mapping spaces given by

$$\mathrm{Hom}(\Delta^n, \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_{\Delta}}(X, Y)) \simeq \mathrm{Hom}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_{\Delta}}(\Delta^n \times X, Y).$$

Definition E.1.5.2. Let X be a simplicial profinite set. We will say that X is *well-presented* if it can be written as the inverse limit of a diagram $\{X_\alpha\}_{\alpha \in A}$ indexed by a partially ordered set A satisfying the following conditions:

(1) The partially ordered set A is filtered. Moreover, for every element $\alpha \in A$, the set $\{\beta \in A : \beta \leqslant \alpha\}$ is finite.

(2) Each X_α is a simplicial set with only finitely many simplices of each dimension, regarded as a simplicial profinite set as in Definition E.1.5.1.

(3) The diagram $\{X_\alpha\}_{\alpha \in A}$ is fibrant with respect to the injective model structure on $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\Delta})$. In particular, each X_α is a Kan complex.

(4) For each index $\alpha \in A$, there exists an integer $n \geqslant 0$ such that the Kan complex X_α is n-coskeletal.

In this case, we will also say that $\{X_\alpha\}_{\alpha \in A}$ is a *good presentation* of X. We let $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})^{\mathrm{wp}}_{\Delta}$ denote the full subcategory of $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_{\Delta}$ spanned by the well-presented simplicial profinite sets.

Theorem E.1.5.3. *The simplicial category $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})^{\mathrm{wp}}_{\Delta}$ is fibrant. That is, for every pair of well-presented simplicial profinite sets X and Y, the mapping space $\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_{\Delta}}(X, Y)$ is a Kan complex. Moreover, there is a canonical equivalence of ∞-categories*

$$\mathrm{N}(\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})^{\mathrm{wp}}_{\Delta}) \simeq \mathcal{S}^{\wedge}_{\pi} .$$

Remark E.1.5.4. For a refinement of Theorem E.1.5.3, we refer the reader to [13].

E.1.6 Preliminaries

We now collect some auxiliary results which will be needed in our proof of Theorem E.1.5.3.

Lemma E.1.6.1. *Let S be a well-founded partially ordered set, and let $u : X \to Y$ be a morphism in the category $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\Delta})$. Then u is a fibration with respect to the injective model structure on $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\Delta})$ if and only if, for every index $s \in S$, the induced map*

$$\theta_s : X(s) \to Y(s) \times_{\varprojlim_{t<s} Y(t)} \varprojlim_{t<s} X(t)$$

is a Kan fibration.

Proof. We first prove the "only if" direction. For every simplicial set K and each $s \in S$, define functors $\underline{K}_{\leqslant s}$ and $\underline{K}_{<s}$ by the formulae

$$\underline{K}_{\leqslant s}(t) = \begin{cases} K & \text{if } t \leqslant s \\ \varnothing & \text{otherwise} \end{cases} \qquad \underline{K}_{<s}(t) = \begin{cases} K & \text{if } t < s \\ \varnothing & \text{otherwise.} \end{cases}$$

Note that the map θ_s is a Kan fibration if and only if, for every trivial cofibration of simplicial sets $K \to L$, the induced map $v^s_{K,L} : \underline{K}_{\leqslant s} \amalg_{\underline{K}_{<s}} \underline{L}_{<s} \to \underline{L}_{\leqslant s}$ has the left lifting property with respect to u. Since each of these maps is a trivial cofibration with respect to the injective model structure on $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\Delta})$, the "only if" direction of the assertion follows immediately.

Conversely, suppose that each of the maps θ_s is a Kan fibration. Using our assumption that S is well-founded, we can choose an ordinal α and an enumeration $S = \{s_\beta\}_{\beta < \alpha}$ such that $s_\beta \leqslant s_\gamma$ implies $\beta \leqslant \gamma$. Let $i : U \to V$ be a trivial cofibration in $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\Delta})$. For each $\beta \leqslant \alpha$, define $U_\beta \in \mathrm{Fun}(S^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\Delta})$ by the formula

$$U_\beta(s_\gamma) = \begin{cases} V(s_\gamma) & \text{if } \gamma < \beta \\ U(s_\gamma) & \text{if } \gamma \geqslant \beta. \end{cases}$$

Then i is a transfinite composition of the morphisms $U_\beta \to U_{\beta+1}$, each of which is a pushout of the morphism $v^s_{U(s),V(s)}$ where $s = s_\beta$. If each θ_s is a Kan fibration, then u has the right lifting property with respect to each of the maps $v^s_{U(s),V(s)}$, and therefore also with respect to the morphism i. □

Lemma E.1.6.2. *Let S be a partially ordered set. Suppose that for each $s \in S$, the set $\{t \in S : t \leqslant s\}$ is finite. Then the collection of fibrations in $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta)$ (which we regard as endowed with the injective model structure) is closed under filtered colimits.*

Proof. This follows immediately from the description of the fibrations in $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta)$ supplied by Lemma E.1.6.1. □

Lemma E.1.6.3. *Let Y be a simplicial set which has only finitely many simplices of each dimension and which is n-coskeletal for some integer n. Then Y is compact when viewed as an object of the category $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})^{\mathrm{op}}_\Delta$.*

Proof. For every simplicial profinite set X, we can write $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta}(X, Y)$ as an inverse limit

$$\varprojlim_{[a]\to[b]} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})}(X_b, Y_a),$$

where the limit is taken over all morphisms $[b] \to [a]$ in $\mathbf{\Delta}$. Since Y is n-coskeletal, we can replace this limit by the limit over the category of all maps $[b] \to [a]$ where $a, b \leqslant n$. Since each Y_a is finite, the construction $X \mapsto \mathrm{Hom}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})}(X_b, Y_a)$ carries filtered inverse limits in $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta$ to filtered colimits of sets. It follows that the construction

$$X \mapsto \mathrm{Hom}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta}(X, Y)$$

also carries filtered limits in $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta$ to filtered colimits of sets. □

Lemma E.1.6.4. *Let A be a filtered partially ordered set. Then there exists a cofinal map of partially ordered sets $f : A' \to A$, where A' is a filtered partially ordered set having the property that for every element $\alpha \in A'$, the set $\{\beta \in A' : \beta \leqslant \alpha\}$ is finite.*

Proof. Take A' to be the collection of all finite subsets of A which contain a largest element, and $f : A' \to A$ to be the function which assigns to every such subset its largest element. □

Lemma E.1.6.5. *Let X be a Kan complex. The following conditions are equivalent:*

(a) *There exists a homotopy equivalence of Kan complexes $X \simeq X'$, where X' has finitely many simplices of each dimension.*

(b) *The set $\pi_0 X$ is finite, and the group $\pi_n(X, x)$ is finite for each integer $n \geqslant 1$ and each vertex $x \in X$.*

Proof. Suppose first that (a) is satisfied. Replacing X by X', we may suppose that X has finitely many simplices of each dimension. It follows immediately that $\pi_0 X$ is finite (since it is a quotient of the set of 0-simplices of X) and that each homotopy group $\pi_n(X, x)$ is finite (since it can be realized as a quotient of a subset of the set of n-simplices of X).

Now suppose that (b) is satisfied. We wish to show that there exists a homotopy equivalence $X \to X'$, where X' has finitely many simplices of each dimension. Writing X as a disjoint union of its connected components, we can reduce to the case where X is connected. We will construct X' as the inverse limit of a tower of Kan complexes

$$\cdots \to X'(3) \to X'(2) \to X'(1)$$

with the following properties:

(i) For every integer $n \geqslant 1$, the map $X \to X'(n)$ exhibits $X'(n)$ as an n-truncation of X.

(ii) For each integer $n \geqslant 1$, the Kan complex $X'(n)$ has only finitely many simplices of each dimension.

(iii) The maps $X'(n) \to X'(n-1)$ are bijective on simplices of dimension $< n$.

The construction proceeds by induction on n. To begin, choose a base point $x \in X$, and let $G = \pi_1(X, x)$. Let EG denote the simplicial set whose n-simplices are maps from $\{0, \ldots, n\}$ into G. Then we can take $X'(1)$ to be the quotient $BG = EG/G$.

To carry out the inductive step, suppose that $X'(n-1)$ has been constructed, and set $A = \pi_n(X, x)$. Let $A[n+1]$ denote the chain complex of abelian groups given by A concentrated in degree $n+1$, and let C_* denote the chain complex of abelian groups

$$\cdots \to 0 \to A \xrightarrow{\text{id}} A \to 0 \to \cdots$$

which is concentrated in degrees n and $n+1$, so that we have a surjection of chain complexes $u : C_* \to A[n+1]$. Let $v : Y \to Z$ be the map of simplicial abelian groups associated to u by the Dold-Kan correspondence. Note that the map $\tau_{\leqslant n} X \to X'(n-1)$ is an n-gerbe banded by A (regarded as a local system of abelian groups on $X'(n-1)$, so that there exists a pullback diagram

$$\begin{array}{ccc} \tau_{\leqslant n} X & \longrightarrow & (Y \times EG)/G \\ \downarrow & & \downarrow \\ X'(n-1) & \longrightarrow & (Z \times EG)/G \end{array}$$

in the ∞-category $\mathcal{S}$. Since the right vertical map in this diagram is a Kan fibration, the fiber product

$$X'(n) = X'(n-1) \times_{(Z \times EG)/G} (Y \times EG)/G$$

satisfies all of our requirements. □

Lemma E.1.6.6. *Let S be a partially ordered set with the property that for each $s \in S$, the set $\{t \in S : t < s\}$ is finite, and suppose that we are given a diagram $X : \mathrm{N}(S)^{\mathrm{op}} \to \mathcal{S}_{\pi}$. Then X is equivalent to the map induced by a functor of ordinary categories $Y : S^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_{\Delta}$ satisfying the following conditions:*

(a) *The diagram Y is fibrant with respect to the injective model structure on the category $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\Delta})$.*

(b) *For each $s \in S$, the simplicial set $Y(s)$ is a Kan complex with finitely many simplices in each dimension.*

(c) *For each $s \in S$, there exists an integer n such that the simplicial set $Y(s)$ is n-coskeletal.*

Proof. Since S is well-founded, we can choose an ordinal α and a transfinite enumeration $S = \{s_\beta\}_{\beta < \alpha}$ such that $s_\beta \leqslant s_\gamma$ implies $\beta \leqslant \gamma$. For each $\beta \leqslant \alpha$, let $S_\beta = \{s_\gamma\}_{\gamma < \beta} \subseteq S$. We will construct Y as the amalgam of a compatible family of diagrams $Y_\beta : S_\beta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_{\Delta}$, satisfying the analogues of conditions (a), (b), and (c). The construction proceeds by induction. Assume that $\beta < \alpha$ and that Y_β has been constructed as a fibrant diagram $S_\beta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_{\Delta}$. Using Lemma E.1.6.1, we deduce that the restriction of Y_β to $\{t \in S : t < s_\beta\}$ is a fibrant diagram. Let K denote the inverse limit $\varprojlim_{t < s_\beta} Y_\beta(t)$, calculated in the ordinary category of simplicial sets. Then K is also the homotopy limit of the diagram $\{Y_\beta(t)\}_{t < s_\beta}$. It follows that K is a Kan complex which is equivalent to $\varprojlim_{t < s_\beta} X(t)$. Consequently, the natural map $X(s_\beta) \to \varprojlim_{t < s_\beta}$ determines a map of Kan complexes $X(s_\beta) \to K$. Since $X(s_\beta)$ is π-finite, Lemma E.1.6.5 implies that there exists a homotopy equivalence of Kan complexes $K' \to X(s_\beta)$, where Y has finitely many simplices of each dimension. Replacing K' by a suitably truncation, we may suppose that K' is n-coskeletal for $n \geqslant 0$. Let f denote the composite map $K' \to X(s_\beta) \to K$. Set $K'' = \mathrm{Fun}(\{0\}, K') \times_{\mathrm{Fun}(\{0\}, K)} \mathrm{Fun}(\Delta^1, K)$. Then evaluation at $\{1\} \subseteq \Delta^1$ determines a Kan fibration $K'' \to K$, which is equivalent to the projection map $X(s_\beta) \to \varprojlim_{t < s_\beta} X(t)$. We complete the proof by taking $Y_{\beta+1}$ to be the functor given on objects by the formula

$$Y_{\beta+1}(t) = \begin{cases} K'' & \text{if } t = s_\beta \\ Y_\beta(t) & \text{otherwise.} \end{cases}$$

□

E.1.7 The Proof of Theorem E.1.5.3

We now turn to the proof of Theorem E.1.5.3. Our first goal is to show that the simplicial category $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})^{\mathrm{wp}}_{\Delta}$ is fibrant. Let X and Y be well-presented simplicial profinite spaces;

we wish to show that the simplicial set $\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta}(X, Y)$ is a Kan complex. Choose good presentations

$$X \simeq \varprojlim_{\alpha \in A} X_\alpha \qquad Y \simeq \varprojlim_{\beta \in B} Y_\beta.$$

Since the diagram $\{Y_\beta\}_{\beta \in B}$ is fibrant (with respect to the injective model structure on $\mathrm{Fun}(B^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta)$), it follows that the diagram of simplicial sets $\{\mathrm{Fun}(X_\alpha, Y_\beta)\}_{\beta \in B}$ is fibrant for each $\alpha \in A$. Applying Lemma E.1.6.2, we deduce that the diagram $\{\varinjlim_{\alpha \in A} \mathrm{Fun}(X_\alpha, Y_\beta)_{\beta \in B}$ is fibrant. Using Lemma E.1.6.3, we can identify this with the diagram of simplicial sets

$$\{\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta}(X, Y_\beta)\}_{\beta \in B}.$$

It follows that

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta}(X, Y) \simeq \varprojlim_{\beta \in B} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta}(X, Y_\beta)$$

is a Kan complex, as desired. Moreover, the above argument also proves the following:

- $(*)$ If $\{Y_\beta\}_{\beta \in B}$ is a good presentation of Y, then Y is a homotopy limit of the diagram $\{Y_\beta\}_{\beta \in B}$ in the simplicial category $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta^{\mathrm{wp}}$.

Let $\mathcal{C}$ denote the simplicial nerve of $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta^{\mathrm{wp}}$. Using $(*)$ and Theorem HTT.4.2.4.1 , we deduce:

- $(*')$ If $\{Y_\beta\}_{\beta \in B}$ is a good presentation of $Y \in \mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta$, then Y is a limit of the diagram $\{Y_\beta\}$ in the ∞-category $\mathcal{C}$.

Choose a fully faithful embedding $f : \mathcal{C} \to \overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ admits small limits, and the functor f preserves all small limits which exist in $\mathcal{C}$ (for example, we can take f to be the Yoneda embedding $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$).

Let $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta^0 \subseteq \mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta$ denote the full subcategory spanned by those Kan complexes which have only finitely many simplices in each degree, which are n-truncated for some integer n (regarded as simplicial profinite sets as in Definition E.1.5.1). Let $\mathcal{C}^0$ denote the simplicial nerve of $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta^0$. Since $\overline{\mathcal{C}}$ admits small filtered limits, the composite functor $\mathcal{C}^0 \hookrightarrow \mathcal{C} \xrightarrow{f} \overline{\mathcal{C}}$ admits an essentially unique extension to a functor $F : \mathrm{Pro}(\mathcal{C}^0) \to \overline{\mathcal{C}}$. We will prove the following:

- $(\star)$ The functor F is fully faithful, and its essential image coincides with the essential image of f.

It follows from Lemma E.1.6.5 that a Kan complex X is homotopy equivalent to an object of $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta^0$ if and only if X is π-finite, so that the inclusion $\mathcal{C}^0 \hookrightarrow \mathcal{S}_\pi$ is an equivalence of ∞-categories. Assuming $(\star)$, we obtain equivalences

$$\mathcal{S}_\pi^\wedge = \mathrm{Pro}(\mathcal{S}_\pi) \leftarrow \mathrm{Pro}(\mathcal{C}^0) \simeq \mathcal{C},$$

thereby completing the proof.

It remains to prove ($\star$). We first show that the essential image of F is contained in the essential image of f. Let X be a profinite space. Then we can write X as the limit of a diagram $\{X_\alpha\}_{\alpha \in A}$ indexed by a filtered partially ordered set A, where each X_α is π-finite. Using Lemma E.1.6.4, we can assume that for each $\alpha \in A$, the set $\{\beta \in A : \beta \leqslant \alpha\}$ is finite. Using Lemma E.1.6.6, we can assume that $\{X_\alpha\}_{\alpha \in A}$ is a diagram in the ordinary category of simplicial sets, which is a good presentation for some simplicial profinite set $\overline{X} \in \mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta$. Using ($*'$), we deduce that $\overline{X}$ is a limit of the diagram $\{X_\alpha\}_{\alpha \in A}$ in the ∞-category $\mathcal{C}$. Since f preserves small limits, it follows that $f(\overline{X})$ is a limit of the diagram $\{f(X_\alpha)\}_{\alpha \in A}$. Since the functor F preserves small filtered limits, we have

$$F(X) \simeq \varprojlim F(X_\alpha) = \varprojlim f(X_\alpha) \simeq f(\overline{X}),$$

so that $F(X)$ belongs to the essential image of f.

We next prove that F is fully faithful. Fix objects $X, Y \in \mathcal{S}_\pi^\wedge$; we wish to show that the canonical map

$$\theta_{X,Y} : \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(X,Y) \to \mathrm{Map}_{\overline{\mathcal{C}}}(F(X), F(Y))$$

is a homotopy equivalence. Since F preserves small filtered limits, the construction $Y \mapsto \theta_{X,Y}$ is compatible with filtered limits. It will therefore suffice to show that $\theta_{X,Y}$ is an equivalence in the special case where Y is π-finite. Using Lemma E.1.6.5, we may assume without loss of generality that Y is n-truncated and has finitely many simplices in each dimension. Write $X = \varprojlim_{\alpha \in A} X_\alpha$ as above. We then have a commutative diagram

$$\begin{array}{ccc} \varinjlim_\alpha \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(X_\alpha, Y) & \longrightarrow & \varinjlim_\alpha \mathrm{Map}_{\overline{\mathcal{C}}}(F(X_\alpha), Y) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(X, Y) & \longrightarrow & \mathrm{Map}_{\overline{\mathcal{C}}}(F(X), F(Y)). \end{array}$$

The upper horizontal map is a homotopy equivalence because the functor F is fully faithful when restricted to $\mathcal{C}^0$ (where it agrees with f). The left vertical map is a homotopy equivalence because Y is π-finite, and the right vertical map can be identified with the map

$$\varinjlim_{\alpha} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta}(X_\alpha, Y) \to \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})_\Delta}(\overline{X}, Y),$$

which is an isomorphism by Lemma E.1.6.3.

To complete the proof of Theorem E.1.5.3, it will suffice to show that for every object $X \in \mathcal{C}$, the object $f(X) \in \overline{\mathcal{C}}$ belongs to the essential image of F. To prove this, choose a good presentation $\{X_\alpha\}_{\alpha \in A}$ for X. It follows from ($*'$) that X is a limit of the diagram $\{X_\alpha\}_{\alpha \in A}$ in $\mathcal{C}$, so that $f(X) \simeq \varprojlim_\alpha f(X_\alpha)$. Since F is a fully faithful embedding which preserves small filtered limits, the essential image of F is closed under small filtered limits. It will therefore suffice to show that $f(X_\alpha)$ belongs to the essential image of F for each $\alpha \in A$. This is clear, since $f(X_\alpha) \simeq F(X_\alpha)$.

E.2 Shape Theory

The ∞-category $\mathcal{S}_\pi$ of π-finite spaces is a bounded ∞-pretopos, whose associated ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{S}_\pi)$ can be identified with the ∞-category $\mathcal{S}$. Using Theorem A.7.5.3 (and the fact that the ∞-topos $\mathcal{S}$ is a final object of $\infty\mathcal{T}\mathrm{op}$, we deduce the existence of a fully faithful embedding

$$\Psi_\pi : \mathcal{S}_\pi^\wedge = \mathrm{Pro}(\mathcal{S}_\pi) \hookrightarrow \infty\mathcal{T}\mathrm{op}_{/\mathcal{S}} \simeq \infty\mathcal{T}\mathrm{op}\,.$$

In this section, we will see that the functor Ψ_π embedding has a very natural interpretation in the language of shape theory: it appears as the right adjoint to the *profinite shape functor* $\mathrm{Sh}_\pi : \infty\mathcal{T}\mathrm{op} \to \mathcal{S}_\pi^\wedge$ (Variant E.2.2.2). It follows that the essential image of Ψ_π ∞-topoi can be regarded as a localization of the ∞-category $\infty\mathcal{T}\mathrm{op}$ of all ∞-topoi. Our main goal is to obtain an explicit description of the associated localization functor $\Psi_\pi \circ \mathrm{Sh}_\pi$, which we will denote by $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{pf}}$ and refer to as *profinite reflection*. As we will see, the ∞-topos $\mathcal{X}^{\mathrm{pf}}$ can be regarded as a "best approximation" to $\mathcal{X}$ that can be recovered from the locally constant constructible objects of $\mathcal{X}$ (for a precise statement, see Theorem E.2.3.2).

E.2.1 Pro-Spaces

We begin with some general remarks about Pro-objects of the ∞-category of spaces.

Definition E.2.1.1. A Pro-*space* is a Pro-object of the ∞-category of spaces: that is, an accessible left-exact functor from the ∞-category $\mathcal{S}$ to itself. We will refer to $\mathrm{Pro}(\mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{S},\mathcal{S})^{\mathrm{op}}$ as the *∞-category of* Pro-*spaces.*.

Remark E.2.1.2. The collection of accessible left-exact functors from $\mathcal{S}$ to itself is closed under composition. Consequently, we can regard the full subcategory $\mathrm{Pro}(\mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{S},\mathcal{S})^{\mathrm{op}}$ as a simplicial monoid (and, in particular, a monoidal ∞-category). We will denote the resulting product on $\mathrm{Pro}(\mathcal{S})$ by

$$\circ : \mathrm{Pro}(\mathcal{S}) \times \mathrm{Pro}(\mathcal{S}) \to \mathrm{Pro}(\mathcal{S}).$$

Note that the unit object $\mathrm{id} \in \mathrm{Pro}(\mathcal{S})$ is also a final object. In particular, for every pair of objects $U, V \in \mathrm{Pro}(\mathcal{S})$, we have a canonical map $U \circ V \to (U \circ \mathrm{id}) \times (\mathrm{id} \circ V) \simeq U \times V$. Beware that this map is generally not an equivalence.

Remark E.2.1.3. Composition with the inclusion functor $i : \mathcal{S}_\pi \hookrightarrow \mathcal{S}$ induces a forgetful functor $\mathrm{Pro}(\mathcal{S}) \to \mathcal{S}_\pi^\wedge$. According to Example A.8.1.8, this forgetful functor is left adjoint to the map $\mathrm{Pro}(i) : \mathcal{S}_\pi^\wedge = \mathrm{Pro}(\mathcal{S}_\pi) \to \mathrm{Pro}(\mathcal{S})$, which is a fully faithful embedding (Proposition A.8.1.9). We can summarize the situation by saying that the forgetful functor $\mathrm{Pro}(\mathcal{S}) \to \mathcal{S}_\pi^\wedge$ exhibits the ∞-category of profinite spaces as a localization of the ∞-category of Pro-spaces. We will sometimes abuse terminology by identifying $\mathcal{S}_\pi^\wedge$ with its essential image in $\mathrm{Pro}(\mathcal{S})$.

Remark E.2.1.4. Since the ∞-category $\mathcal{S}$ of spaces is presentable, Example A.8.1.7 implies that the Yoneda embedding $\mathcal{S} \to \mathrm{Pro}(\mathcal{S})$ admits a right adjoint, given on objects by $\{X_\alpha\} \mapsto \varprojlim X_\alpha$. We will denote this right adjoint by $\mathrm{Mat} : \mathrm{Pro}(\mathcal{S}) \to \mathcal{S}$, and refer to it as the *materialization functor*.

Remark E.2.1.5. The profinite completion functor $X \mapsto X_\pi^\wedge$ factors as a composition

$$\mathcal{S} \xrightarrow{j} \mathrm{Pro}(\mathcal{S}) \xrightarrow{r} \mathrm{Pro}(\mathcal{S}_\pi) = \mathcal{S}_\pi^\wedge,$$

where j denotes the Yoneda embedding and r denotes the restriction functor. Note that j admits a right adjoint (given by the materialization functor $\mathrm{Mat} : \mathrm{Pro}(\mathcal{S}) \to \mathcal{S}$) and that r admits a right adjoint (given by the fully faithful embedding $\mathcal{S}_\pi^\wedge \hookrightarrow \mathrm{Pro}(\mathcal{S})$ of Remark E.2.1.3). It follows that the right adjoint to profinite completion is given by the composition $\mathcal{S}_\pi^\wedge \hookrightarrow \mathrm{Pro}(\mathcal{S}) \xrightarrow{\mathrm{Mat}} \mathcal{S}$, which we will denote also by Mat.

E.2.2 Shape and Profinite Shape

Let $\mathcal{X}$ be an ∞-topos. Then $\mathcal{X}$ admits an essentially unique geometric morphism $q_* : \mathcal{X} \to \mathcal{S}$, which has a left adjoint $q^* : \mathcal{S} \to \mathcal{X}$. The composition $q_* q^* : \mathcal{S} \to \mathcal{S}$ is an accessible left exact functor from the ∞-category $\mathcal{S}$ to itself, which we can identify with a Pro-space $\mathrm{Sh}(\mathcal{X}) \in \mathrm{Pro}(\mathcal{S})$ which we refer to as the *shape* of $\mathcal{X}$ (see §HTT.7.1.6 .

Proposition E.2.2.1. *Let $\Psi : \mathrm{Pro}(\mathcal{S}) \to \infty\mathcal{T}\mathrm{op}$ be the (essentially unique) functor which preserves small filtered limits whose composition with the Yoneda embedding $\mathcal{S} \hookrightarrow \mathrm{Pro}(\mathcal{S})$ is given by $X \mapsto \mathcal{S}_{/X}$ (see Proposition A.8.1.6). Then the functor Ψ admits a left adjoint, given on objects by $\mathcal{X} \mapsto \mathrm{Sh}(\mathcal{X})$.*

Proof. Let $\mathcal{X}$ be an ∞-topos and let Y be a Pro-space; we wish to show that there is a homotopy equivalence $\mathrm{Map}_{\mathrm{Pro}(\mathcal{S})}(\mathrm{Sh}(\mathcal{X}), Y) \simeq \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \Psi(Y))$ which depends functorially on Y. Since both sides commute with filtered limits in Y, we may assume without loss of generality that Y belongs to $\mathcal{S}$ (which, by abuse of notation, we identify with its essential image in $\mathrm{Pro}(\mathcal{S})$). Let $q_* : \mathcal{X} \to \mathcal{S}$ be a geometric morphism of ∞-topoi and let $q^* : \mathcal{S} \to \mathcal{X}$ be its left adjoint. Invoking the universal property of $\Psi(Y) = \mathcal{S}_{/Y}$ given by Proposition **??**, we obtain canonical homotopy equivalences

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \Psi(Y)) = \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{S}_{/Y}) \simeq \Gamma(\mathcal{X}; q^* Y) = \mathrm{Sh}(\mathcal{X})(Y) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{S})}(\mathrm{Sh}(\mathcal{X}), Y).$$

□

It follows from Proposition E.2.2.1 that we can regard the construction $\mathcal{X} \mapsto \mathrm{Sh}(\mathcal{X})$ as a functor from the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi to the ∞-category $\mathrm{Pro}(\mathcal{S})$ of Pro-spaces. Moreover, this functor preserves small colimits.

Variant E.2.2.2. Composing the shape functor $\mathrm{Sh} : \infty\mathcal{T}\mathrm{op} \to \mathrm{Pro}(\mathcal{S})$ with the forgetful functor $\mathrm{Pro}(\mathcal{S}) \to \mathrm{Pro}(\mathcal{S}_\pi) = \mathcal{S}_\pi^\wedge$, we obtain a functor $\mathrm{Sh}_\pi : \infty\mathcal{T}\mathrm{op} \to \mathcal{S}_\pi^\wedge$. If $\mathcal{X}$ is an ∞-topos, we will refer to the profinite space $\mathrm{Sh}_\pi(\mathcal{X})$ as the *profinite shape* of $\mathcal{X}$.

Example E.2.2.3. The construction $X \mapsto \mathcal{S}_{/X}$ determines a fully faithful embedding from the ∞-category of spaces to the ∞-category $\mathcal{T}\mathrm{op}_\infty$ of ∞-topoi (Remark HTT.6.3.5.10). Consequently, for any space X, the profinite shape $\mathrm{Sh}_\pi(\mathcal{S}_{/X})$ of the ∞-topos $\mathcal{S}_{/X}$ can be identified with the profinite completion $X_\pi^\wedge$ of X. In other words, the notion of profinite completion (introduced in Example E.0.7.12) can be regarded as a special case of the notion of profinite shape (introduced in Variant E.2.2.2).

E.2.3 Profinite Reflection

Let $\infty\mathcal{T}\mathrm{op}$ denote the ∞-category of ∞-topoi, and let $\infty\mathcal{T}\mathrm{op}^{\mathrm{pf}}$ denote the full subcategory of $\infty\mathcal{T}\mathrm{op}$ spanned by the profinite ∞-topoi (Definition E.2.4.3). It follows from Theorem E.2.4.1 that $\infty\mathcal{T}\mathrm{op}^{\mathrm{pf}}$ is a localization of $\infty\mathcal{T}\mathrm{op}$: that is, the inclusion functor $\infty\mathcal{T}\mathrm{op}^{\mathrm{pf}} \hookrightarrow \infty\mathcal{T}\mathrm{op}$ admits a left adjoint. We will denote this left adjoint by $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{pf}}$, and we will refer to the profinite ∞-topos $\mathcal{X}^{\mathrm{pf}}$ as the *profinite reflection* of an ∞-topos $\mathcal{X}$.

Example E.2.3.1. Let $X \in \mathcal{S}$ be a space. Then the profinite reflection of the ∞-topos $\mathcal{X}_{/X}$ can be identified with the profinite ∞-topos $\Psi_\pi(X_\pi^\wedge)$ (see Example E.2.2.3), where $X_\pi^\wedge$ denotes the profinite completion of X. In other words, we can regard profinite reflections of ∞-topoi as a generalization of profinite completions of spaces.

We can now state the main result of this section:

Theorem E.2.3.2. *Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}^{\mathrm{lcc}}$ denote the full subcategory of $\mathcal{X}$ spanned by the locally constant constructible objects. Then:*

(1) *The ∞-category $\mathcal{X}^{\mathrm{lcc}}$ is a bounded ∞-pretopos (see Definition A.7.4.1).*

(2) *The inclusion functor $\iota : \mathcal{X}^{\mathrm{lcc}} \to \mathcal{X}$ is an ∞-pretopos morphism (Definition A.6.4.1) .*

(3) *The functor ι induces a geometric morphism of ∞-topoi $\iota_* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{lcc}})$.*

(4) *The geometric morphism ι_* exhibits $\mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{lcc}})$ as a profinite reflection of $\mathcal{X}$.*

Theorem E.2.3.2 immediately implies the following stronger form of Proposition E.2.7.1:

Corollary E.2.3.3. *Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. Then f_* induces an equivalence of profinite spaces $\mathrm{Sh}_\pi(\mathcal{X}) \to \mathrm{Sh}_\pi(\mathcal{Y})$ if and only if the pullback functor f^* restricts to an equivalence of ∞-categories $\mathcal{Y}^{\mathrm{lcc}} \to \mathcal{X}^{\mathrm{lcc}}$.*

Proof of Theorem E.2.3.2. Let $\mathcal{X}^{\mathrm{pf}}$ be the profinite reflection of $\mathcal{X}$. There is an evident geometric morphism $u_* : \mathcal{X} \to \mathcal{X}^{\mathrm{pf}}$ which induces an equivalence of profinite shapes. Applying Proposition E.2.7.1, we deduce that the associated pullback functor u^* induces an equivalence of ∞-categories $(\mathcal{X}^{\mathrm{pf}})^{\mathrm{lcc}} \to \mathcal{X}^{\mathrm{lcc}}$. Since $\mathcal{X}^{\mathrm{pf}}$ is profinite, Proposition **??** implies that we can identify $(\mathcal{X}^{\mathrm{pf}})^{\mathrm{lcc}}$ with the full subcategory of $\mathcal{X}^{\mathrm{pf}}$ spanned by the truncated coherent objects, which is a bounded ∞-pretopos (Example A.7.4.4). This proves (1). Assertion (2) follows from from Proposition E.2.6.1 and assertion (3) from (2) and Proposition A.6.4.4. To prove (4), we observe that there is a commutative diagram of ∞-topoi and geometric morphisms

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{u_*} & \mathcal{X}^{\mathrm{pf}} \\ \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*^{\mathrm{pf}}} \\ \mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{lcc}}) & \longrightarrow & \mathcal{S}\mathrm{hv}((\mathcal{X}^{\mathrm{pf}})^{\mathrm{lcc}}). \end{array}$$

The lower horizontal map is an equivalence by virtue of Proposition E.2.7.1 and the right vertical map is an equivalence by virtue of Corollary E.2.5.7 (since the ∞-topos $\mathcal{X}^{\mathrm{pf}}$ is profinite). Since u_* exhibits $\mathcal{X}^{\mathrm{pf}}$ as a profinite reflection of $\mathcal{X}$ by construction, it follows that the functor ι_* also exhibits $\mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{lcc}})$ as a profinite reflection of $\mathcal{X}$. □

E.2.4 Profinite ∞-Topoi

We now introduce a higher-categorical generalization of the theory of Stone spaces.

Theorem E.2.4.1. *The profinite shape functor* $\mathrm{Sh}_\pi : \infty\mathcal{T}\mathrm{op} \to \mathcal{S}^{\wedge}_{\pi}$ *admits a fully faithful right adjoint* $\Psi_\pi : \mathcal{S}^{\wedge}_{\pi} \to \infty\mathcal{T}\mathrm{op}$.

Proof. The functor Ψ_π can be obtained by composing the fully faithful embedding $\mathcal{S}^{\wedge}_{\pi} = \mathrm{Pro}(\mathcal{S}_\pi) \hookrightarrow \mathrm{Pro}(\mathcal{S})$ with the functor $\Psi : \mathrm{Pro}(\mathcal{S}) \to \infty\mathcal{T}\mathrm{op}$. More concretely, if X is a profinite space given as the limit of a filtered diagram of π-finite spaces $\{X_\alpha\}$, then $\Psi_\pi(X)$ is the limit of the diagram of ∞-topoi $\{\mathcal{S}_{/X_\alpha}\}$. It follows from Theorem **??** (and the observation that $\Psi_\pi(*) \simeq \mathcal{S}$ is a final object of $\infty\mathcal{T}\mathrm{op}$) that the functor Ψ_π is fully faithful. □

Warning E.2.4.2. In the statement of Theorem **??**, the restriction to profinite spaces is essential: the functor $\Psi : \mathrm{Pro}(\mathcal{S}) \to \infty\mathcal{T}\mathrm{op}$ is not fully faithful.

Definition E.2.4.3. We will say that an ∞-topos $\mathcal{X}$ is *profinite* if it belongs to the essential image of the functor $\Psi_\pi : \mathcal{S}^{\wedge}_{\pi} \to \infty\mathcal{T}\mathrm{op}$ of Theorem **??**. We let $\infty\mathcal{T}\mathrm{op}^{\mathrm{pf}}$ denote the full subcategory of $\infty\mathcal{T}\mathrm{op}$ spanned by the profinite ∞-topoi.

Remark E.2.4.4. An ∞-topos $\mathcal{X}$ is profinite if and only if it can be obtained as a limit of a filtered diagram $\{\mathcal{X}_\alpha\}$ in $\infty\mathcal{T}\mathrm{op}$, where each $\mathcal{X}_\alpha$ is equivalent to $\mathcal{S}_{/X_\alpha}$ for some π-finite space X_α.

Remark E.2.4.5. Every profinite ∞-topos $\mathcal{X}$ is bounded and coherent (see Corollary A.8.3.3).

Example E.2.4.6. Let X be a Stone space (Definition A.1.6.8). Then the ∞-topos $\mathcal{S}\mathrm{hv}(X)$ is profinite. To prove this, we first use Theorem E.1.4.1 to write X as the inverse limit of a filtered diagram of finite sets $\{X_\alpha\}$. Let $\mathcal{X}$ denote the profinite ∞-topos $\varprojlim \mathcal{S}_{/X_\alpha}$. Since the collection of 0-localic ∞-topoi is closed under small limits, we see that $\mathcal{X}$ is 0-localic. We may therefore write $\mathcal{X} = \mathcal{S}\mathrm{hv}(\mathcal{U})$, where $\mathcal{U}$ is a locale (Corollary HTT.6.4.2.6). Unwinding the definitions, we can identify X with the topological space of points of $\mathcal{U}$. Since $\mathcal{X}$ is a profinite ∞-topos, it is locally coherent (Remark E.2.4.5), so that the hypercompletion $\mathcal{X}^{\mathrm{hyp}}$ has enough points (Theorem A.4.0.5). It follows that $\mathcal{U}$ is a spatial locale, hence isomorphic to the collection of open subsets of X. We therefore have an equivalence $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(X)$, so that $\mathcal{S}\mathrm{hv}(X)$ is profinite as desired.

Example E.2.4.7. Let G be a profinite group, and let $BG \in \mathcal{S}_\pi^\wedge$ be its (profinite) classifying space (see §E.5). Arguing as in Example E.2.4.6, we see that the associated profinite ∞-topos $\rho(BG)$ is 1-localic, and that its topos of discrete objects can be identified with the category of sets equipped with a continuous action of G. It is therefore natural to think of $\rho(BG)$ as the ∞-category of spaces equipped with a continuous action of G. Beware that this is potentially misleading: the ∞-topos $\rho(BG)$ need not be hypercomplete (see Warning HTT.7.2.2.31), so that the fiber functor $\eta^* : \rho(BG) \to \mathcal{S}$ associated to a base point $\eta \in BG$ (which "forgets the action of G") need not be conservative.

Example E.2.4.8. Let κ be a field, and let $\overline{\kappa}$ denote a separable closure of κ. Then the ∞-category $\mathcal{S}\mathrm{hv}_\kappa^{\text{ét}}$ is 1-localic, and its underlying topos of discrete objects can be identified with the category of sets equipped with a continuous action of the Galois group $G = \mathrm{Gal}(\overline{\kappa}/\kappa)$. It follows that $\mathcal{S}\mathrm{hv}_\kappa^{\text{ét}}$ is a profinite ∞-topos, associated the profinite space BG.

E.2.5 Locally Constant Constructible Sheaves

Every profinite ∞-topos $\mathcal{X}$ is bounded and coherent (Remark E.2.4.5). However, the converse is generally false. For example, if X is a coherent topological space, then $\mathcal{S}\mathrm{hv}(X)$ is a coherent ∞-topos, but $\mathcal{S}\mathrm{hv}(X)$ is profinite if and only if X is a Stone space. To guarantee that an ∞-topos $\mathcal{X}$ is profinite, it is not enough to assume the existence of a large class of coherent objects of $\mathcal{X}$: one needs to assume in addition that such objects are *locally constant* in the following sense:

Definition E.2.5.1. Let $\mathcal{X}$ be an ∞-topos, so that there is an essentially unique geometric morphism $q^* : \mathcal{S} \to \mathcal{X}$. We will say that an object $X \in \mathcal{X}$ is *locally constant constructible* if there exists a finite collection of objects $\{U_i \in \mathcal{X}\}_{1 \leqslant i \leqslant n}$ which cover $\mathcal{X}$, a collection of π-finite spaces $\{Y_i\}_{1 \leqslant i \leqslant n}$, and equivalences equivalences $X \times U_i \simeq q^* Y_i \times U_i$ in the ∞-topos

$\mathcal{X}_{/U_i}$ for $1 \leqslant i \leqslant n$. We let $\mathcal{X}^{\mathrm{lcc}}$ denote the full subcategory of $\mathcal{X}$ spanned by the locally constant constructible objects.

Remark E.2.5.2. Let $\mathcal{X}$ be an ∞-topos. Then the collection of locally constant constructible objects of $\mathcal{X}$ is closed under finite coproducts.

Remark E.2.5.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi, and let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism. Then the pullback functor $f^* : \mathcal{Y} \to \mathcal{X}$ carries locally constant constructible objects of $\mathcal{Y}$ to locally constant constructible objects of $\mathcal{X}$.

Warning E.2.5.4. Let $\mathcal{X}$ be an ∞-topos. Then the condition that an object $X \in \mathcal{X}$ be locally constant constructible is *not* local on $\mathcal{X}$. For example, suppose that $\mathcal{X} = \mathcal{S}_{/Y}$, for some Kan complex Y. Then we can identify objects $X \in \mathcal{X}$ with functors $\chi : Y \to \mathcal{S}$. If χ takes values in the full subcategory $\mathcal{S}_\pi \subseteq \mathcal{S}$ spanned by the π-finite spaces, then there exists a covering of $\mathcal{X}$ by objects U_α such that each $X \times U_\alpha$ is a locally constant constructible object of $\mathcal{X}_{/U_\alpha}$ (take the U_α to be the connected components of Y). However, X is not locally constant constructible unless the essential image of χ has only finitely many homotopy equivalence classes of objects.

