Elliptic Cohomology III: Tempered Cohomology

July 27, 2019

Contents

1 Introduction

Let G be a finite group. We let $\mathrm{Rep}(G)$ denote the *complex representation ring* of G. That is, $\mathrm{Rep}(G)$ is the abelian group generated by symbols $[V]$, where V ranges over the collection of all finite-dimensional complex representations of G, subject to the relation

$$[V] = [V'] + [V'']$$

for every isomorphism of complex representations $V \simeq V' \oplus V''$. It is a free abelian group of finite rank, equipped with a canonical basis consisting of elements $[W]$, where W is an *irreducible* representations of G. We regard $\mathrm{Rep}(G)$ as a commutative ring, whose multiplication is characterized by the formula $[V] \cdot [W] = [V \otimes_{\mathbf{C}} W]$.

If V is a finite-dimensional complex representation of G, we let $\chi_V : G \to \mathbf{C}$ denote the *character* of V, given concretely by the formula

$$\chi_V(g) = \mathrm{Tr}(V \xrightarrow{g} V).$$

The character χ_V is an example of a *class function* on G: that is, it is invariant under conjugation (so $\chi_V(g) = \chi_V(hgh^{-1})$ for all $g, h \in G$). Using the identities

$$\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g) \qquad \chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g),$$

we see that the construction $[V] \mapsto \chi_V$ determines a ring homomorphism

$$\mathrm{Rep}(G) \to \{\text{Class functions } \chi : G \to \mathbf{C}\}.$$

The starting point for the character theory of finite groups is the following result (see Corollary 4.7.8):

Theorem 1.1.1. *Let G be a finite group. Then the characters of the irreducible representations of G form a basis for the vector space of class functions on G. Consequently, the construction $[V] \mapsto \chi_V$ induces an isomorphism of complex vector spaces*

$$\mathbf{C} \otimes_{\mathbf{Z}} \mathrm{Rep}(G) \simeq \{\text{Class functions } \chi : G \to \mathbf{C}\}.$$

Theorem 1.1.1 can be reformulated using the language of equivariant complex K-theory (see [20]). Given a topological space X equipped with an action of G, we let $\mathrm{KU}^0_G(X)$ denote the (0th) G-equivariant complex K-group of X. If X is a finite G-CW complex, then $\mathrm{KU}^0_G(X)$ is a finitely generated abelian group, which can be realized concretely as the Grothendieck group of G-equivariant complex vector bundles on X. In particular, when $X = *$ consists of a single point, we have a canonical isomorphism $\mathrm{KU}^0_G(*) \simeq \mathrm{Rep}(G)$. Theorem 1.1.1 can be generalized as follows (see Corollary 4.7.7):

Theorem 1.1.2. *Let G be a finite group and let X be a finite G-CW complex. For each $g \in G$, let $X^g = \{x \in X : x^g = x\}$ denote the set of fixed points for the action of g. We regard the disjoint union*

$$\coprod_{g \in G} X^g \simeq \{(g, x) \in G \times X : x^g = x\} \subseteq G \times X$$

as equipped with the right action of G given by the formula $(g, x)^h = (h^{-1}gh, x^h)$. Then there is a canonical isomorphism

$$\mathrm{ch}_G : \mathbf{C} \otimes_{\mathbf{Z}} \mathrm{KU}^0_G(X) \to \mathrm{H}^{\mathrm{ev}}((\coprod_{g \in G} X^g)/G; \mathbf{C}),$$

called the equivariant Chern character. Here

$$\mathrm{H}^{\mathrm{ev}}((\coprod_{g \in G} X^g)/G; \mathbf{C}) = \prod_{n \in \mathbf{Z}} \mathrm{H}^{2n}(\coprod_{g \in G} X^g)/G; \mathbf{C})$$

denotes the product of the even cohomology groups of $(\coprod_{g \in G} X^g)/G$ with coefficients in the field $\mathbf{C}$ of complex numbers.

Example 1.1.3. In the special case where $X = *$ consists of a single point, we can identify the quotient $(\coprod_{g \in G} X^g)/G$ appearing in Theorem 1.1.2 with the set of conjugacy classes of elements of G (regarded as a finite set with the discrete topology), so that $\mathrm{H}^{\mathrm{ev}}((\coprod_{g \in G} X^g)/G; \mathbf{C}) \simeq \mathrm{H}^0((\coprod_{g \in G} X^g)/G; \mathbf{C})$ is isomorphic to the vector space of class functions $\chi : G \to \mathbf{C}$. Under this identification, the equivariant Chern character $\mathrm{ch}_G : \mathbf{C} \otimes_{\mathbf{Z}} \mathrm{KU}^0_G(X) \simeq \mathrm{H}^{\mathrm{ev}}((\coprod_{g \in G} X^g)/G; \mathbf{C})$ corresponds to the isomorphism

$$\mathbf{C} \otimes_{\mathbf{Z}} \mathrm{Rep}(G) \simeq \{\text{Class functions } \chi : G \to \mathbf{C}\} \qquad V \mapsto \chi_V$$

of Theorem 1.1.1 (see Example 4.3.9).

Example 1.1.4. When the group G is trivial, the equivariant Chern character of Theorem 1.1.2 specializes to the usual Chern character

$$\mathrm{ch} : \mathbf{C} \otimes_{\mathbf{Z}} \mathrm{KU}^0(X) \to \mathrm{H}^{\mathrm{ev}}(X; \mathbf{C}),$$

which is an isomorphism whenever X is a finite CW complex. In this case, it is not necessary to work over the complex numbers: there is already a canonical isomorphism of rational vector spaces

$$\mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{KU}^0(X) \simeq \mathrm{H}^{\mathrm{ev}}(X; \mathbf{Q}),$$

which induces the isomorphism ch after extending scalars along the inclusion $\mathbf{Q} \hookrightarrow \mathbf{C}$. Beware that this is not true in the equivariant case (even when X is a point): if V is a finite-dimensional representation of G, then the character $\chi_V : G \to \mathbf{C}$ generally does not take values in $\mathbf{Q}$.

Let EG denote a contractible space equipped with a free action of the finite group G. If X is any topological space equipped with a G-action, we let X_{hG} denote the *homotopy orbit space* of X by the action of G, defined as the quotient space $(X \times EG)/G$. The projection map $X \times EG \to X$ induces a homomorphism

$$\zeta : \mathrm{KU}^0_G(X) \to \mathrm{KU}^0_G(X \times EG) \simeq \mathrm{KU}^0(X_{hG}),$$

which we will refer to as the *Atiyah-Segal comparison map*. It is not far from being an isomorphism, by virtue of the following classical result (see Corollary 4.9.3):

Theorem 1.1.5 (Atiyah [1])**.** *Let G be a finite group and let $I_G \subseteq \mathrm{Rep}(G)$ be the augmentation ideal, defined as the kernel of the ring homomorphism*

$$\mathrm{Rep}(G) \to \mathbf{Z} \qquad [V] \mapsto \dim_{\mathbf{C}}(V).$$

For every finite G-CW complex X, the Atiyah-Segal comparison map

$$\zeta : \mathrm{KU}^0_G(X) \to \mathrm{KU}^0(X_{hG})$$

exhibits $\mathrm{KU}^0(X_{hG})$ as the I_G-adic completion of $\mathrm{KU}^0_G(X)$; here we regard $\mathrm{KU}^0_G(X)$ as a module over the representation ring $\mathrm{Rep}(G) \simeq \mathrm{KU}^0_G(*)$.

The conclusion of Theorem 1.1.5 can be simplified by applying a further completion. Fix a prime number p. We say that an element $g \in G$ is *p-singular* if the order of g is a power of p, and we let $G^{(p)} \subseteq G$ denote the subset consisting of p-singular elements. Let $\widehat{\mathrm{KU}}$ denote the p-adic completion of the complex K-theory spectrum KU. Then, after p-adic completion, the Atiyah-Segal comparison map yields a homomorphism

$$\zeta_p : \mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{KU}^0_G(X) \to \widehat{\mathrm{KU}}^0(X_{hG})$$

which is the projection onto a direct factor. After extending scalars to the complex numbers, we can describe this direct factor concretely by the following variant of Theorem 1.1.2:

Theorem 1.1.6. *Fix a prime number p and an embedding $\iota : \mathbf{Z}_p \hookrightarrow \mathbf{C}$. Then there is a canonical isomorphism of complex vector spaces*

$$\widehat{\mathrm{ch}}_G : \mathbf{C} \otimes_{\mathbf{Z}_p} \widehat{\mathrm{KU}}^0(X_{hG}) \to \mathrm{H}^{\mathrm{ev}}((\coprod_{g \in G^{(p)}} X^g)/G; \mathbf{C}).$$

Remark 1.1.7. In the situation of Theorem 1.1.6, the isomorphism $\widehat{\mathrm{ch}}_G$ is related to the equivariant Chern character ch_G of Theorem 1.1.2 by a commutative diagram

$$\begin{tikzcd}
\mathbf{C} \otimes_{\mathbf{Z}} \mathrm{KU}^0_G(X) \arrow[r, "\mathrm{ch}_G", "\sim"'] \arrow[d, "\zeta_p"] & \mathrm{H}^{\mathrm{ev}}((\coprod_{g \in G} X^g)/G; \mathbf{C}) \arrow[d] \\
\mathbf{C} \otimes_{\mathbf{Z}_p} \widehat{\mathrm{KU}}^0(X_{hG}) \arrow[r, "\widehat{\mathrm{ch}}_G", "\sim"'] & \mathrm{H}^{\mathrm{ev}}((\coprod_{g \in G^{(p)}} X^g)/G; \mathbf{C}).
\end{tikzcd}$$

Example 1.1.8. In the situation of Theorem 1.1.6, suppose that G is a p-group. Then one can show that the augmentation ideal $I_G \subseteq \mathrm{Rep}(G)$ of Theorem 1.1.5 satisfies $I_G^n \subseteq p\,\mathrm{Rep}(G)$ for $n \gg 0$. It follows that the completed Atiyah-Segal comparison map $\zeta_p : \mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{KU}^0_G(X) \to \widehat{\mathrm{KU}}^0(X_{hG})$ is an isomorphism. In this case, Theorem 1.1.6 reduces to Theorem 1.1.2 (note also that we have $G^{(p)} = G$ when G is a p-group).

In [5], Hopkins, Kuhn, and Ravenel prove a generalization of Theorem 1.1.6 in the setting of chromatic homotopy theory. To state their result, we will need a bit of notation. Let k be a perfect field of characteristic p and let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group of height $n < \infty$ over k. The formal group $\widehat{\mathbf{G}}_0$ admits a universal deformation $\widehat{\mathbf{G}}$, which is defined over the *Lubin-Tate ring* R (noncanonically isomorphic a power series ring $W(\kappa)[[v_1, \ldots, v_{n-1}]]$). Then $\widehat{\mathbf{G}}$ can be realized as the identity component of a connected p-divisible group $\mathbf{G}$ over R. Let C_0 be the R-algebra classifying isomorphisms of p-divisible groups $(\underline{\mathbf{Q}_p / \mathbf{Z}_p})^n \simeq \mathbf{G}$. Then $\mathrm{Spec}(C_0)$ is a $\mathrm{GL}_n(\mathbf{Z}_p)$-torsor over the affine scheme $\mathrm{Spec}(R_{\mathbf{Q}})$, where $R_{\mathbf{Q}} = R[\frac{1}{p}]$ is the rationalization of R. Let E denote the *Lubin-Tate spectrum* associated to $\widehat{\mathbf{G}}_0$: that is, E is an even periodic ring spectrum equipped with isomorphisms

$$R \simeq \pi_0(E) \qquad \widehat{\mathbf{G}}_0 \simeq \mathrm{Spf}(E^0(\mathbf{CP}^\infty)).$$

Let $E_{\mathbf{Q}} = E[\frac{1}{p}]$ denote the rationalization of E, so that we have an isomorphism $\pi_0(E_{\mathbf{Q}}) \simeq R_{\mathbf{Q}}$. We then have the following:

Theorem 1.1.9 (Hopkins-Kuhn-Ravenel)**.** *Let G be a finite group and let X be a finite G-CW complex. For each homomorphism $\alpha : \mathbf{Z}_p^n \to G$, let $X^\alpha \subseteq X$ denote* *the subspace of X consisting of points which are fixed by the action of the subgroup* $\mathrm{im}(\alpha) \subseteq G$. *Then there is a canonical isomorphism of graded C_0-algebras*

$$C_0 \otimes_R E^*(X_{hG}) \to C_0 \otimes_{R_{\mathbf{Q}}} E^*_{\mathbf{Q}}((\coprod_{\alpha:\mathbf{Z}_p^n \to G} X^\alpha)/G).$$

Example 1.1.10. Let $X = *$ be a single point, so that the homotopy orbit space X_{hG} can be identified with the classifying space $BG = EG/G$. Then the space

$$\coprod_{\alpha:\mathbf{Z}_p^n \to G} X^\alpha$$

appearing in the statement of Theorem 1.1.9 can be identified with the finite set S of n-tuples $(g_1, \ldots, g_n)$ of p-singular elements of G satisfying $g_i g_j = g_j g_i$ for $1 \leqslant i \leqslant j \leqslant n$. It follows that we can identify $C_0 \otimes_R E^0(BG)$ with the module of "higher class functions" $\chi : S \to C_0$ satisfying the identity $\chi(g_1, \ldots, g_n) = \chi(h^{-1}g_1h, \cdots, h^{-1}g_nh)$ for $h \in G$.

Example 1.1.11. Let $k = \mathbf{F}_p$ be the finite field with p-elements, and let $\widehat{\mathbf{G}}_0 = \widehat{\mathbf{G}}_m$ be the formal multiplicative group over k. Then the Lubin-Tate ring R can be identified with $\mathbf{Z}_p$, and C_0 can be identified with the field $\mathbf{Q}_p(\zeta_{p^\infty}) = \bigcup_{m \geqslant 1} \mathbf{Q}_p(\zeta_{p^m})$ obtained from $\mathbf{Q}_p$ by adjoining all p-power roots of unity. The Lubin-Tate spectrum E is then given by the p-adically completed complex K-theory spectrum $\widehat{\mathrm{KU}}$, and the classical Chern character supplies isomorphisms

$$\mathrm{ch} : E^0_{\mathbf{Q}}(Y) \simeq \mathrm{H}^{\mathrm{ev}}(Y; \mathbf{Q}_p).$$

In this case, Theorem 1.1.9 supplies an isomorphism

$$\begin{aligned}
\mathbf{Q}_p(\zeta_{p^\infty}) \otimes_{\mathbf{Z}_p} \widehat{\mathrm{KU}}^0(X_{hG}) &\simeq \mathbf{Q}_p(\zeta_{p^\infty}) \otimes_{\mathbf{Q}_p} \widehat{\mathrm{KU}}^0_{\mathbf{Q}}((\coprod_{\alpha:\mathbf{Z}_p \to G} X^\alpha)/G) \\
&\simeq \mathbf{Q}_p(\zeta_{p^\infty}) \otimes_{\mathbf{Q}_p} \mathrm{H}^{\mathrm{ev}}((\coprod_{g \in G^{(p)}} X^g)/G; \mathbf{Q}_p).
\end{aligned}$$

After extending scalars along an embedding $\iota : \mathbf{Q}_p(\zeta_{p^\infty}) \hookrightarrow \mathbf{C}$, this recovers the isomorphism of Theorem 1.1.6 provided that ι is chosen to satisfy the normalization condition $\iota(\zeta_{p^m}) = \exp(2\pi i/p^m)$.

Remark 1.1.12. In the situation of Theorem 1.1.9, the isomorphism

$$C_0 \otimes_R E^*(X_{hG}) \to C_0 \otimes_{R_{\mathbf{Q}}} E^*_{\mathbf{Q}}((\coprod_{\alpha:\mathbf{Z}_p^n \to G} X^\alpha)/G)$$

is equivariant with respect to the action of the profinite group $\mathrm{GL}_n(\mathbf{Z}_p)$. Passing to fixed points, we obtain an isomorphism

$$E^*(X_{hG})[\frac{1}{p}] \simeq (C_0 \otimes_{R_{\mathbf{Q}}} E^*_{\mathbf{Q}}((\coprod_{\alpha:\mathbf{Z}_p^n \to G} X^\alpha)/G))^{\mathrm{GL}_n(\mathbf{Z}_p)}.$$

Here the fixed points on the right hand side are taken with respect to the simultaneous action of $\mathrm{GL}_n(\mathbf{Z}_p)$ on the coefficient ring C_0 and the space $\coprod_{\alpha:\mathbf{Z}_p^n \to G} X^\alpha$.

Remark 1.1.13. In the statement of Theorem 1.1.9, we can replace the set-theoretic quotient

$$(\coprod_{\alpha:\mathbf{Z}_p^n \to G} X^\alpha)/G$$

by the homotopy orbit space $(\coprod_{\alpha:\mathbf{Z}_p^n \to G} X^\alpha)_{hG}$; the canonical map

$$(\coprod_{\alpha:\mathbf{Z}_p^n \to G} X^\alpha)_{hG} \to (\coprod_{\alpha:\mathbf{Z}_p^n \to G} X^\alpha)/G$$

induces an isomorphism on cohomology with coefficients in $E_{\mathbf{Q}}$, since G is a finite group and the coefficient ring $\pi_*(E_{\mathbf{Q}})$ is a rational vector space.

In the case where the group G is trivial, Theorem 1.1.9 follows from the observation that the comparison map

$$\rho : R_{\mathbf{Q}} \otimes_R E^*(Y) \to E^*_{\mathbf{Q}}(Y)$$

is an isomorphism. This is a much more elementary statement, which is immediate from the flatness of $R_{\mathbf{Q}}$ over R. However, it depends crucially on the assumption that Y is finite. If we instead take $Y = X_{hG}$, where X is a finite G-CW complex, then, after extending scalars from $R_{\mathbf{Q}}$ to C_0, the map ρ factors as a composition

$$C_0 \otimes_R E^*(X_{hG}) \simeq C_0 \otimes_{R_{\mathbf{Q}}} E^*_{\mathbf{Q}}((\coprod_{\alpha:\mathbf{Z}_p^n \to G} X^\alpha)_{hG}) \xrightarrow{\pi} C_0 \otimes_{R_{\mathbf{Q}}} E^*_{\mathbf{Q}}(X_{hG}),$$

where the first map is the isomorphism of Theorem 1.1.9 (and Remark 1.1.13) and the second is induced by the inclusion of X as a summand of the coproduct $\coprod_{\alpha:\mathbf{Z}_p^n \to G} X^\alpha$ (namely, the summand corresponding to the trivial homomorphism $\mathbf{Z}_p^n \to \{1\} \hookrightarrow G$).

From this perspective, we can view Theorem 1.1.9 as measuring the *failure* of the comparison map ρ to be an isomorphism for spaces of the form $Y = X_{hG}$. Note that the cohomology theories E and $E_{\mathbf{Q}}$ have different chromatic heights. The spectrum $E_{\mathbf{Q}}$ is $K(0)$-local, and therefore captures the same information as ordinary cohomology with coefficients in $\mathbf{Q}$. In particular, cohomology with coefficients in $E_{\mathbf{Q}}$ cannot detect the difference between the homotopy orbit space X_{hG} and the set-theoretic quotient X/G when G is a finite group (Remark 1.1.13). By contrast, the Lubin-Tate spectrum E is $K(n)$-local, and therefore has the potential to capture delicate p-torsion information. Theorem 1.1.9 describes the part of this information that survives after inverting the prime number p: roughly speaking, the localization $E^*(X_{hG})[\frac{1}{p}] \simeq R_{\mathbf{Q}} \otimes_R E^*(X_{hG})$ knows not only about the $E_{\mathbf{Q}}$-cohomology of X_{hG}, but also about the $E_{\mathbf{Q}}$-cohomology of "twisted sectors" $(X^{\alpha})_{\mathrm{hZ}(\alpha)}$ (where $Z(\alpha) \subseteq G$ denotes the centralizer of a homomorphism $\alpha : \mathbf{Z}_p^n \to G$).

In [21], Stapleton proves a "transchromatic" generalization of Theorem 1.1.9, which articulates the sort of information which is lost by passing from $K(n)$-local to $K(m)$-local homotopy theory for $0 \leqslant m \leqslant n$. Let $L_{K(m)}E$ denote the $K(m)$-localization of the Lubin-Tate spectrum E defined above, and set $R_m = \pi_0(L_{K(m)}E)$. Let $\mathbf{G}_{R_m}$ denote the p-divisible group over R_m obtained from $\mathbf{G}$ by extending scalars along the canonical map $R = \pi_0(E) \to \pi_0(L_{K(m)}E) = R_m$. Then the p-divisible group $\mathbf{G}_{R_m}$ admits a connected-étale sequence

$$0 \to \mathbf{G}' \to \mathbf{G}_{R_m} \xrightarrow{q} \mathbf{G}'' \to 0,$$

where $\mathbf{G}'$ is a connected p-divisible group of height m and $\mathbf{G}''$ is étale of height $n-m$. Let C_m be universal among those R_m-algebras A which are equipped with a morphism $(\underline{\mathbf{Q}_p/\mathbf{Z}_p})^{n-m} \to \mathbf{G}_A$ of p-divisible groups over A for which the composite map $(\underline{\mathbf{Q}_p/\mathbf{Z}_p})^{n-m} \to \mathbf{G}_A \xrightarrow{q} \mathbf{G}''_A$ is an isomorphism (see §2.7). The main result of [21] can be formulated as follows:

Theorem 1.1.14 (Stapleton)**.** *The commutative ring C_m is flat both as an R-algebra and as an R_m-algebra. Moreover, if G is a finite group and X is a finite G-CW complex, then there is a canonical isomorphism of graded C_m-algebras*

$$C_m \otimes_R E^*(X_{hG}) \simeq C_t \otimes_{R_m} (L_{K(m)}E)^*((\coprod_{\alpha:\mathbf{Z}_p^{n-m}\to G} X^{\alpha})_{hG}).$$

Remark 1.1.15. In the case $m = 0$, the isomorphism of Theorem 1.1.14 reduces to the isomorphism of Theorem 1.1.9.

Remark 1.1.16. As in Remark 1.1.12, one can use Theorem 1.1.14 (and faithfully flat descent) to obtain a description of the groups $R_m \otimes_R E^*(X_{hG})$ in terms of the cohomology theory $L_{K(m)}E$. However, the description is a bit more complicated in the case $m > 0$, because the map $\mathrm{Spec}(C_m) \to \mathrm{Spec}(R_m)$ is not a torsor for a profinite group. To specify an A-valued point of $\mathrm{Spec}(C_m)$ (where A is some commutative R_m-algebra), one must specify not only a trivialization of the étale p-divisible group $\mathbf{G}''_A$, but also a splitting of the sequence

$$0 \to \mathbf{G}'_A \to \mathbf{G}_A \xrightarrow{q} \mathbf{G}''_A \to 0.$$

We refer the reader to [22] for a related discussion.

The goal of this paper is to place all of the results stated above into a more general framework. Fix a prime number p. In [8], we introduced the notion of a *p-divisible group* $\mathbf{G}$ over an $\mathbb{E}_\infty$-ring A. In the case where A is p-complete, we can associate to each p-divisible group $\mathbf{G}$ a formal group $\mathbf{G}^\circ$ over A, which we call the *identity component* of $\mathbf{G}$ ([9]). If A is complex periodic and p-local, we say that a p-divisible group $\mathbf{G}$ over A is *oriented* if, after extending scalars to the p-completion of A, the identity component $\mathbf{G}^\circ$ is identified with the Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_A \simeq \mathrm{Spf}(A^{\mathbf{CP}^\infty})$.

Let $\mathbf{G}$ be an oriented p-divisible group over a p-local $\mathbb{E}_\infty$-ring A. To avoid confusion, let us henceforth use the letter H to denote a finite group. In this paper, we will introduce a functor

$$A^*_{\mathbf{G}}(\bullet//H) : \{H\text{-spaces}\}^{\mathrm{op}} \to \{\text{Graded rings}\}.$$

This functor associates to each H-space X a graded ring $A^*_{\mathbf{G}}(X//H)$, which we will refer to as the *$\mathbf{G}$-tempered cohomology ring of $X//H$* (Construction 4.0.5). Moreover, there is a natural comparison map

$$\zeta_{\mathbf{G}} : A^*_{\mathbf{G}}(X//H) \to A^*(X_{hH}),$$

which we will refer to as the *Atiyah-Segal comparison map*. Let us briefly summarize some of the essential properties of this construction (for a more complete overview, we refer the reader to §4)):

Theorem 1.1.17 (Normalization)**.** *Let A be an $\mathbb{E}_\infty$-ring which $K(n)$-local and let $\mathbf{G}$ be an oriented p-divisible group of height n over A (which is then necessarily equivalent to the Quillen p-divisible group $\mathbf{G}^{\mathcal{Q}}_A$: see Proposition 2.5.6). Then, for any finite group H and any H-space X, the Atiyah-Segal comparison map $A^*_{\mathbf{G}}(X//H) \to A^*(X_{hH})$ is an isomorphism.*

Theorem 1.1.18 (Character Isomorphisms)**.** *Let A be a p-local $\mathbb{E}_\infty$-ring, let $\mathbf{G}_0$ be an oriented p-divisible group over A, and let $\mathbf{G} = \mathbf{G}_0 \oplus (\underline{\mathbf{Q}_p / \mathbf{Z}_p})^n$ for some integer n. Let H be a finite group and let X be an H-space. Then there is a canonical isomorphism of graded rings*

$$\chi : A^*_{\mathbf{G}}(X//H) \simeq A^*_{\mathbf{G}_0}((\coprod_{\alpha:\mathbf{Z}_p^n \to G} X^\alpha)//H).$$

In particular, in the case $n = 1$, we have an isomorphism

$$A^*_{\mathbf{G}}(X//H) \simeq A^*_{\mathbf{G}_0}((\coprod_{h \in H^{(p)}} X^h)//H).$$

Theorem 1.1.19 (Base Change)**.** *Let $f : A \to B$ be a flat morphism of p-local $\mathbb{E}_\infty$-rings and let $\mathbf{G}$ be an oriented p-divisible group over A. Then extending scalars along f determines an oriented p-divisible group over B, which we will also denote by $\mathbf{G}$. For any finite group H and any finite H-space X, we have a canonical isomorphisms*

$$\pi_0(B) \otimes_{\pi_0(A)} A^*_{\mathbf{G}}(X//H) \simeq B^*_{\mathbf{G}}(X//H).$$

Remark 1.1.20. Let us sketch how Theorems 1.1.17, 1.1.19, and 1.1.18 can be combined to recover Theorem 1.1.14 (and therefore also Theorem 1.1.9). Let E be a Lubin-Tate spectrum associated to a formal group of height n and let $\mathbf{G} = \mathbf{G}_E^{\mathcal{Q}}$ denote the associated Quillen p-divisible group (which we now view as a p-divisible group over the $\mathbb{E}_\infty$-ring E, rather than over the ordinary commutative ring $R = \pi_0(E)$). Choose $0 \leqslant m \leqslant n$, let $L_{K(m)}(E)$ denote the $K(m)$-localizaiton of E, and let $\mathbf{G}_{L_{K(m)}(E)}$ denote the p-divisible group over $L_{K(m)}(E)$ obtained from $\mathbf{G}$ by extension of scalars. We then have a connected-étale sequence

$$0 \to \mathbf{G}' \to \mathbf{G}_{L_{K(m)}(E)} \xrightarrow{q} \mathbf{G}'' \to 0,$$

where $\mathbf{G}'$ is an oriented p-divisible group of height m and $\mathbf{G}''$ is an étale p-divisible group of height $n - m$. Let B be universal among those $\mathbb{E}_\infty$-algebras over $L_{K(m)}(E)$ which are equipped with a map $u : (\underline{\mathbf{Q}_p / \mathbf{Z}_p})^{n-m} \to \mathbf{G}_B$ for which the composition $(\underline{\mathbf{Q}_p / \mathbf{Z}_p})^{n-m} \xrightarrow{u} \mathbf{G}_B \xrightarrow{q} \mathbf{G}''_B$ is an equivalence (for a more detailed construction of B, we refer the reader to §2.7). Then B is flat over both E and $L_{K(m)}(E)$, and $\pi_0(B)$ can be identified with the commutative ring C_m appearing in the statement of Theorem 1.1.14. By construction, the p-divisible group $\mathbf{G}_B$ splits as a direct sum $\mathbf{G}'_B \oplus (\underline{\mathbf{Q}_p / \mathbf{Z}_p})^{n-m}$. Consequently, if H is a finite group and X is a finite H-CW complex, we have isomorphisms

$$\begin{aligned} C_m \otimes_R E^*(X_{hH}) &\simeq \pi_0(B) \otimes_{\pi_0(E)} E^*_{\mathbf{G}}(X//H) \\ &\simeq B^*_{\mathbf{G}}(X//H) \\ &\simeq B^*_{\mathbf{G}'}((\coprod_{\alpha:\mathbf{Z}_p^{n-m} \to G} X^\alpha)//H) \\ &\simeq C_m \otimes_{R_m} L_{K(m)}(E)^*((\coprod_{\alpha:\mathbf{Z}_p^{n-m} \to G} X^\alpha)_{hH}) \end{aligned}$$

whose composition is the transchromatic character isomorphism of Theorem 1.1.14.

Theorems 1.1.17 and 1.1.18 are more or less formal: they will follow immediately from our definition of $\mathbf{G}$-tempered cohomology, as will the existence of a natural comparison map

$$\rho_{\mathbf{G}} : \pi_0(B) \otimes_{\pi_0(A)} A^*_{\mathbf{G}}(X//H) \to B^*_{\mathbf{G}}(X//H)$$

in the situation of Theorem 1.1.19. Arguing as in Remark 1.1.20, these ingredients are sufficient to construct the character map

$$C_m \otimes_R E^*(X_{hH}) \to C_m \otimes_{R_m} (L_{K(m)}E)^*((\coprod_{\alpha:\mathbf{Z}_p^{n-m} \to G} X^\alpha)_{hH})$$

appearing in the statement of Theorem 1.1.14. However, to prove that the character map is an isomorphism, we will need need to know that $\rho_{\mathbf{G}}$ is an isomorphism. This is much less obvious. Recall that, if A is any $\mathbb{E}_\infty$-ring, then the A-cohomology groups $A^*(Y)$ of a space Y can be realized as the homotopy groups of a spectrum A^Y (parametrizing *unpointed* maps from Y to A), so that we have isomorphisms $A^*(Y) \simeq \pi_{-*}(A^Y)$. Suppose that $f : A \to B$ is a *flat* morphism of $\mathbb{E}_\infty$-rings. Then, for any space Y, the canonical map of graded rings

$$\pi_0(B) \otimes_{\pi_0(A)} A^*(Y) \to B^*(Y)$$

can be realized as the homotopy of a map of ring spectra $\varphi_Y : B \otimes_A A^Y \to ABY$; here $B \otimes_A A^Y$ denotes the smash product of B with A^Y over A. If Y is a finite CW complex, then the map φ_Y is a homotopy equivalence. However, the map φ_Y is generally not a homotopy equivalence when Y is not finite. In particular, it need not be a homotopy equivalence in the case $Y = X_{hH}$, where H is a finite group and X is a finite H-CW complex. Consequently, Theorem 1.1.19 articulates a property of

G-tempered cohomology which is not shared by the "Borel-equivariant" cohomology theory

$$\{H\text{-spaces}\}^{\mathrm{op}} \to \{\text{Graded rings}\} \qquad X \mapsto A^*(X_{hH}).$$

Let us study the preceding situation in more detail. Let Mod_A denote the ∞-category of A-module spectra. For every space S, we let $\mathrm{LocSys}_A(S) = \mathrm{Fun}(S, \mathrm{Mod}_A)$ denote the ∞-category of Mod_A-valued local systems on S. We will be particularly interested in the case $S = BH$ is the classifying space of a finite group H: objects of the ∞-category $\mathrm{LocSys}_A(BH)$ can be thought of as A-module spectra equipped with an action of H. Pullback along the projection map $BH \to *$ induces a functor $\mathrm{Mod}_A \to \mathrm{Fun}(BH, \mathrm{Mod}_A)$ which carries an A-module spectrum to itself (equipped with the trivial action of H). This functor has both left and right adjoints, which carry an A-module spectrum M equipped with an action of H to the homotopy orbit spectrum M_{hH} and the homotopy fixed point spectrum M^{hH}, respectively. These constructions are related by a canonical map $\mathrm{Nm}_H : M_{hH} \to M^{hH}$, given informally by "averaging" with respect to the action of H. If X is an H-space, then the function spectrum $A^{X_{hH}}$ can be realized the homotopy invariants for the tautological action of H on the function spectrum A^X. It follows that the map $\varphi_{X_{hH}}$ factors as a composition

$$B \otimes_A A^{X_{hH}} \simeq B \otimes_A (A^X)^{hH} \xrightarrow{\psi} (B \otimes_A A^X)^{hH} \xrightarrow{\varphi_X} (B^X)^{hH} \simeq B^{X_{hH}}.$$

If X is a finite H-CW complex, then the map φ_X is a homotopy equivalence. However, the map ψ is generally not a homotopy equivalence: the extension of scalars functor $M \mapsto B \otimes_A M$ usually does not preserve homotopy limits, and therefore need not commute with the operation of taking homotopy invariants with respect to H. However, extension of scalars *does* preserve homotopy colimits, such as the operation of taking homotopy orbits with respect to H. Consequently, the map ψ fits into a commutative diagram

$$\begin{tikzcd}
B \otimes_A (A^X)_{hH} \arrow[r, "\sim"] \arrow[d, "B\otimes_A \mathrm{Nm}_H"] & (B^X)_{hH} \arrow[d, "\mathrm{Nm}_H"] \\
B \otimes_A (A^X)^{hH} \arrow[r] & (B^X)^{hH},
\end{tikzcd}$$

where the upper horizontal map is a homotopy equivalence. We can informally summarize the situation as follows: in the case $Y = X_{hH}$, the (potential) failure of the map φ_Y to be an equivalence is a result of the (potential) failure of the norm maps

$$\mathrm{Nm}_H : (A^X)_{hH} \to (A^X)^{hH} \qquad \mathrm{Nm}_H : (B^X)_{hH} \to (B^X)^{hH}$$

to be equivalences.

Our proof that $\mathbf{G}$-tempered equivariant cohomology satisfies Theorem 1.1.19 will use a variant of the preceding ideas. Let A be a p-local $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be an oriented p-divisible group over A. To every space S, we will associate an ∞-category $\mathrm{LocSys}_{\mathbf{G}}(S)$ whose objects we will refer to as *$\mathbf{G}$-tempered local systems on S* (Definition 5.2.4). This ∞-category is equipped with a forgetful functor

$$\mathrm{LocSys}_{\mathbf{G}}(S) \to \mathrm{LocSys}_A(S) = \mathrm{Fun}(S, \mathrm{Mod}_A)$$

which, in the case $S = BH$, can be viewed as a categorification of the Atiyah-Segal comparison map

$$\zeta : A^*_{\mathbf{G}}(X//H) \to A^*(X_{hH}).$$

More precisely, if H is a finite group, X is a G-space, and $f : BH \to *$ denotes the projection map, then the $\mathbf{G}$-tempered H-equivariant cohomology of X can be described by the formula

$$A^*_{\mathbf{G}}(X//H) \simeq \pi_{-*}(f_*(\mathscr{F})),$$

where $\mathscr{F}$ is a certain $\mathbf{G}$-tempered local system on BH (which is a preimage of the function spectrum A^X under the forgetful functor $\mathrm{LocSys}_{\mathbf{G}}(BH) \to \mathrm{LocSys}_A(BH)$), and $f_* : \mathrm{LocSys}_{\mathbf{G}}(BH) \to \mathrm{LocSys}_{\mathbf{G}}(*) \simeq \mathrm{Mod}_A$ denotes the *right* adjoint to the functor $f^* : \mathrm{LocSys}_{\mathbf{G}}(*) \to \mathrm{LocSys}_{\mathbf{G}}(BH)$ given by pullback along f. To prove Theorem 1.1.19, the essential point is to show that the functor f_* preserves homotopy colimits (and therefore commutes with extension of scalars along a morphism of $\mathbb{E}_\infty$-rings $A \to B$). We will prove this by constructing a *norm map* $\mathrm{Nm}_H : f_! \to f_*$, where $f_!$ denotes the left adjoint to the functor f^*, and showing that the map Nm_H is an equivalence. This is a special case of a much more general assertion (see Theorem 7.2.10):

Theorem 1.1.21 (Tempered Ambidexterity). *Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be an oriented p-divisible group over A, and let $f : S \to S'$ be a map of π-finite spaces (that is, spaces having only finitely many connected components and finitely many nonvanishing homotopy groups, each of which is a finite group). Then the functors $f_!, f_* : \mathrm{LocSys}_{\mathbf{G}}(S) \to \mathrm{LocSys}_{\mathbf{G}}(S')$ are canonically equivalent to one another.*

Remark 1.1.22. Let n be a nonnegative integer, and let $\mathrm{Sp}_{K(n)}$ denote the ∞-category of $K(n)$-local spectra. In [6], we proved that if $f : S \to S'$ is a map of π-finite spaces, then the functors

$$f_!, f_* : \mathrm{LocSys}_{\mathrm{Sp}_{K(n)}}(S) \to \mathrm{LocSys}_{\mathrm{Sp}_{K(n)}}(S')$$

are canonically equivalent to one another. Roughly speaking, Theorem 1.1.21 asserts that this phenomenon persists outside of the $K(n)$-local setting, provided that we work with $\mathbf{G}$-tempered local systems, rather than ordinary local systems.

Let us now outline the contents of this paper. We begin in §2 by reviewing the notion of $\mathbf{P}$-*divisible group* (Definition 2.6.1); here (and throughout this paper) we will use the symbol $\mathbf{P}$ to denote the set $\{2, 3, 5, \cdots\}$ of all prime numbers. If A is an $\mathbb{E}_\infty$-ring, then a $\mathbf{P}$-divisible group $\mathbf{G}$ over A can be identified with a system $\{\mathbf{G}_{(p)}\}_{p \in \mathbf{P}}$ of p-divisible groups $\mathbf{G}_{(p)}$, where p ranges over all prime numbers. We will say that a $\mathbf{P}$-divisible group $\mathbf{G}$ is *oriented* if A if, after extending scalars to the p-completion of A, each of the identity components $\mathbf{G}^{\circ}_{(p)}$ is equipped with an orientation (Definition 2.6.12). Such objects arise naturally in (at least) three ways:

(a) Fix a prime number p, and let A be an $\mathbb{E}_\infty$-ring which is complex periodic and $K(n)$-local, for some $n \geqslant 1$. Then the Quillen formal group $\mathrm{Spf}(A^{\mathbf{CP}^\infty})$ is the identity component of an oriented p-divisible group $\mathbf{G}^{\mathcal{Q}}_A$, which we refer to as the *Quillen p-divisible group* (see §2.4). We can also regard $\mathbf{G}^{\mathcal{Q}}_A$ as an oriented $\mathbf{P}$-divisible group over A, having trivial ℓ-torsion for $\ell \neq p$.

(b) Over any $\mathbb{E}_\infty$-ring A, we can define a $\mathbf{P}$-divisible group $\mu_{\mathbf{P}^\infty} \simeq \bigoplus_{p \in \mathbf{P}} \mu_{p^\infty}$, which we refer to as the *multiplicative* $\mathbf{P}$-*divisible group* (Construction 2.8.1). In the special case where $A = \mathrm{KU}$ is the complex K-theory spectrum, the $\mathbf{P}$-divisible group $\mu_{\mathbf{P}^\infty}$ can be endowed with an orientation (Construction 2.8.6), which arises from the orientation of the formal multiplicative group $\widehat{\mathbf{G}}_m$ over KU ([9]).

(c) Let E be a strict elliptic curve over an $\mathbb{E}_\infty$-ring A. Then E determines a $\mathbf{P}$-divisible group $\mathsf{E}[\mathbf{P}^\infty] \simeq \bigoplus_{p \in \mathbf{P}} \mathsf{E}[p^\infty]$ of *torsion points of* E. Any orientation of X (in the sense of [9]) determines an orientation of the $\mathbf{P}$-divisible group $\mathsf{E}[\mathbf{P}^\infty]$ (Construction 2.9.6).

The entirety of the preceding discussion can be generalized, using oriented $\mathbf{P}$-divisible groups (over general $\mathbb{E}_\infty$-rings) in place of oriented p-divisible groups (over p-local $\mathbb{E}_\infty$-rings). Moreover, working in this generality yields real dividends: when $A = \mathrm{KU}$ and $\mathbf{G} = \mu_{\mathbf{P}^\infty}$ is the multiplicative $\mathbf{P}$-divisible group, there are canonical isomorphisms

$$A^*_{\mathbf{G}}(X//H) \simeq \mathrm{KU}^*_H(X)$$

for any finite group H and any H-space X (Theorem 4.1.2). In other words, our notion of $\mathbf{G}$-tempered equivariant cohomology recovers equivariant complex K-theory (at least for finite groups). We will return to this point in [10] (where we prove a more general result, which applies also to compact Lie groups). Similarly, if $\mathbf{G} = \mathsf{E}[\mathbf{P}^{\infty}]$ is the $\mathbf{P}$-divisible group of torsion on an oriented elliptic curve E over A, then the $\mathbf{G}$-tempered cohomology $A^{*}_{\mathbf{G}}(X//H)$ can be interpreted as the *H-equivariant elliptic cohomology of X* (for the variant of elliptic cohomology associated to the oriented elliptic curve E). We will develop this idea further in [10] (where we will essentially take it as a definition of equivariant elliptic cohomology, at least for finite groups).

To organize our discussion of $\mathbf{G}$-tempered cohomology, it will be convenient to use the formalism of *orbispaces*, which we review in §3. For the purpose of this paper, we define an *orbispace* to be a functor of ∞-categories

$$\mathscr{T}^{\mathrm{op}} \to \mathcal{S},$$

where $\mathcal{S}$ denotes the ∞-category of spaces and $\mathscr{T} \subseteq \mathcal{S}$ is the full subcategory consisting of spaces of the form BH, where H is a finite *abelian* group. The collection of orbispaces can be organized into an ∞-category which we will denote by $\mathcal{OS}$ (Definition 3.1.4), which includes the ∞-category $\mathcal{S}$ of spaces as a full subcategory (we will generally abuse notation by identifying a space X with the orbispace $X^{(-)}$ given by the functor $(T \in \mathscr{T}) \mapsto X^{T}$). We let $\mathrm{Sp}(\mathcal{OS})$ denote the ∞-category of spectrum objects of $\mathcal{OS}$; the objects of $\mathrm{Sp}(\mathcal{OS})$ can be identified with functors

$$E : \mathscr{T}^{\mathrm{op}} \to \mathrm{Sp} \qquad T \mapsto E^{T}.$$

Our starting point is that if A is an $\mathbb{E}_{\infty}$-ring, then there is a fully faithful embedding

$$\{\text{Oriented } \mathbf{P}\text{-divisible groups over } A\}^{\mathrm{op}} \to \mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathrm{CAlg}_{A}),$$

which carries an oriented $\mathbf{P}$-divisible group $\mathbf{G}$ to a functor $A_{\mathbf{G}} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_{A}$, which carries each object $BH \in \mathscr{T}$ to an object $A^{BH}_{\mathbf{G}}$ which corepresents the functor $\mathbf{G}[\widehat{H}]$ of maps from the Pontryagin dual group $\widehat{H} = \mathrm{Hom}(H, \mathbf{Q}/\mathbf{Z})$ into $\mathbf{G}$ (see Theorem 3.5.5). By neglect of structure, we can regard $A_{\mathbf{G}}$ as a spectrum object of $\mathcal{OS}$, representing a cohomology theory

$$A^{*}_{\mathbf{G}} : \mathcal{OS}^{\mathrm{op}} \to \{\text{Graded rings}\}$$

which we will refer to as *$\mathbf{G}$-tempered cohomology.* If H is a finite group, then $\mathbf{G}$-tempered H-equivariant cohomology is defined as the composition

$$\{H\text{-spaces}\}^{\mathrm{op}} \xrightarrow{X \mapsto X//H} \mathcal{OS}^{\mathrm{op}} \xrightarrow{A^{*}_{\mathbf{G}}} \{\text{Graded rings}\};$$

here $X//H$ denotes the formation of the *orbispace quotient* of an H-space X by the action of H (see Construction 3.2.16). In §4, we provide a summary of the formal properties enjoyed $\mathbf{G}$-tempered cohomology in general, including suitable generalizations of Theorems 1.1.17, 1.1.19, and 1.1.18.

In §5, we define the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(S)$ of **G**-*tempered local systems* on any space S (or, more generally, any orbispace S). Roughly speaking, a $\mathbf{G}$-tempered local system $\mathscr{F}$ on S is a rule which assigns, to each object $T \in \mathscr{T}$ and each map $T \to S$, a module $\mathscr{F}(T)$ over the ring spectrum $A^T_{\mathbf{G}}$. These modules are required to depend functorially on T, in the sense that every commutative diagram

$$\begin{tikzcd} T' \arrow[rr] \arrow[dr, "\eta'"'] & & T \arrow[dl, "\eta"] \\ & S & \end{tikzcd}$$

induces a map

$$u : A_{\mathbf{G}}(T') \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T) \to \mathscr{F}(T')$$

which is not too far from being an equivalence (see Definition 5.2.4 for a precise definition, and Theorem 5.5.1 for a convenient reformulation).

Our theory of tempered local systems is essentially controlled by three formal properties, which we establish in §6:

(1) Suppose that the $\mathbb{E}_\infty$-ring A is p-local for some prime number p and that $\mathbf{G}$ is an oriented p-divisible group over A (regarded as a $\mathbf{P}$-divisible group for which the summands $\mathbf{G}_{(\ell)}$ vanish for $\ell \neq p$). We say that a $\mathbf{G}$-tempered local system $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(S)$ is $K(n)$-*local* if, for every object $T \in \mathscr{T}$ and every map $T \to S$, the spectrum $\mathscr{F}(T)$ is $K(n)$-local (Definition 6.1.13). Let $\mathrm{LocSys}^{K(n)}_{\mathbf{G}}(S)$ denote the full subcategory of $\mathrm{LocSys}_{\mathbf{G}}(S)$ spanned by the $K(n)$-local $\mathbf{G}$-tempered local systems on S, and define $\mathrm{LocSys}^{K(n)}_{A}(S) \subseteq \mathrm{LocSys}_{A}(S)$ similarly. In §6.3, we show that the forgetful functor

$$\mathrm{LocSys}^{K(n)}_{\mathbf{G}}(S) \to \mathrm{LocSys}^{K(n)}_{A}(S)$$

is an equivalence when n is equal to the height of the p-divisible group $\mathbf{G}$ (Theorem 6.3.1).

(2) Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A which splits as a direct sum $\mathbf{G}_0 \oplus \underline{\Lambda}$, where Λ is a divisible torsion group whose p-torsion subgroup

$\Lambda[p]$ is finite for each prime number p. In §6.4, we construct a fully faithful embedding of ∞-categories

$$\Phi : \mathrm{LocSys}_{\mathbf{G}}(S) \hookrightarrow \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(S))$$

(Theorem 6.4.9), and in §6.5 we characterize its essential image (Theorem 6.5.13). Here $\widehat{\Lambda} = \mathrm{Hom}(\Lambda, \mathbf{Q}/\mathbf{Z})$ denotes the Pontryagin dual of Λ, and $\mathcal{L}^{\Lambda}(S)$ denotes the *formal loop space* parametrizing maps from the classifying space $B\widehat{\Lambda}$ to S which are compatible with the profinite topology on $\widehat{\Lambda}$ (see Construction 3.4.3 for a precise definition).

(3) Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. For any space (or orbispace S), we can regard $\mathrm{LocSys}_{\mathbf{G}}(S)$ as an A-linear ∞-category. Consequently, if B is an $\mathbb{E}_\infty$-algebra over A, we can consider the ∞-category

$$B \otimes_A \mathrm{LocSys}_{\mathbf{G}}(S) \simeq \mathrm{Mod}_B(\mathrm{LocSys}_{\mathbf{G}}(S))$$

of B-module objects of $\mathrm{LocSys}_{\mathbf{G}}(S)$. In §6.2, we show that this ∞-category can be identified with $\mathrm{LocSys}_{\mathbf{G}_B}(S)$, where $\mathbf{G}_B$ is the oriented $\mathbf{P}$-divisible group obtained from $\mathbf{G}$ by extending scalars along the map $A \to B$ (Remark 6.2.4). In particular, the construction $B \mapsto \mathrm{LocSys}_{\mathbf{G}_B}(S)$ satisfies faithfully flat descent (Proposition 6.2.6).

Properties (1), (2), and (3) can be regarded as categorified versions of Theorems 1.1.17, 1.1.18, and 1.1.19, respectively. In fact, Theorems 1.1.17 and 1.1.18 are easy to deduce from (1) and (2) (though they even easier to establish directly, as we will see in §4). Theorem 1.1.19 does not follow from (3) alone: it requires our tempered ambidexterity theorem, which we prove in §7. Let us briefly outline our approach to the problem, following the ideas introduced in [6]. Let $f : S \to S'$ be a map of π-finite spaces; we wish to construct an equivalence between the functors $f_!, f_* : \mathrm{LocSys}_{\mathbf{G}}(S) \to \mathrm{LocSys}_{\mathbf{G}}(S')$. Working by induction on the number of homotopy groups of the fibers of f, we can assume that we have already constructed an equivalence $\delta_! \simeq \delta_*$, where $\delta : S \to S \times_{S'} S$ is the relative diagonal. Using this equivalence, we can associated to each tempered local system $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(S)$ a comparison map $\mathrm{Nm}_f : f_!(\mathscr{F}) \to f_*(\mathscr{F})$, which we call the *norm map.* The essence of the problem is to show that this map is an equivalence. In §7.6, we establish tempered analogues of Artin and Brauer induction (Theorems 7.6.3 and 7.6.5), which allow us to reduce to the case where S and S' are p-finite spaces (for some fixed prime number p).

Using formal arguments, we can further reduce to the case $S' = *$ is a single point, $S = K(\mathbf{F}_p, m)$ is an Eilenberg-MacLane space, and $\mathscr{F} = \underline{A}_S$ is the unit object of the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(S)$. In this case, the norm map Nm_f determines bilinear form on the (pre)dual of the tempered cohomology ring $A^*_{\mathbf{G}}(S)$, and we must show that this bilinear form is nondegenerate. The proof then rests on a computation of the tempered cohomology ring of Eilenberg-MacLane spaces, which we carry out in §4 (see Theorem 4.4.16).

Note that, to recover Theorem 1.1.14, we do not need the full strength of our tempered ambidexterity theorem: it suffices to establish that **G**-tempered local systems satisfy ambidexterity for maps of the form $f : BH \to *$, where H is a finite group. However, our main result allows us to extend the reach of character theory to π-finite spaces which have nontrivial higher homotopy groups. For example, we have the following (see Corollary 4.8.5):

Corollary 1.1.23. *Let E be the Lubin-Tate spectrum associated to a formal group of height n, andlet C_0 be as in the statement of Theorem 1.1.9. Then then tensor product $C_0 \otimes_{\pi_0(E)} E^0(S)$ is a free C_0-module of finite rank, with a canonical basis indexed by the set of homotopy classes of maps $\mathrm{B}\,\mathbf{Z}_p^n \to S$.*

For some other concrete consequences of Theorem 1.1.21, see §4.8.

Remark 1.1.24. Many of the results of this paper can be interpreted in the language of equivariant stable homotopy theory. Let A be an $\mathbb{E}_\infty$-ring and let **G** be an oriented **P**-divisible group over A. For any finite group H, one can show that the **G**-tempered H-equivariant cohomology functor

$$A^*_{\mathbf{G}}(\bullet//H) : \{H\text{-Spaces}\} \to \{\text{Graded rings}\}$$

is representable by a *genuine* H-spectrum: that is, it is functorial with respect to stable maps of H-spaces (this is not obvious from the definition: it is a special case of our ambidexterity results); we defer a detailed discussion of this point to [10]. However, this observation in some sense misses the point: it follows from Theorem 1.1.21 that our theory of **G**-tempered cohomology has much *more* functoriality than is encoded by the framework of equivariant stable homotopy theory: for example, it has "transfer" maps $\mathrm{tr}_{X/Y} : A^*_{\mathbf{G}}(X) \to A^*_{\mathbf{G}}(Y)$ for every map of spaces $X \to Y$ with π-finite homotopy fibers (see Construction 7.4.1).

Notation and Terminology

Throughout this paper, we will assume that the reader is familiar with the language of ∞-categories developed in [13] and [11], as well as the language of spectral algebraic geometry as developed in [12]. Since we will need to refer to these texts frequently, we adopt the following conventions:

(*HTT*) We will indicate references to [13] using the letters HTT.

(*HA*) We will indicate references to [11] using the letters HA.

(*SAG*) We will indicate references to [12] using the letters SAG.

For example, Theorem HTT.6.1.0.6 refers to Theorem 6.1.0.6 of [13].

We adopt a similar convention for references to the previous papers in this series:

(*AV*) We will indicate references to [8] using the letters AV.

(*Or*) We will indicate references to [9] using the letters Or.

(*Ambi*) We will indicate references to [6] using the letters Ambi.

Throughout this paper, we will adopt the following notational conventions (some of which differ from the established mathematical literature):

differ from those of the texts listed above, or from the established mathematical literature.

- We write $\mathcal{S}$ denote the ∞-category of spaces, Sp for the ∞-category of spectra, and $\mathrm{CAlg} = \mathrm{CAlg}(\mathrm{Sp})$ for the ∞-category of $\mathbb{E}_\infty$-ring spectra (whose objects we will refer to simply as $\mathbb{E}_\infty$-rings).
- If A is a spectrum and X is a space, we let A^X denote the function spectrum parametrizing *unpointed* maps from X into A. This function spectrum is characterized by the existence of homotopy equivalences $\mathrm{Map}_{\mathrm{Sp}}(B, A^X) = \mathrm{Map}_{\mathcal{S}}(X, \mathrm{Map}_{\mathrm{Sp}}(B, A))$, depending functorially on $B \in \mathrm{Sp}$. We write $A^*(X)$ for the A-cohomology groups of the space X, given concretely by the formula $A^*(X) = \pi_{-*}(A^X)$.
- We will generally not distinguish between a category $\mathcal{C}$ and its nerve $\mathrm{N}(\mathcal{C})$. In particular, we regard every category $\mathcal{C}$ as an ∞-category.

- We will generally abuse terminology by not distinguishing between an abelian group M and the associated Eilenberg-MacLane spectrum: that is, we view the ordinary category of abelian groups as a full subcategory of the ∞-category Sp of spectra. Similarly, we regard the ordinary category of commutative rings as a full subcategory of the ∞-category CAlg of $\mathbb{E}_\infty$-rings.

- Let A be an $\mathbb{E}_\infty$-ring. We will refer to A-module spectra simply as *A-modules*. The collection of A-modules can be organized into a stable ∞-category which we will denote by Mod_A and refer to as *the ∞-category of A-modules.* This convention has an unfortunate feature: when A is an ordinary commutative ring, it does not reduce to the usual notion of A-module. In this case, Mod_A is not the abelian category of A-modules but is closely related to it: the homotopy category hMod_A is equivalent to the derived category $D(A)$. *Unless otherwise specified, the term "A-module" will be used to refer to an object of* Mod_A*, even when A is an ordinary commutative ring.* When we wish to consider an A-module M in the usual sense, we will say that M is a *discrete* A-module or an *ordinary* A-module. If M and N are A-modules spectra, we write $\mathrm{Ext}^*_A(M, N)$ for the graded abelian group given by $\mathrm{Ext}^k_A(M, N) = \pi_0 \mathrm{Map}_{\mathrm{Mod}_A}(M, \Sigma^k(N))$.

- Unless otherwise specified, all algebraic constructions we consider in this book should be understood in the "derived" sense. For example, if we are given discrete modules M and N over a commutative ring A, then the tensor product $M \otimes_A N$ denotes the *derived* tensor product $M \otimes^L_A N$. This may not be a discrete A-module: its homotopy groups are given by $\pi_n(M \otimes_A N) \simeq \mathrm{Tor}^A_n(M, N)$. When we wish to consider the usual tensor product of M with N over A, we will denote it by $\mathrm{Tor}^A_0(M, N)$ or by $\pi_0(M \otimes_A N)$.

- If M and N are spectra, we will denote the smash product of M with N by $M \otimes_S N$, rather than $M \wedge N$ (here S denotes the sphere spectrum). More generally, if M and N are modules over an $\mathbb{E}_\infty$-ring A, then we will denote the smash product of M with N over A by $M \otimes_A N$, rather than $M \wedge_A N$. Note that when A is an ordinary commutative ring and the modules M and N are discrete, this agrees with the preceding convention.

Definition 1.1.25. Let X be a space. We say that X is *π-finite* if, for every base point $x \in X$, the homotopy groups $\pi_n(X, x)$ are finite and vanish for $n \gg 0$. Here we include the case $n = 0$ (that is, we require that the set of connected components $\pi_0(X)$ is finite).

Let S be a set of prime numbers. We will say that X is *S-finite* if it is π-finite and S contains every prime number which divides the order of a homotopy group $\pi_n(X, x)$, for any point $x \in X$ and any $n > 0$.

If p is a prime number, we say that X is *p-finite* if it is π-finite and each homotopy group $\pi_n(X, x)$ is a finite p-group (in other words, if it is S-finite for $S = \{p\}$).

Definition 1.1.26. Let A be an $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0(A)$ be a finitely generated ideal. Then:

- An A-module M is *I-nilpotent* if, for each element $t \in I$, the colimit

  $$M[t^{-1}] = \varinjlim(M \xrightarrow{t} M \xrightarrow{t} M \xrightarrow{t} \cdots)$$

  vanishes.

- An A-module M is *I-complete* if, for each element $t \in I$, the limit

  $$\varprojlim(\cdots \rightarrow tM \xrightarrow{t} M \xrightarrow{t} M \xrightarrow{t} M)$$

  vanishes.

- An A-module M is *I-local* if the groups $\mathrm{Ext}^*_A(N, M)$ vanish whenever N is I-nilpotent (equivalently, M is I-local if the groups $\mathrm{Ext}^*_A(M, N)$ vanish whenever N is I-complete). If $I = (t)$ is a principal ideal, this is equivalent to the requirement that the map $t : M \rightarrow M$ is an equivalence.

We refer the reader to Chapter SAG.II.4 for a more detailed discussion of the notions introduced in Definition 1.1.26 (see also [4]).

Warning 1.1.27. Let M be an A-module spectrum and let p be a prime number. We say that M is *p-complete* if it is (p)-complete in the sense of Definition 1.1.26, where $(p) \subseteq \pi_0(A)$ is the principal ideal generated by p. However, we will say that M is *p-local* if it is a module over the localization $A_{(p)}$: in other words, if M is (ℓ)-local for every prime number $\ell \neq p$.

Warning 1.1.28. In this paper, we will use the notation $\widehat{M}$ for two essentially unrelated purposes:

- If M is a module over an $\mathbb{E}_\infty$-ring A, we will sometimes write $\widehat{M}$ to denote the completion of M with respect to a finitely generated ideal $I \subseteq \pi_0(A)$. This will occur most frequently in the special case where $M = A$ and where $I = (p)$, for some prime number p.

- If M is a torsion abelian group, we will sometimes write $\widehat{M}$ to denote the Pontryagin dual group $\mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$. If M is finite, then the Pontryagin dual $\widehat{M}$ is also a finite abelian group (of the same order as M); more generally, $\widehat{M}$ can be regarded as a profinite group (by identifying it with the inverse limit $\varprojlim \widehat{M_0}$, where M_0 ranges over the collection of all finite subgroups of M.

1.2 Acknowledgements

This work was supported by the National Science Foundation under grant DMS-1810917.

2 Orientations and P-Divisible Groups

Let A be an $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}$ be a formal group over A, which we view as a functor

$$\widehat{\mathbf{G}} : \mathrm{CAlg}_{\tau_{\geqslant 0}(A)} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}} .$$

Recall that a *preorientation* of $\widehat{\mathbf{G}}$ is a map of $\mathbf{Z}$-module spectra $e : \Sigma^2(\mathbf{Z}) \to \widehat{\mathbf{G}}(A)$ (Definition Or.4.3.19). Our goal in this section is to study a variant of this definition in the setting of p-divisible groups. In §2.1, we associate to each p-divisible group $\mathbf{G}$ a space $\mathrm{Pre}(\mathbf{G})$, which we will refer to as the *space of preorientations of* $\mathbf{G}$ (Definition 2.1.4). Our theory of preorientations is uniquely determined by the following two assertions, which we will prove in §2.3 and §2.2 respectively:

(a) If $\widehat{A}$ is the p-completion of A and $\mathbf{G}_{\widehat{A}}$ is the p-divisible group over $\widehat{A}$ obtained from $\mathbf{G}$ by extension of scalars, then we have a canonical homotopy equivalence: $\mathrm{Pre}(\mathbf{G}) \xrightarrow{\sim} \mathrm{Pre}(\mathbf{G}_{\widehat{A}})$ (Proposition 2.3.1). Consequently, for the purpose of understanding preorientations of p-divisible groups, there is no harm in restricting our attention to $\mathbb{E}_\infty$-rings which are p-complete.

(b) If A is a p-complete $\mathbb{E}_\infty$-ring, then there is a canonical homotopy equivalence $\mathrm{Pre}(\mathbf{G}) \simeq \mathrm{Pre}(\mathbf{G}^\circ)$ (Proposition 2.2.1). Here $\mathbf{G}^\circ$ denotes the identity component of $\mathbf{G}$ and $\mathrm{Pre}(\mathbf{G}^\circ)$ the space of preorientations of $\mathbf{G}^\circ$ introduced in Definition Or.4.3.19 .

Let us now assume that A is not only p-complete, but $K(n)$-local for some integer $n > 0$. In §Or.4.6 , we constructed a p-divisible group $\mathbf{G}^{\mathcal{Q}}_{A}$ over A, which we refer to as the *Quillen p-divisible group* (Definition Or.4.6.4). In §2.4, we show that giving a preorientation of an arbitrary p-divisible group $\mathbf{G}$ over A is equivalent to giving a morphism of p-divisible groups $e : \mathbf{G}_A^{\mathcal{Q}} \to \mathbf{G}$ (Proposition 2.4.1). In other words, the Quillen p-divisible group $\mathbf{G}_A^{\mathcal{Q}}$ is *universal* among preoriented p-divisible groups over A.

If $\mathbf{G}$ is a p-divisible group over an $\mathbb{E}_\infty$-ring which is p-complete and complex periodic, then we can identify preorientations e of $\mathbf{G}$ with morphisms of formal groups $\iota : \widehat{\mathbf{G}}_A^{\mathcal{Q}} \to \mathbf{G}^\circ$, where $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ denotes the Quillen formal group of A (Construction Or.4.1.13). We will be particularly interested in the case where ι is an equivalence (so that e identifies $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ with the identity component of $\mathbf{G}$). In this case, we will say that e is an *orientation of* $\mathbf{G}$ and that $\mathbf{G}$ is an *oriented p-divisible group over A* (Definition 2.5.1).

For some applications, it is inconvenient to restrict our attention to a single prime number p. In §2.6, we remove this restriction by reviewing the notion of a **P**-*divisible group* over an $\mathbb{E}_\infty$-ring A, where $\mathbf{P} = \{2, 3, 5, \cdots\}$ denotes the set of all prime numbers (Definition 2.6.1). This is essentially just notation: a **P**-divisible group $\mathbf{G}$ can be identified with a family of p-divisible groups $\{\mathbf{G}_{(p)}\}_{p \in \mathbf{P}}$, indexed by the set of all prime numbers p (Remark 2.6.7). We define a *preorientation* e of $\mathbf{G}$ to be a family of preorientations $\{e_p \in \mathrm{Pre}(\mathbf{G}_{(p)})\}_{p \in \mathbf{P}}$ (Definition 2.6.8 and Remark 2.6.9), and we say that e is an *orientation* if A is complex periodic and each e_p induces an orientation of $\mathbf{G}_{(p)}$ after extending scalars to the p-completion of A (Definition 2.6.12). We will be primarily interested in the following pair of examples, which we discuss in §2.8 and §2.9:

- To any $\mathbb{E}_\infty$-ring A, we can associate a **P**-divisible group $\mu_{\mathbf{P}^\infty}$ which we refer to as the *multiplicative* **P**-*divisible group* (Construction 2.8.1). This **P**-divisible group is equipped with a canonical orientation in the case where $A = \mathrm{KU}$ is the periodic complex K-theory spectrum.

- To any strict elliptic curve X over an $\mathbb{E}_\infty$-ring A, we can associate a **P**-divisible group $\mathsf{X}[\mathbf{P}^\infty]$ of *torsion of* X (Construction 2.9.1). Moreover, any orientation of the elliptic curve X (in the sense of Definition Or.7.2.7) determines an orientation of the **P**-divisible group $\mathsf{X}[\mathbf{P}^\infty]$ (Construction 2.9.6.

Remark 2.0.1. Let R_0 be a commutative algebra over $\mathbf{F}_p$ and let $\mathbf{G}_0$ be a p-divisible group over $R_)$. In [9], we proved that if R_0 is Noetherian, F-finite, and $\mathbf{G}_0$ is nonstationary, then $\mathbf{G}_0$ can be "lifted" to an oriented p-divisible group $\mathbf{G}$ over an even complex periodic $\mathbb{E}_\infty$-ring $R^{\mathrm{or}}_{\mathbf{G}_0}$, which we call the *oriented deformation ring* of $\mathbf{G}_0$ (Theorem Or.6.0.3). This result can be used to produce a large class of examples of oriented **P**-divisible groups (to which we can apply the formalism developed in this paper).

2.1 Preorientations of p-Divisible Groups

Let p be a prime number, which we regard as fixed throughout this section. For the reader's convenience, we recall the definition of *p-divisible group* over an $\mathbb{E}_\infty$-ring A (see Definition Or.2.0.2).

Definition 2.1.1. Let A be a connective $\mathbb{E}_\infty$-ring and let CAlg_A denote the ∞-category of $\mathbb{E}_\infty$-algebras over A. A *p-divisible group* over A is a functor

$$\mathbf{G} : \mathrm{CAlg}_A \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$$

which satisfies the following conditions:

(1) For each $B \in \mathrm{CAlg}_A$, the $\mathbf{Z}$-module spectrum $\mathbf{G}(B)$ is p-nilpotent: that is, it satisfies $\mathbf{G}(B)[1/p] \simeq 0$

(2) For every finite abelian p-group M, the functor

$$(B \in \mathrm{CAlg}_A^{\mathrm{cn}}) \mapsto (\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}(A)) \in \mathcal{S})$$

is corepresentable by a finite flat A-algebra.

(3) The map $p : \mathbf{G} \to \mathbf{G}$ is locally surjective with respect to the finite flat topology. In other words, for every object $B \in \mathrm{CAlg}_A^{\mathrm{cn}}$ and every element $x \in \pi_0(\mathbf{G}(B))$, there exists a finite flat map $B \to C$ for which $|\operatorname{Spec}(C)| \to |\operatorname{Spec}(B)|$ is surjective and the image of x in $\pi_0(\mathbf{G}(C))$ is divisible by p.

If A is a nonconnective $\mathbb{E}_\infty$-ring, we define a *p-divisible group over A* to be a p-divisible group over the connective cover $\tau_{\geqslant 0}(A)$, which we view as a functor $\mathbf{G} : \mathrm{CAlg}_{\tau_{\geqslant 0}(A)} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$.

Remark 2.1.2. Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over A. It follows from (1) and (2) that, for any $\mathbb{E}_\infty$-algebra B over A, the canonical map $\mathbf{G}(\tau_{\geqslant 0}(B)) \to \mathbf{G}(B)$ is an equivalence. In other words, $\mathbf{G}$ is a left Kan extension of its restriction to the full subcategory $\mathrm{CAlg}_A^{\mathrm{cn}} \subseteq \mathrm{CAlg}_A$ (so no information is lost by replacing $\mathbf{G}$ by its restriction $\mathbf{G}|_{\mathrm{CAlg}_A^{\mathrm{cn}}}$).

Remark 2.1.3. Let A be an $\mathbb{E}_\infty$-ring which is not necessarily connective, and let $\mathbf{G}$ be a p-divisible group over A. It is not difficult to see that $\mathbf{G}$ is determined by its restriction to $\mathbb{E}_\infty$-algebras over A: that is, by the composite functor

$$\mathrm{CAlg}_A \to \mathrm{CAlg}_{\tau_{\geqslant 0}(A)} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}.$$

However, it is sometimes technically convenient to be able to evaluate $\mathbf{G}$ on objects of $\mathrm{CAlg}_{\tau_{\geqslant 0}(A)}$ which do not admit A-algebra structures (like the ordinary commutative ring $\pi_0(A)$).

We now introduce an analogue of Definition Or.4.3.19 in the setting of p-divisible groups.

Definition 2.1.4. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over A. A *preorientation of* $\mathbf{G}$ is a morphism of $\mathbf{Z}$-module spectra $\Sigma^1(\mathbf{Q}_p / \mathbf{Z}_p) \to \mathbf{G}(A)$. The collection of preorientations of $\mathbf{G}$ are parametrized by a space

$$\mathrm{Pre}(\mathbf{G}) = \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}_p / \mathbf{Z}_p), \mathbf{G}(A)),$$

which we will refer to as the *space of preorientations of* $\mathbf{G}$.

Example 2.1.5. Let A be a commutative ring and let $\mathbf{G}$ be a p-divisible group over A. Then the $\mathbf{Z}$-module spectrum $\mathbf{G}(A)$ is discrete. It follows that the space of preorientations $\mathrm{Pre}(\mathbf{G}) = \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma^1(\mathbf{Q}_p / \mathbf{Z}_p), \mathbf{G}(A))$ is contractible. In other words, $\mathbf{G}$ admits an essentially unique preorientation (given by the zero map $\Sigma^1(\mathbf{Q}_p / \mathbf{Z}_p) \to \mathbf{G}(A)$.

Example 2.1.6. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be an étale p-divisible group over A (Definition Or.2.5.3). Then the $\mathbf{Z}$-module spectrum $\mathbf{G}(B)$ is discrete for *every* object $B \in \mathrm{CAlg}_{\tau_{\geqslant 0}(A)}$ (see Theorem HA.7.5.4.2). It follows that the space $\mathrm{Pre}(\mathbf{G})$ is contractible: that is, $\mathbf{G}$ admits an essentially unique preorientation.

Remark 2.1.7. Let A be an $\mathbb{E}_\infty$-ring and suppose we are given a short exact sequence $0 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 0$ of p-divisible groups over A (Definition Or.2.4.9). We then obtain a fiber sequence of spaces $\mathrm{Pre}(\mathbf{G}') \to \mathrm{Pre}(\mathbf{G}) \to \mathrm{Pre}(\mathbf{G}'')$. In particular, if $\mathbf{G}''$ is étale, then the canonical map $\mathrm{Pre}(\mathbf{G}') \to \mathrm{Pre}(\mathbf{G})$ is a homotopy equivalence: in other words, we can identify preorientations of $\mathbf{G}$ with preorientations of $\mathbf{G}'$.

Remark 2.1.8. Let A be an $\mathbb{E}_\infty$-ring and let $\underline{\mathbf{Q}_p / \mathbf{Z}_p}$ denote the *constant* p-divisible group over A (of height 1) associated to the p-divisible abelian group $\mathbf{Q}_p / \mathbf{Z}_p$. For any p-divisible group $\mathbf{G}$ over A, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{BT}^p(A)}(\underline{\mathbf{Q}_p / \mathbf{Z}_p}, \mathbf{G}) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Q}_p / \mathbf{Z}_p, \mathbf{G}(A)).$$

It follows that the $\mathrm{Pre}(\mathbf{G})$ of preorientations of $\mathbf{G}$ can be identified with the loop space $\Omega \, \mathrm{Map}_{\mathrm{BT}^p(A)}(\underline{\mathbf{Q}_p / \mathbf{Z}_p}, \mathbf{G})$. In particular, homotopy classes of preorientations of $\mathbf{G}$ are classified by the fundamental group $\pi_1 \, \mathrm{Map}_{\mathrm{BT}^p(A)}(\underline{\mathbf{Q}_p / \mathbf{Z}_p}, \mathbf{G})$.

Notation 2.1.9. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over A. For every $\mathbb{E}_\infty$-algebra B over A, we let $\mathbf{G}_B$ denote the p-divisible group over B given by the composite functor

$$\mathrm{CAlg}_{\tau_{\geqslant 0}(B)} \to \mathrm{CAlg}_{\tau_{\geqslant 0}(A)} \xrightarrow{\mathbf{G}} \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}.$$

Then we have a canonical homotopy equivalence

$$\mathrm{Pre}(\mathbf{G}_B) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}_p / \mathbf{Z}_p), \mathbf{G}(B)).$$

In particular, we can regard the construction $B \mapsto \mathrm{Pre}(\mathbf{G}_B)$ as a functor from CAlg_A to the ∞-category $\mathcal{S}$ of spaces, given explicitly by the composition

$$\mathrm{CAlg}_A \xrightarrow{\mathbf{G}} \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}} \xrightarrow{\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}_p / \mathbf{Z}_p), \bullet)} \mathcal{S}.$$

We will need the following elementary observation:

Proposition 2.1.10. *Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over A. Then the functor $(B \in \mathrm{CAlg}_B) \mapsto \mathrm{Pre}(\mathbf{G}_B)$ is corepresentable by an object of the ∞-category CAlg_A. In particular, it commutes with small limits.*

Proof. Replacing A by $\tau_{\geqslant 0}(A)$, we can reduce to the case where A is connective. In this case, we will show that the functor $B \mapsto \mathrm{Pre}(\mathbf{G}_B)$ is corepresentable by a connective $\mathbb{E}_\infty$-algebra over A. By virtue of Notation 2.1.9, we are reduced to showing that the functor $B \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}_p / \mathbf{Z}_p), \mathbf{G}(B))$ is corepresentable by a connective $\mathbb{E}_\infty$-algebra over A. Writing $\mathbf{Q}_p / \mathbf{Z}_p$ as a filtered colimit of finite subgroups of the form $\mathbf{Z} / p^k \mathbf{Z}$, we are reduced to showing that each of the functors

$$\rho_k : \mathrm{CAlg}_A \to \mathcal{S} \qquad \rho_k(B) = \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Z} / p^k \mathbf{Z}), \mathbf{G}(B))$$

is corepresentable by a connective $\mathbb{E}_\infty$-algebra over A. Our assumption that $\mathbf{G}$ is a p-divisible group guarantees that each of the functors $B \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z} / p^k \mathbf{Z}, \mathbf{G}(B))$ is corepresentable by a finite flat A-algebra $A(k)$. Then ρ_k is corepresentable by the suspension of $A(k)$ in the ∞-category $\mathrm{CAlg}_A^{\mathrm{aug}}$ of augmented $\mathbb{E}_\infty$-algebras over A: that is, by the relative tensor product $A \otimes_{A(k)} A$. □

2.2 The p-Complete Case

Let $\mathbf{G}$ be a p-divisible group defined over an $\mathbb{E}_\infty$-ring A which is p-complete, and let $\mathbf{G}^\circ$ denote the formal group given by the identity component of $\mathbf{G}$ (Definition Or.2.0.10). Our goal in this section is to relate preorientations of $\mathbf{G}$ (in the sense of Definition 2.1.4) to preorientations of $\mathbf{G}^\circ$ (in the sense of Definition Or.4.3.19):

Proposition 2.2.1. *Let A be a p-complete $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a p-divisible group over A, and let $\mathbf{G}^\circ$ be its identity component (Definition Or.2.0.10). Then there is a canonical homotopy equivalence* $\mathrm{Pre}(\mathbf{G}) \simeq \mathrm{Pre}(\mathbf{G}^\circ)$.

To prove Proposition 2.2.1, we may assume without loss of generality that A is connective. Recall that, if $\widehat{\mathbf{G}}$ is a formal group over A, then the space $\mathrm{Pre}(\widehat{\mathbf{G}})$ or preorientations of $\widehat{\mathbf{G}}$ can be identified with the mapping space $\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma^2(\mathbf{Z}), \widehat{\mathbf{G}}(A))$ (Remark Or.4.3.20). In the p-complete case, this can be reformulated:

Proposition 2.2.2. *Let $\widehat{\mathbf{G}}$ be a formal group over a connective p-complete $\mathbb{E}_\infty$-ring A. Let $\alpha : \Sigma(\mathbf{Q}_p / \mathbf{Z}_p) \to \Sigma^2(\mathbf{Z})$ denote the map of $\mathbf{Z}$-module spectra determined by the short exact sequence of abelian groups $0 \to \mathbf{Z} \to \mathbf{Z}[1/p] \to \mathbf{Q}_p / \mathbf{Z}_p \to 0$. Then the map*

$$\begin{aligned} \mathrm{Pre}(\widehat{\mathbf{G}}) &\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma^2(\mathbf{Z}), \widehat{\mathbf{G}}(A)) \\ &\xrightarrow{\circ\alpha} \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}_p / \mathbf{Z}_p), \widehat{\mathbf{G}}(A)) \end{aligned}$$

is a homotopy equivalence.

Proof. Without loss of generality, we may assume that A is connective. It will suffice to show that the mapping space $F(A) = \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Z}[1/p]), \widehat{\mathbf{G}}(A))$ is contractible. Since the functor $\widehat{\mathbf{G}}$ is nilcomplete (Proposition Or.1.6.8), we can identify $F(A)$ with the limit $\varprojlim F(\tau_{\leqslant n}(A))$. We are therefore reduced to showing that each $F(\tau_{\leqslant n}(A))$ is contractible. We proceed by induction on n. In the case $n = 0$, the desired result is obvious (since $\widehat{\mathbf{G}}(\tau_{\leqslant n}(A))$ is discrete). To carry out the inductive step, assume that $n > 0$ and set $M = \pi_n(A)$, so that we can regard $\tau_{\leqslant n}(A)$ as a square-zero extension of $\tau_{\leqslant n-1}(A)$ by $\Sigma^n(M)$ (Theorem HA.7.4.1.26). It follows that there exists a pullback diagram

$$\begin{tikzcd}
\tau_{\leqslant n}(A) \arrow[r] \arrow[d] & \pi_0(A) \arrow[d] \\
\tau_{\leqslant n-1}(A) \arrow[r] & \pi_0(A) \oplus \Sigma^{n+1}(M).
\end{tikzcd}$$

Since the functor $\widehat{\mathbf{G}}$ is cohesive (Proposition Or.1.6.8) and $F(\pi_0(A))$ is contractible, we obtain a fiber sequence

$$F(\tau_{\leqslant n}(A)) \to F(\tau_{\leqslant n-1}(A)) \to F(\pi_0(A) \oplus \Sigma^{n+1}(M)).$$

It will therefore suffice to show that the space $F(\pi_(A) \oplus \Sigma^{n+1} M)$ is contractible. Note that we can identify $F(\pi_0(A) \oplus \Sigma^{n+1} M)$ with the zeroth space of the limit of the tower

$$\cdots \xrightarrow{p} Z(\Sigma^{n+1} M) \xrightarrow{p} Z(\Sigma^{n+1} M) \xrightarrow{p} Z(\Sigma^{n+1} M) \xrightarrow{p} Z(\Sigma^{n+1} M),$$

where we define $Z(N) = \Omega \widehat{\mathbf{G}}(\pi_0(A) \oplus N) \in \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$. Note that the construction $N \mapsto Z(N)$ determines an additive functor $\mathrm{Mod}_{\pi_0(A)}^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$, so that the map $p : Z(\Sigma^{n+1} M) \to Z(\Sigma^{n+1} M)$ is induced by the multiplication $p : M \to M$. We therefore obtain a homotopy equivalence $F((\pi_0(A) \oplus \Sigma^{n+1} M) \simeq \Omega^\infty Z(M')$, where M' denotes the limit of the tower

$$\cdots \to \Sigma^{n+1} M \xrightarrow{p} \Sigma^{n+1} M \xrightarrow{p} \Sigma^{n+1} M,$$

formed in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$. We conclude the proof by observing that $M' \simeq 0$, by virtue of our assumption that A is p-complete. □

Proof of Proposition 2.2.1. Without loss of generality, we may assume that A is a connective p-complete $\mathbb{E}_\infty$-ring. Let $\mathbf{G}$ be a p-divisible group over A and let $\mathbf{G}^\circ$ be its identity component. Let $\mathcal{C} \subseteq \mathrm{CAlg}_A^{\mathrm{cn}}$ denote the full subcategory spanned by those connective A-algebras B such that B is truncated and p is nilpotent in $\pi_0(B)$. Then, for each $B \in \mathcal{C}$, we have a canonical fiber sequence

$$\mathbf{G}^\circ(B) \to \mathbf{G}(B) \to \mathbf{G}(B^{\mathrm{red}}).$$

We therefore obtain a natural map $\mathrm{Pre}(\mathbf{G}_B^\circ) \to \mathrm{Pre}(\mathbf{G}_B)$, given by the composition

$$\begin{aligned}
\mathrm{Pre}(\mathbf{G}_B^\circ) &\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma^2(\mathbf{Z}), \mathbf{G}^\circ(B)) \\
&\to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}_p / \mathbf{Z}_p), \mathbf{G}^\circ(B)) \\
&\to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}_p / \mathbf{Z}_p), \mathbf{G}(B)) \\
&\simeq \mathrm{Pre}(\mathbf{G}_B),
\end{aligned}$$

where the first map is given by precomposition with the map $\alpha : \Sigma(\mathbf{Q}_p / \mathbf{Z}_p) \to \mathbf{Z}$ appearing in Proposition 2.2.2 (and is therefore a homotopy equivalence), and the second is a homotopy equivalence by virtue of the fact that $\mathbf{G}(B^{\mathrm{red}})$ is discrete. The resulting homotopy equivalence depends functorially on B, and therefore supplies a homotopy equivalence

$$\varprojlim_{B\in\mathcal{C}} \operatorname{Pre}(\mathbf{G}_B^\circ) \simeq \varprojlim_{B\in\mathcal{C}} \operatorname{Pre}(\mathbf{G}_B).$$

The desired result now follows from the fact that the tautological maps

$$\operatorname{Pre}(\mathbf{G}^\circ) \to \varprojlim_{B\in\mathcal{C}} \operatorname{Pre}(\mathbf{G}_B^\circ) \qquad \operatorname{Pre}(\mathbf{G}) \to \varprojlim_{B\in\mathcal{C}} \operatorname{Pre}(\mathbf{G}_B)$$

are homotopy equivalences (Lemma Or.4.3.16 and Proposition 2.1.10). □

Corollary 2.2.3. *Let A be an $\mathbb{E}_\infty$-ring which is p-complete and complex periodic, and let $\mathbf{G}$ be a p-divisible group over A. Then we have a canonical homotopy equivalence*

$$\operatorname{Pre}(\mathbf{G}) \simeq \operatorname{Map}_{\mathrm{FGroup}(A)}(\widehat{\mathbf{G}}_A^Q, \mathbf{G}^\circ),$$

where $\widehat{\mathbf{G}}_A^Q$ is the Quillen formal group over A (Construction Or.4.1.13).