Proposition E.2.5.5. *Let $\mathcal{X}$ be a coherent ∞-topos and let X be an object of $\mathcal{X}$. If X is locally constant constructible, then X is coherent.*

Proof. Fix an integer $n \geqslant 0$; we will show that X is n-coherent. Let $q^* : \mathcal{S} \to \mathcal{X}$ be a geometric morphism of ∞-topoi. Choose a finite covering $\{U_i\}_{1 \leqslant i \leqslant m}$ of $\mathcal{X}$ and equivalences $X \times U_i \simeq f^* Y_i \times U_i$ in $\mathcal{X}_{/U_i}$, where each Y_i is a π-finite space. Choose an integer k such that each Y_i is k-truncated. Using Corollary A.2.1.5, can reduce to the problem of showing that each of the maps $X \times U_i \simeq q^* Y_i \times U_i \to U_i$ is relatively n-coherent. For this, it will suffice to show that $q^* Y \in \mathcal{X}$ is coherent for each $Y \in \mathcal{S}_\pi$.

Choose an integer k such that Y is k-truncated; we proceed by induction on k. Since the collection of coherent objects of $\mathcal{X}$ is closed under finite coproducts, we may assume that Y is connected. Choose a base point $y \in Y$, so that y induces an effective epimorphism $f : U_i \to f^* Y \times U_i$ in the ∞-topos $\mathcal{X}_{/U_i}$. Note that the fiber product $U_i \times_{f^* Y \times U_i} U_i$ can be identified with $q^*(\{y\} \times_Y \{y\}) \times U_i$, and is therefore an coherent object of $\mathcal{X}$ by our inductive hypothesis. Using Corollary A.2.1.5, we deduce that the map $f : \mathbf{1} \to X$ is relatively coherent, so that X is coherent by virtue of Proposition A.2.1.3. □

For profinite ∞-topoi, Proposition E.2.5.5 admits a converse:

Proposition E.2.5.6. *Let $\mathcal{X}$ be a profinite ∞-topos. Then an object $X \in \mathcal{X}$ is locally constant constructible if and only if it is coherent and n-truncated for some $n \gg 0$.*

Proof. The "only if" direction follows from Proposition E.2.5.5. To prove the converse, write $\mathcal{X}$ as the limit of a filtered diagram of ∞-topoi $\{\mathcal{X}_\alpha\}$, where each $\mathcal{X}_\alpha$ has the form $\mathcal{S}_{/Y}$ for some π-finite space Y. It follows from Proposition A.8.3.2 that the full subcategory $\mathcal{X}^{\mathrm{coh}}_{<\infty} \subseteq \mathcal{X}$ spanned by the truncated coherent objects can be written as a filtered colimit of the subcategories $\mathcal{X}^{\mathrm{coh}}_{\alpha<\infty} \subseteq \mathcal{X}_\alpha$. In particular, every truncated coherent object of $\mathcal{X}$ can obtained as the pullback of a bounded coherent object of some $\mathcal{X}_\alpha$. Using Remark E.2.5.3, we can replace $\mathcal{X}$ by $\mathcal{X}_\alpha$ and thereby reduce to the case where $\mathcal{X} = \mathcal{S}_{/Y}$ for some π-finite space Y.

Unwinding the definitions, we see that a truncated coherent object $X \in \mathcal{X}$ can be identified with a π-finite space X equipped with a map $f : X \to Y$. Since Y is π-finite, the set $\pi_0 Y$ is finite. We can therefore choose a finite collection of points $y_1, \ldots, y_k \in Y$ which represent every connected component of Y, so that objects $\{y_1\}, \ldots, \{y_k\} \in \mathcal{S}_{/Y}$ form a covering of the ∞-topos $\mathcal{S}_{/Y}$. It now suffices to observe that each of the homotopy fibers $X \times_Y \{y_i\} \in \mathcal{S}_{/\{y_i\}} \simeq \mathcal{S}$ is π-finite. □

Corollary E.2.5.7. *Let $\mathcal{X}$ be a profinite ∞-topos. Then:*

(1) *The ∞-category $\mathcal{X}^{\mathrm{lcc}}$ is a bounded ∞-pretopos (see Definition A.7.4.1).*

(2) *The inclusion functor $\iota : \mathcal{X}^{\mathrm{lcc}} \to \mathcal{X}$ is an ∞-pretopos morphism (Definition A.6.4.1) .*

(3) *The functor ι induces an equivalence of ∞-topoi $\iota_* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{lcc}})$ (where we regard $\mathcal{X}^{\mathrm{lcc}}$ as equipped with the effective epimorphism topology of Definition A.6.2.4).*

Proof. Since every profinite ∞-topos $\mathcal{X}$ is bounded and coherent (Remark E.2.4.5), Theorem A.7.5.3 supplies an equivalence of ∞-categories $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$. The desired result now follows from Proposition E.2.5.6. □

E.2.6 Closure Properties of $\mathcal{X}^{\mathrm{lcc}}$

Our next goal is to show that the ∞-category $\mathcal{X}^{\mathrm{lcc}}$ of locally constant constructible sheaves on an ∞-topos $\mathcal{X}$ behaves well even when $\mathcal{X}$ is not coherent.

Proposition E.2.6.1. *Let $\mathcal{X}$ be an ∞-topos. Then the full subcategory $\mathcal{X}^{\mathrm{lcc}} \subseteq \mathcal{X}$ is an ∞-pretopos. Moreover, the inclusion $\mathcal{X}^{\mathrm{lcc}} \hookrightarrow \mathcal{X}$ is a morphism of ∞-pretopoi.*

The proof will require some preliminaries.

Notation E.2.6.2. Let $\mathcal{X}$ be an ∞-topos. For every pair of objects $X, Y \in \mathcal{X}$, the functor $Z \mapsto \mathrm{Map}_{\mathcal{X}}(Z \times X, Y)$ carries small colimits in $\mathcal{X}$ to limits in the ∞-category of spaces, and is therefore representable by an object of $\mathcal{X}$. We will denote this object by $\underline{\mathrm{Map}}_{\mathcal{X}}(X, Y)$.

Let $q_* : \mathcal{X} \to \mathcal{S}$ be the global sections functor (corepresented by a final object of $\mathcal{X}$) and let $q^* : \mathcal{S} \to \mathcal{X}$ be a left adjoint to q_*. Then the functor q^* preserves finite products. Consequently, for every pair of Kan complexes K and L, the canonical map $\mathrm{Fun}(K,L) \times K \to L$ determines a morphism $q^* \mathrm{Fun}(K,L) \times q^* K \to q^* L$, which is adjoint to a map $\phi_{K,L} : q^* \mathrm{Fun}(K,L) \to \underline{\mathrm{Map}}_{\mathcal{X}}(q^* K, q^* L)$.

Lemma E.2.6.3. *Let $\mathcal{X}$ be an ∞-topos, and let $q^* : \mathcal{S} \to \mathcal{X}$ and $\phi_{K,L} : q^* \mathrm{Fun}(K,L) \to \underline{\mathrm{Map}}_{\mathcal{X}}(q^* K, q^* L)$ be defined as above. If K is π-finite and L is truncated, then $\phi_{K,L}$ is an equivalence in $\mathcal{X}$.*

Proof. Since K is π-finite, there exists an integer n such that K is n-truncated. We proceed by induction on n. If $n = 0$, then K is a finite discrete set and both sides can be identified with a product of finitely many copies of $q^* L$. To carry out the inductive step, assume that $n > 0$. Let U_0 be a finite set consisting of one vertex from each connected component of K, and let $U_\bullet$ denote the Čech nerve of the map $U_0 \to K$, so that we have a homotopy equivalence $|U_\bullet| \simeq K$. Then we can identify $\phi_{K,L}$ with the composite map

$$q^* \varprojlim_{[d] \in \mathbf{\Delta}} \mathrm{Fun}(U_d, L) \xrightarrow{\phi'} \varprojlim_{[d] \in \mathbf{\Delta}} q^* \mathrm{Fun}(U_d, L) \xrightarrow{\phi''} \varprojlim_{[d] \in \mathbf{\Delta}} \underline{\mathrm{Map}}_{\mathcal{X}}(q^* U_d, q^* L).$$

The map ϕ'' is an equivalence by virtue of the inductive hypothesis. For each integer m, the map ϕ' fits into a commutative diagram

$$\begin{array}{ccc} q^* \varprojlim_{[d] \in \mathbf{\Delta}} \mathrm{Fun}(U_d, L) & \xrightarrow{\phi'} & \varprojlim_{[d] \in \mathbf{\Delta}} q^* \mathrm{Fun}(U_d, L) \\ \downarrow & & \downarrow \\ q^* \varprojlim_{[d] \in \mathbf{\Delta}_{s, \leq m}} \mathrm{Fun}(U_d, L) & \longrightarrow & \varprojlim_{[d] \in \mathbf{\Delta}_{s, \leq m}} q^* \mathrm{Fun}(U_d, L). \end{array}$$

Since L is truncated, the vertical maps are equivalences for $m \gg 0$, and the lower horizontal map is an equivalence because q^* is left-exact. It follows that ϕ' is an equivalence, as desired. □

Remark E.2.6.4. The hypotheses of Lemma E.2.6.3 are satisfied whenever the Kan complexes K and L are both π-finite. Note that in this case, the Kan complex $\mathrm{Fun}(K,L)$ is also π-finite.

Remark E.2.6.5. Passing to global sections, we see that if K and L are π-finite Kan complexes, then the canonical map $\beta : \mathrm{Map}_{\mathcal{X}}(\mathbf{1}, q^* \mathrm{Map}_{\mathcal{S}}(K,L)) \to \mathrm{Map}_{\mathcal{X}}(q^* K, q^* L)$ is a homotopy equivalence. Note that the map β restricts to a homotopy equivalence $\beta_0 : \mathrm{Map}_{\mathcal{X}}(\mathbf{1}, q^* \mathrm{Map}_{\mathcal{S}^{\simeq}}(K,L)) \to \mathrm{Map}_{\mathcal{X}^{\simeq}}(q^* K, q^* L)$.

It follows from Lemma E.2.6.3 that any finite diagram of locally constant constructible objects of $\mathcal{X}$ can be "made constant" after passing to a finite covering of $\mathcal{X}$:

Proposition E.2.6.6. *Let $\mathcal{X}$ be an ∞-topos, let $q^* : \mathcal{S} \to \mathcal{X}$ be a geometric morphism, let K be a finite simplicial set, and let $\rho : K \to \mathcal{X}^{\mathrm{lcc}}$ be a diagram. Then there exists a finite collection of objects $\{U_i\}_{1 \leqslant i \leqslant n}$ which cover $\mathcal{X}$ and a finite collection of diagrams $\{\rho_i : K \to \mathcal{S}_\pi\}_{1 \leqslant i \leqslant n}$ such that, for $1 \leqslant i \leqslant n$, there is an equivalence $U_i \times \rho \simeq U_i \times q^*(\rho_i)$ in the ∞-category* $\mathrm{Fun}(K, \mathcal{X}_{/U_i})$.

Proof. We proceed by induction on the number of nondegenerate simplices of K. If $K = \varnothing$ there is nothing to prove. Otherwise, we can write K as a coproduct $K_0 \amalg_{\partial \Delta^n} \Delta^n$ for some simplicial subset $K_0 \subseteq K$ and some $n \geqslant 0$. Using our inductive hypothesis, we may assume (after passing to a finite covering of $\mathcal{X}$ if necessary) that $\rho|_{K_0}$ is homotopic to a composition $K_0 \xrightarrow{\rho'_0} \mathcal{S}_\pi \xrightarrow{q^*} \mathcal{X}$ for some functor $\rho'_0 : K_0 \to \mathcal{S}_\pi$; we wish to show that (after passing to a further covering of $\mathcal{X}$ if necessary) that the functor ρ admits a similar factorization. If $n = 0$, this follows immediately from the definition of $\mathcal{X}^{\mathrm{lcc}}$. Let us therefore assume that $n > 0$. Let $L, L' \in \mathcal{S}_\pi$ denote the images under ρ'_0 of the initial and final vertices of $\partial \Delta^n$, so that ρ'_0 and ρ determine a commutative diagram of spaces

$$\begin{array}{ccc} S^{n-2} & \xrightarrow{u_0} & \mathrm{Map}_{\mathcal{S}}(L, L') \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Map}_{\mathcal{X}}(q^* L, q^* L'). \end{array}$$

Using Lemma E.2.6.3, we can identify this diagram with a global section of the object

$$\mathrm{fib}(q^* \mathrm{Map}_{\mathcal{S}}(L, L') \xrightarrow{\delta} q^* \mathrm{Map}_{\mathcal{S}}(L, L')^{S^{n-2}}) \simeq q^* T,$$

where $T \simeq \mathrm{Map}_{\mathcal{S}}(L, L') \times_{\mathrm{Map}_{\mathcal{S}}(L,L')^{S^{n-2}}} \{u_0\}$ is the space of nullhomotopies of u_0. To compete the proof, it suffices to observe that, after passing to a finite covering of $\mathcal{X}$, we can assume that this global section is constant: that is, it is homotopic to the global section determined by some point $t \in T$. This follows from the observation that $\pi_0 T$ is finite. $\square$

Corollary E.2.6.7. *Let $\mathcal{X}$ be an ∞-topos. Then the ∞-pretopos $\mathcal{X}^{\mathrm{lcc}}$ is Boolean (Definition A.6.3.8): that is, for every object $X \in \mathcal{X}^{\mathrm{lcc}}$, the distributive lattice* $\mathrm{Sub}(X)$ *of subobjects of X (in the ∞-category $\mathcal{X}^{\mathrm{lcc}}$) is a Boolean algebra.*

Proof. Let $i : U \hookrightarrow X$ be a (-1)-truncated morphism in $\mathcal{X}^{\mathrm{lcc}}$; we wish to show that U admits a complement in the distributive lattice $\mathrm{Sub}(X)$. Choose a geometric morphism $q^* : \mathcal{S} \to \mathcal{X}$. Using Proposition E.2.6.6, we can assume (after passing to a finite covering of $\mathcal{X}$ if necessary) that $i = q^*(i_0)$ for some map of π-finite spaces $i_0 : U_0 \to X_0$. Replacing U_0 by its essential image in X_0, we may assume that X_0 decomposes as a disjoint union $U_0 \amalg V_0$ for some π-finite space V_0. We now observe that $V = q^* V_0$ is a complement of U in $\mathrm{Sub}(X)$. $\square$

Corollary E.2.6.8. *Let $\mathcal{X}$ be an ∞-topos. Then the full subcategory $\mathcal{X}^{\mathrm{lcc}} \subseteq \mathcal{X}$ is closed under finite limits.*

Proof. Let K be a finite simplicial set and let $p : K \to \mathcal{X}^{\mathrm{lcc}}$ be a diagram; we wish to show that $\varprojlim(p)$ is locally constant constructible. Using Proposition E.2.6.6, we may assume (after passing to a finite cover of $\mathcal{X}$) that the diagram p factors as a composition

$$K \xrightarrow{p_0} \mathcal{S}_\pi \hookrightarrow \mathcal{S} \xrightarrow{q^*} \mathcal{X},$$

where q^* is a geometric morphism. In this case, the left exactness of q^* implies that $\varprojlim(p) \simeq q^*(\varprojlim(p_0))$ is the image under q^* of a π-finite space. □

Corollary E.2.6.9. *Let $\mathcal{X}$ be an ∞-topos and let $X_\bullet$ be a groupoid object of $\mathcal{X}$. If each $X_k \in \mathcal{X}$ is locally constant constructible, then then the geometric realization $|X_\bullet| \in \mathcal{X}$ is locally constant constructible.*

Proof. Choose an integer $n \gg 0$ so that X_0 and X_1 are n-truncated. It follows that each X_k is n-truncated: that is, we can identify $X_\bullet$ with a groupoid object of $\tau_{\leqslant n}\mathcal{X}$. Let K denote the $(n+2)$-skeleton of $\mathrm{N}(\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n+3})$, so that the simplicial object $X_\bullet$ restricts to a diagram $p : K \to \mathcal{X}^{\mathrm{lcc}}$. Since the simplicial set K is finite, we may assume (after passing to a finite covering of $\mathcal{X}$ if necessary) that p factors as a composition

$$K \xrightarrow{p_0} \mathcal{S}_\pi \hookrightarrow \mathcal{S} \xrightarrow{q^*} \mathcal{X}$$

where q^* is a geometric morphism. Replacing p_0 by its composition with the truncation functor $\tau_{\leqslant n} : \mathcal{S}_\pi \to \mathcal{S}_\pi$ if necessary, we may assume that p_0 takes values in the full subcategory of $\mathcal{S}_\pi$ spanned by the n-truncated objects. Since this full subcategory is equivalent to an n-category, the map p_0 extends to a functor $\overline{p}_0 : \mathbf{\Delta}^{\mathrm{op}}_{\leqslant n+3} \to \mathcal{S}_\pi$. Since $X_\bullet$ is a groupoid object, it follows that either the ∞-topos $\mathcal{X}$ is a contractible Kan complex (in which case there is nothing to prove) or the diagram $\overline{p}_0$ is an $(n+3)$-skeletal category object of $\mathcal{S}_\pi$. In the latter case, Theorem A.8.2.3 allows us to extend $\overline{p}_0$ to a category object $X'_\bullet$ of $\mathcal{S}_\pi$, which is easily seen to be a groupoid object. Since $\mathcal{S}_\pi$ is an ∞-pretopos (Remark E.0.7.9), the geometric realization $|X'_\bullet|$ is also π-finite. Note that the simplicial objects $q^* X'_\bullet$ and $X_\bullet$ have equivalent $(n+3)$-skeleta and are therefore equivalent (since they are category objects of $\tau_{\leqslant n}\mathcal{X}$; see Theorem A.8.2.3), so that $|X_\bullet| \simeq |q^* X'_\bullet| \simeq q^*|X'_\bullet|$ lies in the essential image of $q^*|_{\mathcal{S}_\pi}$. □

Proof of Proposition E.2.6.1. Let $\mathcal{X}$ be an ∞-topos. Then the full subcategory $\mathcal{X}^{\mathrm{lcc}} \subseteq \mathcal{X}$ is closed under the formation of finite coproducts (Remark E.2.5.2), finite limits (Corollary E.2.6.8), and geometric realizations of groupoid objects (Corollary E.2.6.9). It follows that $\mathcal{X}^{\mathrm{lcc}}$ is an ∞-pretopos and that the inclusion $\mathcal{X}^{\mathrm{lcc}} \hookrightarrow \mathcal{X}$ is a morphism of ∞-pretopoi (see Remark A.6.1.4). □

E.2.7 Profinite Shape and Locally Constant Sheaves

Our next goal is to show that the process of profinite completion does not lose any information about locally constant constructible sheaves.

Proposition E.2.7.1. *Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. If f_* induces an equivalence of profinite shapes $\mathrm{Sh}_\pi(\mathcal{X}) \to \mathrm{Sh}_\pi(\mathcal{Y})$, then the pullback functor f^* induces an equivalence $\mathcal{Y}^{\mathrm{lcc}} \to \mathcal{X}^{\mathrm{lcc}}$.*

In order to prove Proposition E.2.7.1, it will be convenient to introduce a variant of Definition E.2.5.1.

Definition E.2.7.2. Let $\mathcal{X}$ be an ∞-topos, let $q^* : \mathcal{S} \to \mathcal{X}$ be a geometric morphism, let $\mathcal{K}$ be a full subcategory of $\mathcal{S}$, and let $X \in \mathcal{X}$ be an object. We will say that X is *$\mathcal{K}$-constructible* if there exists a finite collection of objects $\{U_i \in \mathcal{X}\}_{1 \leqslant i \leqslant n}$ which cover $\mathcal{X}$, a finite collection of objects $Y_i \in \mathcal{K}$, and equivalences $X \times U_i \simeq (q^* Y_i) \times U_i$ in $\mathcal{X}_{/U_i}$ for $1 \leqslant i \leqslant n$.

Remark E.2.7.3. In the situation of Definition E.2.7.2, suppose that $\mathcal{K}$ is spanned by finitely many Kan complexes $\{Y_1, \ldots, Y_m\}$. In this case, an object $X \in \mathcal{X}$ is $\mathcal{K}$-constructible if and only if there exists a collection of objects $\{U_\alpha \in \mathcal{X}\}_{\alpha \in A}$ such that $\amalg U_\alpha \to \mathbf{1}$ is an effective epimorphism, and equivalences $X \times U_\alpha \simeq q^* Y_{i(\alpha)} \times U_\alpha$ for some function $i : A \to \{1, \ldots, m\}$ (if this condition is satisfied, we can always arrange that $A = \{1, \ldots, m\}$ by replacing the objects $\{Y_\alpha\}$ by the finite collection $\{\amalg_{i(\alpha)=j} Y_\alpha\}_{1 \leqslant j \leqslant m}$). Consequently, the condition that an object $X \in \mathcal{X}$ be $\mathcal{K}$-constructible can be tested locally on $\mathcal{X}$ (in contrast with Warning **??**).

Remark E.2.7.4. Let $\mathcal{X}$ be an ∞-topos. Then an object $X \in \mathcal{X}$ is locally constant constructible (in the sense of Definition E.2.5.1) if and only if it is $\mathcal{S}_\pi$-constructible (in the sense of Definition E.2.7.2)

Remark E.2.7.5. Let $f^* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi and let $\mathcal{K} \subseteq \mathcal{S}$ be a full subcategory. Then the functor f^* carries $\mathcal{K}$-constructible objects of $\mathcal{X}$ to $\mathcal{K}$-constructible objects of $\mathcal{Y}$.

Example E.2.7.6. Let $\mathcal{K}$ be a full subcategory of $\mathcal{S}$ which is spanned by finitely many Kan complexes $\{Y_1, \ldots, Y_m\}$. Let $\iota_{\mathcal{K}} : \mathcal{K}^{\simeq} \hookrightarrow \mathcal{S}$ denote the inclusion map, which we regard as an object of the ∞-topos $\mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S})$. Then $\iota_{\mathcal{K}}$ is $\mathcal{K}$-constructible: for $1 \leqslant i \leqslant m$, we have $\iota_{\mathcal{K}} \times U_i \simeq (q^* Y_i) \times U_i$, where $q^* : \mathcal{S} \to \mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S})$ is the diagonal inclusion and $U_i : \mathcal{K}^{\simeq} \to \mathcal{S}$ denotes the functor represented by $Y_i \in \mathcal{K}^{\simeq}$.

We will need the following result:

Proposition E.2.7.7. *Let $\mathcal{K}$ be a full subcategory of $\mathcal{S}$ which is spanned by finitely many π-finite spaces and let let $\iota : \mathcal{K}^{\simeq} \hookrightarrow \mathcal{S}$ be the inclusion map. For any ∞-topos $\mathcal{X}$, evaluation on ι induces a fully faithful embedding $e : \mathrm{Fun}^*(\mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S}), \mathcal{X}) \to \mathcal{X}^{\simeq}$ whose essential image is the full subcategory of $\mathcal{X}$ spanned by the $\mathcal{K}$-constructible objects.*

Proof. We first show that e is fully faithful. Suppose we are given geometric morphisms $f^*, g^* : \mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S}) \to \mathcal{X}$; we wish to show that the functor e induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathrm{Fun}^*(\mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S}), \mathcal{X})}(f^*, g^*) \to \mathrm{Map}_{\mathcal{X}^{\simeq}}(f^*(\iota), g^*(\iota)).$$

Let us identify the ∞-topos $\mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S})$ with $\mathcal{S}_{/\mathcal{K}^{\simeq}}$, so that Corollary HTT.6.3.5.6 supplies an equivalence of ∞-categories $\mathrm{Fun}^*(\mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S}), \mathcal{X}) \simeq \mathrm{Map}_{\mathcal{X}}(\mathbf{1}, q^* \mathcal{K}^{\simeq})$, where $q^* : \mathcal{S} \to \mathcal{X}$ is a geometric morphism and $\mathbf{1}$ denotes a final object of $\mathcal{X}$. Under this identification, the geometric morphisms f^* and g^* correspond to maps $f, g : \mathbf{1}_{\mathcal{X}} \to q^* \mathcal{K}^{\simeq}$. Since the assertion that θ is a homotopy equivalence can be tested locally on $\mathcal{X}$, we may assume (after passing to a covering of $\mathcal{X}$ if necessary) that the maps f and g are constant: that is, that they are obtained by applying the functor q^* to inclusions $\{K\} \hookrightarrow \mathcal{K}^{\simeq}$ and $\{L\} \hookrightarrow \mathcal{K}^{\simeq}$ for some π-finite Kan complexes K and L. In this case, the map θ can be identified with the homotopy equivalence $\mathrm{Map}_{\mathcal{X}}(\mathbf{1}, \mathrm{Map}_{\mathcal{S}^{\simeq}}(K, L)) \to \mathrm{Map}_{\mathcal{X}^{\simeq}}(q^*L, q^*K)$ of Remark E.2.6.5.

It remains to prove that the functor e is essentially surjective. Let X be a $\mathcal{K}$-constructible object of $\mathcal{X}$; we wish to show that X belongs to the essential image of e. The first part of the proof shows that this assertion is local on $\mathcal{X}$ (see Remark E.2.7.3). We may therefore assume without loss of generality that $X = q^*K$ for some Kan complex $K \in \mathcal{K}$, in which case the result is obvious. □

Proof of Proposition E.2.7.1. Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. Suppose that f_* induces an equivalence of profinite spaces $\mathrm{Sh}_\pi(\mathcal{X}) \to \mathrm{Sh}_\pi(\mathcal{Y})$. We wish to prove that the pullback functor $f^* : \mathcal{Y}^{\mathrm{lcc}} \to \mathcal{X}^{\mathrm{lcc}}$ is an equivalence of ∞-categories. Note that we can regard $f^* : \mathcal{Y}^{\mathrm{lcc}} \to \mathcal{X}^{\mathrm{lcc}}$ can be regarded as a morphism of ∞-pretopoi (see Proposition E.2.6.1) in which every object is truncated. To show that f^* is an equivalence, it will suffice to show that it satisfies conditions (a) and (b) of Proposition A.9.2.1:

(a) We will show that the functor $f^* : \mathcal{Y}^{\mathrm{lcc}} \to \mathcal{X}^{\mathrm{lcc}}$ is essentially surjective. Fix an object $X \in \mathcal{X}^{\mathrm{lcc}}$. Then we can choose a full subcategory $\mathcal{K} \subseteq \mathcal{S}$ spanned by finitely many π-finite Kan complexes such that X is $\mathcal{K}$-constructible. Let $\iota \in \mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S})$ denote the inclusion functor. It follows from Proposition E.2.7.7 that there exists an (essentially unique) geometric morphism $g_* : \mathcal{X} \to \mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S})$ such that $X \simeq g^*(\iota)$. Since $\mathcal{K}^{\simeq}$ is a π-finite Kan complex, our assumption that $\mathrm{Sh}_\pi(f_*)$ is an equivalence guarantees that the geometric morphism g_* is equivalent to a composition $\mathcal{X} \xrightarrow{f_*} \mathcal{Y} \xrightarrow{g'_*} \mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S})$. It follows that $X \simeq g^*(\iota) \simeq f^*(g'^*\iota) \in f^* \mathcal{Y}^{\mathrm{lcc}}$.

(b) Let $\phi : U \to Y$ be a morphsim in $\mathcal{Y}^{\mathrm{lcc}}$ such that $f^*(\phi)$ is an equivalence; we wish to show that ϕ is an equivalence. Using Corollary E.2.6.7, we see that Y decomposes as a coproduct $U \amalg V$ for some auxiliary object $V \in \mathcal{Y}^{\mathrm{lcc}}$. Since $f^*(\phi)$ is an equivalence, the object $f^*(V)$ is initial in $\mathcal{X}$. Set $V' = \tau_{\leqslant -1} V$. Then V' is a (-1)-truncated object of $\mathcal{Y}^{\mathrm{lcc}}$. Applying Corollary E.2.6.7 again, we deduce that V' is complemented: that is, the final object $\mathbf{1}$ of $\mathcal{Y}$ decomposes as a coproduct $V' \amalg W$. Let T denote the topological space $\{v, w\}$ with the discrete topology, so that the decomposition $\mathbf{1} \simeq V' \amalg W$ is classified by a geometric morphism $g_* : \mathcal{Y} \to \mathcal{S}\mathrm{hv}(T)$. Our assumption $f^* V' \simeq \varnothing$ guarantees that the composite map $\mathcal{X} \xrightarrow{f_*} \mathcal{Y} \xrightarrow{g_*} \mathcal{S}\mathrm{hv}(T)$ factors through the inclusion $i_* : \mathcal{S}\mathrm{hv}(\{w\}) \hookrightarrow \mathcal{S}\mathrm{hv}(T)$. Since the ∞-topoi $\mathcal{S}\mathrm{hv}(T)$ and $\mathcal{S}\mathrm{hv}(\{w\}) \simeq \mathcal{S}$ are profinite, it follows from our assumption that $\mathrm{Sh}_\pi(f_*)$ is an equivalence that the geometric morphism g_* also factors through i_*. In other words, the object $V' \in \mathcal{Y}$ is initial, so that $V \in \mathcal{Y}$ is also initial and the morphism $\phi : U \to U \amalg V \simeq Y$ is an equivalence, as desired.

□

E.3 ∞-Categorical Stone Duality

The theory of Stone duality (Theorems A.1.6.11 and E.1.4.1) supplies equivalences between the following categories:

(*a*) The category $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$ of profinite sets.

(*b*) The category $\mathcal{T}\mathrm{op}_{\mathrm{St}}$ of Stone spaces (a full subcategory of the category $\mathcal{T}\mathrm{op}$ of topological spaces).

(*c*) The opposite of the category BAlg of Boolean algebras (a full subcategory of the category Lat of distributive lattices).

Theorem E.2.4.1 supplies an equivalence between the ∞-category $\mathcal{S}^{\wedge}_{\pi}$ of profinite spaces and the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{pf}}$ of profinite ∞-topoi, which we can regard as an ∞-categorical generalization of the equivalence $(a) \Leftrightarrow (b)$. Our goal in this section is to establish an ∞-categorical version of the equivalence $(b) \Leftrightarrow (c)$. The relevant analogies can be summarized as follows:

Classical Stone Duality	∞-Categorical Stone Duality
Finite Set	π-Finite Space
Profinite Set	Profinite Space
Topological Space	∞-Topos
Coherent Topological Space	Bounded Coherent ∞-Topos
Stone Space	Profinite ∞-Topos
Distributive Lattice	Bounded ∞-Pretopos

Remark E.3.0.8. In the preceding table of analogies, each of the mathematical concepts appearing on the right can be regarded as a generalization of the classical concept which appears on the left. In particular:

- Every finite set S can be regarded as a π-finite space, by equipping S with the discrete topology.
- Every profinite set can be regarded as a profinite space (Remark E.4.1.4).
- Every topological space X determines an ∞-topos $\mathcal{S}\mathrm{hv}(X)$. Moreover, this construction does not lose any information if X is sober (Proposition 1.5.3.5). If X is a Stone space, then $\mathcal{S}\mathrm{hv}(X)$ is a profinite ∞-topos; moreover, the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}}) & \longrightarrow & \mathcal{T}\mathrm{op} \\ \downarrow & & \downarrow{\scriptstyle \mathcal{S}\mathrm{hv}} \\ \mathcal{S}_{\pi}^{\wedge} & \longrightarrow & \infty\mathcal{T}\mathrm{op} \end{array}$$

  commutes up to equivalence, where the upper horizontal map is given by classical Stone duality and the lower horizontal map is the fully faithful embedding of Theorem E.2.4.1 (see Example E.2.4.6).

To complete the picture, we need to determine the ∞-categorical analogue of a Boolean algebra. More precisely, we need to answer the following:

Question E.3.0.9. Let $\mathcal{C}$ be a bounded ∞-pretopos. Under what conditions is the associated ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ profinite?

We will supply three answers to Question E.3.0.9:

(*i*) The ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is profinite if and only if every truncated coherent object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is locally constant constructible (Proposition E.3.1.1).

(*ii*) The ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is profinite if and only if $\mathcal{C}$ is Boolean and of finite breadth (Theorem E.3.2.8).

(*iii*) The ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is profinite if and only if the ∞-category of points $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}(\mathcal{C}), \mathcal{S}) \simeq \mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$ is a Kan complex (Theorem E.3.4.1).

E.3.1 Profinite ∞-Topoi

Our first characterization of the class of profinite ∞-topoi is an easy consequence of Theorem E.2.3.2:

Proposition E.3.1.1. *Let $\mathcal{X}$ be an ∞-topos. Then $\mathcal{X}$ is profinite if and only if it satisfies the following conditions:*

(*a*) *The ∞-topos $\mathcal{X}$ is coherent.*

(*b*) *The ∞-topos $\mathcal{X}$ is bounded.*

(*c*) *Every truncated coherent object of $\mathcal{X}$ is locally constant constructible.*

Proof. The necessity was established by Proposition **??**. For sufficiency, we note that conditions (*a*) and (*b*) imply that the canonical map $u_* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$ is an equivalence (Theorem A.7.5.3). If condition (*c*) is satisfied, then Theorem E.2.3.2 implies that the geometric morphism u_* also exhibits $\mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$ as a profinite reflection of $\mathcal{X}$, so that $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$ is profinite. □

Corollary E.3.1.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be coherent ∞-topoi, and let $f^* : \mathcal{Y} \to \mathcal{X}$ be a geometric morphism. If the ∞-topos $\mathcal{Y}$ is profinite, then the functor f^* carries coherent objects of $\mathcal{Y}$ to coherent objects of $\mathcal{X}$.*

Proof. Let $Y \in \mathcal{Y}$ be a coherent object; we wish to show that $f^*Y \in \mathcal{X}$ is coherent. In other words, we wish to show that f^*Y is n-coherent for each integer $n \geqslant 0$. By virtue of Proposition A.2.4.1, it will suffice to show that $\tau_{\leqslant n+1} f^*Y \simeq f^*(\tau_{\leqslant n+1} Y)$ is n-coherent. Using Corollary A.2.4.4, we can replace Y by $\tau_{\leqslant n+1} Y$ and thereby reduce to the case where Y is truncated. In this case, our assumption that $\mathcal{Y}$ is profinite guarantees that Y is locally constant constructible (Proposition **??**), so that $f^*Y \in \mathcal{X}$ is locally constant constructible (Remark E.2.5.3) and therefore a coherent object of $\mathcal{X}$ by virtue of Proposition E.2.5.5. □

Remark E.3.1.3. It follows from Corollary E.3.1.2 that the ∞-category $\infty\mathcal{T}\mathrm{op}^{\mathrm{pf}}$ of profinite ∞-topoi can be regarded as a full subcategory of both the ∞-category $\infty\mathcal{T}\mathrm{op}$ (whose morphisms are all geometric morphisms) and the ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{coh}}$ of Construction A.7.5.2 (where we require pullback functors to preserve coherent objects). This can be regarded as an ∞-categorical analogue of Remark A.1.6.7.

Proposition E.3.1.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi. If $\mathcal{X}$ is profinite, then the ∞-category* $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ *of geometric morphisms from $\mathcal{X}$ to $\mathcal{Y}$ is an (essentially small) Kan complex.*

Proof. Let us regard $\mathcal{Y}$ as fixed. The collection of those ∞-topoi $\mathcal{X}$ for which $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ is an essentially small Kan complex is closed under small limits in $\infty\mathcal{T}\mathrm{op}$. We may therefore assume without loss of generality that $\mathcal{X} = \mathcal{S}_{/X}$ for some π-finite space X (Remark E.2.4.4), in which case the desired result follows from Corollary HTT.6.3.5.6 . □

Example E.3.1.5. Taking $\mathcal{Y} = \mathcal{S}$ in the statement of Proposition E.3.1.4, we deduce that for any profinite ∞-topos $\mathcal{X}$, the ∞-category $\mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$ is a Kan complex. If $\mathcal{X} = \Psi_\pi(X)$ is the profinite ∞-topos associated to a profinite space X, then this Kan complex can be identified with

$$\mathrm{Fun}^*(\mathcal{X}, \mathcal{S}) \simeq \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{S}, \Psi_\pi(X)) \simeq \mathrm{Map}_{\mathcal{S}^\wedge_\pi}(*, X) \simeq \mathrm{Mat}(X).$$

Theorem E.3.1.6 (Whitehead's Theorem for Profinite Spaces)**.** *The materialization functor* $\mathrm{Mat} : \mathcal{S}^\wedge_\pi \to \mathcal{S}$ *is conservative. That is, if $f : X \to Y$ is a map of profinite spaces, then the following conditions are equivalent:*

(1) *The map f is an equivalence of profinite spaces.*

(2) *The map f induces a homotopy equivalence of spaces* $\mathrm{Mat}(X) \to \mathrm{Mat}(Y)$.

Proof. The implication (1) ⇒ (2) is obvious. Conversely, suppose that $\mathrm{Mat}(f) : \mathrm{Mat}(X) \to \mathrm{Mat}(Y)$ is a homotopy equivalence. Let $\mathcal{X}$ and $\mathcal{Y}$ denote the profinite ∞-topoi associated to X and Y, respectively, so that f induces a geometric morphism $f^* : \mathcal{Y} \to \mathcal{X}$. Combining (2) with Example E.3.1.5, we deduce that f^* induces an equivalence of ∞-categories $\mathrm{Fun}^*(\mathcal{X}, \mathcal{S}) \to \mathrm{Fun}^*(\mathcal{Y}, \mathcal{S})$. Because $\mathcal{X}$ and $\mathcal{Y}$ are bounded coherent ∞-topoi (Remark **??**) and the functor f^* preserves coherent objects (Corollary E.3.1.2), it follows from conceptual completeness (Theorem A.9.0.6) that f^* is an equivalence of ∞-topoi. Since from a profinite space to the associated profinite ∞-topos is fully faithful (Theorem E.2.4.1), it follows that f is an equivalence of profinite spaces. □

E.3.2 ∞-Pretopoi of Finite Breadth

Our next goal is to give a more intrinsic formulation of the criterion of Proposition E.3.1.1. First, we need to introduce some terminology.

Definition E.3.2.1. Let $\mathcal{C}$ be an ∞-pretopos containing an object X and suppose we are given integers $0 \leqslant i < j \leqslant n$. We let $\delta_{i,j} : X^n \to X^{n+1}$ denote the morphism in $\mathcal{C}$ induced by the map of finite sets

$$\{0 < 1 < \cdots < n\} \to \{1 < \cdots < n\} \qquad m \mapsto \begin{cases} m+1 & \text{if } m < i \\ j & \text{if } m = i \\ m & \text{if } m > i. \end{cases}$$

For fixed $n \geqslant 0$, we will say that X has *breadth* $\leqslant n$ if the maps $\{\delta_{i,j} : X^n \to X^{n+1}\}_{0 \leqslant i < j \leqslant n}$ determine a covering for the effective epimorphism topology (Definition A.6.2.4). More generally, we will say that a morphism $u : X \to Y$ in $\mathcal{C}$ has *breadth* $\leqslant n$ if it exhibits X as an object of breadth $\leqslant n$ in the ∞-pretopos $\mathcal{C}_{/Y}$. We will say that $\mathcal{C}$ has *finite breadth* if every morphism in $\mathcal{C}$ has breadth $\leqslant n$ for some integer n.

Example E.3.2.2. Let $\mathcal{C}$ be an ∞-pretopos. A morphism $u : X \to Y$ has breadth $\leqslant 0$ if and only if X is an initial object of $\mathcal{C}$.

Example E.3.2.3. Let $\mathcal{C}$ be an ∞-pretopos. A morphism $u : X \to Y$ has breadth $\leqslant 1$ if and only if the diagonal map $\delta : X \to X \times_Y X$ is an effective epimorphism.

Example E.3.2.4. A morphism $u : X \to Y$ in $\mathcal{S}$ has breadth $\leqslant n$ if and only if, for every point $y \in Y$, the homotopy fiber $X \times_Y \{y\}$ has at most n path components.

Remark E.3.2.5. Let $\mathcal{C}$ be an ∞-pretopos and let $n \geqslant 0$ be an integer. Then an object $C \in \mathcal{C}$ has breadth $\leqslant n$ if and only if the 0-truncation $\tau_{\leqslant 0} C$ has breadth $\leqslant n$.

Remark E.3.2.6. Let $\mathcal{C}$ be an ∞-topos and let $m \geqslant 0$ be an integer. If $u : X \to Y$ is a morphism in $\mathcal{C}$ If an object $C \in \mathcal{C}$ has breadth $\leqslant m$, then it also has breadth $\leqslant n$ whenever $n > m$.

Remark E.3.2.7. Let $f^* : \mathcal{C} \to \mathcal{D}$ be a morphism of ∞-pretopoi. If $u : X \to Y$ is a morphism in $\mathcal{C}$ having breadth $\leqslant n$, then $f^*(u) : f^* X \to f^* Y$ is a morphism in $\mathcal{D}$ having breadth $\leqslant n$.

Definition E.3.2.1 was motivated by the following:

Theorem E.3.2.8. *Let $\mathcal{C}$ be a bounded ∞-pretopos and let $\mathcal{X} = \mathcal{S}\mathrm{hv}(\mathcal{C})$ be the corresponding ∞-topos. The following conditions are equivalent:*

(1) *Every truncated coherent object of $\mathcal{X}$ is locally constant constructible.*

(2) *The ∞-topos $\mathcal{X}$ is profinite.*

(3) *The ∞-pretopos $\mathcal{C}$ is Boolean (Definition A.6.3.8) and of finite breadth.*

Corollary E.3.2.9. *Let $\mathcal{X}$ be an ∞-topos. Then $\mathcal{X}$ is profinite if and only if has the form $\mathcal{S}\mathrm{hv}(\mathcal{C})$, where $\mathcal{C}$ is an ∞-pretopos which is bounded, Boolean, and of finite breadth.*

Proof. Combine Proposition E.3.1.1 with Theorem E.3.2.8. □

Proof of Theorem E.3.2.8. The equivalence of (1) $\Leftrightarrow$ (2) follows from Proposition E.3.1.1. We next show that (1) implies (3). Note that if (1) is satisfied, then the Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{S}\mathrm{hv}(\mathcal{C}) \simeq \mathcal{X}$ induces an equivalence from $\mathcal{C}$ to the full subcategory $\mathcal{X}^{\mathrm{lcc}} \subseteq \mathcal{X}$, so that $\mathcal{C}$ is Boolean by virtue of Corollary E.2.6.7. To complete the proof of (3), it will suffice to show that every morphism $u : X \to Y$ in $\mathcal{X}^{\mathrm{lcc}}$ has finite breadth. Let $q^* : \mathcal{S} \to \mathcal{X}$ be a geometric morphism. Using Proposition E.2.6.6, we can assume (after passing to a finite covering of $\mathcal{X}$ if necessary) that $u = q^*(u_0)$ for some morphism $u_0 : X_0 \to Y_0$ in $\mathcal{S}_\pi$. It will therefore suffice to show that u_0 has finite breadth (Remark E.3.2.7), which follows immediately from Example E.3.2.4.