Proof. Combine Propositions 2.2.1 and Or.4.3.21. □

Corollary 2.2.4. *Let A be a connective $\mathbb{E}_\infty$-ring which is p-complete and 1-truncated. Then, for every p-divisible group $\mathbf{G}$ over A, the space of preorientations* $\operatorname{Pre}(\mathbf{G})$ *is contractible.*

Proof. Combine Proposition 2.2.1 with Example Or.4.3.5. □

2.3 Reduction to the p-Complete Case

Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over A. Proposition 2.2.1 asserts that, if A is p-complete, then giving a preorientation of $\mathbf{G}$ (in the sense of Definition 2.1.4) is equivalent to giving a preorientation of its identity component $\mathbf{G}^\circ$ (in the sense of Definition Or.4.3.19). The general case can be always be reduced to the p-complete case, by virtue of the following result:

Proposition 2.3.1. *Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a p-divisible group over A, and let $\widehat{A}$ be the p-completion of A. Then the map* $\operatorname{Pre}(\mathbf{G}) \to \operatorname{Pre}(\mathbf{G}_{\widehat{A}})$ *of Notation 2.1.9 is a homotopy equivalence.*

Our proof begins with a simple observation:

Lemma 2.3.2. *Let $f : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings which induces an isomorphism $\pi_n(A) \to \pi_n(B)$ for $n > 0$ and let $\mathbf{G}$ be a p-divisible group over A. Then the canonical map* $\operatorname{Pre}(\mathbf{G}) \to \operatorname{Pre}(\mathbf{G}_B)$ *is a homotopy equivalence.*

Proof. Without loss of generality, we may assume that A and B are connective. We then have a homotopy pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
\pi_0(A) \arrow[r] & \pi_0(B).
\end{tikzcd}$$

Applying Proposition 2.1.10, we obtain a pullback diagram of spaces

$$\begin{tikzcd}
\mathrm{Pre}(\mathbf{G}) \arrow[r] \arrow[d] & \mathrm{Pre}(\mathbf{G}_B) \arrow[d] \\
\mathrm{Pre}(\mathbf{G}_{\pi_0(A)}) \arrow[r] & \mathrm{Pre}(\mathbf{G}_{\pi_0(B)}).
\end{tikzcd}$$

The bottom horizontal map is a homotopy equivalence (since both spaces are contractible by virtue of Example 2.1.5). Consequently, the upper horizontal map is a homotopy equivalence as well. □

Proof of Proposition 2.3.1. Let $A' = \tau_{\geqslant 0}(A)$ be the connective cover of A, so that $\mathbf{G}$ is obtained from a p-divisible group $\mathbf{G}'$ over A' by extension of scalars. Let $\widehat{A}'$ be the (p)-completion of A'. Then the map $\widehat{A}' \to \widehat{A}$ induces an isomorphism on π_* for $* > 0$. It follows from Lemma 2.3.2 that the vertical maps in the diagram

$$\begin{tikzcd}
\mathrm{Pre}(\mathbf{G}') \arrow[r] \arrow[d] & \mathrm{Pre}(\mathbf{G}'_{\widehat{A}'}) \arrow[d] \\
\mathrm{Pre}(\mathbf{G}) \arrow[r] & \mathrm{Pre}(\mathbf{G}_{\widehat{A}})
\end{tikzcd}$$

are homotopy equivalences. Consequently, to show that the upper horizontal map is a homotopy equivalence, it will suffice to show that the lower horizontal map is a homotopy equivalence. Replacing A by A' (and $\mathbf{G}$ by $\mathbf{G}'$) we may reduce to the case where A is connective.

For each integer $k \geqslant 0$, let $F_k : \mathrm{CAlg}^{\mathrm{cn}}_A \to \mathcal{S}$ denote the functor given by the formula $F_k(B) = \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Z}/p^k\mathbf{Z}), \mathbf{G}(B))$. Writing $\mathbf{Q}_p/\mathbf{Z}_p$ as a direct limit of finite subgroups $\mathbf{Z}/p^k\mathbf{Z}$, we obtain a canonical equivalence $\mathrm{Pre}(\mathbf{G}_B) \simeq \varprojlim_k F_k(B)$. For each $B \in \mathrm{CAlg}^{\mathrm{cn}}_A$, let $\widehat{B}$ denote the p-completion of B. We will prove Proposition 2.3.1 by showing that the canonical map $\theta^B : \varprojlim_k F_k(B) \to \varprojlim_k F_k(\widehat{B})$ is a homotopy equivalence for each $B \in \mathrm{CAlg}^{\mathrm{cn}}_A$. Note that if B is discrete, then each $F_k(B)$ is contractible.

For $k \geqslant 0$, define a functor $G_k : \mathrm{CAlg}_A^{\mathrm{cn}} \to \mathcal{S}$ by the formula $G_k(B) = \mathrm{fib}(F_k(\widehat{B}) \to F_k(\widehat{\pi_0(B)}))$. Note that the p-completion $\widehat{\pi_0(B)}$ is 1-truncated and p-complete (Corollary SAG.II.4.3.2.4), so that $\mathrm{Pre}(\mathbf{G}_{\widehat{\pi_0(B)}}) \simeq \varprojlim_k F_k(\widehat{\pi_0(B)})$ is contractible (Corollary 2.2.4). It follows that the evident maps $G_k(B) \to F_k(\widehat{B})$ induce a homotopy equivalence $\varprojlim_k G_k(B) \to \varprojlim_k F_k(\widehat{B})$ for each $B \in \mathrm{CAlg}_A^{\mathrm{cn}}$. We can therefore identify θ^B with the limit of maps $\theta_k^B : F_k(B) \to G_k(B)$. We will complete the proof by showing that each θ_k^B is a homotopy equivalence.

Since $\mathbf{G}$ is a p-divisible group, the functor

$$(B \in \mathrm{CAlg}_A^{\mathrm{cn}}) \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}/p^k\,\mathbf{Z}, \mathbf{G}(B))$$

is corepresentable by some finite flat A-algebra C. It follows that the functors F_k and G_k are given concretely by the formulae

$$F_k(B) = \mathrm{fib}(\Omega\,\mathrm{Map}_{\mathrm{CAlg}_A}(C, B) \to \Omega\,\mathrm{Map}_{\mathrm{CAlg}_R}(C, \pi_0(B)))$$

$$G_k(A) = \mathrm{fib}(\Omega\,\mathrm{Map}_{\mathrm{CAlg}_A}(C, \widehat{B}) \to \Omega\,\mathrm{Map}_{\mathrm{CAlg}_R}(C, \widehat{\pi_0(B)})).$$

It follows from these formulae that the functors F_k and G_k are nilcomplete; consequently, it will suffice to show that θ_k^B is a homotopy equivalence under the additional assumption that B is n-truncated for some $n \geqslant 0$. We proceed by induction on n. In the case $n = 0$, the spaces $F_k(B)$ and $G_k(B)$ are both contractible, so there is nothing to prove. To carry out the inductive step, let us suppose that B is n-truncated for some $n \geqslant 0$. Using Theorem HA.7.4.1.26 , we see that B is a square-zero extension of $\tau_{\leqslant n-1}(B)$ by the module $M = \Sigma^n(\pi_n(B))$: that is, there exists a pullback diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & \pi_0(B) \arrow[d] \\
\tau_{\leqslant n-1}(B) \arrow[r] & \pi_0(B) \oplus \Sigma M.
\end{tikzcd}$$

This diagram remains a pullback square after applying the functors F_k and G_k, so we have a pullback diagram

$$\begin{tikzcd}
\theta_k^B \arrow[r] \arrow[d] & \theta_k^{\pi_0(B)} \arrow[d] \\
\theta_k^{\tau_{\leqslant n-1}(B)} \arrow[r] & \theta_k^{\pi_0(B) \oplus \Sigma M}
\end{tikzcd}$$

in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{S})$. Since $\theta_k^{\pi_0(B)}$ and $\theta_k^{\tau_{\leqslant n-1}(B)}$ are homotopy equivalences by our inductive hypothesis, we are reduced to proving that the map

$$\rho = \theta_k^{\pi_0(A) \oplus \Sigma M} : F_k(\pi_0(B) \oplus \Sigma M) \to G_k(\pi_0(B) \oplus \Sigma M)$$

is a homotopy equivalence. Using the formula for G_k given above and the fact that the functor $\mathrm{Map}_{\mathrm{CAlg}_A}(C, \bullet)$ commutes with limits, we obtain a homotopy equivalence $G_k(\pi_0(B) \oplus \Sigma M) \simeq F_k(\pi_0(B) \oplus \Sigma \widehat{M})$, where $\widehat{M}$ denotes the p-adic completion of M. It follows that we have a fiber sequence

$$F_k(\pi_0(B) \oplus \Sigma M) \xrightarrow{\rho} F_k(\pi_0(B) \oplus \Sigma \widehat{M}) \to F_k(\pi_0(B) \oplus N),$$

where N denotes the cofiber of the map $\Sigma M \to \Sigma \widehat{M}$.

Define a functor $H : \mathrm{Mod}^{\mathrm{cn}}_{\pi_0(B)} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}/p^k\mathbf{Z}}$ by the formula

$$H(K) = \mathrm{fib}(\mathbf{G}[p^k]((\pi_0(A) \oplus K) \to \mathbf{G}[p^k](\pi_0(B)).$$

Note that the functor H is additive. Consequently, applying H to the multiplication map $p^k : K \to K$ induces multiplication by p^k on H, which is nullhomotopic by construction. It follows that if multiplication by p is an equivalence from K to itself, then $H(K) \simeq 0$. Applying this observation in the case $K = N$, we deduce that $H(N) \simeq 0$ so that $F_k(\pi_0(B) \oplus N) = \Omega^{\infty+1} H(N)$ is contractible. It follows that ρ is a homotopy equivalence, as desired. □

2.4 The $K(n)$-Local Case

Let A be an $\mathbb{E}_\infty$-ring which is complex periodic and $K(n)$-local, for some $n \geqslant 1$. In §Or.4.6 , we introduced a p-divisible group $\mathbf{G}^{\mathcal{Q}}_A$ which we refer to as the *Quillen p-divisible group of A* (see Definition Or.4.6.4). This p-divisible group is characterized (up to equivalence) by the fact that it is formally connected (with respect to the topology on $\pi_0(A)$ given by the nth Landweber ideal $\mathfrak{I}^A_n$) and that its identity component is the Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ (Theorem Or.4.6.16). For our purposes, this can be reformulated as follows:

Proposition 2.4.1. *Let A be an $\mathbb{E}_\infty$-ring which is complex periodic and $K(n)$-local for some $n > 0$. Then, for any p-divisible group over A, there is a canonical homotopy equivalence*

$$\mathrm{Map}_{\mathrm{BT}^p(A)}(\mathbf{G}^{\mathcal{Q}}_A, \mathbf{G}) \simeq \mathrm{Pre}(\mathbf{G}).$$

Proof. Since $\mathbf{G}^{\mathcal{Q}}_A$ is formally connected, Theorem Or.2.3.12 implies that passage to the identity component induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{BT}^p(A)}(\mathbf{G}^{\mathcal{Q}}_A, \mathbf{G}) \simeq \mathrm{Map}_{\mathrm{FGroup}(A)}(\mathbf{G}^{\mathcal{Q}\circ}_A, \mathbf{G}^{\circ}).$$

Theorem Or.4.6.16 allows us to identify $\mathbf{G}_A^{\mathcal{Q}^\circ}$ with the Quillen formal group $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$, so that the mapping space $\mathrm{Map}_{\mathrm{FGroup}(A)}(\mathbf{G}_A^{\mathcal{Q}^\circ}, \mathbf{G}^\circ)$ can be identified with the space $\mathrm{Pre}(\mathbf{G}^\circ)$ classifying preorientations of $\mathbf{G}^\circ$ (Proposition Or.4.3.21). The desired result now follows from from the homotopy equivalence $\mathrm{Pre}(\mathbf{G}) \simeq \mathrm{Pre}(\mathbf{G}^\circ)$ of Corollary 2.2.3. □

Remark 2.4.2. The homotopy equivalence of Proposition 2.4.1 depends functorially on $\mathbf{G}$. It follows that that the functor

$$(\mathbf{G} \in \mathrm{BT}^p(A)) \mapsto \mathrm{Pre}(A) = \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}_p/\mathbf{Z}_p), \mathbf{G}(A))$$

is *corepresented* by the Quillen formal group $\mathbf{G}_A^{\mathcal{Q}}$. In other words, there exists a preorientation $\eta \in \mathbf{G}_A^{\mathcal{Q}}$ which is universal in the sense that, for any p-divisible group $\mathbf{G}$, evaluation on η induces the homotopy equivalence

$$\mathrm{Map}_{\mathrm{BT}^p(A)}(\mathbf{G}_A^{\mathcal{Q}}, \mathbf{G}) \simeq \mathrm{Pre}(\mathbf{G})$$

of Proposition 2.4.1.

Let us describe the preorientation η more explicitly, without reference to the theory of formal groups. For each finite abelian p-group H, let $\widehat{H} = \mathrm{Hom}(H, \mathbf{Q}/\mathbf{Z})$ denote the Pontryagin dual of H. The Quillen p-divisible group $\mathbf{G}_A^{\mathcal{Q}}$ is charactered by the existence of homotopy equivalences

$$\mathrm{Map}_{\mathrm{CAlg}_{\tau_{\geqslant 0}(A)}}(\tau_{\geqslant 0}(A^{BH}), B) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\widehat{H}, \mathbf{G}_A^{\mathcal{Q}}(B))$$

depending functorially on H and B (which ranges over $\mathbb{E}_\infty$-algebras over the connective cover of A); see Construction Or.4.6.2 . Setting $B = A$ and composing with the natural map

$$BM \to \mathrm{Map}_{\mathrm{CAlg}_A}(A^{BH}, A) \simeq \mathrm{Map}_{\mathrm{CAlg}_{\tau_{\geqslant 0}(A)}}(\tau_{\geqslant 0}(A^{BH}), A),$$

we obtain maps $\rho : BH \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\widehat{H}, \mathbf{G}_A^{\mathcal{Q}}(A))$, depending functorially on H. Here we can regard both sides as *p-torsion objects* of the ∞-category of spaces, in the sense of Definition AV.6.4.2 . The fully faithful embedding $\mathrm{Tors}_p \mathcal{S} \hookrightarrow \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ of Example AV.6.4.11 carries ρ to a morphism of $\mathbf{Z}$-module spectra $\Sigma(\mathbf{Q}_p/\mathbf{Z}_p) \to \mathbf{G}_A^{\mathcal{Q}}(A)$, which we can view as a preorientation of $\mathbf{G}_A^{\mathcal{Q}}$. We leave it to the reader to verify that this agrees with the preorientation constructed implicitly in the proof of Proposition 2.4.1.

2.5 Orientations of p-Divisible Groups

Let A be an $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}$ be a formal group over A. Recall that a preorientation e of $\widehat{\mathbf{G}}$ is said to be an *orientation* if A is complex periodic and e is classified by an equivalence of formal groups $\widehat{\mathbf{G}}_A^{\mathcal{Q}} \to \widehat{\mathbf{G}}$ (Proposition Or.4.3.23). We now adapt this definition to the setting of p-divisible groups.

Definition 2.5.1. Let A be a p-complete $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over A. We will say that a preorientation e of $\mathbf{G}$ is an *orientation* if its image under the homotopy equivalence $\mathrm{Pre}(\mathbf{G}) \simeq \mathrm{Pre}(\mathbf{G}^\circ)$ of Proposition 2.2.1 is an orientation of the identity component $\mathbf{G}^\circ$, in the sense of Definition Or.4.3.9 . We let $\mathrm{OrDat}(\mathbf{G}) \subseteq \mathrm{Pre}(\mathbf{G})$ denote the summand consisting of all orientations of the p-divisible group $\mathbf{G}$.

Remark 2.5.2 (Functoriality)**.** Let $f : A \to A'$ be a morphism of p-complete $\mathbb{E}_\infty$-rings and let $\mathbf{G}$ be a p-divisible group over A. Then the natural map $\mathrm{Pre}(\mathbf{G}) \to \mathrm{Pre}(\mathbf{G}_{A'})$ carries orientations of $\mathbf{G}$ to orientations of $\mathbf{G}_{A'}$.

Remark 2.5.3. Let A be a p-complete $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over A. If A is complex periodic, then giving an orientation of $\mathbf{G}$ is equivalent to choosing an equivalence of formal groups $\widehat{\mathbf{G}}_A^{\mathcal{Q}} \simeq \mathbf{G}^\circ$, where $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ is the Quillen formal group of A. If A is not complex periodic, then the space of orientations $\mathrm{OrDat}(\mathbf{G})$ is empty (Proposition Or.4.3.23).

Remark 2.5.4. Let A be a p-complete $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group of height $\leqslant n$ over A. Suppose that $\mathbf{G}$ admits an orientation (so that A is necessarily complex periodic, by Remark 2.5.3). Then:

- The p-divisible group $\mathbf{G}$ is 1-dimensional (since $\mathbf{G}^\circ$ is equivalent to the 1-dimensional formal group $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$).
- The Quillen p-divisible group $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ has height $\leqslant n$ (since it is equivalent to the identity component of $\mathbf{G}$).

Warning 2.5.5. Let A be a p-complete $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over A. Then $\mathbf{G}$ can be identified with a p-divisible group $\mathbf{G}_0$ over the connective cover $\tau_{\geqslant 0}(A)$, and we can identify preorientations e of $\mathbf{G}$ with preorientations e_0 of $\mathbf{G}_0$. Beware, however, that that e_0 is *never* an orientation (even if e is an orientation), except in the trivial case $A \simeq 0$.

We now prove an analogue of Proposition Or.4.3.23, which allows us to reformulate Definition 2.5.1 without reference to the theory of formal groups. We begin with the $K(n)$-local case.

Proposition 2.5.6. *Let A be an $\mathbb{E}_\infty$-ring which is $K(n)$-local for some $n \geqslant 1$, let $\mathbf{G}$ be a p-divisible group over A, and let e be a preorientation of $\mathbf{G}$. Then e is an orientation if and only if the following conditions are satisfied:*

(0) *The p-divisible group $\mathbf{G}$ is 1-dimensional.*

(1) *The $\mathbb{E}_\infty$-ring A is complex periodic, so that the Quillen p-divisible group $\mathbf{G}_A^{\mathcal{Q}}$ is well-defined (Definition Or.4.6.4).*

(2) *The image of e under the homotopy equivalence $\mathrm{Pre}(\mathbf{G}) \simeq \mathrm{Map}_{\mathrm{BT}^p(A)}(\mathbf{G}_A^{\mathcal{Q}}, \mathbf{G})$ of Proposition 2.4.1 is a monomorphism of p-divisible groups $\mathbf{G}_A^{\mathcal{Q}} \to \mathbf{G}$ (in the sense of Definition Or.2.4.3).*

Proof. Assume first that conditions (0), (1), and (2) are satisfied. Let us abuse notation by identifying e with the map of p-divisible groups $\mathbf{G}_A^{\mathcal{Q}} \to \mathbf{G}$ supplied by Proposition 2.4.1. Using (2), we obtain a short exact sequence of p-divisible groups

$$0 \to \mathbf{G}_A^{\mathcal{Q}} \xrightarrow{e} \mathbf{G} \to \mathbf{H} \to 0$$

(in the sense of Definition Or.2.4.9). Since $\mathbf{G}_A^{\mathcal{Q}}$ and $\mathbf{G}$ are both 1-dimensional, it follows that $\mathbf{H}$ is an étale p-divisible group over A. Consequently, the map e induces an equivalence of identity components $\widehat{\mathbf{G}}_A^{\mathcal{Q}} \simeq (\mathbf{G}_A^{\mathcal{Q}})^\circ \xrightarrow{e} \mathbf{G}^\circ$ and is therefore an orientation in the sense of Definition 2.5.1.

We now prove the converse. Suppose that e is an orientation of $\mathbf{G}$. Then conditions (0) and (1) are automatic (Remarks 2.5.3 and 2.5.4). Let $\mathfrak{I}_n^A \subseteq \pi_0(A)$ be the nth Landweber ideal (Definition Or.4.4.11). We will regard A as an adic $\mathbb{E}_\infty$-ring by equipping $\pi_0(A)$ with the $\mathfrak{I}_n^A$-adic topology. Our assumption that A is $K(n)$-local guarantees that it is complete (as an adic $\mathbb{E}_\infty$-ring); see Proposition Or.4.5.4. The Quillen formal group $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ is the identity component of the Quillen p-divisible group $\mathbf{G}_A^{\mathcal{Q}}$ (Theorem Or.4.6.16). The p-divisible group $\mathbf{G}_A^{\mathcal{Q}}$ is formally connected (essentially by the definition of the ideal $\mathfrak{I}_n^A$), so that $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ is a p-divisible formal group over A in the sense of Definition Or.2.3.14. Since e is an orientation, it follows that $\mathbf{G}^\circ \simeq (\mathbf{G}_A^{\mathcal{Q}})^\circ \simeq \widehat{\mathbf{G}}_A^{\mathcal{Q}}$ is also a p-divisible formal group over A. Applying Proposition Or.2.5.17, we deduce that $\mathbf{G}$ admits a connected-étale sequence

$$0 \to \mathbf{G}' \xrightarrow{i} \mathbf{G} \to \mathbf{G}'' \to 0.$$

Then e factors as a composition $\mathbf{G}_A^{\mathcal{Q}} \xrightarrow{f} \mathbf{G}' \xrightarrow{i} \mathbf{G}$. Since i is a monomorphism of p-divisible groups, it will suffice to show that f is an equivalence. Since $\mathbf{G}_A^{\mathcal{Q}}$ and $\mathbf{G}'$ are formally connected p-divisible groups over A, this is equivalent to the requirement that f induces an equivalence of identity components (Corollary Or.2.3.13), which follows from our assumption that e is an orientation. □

Corollary 2.5.7. *Let A be a p-complete $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a p-divisible group over A, and let e be a preorientation of $\mathbf{G}$. Then e is an orientation if and only if the following conditions are satisfied:*

(0) *The p-divisible group $\mathbf{G}$ is 1-dimensional.*

(1) *The $\mathbb{E}_\infty$-ring A is complex periodic and the classical Quillen formal group $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ has finite height at every point of $|\operatorname{Spec}(A)|$.*

(2) *For each integer $m \geqslant 1$, the image of e under the composite map*

$$\operatorname{Pre}(\mathbf{G}) \to \operatorname{Pre}(\mathbf{G}_{L_{K(m)}A}) \simeq \operatorname{Map}_{\mathrm{BT}^p(A)}(\mathbf{G}^{\mathcal{Q}}_{L_{K(m)}(A)}, \mathbf{G}_{L_{K(m)}(A)})$$

is a monomorphism $f_m : \mathbf{G}^{\mathcal{Q}}_{L_{K(m)}(A)} \to \mathbf{G}_{L_{K(m)}(A)}$ of p-divisible groups over $L_{K(m)}(A)$ (in the sense of Definition Or.2.4.3).

Warning 2.5.8. In the statement of Corollary 2.5.7, the assumption that $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ has finite height at every point of $|\operatorname{Spec}(A)|$ cannot be omitted. Otherwise, we could obtain a counterexample taking A to be any complex periodic $\mathbb{E}_\infty$-algebra over $\mathbf{F}_p$ (and $\mathbf{G}$ to be any 1-dimensional preoriented p-divisible group over A).

Remark 2.5.9. In the statement of Corollary 2.5.7, conditions (0) and (1) do not depend on the preorientation e: they are conditions on A and $\mathbf{G}$ which are necessary for the existence of *any* orientation.

Proof of Corollary 2.5.7. If e is an orientation, then conditions (0) and (1) follow from Remarks 2.5.3 and 2.5.4, while (2) follows from Proposition 2.5.6. Conversely, suppose that (0), (1), and (2) are satisfied. Then the formal group $\mathbf{G}^\circ$ is 1-dimensional. Let $\omega_{\mathbf{G}^\circ}$ denote its dualizing line (Definition Or.4.2.14) and let $\beta_e : \omega_{\mathbf{G}^\circ} \to \Sigma^{-2}(A)$ be the Bott map associated to e (Construction Or.4.3.7). We wish to show that β_e is an equivalence.

Let B be an $\mathbb{E}_\infty$-algebra over A. We will say that B is *good* if it is p-complete and the map $\operatorname{Pre}(\mathbf{G}) \to \operatorname{Pre}(\mathbf{G}_B)$ carries e to an orientation of $\mathbf{G}_B$. Equivalently, B is good if it is p-complete and the morphism β_e becomes an equivalence after extending scalars to B. From this description, we see that the collection of good A-algebras is closed under fiber products.

For each $n \geqslant 0$, let $\mathfrak{I}^A_n$ denote the nth Landweber ideal of A (Definition Or.4.5.1), so that the vanishing locus of $\mathfrak{I}^A_n$ in $|\operatorname{Spec}(A)|$ consists of those points where the classical Quillen formal group $\widehat{\mathbf{G}}^{Q_0}_A$ has height $\geqslant n$. It follows from condition (1) that the union $\bigcup_n \mathfrak{I}^A_n$ is the unit ideal of $\pi_0(A)$. In other words, there exists some integer $n \gg 0$ such that $\mathfrak{I}^A_n = \pi_0(A)$. In particular, A is $\mathfrak{I}^A_n$-local as an A-module. We will complete the proof by establishing the following assertion, for each positive integer m:

($*_m$) Let $B \in \mathrm{CAlg}_A$ be an $\mathbb{E}_\infty$-algebra over A which is p-complete and $\mathfrak{I}^A_m$-local as an A-module (this is equivalent to the requirement that B is $E(m-1)$-local as a spectrum). Then B is good.

The proof of ($*_m$) will proceed by induction on m. In the case $m = 1$, we have $\mathfrak{I}^A_m = (p)$, so any A-algebra which is p-complete and $\mathfrak{I}^A_m$-local must vanish. To carry out the inductive step, assume that assertion ($*_m$) is satisfied and let B be a p-complete $\mathbb{E}_\infty$-algebra over A which is $\mathfrak{I}^A_{m+1}$-local; we will show that B is good. Let $I = \mathfrak{I}^A_m$ denote the mth Landweber ideal of $\pi_0(A)$. Let

$$M \mapsto L_I(M) \qquad M \mapsto M^\wedge_I$$

denote the functors of localization and completion with respect to I. Then we have a pullback diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B^\wedge_I \arrow[d] \\ L_I(B) \arrow[r] & L_I(B^\wedge_I) \end{tikzcd}$$

of $\mathbb{E}_\infty$-algebras over A (since the vertical maps become equivalences after I-localization, and the horizontal maps become equivalences after I-completion). Passing to p-completions (and invoking our assumption that B is p-complete), we obtain a pullback diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B^\wedge_I \arrow[d] \\ L_I(B)^\wedge_{(p)} \arrow[r] & L_I(B^\wedge_I)^\wedge_{(p)}. \end{tikzcd}$$

Here $L_I(B)^\wedge_{(p)}$ and $L_I(B^\wedge_I)^\wedge_{(p)}$ are I-local and p-complete, and therefore good by virtue of our inductive hypothesis. Consequently, to show that B is good, it will suffice to show that $B_I^\wedge$ is good. However, the ring spectrum $B_I^\wedge$ is $K(m)$-local (see Theorem Or.4.5.2), so the unit map $A \to B_I^\wedge$ factors through the $K(m)$-localization $L_{K(m)}(A)$. We are therefore reduced to showing that $L_{K(m)}(A)$ is good, which follows from assumption (2). □

Remark 2.5.10. In the statement of Corollary 2.5.7, we can replace (1) with the following alternate condition:

($1'$) The $\mathbb{E}_\infty$-ring A is complex periodic and $E(m)$-local for some $m \gg 0$.

If A is a complex periodic $\mathbb{E}_\infty$-ring, then conditions (1) and ($1'$) can both be phrased in terms of the Landweber ideals $\mathfrak{I}_n^A$: condition (1) asserts that we have $\mathfrak{I}_n^A = \pi_0(A)$ for $n \gg 0$, while condition ($1'$) asserts that A is $\mathfrak{I}_n^A$-local for $n \gg 0$ (note that A is $E(m)$-local as a spectrum if and only if it is $\mathfrak{I}_{m+1}^A$-local as an A-module). It follows immediately that (1) ⇒ ($1'$). On the other hand, condition ($1'$) is all that was needed in the proof of Corollary 2.5.7.

Beware that it is generally not true that condition ($1'$) implies condition (1) (in the absence of the other assumptions of Corollary 2.5.7). For example, if MP is the periodic complex bordism spectrum, then the canonical map

$$\mathrm{MP}_{(p)} \to L_{E(1)}(\mathrm{MP}_{(p)})$$

induces an isomorphism on π_0. Consequently the classical Quillen formal group of $A = L_{E(1)}(\mathrm{MP}_{(p)})$ coincides with the classical Quillen formal group of $\mathrm{MP}_{(p)}$, and therefore has unbounded height (despite the fact that A is $E(1)$-local). It follows that there cannot exist an oriented p-divisible group over A.

Remark 2.5.11. In the statement of Corollary 2.5.7, we can also replace (1) with the following:

($1''$) The $\mathbb{E}_\infty$-ring A is complex periodic and the smash product $\mathbf{F}_p \otimes_S A$ vanishes.

2.6 P-Divisible Groups

Throughout this paper, we will write $\mathbf{P}$ for the set $\{2, 3, 5, \cdots\}$ of all prime numbers. In §AV.6.5 , we introduced the notion of a $\mathbf{P}$-*divisible group* over an $\mathbb{E}_\infty$-ring. Let us recall the definition in a form which will be convenient for our applications here.

Definition 2.6.1. Let A be a connective $\mathbb{E}_\infty$-ring and let CAlg_A denote the ∞-category of $\mathbb{E}_\infty$-algebras over A. A $\mathbf{P}$-*divisible group* over A is a functor

$$\mathbf{G} : \mathrm{CAlg}_A \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$$

which satisfies the following conditions:

(1) For each $B \in \mathrm{CAlg}_A$, the $\mathbf{Z}$-module spectrum $\mathbf{G}(B)$ is torsion: that is, it satisfies $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{G}(B) \simeq 0$

(2) For every finite abelian group M, the functor

$$(B \in \mathrm{CAlg}_A^{\mathrm{cn}}) \mapsto (\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}(A)) \in \mathcal{S})$$

is corepresentable by a finite flat A-algebra.

(3) For every positive integer n, the map $n : \mathbf{G} \to \mathbf{G}$ is locally surjective with respect to the finite flat topology. In other words, for every object $B \in \mathrm{CAlg}_A^{\mathrm{cn}}$ and every element $x \in \pi_0(\mathbf{G}(B))$, there exists a finite flat map $B \to C$ for which $|\operatorname{Spec}(C)| \to |\operatorname{Spec}(B)|$ is surjective and the image of x in $\pi_0(\mathbf{G}(C))$ is divisible by n.

If A is a nonconnective $\mathbb{E}_\infty$-ring, we define a $\mathbf{P}$-*divisible group over* A to be a $\mathbf{P}$-divisible group over the connective cover $\tau_{\geqslant 0}(A)$, which we view as a functor $\mathbf{G} : \mathrm{CAlg}_{\tau_{\geqslant 0}(A)} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$.

Remark 2.6.2. Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over A. It follows from (1) and (2) that, for any $\mathbb{E}_\infty$-algebra B over A, the canonical map $\mathbf{G}(\tau_{\geqslant 0}(B)) \to \mathbf{G}(B)$ is an equivalence. In other words, $\mathbf{G}$ is a left Kan extension of its restriction to the full subcategory $\mathrm{CAlg}_A^{\mathrm{cn}} \subseteq \mathrm{CAlg}_A$ (so no information is lost by replacing $\mathbf{G}$ by its restriction $\mathbf{G}|_{\mathrm{CAlg}_A^{\mathrm{cn}}}$).

Remark 2.6.3. In the situation of Definition 2.6.1, it suffices to check condition (3) in the special case where $n = p$ is a prime number.

Example 2.6.4. Let p be a prime number and let A be an $\mathbb{E}_\infty$-ring. Then every p-divisible group over A (in the sense of Definition 2.1.1) is a $\mathbf{P}$-divisible group over A (in the sense of Definition 2.6.1).

Example 2.6.4 has a converse:

Construction 2.6.5. For each object $M \in \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ and each prime number p, we let $M_{(p)}$ denote the localization of M at the ideal (p) (given by the formula $M_{(p)} \simeq \mathbf{Z}_{(p)} \otimes_{\mathbf{Z}} M$).

Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over A. For each prime number p, we let $\mathbf{G}_{(p)} : \mathrm{CAlg}_A \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ denote the functor given by the formula $\mathbf{G}_{(p)}(B) = \mathbf{G}(B)_{(p)}$. Then $\mathbf{G}_{(p)}$ is a p-divisible group over A: it satisfies requirements (1), (2), and (3) of Definition 2.1.1 by virtue of the fact that $\mathbf{G}$ satisfies the corresponding requirement of Definition 2.6.1. We refer to $\mathbf{G}_{(p)}$ as the *p-local component of* $\mathbf{G}$.

Notation 2.6.6. Let A be an $\mathbb{E}_\infty$-ring. We let $\mathrm{BT}(A)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$ spanned by the $\mathbf{P}$-divisible groups over A. We will refer to $\mathrm{BT}(A)$ as the *∞-category of $\mathbf{P}$-divisible groups over A*.

Remark 2.6.7. Let A be an $\mathbb{E}_\infty$-ring. Then, for every prime number p, the construction $\mathbf{G} \mapsto \mathbf{G}_{(p)}$ determines a forgetful functor $\mathrm{BT}(A) \to \mathrm{BT}^p(A)$. Moreover, these functors amalgamate to an equivalence of ∞-categories

$$\mathrm{BT}(A) \to \prod_{p \in \mathbf{P}} \mathrm{BT}^p(A),$$

with homotopy inverse given by the construction

$$\{\mathbf{G}_{(p)} \in \mathrm{BT}^p(A)\}_{p \in \mathbf{P}} \mapsto \bigoplus_{p \in \mathbf{P}} \mathbf{G}_{(p)} \in \mathrm{BT}(A).$$

In other words, we can identify a $\mathbf{P}$-divisible group $\mathbf{G}$ over A as a family of p-divisible groups $\{\mathbf{G}_{(p)}\}_{p \in \mathbf{P}}$, where p ranges over the set $\mathbf{P}$ of all prime numbers.

We now introduce a “global” version of Definition 2.1.4:

Definition 2.6.8. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over A. A *preorientation of* $\mathbf{G}$ is a morphism of $\mathbf{Z}$-module spectra $\Sigma(\mathbf{Q}/\mathbf{Z}) \to \mathbf{G}(A)$. The collection of preorientations of $\mathbf{G}$ are parametrized by a space

$$\mathrm{Pre}(\mathbf{G}) = \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}/\mathbf{Z}), \mathbf{G}(A)),$$

which we will refer to as the *space of preorientations of* $\mathbf{G}$.

Remark 2.6.9. The group $\mathbf{Q} / \mathbf{Z}$ splits canonically as a direct sum of local summands $\bigoplus_{p \in \mathbf{P}} \mathbf{Q}_p / \mathbf{Z}_p$. Consequently, if $\mathbf{G}$ is a p-divisible group over an $\mathbb{E}_\infty$-ring A, we have a canonical homotopy equivalence

$$\begin{aligned} \mathrm{Pre}(\mathbf{G}) &= \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q} / \mathbf{Z}), \mathbf{G}(A)) \\ &\simeq \prod_{p \in \mathbf{P}} \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}_p / \mathbf{Z}_p), \mathbf{G}_{(p)}(A)) \\ &\simeq \prod_{p \in \mathbf{P}} \mathrm{Pre}(\mathbf{G}_{(p)}). \end{aligned}$$

In other words, giving a preorientation e of $\mathbf{G}$ (in the sense of Definition 2.6.8) is equivalent to giving a preorientation e_p of the p-local summand $\mathbf{G}_{(p)}$, for each prime number p (in the sense of Definition 2.1.4).

Example 2.6.10. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over A. Then we can also regard $\mathbf{G}$ as a $\mathbf{P}$-divisible group over A (Example 2.6.4). In this case, we can identify preorientations of $\mathbf{G}$ as a p-divisible group (Definition 2.1.4) with preorientations of $\mathbf{G}$ as a $\mathbf{P}$-divisible group (Definition 2.6.8).

Example 2.6.11. Let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Suppose that A is p-local, for some prime number p. Then, for every prime number $\ell \neq p$, the ℓ-local component $\mathbf{G}_{(\ell)}$ is an étale ℓ-divisible group. It follows that the space of preorientations $\mathrm{Pre}(\mathbf{G}_{(\ell)})$ is contractible (Example 2.1.6). Consequently, the product decomposition of Remark 2.6.9 simplifies to a homotopy equivalence $\mathrm{Pre}(\mathbf{G}) \simeq \mathrm{Pre}(\mathbf{G}_{(p)})$. That is, when we are working over a p-local $\mathbb{E}_\infty$-ring A, then we can identify preorientations of a $\mathbf{P}$-divisible group $\mathbf{G}$ (in the sense of Definition 2.6.8) with preorientations of the p-local summand $\mathbf{G}_{(p)}$ (in the sense of Definition 2.1.4).

Definition 2.6.12. Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over A, and let e be a preorientation of $\mathbf{G}$ (Definition 2.6.8), so that e determines a preorientation e_p of the p-local component $\mathbf{G}_{(p)}$ for every prime number p (Remark 2.6.9). We will say that e is an *orientation of* $\mathbf{G}$ if, for every prime number p, the following condition is satisfied:

(∗) Let $\widehat{A}$ denote the p-completion of A and let $\mathbf{G}_{(p),\widehat{A}}$ denote the p-divisible group over $\widehat{A}$ obtained from $\mathbf{G}_{(p)}$ by extending scalars along the canonical map $A \to \widehat{A}$. Then the image of e_p under the homotopy equivalence $\mathrm{Pre}(\mathbf{G}_{(p)}) \xrightarrow{\sim} \mathrm{Pre}(\mathbf{G}_{(p),\widehat{A}})$ of Proposition 2.2.1 is an orientation of $\mathbf{G}_{(p),\widehat{A}}$, in the sense of Definition 2.5.1.

We let $\mathrm{OrDat}(\mathbf{G})$ denote the summand of $\mathrm{Pre}(\mathbf{G})$ consisting of orientations of $\mathbf{G}$.

Example 2.6.13. Let A be a p-local $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over A, and let e be a preorientation of $\mathbf{G}$. Then, for every prime number $\ell \neq p$, the ℓ-completion of A vanishes. It follows that condition $(*)$ of Definition 2.6.12 is automatically satisfied for prime numbers different from p. Consequently, e is an orientation of $\mathbf{G}$ (in the sense of Definition 2.6.12) if and only if its image under the homotopy equivalence

$$\mathrm{Pre}(\mathbf{G}) \simeq \mathrm{Pre}(\mathbf{G}_{(p)}) \simeq \mathrm{Pre}(\mathbf{G}_{(p),\widehat{A}})$$

is an orientation of the p-divisible group $\widehat{\mathbf{G}}_{(p),\widehat{A}}$ (in the sense of Definition 2.5.1). Here $\widehat{A}$ denotes the p-completion of $\mathbf{G}$.

Remark 2.6.14. Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over A, and suppose that $\mathbf{G}$ admits an orientation (in the sense of Definition 2.6.12). Then, for every prime number p, the p-local component $\mathbf{G}_{(p)}$ admits an orientation after extending scalars to the p-completion $\widehat{A}$ of A. It follows that the p-divisible group $\mathbf{G}_{(p),\widehat{A}}$ is 1-dimensional (Remark 2.5.4). In particular, if the p-local component $\mathbf{G}_{(p)}$ vanishes, then the p-completion $\widehat{A}$ must also vanish: that is, the prime number p must be invertible in A.

Example 2.6.15. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over A. Then we can regard $\mathbf{G}$ as a $\mathbf{P}$-divisible group over A (Example 2.6.4), where the ℓ-local component $\mathbf{G}_{(\ell)}$ vanishes for $\ell \neq p$. It follows from Remark 2.6.14 that $\mathbf{G}$ can only admit an orientation (in the sense of Definition 2.6.12) if the $\mathbb{E}_\infty$-ring A is p-local.

Example 2.6.16. Let A be an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$. Then the p-adic completion of A vanishes, for each prime number p. It follows that every $\mathbf{P}$-divisible group $\mathbf{G}$ over A admits an essentially unique preorientation, which is automatically an orientation.

Remark 2.6.17. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A. If $\mathbf{G}$ is étale, then A is an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$. To prove this, it suffices to observe that for every prime number p, the p-divisible group $\mathbf{G}_{(p)}$ becomes both étale and 1-dimensional after extending scalars to the p-completion $A^\wedge_{(p)}$ of A, so we must have $A^\wedge_{(p)} \simeq 0$.

Warning 2.6.18. Let A be an $\mathbb{E}_\infty$-ring. The existence of an oriented $\mathbf{P}$-divisible group $\mathbf{G}$ over A guarantees that the p-completion $A^\wedge_{(p)}$ is complex periodic for every prime number p (Proposition Or.4.3.23). However, it does not guarantee that A itself is complex periodic (Example 2.6.16).

2.7 Splitting of P-Divisible Groups

Let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. We will say that $\mathbf{G}$ is *étale* if, for every prime number p, the p-divisible group $\mathbf{G}_{(p)}$ is étale (in the sense of Definition Or.2.5.3). Equivalently, $\mathbf{G}$ is étale if, for every finite abelian group M, the functor

$$\mathbf{G}[M] : \mathrm{CAlg}_A \to \mathcal{S} \qquad B \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}(B))$$

is corepresentable by an $\mathbb{E}_\infty$-algebra which is finite and étale over A. If the p-divisible groups $\mathbf{G}_{(p)}$ have constant height, this condition guarantees that, after a faithfully flat base change, we can arrange that $\mathbf{G}$ is actually *constant* (see Proposition 2.7.9 below). In this section, we sketch the proof of this (and related) facts and establish some terminology which will be useful later in this paper.

Definition 2.7.1. A *colattice* is an abelian group Λ which satisfies the following conditions:

- The abelian group Λ is torsion; that is, for every element $x \in \Lambda$, there exists a positive integer n such that $nx = 0$.
- For every positive integer n, the map $n : \Lambda \to \Lambda$ is a surjection with finite kernel.

Remark 2.7.2. Let Λ be an abelian group. For each prime number p, we let $\Lambda_{(p)}$ denote the localization of Λ with respect to the prime ideal $(p) \subseteq \mathbf{Z}$. Then Λ is a colattice if and only each each localization $\Lambda_{(p)}$ is isomorphic to $(\mathbf{Q}_p / \mathbf{Z}_p)^n$, for some integer n (which might depend on p).

Example 2.7.3. The abelian group $\mathbf{Q} / \mathbf{Z}$ is a colattice.

Example 2.7.4. For every prime number p, the quotient $\mathbf{Q}_p / \mathbf{Z}_p$ is a colattice.

Construction 2.7.5 (Constant $\mathbf{P}$-Divisible Groups)**.** Let Λ be an abelian group. We let $\underline{\Lambda} : \mathrm{CAlg} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ denote the functor given concretely by the formula $\underline{\Lambda}(B) = \mathrm{Hom}(|\operatorname{Spec}(B)|, \Lambda)$, where the right hand side denotes the set of all locally constant functions from the Zariski spectrum $|\operatorname{Spec}(A)|$ into Λ. If A is an $\mathbb{E}_\infty$-ring, we will generally abuse notation by identifying $\underline{\Lambda}$ with the composition

$$\mathrm{CAlg}_{\tau_{\geqslant 0}(A)} \to \mathrm{CAlg} \xrightarrow{\underline{\Lambda}} \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}.$$

When Λ is a colattice, this functor is a $\mathbf{P}$-divisible group over A, which we will refer to as the *constant* $\mathbf{P}$*-divisible group associated to* Λ.

Remark 2.7.6. Let A be an $\mathbb{E}_\infty$-ring and let Λ be an abelian group. Then the functor $\underline{\Lambda} : \mathrm{CAlg}_{\tau_{\geqslant 0}(A)} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ of Construction 2.7.5 is the sheafification (with respect to the Zariski topology) of the constant functor taking the value Λ. It follows that, if $\mathbf{G}$ is any $\mathbf{P}$-divisible group over A, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_{\tau_{\geqslant 0}(A)}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(\underline{\Lambda}, \mathbf{G}) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Lambda, \mathbf{G}(A)).$$

In particular, if Λ is a colattice, then we have an equivalence $\mathrm{Map}_{\mathrm{BT}(A)}(\underline{\Lambda}, \mathbf{G}) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Lambda, \mathbf{G}(A))$.

Definition 2.7.7. Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over A, and let Λ a colattice. If B is an $\mathbb{E}_\infty$-algebra over A, we say that a map $\rho : \Lambda \to \mathbf{G}(B)$ is a *splitting of* $\mathbf{G}$ *over* B if it induces an equivalence $\underline{\Lambda} \to \mathbf{G}_B$ of $\mathbf{P}$-divisible groups over B.

Let $\rho : \Lambda \to \mathbf{G}(B)$ be a splitting of $\mathbf{G}$ over B. We say that ρ *exhibits* B *as a splitting algebra of* $\mathbf{G}$ if it satisfies the following universal property:

- $(*)$ For every $\mathbb{E}_\infty$-algebra $C \in \mathrm{CAlg}_A$, the induced map

  $$\mathrm{Map}_{\mathrm{CAlg}_A}(B, C) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Lambda, \mathbf{G}(C))$$

  restricts to a homotopy equivalence from $\mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$ to the summand of $\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Lambda, \mathbf{G}(C))$ consisting fo those maps $\Lambda \to \mathbf{G}(C)$ which are splittings of $\mathbf{G}$ over C.

Note that if there exists a map $\rho : \Lambda \to \mathbf{G}(B)$ which exhibits B as a splitting algebra of $\mathbf{G}$, then the $\mathbb{E}_\infty$-algebra B (and the map ρ) are unique up to a contractible space of choices. In this case, we will denote B by $\mathrm{Split}_\Lambda(\mathbf{G})$.

Warning 2.7.8. Our terminology is slightly abusive: a splitting algebra of $\mathbf{G}$ (if it exists) depends not only on $\mathbf{G}$, but also on Λ.

For existence, we have the following:

Proposition 2.7.9. *Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over A, and let Λ be a colattice. Then there exists a splitting algebra* $\mathrm{Split}_\Lambda(\mathbf{G})$ *which is faithfully flat over A if and only if the following conditions are satisfied:*

- (a) *The $\mathbf{P}$-divisible group $\mathbf{G}$ is étale.*

(*b*) *Let p be a prime number and let h be the unique integer for which $\Lambda_{(p)}$ is isomorphic to $(\mathbf{Q}_p / \mathbf{Z}_p)^h$. Then the p-divisible group $\mathbf{G}_{(p)}$ has height h.*

Moreover, if these conditions are satisfied, then $\mathrm{Split}_\Lambda(\mathbf{G})$ *can be realized as a filtered colimit of finite étale A-algebras.*

Proof. Suppose first that there exists a splitting algebra $\mathrm{Split}_\Lambda(\mathbf{G})$ which is faithfully flat over A. Since assertions (a) and (b) can be tested after faithfully flat base change, we can replace A by $\mathrm{Split}_\Lambda(\mathbf{G})$ and thereby reduce to the case where there exists a splitting $\rho : \Lambda \to \mathbf{G}(A)$. In this case, $\mathbf{G}$ is isomorphic to the constant $\mathbf{P}$-divisible group $\underline{\Lambda}$, so assertions (a) and (b) are obvious.

Conversely, suppose that (a) and (b) are satisfied. For each positive integer n, let $\Lambda[n]$ denote the kernel of the map $n : \Lambda \to \Lambda$), let $X_n : \mathrm{CAlg}_A \to \mathcal{S}$ denote the functor given by the formula

$$X_n(B) = \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Lambda[n], \mathbf{G}(B)),$$

and let $X_n^\circ \subseteq X_n$ be the subfunctor whose value on an $\mathbb{E}_\infty$-algebra B is spanned by those maps $\Lambda[n] \to \mathbf{G}_B$ which induce an equivalence of finite flat group schemes $\underline{\Lambda[n]} \to \mathbf{G}_B[n]$. It follows from ($a$) that the functors X_n and X_n° are representable by finite étale A-algebras, and from (b) that these A-algebras are faithfully flat over A. Passing to the inverse limit over n, we conclude that the functor

$$B \mapsto \{\text{Splittings } \rho : \Lambda \to \mathbf{G}(B)\ \}$$

is corepresentable by an $\mathbb{E}_\infty$-algebra $\mathrm{Split}_\Lambda(B)$ which is a filtered colimit of finite étale A-algebras of positive degree (and is therefore faithfully flat over A). □

Remark 2.7.10. In the situation of Proposition 2.7.9, the splitting algebra $\mathrm{Split}_\Lambda(\mathbf{G})$ depends functorially on Λ, and therefore admits an action of the automorphism group $\mathrm{Aut}(\Lambda)$. In fact, we can be more precise: the spectrum $\mathrm{Spec}(\mathrm{Split}_\Lambda(\mathbf{G}))$ can be regarded as a torsor over $\mathrm{Spec}(A)$ (locally trivial for the flat topology) with respect to the profinite group group $\mathrm{Aut}(\Lambda)$.

We will need to consider a more general notion of splitting algebra which applies in a *relative* situation.

Notation 2.7.11. Let A be an $\mathbb{E}_\infty$-ring and let $f : \mathbf{G}_0 \to \mathbf{G}$ be a morphism of $\mathbf{P}$-divisible groups over A. We will say that f is a *monomorphism* if, for every prime number p, the induced map $f_{(p)} : \mathbf{G}_{0(p)} \to \mathbf{G}_{(p)}$ is a monomorphism of p-divisible groups over A (in the sense of Definition Or.2.4.3). In this case, f admits a cofiber in the ∞-category $\mathrm{BT}(A)$, which we will denote by $\mathbf{G}/\mathbf{G}_0$ (see Proposition Or.2.4.8).

Definition 2.7.12. Let A be an $\mathbb{E}_\infty$-ring, let $f : \mathbf{G}_0 \to \mathbf{G}$ be a monomorphism of $\mathbf{P}$-divisible groups over A, and let Λ be a colattice. If B is an $\mathbb{E}_\infty$-algebra over A, we say that a map $\rho : \Lambda \to \mathbf{G}(B)$ is a *splitting of f over B* if the induced map $\Lambda \to (\mathbf{G}/\mathbf{G}_0)(B)$ is a splitting of $\mathbf{G}/\mathbf{G}_0$ over B, in the sense of Definition 2.7.7.

Let $\rho : \Lambda \to \mathbf{G}(B)$ be a splitting of f over B. We say that ρ *exhibits B as a splitting algebra of f* if it satisfies the following universal property:

$(*)$ For every $\mathbb{E}_\infty$-algebra $C \in \mathrm{CAlg}_A$, the induced map

$$\mathrm{Map}_{\mathrm{CAlg}_A}(B, C) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Lambda, \mathbf{G}(C))$$

restricts to a homotopy equivalence from $\mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$ to the summand of $\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Lambda, \mathbf{G}(C))$ consisting fo those maps $\Lambda \to \mathbf{G}(C)$ which are splittings of f over C.

Note that if there exists a map $\rho : \Lambda \to \mathbf{G}(B)$ which exhibits B as a splitting algebra of f, then the $\mathbb{E}_\infty$-algebra B (and the map ρ) are unique up to a contractible space of choices. In this case, we will denote B by $\mathrm{Split}_\Lambda(f)$.

Example 2.7.13. Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over A, and let Λ be a colattice. Then a morphism $\rho : \Lambda \to \mathbf{G}(B)$ is a splitting of $\mathbf{G}$ over B (in the sense of Definition 2.7.7) if and only if it is a splitting of the monomorphism $f : 0 \to \mathbf{G}$ over B (in the sense of Definition 2.7.12). In particular, we can identify the splitting algebra $\mathrm{Split}_\Lambda(\mathbf{G})$ of Definition 2.7.7 (if it exists) with the splitting algebra $\mathrm{Split}_\Lambda(f : 0 \to \mathbf{G})$ of Definition 2.7.12 (if it exists).

Remark 2.7.14. Let A be an $\mathbb{E}_\infty$-ring, let $f : \mathbf{G}_0 \to \mathbf{G}$ be a monomorphism of $\mathbf{P}$-divisible groups over A, and let Λ be a colattice. Then a morphism $\rho : \Lambda \to \mathbf{G}(B)$ is a splitting of f over B if and only if f and ρ together induce an equivalence $\mathbf{G}_{0B} \oplus \underline{\Lambda} \to \mathbf{G}_B$ of $\mathbf{P}$-divisible groups over B.

Proposition 2.7.15. *Let A be an $\mathbb{E}_\infty$-ring, let $f : \mathbf{G}_0 \to \mathbf{G}$ be a monomorphism of $\mathbf{P}$-divisible groups over A, and let Λ be a colattice. Then there exists a splitting algebra $\mathrm{Split}_\Lambda(f)$ which is faithfully flat over A if and only if the following conditions are satisfied:*

(a) *The $\mathbf{P}$-divisible group $\mathbf{G}/\mathbf{G}_0$ is étale.*

(b) *Let p be a prime number and let h be the unique integer for which $\Lambda_{(p)}$ is isomorphic to $(\mathbf{Q}_p / \mathbf{Z}_p)^h$. Then the p-divisible group $(\mathbf{G}/\mathbf{G}_0)_{(p)}$ has height h.*

Proof. As in the proof of Proposition 2.7.9, the necessity of conditions (a) and (b) is clear. To prove that they are sufficient, suppose that (a) and (b) are satisfied. For each positive integer n, let $\Lambda[n]$ denote the kernel of the map $n : \Lambda \to \Lambda$), let $X_n : \mathrm{CAlg}_A \to \mathcal{S}$ denote the functor given by the formula

$$X_n(B) = \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Lambda[n], \mathbf{G}(B)),$$

and let $X_n^\circ \subseteq X_n$ be the subfunctor whose value on an $\mathbb{E}_\infty$-algebra B is spanned by those maps $\Lambda[n] \to \mathbf{G}_B$ which induce an equivalence of finite flat group schemes $\underline{\Lambda[n]} \to (\mathbf{G}/\mathbf{G}_0)_B[n]$. Then X_n and X_n° are representable by finite flat A-algebras, and (b) guarantees these A-algebras are faithfully flat over A. Passing to the inverse limit over n, we conclude that the functor

$$B \mapsto \{\text{Splittings } \rho : \Lambda \to \mathbf{G}(B) \text{ of } f\ \}$$

is corepresentable by an $\mathbb{E}_\infty$-algebra $\mathrm{Split}_\Lambda(B)$ which is a filtered colimit of finite flat A-algebras of positive degree (and is therefore faithfully flat over A). ☐

Remark 2.7.16. In the situation of Proposition 2.7.15, every splitting of the monomorphism $f : \mathbf{G}_0 \to \mathbf{G}$ determines a splitting of the quotient $\mathbf{P}$-divisible group $\mathbf{G}/\mathbf{G}_0$. In particular, the universal splitting of f is classified by a map of splitting algebras $\mathrm{Split}_\Lambda(\mathbf{G}/\mathbf{G}_0) \to \mathrm{Split}_\Lambda(f)$. This map is an equivalence in the case $\mathbf{G}_0 \simeq 0$ (Example 2.7.13). In general, it exhibits $\mathrm{Split}_\Lambda(f)$ as the tensor product of $\mathrm{Split}_\Lambda(\mathbf{G}/\mathbf{G}_0)$ with an auxiliary A-algebra B, where B classifies splittings of the exact sequence

$$0 \to \mathbf{G}_0 \xrightarrow{f} \mathbf{G} \to \mathbf{G}/\mathbf{G}_0 \to 0.$$

Remark 2.7.17. Let $f : \mathbf{G}_0 \to \mathbf{G}$ be a monomorphism of $\mathbf{P}$-divisible groups over an $\mathbb{E}_\infty$-ring A and let Λ be a colattice. Assume that f and Λ satisfy the the hypotheses of Proposition 2.7.15, so that there exists a splitting algebra $\mathrm{Split}_\Lambda(f)$ which is faithfully flat over A. Then, for any morphism of $\mathbb{E}_\infty$-rings $A \to A'$, the relative tensor product $A' \otimes_A \mathrm{Split}_\Lambda(f)$ can be regarded as a splitting algebra for the induced monomorphism $f_{A'} : \mathbf{G}_{0A'} \to \mathbf{G}_{A'}$ of $\mathbf{P}$-divisible groups over A'.

Applying this observation to the maps $A \leftarrow \tau_{\geqslant 0}(A) \to \pi_0(A)$, we deduce that $\pi_0(\mathrm{Split}_\Lambda(f))$ can be identified with a splitting algebra for (f_0, Λ), where $f_0 : \mathbf{G}_{0\pi_0(A)} \to \mathbf{G}_{\pi_0(A)}$ is the underlying map of $\mathbf{P}$-divisible groups over the commutative ring $\pi_0(A)$. This algebra can be characterized in terms of ordinary algebra: it is determined by the fact that it satisfies condition $(*)$ of Definition 2.7.12 whenever $C \in \mathrm{CAlg}^{\heartsuit}_{\pi_0(A)}$ is an ordinary commutative algebra over the commutative ring $\pi_0(A)$.

We can summarize the situation as follows. Let $f : \mathbf{G}_0 \to \mathbf{G}$ is a monomorphism of $\mathbf{P}$-divisible groups over an ordinary commutative ring R. Assume that the quotient $\mathbf{G}/\mathbf{G}_0$ is étale and let Λ be a colattice satisfying the hypotheses of Proposition 2.7.15. Then the splitting algebra $\mathrm{Split}_{\Lambda}(f)$ is flat over R, and in particular an ordinary commutative ring. If A is an $\mathbb{E}_\infty$-ring equipped with an isomorphism $R \simeq \pi_0(A)$, then every lift $\widetilde{f}$ of f to a monomorphism of $\mathbf{P}$-divisible groups over A determines a lift of $\mathrm{Split}_{\Lambda}(f)$ to a flat $\mathbb{E}_\infty$-algebra over A, given by the splitting algebra $\mathrm{Split}_{\Lambda}(\widetilde{f})$.

2.8 Example: The Multiplicative P-Divisible Group

Recall that the *strict multiplicative group* $\mathbf{G}_m : \mathrm{CAlg} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ is the functor characterized by the formula

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}_m(A)) = \mathrm{Map}_{\mathrm{CAlg}}(\Sigma^\infty_+(M), A),$$

where M is any abelian group.

Construction 2.8.1. Let A be an $\mathbb{E}_\infty$-ring. We let $\mu_{\mathbf{P}^\infty}(A)$ denote the fiber of the canonical map

$$u : \mathbf{G}_m(A) \to \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{G}_m(A),$$

formed in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ (that is, it is the connective cover of the usual fiber of u). The construction $A \mapsto \mu_{\mathbf{P}^\infty}(A)$ then defines a functor $\mu_{\mathbf{P}^\infty} : \mathrm{CAlg} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$.

Proposition 2.8.2. *The functor $\mu_{\mathbf{P}^\infty} : \mathrm{CAlg} = \mathrm{CAlg}_S \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ is a $\mathbf{P}$-divisible group over the sphere spectrum S (in the sense of Definition 2.6.1).*

Proof. We must show that $\mu_{\mathbf{P}^\infty}$ satisfies conditions (1), (2), and (3) of Definition 2.6.1. Condition (1) is immediate from the definitions. For condition (2), we observe that for any finite abelian group M, we have a canonical homotopy equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mu_{\mathbf{P}^\infty}(A)) &\simeq \mathrm{fib}(\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}_m(A)) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{G}_m(A)) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}_m(A)) \\ &\simeq \mathrm{Map}_{\mathrm{CAlg}}(\Sigma^\infty_+(M), A). \end{aligned}$$

It follows that the functor $A \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mu_{\mathbf{P}^\infty}(A))$ is corepresentable by the suspension spectrum $\Sigma^\infty_+(M)$, which is a free module over the sphere spectrum S (of rank equal to the order $|M|$ of the group M). Requirement (3) follows from the observation that for any monomorphism $M \hookrightarrow N$ of finite abelian groups, the induced map of suspension spectra $\Sigma^\infty_+(M) \to \Sigma^\infty_+(N)$ is finite flat. □

Definition 2.8.3. We will refer to the functor $\mu_{\mathbf{P}^\infty} : \mathrm{CAlg} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ as the *multiplicative* $\mathbf{P}$*-divisible group over* S. If A is any $\mathbb{E}_\infty$-ring, we will abuse notation by writing $\mu_{\mathbf{P}^\infty}$ for the $\mathbf{P}$-divisible group $(\mu_{\mathbf{P}^\infty})_A$ given by the composition

$$\mathrm{CAlg}_A \to \mathrm{CAlg} \xrightarrow{\mu_{\mathbf{P}^\infty}} \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}};$$

we refer to this composite functor as the *multiplicative* $\mathbf{P}$*-divisible group over* A.

Remark 2.8.4. Let A be an $\mathbb{E}_\infty$-ring and let $\mu_{\mathbf{P}^\infty}$ be the multiplicative $\mathbf{P}$-divisible group over A. Then, for every prime number p, the p-local component $(\mu_{\mathbf{P}^\infty})_{(p)}$ can be identified with the multiplicative p-divisible group μ_{p^∞} over A (see Proposition Or.2.2.11). We therefore have a direct sum decomposition

$$\mu_{\mathbf{P}^\infty}(A) \simeq \bigoplus_{p \in \mathbf{P}} \mu_{p^\infty}(A).$$

Remark 2.8.5. Let A be an $\mathbb{E}_\infty$-ring and let $\mu_{\mathbf{P}^\infty}$ be the multiplicative $\mathbf{P}$-divisible group over A. Then we have canonical homotopy equivalences

$$\begin{aligned} \mathrm{Pre}(\mu_{\mathbf{P}^\infty}) &= \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}/\mathbf{Z}), \mu_{\mathbf{P}^\infty}(A)) \\ &\simeq \mathrm{fib}(\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}/\mathbf{Z}), \mathbf{G}_m(A)) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}/\mathbf{Z}), \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{G}_m(A)) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}/\mathbf{Z}), \mathbf{G}_m(A)) \\ &\simeq \mathrm{Map}_{\mathrm{CAlg}}(\Sigma^\infty_+ K(\mathbf{Q}/\mathbf{Z}, 1), A). \end{aligned}$$

In other words, preorientations of the multiplicative $\mathbf{P}$-divisible group $\mu_{\mathbf{P}^\infty}$ are classified by the $\mathbb{E}_\infty$-ring $\Sigma^\infty_+(K(\mathbf{Q}/\mathbf{Z}, 1))$.

Construction 2.8.6 (The Orientation of $\mu_{\mathbf{P}^\infty}$)**.** Set $R = \Sigma^\infty_+(\mathbf{CP}^\infty) = \Sigma^\infty_+ K(\mathbf{Z}, 2)$. The fiber sequence of $\mathbf{Z}$-module spectra

$$\Sigma(\mathbf{Q}/\mathbf{Z}) \xrightarrow{u} \Sigma^2(\mathbf{Z}) \to \Sigma^2(\mathbf{Q}),$$

and u determines a map of $\mathbb{E}_\infty$-rings $\Sigma^\infty_+ K(\mathbf{Q}/\mathbf{Z}, 1) \to \Sigma^\infty_+ K(\mathbf{Z}, 2) = R$, which classifies a preorientation $e : \Sigma(\mathbf{Q}/\mathbf{Z}) \to \mu_{\mathbf{P}^\infty}(R)$ of the multiplicative $\mathbf{P}$-divisible group over R. However, we get a bit more: there is also a tautological map $\bar{e} : \Sigma^2(\mathbf{Z}) \to \mathbf{G}_m(R)$ which fits into a commutative diagram of fiber sequences

$$\begin{tikzcd} \Sigma(\mathbf{Q}/\mathbf{Z}) \arrow[r] \arrow[d, "e"] & \Sigma^2(\mathbf{Z}) \arrow[r] \arrow[d, "\bar{e}"] & \Sigma^2(\mathbf{Q}) \arrow[d] \\ \mu_{\mathbf{P}^\infty}(R) \arrow[r] & \mathbf{G}_m(R) \arrow[r] & \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{G}_m(R) \end{tikzcd}$$

in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$. Here we can think of $\overline{e}$ as a preorientation of the strict multiplicative group $\mathbf{G}_m$ over R, or equivalently of its formal completion $\widehat{\mathbf{G}}_m$. This preorientation determines a map of R-modules

$$\omega_{\widehat{\mathbf{G}}_m} \simeq R \xrightarrow{\beta_{\overline{e}}} \Sigma^{-2}(R)$$

whose homotopy class determines an element $\beta \in \pi_2(R)$, represented concretely by the composite map

$$S^2 = \mathbf{CP}^1 \hookrightarrow \mathbf{CP}^\infty \to \Omega^\infty \Sigma^\infty(\mathbf{CP}^\infty) \to \Omega^\infty \Sigma^\infty_+(\mathbf{CP}^\infty) = \Omega^\infty(R).$$

Let KU denote the periodic complex K-theory spectrum. Then there is a canonical map of $\mathbb{E}_\infty$-rings

$$\rho : R = \Sigma^\infty_+(\mathbf{CP}^\infty) \to \mathrm{KU},$$

which carries β to an invertible element of $\pi_2(\mathrm{KU})$ (in fact, it induces a homotopy equivalence $R[\beta^{-1}] \simeq \mathrm{KU}$, by a classical theorem of Snaith; see Theorem Or.6.5.1). It follows that, if we regard $\mu_{\mathbf{P}^\infty}$ as a $\mathbf{P}$-divisible group over KU, then the preorientation classified by the composite map

$$\Sigma^\infty_+ K(\mathbf{Q}/\mathbf{Z}) \to \Sigma^\infty_+ K(\mathbf{Z}, 2) = R \xrightarrow{\rho} \mathrm{KU}$$

is an orientation (in the sense of Definition 2.6.12). We will refer to this orientation as the *tautological orientation of* $\mu_{\mathbf{P}^\infty}$ *over* KU.

Remark 2.8.7. Let A be an $\mathbb{E}_\infty$-ring and regard $\mu_{\mathbf{P}^\infty}$ as a $\mathbf{P}$-divisible group over A. For each prime number p, let $A^\wedge_{(p)}$ denote the p-completion of A. Then supplying an orientation e of the multiplicative $\mathbf{P}$-divisible group $\mu_{\mathbf{P}^\infty}$ over A is equivalent to supplying a family of orientations

$$e_p \in \mathrm{OrDat}((\mu_{p^\infty})_{A^\wedge_{(p)}}),$$

where p ranges over all prime numbers, or equivalently to providing an orientation of the formal multiplicative group $\widehat{\mathbf{G}}_m$ over each $A^\wedge_{(p)}$. We therefore obtain a homotopy equivalence

$$\mathrm{OrDat}(\mu_{\mathbf{P}^\infty}) \simeq \prod_{p \in \mathbf{P}} \mathrm{Map}_{\mathrm{CAlg}}(\mathrm{KU}, A^\wedge_{(p)}) \simeq \mathrm{Map}_{\mathrm{CAlg}}(\mathrm{KU}, \widehat{A}),$$

where $\widehat{A} = \prod_{p \in \mathbf{P}} A^{\wedge}_{(p)}$ denotes the profinite completion of A. The pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \widehat{A} \arrow[d] \\
A_{\mathbf{Q}} \arrow[r] & \widehat{A}_{\mathbf{Q}},
\end{tikzcd}$$

then determines a pullback diagram of spaces

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CAlg}}(\mathrm{KU}, A) \arrow[r] \arrow[d] & \mathrm{OrDat}(\mu_{\mathbf{P}^\infty}) \arrow[d, "\chi"] \\
\mathrm{Map}_{\mathrm{CAlg}}(\mathrm{KU}, A_{\mathbf{Q}}) \arrow[r] & \mathrm{Map}_{\mathrm{CAlg}}(\mathrm{KU}, \widehat{A}_{\mathbf{Q}}).
\end{tikzcd}$$

It follows that KU is very close to being *universal* among $\mathbb{E}_\infty$-rings over which there exists an orientation of $\mu_{\mathbf{P}^\infty}$. In particular, every orientation e of $\mu_{\mathbf{P}^\infty}$ determines a map $\chi(e) : \mathrm{KU} \to \widehat{A}_{\mathbf{Q}}$, carrying the Bott element $\beta \in \pi_2(\mathrm{KU})$ to some element $\chi(e)(\beta) \in \pi_2(\widehat{A}_{\mathbf{Q}})$; the orientation e can then be obtained from the tautological orientation of Construction 2.8.6 if and only if $\chi(e)(\beta)$ can be lifted to an element of $\pi_2(A_{\mathbf{Q}})$.

2.9 Example: Torsion of Elliptic Curves

We now consider another natural source of examples of **P**-divisible groups.

Construction 2.9.1. Let A be an $\mathbb{E}_\infty$-ring and let X be a strict abelian variety over A (Definition AV.1.5.1), which we view as a functor

$$\mathrm{CAlg}_{\tau_{\geqslant 0}(A)} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}} .$$

For every object $B \in \mathrm{CAlg}_{\tau_{\geqslant 0}(A)}$, we let $\mathsf{X}[\mathbf{P}^\infty](B)$ denote the fiber of the canonical map

$$u : \mathsf{X}(B) \to \mathbf{Q} \otimes_{\mathbf{Z}} \mathsf{X}(B),$$

formed in the ∞-category $\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ (that is, it is the connective cover of the usual fiber of u). The construction $B \mapsto \mathsf{X}[\mathbf{P}^\infty](B)$ then defines a functor $\mathsf{X}[\mathbf{P}^\infty] : \mathrm{CAlg}_{\tau_{\geqslant 0}(A)} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$.

Proposition 2.9.2. *Let X be a strict abelian variety over an $\mathbb{E}_\infty$-ring A. Then the functor $\mathsf{X}[\mathbf{P}^\infty]$ of Construction 2.9.1 is a **P**-divisible group over A (in the sense of Definition 2.6.1).*

Proof. We must show that $\mathsf{X}[\mathbf{P}^\infty]$ satisfies conditions (1), (2), and (3) of Definition 2.6.1. Condition (1) is immediate from the definitions, while (2) and (3) follow from the observation that for every positive integer n, the multiplication map $\mathsf{X} \xrightarrow{n} \mathsf{X}$ is finite flat (of nonzero degree); see Proposition AV.6.7.3 . $\square$

Remark 2.9.3. Let A be an $\mathbb{E}_\infty$-ring and let X be a strict abelian variety over A. Then, for every prime number p, the p-local component of $\mathsf{X}[\mathbf{P}^\infty]$ can be identified with the p-divisible group $\mathsf{X}[p^\infty]$ associated to X. We therefore have a direct sum decomposition $\mathsf{X}[\mathbf{P}^\infty] \simeq \bigoplus_{p \in \mathbf{P}} \mathsf{X}[p^\infty]$.

Remark 2.9.4. Let A be an $\mathbb{E}_\infty$-ring and let X be a strict abelian variety over A. Then we have canonical homotopy equivalences

$$\begin{aligned}
\mathrm{Pre}(\mathsf{X}[\mathbf{P}^\infty]) &= \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}/\mathbf{Z}), \mathsf{X}[\mathbf{P}^\infty](A)) \\
&\simeq \mathrm{fib}(\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}/\mathbf{Z}), \mathsf{X}(A)) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}/\mathbf{Z}), \mathbf{Q} \otimes_{\mathbf{Z}} \mathsf{X}(A)) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma(\mathbf{Q}/\mathbf{Z}), \mathsf{X}(A)).
\end{aligned}$$

In other words, giving a preorientation of $\mathsf{X}[\mathbf{P}^\infty]$ is equivalent to giving a map $\Sigma(\mathbf{Q}_p/\mathbf{Z}_p) \to \mathsf{X}(A)$.

Remark 2.9.5. Let A be an $\mathbb{E}_\infty$-ring and let X be a strict abelian variety over A. We define a *preorientation of* X to be a map of pointed spaces $S^2 \to \Omega^\infty \mathsf{X}(A)$, or equivalently a map of $\mathbf{Z}$-module spectra $e : \Sigma^2(\mathbf{Z}) \to \mathsf{X}(A)$. Note that giving a preorientation of X is equivalent to giving a preorientation of its formal completion. Moreover, every preorientation e of X determines a preorientation of the $\mathbf{P}$-divisible group $\mathsf{X}[\mathbf{P}^\infty]$, given by the composition $\Sigma(\mathbf{Q}/\mathbf{Z}) \to \Sigma^2(\mathbf{Z}) \xrightarrow{e} \mathsf{X}$.

The following observation provides a rich supply of oriented $\mathbf{P}$-divisible groups (giving non-trivial examples in which we can apply our formalism of tempered cohomology).

Construction 2.9.6. Let A be an $\mathbb{E}_\infty$-ring and let X be a strict elliptic curve over A. Recall that a preorientation e of X is said to be an *orientation* if it determines an orientation of the underlying formal group $\widehat{\mathsf{X}}$ (Definition Or.7.2.7). If this condition is satisfied, then the composite map $\Sigma(\mathbf{Q}/\mathbf{Z}) \to \Sigma^2(\mathbf{Z}) \xrightarrow{e} \mathsf{X}$ determines an orientation of the $\mathbf{P}$-divisible group $\mathsf{X}[\mathbf{P}^\infty]$, in the sense of Definition 2.6.12.

3 Orbispaces

Let G be a compact Lie group. For every G-space X, we let $\mathrm{KU}_G^*(X)$ denote the G-equivariant complex K-theory of X, in the sense of [20]. The construction $X \mapsto \mathrm{KU}_G^*(X)$ determines a cohomology theory on the homotopy category of G-spaces, which is representable by a (genuine) G-spectrum which we denote by KU_G. Moreover, these equivariant spectra are related as the group G varies: for example, if H is a subgroup of G, then the underlying H-spectrum of KU_G can be identified with KU_H. This observation can be summarized by saying that complex K-theory is an example of a *global spectrum*: it has an underlying G-spectrum KU_G for every compact Lie group G, varying functorially with G (for various formalizations of this notion, we refer the reader to [2], [4], and [19]).

The theory of tempered cohomology developed in this paper has a similar feature: given an oriented $\mathbf{P}$-divisible group $\mathbf{G}$ over an $\mathbb{E}_\infty$-ring A, it allows us to construct a family of $\{A_{\mathbf{G},H}\}$ of H-spectra (all having the same underlying spectrum A), where H ranges over the collection of all *finite* groups. The construction of $A_{\mathbf{G},H}$ as a *genuine* H-spectrum is somewhat subtle, and requires the assumption that $\mathbf{G}$ is oriented. However, the underlying *naive* H-spectrum is much easier to define, and makes sense more generally when $\mathbf{G}$ is a *preoriented* $\mathbf{P}$-divisible group over A. This is already enough information to construct a family of cohomology theories

$$A^*_{\mathbf{G},H} : \{H\text{-Spaces}\}^{\mathrm{op}} \to \{\text{Graded abelian groups}\},$$

where H ranges over the collection of finite groups. For our purposes, it will be convenient to assemble this collection of cohomology theories into a single functor

$$A^*_{\mathbf{G}} : \mathcal{OS}^{\mathrm{op}} \to \{\text{Graded abelian groups}\}.$$

Here $\mathcal{OS}$ denotes the ∞-category of functors $\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathcal{S})$, where $\mathcal{S}$ is the ∞-category of spaces and $\mathscr{T} \subsetneq \mathcal{S}$ is the full subcategory spanned by spaces of the form BH, where H is a finite abelian group. We will refer to the objects of $\mathcal{OS}$ as *orbispaces* and to $\mathcal{OS}$ as the *∞-category of orbispaces*.

Warning 3.0.1. Our use of the term *orbispace* is borrowed from the work of Gepner-Henriques ([3]), who associate a *homotopy theory of orbispaces* to every family $\mathcal{F}$ of topological groups. The ∞-category $\mathcal{OS}$ that we consider here is a model for this homotopy theory in the special case where $\mathcal{F}$ is the family of all finite abelian groups. See also [18] and [7] for related discussions.

Our goal in this section is to give a brief overview of the theory of orbispaces, emphasizing the constructions which will play an important role in this paper. The ∞-category $\mathcal{OS}$ can be viewed as an enlargement of the ∞-category $\mathcal{S}$ of spaces. In §3.1, we show that every space X can be promoted to an orbispace in (at least) two ways: we can associate to X the constant functor

$$\underline{X} : \mathscr{T}^{\mathrm{op}} \to \{X\} \hookrightarrow \mathcal{S}$$

taking the value X (Example 3.1.8), or the functor

$$X^{(-)} : \mathscr{T}^{\mathrm{op}} \to \mathcal{S} \qquad T \mapsto X^T$$

represented by X (Example 3.1.6). These constructions determine a fully faithful embeddings of ∞-categories $\mathcal{S} \to \mathcal{OS}$, which are left and right adjoint to the "forgetful" functor $\mathcal{OS} \to \mathcal{S}$ given by evaluation on the final object of $\mathscr{T}$. In general, these embeddings are different (though they coincide on *finite* spaces, by a nontrivial theorem of Miller: see Remark 3.1.14).

In §3.2, we review the relationship between the homotopy theory of orbispaces and equivariant unstable homotopy theory. To every finite group G, we associate a functor of ∞-categories

$$\{G\text{-Spaces}\} \to \mathcal{OS} \qquad X \mapsto X//G,$$

which we refer to as the *orbispace quotient* functor. This construction does not lose very much information: in §3.3, we show that it induces an equivalence from a localization of G-spaces (relative to the family of abelian subgroups of G) to the full subcategory of $\mathcal{OS}_{/BG^{(-)}}$ spanned by orbispaces X equipped with a *representable* morphism $\mathsf{X} \to BG^{(-)}$ (Proposition 3.3.13).

In §3.4, we associate to each torsion abelian group Λ a functor

$$\mathscr{L}^{\Lambda} : \mathcal{OS} \to \mathcal{OS}\,.$$

Roughly speaking, this functor carries an orbispace X to a new orbispace $\mathscr{L}^{\Lambda}(\mathsf{X})$ which parametrizing maps from the classifying space $B\widehat{\Lambda}$ into X which are "continuous" with respect to the profinite topology on the Pontryagin dual group $\widehat{\Lambda} = \mathrm{Hom}(\Lambda, \mathbf{Q}/\mathbf{Z})$ (Construction 3.4.3). We will be particularly interested in the case where $\Lambda = (\mathbf{Q}_p/\mathbf{Z}_p)^n$; in this case, we can think of $\mathscr{L}^{\Lambda}(\mathsf{X})$ as a p-adic version of an iterated free loop space, parametrizing maps from a p-adic torus into X. This construction will play an essential role in our discussion of character theory for tempered cohomology (see §4.3).

We conclude this section by establishing a connection of the theory of orbispaces with the notion of **P**-divisible group introduced in §2.6. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a **P**-divisible group over A. For every finite abelian group H (with Pontryagin dual group $\widehat{H}$) the functor

$$(B \in \mathrm{CAlg}_A) \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\widehat{H}, \mathbf{G}(B))$$

is representable by a finite flat A-algebra which we will denote by $\mathcal{O}_{\mathbf{G}[\widehat{H}]}$. The construction $H \mapsto \mathcal{O}_{\mathbf{G}[\widehat{H}]}$ determines a functor $\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}} \to \mathrm{CAlg}_A$, where $\mathrm{Ab}_{\mathrm{fin}}$ denotes the category of finite abelian groups. In §3.5, we show that choosing a preorientation of $\mathbf{G}$ (in the sense of Definition 2.6.8) is equivalent to factoring this functor as a composition

$$\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}} \xrightarrow{H \mapsto BH} \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A;$$

see Theorem 3.5.5. In particular, every preorientation of $\mathbf{G}$ determines a functor $A_{\mathbf{G}} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$: this will be the representing object for our theory of $\mathbf{G}$-tempered cohomology.

Remark 3.0.2. The notion of orbispace we consider here is defined in terms of the ∞-category $\mathscr{T}$ of classifying spaces BH, where H is a finite abelian group. Many variants of this definition are possible: for example, we could allow all finite groups. For our objectives in this paper, this extra generality serves no purpose. Our theory of $\mathbf{G}$-tempered cohomology already determines an H-equivariant cohomology theory for every finite group H, whose values can be extrapolated (by the process of Kan extension) from the case where H is abelian. Perhaps unexpectedly, this extrapolation procedure gives rise to a theory with excellent properties, at least in the case $\mathbf{G}$ is an *oriented* **P**-divisible group.

3.1 The ∞-Category of Orbispaces

We begin by introducing some definitions.

Notation 3.1.1. Let $\mathcal{S}$ denote the ∞-category of spaces. For every group H, we let BH denote the classifying space of H, which we regard as an object of the ∞-category $\mathcal{S}$. We let $\mathscr{T}$ denote the full subcategory of $\mathcal{S}$ spanned by those objects which are homotopy equivalent to BH, where H is a finite abelian group.

Remark 3.1.2. Let T be an object of the category $\mathscr{T}$: that is, a space which is homotopy equivalent to BH, for some finite abelian group H. Note that the group H is canonically determined by T: it can be recovered as the fundamental group $\pi_1(T)$ (which is canonically independent of the choice of base point, because G is abelian). Moreover, the space T can be recovered up to homotopy equivalence as the classifying space of $H = \pi_1(T)$. Beware, however, that the identification $T \simeq BH$ is not functorial: it depends on a choice of base of T. In particular, the composite functor

$$\mathscr{T} \xrightarrow{\pi_1} \mathrm{Ab}_{\mathrm{fin}} \xrightarrow{H \mapsto BH} \mathscr{T}$$

is not equivalent to the identity functor $\mathrm{id}_{\mathscr{T}}$.

To avoid confusion, we will generally use the notation BH to indicate objects of $\mathscr{T}$ that are equipped with a specified base point (or in situations where it is harmless to choose a base point), and the letter T to denote a generic object of the ∞-category $\mathscr{T}$.

The ∞-category $\mathscr{T}$ of Notation 3.1.1 can be described more concretely.

Remark 3.1.3. Let Group denote the category of groups (with morphisms given by group homomorphisms). Then Group can be viewed as the underlying category of a (strict) 2-category Group^+, which can be described informally as follows:

- The objects of Group^+ are groups.
- If G and H are groups, then a 1-morphism from G to H in Group^+ is a group homomorphism $\varphi : G \to H$.
- If G and H are groups and $\varphi, \psi : G \to H$ are group homomorphisms, then a 2-morphism from φ to ψ in Group^+ is an element $h \in H$ satisfying $\psi(g) = h\varphi(g)h^{-1}$ for each $g \in G$.

Let us abuse notation by identifying the 2-category Group^+ with the associated ∞-category (given by its Duskin nerve). Then the construction $(G \in \mathrm{Group}^+) \mapsto (BG \in \mathcal{S})$ induces an equivalence from the ∞-category Group^+ to the full subcategory of $\mathcal{S}$ spanned by objects of the form BG. It follows that the ∞-category $\mathscr{T}$ of Notation 3.1.1 is equivalent to the full subcategory of Group^+ spanned by the finite abelian groups.

Definition 3.1.4. An *orbispace* is a functor of ∞-categories $\mathsf{X} : \mathscr{T}^{\mathrm{op}} \to \mathcal{S}$. If X is an orbispace, we will denote the value of X on an object $T \in \mathscr{T}^{\mathrm{op}}$ by X^T. We let

$\mathcal{OS}$ denote the ∞-category $\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathcal{S})$. We will refer to $\mathcal{OS}$ as the *∞-category of orbispaces.*

Notation 3.1.5. For any orbispace X, we let $|\mathsf{X}|$ denote the value of X on the final object $\{*\}$ of the ∞-category $\mathscr{T}$. We will refer to $|\mathsf{X}|$ as the *underlying space* of X. The construction $\mathsf{X} \mapsto |\mathsf{X}|$ determines a functor $\mathcal{OS} \to \mathcal{S}$, which we will refer to as the *forgetful functor.*

The forgetful functor of Notation 3.1.5 has left and right adjoints.

Example 3.1.6. Let X be a space. For each object $T \in \mathscr{T}$, we let $X^T = \mathrm{Fun}(T, X) \simeq \mathrm{Map}_{\mathcal{S}}(T, X)$ denote the space parametrizing maps from T into X. Then the construction $T \mapsto X^T$ determines a functor of ∞-categories $\mathscr{T}^{\mathrm{op}} \to \mathcal{S}$, which we can regard as an orbispace. We will denote this orbispace by $X^{(-)}$.

Remark 3.1.7. The functor

$$\mathcal{S} \to \mathcal{OS} \qquad X \mapsto X^{(-)}$$

does not preserve colimits in general. However, it does preserve coproducts: this follows from the observation that each of the spaces $T \in \mathscr{T}$ is connected.

Example 3.1.8. Let X be a space. We let $\underline{X}$ denote the constant functor $\mathscr{T}^{\mathrm{op}} \to \{X\} \hookrightarrow \mathcal{S}$. We will refer to $\underline{X}$ as the *constant orbispace* associated to X.

Note that, if X is any space, then the functor $(T \in \mathscr{T}^{\mathrm{op}}) \mapsto (X^T \in \mathcal{S})$ of Example 3.1.6 is a right Kan extension of its restriction to the full subcategory of $\mathscr{T}^{\mathrm{op}}$ spanned by the contractible space $\{*\} \simeq \Delta^0$. Similarly, the constant functor $\underline{X}$ is a left Kan extension of its restriction to the same subcategory. This immediately implies the following:

Proposition 3.1.9. *Let X be a space and let Y be any orbispace. Then evaluation on the final object $\{*\} \in \mathscr{T}$ induces homotopy equivalences*

$$\mathrm{Map}_{\mathcal{OS}}(\mathsf{Y}, X^{(-)}) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{S}}(|\mathsf{Y}|, X)$$

$$\mathrm{Map}_{\mathcal{OS}}(\underline{X}, \mathsf{Y}) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{S}}(X, |\mathsf{Y}|).$$

Corollary 3.1.10. *The forgetful functor*

$$\mathcal{OS} \to \mathcal{S} \qquad \mathsf{Y} \mapsto |\mathsf{Y}|$$

has both a left adjoint (given by $X \mapsto \underline{X}$) and a right adjoint (given by $X \mapsto X^{(-)}$).

Corollary 3.1.11. *Let X and Y be spaces. Then evaluation on the contractible space $\{e\} \in \mathscr{T}$ induces homotopy equivalences*

$$\mathrm{Map}_{\mathcal{OS}}(Y^{(-)}, X^{(-)}) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{S}}(Y, X) \xleftarrow{\sim} \mathrm{Hom}_{\mathcal{OS}}(\underline{Y}, \underline{X}).$$

Corollary 3.1.12. *The construction $X \mapsto X^{(-)}$ of Example 3.1.6 determines a fully faithful embedding of ∞-categories $\mathcal{S} \hookrightarrow \mathcal{OS}$.*

Corollary 3.1.13. *The construction $X \mapsto \underline{X}$ of Example 3.1.8 induces a fully faithful embedding of ∞-categories $\mathcal{S} \hookrightarrow \mathcal{OS}$.*

Remark 3.1.14 (The Sullivan Conjecture)**.** For any space X, there is a canonical map $\underline{X} \to X^{(-)}$ comparing the orbispaces of Examples 3.1.6 and 3.1.8. When evaluated on an object $T \in \mathscr{T}$, it induces the diagonal embedding $X = \underline{X}^T \to X^T$. In general, this map is not a homotopy equivalence. However, it is a homotopy equivalence when X is finite, by deep theorem of Miller (see [16]).

Example 3.1.15 (The Yoneda Embedding)**.** Let T be an object of of $\mathscr{T}$. Then the orbispace $T^{(-)}$ of Example 3.1.6 is the functor

$$\mathscr{T}^{\mathrm{op}} \to \mathcal{S} \qquad T' \mapsto \mathrm{Map}_{\mathcal{S}}(T', T) = \mathrm{Map}_{\mathscr{T}}(T', T);$$

represented by the object $T \in \mathscr{T}$. In other words, the composition

$$\mathscr{T} \hookrightarrow \mathcal{S} \xrightarrow{X \mapsto X^{(-)}} \mathcal{OS}$$

is the Yoneda embedding for the ∞-category $\mathscr{T}$.

Remark 3.1.16. Let X be an orbispace. Our use of the notation X^T to indicate the value of X on an object $T \in \mathscr{T}^{\mathrm{op}}$ is intended to suggest a point of view: one should view X^T as a parameter space for "maps from T into X." Note that this is literally correct if we identify T with the orbispace $T^{(-)}$ of Example 3.1.6: by Yoneda's lemma, we have a canonical homotopy equivalence $\mathsf{X}^T \simeq \mathrm{Map}_{\mathcal{OS}}(T^{(-)}, \mathsf{X})$.

3.2 Equivariant Homotopy Theory

We now give a brief review of (unstable) equivariant homotopy theory, from the perspective we will adopt in this paper.

Definition 3.2.1. Let G be a group and let BG denote its classifying space. If $\mathcal{C}$ is an ∞-category, we will refer to a functor $BG \to \mathcal{C}$ as a *G-equivariant object of $\mathcal{C}$*. The collection of G-equivariant objects of $\mathcal{C}$ can be organized into an ∞-category $\mathrm{Fun}(BG, \mathcal{C})$, which we will refer to as the *∞-category of G-equivariant objects of $\mathcal{C}$*.

Remark 3.2.2. Let G be a group. For any ∞-category $\mathcal{C}$, evaluation at the base point $*$ of BG determines a forgetful functor $\mathrm{Fun}(BG, \mathcal{C}) \to \mathcal{C}$. We will generally abuse notation by not distinguishing between an object $X \in \mathrm{Fun}(BG, \mathcal{C})$ and its image $X(*) \in \mathcal{C}$ under this forgetful functor. One should think of the functor $X : BG \to \mathcal{C}$ as encoding an action of G on the underlying object $X(*) \in \mathcal{C}$.

Example 3.2.3. Let G be a group. For any ∞-category $\mathcal{C}$, composition with the projection map $BG \to *$ determines a diagonal map $\mathcal{C} \simeq \mathrm{Fun}(*, \mathcal{C}) \to \mathrm{Fun}(BG, \mathcal{C})$. More informally, this functor carries each object $X \in \mathcal{C}$ to itself, equipped with the trivial action of the group G.

Notation 3.2.4. Let G be a group and let $\mathcal{C}$ be an ∞-category which admits small colimits. Then the diagonal map $\mathcal{C} \to \mathrm{Fun}(BG, \mathcal{C})$ of Example 3.2.3 admits a left adjoint $\mathrm{Fun}(BG, \mathcal{C}) \to \mathcal{C}$. If X is an object of $\mathrm{Fun}(BG, \mathcal{C})$, we denote its image under this functor by X_{hG}. We refer to the construction $X \mapsto X_{hG}$ as the *homotopy orbits functor*.

Notation 3.2.5. Let G be a group and let $\mathcal{C}$ be an ∞-category which admits small limits. Then the diagonal map $\mathcal{C} \to \mathrm{Fun}(BG, \mathcal{C})$ of Example 3.2.3 admits a right adjoint $\mathrm{Fun}(BG, \mathcal{C}) \to \mathcal{C}$. If X is an object of $\mathrm{Fun}(BG, \mathcal{C})$, we denote its image under this functor by X^{hG}. We refer to the construction $X \mapsto X^{hG}$ as the *homotopy fixed point functor*.

Example 3.2.6. Let $\mathcal{T}\mathrm{op}$ denote the ordinary category of topological spaces. Then the construction $X \mapsto \mathrm{Sing}_\bullet(X)$ determines a functor $\mathrm{Sing}_\bullet$ from $\mathcal{T}\mathrm{op}$ (regarded as an ordinary category) to $\mathcal{S}$ (regarded as an ∞-category). Passing to G-equivariant objects, we obtain a functor

$$\{\text{Topological spaces with a } G\text{-action}\} = \mathrm{Fun}(BG, \mathcal{T}\mathrm{op}) \to \mathrm{Fun}(BG, \mathcal{S}).$$

Let G be a group. Then the ∞-category $\mathrm{Fun}(BG, \mathcal{S})$ is a setting of the "naive" version of G-equivariant homotopy theory. If X and Y are topological spaces equipped with actions of G and $f : X \to Y$ is a continuous G-equivariant map, then f induces an equivalence $\mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(Y)$ in the ∞-category $\mathrm{Fun}(BG, \mathcal{S})$ if and only if it is a weak homotopy equivalence of the underlying topological spaces. If X and Y are CW complexes, this implies that f admits a homotopy inverse $g : Y \to X$, but does not guarantee that we can choose g to be a G-equivariant map. To model the "genuine" version of G-equivariant homotopy theory, one needs a variant of Definition 3.2.1.

Notation 3.2.7. Let G be a group. We define a category $\mathrm{Orbit}(G)$ as follows:

- The objects of $\mathrm{Orbit}(G)$ are right G-sets of the form $H\backslash G$, where H is a subgroup of G.
- The morphisms in $\mathrm{Orbit}(G)$ are G-equivariant maps.

We will refer to $\mathrm{Orbit}(G)$ as the *orbit category* of the group G.

Remark 3.2.8. Let G be a group. Then we can identify the classifying space BG with the full subcategory of $\mathrm{Orbit}(G)$ spanned by the orbit $G = \{e\}\backslash G$.

Remark 3.2.9. Let G be a group and suppose we are given subgroups $H, H' \subseteq G$. Then giving a map of right G-sets $H\backslash G \to H'\backslash G$ is equivalent to giving an element of $H'\backslash G$ which is fixed by the right action of H. Using this observation, we can define a category $\mathrm{Orbit}'(G)$ which is isomorphic to $\mathrm{Orbit}(G)$ as follows:

- The objects of $\mathrm{Orbit}'(G)$ are the subgroups $H \subseteq G$ (corresponding to the right G-set $H\backslash G \in \mathrm{Orbit}(G)$).
- Given subgroups $H, H' \subseteq G$, a morphism from H to H' in $\mathrm{Orbit}'(G)$ is a coset $uH' \in H'\backslash G$ satisfying $u^{-1}Hu \subseteq H'$.
- Given subgroups $H, H', H'' \subseteq G$, the composition of morphisms
$$uH' \in \mathrm{Hom}_{\mathrm{Orbit}'(G)}(H, H') \qquad vH'' \in \mathrm{Hom}_{\mathrm{Orbit}'(G)}(H', H'')$$
is given by $uvH'' \in \mathrm{Hom}_{\mathrm{Orbit}'(G)}(H, H'')$

Definition 3.2.10. Let G be a finite group. A *G-space* is a functor $X : \mathrm{Orbit}(G)^{\mathrm{op}} \to \mathcal{S}$. We let $\mathcal{S}_G$ denote the functor ∞-category $\mathrm{Fun}(\mathrm{Orbit}(G)^{\mathrm{op}}, \mathcal{S})$; we will refer to $\mathcal{S}_G$ as the *∞-category of G-spaces.*

Remark 3.2.11. Let G be a finite group. Then the inclusion $BG \hookrightarrow \mathrm{Orbit}(G)$ of Remark 3.2.8 determines a forgetful functor

$$\mathcal{S}_G = \mathrm{Fun}(\mathrm{Orbit}(G)^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(BG^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}(BG, \mathcal{S}),$$

which carries a G-space (in the sense of Definition 3.2.10) to a G-equivariant object of $\mathcal{S}$ (in the sense of Definition 3.2.1). Composing with the functor $\mathrm{Fun}(BG, \mathcal{S}) \to \mathcal{S}$ of Remark 3.2.2, we obtain a forgetful functor $\mathcal{S}_G \to \mathcal{S}$, given by evaluation on the G-orbit $G = \{e\}\backslash G \in \mathrm{Orbit}(G)$.

We will often abuse notation by not distinguishing between a G-space $X \in \mathcal{S}_G$, the underlying G-equivariant object $X|_{BG}$, and the underlying space $X(\{e\}\backslash G)$. In particular, if X is a G-space, then we write X_{hG} and X^{hG} for the homotopy orbit and homotopy fixed points of the underlying G-equivariant object of $\mathcal{S}$ (Notation 3.2.4 and Notation 3.2.5).

Remark 3.2.12. Let G be a finite group. Then the ∞-category $\mathcal{S}_G$ is generated, under small colimits, by the image of the Yoneda embedding $\mathrm{Orbit}(G) \hookrightarrow \mathcal{S}_G$. We say that a G-space X is *finite* if it belongs to the subcategory generated by the image of $\mathrm{Orbit}(G)$ under *finite* colimits.

Example 3.2.13. Let X be a topological space equipped with a continuous right action of a finite group G. Then X determines a functor of ordinary categories

$$\mathrm{Orbit}(G)^{\mathrm{op}} \to \{\text{Topological spaces}\},$$

$$H\backslash G \mapsto \mathrm{Hom}_G(H\backslash G, X) = X^H = \{x \in X : (\forall h \in H)[x^h = x]\};$$

here $\mathrm{Hom}_G(H\backslash G, X)$ is the set of G-equivariant maps from $H\backslash G$ into X (equipped with the obvious topology). Composing with the singular complex functor

$$\mathrm{Sing}_\bullet : \mathcal{T}\mathrm{op} \to \mathcal{S},$$

we obtain a functor of ∞-categories

$$\mathrm{Sing}_\bullet^G(X) : \mathrm{Orbit}(G)^{\mathrm{op}} \to \mathcal{S} \qquad H\backslash G \mapsto \mathrm{Sing}_\bullet(X^H),$$

which we can regard as a G-space in the sense of Definition 3.2.10; note that the restriction $\mathrm{Sing}_\bullet^G(X)|_{BG}$ is the G-equivariant object of $\mathcal{S}$ given by Example 3.2.6.

Remark 3.2.14. Let G be a finite group. The construction of Example 3.2.13 induces a functor $\mathrm{Sing}_{\bullet}^{G}$ from the ordinary category of topological spaces with a right action of G to the ∞-category $\mathcal{S}_G$ of G-spaces (Definition 3.2.10). If $f : X \to Y$ is a continuous G-equivariant map, then $\mathrm{Sing}_{\bullet}^{G}(f)$ is an equivalence in the ∞-category $\mathcal{S}_G$ if and only if, for each subgroup $H \subseteq G$, the induced map $X^H \to Y^H$ is a weak homotopy equivalence of topological spaces. In fact, one can say more: by a theorem of Elmendorff, the functor $\mathrm{Sing}_{\bullet}^{G}$ exhibits $\mathcal{S}_G$ as the ∞-category underlying the classical homotopy theory of G-spaces: for example, it induces an equivalence from the homotopy category of G-CW complexes to the homotopy category of the ∞-category $\mathcal{S}_G$. (this is essentially a theorem of Elmendorff; see [15]).

Notation 3.2.15. Let G be a finite group and let Y be a G-space. For each subgroup $H \subseteq G$, we let Y^H denote the object of $\mathcal{S}$ given by evaluating Y on the G-orbit $H\backslash G \in \mathrm{Orbit}(G)^{\mathrm{op}}$. This notation motivated by Example 3.2.13: if $Y = \mathrm{Sing}_{\bullet}^{G}(X)$ for a topological space equipped with a free action of G, then $Y^H = \mathrm{Sing}_{\bullet}(X^H)$ is the singular simplicial set of the subspace $X^H = \{x \in X : (\forall h \in H)[x^h = x]\}$.

We now relate the equivariant homotopy theory of §3.2 to the theory of orbispaces developed in §3.1.

Construction 3.2.16. Let G be a finite group. We let $\mathrm{Orbit}(G)_{\mathrm{ab}}$ denote the full subcategory of $\mathrm{Orbit}(G)$ spanned by those objects of the form $H\backslash G$, where $H \subseteq G$ is an abelian subgroup of G. Let $\mathscr{T}$ be the ∞-category of Notation 3.1.1. Note that if $S \simeq H\backslash G$ is an object of $\mathrm{Orbit}(G)_{\mathrm{ab}}$, then the homotopy orbit space S_{hG} is isomorphic to the classifying space BH, and therefore belongs to $\mathscr{T}$. Consequently, the construction $S \mapsto S_{hG}$ determines a functor $Q : \mathrm{Orbit}(G)_{\mathrm{ab}} \to \mathscr{T}$.

Let $R : \mathcal{S}_G = \mathrm{Fun}(\mathrm{Orbit}(G)^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(\mathrm{Orbit}(G)_{\mathrm{ab}}^{\mathrm{op}}, \mathcal{S})$ be the restriction functor, and let $Q_! : \mathrm{Fun}(\mathrm{Orbit}(G)_{\mathrm{ab}}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathcal{S}) = \mathcal{OS}$ be the functor given by left Kan extension along (the opposite of) Q, Then the composition $Q_! \circ R$ is a functor from the ∞-category $\mathcal{S}_G$ of G-spaces to the ∞-category $\mathcal{OS}$ of orbispaces. We will denote the value of this functor on a G-space X by $X//G$, and refer to it as the *orbispace quotient of X by G.*

Remark 3.2.17. Let G be a finite group and let X be a G-space. The orbispace quotient $X//G$ of Construction 3.2.16 can be described more concretely by the formula

$$(X//G)^{BH} = (\coprod_{\alpha: H \to G} X^{\mathrm{im}(\alpha)})_{hG}.$$

Here H denotes a finite abelian group, and the coproduct is taken over all group homomorphisms $\alpha : H \to G$.

Example 3.2.18. Let G be a finite group and let X be a G-space. Then the image of $X//G$ under the forgetful functor

$$\mathcal{OS} \to \mathcal{S} \qquad \mathsf{Y} \mapsto |\mathsf{Y}|$$

is the homotopy orbit space X_{hG}. Consequently, for each $Y \in \mathcal{S}$, we have canonical homotopy equivalences

$$\mathrm{Map}_{\mathcal{OS}}(X//G, Y^{(-)}) \simeq \mathrm{Map}_{\mathcal{S}}(X_{hG}, Y) \simeq \mathrm{Map}_{\mathcal{S}}(X, Y)^{hG}.$$

$$\mathrm{Map}_{\mathcal{OS}}(\underline{Y}, X//G) \simeq \mathrm{Map}_{\mathcal{S}}(Y, X_{hG}).$$

In particular, we have canonical maps

$$\underline{X_{hG}} \to X//G \to (X_{hG})^{(-)}$$

in the ∞-category of orbispaces (whose composition is the comparison map of Remark 3.1.14).

Remark 3.2.19. Let G be a finite group and let X be a G-space. Then the comparison map $X//G \to (X_{hG})^{(-)}$, when evaluated on an object $BH \in \mathscr{T}$, yields a map of spaces

$$\Phi : (\coprod_{\alpha: H \to G} X^{\mathrm{im}(\alpha)})_{hG} \to (\coprod_{\alpha: H \to G} X^{hH})_{hG};$$

here both coproducts are indexed by the collection of all group homomorphisms $\alpha : H \to G$, and Φ is comprised of individual comparison maps $\Phi_\alpha : X^{\mathrm{im}(\alpha)} \to X^{hH}$.

Example 3.2.20. Let G be a finite group and let $X = *$ be a final object of $\mathcal{S}_G$ (so that X^H is contractible for each subgroup $H \subseteq G$). Then the comparison map $X//G \to (X_{hG})^{(-)} = BG^{(-)}$ of Example 3.2.18 is an equivalence of orbispaces (this follows easily from Remark 3.2.19).

Example 3.2.21. Let $G = \{e\}$ be the trivial group. Then the ∞-category $\mathcal{S}_G$ can be identified with the ∞-category $\mathcal{S}$ of spaces (via the evaluation functor $X \mapsto X^G$). Under this identification, the orbispace quotient construction $X \mapsto X//G$ corresponds to the functor $X \mapsto \underline{X}$ of Example 3.1.8.

3.3 Representable Morphisms of Orbispaces

Let G be a finite group, and consider the orbispace quotient functor

$$\mathcal{S}_G \to \mathcal{OS} \qquad X \mapsto X//G$$

of Construction 3.2.16. This functor fails to be an equivalence of categories for at least two reasons:

(*a*) For any G-space $X \in \mathcal{S}_G$, our definition of orbispace quotient $X//G$ involves only the restriction $X|_{\mathrm{Orbit}(G)^{\mathrm{op}}_{\mathrm{ab}}}$: that is, it depends only on the fixed-point spaces X^H where H is an *abelian* subgroup of G, and ignores the information provided by fixed points for nonabelian groups.

(*b*) The orbispace quotient functor $\mathcal{S}_G \to \mathcal{OS}$ does not preserve final objects: instead, it carries the final object of $\mathcal{S}_G$ to the orbispace $BG^{(-)}$ associated to the classifying space of G (Example 3.2.20). Consequently, for every G-space X, the orbispace quotient $X//G$ comes equipped with an additional datum, given by a structure morphism $X//G \to *//G = BG^{(-)}$.

Our goal in this section is to show that these are essentially the only differences between $\mathcal{S}_G$ and $\mathcal{OS}$. More precisely, we show that the functor $X//G$ induces an equivalence of ∞-categories

$$\begin{tikzcd} \{G\text{-Spaces } X \text{ satisfying } X^H = \varnothing \text{ for } H \text{ is nonabelian}\} \arrow[d, "\sim"] \\ \{\text{Representable orbispace morphisms } f : \mathsf{Y} \to BG^{(-)}\}; \end{tikzcd}$$

see Proposition 3.3.13 below. First, we need to introduce some terminology.

Definition 3.3.1. Let $\mathscr{T}$ be the ∞-category of Notation 3.1.1. We will say that a morphism $f : T_0 \to T$ in $\mathscr{T}$ is a *covering map* if the induced map $\pi_1(T_0) \to \pi_1(T)$ is a monomorphism (of finite abelian groups). We let $\mathrm{Cov}(T)$ denote the full subcategory of $\mathscr{T}_{/T}$ spanned by the covering maps $T_0 \to T$.

Definition 3.3.2. Let T be an object of $\mathscr{T}$. Then we have a canonical equivalence of ∞-categories

$$\Psi : \mathcal{OS}_{/T^{(-)}} \simeq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}_{/T}, \mathcal{S}),$$

given concretely by the formula $\Psi(\mathsf{X})(T_0) = \mathrm{Map}_{\mathcal{OS}_{/T^{(-)}}}(T_0^{(-)}, \mathsf{X})$ (see Corollary HTT.5.1.6.12). We will say that a morphism of orbispaces $\mathsf{X} \to T^{(-)}$ is *representable* if the functor $\Psi(\mathsf{X}) \in \mathrm{Fun}(\mathcal{T}_{/T}^{\mathrm{op}}, \mathcal{S})$ is a left Kan extension of its restriction to the subcategory $\mathrm{Cov}(T)^{\mathrm{op}} \subseteq \mathcal{T}_{/T}^{\mathrm{op}}$ of Definition 3.3.1.

Example 3.3.3. Let T be the final object of $\mathscr{T}$. Then every orbispace X admits an essentially unique map $\mathsf{X} \to T^{(-)}$, which is representable if and only if the orbispace X is a constant functor: that is, if and only if it is equivalent to the functor $\underline{X}$ of Example 3.1.8, for some $X \in \mathcal{S}$.

Remark 3.3.4. Let T be an object of $\mathscr{T}$. Then the collection of representable morphisms $\mathsf{X} \to T^{(-)}$ is closed under the formation of small colimits (in the ∞-category $\mathcal{OS}_{/T^{(-)}}$.

Remark 3.3.5. Let T be an object of $\mathscr{T}$. Then a map of orbispaces $\mathsf{X} \to T^{(-)}$ is representable if and only if X can be written as a colimit (in the ∞-category $\mathcal{OS}_{/T^{(-)}}$ of objects of the form $T_0^{(-)}$, where $T_0 \to T$ is a covering map in $\mathscr{T}$.

Remark 3.3.6. Let $S \to T$ be a morphism in the ∞-category $\mathscr{T}$, and suppose we are given a pullback square of orbispaces

$$\begin{tikzcd} \mathsf{X} \arrow[r] \arrow[d, "f"] & \mathsf{Y} \arrow[d, "g"] \\ S^{(-)} \arrow[r] & T^{(-)}. \end{tikzcd}$$

If g is representable, then f is also representable. To prove this, we can use Remarks 3.3.4 and 3.3.5 to reduce to the case where Y has the form $T_0^{(-)}$, for $T_0 \to T$ is a covering map in $\mathscr{T}$. In this case, Remark 3.1.7 implies that the fiber product $S^{(-)} \times_{T^{(-)}} T_0^{(-)} \simeq (S \times_T T')^{(-)}$ decomposes as a disjoint union of finitely many objects of the form $S_0^{(-)}$, where $S_0 \to S$ is a covering map in $\mathscr{T}$; the desired result then follows from Remark 3.3.4.

Definition 3.3.7. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces. We will say that f is *representable* if, for every object $T \in \mathscr{T}$ and every pullback square

$$\begin{tikzcd} \mathsf{X}_T \arrow[r] \arrow[d, "f'"] & \mathsf{X} \arrow[d, "f"] \\ T^{(-)} \arrow[r] & \mathsf{Y}, \end{tikzcd}$$

the morphism f' is representable (in the sense of Definition 3.3.2).

Example 3.3.8. Let T be an object of $\mathscr{T}$. Then a morphism of orbispaces $\mathsf{X} \to T^{(-)}$ is representable in the sense of Definition 3.3.7 if and only if it is representable in the sense of Definition 3.3.2. The "only if" direction is obvious, and the converse follows from Remark 3.3.6.

Remark 3.3.9. Let Y be an orbispace. Then the collection of representable morphisms $f : \mathsf{X} \to \mathsf{Y}$ is closed under small colimits in the ∞-category $\mathcal{OS}_{/\mathsf{Y}}$ (this is an immediate consequence of Remark 3.3.4, since the formation of pullbacks commutes with the formation of colimits).

Remark 3.3.10. Suppose we are given a pullback diagram of orbispaces

$$\begin{tikzcd}
\mathsf{X}' \arrow[r] \arrow[d, "f'"] & \mathsf{X} \arrow[d, "f"] \\
\mathsf{Y}' \arrow[r] & \mathsf{Y}.
\end{tikzcd}$$

If f is representable, then f' is representable.

Remark 3.3.11. Suppose we are given morphisms of orbispaces $\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z}$. If f and g are representable, then the composition $g \circ f$ is representable. To prove this, we can assume without loss of generality that $\mathsf{Z} = T^{(-)}$, for some $T \in \mathscr{T}$. In this case, we can write Y as a colimit of orbispaces of the form $T'^{(-)}$, where $T' \to T$ is a covering map (Remark 3.3.5). By virtue of Remark 3.3.4, it will suffice to show that each of the composite maps

$$\mathsf{X} \times_{\mathsf{Y}} T'^{(-)} \to T'^{(-)} \to T^{(-)}$$

are representable. Using our representability assumption f, we can write the fiber product $\mathsf{X} \times_{\mathsf{Y}} T'^{(-)}$ as a colimit of orbispaces of the form $T''^{(-)}$, where $T'' \to T'$ is a covering map in $\mathscr{T}$. The desired result now follows from the observation that the composite map $T'' \to T' \to T$ is a covering.

Lemma 3.3.12. *Let G be a finite group, and let $\mathcal{C} \subseteq \mathcal{OS}_{/BG^{(-)}}$ be the smallest full subcategory which is closed under small colimits and contains $BH^{(-)}$, for each abelian subgroup $H \subseteq G$. Then a map of orbispaces $f : \mathsf{X} \to BG^{(-)}$ belongs to $\mathcal{C}$ if and only if f is representable.*

Proof. We first prove the "only if" direction. By virtue of Remark 3.3.9, it will suffice to show that for every subgroup $H \subseteq G$, the map $BH^{(-)} \to BG^{(-)}$ is representable (in fact, it will suffice to prove this when H is abelian, but we will not need this). Form a pullback diagram of orbispaces

$$\begin{tikzcd}
\mathsf{X} \arrow[r] \arrow[d, "f"] & BH^{(-)} \arrow[d] \\
(BH')^{(-)} \arrow[r] & BG^{(-)},
\end{tikzcd}$$

where H' is a finite abelian group. Then $\mathsf{X} = X^{(-)}$, where X is the fiber product $BH' \times_{BG} BH$. Note that X is a finite covering space of BH'. It follows that each connected component X_i of X belongs to $\mathscr{T}$, and the map $X_i \to BH'$ is a covering. The representability of f now follows from Remark 3.3.4 (and Remark 3.1.7).

We now prove the converse. Assume that $f : \mathsf{X} \to BG^{(-)}$ is representable; we wish to show that X belongs to $\mathcal{C}$. Let $\mathscr{T}_{/BG} = \mathscr{T} \times_{\mathcal{S}} \mathcal{S}_{/BG}$ denote the ∞-category whose objects are maps $u : T \to BG$, where T belongs to $\mathscr{T}$. Then the orbispace $BG^{(-)}$ can be realized tautologically as the colimit $\varinjlim_{T \in \mathscr{T}_{/BG}} T^{(-)}$. Let $\mathscr{T}^{\mathrm{cov}}_{/BG} \subseteq \mathscr{T}_{/BG}$ be the full subcategory spanned by those maps $u : T \to BG$ which are covering maps: that is, which are injective on the level of fundamental groups. Then the inclusion $\iota : \mathscr{T}^{\mathrm{cov}}_{/BG} \hookrightarrow \mathscr{T}_{/BG}$ has a left adjoint (carrying an object $T \in \mathscr{T}_{/BG}$ to its Postnikov truncation $\tau_{\leqslant 0}(T)$, formed in the ∞-topos $\mathcal{S}_{/BG}$). It follows that ι is left cofinal, so that the orbispace $BG^{(-)}$ can also be realized as the colimit $\varinjlim_{T \in \mathscr{T}^{\mathrm{cov}}_{/BG}} T^{(-)}$. Consequently, to show that X belongs to $\mathcal{C}$, it will suffice to show that the fiber product $\mathsf{X}_T = T^{(-)} \times_{BG^{(-)}} \mathsf{X}$ belongs to $\mathcal{C}$ for each $T \in \mathscr{T}^{\mathrm{cov}}_{/BG}$. Our assumption that f is representable guarantees that the projection map $\mathsf{X}_T \to T^{(-)}$ is representable, so that we can realize X_T as a colimit of objects of the form $T'^{(-)}$, where $T' \to T$ is a covering map. We now observe that the composite map $T' \to T \to BG$ is also a covering, so that T' is equivalent to BH for some abelian subgroup $H \subseteq G$. $\square$

Proposition 3.3.13 ([3])**.** *Let G be a finite group, and let $\mathcal{S}^{\mathrm{ab}}_G$ denote the full subcategory of $\mathcal{S}_G$ spanned by those G-spaces X such that $X^H = \emptyset$ for every nonabelian subgroup $H \subseteq G$. Then the construction $X \mapsto X//G$ determines a fully faithful embedding*

$$\mathcal{S}^{\mathrm{ab}}_G \hookrightarrow \mathcal{OS}_{/BG^{(-)}},$$

whose essential image is spanned by the representable maps $\mathsf{Y} \to BG^{(-)}$.

Remark 3.3.14. Let G be a finite group. Then an object of $\mathcal{S}_G = \mathrm{Fun}(\mathrm{Orbit}(G)^{\mathrm{op}}, \mathcal{S})$ belongs to the subcategory $\mathcal{S}^{\mathrm{ab}}_G$ of Proposition 3.3.13 if and only if it is a left Kan extension of its extension to the full subcategory $\mathrm{Orbit}(G)^{\mathrm{op}}_{\mathrm{ab}} \subseteq \mathrm{Orbit}(G)^{\mathrm{op}}$ of Construction

3.2.16. It follows that the restriction functor $X \mapsto X|_{\mathrm{Orbit}(G)_{\mathrm{ab}}^{\mathrm{op}}}$ induces an equivalene of ∞-categories $\mathcal{S}_G^{\mathrm{ab}} \simeq \mathrm{Fun}(\mathrm{Orbit}(G)_{\mathrm{ab}}^{\mathrm{op}}, \mathcal{S})$. More informally, the ∞-category $\mathcal{S}_G^{\mathrm{ab}}$ models "G-equivariant homotopy theory relative to the family of abelian subgroups of G."

Proof of Proposition 3.3.13. By virtue of Remark 3.3.14, it will suffice to show that the functor $Q_!$ of Construction 3.2.16 induces a fully faithful embedding

$$\mathrm{Fun}(\mathrm{Orbit}(G)_{\mathrm{ab}}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{OS}_{/BG^{(-)}},$$

whose essential image is the collection of representable maps $\mathsf{X} \to BG^{(-)}$. This follows immediately from the observation that Q induces an equivalence of ∞-categories $\mathrm{Orbit}(G)_{\mathrm{ab}} \simeq \mathscr{T}_{/BG}^{\mathrm{cov}}$, together with the characterization of representable morphisms supplied by Lemma 3.3.12. $\square$

Remark 3.3.15. The discussion of this section can be formulated in the language of *fractured ∞-topoi*, developed in Chapter SAG.VI.1 . By definition, an orbispace is a $\mathcal{S}$-valued presheaf on the ∞-category $\mathscr{T}$, so the ∞-category $\mathcal{OS} = \mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathcal{S})$ is an ∞-topos. The collection of representable morphisms of orbispaces determines a *geometric admissibility structure* on the ∞-category $\mathcal{OS}$ (Definition SAG.VI.1.3.4.1). It follows from Theorem SAG.VI.1.3.4.4 that we can regard $\mathcal{OS}$ as a fractured ∞-topos; moreover, the fracture subcategory of corporeal objects $\mathcal{OS}^{\mathrm{corp}} \subseteq \mathcal{OS}$ can be identified $\mathrm{Fun}(\mathscr{T}^{\mathrm{cov,op}}, \mathcal{S})$; here $\mathscr{T}^{\mathrm{cov}}$ denotes the non-full subcategory of $\mathscr{T}$ whose morphisms are covering maps $T_0 \to T$, and the presheaf ∞-category $\mathrm{Fun}(\mathscr{T}^{\mathrm{cov,op}}, \mathcal{S})$ embeds as a non-full subcategory of the ∞-topos $\mathcal{OS} = \mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathcal{S})$ by means of left Kan extension along the inclusion $\mathscr{T}^{\mathrm{cov,op}} \hookrightarrow \mathscr{T}^{\mathrm{op}}$.

3.4 Formal Loop Spaces

Let $X \in \mathcal{S}$ be a space. We let $\mathcal{L}(X) = \mathrm{Fun}(S^1, X)$ denote the *free loop space of* X, parametrizing maps from the circle $S^1 = K(\mathbf{Z}, 1)$ into X. More generally, for each integer $n \geqslant 0$ we can consider the iterated free loop space

$$\mathcal{L}^n(X) = \begin{cases} X & \text{if } n = 0 \\ \mathcal{L}(\mathcal{L}^{n-1}(X)) & \text{if } n > 0, \end{cases}$$

parametrizing maps from the torus $T^n = K(\mathbf{Z}^n, 1)$ into X. Our goal in this section is to introduce a related construction in the setting of orbispaces.

Notation 3.4.1. The ∞-category of orbispaces $\mathcal{OS} = \mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathcal{S})$ is an ∞-topos; in particular, it is Cartesian closed. Consequently, to any pair of orbispaces X and Y, we can associate an orbispace $\underline{\mathrm{Map}}_{\mathcal{OS}}(\mathsf{Y}, \mathsf{X})$ parametrizing maps from Y to X. More precisely, the orbispace $\underline{\mathrm{Map}}_{\mathcal{OS}}(\mathsf{Y}, \mathsf{X})$ is equipped with an *evaluation map*

$$\mathrm{ev} : \mathsf{Y} \times \underline{\mathrm{Map}}_{\mathcal{OS}}(\mathsf{Y}, \mathsf{X}) \to \mathsf{X}$$

with the following universal property: for every orbispace Z, composition with ev induces a homtopy equivalence

$$\mathrm{Map}_{\mathcal{OS}}(\mathsf{Z}, \underline{\mathrm{Map}}_{\mathcal{OS}}(\mathsf{Y}, \mathsf{X})) \simeq \mathrm{Map}_{\mathcal{OS}}(\mathsf{Y} \times \mathsf{Z}, \mathsf{X}).$$

Remark 3.4.2. Let X and Y be orbispaces. For any object $T \in \mathscr{T}$, we have canonical homotopy equivalences

$$\begin{aligned} \underline{\mathrm{Map}}_{\mathcal{OS}}(\mathsf{Y}, \mathsf{X})^T &\simeq \mathrm{Map}_{\mathcal{OS}}(T^{(-)}, \underline{\mathrm{Map}}_{\mathcal{OS}}(\mathsf{Y}, \mathsf{X})) \\ &\simeq \mathrm{Map}_{\mathcal{OS}}(\mathsf{Y} \times T^{(-)}, \mathsf{X}). \end{aligned}$$

Construction 3.4.3 (Formal Loop Spaces)**.** Let X be an orbispace and let Λ be a torsion abelian group. We define a new orbispace $\mathscr{L}^{\Lambda}(\mathsf{X})$ by the formula

$$\mathscr{L}^{\Lambda}(\mathsf{X}) = \varinjlim_{\Lambda_0 \subseteq \Lambda} \underline{\mathrm{Map}}_{\mathcal{OS}}(B\widehat{\Lambda}_0, \mathsf{X});$$

here the colimit is taken over the collection of all finite subgroups $\Lambda_0 \subseteq \Lambda$, and $\widehat{\Lambda}_0 = \mathrm{Hom}(\Lambda_0, \mathbf{Q}/\mathbf{Z})$ denotes the Pontryagin dual of Λ_0. We will refer to $\mathscr{L}^{\Lambda}(\mathsf{X})$ as the *formal loop space of* X *with respect to* Λ. Concretely, it is characterized by the formula

$$\mathscr{L}^{\Lambda}(\mathsf{X})^T \simeq \varinjlim_{\Lambda_0 \subseteq \Lambda} \mathsf{X}^{B\widehat{\Lambda}_0 \times T}.$$

Example 3.4.4. Let X be an object of $\mathcal{S}$ and let $\underline{X}$ denote the constant orbispace associated to X (Example 3.1.8). Then, for any torsion abelian group Λ, the formal loop space $\mathscr{L}^{\Lambda}(X)$ can be identified with $\underline{X}$.

Example 3.4.5. Let G be a finite group, let $X \in \mathcal{S}_G$ be a G-space. For any torsion abelian group Λ, we can construct a new G-space

$$Y = \coprod_{\alpha : \widehat{\Lambda} \to G} X^{\mathrm{im}(\alpha)},$$

which is essentially characterized by the formula

$$Y^{G_0} = \coprod_{\alpha} X^{G_0 \operatorname{im}(\alpha)};$$

here G_0 denotes a subgroup of G, the coproduct is taken over all continuous group homomorphisms $\alpha : \widehat{\Lambda} \to G$ which are centralized by G_0, and $G_0 \operatorname{im}(\alpha)$ denotes the subgroup of G generated by G_0 together with the image of α. If $X//G$ denotes the orbispace quotient of X by G (Construction 3.2.16), then we have a canonical equivalence of orbispaces

$$\mathcal{L}^H(X//G) \simeq (\coprod_{\alpha:\widehat{\Lambda}\to G} X^{\operatorname{im}(\alpha)})//G.$$

In the special case where G is trivial, this recovers the identification of Example 3.4.4

Remark 3.4.6. Let Λ be a torsion abelian group. Then the functor $\mathcal{L}^{\Lambda} : \mathcal{OS} \to \mathcal{OS}$ preserves small colimits. To prove this, we can assume without loss of generality that Λ is finite, in which case it follows from the description of $\underline{\operatorname{Map}}_{\mathcal{OS}}(B\widehat{\Lambda}, \bullet)$ supplied by Remark 3.4.2.

Let X be an object of $\mathcal{S}$ and let $X^{(-)}$ be the orbispace represented by X. For any torsion abelian group Λ, the underlying space of the orbispace $\mathcal{L}^{\Lambda}(X^{(-)})$ is given by the direct limit $\varinjlim_{\Lambda_0 \subseteq \Lambda} X^{B\widehat{\Lambda}_0}$, taken over the collection of all finite subgroups $\Lambda_0 \subseteq \Lambda$. In particular, we have a canonical map of spaces $|\,\mathcal{L}^{\Lambda}(X^{(-)})| \to X^{B\widehat{\Lambda}}$, which (by virtue of Proposition 3.1.9) can be identified with a map of orbispaces

$$\mathcal{L}^{\Lambda}(X^{(-)}) \to (X^{B\widehat{\Lambda}})^{(-)}.$$

Here $X^{B\widehat{\Lambda}}$ denotes the space $\operatorname{Fun}(B\widehat{\Lambda}, X)$ of all maps from the classifying space $B\widehat{\Lambda}$ into X (where we ignore the profinite topology on the group $\widehat{\Lambda}$). In good cases, this map is an equivalence.

Proposition 3.4.7. *Let X be a π-finite space (Definition 1.1.25) and let Λ be a colattice (Definition 2.7.1). Then the preceding construction induces an equivalence of orbispaces $\mathcal{L}^{\Lambda}(X^{(-)}) \to (X^{B\widehat{\Lambda}})^{(-)}$.*

Proof. Let T be an object of $\mathscr{T}$; we wish to show that the canonical map $\mathcal{L}^{\Lambda}(X^{(-)})^T \to (X^{B\widehat{\Lambda}})^T$ is a homotopy equivalence. Replacing X by X^T, we can reduce to the case where T is contractible. In this case, we wish to show that the canonical map

$$\varinjlim_{\Lambda_0 \subseteq \Lambda} X^{B\widehat{\Lambda}_0} \to X^{B\widehat{\Lambda}}$$

is a homotopy equivalence; here Λ_0 ranges over the collection of all finite subgroups of Λ. Decomposing X as a union of connected components, we may assume without loss of generality that X is connected. Since X is π-finite, there exists an integer n such that X is n-truncated. We proceed by induction on n. In the case $n = 1$, we can identify X with an Eilenberg-MacLane space $BG = K(G, 1)$ for some finite group G. In this case, we are reduced to proving that every group homomorphism $\alpha : \widehat{\Lambda} \to G$ is continuous. This follows from our assumption that Λ is a colattice, since $\ker(\alpha)$ contains the subgroup $m\widehat{\Lambda} \subseteq \widehat{\Lambda}$ for $m = |G|$ (so that α factors through the Pontryagin dual of the finite subgroup $\Lambda[m] \subseteq \Lambda$). To carry out the inductive step, we note that if $n \geqslant 2$ then we have a pullback diagram of π-finite spaces

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X' \arrow[d] \\
\tau_{\leqslant n-1}(X) \arrow[r] & BG,
\end{tikzcd}$$

where $G = \pi_1(X, x)$ is the fundamental group of X (with respect to some choice of base point) and $X' \to BG$ has homotopy fibers $K(M, n+1)$, where M is some finite abelian group with an action of G. Applying our inductive hypothesis to $\tau_{\leqslant n-1}(X)$ and $K(G, 1)$, we are reduced to proving that for every homomorphism $\alpha : \widehat{\Lambda} \to G$ factoring through $\widehat{\Lambda}_0$ for some finite subgroup $\Lambda_0 \subseteq \Lambda$, the canonical map

$$\varprojlim_{\Lambda_1} X'^{B\widehat{\Lambda}_1)} \times_{BG^{B\widehat{\Lambda_1}}} \{\alpha\} \to X'^{B\widehat{\Lambda}} \times_{BG^{B\hat{\Lambda}}} \{\alpha\}$$

is a homotopy equivalence; here Λ_1 ranges over the collection of all finite subgroups of Λ which contain Λ_0. For this, it suffices to show that the map of cohomology rings

$$\theta : \varinjlim_{\Lambda_1} \mathrm{H}^*(B\widehat{\Lambda}_1; M) \to \mathrm{H}^*(B\widehat{\Lambda}; M);$$

is bijective; here we abuse notation by identifying the finite abelian group M with the corresponding local system on the classifying spaces $B\widehat{\Lambda}$ and $B\widehat{\Lambda}_1$. In other words, we are reduced to proving that the cohomology of $\widehat{\Lambda}$ as a *profinite* group (with coefficients in the continuous representation M) agrees with its cohomology as a discrete group. Decomposing M as a direct sum, we may assume that it is a finite abelian p-group for some prime number p. Write $\Lambda = \Lambda' \oplus \Lambda'$, where Λ' is the p-local summand of Λ. In this case, θ is induced by a map

$$\theta' : \varinjlim_{\Lambda_0 \cap \Lambda' \subseteq \Lambda'_1 \subseteq \Lambda'} \mathrm{H}^*(B\widehat{\Lambda}'_1; M) \to \mathrm{H}^*(B\widehat{\Lambda}'; M)$$

by taking fixed points for the action of $B\widehat{\Lambda}''$. It will therefore suffice to prove that θ' is an isomorphism: that is, we can replace Λ by Λ' and thereby reduce to the case where Λ is p-nilpotent. In this case, M admits a finite composition series whose successive quotients carry a *trivial* action of the group $\widehat{\Lambda}$; this allows us to reduce further to the case where $M = \mathbf{F}_p$. We can then identify θ with the canonical map

$$\varinjlim \mathrm{H}^*(B(\mathbf{Z}^r/p^k\,\mathbf{Z}^r); \mathbf{F}_p) \to \mathrm{H}^*(B\,\mathbf{Z}_p^r; \mathbf{F}_p).$$

To show that this map is an isomorphism, we make the stronger claim that the pro-system of homology groups $\{\mathrm{H}_*(B(\mathbf{Z}^r/p^k\,\mathbf{Z}^r); \mathbf{F}_p)\}_{r \geqslant 0}$ is isomorphic to $\mathrm{H}_*(B\,\mathbf{Z}_p^r; \mathbf{F}_p)$ as a pro-system. Using the Künneth formula, we can reduce to the case $r = 1$, where the result follows from a simple calculation (see, for example, Proposition SAG.E.7.5.1). ☐

Variant 3.4.8. Let $\widehat{\mathbf{Z}} = \varprojlim_N \mathbf{Z}/N\,\mathbf{Z}$ denote the profinite completion of $\mathbf{Z}$. Then then inclusion $\mathbf{Z} \hookrightarrow \widehat{\mathbf{Z}}$ induces a map of classifying spaces $u : B\,\mathbf{Z} \to B\widehat{\mathbf{Z}}$. For any space X, precomposition with u induces a map

$$X^{B\widehat{\mathbf{Z}}} \to X^{B\,\mathbf{Z}} = \mathcal{L}(X),$$

where $\mathcal{L}(X)$ is the free loop space of X. We therefore have canonical maps of orbispaces

$$\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(X^{(-)}) \to (X^{B\widehat{\mathbf{Z}}})^{(-)} \to \mathcal{L}(X)^{(-)}.$$

If X is π-finite, then these maps are equivalences: the first by virtue of Proposition 3.4.7, and the second by virtue of the fact that u induces an equivalence of profinite completions (which follows as in the proof of Proposition 3.4.7, using the fact that u induces an isomorphism on cohomology with coefficients in any abelian group with an action of $\widehat{\mathbf{Z}}$). More generally, we have comparison maps $\mathcal{L}^{(\mathbf{Q}/\mathbf{Z})^n}(X^{(-)}) \to \mathcal{L}^n(X)^{(-)}$, which are equivalences when X is π-finite.

Warning 3.4.9. In general, the comparison map $v : \mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(X^{(-)}) \to \mathcal{L}(X)^{(-)}$ of Variant 3.4.8 is not an equivalence. For example, if X is a finite space, then v can be identified with the map $X^{(-)} \to \mathcal{L}(X)^{(-)}$ induced by the identification of X with the subspace of $\mathcal{L}(X)$ given by the constant loops (this follows from Remark 3.1.14 and Example 3.4.4).

In general, the orbispace $\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(X^{(-)})$ need not be of the form $Y^{(-)}$ for any space Y. This is one of the principal motivations for allowing more general orbispaces in our definition of tempered cohomology.

3.5 Preorientations Revisited

Let A be an $\mathbb{E}_\infty$-ring. In §2.6, we introduced the notion of a *preoriented* $\mathbf{P}$*-divisible group* over A (Definition 2.6.8). In this section, we explain a reformulation of this notion which will lead directly to our theories of tempered cohomology (§4) and tempered local systems (§5).

We begin with some general observations. Assume for the moment that A is connective, and let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over A (Definition 2.6.1), which we regard as a functor

$$\mathbf{G} : \mathrm{CAlg}_A \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}} .$$

For every finite abelian group M, we define a functor $\mathbf{G}[M] : \mathrm{CAlg}_A \to \mathcal{S}$ by the formula

$$\mathbf{G}[M](B) = \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}(B)).$$

The $\mathbf{P}$-divisibility of $\mathbf{G}$ guarantees that $\mathbf{G}[M]$ is corepresentable by a finite flat A-algebra that we denote by $\mathcal{O}_{\mathbf{G}[M]}$. The construction $M \mapsto \mathcal{O}_{\mathbf{G}[M]}$ then determines a functor from the category of finite abelian groups $\mathrm{Ab}_{\mathrm{fin}}$ to the ∞-category CAlg_A of $\mathbb{E}_\infty$-algebras over A. In [8], we gave a characterization of those functors which arise in this way:

Definition 3.5.1. Let A be an $\mathbb{E}_\infty$-ring. We will say that a functor $E : \mathrm{Ab}_{\mathrm{fin}} \to \mathrm{CAlg}_A$ is $\mathbf{P}$*-divisible* if it satisfies the following conditions:

(i) The functor E preserves finite coproducts: that is, it carries direct sums of finite abelian groups $\mathscr{T}$ to tensor products in CAlg_A. In particular, the unit map $A \to E(0)$ is an equivalence.

(ii) For every short exact sequence of finite abelian groups $0 \to M' \to M \to M'' \to 0$, the diagram of $\mathbb{E}_\infty$-algebras

$$\begin{tikzcd} E(M') \arrow[r] \arrow[d] & E(0) \arrow[d] \\ E(M) \arrow[r] & E(M'') \end{tikzcd}$$

is a pushout square. Moreover, the vertical maps are finite flat of positive degree.

We let $\mathrm{Fun}^{\mathbf{P}}(\mathrm{Ab}_{\mathrm{fin}}, \mathrm{CAlg}_A)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Ab}_{\mathrm{fin}}, \mathrm{CAlg}_A)$ spanned by $\mathbf{P}$-divisible functors.

Remark 3.5.2. Let A be an $\mathbb{E}_\infty$-ring. Then there is a canonical equivalence of ∞-categories $\mathrm{BT}(A) \simeq \mathrm{Fun}^{\mathbf{P}}(\mathrm{Ab}_{\mathrm{fin}}, \mathrm{CAlg}_A)^{\mathrm{op}}$, which carries a $\mathbf{P}$-divisible group $\mathbf{G}$ to a $\mathbf{P}$-divisible functor E which is essentially characterized by the formula $\mathbf{G}[M] \simeq \mathrm{Spec}(E(M))$. We will review this equivalence below; see also §AV.6.5.

We now introduce a variant of Definition 3.5.1. Let $\mathscr{T}$ be the ∞-category of Notation 3.1.1 (so that the objects of $\mathscr{T}$ are spaces of the form BH, where H is a finite abelian group).

Definition 3.5.3. Let A be an $\mathbb{E}_\infty$-ring. We will say that a functor of ∞-categories $\mathsf{A} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$ is $\mathbf{P}$-*divisible* if the composite functor

$$\mathrm{Ab}_{\mathrm{fin}} \xrightarrow{M \mapsto B\widehat{M}} \mathscr{T}^{\mathrm{op}} \xrightarrow{\mathsf{A}} \mathrm{CAlg}_A$$

is $\mathbf{P}$-divisible, in the sense of Definition 3.5.1. We let $\mathrm{Fun}^{\mathbf{P}}(\mathscr{T}^{\mathrm{op}}, \mathrm{CAlg}_A)$ denote the full subcategory of $\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathrm{CAlg}_A)$ spanned by the $\mathbf{P}$-divisible functors.

Remark 3.5.4. Let A be an $\mathbb{E}_\infty$-ring and let $\mathsf{A} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$ be a functor. Then A is $\mathbf{P}$-divisible (in the sense of Definition 3.5.3) if and only if the following conditions are satisfied:

(a) For each $T \in \mathscr{T}$, the spectrum $\mathsf{A}(T)$ is projective of finite rank as an A-module.

(b) The construction $T \mapsto \pi_0(\mathsf{A}(T))$ determines a $\mathbf{P}$-divisible functor

$$\pi_0(\mathsf{A}) : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_{\pi_0(A)}.$$

Every $\mathbf{P}$-divisible functor $\mathsf{A} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$ determines a $\mathbf{P}$-divisible functor $\mathrm{Ab}_{\mathrm{fin}} \to \mathrm{CAlg}_A$, which we can identify with a $\mathbf{P}$-divisible group $\mathbf{G}$ over A (Remark 3.5.2). The $\mathbf{P}$-divisible group $\mathbf{G}$ is essentially characterized by the formula

$$\mathbf{G}[M] = \mathrm{Spec}(\mathsf{A}(B\widehat{M})).$$

From the $\mathbf{P}$-divisible group $\mathbf{G}$, we can use this formula to determine the value of the functor A on each *object* of the ∞-category $\mathscr{T}$. However, it does not allow us to completely reconstruct A from $\mathbf{G}$, because it only determines the value of A on base-point preserving morphisms of $\mathscr{T}$ (see Remark 3.1.2). To promote a $\mathbf{P}$-divisible group $\mathbf{G}$ to a $\mathbf{P}$-divisible functor $\mathsf{A} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$, we need to supply some additional data. The main result of this section asserts that this additional data can be identified with a choice of preorientation of $\mathbf{G}$:

Theorem 3.5.5. *Let A be an $\mathbb{E}_\infty$-ring. Then the forgetful functor*

$$\mathrm{Fun}^{\mathbf{P}}(\mathscr{T}^{\mathrm{op}}, \mathrm{CAlg}_A)^{\mathrm{op}} \to \mathrm{Fun}^{\mathbf{P}}(\mathrm{Ab}_{\mathrm{fin}}, \mathrm{CAlg}_A)^{\mathrm{op}} \simeq \mathrm{BT}(A)$$

is equivalent to a left fibration, classified by the functor

$$\mathrm{BT}(A) \to \mathcal{S} \qquad \mathbf{G} \mapsto \mathrm{Pre}(\mathbf{G}).$$

Remark 3.5.6. More informally, Theorem 3.5.5 asserts that we can identify $\mathbf{P}$-divisible functors $\mathsf{A} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$ with pairs $(\mathbf{G}, e)$, where $\mathbf{G}$ is a $\mathbf{P}$-divisible group over A and $e : \Sigma(\mathbf{Q}/\mathbf{Z}) \to \mathbf{G}(A)$ is a preorientation of $\mathbf{G}$.

Example 3.5.7 (The $K(n)$-Local Case)**.** Fix a prime number p, and let A be an $\mathbb{E}_\infty$-ring which is $K(n)$-local for some $n > 0$. It follows from Theorem Or.4.6.3 that the functor

$$(T \in \mathscr{T}^{\mathrm{op}}) \mapsto (A^T \in \mathrm{CAlg}_A)$$

is $\mathbf{P}$-divisible (in the sense of Definition 3.5.3). Moreover, the $\mathbf{P}$-divisible group associated to this functor is the Quillen p-divisible group $\mathbf{G}_A^{\mathcal{Q}}$ of Construction Or.4.6.2 (essentially by definition). The equivalence of Theorem 3.5.5 supplies a preorientation on the Quillen p-divisible group $\mathbf{G}_A^{\mathcal{Q}}$, which can be identified with the universal preorientation described in Remark 2.4.2 (this identification will be implicit in our proof of Theorem 3.5.5).

Example 3.5.8 (The Trivial Case)**.** Let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, which we identify with a $\mathbf{P}$-divisible functor $E : \mathrm{Ab}_{\mathrm{fin}} \to \mathrm{CAlg}_A$ (Remark 3.5.2). We can then define a $\mathbf{P}$-divisible functor $\mathsf{A} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$ by the composition

$$\mathscr{T}^{\mathrm{op}} \xrightarrow{T \mapsto \widehat{\pi_1(T)}} \mathrm{Ab}_{\mathrm{fin}} \xrightarrow{E} \mathrm{CAlg}_A .$$

Note that the composition of A with the map $\mathrm{Ab}_{\mathrm{fin}} \xrightarrow{M \mapsto B\widehat{M}} \mathscr{T}^{\mathrm{op}}$ is equivalent to E (Remark 3.1.2). By virtue of Theorem 3.5.5, the functor A supplies preorientation e of the $\mathbf{P}$-divisible group $\mathbf{G}$. This preorientation is given by the zero map $\Sigma(\mathbf{Q}/\mathbf{Z}) \to \mathbf{G}(A)$ (this will again be implicit in our proof of Theorem 3.5.5).

Variant 3.5.9. Let $\mathsf{A}_0 : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}$ be any functor, and let $*$ denote the final object of $\mathscr{T}$. Then $A = \mathsf{A}_0(*)$ is an $\mathbb{E}_\infty$-ring, and A_0 can be promoted to a functor of ∞-categories

$$\mathsf{A} : \mathscr{T}^{\mathrm{op}} \simeq (\mathscr{T}^{\mathrm{op}})_{*/} \xrightarrow{\mathsf{A}_0} \mathrm{CAlg}_{\mathsf{A}_0(*)/} \simeq \mathrm{CAlg}_A .$$

Let $\mathrm{Fun}^{\mathbf{P}}(\mathscr{T}^{\mathrm{op}}, \mathrm{CAlg})$ denote the full subcategory of $\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathrm{CAlg})$ spanned by those functors A_0 for which A is $\mathbf{P}$-divisible, in the sense of Definition 3.5.3. Using Theorem 3.5.5, we can identify objects of $\mathrm{Fun}^{\mathbf{P}}(\mathscr{T}^{\mathrm{op}}, \mathrm{CAlg})$ with triples $(A, \mathbf{G}, e)$ where A is an $\mathbb{E}_\infty$-ring, $\mathbf{G}$ is a $\mathbf{P}$-divisible group over A, and e is an orientation of $\mathbf{G}$.

To prove Theorem 3.5.5, we will need to recall how the equivalence of Remark 3.5.2 is constructed. Let $\mathcal{C}$ be an ∞-category which admits finite limits. A *torsion object of* $\mathcal{C}$ (in the sense of Definition AV.6.4.2) is a functor $X : \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}} \to \mathcal{C}$ which satisfies the following pair of conditions:

- (a) The functor X commutes with finite products; in particular, $X(0)$ is a final object of $\mathcal{C}$.
- (b) For every short exact sequence of abelian groups $0 \to M' \to M \to M'' \to 0$, the diagram
$$\begin{tikzcd} X(M'') \arrow[r] \arrow[d] & X(M) \arrow[d] \\ X(0) \arrow[r] & X(M') \end{tikzcd}$$
is a pullback square in $\mathcal{C}$. In other words, the functor X carries short exact sequences of abelian groups to fiber sequences in $\mathcal{C}$.

We let $\mathrm{Tors}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{C})$ spanned by the torsion objects of $\mathcal{C}$.

Example 3.5.10. Let N be a $\mathbf{Z}$-module spectrum. Then the construction

$$(M \in \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}}) \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, N)$$

determines a functor $\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}} \to \mathcal{S}$ satisfying conditions (a) and (b) above, which we will denote by $N[\mathbf{P}^\infty]$. The construction $N \mapsto N[\mathbf{P}^\infty]$ determines a functor $\mathrm{Mod}_{\mathbf{Z}} \to \mathrm{Tors}(\mathcal{S})$. By virtue of Example AV.6.4.11 , this functor restricts to an equivalence of ∞-categories $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}} \simeq \mathrm{Tors}(\mathcal{S})$. Here $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}}$ denotes the full subcategory of $\mathrm{Mod}_{\mathbf{Z}}$ spanned by those connective $\mathbf{Z}$-module spectra N whose homotopy groups are torsion (that is, $N \otimes_{\mathbf{Z}} \mathbf{Q} \simeq 0$).

Definition 3.5.11. Let $\mathcal{C}$ be an ∞-category which admits finite limits. A *preoriented torsion object* of $\mathcal{C}$ is a functor $X : \mathscr{T} \to \mathcal{C}$ with the property that the composition

$$\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}} \xrightarrow{M \mapsto B\widehat{M}} \mathscr{T} \xrightarrow{X} \mathcal{C}$$

is a torsion object of $\mathcal{C}$. We let $\mathrm{PTors}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathscr{T}, \mathcal{C})$ spanned by the preoriented torsion objects of $\mathcal{C}$. Note that precomposition with the functor $M \mapsto B\widehat{M}$ determines a functor $\mathrm{PTors}(\mathcal{C}) \to \mathrm{Tors}(\mathcal{C})$, which we will refer to as the *forgetful functor*.

Example 3.5.12. Let A be a connective $\mathbb{E}_\infty$-ring and set $\mathcal{C} = \mathrm{Fun}(\mathrm{CAlg}_A, \mathcal{S})$ be the ∞-category of all functors from CAlg_A to $\mathcal{S}$, and let $\mathrm{Spec} : \mathrm{CAlg}_A^{\mathrm{op}} \hookrightarrow \mathcal{C}$ denote the Yoneda embedding. Then composition with Spec induces a fully faithful embedding

$$\mathrm{Fun}^{\mathbf{P}}(\mathrm{Ab}_{\mathrm{fin}}, \mathrm{CAlg}_A)^{\mathrm{op}} \hookrightarrow \mathrm{Tors}(\mathcal{C}).$$

Unwinding the definitions, we obtain a pullback diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Fun}^{\mathbf{P}}(\mathscr{T}^{\mathrm{op}}, \mathrm{CAlg}_A)^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathbf{P}}(\mathrm{Ab}_{\mathrm{fin}}, \mathrm{CAlg}_A)^{\mathrm{op}} \arrow[r, "\sim"] \arrow[d, "\mathrm{Spec}"] & \mathrm{BT}(A) \arrow[d] \\
\mathrm{PTors}(\mathcal{C}) \arrow[r] & \mathrm{Tors}(\mathcal{C}) \arrow[r, "\sim"] & \mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}}),
\end{tikzcd}$$

where the equivalence on the bottom right is provided by Example 3.5.10.

Example 3.5.13. Let $\iota : \mathscr{T} \hookrightarrow \mathcal{S}$ be the inclusion functor. Then ι is a preoriented torsion object of the ∞-category $\mathcal{S}$. Moreover, the image of ι under the forgetful functor $\mathrm{PTors}(\mathcal{S}) \to \mathrm{Tors}(\mathcal{S}) \simeq \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}}$ can be identified with the $\mathbf{Z}$-module spectrum $\Sigma(\mathbf{Q}/\mathbf{Z})$.

Remark 3.5.14. Let $\iota : \mathscr{T} \hookrightarrow \mathcal{S}$ be the inclusion functor. Then ι is a left Kan extension of its restriction to the full subcategory $\{*\} \subseteq \mathscr{T}$ spanned by contractible space $* \simeq \Delta^0$. Consequently, for any functor $X : \mathscr{T} \to \mathcal{S}$, the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathscr{T},\mathcal{S})}(\iota, X) \to \mathrm{Map}_{\mathcal{S}}(\iota(*), X(*)) \simeq X(*)$$

is a homotopy equivalence.

If X is a preoriented torsion object of $\mathcal{S}$ (in the sense of Definition 3.5.11), then $X(*)$ is contractible. It follows that the mapping space $\mathrm{Map}_{\mathrm{PTors}(\mathcal{S})}(\iota, X)$ is also contractible: that is, ι is an initial object of the ∞-category $\mathrm{PTors}(\mathcal{S})$.

We will deduce Theorem 3.5.5 from the following categorical fact:

Proposition 3.5.15. *Let $\iota : \mathscr{T} \hookrightarrow \mathcal{S}$ be the inclusion functor, regarded as a preoriented torsion object of $\mathcal{S}$. Then the forgetful functor The forgetful functor $F : \mathrm{PTors}(\mathcal{S}) \to \mathrm{Tors}(\mathcal{S})$ induces an equivalence of ∞-categories*

$$\mathrm{PTors}(\mathcal{S})_{\iota/} \to \mathrm{Tors}(\mathcal{S})_{F(\iota)/}.$$

Proof of Theorem 3.5.5 from Proposition 3.5.15. Let A be an $\mathbb{E}_\infty$-ring, which we may assume to be connective (without loss of generality). Set $\mathcal{C} = \mathrm{Fun}(\mathrm{CAlg}_A, \mathcal{S})$, so that Example 3.5.12 supplies a pullback diagram σ :

$$\begin{tikzcd}
\mathrm{Fun}^{\mathbf{P}}(\mathscr{T}^{\mathrm{op}}, \mathrm{CAlg}_A)^{\mathrm{op}} \arrow[r] \arrow[d, "U"] & \mathrm{BT}(A) \arrow[d] \\
\mathrm{PTors}(\mathcal{C}) \arrow[r] & \mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}}).
\end{tikzcd}$$

Let $\underline{\iota} \in \mathrm{PTors}(\mathcal{C}) \simeq \mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{PTors}(\mathcal{S}))$ be the constant functor taking the value $\iota \in \mathrm{PTors}(\mathcal{S})$. It follows from Remark 3.5.14 that the mapping space $\mathrm{Map}_{\mathrm{PTors}(\mathcal{C})}(\underline{\iota}, U(\mathsf{A}))$ is contractible for every $\mathbf{P}$-divisible functor $\mathsf{A} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$. We can therefore promote σ to a pullback diagram σ':

$$\begin{tikzcd}
\mathrm{Fun}^{\mathbf{P}}(\mathscr{T}^{\mathrm{op}}, \mathrm{CAlg}_A)^{\mathrm{op}} \arrow[r] \arrow[d, "\overline{U}"] & \mathrm{BT}(A) \arrow[d] \\
\mathrm{PTors}(\mathcal{C})_{\underline{\iota}/} \arrow[r] & \mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}}).
\end{tikzcd}$$

Using Proposition 3.5.15 and Example 3.5.13, we can rewrite σ' as a pullback diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\mathbf{P}}(\mathscr{T}^{\mathrm{op}}, \mathrm{CAlg}_A)^{\mathrm{op}} \arrow[r] \arrow[d, "\overline{U}"] & \mathrm{BT}(A) \arrow[d] \\
\mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}})_{\underline{\Sigma(\mathbf{Q}/\mathbf{Z})}/} \arrow[r] & \mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}}),
\end{tikzcd}$$

where $\underline{\Sigma(\mathbf{Q}/\mathbf{Z})}$ denotes the constant functor $\mathrm{CAlg}_A \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}}$ taking the value $\Sigma(\mathbf{Q}/\mathbf{Z})$. It follows that the upper horizontal map is equivalent to the left fibration classified by the functor

$$(\mathbf{G} \in \mathrm{BT}(A)) \mapsto \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{Mod}_{\mathbf{Z}})}(\underline{\Sigma(\mathbf{Q}/\mathbf{Z})}, \mathbf{G}) \simeq \mathrm{Pre}(\mathbf{G}).$$

□

Proof of Proposition 3.5.15. Let $\Phi : \mathscr{T} \to \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}}$ be the functor given by

$$\Phi(T) = \widehat{\pi_1(T)} = \mathrm{Hom}(\pi_1(T), \mathbf{Q}/\mathbf{Z}) \simeq \mathrm{H}^1(T; \mathbf{Q}/\mathbf{Z}).$$

Then Φ is a left homotopy inverse of the functor

$$\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}} \to \mathscr{T} \qquad M \mapsto B\widehat{M}.$$

Moreover, Φ is a left fibration of ∞-categories, classified by the functor

$$U : \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}} \to \mathcal{S} \qquad U(M) = K(\widehat{M}, 2).$$

Note that U is a torsion object of the ∞-category $\mathcal{S}$, whose image under the equivalence $\mathrm{Tors}(\mathcal{S}) \simeq \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}}$ of Example 3.5.10 is the $\mathbf{Z}$-module spectrum $\Sigma^2(\mathbf{Q}/\mathbf{Z})$. Applying Corollary HTT.5.1.6.12, we see that Φ induces an equivalence of ∞-categories $\mathrm{Fun}(\mathscr{T}, \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{S})_{/U}$, which restricts to an equivalence of full subcategories $\mathrm{PTors}(\mathcal{S}) \simeq \mathrm{Tors}(\mathcal{S})_{/U} \simeq (\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}})_{/\Sigma^2(\mathbf{Q}/\mathbf{Z})}$. Under this equivalence, the forgetful functor

$$\mathrm{PTors}(\mathcal{S})_{\iota/} \to \mathrm{Tors}(\mathcal{S})_{\iota/} \simeq (\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}})_{\Sigma(\mathbf{Q}/\mathbf{Z})/}$$

corresponds to the functor

$$(\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}})_{/\Sigma^2(\mathbf{Q}/\mathbf{Z})} \to (\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Tors}})_{\Sigma(\mathbf{Q}/\mathbf{Z})/}$$

which carries a map of $\mathbf{Z}$-module spectra $u : M \to \Sigma^2(\mathbf{Q}/\mathbf{Z})$ to the fiber $\mathrm{fib}(u)$. This functor is an equivalence of ∞-categories; the inverse equivalence carries a map of $\mathbf{Z}$-module spectra $v : \Sigma(\mathbf{Q}/\mathbf{Z}) \to N$ to the cofiber $\mathrm{cofib}(v)$. □

3.6 Example: Complex K-Theory

Let KU denote the periodic complex K-theory spectrum. Then Construction 2.8.6 supplies an orientation of the multiplicative $\mathbf{P}$-divisible group $\mu_{\mathbf{P}^\infty}$ over KU. By virtue of Theorem 3.5.5, the $\mathbf{P}$-divisible group $\mu_{\mathbf{P}^\infty}$ and its orientation can be encoded by a functor

$$\mathsf{KU} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{KU}}$$

which is $\mathbf{P}$-divisible in the sense of Definition 3.5.3. Our goal in this section is to give an explicit description of this functor and to explain its relationship to the representation theory of finite groups.

Construction 3.6.1. Let $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$ be the groupoid whose objects are finite-dimensional complex vector spaces and whose morphisms are isomorphisms. For any space T, we let $\mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq})$ denote the groupoid of functors from T (or equivalently the fundamental groupoid $\pi_{\leqslant 1}(T)$) into $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$. In other words, $\mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq})$ is the ordinary category whose objects are local systems of finite-dimensional complex vector spaces on T (and whose morphisms are isomorphisms).

We will be exclusively interested in the situation $T = BG$ is the classifying space of a finite group G, so that $\mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq})$ can be identified with the category of finite-dimensional complex representations of G (with morphisms given by isomorphisms). In this case, the standard topology on the field $\mathbf{C}$ determines a topological enrichment of the category $\mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq})$. Let $\mathrm{N}^{\mathrm{hc}}(\mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq}))$ denote the homotopy coherent nerve of $\mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq})$ (as a topologically enriched category).

The formation of direct sums of complex vector spaces determines a symmetric monoidal structure on the categories $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$ and $\mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq})$, and which induces an $\mathbb{E}_\infty$-structure on the space $\mathrm{N}^{\mathrm{hc}}(\mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq}))$. We let $\mathsf{ku}(T)$ denote the connective spectrum given by the group completion of $\mathrm{N}^{\mathrm{hc}}(\mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq}))$.

The formation of tensor products of complex vector spaces determines a second symmetric monoidal structure on the categories $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$ and $\mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq})$, which distributes over the first. This structure endows each $\mathrm{N}^{\mathrm{hc}}(\mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq}))$ with the structure of a commutative algebra object of the ∞-category $\mathrm{CMon}(\mathcal{S})$ of $\mathbb{E}_\infty$-spaces, where we regard $\mathrm{CMon}(\mathcal{S})$ as equipped with the symmetric monoidal structure given by the smash product of $\mathbb{E}_\infty$-spaces (see Proposition AV.3.6.1). Put more informally, $\mathrm{N}^{\mathrm{hc}}(\mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq}))$ is an $\mathbb{E}_\infty$-semiring space, with addition given by direct sum of local systems and multiplication given by the tensor product of local systems. It follows that the group completion $\mathsf{ku}(T)$ inherits the structure of an $\mathbb{E}_\infty$-ring.

Example 3.6.2. When the space T is contractible, the $\mathbb{E}_\infty$-ring $\mathsf{ku}(T)$ can be identified with the connnective complex K-theory spectrum $\mathrm{ku} \simeq \tau_{\geqslant 0}(\mathrm{KU})$ (essentially by construction).

Remark 3.6.3. Let G be a finite group. Then connected components of the space $\mathrm{N}^{\mathrm{hc}}(\mathrm{Fun}(BG, \mathrm{Vect}_{\mathbf{C}}^{\simeq}))$ can be identified with isomorphism classes of finite-dimensional complex representations of G. Passing to group completions, we obtain an isomorphism $\pi_0(\mathsf{ku}(BG)) \simeq \mathrm{Rep}(G)$, where $\mathrm{Rep}(G)$ is the complex representation ring of G.

Remark 3.6.4 (Functoriality)**.** Let T and T' be spaces which are homotopy equivalent to the classifying spaces of finite groups G and G', respectively. For any map $f : T \to T'$, composition with f determines a topologically enriched functor $f^* : \mathrm{Fun}(T', \mathrm{Vect}_{\mathbf{C}}^{\simeq}) \to \mathrm{Fun}(T, \mathrm{Vect}_{\mathbf{C}}^{\simeq})$. This functor is compatible with the formation of direct sums and tensor products, and therefore induces a map of $\mathbb{E}_\infty$-rings $f^* : \mathsf{ku}(T') \to \mathsf{ku}(T)$.

In the special case where G' is the trivial group, we obtain a map of $\mathbb{E}_\infty$-rings $\mathrm{ku} \to \mathsf{ku}(T)$, which exhibits $\mathsf{ku}(T)$ as an $\mathbb{E}_\infty$-algebra over the connective K-theory spectrum ku.

Notation 3.6.5. Let T be a space which is homotopy equivalent to BG, for some finite group G. We let $\mathsf{KU}(T)$ denote the tensor product $\mathrm{KU} \otimes_{\mathrm{ku}} \mathsf{ku}(T)$. In other words, $\mathsf{KU}(T)$ is the $\mathbb{E}_\infty$-algebra over KU which is obtained from $\mathsf{ku}(T)$ by inverting the Bott class $\beta \in \pi_2(\mathrm{ku})$.

Remark 3.6.6. Let G be a finite group, and let $V_1, \ldots, V_n$ be a set of representatives for the collection of all isomorphism classes of irreducible complex representations of G. Then the construction

$$(W_1, \ldots, W_n) \mapsto \bigoplus_{1 \leqslant i \leqslant n} V_i \otimes_{\mathbf{C}} W_i$$

induces an equivalence of topologically enriched categories

$$(\mathrm{Vect}_{\mathbf{C}}^{\simeq})^n \simeq \mathrm{Fun}(BG, \mathrm{Vect}_{\mathbf{C}}^{\simeq}).$$

It follows that the $\mathbb{E}_\infty$-rings $\mathsf{ku}(BG)$ and $\mathsf{KU}(BG)$ are free of rank n when regarded as a module over ku and KU, respectively.

Remark 3.6.7. Let H be a finite abelian group, and let $\widehat{H} = \mathrm{Hom}(H, \mathbf{Q}/\mathbf{Z})$ denote the Pontryagin dual group of H. For each $\lambda \in \widehat{H}$, we let V_λ denote the representation of H whose underlying vector space is $\mathbf{C}$, where H acts by the character

$$H \to \mathbf{C}^\times \qquad h \mapsto \exp(2\pi i \lambda).$$

The construction $\lambda \mapsto [V_\lambda]$ then induces an isomorphism of commutative rings $\mathbf{Z}[\widehat{H}] \xrightarrow{\sim} \mathrm{Rep}(H)$.

For the rest of this section, we specialize Construction 3.6.1 further to the case where T is the classifying space of a finite *abelian* group (we will return to considering nonabelian groups in §4.1). Using Remark 3.6.4, we can regard the constructions $T \mapsto \mathsf{ku}(T)$ and $T \mapsto \mathsf{KU}(T)$ as providing functors

$$\mathsf{ku} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{ku}} \qquad \mathsf{KU} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{KU}}$$

where $\mathscr{T} \subseteq \mathcal{S}$ is the ∞-category of Notation 3.1.1.

Proposition 3.6.8. *The functor* $\mathsf{ku} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{ku}}$ *is* $\mathbf{P}$*-divisible (in the sense of Definition 3.5.3).*

Proof. It follows from Remark 3.6.6 that each $\mathsf{ku}(T)$ is a free module of finite rank over ku. It will therefore suffice to show that the functor

$$\mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_{\pi_0(\mathrm{ku})} \qquad T \mapsto \pi_0(\mathsf{ku}(T))$$

is $\mathbf{P}$-divisible (Remark 3.5.4). Using Remarks 3.6.3 and Remark 3.6.7, we see that this functor is given by $BH \mapsto \mathrm{Rep}(H) \simeq \mathbf{Z}[\widehat{H}]$, and therefore agrees with the $\mathbf{P}$-divisible functor $\mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathbf{Z}}$ associated to the multiplicative $\mathbf{P}$-divisible group $\mu_{\mathbf{P}^\infty}$ of Construction 2.8.1. □

Remark 3.6.9 (Relationship with Construction 2.8.6)**.** Applying Theorem 3.5.5, we can identify the $\mathbf{P}$-divisible functor $\mathsf{ku} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{ku}}$ with a pair $(\mathbf{G}, e)$, where $\mathbf{G}$ is a $\mathbf{P}$-divisible group over ku and e is a preorientation of $\mathbf{G}$. The proof of Proposition 3.6.8 shows that, after extending scalars along the map $\mathrm{ku} \to \pi_0(\mathrm{ku}) \simeq \mathbf{Z}$, there is a canonical isomorphism of $\mathbf{P}$-divisible groups $\gamma_0 : \mathbf{G}_{\mathbf{Z}} \simeq \mu_{\mathbf{P}^\infty}$. Since $\mu_{\mathbf{P}^\infty}$ is Cartier dual to the étale $\mathbf{P}$-divisible group $\underline{\mathbf{Q}/\mathbf{Z}}$, it has no nontrivial deformations: in particular, γ_0 admits an essentially unique lift to an equivalence $\mathbf{G} \simeq \mu_{\mathbf{P}^\infty}$ of $\mathbf{P}$-divisible groups over ku. We can therefore identify the preorientation e with a map of $\mathbb{E}_\infty$-spaces $B(\mathbf{Q}/\mathbf{Z}) \to \mathrm{GL}_1(\mathrm{ku})$, or equivalently with a map of $\mathbb{E}_\infty$-rings $\Sigma^\infty_+ B(\mathbf{Q}/\mathbf{Z}) \to \mathrm{ku}$. Unwinding the constructions, we see that this map factors as a composition

$$\Sigma^\infty_+ B(\mathbf{Q}/\mathbf{Z}) \to \Sigma^\infty_+ \mathbf{CP}^\infty \xrightarrow{\rho} \mathrm{ku}$$

where ρ is induced by the map of $\mathbb{E}_\infty$-spaces

$$\mathbf{CP}^\infty \simeq BU(1) \hookrightarrow \mathrm{N}^{\mathrm{hc}}(\mathrm{Vect}^{\simeq}_{\mathbf{C}}) \to \Omega^\infty(\mathrm{ku}),$$

carrying the canonical generator of $\pi_2(\mathbf{CP}^\infty)$ to the Bott class $\beta \in \pi_2(\mathrm{ku})$.

Combining Proposition 3.6.8 with Remark 3.6.9, we obtain the following:

Corollary 3.6.10. *The construction* $\mathsf{KU} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{KU}}$ *is* $\mathbf{P}$*-divisible (in the sense of Definition 3.5.3). Under the equivalence of Theorem 3.5.5, it corresponds to the multiplicative* $\mathbf{P}$*-divisible group* $\mu_{\mathbf{P}^\infty}$ *over* KU*, equipped with the orientation described in Construction 2.8.6.*

4 Tempered Cohomology

We now introduce the main object of study in this paper.

Notation 4.0.1. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. We let $A_{\mathbf{G}}$ denote the $\mathbf{P}$-divisible functor $\mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$ corresponding to $\mathbf{G}$ under the equivalence Theorem 3.5.5. We will denote the value of $A_{\mathbf{G}}$ on an object $T \in \mathscr{T}$ by $A_{\mathbf{G}}^T$. In particular, if H is a finite abelian group, we have a canonical equivalence $\mathrm{Spec}(A_{\mathbf{G}}^{BH}) = \mathbf{G}[\widehat{H}]$, where $\widehat{H} = \mathrm{Hom}(H, \mathbf{Q}/\mathbf{Z})$ denotes the Pontryagin dual of H.