We now complete the proof by showing that (3) $\Rightarrow$ (1). Assume that (3) is satisfied, let $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C}) = \mathcal{X}$ be the Yoneda embedding, and let $\mathcal{C}_0 = j^{-1}(\mathcal{X}^{\mathrm{lcc}}) \subseteq \mathcal{C}$ be the full subcategory of $\mathcal{C}$ spanned by those object $X \in \mathcal{C}$ for which $j(X) \in \mathcal{X}$ is locally constant constructible. We wish to prove that $\mathcal{C}_0 = \mathcal{C}$. Since $\mathcal{C}_0$ every object of $\mathcal{C}$ is truncated, this is an immediate consequence of the following:

$(*_m)$ Let $u : X \to Y$ be an m-truncated morphism in $\mathcal{C}$. Then $j(X) \in \mathcal{S}\mathrm{hv}(\mathcal{C})_{/j(Y)}$ is locally constant constructible.

The proof of $(*_m)$ proceeds by induction on m. If $m = -2$, then $u : X \to Y$ is an equivalence and there is nothing to prove. Let us therefore assume that $m > -2$. It follows from assumption (3) that the morphism u has breadth $\leqslant k$ for some integer $k \gg 0$; we proceed by induction on k. We first apply Proposition A.6.2.1 to factor u as a composition $X \xrightarrow{u'} Y_0 \xrightarrow{u''} Y$ where u' is an effective epimorphism and u'' is (-1)-truncated. Since $\mathcal{C}$ is Boolean, the morphism u'' determines a splitting $Y \simeq Y_0 \amalg Y_1$. It follows that $j(Y_0)$ is a summand of $j(Y)$, so that $j(X)$ is locally constant constructible as an object of $\mathcal{S}\mathrm{hv}(\mathcal{C})_{/j(Y)}$ if and only if it is locally constant constructible as an object of $\mathcal{S}\mathrm{hv}(\mathcal{C})_{/j(Y_0)}$. We may therefore replace Y by Y_0 and thereby reduce to the case where u is an effective epimorphism in $\mathcal{C}$. In this case, $j(u)$ is an effective epimorphism in $\mathcal{S}\mathrm{hv}(\mathcal{C})_{/j(Y)}$, so that $j(X) \in \mathcal{S}\mathrm{hv}(\mathcal{C})_{/j(Y)}$ is locally constant constructible if and only if $j(X \times_Y X) \in \mathcal{S}\mathrm{hv}(\mathcal{C})_{/j(X)}$ is locally constant constructible. We may therefore replace u by the projection map $u_X : X \times_Y X \to X$ and thereby reduce to the case where u admits a section $s : Y \to X$.

Applying Proposition A.6.2.1, we see that the map s factors as a composition $Y \xrightarrow{s'} U \xrightarrow{s''} X$, where s' is an effective epimorphism and s'' is (-1)-truncated. Since $\mathcal{C}$ is Boolean, the map s'' determines a decomposition $X \simeq U \amalg V$. Because u has breadth $\leqslant k$ and the restriction $u|_U : U \to Y$ admits a section s', the map $u|_V : V \to Y$ has breadth $< k$. It follows from our inductive hypothesis that $j(V) \in \mathcal{S}\mathrm{hv}(\mathcal{C})_{/j(Y)}$ is locally constant constructible. Consequently, to show that $j(X) \simeq j(U) \amalg j(V) \in \mathcal{S}\mathrm{hv}(\mathcal{C})_{/j(Y)}$ is locally constant constructible, it will suffice to show that $j(U)$ is locally constant constructible.

Let $Z_\bullet$ denote the groupoid object of $\mathcal{C}_{/Y}$ given by the Čech nerve of the morphism $s' : Y \to U$. Since U is a subobject of X, it is m-truncated as an object of $\mathcal{C}_{/Y}$. It follows that $Z_a \in \mathcal{C}_{/Y}$ is $(m-1)$-truncated for each $a \geqslant 0$. Applying our inductive hypothesis $(*_{m-1})$, we deduce that $j(Z_a)$ is a locally constant constructible object of $\mathcal{S}\mathrm{hv}(\mathcal{C})_{/j(Y)}$. Applying Corollary E.2.6.9, we conclude that $j(U) \simeq j(|Z_\bullet|) = |j(Z_\bullet)| \in \mathcal{S}\mathrm{hv}(\mathcal{C})_{/j(Y)}$ is also locally constant constructible. □

E.3.3 Digression: Ultraproducts

Before giving our final characterization of the class of profinite ∞-topoi, we need a few facts about ultraproducts and ultrapowers in the ∞-categorical setting.

Definition E.3.3.1. Let S be a set. We let $P(S)$ denote the power set of S: that is, the collection of all subsets of S. An *ultrafilter* on S is a subset $\mathcal{U} \subseteq P(S)$ with the following properties:

- (a) The subset $\mathcal{U} \subseteq P(S)$ is a filter: that is, it is closed upward and stable under finite intersections.
- (b) For every subset $I \subseteq S$, exactly one of the sets I and $S - I$ belongs to $\mathcal{U}$.

Example E.3.3.2. Let S be a set containing an element s. Then the set $\mathcal{U}_s = \{I \in P(S) : s \in I\} \subseteq P(S)$ is an ultrafilter on S. We say that an ultrafilter $\mathcal{U} \subseteq P(S)$ is *principal* if it has the form $\mathcal{U}_s$ for some element $s \in S$.

Remark E.3.3.3. Let S be a set. A subset $\mathcal{U} \subseteq P(S)$ is an ultrafilter if and only if its complement $P(S) - \mathcal{U}$ is a prime ideal, in the sense of Definition A.1.2.1. Consequently, we can identify the set of ultrafilters on S with the spectrum $X = \operatorname{Spec} P(S)$ of the Boolean algebra $P(S)$. For each $s \in S$, the principal ultrafilter $\mathcal{U}_s$ is an isolated point of X, and the construction $s \mapsto \mathcal{U}_s$ determines a homeomorphism from S to a subspace of X. For this reason, X is often referred to as the *Stone-Čech compactification of* X. Note that if S is infinite, then it is not compact when endowed with the discrete topology, so there must exist a nonprincipal ultrafilter on S.

Construction E.3.3.4 (Ultraproducts)**.** Let $\mathcal{E}$ be an ∞-category which admits products and filtered colimits. Suppose we are given a collection of objects $\{E_s\}_{s \in S}$ of $\mathcal{E}$ which is indexed by a set S. This collection of objects determines a functor $F : P(S)^{\mathrm{op}} \to \mathcal{E}$, given on objects by the formula $F(I) = \prod_{s \in I} E_s$ (more precisely, we can regard the collection of objects $\{E_s\}_{s \in S}$ as $F_0 : S \to \mathcal{E}$, where we regard S as a category having only identity morphisms, and we define $F : P(S)^{\mathrm{op}} \to \mathcal{E}$ to be a right Kan extension of F_0). If $\mathcal{U} \subseteq P(S)$ is an ultrafilter, then we can regard $\mathcal{U}^{\mathrm{op}}$ as a direct subset of the partially ordered set $P(S)^{\mathrm{op}}$. We let $\prod_{\mathcal{U}} E_s$ denote the colimit $\varinjlim_{I \in \mathcal{U}^{\mathrm{op}}} F(I)$. We will refer to $\prod_{\mathcal{U}} E_s$ as the *ultraproduct of $\{E_s\}_{s \in S}$ with respect to $\mathcal{U}$.*

Example E.3.3.5. Let $\{E_s\}_{s \in S}$ be a collection of sets indexed by a set S, and suppose that each E_s is nonempty. For every ultrafilter $\mathcal{U} \subseteq P(S)$, the ultraproduct $\prod_{\mathcal{U}} E_s$ (formed in the category $\mathcal{S}$et of sets) can be identified with the quotient of the product $\prod_{s \in S} E_s$ by the equivalence relation $\equiv$ defined as follows:

$$\{x_s\}_{s \in S} \equiv \{y_s\}_{s \in S} \text{ if } \{s \in S : x_s = y_s\} \in \mathcal{U}.$$

Example E.3.3.6. Let $\mathcal{E}$ be an ∞-category which admits products and filtered colimits, let $\{E_s\}_{s \in S}$ be a collection of objects of $\mathcal{E}$ indexed by a set S, and let $\mathcal{U}_t$ be the principal ultrafilter determined by an element $t \in S$. Then the inclusion $\{\{t\}\} \hookrightarrow \mathcal{U}$ is right cofinal, so the ultraproduct $\prod_{\mathcal{U}_t} E_s$ can be identified with E_t.

Remark E.3.3.7. Let $\mathcal{E}$ and $\mathcal{E}'$ be ∞-categories which admit products and filtered colimits. If $F : \mathcal{E} \to \mathcal{E}'$ is any functor which preserves products and filtered colimits, then F commutes with ultraproducts: in other words, for every collection of objects $\{E_s\}_{s \in S}$ of $\mathcal{E}$ and every ultrafilter $\mathcal{U}$ on S, we have a canonical equivalence $F(\prod_{\mathcal{U}} E_s) \simeq \prod_{\mathcal{U}} F(E_s)$ in the ∞-category $\mathcal{E}'$.

Note that Construction E.3.3.4 is functorial: if we fix the ∞-category $\mathcal{E}$, the set S, and the ultrafilter $\mathcal{U} \subseteq P(S)$, then the construction $\{E_s\}_{s \in S} \mapsto \prod_{\mathcal{U}} E_s$ determines a functor $\prod_{\mathcal{U}} : \mathcal{E}^S \to \mathcal{E}$. We will need the following elementary properties of this functor:

Proposition E.3.3.8. *Let S be a set and let $\mathcal{U} \subseteq P(S)$ be an ultrafilter. Then the ultraproduct functor $\prod_{\mathcal{U}} : \mathcal{S}^S \to \mathcal{S}$ is a morphism of ∞-pretopoi. In other words, the functor $\prod_{\mathcal{U}}$ preserves finite limits, finite coproducts, and effective epimorphisms.*

Proof. For each $I \subseteq S$, let $F_I : \mathcal{S}^S \to \mathcal{S}$ denote the functor given by $F_I(\{X_s\}_{s \in S}) = \prod_{s \in I} X_s$. Each of the functors F_I is left exact, and $\prod_{\mathcal{U}}$ can be obtained as a filtered colimit $\varinjlim_{I \in \mathcal{U}^{\mathrm{op}}} F_I$. Since filtered colimits in $\mathcal{S}$ are left exact, it follows that the functor $\prod_{\mathcal{U}}$ is left exact. Each of the functors F_I preserves effective epimorphisms (since the collection of effective epimorphisms in $\mathcal{S}$ is closed under small products), so the functor $\prod_{\mathcal{U}}$ also preserves effective epimorphisms (since the collection of effective epimorphisms in $\mathcal{S}$ is closed under filtered colimits). We will complete the proof by showing that the functor $\prod_{\mathcal{U}}$ preserves finite coproducts. We first observe that $\prod_{\mathcal{U}}$ preserves initial objects: in fact, the functor $F_I : \mathcal{S}^S \to \mathcal{S}$ preserves initial objects for all nonempty subsets $I \subseteq P(S)$, and therefore for all subsets I which belong to the ultrafilter $\mathcal{U}$ (note that the empty set $\varnothing \in P(S)$ is not contained in $\mathcal{U}$).

To complete the proof, it will suffice to show that the functor $\prod_{\mathcal{U}}$ preserves pairwise coproducts. Let $\{X_s\}_{s \in S}$ and $\{Y_s\}_{s \in S}$ be objects of $\mathcal{S}^S$. Since the functor $\prod_{\mathcal{U}}$ is left exact, the natural maps

$$\prod_{\mathcal{U}} X_s \xrightarrow{i} \prod_{\mathcal{U}} (X_s \amalg Y_s) \xleftarrow{j} \prod_{\mathcal{U}} Y_s$$

are (-1)-truncated, and we have

$$\prod_{\mathcal{U}} X_s \times_{\prod_{\mathcal{U}}(X_s \amalg Y_s)} \prod_{\mathcal{U}} Y_s \simeq \prod_{\mathcal{U}} (X_s \times_{X_s \amalg Y_s} Y_s) \simeq \prod_{\mathcal{U}} \varnothing \simeq \varnothing.$$

It follows that the natural map $\rho : (\prod_{\mathcal{U}} X_s) \amalg (\prod_{\mathcal{U}} Y_s) \to \prod_{\mathcal{U}} (X_s \amalg Y_s)$ is (-1)-truncated. To complete the proof, it will suffice to show that ρ is surjective on connected components. Note that every path component of $\prod_{\mathcal{U}} (X_s \amalg Y_s)$ belongs to the image of the natural map $\phi_I : \prod_{s \in I} (X_s \amalg Y_s) \to \prod_{\mathcal{U}} (X_s \amalg Y_s)$ for some $I \in \mathcal{U}$. Decomposing the domain of ϕ_I as a coproduct $\amalg_{I = I_0 \amalg I_1} (\prod_{s \in I_0} X_s) \times (\prod_{s \in I_1} Y_s)$, we see that every path component of $\prod_{\mathcal{U}} (X_s \amalg Y_s)$ belongs to the essential image of the natural map

$$\phi_{I_0, I_1} : (\prod_{s \in I_0} X_s) \times (\prod_{s \in I_1} Y_s) \to \prod_{\mathcal{U}} (X_s \amalg Y_s)$$

for some pair of disjoint subsets $I_0, I_1 \in P(S)$ satisfying $I_0 \amalg I_1 \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, either I_0 or I_1 belongs to $\mathcal{U}$. We conclude by observing that if $I_0 \in \mathcal{U}$, then ϕ_{I_0, I_1} factors through the map i; if I_1 belongs to $\mathcal{U}$, then the map ϕ_{I_0, I_1} factors through j. □

Corollary E.3.3.9 (Łos's Theorem). *Let $\mathcal{C}$ be an ∞-pretopos and let $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$ be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ spanned by those functors which preserve finite limits, finite coproducts, and effective epimorphisms (Definition A.6.4.1). Then the $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$ is closed under the formation of ultraproducts. That is, for every set S and every ultrafilter $\mathcal{U} \subseteq P(S)$, the ultraproduct functor $\prod_{\mathcal{U}} : \mathrm{Fun}(\mathcal{C}, \mathcal{S})^S \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ carries $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})^S$ into $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$.*

Let $\mathcal{E}$ be an ∞-category which admits products and filtered colimits, and let S be a set equipped with an ultrafilter $\mathcal{U} \subseteq P(S)$. Composing the ultraproduct functor $\prod_{\mathcal{U}} : \mathcal{E}^S \to \mathcal{E}$ of Construction E.3.3.4 with the diagonal map $\delta : \mathcal{E} \to \mathcal{E}^S$, we obtain a functor $\mathcal{E} \to \mathcal{E}$, which we will denote by $E \mapsto E^S / \mathcal{U}$. For each object $E \in \mathcal{E}$, we will refer to $E^S / \mathcal{U}$ as the *ultrapower of E by $\mathcal{U}$*.

Remark E.3.3.10. Let $\mathcal{E}$ and $\mathcal{U} \subseteq P(S)$ be as above. Unwinding the definitions, we see that the ultrapower $E^S / \mathcal{U}$ can be described as the filtered colimit $\varinjlim_{I \in \mathcal{U}^{\mathrm{op}}} F(I)$, where $F : P(S)^{\mathrm{op}} \to \mathcal{E}$ is the right Kan extension of the constant functor $F_0 : S \to \mathcal{E}$ taking the value E. Let $F' : P(S)^{\mathrm{op}} \to \mathcal{E}$ denote the constant functor with the value E. Since F is a right Kan extension of $F|_S$, the equality $F'|_S = F|_S$ extends uniquely to a natural transformation $F' \to F$, which induces a map $\gamma : E \simeq \varinjlim_{I \in \mathcal{U}^{\mathrm{op}}} F'(I) \to \varinjlim_{I \in \mathcal{U}^{\mathrm{op}}} F(I) = E^S / \mathcal{U}$.

E.3.4 Profiniteness and Points

We now combine the theory of ultraproducts with the criterion of Theorem E.3.2.8 to establish the following profiniteness criterion:

Theorem E.3.4.1. *Let $\mathcal{X}$ be an ∞-topos. Then $\mathcal{X}$ is profinite if and only if it satisfies the following conditions:*

(a) *The ∞-topos $\mathcal{X}$ is coherent.*

(b) *The ∞-topos $\mathcal{X}$ is bounded.*

(c) *The ∞-category $\operatorname{Fun}^*(\mathcal{X}, \mathcal{S})$ of points of $\mathcal{X}$ is a Kan complex.*

Note that the necessity of conditions (a), (b), and (c) follows from Remark E.2.4.5 and Proposition E.3.1.4. By virtue of Theorem A.7.5.3 and Remark A.9.1.4, the converse is equivalent to the following:

Proposition E.3.4.2. *Let $\mathcal{C}$ be a bounded ∞-pretopos. If the ∞-category $\operatorname{Fun}^*(\mathcal{C}, \mathcal{S})$ is a Kan complex, then the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is profinite.*

Proof. Let $\mathcal{C}$ be a bounded ∞-pretopos having the property that $\operatorname{Fun}^*(\mathcal{C}, \mathcal{S})$ is a Kan complex. We wish to show that the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is profinite. By virtue of Theorem E.3.2.8, it will suffice to show that $\mathcal{C}$ is Boolean and of finite breadth. We begin by showing that $\mathcal{C}$ is Boolean. Fix an object $X \in \mathcal{C}$ and a subobject $U \in \operatorname{Sub}(X)$; we wish to show that U is complemented. Let $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ denote the Yoneda embedding and let V be the largest subobject of $j(X)$ which is disjoint from $j(U)$. Note that if the canonical map $j(U) \amalg V \to j(X)$ is an equivalence, then V is a truncated coherent object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ and is therefore of the form $j(V_0)$ for some $V_0 \in \operatorname{Sub}(X)$ complementary to U. Assume otherwise. Applying Theorem A.4.0.5 to the hypercompletion $\mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}}$, we deduce that there exists a point $\eta^* : \mathcal{S}\mathrm{hv}(\mathcal{C})^{\mathrm{hyp}} \to \mathcal{S}$ and a point $x \in \eta^* j(X)$ which cannot be lifted to either $\eta^* j(U)$ or $\eta^* V$. Set $M = \eta^* \circ j$, and view M as an object of $\operatorname{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S}) \subseteq \operatorname{Pro}(\mathcal{C})^{\mathrm{op}}$, so that we can view x as a morphism $M \to X$ in $\operatorname{Pro}(\mathcal{C})$. Write M as the limit of a filtered diagram $\{M_\alpha\}$ in $\mathcal{C}_{/X}$. Note that if $U \times_X M_\alpha$ were initial in $\mathcal{C}$ for any index $\alpha \in A$, then the map $x : M_\alpha \to X$ would factor through some subobject $V_0 \in \operatorname{Sub}(X)$ disjoint from U, contradicting our assumption that x cannot be lifted to V. It follows that $U \times_X M$ is not initial in $\operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/M}$. Applying Theorem A.4.0.5 again, we deduce that the ∞-topos $\mathcal{S}\mathrm{hv}(\operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/M})^{\mathrm{hyp}}$ has a point μ^* for which the composite map $\operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/M} \hookrightarrow \mathcal{S}\mathrm{hv}(\operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/M})^{\mathrm{hyp}} \xrightarrow{\mu^*} \mathcal{S}$ carries $U \times_X M$ to a nonempty space. Unwinding the definitions, we can identify this composite map with an object $N \in \operatorname{Pro}(\operatorname{Pro}(\mathcal{C})^{\mathrm{cc}}_{/M}) \simeq \operatorname{Pro}(\mathcal{C})_{/M}$ for which the composite map $N \xrightarrow{\pi} M \xrightarrow{x} X$ factor through U. Since the map x itself does not factor through U, the map π cannot be an equivalence, which contradicts our assumption that $\mathrm{Fun}^*(\mathcal{C}, \mathcal{S}) \simeq \mathrm{Pro}(\mathcal{C})^{\mathrm{op}}_{\circ}$ is a Kan complex.

We now show that the ∞-pretopos $\mathcal{C}$ of finite breadth. Fix a morphism $f : X \to Y$ in $\mathcal{C}$; we wish to show that f has breadth $\leqslant n$ for some integer n. Replacing $\mathcal{C}$ by $\mathcal{C}_{/Y}$, we can assume that Y is a final object of $\mathcal{C}$. Note that if the object X does not have breadth $\leqslant n$, then we can use Theorem A.4.0.5 to produce an object $M_n \in \mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$ such that $\pi_0 M_n(X)$ has cardinality $> n$. Assume (for a contradiction) that such an object exists for all $n \geqslant 0$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on the set $\mathbf{Z}_{\geqslant 0}$, and set $M = \prod_{\mathcal{U}} M_n$. Then M is an object of $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$, and we can identify $\pi_0 M(X)$ with the ultraproduct $\prod_{\mathcal{U}} \pi_0 M_n(X)$ (Remark E.3.3.7). It follows that the set $S = \pi_0 M(X)$ is infinite. Let $\mathcal{V}$ be a nonprincipal ultrafilter on S and let $N = M^S/\mathcal{V}$ be the corresponding ultrapower of M, so that the natural map $\gamma : M \to N$ of Remark E.3.3.10 can be regarded as a morphism in the ∞-category $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$. Our assumption that $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{S})$ is a Kan complex guarantees that γ is an equivalence, and therefore induces a bijection of sets $\gamma_0 : \pi_0 M(X) \to \pi_0 N(X)$. Using Remark E.3.3.7, we can identify γ_0 with the diagonal map $S \to S^S/\mathcal{V}$. This is a contradiction: the equivalence class of the identity map $\mathrm{id}_S : S \to S$ does not belong to the image of the map γ_0 (otherwise, the ultrafilter $\mathcal{V}$ would be principal). □

E.4 Truncations of Profinite Spaces

Let $f : X \to Y$ be a map of spaces, and let $n \geqslant -2$ be an integer. Recall that f is said to be *n-truncated* if, for each $y \in Y$, the homotopy fiber $X_y = X \times_Y \{y\}$ is an n-truncated space: that is, if the truncation map $X_y \to \tau_{\leqslant n} X_y$ is an equivalence. We say that f is *$(n+1)$-connective* if each homotopy fiber X_y of f is $(n+1)$-connective: that is, if the truncation $\tau_{\leqslant n} X_y$ is contractible. According to Example HTT.5.2.8.16 , every map of spaces $f : X \to Y$ admits an essentially unique factorization $X \xrightarrow{f'} Z \xrightarrow{f''} Y$, where the map f' is $(n+1)$-connective and f'' is n-truncated. Our goal in this section is to construct an analogous factorization in the case where f is a map of profinite spaces.

E.4.1 Connective and Truncated Morphisms

Throughout this section, we will abuse notation by identifying the ∞-category $\mathcal{S}_\pi$ of π-finite spaces with its essential image in the ∞-category $\mathcal{S}^\wedge_\pi$ of profinite spaces.

Definition E.4.1.1. Let $n \geqslant -2$ be an integer. We will say that a map $f : X \to Y$ of profinite spaces is *n-truncated* if it is given as a filtered limit of morphisms $f_\alpha : X_\alpha \to Y_\alpha$, where each f_α is an n-truncated map of π-finite spaces. If $n \geqslant -1$, we say that f is *n-connective* if it is given as a filtered limit of morphisms $f_\beta : X_\beta \to Y_\beta$, where each f_β is an n-connective map of π-finite spaces.

Let $*$ denote a final object of $\mathcal{S}_\pi^\wedge$. We say that a π-profinite space X is *n-connective* if the constant map $X \to *$ is n-connective, and *n-truncated* if the constant map $X \to *$ is n-truncated. We say that X is *connected* if it is 1-connective, and *simply connected* if it is 2-connective.

Recall that a *factorization system* on an ∞-category $\mathcal{C}$ is a pair (S_L, S_R) with the following properties (Definition HTT.5.2.8.8):

(1) Both S_L and S_R are collections of morphisms of $\mathcal{C}$ which are closed under the formation of retracts.

(2) Every morphism in S_L is left orthogonal to every morphism in S_R (see Definition HTT.5.2.8.1).

(3) Every morphism $f : X \to Y$ in $\mathcal{C}$ can be obtained as a composition $X \xrightarrow{f'} Z \xrightarrow{f''} Y$, where $f' \in S_L$ and $f'' \in S_R$.

We refer the reader to §HTT.5.2.8 for a general discussion of factorization systems in ∞-categories. We can now state the first main result of this section:

Theorem E.4.1.2. *Let $n \geqslant -2$ be an integer. Let $\hat{S}_L$ denote the collection of $(n+1)$-connective morphisms in $\mathcal{S}_\pi^\wedge$, and let $\hat{S}_R$ denote the collection of n-truncated morphisms in $\mathcal{S}_\pi^\wedge$. Then the pair $(\hat{S}_L, \hat{S}_R)$ is a factorization system on the ∞-category $\mathcal{S}_\pi^\wedge$.*

Example E.4.1.3. Let X be a profinite space and let $n \geqslant -2$ be an integer. Theorem E.4.1.2 implies that there is an essentially unique map $f : X \to \tau_{\leqslant n} X$, where f is $(n+1)$-connective and $\tau_{\leqslant n} X$ is n-truncated. The construction $X \mapsto \tau_{\leqslant n} X$ determines a functor from the ∞-category $\mathcal{S}_\pi^\wedge$ to itself, which is left adjoint to the inclusion of n-truncated profinite spaces into $\mathcal{S}_\pi^\wedge$. Alternatively, we can describe $\tau_{\leqslant n}$ as the essentially unique extension of the usual truncation functor on π-finite spaces which commutes with filtered limits.

Remark E.4.1.4. Every finite set can be regarded as a π-finite space. We therefore obtain an inclusion of ∞-categories $\mathcal{S}\mathrm{et}^{\mathrm{fin}} \hookrightarrow \mathcal{S}_\pi$, which induces a fully faithful embedding

$$\mathrm{Pro}(\mathcal{S}^{\mathrm{fin}}) \to \mathrm{Pro}(\mathcal{S}_\pi) = \mathcal{S}_\pi^\wedge$$

whose essential image is the full subcategory of $\mathrm{Pro}(\mathcal{S}_\pi^\wedge)$ spanned by the 0-truncated profinite spaces.

E.4.2 Construction of the Factorization System

Theorem E.4.1.2 is an immediate consequence of the following pair of results:

Proposition E.4.2.1. *Let $n \geqslant -2$ be an integer. Let S_L be the collection of $(n+1)$-connective morphisms between π-finite spaces and let S_R be the collection of n-truncated morphisms between π-finite spaces. Then the pair (S_L, S_R) is a factorization system on the ∞-category $\mathcal{S}_\pi$ of π-finite spaces.*

Proposition E.4.2.2. *Let $\mathcal{C}$ be an essentially small ∞-category which is equipped with a factorization system (S_L, S_R). Let $\mathcal{C}_L$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms belonging to S_L, and $\mathcal{C}_R$ the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms belonging to S_R. Then the inclusions $\mathcal{C}_L \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C}) \hookleftarrow \mathcal{C}_R$ determine fully faithful embeddings*

$$\mathrm{Ind}(\mathcal{C}_L) \hookrightarrow \mathrm{Ind}(\mathrm{Fun}(\Delta^1, \mathcal{C})) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{C})) \hookleftarrow (\mathcal{C}_R)$$

(see Proposition HTT.5.3.5.15). Let $\hat{S}_L$ and $\hat{S}_R$ denote the collections of morphisms in $\mathrm{Ind}(\mathcal{C})$ which belong to the essential images of these embeddings. Then $(\hat{S}_L, \hat{S}_R)$ is a factorization system on $\mathrm{Ind}(\mathcal{C})$.

Proof of Proposition E.4.2.1. According to Example HTT.5.2.8.16 , the collections of $(n+1)$-connective and n-truncated morphisms in $\mathcal{S}$ determine a factorization system on $\mathcal{S}$. The only nontrivial point is to show that if $f : X \to Y$ is a map of π-finite spaces and we factor f as a composition $X \xrightarrow{f'} Z \xrightarrow{f''} Y$ where f' is $(n+1)$-connective and f'' is n-truncated, then the space Z is also π-finite. Since Y is π-finite, it will suffice to show that for each $y \in Y$, the homotopy fiber $Z_y = Z \times_Y \{y\}$ is π-finite. For this, we observe that Z_y is given by the truncation $\tau_{\leqslant n} X_y$, where X_y denotes the homotopy fiber $X \times_Y \{y\}$. □

Proof of Proposition E.4.2.2. Since ∞-categories $\mathrm{Ind}(\mathcal{C}_L)$ and $\mathrm{Ind}(\mathcal{C}_R)$ are idempotent complete, the sets $\hat{S}_L$ and $\hat{S}_R$ are clearly stable under retracts. Let $\mathcal{D}$ denote the full subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{C})$ spanned by those diagrams

$$\begin{array}{ccc} & Z & \\ {}^{f'}\nearrow & & \searrow^{f''} \\ X & \xrightarrow{f} & Y \end{array}$$

where $f' \in S_L$ and $f'' \in S_R$. According to Proposition HTT.5.2.8.17 , the inclusion $\Delta^1 \simeq \Delta^{\{0,2\}} \hookrightarrow \Delta^2$ induces an equivalence of ∞-categories $\mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$. It follows that the induced map $\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathrm{Fun}(\Delta^1, \mathcal{C})) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Ind}(\mathcal{C}))$ is an equivalence. From this, we conclude that every morphism $f : X \to Y$ in $\mathrm{Ind}(\mathcal{C})$ factors as a composition $X \xrightarrow{f'} Z \xrightarrow{f''} Y$, where $f' \in \hat{S}_L$ and $f'' \in \hat{S}_R$.

It remains to prove that every morphism in $\hat{S}_L$ is left orthogonal to every morphism in $\hat{S}_R$. To prove this, suppose we are given a filtered diagram $\{f_\alpha : A_\alpha \to B_\alpha\}$ in $\mathcal{C}_L$ and a filtered diagram $\{g_\beta : X_\beta \to Y_\beta\}$ in $\mathcal{C}_R$, having limits given by morphisms $f : A \to B$ and $g : X \to Y$ in $\mathrm{Ind}(\mathcal{C})$. We wish to show that the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(B, X) & \longrightarrow & \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(A, X) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(B, Y) & \longrightarrow & \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(B, X) \end{array}$$

is a pullback square of spaces. Since the collection of pullback diagrams in $\mathcal{S}$ is closed under filtered colimits and small limits, it suffices to prove that for every pair of indices α and β, the diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(B_\alpha, X_\beta) & \longrightarrow & \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(A_\alpha, X_\beta) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(B_\alpha, Y_\beta) & \longrightarrow & \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(B_\alpha, X_\beta) \end{array}$$

is a pullback square. This follows because f_α is left orthogonal to g_β in the ∞-category $\mathcal{C}$. □

E.4.3 Connectivity of Profinite Completions

We now study the relationship between the connectivity properties of a morphism $f : X \to Y$ of spaces and the induced map of profinite completions $X^\wedge_\pi \to Y^\wedge_\pi$.

Proposition E.4.3.1. *Let $f : X \to Y$ be a map of spaces. If f is n-connective, then the induced map $X^\wedge_\pi \to Y^\wedge_\pi$ is an n-connective map of profinite spaces.*

Proof. By virtue of Theorem E.4.1.2, it will suffice to show that the map $X^\wedge_\pi \to Y^\wedge_\pi$ has the left lifting property with respect to every $(n-1)$-truncated map of profinite space $g : U \to V$. Then g can be written as a filtered limit of $(n-1)$-truncated maps $g_\alpha : U_\alpha \to V_\alpha$. Replacing g by g_α, we can reduce to the case where U and V are π-finite. In this case, the desired property follows immediately from our assumption that f has the left lifting property with respect to g (in the ∞-category of spaces). □

Corollary E.4.3.2. *Let X be an n-connective space. Then the profinite completion $X^\wedge_\pi$ is an n-connective profinite space.*

Corollary E.4.3.2 has a counterpart for truncatedness:

Proposition E.4.3.3. *Let X be a discrete space. Then the profinite completion $X^\wedge_\pi$ is 0-truncated.*

Proof. Set $S = \pi_0 X$. Let $\mathcal{E}$ denote the set of all equivalence relations E on S such that S/E is finite. Then $\{S/E\}_{E \in \mathcal{E}}$ represents a Pro-object in the category of finite sets, and therefore determines a 0-truncated profinite space. We will show that $\{S/E\}_{E \in \mathcal{E}}$ is a profinite completion of X. In other words, we show that for every π-finite space T, the composite map

$$\varinjlim_{E \in \mathcal{E}} \mathrm{Fun}(S/E, T) \xrightarrow{\phi} \mathrm{Fun}(S, T) \xrightarrow{\psi} \mathrm{Fun}(X, T)$$

is a homotopy equivalence. By virtue of Lemma E.1.6.5, we may assume without loss of generality that T has finitely many simplices of each dimension, from which it follows that ϕ is an isomorphism of simplicial sets. The map ψ is a homotopy equivalence by virtue of our assumption that X is discrete. □

Warning E.4.3.4. It is not true in general that if X is an n-truncated space, then the profinite completion $X^{\wedge}_{\pi}$ is n-truncated. For example, suppose that $X = BG$ is the classifying space of a discrete group G (so that X is 1-truncated). Then $\tau_{\leqslant 1} X^{\wedge}_{\pi}$ can be identified with the classifying space for the profinite completion $G^{\wedge}$ of G. The profinite completion $X^{\wedge}_{\pi}$ is 1-truncated if and only if, for every finite abelian group A with an action of G and every integer $m \geqslant 0$, the map of cohomology groups $\mathrm{H}^m_c(G^{\wedge}; M) \to \mathrm{H}^m(G; M)$ is an isomorphism, where the left hand side indicates the profinite group cohomology of $G^{\wedge}$.

E.4.4 Materialization

For applications of Theorem E.4.1.2, it is useful to have some alternate descriptions of the classes of n-connective and n-truncated morphisms in $\mathcal{S}^{\wedge}_{\pi}$. For this, we need a slight digression.

Proposition E.4.4.1. *Let $F : \mathcal{S} \to \mathcal{S}^{\wedge}_{\pi}$ denote the profinite completion functor $F(X) = X^{\wedge}_{\pi}$. Then F admits a right adjoint.*

Proof. Proposition HTT.5.1.3.2 implies that F preserves small colimits. Since the ∞-category $\mathcal{S}$ is presentable, it follows that F admits a right adjoint (Corollary HTT.5.5.2.9 and Remark HTT.5.5.2.10). □

Notation E.4.4.2. The profinite completion functor $X \mapsto X^{\wedge}_{\pi}$ is fully faithful when restricted to the ∞-category $\mathcal{S}_{\pi}$ of π-finite spaces. If X is π-finite, we will generally abuse notation by not distinguishing between X and its profinite completion. In particular, we let $*$ denote the profinite space given by the profinite completion of a one-point space (as an object of $\mathcal{S}^{\wedge}_{\pi} \subseteq \mathrm{Fun}(\mathcal{S}_{\pi}, \mathcal{S})^{\mathrm{op}}$, we can identify $*$ with the inclusion functor $\mathcal{S}_{\pi} \hookrightarrow \mathcal{S}$).

Remark E.4.4.3. For every profinite space X, we have a canonical homotopy equivalence

$$\mathrm{Mat}(X) \simeq \mathrm{Map}_{\mathcal{S}}(*, \mathrm{Mat}(X)) \simeq \mathrm{Map}_{\mathcal{S}^{\wedge}_{\pi}}(*, X).$$

Remark E.4.4.4. Let X and Y be profinite spaces, represented by filtered diagrams of π-finite spaces $\{X_\alpha\}_{\alpha \in A}$, $\{Y_\beta\}_{\beta \in B}$. Then the coproduct $X \amalg Y$ is represented by the filtered diagram $\{X_\alpha \amalg Y_\beta\}_{(\alpha,\beta) \in A \times B}$. It follows that the materialization $\mathrm{Mat}(X \amalg Y)$ is given by

$$\begin{aligned} \mathrm{Mat}(X \amalg Y) &\simeq \varprojlim_{\alpha,\beta} (X_\alpha \amalg Y_\beta) \\ &\simeq (\varprojlim_{\alpha} X_\alpha) \amalg (\varprojlim_{\beta} Y_\beta) \\ &\simeq \mathrm{Mat}(X) \amalg \mathrm{Mat}(Y). \end{aligned}$$

In other words, the materialization functor commutes with finite coproducts.

E.4.5 Connectivity of Materializations

We will see later that a map of profinite spaces $f : X \to Y$ is n-truncated if and only if $\mathrm{Mat}(f)$ is n-truncated (Proposition E.4.6.1), and n-connective if and only if $\mathrm{Mat}(f)$ is n-connective (Corollary E.4.6.3). As a first step, we prove the following:

Proposition E.4.5.1. *Let $f : X \to Y$ be an n-connective map of profinite spaces. Then the induced map $\mathrm{Mat}(X) \to \mathrm{Mat}(Y)$ is also n-connective.*

Remark E.4.5.2. In the special case where $Y = *$, X is discrete, and $n = 0$, we can Proposition E.4.5.1 reduces to Proposition E.1.1.1.

Lemma E.4.5.3. *Let $\mathcal{J}$ be a filtered ∞-category and let $X : \mathcal{J}^{\mathrm{op}} \to \mathcal{S}$ be a diagram of spaces indexed by $\mathcal{J}^{\mathrm{op}}$. Assume that:*

(a) For every object $J \in \mathcal{J}$, the set $\pi_0 X(J)$ is nonempty and finite.

(b) For every object $J \in \mathcal{J}$, every point $\eta \in X(J)$, and every integer $n \geq 1$, the group $\pi_n(X(J), \eta)$ is finite.

Then the limit $\varprojlim_{J \in \mathcal{J}^{\mathrm{op}}} X(J)$ is nonempty.

Proof. According to Proposition HTT.5.3.1.18 , there exists a filtered partially ordered set A and a left cofinal map $\mathrm{N}(A) \to \mathcal{J}$. We may therefore replace $\mathcal{J}$ by $\mathrm{N}(A)$ and thereby assume that $\mathcal{J}$ is the nerve of a filtered partially ordered set.

If $n \geq 0$, we will say that X is *n-truncated* if the space $X(\alpha)$ is n-truncated for each $\alpha \in A$. We first prove that $\varprojlim_{\alpha \in A^{\mathrm{op}}} X(\alpha)$ is nonempty under the additional assumption that X is n-truncated. Our proof proceeds by induction on n; the case $n = 0$ follows from Proposition E.1.1.1.

Suppose that X is n-truncated for $n > 0$. Let $X' : \mathrm{N}(A)^{\mathrm{op}} \to \mathcal{S}$ denote the composition of X with the truncation functor $\tau_{\leq n-1} : \mathcal{S} \to \mathcal{S}$. Our inductive hypothesis implies that the limit $\varprojlim_{\alpha\in A^{\mathrm{op}}} X'(\alpha)$ is nonempty. We will prove that X is nonempty by showing that the map $\theta : \varprojlim_{\alpha\in A^{\mathrm{op}}} X(\alpha) \to \varprojlim_{\alpha\in A^{\mathrm{op}}} X'(\alpha)$ is surjective on connected components. To this end, suppose we are given a point $\eta \in \varprojlim_{\alpha\in A^{\mathrm{op}}} X'(\alpha)$, so that η determines a natural transformation $X'_0 \to X'$, where X'_0 denotes the constant functor $\mathrm{N}(A)^{\mathrm{op}} \to \mathcal{S}$ taking the value Δ^0. Let $X_0 = X \times_{X'} X'_0$. To prove that the homotopy fiber of θ over η is nonempty, we must show that $\varprojlim_{\alpha\in A^{\mathrm{op}}} X_0(\alpha)$ is nonempty. Note that for each $\alpha \in A$, the space $X_0(\alpha)$ is an n-gerbe: that is, it is both n-truncated and n-connective. In particular, since $n > 0$, each of the spaces $X_0(\alpha)$ is connected.