Warning 4.0.2. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. Then, for any object $T \in \mathscr{T}$, there exists an equivalence

$$\mathrm{Spec}(A_{\mathbf{G}}^T) \simeq \mathbf{G}[\widehat{\pi_1(T)}].$$

Beware that this equivalence is not canonical: it depends on a choice of base point of T (which allows us to identify T with the classifying space BH for $H = \pi_1(T)$). Choosing an equivalence $\mathrm{Spec}(A_{\mathbf{G}}^T) \simeq \mathbf{G}[\widehat{\pi_1(T)}]$ which depends functorially on T is equivalent to choosing a nullhomotopy of the preorientation $e : \Sigma(\mathbf{Q}/\mathbf{Z}) \to \mathbf{G}(A)$ (Example 3.5.8).

Construction 4.0.3 (Tempered Function Spectra)**.** Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. Let us abuse notation by identifying the ∞-category $\mathscr{T}$ of Notation 3.1.1 with its essential image under the Yoneda embedding

$$\mathscr{T} \hookrightarrow \mathcal{OS} \qquad T \mapsto T^{(-)}.$$

By virtue of Theorem HTT.5.1.5.6 , the functor $A_{\mathbf{G}} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$ admits an essentially unique extension to a functor $\mathcal{OS}^{\mathrm{op}} \to \mathrm{CAlg}_A$ which preserves small limits (that is, it carries colimits in the ∞-category of orbispaces to limits in the ∞-category CAlg_A). We will abuse notation by denoting this functor also by $A_{\mathbf{G}}$; it carries each orbispace X to an $\mathbb{E}_\infty$-algebra over A which we will denote by $A_{\mathbf{G}}^{\mathsf{X}}$. We will refer to $A_{\mathbf{G}}^{\mathsf{X}}$ as the $\mathbf{G}$-*tempered function spectrum* (parametrizing maps from X to $A_{\mathbf{G}}$).

In the special case where $\mathsf{X} = X^{(-)}$ is the orbispace represented by a space $X \in \mathcal{S}$, we will denote the $\mathbb{E}_\infty$-ring $A_{\mathbf{G}}^{\mathsf{X}}$ simply by $A_{\mathbf{G}}^X$.

Remark 4.0.4. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. Then, for each orbispace X, the spectrum $A_{\mathbf{G}}^{\mathsf{X}}$ is essentially determined (as a spectrum) by the formula

$$\Omega^{\infty-n}(A_{\mathbf{G}}^{\mathsf{X}}) \simeq \mathrm{Map}_{\mathcal{OS}}(\mathsf{X}, \Omega^{\infty-n} A_{\mathbf{G}}).$$

Here we identify $A_{\mathbf{G}}$ with a spectrum object of the ∞-category of orbispaces.

Construction 4.0.5 (Tempered Cohomology)**.** Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. For each orbispace X, we let $A^*_{\mathbf{G}}(\mathsf{X})$ denote the graded-commutative ring given by the formula

$$A^*_{\mathbf{G}}(\mathsf{X}) = \pi_{-*}(A^{\mathsf{X}}_{\mathbf{G}}).$$

We will refer to $A^*_{\mathbf{G}}(\mathsf{X})$ as the $\mathbf{G}$*-tempered cohomology ring of* X.

In the special case where $\mathsf{X} = X^{(-)}$ is the orbispace represented by a space $X \in \mathcal{S}$, we will denote the graded ring $A^*_{\mathbf{G}}(\mathsf{X})$ by $A^*_{\mathbf{G}}(X)$, which we refer to as the $\mathbf{G}$*-tempered cohomology ring of* X.

Our goal in this section is to carry out a detailed study of Constructions 4.0.3 and 4.0.5. We begin in §4.1 with the case where $A = \mathrm{KU}$ is the complex K-theory spectrum and $\mathbf{G} = \mu_{\mathbf{P}^\infty}$ is the multiplicative $\mathbf{P}$-divisible group (endowed with the orientation of Construction 2.8.6). In this case, we will see that Construction 4.0.5 reproduces equivariant complex K-theory (for finite groups). More precisely, for every finite group G and every G-space $X \in \mathcal{S}_G$, we construct a canonical isomorphism

$$\mathrm{KU}^*_H(X) \xrightarrow{\sim} \mathrm{KU}^*_{\mu_{\mathbf{P}^\infty}}(X//G),$$

whose domain is the G-equivariant complex K-theory of X and whose codomain is the $\mu_{\mathbf{P}^\infty}$-tempered cohomology of the orbispace quotient $X//G$ (Corollary 4.1.3). When G is abelian, this is essentially a tautology (by virtue of our description of the orientation of $\mu_{\mathbf{P}^\infty}$ supplied by Corollary 3.6.10). The extension to nonabelian groups articulates an important feature of G-equivariant complex K-theory: it can be formally reconstructed (by a Kan extension procedure) from its behavior with respect to abelian subgroups of G. This observation motivates all of the constructions which appear in this paper: in essence, we are showing that an analogous procedure gives sensible results in other contexts (like the setting of elliptic cohomology).

Remark 4.0.6 (Equivariant Stable Homotopy Theory)**.** Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, and let H be a finite group. Then the construction

$$(X \in \mathcal{S}_H) \mapsto A^*_{\mathbf{G}}(X//H)$$

can be viewed as a cohomology theory defined on the ∞-category of H-spaces $\mathcal{S}_H$. It follows formally that this cohomology theory is representable by a spectrum object of the ∞-category $\mathcal{S}_H$: that is, by a *naive* H-spectrum. In [10], we will show (using ideas developed in this paper; see §7.4) that this naive H-spectrum can be promoted to a *genuine* H-spectrum in the case when $\mathbf{G}$ is an oriented $\mathbf{P}$-divisible group.

Fix a preoriented **P**-divisible group **G** over an $\mathbb{E}_\infty$-ring A. The most essential features of **G**-tempered cohomology can be summarized by the following variants of Theorems 1.1.17, 1.1.18, and 1.1.19:

(a) Let X be an orbispace with underlying space $|\mathsf{X}|$. Then there is a canonical ring homomorphism

$$\zeta : A^*_{\mathbf{G}}(\mathsf{X}) \to A^*(|\mathsf{X}|),$$

which we call the *Atiyah-Segal comparison map* (Construction 4.2.2). In the case where A is $K(n)$-local and **G** is the Quillen p-divisible over A, we show that ζ is an isomorphism (Theorem 4.2.5; this reduces to Theorem 1.1.17 in the case when X is an orbispace quotient $Y//H$).

(b) Suppose that the **P**-divisible group **G** splits as a direct sum $\mathbf{G}_0 \oplus \underline{\Lambda}$, where Λ is a colattice. In §4.3, we associate to every orbispace X a canonical isomorphism

$$\chi : A^*_{\mathbf{G}}(\mathsf{X}) \to A^*_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})),$$

where $\mathcal{L}^{\Lambda}(\mathsf{X})$ denotes the formal loop space of Construction 3.4.3 (Theorem 4.3.2); this reduces to Theorem 1.1.18 in the case where X is an orbispace quotient $Y//H$).

(c) Let B be an $\mathbb{E}_\infty$-algebra over A, and let us abuse notation by identifying **G** with the **P**-divisible group $\mathbf{G}_B$ obtained from **G** by extension of scalars. For every orbispace X, there is a tautological comparison map

$$\theta : B \otimes_A A^{\mathsf{X}}_{\mathbf{G}} \to B^{\mathsf{X}}_{\mathbf{G}}.$$

This map is an equivalence when **G** is oriented and X is representable by a π-finite space X (Theorem 4.7.1, which formally implies Theorem 1.1.19 by arguments that we will outline in §4.7).

Properties (a) and (b) are essentially formal, and we prove them in §4.2 and §4.3, respectively. Assertion (c) is much more difficult. In this section, we prove (c) only in the special case where X is a generalized Eilenberg-MacLane space (with abelian homotopy groups). In this case, we will show that the tempered cohomology ring $A^*_{\mathbf{G}}(X)$ is a projective module of finite rank over the coefficient ring $\pi_{-*}(A)$, which has an explicit description in terms of the arithmetic of the **P**-divisible group **G**. We formulate this description precisely in §4.4 (Theorem 4.4.16) and carry out the proof in §4.5 (making essential use of properties (a) and (b), together with the main results of

[6]). We can then verify assertion (c) by explicitly comparing the tempered cohomology rings $A^*_{\mathbf{G}}(X)$ and $B^*_{\mathbf{G}}(X)$. For a more general π-finite space X, this approach breaks down (it seems unrealistic to hope for an explicit calculation of $A^*_{\mathbf{G}}(X)$ in general). We prove (c) in general in §7 as a consequence of our tempered ambidexterity theorem (Theorem 7.2.10), which ultimately rests on the calculations for Eilenberg-MacLane spaces carried out in this section.

Properties (a), (b), and (c) have many nontrivial consequences, some of which can be formulated without reference to the theory of $\mathbf{G}$-tempered cohomology. As noted in §1, they can be used to recover the "generalized character theory" of Hopkins-Kuhn-Ravenel and Stapleton (as well as the classical character theory of finite groups: see Corollary 4.7.8). For these applications, we do not need the full strength of (c): it suffices to assume that (c) holds for orbispaces X of the form $BG^{(-)}$, where G is a finite group. However, the full strength of assertion (c) allows us to extend the scope of character-theoretic methods. In §4.8, we make this explicit by using (a), (b), and (c) to compute the rationalized Lubin-Tate cohomology of an arbitrary π-finite space X (Corollary 4.8.5). As an application, we show that the Euler characteristic of X with respect to Morava K-theory $K(n)$ (at some prime number p) can be identified with the number of homotopy classes of maps from the p-adic torus $K(\mathbf{Z}_p^n, 1)$ into X (Corollary 4.8.6).

Throughout this section, we view the tempered cohomology theory $\mathsf{X} \mapsto A^*_{\mathbf{G}}(X)$ as a construct which depends on a choice of $\mathbf{P}$-divisible group $\mathbf{G}$ together with a preorientation $e \in \mathrm{Pre}(\mathbf{G}$. By virtue of Theorem 3.5.5, the datum of the pair $(\mathbf{G}, e)$ is equivalent to the datum of the $\mathbf{P}$-divisible functor

$$A_{\mathbf{G}} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$$

of Notation 4.0.1. Note that Constructions 4.0.3 and 4.0.5 are phrased directly in terms of the functor $A_{\mathbf{G}}$ (rather than the $\mathbf{P}$-divisible group $\mathbf{G}$ itself). Consequently, it is possible to adopt a more direct approach to our theory of tempered cohomology (circumventing the formalism of §2) by adopting Definition 3.5.3 as the definition of a preoriented $\mathbf{P}$-divisible group. Beware, however, that many important formal properties of tempered cohomology (like property (c) above) depend on the assumption that $\mathbf{G}$ is an *oriented* $\mathbf{P}$-divisible group. It is therefore desirable to have a criterion for determining if $e \in \mathrm{Pre}(\mathbf{G})$ is an orientation directly in terms of the functor $A_{\mathbf{G}} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$. We establish three such criteria in this section, each based on properties of the Atiyah-Segal comparison map ζ:

- Assume that A is p-complete. Then $\mathbf{G}$ is oriented if and only if A is complex periodic and the Atiyah-Segal comparison map

$$\zeta : A_{\mathbf{G}}^{BC_{p^n}} \to A^{BC_p^n}$$

exhibits $A^{BC_{p^n}}$ as the completion of $A_{\mathbf{G}}^{BC_{p^n}}$ with respect to the augmentation ideal $I_{C_{p^n}} \subseteq A^0_{\mathbf{G}}(BC_{p^n})$, for each $n \geqslant 0$ (Proposition 4.2.12). Here C_{p^n} denotes the cyclic group with p^n elements.

- Let A be any $\mathbb{E}_\infty$-ring. Then $\mathbf{G}$ is oriented if and only if, for every prime power p^n and every $\mathbb{E}_\infty$-algebra B over A, the Atiyah-Segal comparison map

  $$\zeta : B_{\mathbf{G}}^{BC_{p^n}} \to B^{BC_{p^n}}$$

  exhibits $B^{BC_{p^n}}$ as the completion of $B_{\mathbf{G}}^{BC_{p^n}}$ with respect to its augmentation ideal $I_{C_{p^n}} \subseteq A^0_{\mathbf{G}}(BC_{p^n})$ (Proposition 4.2.15).

- Let A be any $\mathbb{E}_\infty$-ring. Then $\mathbf{G}$ is oriented if and only if, for every prime number p, the Atiyah-Segal comparison map

  $$\zeta : A_{\mathbf{G}}^{BC_p} \to A^{BC_p}$$

  exhibits the function spectrum A^{BC_p} as the completion of $A_{\mathbf{G}}^{BC_p}$ with respect to the augmentation ideal $I_{C_p} \subseteq A^0_{\mathbf{G}}(BC_p)$, and the Tate construction A^{tC_p} is I_{C_p}-local (Theorem 4.6.2).

We can roughly paraphrase these results as saying that a preoriented **P**-divisible group **G** is oriented if and only if the theory of **G**-tempered cohomology satisfies an analogue of the Atiyah-Segal completion theorem in a few special cases. In §4.9, we prove a strong converse of this result: if A is Noetherian and **G** is oriented, then our theory of **G**-tempered cohomology satisfies a version of the Atiyah-Segal completion theorem in general (Theorem 4.9.2).

4.1 Equivariant K-Theory as Tempered Cohomology

Throughout this section, we let KU denote the complex K-theory spectrum and $\mu_{\mathbf{P}^\infty}$ the multiplicative **P**-divisible group over KU. We regard $\mu_{\mathbf{P}^\infty}$ as equipped with the orientation of Construction 2.8.6, so that Construction 4.0.3 supplies functors

$$\mathsf{X} \mapsto \mathrm{KU}_{\mu_{\mathbf{P}^\infty}}^{\mathsf{X}} \qquad \mathsf{X} \mapsto \mathrm{KU}^*_{\mu_{\mathbf{P}^\infty}}(\mathsf{X}).$$

It follows from Corollary 3.6.10 that we have equivalences (of $\mathbb{E}_\infty$-algebras over KU) $\mathrm{KU}^{T}_{\mu_{\mathbf{P}^\infty}} \simeq \mathsf{KU}(T)$ depending functorially on $T \in \mathscr{T}$; here $\mathsf{KU}(T)$ is the KU-algebra described in Notation 3.6.5. In particular, if G is a finite abelian group, then the tempered cohomology ring $\mathrm{KU}^{0}_{\mu_{\mathbf{P}^\infty}}(BG)$ can be identified with the representation ring $\mathrm{Rep}(G)$. In this section, we will extend this identification to the case where G is not assumed to be abelian. More generally, for any G-space X, we construct a canonical isomorphism

$$u_X : \mathrm{KU}^*_G(X) \simeq \mathrm{KU}^*_{\mu_{\mathbf{P}^\infty}}(X//G)$$

from the G-equivariant K-theory of X to the $\mu_{\mathbf{P}^\infty}$-tempered cohomology of the orbispace quotient $X//G$ (Theorem 4.1.2). The existence of the isomorphism u_X is more or less tautological in the case where G is abelian; the extension to non-abelian groups G will use the technique of *complex-oriented descent* appearing in the work of Hopkins-Kuhn-Ravenel ([5]).

Let G be a finite group (not necessarily abelian), which we regard as fixed for the remainder of this section. We begin with a brief review of G-equivariant complex K-theory (see [20] for a more detailed exposition). For every G-space X, we let $\mathrm{KU}^*_G(X)$ denote the graded ring given by G-equivariant complex K-theory of X. We then have an isomorphism $\mathrm{KU}^*_G(X) \simeq \pi_{-*}(\mathrm{KU}^X_G)$, where KU^X_G is an $\mathbb{E}_\infty$-algebra over KU which we will refer to as the G-equivariant complex K-theory *spectrum* of X. This construction has the following properties:

(*a*) The construction $X \mapsto \mathrm{KU}^X_G$ determines a functor of ∞-categories $\mathcal{S}^{\mathrm{op}}_G \to \mathrm{CAlg}_{\mathrm{KU}}$, where $\mathcal{S}_G$ denotes the ∞-category of G-spaces (Definition 3.2.10). Moreover, this functor carries small colimits in $\mathcal{S}_G$ to small limits in $\mathrm{CAlg}_{\mathrm{KU}}$.

(*b*) Let $\mathrm{Orbit}(G)$ denote the category of G-orbits, which (by slight abuse of notation) we identify with a full subcategory of $\mathcal{S}_G$. Then the composite functor

$$\mathrm{Orbit}(G)^{\mathrm{op}} \hookrightarrow \mathcal{S}^{\mathrm{op}}_G \xrightarrow{X \mapsto K^X_G} \mathrm{CAlg}_{\mathrm{KU}}$$

is given by the construction $X \mapsto \mathsf{KU}(X_{hG})$; here X_{hG} denotes the homotopy orbit space of X by the action of G and $\mathsf{KU}(X_{hG})$ is the $\mathbb{E}_\infty$-algebra of Notation 3.6.5. In particular, when $X = H\backslash G$ is the quotient of G by a subgroup $H \subseteq G$, we have equivalences

$$\mathrm{KU}^X_G \simeq \mathsf{KU}(BH) \qquad \mathrm{KU}^0_G(X) \simeq \mathrm{Rep}(H).$$

(*c*) Let X be a topological space equipped with a continuous action of G, and let us abuse notation by identifying X with the G-space $\mathrm{Sing}^G_\bullet(X) \in \mathcal{S}_G$ described in Example 3.2.13. Then there is a canonical map of sets

$$\{G\text{-equivariant complex vector bundles on } X\}/\text{isomorphism} \to \mathrm{KU}^0_G(X).$$

$$\mathscr{E} \mapsto [\mathscr{E}]$$

If X is a finite G-space, then this map exhibits $\mathrm{KU}^0_G(X)$ as the Grothendieck group of the commutative monoid of isomorphism classes of G-equivariant complex vector bundles on X.

(*d*) Let X be a topological space equipped with a continuous action of G, let $\mathscr{E}$ be a G-equivariant complex vector bundle of rank r over X, and let $Y = \mathbf{P}(\mathscr{E})$ denote the projectivization of $\mathscr{E}$, so that we have a G-equivariant map $\pi : Y \to X$ which exhibits Y as a fiber bundle over X (whose fibers are homeomorphic to the complex projective space $\mathbf{CP}^{r-1}$). We then have a tautological short exact sequence

$$0 \to \mathcal{O}(-1) \to \pi^*\mathscr{E} \to \mathscr{Q} \to 0$$

of complex vector bundles on Y, where $\mathcal{O}(-1)$ has rank 1. For each integer $d \in \mathbf{Z}$, let $\mathcal{O}(d)$ denote the $(-d)$th tensor power of $\mathcal{O}(-1)$. Let us abuse notation by identifying X and Y with the G-spaces $\mathrm{Sing}^G_\bullet(X), \mathrm{Sing}^G_\bullet(Y) \in \mathcal{S}_G$ of Example 3.2.13. Then the elements $\{[\mathcal{O}(d)]\}_{0 \leqslant d < r}$ form a basis for $\mathrm{KU}^*_G(Y)$ as a graded module over $\mathrm{KU}^*_G(X)$.

Remark 4.1.1. The functor $X \mapsto \mathrm{KU}^X_G$ is characterized by properties (*a*) and (*b*) above: it follows formally that the functor $\mathrm{KU}^{(-)}_G$ can be obtained as a right Kan extension of the functor

$$\mathrm{Orbit}(G)^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{KU}} \qquad X \mapsto \mathsf{KU}(X_{hG})$$

along the Yoneda embedding $\mathrm{Orbit}(G)^{\mathrm{op}} \hookrightarrow \mathcal{S}^{\mathrm{op}}_G$. From this perspective, one can obtain the comparison map

$$\{G\text{-equivariant complex vector bundles on } X\}/\text{isomorphism} \to \mathrm{KU}^0_G(X)$$

of (*c*) by formulating a more refined statement at the level of classifying spaces, and formally extending from the case where X is a G-orbit. The fact that, in good cases, this map exhibits $\mathrm{KU}^0_G(X)$ as the Grothendieck group of complex vector bundles on X

requires additional effort: in essence, one must show that these Grothendieck groups satisfy a form of excision ([20]). For our purposes here, this more refined statement is unnecessary: the construction $\mathscr{E} \mapsto [\mathscr{E}]$ is needed only to formulate property (d) (which follows from the fact that equivariant complex K-theory admits a good theory of Chern classes).

We can now formulate the main result of this section.

Theorem 4.1.2. *Let G be a finite group and let X be a G-space. Then there is a canonical equivalence*

$$\mathrm{KU}_G^X \simeq \mathrm{KU}_{\mu_{\mathbf{P}^\infty}}^{X//G}.$$

Here $X//G$ denotes the orbispace quotient of X by the action of G (Construction 3.2.16), and $\mathrm{KU}_{\mu_{\mathbf{P}^\infty}}^{X//G}$ is the tempered function spectrum of Construction 4.0.3.

Corollary 4.1.3. *Let G be a finite group and let X be a G-space. Then there is a canonical isomorphism of graded rings $\mathrm{KU}_G^*(X) \simeq \mathrm{KU}_{\mu_{\mathbf{P}^\infty}}^*(X//G)$.*

Example 4.1.4. Let G be a finite group. Applying Corollary 4.1.3 in the case where X is a single point (and restricting to degree zero), we obtain a canonical isomorphism $\mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{KU}_{\mu_{\mathbf{P}^\infty}}^0(BG)$.

Proof of Theorem 4.1.2. Let $\mathrm{Orbit}(G)_{\mathrm{ab}}$ denote the full subcategory of $\mathrm{Orbit}(G)$ spanned by G-orbits of the form $H\backslash G$, where $H \subseteq G$ is abelian. Let us abuse notation by identifying $\mathrm{Orbit}(G)_{\mathrm{ab}}$ with its image under the Yoneda embedding $\mathrm{Orbit}(G) \hookrightarrow \mathcal{S}_G$. When X belongs to $\mathrm{Orbit}(G)_{\mathrm{ab}}$, the homotopy X_{hG} is an object of the ∞-category $\mathscr{T}$ (which represents the orbispace quotient $X//G$), so property (b) and Corollary 3.6.10 provide a canonical equivalence

$$u_X : \mathrm{KU}_G^X \simeq \mathsf{KU}(X_{hG}) \simeq \mathrm{KU}_{\mu_{\mathbf{P}^\infty}}^{X//G}.$$

Note that the functor

$$\mathcal{S}_G^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{KU}} \qquad X \mapsto \mathrm{KU}_{\mu_{\mathbf{P}^\infty}}^{X//G}$$

is a right Kan extension of its restriction to $\mathrm{Orbit}(G)_{\mathrm{ab}}^{\mathrm{op}}$. Consequently, the construction $X \mapsto u_X$ admits an essentially unique extension to a natural transformation $u_X : \mathrm{KU}_G^X \to \mathrm{KU}_{\mu_{\mathbf{P}^\infty}}^{X//G}$ defined on the entire ∞-category $\mathcal{S}_G$. We will complete the proof by showing that u_X is an equivalence, for all $X \in \mathcal{S}_G$.

The construction $X \mapsto u_X$ carries colimits in $\mathcal{S}_G$ to limits in the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}_{\mathrm{KU}})$. It will therefore suffice to show that u_X is an equivalence in the special case where X is a G-orbit. Suppose otherwise: then there exists some subgroup $H \subseteq G$ for which the map $u_{H\backslash G}$ is not an equivalence. Among such subgroups, choose one for which the cardinality $|H|$ is as small as possible. The group H cannot be abelian, and therefore admits an irreducible representation V of dimension larger than 1. Then we can identify V with a G-equivariant vector bundle $\mathscr{E}$ on the orbit $X = H\backslash G$. Let $Y_0 = \mathbf{P}(\mathscr{E})$ denote the projectivization of the complex vector bundle $\mathscr{E}$ (regarded as a topological space with an action of G), and let us abuse notation by identifying Y_0 with the object $\mathrm{Sing}^G_\bullet(Y_0) \in \mathcal{S}_G$ given in Example 3.2.13. Let $Y_\bullet$ denote the Čech nerve of the projection map $Y_0 \to X$. We then have a commutative diagram of $\mathbb{E}_\infty$-algebras over KU

$$\begin{tikzcd}
\mathrm{KU}_G^X \arrow[r, "u_X"] \arrow[d, "v"] & \mathrm{KU}_{\mu_{\mathbf{P}^\infty}}(X//G) \arrow[d, "w"] \\
\mathrm{Tot}(\mathrm{KU}_G^{Y_\bullet}) \arrow[r, "\mathrm{Tot}(u_{Y_\bullet})"] & \mathrm{Tot}(\mathrm{KU}_{\mu_{\mathbf{P}^\infty}}^{Y_\bullet//G}).
\end{tikzcd}$$

We will obtain a contradiction by showing that the vertical maps and lower horizontal map in this diagram are equivalences:

- It follows from property (d) above that $\mathrm{KU}_G^{Y_0}$ is a faithfully flat KU_G^X-algebra and that $\mathrm{KU}_G^{Y_\bullet}$ is the cosimplicial KU_G^X-algebra given by the iterated tensor powers of $\mathrm{KU}_G^{Y_0}$. Consequently, the map v is an equivalence virtue of faithfully flat descent.

- Note that the simplicial orbispace $Y_\bullet//G$ can be identified with the Čech nerve (formed in the ∞-category of orbispaces) of the canonical map $\pi : Y_0//G \to X//G$. Consequently, to show that w is an equivalence, it will suffice to show that π is an effective epimorphism of orbispaces. Equivalently, we must show that for every abelian subgroup $A \subseteq G$ and every point $x \in X$ which is fixed by A, we can choose a point $y \in Y_0$ lying over x which is fixed by A. Without loss of generality, we may assume that $x \in H\backslash G$ is the identity coset, so that A is an abelian subgroup of H. In this case, the existence of the point $y \in Y_0^A$ is equivalent to the existence of a 1-dimensional complex subspace $L \subseteq V$ which is fixed by the action of A. This is clear: our assumption that A is abelian guarantees that the representation V decomposes as a direct sum of 1-dimensional representations of A.

- To show that the lower horizontal map is an equivalence, it will suffice to show that the map u_{Y_k} is an equivalence for each $k \geqslant 0$. Writing Y_k as a colimit of G-orbits of the form $H'\backslash G$, we are reduced to the problem of showing that $u_{H'\backslash G}$ is an equivalence whenever there exists a fixed point $y \in Y_k^{H'}$. Let x denote the image of y in the orbit $X = H\backslash G$. Replacing H' by a conjugate subgroup, we may assume that x is the identity coset, so that H' is a subgroup of H. It follows from our minimality assumption that $H' = H$. In this case, the existence of a fixed point $y \in Y_k^H$ lying over the identity coset $x \in H\backslash G$ implies that V contains a 1-dimensional complex subspace $L \subseteq V$ which is invariant under the action of H. Since the representation V is irreducible, it follows that $L = V$, contradicting our assumption that V has dimension > 1.

□

Remark 4.1.5. It follows from Theorem 4.1.2 that, when specialized to the multiplicative **P**-divisible group over KU, our theory of tempered cohomology can be used to reconstruct the equivariant complex K-theory as a *naive* G-spectrum: that is, as a cohomology theory defined on the homotopy category of G-spaces. To recover equivariant complex K-theory as a *genuine* G-spectrum, there is additional work to be done: essentially, one must show that the equivalence $\mathrm{KU}_G^X \simeq \mathrm{KU}_{\mu_{\mathbf{P}^\infty}}^{X//G}$ behaves functorially not only with respect to pullback, but also with respect to transfers. We will return to this point in [10] (see §7.4 for a discussion of transfer maps in the setting of tempered cohomology).

4.2 Atiyah-Segal Comparison Maps

Let G be a finite group and let X be a finite G-space. Then every G-equivariant vector bundle on X determines a vector bundle on the homotopy orbit space X_{hG}. This construction determines a map of K-groups

$$\mathrm{KU}_G^0(X) \to \mathrm{KU}^0(X_{hG}),$$

which is the subject of Atiyah's completion theorem (Theorem 1.1.5). In this section, we describe a variant of this construction in the more general setting of tempered cohomology, and prove a weak version of Atiyah's theorem (Proposition 4.2.8); for a stronger statement, we refer the reader to §4.9.

Let A be an $\mathbb{E}_\infty$-ring. For any space X, we let A^X denote the function spectrum of (unpointed) maps from X into A. The construction $X \mapsto A^X$ determines a functor of ∞-categories $\mathcal{S}^{\mathrm{op}} \to \mathrm{CAlg}_A$, which is determined (up to a contractible space of choices) by the requirement that it preserves small limits and carries the one-point space $*$ to A (Theorem HTT.5.1.5.6). If $\mathbf{G}$ is a preoriented $\mathbf{P}$-divisible group over A, then the functor $X \mapsto A_{\mathbf{G}}^{\underline{X}}$ has the same properties. This proves the following:

Proposition 4.2.1. *Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Then, for any space X, we have a canonical equivalence $A_{\mathbf{G}}^{\underline{X}} \simeq A^X$ of $\mathbb{E}_\infty$-algebras over A; here $\underline{X}$ denotes the constant orbispace associated to X (Example 3.1.8). Passing to homotopy groups, we obtain a canonical isomorphism of graded rings $A_{\mathbf{G}}^*(\underline{X}) \simeq A^*(X)$.*

We now exploit Proposition 4.2.1 to compare our theory of $\mathbf{G}$-tempered cohomology with the usual cohomology theory represented by A.

Construction 4.2.2 (The Atiyah-Segal Comparison Map)**.** Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. Let X be any orbispace, and let $X = |\mathsf{X}|$ denote its underlying space. Then the canonical map of orbispaces $\underline{X} \to \mathsf{X}$ induces a map of $\mathbb{E}_\infty$-algebras $\zeta : A_{\mathbf{G}}^{\mathsf{X}} \to A^{|\mathsf{X}|}$. Passing to homotopy groups, we obtain a map of cohomology rings $A_{\mathbf{G}}^*(\mathsf{X}) \to A^*(|\mathsf{X}|)$, which we will also denote by ζ. We will refer to both of the maps

$$\zeta : A_{\mathbf{G}}^{\mathsf{X}} \to A^{|\mathsf{X}|} \qquad \zeta : A_{\mathbf{G}}^*(\mathsf{X}) \to A^*(|\mathsf{X}|)$$

as the *Atiyah-Segal comparison map.*

In particular, for every object $X \in \mathcal{S}$, the canonical map of orbispaces $\underline{X} \to X^{(-)}$ induces Atiyah-Segal comparison maps

$$\zeta : A_{\mathbf{G}}^{X} \to A^{X} \qquad \zeta : A_{\mathbf{G}}^*(X) \to A^*(X).$$

Example 4.2.3. Let X be a finite space. Then the canonical map $\underline{X} \to X^{(-)}$ is an equivalence of orbispaces (by Miller's theorem; see Remark 3.1.14). It follows that, for any preoriented $\mathbf{P}$-divisible group $\mathbf{G}$ over an $\mathbb{E}_\infty$-ring A, the Atiyah-Segal comparison map $\zeta : A_{\mathbf{G}}^*(X) \to A^*(X)$ is an isomorphism.

Example 4.2.4. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. If H is a finite group and X is an H-space, then we have canonical maps of orbispaces $\underline{X_{hH}} \to X//H \to X_{hH}^{(-)}$, which induce comparison maps

$$A_{\mathbf{G}}^{X_{hH}} \to A_{\mathbf{G}}^{X//H} \to A^{X_{hH}} \qquad A_{\mathbf{G}}^*(X_{hH}) \to A_{\mathbf{G}}^*(X//H) \xrightarrow{\zeta} A^*(X_{hH}).$$

We have the following result (which contains Theorem 1.1.17 as a special case):

Theorem 4.2.5. *Fix a prime number p and a positive integer n. Let A be an $\mathbb{E}_\infty$-ring which is $K(n)$-local, and let $\mathbf{G} = \mathbf{G}_A^{\mathcal{Q}}$ be the Quillen p-divisible group of A (see §2.4). Then, for every orbispace X, the Atiyah-Segal comparison map $\zeta : A_{\mathbf{G}}^{\mathsf{X}} \to A^{|\mathsf{X}|}$ is an equivalence of $\mathbb{E}_\infty$-algebras over A, and therefore induces an isomorphism of graded rings $\zeta : A_{\mathbf{G}}^*(\mathsf{X}) \simeq A^*(|\mathsf{X}|)$.*

Proof. Note that the functor

$$\mathcal{OS}^{\mathrm{op}} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}_A) \qquad \mathsf{X} \mapsto (\zeta : A_{\mathbf{G}}^{\mathsf{X}} \to A^X)$$

preserves small limits. Since the ∞-category of orbispaces is generated (under small colimits) by the image of the Yoneda embedding $\mathscr{T} \hookrightarrow \mathcal{OS}$, it will suffice to prove Theorem 4.2.5 in the special case where $\mathsf{X} = BH^{(-)}$ is the classifying space of a finite abelian group H. In this case, the desired result is immediate from the construction $\mathbf{G}_A^{\mathcal{Q}}$ as an oriented p-divisible group (see Example 3.5.7). ☐

Theorem 4.2.5 has an analogue at height zero:

Variant 4.2.6. Let A be an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$ and let $\mathbf{G} = 0$ be the trivial $\mathbf{P}$-divisible group over A (so that $\mathbf{G}$ admits an essentially unique preorientation). Then, for every orbispace X with underlying space $|\mathsf{X}|$, the Atiyah-Segal comparison map $\zeta : A_{\mathbf{G}}^{\mathsf{X}} \to A^{|\mathsf{X}|}$ is an equivalence of $\mathbb{E}_\infty$-algebras over A, and therefore induces an isomorphism of graded rings $\zeta : A_{\mathbf{G}}^*(\mathsf{X}) \simeq A^*(|\mathsf{X}|)$.

Proof. As in the proof of Theorem 4.2.5, we can reduce to the case where $\mathsf{X} = T^{(-)}$ is representable by an object $T \in \mathscr{T}$. In this case, we are reduced to showing that the unit map $A^*(\{*\}) \to A^*(T)$ is an isomorphism. This is clear, since A is an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$ and the space T is rationally acyclic. ☐

The terminology of Construction 4.2.2 is motivated by the special case where $A = \mathrm{KU}$ is the complex K-theory spectrum and $\mathbf{G} = \mu_{\mathbf{P}^\infty}$ is the multiplicative $\mathbf{P}$-divisible group over A, endowed with the orientation of Construction 2.8.6. If G is a finite group and X is an G-space, then the Atiyah-Segal comparison map $\zeta : A_{\mathbf{G}}^*(X//G) \to A^*(|X//G|)$ can be identified with the map $\mathrm{KU}_G^*(X) \to \mathrm{KU}^*(X_{hG})$ appearing in Theorem 1.1.5. When X is a finite G-complex, this map exhibits $\mathrm{KU}^*(X_{hG})$ as the completion of $\mathrm{KU}_G^*(X)$ with respect to the augmentation ideal in the representation ring $\mathrm{Rep}(G)$. We will show in §4.9 that an analogous phenomenon occurs for *any* oriented $\mathbf{P}$-divisible group, at least when A is Noetherian (Theorem 4.9.2). For the moment, we consider only the special case where G is abelian and X is a single point, in which case the Noetherian assumption on A is unnecessary.

Notation 4.2.7. Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, and let H be a finite group. Then the canonical map $EH \to BH$ determines a surjective ring homomorphism

$$A^0_{\mathbf{G}}(BH) \to A^0_{\mathbf{G}}(EH) \simeq \pi_0(A).$$

We will denote the kernel of this homomorphism by I_H and refer to it as the *augmentation ideal* of the commutative ring $A^0_{\mathbf{G}}(BH)$.

Note that if the group H is abelian, then $A^0_{\mathbf{G}}(BH)$ is a projective module of finite rank as a module over $\pi_0(A)$. In this case, the augmentation ideal I_H is also projective of finite rank as a module over $\pi_0(A)$; in particular, it is finitely generated.

Proposition 4.2.8. *Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A, and let H be a finite abelian group. Then the Atiyah-Segal comparison map $A^{BH}_{\mathbf{G}} \to A^{BH}$ exhibits the function spectrum A^{BH} as the I_H-completion of $A^{BH}_{\mathbf{G}}$, where I_H is the augmentation ideal of Notation 4.2.7.*

The proof of Proposition 4.2.8 will require some preliminaries.

Notation 4.2.9. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. Let $f : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings, and let $\mathbf{G}_B$ denote the preoriented $\mathbf{P}$-divisible group over B obtained from $\mathbf{G}$ by extending scalars along f. For every orbispace X, we will denote the $\mathbb{E}_\infty$-ring $B^{\mathsf{X}}_{\mathbf{G}_B}$ of Construction Construction 4.0.3 simply by $B^{\mathsf{X}}_{\mathbf{G}}$, and we denote the $\mathbf{G}_B$-tempered cohomology ring $B^*_{\mathbf{G}_B}(\mathsf{X})$ simply by $B^*_{\mathbf{G}}(\mathsf{X})$. In the special case where $\mathsf{X} = X^{(-)}$ for some space X (Example 3.1.6), we denote $B^{\mathsf{X}}_{\mathbf{G}}$ and $B^*_{\mathbf{G}}(\mathsf{X})$ by $B^X_{\mathbf{G}}$ and $B^*_{\mathbf{G}}(X)$, respectively.

Remark 4.2.10. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and let B be an $\mathbb{E}_\infty$-algebra over A. For any orbispace X, we have a canonical map of A-modules $A^{\mathsf{X}}_{\mathbf{G}} \to B^{\mathsf{X}}_{\mathbf{G}}$, which extends to a B-linear map $\theta_{\mathsf{X}} : B \otimes_A A^{\mathsf{X}}_{\mathbf{G}} \to B^{\mathsf{X}}_{\mathbf{G}}$. Then:

- (a) If T is an object of $\mathscr{T}$, then the map $\theta_T : B \otimes_A A^T_{\mathbf{G}} \to B^T_{\mathbf{G}}$ is an equivalence.
- (b) If B is perfect as an A-module, then the map $\theta_{\mathsf{X}} : B \otimes_A A^{\mathsf{X}}_{\mathbf{G}} \to B^{\mathsf{X}}_{\mathbf{G}}$ is an equivalence for every orbispace X.

Assertion (a) is immediate from the definition of the $\mathbf{P}$-divisible group $\mathbf{G}_B$. Assertion (b) follows from (a) by writing the orbispace X as a colimit of representable orbispaces.

Lemma 4.2.11. *Let $\mathbf{G}$ be an preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Then, for every orbispace X, the functor*

$$\mathrm{CAlg}_A \to \mathrm{CAlg}_A \qquad B \mapsto B_{\mathbf{G}}^{\mathsf{X}}$$

preserves small limits.

Proof. Writing X as a colimit of representable functors, we may assume that $\mathsf{X} = BH^{(-)}$ for some finite abelian group H. In this case, Remark 4.7.2 implies that for each $B \in \mathrm{CAlg}_A$, the comparison map

$$\rho : B \otimes_A A_{\mathbf{G}}^{\mathsf{X}} \to B_{\mathbf{G}}^{\mathsf{X}}$$

is an equivalence. The desired result now follows from the observation that $A_{\mathbf{G}}^{BH}$ is a finite flat A-module (representing the functor $\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\widehat{H}, \mathbf{G}(\bullet))$, where $\widehat{H}$ is the Pontryagin dual of H). □

Proof of Proposition 4.2.8. Let B be an $\mathbb{E}_\infty$-algebra over A. We will say that B is *good* if the Atiyah-Segal comparison map $\zeta_B : B_{\mathbf{G}}^{BH} \to B^{BH}$ exhibits B^{BH} as the I_H-completion of $B_{\mathbf{G}}^{BH}$ (in other words, B is good if Proposition 4.2.8 is true after replacing A by B; note that the augmentation ideal of $B_{\mathbf{G}}^{0}(BH)$ is generated by the image of I_H). Note that the construction $B \mapsto \zeta_B$ preserves small limits (Lemma 4.2.11). It follows that the collection of good objects of CAlg_A is closed under small limits. We will prove that every object $B \in \mathrm{CAlg}_A$ is good.

The proof proceeds in several steps. Let $m = |H|$ denote the order of the finite group H. We first treat the case where m is invertible in B. In this case, the classifying space BH is acyclic with respect to the spectrum B: that is, evaluation at the base point of BH induces an equivalence $B^{BH} \simeq B$. It follows that we can identify the comparison map ζ_B with the augmentation map $\epsilon : B_{\mathbf{G}}^{BH} \to B$. Since $|H|$ is invertible in $\pi_0(B)$, $B_{\mathbf{G}}^{BH}$ is an étale B-algebra, so that ϵ is the projection onto a direct factor and the result is clear.

Now let B be an arbitrary $\mathbb{E}_\infty$-algebra over A. For each prime number p which divides $n = |H|$, let $B_{(p)}^{\wedge}$ denote the p-completion of B. We then have a pullback square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \prod_{p \mid m} B_{(p)}^{\wedge} \arrow[d] \\ B[\frac{1}{m}] \arrow[r] & (\prod_{p \mid m} B_{(p)}^{\wedge})[\frac{1}{m}], \end{tikzcd}$$

where the algebras on the bottom left and bottom right are good by virtue of the previous step. Consequently, to show that B is good, it will suffice to show that each completion $B^{\wedge}_{(p)}$ is good.

Replacing A by $B^{\wedge}_{(p)}$, we are reduced to proving Proposition 4.2.8 in the special case where A is p-complete for some prime number p. Our assumption that $\mathbf{G}$ is oriented guarantees that A is complex periodic and that the identity component of $\mathbf{G}_{(p)}$ is the Quillen formal group. For each integer n, let $\mathfrak{I}^A_n$ denote the nth Landweber ideal of A (Definition Or.4.4.11). Note that there exists an integer $n \geqslant 0$ such that A is $\mathfrak{I}^A_{n+1}$-local (for example, if n is an upper bound for the height of the p-divisible group $\mathbf{G}_{(p)}$, then we have $\mathfrak{I}^A_{n+1} = \pi_0(A)$). We proceed by induction on n, the case $n = 0$ being trivial (since $\mathfrak{I}^A_{n+1} = (p)$). Let $\widehat{A}$ denote the $\mathfrak{I}^A_n$-completion of A, let B denote the $\mathfrak{I}^A_n$-localization of A, and let $\widehat{B}$ denote the $\mathfrak{I}^A_n$-completion of B. Then we have a pullback square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \widehat{A} \arrow[d] \\
B \arrow[r] & \widehat{B},
\end{tikzcd}$$

where B and $\widehat{B}$ are good by virtue of our inductive hypothesis. Consequently, to show that A is good, it will suffice to show that $\widehat{A}$ is good. Replacing A by $\widehat{A}$, we are reduced to proving Proposition 4.2.8 in the special case where A is $\mathfrak{I}^A_{n+1}$-local and $\mathfrak{I}^A_n$-complete: that is, when A is $K(n)$-local as a spectrum (see Theorem Or.4.5.2).

If A is $K(n)$-local, then the orientation of $\mathbf{G}$ supplies a short exact sequence of p-divisible groups

$$0 \to \mathbf{G}^{\mathcal{Q}}_A \to \mathbf{G}_{(p)} \to \mathbf{G}'' \to 0,$$

where $\mathbf{G}''$ is étale. In this case, the Atiyah-Segal comparison map $\zeta : A^{BH}_{\mathbf{G}} \to A^{BH} = A^{BH}_{\mathbf{G}^{\mathcal{Q}}_A}$ is given by the projection onto a direct factor, where the complementary factor is I_H-local. We are therefore reduced to proving that the function spectrum A^{BH} is I_H-complete as a module over $A^{BH}_{\mathbf{G}}$.

Let $\mathcal{C}$ be the full subcategory of $(\mathcal{OS})_{/BH^{(-)}}$ spanned by those maps $f : \mathsf{X} \to BH^{(-)}$ for which the induced map $A^{BH}_{\mathbf{G}} \to A^{\mathsf{X}}_{\mathbf{G}}$ exhibits $A^{\mathsf{X}}_{\mathbf{G}}$ as an I_H-complete module over $A^{BH}_{\mathbf{G}}$. We wish to show that $\mathcal{C}$ includes the tautological map $\underline{BH} \to BH^{(-)}$. In fact, we claim that $\mathcal{C}$ contains every object of the form $f : \underline{X} \to BH^{(-)}$, where X is a space. Writing X as a homotopy colimit of contractible spaces, we can reduce to the case where X is contractible, in which case f is equivalent to the base point inclusion $\{*\} \to BH$. We are therefore reduced to proving that the augmentation map

$\epsilon : A_{\mathbf{G}}^{BH} \to A$ exhibits A as an I_H-complete module over $A_{\mathbf{G}}^{BH}$, which is immediate from the definition. □

In the statement of Proposition 4.2.8, the assumption that $\mathbf{G}$ is oriented cannot be omitted. For example, if the preorientation $\Sigma(\mathbf{Q}_p / \mathbf{Z}_p) \to \mathbf{G}(A)$ is nullhomotopic, then the Atiyah-Segal comparison map factors as a composition $A_{\mathbf{G}}^{BH} \xrightarrow{\epsilon} A \to A^{BH}$, which cannot exhibit A^{BH} as the I_H-completion of $A_{\mathbf{G}}^{BH}$ except in the trivial case where the order of H is invertible in A. More generally, we will show that a preoriented $\mathbf{P}$-divisible group which satisfies Proposition 4.2.8 (in a sufficiently strong form) is automatically oriented (see Proposition 4.2.15 below). We begin by analyzing the p-complete case.

Proposition 4.2.12. *Let p be a prime number, let A be a p-complete $\mathbb{E}_\infty$-ring, and let $\mathbf{G}$ be a preoriented p-divisible group over A. Then $\mathbf{G}$ is oriented if and only if the following conditions are satisfied:*

(1) *The $\mathbb{E}_\infty$-ring A is complex periodic.*

(2) *For every integer $n \geqslant 0$, the Atiyah-Segal comparison map*

$$\zeta : A_{\mathbf{G}}^{BC_{p^n}} \to A^{BC_{p^n}}$$

exhibits $A^{BC_{p^n}}$ as the completion of $A_{\mathbf{G}}^{BC_{p^n}}$ with respect to the augmentation ideal $I_{C_{p^n}}$.

The proof of Proposition 4.2.12 will require some algebraic preliminaries.

Lemma 4.2.13. *Let R be a connective $\mathbb{E}_\infty$-ring, let M be an R-module which is n-truncated, and let $M_I^\wedge$ denote the completion of M with respect to some finitely generated ideal $I \subseteq \pi_0(R)$. Suppose that, locally on $|\operatorname{Spec}(R)|$, the ideal I can be generated by $\leqslant d$ elements. Then $M_I^\wedge$ is $(n+d)$-truncated.*

Proof. Choose elements $t_1, \ldots, t_m \in \pi_0(R)$ which generate the unit ideal, having the property that each $I[t_i^{-1}] \subseteq \pi_0(R[t_i^{-1}])$ is generated by $\leqslant k$ elements. For nonempty subset $S \subseteq \{1, \ldots, m\}$, let R_S be the R-algebra obtained by inverting the elements $\{t_i\}_{i \in S}$, and set $M_S = R_S \otimes_R M$. Then M can be realized as the limit $\varprojlim_S M_S$, so the I-completion of M is given by $\varprojlim_S (M_S)_I^\wedge$. It will therefore suffice to show that each $(M_S)_I^\wedge$ is $(n+k)$-truncated. Since R_S is flat over R, the module M_S is n-truncated. We can therefore replace R by R_S and M by M_S, and thereby reduce to the case where I is globally generated by $\leqslant d$ elements. In this case, the desired result follows from Proposition SAG.II.4.3.4.4 . □

Lemma 4.2.14. *Let R be a p-complete commutative ring and let $\mathbf{G}$ be p-divisible group over R for which the identity component $\mathbf{G}^{\circ}$ has dimension $d \geqslant 0$. Fix an integer $n > 0$, and write $\mathbf{G}[p^n] = \operatorname{Spec}(H)$, where H is a Hopf algebra which is finite flat over R. Then the augmentation ideal of H is locally generated by $\leqslant d$ elements.*

Proof. Note that the augmentation ideal $I \subseteq H$ is a projective R-module of finite rank, and is therefore finitely generated as an H-module. It will therefore suffice to show that, for any maximal ideal $\mathfrak{m} \subseteq H$, the ideal $I_{\mathfrak{m}} \subseteq H_{\mathfrak{m}}$ is generated by $\leqslant d$ elements. Set $\mathfrak{n} = \mathfrak{m} \cap R$, so that $\mathfrak{n}$ is a maximal ideal of R. By Nakayama's lemma, it will suffice to show that the quotient $I_{\mathfrak{m}}/\mathfrak{n}I_{\mathfrak{m}}$ is generated by $\leqslant d$ elements as a module over $H_{\mathfrak{m}}/\mathfrak{n}H_{\mathfrak{m}}$. We may therefore replace R by the residue field $k = R/\mathfrak{n}$ and thereby reduce to the case where $R = k$ is a field. If $I \neq \mathfrak{m}$, then $I_{\mathfrak{m}}$ is the unit ideal in $H_{\mathfrak{m}}$ and there is nothing to prove. Otherwise, we have $I = \mathfrak{m}$. By Nakayama's lemma, it will suffice to show that the quotient $I/\mathfrak{m}I = I/I^2$ has dimension $\leqslant d$ as a vector space over k. This is clear, since I/I^2 can be identified with the Zariski cotangent space of the formal group $\mathbf{G}^{\circ}$. $\square$

Proof of Proposition 4.2.12. Let A be a p-complete $\mathbb{E}_{\infty}$-ring and let $\mathbf{G}$ be a preoriented p-divisible group over A. If $\mathbf{G}$ is oriented, then conditions (1) and (2) are satisfied by virtue of Propositions 2.5.6 and 4.2.8, respectively. For the converse, assume that (1) and (2) are satisfied; we wish to show that $\mathbf{G}$ is oriented. Let $\widehat{\mathbf{G}}^{\mathcal{Q}}_{A}$ denote the Quillen formal group A (Construction Or.4.1.13) and let $\mathbf{G}^{\circ}$ be the identity component of $\mathbf{G}$ (Definition Or.2.0.10), so that the preorientation of $\mathbf{G}$ can be identified with a map of formal groups $e : \widehat{\mathbf{G}}^{\mathcal{Q}}_{A} \to \mathbf{G}^{\circ}$; we wish to show that e is an equivalence (Proposition Or.4.3.23). Let us abuse notation by identifying $\widehat{\mathbf{G}}^{\mathcal{Q}}_{A}$ and $\mathbf{G}^{\circ}$ with formal groups over the connective cover $\tau_{\geqslant 0}(A)$. Then the underlying formal hyperplanes of $\widehat{\mathbf{G}}^{\mathcal{Q}}_{A}$ and $\mathbf{G}^{\circ}$ can be written as $\operatorname{Spf}(\mathcal{O}_{\widehat{\mathbf{G}}^{\mathcal{Q}}_{A}})$ and $\operatorname{Spf}(\mathcal{O}_{\mathbf{G}^{\circ}})$, respectively, where $\mathcal{O}_{\widehat{\mathbf{G}}^{\mathcal{Q}}_{A}}$ and $\mathcal{O}_{\mathbf{G}^{\circ}}$ are connective adic $\mathbb{E}_{\infty}$-algebras over $\tau_{\geqslant 0}(A)$. Then e induces a map $e^{*} : \mathcal{O}_{\mathbf{G}^{\circ}} \to \mathcal{O}_{\widehat{\mathbf{G}}^{\mathcal{Q}}_{A}}$, and we wish to show that e^{*} is an equivalence of $\mathbb{E}_{\infty}$-algebras over $\tau_{\geqslant 0}(A)$ (it is then automatically a map of adic $\mathbb{E}_{\infty}$-algebras, since the topologies on $\pi_0(\mathcal{O}_{\mathbf{G}^{\circ}})$ and $\pi_0(\mathcal{O}_{\widehat{\mathbf{G}}^{\mathcal{Q}}_{A}})$ are determined by their augmentation ideals). For each $n \geqslant 0$, let $I(n) = I_{C_{p^n}}$ denote the augmentation ideal in the tempered cohomology ring $A^{0}_{\mathbf{G}}(BC_{p^n})$, and let $(\tau_{\geqslant 0}(A_{\mathbf{G}}^{BC_{p^n}}))^{\wedge}_{I(n)}$ denote the $I(n)$-completion of $\tau_{\geqslant 0}(A_{\mathbf{G}}^{C_{p^n}})$. Then

e^* fits into a commutative diagram of $\mathbb{E}_\infty$-algebras

$$\begin{tikzcd}
\mathcal{O}_{\mathbf{G}^\circ} \arrow[r, "e^*"] \arrow[d] & \tau_{\geqslant 0}(A^{\mathbf{CP}^\infty}) \arrow[d] \\
\varprojlim_n (\tau_{\geqslant 0}(A_{\mathbf{G}}^{BC_{p^n}}))^{\wedge}_{I(n)} \arrow[r] & \varprojlim_n \tau_{\geqslant 0}(A^{BC_{p^n}})
\end{tikzcd}$$

where the inverse limits are formed in the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}(A)}$, the left vertical map is an equivalence by the construction of $\mathbf{G}^\circ$, the right vertical map is an equivalence by virtue of our assumption that A is p-complete, and the bottom horizontal map can be realized as a limit (indexed by nonnegative integers n) of the composite maps

$$(\tau_{\geqslant 0}(A_{\mathbf{G}}^{BC_{p^n}}))^{\wedge}_{I(n)} \xrightarrow{\rho(n)} \tau_{\geqslant 0}((A_{\mathbf{G}}^{BC_{p^n}})^{\wedge}_{I(n)}) \xrightarrow{\zeta(n)} \tau_{\geqslant 0}(A^{BC_{p^n}}),$$

where each $\zeta(n)$ is induced by the Atiyah-Segal comparison map determined by the preorientation e and is therefore an equivalence by virtue of assumption (2). Lemma 4.2.14 guarantees that there exists an integer $d \gg 0$ such that each of the ideals $I(n)$ is locally generated by at most d elements, so that the completion $(\tau_{\leqslant -1}(A_{\mathbf{G}}^{BC_{p^n}}))^{\wedge}_{I(n)}$ is $(d-1)$-truncated (Lemma 4.2.13). Using the fiber sequence

$$(\tau_{\geqslant 0}(A_{\mathbf{G}}^{BC_{p^n}}))^{\wedge}_{I(n)} \to (A_{\mathbf{G}}^{BC_{p^n}})^{\wedge}_{I(n)} \to (\tau_{\leqslant -1}(A_{\mathbf{G}}^{BC_{p^n}}))^{\wedge}_{I(n)},$$

we deduce that $\rho(n)$ induces an isomorphism on homotopy groups in degrees $\geqslant d$. Passing to the inverse limit over n, we conclude that the map

$$e^* : \mathcal{O}_{\mathbf{G}^\circ} \to \mathcal{O}_{\hat{\mathbf{G}}^{\mathcal{Q}}_A} = \tau_{\geqslant 0}(A^{\mathbf{CP}^\infty})$$

induces an isomorphism on homotopy groups in degrees $\geqslant d$, and therefore in all degrees (since A is assumed to be complex periodic, and both $\mathcal{O}_{\mathbf{G}^\circ}$ and $\mathcal{O}_{\mathbf{G}^{\mathcal{Q}}_A}$ can be realized as the duals of projective modules over $\tau_{\geqslant 0}(A)$). □

We now prove a variant of Proposition 4.2.12, where we do not assume that the $\mathbb{E}_\infty$-ring A is complex periodic or p-complete.

Proposition 4.2.15. *Let $\mathbf{G}$ be an preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Then $\mathbf{G}$ is oriented if and only if it satisfies the following condition:*

(∗) *For every prime power p^n and every $\mathbb{E}_\infty$-algebra B over A, the Atiyah-Segal comparison map*

$$\zeta : B_{\mathbf{G}}^{BC_{p^n}} \to B^{BC_{p^n}}$$

exhibits $B^{BC_{p^n}}$ as the completion of $B_{\mathbf{G}}^{BC_{p^n}}$ with respect to the augmentation ideal $I_{C_{p^n}}$.

Remark 4.2.16. The statement of Proposition 4.2.15 is potentially confusing, because it does not specify whether we view the ideal $I_{C_{p^n}}$ as an ideal of the tempered cohomology ring $A^0_{\mathbf{G}}(BC_{p^n})$ or the tempered cohomology ring $B^0_{\mathbf{G}}(BC_{p^n})$. However, this does not matter: the augmentation ideal of $B^0_{\mathbf{G}}(BC_{p^n})$ is generated by the image of the augmentation ideal of $A^0_{\mathbf{G}}(BC_{p^n})$.

Remark 4.2.17. We will prove another variant of Proposition 4.2.15 in §4.6 (see Theorem 4.6.2).

Proof of Proposition 4.2.15. If $\mathbf{G}$ is oriented, then it remains oriented after extending scalars along any map $A \to B$, and therefore satisfies condition $(*)$ by virtue of Proposition 4.2.8. Conversely, assume that $(*)$ is satisfied; we wish to show that $\mathbf{G}$ is oriented. Without loss of generality, we may assume that A is p-complete. Let $\mathbf{G}^\circ$ denote the identity component of the p-divisible group $\mathbf{G}_{(p)}$. Factoring A as a direct product, we may assume without loss of generality that the formal group $\mathbf{G}^\circ$ has some constant dimension d. Suppose first that $d \neq 1$. In this case, we claim that the $\mathbb{E}_\infty$-ring A vanishes. Let MP denote the periodic complex bordism spectrum, let B denote the smash product $\mathrm{MP} \otimes_S A$, and let $\widehat{B}$ denote the p-completion of B. Then $\widehat{B}$ is complex periodic, so the preoriented p-divisible group $\mathbf{G}_{(p)\widehat{B}}$ is oriented by virtue of Proposition 4.2.12. Consequently, after extending scalars from A to $\widehat{B}$, the formal group $\mathbf{G}^\circ$ is equivalent to the Quillen formal group of $\widehat{B}$, and therefore has dimension 1. It follows that the ring spectrum $\widehat{B}$ vanishes. Set $M = \mathrm{cofib}(p : A \to A)$ and let $\mathrm{End}_A(M)$ denote the algebra of endomorphisms of M. Then $M \otimes_A B \simeq \mathrm{cofib}(p : B \to B)$ vanishes, so that $\mathrm{End}_B(M \otimes_A B) \simeq \mathrm{MP} \otimes_S \mathrm{End}_A(M)$ vanishes. It follows from the nilpotence theorem that $\mathrm{End}_A(M) \simeq 0$, so that $M \simeq 0$ and therefore the map $p : A \to A$ is an equivalence of A-modules. Since A is assumed to be p-complete, we conclude that $A \simeq 0$ as desired.

We now treat the case where $d = 1$. Let $\omega = \omega_{\mathbf{G}^\circ}$ denote the dualizing line of the formal group $\mathbf{G}^\circ$ (Definition Or.4.2.14), and let $\beta : \omega^\circ_{\mathbf{G}} \to \Sigma^{-2}(A)$ denote the Bott map associated to the preorientation of $\mathbf{G}$ (Construction Or.4.3.7). We wish to show that β is an equivalence. Let N denote the tensor product $M \otimes_A \mathrm{cofib}(\beta)$, where M is defined as above. Since $\mathbf{G}_{(p)\widehat{B}}$ is oriented, the tensor product $\widehat{B} \otimes_A N \simeq B \otimes AN$ vanishes. In particular, the endomorphism algebra $\mathrm{End}_B(B \otimes_A N) \simeq \mathrm{MP} \otimes_S \mathrm{End}_A(N)$ vanishes. Invoking the nilpotence theorem again, we conclude that $\mathrm{End}_A(N) \simeq 0$, so that $N \simeq 0$. By construction, we have a cofiber sequence of A-modules

$$\mathrm{cofib}(\beta) \xrightarrow{p} \mathrm{cofib}(\beta) \to N,$$

so multiplication by p induces an equivalence from $\mathrm{cofib}(\beta)$ to itself. However, the cofiber $\mathrm{cofib}(\beta)$ is a perfect module over the p-complete $\mathbb{E}_\infty$-ring A, and is therefore p-complete. It follows that $\mathrm{cofib}(\beta)$ vanishes, so that the Bott map $\beta : \omega^{\circ}_{\mathbf{G}} \to \Sigma^{-2}(A)$ is an equivalence. ☐

4.3 Character Isomorphisms

Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. Our goal in this section is to describe the $\mathbf{G}$-tempered cohomology functor $\mathsf{X} \mapsto A^*_{\mathbf{G}}(\mathsf{X})$ in the case where $\mathbf{G}$ splits as a direct sum $\mathbf{G}_0 \oplus \underline{\Lambda}$, where $\underline{\Lambda}$ is the constant $\mathbf{P}$-divisible group over A associated to a colattice Λ (see Construction 2.7.5). As an application, we give a construction of the equivariant Chern character appearing in the formulation of Theorem 1.1.2. Our starting point is the following observation:

Proposition 4.3.1. *Let $\mathbf{G}_0$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let Λ be a colattice (Definition 2.7.1), and set $\mathbf{G} = \mathbf{G}_0 \oplus \underline{\Lambda}$. Then the functors*

$$(T \in \mathscr{T}^{\mathrm{op}}) \mapsto (A^T_{\mathbf{G}} \in \mathrm{CAlg}_A) \qquad (T \in \mathscr{T}^{\mathrm{op}}) \mapsto (A^{T^{B\widehat{\Lambda}}}_{\mathbf{G}_0} \in \mathrm{CAlg}_A)$$

are equivalent.

Proof. For each object $T \in \mathscr{T}$, evaluation at the base point of the classifying space $B\widehat{\Lambda}$ determines a map of spaces $\mathrm{ev} : T^{B\widehat{\Lambda}} \to T$. Since T is the classifying space of an abelian group, the evaluation map ev restricts to a homotopy equivalence on each connected component of the mapping space $T^{B\widehat{\Lambda}}$, and therefore induces a homotopy equivalence

$$T^{B\widehat{\Lambda}} \simeq T \times \pi_0(T^{B\widehat{\Lambda}}) = T \times \mathrm{Hom}(\widehat{\Lambda}, \pi_1(T)).$$

We therefore obtain equivalences

$$\begin{aligned} A^{B\widehat{\Lambda}}_{\mathbf{G}_0} &\simeq A^T_{\mathbf{G}_0} \otimes_A A^{\mathrm{Hom}(\widehat{\Lambda}, \pi_1(T))} \\ &\simeq A^T_{\mathbf{G}_0} \otimes_A A^T_{\underline{\Lambda}} \\ &\simeq A^T_{\mathbf{G}}, \end{aligned}$$

depending functorially on T. ☐

Theorem 4.3.2. *Let $\mathbf{G}_0$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let Λ be a colattice, and set $\mathbf{G} = \mathbf{G}_0 \oplus \underline{\Lambda}$. Then, for any orbispace X, there is a canonical equivalence $A^{\mathsf{X}}_{\mathbf{G}} \simeq A^{\mathcal{L}^{\Lambda}(\mathsf{X})}_{\mathbf{G}_0}$.*

Proof. By definition, the functor

$$F : \mathcal{OS}^{\mathrm{op}} \to \mathrm{CAlg}_A \qquad \mathsf{X} \mapsto A_{\mathbf{G}}^{\mathsf{X}}$$

is characterized (up to equivalence) by the following properties:

(a) The composition of F with the Yoneda embedding $\mathscr{T}^{\mathrm{op}} \hookrightarrow \mathcal{OS}^{\mathrm{op}}$ is equivalent to the functor $A_{\mathbf{G}}$.

(b) The functor F carries small colimits of orbispaces to limits in the ∞-category CAlg_A.

It will therefore suffice to show that the functor

$$\mathcal{OS}^{\mathrm{op}} \xrightarrow{\mathcal{L}^{\Lambda}} \mathcal{OS}^{\mathrm{op}} \xrightarrow{\mathsf{X} \mapsto A_{\mathbf{G}_0}^{\mathsf{X}}} \mathrm{CAlg}_A$$

also has properties (a) and (b). Property (a) follows from Proposition 4.3.1 (and Proposition 3.4.7), while (b) follows from the fact that the formal loop functor $\mathcal{L}^{\Lambda} : \mathcal{OS} \to \mathcal{OS}$ preserves small colimits (Remark 3.4.6). $\square$

Notation 4.3.3 (Character Maps)**.** Let $\mathbf{G}_0$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let Λ be a colattice (Definition 2.7.1), and set $\mathbf{G} = \mathbf{G}_0 \oplus \underline{\Lambda}$. For any orbispace X, we let

$$\chi : A_{\mathbf{G}}^{\mathsf{X}} \to A_{\mathbf{G}_0}^{\mathcal{L}^{\Lambda}(\mathsf{X})}$$

denote the equivalence constructed in the proof of Theorem 4.3.2. We will refer to χ as the *character map*. Passing to homotopy groups, we obtain an isomorphism of tempered cohomology rings

$$\chi : A_{\mathbf{G}}^{*}(\mathsf{X}) \to A_{\mathbf{G}_0}^{*}(\mathcal{L}^{\Lambda}(\mathsf{X}))$$

which we will also refer to as the character map (and denote by the same symbol χ).

From Theorem 4.3.2 we obtain the following stronger version of Theorem 1.1.18:

Corollary 4.3.4. *Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A which splits as a direct sum $\mathbf{G}_0 \oplus \underline{\Lambda}$. Let H be a finite group and let $X \in \mathcal{S}_H$ be an H-space, and let $Y = \coprod_{\alpha : \widehat{\Lambda} \to H} X^{\mathrm{im}(\alpha)}$ be the H-space appearing in Example 3.4.5. Then there is a canonical equivalence $\chi : A_{\mathbf{G}}^{X//H} \simeq A_{\mathbf{G}_0}^{Y//H}$ of $\mathbb{E}_\infty$-algebras over A. In particular, there is a canonical isomorphism of tempered cohomology ringstempered cohomology rings*

$$\chi : A_{\mathbf{G}}^{*}(X//H) \simeq A_{\mathbf{G}_0}^{*}((\coprod_{\alpha : \widehat{\Lambda} \to H} X^{\mathrm{im}(\alpha)})//H).$$

Proof. Combine Theorem 4.3.2 with Example 3.4.5. □

We now specialize Theorem 4.3.2 to the setting of complex K-theory.

Notation 4.3.5. Let KU denote the complex K-theory spectrum. We let $\mathrm{KU}_{\mathbf{Q}}$ denote the $\mathbb{E}_\infty$-ring given by the smash product $\mathbf{Q} \otimes_S \mathrm{KU}$. The homotopy ring of this smash product is given by

$$\pi_*(\mathrm{KU}_{\mathbf{Q}}) \simeq \mathbf{Q} \otimes_{\mathbf{Z}} \pi_*(\mathrm{KU}) \simeq \mathbf{Q}[\beta^{\pm 1}],$$

where β denotes the Bott element of $\pi_2(\mathrm{KU})$. It follows that, as an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$, $\mathrm{KU}_{\mathbf{Q}}$ is freely generated by an invertible element of homological degree 2. The spectrum $\mathrm{KU}_{\mathbf{Q}}$ represents the 2-periodic version of rational cohomology, whose value on a space X is given concretely by the formula

$$\mathrm{KU}^*_{\mathbf{Q}}(X) = \mathrm{H}^*(X; \mathbf{Q})((\beta^{-1}))$$

The canonical map $\mathrm{KU} \to \mathbf{Q} \otimes_S \mathrm{KU} = \mathrm{KU}_{\mathbf{Q}}$ induces a map of cohomology theories, which (when evaluated on a space X) is the classical Chern character map

$$\mathrm{ch} : \mathrm{KU}^*(X) \to \mathrm{H}^*(X; \mathbf{Q})((\beta^{-1})).$$

Replacing $\mathbf{Q}$ by the larger field $\mathbf{C}$ of complex numbers in the above discussion, we obtain complexified K-theory spectrum $\mathrm{KU}_{\mathbf{C}} = \mathbf{C} \otimes_S \mathrm{KU}$, and complexified Chern character $\mathrm{ch} : \mathrm{KU}^*(X) \to \mathrm{H}^*(X; \mathbf{C})((\beta^{-1}))$.

Construction 4.3.6 (The Orbispace Chern Character)**.** Let KU denote the complex K-theory spectrum. Let $\mu_{\mathbf{P}^\infty}$ denote the multiplicative $\mathbf{P}$-divisible group, which we regard as an oriented $\mathbf{P}$-divisible group over KU (Construction 2.8.6). After extending scalars to the complexification $\mathrm{KU}_{\mathbf{C}} = \mathbf{C} \otimes_S \mathrm{KU}$, we have an equivalence of $\mathbf{P}$-divisible groups

$$\exp : \underline{\mathbf{Q}/\mathbf{Z}} \to \mu_{\mathbf{P}^\infty} \qquad \lambda \mapsto \exp(2\pi i \lambda).$$

For any orbispace X, Theorem 4.3.2 and Variant 4.2.6 supply equivalences

$$(\mathrm{KU}_{\mathbf{C}})^{\mathsf{X}}_{\mu_{\mathbf{P}^\infty}} \simeq (\mathrm{KU}_{\mathbf{C}})^{\mathsf{X}}_{\underline{\mathbf{Q}/\mathbf{Z}}} \simeq \mathrm{KU}_{\mathbf{C}}^{|\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(\mathsf{X})|}$$

Composing with the tautological map $\mathrm{KU}^{\mathsf{X}}_{\mu_{\mathbf{P}^\infty}} \to (\mathrm{KU}_{\mathbf{C}})^{\mathsf{X}}_{\mu_{\mathbf{P}^\infty}}$ and passing to homotopy groups, we obtain a map

$$\mathrm{ch} : \mathrm{KU}^*_{\mu_{\mathbf{P}^\infty}}(\mathsf{X}) \to \mathrm{H}^*(|\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(\mathsf{X})|; \mathbf{C})((\beta^{-1})),$$

which we will refer to as the *orbispace Chern character*.

Example 4.3.7. Let X be a space and let $\mathsf{X} = \underline{X}$ be the constant orbispace associated to X. Then the orbispace Chern character map

$$\mathrm{ch} : \mathrm{KU}^*_{\mu_{\mathbf{P}^\infty}}(\mathsf{X}) \to \mathrm{H}^*(|\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(\mathsf{X})|; \mathbf{C})((\beta^{-1}))$$

reduces to the classical (complexified) Chern character of Notation 4.3.5.

Example 4.3.8 (The Equivariant Chern Character)**.** Let G be a finite group and let X be a G-space. Combining the orbispace Chern character of Construction 4.3.6 to the orbispace quotient $\mathsf{X} = X//G$ with Theorem 4.1.2 (and using the description of $\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(\mathsf{X})$ supplied by Example 3.4.5), we obtain a map

$$\mathrm{ch}_G : \mathrm{KU}^*_G(X) \simeq \mathrm{KU}^*_{\mu_{\mathbf{P}^\infty}}(X//G) \to= \mathrm{H}^*((\coprod_{g \in G} X^g)_{hG}; \mathbf{C})((\beta^{-1}))$$

which we will refer to as the *equivariant Chern character*.

Example 4.3.9. Let G be a finite group. Applying Example 4.3.8 in the case where X is a point (and restricting to cohomological degree zero), we obtain a map

$$\mathrm{ch}_G : \mathrm{Rep}(G) \to \{\text{Class functions } f : G \to \mathbf{C}\}.$$

We claim that this map carries the class $[V]$ of a representation V to the character

$$\chi_V : G \to \mathbf{C} \qquad \chi(g) = \mathrm{tr}(g|_V).$$

By functoriality, it suffices to prove this when G is abelian (or even when G is a cyclic group, since a class function on G is determined by its restriction to cyclic subgroups of G). In this case, we may assume without loss of generality that V is a 1-dimensional representation of G, whose character is given by $\chi_V(g) = \exp(2\pi i \lambda(g))$ for some element λ of the Pontryagin dual group $\widehat{G}$. The desired equality now follows from fact that the isomorphism of $\mathbf{P}$-divisible groups $\underline{\mathbf{Q}/\mathbf{Z}} \simeq \mu_{\mathbf{P}^\infty}$ over $\mathrm{KU}_{\mathbf{C}} = \mathbf{C} \otimes_S \mathrm{KU}$ is also given by the exponential map $\lambda \mapsto \exp(2\pi i \lambda)$.

4.4 Tempered Cohomology of Eilenberg-MacLane Spaces

For every finite abelian group H, let $\widehat{H} = \mathrm{Hom}(H, \mathbf{Q}/\mathbf{Z})$ denote the Pontryagin dual group of H. If $\mathbf{G}$ is a $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, we let $\mathbf{G}[\widehat{H}]$ denote the functor

$$\mathrm{CAlg}_A \to \mathcal{S} \qquad B \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\widehat{H}, \mathbf{G}(B)).$$

If $\mathbf{G}$ is preoriented, then the tempered cohomology theory $A_{\mathbf{G}}$ is related to $\mathbf{G}$ by the existence of equivalences

$$\operatorname{Spec}(A_{\mathbf{G}}^{K(H,1)}) \simeq \mathbf{G}[\widehat{H}],$$

depending functorially on H. If $\mathbf{G}$ is *oriented*, then there is an analogous description of the tempered cohomology of Eilenberg-MacLane spaces $K(H,d)$ for *every* nonnegative integer d:

Theorem 4.4.1. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $d \geqslant 0$ be an integer. Then there exists a $\mathbf{P}$-divisible group $\mathbf{G}^{(d)}$ over A equipped with equivalences*

$$\operatorname{Spec}(A_{\mathbf{G}}^{K(H,d)}) \simeq \mathbf{G}^{(d)}[\widehat{H}],$$

depending functorially on H.

Remark 4.4.2. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $d \geqslant 0$ be an integer. Theorem 4.4.1 is equivalent to the assertion that the functor

$$\mathrm{Ab}_{\mathrm{fin}} \to \mathrm{CAlg}_A \qquad M \mapsto A_{\mathbf{G}}^{K(\widehat{M},d)}$$

is $\mathbf{P}$-divisible, in the sense of Definition 3.5.1. More concretely, this is equivalent to the following three assertions;

(a) For every finite abelian group H, the tempered function spectrum $A_{\mathbf{G}}^{K(H,d)}$ is a projective A-module of finite rank.

(b) For every pair of finite abelian groups H and H', the canonical map

$$A_{\mathbf{G}}^{K(H,d)} \otimes_A A_{\mathbf{G}}^{K(H',d)} \to A_{\mathbf{G}}^{K(H \times H',d)}$$

is an equivalence.

(c) For every short exact sequence of finite abelian groups

$$0 \to H' \to H \to H'' \to 0,$$

the associated diagram of tempered function spectra

$$\begin{tikzcd}
A_{\mathbf{G}}^{K(H'',d)} \arrow[r] \arrow[d] & A_{\mathbf{G}}^{K(H,d)} \arrow[d] \\
A \arrow[r] & A_{\mathbf{G}}^{K(H',d)}
\end{tikzcd}$$

is a pushout diagram, and the horizontal maps are finite flat of nonzero degree.

Example 4.4.3. Theorem 4.4.1 holds for $d = 1$, and the $\mathbf{P}$-divisible group $\mathbf{G}^{(1)}$ can be identified with $\mathbf{G}$. This is essentially immediate from the construction of $\mathbf{G}$-tempered cohomology (and does not require the assumption that $\mathbf{G}$ is oriented).

Example 4.4.4. Theorem 4.4.1 holds for $d = 0$, and $\mathbf{G}^{(0)}$ can be identified with the constant $\mathbf{P}$-divisible group $\underline{\mathbf{Q}/\mathbf{Z}}$.

Remark 4.4.5. In the situation of Theorem 4.4.1, one can think of the $\mathbf{P}$-divisible group $\mathbf{G}^{(d)}$ as a kind of "dth exterior power" of $\mathbf{G}$ (see Remark 4.4.18 below).

Warning 4.4.6. In the situation of Theorem 4.4.1, the $\mathbf{P}$-divisible groups $\mathbf{G}^{(d)}$ are not generally not oriented for $d \neq 1$ (for example, they are generally not 1-dimensional after completing at some prime number p). However, they carry an analogous structure: since the functor $X \mapsto \mathrm{Spec}(A^{X}_{\mathbf{G}})$ is functorial for *unpointed* maps between Eilenberg-MacLane spaces, each $\mathbf{G}^{(d)}$ is equipped with a map $e_d : \Sigma^d(\mathbf{Q}/\mathbf{Z}) \to \mathbf{G}^{(d)}(A)$, which specializes to the preorientation of $\mathbf{G} = \mathbf{G}^{(1)}$ when $d = 1$ (this follows from a variant of Theorem 3.5.5).

Our goal for the rest of this section is to formulate a more precise version of Theorem 4.4.1 (which we will prove in §4.5). We begin with a few general remarks. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. For every pair of orbispaces X and Y, the projection maps $\mathsf{X} \leftarrow \mathsf{X} \times \mathsf{Y} \rightarrow \mathsf{Y}$ determine morphisms of $\mathbb{E}_\infty$-algebras $A^{\mathsf{X}}_{\mathbf{G}} \to A^{\mathsf{X}\times\mathsf{Y}}_{\mathbf{G}} \leftarrow A^{\mathsf{Y}}_{\mathbf{G}}$, which we can assemble into a single map

$$m : A^{\mathsf{X}}_{\mathbf{G}} \otimes_A A^{\mathsf{Y}}_{\mathbf{G}} \to A^{\mathsf{X}\times\mathsf{Y}}_{\mathbf{G}}.$$

Proposition 4.4.7 (Tempered Künneth Formula)**.** *Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Let X and Y be orbispaces. If either $A^{\mathsf{X}}_{\mathbf{G}}$ or $A^{\mathsf{Y}}_{\mathbf{G}}$ is perfect as an A-module spectrum, then the multiplication map*

$$m : A^{\mathsf{X}}_{\mathbf{G}} \otimes_A A^{\mathsf{Y}}_{\mathbf{G}} \to A^{\mathsf{X}\times\mathsf{Y}}_{\mathbf{G}}.$$

is an equivalence.

Proof. Assume that $A^{\mathsf{Y}}_{\mathbf{G}}$ is perfect as an A-module spectrum. Regarding the orbispace Y as fixed and allowing X to vary, we note that the functors

$$\mathsf{X} \mapsto A^{\mathsf{X}}_{\mathbf{G}} \otimes_A A^{\mathsf{Y}}_{\mathbf{G}} \qquad \mathsf{X} \mapsto A^{\mathsf{X}\times\mathsf{Y}}_{\mathbf{G}}$$

carry colimits in the ∞-category $\mathcal{OS}$ to limits in the ∞-category CAlg_A. Since the ∞-category of orbispaces is generated under small colimits by the image of the Yoneda embedding, we may assume without loss of generality that $\mathsf{X} = T^{(-)}$ for some $T \in \mathscr{T}$. Under this assumption, we claim that m is an equivalence for *every* orbispace Y. Repeating the above argument (with $\mathsf{X} = T^{(-)}$ fixed and allowing Y to vary), we can reduce to the case where $\mathsf{Y} = T'^{(-)}$ for some object $T' \in \mathscr{T}$. In this case, the desired result follows from the definition of a $\mathbf{P}$-divisible functor $\mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A$ (Definition 3.5.3). □

Corollary 4.4.8. *Let* $\mathbf{G}$ *be a preoriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. Let* X *and* Y *be orbispaces. If either* X *or* Y *is representable by an object of* $\mathscr{T}$*, then the multiplication map*

$$m : A_{\mathbf{G}}^{\mathsf{X}} \otimes_A A_{\mathbf{G}}^{\mathsf{Y}} \to A_{\mathbf{G}}^{\mathsf{X}\times\mathsf{Y}}.$$

is an equivalence.

Under a suitable flatness assumption, Proposition 4.4.7 supplies a Künneth formula at the level of tempered cohomology *groups*.

Corollary 4.4.9. *Let* $\mathbf{G}$ *be a preoriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. Let* X *and* Y *be orbispaces, and suppose that* $A_{\mathbf{G}}^{\mathsf{X}}$ *is a projective* A*-module of finite rank. Then the multiplication map of Proposition 4.4.7 induces an isomorphism*

$$A_{\mathbf{G}}^{0}(\mathsf{X}) \otimes_{\pi_0(A)} A_{\mathbf{G}}^{*}(\mathsf{Y}) \to A_{\mathbf{G}}^{*}(\mathsf{X} \times \mathsf{Y}).$$

Corollary 4.4.10. *Let* $\mathbf{G}$ *be a preoriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. Let* X *and* Y *be orbispaces. If* $\mathsf{X} = T^{(-)}$ *is the orbispace represented by an object* $T \in \mathscr{T}$*, then the multiplication map of Proposition 4.4.7 induces an isomorphism*

$$A_{\mathbf{G}}^{0}(\mathsf{X}) \otimes_{\pi_0(A)} A_{\mathbf{G}}^{*}(\mathsf{Y}) \to A_{\mathbf{G}}^{*}(\mathsf{X} \times \mathsf{Y}).$$

We now recall some algebraic constructions from [6].

Notation 4.4.11. Let R be a commutative ring and let G be a finite flat commutative group scheme over R. For every integer $d \geqslant 0$, we let $\mathrm{Skew}_G^{(d)}$ denote the group scheme over R given in Definition Ambi.3.2.9 , so that we can identify R-valued points of $\mathrm{Skew}_G^{(d)}$ with maps

$$G \times_{\mathrm{Spec}(R)} \cdots \times_{\mathrm{Spec}(R)} G \to \mathbf{G}_m$$

which are multilinear and skew-symmetric (in particular, $\mathrm{Skew}_G^{(1)}$ is the Cartier dual of G). If H is the R-linear dual of the ring of functions on G, then we can identify $\mathrm{Skew}_G^{(d)}$ with a closed subscheme of the affine space $\mathrm{Spec}(\mathrm{Sym}_R^*(H^{\otimes d}))$ (which parametrizes *all* maps from $G \times_{\mathrm{Spec}(R)} \cdots \times_{\mathrm{Spec}(R)} G$ to the affine line).

We let $\mathrm{Alt}^{(d)}_G \subseteq \mathrm{Skew}^{(d)}_G$ denote the subgroup scheme given by Construction Ambi.3.2.11 (so that $\mathrm{Alt}^{(d)}_G = \mathrm{Skew}^{(d)}_G$ whenever multiplication by 2 is an isomorphism from G to itself).

Definition 4.4.12. Let p be a prime number and let X be a p-finite space. We will say that X is *split* if it can be written as a finite product of spaces of the form $K(H, m)$, where H is a finite abelian p-group and m is a nonnegative integer. In other words, X is split if it is a generalized Eilenberg-MacLane space: that it, if it has the form $\Omega^{\infty}(M)$, where $M \in \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ is a $\mathbf{Z}$-module spectrum (which is necessarily truncated with p-power torsion homotopy groups, since X is p-finite). We let $\mathcal{S}^{\diamond}$ denote the full subcategory of $\mathcal{S}$ spanned by the split p-finite spaces.