Let $\mathcal{B}$ denote the collection of all finite subsets $B \subseteq A$ which contain a largest element. Let K denote the simplicial subset of $\mathrm{N}(A)$ given by the union of all the vertices. For each $B \in \mathcal{B}$, let $K_B \subseteq \mathrm{N}(A)$ denote the union $K \cup \mathrm{N}(B)$. Regard $\mathcal{B}$ as a partially ordered set with respect to inclusions, and define a functor $Y : \mathrm{N}(\mathcal{B})^{\mathrm{op}} \to \mathcal{S}$ by the formula $Y(B) = \varprojlim(X_0|_{K_B^{\mathrm{op}}})$ (see §HTT.4.2.3). Using Proposition HTT.4.2.3.8 , we obtain a homotopy equivalence $\varprojlim_{B\in\mathcal{B}^{\mathrm{op}}} Y(B) \simeq \varprojlim_{\alpha\in A} X_0(\alpha)$. It will therefore suffice to show that $\varprojlim_{B\in\mathcal{B}^{\mathrm{op}}} Y(B)$ is nonempty. Let $M = \varprojlim(X_0|_K) = \prod_{\alpha\in A} X_0(\alpha)$ and let $Z : \mathrm{N}(\mathcal{B}^{\mathrm{op}}) \to \mathcal{S}$ be the constant functor taking the value M. Note that $\mathcal{B}$ is filtered, so that $\varprojlim_{B\in\mathcal{B}^{\mathrm{op}}} Z(B) \simeq M$. We have an evident natural transformation of functors $Y \to Z$ which induces a map $\theta' : \varprojlim_{B\in\mathcal{B}^{\mathrm{op}}} Y(B) \to \varprojlim_{B\in\mathcal{B}^{\mathrm{op}}} Z(B) \simeq M$. Since M is nonempty, it will suffice to show that the homotopy fibers of θ' are nonempty.

Choose a point $\zeta \in M$, corresponding to a collection of points $\{\zeta_\alpha \in X_0(\alpha)\}_{\alpha\in A}$. The point ζ determines a natural transformation of functors $Z_0 \to Z$, where $Z_0 : \mathrm{N}(\mathcal{B})^{\mathrm{op}} \to \mathcal{S}$ is the constant functor taking the value Δ^0. Let $Y_0 = Y \times_Z Z_0$, so that the homotopy fiber of θ' over the point ζ is given by $\varprojlim_{B\in\mathcal{B}^{\mathrm{op}}} Y_0(B)$. Fix an element $B \in \mathcal{B}$, so that B is a subset of A which contains a largest element β. We have homotopy equivalences $Y(B) \simeq X_0(\beta) \times \prod_{\alpha\notin B} X_0(\alpha)$ and $Z(B) \simeq \prod_{\alpha\in A} X_0(\alpha)$. For each $\alpha \in B$, let ζ'_α denote the image of ζ_β under the map $X_0(\beta) \to X_0(\alpha)$. Unwinding the definitions, we see that $Y_0(B)$ can be identified with the product over all $\alpha \in B - \{\alpha\}$ of the space of paths joining ζ_α with ζ'_α in $X_0(\alpha)$. Since each $X_0(\alpha)$ is a connected n-truncated space with finite homotopy groups, we conclude that $Y_0(B)$ is a nonempty $(n-1)$-truncated space with finite homotopy groups. Since $\mathcal{B}$ is filtered, it follows from the inductive hypothesis that $\varprojlim_{B\in\mathcal{B}^{\mathrm{op}}} Y_0(B)$ is nonempty.

We now treat the case of a general functor $X : \mathrm{N}(A)^{\mathrm{op}} \to \mathcal{S}$. For each integer n, let $\tau_{\leqslant n}X$ denote the composition of X with the truncation functor $\tau_{\leqslant n} : \mathcal{S} \to \mathcal{S}$. Then X is the limit of the tower

$$\cdots \to \tau_{\leqslant 2}X \to \tau_{\leqslant 1}X \to \tau_{\leqslant 0}X,$$

so that $\varprojlim_{\alpha\in A^{\mathrm{op}}} X(\alpha)$ is given by the limit of the tower of spaces $\{\varprojlim_{\alpha\in A^{\mathrm{op}}} \tau_{\leqslant n}X(\alpha)\}_{n\geqslant 0}$. The above arguments show that $\varprojlim_{\alpha\in A^{\mathrm{op}}} \tau_{\leqslant 0}X(\alpha)$ is nonempty and that each of the transition maps $\varprojlim_{\alpha\in A^{\mathrm{op}}} \tau_{\leqslant n}X(\alpha) \to \varprojlim_{\alpha\in A^{\mathrm{op}}} \tau_{\leqslant n-1}X(\alpha)$ has nonempty homotopy fibers, from which it immediately follows that the tower $\{\varprojlim_{\alpha \in A^{\mathrm{op}}} \tau_{\leqslant n} X(\alpha)\}_{n \geqslant 0}$ has nonempty limit. $\square$

Corollary E.4.5.4. *Let $\mathcal{J}$ and X be as in Lemma E.4.5.3, and let $n \geqslant 0$ be an integer such that each $X(J)$ is n-connective. Then $X = \varprojlim_{J \in \mathcal{J}} X(J)$ is n-connective.*

Proof. We proceed by induction on n. If $n = 0$, the desired result follows from Lemma E.4.5.3. Assume therefore that $n > 0$. The inductive hypothesis implies that X is nonempty. It will therefore suffice to show that, for every pair of points $\eta, \eta' \in X$, the path space $\{\eta\} \times_X \{\eta'\}$ is $(n-1)$-connective. Let $X_0, X_1 : \mathcal{J}^{\mathrm{op}} \to \mathcal{S}$ denote the constant functor taking the value Δ^0, so that η and η' determine natural transformations $X_0 \to X \leftarrow X_1$ and we have a homotopy equivalence $\{\eta\} \times_X \{\eta'\} \simeq \varprojlim_{J \in \mathcal{J}^{\mathrm{op}}} (X_0 \times_X X_1)(J)$. Since $X_0 \times_X X_1$ takes $(n-1)$-connective values, the inductive hypothesis implies that $\{\eta\} \times_X \{\eta'\}$ is $(n-1)$-connective. $\square$

Remark E.4.5.5. Let X be a profinite space, let $n \geqslant 0$ be an integer, and let $f : X \to \tau_{\leqslant n} X$ be an n-truncation of X. Then $\mathrm{Mat}(f) : \mathrm{Mat}(X) \to \mathrm{Mat}(\tau_{\leqslant n} X)$ is a map from $\mathrm{Mat}(X)$ to an n-truncated space. Since f is $(n+1)$-connective, Corollary E.4.5.4 implies that $\mathrm{Mat}(f)$ is $(n+1)$-connective: that is, $\mathrm{Mat}(f)$ induces an equivalence $\tau_{\leqslant n} \mathrm{Mat}(X) \simeq \mathrm{Mat}(\tau_{\leqslant n} X)$.

Proof of Proposition E.4.5.1. Write f as a filtered limit of n-connective maps $f_\alpha : X_\alpha \to Y_\alpha$ between p-finite spaces. Let $\eta \in \mathrm{Mat}(Y)$, so that η determines a compatible family of points $\eta_\alpha \in Y_\alpha$. We wish to prove that the homotopy fiber $\mathrm{Mat}(X)_\eta = \mathrm{Mat}(X) \times_{\mathrm{Mat}(Y)} \{\eta\}$ is n-connective. This homotopy fiber is given as the limit of a filtered system n-connective, p-finite spaces $X_\alpha \times_{Y_\alpha} \{\eta_\alpha\}$, and is therefore n-connective by Corollary E.4.5.4. $\square$

E.4.6 Truncatedness of Materializations

We now prove an analogue of Proposition E.4.5.1 for truncated morphisms of profinite spaces.

Proposition E.4.6.1. *Let $f : X \to Y$ be a morphism of profinite spaces and let $n \geqslant -2$ be an integer. The following conditions are equivalent:*

(1) *The morphism f is n-truncated, in the sense of Definition E.4.1.1.*

(2) *The morphism f exhibits X as an n-truncated object of the ∞-category $(\mathcal{S}_\pi^\wedge)_{/Y}$: that is, for every profinite space Z, the map of spaces $\mathrm{Map}_{\mathcal{S}_\pi^\wedge}(Z, X) \to \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(Z, Y)$ is n-truncated.*

(3) *The induced map of materializations $\mathrm{Mat}(X) \to \mathrm{Mat}(Y)$ is n-truncated.*

Proof. Suppose first that condition (1) is satisfied: that is, f is given as the limit of a filtered diagram of n-truncated morphisms $f_\alpha : X_\alpha \to Y_\alpha$ between π-finite spaces. We will prove that (2) is satisfied. Let Z be an arbitrary π-profinite space. Then the map $\theta : \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(Z, X) \to \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(Z, Y)$ is a filtered limit of maps

$$\theta_\alpha : \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(Z, X_\alpha) \to \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(Z, Y_\alpha).$$

Consequently, to prove that θ is n-truncated, it will suffice to show that each θ_α is n-truncated. Write Z as a filtered limit of π-finite spaces Z_β, so that θ_α is a filtered colimit of maps

$$\theta_{\alpha,\beta} : \mathrm{Map}_{\mathcal{S}}(Z_\beta, X_\alpha) \to \mathrm{Map}_{\mathcal{S}}(Z_\beta, Y_\alpha).$$

It will therefore suffice to show that each $\theta_{\alpha,\beta}$ is n-truncated, which follows immediately from our assumption that f_α is n-truncated.

The implication (2) ⇒ (3) is obvious. We will complete the proof by showing that (3) ⇒ (1). Using Theorem E.4.1.2, we can factor f as a composition $X \xrightarrow{f'} Z \xrightarrow{f''} Y$ where f' is $(n+1)$-connective and f'' is n-truncated. The first part of the proof shows that the map of materializations $\mathrm{Mat}(Z) \to \mathrm{Mat}(Y)$ is n-truncated, so that f' induces an n-truncated map $\mathrm{Mat}(X) \to \mathrm{Mat}(Z)$. Proposition E.4.5.1 implies that $\mathrm{Mat}(X) \to \mathrm{Mat}(Z)$ is $(n+1)$-connective. It follows that the map from $\mathrm{Mat}(X)$ to $\mathrm{Mat}(Z)$ is a homotopy equivalence. Using Theorem E.3.1.6, we deduce that f' is an equivalence of profinite spaces, so that f is n-truncated as desired. □

Corollary E.4.6.2. *Let $n \geqslant -2$ be an integer, and let $X \in \mathcal{S}_\pi^\wedge$ be a profinite space. The following conditions are equivalent:*

(1) *The profinite space X belongs to the essential image of the localization functor $\tau_{\leqslant n} : \mathcal{S}_\pi^\wedge \to \mathcal{S}_\pi^\wedge$ of Example E.4.1.3.*

(2) *The profinite space X is n-truncated, in the sense of Definition E.4.1.1.*

(3) *For every profinite space Y, the mapping space $\mathrm{Map}_{\mathcal{S}_\pi^\wedge}(Y, X)$ is n-truncated.*

(4) *The space $\mathrm{Mat}(X)$ is n-truncated.*

Corollary E.4.6.3. *Let $f : X \to Y$ be a morphism of profinite spaces, and let $n \geqslant -1$ be an integer. The following conditions are equivalent:*

(1) *The morphism f is n-connective, in the sense of Definition E.4.1.1.*

(2) *The induced map of materializations $\mathrm{Mat}(X) \to \mathrm{Mat}(Y)$ is n-connective.*

Proof. The implication (1) ⇒ (2) follows from Proposition E.4.5.1. Assume that (2) is satisfied; we will prove (1). Using Theorem E.4.1.2, we can factor f as a composition

$$X \xrightarrow{f'} Z \xrightarrow{f''} Y$$

where f' is n-connective and f'' is $(n-1)$-truncated. Using Propositions E.4.6.1 and E.4.5.1, we conclude that $\mathrm{Mat}(f')$ is n-connective and $\mathrm{Mat}(f'')$ is $(n-1)$-truncated. Since $\mathrm{Mat}(f)$ is n-connective, we conclude that $\mathrm{Mat}(f'')$ is an equivalence. Applying Theorem E.3.1.6, we deduce that f'' is an equivalence, so that $f \simeq f'' \circ f'$ is n-connective. □

E.5 Profinite Classifying Spaces

Let X be a space equipped with a base point $x \in X$, and let ΩX denote the loop space of X (based at the point x). Then composition of loops determines a multiplication map $\Omega X \times \Omega X \to \Omega X$, which is unital and associative up to coherent homotopy. We can summarize the situation by saying that the construction $X \mapsto \Omega X$ determines a functor from the ∞-category $\mathcal{S}_*$ of pointed spaces to the ∞-category $\mathcal{G}\mathrm{rp}(\mathcal{S})$ of group objects in the ∞-category $\mathcal{S}$ of spaces. This functor restricts to an equivalence of ∞-categories $\mathcal{S}_*^{\geqslant 1} \simeq \mathcal{G}\mathrm{rp}(\mathcal{S})$, where $\mathcal{S}_*^{\geqslant 1}$ denotes the full subcategory of $\mathcal{S}_*$ spanned by the connected pointed spaces (Lemma HTT.7.2.2.10). Our goal in this section is to prove the following:

Theorem E.5.0.4. *The functor $\Omega : (\mathcal{S}_\pi^\wedge)_* \to \mathcal{G}\mathrm{rp}(\mathcal{S}_\pi^\wedge)$ induces an equivalence of ∞-categories $(\mathcal{S}_\pi^\wedge)_*^{\geqslant 1} \simeq \mathcal{G}\mathrm{rp}(\mathcal{S}_\pi^\wedge)$.*

E.5.1 Profinite Groups

It is not difficult to show that $(\mathcal{S}_\pi^\wedge)_*$ can be identified with the ∞-category of Pro-objects of $(\mathcal{S}_\pi)_*^{\geqslant 1}$, which (by Lemma HTT.7.2.2.10) can be identified with the ∞-category of group objects of $\mathcal{S}_\pi$. Consequently, Theorem E.5.0.4 is equivalent to the assertion that the canonical map $\mathrm{Pro}(\mathcal{G}\mathrm{rp}(\mathcal{S}_\pi)) \to \mathcal{G}\mathrm{rp}(\mathrm{Pro}(\mathcal{S}_\pi))$ is an equivalence of ∞-categories. This can be regarded as a generalization of a classical fact about profinite groups, which we now review.

Definition E.5.1.1. A *profinite group* is a topological group G whose underlying topological space is a Stone space (see Definition E.1.3.2). We let $\mathrm{Grp}_{\mathrm{St}}$ denote the category whose objects are profinite groups and whose morphisms are continuous group homomorphisms.

Example E.5.1.2. Every finite group G can be regarded as a profinite group (when endowed with the discrete topology). That is, we can regard the category $\mathrm{Grp}_{\mathrm{fin}}$ of finite groups as a full subcategory of $\mathrm{Grp}_{\mathrm{St}}$.

Proposition E.5.1.3. *The inclusion $\mathrm{Grp}_{\mathrm{fin}} \hookrightarrow \mathrm{Grp}_{\mathrm{St}}$ extends to an equivalence of categories*

$$\mathrm{Pro}(\mathrm{Grp}_{\mathrm{fin}}) \simeq \mathrm{Grp}_{\mathrm{St}} .$$

Proof. Since the category of Stone spaces is closed under limits (in the larger category of all topological spaces; see Lemma E.1.4.2), the category $\mathrm{Grp}_{\mathrm{St}}$ is closed under limits in the category of all topological groups. It follows that $\mathrm{Grp}_{\mathrm{St}}$ admits small limits, so that the inclusion $\mathrm{Grp}_{\mathrm{fin}} \hookrightarrow \mathrm{Grp}_{\mathrm{St}}$ extends to a functor $F : \mathrm{Pro}(\mathrm{Grp}_{\mathrm{fin}}) \to \mathrm{Grp}_{\mathrm{St}}$ which commutes with filtered limits (Proposition HTT.5.3.5.10). We first claim that F is fully faithful. According to Proposition HTT.5.3.5.11 , it will suffice to show that every finite group G is compact when viewed as an object of $\mathrm{Grp}_{\mathrm{St}}{}^{\mathrm{op}}$. In other words, we claim that if we are given a filtered system of profinite groups H_α having limit H, then the canonical map

$$\varinjlim \mathrm{Hom}_{\mathrm{Grp}_{\mathrm{St}}}(H_\alpha, G) \to \mathrm{Hom}_{\mathrm{Grp}_{\mathrm{St}}}(H, G)$$

is bijective. Theorem E.1.4.1 implies that G is compact when viewed as an object of the category $\mathcal{T}\mathrm{op}_{\mathrm{St}}^{\mathrm{op}}$. Consequently, every continuous group homomorphism $f : H \to G$ factors as a composition $H \to H_\alpha \xrightarrow{f'} G$, for some index α. We must show that it is possible to choose α so that f' is a group homomorphism. To prove this, we consider the pair of maps

$$u, v : H_\alpha \times H_\alpha \to G$$

given by $u(x, y) = f'(xy)$, $v(x, y) = f'(x)f'(y)$. Then u and v induce the same map from $H \times H$ into G. Using Theorem E.1.4.1 we conclude that there exists a map of indices $\beta \to \alpha$ such that u and v agree on $H_\beta \times H_\beta$. Replacing α by β, we may assume that $u = v$ so that f' is a group homomorphism, as desired.

It remains to prove that the functor F is essentially surjective. Fix a profinite group G, and let S be the partially ordered set of open normal subgroups $G_0 \subseteq G$. Then S^{op} is filtered (since the collection of open normal subgroups is closed under finite intersections). We may therefore view the inverse system $\{G/G_0\}_{G_0 \in A}$ as an object $\overline{G} \in \mathrm{Pro}(\mathrm{Grp}_{\mathrm{fin}})$. We will complete the proof by showing that the natural map $\phi : G \to F(\overline{G})$ is an isomorphism of profinite groups.

Note that every nonempty open subset of $F(\overline{G})$ contains the inverse image of some element of G/G_0, where G_0 is an open normal subgroup of G. From this we immediately deduce that ϕ has dense image. Since G is compact and $F(\overline{G})$ is Hausdorff, it follows that ϕ is a quotient map: that is, ϕ induces an isomorphism of profinite groups $G/\ker(\phi) \to F(\overline{G})$. We will complete the proof by showing that $\ker(\phi)$ is trivial.

Choose a non-identity element $x \in G$; we wish to show that there exists an open normal subgroup of G which does not contain x. Since G is a Stone space, there exists a closed and open subset $Y \subseteq G$ which contains the identity element but does not contain x. Let $Y^+ = \{gyg^{-1} : g \in G, y \in Y\}$. Then Y^+ image of a continuous map $G \times Y \to G$. Since G is compact, we conclude that Y^+ is compact and therefore a closed subset of G. As a union of conjugates of Y, Y^+ is also an open subset of G. Let $G_0 = \{g \in G : gY^+ = Y^+\}$. Then G_0 is a subgroup of G which does not contain x. Since Y^+ is conjugation-invariant, the subgroup G_0 is normal. Moreover, the complement of G_0 is given by the image of $(G - Y^+) \times Y^+$ under the continuous map $(g, h) \mapsto (gh^{-1})$. Since the product is a compact set, we conclude that $G - G_0$ is compact, so that G_0 is an open subgroup of G. □

E.5.2 Homotopy Groups of Profinite Spaces

Profinite groups arise naturally when studying algebraic invariants of profinite spaces.

Definition E.5.2.1. Let X be a profinite space. We let $\pi_0 X$ denote the set $\pi_0 \operatorname{Mat}(X)$ of path components of the materialization $\operatorname{Mat}(X)$. For each integer $n \geqslant 1$ and each point $x \in \operatorname{Mat}(X)$, we let $\pi_n(X, x)$ denote the homotopy group $\pi_n(\operatorname{Mat}(X), x)$. We will refer to the groups $\pi_n(X, x)$ as the *homotopy groups of* X (with base point x).

The next result enables us to compute the homotopy groups of a profinite space:

Proposition E.5.2.2. *The construction $X \mapsto \pi_0 X$ determines a functor $\mathcal{S}_\pi^\wedge \to \mathcal{S}\mathrm{et}$ which preserves filtered limits.*

Proof. Let $F : \mathcal{S}_\pi^\wedge \to \mathcal{S}\mathrm{et}$ be the functor given by $F(X) = \pi_0 X$. Let us abuse notation by identifying the ∞-category $\mathcal{S}_\pi$ of π-finite spaces with a full subcategory of the ∞-category $\mathcal{S}_\pi^\wedge$ of profinite spaces, so that $F_0 = F|_{\mathcal{S}_\pi}$ is the functor given by $F_0(T) = \pi_0 T$. Let $F' : \mathcal{S}_\pi^\wedge \to \mathcal{S}\mathrm{et}$ be a right Kan extension of F_0, so that the identification $F'|_{\mathcal{S}_\pi} = F|_{\mathcal{S}_\pi}$ extends to a natural transformation $u : F \to F'$. Since F' commutes with filtered limits, it will suffice to show that u is an equivalence. To this end, consider a profinite space K, which we can assume is given by the limit of a diagram $X : \mathcal{J}^{\mathrm{op}} \to \mathcal{S}_\pi$ for some filtered ∞-category $\mathcal{J}$. We wish to show that the canonical map

$$\theta : \pi_0(\varprojlim_{J \in \mathcal{J}^{\mathrm{op}}} X(J)) \to \varprojlim_{J \in \mathcal{J}^{\mathrm{op}}} \pi_0 X(J)$$

is a bijection. Choose a point $\eta \in \varprojlim_{J \in \mathcal{J}^{\mathrm{op}}} \pi_0 X(J)$. Then η determines, for each $J \in \mathcal{J}$, a connected component $X_\eta(J)$ of $X(J)$. We can regard X_η itself as a functor $\mathcal{J}^{\mathrm{op}} \to \mathcal{S}_\pi$. Note that $\theta^{-1}\{\eta\}$ can be identified with the set of path components of the limit $V_\eta = \varprojlim_{J \in \mathcal{J}^{\mathrm{op}}} X_\eta(J)$. To prove that θ is a bijection, it will suffice to show that each of the spaces V_η is connected. This follows from Corollary E.4.5.4, since each $X_\eta(J)$ is connected by construction. □

We now use Proposition E.5.2.2 to deduce an analogous statement for homotopy groups π_n for $n > 0$.

Definition E.5.2.3. Let X be a profinite space. A *point* of X is a point of the materialization $\operatorname{Mat}(X)$. By virtue of Remark E.4.4.3, we can identify points of X with maps $* \to X$ in the ∞-category $\mathcal{S}_\pi^\wedge$ of profinite spaces. We let $(\mathcal{S}_\pi^\wedge)_*$ denote the ∞-category of pointed objects of $\mathcal{S}_\pi^\wedge$: that is, the ∞-category of profinite spaces X equipped with a point $* \to X$.

Corollary E.5.2.4. *For each integer $n \geqslant 1$, let $\pi_n : (\mathcal{S}_\pi^\wedge)_*$ denote the functor which carries a pointed profinite space (X, x) to the homotopy group $\pi_n(X, x)$. Then the functor π_n commutes with filtered limits.*

Proof. This follows from Proposition E.5.2.2, since the functor $X \mapsto \Omega(X)$ preserves limits. □

Remark E.5.2.5. Let X be a profinite space. Then we can write X as the limit of a diagram $\{X_\alpha\}$, where each X_α is π-finite. Using Proposition E.5.2.2, we deduce that $\pi_0 X$ can be identified with the Stone space associated to the profinite set given by the diagram $\{\pi_0 X_\alpha\}$. In particular, the inverse limit topology endows $\pi_0 X$ with the structure of a Stone space.

Let x be a point of X, which we can identify with a compatible family of points $x_\alpha \in X_\alpha$. For each integer $n \geqslant 1$, Corollary E.5.2.4 supplies an isomorphism $\pi_n(X, x) \simeq \varprojlim_\alpha \pi_n(X_\alpha, x_\alpha)$, so that we can view $\pi_n(X, x)$ as a profinite group.

Remark E.5.2.6. Let X be a space, and let $X_\pi^\wedge$ be its profinite completion (Example E.0.7.12). Then $\pi_0 X_\pi^\wedge$ can be identified with the Stone-Čech compactification of the set $\pi_0 X$ (Remark E.1.4.4). In particular, for every space X, we have a canonical map of profinite spaces

$$X_\pi^\wedge \to \beta(\pi_0 X)$$

(where we identify the Stone-Čech compactification of $\pi_0 X$ with the corresponding profinite space, under the fully faithful embedding of Remark E.4.1.4). If X is discrete, then this map is an equivalence (Proposition E.4.3.3).

E.5.3 Digression: Strong n-Truncations

We now introduce a technical notion which will play an important role in our proof of Theorem E.5.0.4.

Definition E.5.3.1. Let X be a space and let $n \geqslant 1$ be an integer. We will say that X is *strongly n-truncated* if it satisfies the following conditions:

- (a) The space X is n-truncated. That is, the groups $\pi_m(X, x)$ are trivial for every base point $x \in X$ and every integer $m > n$.
- (b) For every choice of base point $x \in X$, the group $\pi_1(X, x)$ acts trivially on $\pi_n(X, x)$.

Remark E.5.3.2. For every integer $n \geqslant 1$, the inclusion from the ∞-category of strongly n-truncated spaces to the ∞-category of spaces admits a left adjoint $\tau^s_{\leqslant n}$. To every space X, this left adjoint assigns another space $\tau^s_{\leqslant n} X$, which we will refer to as the *strong n-truncation of X*. It is characterized up to equivalence by the requirement that there is a map $f : X \to \tau^s_{\leqslant n} X$ which is bijective on connected components, and the homotopy groups of $\tau^s_{\leqslant n} X$ are given by

$$\pi_m(\tau^s_{\leqslant n} X, f(x)) \simeq \begin{cases} \pi_m(X,x) & \text{if } m < n \\ \pi_n(X,x)_{\pi_1(X,x)} & \text{if } m = n \\ 0 & \text{if } m > n. \end{cases}$$

Here $\pi_n(X,x)_{\pi_1(X,x)}$ denotes the group of coinvariants for the action of $\pi_1(X,x)$ on $\pi_n(X,x)$ (if $n = 1$, this is the abelianization of $\pi_1(X,x)$).

Remark E.5.3.3. Let $n \geqslant 1$ be an integer. The strong truncation functor $\tau^s_{\leqslant n}$ of Remark E.5.3.2 carries π-finite spaces to π-finite spaces. It therefore admits an essentially unique extension to a functor $\mathcal{S}^\wedge_\pi \to \mathcal{S}^\wedge_\pi$ which commutes with filtered limits. We will abuse notation by denoting this functor also by $\tau^s_{\leqslant n}$.

Let X be a profinite space, and write X as the limit of a filtered diagram of π-finite spaces X_α. By definition, $\tau^s_{\leqslant n} X$ is given by the limit of the diagram $\{\tau^s_{\leqslant n} X_\alpha\}$ (in the ∞-category $\mathcal{S}^\wedge_\pi$ of profinite spaces).

Proposition E.5.3.4. *Let X be a profinite space, let $n \geqslant 1$ be an integer, and let $\theta : X \to \tau^s_{\leqslant n} X$ be the canonical map. Then:*

- (a) *The map θ induces a bijection $\pi_0 X \to \pi_0 \tau^s_{\leqslant n} X$.*
- (b) *For each point $x \in X$, the map $\pi_m(X,x) \to \pi_m(\tau^s_{\leqslant n} X, \theta(x))$ is an isomorphism for $m < n$.*
- (c) *For each point $x \in X$, let K denote the kernel of the canonical map $\pi_n(X,x) \to \pi_n(X,x)_{\pi_1(X,x)}$, and let $\overline{K}$ denote the closure of K in $\pi_n(X,x)$ (with respect to the profinite topology described in Remark E.5.2.5). Then θ induces a surjection $\pi_n(X,x) \to \pi_n(\tau^s_{\leqslant n} X, \theta(x))$ with kernel $\overline{K}$.*
- (d) *For each point $x \in X$ and each $m > n$, the group $\pi_m(\tau^s_{\leqslant n} X, \theta(x))$ vanishes.*

Proof. Write X as the limit of a diagram of π-finite spaces $\{X_\alpha\}_{\alpha \in A}$ indexed by a filtered partially ordered set A. Since each of the maps $X_\alpha \to \tau^s_{\leqslant n} X_\alpha$ is bijective on connected components, assertion (a) follows from Proposition E.5.2.2. Similarly, (b) and (d) follow from Corollary E.5.2.4. It remains to prove (c). Let x be a point of X, given by a compatible family of points $x_\alpha \in X_\alpha$. For each $\alpha \in A$, let M_α denote the coinvariants for the action of $\pi_1(X_\alpha, x_\alpha)$ on $\pi_n(X_\alpha, x_\alpha)$, and set $M = \varprojlim_\alpha M_\alpha$. Then Corollary E.5.2.4 supplies an isomorphism $\pi_n(\tau^s_{\leqslant n} X, \theta(x)) \simeq M$. The natural map $\nu : \pi_n(X,x) \to M$ is an inverse limit of surjective homomorphisms of finite groups $\nu_\alpha : \pi_n(X_\alpha, x_\alpha) \to M_\alpha$, and therefore a surjection (Proposition E.1.1.1). It remains to prove that $\ker(\nu) = \overline{K}$. It is clear that $\ker(\nu)$ is a closed subgroup of $\pi_n(X,x)$ which contains K. To complete the proof, it will suffice to show that K is dense in $\ker(\nu)$. To this end, choose an element $u \in \ker(\nu) \subseteq \pi_n(X,x)$, given by a compatible family of elements $u_\alpha \in \pi_n(X_\alpha, x_\alpha)$. We wish to show that for each $\alpha \in A$, u_α belongs to the image of the map $K \to \pi_n(X_\alpha, x_\alpha)$.

For each index $\beta \in A$, the action of $\pi_1(X_\beta, x_\beta)$ on $\pi_n(X_\beta, x_\beta)$ is given by a map

$$a_\beta : \pi_1(X_\beta, x_\beta) \times \pi_n(X_\beta, x_\beta) \to \pi_n(X_\beta, x_\beta).$$

The inverse limit of these maps determines an action map

$$a : \pi_1(X,x) \times \pi_n(X,x) \to \pi_n(X,x).$$

Let K_β denote the kernel of the map $\pi_n(X_\beta, x_\beta) \to M_\beta$. Let $K_\beta^{(1)} \subseteq K_\beta$ denote the subset of $\pi_n(X_\beta, x_\beta)$ consisting of elements of the form $a_\beta(g,h)h^{-1}$. Then $K_\beta^{(1)}$ generates the group K_β. For each integer $m \geqslant 1$, let $K_\beta^{(m)}$ denote the subset of $\pi_n(X_\beta, x_\beta)$ consisting of m-fold products of elements of $K_\beta^{(1)}$. Since $\pi_n(X_\beta, x_\beta)$ is finite, we have $K_\beta = K_\beta^{(m)}$ for m sufficiently large (depending on β).

For each $\beta \geqslant \alpha$ and each integer $m > 0$, let $L_\beta^{(m)}$ denote the image of $K_\beta^{(m)}$ in $\pi_n(X_\alpha, x_\alpha)$. Then $L_\beta^{(m)}$ is the subset of $\pi_n(X_\alpha, x_\alpha)$ generated by all m-fold products of elements of $L_\beta^{(1)}$. Let m_0 denote the order of the finite group $\pi_n(X_\alpha, x_\alpha)$, so that $L_\beta^{(m)} = L_\beta^{(m_0)}$ for $m \geqslant N$. Since $u_\beta \in K_\beta = \bigcup_m K_\beta^{(m)}$, we have $u_\alpha \in \bigcup_m L_\beta^{(m)} = L_\beta^{(m_0)}$.

For each $\beta \geqslant \alpha$, let $S_\beta \subseteq \pi_1(X_\beta, x_\beta)^{m_0} \times \pi_n(X_\beta, x_\beta)^{m_0}$ denote the subset consisting of those tuples $(g_1, \ldots, g_{m_0}, h_1, \ldots, h_{m_0})$ such that

$$a_\beta(g_1,h_1)h_1^{-1} a_\beta(g_2,h_2)h_2^{-1} \cdots a_\beta(g_{m_0}, h_{m_0})h_{m_0}^{-1}$$

is a preimage of u_α under the map $\pi_n(X_\beta, x_\beta) \to \pi_n(X_\alpha, x_\alpha)$. It follows from the above reasoning that each of the sets S_β is nonempty. Applying Proposition E.1.1.1, we deduce that $\varprojlim_{\beta \geqslant \alpha} S_\beta$ is nonempty. Using Corollary E.5.2.4, we can identify an element of the inverse limit with a tuple $(g_1, \ldots, g_{m_0}, h_1, \ldots, h_{m_0}) \in \pi_1(X,x)^{m_0} \times \pi_n(X,x)^{m_0}$. Set

$$v = a(g_1,h_1)h_1^{-1} \cdots a(g_{m_0}, h_{m_0})h_{m_0}^{-1}.$$

By construction, v is an element of K whose image in $\pi_n(X_\alpha, x_\alpha)$ coincides with u_α. □

Corollary E.5.3.5. *Let X be a profinite space and let $n \geqslant 1$ be an integer such that* $\mathrm{Mat}(X)$ *is strongly n-truncated. Then the canonical map $\theta : X \to \tau^s_{\leqslant n} X$ is an equivalence in $\mathcal{S}^\wedge_\pi$.*

Proof. It follows from Proposition E.5.3.4 that $\mathrm{Mat}(\theta)$ is a homotopy equivalence. Applying Theorem E.3.1.6, we deduce that θ is an equivalence of profinite spaces. □

E.5.4 Families of Abelian Groups

Let X be a space which is strongly n-truncated (Definition E.5.3.1). Then, for any base point $x \in X$, the fundamental group $\pi_1(X, x)$ acts trivially on the homotopy group $\pi_n(X, x)$. It follows that $\pi_n(X, x)$ is an abelian group which depends only on the image of x in $\pi_0 X$. We now axiomatize this structure.

Definition E.5.4.1. A *family of abelian groups* consists of the following data:

(a) A map of sets $\phi : A \to S$.

(b) An abelian group structure on each fiber $A_s = \phi^{-1}\{s\}$.

We will generally abuse notation and indicate a family of abelian groups simply as a map $\phi : A \to S$. We will say that a family of abelian groups $\phi : A \to S$ is *finite* if A and S are finite.

If $\phi : A \to S$ and $\phi' : A' \to S'$ are families of abelian groups, then a *morphism of families of abelian groups* from ϕ to ϕ' is a commutative diagram of sets

$$\begin{array}{ccc} A & \xrightarrow{\phi} & S \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f} \\ A' & \xrightarrow{\phi'} & S' \end{array}$$

such that, for every element $s \in S$, the induced map $F_s : A_s \to A'_{f(s)}$ is a group homomorphism.

We let Fam denote the category whose objects are families of abelian groups, and $\mathrm{Fam}_{\mathrm{fin}}$ the full subcategory of Fam spanned the finite families of abelian groups.

Remark E.5.4.2. Let $\phi : A \to S$ be a family of abelian groups. For every element $s \in S$, let $e(s)$ denote the identity element of the abelian group A_s. Then the construction $s \mapsto e(s)$ determines a map from S to A, which we will refer to as the *zero section* of ϕ.

Notation E.5.4.3. If $\phi : A \to S$ is a family of abelian groups, then we can identify A with an abelian group object of the topos $\mathcal{S}\mathrm{et}_{/S}$. For each $n \geqslant 0$, we let $K(\phi, n)$ denote the associated Eilenberg-MacLane object of the ∞-topos $\mathcal{S}_{/S}$. More concretely, we have $K(\phi, n) = \amalg_{s \in S} K(A_s, n)$, where A_s denotes the abelian group $\phi^{-1}\{s\}$. Note that if $\phi : A \to S$ is a finite family of abelian groups, then the space $K(\phi, n)$ is π-finite.

Example E.5.4.4. Let X be a strongly n-truncated space. For each path component $\eta \in \pi_0 X$, let $\pi_n(X, \eta)$ denote the homotopy group $\pi_n(X, x)$, where $x \in X$ is a representative of η. Our assumption that $\pi_1(X, x)$ acts trivially on $\pi_n(X, x)$ guarantees that the abelian group $\pi_n(X, \eta)$ is independent of the choice of x, up to *canonical* isomorphism. Then the map $\amalg_{\eta \in \pi_0 X} \pi_n(X, \eta) \to \pi_0 X$ is a family of abelian groups, in the sense of Definition E.5.4.1. Let us denote this family by $\pi_n X$.

The map $X \to \tau_{\leqslant n-1} X$ is n-connective and n-truncated. In the terminology of §HTT.7.2.2 , we can regard X as an n-gerbe over $\tau_{\leqslant n-1} X$, banded by a local system of abelian groups on $\tau_{\leqslant n-1} X$. By assumption, this local system is trivial: that is, it can be written (canonically) as the pullback of $\pi_n(X)$, which we regard as a local system of abelian groups on $\pi_0(X)$. We therefore have a pullback diagram of spaces σ_X :

$$\begin{array}{ccc} X & \longrightarrow & \pi_0 X \\ \downarrow & & \downarrow \\ \tau_{\leqslant n-1} X & \longrightarrow & K(\pi_n(X), n+1). \end{array}$$

Note that σ_X depends functorially on X, and that the construction $X \mapsto \sigma_X$ commutes with products.

The category $\mathrm{Fam}_{\mathrm{fin}}$ admits finite products, so it makes sense to consider group objects in $\mathrm{Fam}_{\mathrm{fin}}$. The collection of group objects in $\mathrm{Fam}_{\mathrm{fin}}$ forms a category, which we will denote by $\mathcal{G}\mathrm{rp}(\mathrm{Fam}_{\mathrm{fin}})$. We will need the following fact, whose proof we defer until the end of this section:

Proposition E.5.4.5. *Every group object of* $\mathrm{Pro}(\mathrm{Fam}_{\mathrm{fin}})$ *can be written as a filtered inverse limit of group objects of* $\mathrm{Fam}_{\mathrm{fin}}$.

E.5.5 Digression: Effective Epimorphisms of Profinite Spaces

Recall that if $f : U \to X$ is a map of spaces which is surjective on connected components, then f is an effective epimorphism: that is, we can recover X as the geometric realization of the simplicial space $U_\bullet$ given as the Čech nerve of f. We now establish an analogous statement in the setting of profinite homotopy theory.

Notation E.5.5.1. Let $\mathbf{\Delta}_+$ denote the augmented simplex category. We have fully faithful embeddings $\mathbf{\Delta}^{\mathrm{op}} \overset{j}{\hookrightarrow} \mathbf{\Delta}_+^{\mathrm{op}} \overset{i}{\hookleftarrow} \Delta^1$ where i carries Δ^1 to the morphism $[0] \to [-1]$ in $\mathbf{\Delta}_+^{\mathrm{op}}$.

For any ∞-category $\mathcal{C}$, the embeddings i and j determine restriction functors

$$\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}) \overset{j^*}{\leftarrow} \mathrm{Fun}(\mathbf{\Delta}_+^{\mathrm{op}}, \mathcal{C}) \overset{i^*}{\to} \mathrm{Fun}(\Delta^1, \mathcal{C}).$$

If $\mathcal{C}$ admits fiber products, then the functor i^* admits a right adjoint i_*, which carries a morphism $f : U \to X$ in $\mathcal{C}$ to the augmented simplicial object of $\mathcal{C}$ given by the Čech nerve of f. If the ∞-category $\mathcal{C}$ admits a geometric realizations of simplicial objects, then the functor j^* admits a left adjoint $j_!$, given by $(j_! X_\bullet)_{-1} = |X_\bullet|$. Note that if $\mathcal{C}$ admits finite limits, then the functor i_* takes values in the ∞-category $\mathcal{G}\mathrm{pd}(\mathcal{C})$ of groupoid objects of

$\mathcal{C}$ (see Proposition HTT.6.1.2.11). If $\mathcal{C}$ admits finite limits and geometric realizations, we obtain a pair of adjoint functors

$$\mathcal{G}\mathrm{pd}(\mathcal{C}) \underset{j^* i_*}{\overset{i^* j_!}{\rightleftarrows}} \mathrm{Fun}(\Delta^1, \mathcal{C}).$$

Remark E.5.5.2. Let $\mathcal{C}$ be an ∞-category which admits finite limits and geometric realizations of simplicial objects. Recall that a *group object* of $\mathcal{C}$ is a groupoid object $U_\bullet$ of $\mathcal{C}$ such that U_0 is a final object of $\mathcal{C}$. We let $\mathcal{G}\mathrm{rp}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$ denote the full subcategory spanned by the group objects of $\mathcal{C}$, and $\mathcal{C}_* \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$ the ∞-category of pointed objects of $\mathcal{C}$ (that is, the full subcategory $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms $U \to X$ where U is a final object of $\mathcal{C}$). Then the adjunction of Notation E.5.5.1 restricts to a pair of adjoint functors

$$\mathcal{G}\mathrm{rp}(\mathcal{C}) \underset{\Omega}{\overset{B}{\rightleftarrows}} \mathcal{C}_*.$$

Proposition E.5.5.3. *Let $\mathcal{E}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{S}_\pi^\wedge)$ spanned by those morphisms $f : U \to X$ of profinite spaces which induce a surjection $\pi_0 U \to \pi_0 X$. Then the functor $j^* i_* : \mathrm{Fun}(\Delta^1, \mathcal{S}_\pi^\wedge) \to \mathcal{G}\mathrm{pd}(\mathcal{S}_\pi^\wedge)$ of Notation E.5.5.1 is fully faithful when restricted to $\mathcal{C}$.*

Corollary E.5.5.4. *Let $(\mathcal{S}_\pi^\wedge)_*$ denote the ∞-category of pointed objects of $\mathcal{S}_\pi^\wedge$, and let $(\mathcal{S}_\pi^\wedge)_*^{\geqslant 1}$ denote the full subcategory of $(\mathcal{S}_\pi^\wedge)_*$ spanned by the pointed connected profinite spaces. Then the functor Ω of Remark E.5.5.2 restricts to a fully faithful embedding $(\mathcal{S}_\pi^\wedge)_*^{\geqslant 1} \hookrightarrow \mathcal{G}\mathrm{rp}(\mathcal{S}_\pi^\wedge)$.*

Warning E.5.5.5. In §E.5, we will show that the functor $\Omega : (\mathcal{S}_\pi^\wedge)_*^{\geqslant 1} \to \mathcal{G}\mathrm{rp}(\mathcal{S}_\pi^\wedge)$ is an equivalence of ∞-categories: that is, every group object in the ∞-category of profinite spaces can be identified with the loop space of a profinite space (Theorem E.5.0.4). However, the fully faithful embedding $\mathcal{E} \hookrightarrow \mathcal{G}\mathrm{pd}(\mathcal{S}_\pi^\wedge)$ of Proposition E.5.5.3 is not essentially surjective. To see this, let X be an arbitrary compact Hausdorff space, and let U be a Stone space equipped with a surjective map $f : U \to X$. Then the Čech nerve $U_\bullet$ of f is a groupoid object in the category of Stone spaces, which we can view as a groupoid object in the ∞-category of discrete objects of $\mathcal{S}_\pi^\wedge$. However, this groupoid is not effective unless X is also a Stone space.

Proof of Proposition E.5.5.3. In what follows, we will abuse notation by identifying $\mathcal{S}_\pi$ with its essential image in $\mathcal{S}_\pi^\wedge$. Let $f : U \to X$ be a map of profinite spaces which induces a surjection $\pi_0 U \to \pi_0 X$, and let $U_\bullet$ denote the Čech nerve of U. We wish to prove that the canonical map $|U_\bullet| \to X$ is an equivalence in the ∞-category of profinite spaces. Equivalently, we wish to show that the canonical map

$$\phi : \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(X, T) \to \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(|U_\bullet|, T) \simeq \varprojlim \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(U_\bullet, T)$$

is a homotopy equivalence.

Using Proposition HTT.5.3.5.15 , we can write f as a limit of maps $\{f_\alpha : U_\alpha \to X_\alpha\}_{\alpha \in A}$ indexed by a filtered partially ordered set A, where each U_α and X_α is π-finite. For each index α, let $X^0_\alpha \subseteq X_\alpha$ denote the union of those connected components of X_α which contain vertices in the image of f_α. We first claim that the canonical map $\varprojlim X^0_\alpha \to \varprojlim X_\alpha$ is an equivalence in $\mathcal{S}^\wedge_\pi$. Note that we have a pullback diagram

$$\begin{array}{ccc} \varprojlim X^0_\alpha & \longrightarrow & \varprojlim X_\alpha \\ \downarrow & & \downarrow \\ \varprojlim \pi_0 X^0_\alpha & \xrightarrow{\theta} & \varprojlim \pi_0 X_\alpha. \end{array}$$

It will therefore suffice to show that θ is an equivalence in $\mathcal{S}^\wedge_\pi$. According to Corollary E.1.4.3, it will suffice to show that θ induces a bijection on the underlying sets. It is clear that θ is injective (since it is an inverse limit of injections). The surjectivity follows from our assumption that the composite map

$$\pi_0 U \simeq \varprojlim \pi_0 U_\alpha \to \varprojlim \pi_0 X^0_\alpha \to \varprojlim \pi_0 X_\alpha \simeq \pi_0 X$$

is surjective. We may therefore replace each X_α with X^0_α, and thereby reduce to the case where each of the maps f_α is surjective on connected components.

For each index $\alpha \in A$, let $U_{\alpha\bullet}$ denote the Čech nerve of f_α, so that $U_\bullet \simeq \varprojlim_\alpha U_{\alpha\bullet}$. It follows that we can identify ϕ with the composition

$$\varinjlim_{\alpha \in A} \mathrm{Map}_{\mathcal{S}_\pi}(X_\alpha, T) \xrightarrow{\phi'} \varinjlim_{\alpha \in A} \varprojlim_{[n] \in \mathbf{\Delta}} \mathrm{Map}_{\mathcal{S}_\pi}(U_{\alpha n}, T) \xrightarrow{\phi''} \varprojlim_{[n] \in \mathbf{\Delta}} \varinjlim_{\alpha \in A} \mathrm{Map}_{\mathcal{S}_\pi}(U_{\alpha n}, T).$$

Since each f_α is an effective epimorphism, the canonical maps $|U_{\alpha\bullet}| \to X_\alpha$ are homotopy equivalences, so that ϕ' is a homotopy equivalence. We will complete the proof by showing that ϕ'' is a homotopy equivalence. To this end, we note that for every integer $m \geqslant 0$, the map ϕ'' fits into a commutative diagram

$$\begin{array}{ccc} \varinjlim_{\alpha \in A} \varprojlim_{[n] \in \mathbf{\Delta}} \mathrm{Map}_{\mathcal{S}_\pi}(U_{\alpha n}, T) & \longrightarrow & \varprojlim_{[n] \in \mathbf{\Delta}} \varinjlim_{\alpha \in A} \mathrm{Map}_{\mathcal{S}_\pi}(U_{\alpha n}, T) \\ \downarrow & & \downarrow \\ \varinjlim_{\alpha \in A} \varprojlim_{[n] \in \mathbf{\Delta}_{s, \leqslant m}} \mathrm{Map}_{\mathcal{S}_\pi}(U_{\alpha n}, T) & \xrightarrow{\psi_m} & \varprojlim_{[n] \in \mathbf{\Delta}_{s, \leqslant m}} \varinjlim_{\alpha \in A} \mathrm{Map}_{\mathcal{S}_\pi}(U_{\alpha n}, T); \end{array}$$

here $\mathbf{\Delta}_{s, \leqslant m}$ denotes the subcategory of $\mathbf{\Delta}$ whose objects have the form $[n]$ for $n \leqslant m$ and whose morphisms are required to be injective. Since the simplicial sets $\mathrm{N}(\mathbf{\Delta}_{\leqslant m})$ are finite and A is filtered, the maps ψ_m are homotopy equivalences. The space T is π-finite and therefore truncated. It follows that the vertical maps are homotopy equivalences for $m \gg 0$ (see the proof of Lemma HA.1.3.3.10), so that ϕ'' is a homotopy equivalence as desired. $\square$

E.5.6 The Proof of Theorem E.5.0.4

We now explain how to deduce Theorem E.5.0.4 from Proposition E.5.4.5. Let G be a group object of $\mathcal{S}_\pi^\wedge$; we wish to show that G belongs to the essential image of the functor $\Omega : (\mathcal{S}_\pi^\wedge)_*^{\geqslant 1} \to \mathcal{G}\mathrm{rp}(\mathcal{S}_\pi^\wedge)$. Note that $(\mathcal{S}_\pi^\wedge)_*^{\geqslant 1}$ is closed under filtered limits in $(\mathcal{S}_\pi^\wedge)_*$ (Proposition E.5.2.2). Since the functor Ω is fully faithful (Corollary E.5.5.4) and commutes with filtered limits, it follows that the essential image of Ω is closed under filtered limits in $\mathcal{G}\mathrm{rp}(\mathcal{S}_\pi^\wedge)$. Writing G as the inverse limit of its Postnikov towe

$$\cdots \to \tau_{\leqslant 2} G \to \tau_{\leqslant 1} G \to \tau_{\leqslant 0} G,$$

we can reduce to the case where G is n-truncated for some integer n.

We now proceed by induction on n. In the case where $n = 0$, we can identify G with a group object in the ordinary category of profinite sets. Using Proposition E.5.1.3, we deduce that G can be written as a filtered limit of group objects in the category of finite sets. Since the essential image of Ω is closed under filtered limits, we can assume that G is a finite group. Then the classifying space BG is π-finite, and we have $G \simeq \Omega(BG)$.

Suppose now that $n \geqslant 1$, and assume that $\tau_{\leqslant n-1} G$ belongs to the essential image of Ω. Note that $\mathrm{Mat}(G)$ can be regarded as a group object of the ∞-category of spaces. It follows that for each point $x \in G$, the fundamental group $\pi_1(G, x)$ acts trivially on the homotopy groups $\pi_m(G, x)$. In particular, $\mathrm{Mat}(G)$ is strongly n-truncated. Applying Corollary E.5.3.5, we deduce that the canonical map $G \to \tau^s_{\leqslant n} G$ is an equivalence.

Let $U : \mathrm{Fam}_{\mathrm{fin}} \to \mathcal{S}_\pi$ be the functor given by $U(\phi : A \to S) = K(\phi, n+1)$ (see Notation E.5.4.3). Then U admits an essentially unique extension to a functor $\widehat{U} : \mathrm{Pro}(\mathrm{Fam}_{\mathrm{fin}}) \to \mathcal{S}_\pi^\wedge$ which commutes with filtered limits.

Let $\mathcal{C}$ denote the full subcategory of $\mathcal{S}$ spanned by those π-finite spaces which are strongly n-truncated. For each object $X \in \mathcal{C}$, define $\pi_n X \in \mathrm{Fam}_{\mathrm{fin}}$ as in Example E.5.4.4. The construction $X \mapsto \pi_n X$ admits an essentially unique extension to a functor $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathrm{Fam}_{\mathrm{fin}})$ which commutes with filtered limits. We will denote this functor also by $X \mapsto \pi_n X$.

For each $X \in \mathcal{C}$, let σ_X denote the pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & \pi_0 X \\ \downarrow & & \downarrow \\ \tau_{\leqslant n-1} X & \longrightarrow & K(\pi_n X, n+1) \end{array}$$

appearing in Example E.5.4.4. Then the construction $X \mapsto \sigma_X$ admits an essentially unique extension to a functor $\widehat{\sigma} : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{S}_\pi^\wedge)$ which commutes with filtered limits. In particular, it carries group objects of $\mathrm{Pro}(\mathcal{C})$ to pullback diagrams $\mathcal{G}\mathrm{rp}(\mathcal{S}_\pi^\wedge)$. Applying this functor to G, we obtain a pullback square

$$\begin{array}{ccc} G & \longrightarrow & \tau_{\leqslant 0}G \\ \downarrow & & \downarrow f \\ \tau_{\leqslant n-1}G & \xrightarrow{f'} & \widehat{U}(\pi_n G). \end{array}$$

Since the functor $\pi_n : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathrm{Fam}_{\mathrm{fin}})$ commutes with finite products, we can regard $\pi_n G$ as a group object of $\mathrm{Pro}(\mathrm{Fam}_{\mathrm{fin}})$. Applying Proposition E.5.4.5, we can write $\pi_n G$ as a filtered limit of objects $\phi_\alpha \in \mathcal{G}\mathrm{rp}(\mathrm{Fam}_{\mathrm{fin}})$. Then each $\widehat{U}(\phi_\alpha) = K(\phi_\alpha, n+1)$ is a group object of $\mathcal{S}_\pi$, and therefore belongs to the essential image of Ω. It follows that $\widehat{U}(\pi_n G)$ belongs to the essential image of Ω. Write $\widehat{U}(\pi_n G) \simeq \Omega(X)$, where X is a pointed connected profinite space. By the inductive hypothesis, we can also write

$$\tau_{\leqslant 0}G \simeq \Omega(Y) \qquad \tau_{\leqslant n-1}G \simeq \Omega(Z)$$

for some pointed profinite spaces Y and Z. Using Proposition E.5.5.3, we see that f and f' are induced by pointed maps $\overline{f} : Y \to X$ and $\overline{f}' : Z \to X$. It follows from Proposition E.5.2.2 that the map f induces an isomorphism on connected components, so that $\overline{f}$ is an isomorphism on fundamental groups. Consequently, the fiber product $Y \times_X Z$ is also connected. It follows that $G \simeq \Omega(Y \times_X Z)$ belongs to the essential image of Ω, as desired.

E.5.7 The Proof of Proposition E.5.4.5

The proof of Proposition E.5.4.5 will require some preliminaries.

Definition E.5.7.1. A *topological family of abelian groups* is a family of abelian groups $\phi : A \to S$, together with topologies on the sets A and S, for which the map ϕ is continuous, the zero section $S \to A$ is continuous, and the addition on each A_s is given by a continuous map $a : A \times_S A \to A$. We let $\mathrm{Fam}_{\mathcal{T}\mathrm{op}}$ denote the category whose objects are topological families of abelian groups, and whose morphisms are maps of families of abelian groups

$$\begin{array}{ccc} A & \xrightarrow{\phi} & S \\ \downarrow & & \downarrow \\ A' & \xrightarrow{\phi'} & S' \end{array}$$

where the vertical maps are continuous.

Remark E.5.7.2. We can obtain a variant of Definition E.5.7.1 by requiring also that the inverse maps on the groups A_s be given by a continuous map $A \to A$ (so that each A_s has the structure of a topological group). This additional condition will automatically be satisfied in cases of interest to us here (for example, if A and S are Stone spaces, or if ϕ is a group object of $\mathrm{Fam}_{\mathcal{T}\mathrm{op}}$).

If $\phi : A \to S$ is a finite family of abelian groups, we will view ϕ as a topological family of abelian groups where A and S are equipped with the discrete topology. This construction determines a fully faithful embedding $\iota : \mathrm{Fam}_{\mathrm{fin}} \hookrightarrow \mathrm{Fam}_{\mathcal{T}\mathrm{op}}$. Since the category $\mathrm{Fam}_{\mathcal{T}\mathrm{op}}$ admits filtered limits, ι admits an essentially unique extension to a functor $\widehat{\iota} : \mathrm{Pro}(\mathrm{Fam}_{\mathrm{fin}}) \to \mathrm{Fam}_{\mathcal{T}\mathrm{op}}$ which commutes with filtered limits.

Lemma E.5.7.3. *The functor* $\widehat{\iota} : \mathrm{Pro}(\mathrm{Fam}_{\mathrm{fin}}) \to \mathrm{Fam}_{\mathcal{T}\mathrm{op}}$ *is fully faithful.*

Proof. Suppose we are given a pair of objects $\phi, \psi \in \mathrm{Pro}(\mathrm{Fam}_{\mathrm{fin}})$, which are given as limits of finite families of abelian groups

$$\{\phi_\alpha : A_\alpha \to S_\alpha\} \qquad\qquad \{\psi_\beta : B_\beta \to T_\beta\}$$

indexed by filtered partially ordered sets. We wish to prove that the canonical map $\theta : \mathrm{Hom}_{\mathrm{Pro}(\mathrm{Fam}_{\mathrm{fin}})}(\phi, \psi) \to \mathrm{Hom}_{\mathrm{Fam}_{\mathcal{T}\mathrm{op}}}(\widehat{\iota}(\phi), \widehat{\iota}(\psi))$ is a bijection. Unwinding the definitions, we can write θ as an inverse limit of maps $\theta_\beta : \mathrm{Hom}_{\mathrm{Pro}(\mathrm{Fam}_{\mathrm{fin}})}(\phi, \psi_\beta) \to \mathrm{Hom}_{\mathrm{Fam}_{\mathcal{T}\mathrm{op}}}(\widehat{\iota}(\phi), \iota(\psi_\beta))$. Replacing ψ by ψ_β, we may reduce to the case where $\psi \in \mathrm{Fam}_{\mathrm{fin}}$. In this case, θ can be identified with the canonical map $\varinjlim_\alpha \mathrm{Hom}_{\mathrm{Fam}}(\phi_\alpha, \psi) \to \mathrm{Hom}_{\mathrm{Fam}_{\mathcal{T}\mathrm{op}}}(\varprojlim \phi_\alpha, \psi)$. We first show that θ is injective. Suppose we are given an index α and a pair of maps $u, v : \phi_\alpha \to \psi$ which induce the same map from $\varprojlim \phi_\alpha$ to ψ. Let $E \subseteq A_\alpha$ be the subset consisting of elements on which u and v do not agree. Then E is a finite set which is disjoint from the image of the map $\varprojlim_{\alpha' \geqslant \alpha} A_{\alpha'} \to A_\alpha$. Using Proposition E.1.1.1, we see that E is disjoint from the image of the map $A_{\alpha'} \to A_\alpha$ for some $\alpha' \geqslant \alpha$. Then u and v have the same image in $\mathrm{Hom}_{\mathrm{Fam}}(\phi_{\alpha'}, \psi)$, hence the same image in $\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Fam})}(\phi, \psi)$.

Set $A = \varprojlim A_\alpha$ and $S = \varprojlim S_\alpha$, so that $\widehat{\iota}(\phi)$ is a topological family of abelian groups $A \to S$. Let $\psi : B \to T$ be a finite family of abelian groups, and let $u \in \mathrm{Hom}_{\mathrm{Fam}_{\mathcal{T}\mathrm{op}}}(\widehat{\iota}(\phi), \iota(\psi))$, given by a diagram of continuous maps

$$\begin{array}{ccc} A & \longrightarrow & S \\ \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u''} \\ B & \longrightarrow & T. \end{array}$$

Applying Theorem E.1.4.1, we deduce that there exists an index α such that u' and u'' factor through maps $u'_\alpha : A_\alpha \to B$ and $u''_\alpha : S_\alpha \to T$. For each $\alpha' \geqslant \alpha$, let $u'_{\alpha'} : A_{\alpha'} \to B$ and $u''_{\alpha'} : S_{\alpha'} \to T$ denote the compositions of u'_α and u''_α with the projection maps $A_{\alpha'} \to A_\alpha$ and $S_{\alpha'} \to S_\alpha$.

Enlarging α if necessary, we may suppose that the diagram

$$\begin{array}{ccc} A_\alpha & \longrightarrow & S \\ \downarrow{\scriptstyle u'_\alpha} & & \downarrow{\scriptstyle u''_\alpha} \\ B & \longrightarrow & T. \end{array}$$

commutes. Let $E \subseteq A_\alpha \times_{S_\alpha} A_\alpha$ be the set consisting of those pairs (a, b) such that $u'_\alpha(a+b) \neq u'_\alpha(a) + u'_\alpha(b)$. Since u is a map of families of abelian groups, E does not intersect the image of the projection map $A \times_S A \to A_\alpha \times_{S_\alpha} A_\alpha$. Using Proposition E.1.1.1, we conclude that there exists $\alpha' \geqslant \alpha$ such that E does not intersect the image of the projection map $A_{\alpha'} \times_{S_{\alpha'}} A_{\alpha'} \to A_\alpha \times_{S_\alpha} A_\alpha$. Replacing α by α', we may suppose that $E = \emptyset$. In this case, each of the maps $A_\alpha \times_{S_\alpha} \{s\} \to B \times_T \{u'_\alpha(s)\}$ is a function between finite abelian groups which preserves addition, and therefore a group homomorphism. It follows that (u'_α, u''_α) is a morphism from ϕ_α to ψ in the category Fam, so that u belongs to the image of θ. □

Example E.5.7.4. Let G be a group, and let M be an abelian group equipped with a (right) action of G, given by $(x \in M, g \in G) \mapsto x^g$. Let $G \ltimes M$ denote the semidirect product of G and A: that is, the product $G \times M$, equipped with the group structure given by $(g, x)(g', x') = (gg', x^{g'} + x')$. Each fiber of the projection map $\phi : G \ltimes M \to G$ can be identified with the abelian group M, so we can view ϕ as a family of abelian groups. Moreover, the multiplication on the groups $G \ltimes M$ and G exhibit ϕ as a group object in the category Fam.

Remark E.5.7.5. Suppose that $\phi : A \to S$ is a group object in the category Fam. In particular, we can regard A and S as groups, so that the map ϕ and the zero section $e : S \to A$ are group homomorphisms. For each $s \in S$, let A_s denote the abelian group $\phi^{-1}\{s\}$. Then the multiplication on ϕ determines group homomorphisms $A_s \times A_{s'} \to A_{ss'}$. Let 1 denote the identity element of S. Then $e(1)$ is the identity element of A, and also the unit element of A_1. Since the multiplication on A induces a group homomorphism $A_1 \times A_1 \to A_1$, it follows that the inclusion $A_1 \hookrightarrow A$ is a group homomorphism: that is, we can identify A_1 with the kernel $\ker(\phi)$ not only as a set, but as a group. We have an exact sequence

$$0 \to A_1 \to A \xrightarrow{\phi} S \to 0.$$

Since A_1 is abelian, the group S acts on A_1 by conjugation. Moreover, this sequence is split by the map $e : S \to A$, so we obtain an isomorphism of groups $S \ltimes A_1 \simeq A$, given by $(s, x) \mapsto e(s)x$. We can summarize the situation as follows: every group object of Fam has the form $G \ltimes M \to G$, where G is a group and M is an abelian group with an action of G, as in Example E.5.7.4.

If ϕ is a group object in the category $\mathrm{Fam}_{\mathcal{T}\mathrm{op}}$ of topological families of abelian groups, the same argument shows that we can write ϕ as the projection map $\phi : G \ltimes M \to G$, where G is a topological group and M is a topological abelian group equipped with a continuous action of G.

Lemma E.5.7.6. *Let M be a profinite group equipped with a continuous action of a profinite group G. Then the collection of open G-invariant normal subgroups of M forms a neighborhood basis for the identity element of M.*

Proof. Let U be an open subset of M containing the identity element. We wish to show that U contains an open G-invariant normal subgroup of M. It follows from Proposition E.5.1.3 that U contains an open normal subgroup of M. Shrinking U if necessary, we may suppose that U is an open normal subgroup.

For each $g \in G$, let U^g denote the image of U under the group homomorphism from M to itself determined by g, and set $V = \bigcap_{g \in G} U^g$. Then V is a G-invariant normal subgroup of M which is contained in U. We will complete the proof by showing that V is open. This is clear, since the complement of V is the image of a continuous map $G \times (M - U) \to M$ whose domain is compact. □

Proof of Proposition E.5.4.5. Suppose that ϕ is a group object of $\mathrm{Pro}(\mathrm{Fam}_{\mathrm{fin}})$. Let $\widehat{\iota} : \mathrm{Pro}(\mathrm{Fam}_{\mathrm{fin}}) \to \mathrm{Fam}_{\mathcal{T}\mathrm{op}}$ be the fully faithful embedding of Lemma E.5.7.3. Then $\widehat{\iota}(\phi)$ is a group object of $\mathrm{Fam}_{\mathcal{T}\mathrm{op}}$. According to Remark E.5.7.5, $\widehat{\iota}(\phi)$ has the form $G \ltimes M \to M$, where G is a topological group and M is a topological abelian group equipped with an action of G. Note that each of these groups is profinite.

Let A denote the set of all pairs (G_0, M_0), where M_0 is an open G-invariant subgroup of M and G_0 is an open normal subgroup of G which acts trivially on the quotient M/M_0. We regard A as a partially ordered set by reverse inclusion. Since the set of such pairs (G_0, M_0) is closed under finite intersections, A is a filtered partially ordered set. For each pair $(G_0, M_0) \in A$, let ϕ_{G_0, M_0} denote the projection map

$$G/G_0 \ltimes M/M_0 \to M/M_0,$$

regarded as a group object of $\mathrm{Fam}_{\mathrm{fin}}$ (see Example E.5.7.4). We claim that ϕ is represented by the inverse limit of the system $\{\phi_{(G_0,M_0)}\}_{(G_0,M_0) \in A}$. To prove this, it will suffice (by virtue of Lemma E.5.7.3) to show that the map

$$\widehat{\iota}(\phi) \to \varprojlim_{(G_0,M_0) \in A} \phi_{(G_0,M_0)}$$

is an isomorphism of topological families of abelian groups. Equivalently, we must show that the maps

$$\alpha : G \to \varprojlim_{(G_0,M_0) \in A} G/G_0 \qquad \beta : M \to \varprojlim_{(G_0,M_0) \in A} M/M_0$$

are isomorphisms of profinite groups. By construction, these maps have dense image and are therefore surjective. We will complete the proof by showing that α and β are injective.

We first show that α is injective. Let g be a nontrivial element of G. It follows from Proposition E.5.1.3 that there exists an open normal subgroup $G_0 \subseteq G$ which does not contain g. Then (G_0, M) is an element of A, and the image of g in G/G_0 is nontrivial. It follows that $\alpha(g)$ is nontrivial.

We now show that β is injective. Let x be a nonzero element of M. It follows from Lemma E.5.7.6 that there exists a G-invariant open subgroup $M_0 \subseteq M$ which does not contain x. Then G acts continuously on the finite set M/M_0. Let $G_0 \subseteq G$ be the kernel of this action. Then (G_0, M_0) is an element of A, and x has nonzero image in M/M_0. It follows that $\beta(x)$ is nonzero. □

E.6 Universality of Colimits

Recall that an ∞-category $\mathcal{X}$ is an ∞-*topos* if it satisfies the ∞-categorical version of Giraud's axioms (see Theorem HTT.6.1.0.6):

(*i*) The ∞-category $\mathcal{X}$ is presentable.

(*ii*) Coproducts in $\mathcal{X}$ are disjoint.

(*iii*) Every groupoid object of $\mathcal{X}$ is effective.

(*iv*) Colimits in $\mathcal{X}$ are universal.

Roughly speaking, these axioms assert that the homotopy theory of $\mathcal{X}$ behaves much like the classical homotopy theory of spaces. In this section, we will investigate the extent to which these axioms are satisfied by the ∞-category $\mathcal{S}_\pi^\wedge$ of profinite spaces. We can summarize the situation as follows:

(*i'*) The ∞-category $\mathcal{S}_\pi^\wedge$ of profinite spaces is not presentable (in fact, it has no small objects other than the initial object). However, the opposite ∞-category $(\mathcal{S}_\pi^\wedge)^{\mathrm{op}}$ is presentable. In particular, the ∞-category $\mathcal{S}_\pi^\wedge$ admits all small limits and colimits.

(*ii'*) Coproducts are disjoint in the ∞-category $\mathcal{S}_\pi^\wedge$ of profinite spaces. To see this, suppose we are given a pair of profinite spaces X and Y, represented by filtered diagrams of π-finite spaces $\{X_\alpha\}_{\alpha \in A}, \{Y_\beta\}_{\beta \in B}$. Then the diagram of profinite spaces σ :

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \amalg Y \end{array}$$

can be written as a filtered limit of diagrams $\sigma_{\alpha,\beta}$:

$$\begin{array}{ccc} \emptyset & \longrightarrow & X_\alpha \\ \downarrow & & \downarrow \\ Y_\beta & \longrightarrow & X_\alpha \amalg Y_\beta. \end{array}$$

Since coproducts are disjoint in the ∞-category $\mathcal{S}$ of spaces, each of the diagrams $\sigma_{\alpha,\beta}$ is a pullback square. It follows immediately that σ is also a pullback square.

(iii') Not every groupoid object of $\mathcal{S}_\pi^\wedge$ is effective (see Warning E.5.5.5). Nevertheless, in §E.5, we proved that every group object of $\mathcal{S}_\pi^\wedge$ can be recovered as the loop space of its classifying space (Theorem E.5.0.4). In other words, every group object G of $\mathcal{S}_\pi^\wedge$ is effective when viewed as a groupoid object of $\mathcal{S}_\pi^\wedge$.

(iv') Colimits are generally not universal in the ∞-category $\mathcal{S}_\pi^\wedge$ of profinite spaces (see Warning E.6.0.9). Nevertheless, we will prove in this section that colimits indexed by finite diagrams are universal, and that geometric realizations are universal (Theorem E.6.3.1).

Before discussing (iv') in more detail, let us consider another fundamental feature of classical homotopy theory. If $\mathcal{X}$ is an ∞-category satisfying axioms (i) and (iv), then conditions (ii) and (iii) are equivalent to the following:

(v) The construction $(X \in \mathcal{X}) \mapsto \mathcal{X}_{/X}$ determines a functor $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which commutes with small limits.

In the particular case where $\mathcal{X} = \mathcal{S}$ is the ∞-category of spaces, axiom (v) supplies an equivalence of ∞-categories $\mathrm{Fun}(X, \mathcal{S}) \simeq \mathcal{S}_{/X}$ for any space X. We can describe the inverse equivalence informally as follows: it assigns to a map of spaces $Y \to X$ the functor $X \to \mathcal{S}$ which carries a point $x \in X$ to the homotopy fiber $Y_x = Y \times_X \{x\}$. Our first goal will be to show that an analogous statement holds in the profinite setting, provided that the space X is π-finite:

Theorem E.6.0.7. *Let X be a π-finite space, and let $\varinjlim : \mathrm{Fun}(X, \mathcal{S}_\pi^\wedge) \to \mathcal{S}_\pi^\wedge$ be the colimit functor (a left adjoint to the diagonal embedding $\mathcal{S}_\pi^\wedge \to \mathrm{Fun}(X, \mathcal{S}_\pi^\wedge)$). Then:*

(1) *Let e_0 denote a final object of $\mathrm{Fun}(X, \mathcal{S}_\pi^\wedge)$ (that is, the constant functor from X to $\mathcal{S}_\pi^\wedge$ taking the value $* \in \mathcal{S}_\pi^\wedge$). Then there is a canonical equivalence $\varinjlim(e_0) \simeq X$ in $\mathcal{S}_\pi^\wedge$.*

(2) *The functor $\varinjlim$ induces an equivalence of ∞-categories*

$$F : \mathrm{Fun}(X, \mathcal{S}_\pi^\wedge) \simeq \mathrm{Fun}(X, \mathcal{S}_\pi^\wedge)_{/e_0} \to (\mathcal{S}_\pi^\wedge)_{/X}.$$

Corollary E.6.0.8. *Let $f : X \to Y$ be a map of π-finite spaces, and let $f^* : (\mathcal{S}_\pi^\wedge)_{/Y} \to (\mathcal{S}_\pi^\wedge)_{/X}$ be the functor given by $f^*Z = X \times_Y Z$. Then f^* preserves small colimits.*

Proof. Using Theorem E.6.0.7, we can identify f^* with the functor $\mathrm{Fun}(Y, \mathcal{S}_\pi^\wedge) \to \mathrm{Fun}(X, \mathcal{S}_\pi^\wedge)$ given by composition with f. □

Warning E.6.0.9. The conclusion of Corollary E.6.0.8 generally fails if we allow f to be a map of profinite spaces, rather than a map of π-finite spaces. For example, suppose we are given a pair of set A and B, which we regard as discrete spaces. Let $X = A^{\wedge}_{\pi}$ be the profinite completion of A: then X is a 0-truncated profinite space, and set $Y = *$. Then the canonical map $\amalg_{b \in B}(X \times \{b\}) \rightarrow X \times \amalg_{b \in B}\{b\}$ is generally not an equivalence of profinite spaces. Using Remark E.5.2.6, we can identify the left hand side with the Stone-Čech compactification of $A \times B$ and the right hand side with product of the Stone-Čech compactifications of A and B. These are not isomorphic unless either A or B is finite.

E.6.1 Diagrams Indexed by π-Finite Spaces

Let X be a Kan complex. We will say that X is *almost π-finite* if, for each vertex $x \in X$ and each $n \geqslant 0$, the set $\pi_n(X, x)$ is finite (by virtue of Lemma E.1.6.5, this is equivalent to the requirement that there exists a homotopy equivalence of Kan complexes $X \simeq X'$, where X' has finitely many simplices of each dimension). We now summarize some elementary facts about diagrams indexed by almost π-finite Kan complexes which will be needed in the proof of Theorem E.6.0.7.

Lemma E.6.1.1. *Let X be a Kan complex which is almost π-finite, let $\mathcal{C}$ be an essentially small ∞-category which admits finite limits, and suppose that every object of $\mathcal{C}$ is truncated. Then:*

(1) *Every functor $X \rightarrow \mathcal{C}$ admits a limit in $\mathcal{C}$.*

(2) *The Yoneda embedding $j : \mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ preserves X-indexed limits.*

Proof. Fix a diagram $q : X \rightarrow \mathcal{C}$. Since X has only finitely many connected components, it follows that there exists an integer n such that $q(x)$ is n-truncated, for each point $x \in X$. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ denote the full subcategory spanned by the n-truncated objects, so that q factors through $\mathcal{C}_0$. Replacing $\mathcal{C}$ by $\mathcal{C}_0$, we may reduce to the case where $\mathcal{C}$ is equivalent to an $(n+1)$-category.

Since $\mathcal{C}$ admits finite limits, $\mathrm{Pro}(\mathcal{C})$ admits all small limits. Consequently, the diagram $j \circ q$ admits a limit $C \in \mathrm{Pro}(\mathcal{C})$. To complete the proof, it will suffice to show that C belongs to the essential image of j. Without loss of generality we can assume that X has only finitely many simplices of each dimension. Let K be the $(n+1)$-skeleton of X. Since $\mathrm{Pro}(\mathcal{C})$ is equivalent to an $(n+1)$-category, the restriction map $\varprojlim(j \circ q) \rightarrow \varprojlim(j \circ q|_K)$ is an equivalence in $\mathrm{Pro}(\mathcal{C})$. It will therefore suffice to show that $\varprojlim(j \circ q|_K)$ belongs to the essential image of j, which follows because j preserves finite limits (Proposition HTT.5.3.5.14). □

Lemma E.6.1.2. *Let $\mathcal{C}$ be an essentially small ∞-category which admits finite limits and let X be a Kan complex which is almost π-finite. If every object of $\mathcal{C}$ is truncated, then the canonical map $\theta : \mathrm{Pro}(\mathrm{Fun}(X, \mathcal{C})) \to \mathrm{Fun}(X, \mathrm{Pro}(\mathcal{C}))$ is fully faithful.*

Proof. Suppose we are given objects $\mu, \nu \in \mathrm{Pro}(\mathrm{Fun}(X, \mathcal{C}))$, represented by filtered diagrams

$$\{\mu_\alpha \in \mathrm{Fun}(X, \mathcal{C})\}_{\alpha \in A} \qquad \{\nu_\beta \in \mathrm{Fun}(X, \mathcal{C})\}_{\beta \in B}.$$

We wish to show that the canonical map

$$\theta_{\mu,\nu} : \mathrm{Map}_{\mathrm{Pro}(\mathrm{Fun}(X,\mathcal{C}))}(\mu, \nu) \to \mathrm{Map}_{\mathrm{Fun}(X,\mathrm{Pro}(\mathcal{C}))}(\theta(\mu), \theta(\nu))$$

is a homotopy equivalence. Unwinding the definitions, we can identify $\theta_{\mu,\nu}$ with the map

$$\varprojlim_{\beta \in B} \varinjlim_{\alpha \in A} \varprojlim_{x \in X} \mathrm{Map}_{\mathcal{C}}(\mu_\alpha(x), \mu_\beta(x)) \to \varprojlim_{x \in X} \varprojlim_{\beta \in B} \varinjlim_{\alpha \in A} \mathrm{Map}_{\mathcal{C}}(\mu_\alpha(x), \mu_\beta(x)).$$

Without loss of generality, we may suppose that X has only finitely many simplices of each dimension. For each pair of indices $\alpha \in A$, $\beta \in B$, let $u_{\alpha,\beta} : X \to \mathcal{S}$ denote the diagram given by $v \mapsto \mathrm{Map}_{\mathcal{C}}(\mu_\alpha(x), \mu_\beta(x))$. We are then reduced to proving that for each index $\beta \in B$, the canonical map

$$\varinjlim_{\alpha \in A} \varprojlim_{x \in X} (u_{\alpha,\beta}(x)) \to \varprojlim_{x \in X} (\varinjlim_{\alpha \in A} u_{\alpha,\beta}(x))$$

is a homotopy equivalence. Since every object of $\mathcal{C}$ is truncated, we can choose an integer n such that $\mu_\beta(x)$ is n-truncated for every vertex $x \in X$. In this case, each of the diagrams $u_{\alpha,\beta}$ takes values in the full subcategory $\tau_{\leq n} \mathcal{S}$. Let K be the $(n+1)$-skeleton of X, so that we have a commutative diagram

$$\begin{array}{ccc} \varinjlim_{\alpha \in A} \varprojlim (u_{\alpha,\beta}) & \longrightarrow & \varprojlim (\varinjlim_{\alpha \in A} u_{\alpha,\beta}) \\ \downarrow & & \downarrow \\ \varinjlim_{\alpha \in A} \varprojlim (u_{\alpha,\beta}|_K) & \longrightarrow & \varprojlim (\varinjlim_{\alpha \in A} u_{\alpha,\beta}|_K) \end{array}$$

where the vertical maps are homotopy equivalences. It will therefore suffice to show that the bottom horizontal map is a homotopy equivalence, which follows because filtered colimits in the ∞-category $\mathcal{S}$ are left exact. □

Lemma E.6.1.3. *Let X be a Kan complex which is almost π-finite. Then the canonical map $\theta : \mathrm{Pro}(\mathrm{Fun}(X, \mathcal{S}_\pi)) \to \mathrm{Fun}(X, \mathcal{S}_\pi^\wedge)$ is an equivalence of ∞-categories.*

Proof. It follows from Lemma E.6.1.2 that θ is fully faithful. Let $\mathcal{E} \subseteq \mathrm{Fun}(X, \mathcal{S}_\pi^\wedge)$ denote the essential image of θ. Since θ preserves small limits, the full subcategory $\mathcal{E}$ is closed under small limits in $\mathrm{Fun}(X, \mathcal{S}_\pi^\wedge)$. Fix a functor $\lambda : X \to \mathcal{S}_\pi^\wedge$; we wish to prove that $\lambda \in \mathcal{E}$. For each integer n, let $\lambda_n : X \to \mathcal{S}_\pi^\wedge$ be the functor given by $\lambda_n(x) = \tau_{\leqslant n}\lambda(x)$. Then λ is a limit of the diagram $\{\lambda_n\}_{n \geqslant 0}$. It will therefore suffice to show that each λ_n belongs to $\mathcal{E}$. Replacing λ by λ_n, we may suppose that λ takes values in the full subcategory $\tau_{\leqslant n}\mathcal{S}_\pi^\wedge$.

Without loss of generality, we can assume that X has only finitely many simplices of each dimension. Let K be the $(n+1)$-skeleton of X. Using Variant HTT.4.2.3.16 , we can choose a right cofinal map $\mathrm{N}(A) \to K$, where A is a finite partially ordered set. Let f denote the composite map $\mathrm{N}(A) \to K \subseteq X$. Let $f^* : \mathrm{Fun}(X, \mathcal{S}_\pi^\wedge) \to \mathrm{Fun}(\mathrm{N}(A), \mathcal{S}_\pi^\wedge)$ denote the functor given by composition with f, and let $f_* : \mathrm{Fun}(\mathrm{N}(A), \mathcal{S}_\pi^\wedge) \to \mathrm{Fun}(X, \mathcal{S}_\pi^\wedge)$ denote a right adjoint to f^*, given by right Kan extension along f. For each point $x \in X$, let $\mathrm{N}(A)_{x/}$ denote the fiber product $\mathrm{N}(A) \times_X X_{x/}$, so that we have

$$(f_* f^* \lambda)(x) = \varprojlim_{a \in \mathrm{N}(A)_{x/}} \lambda(f(a)).$$

Since X is a Kan complex, the diagram

$$\begin{array}{ccc} \mathrm{N}(A)_{x/} & \longrightarrow & X_{x/} \\ \downarrow & & \downarrow \\ \mathrm{N}(A) & \longrightarrow & X \end{array}$$

is a homotopy pullback square with respect to the usual model structure on the category of simplicial sets. Since $\mathrm{N}(A)$ is weakly homotopy equivalent to the $(n+1)$-skeleton of X, it follows that (a fibrant replacement for) $\mathrm{N}(A)_{x/}$ is $(n+1)$-connective. Because λ takes values in n-truncated objects of $\mathcal{S}_\pi^\wedge$, we deduce that the unit map $\lambda \to (f_* f^* \lambda)$ is an equivalence. We may therefore identify λ with $f_*\mu$, where $\mu : \mathrm{N}(A) \to \mathcal{S}_\pi^\wedge$ is a functor taking n-truncated values.

Using Proposition HTT.5.3.5.15 , we can write μ as a filtered limit of functors $\{\mu_\beta : \mathrm{N}(A) \to \mathcal{S}_\pi\}$. It follows that $\lambda \simeq \varprojlim_\beta f_*\mu_\beta$. Since $\mathcal{E}$ is closed under small limits, it will suffice to prove that each of the functors $f_*\mu_\beta$ belongs to $\mathcal{E}$. In fact, we claim that each of the functors $f_*\mu_\beta$ belongs to $\mathrm{Fun}(X, \mathcal{S}_\pi)$. To prove this, it will suffice to show that for each point $x \in X$, the limit $\varprojlim_{a \in \mathrm{N}(A)_{x/}} \mu_\beta(a)$ belongs to the full subcategory $\mathcal{S}_\pi \subseteq \mathcal{S}_\pi^\wedge$. Let $\overline{\mu}_\beta$ denote the restriction of μ_β to $\mathrm{N}(A)_{x/}$, and let ν be a right Kan extension of $\overline{\mu}_\beta$ along the projection $\mathrm{N}(A)_{x/} \to \mathrm{N}(A)$. Since $\mathcal{S}_\pi$ is closed under finite limits in $\mathcal{S}_\pi^\wedge$, it will suffice to show that ν takes values in $\mathcal{S}_\pi$. Unwinding the definitions, we see that for each $a \in A$, $\nu(a)$ is given by the limit of the constant diagram $\{x\} \times_X \{f(a)\} \to \mathcal{S}_\pi^\wedge$ taking the value $\mu_\beta(a)$. This follows from Lemma E.6.1.1, since the path space $\{x\} \times_X \{f(a)\}$ satisfies the requirement of Lemma E.6.1.1. $\square$

E.6.2 The Proof of Theorem E.6.0.7

Let X be a π-finite space and let $e_0 \in \mathrm{Fun}(X, \mathcal{S}_\pi^\wedge)$ be the constant functor taking the value $*$. Since the Yoneda embedding $j : \mathcal{S}_\pi \hookrightarrow \mathcal{S}_\pi^\wedge$ preserves all colimits which exist in $\mathcal{S}_\pi$ (Proposition HTT.5.1.3.2), we have a canonical identification $X \simeq \varinjlim e_0$, which induces a functor

$$F : \mathrm{Fun}(X, \mathcal{S}_\pi^\wedge) \simeq \mathrm{Fun}(X, \mathcal{S}_\pi^\wedge)_{/e_0} \xrightarrow{\varinjlim} (\mathcal{S}_\pi^\wedge)_{/X}.$$

We wish to show that F is an equivalence of ∞-categories. We begin by showing that the functor F preserves filtered limits. Let $\{\mu_\alpha\}_{\alpha \in A}$ be a filtered diagram in $\mathrm{Fun}(X, \mathcal{S}_\pi^\wedge)$; we wish to show that the canonical map $F(\varprojlim_{\alpha \in A} \mu_\alpha) \to \varprojlim_{\alpha \in A} F(\mu_\alpha)$ is an equivalence of profinite spaces. Equivalently, we wish to show that for every π-finite space T, the canonical map

$$\theta : \varinjlim_{\alpha} \varprojlim_{x \in X} \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(\mu_\alpha(x), T) \to \varprojlim_{x \in X} \varinjlim_{\alpha} \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(\mu_\alpha(x), T)$$

is a homotopy equivalence. Using Lemma E.1.6.5, we may assume without loss of generality that X has finitely many simplices of each dimension. Choose an integer n such that T is n-truncated, and let K denote the $(n+1)$-skeleton of X. Then θ can be identified with the upper horizontal map in the diagram

$$\begin{array}{ccc} \varprojlim_{x \in X} \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(\mu_\alpha(x), T) & \longrightarrow & \varprojlim_{x \in X} \varinjlim_{\alpha} \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(\mu_\alpha(x), T) \\ \downarrow & & \downarrow \\ \varprojlim_{x \in K} \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(\mu_\alpha(x), T) & \longrightarrow & \varprojlim_{x \in K} \varinjlim_{\alpha} \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(\mu_\alpha(x), T) \end{array}$$

where the vertical maps are homotopy equivalences. It therefore suffices to show that the bottom horizontal map is a homotopy equivalence, which follows because filtered colimits in $\mathcal{S}$ are left exact.