Notation 4.4.13. Let p be a prime number and let $\mathbf{G}$ be a preoriented p-divisible group over an $\mathbb{E}_{\infty}$-ring A which satisfies the following condition:

(F) For every split p-finite space X, the tempered function spectrum $A^X_{\mathbf{G}}$ is a perfect A-module.

For every split p-finite space X, let $A^{\mathbf{G}}_*(X)$ denote the graded abelian group given by the formula

$$A^{\mathbf{G}}_*(X) = \pi_*((A^X_{\mathbf{G}})^{\vee}) = \mathrm{Ext}^{-*}_A(A^X_{\mathbf{G}}, A),$$

where $(A^X_{\mathbf{G}})^{\vee}$ denotes the A-linear dual of $A^X_{\mathbf{G}}$. We will refer to the groups $A^{\mathbf{G}}_*(X)$ as the *$\mathbf{G}$-tempered homology groups of X*.

Remark 4.4.14. Let p be a prime number and let $\mathbf{G}$ be a preoriented p-divisible group over an $\mathbb{E}_{\infty}$-ring A which satisfies condition (F) of Notation 4.4.13. Then, if X and Y are split p-finite spaces, the canonical map

$$A^X_{\mathbf{G}} \otimes_A A^Y_{\mathbf{G}} \to A^{X \times Y}_{\mathbf{G}}$$

is an equivalence (in fact, it suffices to assume that either one of the spaces X and Y is p-finite; see Proposition 4.4.7). It follows that the construction $X \mapsto (A^X_{\mathbf{G}})^{\vee}$ determines a symmetric monoidal functor from the ∞-category $\mathcal{S}^{\diamond}$ (with the symmetric monoidal structure given by Cartesian product) to the ∞-category Mod_A (with symmetric monoidal structure gives by $\otimes_A$). Passing to homotopy groups, we deduce that the tempered homology functor $X \mapsto A^{\mathbf{G}}_*(X)$ is *lax* symmetric monoidal (as a functor from the ∞-category $\mathcal{S}^{\diamond}$ to the ordinary category of graded $\pi_*(A)$-modules). In particular, the functor

$$\mathcal{S}^{\diamond} \to \mathrm{Mod}^{\heartsuit}_{\pi_0(A)} \qquad X \mapsto A^{\mathbf{G}}_0(X)$$

is also lax symmetric monoidal: that is, for every pair of split π-finite spaces X and Y, we have a canonical map

$$A_0^{\mathbf{G}}(X) \otimes_{\pi_0(A)} A_0^{\mathbf{G}}(Y) \to A_0^{\mathbf{G}}(X \times Y).$$

Construction 4.4.15. Let p be a prime number and let $\mathbf{G}$ be a preoriented p-divisible group over an $\mathbb{E}_\infty$-ring A which satisfies condition (F) of Notation 4.4.13. We let $\mathbf{G}^\heartsuit$ denote the underlying $\mathbf{P}$-divisible group over the ordinary commutative ring $\pi_0(A)$ and $\mathbf{G}^\heartsuit[p^t]$ the finite flat group scheme of p^t-torsion points $\mathbf{G}^\heartsuit$.

For every pair of nonnegative integers $d, t \geqslant 0$, we view the the Eilenberg-MacLane space $K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$ as a commutative monoid object of the ∞-category $\mathcal{S}^\diamond$ of split p-finite spaces. Applying Remark 4.4.14, we see that the $\mathbf{G}$-tempered homology group $A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))$ inherits the structure of a commutative algebra over $\pi_0(A)$. We denote its spectrum by $\mathrm{Spec}(A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d)))$, which we view as an affine scheme over $\pi_0(A)$. In the special case $d = 1$, we can view $\mathrm{Spec}(A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))$ as a finite flat group scheme over $\pi_0(A)$: it is the Cartier dual of the finite flat group scheme $\mathbf{G}^\heartsuit[p^t]$. In particular, each $A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, 1))$ has the structure of a (commutative and cocommutative) Hopf algebra over $\pi_0(A)$. The iterated cup product is classified by a map of split p-finite spaces $K(\mathbf{Z}/p^t\,\mathbf{Z}, 1)^d \to K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$ which induces a map of $\pi_0(A)$-modules

$$A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, 1))^{\otimes d} \to A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))$$

which extends to a map of $\pi_0(A)$-algebras

$$\mathrm{Sym}^*_{\pi_0(A)}(A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, 1))^{\otimes d}) \to A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d)).$$

Using the multilinearity and skew-symmetry of the cup product, we obtain a map of affine schemes

$$\rho_{d,t} : \mathrm{Spec}(A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))) \to \mathrm{Skew}^{(d)}_{\mathbf{G}^\heartsuit[p^t]}.$$

Theorem 4.4.16. *Let $\mathbf{G}$ be a preoriented p-divisible group over an $\mathbb{E}_\infty$-ring A which is oriented over the p-completion of A. Then:*

(1) *For every split p-finite space X, the tempered function spectrum $A_{\mathbf{G}}^X$ is a projective A-module of finite rank (in particular, $\mathbf{G}$ satisfies condition (F) of Notation 4.4.13).*

(2) *For every pair of integers $d, t \geqslant 0$, the map*

$$\rho_{d,t} : \mathrm{Spec}(A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))) \to \mathrm{Skew}^{(d)}_{\mathbf{G}^\heartsuit[p^t]}$$

of Construction 4.4.15 induces an isomorphism of $\mathrm{Spec}(A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\mathbf{Z}, d)))$ *with the subscheme* $\mathrm{Alt}^{(d)}_{\mathbf{G}^\heartsuit[p^t]} \subseteq \mathrm{Skew}^{(d)}_{\mathbf{G}^\heartsuit[p^t]}$.

To deduce Theorem 4.4.1 from Theorem 4.4.16, we will need one more elementary observation.

Lemma 4.4.17. *Let* $\mathbf{G}$ *be a preoriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*, let* S *be a set of prime numbers, and regard* $\mathbf{G}_{(S)} = \bigoplus_{p \in S} \mathbf{G}_{(p)}$ *as a direct factor of* $\mathbf{G}$ *(so that* $\mathbf{G}_{(S)}$ *inherits a preorientation). Then, for every* S*-finite space* X *(see Definition 1.1.25), the canonical map* $A^X_{\mathbf{G}_{(S)}} \to A^X_{\mathbf{G}}$ *is an equivalence.*

Proof. Let $\mathscr{T}_{(S)}$ denote the full subcategory of $\mathscr{T}$ spanned by those spaces of the form BH, where H is a finite abelian group whose prime divisors belong to S. By construction, the map of preoriented $\mathbf{P}$-divisible groups $\mathbf{G} \to \mathbf{G}_{(S)}$ induces an equivalence $A^T_{\mathbf{G}_{(S)}} \to A^T_{\mathbf{G}}$ for each object $T \in \mathscr{T}$. It follows that the map $A^{\mathsf{X}}_{\mathbf{G}_{(S)}} \to A^{\mathsf{X}}_{\mathbf{G}}$ is an equivalence whenever $\mathsf{X} : \mathscr{T}^{\mathrm{op}} \to \mathcal{S}$ is an orbispace which is a left Kan extension of its restriction to the full subcategory $\mathscr{T}^{\mathrm{op}}_{(S)} \subseteq \mathscr{T}^{\mathrm{op}}$. We conclude by observing that this condition is satisfied in the case where $\mathsf{X} = X^{(-)}$ is representable by an S-finite space X. □

Proof of Theorem 4.4.1 from Theorem 4.4.16. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. We wish to show that $\mathbf{G}$ satisfies conditions (a), (b), and (c) of Remark 4.4.2. Note that if p is a prime number, then the canonical map $A^X_{\mathbf{G}_{(p)}} \to A^X_{\mathbf{G}}$ is an equivalence for any p-finite space X (Lemma 4.4.17). Applying Theorem 4.4.16, we deduce that $A^{K(H,d)}_{\mathbf{G}}$ is a projective A-module of finite rank whenever H is a finite p-group. Assertions (a) and (b) now follow from the Künneth formula of Proposition 4.4.7. To prove (c), it will suffice to show that for every short exact sequence of finite abelian groups

$$0 \to H' \to H \to H'' \to 0,$$

the resulting sequence of finite flat group schemes

$$0 \to \mathrm{Spec}(A^0_{\mathbf{G}}(K(H', d))) \to \mathrm{Spec}(A^0_{\mathbf{G}}(K(H, d))) \to \mathrm{Spec}(A^0_{\mathbf{G}}(K(H'', d))) \to 0$$

is also short exact. Using (b), we can reduce to the case where H is a cyclic group of prime power order. Passing to Cartier duals and applying Theorem 4.4.16 to the p-divisible group $\mathbf{G}_{(p)}$, we are reduced to proving the exactness of sequences of the form

$$0 \to \mathrm{Alt}^{(d)}_{\mathbf{G}^\heartsuit_{(p)}[p^t]} \to \mathrm{Alt}^{(d)}_{\mathbf{G}^\heartsuit_{(p)}[p^{t+t'}]} \to \mathrm{Alt}^{(d)}_{\mathbf{G}^\heartsuit_{(p)}[p^{t'}]} \to 0,$$

which follows from Corollary Ambi.3.5.4 (since our assumption that $\mathbf{G}$ is oriented guarantees that the p-divisible group $\mathbf{G}^{\heartsuit}_{(p)}$ has dimension 1). $\square$

Remark 4.4.18. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let d be a nonnegative integer, and let $\mathbf{G}^{(d)}$ be the $\mathbf{P}$-divisible group which appears in the statement of Theorem 4.4.1. Then Theorem 4.4.16 supplies a complete description of the underlying classical $\mathbf{P}$-divisible group $\mathbf{G}^{(d)\heartsuit}$ over the commutative ring $\pi_0(A)$. In particular, it implies the following:

- If p is a prime number and the p-divisible group and the p-local summand $\mathbf{G}_{(p)}$ has height n, then the p-local summand $\mathbf{G}^{(d)}_{(p)}$ has height $\binom{n}{d}$ and dimension $\binom{n-1}{d-1}$ (see Corollary Ambi.3.5.4). In particular, the p-divisible group $\mathbf{G}^{(d)}_{(p)}$ vanishes for $d > n$.

- For every perfect field κ of characteristic p and every map $x : \mathrm{Spec}(\kappa) \rightarrow \mathrm{Spec}(\pi_0(A))$, the Dieudonné module of $\mathbf{G}^{(d)}$ at the point x can be identified with the dth exterior power of the Dieudonné module of $\mathbf{G}$ at the point x (see Theorem Ambi.3.3.1).

4.5 The Proof of Theorem 4.4.16

We devote this section to the proof of Theorem 4.4.16. Let $\mathbf{G}$ be a preoriented p-divisible group over an $\mathbb{E}_\infty$-ring A which is oriented after extending scalars to the p-completion of A. For every $\mathbb{E}_\infty$-algebra B over A, we let $\mathbf{G}_B$ denote the oriented p-divisible group obtained from $\mathbf{G}$ by extending scalars from A to B and $\mathbf{G}^{\heartsuit}_B$ the underlying classical p-divisible group over the commutative ring $\pi_0(B)$. We will say that B is *good* if it satisfies the following conditions:

($T1$) For every split p-finite space X, the tempered function spectrum $B^X_{\mathbf{G}}$ is a projective B-module of finite rank.

($T2$) For every pair of integers $d, t \geqslant 0$, the map

$$\rho^B_{d,t} : \mathrm{Spec}(B^{\mathbf{G}}_0(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))) \rightarrow \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}_B[p^t]}$$

of Construction 4.4.15 induces an isomorphism of schemes

$$\mathrm{Spec}(B^{\mathbf{G}}_0(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))) \simeq \mathrm{Alt}^{(d)}_{\mathbf{G}^{\heartsuit}_B[p^t]}.$$

To prove Theorem 4.4.16, we must show that A is good. Note that when A is a Lubin-Tate spectrum and $\mathbf{G}$ is the Quillen p-divisible group of A, this is one of the main theorems of [6] (Theorem Ambi.3.4.1). Our strategy is to reduce to the Lubin-Tate case by showing that the collection of good A-algebras has strong closure properties. We first observe that class of good A-algebras is closed under finite products. Consequently, we may assume without loss of generality that the p-divisible group $\mathbf{G}$ has some fixed height n. Proceeding by induction on n, we may assume that Theorem 4.4.16 holds for preoriented p-divisible groups of height $< n$. The case $n = 0$ is trivial (since the p-divisible group $\mathbf{G}$ vanishes and p is invertible in the commutative ring $\pi_0(A)$). We will therefore assume that $n > 0$.

Lemma 4.5.1. *Let $B \to B'$ be a morphism of $\mathbb{E}_\infty$-algebras over A and let X be a split p-finite space. If B and B' are good, then the canonical map $\theta : B' \otimes_B B^X_{\mathbf{G}} \to B'^X_{\mathbf{G}}$ is an equivalence.*

Proof. By virtue of the Künneth formula of Proposition 4.4.7, we may assume without loss of generality that X is an Eilenberg-MacLane space $K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$. Note that the domain and codomain of θ are projective B'-modules of finite rank. It will therefore suffice to show that the B'-linear dual of θ induces an isomorphism of commutative rings

$$B'^{\mathbf{G}}_0(X) \to \pi_0(B') \otimes_{\pi_0(B)} B^{\mathbf{G}}_0(X),$$

which follows from the description supplied by $(T2)$. □

Lemma 4.5.2. *Let B be an $\mathbb{E}_\infty$-algebra over A and let $B^\bullet$ be a flat hypercovering of B. If each B^k is good, then B is good.*

Proof. Let $\mathcal{C} = \varprojlim \mathrm{Mod}_{B^\bullet}$ denote the ∞-category of cosimplicial spectra $M^\bullet$ which are modules over $B^\bullet$, for which the canonical map $B^d \otimes_{B^{d'}} M^{d'} \to M^d$ are equivalences. According to Corollary SAG.D.7.7.7 , the canonical map

$$\mathrm{Mod}_B \to \varprojlim \mathrm{Mod}_{B^\bullet} = \mathcal{C}$$

is an equivalence of ∞-categories, with a homotopy inverse given by the functor $M^\bullet \mapsto \varprojlim M^\bullet$. Let X be a split p-finite space. Using Lemma 4.5.1, we see that the cosimplicial spectrum $B^{\bullet X}_{\mathbf{G}}$ can be identified with an object of $\mathcal{C}$, whose image in Mod_B is given by the totalization $\mathrm{Tot}(B^{\bullet X}_{\mathbf{G}}) \simeq \mathrm{Tot}(B^\bullet)^X_{\mathbf{G}} \simeq A^X_{\mathbf{G}}$ (where the first equivalence is supplied by Lemma 4.2.11). It follows that the canonical map $B^0 \otimes_B B^X_{\mathbf{G}} \to B^{0X}_{\mathbf{G}}$ is an equivalence for every split p-finite space X.

Since B^0 satisfies conditions $(T1)$, $B^{0X}_{\mathbf{G}}$ is a projective module of finite rank over B^0 for every split p-finite space X. Using faithfully flat descent, we deduce that $B^X_{\mathbf{G}}$ is a projective B-module of finite rank. This proves $(T1)$. To prove $(T2)$, it suffices to observe that the map of affine schemes

$$\rho^B_{d,t} : \mathrm{Spec}(B^{\mathbf{G}}_0(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))) \to \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}_B[p^t]}$$

factors through an isomorphism

$$\mathrm{Spec}(B^{\mathbf{G}}_0(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))) \simeq \mathrm{Alt}^{(d)}_{\mathbf{G}^{\heartsuit}_B[p^t]}$$

if and only if it does so after extending scalars along the faithfully flat map of commutative rings $\pi_0(B) \to \pi_0(B^0)$. □

Suppose that B is an $\mathbb{E}_\infty$-algebra over A which satisfies condition $(T1)$. If p is odd, then we have $\mathrm{Alt}^{(d)}_{\mathbf{G}^{\heartsuit}_B[p^t]} = \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}_B[p^t]}$, so Construction 4.4.15 directly supplies maps

$$\rho^B_{d,t} : \mathrm{Spec}(B^{\mathbf{G}}_0(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))) \to \mathrm{Alt}^{(d)}_{\mathbf{G}^{\heartsuit}_B[p^t]}$$

that we wish to prove are isomorphisms. When $p = 2$, the situation is a bit more complicated: it is not immediately obvious that the maps $\rho^B_{d,t}$ factor through $\mathrm{Alt}^{(d)}_{\mathbf{G}^{\heartsuit}_B[p^t]}$. To address this point, we will need some auxiliary constructions.

Notation 4.5.3. Fix an integer $d \geqslant 0$. Then we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}[p^3]} \arrow[r] \arrow[d] & \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}[p^2]} \arrow[r] \arrow[d] & \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}[p]} \arrow[r] \arrow[d] & \mathrm{Spec}(\pi_0(A)) \arrow[d, equal] \\
\cdots \arrow[r] & \mathrm{Alt}^{(d)}_{\mathbf{G}^{\heartsuit}[p^3]} \arrow[r] & \mathrm{Alt}^{(d)}_{\mathbf{G}^{\heartsuit}[p^2]} \arrow[r] & \mathrm{Alt}^{(d)}_{\mathbf{G}^{\heartsuit}[p]} \arrow[r] & \mathrm{Spec}(\pi_0(A))
\end{tikzcd}$$

of commutative group schemes over the commutative ring $\pi_0(A)$ where the vertical maps are monomorphisms. It follows from Corollary Ambi.3.5.4 that the upper horizontal maps in this diagram are finite flat of degree $p^{\binom{n}{d}}$. In particular, each $\mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}[p^t]}$ can be written as the spectrum of a commutative ring $R_{d,t}$ which is finite flat of degree $p^{t\binom{n}{d}}$ over the commutative ring $\pi_0(A)$. Consequently, the vertical maps in the preceding diagram are closed immersions. Write $\mathrm{Alt}^{(d)}_{\mathbf{G}^{\heartsuit}[p^t]} = \mathrm{Spec}(\overline{R}_{d,t})$ for some commutative algebra $\overline{R}_{d,t}$ (which need not be finite over $\pi_0(A)$ when $p = 2$), so that we have a commutative diagram of rings

$$\begin{tikzcd}
\cdots & R_{d,3} \arrow[l] & R_{d,2} \arrow[l] & R_{d,1} \arrow[l] & R_{d,0} \arrow[l] \\
\cdots \arrow[r, leftrightarrow] & \overline{R}_{d,3} \arrow[u] & \overline{R}_{d,2} \arrow[l] \arrow[u] & \overline{R}_{d,1} \arrow[l] \arrow[u] & \overline{R}_{d,0} \arrow[l] \arrow[u, "\sim"]
\end{tikzcd}$$

where the vertical maps exhibit each $R_{d,t}$ as the quotient of $\overline{R}_{d,t}$ by an ideal $J_{d,t} \subseteq \overline{R}_{d,t}$. Note that the exact sequences

$$0 \to J_{d,t} \to \overline{R}_{d,t} \to R_{d,t} \to 0$$

are automatically split in the category of $\pi_0(A)$-modules (since each $R_{d,t}$ is a projective module over $\pi_0(A)$). Choose a collection of splittings $s_{d,t} : R_{d,t} \to \overline{R}_{d,t}$ (so that each $s_{d,t}$ is a map of $\pi_0(A)$-modules). We assume that these splittings are chosen to be compatible as t varies, in the sense that each of the diagrams

$$\begin{tikzcd}
R_{d,t+1} \arrow[d, "s_{d,t+1}"] & R_{d,t} \arrow[l] \arrow[d, "s_{d,t}"] \\
\overline{R}_{d,t+1} & \overline{R}_{d,t} \arrow[l]
\end{tikzcd}$$

is commutative (note that it is always possible to arrange this, since the maps $R_{d,t} \to R_{d,t+1}$ are split monomorphisms in the category of $\pi_0(A)$-modules).

For each $d, t \geqslant 0$, fix an A-module $A_{d,t}$ which is projective of finite rank and an isomorphism $\pi_0(A_{d,t}) \simeq R_{d,t}$. Note that $A_{d,t}$ exists and is unique up to isomorphism as an object of the homotopy category of Mod_A (Corollary HA.7.2.2.19). We can even regard $A_{d,t}$ as a commutative algebra object of the homotopy category hMod_A, but we will not need this: we regard $A_{d,t}$ only as a module over A.

Remark 4.5.4. In the situation of Notation 4.5.3, each of the transition maps $\mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}[p^{t+1}]} \to \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}[p^{t}]}$ factors through the closed subscheme $\mathrm{Alt}^{(d)}_{\mathbf{G}[p^t]} \subseteq \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}[p^{t}]}$. This is tautological when p is odd, and follows from Lemma Ambi.3.3.8 when $p = 2$. In other words, each of the ring homomorphisms $\overline{R}_{d,t} \to \overline{R}_{d,t+1}$ annihilates the ideal $I_{d,t} \subseteq \overline{R}_{d,t}$.

Construction 4.5.5. Let B be an $\mathbb{E}_\infty$-algebra over A which satisfies the following weaker version of condition $(T1)$:

($T1'$) For every split p-finite space X, the tempered function spectrum $B_{\mathbf{G}}^{X}$ is a perfect B-module.

Under this assumption, we can apply Construction 4.4.15 to obtain maps

$$\rho_{d,t}^{B} : \mathrm{Spec}(B_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d)) \to \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}_{B}[p^t]},$$

which are classified by $\pi_0(A)$-algebra homomorphisms $\overline{R}_{d,t} \to B_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))$. For each $d, t \geqslant 0$, the composite map

$$R_{d,t} \xrightarrow{s_{d,t}} \overline{R}_{d,t} \to B_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))$$

can be lifted to a map of A-module spectra $\psi_{d,t} : A_{d,t} \to (B_{\mathbf{G}}^{K(\mathbf{Z}/p^t\,\mathbf{Z},d)})^{\vee}$, which is uniquely determined up to homotopy.

Lemma 4.5.6. *Let B be an $\mathbb{E}_\infty$-algebra over A. Then B is good if and only if it satisfies condition ($T1'$) together with the following:*

($T2'$) *For each $d, t \geqslant 0$, the map $\psi_{d,t} : A_{d,t} \to (B_{\mathbf{G}}^{K(\mathbf{Z}/p^t\,\mathbf{Z},d)})^{\vee}$ extends to an equivalence $B \otimes_A A_{d,t} \simeq (B_{\mathbf{G}}^{K(\mathbf{Z}/p^t\,\mathbf{Z},d)})^{\vee}$.*

Proof. It is easy to see that if B is good, then conditions ($T1'$) and ($T2'$) are satisfied (note that ($T1'$) is a weaker version of ($T1$) and ($T2'$) is a weaker version of ($T2$)). Conversely, suppose that B satisfies ($T1'$) and ($T2'$); we wish to show that it also satisfies ($T1$) and ($T2$). Without loss of generality, we may assume that $B = A$, so that ($T2'$) asserts that $(A_{\mathbf{G}}^{K(\mathbf{Z}/p^t\,\mathbf{Z},d)})^{\vee} \simeq A_{d,t}$ is a projective module of finite rank over A. Combining this with ($T1'$), we conclude that the tempered function spectrum $A_{\mathbf{G}}^{K(\mathbf{Z}/p^t\,\mathbf{Z},d)}$ itself is a projective module of finite rank over A. Applying the Künneth formula of Proposition 4.4.7, we deduce that $A_{\mathbf{G}}^{X}$ is projective of finite rank for every split p-finite space X. This proves ($T1$). To prove ($T2$), let us identify each of the maps $\rho_{d,t} : \mathrm{Spec}(A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))) \to \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}[p^t]}$ with a ring homomorphism $u_{d,t} : \overline{R}_{d,t} \to A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d))$. We wish to show that $u_{d,t}$ annihilates the ideal $I_{d,t}$ of Notation 4.5.3 and induces an isomorphism

$$\overline{R}_{d,t}/I_{d,t} \simeq R_{d,t} \to A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d)).$$

The second assertion is immediate from assumption ($T2'$). To prove the first, we observe that there is a commutative diagram of $\pi_0(A)$-modules

$$\begin{tikzcd}
R_{d,t} \arrow[r, "s_{d,t}"] \arrow[d] & \overline{R}_{d,t} \arrow[r, "u_{d,t}"] \arrow[d] & A_0^{\mathbf{G}}(K(\mathbf{Z}/p^t\,\mathbf{Z}, d)) \arrow[d] \\
R_{d,t+1} \arrow[r, "s_{d,t+1}"] & \overline{R}_{d,t+1} \arrow[r, "u_{d,t+1}"] & A_0^{\mathbf{G}}(K(\mathbf{Z}/p^{t+1}\,\mathbf{Z}, d))
\end{tikzcd}$$

where the left vertical map is a monomorphism, the middle vertical map annihilates the ideal $I_{d,t}$ (Remark 4.5.4), and the horizontal composites are isomorphisms (by assumption $(T2')$). It follows that the right vertical map is also a monomorphism, so that $u_{d,t}$ must also annihilate the ideal $I_{d,t}$. $\square$

Lemma 4.5.7. *Let B be an $\mathbb{E}_\infty$-algebra over A, let $I \subseteq \pi_0(B)$ be a finitely generated ideal. Suppose that, for each element $x \in I$, the localization $B[x^{-1}]$ is good. If B is I-local, then B is good.*

Proof. Choose a finite sequence $x_1, \ldots, x_m \in I$ of generators for the ideal I. For every subset $J \subseteq \{1, \ldots, m\}$, set $x_J = \prod_{j \in J} x_j$, and set $B_J = B[x_J^{-1}]$. Let P denote the partially ordered set of all nonempty subsets of $\{1, \ldots, m\}$. For each $1 \leqslant j \leqslant m$, let P_j denote the set $\{J \in P : j \in J\}$.

Our assumption that B is I-local implies that the canonical map $B \to \varprojlim_{J \in P} B_J$ is an equivalence. In particular, for every space X, the canonical map $B^X_{\mathbf{G}} \to \varprojlim_{J \in P} B^X_{J\mathbf{G}}$ is an equivalence (Lemma 4.2.11). Fix an element $j \in \{1, \ldots, m\}$, so that we have an equivalence

$$B^X_{\mathbf{G}}[x_j^{-1}] \to \varprojlim_{J \in P} B^X_{J\mathbf{G}}[x_j^{-1}].$$

By assumption, for each $J \in P$, the A-algebras B_J and $B_{J \cup \{j\}}$ are good. Applying Lemma 4.5.1, we conclude that for every split p-finite space X, the canonical map

$$B^X_{J\mathbf{G}}[x_j^{-1}] \simeq B_{J \cup \{j\}} \otimes_{B_J} B^X_{J\mathbf{G}} \to (B_{J \cup \{j\}})^X_{\mathbf{G}} = (B_{J \cup \{j\}})^X_{\mathbf{G}}[x_j^{-1}]$$

is an equivalence. It follows that the functor $J \mapsto B^X_{J\mathbf{G}}[x_j^{-1}]$ is a right Kan extension of its restriction to P_j. Since P_j contains the set $\{j\}$ as an initial object, we conclude that the restriction map

$$\varprojlim_{J \in P} B^X_{J\mathbf{G}}[x_j^{-1}] \to B^X_{\{j\}\mathbf{G}}[x_j^{-1}] \simeq B[x_j^{-1}]^X_{\mathbf{G}}$$

is an equivalence. It follows that the natural map $B^X_{\mathbf{G}}[x_j^{-1}] \to B[x_j^{-1}]^X_{\mathbf{G}}$ is an equivalence for $1 \leqslant j \leqslant m$.

We now show that B satisfies the criterion of Lemma 4.5.6. We first verify condition $(T1')$. Let X be a split p-finite space; we wish to prove that $B^X_{\mathbf{G}}$ is a compact object of the ∞-category Mod_B. Equivalently, we wish to show that for every filtered diagram $\{M_\alpha\}$ in Mod_B having colimit M, the canonical map

$$\theta : \varinjlim \mathrm{Map}_{\mathrm{Mod}_B}(B^X_{\mathbf{G}}, M_\alpha) \to \mathrm{Map}_{\mathrm{Mod}_B}(B^X_{\mathbf{G}}, M)$$

is a homotopy equivalence. Since filtered colimits in $\mathcal{S}$ commute with finite limits, we can write θ as the limit of a diagram of maps

$$\theta_J : \varinjlim \mathrm{Map}_{\mathrm{Mod}_B}(B^X_{\mathbf{G}}, B_J \otimes_B M_\alpha) \to \mathrm{Map}_{\mathrm{Mod}_B}(B^X_{\mathbf{G}}, B_J \otimes_B M),$$

where J ranges over the nonempty subsets of $\{1, \ldots, m\}$. Using the first part of the proof, we can identify each θ_J with the canonical map

$$\varinjlim \mathrm{Map}_{\mathrm{Mod}_{B_J}}(B^X_{J\mathbf{G}}, B_J \otimes_B M_\alpha) \to \mathrm{Map}_{\mathrm{Mod}_{B_J}}(B^X_{J\mathbf{G}}, B_J \otimes_B M),$$

which is an equivalence by virtue of our assumption that each B_J satisfies $(T1')$.

We now verify $(T2')$. Choose $d, t \geqslant 0$, and let $\psi_{d,t} : A_{d,t} \to (B^X_{\mathbf{G}})^\vee$ be as in Construction 4.5.5. We wish to show that $\psi_{d,t}$ induces an equivalence $B \otimes_A A_{d,t} \to (B^X_{\mathbf{G}})^\vee$. Since B is I-local, it will suffice to show that this map becomes an equivalence after tensoring both sides with B_J, for $J \in P$. This follows from our assumption that B_J satisfies $(T2')$. ☐

Lemma 4.5.8. *Suppose that $B \in \mathrm{CAlg}_A$ has the property that $L_{K(n)}B \simeq 0$. Then B is good.*

Proof. Let $R = \pi_0(A)/(p)$, so that $\mathbf{G}$ determines a p-divisible group $\mathbf{G}_R$ over the commutative $\mathbf{F}_p$-algebra R having identity component $\mathbf{G}^\circ_R$. Let $J \subseteq R$ be the nth Landweber ideal of the formal group $\mathbf{G}^\circ_R$, and let $I \subseteq \pi_0(A)$ be the inverse image of the ideal J. Then I is a finitely generated ideal of $\pi_0(A)$ which contains p, and the image of I in $\pi_0(A^\wedge_{(p)})$ generates the nth Landweber ideal in the complex periodic $\mathbb{E}_\infty$-ring $A^\wedge_{(p)}$. Since $\mathbf{G}$ has height $\leqslant n$, the ring spectrum A is $E(n)$-local. It follows that, for any A-module spectrum M, we can identify the I-completion $M^\wedge_I$ with the $K(n)$-localization of M. In particular, our hypothesis guarantees that the completion $B^\wedge_I$ vanishes: that is, the algebra B is local with respect to I. Consequently, to show that B is good, it will suffice to show that $B[x^{-1}]$ is good, for each element $x \in I$ (Lemma 4.5.7). We may therefore replace B by $B[x^{-1}]$ and thereby reduce to the case where I generates the unit ideal of B.

Let $\mathbf{G}^\heartsuit_B[p]$ denote the p-torsion subgroup of the p-divisible group $\mathbf{G}^\heartsuit_B$, which we regard as a finite flat group scheme over the commutative ring $\pi_0(B)$. Let $\mathbf{G}^\heartsuit_B[p]^\circ$ denote the quasi-compact open subscheme of $\mathbf{G}^\heartsuit_B[p]$ obtained by removing the zero section. Our assumption that I generates the unit ideal of $\pi_0(B)$ guarantees that the map $\mathbf{G}^\heartsuit_B[p]^\circ \to \mathrm{Spec}(\pi_0(B))$ is surjective. Choose an étale surjection of schemes $U \to \mathbf{G}^\heartsuit_B[p]^\circ$, where U is affine. Then the map $U \to \mathbf{G}^\heartsuit_B[p] \simeq \mathrm{Spec}(B^0_{\mathbf{G}}(BC_p))$ is surjective. Invoking Theorem HA.7.5.0.6 , we can write $U = \mathrm{Spec}(\pi_0(B'))$, where B' is an $\mathbb{E}_\infty$-algebra which is étale over $B_{\mathbf{G}}^{BC_p}$ and faithfully flat over B. Let B'' denote the direct limit of the sequence

$$B' \to B' \otimes_{B_{\mathbf{G}}^{BC_p}} B_{\mathbf{G}}^{BC_{p^2}} \to B' \otimes_{B_{\mathbf{G}}^{BC_p}} B_{\mathbf{G}}^{BC_{p^3}} \to \cdots .$$

Each term in this sequence is faithfully flat over B, so that B'' is also faithfully flat over B. By virtue of Lemma 4.5.2, it will suffice to show that every $\mathbb{E}_\infty$-algebra C over B'' is good. Replacing A by C, we are reduced to proving that A is good in the special case where the p-divisible group $\mathbf{G}$ splits as a direct sum $\mathbf{G}_0 \oplus \underline{\mathbf{Q}_p / \mathbf{Z}_p}$. In this case, $\mathbf{G}_0$ is an oriented p-divisible group of height $n-1$, and therefore satisfies Theorem 4.4.16 by virtue of our inductive hypothesis. Moreover, for every p-finite space X, Theorem 4.3.2 supplies an equivalence of tempered function spectra

$$A_{\mathbf{G}}^{X} \simeq A_{\mathbf{G}_0}^{\mathcal{L}(X)},$$

where $\mathcal{L}(X) = \mathcal{L}^{\mathbf{Z}_p}(X) = X^{B\,\mathbf{Z}_p}$ is the free loop space of X. If X is a split p-finite space, then $\mathcal{L}(X)$ is also a split p-finite space. Our inductive hypothesis then guarantees that $A_{\mathbf{G}_0}^{\mathcal{L}(X)}$ is a projective A-module of finite rank, so that $A_{\mathbf{G}}^{X}$ is also a projective A-module of finite rank: that is, A satisfies condition $(T1)$.

We will complete the proof by showing that A satisfies $(T2)$. Fix integers $d, t \geqslant 0$, and set $X = K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$. Set $Y = K(\mathbf{Z}/p^t\,\mathbf{Z}, 1)$, so that the iterated cup product is classified by a pair of maps

$$m_d : Y^d \to X \qquad m_{d-1} : Y^{d-1} \to \Omega X.$$

These maps are multilinear and skew symmetric up to homotopy, and therefore induce maps of $\pi_0(A)$-schemes

$$\rho_{d,t} : \mathrm{Spec}(A_0^{\mathbf{G}}(X)) \to \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}[p^t]}$$

$$\rho^{+}_{d,t} : \mathrm{Spec}(A_0^{\mathbf{G}_0}(X)) \to \mathrm{Skew}^{(d)}_{\mathbf{G}_0^{\heartsuit}[p^t]}$$

$$\rho^{-}_{d,t} : \mathrm{Spec}(A_0^{\mathbf{G}_0}(\Omega(X)) \to \mathrm{Skew}^{(d-1)}_{\mathbf{G}_0^{\heartsuit}[p^t]}$$

Our inductive hypothesis implies that the maps $\rho^{+}_{d,t}$ and $\rho^{-}_{d,t}$ are closed immersions, having schematic images $\mathrm{Alt}^{(d)}_{\mathbf{G}_0^{\heartsuit}[p^t]} \subseteq \mathrm{Skew}^{(d)}_{\mathbf{G}_0^{\heartsuit}[p^t]}$ and $\mathrm{Alt}^{(d-1)}_{\mathbf{G}_0^{\heartsuit}[p^t]} \subseteq \mathrm{Skew}^{(d-1)}_{\mathbf{G}_0^{\heartsuit}[p^t]}$, respectively. We wish to prove that $\rho_{d,t}$ is a closed immersion with schematic image $\mathrm{Alt}^{(d)}_{\mathbf{G}^{\heartsuit}[p^t]} \subseteq \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}[p^t]}$.

The splitting of p-divisible groups $\mathbf{G} \simeq \mathbf{G}_0 \oplus \underline{\mathbf{Q}_p / \mathbf{Z}_p}$ determines an isomorphism of finite flat group schemes

$$\mathbf{G}^{\heartsuit}[p^t] \simeq \mathbf{G}_0^{\heartsuit}[p^t] \oplus \underline{\mathbf{Z} / p^t \, \mathbf{Z}}$$

over the commutative ring $\pi_0(A)$. Applying Remark Ambi.3.2.21 , we obtain an isomorphism of $\pi_0(A)$-schemes

$$\beta : \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}[p^t]} \simeq \mathrm{Skew}^{(d)}_{\mathbf{G}_0^{\heartsuit}[p^t]} \times_{\mathrm{Spec}(\pi_0(A))} \mathrm{Skew}^{(d-1)}_{\mathbf{G}_0^{\heartsuit}[p^t]},$$

given by a pair of projection maps

$$\beta^{+} : \mathrm{Skew}^{(d)}_{\mathbf{G}^{\heartsuit}[p^t]} \to \mathrm{Skew}^{(d)}_{\mathbf{G}_0^{\heartsuit}[p^t]} \qquad \beta^{-} : \mathrm{Skew}^{(d)}_{\mathbf{G}_0^{\heartsuit}[p^t]} \to \mathrm{Skew}^{(d-1)}_{\mathbf{G}'_0[p^t]} .$$

Moreover, Remark Ambi.3.2.21 implies that β restricts to an isomorphism

$$\mathrm{Alt}^{(d)}_{\mathbf{G}^{\heartsuit}[p^t]} \simeq \mathrm{Alt}^{(d)}_{\mathbf{G}_0^{\heartsuit}[p^t]} \times_{\mathrm{Spec}(\pi_0(A))} \mathrm{Alt}^{(d-1)}_{\mathbf{G}_0^{\heartsuit}[p^t]} .$$

Note that we have canonical maps

$$X \to \mathcal{L}(X) \qquad \Omega(X) \to \mathcal{L}(X),$$

where the first is given by precomposition with the projection map $B\,\mathbf{Z}_p \to *$ and the second by the identification of $\Omega(X)$ with the space of pointed maps from $B\,\mathbf{Z}_p$ into X. Using the addition law on $\mathcal{L}(X)$, we can amalgamate these maps to a homotopy equivalence $\zeta : X \times \Omega(X) \to \mathcal{L}(X)$. Our inductive hypothesis then supplies a Künneth decomposition

$$A^{X}_{\mathbf{G}} \simeq A^{\mathcal{L}(X)}_{\mathbf{G}_0} \simeq A^{X}_{\mathbf{G}_0} \otimes_A A^{\Omega(X)}_{\mathbf{G}}.$$

Since both tensor factors are flat over A, this gives an isomorphism of affine schemes

$$\gamma : \mathrm{Spec}(A_0^{\mathbf{G}}(X)) \simeq \mathrm{Spec}(A_0^{\mathbf{G}_0}(X)) \times_{\mathrm{Spec}(\pi_0(A))} \mathrm{Spec}(A_0^{\mathbf{G}_0}(\Omega(X)))$$

given by a pair of projection maps

$$\gamma^{+} : \mathrm{Spec}(A_0^{\mathbf{G}}(X)) \to \mathrm{Spec}(A_0^{\mathbf{G}_0}(X)) \qquad \gamma^{-} : \mathrm{Spec}(A_0^{\mathbf{G}}(X)) \to \mathrm{Spec}(A_0^{\mathbf{G}_0}(\Omega(X))).$$

To complete the proof that $\rho_{d,t}$ is a closed immersion with image $\mathrm{Alt}^{(d)}_{\mathbf{G}^{\heartsuit}[p^t]}$, it will suffice to verify the following:

(*a*) The diagram of $\pi_0(A)$-schemes

$$\begin{tikzcd}
\mathrm{Spec}(A_0^{\mathbf{G}}(X)) \arrow[r, "\gamma^+"] \arrow[d, "\rho_{d,t}"] & \mathrm{Spec}(A_0^{\mathbf{G}_0}(X)) \arrow[d, "\rho_{d,t}^+"] \\
\mathrm{Skew}^{(d)}_{\mathbf{G}^\heartsuit[p^t]} \arrow[r, "\beta^+"] & \mathrm{Skew}^{(d)}_{\mathbf{G}_0^\heartsuit[p^t]}
\end{tikzcd}$$

commutes.

(*b*) The diagram of $\pi_0(A)$-schemes

$$\begin{tikzcd}
\mathrm{Spec}(A_0^{\mathbf{G}}(X)) \arrow[r, "\gamma^-"] \arrow[d, "\rho_{d,t}"] & \mathrm{Spec}(A_0^{\mathbf{G}_0}(\Omega(X))) \arrow[d, "\rho_{d,t}^-"] \\
\mathrm{Skew}^{(d)}_{\mathbf{G}^\heartsuit[p^t]} \arrow[r, "\beta^-"] & \mathrm{Skew}^{(d-1)}_{\mathbf{G}_0^\heartsuit[p^t]}
\end{tikzcd}$$

commutes.

Assertion (*a*) follows immediately from fact that the diagram of spaces

$$\begin{tikzcd}
Y^d \arrow[r] \arrow[d, "m_d"] & X \arrow[d] \\
\mathcal{L}(Y)^d \arrow[r, "\mathcal{L}(m)"] & \mathcal{L}(X)
\end{tikzcd}$$

commutes up to homotopy (where the vertical maps are given by the diagonal embeddings). Similarly, (*b*) follows from the homotopy commutativity of the diagram

$$\begin{tikzcd}
\Omega(Y) \times Y^{d-1} \arrow[r, "\nu"] \arrow[d] & \Omega(X) \arrow[d] \\
\mathcal{L}(Y)^d \arrow[r, "\mu_d"] & \mathcal{L}(X),
\end{tikzcd}$$

where $\nu : K(\mathbf{Z}/p^t\,\mathbf{Z}, 0) \times K(\mathbf{Z}/p^t\,\mathbf{Z}, 1)^{d-1} \to K(\mathbf{Z}/p^t\,\mathbf{Z}, d-1)$ classifies the cup product (together with a careful inspection of the definition of the isomorphism $\rho_{d,t}$). □

Lemma 4.5.9. *Let B be an $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0(B)$ be a finitely generated ideal, and let M be a B-module. Suppose that there exists a pullback diagram of B-modules*

$$\begin{tikzcd}
M \arrow[r, "\psi'"] \arrow[d, "\phi'"] & M_0 \arrow[d, "\phi"] \\
M_1 \arrow[r, "\psi"] & M_{01}
\end{tikzcd}$$

in Mod_A *with the following properties:*

- (a) *The B-module M_0 is I-complete, and therefore admits the structure of a module over the I-completion $B_0 = B^{\wedge}_I$.*

- (b) *The B-module M_1 is I-local, and therefore admits the structure of a module over the I-localization $B_1 = L_I(B)$.*

- (c) *Set $B_{01} = L_I(B^{\wedge}_I)$. Then M_{01} admits the structure of an B_{01}-module, and the maps ϕ and ψ induce equivalences*

$$B_{01} \otimes_{B_0} M_0 \to M_{01} \leftarrow B_{01} \otimes_{B_1} M_1.$$

Then the maps ϕ' and ψ' induce equivalences $B_0 \otimes_B M \simeq M_0$ and $B_1 \otimes_B M \simeq M_1$. Moreover, if M_0 and M_1 are perfect as modules over B_0 and B_1, respectively, then M is perfect over B.

Proof. We first show that ϕ' induces an equivalence $B_1 \otimes_B M \to M_1$. Note that the left hand side can be identified with $L_I(M)$. It will therefore suffice to show that the map $M \to M_1$ becomes an equivalence after applying the functor L_I. Since σ is a pullback diagram, this is equivalent to the requirement that the map $M_0 \to M_{01}$ becomes an equivalence after applying the functor L_I, which follows from assumption (c).

We next claim that ψ' induces an equivalence $\mu : B_0 \otimes_B M \to M_0$. To prove this, it will suffice to show that μ becomes an equivalence after applying the localization functor L_I or the completion functor $(-)^{\wedge}_I$. Using the first step of the proof, we deduce that $L_I(\mu)$ can be identified with the canonical map $B_{01} \otimes_{B_1} M_1 \to M_{01}$, which is an equivalence by virtue of (c). We are therefore reduced to proving that μ induces an equivalence after I-completion. Since unit map $\nu : M \to B_0 \otimes_B M$ is an equivalence after I-completion, it will suffice to show that the I-completion of the composite map $\psi' = \mu \circ \nu$ is an equivalence. Since σ is a pullback diagram, this is equivalent to the assertion that ψ induces an equivalence $(M_1)^{\wedge}_I \to (M_{01})^{\wedge}_I$, which is clear (since both completions vanish).

To prove that M is perfect as an B-module, it will suffice to show that the functor $N \mapsto \mathrm{Map}_{\mathrm{Mod}_B}(M, N)$ commutes with filtered colimits. For each $N \in \mathrm{Mod}_B$ we have a pullback diagram

$$\begin{tikzcd}
N \arrow[r] \arrow[d] & B_0 \otimes_B N \arrow[d] \\
B_1 \otimes_B N \arrow[r] & B_{01} \otimes_B N,
\end{tikzcd}$$

and therefore a pullback diagram of spaces σ :

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Mod}_B}(M, N) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Mod}_A}(M, B_0 \otimes_B N) \arrow[d] \\
\mathrm{Map}_{\mathrm{Mod}_B}(M, B_1 \otimes_B N) \arrow[r] & \mathrm{Map}_{\mathrm{Mod}_A}(M, B_{01} \otimes_B N).
\end{tikzcd}$$

It will therefore suffice to prove that the functors

$$N \mapsto \mathrm{Map}_{\mathrm{Mod}_B}(M, B_0 \otimes_B N) \simeq \mathrm{Map}_{\mathrm{Mod}_{B_0}}(M_0, B_0 \otimes_B N)$$

$$N \mapsto \mathrm{Map}_{\mathrm{Mod}_B}(M, B_1 \otimes_B N) \simeq \mathrm{Map}_{\mathrm{Mod}_{B_1}}(M_1, B_1 \otimes_B N)$$

$$N \mapsto \mathrm{Map}_{\mathrm{Mod}_{B_{01}}}(M, B_{01} \otimes_B N) \simeq \mathrm{Map}_{\mathrm{Mod}_{B_1}}(M_1, B_{01} \otimes_B N)$$

preserve filtered colimits, which follows from assumptions (a) and (b). □

Lemma 4.5.10. *Let $B \in \mathrm{CAlg}_A$. If $L_{K(n)}(B)$ is good, then B is good.*

Proof. Let $I \subseteq \pi_0(B)$ be as in the proof of Lemma 4.5.8. Set $B_0 = B^\wedge_I \simeq L_{K(n)}(B)$, $B_1 = L_I(B)$, and $B_{01} = L_I(B^\wedge_I)$. Our hypothesis guaranees that B_0 is good, and Lemma 4.5.8 guarantees that B_1 and B_{01} are good. For every split p-finite space X, Lemma 4.2.11 supplies a pullback diagram σ_X of tempered function spectra

$$\begin{tikzcd}
B^X_{\mathbf{G}} \arrow[r] \arrow[d] & (B_0)^X_{\mathbf{G}} \arrow[d] \\
(B_1)^X_{\mathbf{G}} \arrow[r] & (B_{01})^X_{\mathbf{G}}.
\end{tikzcd}$$

Using Lemma 4.5.1, we see that this diagram satisfies the hypotheses of Lemma 4.5.9. This allows us to draw three conclusions:

(i) For every split p-finite space X, the tempered function spectrum $B^X_{\mathbf{G}}$ is a perfect B-module.

(ii) For every split p-finite space X, the canonical map $B_0 \otimes_B B^X_{\mathbf{G}} \to (B_0)^X_{\mathbf{G}}$ is an equivalence.

(iii) For every split p-finite space X, the canonical map $B_1 \otimes_B B^X_{\mathbf{G}} \to (B_1)^X_{\mathbf{G}}$ is an equivalence.

We will complete the proof by showing that B satisfies criterion $(T2')$ of Lemma 4.5.6. Set $X = K(\mathbf{Z}/p^t\mathbf{Z}, d)$, and let $\psi_{d,t} : A_{d,t} \to (B^X_{\mathbf{G}})^\vee$ be the map of Construction 4.5.5; we wish to show that $\psi_{d,t}$ induces an equivalence of B-modules $\theta : B \otimes_A A_{d,t} \to (B^X_{\mathbf{G}})^\vee$. Since B_0 and B_1 are good, it follows from (ii) and (iii) that θ becomes an equivalence after extending scalars from B to B_0 or B_1 (hence also after extending scalars from B to B_{01}). Since B is the fiber product of B_0 and B_1 over B_{01}, it follows that θ is an equivalence. □

Proof of Theorem 4.4.16. We wish to prove that A is good. By virtue of Lemma 4.5.10, we can replace A by $L_{K(n)}(A)$ and thereby reduce to the case where A is $K(n)$-local. Then A is p-complete, so $\mathbf{G}$ is oriented and A is complex periodic. Let E be a Lubin-Tate spectrum of height n, and let A^0 denote the smash product $A \otimes_S E$. Let $A^\bullet$ be the cosimplicial A-algebra given by the iterated tensor powers of A^0 over A. Since E is Landweber exact, A^0 is faithfully flat over A. By virtue of Lemma 4.5.2, it will suffice to show that each A^k is good. Applying Lemma 4.5.10 again, we are reduced to proving that the localization $L_{K(n)}(A^k)$ is good. We may therefore replace A by $L_{K(n)}(A)$ and thereby reduce to the case where A is a $K(n)$-local algebra over the Lubin-Tate spectrum E. Then $\mathbf{G}$ is an oriented p-divisible group of height n over A and therefore equivalent to the Quillen p-divisible group $\mathbf{G}^{\mathcal{Q}}_A$ of Construction Or.4.6.2 . Applying Theorem 4.2.5, we deduce that the Atiyah-Segal completion map $\zeta : A^X_{\mathbf{G}} \to A^X$ is an equivalence for every space X. Using Corollary Ambi.5.4.7 , we deduce that the map $A \otimes_E E^X \to A^X$ is an equivalence whenever X is π-finite. We can therefore replace A by the Lubin-Tate spectrum E, in which case the desired result follows from Theorem Ambi.3.4.1 . □

4.6 The Tate Construction

Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. According to Proposition 4.2.15, $\mathbf{G}$ is oriented if and only if the Atiyah-Segal comparison map

$$\zeta : B^{BC_{p^n}}_{\mathbf{G}} \to B^{BC_{p^n}}$$

exhibits $B^{BC_{p^n}}$ as the completion of $B^{BC_{p^n}}_{\mathbf{G}}$ with respect to the augmentation ideal $I_{C_{p^n}}$ for every prime power p^n and every $\mathbb{E}_\infty$-algebra B over A. Our goal in this section is to supply a variant of this criterion, which only needs to be checked in the special case where $B = A$ and $n = 1$. The proof is based on a locality property of the Tate construction A^{tC_p} (Proposition 4.6.8) which will play an essential role in the theory of $\mathbf{G}$-tempered local systems we introduce in §5.

Notation 4.6.1. Let A be an $\mathbb{E}_\infty$-ring, let H be a finite group, and let $M : BH \to \mathrm{Mod}_A$ be a local system of A-modules on the classifying space BH. We let M^{hH} denote the associated homotopy fixed point spectrum (that is, the limit of the diagram M) and M_{hH} the homotopy orbit spectrum (that is, the colimit of the diagram M). We let M^{tH} denote the Tate construction on M: that is, the cofiber of the norm map $\mathrm{Nm} : M_{hH} \to M^{hH}$ (see Example Ambi.4.4.14). Note that since $\mathrm{Fun}(BH, \mathrm{Mod}_A)$ is an A^{BH}-linear ∞-category, we can regard Nm as a morphism of A^{BH}-modules, so that the Tate construction M^{tH} inherits the structure of a module over A^{BH}.

In particular, if $\mathbf{G}$ is a preoriented $\mathbf{P}$-divisible group over A, then we can view $M_{hH} \to M^{hH} \to M^{tH}$ as a fiber sequence of modules over the tempered function spectrum $A_{\mathbf{G}}^{BH}$ (via the Atiyah-Segal comparison map $\zeta : A_{\mathbf{G}}^{BH} \to A^{BH}$).

Theorem 4.6.2. *Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. Then $\mathbf{G}$ is oriented if and only if, for every prime number p, the following conditions are satisfied:*

($*_p$) *The Atiyah-Segal comparison map $\zeta : A_{\mathbf{G}}^{BC_p} \to A^{BC_p}$ exhibits A^{BC_p} as the completion of $A_{\mathbf{G}}^{BC_p}$ with respect to the augmentation ideal $I_{C_p} \subseteq A_{\mathbf{G}}^0(BC_p)$.*

($*'_p$) *The Tate construction A^{tC_p} is I_{C_p}-local when viewed as a module over the tempered function spectrum $A_{\mathbf{G}}^{BC_p}$.*

Remark 4.6.3. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let H be a finite group, and assume that the augmentation ideal $I_H \subseteq A_{\mathbf{G}}^0(BH)$ is finitely generated (this is satisfied automatically if H is abelian or A is Noetherian). Viewing M as a local system of $A_{\mathbf{G}}^{BH}$-modules on the classifying space BH, we note that the value of M on each point $x \in BH$ is both I_H-nilpotent and I_H-complete (since the action of $A_{\mathbf{G}}^{BH}$ on M_x factors through the evaluation map $A_{\mathbf{G}}^{BH} \to A_{\mathbf{G}}^{\{x\}} \simeq A$, which annihilates the ideal I_H). It follows that the homotopy orbit spectrum M_{hH} also I_H-nilpotent (since the collection of I_H-nilpotent objects of $\mathrm{Mod}_{A_{\mathbf{G}}^{H}}$ is closed under colimits) and the homotopy fixed point spectrum M^{hH} is I_H-complete (since the collection of I_H-complete objects of $\mathrm{Mod}_{A_{\mathbf{G}}^{BH}}$ is closed under limits).

Remark 4.6.4. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Then, for every prime number p, the function spectrum A^{BC_p} can be identified with the homotopy fixed point spectrum A^{hC_p}, where we endow A with the trivial action of C_p. It follows that A^{BC_p} is automatically I_{C_p}-complete when viewed as a module over $A_{\mathbf{G}}^{BC_p}$. Consequently, assertion ($*_p$) of Theorem 4.6.2 is equivalent to the requirement that the fiber of the Atiyah-Segal comparison map $\zeta : A_{\mathbf{G}}^{BC_p} \to A^{BC_p}$ is I_{C_p}-local.

Remark 4.6.5. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. Then conditions $(*_\ell)$ and $(*'_\ell)$ of Theorem 4.6.2 are automatically satisfied for any prime number ℓ which is invertible in $\pi_0(A)$. In particular, if the $\mathbb{E}_\infty$-ring A is p-local, then Theorem 4.6.2 asserts that $\mathbf{G}$ is oriented if and only if it satisfies conditions $(*_p)$ and $(*'_p)$.

Remark 4.6.6 (Condition $(*_p)$ and Equivariant Stable Homotopy Theory)**.** Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A which satisfies condition $(*_p)$ of Theorem 4.6.2. Then, if M is any $A_{\mathbf{G}}^{BC_p}$-module which is I_{C_p}-nilpotent, composition with ζ induces an isomorphism

$$\mathrm{Ext}^*_{A_{\mathbf{G}}^{BC_p}}(M, A_{\mathbf{G}}^{BC_p}) \to \mathrm{Ext}^*_{A_{\mathbf{G}}^{BC_p}}(M, A^{BC_p}).$$

Applying this observation in the special case $M = A_{hC_p}$, we deduce that the norm map $\mathrm{Nm} : A_{hC_p} \to A^{hC_p} = A^{BC_p}$ admits an essentially unique factorization as a composition

$$A_{hC_p} \xrightarrow{\mathrm{Nm}_0} A_{\mathbf{G}}^{BC_p} \xrightarrow{\zeta} A^{BC_p}.$$

This factorization equips A with the structure of a *genuine C_p-spectrum.* More precisely, it allows us to construct a C_p-spectrum with underlying spectrum is A (equipped with the trivial action of C_p), "genuine" fixed point spectrum is $A_{\mathbf{G}}^{BC_p}$, and geometric fixed point spectrum $\Phi^{C_p}(A)$ given by the the cofiber of the map $\mathrm{Nm}_0 : A_{hC_p} \to A_{\mathbf{G}}^{BC_p}$. We then have a homotopy pullback diagram of spectra

$$\begin{tikzcd}
A_{\mathbf{G}}^{BC_p} \arrow[r, "\zeta"] \arrow[d] & A^{hC_p} \arrow[d] \\
\Phi^{C_p}(A) \arrow[r] & A^{tC_p},
\end{tikzcd}$$

which we will refer to as the *equivariant fracture square.*

Remark 4.6.7 (Condition $(*'_p)$ and Equivariant Stable Homotopy Theory)**.** Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. Fix a prime number p and let $I = I_{C_p} \subseteq \pi_0(A_{\mathbf{G}}^{BC_p})$ be the augmentation ideal of Notation 4.2.7. For every $A_{\mathbf{G}}^{BC_p}$-module M, we will denote the I-completion of M by $M_I^\wedge$ and the I-localization of M by $L_I(M)$, so that we have a pullback diagram of $A_{\mathbf{G}}^{BC_p}$-modules σ_M:

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & M_I^\wedge \arrow[d] \\
L_I(M) \arrow[r] & L_I(M_I^\wedge)
\end{tikzcd}$$

which we will refer to as the *algebraic fracture square*.

Suppose that $\mathbf{G}$ satisfies condition $(*_p)$ of Theorem 4.6.2. Taking $M = A_{\mathbf{G}}^{BC_p}$, we conclude that the completion $M_I^\wedge$ can be identified with the function spectrum A^{BC_p}. Moreover, the homotopy orbit spectrum A_{hC_p} is automatically I_{C_p}-nilpotent (Remark 4.6.3). It follows that there exists a commutative diagram of $A_{\mathbf{G}}^{BC_p}$-modules

$$\begin{tikzcd}
A_{\mathbf{G}}^{BC_p} \arrow[r, "\zeta"] \arrow[d] & A^{BC_p} \arrow[d] \\
L_I(A_{\mathbf{G}}^{BC_p}) \arrow[r] \arrow[d, "u"] & L_I(A^{BC_p}) \arrow[d, "v"] \\
\Phi^{C_p}(A) \arrow[r] & A^{tC_p},
\end{tikzcd}$$

where the upper square is the algebraic fracture square of M and the outer rectangle is the equivariant fracture square of Remark 4.6.6. In particular, the lower square is also a pushout diagram, so that u is an equivalence if and only if v is an equivalence. The following conditions are equivalent:

- The $\mathbf{P}$-divisible group $\mathbf{G}$ satisfies condition $(*'_p)$ of Theorem 4.6.2: that is, the Tate construction A^{tC_p} is I_{C_p}-local as a module over $A_{\mathbf{G}}^{BC_p}$.

- The morphisms u and v appearing in the above diagram are invertible: in other words, the algebraic fracture square of $M = A_{\mathbf{G}}^{BC_p}$ agrees with the equivariant fracture square of Remark 4.6.6.

- The spectrum $\Phi^{C_p}(A)$ is I_{C_p}-local. Equivalently, the map $\mathrm{Nm}_0 : A_{hC_p} \to A_{\mathbf{G}}^{BC_p}$ identifies A_{hC_p} with the local cohomology spectrum $\Gamma_{I_{C_p}}(A_{\mathbf{G}}^{BC_p})$ (this can be viewed as a "homological" version of the condition $(*_p)$ of Theorem 4.6.2).

The proof of Theorem 4.6.2 will require some preliminaries. We first show that every oriented $\mathbf{P}$-divisible group satisfies condition $(*'_p)$.

Proposition 4.6.8. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. Let* C_p *be a cyclic group of order* p*, for some prime number* p*, and let* $M \in \mathrm{Fun}(BC_p, \mathrm{Mod}_A)$ *be* C_p*-equivariant object of the* ∞*-category* Mod_A*. Then the Tate construction* M^{tC_p} *is* I_{C_p}*-local when viewed as a module over* $A_{\mathbf{G}}^{BC_p}$*.*

Remark 4.6.9. In the situation of Proposition 4.6.8, the homotopy fixed point spectrum M^{hC_p} is automatically complete with respect to the augmentation ideal I_{C_p}

(Remark 4.6.3). Consequently, Proposition 4.6.8 is equivalent to the assertion that the norm map $\mathrm{Nm} : M_{hC_p} \to M^{hC_p}$ exhibits M^{hC_p} as the I_{C_p}-completion of M_{hC_p} (or dually that it induces an equivalence equivalence $M_{hC_p} \simeq \Gamma_{I_{C_p}}(M^{hC_p})$.

Remark 4.6.10. The assertion of Proposition 4.6.8 is *a priori* stronger than condition $(*_p')$ of Theorem 4.6.2, since it applies to any C_p-equivariant object M of the ∞-category Mod_A, rather than just to A itself (endowed with the trivial action of C_p). However, it is actually equivalent to $(*_p')$. The Tate construction can be viewed as a lax symmetric monoidal functor from $\mathrm{Fun}(BC_p, \mathrm{Mod}_A)$ to Mod_A. Consequently, A^{tC_p} has the structure of an $\mathbb{E}_\infty$-algebra over A, and M^{tC_p} has the structure of a module over A^{tC_p}. In particular, if A^{tC_p} is I_{C_p}-local, then so is M^{tC_p}.

The proof of Proposition 4.6.8 will make use of the following elementary observation:

Lemma 4.6.11. *Let H be a finite group and suppose we are given a map of spaces $f : X \to BH$. Let $\mathscr{F} \in \mathrm{Fun}(X, \mathrm{Sp})$ be a local system of spectra on X and let $f_* \mathscr{F} \in \mathrm{Fun}(BH, \mathrm{Sp})$ denote its pushforward to BH (that is, the right Kan extension of $\mathscr{F}$ along f). If X is a finite space, then the Tate construction $(f_* \mathscr{F})^{tH}$ vanishes.*

Proof. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(BH, \mathrm{Sp})$ spanned by those objects M with $M^{tH} \simeq 0$. Since the construction $M \mapsto M^{tH}$ is exact, $\mathcal{C}$ is closed under finite limits in $\mathrm{Fun}(BH, \mathrm{Sp})$. We wish to prove that $f_* \mathscr{F} \in \mathcal{C}$. For each $x \in X$, let $i_x : \{x\} \to X$ denote the inclusion map. Using the equivalence $\mathscr{F} \simeq \varprojlim_{x \in X} i_{x*} i_x^* \mathscr{F}$ (and the finiteness of X), we can reduce to the case where $\mathscr{F}$ has the form $i_{x*} \mathscr{F}'$, for some $\mathscr{F}' \in \mathrm{Fun}(\{x\}, \mathrm{Sp}) \simeq \mathrm{Sp}$. We may therefore replace X by $\{x\}$ and thereby reduce to the case where X is a point, in which case the desired result follows from Example HA.6.1.6.26 . □

Proof of Proposition 4.6.8. Let $\widehat{M}$ be the p-completion of M and let N denote the fiber of the canonical map $M \to \widehat{M}$. We then have a fiber sequence of Tate constructions

$$N^{tC_p} \to M^{tC_p} \to \widehat{M}^{tC_p},$$

where the first term vanishes because p acts invertibly on N. We may therefore replace M by $\widehat{M}$ and thereby reduce to the case where M is a module over the p-completion of A. In this case, we can replace A by its p-completion and thereby reduce to the case where A is p-complete. We may also replace the $\mathbf{P}$-divisible group $\mathbf{G}$ by its p-local summand $\mathbf{G}_{(p)}$ (since this does not change the tempered function spectrum $A_{\mathbf{G}}^{BC_p}$), and thereby reduce to the case where $\mathbf{G}$ is an oriented p-divisible group.

For each $n \geqslant 0$, let $\mathfrak{I}_n^A \subseteq \pi_0(A)$ denote the nth Landweber ideal of A (Definition Or.4.4.11) Note that every A-module is $\mathfrak{I}_{n+1}^A$-local (or equivalently $E(n)$-local, where $E(n)$ denotes the nth Johnson-Wilson spectrum) for $n \gg 0$; in fact, it suffices to take n to be any upper bound for the height of the p-divisible group $\mathbf{G}$ (since we then have $\mathfrak{I}_{n+1}^A = \pi_0(A)$). It will therefore suffice to prove the following:

$(*_n)$ If M is $\mathfrak{I}_n^A$-local, then the Tate construction M^{tC_p} is I_{C_p}-local.

We proceed by induction on n. If $n = 1$, then the assumption that M is $\mathfrak{I}_n^A$-local guarantees that p acts invertibly on M, so that M^{tC_p} vanishes. To carry out the inductive step, assume that $(*_n)$ holds for some $n \geqslant 1$ and that M is $\mathfrak{I}_{n+1}^A$-local. Let $\widehat{M}$ denote the completion of M for the ideal $\mathfrak{I}_n^A$, and let LM and $L\widehat{M}$ denote the localizations of M and $\widehat{M}$ with respect to $\mathfrak{I}_n^A$. We then have a pullback square

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & \widehat{M} \arrow[d] \\
LM \arrow[r] & L\widehat{M}
\end{tikzcd}$$

of C_p-equivariant objects of Mod_A, which induces a pullback square of Tate constructions

$$\begin{tikzcd}
M^{tC_p} \arrow[r] \arrow[d] & \widehat{M}^{tC_p} \arrow[d] \\
(LM)^{tC_p} \arrow[r] & (L\widehat{M})^{tC_p}.
\end{tikzcd}$$

Our inductive hypothesis then guarantees that $(LM)^{tC_p}$ and $(L\widehat{M})^{tC_p}$ are I_{C_p}-local. Consequently, to show that M^{tC_p} is I_{C_p}-local, it will suffice to show that $\widehat{M}^{tC_p}$ is I_{C_p}-local. We may therefore replace M by $\widehat{M}$ and thereby reduce to the case where M is $\mathfrak{I}_{n+1}^A$-local and $\mathfrak{I}_n^A$-complete: that is, the case where M is $K(n)$-local as a spectrum (see Proposition Or.4.5.4). Replacing A by its $K(n)$-localization, we can assume also that A is $K(n)$-local. In this case, our orientation of $\mathbf{G}$ supplies a short exact sequence of p-divisible groups

$$0 \to \mathbf{G}_A^{\mathcal{Q}} \to \mathbf{G} \to \mathbf{G}_{\text{ét}} \to 0,$$

where $\mathbf{G}_A^{\mathcal{Q}}$ is the Quillen p-divisible group of A (Proposition 2.5.6). In particular, the underlying map of finite flat group scheme $\mathbf{G}_A^{\mathcal{Q}}[p] \to \mathbf{G}[p]$ is a strict monomorphism, so that the Atiyah-Segal comparison map $\zeta : A_{\mathbf{G}}^*(BC_p) \to A^*(BC_p)$ is surjective.

We have a short exact sequence of abelian groups $0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to C_p \to 0$ which induces a fiber sequence of spaces $BC_p \xrightarrow{\phi} \mathbf{CP}^\infty \xrightarrow{p} \mathbf{CP}^\infty$. Let $\overline{e} \in A^2(\mathbf{CP}^\infty)$

be a complex orientation of A. We can then choose an element $e \in A^2_{\mathbf{G}}(BC_p)$ satisfying $\zeta(e) = \phi^*(\overline{e})$ in $A^2(BC_p)$. Note that e is annihilated by the pullback map $A^2_{\mathbf{G}}(BC_p) \to A^2_{\mathbf{G}}(EC_p) \simeq \pi_{-2}(A)$, so that $eA^{-2}_{\mathbf{G}}(BC_p)$ is contained in the augmentation ideal I_{C_p}. Consequently, to show that M^{tC_p} is I_{C_p}-local, it will suffice to show that multiplication by e induces an equivalence $\theta : \Sigma^{-2} M^{tC_p} \to M^{tC_p}$.

To prove this, form a pullback diagram of spaces

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f'"] & * \arrow[d, "f"] \\
BC_p \arrow[r, "\phi"] & \mathbf{CP}^{\infty},
\end{tikzcd}$$

and let $\underline{A}_X \in \mathrm{Fun}(X, \mathrm{Mod}_A)$ denote the constant local system on X with the value A. The cofiber of θ is then given by the Tate construction Q^{tC_p}, where Q is the C_p-equivariant object of Mod_A given by

$$\begin{aligned}
M \otimes_A \phi^* \operatorname{cofib}(\overline{e} : \Sigma^{-2}\underline{A}_{\mathbf{CP}^\infty} \to \underline{A}_{\mathbf{CP}^\infty}) &\simeq M \otimes_A \phi^* f_*(A)) \\
&\simeq M \otimes_A f'_* \underline{A}_X \\
&\simeq f'_* f'^* M.
\end{aligned}$$

The vanishing of Q^{tC_p} now follows from Lemma 4.6.11, since X is homotopy equivalent to a circle. □

Lemma 4.6.12. *Let p be a prime number and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A which satisfies conditions $(*_p)$ and $(*'_p)$ of Theorem 4.6.2. Then, for every module M over the tempered function spectrum $A^{BC_p}_{\mathbf{G}}$, the canonical map*

$$\theta_M : M \to (M \otimes_{A^{BC_p}_{\mathbf{G}}} A^{EC_p}_{\mathbf{G}})^{hC_p}$$

exhibits $(M \otimes_{A^{BC_p}_{\mathbf{G}}} A^{EC_p}_{\mathbf{G}})^{hC_p}$ as the I_{C_p}-completion of M.

Proof. Set $I = I_{C_p}$ and let $\mathrm{Mod}^{\mathrm{Cpl}(I)}_{A^{BC_p}_{\mathbf{G}}}$ denote the full subcategory of $\mathrm{Mod}_{A^{BC_p}_{\mathbf{G}}}$ spanned by the $A^{BC_p}_{\mathbf{G}}$-modules which are I-complete. Then the construction

$$M \mapsto (M \otimes_{A^{BC_p}_{\mathbf{G}}} A^{EC_p}_{\mathbf{G}})^{hC_p}$$

determines a functor $F : \mathrm{Mod}_{A^{BC_p}_{\mathbf{G}}} \to \mathrm{Mod}^{\mathrm{Cpl}(I)}_{A^{BC_p}_{\mathbf{G}}}$. Let $\mathcal{C} \subseteq \mathrm{Mod}_{A^{BC_p}_{\mathbf{G}}}$ be the full subcategory spanned by those objects M for which the map $\theta_M : M \to F(M)$ exhibits

$F(M)$ as an I_{C_p}-completion of M. It follows from assumption $(*_p)$ $\mathcal{C}$ contains the tempered function spectrum $A_{\mathbf{G}}^{BC_p}$ (and all of its shifts). Consequently, to show that $\mathcal{C} = \mathrm{Mod}_{A_{\mathbf{G}}^{BC_p}}$, it will suffice to show that $\mathcal{C}$ is closed under small colimits. For this, it will suffice to show that the functor F preserves small colimits. Using assumption $(*'_p)$ (and Remark 4.6.10), we can factor F as a composition

$$\mathrm{Mod}_{A_{\mathbf{G}}^{BC_p}} \xrightarrow{F'} \mathrm{Mod}_{A_{\mathbf{G}}^{BC_p}} \xrightarrow{F''} \mathrm{Mod}_{A_{\mathbf{G}}^{BC_p}}^{\mathrm{Cpl}(I)},$$

where F'' is the functor of completion with respect to I (which preserves small colimits, since it is left adjoint to the inclusion) and F' given by the construction $M \mapsto (M \otimes_{A_{\mathbf{G}}^{BC_p}} A_{\mathbf{G}}^{EC_p})_{hC_p}$ (which also preserves small colimits). $\square$

Lemma 4.6.13. *Let p be a prime number and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A which satisfies conditions $(*_p)$ and $(*'_p)$ of Theorem 4.6.2, for some prime number p. Then, for every nonnegative integer n, the Atiyah-Segal comparison map*

$$\zeta : A_{\mathbf{G}}^{BC_{p^n}} \to A^{BC_{p^n}}$$

exhibits $A^{BC_{p^n}}$ as the completion of $A_{\mathbf{G}}^{BC_{p^n}}$ with respect to the augmentation ideal $I_{C_{p^n}}$.

Proof. We proceed by induction on n, the case $n = 0$ being trivial. To carry out the inductive step, we observe that the short exact sequence of abelian groups $0 \to C_{p^{n-1}} \to C_{p^n} \to C_p$ provides a factorization of ζ as a composition

$$A_{\mathbf{G}}^{BC_{p^n}} \xrightarrow{\zeta'} (A_{\mathbf{G}}^{BC_{p^{n-1}}})^{hC_p} \xrightarrow{\zeta''} (A^{BC_{p^{n-1}}})^{hC_p} \simeq A^{BC_{p^n}}.$$

It follows from Lemma 4.6.12 that the fiber $\mathrm{fib}(\zeta')$ is local with respect to the augmentation ideal $I_{C_p} \subseteq A_{\mathbf{G}}^0(BC_p)$, hence also with respect to the augmentation ideal $I_{C_{p^n}} \subseteq A_{\mathbf{G}}^0(BC_{p^n})$ (which contains the image of I_{C_p}). Since $I_{C_{p^{n-1}}}$ is generated by the image of $I_{C_{p^n}}$, our inductive hypothesis guarantees that $\mathrm{fib}(\zeta'')$ is also $I_{C_{p^n}}$-local. It follows that $\mathrm{fib}(\zeta)$ is $I_{C_{p^n}}$-local, and therefore exhibits $A^{BC_{p^n}}$ as the $I_{C_{p^n}}$-completion of $A_{\mathbf{G}}^{BC_{p^n}}$ (since $A^{BC_{p^n}}$ is automatically $I_{C_{p^n}}$-local, by virtue of Remark 4.6.3). $\square$

Lemma 4.6.14. *Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let B be an $\mathbb{E}_\infty$-algebra over A, and let p be a prime number. If $\mathbf{G}$ satisfies conditions $(*_p)$ and $(*'_p)$ of Theorem 4.6.2, then the preoriented $\mathbf{P}$-divisible group $\mathbf{G}_B$ also satisfies conditions $(*_p)$ and $(*'_p)$.*

Proof. Condition $(*'_p)$ follows from Remark 4.6.10. To prove $(*_p)$, we must show that the Atiyah-Segal comparison map $\zeta : B_{\mathbf{G}}^{BC_p} \to B^{BC_p}$ exhibits B^{BC_p} as the completion of $B_{\mathbf{G}}^{BC_p}$ with respect to the augmentation ideal $I_{C_p} \subseteq A^0_{\mathbf{G}}(BC_p)$ (or equivalently, with respect to the ideal that it generates in $B^0_{\mathbf{G}}(BC_p)$). Note that we have a commutative diagram

$$\begin{tikzcd}
B \otimes_A A_{\mathbf{G}}^{BC_p} \arrow[r] \arrow[d] & B \otimes_A A^{BC_p} \arrow[d] \\
B_{\mathbf{G}}^{BC_p} \arrow[r] & B^{BC_p},
\end{tikzcd}$$

where the left vertical map is an equivalence and the upper horizontal map induces an equivalence after completion with respect to I_{C_p} (by virtue of assumption (2)). It will therefore suffice to show that the right vertical map also induces an equivalence after completion with respect to I_{C_p}. To prove this, we observe that it fits into a commutative diagram of fiber sequences

$$\begin{tikzcd}
B \otimes_A A_{hC_p} \arrow[r, "\mathrm{Nm}"] \arrow[d, "\sim"] & B \otimes_A A^{hC_p} \arrow[r] \arrow[d] & B \otimes_A A^{tC_p} \arrow[d] \\
B_{hC_p} \arrow[r, "\mathrm{Nm}"] & B^{hC_p} \arrow[r] & B^{tC_p},
\end{tikzcd}$$

where the left vertical map is an equivalence. It will therefore suffice to show that the right vertical map becomes an equivalence after completion with respect to I_{C_p}. In fact, both $B \otimes_A A^{tC_p}$ and B^{tC_p} vanish after completion with respect to I_{C_p}, by virtue of our assumption that $\mathbf{G}$ satisfies $(*'_p)$. □

Proof of Theorem 4.6.2. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. If $\mathbf{G}$ is oriented and p is a prime number, then $\mathbf{G}$ satisfies condition $(*_p)$ (Proposition 4.2.8) and condition $(*'_p)$ (Proposition 4.6.8). Conversely, suppose that $\mathbf{G}$ satisfies conditions $(*_p)$ an $(*'_p)$ for every prime number p. Then, for every $\mathbb{E}_\infty$-algebra B over A, the preoriented $\mathbf{P}$-divisible group $\mathbf{G}_B$ has the same property (Lemma 4.6.14). Applying Lemma 4.6.13, we deduce that the Atiyah-Segal comparison maps

$$\zeta : B_{\mathbf{G}}^{BC_{p^n}} \to B^{BC_{p^n}}$$

exhibit each $B^{BC_{p^n}}$ as the completion of $B_{\mathbf{G}}^{BC_{p^n}}$ with respect to the augmentation ideal $I_{C_{p^n}}$, so that $\mathbf{G}$ is oriented by virtue of Proposition 4.2.15. □

4.7 Base Change and Finiteness

Let $f : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings. Then, for every space X, f induces an A-linear map of (unpointed) function spectra $f^X : A^X \to B^X$, which extends to a B-algebra map $B \otimes_A A^X \to B^X$. This map is an equivalence if the space X is finite, or if B is perfect as an A-module. However, it is rarely an equivalence in general. In essence, our theory of tempered cohomology is designed to correct this problem: it provides a replacement for the function spectrum A^X, which is more likely to be compatible with extension of scalars. If $\mathbf{G}$ is an oriented $\mathbf{P}$-divisible group over A, then Theorem 4.4.16 (along with Lemma 4.5.1) implies that the canonical map $B \otimes_A A^X_{\mathbf{G}} \to B^X_{\mathbf{G}}$ is an equivalence when $X = K(H, d)$ is an Eilenberg-MacLane space associated to a finite abelian p-group H. In fact, we have the following more general result:

Theorem 4.7.1 (Base Change for Tempered Cohomology)**.** *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* X *be a* π*-finite space. Then, for every map of* $\mathbb{E}_\infty$*-rings* $A \to B$*, the canonical map* $A^X_{\mathbf{G}} \to B^X_{\mathbf{G}}$ *extends to an equivalence*

$$\rho : B \otimes_A A^X_{\mathbf{G}} \to B^X_{\mathbf{G}}$$

of $\mathbb{E}_\infty$*-algebras over* B*.*

We will give a proof of Theorem 4.7.1 in §7 (see Corollary 7.3.12).

Remark 4.7.2. In the special case where $X = BH$ is the classifying space of a finite abelian group H, Theorem 4.7.1 is a tautology: it follows immediately from the definition of the $\mathbf{P}$-divisible group $\mathbf{G}_B$, and does not require the assumption that $\mathbf{G}$ is oriented.

Let us collect some consequences of Theorem 4.7.1.

Corollary 4.7.3. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* X *be a* π*-finite space. Let* $f : A \to B$ *be a morphism of* $\mathbb{E}_\infty$*-rings. If either* $A^X_{\mathbf{G}}$ *or* B *is flat as an* A*-module spectrum, then the comparison map of Theorem 4.7.1 induces an isomorphism of* $\mathbf{G}$*-tempered cohomology rings*

$$\pi_0(B) \otimes_{\pi_0(A)} A^*_{\mathbf{G}}(X) \simeq B^*_{\mathbf{G}}(X).$$

Proof. Combine Theorem 4.7.1 with Proposition HA.7.2.2.13. □

Remark 4.7.4. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. If B is a finite flat $\mathbb{E}_\infty$-algebra over A, then the comparison map $\pi_0(B) \otimes_{\pi_0(A)} A^*_{\mathbf{G}}(\mathsf{X}) \simeq B^*_{\mathbf{G}}(\mathsf{X})$ is an equivalence for *every* orbispace X (see Remark 4.2.10).

Corollary 4.7.5. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. Let* H *be a finite group and let* X *be a finite* H*-space. Then, for every* $\mathbb{E}_\infty$*-ring* B*, the comparison map* $\rho : B \otimes_A A^{X//H}_{\mathbf{G}} \to B^{X//H}_{\mathbf{G}}$ *is an equivalence of* $\mathbb{E}_\infty$*-algebras over* B*.*

Proof. The constructions $X \mapsto B \otimes_A A^{X//H}_{\mathbf{G}}$ and $X \mapsto B^{X//H}_{\mathbf{G}}$ carry finite colimits in the ∞-category $\mathcal{S}_H$ of H-spaces to finite limits in the ∞-category CAlg_B of $\mathbb{E}_\infty$-algebras over B. It will therefore suffice to prove Corollary 4.7.5 in the special case where X is an H-space of the form $H_0 \backslash H$, where H_0 is a subgroup of H. In this case, the orbispace quotient $X//H$ can be identified with the $BH_0^{(-)}$, and the desired result is a special case of Theorem 4.7.1. □

From Corollary 4.7.5, we immediately deduce the following slightly stronger form of Theorem 1.1.19:

Corollary 4.7.6. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. Let* H *be a finite group, let* X *be a finite* H*-space, and let* B *be an* $\mathbb{E}_\infty$*-algebra over* A*. If either* $A^{X//H}_{\mathbf{G}}$ *or* B *is flat as an* A*-module spectrum, then the comparison map of Corollary 4.7.5 induces an isomorphism of* $\mathbf{G}$*-tempered cohomology rings*

$$\pi_0(B) \otimes_{\pi_0(A)} A^*_{\mathbf{G}}(X//H) \to B^*_{\mathbf{G}}(X//H).$$

Proof. Combine Corollary 4.7.5 with Proposition HA.7.2.2.13 . □

Corollary 4.7.7. *Let* H *be a finite group and let* X *be a finite* H*-space. Then the equivariant Chern character of Example 4.3.8 induces an isomorphism of complex vector spaces*

$$\mathrm{ch}_H : \mathbf{C} \otimes_{\mathbf{Z}} \mathrm{KU}^*_H(X) \to \mathrm{H}^*((\coprod_{h \in H} X^h)_{hH}; \mathbf{C})((\beta^{-1})).$$

Proof. Apply Corollary 4.7.6 in the case where $A = \mathrm{KU}$ is complex K-theory, $\mathbf{G} = \mu_{\mathbf{P}^\infty}$ is the multiplicative $\mathbf{P}$-divisible group over KU (endowed with the orientation of Construction 2.8.6), and $B = \mathbf{C} \otimes_S \mathrm{KU}$ is the complexification of KU. □

Corollary 4.7.8. *Let* H *be a finite group. Then the construction* $[V] \mapsto \chi_V$ *induces an isomorphism of complex vector spaces*

$$\chi : \mathbf{C} \otimes_{\mathbf{Z}} \mathrm{Rep}(H) \to \{\text{Class functions } H \to \mathbf{C}\}.$$

Proof. Combine Corollary 4.7.7 with Example 4.3.9. ☐

If $\mathbf{G}$ is an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and $X = K(H, d)$ is an Eilenberg-MacLane space associated to a finite group H, then Theorem 4.4.1 implies that the tempered function spectrum $A_{\mathbf{G}}^{X}$ is a projective A-module of finite rank. Using Theorem 4.7.1, we can prove a weak version of this assertion for π-finite spaces in general:

Proposition 4.7.9. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let X be a π-finite space. Then $A_{\mathbf{G}}^{X}$ is perfect as an A-module spectrum.*

Corollary 4.7.10. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let H be a finite group, and let X be a finite H-space. Then $A_{\mathbf{G}}^{X//H}$ is perfect as an A-module spectrum.*

Proof. As in the proof of Corollary 4.7.5, we can reduce to the case where $X = H_0 \backslash H$ is an orbit of H, in which case the orbispace quotient $X//H$ can be identified with $BH_0^{(-)}$ and the result follows from Proposition 4.7.9. ☐

Corollary 4.7.11. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Let X and Y be orbispaces. Suppose that $\mathsf{X} = X^{(-)}$ for a π-finite space X. Then the multiplication map*

$$m : A_{\mathbf{G}}^{\mathsf{X}} \otimes_A A_{\mathbf{G}}^{\mathsf{Y}} \to A_{\mathbf{G}}^{\mathsf{X} \times \mathsf{Y}}.$$

is an equivalence.

Proof. Combine Propositions 4.4.7 and 4.7.9. ☐

The proof of Proposition 4.7.9 will make use of the following general observation:

Lemma 4.7.12. *Let A be an $\mathbb{E}_\infty$-ring and let M be an A-module spectrum. Suppose that the functor*

$$\mathrm{CAlg}_A \to \mathrm{Mod}_A \qquad B \mapsto B \otimes_A M$$

preserves small limits. Then M is perfect.

Proof. Specializing to A-algebras of the form $A \oplus N$ for $N \in \mathrm{Mod}_A$, we deduce that the functor

$$\mathrm{Mod}_A \to \mathrm{Mod}_A \qquad N \mapsto N \otimes_A M$$

preserves small limits, and therefore admits a left adjoint (Corollary HTT.5.5.2.9). It follows that M is dualizable and therefore perfect as an A-module. ☐

Proof of Proposition 4.7.9. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let X be a π-finite space. Then the functor

$$\mathrm{CAlg}_A \to \mathrm{Mod}_A \qquad B \otimes_A A^X_{\mathbf{G}}$$

can be identified with $B \mapsto B^X_{\mathbf{G}}$ (by virtue of Theorem 4.7.1), and therefore preserves small limits (Lemma 4.2.11). Applying Lemma 4.7.12, we conclude that $A^X_{\mathbf{G}}$ is a perfect A-module spectrum. $\square$

We say that an $\mathbb{E}_\infty$-ring A is *Noetherian* if $\pi_0(A)$ is Noetherian and each homotopy group $\pi_n(A)$ is finitely generated as a module over $\pi_0(A)$. If A is Noetherian and M is a perfect A-module spectrum, then each homotopy group $\pi_n(M)$ is finitely generated as a module over $\pi_0(A)$. We therefore obtain the following:

Corollary 4.7.13. *Let A be a Noetherian $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A. Then:*

- (a) *If X is a π-finite space, then each of the tempered cohomology groups $A^n_{\mathbf{G}}(X)$ is finitely generated as a module over $\pi_0(A)$.*
- (b) *If H is a finite group and X is a finite H-space, then each of the tempered cohomology groups $A^n_{\mathbf{G}}(X//H)$ is finitely generated as a module over $\pi_0(A)$.*

4.8 Application: Character Theory for π-Finite Spaces

We now combine the results of §4.2, §4.3, and §4.7. We begin by studying the rational version of tempered cohomology.

Proposition 4.8.1. *Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A. Assume that, for every prime number p, the p-divisible group $\mathbf{G}_{(p)}$ has some constant height h_p, and set $\Lambda = \bigoplus_{p \in \mathbf{P}} (\mathbf{Q}_p / \mathbf{Z}_p)^{h_p}$. Then:*

- (a) *The $\mathbf{P}$-divisible group $\mathbf{G}$ admits a splitting algebra $B = \mathrm{Split}_\Lambda(\mathbf{G})$ (see Definition 2.7.7) which is faithfully flat over the rationalization $A_{\mathbf{Q}} = \mathbf{Q} \otimes_S A$.*
- (b) *Let X be a π-finite space and let S be the finite set of all homotopy classes of maps from the classifying space $B\widehat{\Lambda}$ into X, where $\widehat{\Lambda}$ is the Pontryagin dual of Λ. Then there is a canonical equivalence*

$$B \otimes_A A^X_{\mathbf{G}} \simeq \prod_{s \in S} B$$

of $\mathbb{E}_\infty$-algebras over B, which induces an isomorphism of graded rings

$$\pi_0(B) \otimes_{\pi_0(A)} A^*_{\mathbf{G}}(X) \simeq \prod_{s \in S} B^*(\{s\}) = B^*(S)$$

Proof. Note that, if $\mathbf{G}$ admits a splitting $\Lambda \to \mathbf{G}(A')$ for an $\mathbb{E}_\infty$-algebra A' over A, then the $\mathbf{P}$-divisible group $\mathbf{G}_{A'}$ is étale. The existence of an orientation of $\mathbf{G}$ then guarantees that A' is an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$ (Remark 2.6.17). It follows that we can identify a splitting algebras of $\mathbf{G}$ with a splitting algebras of $\mathbf{G}_{A_\mathbf{Q}}$, which exists (and is faithfully flat over $A_\mathbf{Q}$) by virtue of Proposition 2.7.9. This proves (a). To prove (b), let $\mathbf{G}_0 = 0$ denote the trivial $\mathbf{P}$-divisible group over $A_\mathbf{Q}$. Then the tautological splitting of $\mathbf{G}$ over B can be regarded as a splitting of the monomorphism $\mathbf{G}_0 \to \mathbf{G}_{A_\mathbf{Q}}$. We have a diagram of equivalences

$$B \otimes_A A^X_{\mathbf{G}} \xrightarrow{\sim} B^X_{\mathbf{G}} \xrightarrow{\sim} B^{\mathcal{L}^\Lambda(X^{(-)})}_{\mathbf{G}_0} \xleftarrow{\sim} B^{X^{B\hat{\Lambda}}}_{\mathbf{G}_0} \xrightarrow{\sim} B^{X^{B\hat{\Lambda}}} \xleftarrow{\sim} B^S$$

where the first map is supplied by Theorem 4.7.1, the second by Theorem 4.3.2, the third by Proposition 3.4.7, the fourth by Variant 4.2.6, and the fifth by the observation that the projection map $X^{B\hat{\Lambda}} \to \pi_0(X^{B\hat{\Lambda}}) = S$ induces an isomorphism on rational cohomology (since $X^{B\hat{\Lambda}}$ is a π-finite space). Passing to homotopy groups (and invoking the fact that B is flat over $A_\mathbf{Q}$, hence over A), we obtain the isomorphism of graded rings

$$\pi_0(B) \otimes_{\pi_0(A)} A^*_{\mathbf{G}}(X) \simeq \prod_{s \in S} B^*(\{s\}) = B^*(S).$$

□

Remark 4.8.2. In the situation of Proposition 4.8.1, the isomorphism of graded rings

$$\pi_0(B) \otimes_{\pi_0(A)} A^*_{\mathbf{G}}(X) \simeq B^*(S)$$

is equivariant with respect to the action of the profinite group $\mathrm{Aut}(\Lambda)$; here $\mathrm{Aut}(\Lambda)$ acts on the left hand side via its action on $B = \mathrm{Split}_\Lambda(\mathbf{G})$, and on the right hand side by combining its action on B and on the finite set $S = \pi_0(X^{B\hat{\Lambda}})$. Note that $\pi_0(B)$ can be regarded as a (profinite) Galois extension of $\pi_0(A_\mathbf{Q})$ with Galois group $\mathrm{Aut}(\Lambda)$ (Remark 2.7.10). Passing to invariants, we obtain an isomorphism of graded rings

$$\mathbf{Q} \otimes_\mathbf{Z} A^*_{\mathbf{G}}(X) \simeq B^*(S)^{\mathrm{Aut}(\Lambda)},$$

where the right hand side denotes the fixed points for the action of $\mathrm{Aut}(\Lambda)$ on the graded ring $B^*(S)$.

Remark 4.8.3. In the situation of Proposition 4.8.1, the existence of an isomorphism

$$\pi_0(B) \otimes_{\pi_0(A)} A^*_{\mathbf{G}}(X) \simeq \prod_{s \in S} B^*(\{s\})$$

guarantees that the tensor product $\pi_0(B) \otimes_{\pi_0(A)} A^*_{\mathbf{G}}(X)$ is a finitely generated free module over the coefficient ring $\pi_{-*}(B)$, having a canonical basis parametrized by the set of homotopy classes of maps $B\widehat{\Lambda} \to X$.

Corollary 4.8.4. *Let A be an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$ and let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over A. Assume that, for every prime number p, the p-divisible group $\mathbf{G}_{(p)}$ has some constant height h_p, and set $\Lambda = \bigoplus_{p \in \mathbf{P}} (\mathbf{Q}_p / \mathbf{Z}_p)^{h_p}$. Let X be a π-finite space. Then the graded ring $\mathbf{Q} \otimes_{\mathbf{Z}} A^*_{\mathbf{G}}(X)$ is a projective module over the coefficient ring $\pi_{-*}(A_{\mathbf{Q}})$, with rank equal to the number of homotopy classes of maps from $B\widehat{\Lambda}$ into X.*

Proof. Combine Remark 4.8.3 with the faithful flatness of the map $\pi_0(A_{\mathbf{Q}}) \to \pi_0(\mathrm{Split}_\Lambda(\mathbf{G}))$. □

Corollary 4.8.5. *Let A be an $\mathbb{E}_\infty$-ring which is complex periodic and $K(n)$-local for some $n > 0$. Set $\Lambda = (\mathbf{Q}_p / \mathbf{Z}_p)^n$. Then:*

- (a) *The Quillen p-divisible group $\mathbf{G}^{\mathcal{Q}}_A$ admits a splitting algebra $B = \mathrm{Split}_\Lambda(\mathbf{G}^{\mathcal{Q}}_A)$ which is faithfully flat over the rationalization $A_{\mathbf{Q}} = \mathbf{Q} \otimes_S A$.*
- (b) *Let X be a π-finite space and let S be the finite set of all homotopy classes of maps from the classifying space $B\,\mathbf{Z}_p^n$ into X. Then there is a canonical equivalence*

  $$B \otimes_A A^X \simeq \prod_{s \in S} B$$

  of $\mathbb{E}_\infty$-algebras over B, which induces an isomorphism of graded rings

  $$\pi_0(B) \otimes_{\pi_0(A)} A^*(X) \simeq \prod_{s \in S} B^*(\{s\}) = B^*(S)$$

Proof. Combine Proposition 4.8.1 with Theorem 4.2.5. □

Let $K(n)$ denote the nth Morava K-theory (for some fixed prime number p). We say that a space X is *$K(n)$-finite* if each of the groups $K(n)^i(X)$ is finite-dimensional as a vector space over the field $\kappa = \pi_0(K(n))$. In this case, we refer to the difference

$$\chi_{K(n)}(X) = \dim_\kappa(K(n)^0(X)) - \dim_\kappa(K(n)^1(X))$$

as the *$K(n)$-Euler characteristic* of X.

Corollary 4.8.6. *Fix a prime number p and an integer $n > 0$, and let X be a π-finite space. Then X is $K(n)$-finite, and the $K(n)$-Euler characteristic $\chi_{K(n)}(X)$ is equal to the number of homotopy classes of maps $B\,\mathbf{Z}_p^n \to X$. In particular, we have $\chi_{K(n)}(X) \geqslant 0$, with equality if and only if X is empty.*

Proof. Let E be the Lubin-Tate spectrum associated to a formal group of height n over a perfect field κ, Without loss of generality, we may assume that $K(n)$ is the Morava K-theory associated to E. Let L denote the fraction field of the Lubin-Tate ring $\pi_0(E)$, and let $\mathbf{G}$ be the Quillen p-divisible group of E. Then the Atiyah-Segal comparison map $\zeta : E^X_{\mathbf{G}} \to E^X$ is an equivalence (Theorem 4.2.5), so that E^X is perfect as an E-module spectrum (Proposition 4.7.9). Let r be the number of homotopy classes of maps $B\,\mathbf{Z}_p^n \to X$. According to Corollary 4.8.5, the tensor product $\mathbf{Q} \otimes \pi_*(E^X)$ is a free module over $\mathbf{Q} \otimes \pi_*(E)$, of rank equal to r. In particular, we have

$$\dim_L(L \otimes_{\pi_0(E)} \pi_0(E^X)) = r \qquad \dim_L(L \otimes_{\pi_0(E)} \pi_{-1}(E^X)) = 0.$$

It will therefore suffice to prove the following general assertion:

$(*)$ Let M be a perfect module over the Lubin-Tate spectrum E. Then the integers

$$\chi_L(M) = \dim_L(L \otimes_{\pi_0(E)} \pi_0(M)) - \dim_L(L \otimes_{\pi_0(E)} \pi_{-1}(M))$$

$$\chi_\kappa(M) = \dim_\kappa(\pi_0(K(n) \otimes_E M)) - \dim_\kappa(\pi_{-1}(K(n) \otimes_E M))$$

are the same.

To prove $(*)$, let d denote the projective dimension of $\pi_0(M) \oplus \pi_{-1}(M)$ as a module over $\pi_0(E)$ (which is necessarily finite, since $\pi_0(E)$ is a regular local ring). We proceed by induction on d. If $d = 0$, then M can be written as a finite sum of copies of E and its suspension $\Sigma(E)$; in this case, the equality asserted by $(*)$ is clear. To treat the case $d > 0$, we observe that our assumption that M is perfect guarantees that the homotopy groups of M are finitely generated as modules over $\pi_0(E)$. Choose a fiber sequence $M' \to P \xrightarrow{u} M$ where P is a sum of copies of E and its suspension $\Sigma(E)$, and u induces a surjection $\pi_*(P) \to \pi_*(M)$. In this case, we have a short exact sequence of homotopy groups $0 \to \pi_*(M') \to \pi_*(P) \to \pi_*(M) \to 0$. It follows that the homotopy groups of M' and P have projective dimension $< d$ over the Lubin-Tate ring $\pi_0(E)$, so that our inductive hypothesis (and the additivity of the Euler characteristics defined in $(*)$) supplies an identity

$$\chi_L(M) = \chi_L(P) - \chi_L(M') = \chi_\kappa(P) - \chi_\kappa(M') = \chi_\kappa(M).$$

□

4.9 Application: The Completion Theorem

Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. For every finite abelian group H, Proposition 4.2.8 implies that the Atiyah-Segal comparison map

$$\zeta : A^{BH}_{\mathbf{G}} \to A^{BH}$$

exhibits A^{BH} as the completion of $A^{BH}_{\mathbf{G}}$ with respect to the augmentation ideal I_H. Our goal in this section is to prove a more general version of this result (Theorem 4.9.2), where we replace the classifying space BH by an orbispace quotient $X//H$ and we drop the assumption that H is abelian. Here we potentially encounter a technical problem: when H is not abelian, it is not clear that the augmentation ideal $I_H \subseteq A^0_{\mathbf{G}}(BH)$ is finitely generated. To address this point, we will assume that the $\mathbb{E}_\infty$-ring A is Noetherian. This guarantees that $A^0_{\mathbf{G}}(BH)$ is finitely generated as a module over $\pi_0(A)$ (Corollary 4.7.13), and therefore a Noetherian ring.

Remark 4.9.1. Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, and let H be a finite group. For every H-space X, the canonical map of orbispaces

$$X//H \to *//H = BH^{(-)}$$

induces a homomorphism of tempered cohomology rings $A^*_{\mathbf{G}}(BH) \to A^*_{\mathbf{G}}(X//H)$. In particular, we can view each tempered cohomology group $A^n_{\mathbf{G}}(X//H)$ as a module over the commutative ring $A^0_{\mathbf{G}}(BH)$. If A is Noetherian, $\mathbf{G}$ is oriented, and X is a finite H-space, then $A^n_{\mathbf{G}}(X//H)$ is finitely generated as a module over $A^0_{\mathbf{G}}(BH)$ (since it is already finitely generated as a module over $\pi_0(A)$, by virtue of Corollary 4.7.13).

We can now state our main result.

Theorem 4.9.2 (Tempered Atiyah-Segal Completion Theorem)**.** *Let A be a Noetherian $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A, let H be a finite group, and let $I_H \subseteq A^0_{\mathbf{G}}(BH)$ be the augmentation ideal of Notation 4.2.7. Then, for every finite H-space X, the Atiyah-Segal comparison map*

$$\zeta : A^*_{\mathbf{G}}(X//H) \to A^*(X_{hH})$$

exhibits each $A^n(X_{hH})$ as the I_H-adic completion of $A^n_{\mathbf{G}}(X//H)$. That is, it induces an isomorphism

$$A^n(X_{hH}) \simeq \varprojlim_m A^n_{\mathbf{G}}(X//H)/I^m_H A^n_{\mathbf{G}}(X//H)$$

.

Corollary 4.9.3 (Atiyah)**.** *Let H be a finite group and let X be a finite H-space. Then the comparison map $\zeta : \mathrm{KU}^*_H(X) \to \mathrm{KU}^*(X_{hH})$ exhibits $\mathrm{KU}^*(X_{hH})$ as the completion of $\mathrm{KU}^*_H(X)$ with respect to the augmentation ideal of $\mathrm{Rep}(H)$.*

Proof. Combine Theorem 4.9.2 with Theorem 4.1.2. □

Remark 4.9.4. For a general version of Atiyah's completion theorem in equivariant stable homotopy theory (closely related to our Theorem 4.9.2), we refer the reader to the work of Greenlees-May ([4]).

Theorem 4.9.2 is a consequence of a more basic spectrum-level completion theorem, which does not require the finiteness of X.

Theorem 4.9.5. *Let A be a Noetherian $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A, and let H be a finite group. Let X be an H-space, and regard $A_{\mathbf{G}}^{X//H}$ as a module spectrum over the $\mathbb{E}_\infty$-ring $A_{\mathbf{G}}^{BH}$. Then the Atiyah-Segal comparison map $\zeta : A_{\mathbf{G}}^{X//H} \to A^{X_{hH}}$ exhibits $A^{X_{hH}}$ as the I_H-completion of $A_{\mathbf{G}}^{X//H}$, where I_H is the augmentation ideal of Notation 4.2.7.*

Proof of Theorem 4.9.2 from Theorem 4.9.5. Let M be a module spectrum over the $\mathbb{E}_\infty$-ring $A_{\mathbf{G}}^{BH}$, and let $\widehat{M}$ denote the I_H-completion of M. If each homotopy group of M is finitely generated as a module over $A^0_{\mathbf{G}}(BH)$, then the canonical map $\pi_*(M) \to \pi_*(\widehat{M})$ exhibits each $\pi_n(\widehat{M})$ as the classical I_H-adic completion of $\pi_n(M)$ (Corollary SAG.II.4.3.6.6). When X is a finite H-space, this finiteness hypothesis is satisfied in the case $M = A_{\mathbf{G}}^{X//H}$ (Remark 4.9.1). In this case, Theorem 4.9.5 allows us to identify $\widehat{M}$ with the function spectrum $A^{X_{hH}}$, so that the cohomology groups $A^n(X_{hH}) \simeq \pi_{-n}(\widehat{M})$ are the classical I_H-adic completions of the tempered cohomology groups $A^n_{\mathbf{G}}(X//H) \simeq \pi_{-n}(M)$. □

We will reduce the general case of Theorem 4.9.5 to the abelian case using the following:

Lemma 4.9.6. *Let A be a Noetherian $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A. Let H be a finite group and let $H_0 \subseteq H$ be an abelian subgroup. Let $I_H \subseteq A^0_{\mathbf{G}}(BH)$ be the augmentation ideal of Notation 4.2.7, and define $I_{H_0} \subseteq A^0_{\mathbf{G}}(BH_0)$ similarly. Then there exists an integer $m \gg 0$ such that $I^m_{H_0} \subseteq I_H A^0_{\mathbf{G}}(BH_0) \subseteq I_{H_0}$.*

Remark 4.9.7. Lemma 4.9.6 admits an algebro-geometric interpretation. The commutative diagram of spaces

$$\begin{tikzcd} EH_0 \arrow[r, "\sim"] \arrow[d] & EH \arrow[d] \\ BH_0 \arrow[r] & BH \end{tikzcd}$$

determines a commutative diagram of affine schemes

$$\begin{tikzcd} \operatorname{Spec}(\pi_0(A)) \arrow[r, "\mathrm{id}"] \arrow[d, "i_0"] & \operatorname{Spec}(\pi_0(A)) \arrow[d, "i"] \\ \operatorname{Spec}(A^0_{\mathbf{G}}(BH_0)) \arrow[r, "f"] & \operatorname{Spec}(A^0_{\mathbf{G}}(BH)), \end{tikzcd}$$

where i and i_0 are closed immersions. Lemma 4.9.6 is equivalent to the assertion that this diagram is a pullback square at the level of the underlying topological spaces: that is, a point x of the space $|\operatorname{Spec}(A^0_{\mathbf{G}}(BH_0))|$ belongs to the image of i_0 if and only if $f(x)$ belongs to the image of i.

Proof of Lemma 4.9.6. We use the formulation of Remark 4.9.7. Let x be a point of the Zariski spectrum $|\operatorname{Spec}(A^0_{\mathbf{G}}(BH_0))|$ which does not belong to the image of i_0; we will show that $f(x) \in |\operatorname{Spec}(A^0_{\mathbf{G}}(BH))|$ does not belong to the image of i. Let $\mathfrak{p} \subseteq \pi_0(A)$ be the prime ideal corresponding to the image of x in $|\operatorname{Spec}(\pi_0(A))|$, let $A_{\mathfrak{p}}$ be the localization of A with respect to $\mathfrak{p}$, and let $\widehat{A}$ be the completion of $A_{\mathfrak{p}}$. Since A is Noetherian, $\widehat{A}$ is flat over A. It follows that the natural maps

$$\pi_0(\widehat{A}) \otimes_{\pi_0(A)} A^0_{\mathbf{G}}(BH_0) \to \widehat{A}^0_{\mathbf{G}}(BH_0) \qquad \pi_0(\widehat{A}) \otimes_{\pi_0(A)} A^0_{\mathbf{G}}(BH) \to \widehat{A}^0_{\mathbf{G}}(BH)$$

are isomorphisms (Corollary 4.7.3). We may therefore replace A by $\widehat{A}$ and thereby reduce to the case where $\pi_0(A)$ is complete local Noetherian ring and x lies over the closed point of $|\operatorname{Spec}(\pi_0(A))|$.

Let κ be the residue field of the local ring $\pi_0(A)$. If κ has characteristic p, then the p-divisible group $\mathbf{G}_{(p)}$ admits a connected-étale sequence

$$0 \to \mathbf{G}_0 \to \mathbf{G}_{(p)} \to \mathbf{G}'' \to 0$$

where $\mathbf{G}''$ is étale and the closed fiber of $\mathbf{G}_0$ is connected (Corollary Or.2.5.22). If κ has characteristic zero, set $\mathbf{G}_0 = 0$. In either case, we have a monomorphism of $\mathbf{P}$-divisible groups $f : \mathbf{G}_0 \to \mathbf{G}$ for which the quotient $\mathbf{G}/\mathbf{G}_0$ is étale. Let $\widehat{H}_0$ denote the Pontryagin dual of the finite abelian group H_0. We then have a short exact sequence

$$0 \to \mathbf{G}_0[\widehat{H}_0] \to \mathbf{G}[\widehat{H}_0] \xrightarrow{q} (\mathbf{G}/\mathbf{G}_0)[\widehat{H}_0] \to 0$$

of finite flat group schemes over A, where first term has connected fiber over the closed point of $|\operatorname{Spec}(\pi_0(A))|$ and the third term is étale over A, and the middle term has underlying topological space $|\operatorname{Spec}(A^0_{\mathbf{G}}(BH_0))|$. Consequently, our assumption that x does not belong to the image of i_0 guarantees that that its image under q does not belong to the zero section of $(\mathbf{G}/\mathbf{G}_0)[\widehat{H}_0]$.

Since $|\operatorname{Spec}(\pi_0(A))|$ is connected, the ℓ-divisible groups $(\mathbf{G}/\mathbf{G}_0)_{(\ell)}$ each have some constant height h_ℓ. Let Λ be the colattice given by the sum $\bigoplus_{\ell \in \mathbf{P}} (\mathbf{Q}_\ell / \mathbf{Z}_\ell)^{h_\ell}$. Applying Proposition 2.7.15, we deduce that f admits a splitting algebra $B = \operatorname{Split}_\Lambda(f)$ which is faithfully flat over A. Using Corollary 4.7.3 and Theorem 4.3.2, we obtain canonical isomorphisms

$$\pi_0(B) \otimes_{\pi_0(A)} A^0_{\mathbf{G}}(BH_0) \simeq B^0_{\mathbf{G}}(BH_0) \simeq \prod_{\alpha : \widehat{\Lambda} \to H_0} B^0_{\mathbf{G}_0}(BH_0)$$

$$\pi_0(B) \otimes_{\pi_0(A)} A^0_{\mathbf{G}}(BH) \simeq B^0_{\mathbf{G}}(BH) \simeq \prod_{\alpha : \widehat{\Lambda} \to H} B^0_{\mathbf{G_0}}(BZ(\alpha))$$

where $\widehat{\Lambda}$ denotes the Pontryagin dual of Λ, the second product is indexed by the collection of all conjugacy classes of homomorphisms $\alpha : \widehat{\Lambda} \to H$, and $Z(\alpha) \subseteq H$ denotes the centralizer of the image of α. Since $\pi_0(B)$ is faithfully flat over $\pi_0(A)$, we can lift x to a point $\widetilde{x}$ of the affine scheme

$$\operatorname{Spec}(\pi_0(B) \otimes_{\pi_0(A)} A^0_{\mathbf{G}}(BH_0)) \simeq \coprod_{\alpha : \widehat{\Lambda} \to H_0} \operatorname{Spec}(B^0_{\mathbf{G}_0}(BH_0)).$$

Our assumption that $q(x)$ is not contained in the zero section of $(\mathbf{G}/\mathbf{G}_0)[\widehat{H}_0]$ guarantees that $\widetilde{x}$ belongs to a component of the right hand side which corresponds to a *nontrivial* homomorphism $\alpha : \widehat{\Lambda} \to H_0$. Then the image of $\widetilde{x}$ in the fiber product

$$\operatorname{Spec}(\pi_0(B) \otimes_{\pi_0(A)} A^0_{\mathbf{G}}(BH)) \simeq \coprod_{\alpha : \widehat{\Lambda} \to H} \operatorname{Spec}(B^0_{\mathbf{G}_0}(BZ(\alpha))).$$

is contained in a summand which corresponds to a conjugacy class of nonzero maps $\widehat{\Lambda} \to H$. In particular, it is contained in the inverse image of $\operatorname{im}(i)$, so that $f(x)$ cannot be contained in the image of i. $\square$

Proof of Theorem 4.9.5. Let A be a Noetherian $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A, and let H be a finite group. We wish to show that, for every H-space X, the Atiyah-Segal comparison map $\zeta : A_{\mathbf{G}}^{X//H} \to A^{X_{hH}}$ exhibits $A^{X_{hH}}$ as the I_H-completion of $A_{\mathbf{G}}^{X//H}$. Let us regard the H-space X as a functor of ∞-categories $\mathrm{Orbit}(H)^{\mathrm{op}} \to \mathcal{S}$. Let $\mathrm{Orbit}(H)_{\mathrm{ab}} \subseteq \mathrm{Orbit}(H)$ be the full subcategory defined in Construction 3.2.16. Note that replacing X by the left Kan extension of $X|_{\mathrm{Orbit}(H)_{\mathrm{ab}}^{\mathrm{op}}}$ does not change the orbispace quotient $X//H$ or the homotopy orbit space X_{hH}; we may therefore assume without loss of generality that X is a left Kan extension of its restriction to $\mathrm{Orbit}(H)_{\mathrm{ab}}^{\mathrm{op}}$. In this case, we can write X as a colimit of H-spaces which are represented by orbits of the form H/H_0, where H_0 is an abelian subgroup of H. Since the constructions $X \mapsto A_{\mathbf{G}}^{X//H}$ and $X \mapsto A^{X_{hH}}$ carry colimits of H-spaces to limits in CAlg_A (and the I_H-completion functor commutes with limits), it will suffice to prove Theorem 4.9.5 in the special case where X has the form H/H_0. In this case, we are reduced to proving that the Atiyah-Segal comparison map $\zeta : A_{\mathbf{G}}^{BH_0} \to A^{BH_0}$ exhibits A^{BH_0} as the completion of $A_{\mathbf{G}}^{BH_0}$ with respect to the augmentation ideal $I_H \subseteq A_{\mathbf{G}}^0(BH)$. Equivalently, we wish to show that ζ exhibits A^{BH_0} as the completion of $A_{\mathbf{G}}^{BH_0}$ with respect to the ideal $I_H A_{\mathbf{G}}^0(BH_0) \subseteq A_{\mathbf{G}}^0(BH_0)$. By virtue of Lemma 4.9.6, we can replace H by H_0 and thereby reduce to the situation treated in Proposition 4.2.8. □

5 Tempered Local Systems

Let A be an $\mathbb{E}_\infty$-ring. For any space X, we let $\mathrm{LocSys}_A(X)$ denote the ∞-category $\mathrm{Fun}(X, \mathrm{Mod}_A)$ of local systems of A-modules on X, and we let $\underline{A}_X \in \mathrm{LocSys}_A(X)$ denote the constant local system taking the value $A \in \mathrm{Mod}_A$. The construction $X \mapsto \mathrm{LocSys}_A(X)$ can be regarded as a categorification of the functor $X \mapsto A^*(X)$ in the following sense: for any space X, we have a canonical isomorphism of graded rings

$$A^*(X) \simeq \mathrm{Ext}^*_{\mathrm{LocSys}_A(X)}(\underline{A}_X, \underline{A}_X).$$

Our goal in this section is to show that if $\mathbf{G}$ is an oriented $\mathbf{P}$-divisible group over A, then our theory of tempered cohomology $X \mapsto A_{\mathbf{G}}^*(X)$ admits an analogous categorification. More precisely, we will associate to each space X an ∞-category $\mathrm{LocSys}_{\mathbf{G}}(X)$, whose objects we refer to as $\mathbf{G}$-*tempered local systems on* X. This stable ∞-category contains a distinguished object which we will denote by $\underline{A}_X$, and the tempered cohomology ring $A_{\mathbf{G}}^*(X)$ can be recovered as the endomorphism ring $\mathrm{Ext}^*_{\mathrm{LocSys}_{\mathbf{G}}(X)}(\underline{A}_X, \underline{A}_X)$ (Remark 5.1.20).

Before giving a formal definition, let us begin by describing some of the essential features of our construction:

(a) For any space X, there is a forgetful functor $U : \mathrm{LocSys}_{\mathbf{G}}(X) \to \mathrm{LocSys}_A(X)$, which can be regarded as a categorification of the Atiyah-Segal comparison map on tempered cohomology (Construction 4.2.2).

(b) To every $\mathbf{G}$-tempered local system $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(X)$, we can associate a spectrum $\Gamma_{\mathbf{G}}(X; \mathscr{F})$ of *tempered global sections* of $\mathscr{F}$, which is a module over the tempered function spectrum $A^X_{\mathbf{G}}$ of Construction 4.0.3.

(c) Let $\mathscr{F}$ is a $\mathbf{G}$-tempered local system on X, let $U(\mathscr{F})$ denotes the underlying local system of $\mathscr{F}$, and let $\Gamma(X; U(\mathscr{F}))$ denote the spectrum of global sections of $U(\mathscr{F})$ (in other words, the homotopy limit of the functor $U(\mathscr{F}) : X \to \mathrm{Mod}_A$). Then there is a comparison map

$$\zeta_{\mathscr{F}} : \Gamma_{\mathbf{G}}(X; \mathscr{F}) \to \Gamma(X; U(\mathscr{F})),$$

which we will refer to as the *Atiyah-Segal comparison map with coefficients in* $\mathscr{F}$ (in the special case $\mathscr{F} = \underline{A}_X$, it specializes to the Atiyah-Segal comparison map of Construction 4.0.3).

(d) Suppose that $X = BH$ is the classifying space of a finite abelian group H, and let $\mathscr{F}$ be a $\mathbf{G}$-tempered local system on X. Then the comparison map

$$\zeta_{\mathscr{F}} : \Gamma_{\mathbf{G}}(X; \mathscr{F}) \to \Gamma(X; U(\mathscr{F}))$$

of (c) exhibits $\Gamma(X; U(\mathscr{F}))$ as the completion of $\Gamma_{\mathbf{G}}(X; \mathscr{F})$ with respect to the augmentation ideal $I_H \subseteq A^0_{\mathbf{G}}(BH)$ of Notation 4.2.7.

The simplest nontrivial case to consider is where $X = BC_p$ is the classifying space of the the cyclic group $C_p = \mathbf{Z}/p\mathbf{Z}$ of order p, for some prime number p. In this case, properties (a) through (d) provide a complete description of the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(X)$. More precisely, suppose we are given a local system $\mathscr{F}_0 \in \mathrm{LocSys}_A(X)$, which we can view an object of $\mathrm{LocSys}_A(X)$ as an A-module spectrum M equipped with an action of C_p (in the "naive" sense of Definition 3.2.1). Then the global sections spectrum $\Gamma(X; \mathscr{F}_0)$ can be identified with the homotopy fixed point spectrum M^{hC_p}. This homotopy fixed point has the structure of a module over the $\mathbb{E}_\infty$-ring A^{BC_p}, and can therefore also be viewed as a module over the tempered function spectrum $A^{BC_p}_{\mathbf{G}}$

by means of the Atiyah-Segal comparison map $\zeta : A_{\mathbf{G}}^{BC_p} \to A^{BC_p}$. Promoting $\mathscr{F}_0$ to a *tempered* local system $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(X)$ then equivalent to choosing an $A_{\mathbf{G}}^{BC_p}$-module $N = \Gamma_{\mathbf{G}}(X; \mathscr{F})$ equipped with an $A_{\mathbf{G}}^{BC_p}$-linear map $\zeta : N \to M^{hC_p}$ which exhibits M^{hC_p} as the completion of N with respect to the augmentation ideal $I_{C_p} \subseteq A_{\mathbf{G}}^0(BC_p)$ (see Example 5.4.5).

Let us now consider the more general situation where $X = BH$ is the classifying space of a finite abelian group H. In this case, a tempered local system $\mathscr{F}$ on X generally cannot be recovered from the data (a), (b), and (c) alone. For every subgroup $H_0 \subseteq H$, we can restrict $\mathscr{F}$ to a tempered local system on the classifying space BH_0, which has a tempered global section spectrum $\Gamma_{\mathbf{G}}(BH_0; \mathscr{F}|_{BH_0})$ (which is a module over the tempered function spectrum $A_{\mathbf{G}}^{BH_0}$). This tempered function spectrum then carries an action of the quotient group H/H_0 (which acts by deck transformations on the finite covering map $BH_0 \to BH$). This construction recovers the datum of the underlying local system $U(\mathscr{F})$ in the special case where $H_0 = \{0\}$, and the datum of the module $\Gamma_{\mathbf{G}}(X; \mathscr{F})$ in the special case $H_0 = H$. To reconstruct a general tempered local system on $X = BH$, one must specify *all* of the spectra $\Gamma_{\mathbf{G}}(BH_0; \mathscr{F}|_{BH_0})$, along with relative versions of the comparison maps (c) (which we require to satisfy a suitable generalization of (d): see Definition 5.2.4).

As with the theory of $\mathbf{G}$-tempered cohomology, we will define the notion of $\mathbf{G}$-tempered local system on a general spaces X by a Kan extension procedure. Roughly speaking, to give a $\mathbf{G}$-tempered local system $\mathscr{F}$ on X, one must give a compatible family of $\mathbf{G}$-tempered local systems $\{\mathscr{F}|_T \in \mathrm{LocSys}_{\mathbf{G}}(T)\}_{T \to \mathscr{T}_{/X}}$, indexed by the collection of all maps $T \to X$ where T is the classifying space of a finite abelian group. Note that the role of X here is a bit indirect: the input to the construction is really the orbispace

$$X^{(-)} : \mathscr{T}^{\mathrm{op}} \to \mathcal{S} \qquad T \mapsto \mathrm{Map}_{\mathcal{S}}(T, X)$$

represented by X (Example 3.1.6). For various applications, it will be consider a more general construction $\mathsf{X} \mapsto \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ whose input is an arbitrary orbispace X (through we will ultimately be most interested in the special case where $\mathsf{X} = X^{(-)}$ is the orbispace represented by a π-finite space X).

Let us now outline the contents of this section. We begin in §5.1 by associating to each orbispace X an ∞-category $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ of $\mathbf{G}$-*pretempered local systems* (Construction 5.1.3), where we do not require the analogue of the Atiyah-Segal completion theorem: in the special case where $X = BC_p$ is the classifying space of a cyclic group $C_p = \mathbf{Z}/p\,\mathbf{Z}$ of prime order, an object of $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ can be identified with a triple (M, N, ζ), where M is an A-module spectrum equipped with an action of

C_p and $\zeta : N \to M^{hC_p}$ is a morphism of modules over the tempered function spectrum $A_{\mathbf{G}}^{BC_p}$ which is not required to satisfy any additional conditions (see Proposition 5.1.12 for a more general description, which applies to the classifying space of any finite abelian group). In §5.2, we define the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ to be a certain full subcategory of $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ (Definition 5.2.4). The definition makes sense in general for any *preoriented* $\mathbf{P}$-divisible group $\mathbf{G}$. However, to show that it has good properties (and to guarantee a good supply of examples of tempered local systems), we will need to assume that $\mathbf{G}$ is oriented. In §5.3, we illustrate this point by showing that if $\mathbf{G}$ is oriented, then the full subcategory $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \subseteq \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ is closed under colimits: that is, colimits of $\mathbf{G}$-tempered local systems can be computed *levelwise*. In §5.4, we use similar techniques to analyze the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(T)$ in the case where T is the classifying space of a finite abelian group: under the assumption that $\mathbf{G}$ is oriented, we show that the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(T)$ admits a concrete description which generalizes the discussion above in the case $T = BC_p$ (Proposition 5.4.2). The theory of tempered local systems in general is controlled by its behavior on the objects of $\mathscr{T}$: for every orbispace X we have a canonical equivalence of ∞-categories

$$\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \simeq \varprojlim_{T \to \mathsf{X}} \mathrm{LocSys}_{\mathbf{G}}(T),$$

where T ranges over classifying spaces of finite abelian groups (Remark 5.2.11). In fact, we do not even need to allow *all* finite abelian groups: in §5.6, we show that it suffices to allow finite abelian groups $H \simeq \bigoplus_{p \in \mathbf{P}} H_{(p)}$ with the property that each $H_{(p)}$ can be generated by at most h_p elements, where h_p is (any upper bound for) the height of the p-divisible group $\mathbf{G}_{(p)}$ (Theorem 5.6.2). In particular, if $\mathbf{G}$ is a p-divisible group, then we can take T to range over classifying spaces of finite abelian p-groups (Example 5.6.5).

For any orbispace X, the inclusion of stable ∞-categories

$$\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \hookrightarrow \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$$

admits a left adjoint L (Proposition 5.2.12). Consequently, we can identify $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ as the quotient of $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ by a stable subcategory $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{nul}}(\mathsf{X}) \subseteq \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ (namely, the stable subcategory annihilated by the functor L). In §5.7, we give an explicit description of the subcategory $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{nul}}(\mathsf{X})$ (assuming that $\mathbf{G}$ is oriented) in terms of the geometry of the $\mathbf{P}$-divisible group $\mathbf{G}$ (Theorem 5.7.3). We apply this result in §5.8 to define construct a *tensor product* of $\mathbf{G}$-tempered local systems, by localizing the "levelwise" tensor product on $\mathbf{G}$-pretempered local systems.

Remark 5.0.1 (Relationship with Equivariant Stable Homotopy Theory)**.** Let $X = BC_p$ be the classifying space of a cyclic group of prime order. As indicated above, we can identify objects $\mathscr{F} \in \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(X)$ with triples (M, N, ζ), where M is an A-module spectrum equipped with an action of C_p and $\zeta : N \to M^{hC_p}$ is a morphism of $A_{\mathbf{G}}^{BC_p}$-modules. From this data, we can assemble a "naive" C_p-spectrum (that is, spectrum object of the ∞-category $\mathcal{S}_{C_p}$ of Definition 3.2.10), having underlying spectrum M and C_p-fixed point spectrum N (so that ζ plays the role of the comparison map of genuine and homotopy fixed points). If $\mathscr{F}$ belongs to the subcategory $\mathrm{LocSys}_{\mathbf{G}}(X)$, then this "naive" C_p-spectrum can be promoted to a "genuine" C_p-spectrum: that is, we can complete the following diagram:

$$\begin{tikzcd}
M_{hC_p} \arrow[rr, "\mathrm{Norm}"] \arrow[dr, dashed] & & M^{hC_p} \\
& N. \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

To see this, note that if ζ exhibits M^{hC_p} as the completion of N with respect to the augmentation ideal $I_{C_p} \subseteq A^0_{\mathbf{G}}(BC_p)$, then it induces an equivalence $\Gamma_{I_{C_p}}(N) \simeq \Gamma_{I_{C_p}}(M^{hC_p})$, whose codomain can be identified with the homotopy orbit spectrum M_{hC_p}. In the case of the constant local system, this recovers the construction described in Remark 4.6.7.

More generally, if H is any finite group, then our theory of $\mathbf{G}$-tempered local systems on the classifying space BH can be formulated in terms of H-equivariant stable homotopy theory. We will return to this point in [10].

5.1 Pretempered Local Systems

We begin by introducing some notation.

Notation 5.1.1. Let $\mathscr{T}$ be the ∞-category introduced in Notation 3.1.1 and let $\mathcal{OS} = \mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathcal{S})$ denote the ∞-category of orbispaces. For each orbispace $\mathsf{X} \in \mathcal{OS}$, we let $\mathscr{T}_{/\mathsf{X}}$ denote the fiber product $\mathscr{T} \times_{\mathcal{OS}} \mathcal{OS}_{/\mathsf{X}}$. More informally, $\mathscr{T}_{/\mathsf{X}}$ is the ∞-category whose objects are pairs (T, η), where T is an object of $\mathscr{T}$ and $\eta : T^{(-)} \to \mathsf{X}$ is a map of orbispaces, or equivalently a point of the space X^T. We will write $\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}$ for the opposite of the ∞-category $\mathscr{T}_{/\mathsf{X}}$. We will generally abuse notation by identifying an object (T, η) of ∞-category $\mathscr{T}_{/\mathsf{X}}$ with the underlying object $T \in \mathscr{T}$.

Notation 5.1.2. Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, so that $\mathbf{G}$ determines a functor

$$A_{\mathbf{G}} : \mathscr{T}^{\mathrm{op}} \to \mathrm{CAlg}_A \qquad T \mapsto A_{\mathbf{G}}^T$$

(see Notation 4.0.1). If X is an orbispace, we let $\underline{A}_{\mathsf{X}}$ denote the composite functor

$$\mathscr{T}_{/\mathsf{X}}^{\mathrm{op}} \to \mathscr{T}^{\mathrm{op}} \xrightarrow{A_{\mathbf{G}}} \mathrm{CAlg},$$

which we view as a commutative algebra in the symmetric monoidal ∞-category $\mathrm{Fun}(\mathscr{T}_{/\mathsf{X}}^{\mathrm{op}}, \mathrm{Sp})$. We let $\mathrm{Mod}_{\underline{A}_{\mathsf{X}}} = \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}(\mathrm{Fun}(\mathscr{T}_{/\mathsf{X}}^{\mathrm{op}}, \mathrm{Sp}))$ denote the ∞-category of $\underline{A}_{\mathsf{X}}$-module objects of $\mathrm{Fun}(\mathscr{T}_{/\mathsf{X}}^{\mathrm{op}}, \mathrm{Sp})$.

More informally, an object $\mathscr{F} \in \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}$ is a rule which associates to each object T of $\mathscr{T}_{/\mathsf{X}}$ a module $\mathscr{F}(T)$ for the $\mathbf{G}$-tempered function spectrum $A_{\mathbf{G}}^T$, and associates to each morphism $\alpha : T' \to T$ in $\mathscr{T}_{/\mathsf{X}}$ an $A_{\mathbf{G}}^T$-linear map $\mathscr{F}(T) \to \mathscr{F}(T')$ (compatible with composition up to coherent homotopy).

Construction 5.1.3 (Pretempered Local Systems)**.** Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. Let X be an orbispace, and let $\underline{A}_{\mathsf{X}}$ be as in Notation 5.1.2. A $\mathbf{G}$-*pretempered local system* is an $\underline{A}_{\mathsf{X}}$-module object of the functor ∞-category $\mathrm{Fun}(\mathscr{T}_{/\mathsf{X}}^{\mathrm{op}}, \mathrm{Sp})$ which satisfies the following condition:

- (A) Let $\alpha : T' \to T$ be a morphism in $\mathscr{T}_{/\mathsf{X}}$ with connected homotopy fibers (that is, α induces a surjection of fundamental groups $\pi_1(T') \twoheadrightarrow \pi_1(T)$). Then $\mathscr{F}(\alpha)$ induces an equivalence $A_{\mathbf{G}}^{T'} \otimes_{A_{\mathbf{G}}^T} \mathscr{F}(T) \to \mathscr{F}(T')$ of $A_{\mathbf{G}}^{T'}$-modules.

We will write $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ to denote the full subcategory of $\mathrm{Mod}_{\underline{A}_{\mathsf{X}}}$ spanned by the $\mathbf{G}$-pretempered local systems.

Variant 5.1.4. Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, and let X be a space. We define a $\mathbf{G}$-*pretempered local system on* X to be a $\mathbf{G}$-pretempered local system on the orbispace $X^{(-)}$ represented by X (Example 3.1.6). We let $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(X) = \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(X^{(-)})$ denote the ∞-category of $\mathbf{G}$-pretempered local systems on X.

Example 5.1.5. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let X be an orbispace. Then $\underline{A}_{\mathsf{X}}$ is a $\mathbf{G}$-pretempered local system on X (when viewed as a module over itself). We will refer to $\underline{A}_{\mathsf{X}}$ as the *trivial* $\mathbf{G}$-*pretempered local system.* In the special case where $\mathsf{X} = X^{(-)}$ is the orbispace represented by a space X, we will denote $\underline{A}_{\mathsf{X}}$ by $\underline{A}_X$.

Remark 5.1.6 (Pullback of $\mathbf{G}$-Pretempered Local Systems)**.** Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of orbispaces. Then composition with f induces a functor of ∞-categories $F : \mathscr{T}_{/\mathsf{X}} \to \mathscr{T}_{/\mathsf{Y}}$ which is compatible with the projection to $\mathscr{T}$. Precomposition with F then induces a functor $f^* : \mathrm{Mod}_{\underline{A}_{\mathsf{Y}}} \to \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}$ which restricts to a functor of full subcategories

$$f^* : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}).$$

If $\mathscr{F}$ is a $\mathbf{G}$-pretemprered local system on Y, then $f^* \mathscr{F}$ is a $\mathbf{G}$-pretempered local system on X which we will refer to as the *pullback of $\mathscr{F}$ along f*. Concretely, it is given by the formula

$$(f^* \mathscr{F})(T^{(-)} \xrightarrow{\eta} \mathsf{X}) = \mathscr{F}(T^{(-)} \xrightarrow{f \circ \eta} \mathsf{Y}).$$

Remark 5.1.7. Let X be an orbispace, and suppose we are given a family of maps $\{f_\alpha : \mathsf{X}_\alpha \to \mathsf{X}\}$ with the property that, for every object $T \in \mathscr{T}$, the induced map

$$\coprod_{\alpha} \pi_0(\mathsf{X}^T_\alpha) \to \pi_0(\mathsf{X}^T)$$

is surjective. Let $\mathscr{F}$ be an $\underline{A}_{\mathsf{X}}$-module object of the functor ∞-category $\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}, \mathrm{Sp})$. If each pullback $f^*_\alpha \mathscr{F}$ is a $\mathbf{G}$-pretempered local system on X_α, then $\mathscr{F}$ is a $\mathbf{G}$-pretempered local system on X.

In the sequel, we will need a more refined version of Remark 5.1.6, which allows us to view the construction $\mathsf{X} \mapsto \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ as a functor of ∞-categories.

Construction 5.1.8. Let $\mathrm{Mod}(\mathrm{Sp})$ denote the ∞-category whose objects are pairs (B, M), where B is an $\mathbb{E}_\infty$-ring and M is a B-module spectrum. The construction $(B, M) \mapsto B$ then determines a forgetful functor $q : \mathrm{Mod}(\mathrm{Sp}) \to \mathrm{CAlg}(\mathrm{Sp}) = \mathrm{CAlg}$.

Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and let $\overline{\mathscr{T}}$ be an ∞-category equipped with a functor $\overline{\mathscr{T}} \to \mathscr{T}$. We let $\mathrm{Fun}_{\mathrm{CAlg}}(\overline{\mathscr{T}}^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp}))$ denote the ∞-category given by the fiber product

$$\mathrm{Fun}(\overline{\mathscr{T}}^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp})) \times_{\mathrm{Fun}(\overline{\mathscr{T}}^{\mathrm{op}}, \mathrm{CAlg})} \{A_{\mathbf{G}}\},$$

so that the objects of $\mathrm{Fun}_{\mathrm{CAlg}}(\overline{\mathscr{T}}^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp}))$ can be identified with functors $\mathscr{F}$ which fit into a commutative diagram

$$\begin{tikzcd}
\overline{\mathscr{T}}^{\mathrm{op}} \arrow[r, "\mathscr{F}"] \arrow[d] & \mathrm{Mod}(\mathrm{Sp}) \arrow[d] \\
\mathscr{T}^{\mathrm{op}} \arrow[r, "A_{\mathbf{G}}"] & \mathrm{CAlg}.
\end{tikzcd}$$

Note that, if $\overline{\mathscr{T}} \to \mathscr{T}$ is a right fibration classified by a functor $\mathsf{X} : \mathscr{T}^{\mathrm{op}} \to \mathcal{S}$, then $\overline{\mathscr{T}}$ is equivalent to the ∞-category $\mathscr{T}_{/\mathsf{X}}$ of Notation 5.1.1. In this case, we obtain an equivalence of ∞-categories $\mathrm{Mod}_{\underline{A}_{\mathsf{X}}} \simeq \mathrm{Fun}_{\mathrm{CAlg}}(\overline{\mathscr{T}}^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp}))$. We let $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\overline{\mathscr{T}})$ denote the essential image of $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ under this equivalence.

Let $\mathcal{Q}$ denote the ordinary category whose objects are simplicial sets $\overline{\mathscr{T}}$ equipped with a right fibration $\overline{\mathscr{T}} \to \mathscr{T}$. We view $\mathcal{Q}$ as a simplicially enriched category, with

$$\mathrm{Hom}_{\mathcal{Q}}(\overline{\mathscr{T}}, \overline{\mathscr{T}}')_{\bullet} = \mathrm{Hom}_{(\mathcal{S}\mathrm{et}_{\Delta})_{/\mathscr{T}}}(\overline{\mathscr{T}} \times (\Delta^{\bullet})^{\mathrm{op}}, \overline{\mathscr{T}}').$$

Then the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\mathcal{Q})$ is an ∞-category. Moreover, the construction $\overline{\mathscr{T}} \mapsto \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\overline{\mathscr{T}})$ determines a simplicially enriched functor from $\mathcal{Q}^{\mathrm{op}}$ to the category of (large) simplicial sets. Passing to homotopy coherent nerves, we obtain a functor of ∞-categories

$$\theta : \mathrm{N}^{\mathrm{hc}}(\mathcal{Q})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty} \qquad \overline{\mathscr{T}} \mapsto \mathrm{Fun}_{\mathrm{CAlg}}(\overline{\mathscr{T}}^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp})).$$

Note that there is a canonical equivalence of ∞-categories

$$\psi : \mathcal{OS} = \mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{N}^{\mathrm{hc}}(\mathcal{Q}),$$

which carries each orbispace X to a right fibration $\overline{\mathscr{T}} \to \mathscr{T}$ which is classified by the functor $\mathsf{X} : \mathscr{T}^{\mathrm{op}} \to \mathcal{S}$ and can therefore be identified with the ∞-category $\mathscr{T}_{/\mathsf{X}}$ of Notation 5.1.1 (see §HTT.5.1.1). Composing this equivalence with θ, we obtain a functor $\mathcal{OS}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$. We will abuse notation by denoting this functor by $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\bullet) : \mathcal{OS}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$. By construction, its value on an orbispace X is equivalent to the ∞-category $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ of Construction 5.1.3, and its value on a morphism of orbispaces is given by the construction of Remark 5.1.6.

Proposition 5.1.9. *Let A be an $\mathbb{E}_{\infty}$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. Then the functor*

$$\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\bullet) : \mathcal{OS}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$$

of Construction 5.1.8 preserves small limits.

Proof. Let us abuse notation by identifying $\mathcal{OS} = \mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ with the ∞-category $\mathrm{N}^{\mathrm{hc}}(\mathcal{Q})$ appearing in Construction 5.1.8. It follows from Theorem HTT.2.2.1.2 that the functor

$$\mathrm{N}^{\mathrm{hc}}(\mathcal{Q})^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty} \qquad \overline{\mathscr{T}} \mapsto \mathrm{Fun}_{\mathrm{CAlg}}(\overline{\mathscr{T}}^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp}))$$

preserves small limits. We wish to show that the subfunctor $\overline{\mathscr{T}} \mapsto \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\overline{\mathscr{T}})$ has the same property. To prove this, suppose we are given a diagram $\{\overline{\mathscr{T}}_\alpha\}$ in $\mathrm{N}^{\mathrm{hc}}(\mathcal{Q})^{\mathrm{op}}$ having a colimit $\overline{\mathscr{T}}$. We then obtain a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\overline{\mathscr{T}}) \arrow[r, "\theta"] \arrow[d] & \varprojlim_\alpha \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\overline{\mathscr{T}}_\alpha) \arrow[d] \\
\mathrm{Fun}_{\mathrm{CAlg}}(\overline{\mathscr{T}}^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp})) \arrow[r, "\theta'"] & \varprojlim_\alpha \mathrm{Fun}_{\mathrm{CAlg}}(\overline{\mathscr{T}}^{\mathrm{op}}_\alpha, \mathrm{Mod}(\mathrm{Sp})),
\end{tikzcd}$$

where θ' is an equivalence of ∞-categories and the vertical maps are inclusions of full subcategories. To show that the upper horizontal map is an equivalence, it will suffice to show that the diagram is a pullback square, which follows immediately from Remark 5.1.7. $\square$

Warning 5.1.10. The functor

$$\mathcal{S}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty \qquad X \mapsto \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(X)$$

generally does not carry colimits of spaces to limits of ∞-categories. However, it does carry coproducts in $\mathcal{S}$ to products of ∞-categories (see Remark 3.1.7).

It follows from Proposition 5.1.9 that the functor $\mathsf{X} \mapsto \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ is determined by its restriction along the Yoneda embedding

$$\mathscr{T} \to \mathcal{OS} \qquad T \mapsto T^{(-)}.$$

We now describe this restriction more explicitly.

Notation 5.1.11. Let T be an object of $\mathscr{T}$. We let $\mathrm{Cov}(T)$ denote the category of connected covering spaces $T_0 \to T$. Note that if we fix a base point $t \in T$, then the construction

$$T_0 \mapsto T_0 \times_T \{t\}$$

induces an equivalence of categories $\mathrm{Cov}(T) \to \mathrm{Orbit}(\pi_1(T))$, where $\mathrm{Orbit}(\pi_1(T))$ denotes the orbit category of the finite abelian group $\pi_1(T)$ (Notation 3.2.7). We will identify $\mathrm{Cov}(T)$ with a full subcategory of the ∞-category $\mathscr{T}_{/T} \simeq \mathscr{T}_{/T^{(-)}}$ of Notation 5.1.1, spanned by those maps $T_0 \to T$ for which the induced map $\pi_1(T_0) \to \pi_1(T)$ is a monomorphism of finite abelian groups.

If $\mathbf{G}$ is a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, we let $A_{\mathbf{G},T}$ denote the composite functor

$$\mathrm{Cov}(T)^{\mathrm{op}} \hookrightarrow \mathscr{T}^{\mathrm{op}}_{/T} \to \mathscr{T}^{\mathrm{op}} \xrightarrow{A_{\mathbf{G}}} \mathrm{CAlg},$$

which we regard as a commutative algebra object of ∞-category $\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp})$.

Proposition 5.1.12. *Let* $\mathbf{G}$ *be a preoriented* p*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*, and let* T *be an object of* $\mathscr{T}$*. Then composition with the inclusion* $\mathrm{Cov}(T) \hookrightarrow \mathscr{T}_{/T}$ *induces an equivalence of* ∞*-categories*

$$\phi : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(T) \to \mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp})).$$

Proof. Let $q : \mathrm{Mod}(\mathrm{Sp}) \to \mathrm{CAlg}$ be as in Construction 5.1.8, so that we can identify ϕ with the restriction map

$$\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(T) \hookrightarrow \mathrm{Fun}_{\mathrm{CAlg}}(\mathscr{T}^{\mathrm{op}}_{/T}, \mathrm{Mod}(\mathrm{Sp})) \to \mathrm{Fun}_{\mathrm{CAlg}}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp})).$$

By virtue of Proposition HTT.4.3.2.15 , it will suffice to prove the following:

- $(*)$ Let $\mathscr{F} \in \mathrm{Fun}_{\mathrm{CAlg}}(\mathscr{T}^{\mathrm{op}}_{/T}, \mathrm{Mod}(\mathrm{Sp}))$. Then $\mathscr{F}$ is a $\mathbf{G}$-pretempered local system on T if and only if it is a q-left Kan extension of the restriction $\mathscr{F}\,|_{\mathrm{Cov}(T)^{\mathrm{op}}}$.

To prove this, we first note that the inclusion $\mathrm{Cov}(T) \hookrightarrow \mathscr{T}_{/T}$ admits a left adjoint U. Concretely, U carries an object $T' \in \mathscr{T}_{/T}$ to another object $U(T') \in \mathscr{T}_{/T}$, which is characterized up to equivalence by the existence of a diagram

$$T' \xrightarrow{\mu_{T'}} U(T') \xrightarrow{\nu_{T'}} T,$$

where μ is surjective on fundamental groups and ν is injective on fundamental groups. It follows that $\mathscr{F} \in \mathrm{Fun}_{\mathrm{CAlg}}(\mathscr{T}_{/T}{}^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp}))$ is a q-left Kan extension of $\mathscr{F}\,|_{\mathrm{Cov}(T)^{\mathrm{op}}}$ if and only if, for every object $T' \in \mathscr{T}_{/T}$, it carries $\mu_{T'}$ to a q-coCartesian morphism of the ∞-category $\mathrm{Mod}(\mathrm{Sp})$. More concretely, this amounts to the condition that $\mathscr{F}$ induces an equivalence of $A^{T'}_{\mathbf{G}}$-modules $A^{T'}_{\mathbf{G}} \otimes_{A^{U(T')}_{\mathbf{G}}} \mathscr{F}(U(T')) \to \mathscr{F}(T')$. The "only if" direction of $(*)$ follows immediately from the definitions. For the converse, suppose that $\mathscr{F}(\mu_{T'})$ is an equivalence for each object $T' \in \mathscr{T}_{/T}$; we wish to show that $\mathscr{F}$ satisfies condition (A) of Construction 5.1.3. Let $\alpha : T'' \to T'$ be a morphism in $\mathscr{T}_{/T}$ with connected homotopy fibers. Then α induces an homotopy equivalence $U(\alpha) : U(T'') \to U(T')$. It follows that we can identify $\mu_{T''}$ with the composition $\mu_{T'} \circ \alpha$. Since $\mathscr{F}$ carries $\mu_{T'}$ and $\mu_{T''}$ to q-coCartesian morphisms of $\mathrm{Mod}(\mathrm{Sp})$, it must also carry α to a q-coCartesian morphism of $\mathrm{Mod}(\mathrm{Sp})$ (Proposition HTT.2.4.1.7). □

Example 5.1.13. If $T \in \mathscr{T}$ is contractible, then $\mathrm{Cov}(T)$ is equivalent to the category with a single object and a single morphism. Applying Proposition 5.1.12, we deduce that the evaluation functor $\mathscr{F} \mapsto \mathscr{F}(T)$ is an equivalence of ∞-categories $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(T) \simeq \mathrm{Mod}_A$ (for any preoriented $\mathbf{P}$-divisible group $\mathbf{G}$ over A).

Example 5.1.14. Let p be a prime number and let $T = BC_p$ be the classifying space of the cyclic group C_p of order p. Then, up to isomorphism, the category $\mathrm{Cov}(T)$ has exactly two objects:

- The covering map $EC_p \to BC_p$, whose automorphism group is the cyclic group C_p.
- The space $T = BC_p$ itself, which is a final object of $\mathrm{Cov}(T)$.

The object EC_p spans a full subcategory of $\mathrm{Cov}(T)$ which we can identify with the classifying space BC_p itself, so that the entire category $\mathrm{Cov}(T)$ can be identified with the cone $(BC_p)^{\rhd}$.

Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Then the ∞-category $\mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$ is easy to describe: its objects can be identified with pairs $(M, \zeta : N \to M^{hC_p})$ where M is a C_p-equivariant object of the ∞-category Mod_A, N is a module over the tempered function spectrum $A_{\mathbf{G}}^{BC_p}$, and $\zeta : N \to M^{hC_p}$ is a morphism of $A_{\mathbf{G}}^{BC_p}$-modules (where we regard M^{hC_p} as a module over $A_{\mathbf{G}}^{BC_p}$ via the Atiyah-Segal comparison map $A_{\mathbf{G}}^{BC_p} \to A^{BC_p}$).

Variant 5.1.15. Let X be any orbispace. Then the underlying space $|\mathsf{X}|$ of Notation 3.1.5 can be identified with a full subcategory of the ∞-category $\mathscr{T}_{/\mathsf{X}}$, spanned by those objects $T \in \mathscr{T}_{/\mathsf{X}}$ whose underlying space is contractible. If $\mathbf{G}$ is a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, then precomposition with the inclusion functor $|\mathsf{X}| \hookrightarrow \mathscr{T}_{/\mathsf{X}}$ supplies a forgetful functor

$$\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X}) \to \mathrm{Fun}(|\mathsf{X}|, \mathrm{Mod}_A) = \mathrm{LocSys}_A(|\mathsf{X}|).$$

In the special case where $\mathsf{X} = \underline{X}$ is the constant orbispace associated to a space X (Example 3.1.8), this forgetful functor supplies an equivalence of categories

$$\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\underline{X}) \simeq \mathrm{LocSys}_A(|\mathsf{X}|).$$

To prove this, we can use Proposition 5.1.9 to reduce to the case where X is contractible, in which case it follows from Example 5.1.13.

Remark 5.1.16. Let X be a space. Applying Variant 5.1.15 to the representable orbispace $\mathsf{X} = X^{(-)}$ of Example 3.1.6, we obtain a forgetful functor $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(X) \to \mathrm{LocSys}_A(X)$ from $\mathbf{G}$-pretempered local systems on X to the ∞-category of local systems of A-modules on X. Under the identification of Variant 5.1.15, this forgetful functor is given by pullback along the map of orbispaces $\underline{X} \to X^{(-)}$. In particular, it is an equivalence if X is a finite space (Remark 3.1.14). However, it is not an equivalence in general.

Example 5.1.17. Let p be a prime number, let $T = BC_p$ be the classifying space of the cyclic group C_p of order p, and let $T_0 = EC_p$ be its universal cover. Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathbf{G}$-pretempered local systems on T, and let $\mathscr{F}_0, \mathscr{G}_0 \in \mathrm{LocSys}_A(T)$ denote their images under the forgetful functor of Remark 5.1.16. Using the analysis of Example 5.1.14, we obtain a pullback diagram of spaces

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(T)}(\mathscr{F}, \mathscr{G}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Mod}_{A^T_{\mathbf{G}}}}(\mathscr{F}(T), \mathscr{G}(T)) \arrow[d] \\
\mathrm{Map}_{\mathrm{LocSys}_A(T)}(\mathscr{F}_0, \mathscr{G}_0) \arrow[r] & \mathrm{Map}_{\mathrm{Mod}_{A^T_{\mathbf{G}}}}(\mathscr{F}(T), \mathscr{G}(T_0)^{hC_p}).
\end{tikzcd}$$

In particular, we have a fiber sequence of mapping spaces

$$\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(T)}(\mathscr{F}, \mathscr{G}) \to \mathrm{Map}_{\mathrm{LocSys}_A(T)}(\mathscr{F}_0, \mathscr{G}_0) \to \mathrm{Map}_{\mathrm{Mod}_{A^T_{\mathbf{G}}}}(\mathscr{F}(T), \mathrm{cofib}(\zeta))$$

where ζ denotes the canonical map $\mathscr{G}(T) \to \mathscr{G}(T_0)^{hC_p}$.

Corollary 5.1.18. *Let $\mathbf{G}$ be a preoriented p-divisible group over an $\mathbb{E}_\infty$-ring A, and let T be an object of $\mathscr{T}$. Then:*

1. *The ∞-category $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(T)$ is stable and presentable.*
2. *For each object $T' \in \mathscr{T}_{/T}$, the evaluation functor*

$$\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(T) \to \mathrm{Mod}_{A^{T'}_{\mathbf{G}}} \qquad \mathscr{F} \mapsto \mathscr{F}(T')$$

preserves small limits and colimits.

Proof. Since the ∞-category Mod_A is stable and presentable, the functor ∞-category $\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Mod}_A)$ has the same properties. Proposition 5.1.12 allows us to identify $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(T)$ with the ∞-category of $A_{\mathbf{G},T}$-module objects of $\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Mod}_A)$. It is therefore also stable (by virtue of Proposition HA.7.1.1.4) and presentable (by virtue of Corollary HA.4.2.3.7). This proves (1). To prove (2), let T' be an object of $\mathscr{T}$ and let $U(T')$ be as in the proof of Proposition 5.1.12. Then the functor $\mathscr{F} \mapsto \mathscr{F}(T')$ is given by the composition

$$\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(T) \xrightarrow{\mathscr{F} \mapsto \mathscr{F}(U(T'))} \mathrm{Mod}_{A^{U(T')}_{\mathbf{G}}} \xrightarrow{A^{T'}_{\mathbf{G}} \otimes_{A^{U(T')}_{\mathbf{G}}} \bullet} \mathrm{Mod}_{A^{T'}_{\mathbf{G}}}.$$

Since $A^{T'}_{\mathbf{G}}$ is finite flat over $A^{U(T')}_{\mathbf{G}}$, the second functor preserves small limits and colimits. We can therefore replace T' by $U(T')$ and thereby reduce to the case where

T' belongs to $\mathrm{Cov}(T)$, in which case the desired result follows from the fact that the forgetful functor $\mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp})) \to \mathrm{Mod}_A$ preserves small limits and colimits (Corollaries HA.4.2.3.3 and HA.4.2.3.5). □

Corollary 5.1.19. *Let* $\mathbf{G}$ *be a preoriented* p*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. Then:*

(1) *For every orbispace* X*, the* ∞*-category* $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ *is stable and presentable.*

(2) *For every orbispace* X *and every object* $T \in \mathscr{T}_{/\mathsf{X}}$*, the evaluation functor*

$$\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{Mod}_{A^T_{\mathbf{G}}} \qquad \mathscr{F} \mapsto \mathscr{F}(T)$$

preserves small limits and colimits.

(3) *For every morphism of orbispaces* $f : \mathsf{X} \to \mathsf{Y}$*, the pullback functor* $f^* : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ *preserves small limits and colimits.*

Proof. To prove (1), we observe that every orbispace X can be written as a small colimit $\varinjlim_\alpha \mathsf{X}_\alpha$, where each X_α is representable by an object $T^{(-)}_\alpha \in \mathscr{T}$ (in fact, it has a canonical representation in this form, where the diagram is indexed by the ∞-category $\mathscr{T}_{/\mathsf{X}}$). Then Proposition 5.1.9 supplies an equivalence of ∞-categories $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \simeq \varprojlim_\alpha \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}_\alpha)$. Corollary 5.1.19 implies that each of the ∞-categories $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}_\alpha)$ is stable and presentable and that each of the transition functors $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}_\alpha) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}_\beta)$ preserves small limits and colimits. It follows that $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ is also stable (Theorem HA.1.1.4.4) and presentable (Proposition HTT.5.5.3.13), and that the pullback functors $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}_\alpha)$ preserve small limits and colimits. This immediately implies (1) and (2), and the implication (2) ⇒ (3) follows from the definition of pullback for $\mathbf{G}$-pretempered local systems. □

Remark 5.1.20 (Relationship with $\mathbf{G}$-Tempered Cohomology)**.** Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. For every orbispace X, the ∞-category $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ is a presentable A-linear ∞-category. In particular, to every pair of objects $\mathscr{F}, \mathscr{G} \in \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$, we can associate an A-module spectrum $\underline{\mathrm{Map}}(\mathscr{F}, \mathscr{G})$ which classifies maps from $\mathscr{F}$ into $\mathscr{G}$ in the following sense: for every A-module spectrum M, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_A}(M, \underline{\mathrm{Map}}(\mathscr{F}, \mathscr{G})) \simeq \mathrm{Map}_{\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})}(M \otimes_A \mathscr{F}, \mathscr{G}).$$

In the special case $\mathscr{F} = \mathscr{G} = \underline{A}_{\mathsf{X}}$, we obtain an associative algebra $\underline{\mathrm{Map}}(\underline{A}_{\mathsf{X}}, \underline{A}_{\mathsf{X}})$. It is not difficult see that the construction $\mathsf{X} \mapsto \underline{\mathrm{Map}}(\underline{A}_{\mathsf{X}}, \underline{A}_{\mathsf{X}})$ determines a functor $\mathcal{OS}^{\mathrm{op}} \to$

Alg_A. This functor carries colimits of orbispaces to limits in Alg_A (Proposition 5.1.9) and carries each representable orbispace $T^{(-)}$ to the tempered function spectrum $A^T_{\mathbf{G}}$ (by Proposition 5.1.12). It follows that we can functorially identify $\underline{\mathrm{Map}}(\underline{A}_{\mathsf{X}}, \underline{A}_{\mathsf{X}})$ with the $\mathbf{G}$-tempered function spectrum $A^{\mathsf{X}}_{\mathbf{G}}$ of Construction 4.0.3. Passing to homotopy groups, we obtain a canonical isomorphism

$$A^*_{\mathbf{G}}(\mathsf{X}) \simeq \mathrm{Ext}^*_{\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})}(\underline{A}_{\mathsf{X}}, \underline{A}_{\mathsf{X}}),$$

depending functorially on X.

5.2 The ∞-Category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$

Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. In this section, we associate to each orbispace X a full subcategory $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \subseteq \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$, whose objects we will refer to as $\mathbf{G}$-*tempered local systems* on X. First, we need to establish some notation.

Notation 5.2.1. Let T be an object of the ∞-category $\mathscr{T}$: that is, a space which is homotopy equivalent to BH, for some finite abelian group H. Let $f : T_0 \to T$ be a map which exhibits T_0 as a connected covering space of T. Then T_0 is homotopy equivalent to the classifying space BH_0, where $H_0 \subseteq H$ is the subgroup given by the image of the map $\pi_1(T_0) \to \pi_1(T)$. In particular, T_0 is also an object of $\mathscr{T}$. We let $\mathrm{Aut}(T_0/T)$ denote the group of deck transformations of the covering $T_0 \to T$. Then $\mathrm{Aut}(T_0/T)$ can be identified with the quotient group H/H_0: in particular, it is also a finite abelian group.

Let X be an orbispace, and suppose we are given an object $\overline{T} \in \mathscr{T}_{/\mathsf{X}}$ corresponding to a pair $(T, \eta : T^{(-)} \to \mathsf{X})$. Then we can lift f to a morphism $\overline{T}_0 \to \overline{T}$ in the ∞-category $\mathscr{T}_{/\mathsf{X}}$, where $\overline{T}'$ is the pair $(T_0, f \circ \eta : T'^{(-)} \to \mathsf{X})$. Moreover, the automorphism group $\mathrm{Aut}(T_0/T)$ acts on the object $\overline{T}_0$.

Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Then f induces a surjective homomorphism of tempered cohomology rings $A^0_{\mathbf{G}}(T) \to A^0_{\mathbf{G}}(T_0)$. We will denote the kernel of this homomorphism by $I(T_0/T) \subseteq A^0_{\mathbf{G}}(T)$ and refer to it as the *relative augmentation ideal* of the map $T_0 \to T$. Note that $I(T_0/T)$ is a projective module of finite rank over the commutative ring $\pi_0(A)$, and is therefore finitely generated as an ideal of the ring $A^0_{\mathbf{G}}(T_0)$.

Remark 5.2.2. Let H be a finite abelian group. Then the canonical map $EH \to BH$ is a covering, and the relative augmentation ideal $I(EH/BH)$ Notation 5.2.1 coincides with the augmentation ideal I_H of Notation 4.2.7.

Remark 5.2.3. Let $T_0 \to T$ be a covering map in the ∞-category $\mathscr{T}$, and let $T' \to T$ be a morphism in $\mathscr{T}$ with connected homotopy fibers (so that the map $\pi_1(T') \to \pi_1(T)$ is surjective). Then the fiber product $T'_0 = T_0 \times_T T'$ is a connected covering space of T', and therefore also belongs to $\mathscr{T}$. If $\mathbf{G}$ is a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, then the pullback diagram

$$\begin{tikzcd}
T'_0 \arrow[r] \arrow[d] & T' \arrow[d] \\
T_0 \arrow[r] & T
\end{tikzcd}$$

induces a pullback diagram of affine schemes

$$\begin{tikzcd}
\mathrm{Spec}(A^0_{\mathbf{G}}(T'_0)) \arrow[r] \arrow[d] & \mathrm{Spec}(A^0_{\mathbf{G}}(T')) \arrow[d] \\
\mathrm{Spec}(A^0_{\mathbf{G}}(T_0)) \arrow[r] & \mathrm{Spec}(A^0_{\mathbf{G}}(T)),
\end{tikzcd}$$

where the horizontal maps are closed immersions and the vertical maps are finite flat. It follows that the relative augmentation ideal $I(T'_0/T') \subseteq A^0_{\mathbf{G}}(T')$ is equal to $I(T_0/T)A^0_{\mathbf{G}}(T')$: that is, it is generated by the image of the relative augmentation ideal $I(T_0/T) \subseteq A^0_{\mathbf{G}}(T)$.

Definition 5.2.4 (Tempered Local Systems)**.** Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, let X be an orbispace, and let $\mathscr{F} \in \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ be a $\mathbf{G}$-pretempered local system on X. We say that $\mathscr{F}$ is a $\mathbf{G}$-*tempered local system* if it satisfies the following additional condition:

(B) Let T be an object of $\mathscr{T}_{/\mathsf{X}}$ and let T_0 be a connected covering space of T. Then the canonical map $\mathscr{F}(T) \to \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}$ exhibits $\mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}$ as an $I(T_0/T)$-completion of $\mathscr{F}(T)$, where $I(T_0/T)$ is the relative augmentation ideal of Notation 5.2.1.

We let $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ denote the full subcategory of $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ spanned by the $\mathbf{G}$-tempered local systems on X.

Remark 5.2.5. In the case where $\mathbf{G}$ is oriented, we will give alternate characterization of the class of tempered local systems in §5.5; see Theorem 5.5.1.

Variant 5.2.6. Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, and let X be a space. We define a $\mathbf{G}$-*tempered local system on* X is a $\mathbf{G}$-tempered local system on the orbispace $X^{(-)}$ represented by X (Example 3.1.6). We let $\mathrm{LocSys}_{\mathbf{G}}(X) = \mathrm{LocSys}_{\mathbf{G}}(X^{(-)})$ denote the ∞-category of $\mathbf{G}$-tempered local systems on X.

Remark 5.2.7. In the situation of axiom (B) of Definition 5.2.4, note that any $A_{\mathbf{G}}^{T_0}$-module M is automatically $I(T_0/T)$-complete when viewed as a $A_{\mathbf{G}}^{T}$-module (since the homotopy groups of M are annihilated by the relative augmentation ideal $I(T_0/T)$). Since the collection of $I(T_0/T)$-complete $A_{\mathbf{G}}^{T}$-modules is closed under limits, it follows that the homotopy fixed point spectrum $\mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}$ is automatically $I(T_0/T)$-complete. We may therefore replace (B) by the following *a priori* weaker condition:

(B') The fiber of the canonical map $\mathscr{F}(T) \to \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}$ is $I(T_0/T)$-local.

Remark 5.2.8. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let X be an orbispace, and let $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ be a $\mathbf{G}$-pretempered local system on X. Then condition (B) of Definition 5.2.4 is equivalent to the following *a priori weaker* condition:

(B'') Let T be an object of $\mathscr{T}_{/\mathsf{X}}$ and let T_0 be a connected covering space of T for which the automorphism group $\mathrm{Aut}(T_0/T)$ is isomorphic to C_p for some prime number p. Then the fiber of the canonical map $\mathscr{F}(T) \to \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}$ is $I(T_0/T)$-local.

The implication $(B) \Rightarrow (B'')$ is immediate. To prove the reverse implication, we note that every covering map $T_0 \to T$ in $\mathscr{T}$ factors as a composition

$$T_0 \to T_1 \to \cdots \to T_n = T,$$

where each T_{k-1} is a connected p-fold covering space of T_k for some prime number p (which might depend on k). It follows that the fiber of the canonical map $\mathscr{F}(T) \to \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}$ can be written as a composition of maps

$$\xi_k : \mathscr{F}(T_k)^{\mathrm{hAut}(T_k/T)} \to \mathscr{F}(T_{k-1})^{\mathrm{hAut}(T_{k-1}/T)}.$$

By virtue of Remark 5.2.7, it will suffice to show that each of the fibers $\mathrm{fib}(\xi_k)$ is $I(T_0/T)$-local. Note that the $\mathrm{fib}(\xi_k)$ can be identified with the homotopy fixed points for the action of $\mathrm{Aut}(T_k/T)$ on $\mathrm{fib}(\theta_k)$, where θ_k denotes the canonical map $\mathscr{F}(T_k) \to \mathscr{F}(T_{k-1})^{\mathrm{hAut}(T_{k-1}/T_k)}$. Assumption (B'') guarantees that $\mathrm{fib}(\theta_k)$ is $I(T_{k-1}/T_k)$-local when regarded as an $A_{\mathbf{G}}^{T_k}$-module spectrum, and therefore $I(T_{k-1}/T)$-local when regarded as an $A_{\mathbf{G}}^{T}$-module spectrum. It now suffices to observe that every $A_{\mathbf{G}}^{T}$-module which is $I(T_{k-1}/T)$-local is also $I(T_0/T)$-local (since $I(T_{k-1}/T)$ is contained in $I(T_0/T)$).

Remark 5.2.9. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_{\infty}$-ring A and let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of orbispaces. Then the pullback functor $f^* : \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ of Remark 5.1.6 carries $\mathbf{G}$-tempered local systems on Y to $\mathbf{G}$-tempered local systems on X and therefore restricts to a functor $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, which we will also denote by f^*.

Remark 5.2.10. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_{\infty}$-ring A and let $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ be a $\mathbf{G}$-pretempered local system on an orbispace X. Suppose that there exists a collection of orbispace morphisms $f_{\alpha} : \mathsf{X}_{\alpha} \to \mathsf{X}$ with the following properties:

- For every object $T \in \mathscr{T}$, the induced map $\coprod_{\alpha} \pi_0(\mathsf{X}_{\alpha}^{T}) \to \pi_0(\mathsf{X})$ is surjective.
- Each pullback $f_{\alpha}^* \mathscr{F}$ is a $\mathbf{G}$-tempered local system on X_{α}.

Then $\mathscr{F}$ is a $\mathbf{G}$-tempered local system on X.

Remark 5.2.11. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_{\infty}$-ring A. Then the construction $\mathsf{X} \mapsto \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ determines a functor

$$\mathrm{LocSys}_{\mathbf{G}}(\bullet) : \mathcal{OS}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$$

which carries (small) colimits of orbispaces to (small) limits of ∞-categories: this follows from Proposition 5.1.9 and Remark 5.2.10.

We now summarize some formal properties of Definition 5.2.4:

Proposition 5.2.12. *Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_{\infty}$-ring and let X be an orbispace. Then $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is a presentable stable ∞-category. Moreover, the inclusion functor $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \hookrightarrow \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ admits a left adjoint $L : \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$.*

Proof. Choose a set of representatives for all equivalence classes of pairs $u = (T, f : T_0 \to T)$, where T is an object of $\mathscr{T}_{/\mathsf{X}}$ and $f : T_0 \to T$ exhibits T_0 as a connected covering space of T. For every such pair u, let $\phi_u : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{Mod}_{A_{\mathbf{G}}^{T}}$ be the functor given by

$$\phi_u(\mathscr{F}) = \mathrm{fib}(\mathscr{F}(T) \to \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}).$$

The functor ϕ_u is accessible and preserves small limits. Let $\mathcal{C}_u$ denote the full subcategory of $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ spanned by those objects $\mathscr{F}$ such that $\phi_u(\mathscr{F})$ is $I(T_0/T)$-local. Applying Lemma HTT.5.5.4.17 , we deduce that $\mathcal{C}_u$ is a strongly reflective subcategory of $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$. Note that $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is given by the intersection $\bigcap_u \mathcal{C}_u$. Applying Lemma HTT.5.5.4.18 , we deduce that $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is a strongly reflective subcategory of $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(X)$: that is, the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is presentable and the inclusion $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \hookrightarrow \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ admits a left adjoint. The stability of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ follows from the observation that it closed under suspensions and limits in the stable ∞-category $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$. □

Corollary 5.2.13. *Let* $\mathbf{G}$ *be a preoriented p-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* X *be an orbispace. Then:*

- *For each object* $T \in \mathscr{T}_{/\mathsf{X}}$*, the evaluation functor*

$$(\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})) \mapsto (\mathscr{F}(T) \in \mathrm{Mod}_{A^T_{\mathbf{G}}})$$

  preserves small limits.