We next prove that F is fully faithful. Using Lemma E.6.1.3 and Proposition HTT.5.3.5.11 , we are reduced to proving that the restriction $F_0 = F|_{\mathrm{Fun}(X, \mathcal{S}_\pi)}$ is fully faithful, and that F_0 takes values in the full subcategory of $(\mathcal{S}_\pi^\wedge)_{/X}$ spanned by the cocompact objects (that is, the full subcategory spanned by those objects which are compact in the opposite ∞-category $(\mathcal{S}_\pi^\wedge)_{/X}^{\mathrm{op}}$). Let $\mu, \nu : X \to \mathcal{S}_\pi \subseteq \mathcal{S}$ be functors, classifying Kan fibrations $Y_\mu \to X \leftarrow Y_\nu$ with essentially small fibers. Then $F(\mu)$ and $F(\nu)$ can be identified with the objects of $(\mathcal{S}_\pi)_{/X} \subseteq (\mathcal{S}_\pi^\wedge)_{/X}$ given by Y_μ and Y_ν (see Proposition HTT.3.3.4.5). From this description it is easy to see that the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(X, \mathcal{S}_\pi^\wedge)}(\mu, \nu) \to \mathrm{Map}_{(\mathcal{S}_\pi^\wedge)_{/X}}(F(\mu), F(\nu))$$

is a homotopy equivalence (both sides can be identified with the mapping space $\mathrm{Fun}_X(Y_\mu, Y_\nu)$), and that $F(\mu)$ and $F(\nu)$ are cocompact.

Note that the functor F admits a right adjoint G, which carries a map of profinite spaces $Y \to X$ to the functor $G(Y) \in \operatorname{Fun}(X, \mathcal{S}_{\pi}^{\wedge})$ given on objects by the formula $G(Y)(x) = \{x\} \times_X Y$. To complete the proof that F is an equivalence of ∞-categories, it will suffice to show that G is conservative. That is, we must show that if we are given a diagram of profinite spaces

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

which induces an equivalence $\{x\} \times_X Y \to \{x\} \times_X Y'$ for every point $x \in X$, then g is an equivalence. This follows immediately from the characterization of equivalences in $\mathcal{S}_{\pi}^{\wedge}$ supplied by Theorem E.3.1.6.

E.6.3 Universality of Colimits

Using Theorem **??**, we can establish the universality of a large class of colimits in the ∞-category $\mathcal{S}_{\pi}^{\wedge}$ of profinite spaces.

Theorem E.6.3.1. *Let $f : X \to Y$ be a map of profinite spaces, and let $f^* : (\mathcal{S}_{\pi}^{\wedge})_{/Y} \to (\mathcal{S}_{\pi}^{\wedge})_{/X}$ be the functor given by $f^*(Y') = X \times_Y Y'$. Let K be a simplicial set having only finitely many simplices of each dimension. Then the functor f^* preserves K-indexed colimits.*

Corollary E.6.3.2. *Let $f : X \to Y$ be a map of profinite spaces, and let $f^* : (\mathcal{S}_{\pi}^{\wedge})_{/Y} \to (\mathcal{S}_{\pi}^{\wedge})_{/X}$ be the functor given by $f^*(Y') = X \times_Y Y'$. Then f preserves geometric realizations of simplicial objects.*

Corollary E.6.3.3. *Let K be a simplicial set with only finitely many simplices in each degree. Then the Cartesian product functor $\mathcal{S}_{\pi}^{\wedge} \times \mathcal{S}_{\pi}^{\wedge} \to \mathcal{S}_{\pi}^{\wedge}$ preserves K-indexed colimits separately in each variable. In particular, it preserves geometric realizations of simplicial objects.*

The proof of Theorem E.6.3.1 will require some preliminaries.

Lemma E.6.3.4. *Let K be a simplicial set with only finitely many simplices of each dimension, and let $\varinjlim : \operatorname{Fun}(K, \mathcal{S}_{\pi}^{\wedge}) \to \mathcal{S}_{\pi}^{\wedge}$ denote the colimit functor. Then $\varinjlim$ commutes with filtered limits.*

Proof. Let $\{\mu_\alpha : K \to \mathcal{S}_{\pi}^{\wedge}\}_{\alpha \in A}$ be a filtered diagram in $\operatorname{Fun}(K, \mathcal{S}_{\pi}^{\wedge})$. We wish to show that the canonical map $\varinjlim(\varprojlim_{\alpha \in A} \mu_\alpha) \to \varprojlim_{\alpha \in A}(\varinjlim \mu_\alpha)$ is an equivalence of profinite spaces. Equivalently, we wish to show that for every π-finite space T, the canonical map

$$\theta : \varinjlim_{\alpha \in A} \varprojlim_{v \in K} \operatorname{Map}_{\mathcal{S}_{\pi}^{\wedge}}(\mu_\alpha(v), T) \to \varprojlim_{v \in K} \varinjlim_{\alpha \in A} \operatorname{Map}_{\mathcal{S}_{\pi}^{\wedge}}(\mu_\alpha(v), T)$$

is a homotopy equivalence. Choose an integer n such that T is n-truncated, and let K_0 denote the $(n+1)$-skeleton of K. Then the vertical maps in the diagram

$$\begin{array}{ccc} \varinjlim_{\alpha \in A} \varprojlim_{v \in K} \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(\mu_\alpha(v), T) & \xrightarrow{\theta} & \varprojlim_{v \in K} \varinjlim_{\alpha \in A} \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(\mu_\alpha(v), T) \\ \downarrow & & \downarrow \\ \varinjlim_{\alpha \in A} \varprojlim_{v \in K_0} \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(\mu_\alpha(v), T) & \xrightarrow{\theta'} & \varprojlim_{v \in K_0} \varinjlim_{\alpha \in A} \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(\mu_\alpha(v), T) \end{array}$$

are homotopy equivalences. We are therefore reduced to proving that θ' is a homotopy equivalence, which follows because filtered colimits are left exact in the ∞-category $\mathcal{S}$. □

We are now in a position to prove a weak form of Theorem E.6.3.1:

Lemma E.6.3.5. *Let $f : X \to Y$ be a map of profinite spaces, and let $f^* : (\mathcal{S}_\pi^\wedge)_{/Y} \to (\mathcal{S}_\pi^\wedge)_{/X}$ be the functor given by $f^*(Y') = X \times_Y Y'$. Then the functor f^* preserves finite colimits.*

Proof. Since the functor f^* preserves initial objects, it will suffice to show that f^* preserves pushout squares (Corollary HTT.4.4.2.5). To prove this, suppose we are given a diagram σ :

$$\begin{array}{ccccc} & & W & & \\ & \swarrow & & \searrow & \\ & X & & Y & \\ & & \searrow & \swarrow & \\ Z' & \longrightarrow & Z & & \end{array}$$

in the ∞-category $\mathcal{S}_\pi^\wedge$. We wish to prove that the canonical map

$$\theta : (Z' \times_Z X) \amalg_{Z' \times_Z W} (Z' \times_Z Y) \to Z' \times_Z (X \amalg_W Y)$$

is an equivalence of profinite spaces. Using Proposition HTT.5.3.5.15 , we can write σ as a filtered limit of diagrams σ_α:

$$\begin{array}{ccccc} & & W_\alpha & & \\ & \swarrow & & \searrow & \\ & X_\alpha & & Y_\alpha & \\ & & \searrow & \swarrow & \\ Z'_\alpha & \longrightarrow & Z_\alpha & & \end{array}$$

of π-finite spaces. Using Lemma E.6.3.4, we see that θ can be written as a filtered limit of maps

$$\theta_\alpha : (Z'_\alpha \times_{Z_\alpha} X_\alpha) \amalg_{(Z'_\alpha \times_{Z_\alpha} W_\alpha)} (Z'_\alpha \times_{Z_\alpha} Y_\alpha) \to Z'_\alpha \times_{Z_\alpha} (X_\alpha \amalg_{W_\alpha} Y_\alpha).$$

Each of these maps is an equivalence by virtue of Corollary E.6.0.8. □

Proof of Theorem E.6.3.1. Let K be a simplicial set with only finitely many vertices of each dimension, and suppose we are given a diagram $q : K \to (\mathcal{S}_{\pi}^{\wedge})_{/X}$. We wish to show that the canonical map $\varinjlim_{v \in K} Y \times_X q(v) \to Y \times_X \varinjlim_{v \in K} q(v)$ is an equivalence of profinite spaces. For this, it will suffice to show that for each integer $n \geqslant 0$, the induced map $\tau_{\leqslant n}(\varinjlim_{v \in K} Y \times_X q(v)) \to \tau_{\leqslant n}(Y \times_X \varinjlim_{v \in K} q(v))$ is an equivalence. Let K_0 denote the $(n+2)$-skeleton of K, so that we have a commutative diagram

$$\begin{array}{ccc} \tau_{\leqslant n}(\varinjlim_{v \in K_0} Y \times_X q(v)) & \longrightarrow & \tau_{\leqslant n}(Y \times_X \varinjlim_{v \in K_0} q(v)) \\ \downarrow \phi & & \downarrow \psi \\ \tau_{\leqslant n}(\varinjlim_{v \in K} Y \times_X q(v)) & \longrightarrow & \tau_{\leqslant n}(Y \times_X \varinjlim_{v \in K} q(v)). \end{array}$$

It follows from Lemma E.6.3.5 that the lower horizontal map in this diagram is an equivalence. It will therefore suffice to show that the the maps ϕ and ψ are equivalences. For ϕ, this is clear: the ∞-category of n-truncated objects of $\mathcal{S}_{\pi}^{\wedge}$ is equivalent to an $(n+1)$-category, so the colimit of a K-indexed diagram in $\tau_{\leqslant n} \mathcal{S}_{\pi}^{\wedge}$ is equivalent to the colimit of its restriction to K_0. The same argument shows that the map $\tau_{\leqslant n+1} \varinjlim_{v \in K_0} q(v) \to \tau_{\leqslant n+1} \varinjlim_{v \in K} q(v)$ is an equivalence, so the the map $\varinjlim_{v \in K_0} q(v) \to \varinjlim_{v \in K} q(v)$ is $(n+1)$-connective. It follows that the map $Y \times_X \varinjlim_{v \in K_0} q(v) \to Y \times_X \varinjlim_{v \in K} q(v)$ is also $(n+1)$-connective, so that ψ is an equivalence as desired. □

E.6.4 Digression: Bar Constructions

Let Y be a connected profinite space with a base point $y \in Y$. According to Theorem E.5.0.4, Y can be functorially recovered from its loop space $G = \Omega Y$, regarded as a group object of $\mathcal{S}_{\pi}^{\wedge}$. We will prove below that the ∞-category $(\mathcal{S}_{\pi}^{\wedge})_{/Y}$ of profinite spaces over Y can be identified with the ∞-category of profinite spaces equipped with an action of G (Theorem E.6.5.1). First, we need some general remarks about group actions in the ∞-categorical setting.

Notation E.6.4.1. Let $\mathcal{C}$ be an ∞-category which admits finite limits and geometric realizations of simplicial objects, and suppose that Cartesian product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations separately in each variable. We will regard $\mathcal{C}$ as endowed with the Cartesian symmetric monoidal structure, so that every group object G of $\mathcal{C}$ can be identified with an associative algebra object of $\mathcal{C}$. In this case, we can think of $\mathrm{LMod}_G(\mathcal{C})$ as the ∞-category of objects of $\mathcal{C}$ equipped with a left action of G, $\mathrm{RMod}_G(\mathcal{C})$ as the ∞-category of spaces equipped with a right action of G. It follows from Theorem HA.4.4.2.8 the relative tensor product functor $\otimes_G : \mathrm{RMod}_G(\mathcal{C}) \times \mathrm{LMod}_G(\mathcal{C}) \to \mathcal{C}$ is well-defined, and is computed by the two-sided bar construction

$$X \otimes_G Y = |\mathrm{Bar}_G(X, Y)_\bullet| \qquad \mathrm{Bar}_G(X, Y)_n = X \times G^n \times Y.$$

Example E.6.4.2. Let $\mathcal{C}$ be an ∞-category satisfying the hypotheses of Notation E.6.4.1, and let G be a group object of $\mathcal{C}$. Using Corollary HA.4.2.3.3 , we see that the final object $* \in \mathcal{C}$ can be regarded as a (left or right) G-module in an essentially unique way. We let BG denote the relative tensor product $* \otimes_G *$. Unwinding the definitions, we see that the two-sided bar construction $\mathrm{Bar}_G(*, *)_\bullet$ agrees with G (as a simplicial object of $\mathcal{C}$), so that the construction $G \mapsto BG$ is left adjoint to the functor $\Omega : \mathcal{C}_* \to \mathcal{G}\mathrm{rp}(\mathcal{C})$ of Remark E.5.5.2. We will refer to BG as the *classifying space* of G.

Construction E.6.4.3. Let $\mathcal{C}$ be an ∞-category satisfying the hypotheses of Notation E.6.4.1, and let G be a group object of $\mathcal{C}$. The construction $X \mapsto X \otimes_G *$ determines a functor $\mathrm{RMod}_G(\mathcal{C}) \to \mathcal{C}$, which carries the final object of $\mathrm{RMod}_G(\mathcal{C})$ to the classifying space BG. It therefore induces a functor $\mathrm{RMod}_G(\mathcal{C}) \simeq \mathrm{RMod}_G(\mathcal{C})_{/*} \to \mathcal{C}_{/BG}$, which we will denote by $X \mapsto X/G$. We will refer to X/G as the *quotient of X by G*.

Proposition E.6.4.4. *Let $\mathcal{C}$ be an ∞-category which admits finite limits and geometric realizations of simplicial objects, and assume that the Cartesian product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects. Let G be a group object of $\mathcal{C}$, and let $U : \mathrm{RMod}_G(\mathcal{C}) \to \mathcal{C}_{/BG}$ be the functor given by $X \mapsto X/G$. Then U admits a right adjoint V. Moreover, the composite functor $\mathcal{C}_{/BG} \xrightarrow{V} \mathrm{RMod}_G(\mathcal{C}) \to \mathcal{C}$ is given on objects by $V(f : Y \to BG) = \mathrm{fib}(f)$.*

Proof. Let $\mathcal{E}_0$ denote the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}} \times \Delta^1, \mathcal{C}) \times_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}} \times \{1\}, \mathcal{C})} \{G\}$ spanned by those morphisms $Z \to G$ between simplicial objects of $\mathcal{C}$ which satisfy the following condition:

($*$) For each $n \geqslant 0$, the inclusion $\{0\} \hookrightarrow [n]$ induces a map $Z_n \to Z_0$ which exhibits Z_n as a product of Z_0 with G_n.

Proposition HA.4.2.2.9 supplies an equivalence of ∞-categories $\mathrm{RMod}_G(\mathcal{C}) \to \mathcal{E}_0$ which carries a right G-module X to the map $\mathcal{C}$ given by the two-sided bar construction $\mathrm{Bar}_G(X, *)_\bullet$, where $*$ denotes the final object of $\mathcal{C}$, which we regard as a left G-module (together with the evident map of simplicial objects $\mathrm{Bar}_G(X, *)_\bullet \to \mathrm{Bar}\, G(*, *)_\bullet \simeq G$).

Let $\overline{G} : \mathbf{\Delta}_+^{\mathrm{op}} \to \mathcal{C}$ be a colimit of the simplicial object G of $\mathcal{C}$ (so that $\overline{G}$ is an augmented simplicial object of $\mathcal{C}$ with $\overline{G}_{-1} = BG$). Let $K_+ \subseteq \mathbf{\Delta}_+^{\mathrm{op}} \times \Delta^1$ denote the full subcategory obtained by removing the objects $([n], 0)$ for $n \geqslant 0$. Let $\mathcal{E}_-$ denote the full subcategory of $\mathrm{Fun}(K_-, \mathcal{C}) \times_{\mathrm{Fun}(\mathbf{\Delta}_+^{\mathrm{op}} \times \{1\}, \mathcal{C})} \{\overline{G}\}$ spanned by those functors which satisfy ($*$), let $\mathcal{E}$ denote the inverse image of $\mathcal{E}_-$ in $\mathrm{Fun}(\mathbf{\Delta}_+^{\mathrm{op}} \times \Delta^1, \mathcal{C})$, and let $\mathcal{E}_+$ denote the fiber product $\mathrm{Fun}(K_+, \mathcal{C}) \times_{\mathrm{Fun}(\mathbf{\Delta}_+^{\mathrm{op}} \times \{1\}, \mathcal{C})} \{\overline{G}\}$.

We have evident restriction maps $\mathcal{E}_- \xleftarrow{u^*} \mathcal{E} \xrightarrow{v^*} \mathcal{E}_+$. Note that the diagram of ∞-categories

$$\begin{array}{ccc} \mathbf{\Delta}^{\mathrm{op}} \times \{1\} & \longrightarrow & \mathbf{\Delta}_+^{\mathrm{op}} \times \{1\} \\ \downarrow & & \downarrow \\ \mathbf{\Delta}^{\mathrm{op}} \times \Delta^1 & \longrightarrow & \mathcal{E}_- \end{array}$$

is a homotopy pushout square. The restriction map $\mathcal{E}_- \to \mathcal{E}_0$ is trivial Kan fibration, so that we have an equivalence of ∞-categories $\mathcal{E}_- \simeq \mathrm{RMod}_G(\mathcal{C})$. The ∞-category $\mathcal{E}_+$ is isomorphic as a simplicial set to $\mathcal{C}^{/BG}$, hence equivalent to $\mathcal{C}_{/BG}$.

Since $\mathcal{C}$ admits geometric realizations of simplicial objects, the functor u^* admits a left adjoint $u_!$, given by left Kan extension along the inclusion $\mathcal{E}_- \hookrightarrow \mathcal{E}$. Unwinding the definitions, we see that the composite functor see that the composite functor

$$\mathrm{RMod}_G(\mathcal{C}) \simeq \mathcal{E}_- \xrightarrow{u_!} \mathcal{E} \xrightarrow{v^*} \mathcal{E}_+ \simeq \mathcal{C}_{/BG}$$

$$(X \in \mathrm{RMod}_G(\mathcal{C})) \mapsto |\operatorname{Bar}_G(X, *)_\bullet| = X/G.$$

Since $\mathcal{C}$ admits finite limits, the functor v^* admits a right adjoint v_* given by right Kan extension along the inclusion $\mathcal{E}_+ \hookrightarrow \mathcal{E}$. It follows that U admits a right adjoint V, given by the composition

$$\mathcal{C}_{/BG} \simeq \mathcal{E}_+ \xrightarrow{v_*} \mathcal{E} \xrightarrow{u^*} \mathcal{E}_- \simeq \mathrm{RMod}_G(\mathcal{C}).$$

More concretely, this functor is given by

$$(f : Y \to \overline{G}_{-1}) \mapsto \overline{G}_0 \times_{\overline{G}_{-1}} Y = \operatorname{fib}(f)$$

(endowed with appropriate action of G). □

E.6.5 Bar Construction

According to Corollary E.6.3.3, the formation of Cartesian products of profinite spaces determines a functor

$$\mathcal{S}_\pi^\wedge \times \mathcal{S}_\pi^\wedge \to \mathcal{S}_\pi^\wedge \qquad (X, Y) \mapsto X \times Y$$

which preserves preserves geometric realizations of simplicial object. It follows that the ∞-category $\mathcal{S}_\pi^\wedge$ of profinite spaces satisfies the hypotheses of Notation E.6.4.1. In particular, if G is a group object of the ∞-category $\mathcal{S}_\pi^\wedge$ and X is a profinite space equipped with a right action of G, then we can define the quotient X/G as in Construction E.6.4.3. We can now formulate our main result:

Theorem E.6.5.1. *Let G be a group object of the ∞-category $\mathcal{S}_\pi^\wedge$. Then the construction $X \mapsto X/G$ determines an equivalence of ∞-categories $U : \mathrm{RMod}_G(\mathcal{S}_\pi^\wedge) \to (\mathcal{S}_\pi^\wedge)_{/BG}$.*

Proof. According to Proposition E.6.4.4, the functor U admits a right adjoint V. We first claim that V is fully faithful: that is, for every object $Y \in (\mathcal{S}^{\wedge}_{\pi})_{/BG}$, the counit map $v : (U \circ V)(Y) \to Y$ is an equivalence of profinite spaces. Let us identify G with a simplicial object $K_{\bullet}$ of $\mathcal{C}$. The proof of Proposition E.6.4.4 shows that v can be identified with the composite map $|K_{\bullet} \times_{BG} Y| \xrightarrow{v'} |K_{\bullet}| \times_{BG} Y \xrightarrow{v''} Y$. The map v'' is evidently an equivalence, and v' is an equivalence by virtue of Corollary E.6.3.2 (the statement that v is an equivalence can also be deduced directly from Proposition E.5.5.3).

We now complete the proof by showing that the functor V is essentially surjective. Using Corollaries E.6.3.3 and HA.4.2.3.5 , we see that the forgetful functor $f : \mathrm{RMod}_G(\mathcal{S}^{\wedge}_{\pi}) \to \mathcal{S}^{\wedge}_{\pi}$ preserves geometric realizations of simplicial objects. Corollary E.6.3.2 and Proposition E.6.4.4 imply that the composite functor $fV : (\mathcal{S}^{\wedge}_{\pi})_{/BG} \to \mathcal{S}^{\wedge}_{\pi}$ also preserves geometric realizations of simplicial objects. Because f is conservative, this implies that V preserves geometric realizations of simplicial objects. Since V is fully faithful, the essential image of V is closed under geometric realizations in $\mathrm{RMod}_G(\mathcal{S}^{\wedge}_{\pi})$. By virtue of Proposition HA.4.7.3.14 , we are reduced to proving that the essential image of V contains all *free* right G-modules: that is, right G-modules of the form $Y \times G$, where Y is a profinite space. This is clear, since $Y \times G$ can be identified with $V(Y)$, where $Y \to BG$ is nullhomotopic. □

Remark E.6.5.2. Let X and Y be connected pointed profinite spaces, with loop spaces $G = \Omega X$ and $H = \Omega Y$. Then we can regard the ∞-categories

$$(\mathcal{S}^{\wedge}_{\pi})_{/X} \simeq \mathrm{RMod}_G(\mathcal{S}^{\wedge}_{\pi}) \qquad (\mathcal{S}^{\wedge}_{\pi})_{/Y} \simeq \mathrm{RMod}_H(\mathcal{S}^{\wedge}_{\pi})$$

as tensored over $\mathcal{S}^{\wedge}_{\pi}$ (which we regard as a symmetric monoidal ∞-category via the Cartesian product). Let $\mathrm{LinFun}_{\mathcal{S}^{\wedge}_{\pi}}(\mathcal{S}^{\wedge}_{\pi}, (\mathcal{S}^{\wedge}_{\pi})_{/X})(\mathcal{S}^{\wedge}_{\pi})_{/Y}$ denote the ∞-category of $\mathcal{S}^{\wedge}_{\pi}$-linear functors from $(\mathcal{S}^{\wedge}_{\pi})_{/X}$ to $(\mathcal{S}^{\wedge}_{\pi})_{/Y}$ which commute with geometric realizations of simplicial objects. Then Theorem HA.4.8.4.1 supplies an equivalence of ∞-categories

$$\mathrm{LinFun}^{\mathcal{S}^{\wedge}_{\pi}}_{\mathcal{S}^{\wedge}_{\pi}}((\mathcal{S}^{\wedge}_{\pi})_{/X}, (\mathcal{S}^{\wedge}_{\pi})_{/Y}) \simeq {}_G\mathrm{BMod}_H(\mathcal{S}^{\wedge}_{\pi}).$$

Using Theorem E.6.5.1, we can identify the left hand side with the ∞-category $(\mathcal{S}^{\wedge}_{\pi})_{/X \times Y}$. We can describe this equivalence more concretely as follows: it associates to each correspondence $M \to X \times Y$ the functor $(\mathcal{S}^{\wedge}_{\pi})_{/X} \to (\mathcal{S}^{\wedge}_{\pi})_{/Y}$ given by $X' \mapsto M \times_X X'$.

E.7 Profinite Spaces of Finite Type

In this section, we will study the following finiteness condition in profinite homotopy theory:

Definition E.7.0.3. Let X be a profinite space. We will say that X is of *finite type* if, for every π-finite space Y, the mapping space $\mathrm{Map}_{\mathcal{S}^{\wedge}_{\pi}}(X, Y)$ is π-finite.

Remark E.7.0.4. Let X be a profinite space. If X is of finite type, then $\pi_0 X$ is finite. It follows that a profinite space is of finite type if and only if it can be written as a disjoint union of finitely many connected profinite spaces of finite type.

Our main result asserts that for simply connected profinite spaces, Definition E.7.0.3 admits a number of reformulations:

Theorem E.7.0.5. *Let X be a simply connected profinite space. The following conditions are equivalent:*

(1) *The profinite space X is of finite type.*

(2) *For every prime number p and every integer $n \geqslant 0$, the cohomology group $\mathrm{H}^n(X; \mathbf{F}_p)$ is finite (see Definition E.7.1.1).*

(3) *For every prime number p and every integer $n \geqslant 2$, the cokernel of the map $\pi_n X \xrightarrow{p} \pi_n X$ is finite.*

E.7.1 Cohomology of Profinite Spaces

Our first step is to introduce the definitions of the cohomology groups $\mathrm{H}^n(X; \mathbf{F}_p)$ which appear in the statement of Theorem E.7.0.5. We begin by reviewing the definition of cohomology groups in classical homotopy theory.

Definition E.7.1.1. If X is a Kan complex and M is an abelian group, we let $C^*(X; M)$ denote the chain complex of M-valued cochains on X. The construction $X \mapsto C^*(X; M)$ carries homotopy equivalences of Kan complexes to quasi-isomorphisms of chain complexes, and therefore induces a functor of ∞-categories $\mathcal{S}^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$ which we will denote by $C^*(\bullet, M)$.

Applying Proposition HTT.5.3.5.10 , we deduce that there is an essentially unique functor $(\mathcal{S}_\pi^\wedge)^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$ which commutes with filtered colimits and coincides with $C^*(\bullet; M)$ on the full subcategory $\mathcal{S}_\pi \subseteq \mathcal{S}_\pi^\wedge$ of π-finite spaces. We will abuse notation by denoting this functor also by $X \mapsto C^*(X; M)$.

If X is a profinite space and $n \geqslant 0$ is an integer, we let $\mathrm{H}^n(X; M)$ denote the homotopy group $\pi_{-n} C^*(X; M)$. We will refer to $\mathrm{H}^n(X; M)$ as the *nth cohomology group of X with coefficients in M*.

Remark E.7.1.2. Let M be an abelian group, and let X be a profinite space represented by a filtered diagram of π-finite spaces $\{X_\alpha\}$. By definition, $C^*(X; M)$ is given by the filtered colimit $\varinjlim C^*(X_\alpha; M)$. In particular, we have canonical isomorphisms $\mathrm{H}^n(X; M) = \varinjlim \mathrm{H}^n(X_\alpha; M)$.

Example E.7.1.3. Let X be a profinite space and let M be an abelian group. Using Remark E.7.1.2 and Proposition E.5.2.2, we see that $\mathrm{H}^0(X; M)$ can be identified with the set of locally constant M-valued functions on $\pi_0 X$.

Example E.7.1.4. Let G be a profinite group, and let $BG \in \mathcal{S}_{\pi}^{\wedge}$ be the classifying space of G (Example E.6.4.2). For each integer n, we can identify $\mathrm{H}^n(BG; M)$ with the profinite group cohomology of G with coefficients in M (see, for example, [189]).

Remark E.7.1.5. Let M be an abelian group. Then the functor $(\mathcal{S}_{\pi}^{\wedge})^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$ $\quad X \mapsto C^*(X; M)$ is a left Kan extension of its restriction to the full subcategory $\mathcal{S}_{\pi}^{\mathrm{op}} \subseteq (\mathcal{S}_{\pi}^{\wedge})^{\mathrm{op}}$. Because the functor $X \mapsto C^*(\mathrm{Mat}(X); M)$ agrees with $X \mapsto C^*(X; M)$ when X is π-finite, we obtain a comparison map $C^*(X; M) \to C^*(\mathrm{Mat}(X); M)$, depending functorially on X and M. In particular, we obtain natural maps $\mathrm{H}^n(X; M) \to \mathrm{H}^n(\mathrm{Mat}(X); M)$. If X is a profinite space, we can think of $\mathrm{H}^n(X; M)$ as a continuous version of the usual cohomology of the materialization $\mathrm{Mat}(X)$. For example, $\mathrm{H}^0(\mathrm{Mat}(X); M)$ can be identified with the set of all M-valued functions on the set $\pi_0 X$, while $\mathrm{H}^0(X; M)$ can be identified with the set of M-valued functions on $\pi_0 X$ which are continuous with respect to the Stone space topology on $\pi_0 X$ (Example E.7.1.3).

Remark E.7.1.6. Let M be a finite abelian group and let $n \geqslant 0$ be an integer, so that the Eilenberg-MacLane space $K(M, n)$ is π-finite. Then the functor $X \mapsto \pi_0 \mathrm{Map}_{\mathcal{S}_{\pi}^{\wedge}}(X, K(M, n))$ agrees with with the functor $X \mapsto \mathrm{H}^n(X; M)$ on π-finite spaces and commutes with filtered colimits, and therefore agrees with the functor $X \mapsto \mathrm{H}^n(X; M)$ on all profinite spaces. In other words, the functor $X \mapsto \mathrm{H}^n(X; M)$ is representable (in the homotopy category of profinite spaces) by the Eilenberg-MacLane space $K(M, n)$.

Remark E.7.1.7. Let M be an abelian group, let X be a space, and let $X_{\pi}^{\wedge}$ be the profinite completion of X. Composing the map $C^*(X_{\pi}^{\wedge}; M) \to C^*(\mathrm{Mat}(X_{\pi}^{\wedge}); M)$ of Remark E.7.1.5 with pullback along the unit map $X \to \mathrm{Mat}(X_{\pi}^{\wedge})$, we obtain a canonical map $C^*(X_{\pi}^{\wedge}; M) \to C^*(X; M)$. If the abelian group M is finite, then this map is an equivalence. To prove this, it suffices to show that for each $n \geqslant 0$, the induced map $\mathrm{H}^n(X_{\pi}^{\wedge}; M) \to \mathrm{H}^n(X; M)$ is an isomorphism. This is clear: both sides can be identified with the set of homotopy classes of maps from X into the Eilenberg-MacLane space $K(M, n)$ (see Remark E.7.1.6).

E.7.2 The Künneth Formula

Let κ be a commutative ring. Then we can regard the construction $X \mapsto C^*(X; \kappa)$ as a functor from $\mathcal{S}^{\mathrm{op}}$ to the ∞-category of $\mathbb{E}_{\infty}$-algebras over κ. Consequently, the construction Definition E.7.1.1 can be refined to obtain a functor $C^*(\bullet; \kappa) : (\mathcal{S}_{\pi}^{\wedge})^{\mathrm{op}} \to \mathrm{CAlg}_{\kappa}$.

Proposition E.7.2.1. *Let κ be a field, and suppose we are given a pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \phi \\ Y' & \longrightarrow & Y \end{array}$$

of profinite spaces. If Y is simply connected, then the diagram σ :

$$\begin{array}{ccc} C^*(X';\kappa) & \longleftarrow & C^*(X;\kappa) \\ \uparrow & & \uparrow \\ C^*(Y';\kappa) & \longleftarrow & C^*(Y;\kappa) \end{array}$$

is a pushout square in CAlg_κ.

Proof. Using Proposition HTT.5.3.5.15 , we can write the diagram $X \xrightarrow{\phi} Y \xleftarrow{\psi} Y'$ as a filtered limit of diagrams $X_\alpha \xrightarrow{\phi_\alpha} Y_\alpha \xleftarrow{\psi_\alpha} Y'_\alpha$ of π-finite spaces. Since Y is simply connected, we may assume without loss of generality that each Y_α is simply connected. It follows that σ is a filtered colimit of diagrams σ_α :

$$\begin{array}{ccc} C^*(X_\alpha \times_{Y_\alpha} Y'_\alpha;\kappa) & \longleftarrow & C^*(X_\alpha;\kappa) \\ \uparrow & & \uparrow \\ C^*(Y'_\alpha;\kappa) & \longleftarrow & C^*(Y_\alpha;\kappa). \end{array}$$

We may therefore reduce to the case where X, Y, and Y' (and therefore also X') are π-finite. For each $y \in Y$, let X_y denote the homotopy fiber of ϕ over the point y. Using Corollary **??** (and the simple-connectivity of Y), we are reduced to proving that the cohomology groups $\mathrm{H}^n(X_y;\kappa)$ are finite-dimensional as vector spaces over κ, for each $n \geqslant 0$. Using Lemma E.1.6.5, we may assume that X_y has only finitely many simplices of each dimension, in which case the result is obvious. □

Corollary E.7.2.2. *Let X and Y be simply connected profinite spaces. For every field κ, the canonical map*

$$C^*(X;\kappa) \otimes_\kappa C^*(Y;\kappa) \to C^*(X \times Y;\kappa)$$

is an equivalence. In particular, we have a canonical isomorphism

$$\mathrm{H}^*(X;\kappa) \otimes_\kappa \mathrm{H}^*(Y;\kappa) \to \mathrm{H}^*(X \times Y;\kappa).$$

Remark E.7.2.3. Let κ be a field, and suppose we are given a pullback diagram of profinite spaces

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \phi \\ Y' & \longrightarrow & Y \end{array}$$

where Y is simply connected. Combining Propositions E.7.2.1 and HA.7.2.1.19 , we obtain a spectral sequence $\{E_r^{p,q}, d_r\}_{r \geqslant 2}$ converging to $\mathrm{H}^*(X'; \kappa)$, whose second page is given by $\mathrm{Tor}_*^{\mathrm{H}^*(Y;\kappa)}(\mathrm{H}^*(Y'; \kappa), \mathrm{H}^*(X; \kappa))$; when Y is a point, this spectral sequence degenerates and yields the isomorphism of Corollary E.7.2.2.

E.7.3 Singular Elements of Profinite Abelian Groups

We now discuss the finiteness condition that appears in condition (3) of Theorem E.7.0.5.

Definition E.7.3.1. Let M be a finite abelian group, and let p be a prime number. We will say that an element $x \in M$ is *p-singular* if $p^d x = 0$ for $d \gg 0$. If M is a profinite abelian group, we will say that an element $x \in M$ is *topologically p-singular* if the sequence $x, px, p^2x, \ldots$ converges to the identity element of M.

Remark E.7.3.2. Let M be a profinite abelian group. An element $x \in M$ is topologically p-singular if and only if, for every open subgroup $N \subseteq M$, the image of x is a p-singular element of the finite abelian group M/N. In particular, the collection of topologically p-singular elements of M forms a closed subgroup $M_p \subseteq M$.

Remark E.7.3.3. Let M be a finite abelian group. Then M is isomorphic to the product $\prod_p M_p$, where p ranges over all prime numbers and M_p denotes the subgroup of M consisting of p-singular elements.

Remark E.7.3.4. Let M be a profinite abelian group. Combining Remarks E.7.3.2 and E.7.3.3, we deduce that M factors canonically as a product $\prod_p M_p$, where p ranges over all prime numbers and M_p denotes the subgroup of p consisting of topologically p-singular elements. Note that each factor M_p can be regarded as a module over the ring $\mathbf{Z}_p = \varprojlim_{d \geqslant 0} \mathbf{Z} / p^d \mathbf{Z}$ of p-adic integers.

Proposition E.7.3.5. *Let M be a profinite abelian group. The following conditions are equivalent:*

(1) *For every prime number p, the subgroup $M_p \subseteq M$ of topologically p-singular elements is finitely generated as a module over the ring $\mathbf{Z}_p$ of p-adic integers.*

(2) *For every prime number p, the subgroup $M_p \subseteq M$ of topologically p-singular elements is isomorphic (as an abstract group) to a finite product of groups of the form $\mathbf{Z}_p$ and $\mathbf{Z}_p / p^d \mathbf{Z}_p$.*

(3) *For every prime number p, there are only finitely many group homomorphisms from M to $\mathbf{F}_p$.*

(4) *For each prime number p, there are only finitely many continuous group homomorphisms from M to $\mathbf{F}_p$.*

(5) *For every prime number p, the quotient M/pM is finite.*

(6) *For every prime number p, the multiplication map $p : M \to M$ has open image.*

Definition E.7.3.6. Let M be a profinite abelian group. We will say that M is *of finite type* if it satisfies the equivalent conditions of Proposition E.7.3.5. If X is a simply connected profinite space, then we will say that X is *of finite type* if the homotopy groups $\{\pi_m X\}_{m \geqslant 2}$ are of finite type (when regarded as profinite abelian groups).

Remark E.7.3.7. Let M be a profinite abelian group. We will say that M is *topologically finitely generated* if there exists a finitely generated subgroup of M which is dense in M. In this case, M can be written as a quotient of $\widehat{\mathbf{Z}}^d$ for some $d \geqslant 0$, where $\widehat{\mathbf{Z}} \simeq \varprojlim \mathbf{Z}/n\mathbf{Z}$ denotes the profinite completion of $\mathbf{Z}$. It follows that if M is topologically finitely generated, then M is of finite type. However, the converse fails. For example, the profinite group $\prod_p (\mathbf{Z}/p\mathbf{Z})^p$ is of finite type, but is not topologically finitely generated.

Proof of Proposition E.7.3.5. The implication (1) $\Rightarrow$ (2) follows from the structure theory of finitely generated modules over the discrete valuation ring $\mathbf{Z}_p$. Since the inclusion $M_p \hookrightarrow M$ induces an isomorphism $M_p/pM_p \simeq M/pM$, the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. Note that the maps $p : M \to M$ are closed, so that each the quotient M/pM inherits the structure of a compact Hausdorff topological group. In particular, M/pM is finite if and only if the identity element comprises an open subset of M/pM, which is equivalent to the requirement that pM is open in M. This proves (5) $\Leftrightarrow$ (6).

We next prove that (4) $\Rightarrow$ (5). Write M as an inverse limit $\varprojlim M_\alpha$, where each M_α is a finite abelian group. Let $\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p}$ denote the category of finite-dimensional vector spaces over the finite field $\mathbf{F}_p$. Then we can identify M/pM with the inverse limit of the diagram $\{M_\alpha/pM_\alpha\}$ in $\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p}$, representing a pro-object of $\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p}$. Vector space duality determines an equivalence of categories $(\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p})^{\mathrm{op}} \simeq \mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p}$, which extends to an equivalence $\mathrm{Pro}(\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p})^{\mathrm{op}} \simeq \mathrm{Ind}(\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p}) \simeq \mathrm{Vect}_{\mathbf{F}_p}$. The image of the diagram $\{M_\alpha/pM_\alpha\}$ under this equivalence can be identified with the set $\mathrm{Hom}_c(M, \mathbf{F}_p)$ of continuous group homomorphisms from M to $\mathbf{F}_p$. Condition (4) implies that $\mathrm{Hom}_c(M, \mathbf{F}_p) \in \mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p} \subseteq \mathrm{Vect}_{\mathbf{F}_p}$: that is, the Ind-object obtained by applying vector space duality to $\{M_\alpha/pM_\alpha\}$ is constant. It follows that the Pro-system $\{M_\alpha/pM_\alpha\}$ is also constant, so that M/pM is a retract of some M_α/pM_α, and is in particular finite.