- *For every map of orbispaces* $f : \mathsf{Y} \to \mathsf{X}$*, the pullback functor* $f^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ *preserves small limits.*

Proof. Combine Proposition 5.2.12 with Corollary 5.1.19. □

5.3 Colimits of Tempered Local Systems

The ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ can be defined for any preoriented $\mathbf{P}$-divisible group $\mathbf{G}$ and any orbispace X. However, it is particularly well-behaved when the $\mathbf{P}$-divisible group $\mathbf{G}$ is oriented.

Theorem 5.3.1. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* X *be an orbispace. Then the full subcategory* $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \subseteq \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ *is closed under small colimits.*

Corollary 5.3.2. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. Then: Then:*

(1) *For every orbispace* X *and every object* $T \in \mathscr{T}_{/\mathsf{X}}$*, the evaluation functor*

$$\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{Mod}_{A^T_{\mathbf{G}}} \qquad \mathscr{F} \mapsto \mathscr{F}(T)$$

preserves small colimits.

(2) *For every map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$, the pullback functor $f^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ preserves small colimits.*

Proof. Combine Theorem 5.3.1 with Corollary 5.1.19. ☐

Corollary 5.3.3. *Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A, and let X be an orbispace. Then the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is compactly generated.*

Proof. For each object $T \in \mathscr{T}_{/\mathsf{X}}$, let $e_T : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{Mod}_{A^T_{\mathbf{G}}}$ denote the evaluation functor given by $e_T(\mathscr{F}) = \mathscr{F}(T)$. It follows from Corollaries 5.2.13 and 5.3.2 that e_T preserves small limits and colimits. Applying Corollary HTT.5.5.2.9 , we deduce that e_T admits a left adjoint F_T. Using Proposition HTT.5.5.7.2 , we conclude that F_T carries compact objects of $\mathrm{Mod}_{A^T_{\mathbf{G}}}$ to compact objects of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ spanned by objects of the form $F_T(\Sigma^n A^T_{\mathbf{G}})$, where T is an object of $\mathscr{T}_{/\mathsf{X}}$ and n is an integer. Let $\overline{\mathcal{C}}$ denote the full subcategory of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ generated by $\mathcal{C}$ under small colimits. Then $\overline{\mathcal{C}}$ is compactly generated, so Corollary HTT.5.5.2.9 implies that the inclusion $\overline{\mathcal{C}} \hookrightarrow \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ admits a right adjoint U. To prove that $\overline{\mathcal{C}} = \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, it will suffice to show that U is conservative. Let α be a morphism in $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ such that $U(\alpha)$ is an equivalence. Then $U(\mathrm{fib}(\alpha)) \simeq 0$, so that

$$\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(F_T(\Sigma^n A^T_{\mathbf{G}}), \mathrm{fib}(\alpha)) \simeq \Omega^{\infty-n}\,\mathrm{fib}(\alpha)(T)$$

is contractible for every object $T \in \mathscr{T}_{/\mathsf{X}}$ and every integer n. It follows that $\mathrm{fib}(\alpha) \simeq 0$, so that α is an equivalence. ☐

Proof. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let X be an orbispace, and let $\{\mathscr{F}_\alpha\}$ be a diagram taking values in the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$. Let $\mathscr{F} = \varinjlim_\alpha \mathscr{F}_\alpha$, where the colimit is formed in the larger ∞-category $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ of $\mathbf{G}$-pretempered local systems. We wish to show that $\mathscr{F}$ is $\mathbf{G}$-tempered. We will prove this by verifying condition (B'') of Remark 5.2.8. Let T be an object of $\mathscr{T}_{/\mathsf{X}}$ and let T_0 be a connected covering space of T for which the automorphism group $\mathrm{Aut}(T_0/T)$ is isomorphic to the cyclic group C_p, for some prime number p. We wish to show that the fiber of the canonical map $\xi : \mathscr{F}(T) \to \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}$ is $I(T_0/T)$-local.

For each index α, our assumption that $\mathscr{F}_\alpha$ is $\mathbf{G}$-tempered guarantees that the natural map $\xi_\alpha : \mathscr{F}_\alpha(T) \to \mathscr{F}_\alpha(T_0)^{\mathrm{hAut}(T_0)}$ has $I(T_0/T)$-local fiber. Note that the map ξ factors as a composition

$$\varinjlim_\alpha \mathscr{F}_\alpha(T) \xrightarrow{\varinjlim \xi_\alpha} \varinjlim_\alpha \mathscr{F}_\alpha(T_0)^{\mathrm{hAut}(T_0/T)} \xrightarrow{\theta} (\varinjlim \mathscr{F}_\alpha(T_0))^{\mathrm{hAut}(T_0/T)}.$$

Since the collection of $I(T_0/T)$-local $A_{\mathbf{G}}^{T}$-modules is closed under colimits, the fiber of the map $\varinjlim_{\alpha} \xi_{\alpha})$ is $I(T_0/T)$-local (since the collection of J-local $\mathscr{O}_{\mathbf{G}}(T)$-modules is closed under small colimits). It will therefore suffice to show that $\mathrm{fib}(\theta)$ is $I(T_0/T)$-local.

By assumption, we have, we have a pullback diagram of spaces

$$\begin{tikzcd}
T_0 \arrow[r] \arrow[d] & EC_p \arrow[d] \\
T \arrow[r] & BC_p
\end{tikzcd}$$

which induces a pushout diagram of tempered function spectra

$$\begin{tikzcd}
A_{\mathbf{G}}^{T_0} \arrow[r] \arrow[d] & A_{\mathbf{G}}^{EC_p} \arrow[d] \\
A_{\mathbf{G}}^{T} \arrow[r] & A_{\mathbf{G}}^{BC_p},
\end{tikzcd}$$

where the horizontal maps are finite flat. It follows that the relative augmentation ideal $I(T_0/T)$ is generated by the image of the augmentation ideal $I_{C_p} \subseteq A_{\mathbf{G}}^{0}(BC_p)$. It will therefore suffice to show that the fiber of θ is I_{C_p}-local when viewed as a module over $A_{\mathbf{G}}^{BC_p}$.

Note that θ fits into a commutative diagram of fiber sequences

$$\begin{tikzcd}
\varinjlim_{\alpha}(\mathscr{F}_{\alpha}(T_0)_{hC_p}) \arrow[r] \arrow[d, "\theta'"] & \varinjlim_{\alpha}(\mathscr{F}_{\alpha}(T_0)^{hC_p}) \arrow[r] \arrow[d, "\theta"] & \varinjlim_{\alpha}(\mathscr{F}_{\alpha}(T_0)^{tC_p}) \arrow[d, "\theta''"] \\
(\varinjlim_{\alpha} \mathscr{F}_{\alpha}(T_0))_{hC_p} \arrow[r] & (\varinjlim_{\alpha} \mathscr{F}_{\alpha}(T_0))^{hC_p} \arrow[r] & (\varinjlim_{\alpha} \mathscr{F}_{\alpha}(T_0))^{tC_p}.
\end{tikzcd}$$

Here the map θ' is an equivalence, so we have an equivalence of fibers $\mathrm{fib}(\theta) \simeq \mathrm{fib}(\theta'')$. It will therefore suffice to show that $\mathrm{fib}(\theta'')$ is I_{C_p}-local. In fact, both the domain and codomain of θ'' are I_{C_p}-local, by virtue Proposition 4.6.8. □

5.4 Tempered Local Systems on Classifying Spaces

Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_{\infty}$-ring A, and let T be an object of $\mathscr{T}$. According to Proposition 5.2.12, a $\mathbf{G}$-pretempered local system $\mathscr{F}$ on T can be recovered (functorially) from its restriction $\mathscr{F}_0 = \mathscr{F}|_{\mathrm{Cov}(T)^{\mathrm{op}}}$ to the category $\mathrm{Cov}(T)$ of connected covering spaces of T. Our goal in this section is to show that, if $\mathbf{G}$ is oriented, then the condition that $\mathscr{F}$ is tempered has a simple formulation in terms of $\mathscr{F}_0$.

Notation 5.4.1. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let T be an object of $\mathscr{T}$, and let $A_{\mathbf{G},T} : \mathrm{Cov}(T)^{\mathrm{op}} \to \mathrm{CAlg}$ be the functor given by $(T_0 \in \mathrm{Cov}(T)) \mapsto A_{\mathbf{G}}^{T_0}$. Let $\mathscr{F}_0$ be an $A_{\mathbf{G},T}$-module object of $\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp})$. We will say that $\mathscr{F}_0$ is *tempered* if it satisfies the following condition:

- (B_0') Let T_1 be a connected covering space of T, and let T_0 be a connected covering space of T_0. Then the fiber of the canonical map $\mathscr{F}(T_1) \to \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T_1)}$ is $I(T_0/T_1)$-local.

We let

$$\mathrm{Mod}^{\mathrm{tem}}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp})) \subseteq \mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$$

denote the full subcategory spanned by the tempered $A_{\mathbf{G},T}$-modules.

Proposition 5.4.2. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let T be an object of $\mathscr{T}$. Then the equivalence*

$$\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(T) \simeq \mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$$

of Proposition 5.1.12 restricts to an equivalence of ∞-categories

$$\mathrm{LocSys}_{\mathbf{G}}(T) \to \mathrm{Mod}^{\mathrm{tem}}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp})).$$

In other words, a $\mathbf{G}$-pretempered local system $\mathscr{F} \in \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(T)$ is $\mathbf{G}$-tempered (in the sense of Definition 5.2.4) if and only if the restriction $\mathscr{F}_0 = \mathscr{F}|_{\mathrm{Cov}(T)^{\mathrm{op}}}$ is tempered (in the sense of Notation 5.4.1).

Before giving the proof of Proposition 5.4.2, let us note some of its consequences.

Corollary 5.4.3. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $\mathsf{X} = \underline{X}$ be a constant orbispace (Example 3.1.8). Then every $\mathbf{G}$-pretempered local system on X is $\mathbf{G}$-tempered. Consequently, the restriction procedure of Variant 5.1.15 determines an equivalence of ∞-categories $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \simeq \mathrm{LocSys}_A(X)$.*

Proof. Using Remark 5.2.10, we can reduce to the case where X is contractible, in which case the desired result follows from Proposition 5.4.2 (note that condition (B_0') of Notation 5.4.1) is vacuous when the space T is contractible). □

Corollary 5.4.4. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*, let* X *be an orbispace, and let* $\underline{A}_{\mathsf{X}}$ *be the trivial* $\mathbf{G}$*-pretempered local system on* X*. Then* $\underline{A}_{\mathsf{X}}$ *is* $\mathbf{G}$*-tempered.*

Proof. Since the collection of $\mathbf{G}$-tempered local systems is stable under pullback (Remark 5.2.9), we can assume without loss of generality that X is the final object of $\mathcal{OS}$. In this case, the desired result follows from Corollary 5.4.3. □

Example 5.4.5. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let p be a prime number. Using Example 5.1.14, we can identify the objects of $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(BC_p)$ with pairs $(M, \zeta : N \to M^{hC_p})$, where M is a C_p-equivariant object of the ∞-category Mod_A and ζ is a morphism of $A_{\mathbf{G}}^{BC_p}$-modules. Under this identification, $\mathrm{LocSys}_{\mathbf{G}}(BC_p)$ corresponds to the full subcategory spanned by those pairs $(M, \zeta : N \to M^{hC_p})$ where ζ exhibits M^{hC_p} as the completion of N with respect to the augmentation ideal $I_{C_p} \subseteq A^0_{\mathbf{G}}(BC_p)$.

Proof of Proposition 5.4.2. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let T be an object of $\mathscr{T}$, and let $\mathscr{F}$ be a $\mathbf{G}$-pretempered local system on T. It follows immediately from the definitions that if $\mathscr{F}$ is $\mathbf{G}$-tempered (in the sense of Definition 5.2.4), then the restriction $\mathscr{F}_0 = \mathscr{F}|_{\mathrm{Cov}(T)^{\mathrm{op}}}$ is tempered (in the sense of Notation 5.4.1). Conversely, assume that $\mathscr{F}_0$ is tempered; we will show that $\mathscr{F}$ satisfies condition (B'') of Remark 5.2.8. Choose any morphism $T'' \to T$ in $\mathscr{T}$, and let $\beta : T''_0 \to T''$ exhibit T''_0 as a connected covering space of T'' whose automorphism group $\mathrm{Aut}(T''_0/T'')$ is cyclic of order p, for some prime number p. We wish to show that the fiber of the canonical map

$$\theta : \mathscr{F}(T'') \to \mathscr{F}(T''_0)^{\mathrm{hAut}(T''_0/T'')}$$

is local with respect to the ideal the ideal $I(T''_0/T'') \subseteq A^0_{\mathbf{G}}(T'')$.

Form a commutative diagram in $\mathscr{T}$

$$\begin{tikzcd}
T''_0 \arrow[r] \arrow[d, "\beta"] & T'_0 \arrow[d, "\gamma"] \\
T'' \arrow[r] \arrow[dr, "\alpha"'] & T' \arrow[d] \\
 & T
\end{tikzcd}$$

where the vertical maps are finite coverings the horizontal maps have connected homotopy fibers. Since β has degree p, the map γ has degree either 1 or p. We consider these cases separately:

(1) Suppose that γ has degree p: that is, the upper square in the preceding diagram above is a pullback. Then we can identify $\mathrm{Aut}(T''_0/T'')$ with $\mathrm{Aut}(T'_0/T')$, and $A_{\mathbf{G}}^{T''_0}$ with the tensor product $A_{\mathbf{G}}^{T''} \otimes_{A_{\mathbf{G}}^{T'}} A_{\mathbf{G}}^{T'_0}$. Invoking our assumption that $\mathscr{F}$ is $\mathbf{G}$-pretempered, we can identify θ with the natural map

$$\begin{aligned} A_{\mathbf{G}}^{T''} \otimes_{A_{\mathbf{G}}^{T'}} \mathscr{F}(T') &\to (A_{\mathbf{G}}^{T''_0} \otimes_{A_{\mathbf{G}}^{T'_0}} \mathscr{F}(T'_0))^{\mathrm{hAut}(T''_0/T'')} \\ &\simeq (A_{\mathbf{G}}^{T''} \otimes_{A_{\mathbf{G}}^{T'}} \mathscr{F}(T'_0))^{\mathrm{hAut}(T'_0/T')} \\ &\simeq A_{\mathbf{G}}^{T''} \otimes_{A_{\mathbf{G}}^{T'}} \mathscr{F}(T'_0)^{\mathrm{hAut}(T'_0/T')}; \end{aligned}$$

here the second equivalence follows from the observation that $A_{\mathbf{G}}^{T''}$ is finite flat as a module over $A_{\mathbf{G}}^{T'}$. It follows that the fiber of θ is given by $A_{\mathbf{G}}^{T''} \otimes_{A_{\mathbf{G}}^{T'}} \mathrm{fib}(\mu)$, where μ denotes the canonical map $\mathscr{F}(T') \to \mathscr{F}(T'_0)^{\mathrm{hAut}(T'_0/T')}$. Our assumption that $\mathscr{F}_0$ is tempered guarantees that $\mathrm{fib}(\mu)$ is local with respect to the ideal $I(T'_0/T')$, so that $A_{\mathbf{G}}^{T''} \otimes_{A_{\mathbf{G}}^{T'}} \mathrm{fib}(\mu)$ is local with respect to the ideal $I(T'_0/T')A_{\mathbf{G}}^0(T'') = I(T''_0/T'')$ (see Remark 5.2.3).

(2) Suppose that γ has degree 1: that is, T'_0 is isomorphic to T'. In this case, our assumption that $\mathscr{F}$ is $\mathbf{G}$-pretempered allows us to identify θ with the canonical map

$$A_{\mathbf{G}}^{T''} \otimes_{A_{\mathbf{G}}^{T'}} \mathscr{F}(T') \to (A_{\mathbf{G}}^{T''_0} \otimes_{A_{\mathbf{G}}^{T'}} \mathscr{F}(T'))^{\mathrm{hAut}(T''_0/T'')}.$$

Since T''_0 is a connected cyclic p-fold covering map of T'', there is a pullback diagram of spaces

$$\begin{tikzcd} T''_0 \arrow[r] \arrow[d] & EC_p \arrow[d] \\ T'' \arrow[r] & BC_p, \end{tikzcd}$$

where the horizontal maps have connected homotopy fibers. It follows that we can identify $A_{\mathbf{G}}^{T''_0}$ with the tensor product $A_{\mathbf{G}}^{T''} \otimes_{A_{\mathbf{G}}^{BC_p}} A_{\mathbf{G}}^{EC_p}$. Set $M = A_{\mathbf{G}}^{T''} \otimes_{A_{\mathbf{G}}^{T'}} \mathscr{F}(T')$ and regard M as module over the tempered function spectrum $A_{\mathbf{G}}^{BC_p}$. Then θ can be identified with the natural map $M \to (A_{\mathbf{G}}^{EC_p} \otimes_{A_{\mathbf{G}}^{BC_p}} M)^{hC_p}$. Applying Lemma 4.6.12 (and Theorem 4.6.2), we deduce that $\mathrm{fib}(\theta)$ is I_{C_p}-local when viewed as an $A_{\mathbf{G}}^{BC_p}$-module spectrum. It is therefore $I(T''_0/T'') = I_{C_p}A_{\mathbf{G}}^0(T'')$ when viewed as an $A_{\mathbf{G}}^{T''}$-module spectrum.

□

5.5 Recognition Principle for Tempered Local Systems

We now provide an alternate characterization of tempered local systems for oriented **P**-divisible groups.

Theorem 5.5.1. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* $\mathscr{F}$ *be a* $\mathbf{G}$*-pretempered local system on an orbispace* X*. Then* $\mathscr{F}$ *is* $\mathbf{G}$*-tempered if and only if it satisfies the following condition:*

- $(*)$ *Let* $T \in \mathscr{T}_{/\mathsf{X}}$*, let* $T_0 \in \operatorname{Cov}(T)$ *be a connected covering space for which the automorphism group* $\operatorname{Aut}(T_0/T)$ *is a cyclic group of order* p*, and let* M *denote the cofiber of the multiplication map* $A_{\mathbf{G}}^{T_0} \otimes_{A_{\mathbf{G}}^{T}} \mathscr{F}(T) \to \mathscr{F}(T_0)$*. Then multiplication by* p *is an equivalence from* M *to itself, and the action of* $\operatorname{Aut}(T_0/T)$ *on* $\pi_*(M)$ *has no nonzero fixed points.*

Before giving the proof of Theorem 5.5.1, let us note some of its consequences.

Corollary 5.5.2. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*, let* X *be an orbispace, and let* $\mathscr{F}$ *be an object of* $\operatorname{Mod}_{\underline{A}_{\mathsf{X}}}$*. Suppose that, for every morphism* $T' \to T$ *in* $\mathscr{T}_{/\mathsf{X}}$*, the induced map* $A_{\mathbf{G}}^{T'} \otimes_{A_{\mathbf{G}}^{T}} \mathscr{F}(T) \to \mathscr{F}(T')$ *is an equivalence. Then* $\mathscr{F}$ *is a* $\mathbf{G}$*-tempered local system on* X*.*

Corollary 5.5.2 admits a weak converse:

Corollary 5.5.3. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*, let* p *be a prime number, and let* X *be an orbispace which satisfies the following condition:*

- $(*)$ *For every finite abelian group* H*, the canonical map* $\mathsf{X}^{BH_{(p)}} \to \mathsf{X}^{BH}$ *is surjective on connected components.*

Let $\mathscr{F}$ *be an object of the* ∞*-category* $\operatorname{Mod}_{\underline{A}_{\mathsf{X}}}$ *which is* p*-nilpotent (that is, the localization* $\mathscr{F}[1/p]$ *vanishes). Then* $\mathscr{F}$ *is a* $\mathbf{G}$*-tempered local system if and only if, for every morphism* $T' \to T$ *in* $\mathscr{T}_{/\mathsf{X}}$*, the induced map* $A_{\mathbf{G}}^{T'} \otimes_{A_{\mathbf{G}}^{T}} \mathscr{F}(T) \to \mathscr{F}(T')$ *is an equivalence.*

Remark 5.5.4. Condition $(*)$ of Corollary 5.5.3 is automatic in the following cases:

- The orbispace X has the form $X^{(-)}$, where X is a p-finite space.
- The orbispace X has the form $X//H$, where H is a finite p-group and X is an H-space.

Proof of Corollary 5.5.3. The "if" direction of Corollary 5.5.3 follows from Corollary 5.5.2. Conversely, suppose that $\mathscr{F}$ is $\mathbf{G}$-tempered. Let us say that a morphism $f : T' \to T$ in $\mathscr{T}_{/\mathsf{X}}$ is *good* if the induced map $\theta : A_{\mathbf{G}}^{T'} \otimes_{A_{\mathbf{G}}^{T}} \mathscr{F}(T) \to \mathscr{F}(T')$ is an equivalence. Note that the collection of good morphisms in $\mathscr{T}_{/\mathsf{X}}$ is closed under composition. We wish to show that every morphism $f : T' \to T$ in $\mathscr{T}_{/\mathsf{X}}$ is good. Since $\mathscr{F}$ is $\mathbf{G}$-pretempered, this condition is automatic when f has connected homotopy fibers. In general, the morphism f factors as a composition $T' \xrightarrow{f'} T_0 \xrightarrow{f''} T$, where f'' is a covering map and f' has connected homotopy fibers. It will therefore suffice to show that every covering map $T_0 \to T$ is good. Proceeding by induction on the order of the finite group $\mathrm{Aut}(T_0/T)$, we can reduce to the case where $\mathrm{Aut}(T_0/T)$ is a cyclic group of prime order. If $\mathrm{Aut}(T_0/T)$ has order p, then Theorem 5.5.1 guarantees that multiplication by p induces an equivalence from the cofiber $\mathrm{cofib}(\theta)$ to itself. Since $\mathscr{F}$ is p-nilpotent, it follows that θ is an equivalence. To handle the case where $\mathrm{Aut}(T_0/T)$ has order prime to p, we apply hypothesis $(*)$ to factor the map $T \to \mathsf{X}$ as a composition $T \xrightarrow{g} T_{(p)} \to \mathsf{X}$, where $\pi_1(T_{(p)})$ is the p-local factor of $\pi_1(T)$. Then both g and $g|_{T_0}$ have connected homotopy fibers, and are therefore good (since $\mathscr{F}$ is $\mathbf{G}$-pretempered). It follows that the covering map $T_0 \to T$ is also good. □

Corollary 5.5.5. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*, let* p *be a prime number, and let* T *be the classifying space of a finite abelian* p*-group. Then evaluation on* T *induces an equivalence of* ∞*-categories*

$$\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(p)}(T) \simeq \mathrm{Mod}_{A_{\mathbf{G}}^{T}}^{\mathrm{Nil}(p)} .$$

Here $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(p)}(T)$ *denotes the full subcategory of* $\mathrm{LocSys}_{\mathbf{G}}(T)$ *spanned by the* p*-nilpotent objects, and* $\mathrm{Mod}_{A_{\mathbf{G}}^{T}}^{\mathrm{Nil}(p)} \subseteq \mathrm{Mod}_{A_{\mathbf{G}}^{T}}$ *is defined similarly.*

The proof of Theorem 5.5.1 will require the following:

Lemma 5.5.6. *Let* p *be a prime number and let* M *be a* C_p*-equivariant object of the* ∞*-category of spectra. Suppose that the* p*-completion of* M *is* $E(n)$*-local for some* $n \gg 0$*. Then* M^{hC_p} *vanishes if and only if the map* $p : M \to M$ *is invertible and the action of* C_p *on* $\pi_*(M)$ *has no nonzero fixed points.*

Proof. Note that if the map $p : M \to M$ is an equivalence, then the cohomology of C_p with coefficients in $\pi_*(M)$ vanishes in degrees > 0. It follows that the canonical map $M^{hC_p} \to M$ induces an isomorphism from $\pi_*(M^{hC_p})$ to the fixed points for the action of C_p on $\pi_*(M)$. This proves the "if" direction of Lemma 5.5.6. Conversely, suppose that the homotopy fixed point spectrum M^{hC_p} vanishes; we will complete the proof by showing that the map $p : M \to M$ is an equivalence. Let N denote the p-completion $M^{\wedge}_{(p)}$; we wish to show that $N \simeq 0$. By assumption, there exists some integer n for which the spectrum N is $E(n)$-local. Choose n to be as small as possible; we will complete the proof by showing that $n = 0$. Assume otherwise, so that $n > 0$ and the localization $L_{K(n)}(N)$ does not vanish. It follows from Theorem Ambi.5.4.3 that the homotopy fixed point spectrum $L_{K(n)}(N)^{hC_p}$ also does not vanish. This contradicts the vanishing of M^{hC_p}, since the p-completion functor $M \mapsto M^{\wedge}_{(p)}$ commutes with limits and the $K(n)$-localization functor $L_{K(n)}$ commutes with limits when restricted to $E(n)$-local spectra. □

Proof of Theorem 5.5.1. Let $\mathscr{F}$ be a $\mathbf{G}$-pretempered local system on an orbispace X. Suppose we are given an object $T \in \mathscr{T}_{/\mathsf{X}}$, a connected covering space $T_0 \in \mathrm{Cov}(T)$, and an isomorphism of finite groups $\mathrm{Aut}(T_0/T) \simeq C_p$ for some prime number p. We will prove that the following assertions are equivalent:

- (a) The fiber of the comparison map $\theta : \mathscr{F}(T) \to \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}$ is $I(T_0/T)$-local.
- (b) Let $\rho : A^{T_0}_{\mathbf{G}} \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T) \to \mathscr{F}(T_0)$ be the canonical map. Then multiplication by p induces an equivalence from $\mathrm{fib}(\rho)$ to itself, and the abelian group $\pi_*(\mathrm{fib}(\rho))$ contains no nonzero elements which are fixed by the action of the cyclic group $\mathrm{Aut}(T_0/T)$.

Allowing T and T_0 to vary, this will show that $\mathscr{F}$ is a $\mathbf{G}$-tempered local system if and only if it satisfies condition $(*)$ of Theorem 5.5.1 (see Remark 5.2.8).

Note that the map θ factors as a composition

$$\begin{aligned} \mathscr{F}(T) &\xrightarrow{\theta'} (A^{T_0}_{\mathbf{G}})^{\mathrm{hAut}(T_0/T)} \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T) \\ &\xrightarrow{\theta''} (A^{T_0}_{\mathbf{G}} \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T))^{\mathrm{hAut}(T_0/T)} \\ &\xrightarrow{\theta'''} \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}, \end{aligned}$$

where θ''' is obtained from ρ by passing to homotopy fixed points for the action of $\mathrm{Aut}(T_0/T)$. Note that the fiber $\mathrm{fib}(\theta')$ is given by the tensor product

$$\mathrm{fib}(A^T_{\mathbf{G}} \to (A^{T_0}_{\mathbf{G}})^{\mathrm{hAut}(T_0/T)}) \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T),$$

which is $I(T_0/T)$-local because the first factor is $I(T_0/T)$-local (note that $\underline{A}_{\mathsf{X}}$ is a $\mathbf{G}$-tempered local system; see Corollary 5.4.4). The map θ'' fits into a commutative diagram of fiber sequences

$$\begin{tikzcd}
(A^{T_0}_{\mathbf{G}})_{\mathrm{hAut}(T_0/T)} \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T) \arrow[r] \arrow[d] & (A^{T_0}_{\mathbf{G}} \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T))_{\mathrm{hAut}(T_0/T)} \arrow[d] \\
(A^{T_0}_{\mathbf{G}})^{\mathrm{hAut}(T_0/T)} \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T) \arrow[r, "\theta''"] \arrow[d] & (A^{T_0}_{\mathbf{G}} \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T))^{\mathrm{hAut}(T_0/T)} \arrow[d] \\
(A^{T_0}_{\mathbf{G}})^{t\,\mathrm{Aut}(T_0/T)} \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T) \arrow[r] & (A^{T_0}_{\mathbf{G}} \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T))^{t\,\mathrm{Aut}(T_0/T)}
\end{tikzcd}$$

where the upper horizontal map is an equivalence and the lower horizontal map has $I(T_0/T)$-local domain and codomain (Proposition 4.6.8). It follows that $\mathrm{fib}(\theta'')$ is also $I(T_0/T)$-local. Note that spectra $A^{T_0}_{\mathbf{G}} \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T)$ and $\mathscr{F}(T_0)$ are both $A^{T_0}_{\mathbf{G}}$-modules, and therefore $I(T_0/T)$-complete when viewed as modules over $A^T_{\mathbf{G}}$. Passing to homotopy fixed points, we deduce that the domain and codomain of θ''' are both $I(T_0/T)$-complete. Consequently, the fiber $\mathrm{fib}(\theta''')$ is $I(T_0/T)$-local if and only if it vanishes. It follows that (a) can be restated as follows:

- (a') The map $\theta''' : (A^{T_0}_{\mathbf{G}} \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T))^{\mathrm{hAut}(T_0/T)} \to \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}$ is an equivalence.

The equivalence of (a') and (b) now follows from Lemma 5.5.6. □

5.6 Extrapolation from Small Groups

Let A be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A. It follows from Remark 5.2.11 that for any orbispace X, the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ can be identified with the inverse limit

$$\varprojlim_{T \to \mathsf{X}} \mathrm{LocSys}_{\mathbf{G}}(T),$$

indexed by the collection of all objects $T \in \mathscr{T}$ equipped with a map of orbispaces $T^{(-)} \to \mathsf{X}$. We now formulate a refinement of this result.

Notation 5.6.1. Let $\vec{h} = \{h_p\}_{p \in \mathbf{P}}$ be a collection of nonnegative integers, indexed by the set $\mathbf{P}$ of all prime numbers. We let $\mathscr{T}(\leqslant \vec{h})$ denote the full subcategory of $\mathscr{T}$ spanned by those spaces of the form BH, where H is a finite abelian group with the following additional property:

- $(*)$ For each prime number $p \in \mathbf{P}$, the quotient H/pH has dimension $\leqslant h_p$ when regarded as a vector space over the finite field $\mathbf{F}_p$.

If $\mathbf{G}$ is a $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, we will say that $\mathbf{G}$ *has height* $\leqslant \vec{h}$ if, for each prime number p, the p-local summand $\mathbf{G}_{(p)}$ has height $\leqslant h_p$.

Theorem 5.6.2. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Let $\vec{h} = \{h_p\}_{p \in \mathbf{P}}$ be a collection of nonnegative integers such that $\mathbf{G}$ has height $\leqslant \vec{h}$. Then the functor*

$$\mathcal{OS}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty \qquad \mathsf{X} \mapsto \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$$

is a right Kan extension of its restriction to $\mathscr{T}(\leqslant \vec{h})^{\mathrm{op}} \subseteq \mathcal{OS}^{\mathrm{op}}$. In other words, for every orbispace X, the canonical map

$$\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \varprojlim_{T \to \mathsf{X}} \mathrm{LocSys}_{\mathbf{G}}(T)$$

is an equivalence of ∞-categories, where T ranges over objects of $\mathscr{T}(\leqslant \vec{h})$ equipped with a map of orbispace $T^{(-)} \to \mathsf{X}$.

Let us first note some consequences of Theorem 5.6.2.

Corollary 5.6.3. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Let $\vec{h} = \{h_p\}_{p \in \mathbf{P}}$ be a collection of nonnegative integers such that $\mathbf{G}$ has height $\leqslant \vec{h}$. Then the functor*

$$\mathcal{OS}^{\mathrm{op}} \to \mathrm{CAlg}_A \qquad \mathsf{X} \mapsto A_{\mathbf{G}}^{\mathsf{X}}$$

is a right Kan extension of its restriction to $\mathscr{T}(\leqslant \vec{h})^{\mathrm{op}} \subseteq \mathcal{OS}^{\mathrm{op}}$. In other words, for every orbispace X, the canonical map

$$A_{\mathbf{G}}^{\mathsf{X}} \to \varprojlim_{T \to \mathsf{X}} A_{\mathbf{G}}^{T}$$

is an equivalence of $\mathbb{E}_\infty$-algebras, where T ranges over objects of $\mathscr{T}(\leqslant \vec{h})$ equipped with a map of orbispace $T^{(-)} \to \mathsf{X}$.

Proof. Let X be an orbispace and let $\underline{A}_{\mathsf{X}} \in \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ be as in Example 5.1.5. Since $\mathbf{G}$ is oriented, $\underline{A}_{\mathsf{X}}$ is $\mathbf{G}$-tempered (Corollary 5.4.4). Combining Theorem 5.6.2 with Remark 5.1.20, we obtain equivalences

$$\begin{aligned} A_{\mathbf{G}}^{\mathsf{X}} &\simeq \underline{\mathrm{Map}}(\underline{A}_{\mathsf{X}}, \underline{A}_{\mathsf{X}}) \\ &\simeq \varprojlim_{f : T \to \mathsf{X}} \underline{\mathrm{Map}}(f^* \underline{A}_{\mathsf{X}}, f^* \underline{A}_{\mathsf{X}}) \\ &\simeq \varprojlim_{f : T \to \mathsf{X}} A_{\mathbf{G}}^{T}; \end{aligned}$$

here the limit is taken over objects $T \in \mathscr{T}(\vec{h})$ equipped with a map of orbispaces $f : T^{(-)} \to \mathsf{X}$. □

Remark 5.6.4. In §7.6, we will discuss some more concrete variants of Corollary 5.6.3 which can be used to obtain information about the $\mathbf{G}$-tempered cohomology ring $A^*_{\mathbf{G}}(\mathsf{X})$: see Theorems 7.6.3 and 7.6.5.

Example 5.6.5. Let p be a prime number and let $\mathbf{G}$ be an oriented p-divisible group over an $\mathbb{E}_\infty$-ring A. It follows from Theorem 5.6.2 and Corollary 5.6.3 that, for any orbispace X, the canonical maps

$$\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \varprojlim_{T \to \mathsf{X}} \mathrm{LocSys}_{\mathbf{G}}(T) \qquad A^{\mathsf{X}}_{\mathbf{G}} \to \varprojlim_{T \to \mathsf{X}} A^{T}_{\mathbf{G}}$$

are equivalences, where both limits are taken over the collection of maps $T^{(-)} \to \mathsf{X}$ where T is the classifying space of a finite abelian p-group.

Example 5.6.6. Let $\mu_{\mathbf{P}^\infty}$ be the multiplicative $\mathbf{P}$-divisible group, viewed as an oriented $\mathbf{P}$-divisible group over the complex K-theory spectrum KU (Construction 2.8.6). It follows from Theorem 5.6.2 and Corollary 5.6.3 that, for any orbispace X, the canonical maps

$$\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \varprojlim_{T \to \mathsf{X}} \mathrm{LocSys}_{\mathbf{G}}(T) \qquad \mathrm{KU}^{\mathsf{X}}_{\mathbf{G}} \to \varprojlim_{T \to \mathsf{X}} \mathrm{KU}^{T}_{\mathbf{G}}$$

are equivalences, where both limits are taken over the collection of maps $T^{(-)} \to \mathsf{X}$ where T is the classifying space of a finite cyclic group.

Example 5.6.7. Let A be an $\mathbb{E}_\infty$-ring, let E be an oriented elliptic curve over A, and let $\mathsf{E}[\mathbf{P}^\infty]$ denote the torsion subgroup of E, regarded as an oriented $\mathbf{P}$-divisible group as in Construction 2.9.6. It follows from Theorem 5.6.2 and Corollary 5.6.3 that, for any orbispace X, the canonical maps

$$\mathrm{LocSys}_{\mathsf{E}[\mathbf{P}^\infty]}(\mathsf{X}) \to \varprojlim_{T \to \mathsf{X}} \mathrm{LocSys}_{\mathsf{E}[\mathbf{P}^\infty]}(T) \qquad A^{\mathsf{X}}_{\mathsf{E}[\mathbf{P}^\infty]} \to \varprojlim_{T \to \mathsf{X}} A^{T}_{\mathsf{E}[\mathbf{P}^\infty]}$$

are equivalences, where both limits are taken over the collection of maps $T^{(-)} \to \mathsf{X}$ where T is the classifying space of a finite abelian group that can be generated by two elements.

The proof of Theorem 5.6.2 is based on a reformulation of condition (B) appearing in the definition of $\mathbf{G}$-tempered local system (Definition 5.2.4). First, we need a bit of terminology. Let R be an $\mathbb{E}_\infty$-ring and let $K \subseteq |\operatorname{Spec}(R)|$ be a cocompact closed subset (that is, a closed subset with quasi-compact complement). Then K

can be realized as the vanishing locus of a finitely generated ideal $I \subseteq \pi_0(R)$. We will say that an R-module spectrum M is *K-complete* (*K-local*, *K-nilpotent*) if it is I-complete (I-local, I-nilpotent), in the sense of Definition SAG.II.4.3.1.1 (Definition SAG.II.4.2.4.1 , Definition SAG.II.4.1.1.6). We say that a morphism of R-modules $M \to \widehat{M}$ *exhibits* $\widehat{M}$ *as the completion of* M *along* K if it exhibits $\widehat{M}$ as the completion of M with respect to I. We will be particularly interested in the case where R is a tempered function spectrum $A^T_{\mathbf{G}}$; in this case, there are several closed subsets of $|\operatorname{Spec}(R)|$ of geometric interest.

Notation 5.6.8. Let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. For each finite abelian group M, we let $|\mathbf{G}[M]|$ denote the underlying topological space of the finite flat A-scheme representing the functor

$$\mathrm{CAlg}_A \to \mathcal{S} \qquad B \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}(B)).$$

Note that for every subgroup $M_0 \subseteq M$, the canonical map $\iota_{M_0} : |\mathbf{G}[M/M_0]| \to |\mathbf{G}[M]|$ is a closed embedding. We let $|\mathbf{G}[M]|^{\deg} \subseteq |\mathbf{G}[M]|$ denote the union of the images of the maps ι_{M_0}, where M_0 ranges over all nontrivial subgroups of M. More informally, $|\mathbf{G}[H]|^{\deg}$ is the closed subset of $|\mathbf{G}[M]|$ which parametrizes maps $M \to \mathbf{G}$ which are *degenerate* in the sense that they annihilate some nonzero subgroup of M (at the level of geometric points).

Now suppose that $\mathbf{G}$ is equipped with a preorientation. Let T be an object of $\mathscr{T}$, and let $M = \widehat{\pi_1(T)}$ be the Pontryagin dual of the finite abelian group $\pi_1(T)$, so that we can identify $|\mathbf{G}[M]|$ with the Zariski spectrum $|\operatorname{Spec}(A^T_{\mathbf{G}})|$. We let $|\operatorname{Spec}(A^T_{\mathbf{G}})|^{\deg}$ denote the image of $|\mathbf{G}[M]|^{\deg}$ under this identification.

Theorem 5.6.9. *Let* $\mathbf{G}$ *be a preoriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* T *be an object of* $\mathscr{T}$*. Then an object* $\mathscr{F} \in \mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$ *is tempered (in the sense of Notation 5.4.1) if and only if it satisfies the following condition, for every object* $T' \in \mathrm{Cov}(T)$*:*

(∗) *Let* $\mathrm{Cov}^\circ(T') \subsetneq \mathrm{Cov}(T')$ *denote the full subcategory of* $\mathrm{Cov}(T')$ *spanned by those connected covering maps* $T'' \to T'$ *which are not homotopy equivalences. Then the canonical map*
$$\mathscr{F}(T') \to \varprojlim_{T'' \in \mathrm{Cov}^\circ(T')^{\mathrm{op}}} \mathscr{F}(T'')$$
exhibits $\varprojlim_{T'' \in \mathrm{Cov}^\circ(T')^{\mathrm{op}}} \mathscr{F}(T'')$ *as the completion of* $\mathscr{F}(T')$ *along the closed subset* $|\operatorname{Spec}(A^{T'}_{\mathbf{G}})|^{\deg} \subseteq |\operatorname{Spec}(A^{T'}_{\mathbf{G}})|$.

Remark 5.6.10. In the situation of Theorem 5.6.9, the limit $\varprojlim_{T'' \in \mathrm{Cov}^{\circ}(T')^{\mathrm{op}}} \mathscr{F}(T'')$ is automatically complete for the closed subset $|\operatorname{Spec}(A^{T'}_{\mathbf{G}})|^{\deg} \subseteq |\operatorname{Spec}(A^{T'}_{\mathbf{G}})|$. Consequently, condition $(*)$ is satisfied if and only if the fiber of the map $\mathscr{F}(T') \to \varprojlim_{T'' \in \mathrm{Cov}^{\circ}(T')^{\mathrm{op}}} \mathscr{F}(T'')$ is local with respect to $|\operatorname{Spec}(A^{T'}_{\mathbf{G}})|^{\deg}$ (that is, it arises from a quasi-coherent sheaf on the complement of $|\operatorname{Spec}(A^{T'}_{\mathbf{G}})|^{\deg}$).

Corollary 5.6.11. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_{\infty}$-ring A, let X be an orbispace, and let $\mathscr{F} \in \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ be a $\mathbf{G}$-pretempered local system on X. Then $\mathscr{F}$ is $\mathbf{G}$-tempered if and only if, for each $T \in \mathscr{T}_{/\mathsf{X}}$, the following condition is satisfied:*

- $(*')$ *Let $\mathrm{Cov}^{\circ}(T) \subsetneq \mathrm{Cov}(T)$ denote the full subcategory of $\mathrm{Cov}(T)$ spanned by those connected covering maps $T' \to T$ which are not homotopy equivalences. Then the canonical map*
$$\mathscr{F}(T) \to \varprojlim_{T' \in \mathrm{Cov}^{\circ}(T)^{\mathrm{op}}} \mathscr{F}(T')$$
exhibits $\varprojlim_{T' \in \mathrm{Cov}^{\circ}(T')^{\mathrm{op}}} \mathscr{F}(T')$ as the completion of $\mathscr{F}(T)$ along the closed subset $|\operatorname{Spec}(A^{T}_{\mathbf{G}})|^{\deg} \subseteq |\operatorname{Spec}(A^{T}_{\mathbf{G}})|$.

Proof of Theorem 5.6.2 from Theorem 5.6.9. Let $\mathrm{LocSys}'_{\mathbf{G}} : \mathcal{OS}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ be a right Kan extension of the functor

$$\mathscr{T}(\leqslant \vec{h})^{\mathrm{op}} \xrightarrow{T \mapsto T^{(-)}} \mathcal{OS}^{\mathrm{op}} \xrightarrow{\mathrm{LocSys}_{\mathbf{G}}} \widehat{\mathcal{C}\mathrm{at}}_{\infty},$$

given informally by $\mathrm{LocSys}'_{\mathbf{G}}(\mathsf{X}) = \varprojlim_{T \to \mathsf{X}} \mathrm{LocSys}_{\mathbf{G}}(T)$ where the limit is taken over objects $T \in \mathscr{T}(\leqslant \vec{h})$. We wish to show that for every orbispace X, the canonical map $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}'_{\mathbf{G}}(\mathsf{X})$ is an equivalence of ∞-categories. By the transitivity of Kan extensions (Proposition HTT.4.3.2.8), it will suffice to prove this in the special case where $\mathsf{X} = T^{(-)}$ is representable by an object $T \in \mathscr{T}$. We proceed by induction on the order of the finite group $\pi_1(T)$. If T belongs to $\mathscr{T}(\leqslant \vec{h})$, there is nothing to prove. Otherwise, let $\mathscr{T}^{\circ}_{/T} \subseteq \mathscr{T}_{/T}$ be the full subcategory spanned by those maps $T' \to T$ which are not surjective on fundamental groups. Then $\mathscr{T}^{\circ}_{/T}$ contains every map $T' \to T$ where T' belongs to $\mathscr{T}(\leqslant \vec{h})$. It follows that we can identify $\mathrm{LocSys}'_{\mathbf{G}}(T)$ with the limit $\varprojlim_{T' \in (\mathscr{T}^{\circ}_{/T})^{\mathrm{op}}} \mathrm{LocSys}'_{\mathbf{G}}(T')$. Let $\mathrm{Cov}^{\circ}(T)$ be as in Theorem 5.6.9, so that we can regard $\mathrm{Cov}^{\circ}(T)$ as a full subcategory of $\mathscr{T}^{\circ}_{/T}$. Moreover, the inclusion functor $\mathrm{Cov}^{\circ}(T) \hookrightarrow \mathscr{T}^{\circ}_{/T}$ has a left adjoint, and is therefore left cofinal. We have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{LocSys}_{\mathbf{G}}(T) \arrow[r] \arrow[d] & \varprojlim_{T' \in \mathrm{Cov}^{\circ}(T)^{\mathrm{op}}} \mathrm{LocSys}_{\mathbf{G}}(T') \arrow[d] \\
\mathrm{LocSys}'_{\mathbf{G}}(T) \arrow[r] & \varprojlim_{T' \in \mathrm{Cov}^{\circ}(T)^{\mathrm{op}}} \mathrm{LocSys}'_{\mathbf{G}}(T'),
\end{tikzcd}$$

where the bottom horizontal map is an equivalence (by the preceding argument) and the right vertical map is an equivalence (by our inductive hypothesis). It therefore suffice to show that the upper horizontal map is an equivalence of ∞-categories.

Let $q : \mathrm{Mod}(\mathrm{Sp}) \to \mathrm{CAlg}$ be as in Construction 5.1.8. We then have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{LocSys}_{\mathbf{G}}(T) \arrow[r] \arrow[d, "\iota"] & \varprojlim_{T' \in \mathrm{Cov}^{\circ}(T)^{\mathrm{op}}} \mathrm{LocSys}_{\mathbf{G}}(T') \arrow[d, "\iota^{\circ}"] \\
\mathrm{Fun}_{\mathrm{CAlg}}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp})) \arrow[r] & \mathrm{Fun}_{\mathrm{CAlg}}(\mathrm{Cov}^{\circ}(T)^{\mathrm{op}}, \mathrm{Sp})
\end{tikzcd}$$

where the horizontal maps are given by restriction and the vertical maps are fully faithful embeddings. Moreover, Theorem 5.6.9 implies that an object $\mathscr{F}$ of the ∞-category $\mathrm{Fun}_{\mathrm{CAlg}}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp}))$ belongs to the essential image of ι if and only if $\mathscr{F}|_{\mathrm{Cov}^{\circ}(T)^{\mathrm{op}}}$ belongs to the essential image of ι° and the canonical map

$$\theta : \mathscr{F}(T) \to \varprojlim_{T' \in \mathrm{Cov}^{\circ}(T)^{\mathrm{op}}} \mathscr{F}(T')$$

exhibits $\varprojlim_{T' \in \mathrm{Cov}^{\circ}(T)^{\mathrm{op}}} \mathscr{F}(T')$ as the completion of $\mathscr{F}(T)$ along the closed subset $|\operatorname{Spec}(A^{T}_{\mathbf{G}})|^{\mathrm{deg}} \subseteq |\operatorname{Spec}(A^{T}_{\mathbf{G}})|$. Our assumption that T does not belong to $\mathscr{T}(\leqslant \vec{h})$ guarantees that there exists some prime number p for which the quotient $\pi_1(T)/p\pi_1(T)$ has dimension strictly larger than the height of the p-divisible group $\mathbf{G}_{(p)}$. It follows that $|\operatorname{Spec}(A^{T}_{\mathbf{G}})|^{\mathrm{deg}}$ is equal to $|\operatorname{Spec}(A^{T}_{\mathbf{G}})|$. Consequently, a functor a functor $\mathscr{F} \in \mathrm{Fun}_{\mathrm{CAlg}}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp}))$ belongs to the essential image of ι if and only if $\mathscr{F}|_{\mathrm{Cov}^{\circ}(T)^{\mathrm{op}}}$ belongs to the essential image of ι° and the map θ is an equivalence: that is, $\mathscr{F}$ is a q-right Kan extension of its restriction to the subcategory $\mathrm{Cov}^{\circ}(T)^{\mathrm{op}} \subseteq \mathrm{Cov}(T)^{\mathrm{op}}$. The desired result now follows from Proposition HTT.4.3.2.15 . □

The proof of Theorem 5.6.9 will require the following general fact about completions:

Proposition 5.6.12. *Let R be an $\mathbb{E}_\infty$-ring. Let S be a finite partially ordered set, let $\{K_s\}_{s \in S}$ be a collection of cocompact closed subsets of $|\operatorname{Spec}(R)|$ parametrized by s, and let $M : S^{\mathrm{op}} \to \operatorname{Mod}_R$ be a diagram of R-modules parametrized by S^{op}. Assume that the following conditions are satisfied:*

- (a) *The partially ordered set S is a lower semilattice. That is, S contains a greatest element $\mathbf{1}$, and every pair of elements $s, s' \in S$ have a greatest lower bound $s \wedge s' \in S$.*
- (b) *The construction $s \mapsto K_s$ is a homomorphism of lower semilattices. That is, we have $K_{\mathbf{1}} = |\operatorname{Spec}(R)|$, and $K_{s \wedge s'} = K_s \cap K_{s'}$ for all $s, s' \in S$.*
- (c) *For $s \leqslant s' < \mathbf{1}$, the map $M(s') \to M(s)$ exhibits $M(s)$ as the completion of $M(s')$ along the closed subset K_s. In particular, each $M(s)$ is K_s-compete.*

Let $K = \bigcup_{s < \mathbf{1}} K_s$. Then the following conditions are equivalent:

- (1) *For each $s \in S$, the map $M(\mathbf{1}) \to M(s)$ exhibits $M(s)$ as the completion of $M(\mathbf{1})$ along K_s.*
- (2) *Let $M' = \varprojlim_{s < \mathbf{1}} M(s)$. Then the canonical map $M(\mathbf{1}) \to M'$ exhibits M' as the completion of $M(\mathbf{1})$ along K.*

Proof. Assume first that (2) is satisfied, and choose $s \in S$. We wish to prove that the map $M(\mathbf{1}) \to M(s)$ exhibits $M(s)$ as the completion of $M(\mathbf{1})$ along K_s. We may assume that $s = \mathbf{1}$ (otherwise there is nothing to prove). For $t \neq \mathbf{1}$, condition (c) implies that $M(t)$ is K_t-complete. Let $M(t)^{\wedge}_{K_s}$ denote the completion of $M(t)$ along K_s. Then $M(t)^{\wedge}_{K_s}$ is also the completion of $M(t)$ along the intersection $K_s \cap K_t$, which is equal to $K_{s \wedge t}$ by virtue of (b). Applying (c), we conclude that the canonical map $M(t) \to M(s \wedge t)$ exhibits $M(s \wedge t)$ as the completion of M_t along K_s. Since completion along K_s commutes with with limits, we obtain an equivalence

$$M'^{\wedge}_{K_s} \simeq \varprojlim_{t \neq \mathbf{1}} M(t)^{\wedge}_{K_s} \simeq \varprojlim_{t \neq \mathbf{1}} M(s \wedge t) \simeq M_s.$$

In other words, the canonical map $M' \to M_s$ exhibits M_s as the completion of M' along K_s. It will therefore suffice to show that the natural map $M \to M'$ induces an equivalence after completion along K_s. This follows immediately from assumption (2), since K_s is contained in K.

The implication (1) $\Rightarrow$ (2) can be rephrased as follows:

(∗) Let N be an R-module. Then the canonical map $N \to \varprojlim_{t \neq \mathbf{1}} N^{\wedge}_{K_t}$ exhibits $\varprojlim_{t \neq \mathbf{1}} N^{\wedge}_{K_t}$ as the completion of N along K.

To prove (∗), we note that for each $s \in S$ there is a fiber sequence

$$N' \to N \to N^{\wedge}_{K_s},$$

where N' is K_s-local. It therefore suffices to prove that condition (∗) holds for N" and $N^{\wedge}_{K_s}$ individually. Moreover, if N is K_s-complete (K_t-local) for some other index $t \in S$, then N' and $N^{\wedge}_{K_s}$ are K_t-complete (K_t-local). Applying this observation repeatedly, we may reduce to the case where N is either K_s-local or K_s-complete for *every* value $s \in S$.

Let $S' \subseteq S$ be the collection of those elements $s \in S$ for which N is K_s-complete. Then the completions $N^{\wedge}_{K_s}$ vanish for $s \notin S'$. It follows that the functor $(S \backslash \{\mathbf{1}\})^{\mathrm{op}} \to \mathrm{Mod}_R$ given by $s \mapsto N^{\wedge}_{K_s}$ is a right Kan extension of its restriction to $(S' \backslash \{\mathbf{1}\})^{\mathrm{op}}$, so that

$$\varprojlim_{t \in S \backslash \{\mathbf{1}\}} N^{\wedge}_{K_t} \simeq \varprojlim_{t \in S' \backslash \{\mathbf{1}\}} N^{\wedge}_{K_t}.$$

Using conditions (a) and (b), we see that S' is closed under finite meets in S. Since S' is finite, it has a smallest element s. There are two cases to consider:

- Suppose that $s \neq \mathbf{1}$. Then $K_s \subseteq K$. Since $s \in S'$, the R-module N is K_s-complete and therefore also K-complete. We are therefore reduced to proving that the canonical map

  $$N \to \varprojlim_{t \in S \backslash \{\mathbf{1}\}} N^{\wedge}_{K_t} \simeq \varprojlim_{t \in S' \backslash \{\mathbf{1}\}} N^{\wedge}_{K_t} \simeq N^{\wedge}_{K_s}$$

  is an equivalence, which is clear.

- Suppose that $s = \mathbf{1}$, so that the completion $N^{\wedge}_{K_t}$ vanishes for $t \neq \mathbf{1}$. Then N is local with respect to the closed subset $K \subseteq |\operatorname{Spec}(R)|$, so that the canonical map $N \to \varprojlim_{t \neq \mathbf{1}} N^{\wedge}_{K_t} \simeq 0$ exhibits $\varprojlim_{t \neq \mathbf{1}} N^{\wedge}_{K_t} \simeq 0$ as the completion of N along K.

□

Proof of Theorem 5.6.9. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_{\infty}$-ring A, let T be an object of $\mathscr{T}$, and let $\mathscr{F}$ be an object of $\mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$. Assume first that $\mathscr{F}$ is tempered; we wish to show that $\mathscr{F}$ satisfies condition (∗) of

Theorem 5.6.9. Let T' be a connected covering space of T, so that the restriction of $\mathscr{F}$ to $\mathrm{Cov}(T')$ determines a functor $M_0 : \mathrm{Cov}(T')^{\mathrm{op}} \to \mathrm{Mod}_{A_{\mathbf{G}}^{T'}}$. Let S denote the collection of subgroups of $\pi_1(T')$, ordered by inclusion. We have an evident functor $\mathrm{Cov}(T') \to S$, which carries a covering space $T'' \to T'$ to the image of the induced homomorphism $\pi_1(T'') \hookrightarrow \pi_1(T')$. Let $M : S^{\mathrm{op}} \to \mathrm{Mod}_{A_{\mathbf{G}}^{T'}}$ be the right Kan extension of M_0 along the map $\mathrm{Cov}(T')^{\mathrm{op}} \to S^{\mathrm{op}}$. Unwinding the definitions, we see that if $H \subseteq \pi_1(T')$ is the image of the fundamental group of some covering space T'' of T', then $M(H)$ can be identified with the homotopy fixed point spectrum $\mathscr{F}(T'')^{\mathrm{hAut}(T''/T')}$.

For every subgroup $H \subseteq \pi_1(T')$, let $K_H \subseteq |\mathrm{Spec}(A_{\mathbf{G}}^{T'})|$ denote the image of the closed embedding $|\mathrm{Spec}(A_{\mathbf{G}}^{T''})| \hookrightarrow |\mathrm{Spec}(A_{\mathbf{G}}^{T'})|$, where T'' is the connected covering space of T with fundamental group H. We claim that the constructions

$$H \mapsto K_H \qquad H \mapsto M(H)$$

satisfy hypotheses (a) through (c) of Proposition 5.6.12:

(a) As a partially ordered set, S is a lower semi-lattice. This is clear, since the intersection of a finite collection of subgroups of $\pi_1(T')$ is again a subgroup of $\pi_1(T')$.

(b) The construction $H \mapsto K_H$ is a homomorphism of lower semilattices. At the level of geometric points, this follows from the observation that a group homomorphism $\widehat{\pi_1(T')} \to \mathbf{G}(\kappa)$ factors through the Pontryagin dual of an intersection of subgroups $H \cap H'$ if and only if it factors through both $\widehat{H}$ and $\widehat{H}'$.

(c) For $H \subseteq H' \subsetneq \pi_1(T)$, the map $M(H') \to M(H)$ exhibits $M(H)$ as the completion of $M(H')$ along the closed subset $K_H \subseteq |\mathrm{Spec}(A_{\mathbf{G}}^{T'})|$. Write $H = \pi_1(T'')$ and $H' = \pi_1(T''')$ for covering maps $T''' \to T'' \to T'$. Unwinding the definitions, we wish to show that the map

$$\mathscr{F}(T'')^{\mathrm{hAut}(T''/T')} \to \mathscr{F}(T''')^{\mathrm{hAut}(T'''/T')}$$

exhibits $\mathscr{F}(T''')^{\mathrm{hAut}(T'''/T')}$ as a completion of $\mathscr{F}(T'')^{\mathrm{hAut}(T''/T')}$ with respect to the relative augmentation ideal $I(T'''/T'')$. Since the formation of completions completions commutes with limits, it will suffice to show that the map $\beta : \mathscr{F}(T'') \to \mathscr{F}(T''')^{\mathrm{hAut}(T'''/T'')}$ exhibits $\mathscr{F}(T''')^{\mathrm{hAut}(T'''/T'')}$ as the $I(T'''/T'')$-completion of $\mathscr{F}(T'')$, which follows from our assumption that $\mathscr{F}$ is tempered.

Note that the verification of condition (c) does not require the assumption that H is a proper subgroup of $\pi_1(T')$. It follows that the functor $M : S^{\mathrm{op}} \to \mathrm{Mod}_{A_{\mathbf{G}}^{T'}}$ satisfies condition (1) of Proposition 5.6.12. It therefore also satisfies condition (2): that is, the canonical map

$$\mathscr{F}(T') = M(\pi_1(T')) \to \varprojlim_{H \subsetneq \pi_1 T')} M(H) \simeq \varprojlim_{T'' \in \mathrm{Cov}^{\circ}(T')^{\mathrm{op}}} \mathscr{F}(T'')$$

exhibits $\varprojlim_{T'' \in \mathrm{Cov}^{\circ}(T')^{\mathrm{op}}} \mathscr{F}(T'')$ as the completion of $\mathscr{F}(T')$ along the closed set

$$\bigcup_{H \subsetneq \pi_1(T')} K_H = |\operatorname{Spec}(A_{\mathbf{G}}^{T'})|^{\deg}$$

of Notation 5.6.8.

Suppose now that $\mathscr{F}$ satisfies condition $(*)$ of Theorem 5.6.9; we wish to show that it also satisfies condition (B_0') of Notation 5.4.1. Let T' be a connected covering space of T and let T'' be a connected covering space of T'. We will show that the map $\mathscr{F}(T') \to \mathscr{F}(T'')^{\mathrm{hAut}(T''/T')}$ exhibits $\mathscr{F}(T'')^{\mathrm{hAut}(T''/T')}$ as an $I(T''/T')$-completion of $\mathscr{F}(T')$. We proceed by induction on the order of the finite group $\pi_1(T)$. Let S, $\{M_H\}_{M \in S}$, and $\{K_H\}_{H \in S}$ be defined as in the first part of the proof. Then the data

$$H \mapsto K_H \qquad H \mapsto M_H$$

satisfies conditions (a) through (c) of Proposition 5.6.12 (the proof is exactly as above, except that (c) follows from the inductive hypothesis rather than our assumption that $\mathscr{F}$ arises from a **G**-tempered local system). Hypothesis $(*)$ of Theorem 5.6.9 then guarantees that the map

$$M(\pi_1(T')) \simeq \mathscr{F}(T') \to \varprojlim_{T'' \in \mathrm{Cov}^{\circ}(T')^{\mathrm{op}}} \mathscr{F}(T'') \simeq \varprojlim_{H \subsetneq \pi_1(T')} M(H)$$

exhibits $\varprojlim_{H \subsetneq \pi_1(T')} M(H)$ as the completion of $M(\pi_1(T'))$ along $|\operatorname{Spec}(A_{\mathbf{G}}^{T'})|^{\deg}$. Applying Proposition 5.6.12, we deduce that if $H \subseteq \pi_1(T')$ is the fundamental group of a connected covering space T'' of T', then the canonical map

$$\mathscr{F}(T') \simeq M(\pi_1(T')) \to M(H) \simeq \mathscr{F}(T'')^{\mathrm{hAut}(T''/T')}$$

exhibits $\mathscr{F}(T'')^{\mathrm{hAut}(T''/T')}$ as an $I(T''/T')$-completion of $\mathscr{F}(T')$, as desired. □

5.7 Digression: The ∞-Category $\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}}(\mathsf{X})$

Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let X be an orbispace. Then the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is a localization of the stable ∞-category $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$: that is, the inclusion functor $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \hookrightarrow \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ admits a left adjoint $L : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ (Proposition 5.2.12). It follows that the ∞-category $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ admits a semi-orthogonal decomposition $({}^{\perp}\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}), \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}))$ (see Proposition SAG.II.4.2.1.4); here ${}^{\perp}\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ denotes the full subcategory of $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ spanned by those $\mathbf{G}$-pretempered local systems $\mathscr{F}$ satisfying $L\mathscr{F} \simeq 0$ (or, equivalently, the full subcategory spanned by those objects $\mathscr{F}$ satisfying $\mathrm{Ext}^*_{\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})}(\mathscr{F}, \mathscr{G}) \simeq 0$ whenever $\mathscr{G}$ is $\mathbf{G}$-tempered). Our goal in this section is to give an explicit description of the subcategory ${}^{\perp}\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ in the special case where $\mathbf{G}$ is oriented (Theorem 5.7.3).

Definition 5.7.1. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and let $\mathscr{F} \in \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ be a $\mathbf{G}$-pretempered local system on an orbispace X. We will say that $\mathscr{F}$ is *null* if, for every object $T \in \mathscr{T}_{/\mathsf{X}}$, the $A^T_{\mathbf{G}}$-module $\mathscr{F}(T)$ is $|\operatorname{Spec}(A^T_{\mathbf{G}})|^{\mathrm{deg}}$-nilpotent. We let $\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}}(\mathsf{X})$ denote the full subcategory of $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ spanned by the null $\mathbf{G}$-pretempered local systems on X.

Remark 5.7.2. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $\mathscr{F} \in \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ be a $\mathbf{G}$-pretempered local system on an orbispace X. Then:

- If $\mathscr{F}$ is null and $f : \mathsf{Y} \to \mathsf{X}$ is any map of orbispaces, then the pullback $f^*\mathscr{F}$ is null.
- Suppose that there exists a collection of maps $\{f_\alpha : \mathsf{X}_\alpha \to \mathsf{X}\}$ which induces a surjection $\coprod_\alpha \pi_0(\mathsf{X}^T_\alpha) \to \pi_0(\mathsf{X}^T)$, for each $T \in \mathscr{T}$. If each pullback $f^*_\alpha \mathscr{F}$ is null, then $\mathscr{F}$ is null.

Our main result can be stated as follows:

Theorem 5.7.3. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Then, for any orbispace X, the subcategories $(\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}}(\mathsf{X}), \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}))$ determine a semi-orthogonal decomposition of $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$. In other words, a $\mathbf{G}$-pretempered local system $\mathscr{F}$ on X is null if and only if it is annihilated by the localization functor $L : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ of Proposition 5.2.12.*

Remark 5.7.4. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. For any orbispace X, the inclusion functor $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \hookrightarrow \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ preserves small colimits

(Theorem 5.3.1), and therefore admits a right adjoint $L' : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ (Corollary HTT.5.5.2.9). It follows that $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ admits a semi-orthogonal decomposition $(\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}), \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})^{\perp})$, where $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})^{\perp} \subseteq \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is the *right* orthogonal to the subcategory $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$: that is, the full subcategory spanned by those objects which are annihilated by the functor L'. We do not know an analogue of Theorem 5.7.3 for the subcategory $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})^{\perp}$.

Before giving the proof of Theorem 5.7.3, let us note some consequences.

Corollary 5.7.5. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_{\infty}$*-ring* A *and let* $f : \mathsf{X} \to \mathsf{Y}$ *be a morphism of orbispaces. Then the diagram of* ∞*-categories*

$$\begin{tikzcd}
\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \arrow[r] \arrow[d, "f^*"] & \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{Y}) \arrow[d, "f^*"] \\
\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \arrow[r] & \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{Y})
\end{tikzcd}$$

is left adjointable. That is, if

$$L_{\mathsf{X}} : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \qquad L_{\mathsf{Y}} : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$$

denote left adjoints to the inclusion maps, then the evident natural transformation $L_{\mathsf{X}} \circ f^* \to f^* \circ L_{\mathsf{Y}}$ *is an equivalence of functors from* $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{Y})$ *to* $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$.

Proof. Let $\mathscr{F}$ be a $\mathbf{G}$-pretempered local system Y, so that we have a fiber sequence $\mathscr{F}' \to \mathscr{F} \xrightarrow{\alpha} L_{\mathsf{Y}} \mathscr{F}$ in the ∞-category $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{Y})$. Pulling back along f, we obtain a fiber sequence

$$f^* \mathscr{F}' \to f^* \mathscr{F} \xrightarrow{f^*(\alpha)} f^* L_{\mathsf{Y}} \mathscr{F} .$$

We wish to show that $f^*(\alpha)$ exhibits $f^* L_{\mathsf{Y}} \mathscr{F}$ as a $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$-localization of $f^* \mathscr{F}$. Since $f^* L_{\mathsf{Y}} \mathscr{F}$ is $\mathbf{G}$-tempered, it will suffice (by virtue of Theorem 5.7.3) to show that the pullback $f^* \mathscr{F}'$ is null. This follows from Remark 5.7.2, since $\mathscr{F}'$ is null (Theorem 5.7.3). □

Remark 5.7.6. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_{\infty}$-ring A, let $\mathscr{F} \in \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ be a $\mathbf{G}$-pretempered local system on an orbispace X, and let $\alpha : \mathscr{F} \to L \mathscr{F}$ be a morphism which exhibits $L \mathscr{F}$ as a $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$-localization of $\mathscr{F}$. Then the forgetful functor $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{A}(|\mathsf{X}|)$ of Variant 5.1.15 carries α to an equivalence in $\mathrm{LocSys}_{A}(|\mathsf{X}|)$. In other words, replacing a $\mathbf{G}$-pretempered local system $\mathscr{F}$ with the associated $\mathbf{G}$-tempered local system does not change the underlying local system of $\mathscr{F}$. To prove this, it suffices to observe that $\mathscr{G} = \mathrm{fib}(\alpha)$ is null (Theorem 5.7.3), so that $\mathscr{G}(T)$ vanishes whenever T is contractible (since the topological space $|\operatorname{Spec}(A_{\mathbf{G}}^{T})|^{\deg}$ is empty when T is contractible).

The proof of Theorem 5.7.3 will require some preliminaries. We begin by observing that for each orbispace X, Proposition 5.1.9 supplies an equivalence of ∞-categories

$$\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X}) \simeq \varprojlim_{T \in \mathscr{T}_{/\mathsf{X}}^{\mathrm{op}}} \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(T).$$

By virtue of Remarks 5.2.11 and 5.7.2, this restricts to equivalences of full subcategories

$$\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{nul}}(\mathsf{X}) \simeq \varprojlim_{T \in \mathscr{T}_{/\mathsf{X}}^{\mathrm{op}}} \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{nul}}(T) \qquad \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \simeq \varprojlim_{T \in \mathscr{T}_{/\mathsf{X}}^{\mathrm{op}}} \mathrm{LocSys}_{\mathbf{G}}(T).$$

Consequently, to show that the pair $(\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{nul}}(\mathsf{X}), \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}))$ is a semi-orthogonal decomposition of the stable ∞-category $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$, it will suffice to establish the following special case of Theorem 5.7.3:

Proposition 5.7.7. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Then, for every object $T \in \mathscr{T}$, the subcategories $(\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{nul}}(T), \mathrm{LocSys}_{\mathbf{G}}(T))$ determine a semi-orthogonal decomposition of the stable ∞-category $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(T)$.*

Our next step is to describe the ∞-category $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{nul}}(T)$ more concretely.

Notation 5.7.8. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let T be an object of $\mathscr{T}$, and let $\mathscr{F}$ be an $A_{\mathbf{G},T}$-module object of $\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp})$. We will say that $\mathscr{F}$ is *null* if, for every connected covering space T_0 of T, the spectrum $\mathscr{F}(T_0)$ is $|\operatorname{Spec}(A_{\mathbf{G}}^{T_0})|^{\deg}$-nilpotent, when viewed as a module over the tempered function spectrum $A_{\mathbf{G}}^{T_0}$. We let $\mathrm{Mod}_{A_{\mathbf{G},T}}^{\mathrm{nul}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$ denote the full subcategory of $\mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$ spanned by the null $A_{\mathbf{G},T}$-modules.

Lemma 5.7.9. *Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let T be an object of $\mathscr{T}$. Then the equivalence $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(T) \simeq \mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$ of Proposition 5.1.12 restricts to an equivalence of ∞-categories $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{nul}}(T) \simeq \mathrm{Mod}_{A_{\mathbf{G},T}}^{\mathrm{nul}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$. In other words, a $\mathbf{G}$-pretempered local system $\mathscr{F}$ on T is null (in the sense of Definition 5.7.1) if and only if the restriction $\mathscr{F}_0 = \mathscr{F}|_{\mathrm{Cov}(T)^{\mathrm{op}}}$ is null (in the sense of Notation 5.7.8).*

Proof. It follows immediately from the definition that if $\mathscr{F} \in \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(T)$ is null, then $\mathscr{F}_0 = \mathscr{F}|_{\mathrm{Cov}(T)^{\mathrm{op}}}$ is null. Conversely, suppose that $\mathscr{F}_0$ is null; we wish to show that $\mathscr{F}$ is null. In other words, we wish to show that for each morphism $\alpha : T'' \to T$ in $\mathscr{T}$, the $A^{T''}_{\mathbf{G}}$-module $\mathscr{F}(T'')$ is $|\operatorname{Spec}(A^{T''}_{\mathbf{G}})|^{\deg}$-nilpotent. Note that the map α factors as a composition $T'' \xrightarrow{\beta} T' \xrightarrow{\gamma} T$, where γ exhibits T' as a connected covering space of T and β has connected homotopy fibers. Since $\mathscr{F}_0$ is null, the spectrum $\mathscr{F}(T')$ is $|\operatorname{Spec}(A^{T'}_{\mathbf{G}})|^{\deg}$-nilpotent, and our assumption that $\mathscr{F}$ is $\mathbf{G}$-pretempered supplies an equivalence $\mathscr{F}(T'') \simeq A^{T''}_{\mathbf{G}} \otimes_{A^{T'}_{\mathbf{G}}} \mathscr{F}(T')$. It follows that $\mathscr{F}(T'')$ is also $|\operatorname{Spec}(A^{T'}_{\mathbf{G}})|^{\deg}$-nilpotent when viewed as a module over $A^{T'}_{\mathbf{G}}$. It now suffices to observe that the map of Zariski spectra $|\operatorname{Spec}(A^{T''}_{\mathbf{G}})| \to |\operatorname{Spec}(A^{T'}_{\mathbf{G}})|$ carries the closed subset $|\operatorname{Spec}(A^{T''}_{\mathbf{G}})|^{\deg}$ into $|\operatorname{Spec}(A^{T'}_{\mathbf{G}})|^{\deg}$ (see Remark 5.2.3). ☐

Using Proposition 5.1.12, Proposition 5.4.2, and Lemmas 5.7.9, we see that Proposition 5.7.7 reduces to the following result (which no longer requires the assumption that $\mathbf{G}$ is oriented):

Proposition 5.7.10. *Let* $\mathbf{G}$ *be a preoriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* T *be an object of* $\mathscr{T}$*. Then the pair of subcategories*

$$(\mathrm{Mod}^{\mathrm{nul}}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp})), \mathrm{Mod}^{\mathrm{tem}}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp})))$$

is is a semi-orthogonal decomposition of the stable ∞*-category*

$$\mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp})).$$

The proof of Proposition 5.7.10 will require some preliminaries.

Notation 5.7.11. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let T be an object of $\mathscr{T}$, and let M be an $A^T_{\mathbf{G}}$-module spectrum. We let $M^!_T \in \mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$ denote the functor given informally by the formula

$$M^!_T(T_0) = \begin{cases} M & \text{if } T_0 \simeq T \\ 0 & \text{otherwise.} \end{cases}$$

More precisely, if $q : \mathrm{Mod}(\mathrm{Sp}) \to \mathrm{CAlg}$ is the forgetful functor, then we view $M^!_T$ as a functor from $\mathrm{Cov}(T)^{\mathrm{op}}$ to $\mathrm{Mod}(\mathrm{Sp})$ fitting in to a commutative diagram

$$\begin{tikzcd} \mathrm{Cov}(T)^{\mathrm{op}} \arrow[r, "M^!_T"] \arrow[d] & \mathrm{Mod}(\mathrm{Sp}) \arrow[d, "q"] \\ \mathscr{T}^{\mathrm{op}} \arrow[r, "A_{\mathbf{G}}"] & \mathrm{CAlg}, \end{tikzcd}$$

such that $M^!_T(T) = M$ and $M^!_T$ is a q-right Kan extension of its restriction to $\{T\} \subseteq \mathrm{Cov}(T)^{\mathrm{op}}$.

More generally, if $\overline{T} \in \mathrm{Cov}(T)$ is a connected covering space of T and M is a module over the tempered function spectrum $A^{\overline{T}}_{\mathbf{G}}$, we let $M^!_{\overline{T}/T}$ denote the object of $\mathrm{Mod}_{A_{\mathbf{G},T'}}(\mathrm{Fun}(\mathrm{Cov}(T')^{\mathrm{op}}, \mathrm{Sp}))$ given by the q-left Kan extension of $M^!_{\overline{T}}$ along the forgetful functor $\mathrm{Cov}(\overline{T})^{\mathrm{op}} \to \mathrm{Cov}(T)^{\mathrm{op}}$. More explicitly, if T_0 is a connected covering space of T, then the spectrum $M^!_{\overline{T}/T}(T_0)$ vanishes unless T_0 is isomorphic to $\overline{T}$, in which case it is equivalent to a direct sum of copies of M (indexed by the set of all isomorphisms of T_0 with $\overline{T}$ in the category $\mathrm{Cov}(T)$).

Remark 5.7.12. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let T be an object of $\mathscr{T}$, and let M be an $A^T_{\mathbf{G}}$-module spectrum. Then, for any object $\mathscr{F} \in \mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{A_{\mathbf{G},T}}}(M^!_T, \mathscr{F}) \simeq \mathrm{Map}_{\mathrm{Mod}_{A^T_{\mathbf{G}}}}(M, \mathrm{fib}(\mathscr{F}(T) \to \varprojlim_{T_0 \in \mathrm{Cov}^\circ(T)^{\mathrm{op}}} \mathscr{F}(T_0))).$$

More generally, if $\overline{T}$ is a connected covering space of T and M is an $A^{\overline{T}}_{\mathbf{G}}$-module spectrum, then we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{A_{\mathbf{G},T}}}(M^!_{\overline{T}/T}, \mathscr{F}) \simeq \mathrm{Map}_{\mathrm{Mod}_{A^T_{\mathbf{G}}}}(M, \mathrm{fib}(\mathscr{F}(\overline{T}) \to \varprojlim_{\overline{T}_0 \in \mathrm{Cov}^\circ(\overline{T})^{\mathrm{op}}} \mathscr{F}(\overline{T}_0))).$$

Remark 5.7.13. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let T be an object of $\mathscr{T}$, and let $\mathscr{F}$ be an $A_{\mathbf{G},T}$-module object of $\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp})$. Then $\mathscr{F}$ is tempered (in the sense of Notation 5.4.1) if and only if it satisfies the following condition:

- $(*)$ For every connected covering space $\overline{T}$ of T and every $A^{\overline{T}}_{\mathbf{G}}$-module M which is $|\mathrm{Spec}(A^{\overline{T}}_{\mathbf{G}})|^{\mathrm{deg}}$-nilpotent, the mapping space $\mathrm{Map}_{\mathrm{Mod}_{A_{\mathbf{G},T}}}(M^!_{\overline{T}/T}, \mathscr{F})$ is contractible.

This follows by combining the calculation of Remark 5.7.12 with the criterion of Theorem 5.6.9.

Proof of Proposition 5.7.10. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let T be an object of $\mathscr{T}$. Let $\mathcal{C} \subseteq \mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$ be the smallest stable subcategory which is closed under small colimits and contains every object of the form $M^!_{\overline{T}/T}$, where $\overline{T}$ is a connected covering space of T and M is a $A^{\overline{T}}_{\mathbf{G}}$-module spectrum which is $|\mathrm{Spec}(A^{\overline{T}}_{\mathbf{G}})|^{\mathrm{deg}}$-nilpotent. It follows from Proposition HA.1.4.4.11

that the stable ∞-category $\mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$ admits a semi-orthogonal decomposition $(\mathcal{C}, \mathcal{C}^{\perp})$, where $\mathcal{C}^{\perp}$ denotes the full subcategory spanned by those objects $\mathscr{F}$ for which the mapping space $\mathrm{Map}_{\mathrm{Mod}_{A_{\mathbf{G},T}}}(\mathscr{G}, \mathscr{F})$ is contractible for each $\mathscr{G} \in \mathcal{C}$. Remark 5.7.13 shows that $\mathcal{C}^{\perp} = \mathrm{Mod}^{\mathrm{tem}}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$. We will complete the proof by showing that $\mathcal{C} = \mathrm{Mod}^{\mathrm{nul}}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$. The inclusion

$$\mathcal{C} \subseteq \mathrm{Mod}^{\mathrm{nul}}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$$

is clear, since $\mathrm{Mod}^{\mathrm{nul}}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$ is a stable subcategory of the ∞-category $\mathrm{Mod}_{A_{\mathbf{G},T}}(\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp}))$ which is closed under small colimits and contains $M^{!}_{\overline{T}/T}$ whenever $M \in \mathrm{Mod}_{A^{\overline{T}}_{\mathbf{G}}}$ is $|\operatorname{Spec}(A^{\overline{T}}_{\mathbf{G}})|^{\mathrm{deg}}$-nilpotent. Conversely, suppose that $\mathscr{F}$ is an $A_{\mathbf{G},T}$-module object of $\mathrm{Fun}(\mathrm{Cov}(T)^{\mathrm{op}}, \mathrm{Sp})$ which is null; we wish to show that $\mathscr{F}$ belongs to $\mathcal{C}$. Note that $\mathscr{F}$ fits into a fiber sequence $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$, where $\mathscr{F}'$ belongs to $\mathcal{C}$ and $\mathscr{F}''$ is tempered. We will complete the proof by showing that $\mathscr{F}'' \simeq 0$. Suppose otherwise: then there exists some connected covering space $\overline{T}$ of T such that $\mathscr{F}''(\overline{T})$ is not zero. Choose $\overline{T}$ so that the fundamental group $\pi_1(\overline{T})$ is as small as possible. It then follows that $\mathscr{F}''(\overline{T}_0) \simeq 0$ for every connected covering space $\overline{T}_0$ of $\overline{T}$ which is not isomorphic to $\overline{T}$. Consequently, the limit $\varprojlim_{\overline{T}_0 \in \mathrm{Cov}^{\circ}(\overline{T})^{\mathrm{op}}} \mathscr{F}''(\overline{T}_0)$ vanishes. Since $\mathscr{F}''$ is tempered, Theorem 5.6.9 implies that $\mathscr{F}''(\overline{T})$ is $|\operatorname{Spec}(A^{\overline{T}}_{\mathbf{G}})|^{\mathrm{deg}}$-local. On the other hand, $\mathscr{F}''$ is null (since both $\mathscr{F}'$ and $\mathscr{F}$ are null), so that $\mathscr{F}''(\overline{T})$ is also $|\operatorname{Spec}(A^{\overline{T}}_{\mathbf{G}})|^{\mathrm{deg}}$-nilpotent. It follows that $\mathscr{F}''(\overline{T})$ vanishes, contradicting our choice of $\overline{T}$. $\square$

5.8 Tensor Products of Tempered Local Systems

We now exploit Theorem 5.7.3 to construct a tensor product operation in the setting of tempered local systems.

Notation 5.8.1. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_{\infty}$-ring A. For each orbispace X, we let $\underline{A}_{\mathsf{X}}$ denote the composite functor

$$\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}} \to \mathscr{T}^{\mathrm{op}} \xrightarrow{A_{\mathbf{G}}} \mathrm{CAlg},$$

which we view as a commutative algebra object of $\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}, \mathrm{Sp})$. Then the ∞-category $\mathrm{Mod}_{\underline{A}_{\mathsf{X}}} = \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}(\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}, \mathrm{Sp}))$ inherits a symmetric monoidal structure, given by the formation of relative tensor product over $\underline{A}_{\mathsf{X}}$ (see §HA.4.5.2). We will denote this relative tensor product operation by

$$\overline{\otimes} : \mathrm{Mod}_{\underline{A}_{\mathsf{X}}} \times \mathrm{Mod}_{\underline{A}_{\mathsf{X}}} \to \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}.$$

Concretely, it is given by the formula $(\mathscr{F} \mathbin{\overline{\otimes}} \mathscr{G})(T) = \mathscr{F}(T) \otimes_{A^T_{\mathbf{G}}} \mathscr{G}(T)$.

From the levelwise description of the tensor product $\overline{\otimes}$, we immediately deduce the following:

Proposition 5.8.2. *Let A be an $\mathbb{E}_\infty$-ring, and let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A. For every orbispace X, the full subcategory $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \subseteq \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}$ contains the unit object $\underline{A}_{\mathsf{X}}$ and is closed under the tensor product functor $\overline{\otimes}$ of Notation 5.8.1. In particular, $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ inherits the structure of a symmetric monoidal ∞-category.*

Remark 5.8.3 (Functoriality)**.** In the situation of Proposition 5.8.2, let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of orbispaces. Then the pullback functor $f^* : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ of Remark 5.1.6 is given by precomposition with the forgetful functor $\mathscr{T}_{/\mathsf{X}} \to \mathscr{T}_{/\mathsf{Y}}$, and can therefore be promoted to a symmetric monoidal functor: that is, it commutes with the tensor product operation $\overline{\otimes}$ of Notation 5.8.1.