We now complete the proof by showing that (5) $\Rightarrow$ (1). Replacing M by M_p, we may suppose that M is a p-profinite abelian group (see Definition **??**). Choose a basis $\{\overline{x}_i\}_{1 \leqslant i \leqslant n}$ for the $\mathbf{F}_p$-vector space M/pM. Each $\overline{x}_i$ can be lifted to an element $x_i \in M$, which determines a group homomorphism $\phi_i : \mathbf{Z} \to M$. Since M is a profinite p-group, the map ϕ_i factors through the p-profinite completion $\mathbf{Z}_p$ of $\mathbf{Z}$. We therefore obtain a sequence of continuous maps $\overline{\phi}_i : \mathbf{Z}_p \to M$. Since M is abelian, we can add these homomorphisms to obtain a continuous group homomorphism $\overline{\phi} : \mathbf{Z}_p^n \to M$. We claim that $\overline{\phi}$ is surjective. Since $\mathbf{Z}_p^n$ is compact, it will suffice to show that the image of $\overline{\phi}$ is dense. In other words, we must show that if $\psi : M \to N$ is a continuous surjection for some finite group N, then the composite map $\psi \circ \overline{\phi} : \mathbf{Z}_p^n \to N$ is surjective. Since the action of p is nilpotent on N, it suffices to show that the composite map $\theta : \mathbf{Z}_p^n \to N/pN$ is surjective (by Nakayama's lemma). This is clear, since θ is a composition of surjections $\mathbf{Z}_p^n \to \mathbf{F}_p^n \simeq M/pM \to N/pN$. □

E.7.4 The Hurewicz Theorem

We now begin to connect Definition E.7.3.6 with finiteness hypotheses on profinite spaces.

Proposition E.7.4.1. *Let X be a profinite space which is n-connective for some $n \geqslant 1$, and let M be an abelian group. There are canonical isomorphisms*

$$\mathrm{H}^m(X; M) \simeq \begin{cases} M & \text{if } m = 0 \\ 0 & \text{if } 0 < m < n \\ \mathrm{Hom}_c(\pi_n X, M) & \text{if } m = n. \end{cases}$$

Here $\mathrm{Hom}_c(\pi_n X, M)$ denotes the collection of continuous group homomorphisms from $\pi_n(X, x)$ to M (where we regard M as endowed with the discrete topology), where x is an arbitrarily chosen base point of X.

Corollary E.7.4.2. *Let X be a profinite space which is n-connective for some $n \geqslant 2$. The following conditions are equivalent:*

(1) *The profinite abelian group $\pi_n X$ is of finite type.*

(2) *The cohomology group $\mathrm{H}^n(X; \mathbf{F}_p)$ is finite, for every prime number p.*

Proof of Proposition E.7.4.1. Choose a point $x \in X$. Since X is n-connective, we can write X as the limit of a filtered diagram $\{X_\alpha\}$ of n-connective π-finite spaces. Let x_α denote the image of x in X_α, and let $\pi_n X_\alpha$ denote the finite group $\pi_n(X_\alpha, x_\alpha)$. Since each X_α is n-connective, the Hurewicz and universal coefficient theorems of classical homotopy theory supply isomorphisms

$$\mathrm{H}^m(X_\alpha; M) \simeq \begin{cases} M & \text{if } m = 0 \\ 0 & \text{if } 0 < m < n \\ \mathrm{Hom}(\pi_n X_\alpha, M) & \text{if } m = n. \end{cases}$$

Since $\mathrm{H}^*(X; M) \simeq \varinjlim \mathrm{H}^*(X_\alpha; M)$, we obtain canonical isomorphisms

$$\mathrm{H}^m(X; M) \simeq \begin{cases} M & \text{if } m = 0 \\ 0 & \text{if } 0 < m < n \\ \varinjlim \mathrm{Hom}(\pi_n X_\alpha, \mathbf{F}_p) & \text{if } m = n. \end{cases}$$

It now suffices to observe that the profinite group $\pi_n X$ is given as the limit of the filtered system of finite groups $\pi_n X$, so that $\mathrm{Hom}_c(\pi_n X, M) \simeq \varinjlim \mathrm{Hom}(\pi_n X_\alpha, M)$. □

E.7.5 A Convergence Theorem

The main ingredient in our proof of Theorem E.7.0.5 is the following technical result, which will also play an important role in §E.8:

Proposition E.7.5.1. *Let $n \geqslant 1$, let p be a prime number, and consider the tower of spaces $\{K(\mathbf{Z}/p^a\mathbf{Z}, n)\}_{a \geqslant 0}$ having limit $K(\mathbf{Z}_p, n) \in \mathcal{S}$. For every integer $m \geqslant 0$, the pro-system of abelian groups $\mathrm{H}_m(K(\mathbf{Z}/p^a\mathbf{Z}, n); \mathbf{F}_p)$ is equivalent to the constant pro-system with value $\mathrm{H}_m(K(\mathbf{Z}_p, n); \mathbf{F}_p)$.*

Remark E.7.5.2. If M is a finite abelian group, then the Eilenberg-MacLane spaces $K(M, n)$ can be represented by Kan complexes having only finitely many simplices of each dimension. It follows immediately that the groups $\mathrm{H}_m(K(M, n); \mathbf{F}_p)$ are finite for every prime number p and every integer $m \geqslant 0$. Proposition E.7.5.1 implies that the tower of abelian groups $\{\mathrm{H}_m(K(\mathbf{Z}/p^d\mathbf{Z}, n); \mathbf{F}_p)\}_{d \geqslant 0}$ is equivalent (as a Pro-abelian group) to the abelian group $\mathrm{H}_m(K(\mathbf{Z}_p, n); \mathbf{F}_p)$. It follows that the group $\mathrm{H}_m(K(\mathbf{Z}_p, n); \mathbf{F}_p)$ is isomorphic to a retract of $K(\mathbf{Z}/p^d\mathbf{Z}, n); \mathbf{F}_p)$ for $d \gg 0$. In particular, $\mathrm{H}_m(K(\mathbf{Z}_p, n); \mathbf{F}_p)$ is a finite group.

For the proof of Proposition E.7.5.1, we will need the following standard fact:

Lemma E.7.5.3. *Let M be an abelian group, and let p be a prime number for which the map $p : M \to M$ is an isomorphism. Then, for every integer $n \geqslant 1$, the augmentation map $C_*(K(M, n); \mathbf{F}_p) \to \mathbf{F}_p$ is an equivalence.*

Proof. For every abelian group N, the classifying space BN can be constructed as a simplicial abelian group, so that the homology $\mathrm{H}_*(BN; \mathbf{F}_p)$ inherits the structure of a graded commutative ring. Note that we have canonical isomorphisms $\mathrm{H}_1(BN; \mathbf{F}_p) \simeq (\pi_1 BN)/p(\pi_1 BN) \simeq N/pN$. If N is a free abelian group of finite rank, an elementary calculation shows that this extends to an isomorphism of $\mathrm{H}_*(BN; \mathbf{F}_p)$ with an exterior algebra (over $\mathbf{F}_p$) on the vector space N/pN. More generally, if N is a torsion-free abelian group, then writing N as a union of its finitely generated submodules, we obtain an isomorphism $\mathrm{H}_*(BN; \mathbf{F}_p) \simeq \bigwedge^*(N/pN)$.

Choose an exact sequence of $\mathbf{Z}[p^{-1}]$-modules $0 \to Q \to P \to M \to 0$, where P is a free $\mathbf{Z}[p^{-1}]$-module. Then P and Q are torsion-free, and the quotients P/pP and Q/pQ are trivial. It follows from the above arguments that $C_*(BP; \mathbf{F}_p) \simeq C_*(BQ; \mathbf{F}_p) \simeq \kappa$. Note that we can identify BM with the quotient of BP by the action of BQ, so that

$$C_*(BM; \mathbf{F}_p) \simeq C_*(BP; \mathbf{F}_p) \otimes_{C_*(BQ; \mathbf{F}_p)} \mathbf{F}_p \simeq \mathbf{F}_p \otimes_{\mathbf{F}_p} \mathbf{F}_p \simeq \mathbf{F}_p.$$

This completes the proof in the special case $n = 1$.

We handle the general case using induction on n. Assume that the counit map $C_*(K(M,n);\mathbf{F}_p) \to \mathbf{F}_p$ is an equivalence. We can regard $K(M,n)$ as a simplicial abelian group, whose classifying space can be identified with $K(M,n+1)$. We therefore have a canonical equivalence

$$C_*(K(M,n+1);\mathbf{F}_p) \simeq \mathbf{F}_p \otimes_{C_*(K(M,n);\mathbf{F}_p)} \mathbf{F}_p \simeq F_p \otimes_{\mathbf{F}_p} \mathbf{F}_p \simeq \mathbf{F}_p,$$

as desired. □

Proof of Proposition E.7.5.1. We first treat the case $n = 1$. We have a fiber sequence of simply connected spaces $K(\mathbf{Z},1) \to K(\mathbf{Z}_p,1) \to K(\mathbf{Z}_p/\mathbf{Z},1)$. Since multiplication by p is invertible on $\mathbf{Z}_p/\mathbf{Z}$, we have $\mathrm{H}_m(K(\mathbf{Z}_p/\mathbf{Z},1);\mathbf{F}_p) \simeq 0$ for $m > 0$. Using the Serre spectral sequence, we deduce that the map $K(\mathbf{Z},1) \to K(\mathbf{Z}_p,1)$ induces an isomorphism on homology groups with coefficients in $\mathbf{F}_p$. It will therefore suffice to show that, for each $m \geqslant 0$, the pro-system $\{\mathrm{H}_m(K(\mathbf{Z}/p^a\mathbf{Z},1);\mathbf{F}_p)\}_{a \geqslant 0}$ is equivalent to the constant pro-system taking the value

$$\mathrm{H}_m(K(\mathbf{Z},1);\mathbf{F}_p) \simeq \begin{cases} \mathbf{F}_p & \text{if } m \in \{0,1\} \\ 0 & \text{otherwise.} \end{cases}$$

For each $a \geqslant 1$, let $\epsilon_a \in \mathrm{H}^1(K(\mathbf{Z}/p^a\mathbf{Z},1);\mathbf{F}_p)$ classify the unit map $\mathbf{Z}/p^a\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \simeq \mathbf{F}_p$, and let $\eta_a \in \mathrm{H}^2(K(\mathbf{Z}/p^a\mathbf{Z},1);\mathbf{F}_p)$ classify the central extension $0 \to \mathbf{F}_p \to \mathbf{Z}/p^{a+1}\mathbf{Z} \to \mathbf{Z}/p^a\mathbf{Z} \to 0$. Then $\mathrm{H}^*(K(\mathbf{Z}/p^a\mathbf{Z},1);\mathbf{F}_p)$ has a basis given by the products η^i and $\epsilon_a\eta_a^i$ for $i \geqslant 0$. Note that the image of η_a in $\mathrm{H}^2(K(\mathbf{Z}/p^{a+1}\mathbf{Z},1);\mathbf{F}_p)$ is zero (since the central extension classified by η_a splits over $\mathbf{Z}/p^{a+1}\mathbf{Z}$), and the image of ϵ_a in $\mathrm{H}^1(K(\mathbf{Z}/p^{a+1}\mathbf{Z},1))$ is ϵ_{a+1}. It follows that the maps

$$\mathrm{H}^m(K(\mathbf{Z}/p\mathbf{Z},1);\mathbf{F}_p) \to \mathrm{H}^m(K(\mathbf{Z}/p^2\mathbf{Z},1);\mathbf{F}_p) \to \mathrm{H}^m(K(\mathbf{Z}/p^3\mathbf{Z},1);\mathbf{F}_p) \to \cdots$$

are isomorphisms for $m \leqslant 1$ and zero for $m > 1$. Passing to dual spaces, we conclude that the tower

$$\cdots \to \mathrm{H}_m(K(\mathbf{Z}/p^3\mathbf{Z},1);\mathbf{F}_p) \to \mathrm{H}_m(K(\mathbf{Z}/p^2\mathbf{Z},1);\mathbf{F}_p) \to \mathrm{H}_m(K(\mathbf{Z}/p\mathbf{Z},1);\mathbf{F}_p)$$

consists of isomorphisms for $m \leqslant 1$ and zero maps for $m > 1$. We now complete the proof (in the case $n = 1$) by observing that the maps $\mathrm{H}_m(K(\mathbf{Z},1);\mathbf{F}_p) \to \mathrm{H}_m(K(\mathbf{Z}/p\mathbf{Z},1);\mathbf{F}_p)$ are isomorphisms for $m \leqslant 1$.

We now treat the general case. Since $\mathrm{Mod}_{\mathbf{F}_p}$ is a presentable symmetric monoidal ∞-category, there is a unique colimit-preserving symmetric monoidal functor $\mathcal{S} \to \mathrm{Mod}_{\mathbf{F}_p}$. Let us denote this functor by $X \mapsto C_*(X)$. Note that the homology groups of a space X are given by the formula $\mathrm{H}_m(X;\mathbf{F}_p) = \pi_m C_*(X)$. It will therefore suffice to prove the following:

($*$) For every integer m, the Pro-object $\{\tau_{\leqslant m} C_*(K(\mathbf{Z}/p^a\mathbf{Z}, n))\}_{a \geqslant 0}$ of $\mathrm{Mod}_{\mathbf{F}_p}$ is equivalent to the constant $\mathbf{F}_p$-module spectrum $\tau_{\leqslant m} C_*(K(\mathbf{Z}_p, n))$.

Since the functor $X \mapsto \tau_{\leqslant m} C_*(X)$ is a successive extension of the functors $X \mapsto \mathrm{H}_a(X; \mathbf{F}_p)$ for $0 \leqslant a \leqslant m$, the proof given above shows that ($*$) holds when $n = 1$. To prove ($*$) in general, we proceed by induction on n. Suppose that $n > 1$. Let $X_\bullet$ be a Čech nerve of the map $* \to K(\mathbf{Z}_p, n)$, so that $X_\bullet$ is a group object of $\mathcal{S}$. Since C_* is a symmetric monoidal functor, we deduce that $C_*(X_1)$ is an associative algebra object of $\mathrm{Mod}_{\mathbf{F}_p}$; here $X_1 = * \times_{K(\mathbf{Z}_p, n)} * \simeq K(\mathbf{Z}_p, n-1)$. Since $K(\mathbf{Z}_p, n)$ is connected, the canonical map $|X_\bullet| \to K(\mathbf{Z}_p, n)$ is an equivalence. Because C_* preserves colimits, we deduce that $C_*(K(\mathbf{Z}_p, n))$ is given by $|C_*(X_\bullet)|$. Unwinding the definitions, we see that this geometric realization corresponds to the bar construction $\mathbf{F}_p \otimes_{C_*(K(\mathbf{Z}_p, n-1))} \mathbf{F}_p$. Similar reasoning yields an equivalence $C_*(K(\mathbf{Z}/p^a\mathbf{Z}, n)) \simeq \mathbf{F}_p \otimes_{C_*(K(\mathbf{Z}/p^a\mathbf{Z}, n-1))} \mathbf{F}_p$ for each $a \geqslant 0$. We have a commutative diagram of Pro-objects

$$\begin{array}{ccc}
\mathbf{F}_p \otimes_{C_*(K(\mathbf{Z}_p, n-1))} \mathbf{F}_p & \longrightarrow & \{\mathbf{F}_p \otimes_{C_*(K(\mathbf{Z}/p^a\mathbf{Z}, n-1))} \mathbf{F}_p\} \\
\downarrow & & \downarrow \\
C_*(K(\mathbf{Z}_p, n)) & \longrightarrow & \{C_*(K(\mathbf{Z}/p^a\mathbf{Z}, n))\}
\end{array}$$

where the vertical maps are equivalences. Using the inductive hypothesis, we deduce that upper horizontal map becomes an equivalence after applying the truncation functor $\tau_{\leqslant m}$. It follows that the lower horizontal map becomes an equivalence after applying the truncation functor $\tau_{\leqslant m}$ as well, which proves ($*$). □

E.7.6 Profinite Eilenberg-MacLane Spaces

We now discuss Eilenberg-MacLane spaces in the setting of profinite homotopy theory.

Notation E.7.6.1. Let $n \geqslant 1$ be an integer, and let G be a profinite group (which we assume to be abelian if $n \geqslant 2$). We let $\widehat{K}(G, n)$ denote the profinite space represented by the filtered diagram $\{K(G/U, n)\}$, where U ranges over all open normal subgroups of G.

Suppose that X is a simply connected profinite space, so that X can be represented by a filtered diagram $\{X_\alpha\}$ of simply connected π-finite spaces X_α. Choosing a base point $x \in X$, we obtain a collection of fiber sequences $\tau_{\leqslant n} X_\alpha \to \tau_{\leqslant n-1} X_\alpha \to K(\pi_n X_\alpha, n+1)$, depending functorially on α. Using Remark E.5.2.5, we see that the filtered diagram $\{\pi_n X_\alpha\}$ is isomorphic (as a Pro-object) to the diagram $\{\pi_n X/U\}$, where U ranges over all open subgroups of $\pi_n X$. Passing to the inverse limit over α, we obtain a fiber sequence of profinite spaces $\tau_{\leqslant n} X \to \tau_{\leqslant n-1} X \to \widehat{K}(\pi_n X, n+1)$.

Proposition E.7.6.2. *Let M be a profinite abelian group, let p be a prime number, and let $M_p \subseteq M$ be the subgroup consisting of topologically p-singular elements (see Definition E.7.3.1). For every integer $n \geqslant 1$, the canonical map*

$$C^*(\widehat{K}(M,n); \mathbf{F}_p) \to C^*(\widehat{K}(M_p,n); \mathbf{F}_p)$$

is an equivalence.

Proof. Writing M as an inverse limit of finite abelian groups, we can reduce to the case where M is finite. In this case, we can write $M = M_p \times M'$, where M' is a subgroup of M on which p acts invertibly. We then have an equivalence

$$C_*(K(M,n); \mathbf{F}_p) \simeq C_*(K(M_p,n); \mathbf{F}_p) \otimes_{\mathbf{F}_p} C_*(K(M',n); \mathbf{F}_p),$$

so that the desired result follows from Lemma E.7.5.3. □

Proposition E.7.6.3. *Let X be a simply connected profinite space, and suppose that the profinite groups $\pi_m X$ are of finite type for $m \geqslant 2$. Then the cohomology groups $\mathrm{H}^n(X; M)$ are finite for every integer $n \geqslant 0$ and every finite abelian group M.*

Proof. Every exact sequence of abelian groups $0 \to M' \to M \to M'' \to 0$ determines a fiber sequence $C^*(X; M') \to C^*(X; M) \to C^*(X; M'')$, hence a long exact sequence of cohomology groups

$$\cdots \to \mathrm{H}^{n-1}(X; M'') \to \mathrm{H}^n(X; M') \to \mathrm{H}^n(X; M) \to \mathrm{H}^n(X; M'') \to \mathrm{H}^{n+1}(X; M') \to \cdots$$

It will therefore suffice to prove finiteness after replacing M by M' and M''. Choosing an appropriate filtration of M, we can reduce to the case where $M = \mathbf{F}_p$ for some prime number p.

Note that the canonical map $\mathrm{H}^n(\tau_{\leqslant n} X; \mathbf{F}_p) \to \mathrm{H}^n(X; \mathbf{F}_p)$ is an isomorphism for every integer n. Consequently, to prove the finiteness of $\mathrm{H}^n(X; \mathbf{F}_p)$, we may replace X by $\tau_{\leqslant n} X$, and thereby reduce to the case where X is n-truncated. We proceed by induction on n, the case $n \leqslant 1$ being trivial (since X is assumed to be simply connected). To carry out the inductive step, set $M = \pi_n X$, so that Notation E.7.6.1 supplies a fiber sequence

$$X \to \tau_{\leqslant n-1} X \to \widehat{K}(M, n+1).$$

Applying Proposition E.7.2.1, we obtain a pushout diagram

$$\begin{array}{ccc} C^*(\widehat{K}(M,n+1); \mathbf{F}_p) & \longrightarrow & \mathbf{F}_p \\ \downarrow & & \downarrow \\ C^*(\tau_{\leqslant n-1} X; \mathbf{F}_p) & \longrightarrow & C^*(X; \mathbf{F}_p) \end{array}$$

of $\mathbb{E}_\infty$-algebras over $\mathbf{F}_p$. Using the inductive hypothesis and Lemma **??**, we are reduced to proving that $C^*(\widehat{K}(M, n+1); \mathbf{F}_p)$ is locally finite. Using Proposition E.7.6.2, we see that this is equivalent to the local finiteness of $C^*(\widehat{K}(M_p, n+1); \mathbf{F}_p)$. Since X is of finite type, we can write M_p as product of finitely many factors isomorphic to $\mathbf{Z}_p$ or $\mathbf{Z}/p^d\mathbf{Z}$. Using Proposition E.7.2.1 again, we are reduced to proving that the $\mathbb{E}_\infty$-algebras $C^*(\widehat{K}(\mathbf{Z}_p, n+1); \mathbf{F}_p)$ and $C^*(K(\mathbf{Z}/p^d\mathbf{Z}, n+1); \mathbf{F}_p)$ are locally finite, which follows from Remark E.7.5.2. □

E.7.7 The Proof of Theorem E.7.0.5

Let X be a simply connected profinite space. We wish to prove that the following conditions are equivalent:

(1) The profinite space X is of finite type.

(2) For every prime number p and every integer $n \geqslant 0$, the cohomology group $\mathrm{H}^n(X; \mathbf{F}_p)$ is finite.

(3) For every prime number p and every integer $n \geqslant 2$, the cokernel of the map $\pi_n X \xrightarrow{p} \pi_n X$ is finite.

The implicatoin (1) ⇒ (2) follows from Remark E.7.1.6. We next show that (2) implies (1). Assume that X satisfies (2) and let Y be a π-finite space; We wish to show that the mapping space $\mathrm{Map}_{\mathcal{S}_\pi^\wedge}(X, Y)$ is π-finite. Since X is simply connected, the diagonal embedding $\tau_{\leqslant 1} Y \to \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(X, \tau_{\leqslant 1} Y)$ is a homotopy equivalence. It follows that $\mathrm{Map}_{\mathcal{S}_\pi^\wedge}(X, \tau_{\leqslant 1} Y)$ is π-finite. Consequently, we are reduced to proving that the homotopy fibers of the map $\mathrm{Map}_{\mathcal{S}_\pi^\wedge}(X, Y) \to \mathrm{Map}_{\mathcal{S}_\pi^\wedge}(X, \tau_{\leqslant 1} Y)$ are π-finite. Since X is simply-connected, every map $X \to \tau_{\leqslant 1} Y$ is homotopic to a constant map. We may therefore replace Y by one of the homotopy fibers of the map $Y \to \tau_{\leqslant 1} Y$, and thereby reduce to the case where Y is simply connected.

Since Y is π-finite, it is n-truncated for some integer n. We proceed by induction on n, the case $n \leqslant 1$ being trivial. Since Y is simply connected, there exists a fiber sequence $Y \to \tau_{\leqslant n-1} Y \to K(M, n+1)$, where $M = \pi_n Y$. The inductive hypothesis implies that the mapping space $\mathrm{Map}_{\mathcal{S}_\pi^\wedge}(X, \tau_{\leqslant n-1} Y)$ is π-finite. It will therefore suffice to show that the mapping space $\mathrm{Map}_{\mathcal{S}_\pi^\wedge}(X, K(M, n+1))$ is π-finite. Using Remark E.7.1.6, we are reduced to proving that the cohomology groups $\mathrm{H}^d(X; M)$ are finite for every integer d. The collection of finite abelian groups M which satisfy this condition are closed under extension. We may therefore reduce to the case where M is a cyclic group of prime order, in which case the desired result follows from (2). This completes the proof that (2) ⇒ (1).

The implication (3) ⇒ (2) follows from Proposition E.7.6.3. We will complete the proof of Theorem E.7.0.5 by showing that (2) ⇒ (3). Let X be a simply connected profinite space, and suppose that the cohomology group $\mathrm{H}^n(X; \mathbf{F}_p)$ is finite for every integer n and every prime number p. We wish to prove that each homotopy group $\pi_m X$ is of finite type as a profinite abelian group. We proceed by induction on m, the case $m \leqslant 1$ being trivial. To carry out the inductive step, assume that $\pi_d X$ is of finite type for $d < m$. Choose a base point $x \in X$, and form a fiber sequence of profinite spaces

$$Y \to X \to \tau_{\leqslant m-1} X.$$

For every prime number p, Proposition E.7.2.1 supplies a pushout diagram of $\mathbb{E}_\infty$-algebras

$$\begin{array}{ccc} C^*(\tau_{\leqslant m-1}X; \mathbf{F}_p) & \longrightarrow & \mathbf{F}_p \\ \downarrow & & \downarrow \\ C^*(X; \mathbf{F}_p) & \longrightarrow & C^*(Y; \mathbf{F}_p). \end{array}$$

Since the homotopy groups of $\tau_{\leqslant m-1} X$ are of finite type, Proposition E.7.6.3 guarantees that the $\mathbb{E}_\infty$-algebra $C^*(\tau_{\leqslant m-1}X; \mathbf{F}_p)$ is locally finite. Because $C^*(X; \mathbf{F}_p)$ is also locally finite, Lemma **??** implies that $C^*(Y; \mathbf{F}_p)$ is locally finite. In particular, the group $\mathrm{H}^m(Y; \mathbf{F}_p)$ is finite. Proposition E.7.4.1 supplies an isomorphism

$$\mathrm{H}^m(Y; \mathbf{F}_p) \simeq \mathrm{Hom}_c(\pi_m Y; \mathbf{F}_p) \simeq \mathrm{Hom}_c(\pi_m X; \mathbf{F}_p).$$

It follows that $\mathrm{Hom}_c(\pi_m X; \mathbf{F}_p)$ is finite for every prime number p, so that the profinite abelian group $\pi_m X$ is of finite type as desired(Proposition E.7.3.5).

We close this section by mentioning a simple consequence of Theorem E.7.0.5:

Corollary E.7.7.1. *Let X be a simply connected space, and suppose that the cohomology groups $\mathrm{H}^n(X; \mathbf{F}_p)$ are finite for every integer $n \geqslant 2$ and every prime number p. Then the profinite completion $X^\wedge_\pi$ is a simply connected profinite space of finite type.*

Proof. Combine Theorem E.7.0.5 with Remark E.7.1.7 and Corollary E.4.3.2. □

E.8 Materialization

In §E.3, we proved that the materialization functor $\mathrm{Mat} : \mathcal{S}^\wedge_\pi \to \mathcal{S}$ is conservative (Theorem E.3.1.6). In this section, we will show that the materialization functor is fully faithful when restricted to profinite spaces which are simply connected and of finite type. Moreover, we show that the essential image of this restriction admits a simple algebraic description (Theorem E.8.2.1). Moreover, a space Y belongs to the essential image of this restriction if and only if the homotopy groups of Y satisfy a purely algebraic condition (Theorem E.8.2.1).

E.8.1 Congruence Completion of Abelian Groups

We begin by reviewing some elementary algebra.

Definition E.8.1.1. Let M be an abelian group. We will say that M is *congruence-finite* if, for every positive integer d, the kernel and cokernel of the map $M \xrightarrow{d} M$ are finite abelian groups. If M is a congruence-finite abelian group, we let $\widehat{M}$ denote the inverse limit $\varprojlim_d M/dM$, where d ranges over all positive integers. We will refer to $\widehat{M}$ as the *congruence completion* of M. We will say that M is *congruence-separated* if it is congruence-finite, and the canonical map $\theta : M \to \widehat{M}$ is injective. We will say that M is *congruence-complete* if M is congruence-finite and θ is an isomorphism.

Remark E.8.1.2. For every congruence-finite abelian group M, we will regard the congruence-completion $\widehat{M} = \varprojlim M/dM$ as a profinite abelian group, equipped with the inverse limit topology.

Remark E.8.1.3. Let M be an abelian group. Then M is congruence-finite if and only if, for every prime number p, the kernel and cokernel of the map $M \xrightarrow{p} M$ are finite abelian groups.

Remark E.8.1.4. The group $\mathbf{Z}$ is congruence-separated. Its congruence completion $\widehat{\mathbf{Z}} = \varprojlim \mathbf{Z}/N\mathbf{Z}$ is a commutative ring, which is isomorphic to the product $\prod_p \mathbf{Z}_p$.

Example E.8.1.5. Let M be a finitely generated abelian group. Then M is congruence-separated. Moreover, the congruence-completion of M is isomorphic to the tensor product $\widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} M$.

The relevance of Definition E.8.1.1 to our investigation stems from the following characterization of the class of congruence-complete abelian groups:

Proposition E.8.1.6. *Let* Ab *denote the category of abelian groups, and let* $\mathcal{C}$ *denote the category of profinite abelian groups of finite type. Then the forgetful functor* $\theta : \mathcal{C} \to \mathrm{Ab}$ *is fully faithful, and its essential image consists of those abelian groups which are congruence-complete.*

Proof. Let M and N be profinite abelian groups. To prove that θ is fully faithful, it will suffice to show that if M is of finite type, then every group homomorphism $f : M \to N$ is continuous. Let $N' \subseteq N$ be an open subgroup, so that N/N' is finite. Choose an integer $d > 0$ which annihilates N/N'. Then $f^{-1}(N')$ contains the subgroup $dM \subseteq M$. Since M has finite type, Proposition E.7.3.5 implies that dM is a closed subgroup of finite index in M, hence open. It follows that $f^{-1}(N')$ is also an open subgroup of M. Allowing N' to vary over all open subgroups of N, we deduce that f is continuous. This completes the proof that θ is fully faithful.

We next prove that if M is a profinite abelian group of finite type, then M is congruence-complete (when regarded as an abstract abelian group). We first claim that M is congruence-finite. For this, it will suffice to show that for every prime number p, the map $M \xrightarrow{p} M$ has finite kernel and cokernel (Remark E.8.1.3). Without loss of generality, we may replace M by the subgroup $M_p \subseteq M$ of topologically p-singular elements. In this case, M is a finitely generated module over $\mathbf{Z}_p$ (Proposition E.7.3.5) and the result is obvious.

Arguing as above, we see that the subgroup $dM \subseteq M$ is open for each $d > 0$, so that the projection maps $M \to M/dM$ are continuous. It follows that the natural map $f : M \to \widehat{M}$ is continuous. The image of f is evidently dense in $\widehat{M}$. Since M is compact, we conclude that f is surjective. To prove the injectivity of M, it will suffice to show that every element $x \in \bigcap_d dM$ is zero. Since M is a profinite abelian group, it will suffice to show that the image of x vanishes in each quotient M/M', where M' is an open subgroup of M. The group M/M' is finite, hence annihilated by some integer $d > 0$. Then $dM \subseteq M'$, so that $x \mapsto 0 \in M/M'$ as desired.

Conversely, suppose we are given a congruence-complete abelian group N; we wish to show that N belongs to the essential image of θ. Since N is congruence-complete, we have an isomorphism of abstract groups $u : N \simeq \widehat{N}$. It will therefore suffice to show that the profinite abelian group $\widehat{N}$ has finite type. Using Proposition E.7.3.5, we are reduced to proving that $\widehat{N}/p\widehat{N}$ is finite for every prime number p. This is clear, since u induces an isomorphism $\widehat{N}/p\widehat{N} \simeq N/pN$. □

E.8.2 Materialization for Profinite Spaces of Finite Type

We can now formulate the main result of this section:

Theorem E.8.2.1. *Let $(\mathcal{S}^{\wedge}_{\pi})_{\mathrm{ft}}$ denote the full subcategory of $\mathcal{S}^{\wedge}_{\pi}$ spanned by those profinite spaces which are simply connected and of finite type. Then the materialization functor $\mathrm{Mat} : \mathcal{S}^{\wedge}_{\pi} \to \mathcal{S}$ restricts to a fully faithful embedding $\mathrm{Mat}_{\mathrm{ft}} : \mathcal{S}^{\wedge}_{\pi})_{\mathrm{ft}} \to \mathcal{S}$. Moreover, a space X belongs to the essential image of $\mathrm{Mat}_{\mathrm{ft}}$ if and only if it is simply connected and each homotopy group $\pi_n X$ is congruence-complete.*

Remark E.8.2.2. Theorem E.8.2.1 can be regarded as a homotopy-theoretic analogue of Proposition E.8.1.6.

We will deduce Theorem E.8.2.1 from the following pair of assertions, which we prove later in this section:

Proposition E.8.2.3. *Let X be a profinite space. If X is simply connected and of finite type, then the counit map $v : \mathrm{Mat}(X)^{\wedge}_{\pi} \to X$ is an equivalence of profinite spaces (that is, X can be recovered as the profinite completion of its materialization).*

Proposition E.8.2.4. *Let X be a simply connected space, and suppose that each homotopy group $\pi_n X$ is congruence-separated. Then:*

(a) *For every prime number p and every integer $n \geqslant 0$, the cohomology group $\mathrm{H}^n(X; \mathbf{F}_p)$ is finite.*

(b) *The unit map $X \to \mathrm{Mat}(X^{\wedge}_{\pi})$ induces maps $\pi_n X \to \pi_n X^{\wedge}_{\pi}$ which exhibit each $\pi_n X^{\wedge}_{\pi}$ as a congruence-completion of $\pi_n X$.*

Proof of Theorem E.8.2.1. Let X and Y be profinite spaces. Then the canonical map

$$\mathrm{Map}_{\mathcal{S}^{\wedge}_{\pi}}(X, Y) \to \mathrm{Map}_{\mathcal{S}}(\mathrm{Mat}(X), \mathrm{Mat}(Y)) \simeq \mathrm{Map}_{\mathcal{S}^{\wedge}_{\pi}}(\mathrm{Mat}(X)^{\wedge}_{\pi}, Y)$$

is given by composition with the counit map $v : \mathrm{Mat}(X)^{\wedge}_{\pi} \to X$. If X is simply connected and of finite type, then v is an equivalence (Proposition E.8.2.3). It follows that Mat is fully faithful when restricted to $(\mathcal{S}^{\wedge}_{\pi})_{\mathrm{ft}}$.

It remains to describe the essential image of the functor $\mathrm{Mat}|_{(\mathcal{S}^{\wedge}_{\pi})_{\mathrm{ft}}}$. Note that if X is a simply connected profinite space of finite type, then each homotopy group $\pi_n X$ is a profinite abelian group of finite type (Theorem E.7.0.5), so that $\pi_n \mathrm{Mat}(X)$ is congruence-complete when viewed as an abstract abelian group (Proposition E.8.1.6). Conversely, suppose that Z is a simply connected space whose homotopy groups are congruence-complete; we wish to show that there exists a simply connected profinite space X of finite type and an equivalence $Z \simeq \mathrm{Mat}(X)$. Let $X = Z^{\wedge}_{\pi}$ be the profinite completion of Z. Then X is simply connected (Corollary E.4.3.2). Using Proposition E.8.2.4 and Remark E.7.1.7, we deduce that the cohomology groups $\mathrm{H}^n(X; \mathbf{F}_p)$ are finite, so that X is of finite type (Theorem E.7.0.5). To complete the proof, it will suffice to show that the unit map $u : Z \to \mathrm{Mat}(X)$ is a homotopy equivalence. Since the domain and codomain of u are simply connected, this is equivalent to the assertion that u induces isomorphisms $\pi_n Z \to \pi_n \mathrm{Mat}(X)$ for $n \geqslant 2$. This follows from Proposition E.8.2.4, since the homotopy groups of Z are assumed to be congruence-complete. □

E.8.3 The Proof of Proposition E.8.2.3

We will deduce Proposition E.8.2.3 from the following:

Lemma E.8.3.1. *Let X be a simply connected profinite space of finite type. Then, for every prime number p, the map of cohomology rings $\theta : \mathrm{H}^*(X; \mathbf{F}_p) \to \mathrm{H}^*(\mathrm{Mat}(X); \mathbf{F}_p)$ is an isomorphism.*

Proof. We will say that a profinite space X is *p-good* if the map $\theta : \mathrm{H}^*(X; \mathbf{F}_p) \to \mathrm{H}^*(\mathrm{Mat}(X); \mathbf{F}_p)$ of Remark E.7.1.5 is an isomorphism. Suppose we are given a pullback diagram of profinite spaces σ :

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y. \end{array}$$

Choose a base point $y \in Y$, and let X_y denote the fiber product $X \times_Y \{y\}$. If Y is simply connected and $C^*(\mathrm{Mat}(X_y); \mathbf{F}_p)$ is locally finite, then Corollary **??** implies that the induced diagram

$$\begin{array}{ccc} C^*(\mathrm{Mat}(X'); \mathbf{F}_p) & \longleftarrow & C^*(\mathrm{Mat}(X); \mathbf{F}_p) \\ \uparrow & & \uparrow \\ C^*(\mathrm{Mat}(Y'); \mathbf{F}_p) & \longleftarrow & C^*(\mathrm{Mat}(Y); \mathbf{F}_p) \end{array}$$

is a pushout square of $\mathbb{E}_\infty$-algebras over $\mathbf{F}_p$. Similarly, Proposition E.7.2.1 implies that the diagram

$$\begin{array}{ccc} C^*(X'; \mathbf{F}_p) & \longleftarrow & C^*(X; \mathbf{F}_p) \\ \uparrow & & \uparrow \\ C^*(Y'; \mathbf{F}_p) & \longleftarrow & C^*(Y; \mathbf{F}_p) \end{array}$$

is a pushout square of $\mathbb{E}_\infty$-algebras over $\mathbf{F}_p$. This proves the following:

$(*)$ If the profinite spaces X, Y, and Y' are p-good, the cochain complex $C^*(\mathrm{Mat}(X_y); \mathbf{F}_p)$ is locally finite, and the profinite space Y is simply connected, then X' is also p-good.

Note that for $n \geqslant 1$, the profinite space $\widehat{K}(\mathbf{Z}_p, n)$ is p-good (Proposition E.7.5.1) and that the $\mathbb{E}_\infty$-algebra $C^*(\widehat{K}(\mathbf{Z}_p, n); \mathbf{F}_p)$ is locally finite (Remark E.7.5.2). Taking $X = \widehat{K}(\mathbf{Z}_p, n)$ for $n \geqslant 1$ and $Y = *$, we conclude that if a simply connected profinite space Z is p-good, then the product $Z \times \widehat{K}(\mathbf{Z}_p, n)$ is also p-good for $n \geqslant 2$. Similarly, if Z is simply connected and p-good, then the product $Z \times K(\mathbf{Z}/p^d \mathbf{Z}, n)$ is p-good for any $n \geqslant 2$ and any $d \geqslant 0$. Combining these observations with Proposition E.7.3.5, we deduce that $\widehat{K}(M, n)$ is p-good for $n \geqslant 2$ and any finitely generated $\mathbf{Z}_p$-module M. More generally, if M is any profinite abelian group, then Lemma E.7.5.3 and Proposition E.7.6.3 supply isomorphisms

$$\mathrm{H}^*(K(M, n); \mathbf{F}_p) \simeq \mathrm{H}^*(K(M_p, n); \mathbf{F}_p) \qquad \mathrm{H}^*(\widehat{K}(M, n); \mathbf{F}_p) \simeq \mathrm{H}^*(\widehat{K}(M_p, n); \mathbf{F}_p),$$

where M_p denotes the closed subgroup of topologically p-singular elements of M. It follows that $\widehat{K}(M, n)$ is p-good if and only if $\widehat{K}(M_p, n)$ is p-good. In particular, if M is a profinite abelian group of finite type, then $\widehat{K}(M, n)$ is p-good for all $n \geqslant 2$.

Let X be a simply connected profinite space, and write $X = \varprojlim \{\tau_{\leqslant n} X\}$. The canonical map $\mathrm{Mat}(X) \to \mathrm{Mat}(\tau_{\leqslant n} X)$ has $(n+1)$-connective homotopy fibers, and therefore induces isomorphisms

$$\mathrm{H}^m(\mathrm{Mat}(\tau_{\leqslant n} X); \mathbf{F}_p) \simeq \mathrm{H}^m(\mathrm{Mat}(X); \mathbf{F}_p)$$

for $m \leqslant n$. We therefore have a commutative diagram

$$\begin{array}{ccc} \varinjlim \mathrm{H}^*(\tau_{\leqslant n} X; \mathbf{F}_p) & \longrightarrow & \varinjlim \mathrm{H}^*(\mathrm{Mat}(\tau_{\leqslant n} X); \mathbf{F}_p) \\ \downarrow & & \downarrow \\ \mathrm{H}^*(X; \mathbf{F}_p) & \longrightarrow & \mathrm{H}^*(\mathrm{Mat}(X); \mathbf{F}_p). \end{array}$$

where the vertical maps are isomorphisms. Consequently, to prove that X is p-good, it suffices to show that each truncation $\tau_{\leqslant n} X$ is p-good.

Assume now that X has finite type. We prove by induction on n that the truncation $\tau_{\leqslant n} X$ is good, the case $n = 1$ being trivial. If $n > 1$, then Notation E.7.6.1 supplies a pullback diagram of profinite spaces

$$\begin{array}{ccc} \tau_{\leqslant n} X & \longrightarrow & * \\ \downarrow & & \downarrow \phi \\ \tau_{\leqslant n-1} X & \longrightarrow & \widehat{K}(M, n+1) \end{array}$$

where $M = \pi_n X$ is a profinite abelian group of finite type. Since $\tau_{\leqslant n-1} X$ and $\widehat{K}(M, n+1)$ are p-good, we are reduced to proving that the fiber $F = \mathrm{fib}(\phi) \simeq \widehat{K}(M, n)$ has the property that $C^*(\mathrm{Mat}(F); \mathbf{F}_p)$ is of finite type. Equivalently, we must show that the homology groups $\mathrm{H}_*(K(M, n); \mathbf{F}_p)$ are finite in each degree. Since M is of finite type, it can be written as a finite product of factors which are isomorphic to $\mathbf{Z}_p$, $\mathbf{Z}/p^d\,\mathbf{Z}$, and abelian groups on which p acts by an isomorphism. Using the Künneth formula and Lemma E.7.5.3, we are reduced to proving that the homology groups

$$\mathrm{H}_*(K(\mathbf{Z}_p, n); \mathbf{F}_p) \qquad \mathrm{H}_*(K(\mathbf{Z}/p^d\,\mathbf{Z}, n); \mathbf{F}_p)$$

are finite in each degree, which follows from Remark E.7.5.2. □

Proof of Proposition E.8.2.3. Since X and $\mathrm{Mat}(X)^\wedge_\pi$ are simply connected, it will suffice to show that composition with v induces a homotopy equivalence

$$\theta_Y : \mathrm{Map}_{\mathcal{S}^\wedge_\pi}(X, Y) \to \mathrm{Map}_{\mathcal{S}^\wedge_\pi}(\mathrm{Mat}(X)^\wedge_\pi, Y)$$

for every simply connected profinite space Y. Writing Y as the limit of a filtered diagram of simply connected π-finite spaces, we can reduce to the case where Y is π-finite. Then Y is n-truncated for some integer n. We proceed by induction on n, the case $n \leqslant 1$ being trivial. In general, we have a fiber sequence $Y \to \tau_{\leqslant n-1} Y \to K(M, n+1)$ where $M = \pi_n Y$. Since $\theta_{\tau_{\leqslant n-1} Y}$ is a homotopy equivalence by the inductive hypothesis, we are reduced to proving that $\theta_{K(M,n+1)}$ is an homotopy equivalence. For this, it suffices to show that v induces an isomorphism $\mathrm{H}^*(X; M) \to \mathrm{H}^*(\mathrm{Mat}(X)^\wedge_\pi; M)$ (see Remark E.7.1.6). Since M

is finite, Remark E.7.1.7 allows us to identify this with the canonical map $\mathrm{H}^*(X; M) \to \mathrm{H}^*(\mathrm{Mat}(X); M)$. The collection of those finite abelian groups M for which this map is an isomorphism is closed under extension (as in the proof of Proposition E.7.6.3). We may therefore reduce to the case where $M = \mathbf{F}_p$ for some prime number p, in which case the desired result follows from Lemma E.8.3.1. □

Remark E.8.3.2. Let X be a simply connected space, and suppose that the cohomology group $\mathrm{H}^n(X; \mathbf{F}_p)$ is finite for every prime number p and every integer $n \geqslant 0$. Then the canonical map $\theta : X \to \mathrm{Mat}(X^\wedge_\pi)$ exhibits $\mathrm{Mat}(X^\wedge_\pi)$ as a localization of X with respect to the family of homology theories $\{\mathrm{H}_*(\bullet; \mathbf{F}_p)\}$, in the ∞-category $\mathcal{S}^{\geqslant 2}$ of simply connected spaces. More precisely, we have the following:

- (a) The map θ induces an isomorphism on homology groups $\mathrm{H}_*(X; \mathbf{F}_p) \to \mathrm{H}_*(\mathrm{Mat}(X^\wedge_\pi); \mathbf{F}_p)$ for every prime number p.

- (b) Let $f : Y \to Z$ be a map of simply connected spaces which induces an isomorphism $\mathrm{H}_*(Y; \mathbf{F}_p) \to \mathrm{H}_*(Z; \mathbf{F}_p)$ for every prime number p. Then composition with f induces a homotopy equivalence $\mathrm{Map}_{\mathcal{S}}(Z, \mathrm{Mat}(X^\wedge_\pi)) \to \mathrm{Map}_{\mathcal{S}}(Y, \mathrm{Mat}(X^\wedge_\pi))$.

Assertion (a) is an immediate consequence of Lemma E.8.3.1 and Remark E.7.1.7. To prove (b), it suffices to show that f induces an equivalence of simply connected profinite spaces $Y^\wedge_\pi \to Z^\wedge_\pi$. Equivalently, we must show that for every simply connected π-finite space T, composition with f induces a homotopy equivalence $\mu_T : \mathrm{Map}_{\mathcal{S}}(Z, T) \to \mathrm{Map}_{\mathcal{S}}(Y, T)$. Choose an integer n such that T is n-truncated. We proceed by induction on n, the case $n \geqslant 1$ being trivial. To handle the inductive step, we note that there is a fiber sequence $T \to \tau_{\leqslant n-1} T \to K(M, n+1)$ where $M = \pi_n T$. Since the inductive hypothesis guarantees that $\mu_{\tau_{\leqslant n-1} T}$ is a homotopy equivalence, we are reduced to proving that $\mu_{K(M,n+1)}$ is a homotopy equivalence. For this, it suffices to show that f induces an isomorphism $\mathrm{H}^*(Z; M) \to \mathrm{H}^*(Y; M)$. The collection of those finite abelian groups M which satisfy this condition is closed under extension; we may therefore reduce to the case where $M = \mathbf{F}_p$ for some prime number p, in which case the desired result follows immediately from our hypothesis that f induces an isomorphism on $\mathbf{F}_p$-homology.

E.8.4 The Proof of Proposition E.8.2.4

We begin with some auxiliary algebraic results.

Lemma E.8.4.1. *Let M be a congruence-finite abelian group, let d be a positive integer, and let x be an element of the kernel of the projection map $\widehat{M} \to M/dM$. Then we can write $x = dy$ for some $y \in \widehat{M}$.*

Proof. For every positive integer d', let x_d denote the image of x in $\ker(M/dd'M \to M/dM)$, and let $S_{d'}$ denote the preimage of x under the surjection $M/d'M \xrightarrow{d} \ker(M/dd'M \to M/dM)$. Then $\{S_{d'}\}_{d'>0}$ is a filtered diagram of nonempty finite sets. Applying Proposition E.1.1.1, we deduce that $\varprojlim S_{d'}$ is nonempty. □

Lemma E.8.4.2. *Let M be a congruence-finite abelian group, let d be a positive integer, and let $x \in \widehat{M}$ be an element satisfying $dx = 0$. Then x is the image of an element $y \in M$ satisfying $dy = 0$.*

Proof. For every positive integer d', let $S_{d'}$ denote the collection of those elements $y \in M$ such that $dy = 0$ and y represents the image of x in $M/dd'M$. We wish to show that the inverse limit $\varprojlim S_{d'}$ is nonempty. Since the map $M \xrightarrow{d} M$ has finite kernel, each $S_{d'}$ is a finite set. By virtue of Proposition E.1.1.1, it will suffice to show that each $S_{d'}$ is nonempty. To prove this, choose an arbitrary element $y \in M$ representing the image of x in $M_{dd'}M$. Since $dx = 0$, we can write $dy = dd'z$ for some $z \in M$. Then $y - d'z$ belongs to $S_{d'}$. □

Lemma E.8.4.3. *Let M be an abelian group. The following conditions are equivalent:*

(1) *The group M is congruence-separated.*

(2) *The group M is congruence-finite, and the canonical map $u : M \to \widehat{M}$ is an injection whose cokernel is a rational vector space.*

(3) *There exists an exact sequence of abelian groups $0 \to M \to N \to V \to 0$ where N is congruence-complete and V is a rational vector space.*

Proof. Suppose first that (1) is satisfied. Then M is congruence-finite and the map $u : M \to \widehat{M}$ is injective. Let $V = \operatorname{coker}(u)$. We claim that V is a rational vector space. To prove this, it will suffice to show that for every positive integer d, the map $d : V \to V$ is bijective. To prove surjectivity, fix an element $\overline{x} \in V$, represented by an element $x \in \widehat{M}$. Let x_d denote the image of x in M/dM, and let $x' \in M$ be a representative of x_d. Replacing x by $x - u(x')$, we may assume that $x_d = 0$. Applying Lemma E.8.4.1, we can write $x = dy$ for some $y \in \widehat{M}$, so that $\overline{x}$ belongs to the image of the map $d : V \to V$.

To prove injectivity, suppose that $\overline{x} \in V$ satisfies $d\overline{x} = 0$. Let $x \in \widehat{M}$ be a representative of $\overline{x}$, so that $dx = u(y)$ for some $y \in M$. Note that dx belongs to the kernel of the map $\widehat{M} \to M/dM$, so that $y \in dM$. Write $y = dx'$ for some $x' \in M$. Replacing x by $x - u(x')$, we can reduce to the case where $dx = 0$. Applying Lemma E.8.4.2, we deduce that x belongs to the image of u, so that $\overline{x} = 0$. This completes the proof that (1) $\Rightarrow$ (2).

To prove that (2) $\Rightarrow$ (3), we will show the congruence completion $\widehat{M}$ is congruence-complete. For this, it suffices to show that $\widehat{M}$ is a profinite abelian group of finite type (Proposition E.8.1.6). Let p be a prime number; we wish to show that $\widehat{M}/p\widehat{M}$ is finite (see

Proposition E.7.3.5). This is clear, since there exists an exact sequence of abelian groups $M/pM \to \widehat{M}/p\widehat{M} \to V/pV$ where M/pM is finite and $V/pV \simeq 0$.

We now show that (3) $\Rightarrow$ (1). Choose an exact sequence $0 \to M \xrightarrow{\phi} N \to V \to 0$ where N is congruence-complete and V is a rational vector space. For every positive integer d, we have $\mathrm{Tor}_0(V, \mathbf{Z}/d\mathbf{Z}) \simeq \mathrm{Tor}_1(V, \mathbf{Z}/d\mathbf{Z}) \simeq 0$, so that ϕ induces an isomorphism $M/dM \to N/dN$. It follows that the lower horizontal map in the diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow{\scriptstyle\psi} & & \downarrow \\ \widehat{M} & \longrightarrow & \hat{N} \end{array}$$

is an isomorphism. Since N is congruence-complete, the right vertical map is also an isomorphism. Since ϕ is injective, we conclude that ψ is also injective: that is, M is congruence-separated. □

Remark E.8.4.4. The proof of the implication (3) $\Rightarrow$ (1) in Lemma E.8.4.3 shows that if M is a congruence-finite abelian group and we are given any injection $u : M \hookrightarrow N$, where N is congruence-complete and N/M is a rational vector space, then u induces an isomorphism of N with the congruence-completion of M.

Lemma E.8.4.5. *Let M be a congruence-separated abelian group, and let p be a prime number. Then the cohomology groups $\mathrm{H}^m(K(M,n); \mathbf{F}_p)$ are finite for $n \geqslant 2$.*

Proof. Lemma E.8.4.3 supplies an exact sequence $0 \to M \to \widehat{M} \to V \to 0$, where V is a rational vector space. We therefore have a fiber sequence of Eilenberg-MacLane spaces $K(V, n-1) \to K(M,n) \to K(\widehat{M}, n)$. Since $\mathrm{H}^d(K(V, n-1); \mathbf{F}_p) \simeq 0$ for $d > 0$ (Lemma E.7.5.3), the map $\mathrm{H}^*(K(\widehat{M}, n); \mathbf{F}_p) \to \mathrm{H}^*(K(M,n); \mathbf{F}_p)$ is an isomorphism. It will therefore suffice to show that $\mathrm{H}^*(K(\widehat{M}, n); \mathbf{F}_p)$ is finite dimensional in each degree. Let us regard $\widehat{M}$ as a profinite abelian group, and let $\widehat{K}(\widehat{M}, n)$ be the profinite space introduced in Notation E.7.6.1. Since $\widehat{M}$ has finite type, Lemma E.8.3.1 supplies an isomorphism $\mathrm{H}^*(\widehat{K}(\widehat{M}, n); \mathbf{F}_p) \simeq \mathrm{H}^*(K(\widehat{M}, n); \mathbf{F}_p)$. We are therefore reduced to proving that $\mathrm{H}^*(K(\widehat{M}, n); \mathbf{F}_p)$ is finite-dimensional in each degree, which is a special case of Proposition E.7.6.3. □

Lemma E.8.4.6. *Let X be a simply connected space, and suppose that each homotopy group $\pi_m X$ is congruence-separated. Then for every prime number p and every $n \geqslant 0$, the cohomology group $\mathrm{H}^n(X; \mathbf{F}_p)$ is finite.*

Proof. Replacing X by $\tau_{\leqslant n} X$, we may assume that X is m-truncated for some integer m. We proceed by induction on m, the case $m \leqslant 1$ being trivial. To carry out the inductive step, we note that there is a pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow u \\ \tau_{\leqslant m-1}X & \longrightarrow & K(\pi_m X, m+1). \end{array}$$

The homotopy fibers of u can be identified with $K(\pi_m X, m)$, so that $C^*(\mathrm{fib}(u); \mathbf{F}_p)$ is locally finite by virtue of Lemma E.8.4.5. Applying Corollary **??**, we obtain an equivalence

$$C^*(X; \mathbf{F}_p) \simeq C^*(\tau_{\leqslant m-1}X; \mathbf{F}_p) \otimes_{C^*(K(\pi_m X, m+1); \mathbf{F}_p)} \mathbf{F}_p.$$

Using the inductive hypothesis and Lemma **??**, we are reduced to proving that $C^*(K(\pi_m X, m+1); \mathbf{F}_p)$ is locally finite, which follows from Lemma E.8.4.5. □

Lemma E.8.4.7. *Let M be a congruence-separated abelian group, and let V be a rational vector space. Then every group homomorphism $V \to M$ is trivial.*

Proof. Since M is congruence-separated, the natural map from M to its congruence-completion $\widehat{M}$ is injective. It will therefore suffice to show that every map $f : V \to \widehat{M}$ is trivial. Writing $\widehat{M}$ as the inverse limit $\varprojlim M/dM$, we are reduced to proving that every map $V \to M/dM$ is trivial. This is clear (since d acts invertibly on V). □

Lemma E.8.4.8. *Let X be a connected space with base point $x \in X$. Assume that $\pi_1(X, x)$ is abelian, and that for each $n \geqslant 2$ the action of $\pi_1(X, x)$ on $\pi_n(X, x)$ is nilpotent (that is, $\pi_n(X, x)$ admits a finite filtration by $\pi_1(X, x)$-invariant submodules whose successive quotients are acted on trivially by $\pi_1(X, x)$). Let p be a prime number. Then the following conditions are equivalent:*

- (a) *Each homotopy group $\pi_n(X, x)$ is a module over $\mathbf{Z}[p^{-1}]$.*
- (b) *The cohomology groups $\mathrm{H}^n(X; \mathbf{F}_p)$ vanish for $n > 0$.*

Proof. Suppose first that (a) is satisfied. We wish to prove that $\mathrm{H}^n(X; \mathbf{F}_p) \simeq 0$ for $n > 0$. Replacing X by $\tau_{\leqslant n}X$, we can reduce to the case where X is m-truncated for some integer m. We proceed by induction on m, the case $m = 0$ being trivial. To carry out the inductive step, we observe that there is a fiber sequence $K(\pi_m(X, x), m) \to X \to \tau_{\leqslant m-1}X$. Using assumption (a) and Lemma E.7.5.3, we deduce that $\mathrm{H}^d(K(\pi_m(X, x), m); \mathbf{F}_p) \simeq 0$ for $d > 0$, so that the restriction map $\mathrm{H}^*(\tau_{\leqslant m-1}X; \mathbf{F}_p) \to \mathrm{H}^*(X; \mathbf{F}_p)$ is an isomorphism. The desired vanishing now follows from the inductive hypothesis.

We now prove that $(b) \Rightarrow (a)$. Suppose, for a contradiction, that (a) is not satisfied. Then there exists some smallest integer n such that the action of p on $\pi_n(X, x)$ is not invertible. Let $Y = \tau_{\leqslant n-1}X$, let $y \in Y$ denote the image of the point x, and let F denote the homotopy fiber of the map $X \to Y$ over the point y. Then Y satisfies condition (a). The first part of the proof shows that $\mathrm{H}^m(Y; \mathbf{F}_p) \simeq 0$ for $m > 0$. Using the Serre spectral sequence, we deduce the following:

(a) The cohomology group $\mathrm{H}^n(X; \mathbf{F}_p)$ can be identified with the group of invariants for the action of $\pi_1(Y, y)$ on $\mathrm{H}^n(F; \mathbf{F}_p)$.

(b) There is a surjection from $\mathrm{H}^{n+1}(X; \mathbf{F}_p)$ to the group of invariants for the action of $\pi_1(Y, y)$ on $\mathrm{H}^{n+1}(F; \mathbf{F}_p)$.

Since F is n-connective, we have $\mathrm{H}^n(F; \mathbf{F}_p) \simeq \mathrm{Hom}(\pi_n F; \mathbf{F}_p) \simeq \mathrm{Hom}(\pi_n(X, x); \mathbf{F}_p)$. It follows that the action of $\pi_1(Y, y)$ on $\mathrm{H}^n(F; \mathbf{F}_p)$ is nilpotent. Since $\mathrm{H}^n(X; \mathbf{F}_p) \simeq 0$, assertion (a) implies that $\mathrm{H}^n(F; \mathbf{F}_p) \simeq 0$. It follows that the map $\pi_n(X, x) \xrightarrow{p} \pi_n(X, x)$ is surjective.

We have a fiber sequence $F' \to F \to K(\pi_n(X, x), n)$. Since F' is $(n+1)$-connective, the induced map

$$\mathrm{Ext}^1_{\mathbf{Z}}(\pi_n(X, x); \mathbf{F}_p) \simeq \mathrm{H}^{n+1}(K(\pi_n(X, x), n); \mathbf{F}_p) \to \mathrm{H}^{n+1}(F; \mathbf{F}_p)$$

is injective. Using (b) and our assumption that $\mathrm{H}^{n+1}(X; \mathbf{F}_p) \simeq 0$, we conclude that $\mathrm{Ext}^1_{\mathbf{Z}}(\pi_n(X, x); \mathbf{F}_p)$ does not contain any nonzero elements which are invariant under the action of $\pi_1(Y, y) \simeq \pi_1(X, x)$. Since the action of $\pi_1(X, x)$ on $\mathrm{Ext}^1_{\mathbf{Z}}(\pi_n(X, x); \mathbf{F}_p)$ is nilpotent, we conclude that $\mathrm{Ext}^1_{\mathbf{Z}}(\pi_n(X, x); \mathbf{F}_p) \simeq 0$: that is, the multiplication map $\pi_n(X, x) \xrightarrow{p} \pi_n(X, x)$ is injective. □

Proof of Proposition E.8.2.4. Let X be a simply connected space, and suppose that each homotopy group of X is congruence-separated. Choose a base point $x \in X$. Let $X^\wedge_\pi$ be the profinite completion of X. Using Lemma E.8.4.6 and Remark E.7.1.7, we deduce that each cohomology group $\mathrm{H}^n(X^\wedge_\pi; \mathbf{F}_p)$ is finite, so that $X^\wedge_\pi$ is of finite type (Theorem E.7.0.5). Let u denote the unit map $X \to \mathrm{Mat}(X^\wedge_\pi)$, and let F denote the homotopy fiber of u (taken over the image of x), and let $\overline{x}$ denote the base point of F determined by x. Since the domain and codomain of u are simply-connected, F is a connected space with abelian fundamental group $\pi_1(F, \overline{x})$, and the action of $\pi_1(F, \overline{x})$ on the higher homotopy groups $\pi_n(F, \overline{x})$ is nilpotent.

Let p be a prime number, and let κ denote the finite field $\mathbf{F}_p$. Using Remark E.7.1.7 and Lemma E.8.3.1, we conclude that u induces an isomorphism $\mathrm{H}^*(\mathrm{Mat}(X^\wedge_\pi; \kappa) \to \mathrm{H}^*(X; \kappa)$. Let $\mathscr{F} \in \mathrm{Fun}(\mathrm{Mat}(X^\wedge_\pi); \mathrm{Mod}_\kappa)$ denote the direct image of the constant local system $\underline{\kappa}_X$, and let $\theta : \underline{\kappa}_{\mathrm{Mat}(X^\wedge_\pi)} \to \mathscr{F}$ be the unit map. Then θ induces an equivalence

$$C^*(\mathrm{Mat}(X^\wedge_\pi); \kappa) \to C^*(\mathrm{Mat}(X^\wedge_\pi); \mathscr{F}) \simeq C^*(X; \kappa).$$

It follows that $C^*(\mathrm{Mat}(X^\wedge_\pi); \mathrm{cofib}(\theta)) \simeq 0$. Since $\mathrm{Mat}(X^\wedge_\pi)$ is simply connected, we conclude that $\mathrm{cofib}(\theta)$ is an equivalence. It follows that $\mathrm{H}^m(F; \kappa) \simeq 0$ for $m > 0$. Applying Lemma E.8.4.8, we conclude that each homotopy group of F is a module over $\mathbf{Z}[p^{-1}]$. Since this holds for every prime number p, we conclude that the homotopy groups of F are rational vector spaces.

We have a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F, \overline{x}) \xrightarrow{\phi} \pi_n(X, x) \to \pi_n(\mathrm{Mat}(X_\pi^\wedge), u(x)) \to \pi_{n-1}(F, \overline{x}) \xrightarrow{\psi} \pi_{n-1}(X, x) \to \cdots$$

Using Lemma E.8.4.7, we see that the maps ϕ and ψ are trivial, so that we have a short exact sequence

$$0 \to \pi_n(X, x) \to \pi_n(\mathrm{Mat}(X_\pi^\wedge), u(x)) \to \pi_{n-1}(F, \overline{x}) \to 0.$$

Applying Remark E.8.4.4, we deduce that u exhibits $\pi_n(\mathrm{Mat}(X_\pi^\wedge), u(x))$ as the congruence-completion of $\pi_n(X, x)$. □

E.9 The Arithmetic Square

Let M be a finitely generated abelian group, let $\widehat{M} \simeq \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} M$ denote the congruence completion of M (Definition E.8.1.1), and let $M_{\mathbf{Q}} \simeq \mathbf{Q} \otimes_{\mathbf{Z}} M$ denote the rational vector space obtained by tensoring M with the field of rational numbers. Then there is a pullback diagram of abelian groups

$$\begin{array}{ccc} M & \longrightarrow & \widehat{M} \\ \downarrow & & \downarrow \\ M_{\mathbf{Q}} & \xrightarrow{\phi} & \widehat{M}_{\mathbf{Q}}. \end{array}$$

Consequently, we can recover M (up to canonical isomorphism) from its congruence completion $\widehat{M}$, the rational vector space $M_{\mathbf{Q}}$, and the map ϕ. Our goal in this section is to establish the following homotopy-theoretic analogue:

Theorem E.9.0.9 (Sullivan's Arithmetic Square)**.** *Let X be a simply connected space, and assume that the cohomology group $\mathrm{H}^n(X; \mathbf{F}_p)$ is finite for each prime number p and each $n \geqslant 0$. Then the diagram of spaces*

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Mat}(X_\pi^\wedge) \\ \downarrow & & \downarrow \\ X_{\mathbf{Q}} & \longrightarrow & \mathrm{Mat}(X_\pi^\wedge)_{\mathbf{Q}} \end{array}$$

is a pullback square in the ∞-category $\mathcal{S}$.

Remark E.9.0.10. In the statement of Theorem E.9.0.9, the symbols $X_{\mathbf{Q}}$ and $\mathrm{Mat}(X_\pi^\wedge)_{\mathbf{Q}}$ denote rationalizations of the space X and $\mathrm{Mat}(X_\pi^\wedge)$, respectively. We refer the reader to §**??** for a review of rational homotopy theory.

E.9.1 Example: Spaces with Finitely Generated Homotopy Groups

Let us first consider the special case of Theorem E.9.0.9 in which X is a simply connected space whose homotopy groups are finitely generated (note that any such space satisfies the hypotheses of Theorem E.9.0.9: see Example E.8.1.5 and Proposition E.8.2.4). Form a pullback diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{Mat}(X_{\pi}^{\wedge}) \\ \downarrow & & \downarrow \\ X_{\mathbf{Q}} & \longrightarrow & \mathrm{Mat}(X_{\pi}^{\wedge})_{\mathbf{Q}}. \end{array}$$

To prove Theorem E.9.0.9, we must show that the canonical map $\rho : X \to Y$ is a homotopy equivalence. This can be established by direct calculation. Fix an integer $n \geqslant 1$, a base point $x \in X$, and let $y = \rho(x)$ be the image of x in Y. Set $M = \pi_n X$ and $N = \pi_{n+1} X$, and let $\widehat{M}$ and $\widehat{N}$ denote the congruence-completions of M and N. We then have canonical isomorphisms

$$\pi_n \mathrm{Mat}(X_{\pi}^{\wedge}) \simeq \widehat{M} \qquad \pi_n \mathrm{Mat}(X_{\pi}^{\wedge})_{\mathbf{Q}} \simeq \widehat{M}_{\mathbf{Q}} \qquad \pi_n X_{\mathbf{Q}} \simeq M_{\mathbf{Q}}$$

$$\pi_{n+1} \mathrm{Mat}(X_{\pi}^{\wedge}) \simeq \widehat{N} \qquad \pi_{n+1} \mathrm{Mat}(X_{\pi}^{\wedge})_{\mathbf{Q}} \simeq \widehat{N}_{\mathbf{Q}} \qquad \pi_{n+1} X_{\mathbf{Q}} \simeq N_{\mathbf{Q}}.$$

It follows that the homotopy group $\pi_n(Y, y)$ fits into a long exact sequence

$$\widehat{N} \oplus N_{\mathbf{Q}} \xrightarrow{\phi} \widehat{N}_{\mathbf{Q}} \to \pi_n(Y, y) \to \widehat{M} \oplus M_{\mathbf{Q}} \xrightarrow{\psi} \widehat{M}_{\mathbf{Q}}.$$

Since M and N are finitely generated abelian groups, an elementary calculation shows that ϕ is surjective, so that the diagram

$$\begin{array}{ccc} \pi_n(Y, y) & \longrightarrow & \pi_n \mathrm{Mat}(X_{\pi}^{\wedge}) \\ \downarrow & & \downarrow \\ \pi_n X_{\mathbf{Q}} & \longrightarrow & \pi_n \mathrm{Mat}(X_{\pi}^{\wedge})_{\mathbf{Q}}. \end{array}$$

is a pullback square. Consequently, to show that the map ρ induces an isomorphism $\pi_n(X, x) \to \pi_n(Y, y)$, it will suffice to show that the diagram

$$\begin{array}{ccc} \pi_n(X, x) & \longrightarrow & \pi_n \mathrm{Mat}(X_{\pi}^{\wedge}) \\ \downarrow & & \downarrow \\ \pi_n X_{\mathbf{Q}} & \longrightarrow & \pi_n \mathrm{Mat}(X_{\pi}^{\wedge})_{\mathbf{Q}} \end{array}$$

is also a pullback square. In other words, we are reduced to showing that the diagram

$$\begin{array}{ccc} M & \longrightarrow & \widehat{M} \\ \downarrow & & \downarrow \\ M_{\mathbf{Q}} & \longrightarrow & \widehat{M}_{\mathbf{Q}} \end{array}$$

is a pullback, which is a purely algebraic consequence of our assumption that M is a finitely generated abelian group.

E.9.2 The Main Lemma

There are many examples of spaces X which satisfy the hypotheses of Theorem E.9.0.9 but whose homotopy groups are *not* finitely generated: for example, we can take $X = \mathrm{Mat}(Y)$ for any simply connected profinite space Y of finite type (Proposition E.8.2.3). To establish Theorem E.9.0.9 in general, we will need a slightly more sophisticated argument. First, let us introduce a bit of terminology.

Definition E.9.2.1. Let M be an abelian group, and let $M_{\mathbf{Q}} = \mathbf{Q} \otimes_{\mathbf{Z}} M$ denote its rationalization. We will say that a subset $U \subseteq M_{\mathbf{Q}}$ is *congruence-open* if, for every element $x \in U$, there exists an integer $n > 0$ such that $x + nM \subseteq U$. The collection of congruence-open sets determines a topology on $M_{\mathbf{Q}}$, which we will refer to as the *congruence topology.*

The main ingredient in our proof of Theorem E.9.0.9 is the following:

Lemma E.9.2.2. *Let Y be a simply connected profinite space of finite type, let Z be a simply connected rational space, and let $\phi : Z \to \mathrm{Mat}(Y)_{\mathbf{Q}}$ be a map satisfying the following technical condition:*

$(*)$ *The image of the map $\pi_2 Z \to \pi_2 \mathrm{Mat}(Y)_{\mathbf{Q}} \simeq (\pi_2 Y)_{\mathbf{Q}}$ is dense with respect to the congruence topology of Definition E.9.2.1.*

Form a pullback diagram σ :

$$\begin{array}{ccc} \overline{Z} & \xrightarrow{\psi} & \mathrm{Mat}(Y) \\ \downarrow{\scriptstyle \psi'} & & \downarrow \\ Z & \xrightarrow{\phi} & \mathrm{Mat}(Y)_{\mathbf{Q}}. \end{array}$$

Then:

(a) *The space $\overline{Z}$ is simply connected.*

(b) *The map ψ induces isomorphisms $\mathrm{H}^*(\mathrm{Mat}(Y); \mathbf{F}_p) \to \mathrm{H}^*(\overline{Z}; \mathbf{F}_p)$ for every prime number p.*

(c) *The cohomology group $\mathrm{H}^n(\overline{Z}; \mathbf{F}_p)$ is finite for every integer $n \geqslant 0$ and every prime number p.*

(d) *The map ψ' induces an isomorphism $\mathrm{H}^*(Z; \mathbf{Q}) \to \mathrm{H}^*(\overline{Z}; \mathbf{Q})$.*

Proof. Since $\mathrm{Mat}(Y)$ and Z are connected and $\mathrm{Mat}(Y)_{\mathbf{Q}}$ is simply connected, we immediately see that $\overline{Z}$ is connected. Choose a base point $z \in \overline{Z}$, so that we have an exact sequence

$$\pi_2(Z, \psi'(z)) \oplus \pi_2(Y, \psi(z)) \xrightarrow{\mu} \pi_2(\mathrm{Mat}(Y)_{\mathbf{Q}}, (\phi\psi')(z)) \to \pi_1(\overline{Z}, z) \to 0.$$

To prove (a), it suffices to show that the map μ is surjective. This is equivalent to the surjectivity of the map $\pi_2 Z \to \mathrm{coker}(\pi_2 Y \to (\pi_2 Y)_{\mathbf{Q}})$, which is a reformulation of assumption $(*)$.

Let κ denote the finite field $\mathbf{F}_p$ for some prime number p, let $\underline{\kappa}_Z$ denote the constant local system on Z with value κ, and let $\mathscr{F} \in \mathrm{Fun}(\mathrm{Mat}(Y)_{\mathbf{Q}}; \mathrm{Mod}_\kappa)$ denote the direct image of $\underline{\kappa}_Z$ under the map ϕ. We have a fiber sequence of local systems $\underline{\kappa}_{\mathrm{Mat}(Y)_{\mathbf{Q}}} \xrightarrow{u} \mathscr{F} \to \mathscr{F}'$. Since Z and $\mathrm{Mat}(Y)_{\mathbf{Q}}$ are simply connected rational spaces, Lemma E.8.4.8 implies that the map ϕ induces an equivalence

$$C^*(\mathrm{Mat}(Y)_{\mathbf{Q}}; \kappa) \to C^*(Z; \kappa) \simeq C^*(\mathrm{Mat}(Y)_{\mathbf{Q}}; \mathscr{F}),$$

so that $C^*(\mathrm{Mat}(Y)_{\mathbf{Q}}; \mathscr{F}') \simeq 0$. Note that $\pi_n \mathscr{F}' \simeq 0$ for $n > 0$. Consequently, if $\mathscr{F}' \neq 0$, there is a largest integer m such that $\pi_m \mathscr{F}' \neq 0$. Since $\mathrm{Mat}(Y)_{\mathbf{Q}}$ is simply connected, it would then follow that $\pi_m C^*(\mathrm{Mat}(Y)_{\mathbf{Q}}; \mathscr{F}') \neq 0$, and we would obtain a contradiction. It follows that $\mathscr{F}' \simeq 0$: that is, the unit map u is an equivalence of local systems on $\mathrm{Mat}(Y)_{\mathbf{Q}}$. Since the diagram σ is a pullback square, it follows that $\underline{\kappa}_{\mathrm{Mat}(Y)}$ is the direct image of the constant local system $\underline{\kappa}_{\overline{Z}}$, so that ψ' induces an isomorphism $\mathrm{H}^*(\mathrm{Mat}(Y); \kappa) \to \mathrm{H}^*(\overline{Z}; \kappa)$ as desired. This proves (b). Assertion (c) follows from (b), Lemma E.8.3.1, and Theorem E.7.0.5.

We now prove (d). Let $\mathscr{G} \in \mathrm{Fun}(\mathrm{Mat}(Y)_{\mathbf{Q}}; \mathrm{Mod}_{\mathbf{Q}})$ denote the direct image of the constant local system $\underline{\mathbf{Q}}_{\mathrm{Mat}(Y)}$ along the map $\nu : \mathrm{Mat}(Y) \to \mathrm{Mat}(Y)_{\mathbf{Q}}$, and let $u' : \underline{\mathbf{Q}}_{\mathrm{Mat}(Y)_{\mathbf{Q}}} \to \mathscr{G}$ denote the unit map. Since ν is a rational homotopy equivalence, the map u' induces an equivalence

$$C^*(\mathrm{Mat}(Y)_{\mathbf{Q}}; \mathbf{Q}) \to C^*(\mathrm{Mat}(Y); \mathbf{Q}) \simeq C^*(\mathrm{Mat}(Y)_{\mathbf{Q}}; \mathscr{G}).$$

It follows that $C^*(\mathrm{Mat}(Y)_{\mathbf{Q}}; \mathrm{cofib}(u')) \simeq 0$. Arguing as above, we conclude that $\mathrm{cofib}(u') \simeq 0$ so that u' is an equivalence. Since σ is a pullback square, we conclude that $\underline{\mathbf{Q}}_Z$ is the direct image of the constant local system $\underline{\mathbf{Q}}_{\overline{Z}}$ along ψ', so that ψ' induces an isomorphism $\mathrm{H}^*(Z; \mathbf{Q}) \to \mathrm{H}^*(\overline{\mathbf{Z}}; \mathbf{Q})$. □

E.9.3 The Proof of Theorem E.9.0.9

In the proof of Theorem E.9.0.9, we will need the following standard homotopy-theoretic fact:

Lemma E.9.3.1. *Let $f : X \to Y$ be a map of simply connected spaces, and suppose that f satisfies the following conditions:*

(1) *For every prime number p, the map f induces an isomorphism* $\mathrm{H}^*(Y;\mathbf{F}_p) \to \mathrm{H}^*(X;\mathbf{F}_p)$.

(2) *The map f induces an isomorphism* $\mathrm{H}^*(Y;\mathbf{Q}) \to \mathrm{H}^*(X;\mathbf{Q})$.

Then f is a homotopy equivalence.

Proof. Since X and Y are simply connected, it will suffice to show that the natural map $\phi_{\mathbf{Z}} : \mathrm{H}_*(X;\mathbf{Z}) \to \mathrm{H}_*(Y;\mathbf{Z})$ is an isomorphism. Using the exact sequence of abelian groups $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$, we are reduced to showing that the maps

$$\phi_{\mathbf{Q}} : \mathrm{H}_*(X;\mathbf{Q}) \to \mathrm{H}_*(Y;\mathbf{Q}) \qquad \phi_{\mathbf{Q}/\mathbf{Z}} : \mathrm{H}_*(X;\mathbf{Q}/\mathbf{Z}) \to \mathrm{H}_*(Y;\mathbf{Q}/\mathbf{Z})$$

are isomorphisms. For the map $\phi_{\mathbf{Q}}$, this follows immediately from assumption (2). For the map $\phi_{\mathbf{Q}/\mathbf{Z}}$, we can write $\mathbf{Q}/\mathbf{Z}$ as a direct limit of finite abelian groups and thereby reduce to the problem of showing that the map $\phi_M : \mathrm{H}_*(X;M) \to \mathrm{H}_*(Y;M)$ is an equivalence when M is finite. Writing M as a successive extension of cyclic groups of prime order, we can reduce to the case $M = \mathbf{F}_p$, in which case the desired result follows from assumption (1). □

Proof of Theorem E.9.0.9. Let X be a simply connected space such that each cohomology group $\mathrm{H}^n(X;\mathbf{F}_p)$ is finite. Using Remark E.7.1.7 and Proposition E.7.3.5, we deduce that the profinite completion $X_\pi^\wedge$ has finite type. Note that for every finite abelian group M, we have

$$\mathrm{Hom}_c(\pi_2 X_\pi^\wedge; M) \simeq \mathrm{H}^2(X_\pi^\wedge; M) \simeq \mathrm{H}^2(X;M) \simeq \mathrm{Hom}(\pi_2 X; M),$$

so that the map $\pi_2 X \to \pi_2 X_\pi^\wedge$ has dense image. It follows that the map $\pi_2 X_{\mathbf{Q}} \to \pi_2 \mathrm{Mat}(X_\pi^\wedge)_{\mathbf{Q}}$ also has dense image (where we endow $\pi_2 \mathrm{Mat}(X_\pi^\wedge)_{\mathbf{Q}}$ with the congruence topology): that is, the map $\phi : X_{\mathbf{Q}} \to \mathrm{Mat}(X_\pi^\wedge)_{\mathbf{Q}}$ satisfies hypothesis $(*)$ of Lemma E.9.2.2. Set $Y = X_{\mathbf{Q}} \times_{\mathrm{Mat}(X_\pi^\wedge)_{\mathbf{Q}}} \mathrm{Mat}(X_\pi^\wedge)$. Applying Lemma E.9.2.2, we deduce the following:

(a) The space Y is simply connected.

(b) For every prime number p, the canonical map $\mathrm{H}^*(\mathrm{Mat}(X_\pi^\wedge);\mathbf{F}_p) \to \mathrm{H}^*(Y;\mathbf{F}_p)$ is an isomorphism.

(d) The map $\mathrm{H}^*(X_{\mathbf{Q}};\mathbf{Q}) \to \mathrm{H}^*(Y;\mathbf{Q})$ is an isomorphism.

The commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Mat}(X_\pi^\wedge) \\ \downarrow & & \downarrow \\ X_{\mathbf{Q}} & \longrightarrow & \mathrm{Mat}(X_\pi^\wedge)_{\mathbf{Q}} \end{array}$$

determines a map of spaces $f : X \to Y$. The composite map

$$\mathrm{H}^*(X_{\mathbf{Q}}; \mathbf{Q}) \to \mathrm{H}^*(Y; \mathbf{Q}) \to \mathrm{H}^*(X; \mathbf{Q})$$

is an isomorphism by construction, and the composite map

$$\mathrm{H}^*(\mathrm{Mat}(X^{\wedge}_{\pi}); \mathbf{F}_p) \to \mathrm{H}^*(Y; \mathbf{F}_p) \to \mathrm{H}^*(X; \mathbf{F}_p)$$

is an isomorphism for every prime number p by virtue of Lemma E.8.3.1 and Remark E.7.1.7. Combining these observations with (b) and (d), we conclude that f induces isomorphisms

$$\mathrm{H}^*(Y; \mathbf{Q}) \to \mathrm{H}^*(X; \mathbf{Q}) \qquad \mathrm{H}^*(Y; \mathbf{F}_p) \to \mathrm{H}^*(X; \mathbf{F}_p),$$

so that Lemma E.9.3.1 guarantees that f is a homotopy equivalence. □

Remark E.9.3.2. The construction $Y \mapsto \mathrm{Mat}(Y)_{\mathbf{Q}}$ determines a functor $\mathcal{S}^{\wedge}_{\pi} \to \mathcal{S}$. Let $\mathcal{C}$ denote the full subcategory of the fiber product $\mathrm{Fun}(\Delta^1, \mathcal{S}) \times_{\mathrm{Fun}(\{1\}, \mathcal{S})} \mathcal{S}^{\wedge}_{\pi}$ spanned by those triples $(Y, Z, \phi : Z \to \mathrm{Mat}(Y)_{\mathbf{Q}})$ where Y is a simply connected profinite space of finite type, Z is a simply connected rational space, and ϕ is a map satisfying hypothesis $(*)$ of Lemma E.9.2.2. Let $\mathcal{C}'$ denote the full subcategory of $\mathcal{S}$ spanned by the simply connected spaces X for which the cohomology groups $\mathrm{H}^n(X; \mathbf{F}_p)$ are finite dimensional for all prime numbers p and all $n \geqslant 0$. It follows from Lemma E.9.2.2 that the construction $(Y, Z, \phi) \mapsto Z \times_{\mathrm{Mat}(Y)_{\mathbf{Q}}} \mathrm{Mat}(Y)$ determines a functor $G : \mathcal{C} \to \mathcal{C}'$. The proof of Theorem E.9.0.9 shows that the construction $X \mapsto (X^{\wedge}_{\pi}, X_{\mathbf{Q}}, \phi : X_{\mathbf{Q}} \to \mathrm{Mat}(X^{\wedge}_{\pi})_{\mathbf{Q}})$ determines a functor $F : \mathcal{C}' \to \mathcal{C}$, which is easily seen to be left adjoint to G. Theorem E.9.0.9 shows that the unit map $u : \mathrm{id}_{\mathcal{C}'} \to G \circ F$ is an equivalence of functors from $\mathcal{C}'$ to itself. We claim that the counit map $v : F \circ G \to \mathrm{id}_{\mathcal{C}}$ is also an equivalence. In other words, we claim that for every object $(Y, Z, \phi) \in \mathcal{C}$, the canonical maps

$$f : (Z \times_{\mathrm{Mat}(Y)_{\mathbf{Q}}} \mathrm{Mat}(Y))^{\wedge}_{\pi} \to Y \qquad g : (Z \times_{\mathrm{Mat}(Y)_{\mathbf{Q}}} \mathrm{Mat}(Y))_{\mathbf{Q}} \to Z$$

are equivalences. This follows from assertions (b) and (d) of Lemma E.9.2.2, respectively. It follows that F and G are mutually inverse equivalences of ∞-categories.

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