Proposition 5.8.4. *Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, and let X be an orbispace. Then the full subcategory $\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}}(\mathsf{X}) \subseteq \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ is a tensor ideal. That is, if $\mathscr{F}$ belongs to $\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}}(\mathsf{X})$ and $\mathscr{G}$ belongs to $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$, then $\mathscr{F} \mathbin{\overline{\otimes}} \mathscr{G}$ belongs to $\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}}(\mathsf{X})$.*

Proof. For each object $T \in \mathscr{T}_{/\mathsf{X}}$, we have $(\mathscr{F} \mathbin{\overline{\otimes}} \mathscr{G})(T) = \mathscr{F}(T) \otimes_{A^T_{\mathbf{G}}} \mathscr{G}(T)$. Since $\mathscr{F}$ is null, $\mathscr{F}(T)$ is $|\operatorname{Spec}(A^T_{\mathbf{G}})|^{\deg}$-nilpotent when viewed as an $A^T_{\mathbf{G}}$-module. It follows that the tensor product $\mathscr{F}(T) \otimes_{A^T_{\mathbf{G}}} \mathscr{G}(T)$ is also $|\operatorname{Spec}(A^T_{\mathbf{G}})|^{\deg}$-nilpotent. $\square$

Corollary 5.8.5. *Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A, let X be an orbispace, and let $L : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ be a left adjoint to the inclusion (Proposition 5.2.12). Then the localization functor L is compatible with the symmetric monoidal structure of Proposition 5.8.2. That is, if $\alpha : \mathscr{F} \to \mathscr{F}'$ is a morphism in $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ for which $L(\alpha) : L(\mathscr{F}) \to L(\mathscr{F}')$ is an equivalence, and $\mathscr{G}$ is any object of $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$, then the induced map $L(\mathscr{F} \mathbin{\overline{\otimes}} \mathscr{G}) \to L(\mathscr{F}' \mathbin{\overline{\otimes}} \mathscr{G})$ is an equivalence.*

Proof. Combine Proposition 5.8.4 with Theorem 5.7.3. $\square$

Corollary 5.8.6. *Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A, and let X be an orbispace. Then there is an essentially unique symmetric monoidal structure on the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ for which the localization functor $L : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is symmetric monoidal.*

Proof. Combine Corollary 5.8.5 with Proposition HA.2.2.1.9. □

Construction 5.8.7 (The Tempered Tensor Product)**.** Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. For any orbispace X, we will regard $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ as equipped with the symmetric monoidal structure of Corollary 5.8.6. We will denote the underlying tensor product of this symmetric monoidal structure by

$$\otimes : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \times \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}} \qquad (\mathscr{F}, \mathscr{G}) \mapsto \mathscr{F} \otimes \mathscr{G}\,.$$

Concretely, it is given by the formula $\mathscr{F} \otimes \mathscr{G} = L(\mathscr{F} \overline{\otimes} \mathscr{G})$, where $L : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is left adjoint to the inclusion functor.

Remark 5.8.8. In the situation of Construction 5.8.7, the unit object of the symmetric monoidal ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is the trivial $\mathbf{G}$-tempered local system $\underline{A}_{\mathsf{X}}$ of Example 5.1.5 (which is $\mathbf{G}$-tempered by virtue of Corollary 5.4.4).

Remark 5.8.9. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and let $\mathscr{F}$ and $\mathscr{G}$ be $\mathbf{G}$-tempered local systems on an orbispace X. It follows from Theorem 5.7.3 that the tensor product $\mathscr{F} \otimes \mathscr{G}$ of Construction 5.8.7 can be characterized as follows:

- There exist $A^T_{\mathbf{G}}$-linear maps $\mu_T : \mathscr{F}(T) \otimes_{A^T_{\mathbf{G}}} \mathscr{G}(T) \to (\mathscr{F} \otimes \mathscr{G})(T)$, depending functorially on $T \in \mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}$.
- For each $T \in \mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}$, the fiber $\mathrm{fib}(\mu_T)$ is $|\,\mathrm{Spec}(A^T_{\mathbf{G}})|^{\mathrm{deg}}$-nilpotent (when regarded as an $A^T_{\mathbf{G}}$-module).

Remark 5.8.10. In the situation of Remark 5.8.9, the map μ_T Is an equivalence whenever T is contractible (Remark 5.7.6). It follows that the forgetful functor $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_A(|\mathsf{X}|)$ (see Variant 5.1.15) is symmetric monoidal: that is, it carries the tensor products of tempered local systems (given by Construction 5.8.7) to the pointwise tensor product of Mod_A-valued local systems on $|\mathsf{X}|$.

Warning 5.8.11. In the situation of Remark 5.8.9, the map $\mu_T : \mathscr{F}(T) \otimes_{A^T_{\mathbf{G}}} \mathscr{G}(T) \to (\mathscr{F} \otimes \mathscr{G})(T)$ is generally not an equivalence when T is not contractible. That is, the tensor product of tempered local systems cannot be computed levelwise.

Example 5.8.12. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let X be a space. Applying Remark 5.7.6 and Corollary 5.4.3 to the constant orbispace $\underline{X}$, we obtain an equivalence of symmetric monoidal ∞-categories $\mathrm{LocSys}_{\mathbf{G}}(\underline{X}) \simeq \mathrm{LocSys}_A(X)$. In particular, when $X \simeq *$ is contractible, we obtain an equivalence of symmetric monoidal ∞-categories $\mathrm{LocSys}_{\mathbf{G}}(*) \simeq \mathrm{Mod}_A$.

Note that if we are given a map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$, then the symmetric monoidal pullback functor $f^* : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ automatically restricts to a *lax* symmetric monoidal functor $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, which we will also denote by f^*.

Proposition 5.8.13. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. For every map of orbispaces* $f : \mathsf{X} \to \mathsf{Y}$*, the pullback functor* $f^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ *is symmetric monoidal.*

Proof. It follows from Remark 5.8.8 that the pullback functor f^* preserves unit objects. We will complete the proof by showing that for every pair of objects $\mathscr{F}, \mathscr{F}' \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$, the canonical map $\alpha : f^* \mathscr{F} \otimes f^* \mathscr{F}' \to f^*(\mathscr{F} \otimes \mathscr{F}')$ is an equivalence. We have a commutative diagram

$$\begin{tikzcd}
f^* \mathscr{F} \overline{\otimes} f^* \mathscr{F}' \arrow[r] \arrow[d] & f^*(\mathscr{F} \overline{\otimes} \mathscr{F}') \arrow[d] \\
f^* \mathscr{F} \otimes f^* \mathscr{F}' \arrow[r, "\alpha"] & f^*(\mathscr{F} \otimes \mathscr{F}')
\end{tikzcd}$$

in the ∞-category $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$. Note that the upper horizontal map is an equivalence, and that the fibers of the vertical maps belong to $\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}}(\mathsf{X})$. It follows that $\mathrm{fib}(\alpha) \in \mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}}(\mathsf{X}) \cap \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, so that $\mathrm{fib}(\alpha) \simeq 0$ and α is an equivalence. ☐

Proposition 5.8.14. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*, let* $f : \mathsf{X} \to \mathsf{Y}$ *be a map of orbispaces, and let* $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ *and* $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ *be* $\mathbf{G}$*-tempered local systems on* X *and* Y*, respectively. If* $\mathsf{Y} \simeq *$ *is a final object of* $\mathcal{OS}$*, then the canonical map*

$$\theta : f^* \mathscr{G} \overline{\otimes} \mathscr{F} \to f^* \mathscr{G} \otimes \mathscr{F}$$

is an equivalence in $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$*. In other words, the map* $\mu_T : \mathscr{G}(T) \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T) \to (f^* \mathscr{G} \otimes \mathscr{F})(T)$ *of Remark 5.8.9 is an equivalence for each object* $T \in \mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}$*.*

Proof. Let us regard $\mathscr{F}$ as fixed, and let $\mathcal{C}$ denote the full subcategory of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ spanned by those objects $\mathscr{G}$ for which the morphism θ is an equivalence. Then $\mathcal{C}$ is a stable subcategory of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$, and it follows from Theorem 5.3.1 that $\mathcal{C}$ is closed under small colimits. Since $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ is equivalent to Mod_A as a symmetric monoidal ∞-category (Example 5.8.12), to prove that $\mathcal{C} = \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ it will suffice to show that $\mathcal{C}$ contains the unit object of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$, which is immediate. ☐

Remark 5.8.15. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces. It follows from Proposition 5.8.13 that the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is tensored over $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$, with action given concretely by the construction

$$\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \times \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \qquad (\mathscr{G}, \mathscr{F}) \mapsto (f^* \mathscr{G}) \otimes \mathscr{F} .$$

Proposition 5.8.14 makes this action explicit in the special case where $\mathsf{Y} = *$ is a final object of $\mathcal{OS}$. In this case, we can identify $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ with the ∞-category Mod_A (Example 5.8.12). Then $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ inherits an action of Mod_A, which we will denote by

$$\mathrm{Mod}_A \times \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \qquad (M, \mathscr{F}) \mapsto M \otimes_A \mathscr{F} .$$

Proposition 5.8.14 asserts that this action is computed levelwise: that is, it is given on objects by the formula $(M \otimes_A \mathscr{F})(T) = M \otimes_A \mathscr{F}(T)$.

6 Analysis of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$

Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. In §5, we introduced the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ of $\mathbf{G}$*-tempered local systems* on an orbispace X (Definition 5.2.4). Our goal in this section is to develop an arsenal of tools for working with $\mathbf{G}$-tempered local systems, which can often be used to translate questions about $\mathbf{G}$-tempered local systems on orbispaces to questions about ordinary local systems on spaces.

To simplify the discussion, let us assume for the moment that the $\mathbb{E}_\infty$-ring A is p-local for some prime number p and that $\mathbf{G}$ is a p-divisible group of some fixed height $h \geqslant 0$. We will say that a $\mathbf{G}$-tempered local system $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is $K(n)$*-local* if, for each object $T \in \mathscr{T}_{/\mathsf{X}}$, the spectrum $\mathscr{F}(T)$ is $K(n)$-local; here $K(n)$ denotes the nth Morava K-theory (at the prime p). The collection of $K(n)$-local $\mathbf{G}$-tempered local systems span a full subcategory $\mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X}) \subseteq \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$. In §6.1, we show that the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ admits a semi-orthogonal decomposition by the subcategories $\{\mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X})\}_{0 \leqslant n \leqslant h}$ (Corollary 6.1.17). Consequently, the problem of understanding the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ can be partially reduced to the problem of understanding the subcategories $\mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X})$.

In §6.3, we study the ∞-category $\mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X})$ in the special case where $n = h$ is the height of the p-divisible group $\mathbf{G}$. In this case, we show that the forgetful functor $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_A(|\mathsf{X}|)$ of Variant 5.1.15 restricts to an equivalence of full subcategories $\mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X}) \simeq \mathrm{LocSys}_A^{K(n)}(|\mathsf{X}|)$, where $\mathrm{LocSys}_A^{K(n)}(|\mathsf{X}|) \simeq \mathrm{Fun}(|\mathsf{X}|, \mathrm{Mod}_A^{K(n)})$

denotes the ∞-category of local systems on $|\mathsf{X}|$ with values in the ∞-category $\mathrm{Mod}_A^{K(n)}$ of $K(n)$-local A-modules (Theorem 6.3.1). This can be regarded as a categorification of Theorem 4.2.5, which asserts that the Atiyah-Segal comparison map $\zeta : A^*_{\mathbf{G}}(\mathsf{X}) \to A^*(|\mathsf{X}|)$ is an isomorphism in the case where A is $K(n)$-local and $\mathbf{G} = \mathbf{G}_A^{\mathcal{Q}}$ is the Quillen p-divisible group of A.

To understand the ∞-categories $\mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X})$ for $n < h$, it will be convenient to enlarge the $\mathbb{E}_\infty$-ring A. To every $\mathbb{E}_\infty$-algebra B over A, we can associate a p-divisible group $\mathbf{G}_B$ over B, obtained from $\mathbf{G}$ by extensions of scalars. In §6.2, we study the relationship between the ∞-categories $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ and $\mathrm{LocSys}_{\mathbf{G}_B}(\mathsf{X})$. In the special case where $B = L_{K(n)}(A)$ is the $K(n)$-localization of A, we show that the subcategories $\mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X})$ and $\mathrm{LocSys}_{\mathbf{G}_B}^{K(n)}(\mathsf{X})$ are equivalent (Corollary 6.2.8). We may therefore assume without loss of generality that the $\mathbb{E}_\infty$-ring A is $K(n)$-local. In this case, the orientation of $\mathbf{G}$ determines a short exact sequence

$$0 \to \mathbf{G}_0 \xrightarrow{\iota} \mathbf{G} \to \mathbf{G}_{\text{ét}} \to 0,$$

where $\mathbf{G}_0 = \mathbf{G}_A^{\mathcal{Q}}$ is the Quillen p-divisible group of A (see Corollary 2.5.7) and $\mathbf{G}_{\text{ét}}$ is an étale p-divisible group of height $h - n$. Set $\Lambda = (\mathbf{Q}_p / \mathbf{Z}_p)^{h-n}$ and let $B = \mathrm{Split}_\Lambda(\iota)$ be a splitting algebra of ι (Definition 2.7.12). Then B is a faithfully flat A-algebra (Proposition 2.7.15), so that A can be identified with the totalization fo the cosimplicial A-algebra

$$B^\bullet = (B \rightrightarrows B \otimes_A B \rightrightarrows\!\!\!\!\rightarrow \cdots).$$

According to Proposition 6.2.6, the theory of $\mathbf{G}$-tempered local systems satisfies faithfully flat descent: that is, we can identify $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ with the totalization of the cosimplicial ∞-category $\mathrm{LocSys}_{\mathbf{G}_{B^\bullet}}(\mathsf{X})$. Consequently, various questions about the structure of the ∞-category $\mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X})$ can be addressed after extending scalars along the maps $A \to B^m$, so that the p-divisible group $\mathbf{G}$ splits as a direct sum $\mathbf{G}_0 \oplus \underline{\Lambda}$. Beware that the A-algebras B^m are essentially never $K(n)$-local (so that, after extending scalars, we cannot identify $\mathbf{G}_0$ with the Quillen p-divisible group of B^m), but (if desired) this can rectified by replacing each B^m by its $K(n)$-localization.

Let $\mathcal{L}^\Lambda(\mathsf{X})$ denote the formal loop space of X given by Construction 3.4.3. In §6.4, we construct a fully faithful embedding of ∞-categories

$$\Phi : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \hookrightarrow \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^\Lambda(\mathsf{X})),$$

which can be regarded as a categorification of the character isomorphism $A^*_{\mathbf{G}}(\mathsf{X}) \simeq A^*_{\mathbf{G}_0}(\mathcal{L}^\Lambda(\mathsf{X}))$ of Theorem 4.3.2 (see Theorem 6.4.1). In §6.5 we identify the essential image of Φ with the full subcategory $\mathrm{LocSys}^{\mathrm{iso}}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$ of *isotropic* local systems on $\mathcal{L}^{\Lambda}(\mathsf{X})$ (Definition 6.5.8). For $m > 0$, every $K(m)$-local object of $\mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$ is automatically isotropic (Corollary 6.5.16), and the embedding above restricts to an equivalence of ∞-categories

$$\mathrm{LocSys}^{K(m)}_{\mathbf{G}}(\mathsf{X}) \simeq \mathrm{LocSys}^{K(m)}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$$

(beware that the situation is a bit more complicated if $m = 0$, or if $\mathbf{G}$ is a $\mathbf{P}$-divisible group with nonvanishing components $\mathbf{G}_{(\ell)}$ for $\ell \neq p$). Consequently, for the purpose of understanding the ∞-category $\mathrm{LocSys}^{K(n)}_{\mathbf{G}}(\mathsf{X})$ when $n > 0$, we can replace $\mathbf{G}$ by the p-divisible group $\mathbf{G}_0$ (at the cost of replacing X by the more complicated orbispace $\mathcal{L}^{\Lambda}(\mathsf{X})$), thereby reducing to the situation studied in §6.3.

6.1 Localization and Completions of Tempered Local Systems

Let A be an $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0(A)$ be a finitely generated ideal. We let $\mathrm{Mod}^{\mathrm{Nil}(I)}_A$, $\mathrm{Mod}^{\mathrm{Loc}(I)}_A$, and $\mathrm{Mod}^{\mathrm{Cpl}(I)}_A$ denote the full subcategories of Mod_A spanned by those A-modules which are I-nilpotent, I-local, and I-complete, respectively (see Chapter SAG.II.4). The ∞-category Mod_A then admits a pair of semi-orthogonal decompositions $(\mathrm{Mod}^{\mathrm{Nil}(I)}_A, \mathrm{Mod}^{\mathrm{Loc}(I)}_A)$ and $(\mathrm{Mod}^{\mathrm{Loc}(I)}_A, \mathrm{Mod}^{\mathrm{Cpl}(I)}_A)$. In particular, for every A-module M, there are essentially unique fiber sequences

$$M' \to M \to M^{\wedge}_I \qquad \Gamma_I(M) \to M \to M''$$

where $M^{\wedge}_I$ is I-complete, $\Gamma_I(M)$ is I-nilpotent, and M' and M'' are I-local. Our goal in this section is to establish a generalization of this picture, where we replace Mod_A with the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ of $\mathbf{G}$-tempered local systems on an orbispace X. In this situation, we can make sense of the sequence on the left for *any* preoriented $\mathbf{P}$-divisible group $\mathbf{G}$ over A (Corollary 6.1.6), and the sequence on the right under the assumption that $\mathbf{G}$ is oriented (Corollary 6.1.10).

Definition 6.1.1. Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, let X be an orbispace, and let $I \subseteq \pi_0(A)$ be a finitely generated ideal. We will say that a $\mathbf{G}$-tempered local system $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is *I-nilpotent* (*I-local*, *I-complete*) if, for every object $T \in \mathscr{T}_{/\mathsf{X}}$, the spectrum $\mathscr{F}(T)$ is I-nilpotent (I-local, I-complete) when viewed as an A-module. We let $\mathrm{LocSys}^{\mathrm{Nil}(I)}_{\mathbf{G}}(\mathsf{X})$ ($\mathrm{LocSys}^{\mathrm{Loc}(I)}_{\mathbf{G}}(\mathsf{X})$, $\mathrm{LocSys}^{\mathrm{Cpl}(I)}_{\mathbf{G}}(\mathsf{X})$) denote the full subcategory of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ spanned by those objects are which are I-nilpotent (I-local, I-complete).

Warning 6.1.2. Let A be an $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0(A)$ be a finitely generated ideal, and let $\mathcal{C}$ be a presentable A-linear stable ∞-category. We say that an object $C \in \mathcal{C}$ is *I-nilpotent* if, for each element $x \in I$, the colimit of the diagram

$$C \xrightarrow{x} C \xrightarrow{x} C \xrightarrow{x} C \xrightarrow{x} \cdots$$

vanishes (Definition SAG.II.4.1.1.6), and that C is *I-complete* if, for each element $x \in I$, the limit of the diagram

$$\cdots \xrightarrow{x} C \xrightarrow{x} C \xrightarrow{x} C \xrightarrow{x} C$$

vanishes (see Corollary SAG.II.4.3.3.3). In the situation of Definition 6.1.1, a $\mathbf{G}$-tempered local system $\mathscr{F}$ is I-complete in the sense of Definition 6.1.1 if and only if it is I-complete when viewed as an object of the A-linear ∞-category $\mathcal{C} = \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$: this follows from the fact that the evaluation functors $\mathscr{F} \mapsto \mathscr{F}(T)$ are jointly conservative and preserve small limits (Corollary 5.2.13). If $\mathbf{G}$ is oriented, then $\mathscr{F}$ is I-nilpotent in the sense of Definition 6.1.1 if and only if it is I-nilpotent when viewed as an object of $\mathcal{C} = \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ (since the evaluation functors $\mathscr{F} \mapsto \mathscr{F}(T)$ also preserve small colimits when $\mathbf{G}$ is oriented; see Corollary 5.3.2). Beware that this is generally not true if $\mathbf{G}$ is only assumed to be preoriented.

Remark 6.1.3 (Functoriality)**.** Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let $I \subseteq \pi_0(A)$ be a finitely generated ideal, and let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces. If $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ is I-nilpotent (I-local, I-complete), then the pullback $f^*(\mathscr{F}) \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is also I-nilpotent (I-local, I-complete).

Proposition 6.1.4. *Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, and let X be an orbispace. Then the inclusion functor*

$$\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Cpl}(I)}(\mathsf{X}) \hookrightarrow \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$$

admits a left adjoint. Moreover, if $\alpha : \mathscr{F} \to \mathscr{F}'$ is a morphism in $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, then the following conditions are equivalent:

(1) *The morphism α exhibits $\mathscr{F}'$ as a $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Cpl}(I)}(\mathsf{X})$-localization of $\mathscr{F}$.*

(2) *For every object $T \in \mathscr{T}_{/\mathsf{X}}$, the induced map $\alpha(T) : \mathscr{F}(T) \to \mathscr{F}'(T)$ exhibits $\mathscr{F}'(T)$ as an I-completion of $\mathscr{F}(T)$ in the ∞-category Mod_A.*

Proof. Fix an element $x \in I$. For each object $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, let $\Theta_x(\mathscr{F})$ denote the limit of the tower $\cdots \xrightarrow{x} \mathscr{F} \xrightarrow{x} \mathscr{F} \xrightarrow{x} \mathscr{F}$, and let $\mathscr{F}^{\wedge}_{(x)}$ denote the cofiber of the canonical map $\Theta_x(\mathscr{F}) \to \mathscr{F}$. By construction, x acts by an equivalence on $\Theta_x(\mathscr{F})$. It follows that multiplication by x acts by a homotopy equivalence from $\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\Theta_x(\mathscr{F}), \mathscr{G})$ to itself, for every object $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$. In particular, the canonical map

$$\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\Theta_x(\mathscr{F}), \Theta_x(\mathscr{G})) \to \mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\Theta_x(\mathscr{F}), \mathscr{G})$$

is a homotopy equivalence. If $\mathscr{G}$ is I-complete, then it is annihilated by the functor Θ_x, so that the mapping space $\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\Theta_x(\mathscr{F}), \mathscr{G})$ is contractible. It follows that the canonical map

$$\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\mathscr{F}^{\wedge}_{(x)}, \mathscr{G}) \to \mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\mathscr{F}, \mathscr{G})$$

is a homotopy equivalence.

Choose a finite collection of generators $x_1, \ldots, x_n \in \pi_0(A)$ for the ideal I, and let $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$. Let α denote the composite map

$$\mathscr{F} \to \mathscr{F}^{\wedge}_{(x_1)} \to (\mathscr{F}^{\wedge}_{(x_1)})^{\wedge}_{(x_2)} \to \cdots ((\mathscr{F}^{\wedge}_{(x_1)}) \cdots)^{\wedge}_{(x_n)} \mathscr{F} = \mathscr{F}'$$

Corollary 5.2.13 implies that for every object $T \in \mathscr{T}_{/\mathsf{X}}$, the canonical map $\mathscr{F}(T) \to \mathscr{F}'(T)$ exhibits $\mathscr{F}'(T)$ as an I-completion of $\mathscr{F}(T)$, so that $\mathscr{F}'$ is I-complete. It follows from the above analysis that α exhibits $\mathscr{F}'$ as a $\mathrm{LocSys}^{\mathrm{Cpl}(I)}_{\mathbf{G}}(\mathsf{X})$-localization of $\mathscr{F}$. This completes the proof that $\mathrm{LocSys}^{\mathrm{Cpl}(I)}_{\mathbf{G}}(\mathsf{X})$ is a localization of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, and proves that (1) ⇒ (2).

We now complete the proof by showing that (2) ⇒ (1). Let $\beta : \mathscr{F} \to \mathscr{G}$ be a morphism in $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ satisfying condition (2). Then $\mathscr{G}$ is I-complete, so that β factors as a composition $\mathscr{F} \xrightarrow{\alpha} \mathscr{F}' \xrightarrow{\gamma} \mathscr{G}$, where α is defined as above. Then α and β both satisfy condition (2). It follows that for every object $T \in \mathscr{T}_{/\mathsf{X}}$, the induced map $\mathscr{F}'(T) \to \mathscr{G}(T)$ is an equivalence. We conclude that γ is an equivalence, so that $\beta = \gamma \circ \alpha$ exhibits $\mathscr{G}$ as a $\mathrm{LocSys}^{\mathrm{Cpl}(I)}_{\mathbf{G}}(\mathsf{X})$-localization of $\mathscr{F}$. ☐

Notation 6.1.5 (Completion with Respect to an Ideal)**.** Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let X be an orbispace, and let $I \subseteq \pi_0(A)$ be a finitely generated ideal. For each $\mathbf{G}$-pretempered local system $\mathscr{F}$ on X, we let $\mathscr{F}^{\wedge}_{I}$ denote the image of $\mathscr{F}$ under the functor $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{Cpl}(I)}_{\mathbf{G}}(\mathsf{X})$ which is left adjoint to the inclusion. More informally, $\mathscr{F}^{\wedge}_{I}$ is the $\mathbf{G}$-tempered local system on X given by the formula $\mathscr{F}^{\wedge}_{I}(T) = \mathscr{F}(T)^{\wedge}_{I}$. We will refer to $\mathscr{F}^{\wedge}_{I}$ as the *I-completion of $\mathscr{F}$*.

In the situation of Notation 6.1.5, the completion $\mathscr{F}_I^\wedge \simeq 0$ vanishes if and only if the $\mathbf{G}$-tempered local system $\mathscr{F}$ is I-local, in the sense of Definition 6.1.1. We therefore obtain the following:

Corollary 6.1.6. *Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, and let X be an orbispace. For every finitely generated ideal $I \subseteq \pi_0(A)$, the pair of stable subcategories $(\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Loc}(I)}(\mathsf{X}), \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Cpl}(I)}(\mathsf{X}))$ determine a semi-orthogonal decomposition of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$. In particular, every $\mathbf{G}$-tempered local system $\mathscr{F}$ determines an (essentially unique) fiber sequence $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}_I^\wedge$, where $\mathscr{F}'$ is I-local and $\mathscr{F}_I^\wedge$ is I-complete.*

Proposition 6.1.7. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let $I \subseteq \pi_0(A)$ be a finitely generated ideal, and let X be an orbispace. Then the inclusion $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(I)}(\mathsf{X}) \hookrightarrow \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ admits a right adjoint. Moreover, if $\alpha : \mathscr{F}' \to \mathscr{F}$ is a morphism in $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, then the following conditions are equivalent:*

(1) *The morphism α exhibits $\mathscr{F}'$ as a $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(I)}(\mathsf{X})$-colocalization of $\mathscr{F}$.*

(2) *For every object $T \in \mathscr{T}_{/\mathsf{X}}$, the morphism $\alpha(T)$ induces an equivalence of A-module spectra $\mathscr{F}'(T) \simeq \Gamma_I \mathscr{F}(T)$.*

Proof. We proceed as in the proof of Proposition 6.1.4. Fix an element $x \in I$. For each object $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, let $\mathscr{F}[x^{-1}]$ denote the colimit of the sequence

$$\mathscr{F} \xrightarrow{x} \mathscr{F} \xrightarrow{x} \mathscr{F} \xrightarrow{x} \cdots,$$

and let $\Gamma_{(x)} \mathscr{F}$ denote the fiber of the canonical map $\mathscr{F} \to \mathscr{F}[x^{-1}]$. By construction, x acts by an equivalence on $\mathscr{F}[x^{-1}]$. It follows that multiplication by x acts by a homotopy equivalence from $\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\mathscr{G}, \mathscr{F}[x^{-1}])$ to itself, for every object $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$. In particular, the canonical map

$$\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\mathscr{G}[x^{-1}], \mathscr{F}[x^{-1}]) \to \mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\mathscr{G}, \mathscr{F}[x^{-1}])$$

is a homotopy equivalence. If $\mathscr{G}$ is I-nilpotent, then $\mathscr{G}[x^{-1}] \simeq 0$, so that the mapping space $\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\mathscr{G}, \mathscr{F}[x^{-1}])$ is contractible. It follows that the natural map

$$\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\mathscr{G}, \Gamma_{(x)} \mathscr{F}) \to \mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\mathscr{G}, \mathscr{F})$$

is a homotopy equivalence.

Choose a finite collection of generators $x_1, \ldots, x_n \in \pi_0(A)$ for the ideal I, and let $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$. Let α denote the composite map

$$\mathscr{F}' = \Gamma_{(x_n)}(\cdots(\Gamma_{(x_1)}\mathscr{F})) \to \cdots \to \Gamma_{(x_1)}\mathscr{F} \to \mathscr{F}.$$

Corollary 5.3.2 implies that for every object $T \in \mathscr{T}_{/\mathsf{X}}$, α induces an equivalence $\mathscr{F}'(T) \simeq \Gamma_I \mathscr{F}(T)$, so that $\mathscr{F}'$ is I-nilpotent. It follows from the above analysis that α exhibits $\mathscr{F}'$ as a $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(I)}(\mathsf{X})$-colocalization of $\mathscr{F}$. This completes the proof that $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(I)}(\mathsf{X})$ is a colocalization of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, and shows that $(1) \Rightarrow (2)$.

We now complete the proof by showing that $(2) \Rightarrow (1)$. Let $\beta : \mathscr{G} \to \mathscr{F}$ be a morphism in $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ satisfying condition (2). Then $\mathscr{G}$ is I-nilpotent, so that β factors as a composition

$$\mathscr{G} \xrightarrow{\gamma} \mathscr{F}' \xrightarrow{\alpha} \mathscr{F},$$

where α is defined as above. Then α and β both satisfy condition (2). It follows that for every object $T \in \mathscr{T}_{/\mathsf{X}}$, the induced map $\mathscr{G}(T) \to \mathscr{F}'(T)$ is an equivalence. We conclude that γ is an equivalence, so that $\beta = \alpha \circ \gamma$ exhibits $\mathscr{G}$ as a $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(I)}(\mathsf{X})$-colocalization of $\mathscr{F}$. □

Notation 6.1.8. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let X be an orbispace, and let $I \subseteq \pi_0(A)$ be a finitely generated ideal. We let

$$\Gamma_I : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(I)}(\mathsf{X})$$

denote a right adjoint to the inclusion functor, whose existence is asserted by Proposition 6.1.7. More informally, the functor Γ_I carries each $\mathbf{G}$-tempered local system $\mathscr{F}$ to an I-nilpotent $\mathbf{G}$-tempered local system $\Gamma_I \mathscr{F}$, given informally by the formula $(\Gamma_I \mathscr{F})(T) = \Gamma_I(\mathscr{F}(T))$.

Warning 6.1.9. In the situation of Notation 6.1.8, suppose that we assume only that $\mathbf{G}$ is a preoriented $\mathbf{P}$-divisible group over A. Then, to every $\mathbf{G}$-tempered local system $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, we can associate a $\mathbf{G}$-pretempered local system $\Gamma_I \mathscr{F}$ by the formula $(\Gamma_I \mathscr{F})(T) = \Gamma_I(\mathscr{F}(T))$. However, this formula need not define a $\mathbf{G}$-tempered local system unless $\mathbf{G}$ is oriented.

In the situation of Notation 6.1.8, the $\mathbf{G}$-tempered local system $\Gamma_I \mathscr{F}$ vanishes if and only if $\mathscr{F}$ is I-local. This proves the following:

Corollary 6.1.10. *Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A, and let X be an orbispace. For every finitely generated ideal $I \subseteq \pi_0(A)$,* *the pair of stable subcategories* $(\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(I)}(\mathsf{X}), \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Loc}(I)}(\mathsf{X}))$ *determine a semi-orthogonal decomposition of* $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$*. In particular, every* $\mathbf{G}$*-tempered local system* $\mathscr{F}$ *determines an (essentially unique) fiber sequence* $\Gamma_I \mathscr{F} \to \mathscr{F} \to \mathscr{F}''$*, where* $\Gamma_I \mathscr{F}$ *is* I*-nilpotent and* $\mathscr{F}''$ *is* I*-local.*

Combining Corollaries 6.1.6 and 6.1.10 with Proposition HA.A.8.20 , we obtain the following:

Corollary 6.1.11. *Let* A *be an* $\mathbb{E}_\infty$*-ring, let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over* A*, and let* X *be an orbispace. Then, for every finitely generated ideal* $I \subseteq \pi_0(A)$*, the* ∞*-category* $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ *is a recollement of the full subcategories*

$$\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Loc}(I)}(\mathsf{X}), \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Cpl}(I)}(\mathsf{X}) \subseteq \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}),$$

in the sense of Definition HA.A.8.1 .

Corollary 6.1.12. *Let* A *be an* $\mathbb{E}_\infty$*-ring, let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over* A*, and let* X *be an orbispace. Then, for every finitely generated ideal* $I \subseteq \pi_0(A)$*, the functor of* I*-completion determines an equivalence of* ∞*-categories*

$$\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(I)}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Cpl}(I)}(\mathsf{X}).$$

We now specialize to a particularly important case.

Definition 6.1.13. Let p be a prime number, let A be a p-local $\mathbb{E}_\infty$-ring, and let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A. Let $\mathscr{F}$ be a $\mathbf{G}$-tempered local system on an orbispace X. We will say that $\mathscr{F}$ is *$K(n)$-local* if, for each object $T \in \mathscr{T}_{/\mathsf{X}}$, the spectrum $\mathscr{F}(T)$ is $K(n)$-local (here $K(n)$ denotes the nth Morava K-theory spectrum at the prime p). We let $\mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X})$ denote the full subcategory of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ spanned by the $K(n)$-local $\mathbf{G}$-tempered local systems on X.

We say that $\mathscr{F}$ is *$E(n)$-local* if, for each object $T \in \mathscr{T}_{/\mathsf{X}}$, the spectrum $\mathscr{F}(T)$ is $E(n)$-local (where $E(n)$ denotes the nth Johnson-Wilson spectrum at the prime p). We let $\mathrm{LocSys}_{\mathbf{G}}^{E(n)}(\mathsf{X})$ denote the full subcategory of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ spanned by the $E(n)$-local $\mathbf{G}$-tempered local systems on X.

Remark 6.1.14. Let p be a prime number, let A be a p-complete $\mathbb{E}_\infty$-ring, and let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A. Then A is complex periodic. For each $m \geqslant 0$, we let $\mathfrak{I}_m^A \subseteq \pi_0(A)$ denote the mth Landweber ideal of A (Definition Or.4.5.1). Then:

- A $\mathbf{G}$-tempered local system $\mathscr{F}$ is $E(n)$-local if and only if it is $\mathfrak{J}^A_{n+1}$-local. That is, we have
$$\mathrm{LocSys}^{E(n)}_{\mathbf{G}}(\mathsf{X}) = \mathrm{LocSys}^{\mathrm{Loc}(\mathfrak{J}^A_{n+1})}_{\mathbf{G}}(\mathsf{X}).$$
- A $\mathbf{G}$-tempered local system is $K(n)$-local if and only if it is both $\mathfrak{J}^A_{n+1}$-local and $\mathfrak{J}^A_n$-complete. That is, we have
$$\mathrm{LocSys}^{K(n)}_{\mathbf{G}}(\mathsf{X}) = \mathrm{LocSys}^{\mathrm{Loc}(\mathfrak{J}^A_{n+1})}_{\mathbf{G}}(\mathsf{X}) \cap \mathrm{LocSys}^{\mathrm{Cpl}(\mathfrak{J}^A_n)}_{\mathbf{G}}(\mathsf{X}).$$

Notation 6.1.15. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over a p-local $\mathbb{E}_\infty$-ring A and let X be an orbispace. It follows from Corollaries 6.1.6 and 6.1.10 that the inclusion functors
$$\mathrm{LocSys}^{E(n)}_{\mathbf{G}}(\mathsf{X}) \subseteq \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \qquad \mathrm{LocSys}^{K(n)}_{\mathbf{G}}(\mathsf{X} \subseteq \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$$
admits left adjoints, which we will denote by $L_{E(n)} : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}^{E(n)}_{\mathbf{G}}(\mathsf{X})$ and $L_{K(n)} : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}^{K(n)}_{\mathbf{G}}(\mathsf{X})$, respectively. Concretely, these functors are given by the formulae
$$(L_{E(n)}\mathscr{F})(T) = L_{E(n)}(\mathscr{F}(T)) \qquad (L_{K(n)}\mathscr{F})(T) = L_{K(n)}(\mathscr{F}(T))$$
for $T \in \mathscr{T}_{/\mathsf{X}}$.

Proposition 6.1.16. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over a p-local $\mathbb{E}_\infty$-ring A, and let X be an orbispace. Then, for each $n \geqslant 1$, the stable ∞-category $\mathrm{LocSys}^{E(n)}_{\mathbf{G}}(\mathsf{X})$ is a recollement of the full subcategories*
$$\mathrm{LocSys}^{E(n-1)}_{\mathbf{G}}(\mathsf{X}), \mathrm{LocSys}^{K(n)}_{\mathbf{G}}(\mathsf{X}) \subseteq \mathrm{LocSys}^{E(n)}_{\mathbf{G}}(\mathsf{X}).$$

Proof. Let $I = \mathfrak{J}^A_n$ be the nth Landweber ideal of A. For $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, Corollaries 6.1.6 and 6.1.10 supply fiber sequences
$$\mathscr{F}' \to \mathscr{F} \to \mathscr{F}^\wedge_I \qquad \Gamma_I\mathscr{F} \to \mathscr{F} \to \mathscr{F}''$$
where $\mathscr{F}'$ and $\mathscr{F}''$ are I-local (that is, $E(n-1)$-local). If $\mathscr{F}$ is $E(n)$-local, then $\mathscr{F}^\wedge_I$ and $\Gamma_I\mathscr{F}$ are also $E(n)$-local (so that $\mathscr{F}^\wedge_I$ is $K(n)$-local). □

Corollary 6.1.17. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over a p-local $\mathbb{E}_\infty$-ring A, and suppose that the p-divisible group $\mathbf{G}_{(p)}$ has height $\leqslant h$ for some nonnegative* *integer h. Then, for any orbispace X, the stable ∞-category* $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ *admits a semi-orthogonal decomposition by full subcategories*

$$(\mathrm{LocSys}^{K(0)}_{\mathbf{G}}(\mathsf{X}), \mathrm{LocSys}^{K(1)}_{\mathbf{G}}(\mathsf{X}), \cdots, \mathrm{LocSys}^{K(h)}_{\mathbf{G}}(\mathsf{X})).$$

In particular, every $\mathbf{G}$-tempered local system $\mathscr{F}$ on X admits a canonical filtration

$$0 = \mathscr{F}(-1) \to \mathscr{F}(0) \to \mathscr{F}(1) \to \cdots \to \mathscr{F}(h) = \mathscr{F},$$

where each cofiber $\mathscr{F}(n)/\mathscr{F}(n-1)$ is $K(n)$-local.

6.2 Change of Ring

Let $\phi : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings. If $\mathbf{G}$ is a preoriented $\mathbf{P}$-divisible group over A, we let $\mathbf{G}_B$ denote the preoriented $\mathbf{P}$-divisible group over B obtained from $\mathbf{G}$ by extension of scalars along ϕ. For each orbispace X, we let $\underline{A}_{\mathsf{X}}, \underline{B}_{\mathsf{X}} : \mathscr{T}^{\mathrm{op}}_{/\mathsf{X}} \to \mathrm{CAlg}$ denote the functors given by Notation 5.1.2, so that we have an equivalence of functors $\underline{B}_{\mathsf{X}}(\bullet) \simeq B \otimes_A \underline{A}_{\mathsf{X}}(\bullet)$. There is an evident restriction of scalars functor

$$\phi_* : \mathrm{Mod}_{\underline{B}_{\mathsf{X}}} \to \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}.$$

In what follows, we will generally abuse notation by identifying an object $\mathscr{G} \in \mathrm{Mod}_{\underline{B}_{\mathsf{X}}}$ with its image under ϕ_*. The functor ϕ_* admits a left adjoint ϕ^*. For each object $\mathscr{F} \in \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}$, we denote $\phi^* \mathscr{F}$ by $B \otimes_A \mathscr{F}$; concretely, it is given by the formula

$$(B \otimes_A \mathscr{F})(T) = B^T_{\mathbf{G}} \otimes_{A^T_{\mathbf{G}}} \mathscr{F}(T) \simeq B \otimes_A \mathscr{F}(T)$$

for $T \in \mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}$.

Proposition 6.2.1. *Let $\phi : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, and let X be an orbispace. Then:*

- (a) *An object $\mathscr{G} \in \mathrm{Mod}_{\underline{B}_{\mathsf{X}}}$ is a $\mathbf{G}_B$-pretempered local system on X if and only $\phi_* \mathscr{G}$ is a $\mathbf{G}$-pretempered local system on X.*
- (b) *An object $\mathscr{G} \in \mathrm{Mod}_{\underline{B}_{\mathsf{X}}}$ is a $\mathbf{G}_B$-tempered local system on X if and only $\phi_* \mathscr{G}$ is a $\mathbf{G}$-tempered local system on X.*
- (c) *If $\mathscr{F} \in \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}$ is a $\mathbf{G}$-pretempered local system on X, then $\phi^* \mathscr{F} = B \otimes_A \mathscr{F}$ is a $\mathbf{G}_B$-pretempered local system on X.*

(*d*) *If* $\mathbf{G}$ *is oriented and* $\mathscr{F}$ *is a* $\mathbf{G}$*-tempered local system on* X*, then* $\phi^* \mathscr{F} = B \otimes_A \mathscr{F}$ *is a* $\mathbf{G}_B$*-tempered local system on* X*.*

Proof. Assertions (*a*), (*b*), and (*c*) follow immediately from the definitions. To prove (*d*), it will suffice (by virtue of (*b*)) to show that if $\mathbf{G}$ is oriented, $\mathscr{F}$ is a $\mathbf{G}$-tempered local system on X, and M is an A-module spectrum, then the relative tensor product $M \otimes_A \mathscr{F}$ (given by $(T \in \mathscr{T}_{/\mathsf{X}}^{\mathrm{op}}) \mapsto M \otimes_A \mathscr{F}(T)$) is also a $\mathbf{G}$-tempered local system on X, which follows from Remark 5.8.15. $\square$

Remark 6.2.2. In the situation of Proposition 6.2.1, assume that $\mathbf{G}$ is oriented. Then:

(1) If $\mathscr{G} \to \mathscr{G}'$ is a morphism in $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}_B}(\mathsf{X})$ which exhibits $\mathscr{G}'$ as a $\mathrm{LocSys}_{\mathbf{G}_B}(\mathsf{X})$-localization of $\mathscr{G}$, then the induced map $\phi_* \mathscr{G} \to \phi_* \mathscr{G}'$ exhibits $\phi_* \mathscr{G}'$ as a $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$-localization of $\phi_* \mathscr{G}$.

(2) If $\mathscr{F} \to \mathscr{F}'$ is a morphism in $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ which exhibits $\mathscr{F}'$ as a $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$-localization of $\mathscr{F}$, then the induced map

$$B \otimes_A \mathscr{F} = \phi^* \mathscr{F} \to \phi^* \mathscr{F}' = B \otimes_A \mathscr{F}'$$

exhibits $B \otimes_A \mathscr{F}'$ as a $\mathrm{LocSys}_{\mathbf{G}_B}(\mathsf{X})$-localization of $B \otimes_A \mathscr{F}$.

These assertions follow from Theorem 5.7.3, combined with the observation that the adjoint functors

$$\begin{tikzcd} \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \arrow[r, shift left, "\phi^*"] & \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}_B}(\mathsf{X}) \arrow[l, shift left, "\phi_*"] \end{tikzcd}$$

carry $\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}}(\mathsf{X})$ into $\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}_B}(\mathsf{X})$ and vice-versa.

In the situation of Proposition 6.2.1, the extension of scalars functor

$$\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}_B}(\mathsf{X}) \qquad \mathscr{F} \mapsto B \otimes_A \mathscr{F}$$

is symmetric monoidal with respect to the levelwise symmetric monoidal structure on the ∞-categories $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ and $\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}_B}(\mathsf{X})$ (given by the tensor product $\overline{\otimes}$ of Notation 5.8.1). If $\mathbf{G}$ is oriented, then it restricts to a lax symmetric monoidal functor

$$\phi^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}_B}(\mathsf{X}) \qquad \mathscr{F} \mapsto B \otimes_A \mathscr{F}.$$

In fact, we can say more:

Proposition 6.2.3. *Let $\phi : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings, let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over A, and let X be an orbispace. Then the lax symmetric monoidal functor*

$$\phi^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}_B}(\mathsf{X}) \qquad \mathscr{F} \mapsto B \otimes_A \mathscr{F}$$

is symmetric monoidal.

Proof. Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathbf{G}$-tempered local systems on X; we wish to show that the canonical map $\theta : (\phi^* \mathscr{F}) \otimes (\phi^* \mathscr{G}) \to \phi^*(\mathscr{F} \otimes \mathscr{G})$ is an equivalence in the ∞-category $\mathrm{LocSys}_{\mathbf{G}_B}(\mathsf{X})$. Unwinding the definitions, we see that θ fits into a commutative diagram

$$\begin{tikzcd}
(\phi^* \mathscr{F}) \overline{\otimes} (\phi^* \mathscr{G}) \arrow[r, "\sim"] \arrow[d] & \phi^*(\mathscr{F} \overline{\otimes} \mathscr{G}) \arrow[d] \\
(\phi^* \mathscr{F}) \otimes (\phi^* \mathscr{G}) \arrow[r, "\theta"] & \phi^*(\mathscr{F} \otimes \mathscr{G})
\end{tikzcd}$$

where the upper horizontal map is an equivalence, and the left vertical map exhibits $(\phi^* \mathscr{F}) \otimes (\phi^* \mathscr{G})$ as a $\mathrm{LocSys}_{\mathbf{G}_B}(\mathsf{X})$-localization of the levelwise tensor product $(\phi^* \mathscr{F}) \overline{\otimes} (\phi^* \mathscr{G})$. It will therefore suffice to show that the right vertical map exhibits $\phi^*(\mathscr{F} \otimes \mathscr{G})$ as a $\mathrm{LocSys}_{\mathbf{G}_B}(\mathsf{X})$-localization of $\phi^*(\mathscr{F} \overline{\otimes} \mathscr{G})$, which follows from Remark 6.2.2. $\square$

Remark 6.2.4. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and let X be an orbispace. Then we can regard $\mathcal{C} = \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}$ as an A-linear ∞-category. If B is an $\mathbb{E}_\infty$-algebra over A, then $\mathcal{C}_B = \mathrm{Mod}_{\underline{B}_{\mathsf{X}}}$ can then be identified with the ∞-category $B \otimes_A \mathcal{C} \simeq \mathrm{Mod}_B(\mathcal{C})$ of B-module objects of $\mathcal{C}$. By virtue of Proposition 6.2.1, this identification restricts to equivalences

$$\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}_B}(\mathsf{X}) \simeq B \otimes_A \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \simeq \mathrm{Mod}_B(\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}))$$

$$\mathrm{LocSys}_{\mathbf{G}_B}(\mathsf{X}) \simeq B \otimes_A \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \simeq \mathrm{Mod}_B(\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})).$$

Example 6.2.5. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $I \subseteq \pi_0(A)$ be a finitely generated ideal, and let $B = L_I(A)$ denote the I-localization of A (so that B is I-local as an A-module, and the fiber of the map $A \to B$ is I-nilpotent). Then, for any A-linear ∞-category $\mathcal{C}$, we can identify $\mathrm{Mod}_B(\mathcal{C})$ with the full subcategory of $\mathcal{C}$ spanned by the I-local objects. In particular, for any orbispace X, the forgetful functor $\mathrm{LocSys}_{\mathbf{G}_B}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is a fully faithful embedding, whose essential image is the subcategory $\mathrm{LocSys}^{\mathrm{Loc}(I)}_{\mathbf{G}}(\mathsf{X}) \subseteq \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ appearing in Definition 6.1.1.

Let A be complex periodic and p-local (for some prime number p). Applying the above analysis in the case where $I = \mathfrak{I}^A_{n+1}$ is the $(n+1)$st Landweber ideal of A, we obtain an equivalence $\mathrm{LocSys}^{E(n)}_{\mathbf{G}}(\mathsf{X}) = \mathrm{LocSys}_{\mathbf{G}_B}(\mathsf{X})$, where $B = L_I(A) = L_{E(n)}(A)$ is the $E(n)$-localization of A.

Proposition 6.2.6 (Faithfully Flat Descent)**.** *Let A be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A, and let $A^\bullet$ be a flat hypercovering of A (see Definition SAG.D.6.1.4). Then, for any orbispace X, extension of scalars induces an equivalence of ∞-categories*

$$\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{Tot}(\mathrm{LocSys}_{\mathbf{G}_{A^\bullet}}(\mathsf{X})).$$

Proof. Since $\mathbf{G}$ is oriented, the ∞-category $\mathcal{C} = \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is compactly generated (Corollary 5.3.3). By virtue of Remark 6.2.4, we are reduced to proving that the canonical map $\mathcal{C} \to \mathrm{Tot}(A^\bullet \otimes_A \mathcal{C})$ is an equivalence, which is a special case of Corollary SAG.D.7.7.7 . ☐

Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $I \subseteq \pi_0(A)$ be a finitely generated ideal. Let B be an $\mathbb{E}_\infty$-algebra over A and let $J = I\pi_0(B)$ be the ideal generated by the image of I. Then a B-module spectrum M is J-nilpotent (J-local, J-complete) if and only if it is I-nilpotent (I-local, I-complete) when viewed as an A-module. It follows that, for any orbispace X, the forgetful functor $\mathrm{LocSys}_{\mathbf{G}_B}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ restricts to functors

$$\mathrm{LocSys}^{\mathrm{Nil}(J)}_{\mathbf{G}_B}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{Nil}(I)}_{\mathbf{G}}(\mathsf{X})$$

$$\mathrm{LocSys}^{\mathrm{Loc}(J)}_{\mathbf{G}_B}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{Loc}(I)}_{\mathbf{G}}(\mathsf{X})$$

$$\mathrm{LocSys}^{\mathrm{Cpl}(J)}_{\mathbf{G}_B}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{Cpl}(I)}_{\mathbf{G}}(\mathsf{X}).$$

Proposition 6.2.7. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $I \subseteq \pi_0(A)$ be a finitely generated ideal. Let B be an $\mathbb{E}_\infty$-algebra over A, let $J = I\pi_0(B)$ denote the ideal generated by I, and suppose that the map of completions $A^\wedge_I \to B^\wedge_J$ is an equivalence. Then, for any orbispace X, the restriction functors*

$$\mathrm{LocSys}^{\mathrm{Nil}(J)}_{\mathbf{G}_B}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{Nil}(I)}_{\mathbf{G}}(\mathsf{X}) \qquad \mathrm{LocSys}^{\mathrm{Cpl}(J)}_{\mathbf{G}_B}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{Cpl}(I)}_{\mathbf{G}}(\mathsf{X})$$

are equivalences of ∞-categories.

Proof. Let $\phi : A \to B$ exhibit B as an $\mathbb{E}_\infty$-algebra over A. We will show that the forgetful functor $\phi_* : \mathrm{LocSys}_{\mathbf{G}_B}^{\mathrm{Nil}(J)}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(I)}(\mathsf{X})$ is an equivalence; the analogous assertion for complete objects is then a formal consequence (see Corollary 6.1.12). Note that ϕ_* admits a left adjoint $\phi^* : \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(I)}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}_B}^{\mathrm{Nil}(J)}(\mathsf{X})$. Since ϕ_* is conservative, it will suffice to show that the unit map $\mathrm{id} \to \phi_* \phi^*$ is an equivalence of functors from $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Nil}(I)}(\mathsf{X})$ to itself. In other words, it will suffice to show that if $\mathscr{F}$ is I-nilpotent, then the canonical map $\mathscr{F} \to B \otimes_A \mathscr{F}$ is an equivalence of $\mathbf{G}$-tempered local systems on X. Fix an object $T \in \mathscr{T}_{/\mathsf{X}}^{\mathrm{op}}$; we wish to show that the canonical map $\theta : \mathscr{F}(T) \to B \otimes_A \mathscr{F}(T)$ is an equivalence. This is clear: the homotopy fiber $\mathrm{fib}(\theta)$ can be identified with the tensor product $\mathrm{fib}(\phi) \otimes_A \mathscr{F}(T)$, which vanishes because $\mathrm{fib}(\phi)$ is I-local and $\mathscr{F}(T)$ is I-nilpotent. □

Corollary 6.2.8. *Let p be a prime number, let $\phi : A \to B$ be a map of p-local $\mathbb{E}_\infty$-rings, and let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over A. Let n be a nonnegative integer for which ϕ induces an equivalence $L_{K(n)}(A) \to L_{K(n)}(B)$. Then, for any orbispace X, the forgetful functor*

$$\mathrm{LocSys}_{\mathbf{G}_B}^{K(n)}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X})$$

is an equivalence of ∞-categories.

Proof. When $n = 0$, this follows from Example 6.2.5. Let us therefore assume that $n > 0$. In this case, we can apply Proposition 6.2.7 (with $I = (p)$) to reduce to the case where A and B are p-complete. Our assumption that $\mathbf{G}$ is oriented then guarantees that A is complex periodic (so that B is also complex periodic). Using Example 6.2.5 again, we can replace A by $L_{E(n)}(A)$ and thereby reduce to the case where A is $E(n)$-local. In this case, a $\mathbf{G}$-tempered local system $\mathscr{F}$ on X is $K(n)$-local if and only if it is $\mathfrak{I}_n^A$-complete, where $\mathfrak{I}_n^A$ denotes the nth Landweber ideal of $\pi_0(A)$ (Remark 6.1.14). The desired result now follows from Proposition 6.2.7. □

6.3 The Infinitesimal Case

Let p be a prime number, which we regard as fixed throughout this section. Let A be a p-local $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be an oriented p-divisible group over A. For any orbispace X, Corollary 6.1.17 asserts that the stable ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ admits a semi-orthogonal decomposition by the full subcategories $\{\mathrm{LocSys}_{\mathbf{G}}^{K(m)}(\mathsf{X})\}_{m \geqslant 0}$. The last of these subcategories admits a more concrete description:

Theorem 6.3.1. *Let* $\mathbf{G}$ *be an oriented* p*-divisible group of height* n *over a* p*-local* $\mathbb{E}_\infty$*-ring* A*, and let* X *be an orbispace with underlying space* $|\mathsf{X}|$*. Then the forgetful functor* $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_A(|\mathsf{X}|)$ *of Variant 5.1.15 restricts to an equivalence of* ∞*-categories*

$$\mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X}) \simeq \mathrm{LocSys}_A^{K(n)}(|\mathsf{X}|).$$

Here $\mathrm{LocSys}_A^{K(n)}(|\mathsf{X}|) = \mathrm{Fun}(|\mathsf{X}|, \mathrm{Mod}_A^{K(n)})$ *denotes the full subcategory of* $\mathrm{LocSys}_A(|\mathsf{X}|)$ *spanned by those local systems of* A*-modules on* $|\mathsf{X}|$ *which take* $K(n)$*-local values.*

Remark 6.3.2. To prove Theorem 6.3.1, we are free to replace A by its $K(n)$-localization $L_{K(n)}(A)$ (see Corollary 6.2.8). If $n > 0$, the orientation of $\mathbf{G}$ then determines an equivalence $\mathbf{G}_A^{\mathcal{Q}} \simeq \mathbf{G}$, where $\mathbf{G}_A^{\mathcal{Q}}$ is the Quillen p-divisible group of A (Proposition 2.5.6).

Remark 6.3.3. Let A be an $\mathbb{E}_\infty$-ring which is $K(n)$-local and complex periodic, and let $\mathbf{G} = \mathbf{G}_A^{\mathcal{Q}}$ be the Quillen p-divisible group of A. For any orbispace X, the forgetful functor $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_A(|\mathsf{X}|)$ of Variant 4.9.6 carries the trivial $\mathbf{G}$-tempered local system $\underline{A}_{\mathsf{X}}$ to the trivial local system $\underline{A}_{|\mathsf{X}|}$. Since $\underline{A}_{\mathsf{X}}$ is $K(n)$-local, Theorem 6.3.1 implies that the induced map

$$\zeta : \mathrm{Ext}^*_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}(\underline{A}_{\mathsf{X}}, \underline{A}_{\mathsf{X}}) \simeq \mathrm{Ext}^*_{\mathrm{LocSys}_A(|\mathsf{X}|)}(\underline{A}_{|\mathsf{X}|}, \underline{A}_{|\mathsf{X}|})$$

is an isomorphism. Combining this observation with Remark 5.1.20, we recover the statement that the Atiyah-Segal comparison map $\zeta : A^*_{\mathbf{G}}(\mathsf{X}) \to A^*(|\mathsf{X}|)$ is an isomorphism. In other words, we can regard Theorem 6.3.1 as a categorified version of Theorem 4.2.5.

Proof of Theorem 6.3.1. Without loss of generality, we may assume that the $\mathbb{E}_\infty$-ring A is $K(n)$-local and that $\mathbf{G} = \mathbf{G}_A^{\mathcal{Q}}$ is the Quillen p-divisible group of A (Remark 6.3.2). Let us abuse notation by identifying $|\mathsf{X}|$ with the full subcategory of $\mathscr{T}_{/\mathsf{X}}^{\mathrm{op}}$ spanned by those objects $T \to \mathsf{X}$ where T is contractible. Let $\mathcal{C} \subseteq \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}$ denote the full subcategory spanned by those $\underline{A}_{\mathsf{X}}$-modules $\mathscr{F}$ which are right Kan extensions of their restriction to $|\mathsf{X}|$, and let $\mathcal{C}^{K(n)}$ denote the full subcategory of $\mathcal{C}$ spanned by those objects $\mathscr{F}$ for which the spectrum $\mathscr{F}(T)$ is $K(n)$-local for $T \in |\mathsf{X}|$. Applying Proposition HTT.4.3.2.15 (to the fibration $q : \mathrm{Mod}(\mathrm{Sp}) \to \mathrm{CAlg}$ of Construction 5.1.8), we deduce that the restriction functors

$$\mathcal{C} \to \mathrm{LocSys}_A(\mathsf{X}) \qquad \mathscr{F} \mapsto \mathscr{F}|_{|\mathsf{X}|}$$

$$\mathcal{C}^{K(n)} \to \mathrm{LocSys}_A^{K(n)}(\mathsf{X}) \qquad \mathscr{F} \mapsto \mathscr{F}|_{|\mathsf{X}|}$$

are equivalences of ∞-categories. It will therefore suffice to show that $\mathcal{C}^{K(n)} = \mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X})$.

Unwinding the definitions, we see that a module $\mathscr{F} \in \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}$ belongs to $\mathcal{C}^{K(n)}$ if and only if it satisfies the following conditions:

- (a) For each $T \in |\mathsf{X}|$, the spectrum $\mathscr{F}(T)$ is $K(n)$-local.
- (b) For each object $T \in \mathscr{T}_{/\mathsf{X}}$ having universal cover $T_0 \in \mathrm{Cov}(T)$, the canonical map $\mathscr{F}(T) \to \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}$ is an equivalence.

Suppose first that $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X})$; we wish to show that $\mathscr{F}$ satisfies (a) and (b). Condition (a) is obvious. To prove (b), it suffices to show that $\mathscr{F}(T)$ is $I(T_0/T)$-complete when viewed as a module over $A_{\mathbf{G}}^T = A^T$. We will assume that $n > 0$ (otherwise there is nothing to prove), so that A is complex periodic. Let $\mathfrak{I}_n^A \subseteq \pi_0(A)$ denote the nth Landweber ideal of A. Then the inverse image of the vanishing locus of $\mathfrak{I}_n^A$ under the map $|\operatorname{Spec}(A_{\mathbf{G}}^T)| \to |\operatorname{Spec}(A)|$ is contained in the zero section: that is, some power of the ideal $I(T_0/T)$ is contained in $\mathfrak{I}_n^A A_{\mathbf{G}}^0(T)$. Consequently, to show that $\mathscr{F}(T)$ is $I(T_0/T)$-local, it suffices to show that it is $\mathfrak{I}_n^A$-local, or equivalently that it is $K(n)$-local (which follows by assumption).

Now suppose that $\mathscr{F}$ satisfies (a) and (b). It follows immediately that for each $T \in \mathscr{T}_{/\mathsf{X}}$, the spectrum $\mathscr{F}(T)$ is $K(n)$-local. We will show that $\mathscr{F}$ is a $\mathbf{G}$-tempered local system. We first verify condition (B) of Definition 5.2.4. Let T be any object of $\mathscr{T}_{/\mathsf{X}}$ and let $T_0 \in \mathrm{Cov}(T)$ be a connected covering space of T; we wish to show that the map $\theta : \mathscr{F}(T) \to \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)}$ has $I(T_0/T)$-local fiber. To prove this, let T_1 be a universal cover of T_0. We then have a commutative diagram

$$\begin{tikzcd}
\mathscr{F}(T) \arrow[rr, "\theta"] \arrow[dr] & & \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)} \arrow[dl] \\
& \mathscr{F}(T_1)^{\mathrm{hAut}(T_1/T)} &
\end{tikzcd}$$

where the vertical maps are equivalences by virtue of assumption (b). It follows that θ is an equivalence (so that $\mathrm{fib}(\theta) \simeq 0$ is automatically $I(T_0/T)$-local).

It remains to prove that $\mathscr{F}$ is a $\mathbf{G}$-pretempered local system on X. Fix a map $u : T' \to T$ in $\mathscr{T}_{/\mathsf{X}}$ with connected homotopy fibers; we wish to show that the canonical map $\rho : A_{\mathbf{G}}^{T'} \otimes_{A_{\mathbf{G}}^{T}} \mathscr{F}(T) \to \mathscr{F}(T')$ is an equivalence. Form a pullback square

$$\begin{tikzcd}
T_0' \arrow[r] \arrow[d] & T_0 \arrow[d] \\
T' \arrow[r] & T,
\end{tikzcd}$$

where T_0 is a universal cover of T. Using condition (b), we can identify ρ with the map

$$\rho : A_{\mathbf{G}}^{T'} \otimes_{A_{\mathbf{G}}^{T}} \mathscr{F}(T_0)^{\mathrm{hAut}(T_0/T)} \to \mathscr{F}(T_0')^{\mathrm{hAut}(T_0'/T')}.$$

Since $A_{\mathbf{G}}^{T'}$ is finite and flat as a module over $A_{\mathbf{G}}^{T}$, the functor $M \mapsto A_{\mathbf{G}}^{T'} \otimes A_{\mathbf{G}}^{T} M$ commutes with limits. Consequently, it will suffice to show that the natural map

$$\rho_0 : A_{\mathbf{G}}^{T'} \otimes_{A_{\mathbf{G}}^{T}} \mathscr{F}(T_0) \simeq A_{\mathbf{G}}^{T_0'} \otimes_{A_{\mathbf{G}}^{T_0}} \mathscr{F}(T_0) \to \mathscr{F}(T_0')$$

is an equivalence. In other words, we can replace T by T_0 (and T' by T_0') and thereby reduce to the problem of showing that ρ is an equivalence in the special case T is contractible. Setting $M = \mathscr{F}(T)$, we are reduced to the problem of showing that the canonical map $A^{T'} \otimes_A M \to M^{T'}$ is an equivalence. It follows from Proposition Ambi.5.4.6 that this map is an equivalence after $K(n)$-localization, and is therefore an equivalence (since both $A^{T'} \otimes_A M$ and $M^{T'}$ are $K(n)$-local). □

6.4 Categorified Character Theory

Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A which splits as a direct sum $\mathbf{G}_0 \oplus \underline{\Lambda}$, where $\underline{\Lambda}$ is the constant $\mathbf{P}$-divisible group associated to a colattice Λ (Construction 2.7.5). For any orbispace X, Theorem 4.3.2 supplies an equivalence of $\chi : A_{\mathbf{G}}^{\mathsf{X}} \simeq A_{\mathbf{G}_0}^{\mathcal{L}^\Lambda(\mathsf{X})}$. When $\mathbf{G}$ is oriented, this result has a counterpart for tempered local systems:

Theorem 6.4.1. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A which decomposes as a sum $\mathbf{G}_0 \oplus \underline{\Lambda}$, for some colattice Λ. For any orbispace X, there exists a symmetric monoidal fully faithful embedding*

$$\Phi : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^\Lambda(\mathsf{X})).$$

The remainder of this section is devoted to the proof of Theorem 6.4.1. In what follows, we fix a preoriented $\mathbf{P}$-divisible group $\mathbf{G}_0$ over an $\mathbb{E}_\infty$-ring A, a colattice Λ

with Pontryagin dual $\widehat{\Lambda}$, and an orbispace X. Let $\mathbf{G}$ denote the direct sum $\mathbf{G}_0 \oplus \underline{\Lambda}$, which we also regard as a preoriented $\mathbf{P}$-divisible group over A. Our first step is to construct a functor $\Psi : \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ which will be right adjoint to the embedding Φ of Theorem 6.4.1 (at least in the case where $\mathbf{G}$ is oriented), but is much easier to describe.

Notation 6.4.2. For each object $T \in \mathscr{T}$, we will identify the set of connected components $\pi_0(\mathcal{L}^{\Lambda}(T)) = \pi_0(T^{B\widehat{\Lambda}})$ with the set $\mathrm{Hom}(\widehat{\Lambda}, \pi_1(T))$ of group homomorphisms from $\widehat{\Lambda}$ to the finite group $\pi_1(T)$. If $\alpha : \widehat{\Lambda} \to \pi_1(T)$ is a group homomorphism, we will write $\mathcal{L}^{\Lambda}(T)_\alpha$ for the corresponding connected component of $\mathcal{L}^{\Lambda}(T)$ (so that $\mathcal{L}^{\Lambda}(T)_\alpha$ is homotopy equivalent to T, by evaluation at the base point of the classifying space $B\widehat{\Lambda}$).

Construction 6.4.3 (The Functor $\overline{\Psi}$)**.** For every functor $\mathscr{G} : \mathscr{T}^{\mathrm{op}}_{/\mathcal{L}^{\Lambda}(\mathsf{X})} \to \mathrm{Sp}$, we define a functor $\overline{\Psi}(\mathscr{G}) : \mathscr{T}^{\mathrm{op}}_{/\mathsf{X}} \to \mathrm{Sp}$ by the formula

$$\overline{\Psi}(\mathscr{G})(T) = \prod_{\alpha : \widehat{\Lambda} \to \pi_1(T)} \mathscr{G}(\mathcal{L}^{\Lambda}(T)_\alpha).$$

The construction $\mathscr{G} \mapsto \overline{\Psi}(\mathscr{G})$ then determines a lax symmetric monoidal functor

$$\overline{\Psi} : \mathrm{Fun}(\mathscr{T}^{\mathrm{op}}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}, \mathrm{Sp}) \to \mathrm{Fun}(\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}, \mathrm{Sp});$$

see Construction 6.4.10 below for a more precise description of this functor. In particular, the functor $\overline{\Psi}$ carries commutative algebra objects of $\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}, \mathrm{Sp})$ to commutative algebra objects of $\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}, \mathrm{Sp})$. By virtue of Proposition 2.7.15, it carries the trivial $\mathbf{G}_0$-pretempered local system $\underline{A}_{\mathcal{L}^{\Lambda}(\mathsf{X})}$ on $\mathcal{L}^{\Lambda}(\mathsf{X})$ to the trivial $\mathbf{G}$-pretempered local system $\underline{A}_{\mathsf{X}}$ on X. It follows that $\overline{\Psi}$ also determines a functor

$$\mathrm{Mod}_{\underline{A}_{\mathcal{L}^{\Lambda}(\mathsf{X})}} \to \mathrm{Mod}_{\underline{A}_{\mathcal{L}^{\Lambda}(\mathsf{X})}},$$

which we will also denote by $\overline{\Psi}$.

Remark 6.4.4. Let $\underline{A}_{\mathsf{X}}$ be the trivial $\mathbf{G}$-tempered local system on X, which we view as a commutative algebra object of the functor ∞-category $\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}, \mathrm{Sp})$. Let $\mathscr{F}$ be an $\underline{A}_{\mathsf{X}}$-module. For each object $T \in \mathscr{T}_{/\mathsf{X}}$, we can view $\mathscr{F}(T)$ as a module over the ring spectrum

$$A^T_{\mathbf{G}} = A^{\mathcal{L}^{\Lambda}(T)}_{\mathbf{G}_0} = \prod_{\alpha : \widehat{\Lambda} \to \pi_1(T)} A^{\mathcal{L}^{\Lambda}(T)_\alpha}_{\mathbf{G}_0}.$$

For each homomorphism $\alpha : \widehat{\Lambda} \to \pi_1(T)$, we define

$$\mathscr{F}(T)_\alpha = A_{\mathbf{G}_0}^{\mathcal{L}^\Lambda(T)_\alpha} \otimes_{A_{\mathbf{G}}^T} \mathscr{F}(T).$$

Then $\mathscr{F}(T)$ factors as a product $\prod_{\alpha:\widehat{\Lambda}\to\pi_1(T)} \mathscr{F}(T)_\alpha$, where each factor $\mathscr{F}(T)_\alpha$ is a module over the tempered function spectrum $A_{\mathbf{G}_0}^{\mathcal{L}^\Lambda(T)_\alpha} \simeq A_{\mathbf{G}_0}^T$.

Unwinding the definitions, we see that:

(*a*) The $\underline{A}_{\mathsf{X}}$-module $\mathscr{F}$ is a $\mathbf{G}$-pretempered local system on X if and only if, for each object $T \in \mathscr{T}_{/\mathsf{X}}$, each morphism $f : T' \to T$ with connected homotopy fibers, and each homomorphism $\alpha : \widehat{\Lambda} \to \pi_1(T')$, the canonical map

$$A_{\mathbf{G}_0}^{T'} \otimes_{A_{\mathbf{G}_0}^T} \mathscr{F}(T)_\alpha \to \mathscr{F}(T')_{\alpha'}$$

is an equivalence. Here we abuse notation by identifying α with its image in $\mathrm{Hom}(\widehat{\Lambda}, \pi_1(T))$.

(*b*) An object $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X})$ is $\mathbf{G}$-tempered if and only if, for each object $T \in \mathscr{T}_{/\mathsf{X}}$, each connected covering space $T_0 \in \mathrm{Cov}(T)$, and each homomorphism $\alpha : \widehat{\Lambda} \to \pi_1(T_0)$, the canonical map

$$\mathscr{F}(T)_\alpha \to \mathscr{F}(T_0)_\alpha^{\mathrm{hAut}(T_0/T)}$$

exhibits $\mathscr{F}(T_0)_\alpha^{\mathrm{hAut}(T_0/T)}$ as the completion of $\mathscr{F}(T)_\alpha$ with respect to the augmentation ideal $I(T_0/T) \subseteq A_{\mathbf{G}_0}^0(T)$; here we abuse notation by identifying α with its image in $\mathrm{Hom}(\widehat{\Lambda}, \pi_1(T))$.

Remark 6.4.5. Let $\mathscr{G}$ be an $\underline{A}_{\mathcal{L}^\Lambda(\mathsf{X})}$-module object of $\mathrm{Fun}(\mathscr{T}_{/\mathcal{L}^\Lambda(\mathsf{X})}^{\mathrm{op}}, \mathrm{Sp})$. Using the conventions of Remark 6.4.4, we see that the $\underline{A}_{\mathsf{X}}$-module $\overline{\Psi}(\mathscr{G})$ of Construction 6.4.3 is given by the formula $\overline{\Psi}(\mathscr{G})(T)_\alpha = \mathscr{G}(\mathcal{L}^\Lambda(T)_\alpha)$; here T denotes an object of $\mathscr{T}_{/\mathsf{X}}$ and α any homomorphism from $\widehat{\Lambda}$ to $\pi_1(T)$.

Combining Remarks 6.4.4 and 6.4.5, we obtain the following:

Proposition 6.4.6. *For every $\mathbf{G}_0$-pretempered local system $\mathscr{G}$ on the formal loop space $\mathcal{L}^\Lambda(\mathsf{X})$, the $\underline{A}_{\mathsf{X}}$-module $\overline{\Psi}(\mathscr{G})$ of Construction 6.4.3 is a $\mathbf{G}$-pretempered local system on X. If $\mathscr{G}$ is a $\mathbf{G}_0$-tempered local system on $\mathcal{L}^\Lambda(\mathsf{X})$, then $\overline{\Psi}(\mathscr{G})$ is $\mathbf{G}$-tempered local system on X.*

Notation 6.4.7 (The Functor Ψ)**.** It follows from Proposition 6.4.6 that the construction $\mathscr{G} \mapsto \overline{\Psi}(\mathscr{G})$ restricts to functors

$$\Psi^{\mathrm{pre}} : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \qquad \Psi : \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}).$$

The lax symmetric monoidal structure on $\overline{\Psi}$ then determines a lax symmetric monoidal structure on the functor Ψ^{pre} (with respect to the levelwise tensor product $\overline{\otimes}$ of Notation 5.8.1). If $\mathbf{G}$ is oriented, it determines a lax symmetric monoidal structure on Ψ (with respect to the tempered tensor product of Construction 5.8.7).

Remark 6.4.8. The lax symmetric monoidal functor Ψ^{pre} is actually symmetric monoidal. However, the functor Ψ is not symmetric monoidal.

We can now formulate Theorem 6.4.1 more precisely:

Theorem 6.4.9. *If $\mathbf{G}$ is an oriented $\mathbf{P}$-divisible group, then the functor Ψ of Notation 6.4.7 has a fully faithful left adjoint $\Phi : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$. Moreover, the lax symmetric monoidal structure on Ψ induces a symmetric monoidal structure on Φ.*

We now construct the left adjoint Φ appearing in Theorem 6.4.9.

Construction 6.4.10. Let $\mathcal{M}^{+} \to \Delta^1$ be a Cartesian fibration which classifies the formal loop functor $\mathcal{L}^{\Lambda} : \mathcal{OS}_{/\mathcal{L}^{\Lambda}(\mathsf{X})} \to \mathcal{OS}_{/\mathsf{X}}$, so that the fibers are given by

$$\mathcal{M}^{+}_0 = \mathcal{OS}_{/\mathcal{L}^{\Lambda}(\mathsf{X})} \qquad \mathcal{M}^{+}_1 = \mathcal{OS}_{/\mathsf{X}}\,.$$

We let $\mathcal{M}$ denote the full subcategory of $\mathcal{M}^{+}$ spanned by the objects which belong either to the full subcategory $\mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})} \subset \mathcal{OS}_{/\mathcal{L}^{\Lambda}(\mathsf{X})} = \mathcal{M}^{+}_0$ or to $\mathscr{T}_{/\mathsf{X}} \subset \mathcal{OS}_{/\mathsf{X}} \simeq \mathcal{M}^{+}_0$. More informally, $\mathcal{M}$ is an ∞-category equipped with a functor $\mathcal{M} \to \Delta^1$ having fibers $\mathcal{M}_0 = \mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}$, $\mathcal{M}_1 = \mathscr{T}_{/\mathsf{X}}$, with morphisms given by

$$\mathrm{Map}_{\mathcal{M}}(T_0, T_1) = \mathrm{Map}_{\mathcal{OS}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}}(T_0, \mathcal{L}^{\Lambda}(T_1)) = \coprod_{\alpha : \widehat{\Lambda} \to \pi_1(T_1)} \mathrm{Map}_{\mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}}(T_0, \mathcal{L}^{\Lambda}(T_1)_{\alpha})$$

for $T_0 \in \mathcal{M}_0$, $T_1 \in \mathcal{M}_1$.

Note that we have an evident retraction r of $\mathcal{M}^{+}$ onto the subcategory $\mathcal{M}^{+}_0$ (whose restriction to $\mathcal{M}^{+}_1$ is the formal loop functor $\mathcal{L}^{\Lambda} : \mathcal{OS}_{/\mathcal{L}^{\Lambda}(\mathsf{X})} \to \mathcal{OS}_{/\mathsf{X}}$). Let $A_{\mathcal{M}}$ denote the opposite of the composite functor

$$\mathcal{M} \hookrightarrow \mathcal{M}^{+} \xrightarrow{r} \mathcal{M}^{+}_0 \to \mathcal{OS} \xrightarrow{A_{\mathbf{G_0}}} \mathrm{CAlg}^{\mathrm{op}}\,.$$

We regard $A_{\mathcal{M}}$ as a functor from $\mathcal{M}^{\mathrm{op}}$ to CAlg, whose restriction to $\mathcal{M}_0^{\mathrm{op}} \simeq \mathscr{T}^{\mathrm{op}}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}$ is the trivial $\mathbf{G}_0$-tempered local system $\underline{A}_{\mathcal{L}^{\Lambda}(\mathsf{X})}$, and whose restriction to $\mathcal{M}_1^{\mathrm{op}} \simeq \mathscr{T}_{/\mathsf{X}}$ is the trivial $\mathbf{G}$-tempered local system $\underline{A}_{\mathsf{X}}$.

Let $\iota_0 : \mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})} = \mathcal{M}_0 \hookrightarrow \mathcal{M}$ and $\iota_1 : \mathscr{T}_{/\mathsf{X}} = \mathcal{M}_1 \hookrightarrow \mathcal{M}$ be the inclusion maps, and let $q : \mathrm{Mod}(\mathrm{Sp}) \to \mathrm{CAlg}$ be the fibration of Construction 5.1.8. Unwinding the defintions, we see that the functor $\overline{\Psi}$ of Construction 6.4.3 is given (on $\underline{A}_{\mathcal{L}^{\Lambda}(\mathsf{X})}$-modules) by the composition

$$\begin{aligned}
\mathrm{Mod}_{\underline{A}_{\mathcal{L}^{\Lambda}(\mathsf{X})}} &\simeq \mathrm{Fun}_{\mathrm{CAlg}}(\mathscr{T}^{\mathrm{op}}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}, \mathrm{Mod}(\mathrm{Sp}))\\
&\xrightarrow{\iota_{0*}} \mathrm{Fun}_{\mathrm{CAlg}}(\mathcal{M}^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp}))\\
&\xrightarrow{\iota_1^*} \mathrm{Fun}_{\mathrm{CAlg}}(\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}, \mathrm{Mod}(\mathrm{Sp}))\\
&\simeq \mathrm{Mod}_{\underline{A}_{\mathsf{X}}},
\end{aligned}$$

where ι_1^* is given by precomposition with ι_1 and ι_{0*} is given by q-right Kan extension along ι_0. It follows that $\overline{\Psi}$ admits a left adjoint $\overline{\Phi}$, given by the composition

$$\begin{aligned}
\mathrm{Mod}_{\underline{A}_{\mathsf{X}}} &\simeq \mathrm{Fun}_{\mathrm{CAlg}}(\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}, \mathrm{Mod}(\mathrm{Sp}))\\
&\xrightarrow{\iota_{1!}} \mathrm{Fun}_{\mathrm{CAlg}}(\mathcal{M}^{\mathrm{op}}, \mathrm{Mod}(\mathrm{Sp}))\\
&\xrightarrow{\iota_0^*} \mathrm{Fun}_{\mathrm{CAlg}}(\mathscr{T}^{\mathrm{op}}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}, \mathrm{Mod}(\mathrm{Sp}))\\
&\simeq \mathrm{Mod}_{\underline{A}_{\mathcal{L}^{\Lambda}(\mathsf{X})}},
\end{aligned}$$

where ι_0^* is given by precomposition with ι_0 and $\iota_{1!}$ is given by q-left Kan extension along ι_1.

More informally: if $\mathscr{F}$ is an $\underline{A}_{\mathbf{G}}$-module object of $\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}, \mathrm{Sp})$, then $\overline{\Phi}(\mathscr{F})$ is given by the formula $\overline{\Phi}(\mathscr{F})(T) = \varinjlim_{T'}(A^{T}_{\mathbf{G}_0} \otimes_{A^{T'}_{\mathbf{G}}} \mathscr{F}(T'))$, where the colimit is indexed by the opposite of the ∞-category $\mathscr{T}_{/\mathsf{X}} \times_{\mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}} \mathscr{T}_{T//\mathcal{L}^{\Lambda}(\mathsf{X})}$.

Remark 6.4.11. Let T be an object of $\mathscr{T}$. By definition, the space $\mathcal{L}^{\Lambda}(\mathsf{X})^{T} = \mathrm{Map}_{\mathcal{OS}}(T, \mathcal{L}^{\Lambda}(\mathsf{X}))$ can be identified with the filtered colimit

$$\varinjlim_{\Lambda' \subseteq \Lambda} \mathrm{Map}_{\mathcal{OS}}(T \times B\widehat{\Lambda}', \mathsf{X}),$$

indexed by the collection of all finite subgroups $\Lambda' \subseteq \Lambda$ (and $\widehat{\Lambda}'$ denotes the Pontryagin dual group $\mathrm{Hom}(\Lambda', \mathbf{Q}/\mathbf{Z})$). If $\Lambda_0 \subseteq \Lambda$ is a finite subgroup, we will say that a map of orbispaces $f : T \to \mathcal{L}^{\Lambda}(\mathsf{X})$ is *represented by* a map $f_0 : T \times B\widehat{\Lambda}_0 \to \mathsf{X}$ if it is the image of f_0 under the composite map

$$\mathrm{Map}_{\mathcal{OS}}(T \times B\widehat{\Lambda}_0, \mathsf{X}) \to \varinjlim_{\Lambda' \subseteq \Lambda} \mathrm{Map}_{\mathcal{OS}}(T \times B\widehat{\Lambda}', \mathsf{X}) \simeq \mathrm{Map}_{\mathcal{OS}}(T, \mathcal{L}^{\Lambda}(\mathsf{X})).$$

If this condition is satisfied, then the construction $\Lambda' \mapsto T \times B\widehat{\Lambda'}$ determines a right cofinal functor

$$\{\text{Finite subgroups } \Lambda' \subseteq \Lambda \text{ containing } \Lambda_0\} \to \mathscr{T}_{/\mathsf{X}} \times_{\mathscr{T}_{/\mathcal{L}^\Lambda(\mathsf{X})}} \mathscr{T}_{T//\mathcal{L}^\Lambda(\mathsf{X})}$$

If $\mathscr{F}$ is a $\underline{A}_{\mathbf{G}}$-module object of $\operatorname{Fun}(\mathscr{T}_{/\mathsf{X}}^{\mathrm{op}}, \mathrm{Sp})$, we therefore obtain an equivalence

$$\overline{\Phi}(\mathscr{F})(T) = \varinjlim_{\Lambda_0 \subseteq \Lambda' \subseteq \Lambda} (A^T_{\mathbf{G}_0} \otimes_{A^{T \times B\widehat{\Lambda'}}_{\mathbf{G}}} \mathscr{F}(T \times B\widehat{\Lambda'})).$$

Example 6.4.12. Let $\mathscr{F}$ be a $\mathbf{G}$-pretempered local system on X, let T be an object of $\mathscr{T}$, and let $f : T \to \mathcal{L}^\Lambda(\mathsf{X})$ be a map of orbispaces which is represented by $f_0 : T \times B\widehat{\Lambda}_0 \to \mathsf{X}$, for some finite subgroup $\Lambda_0 \subseteq \Lambda$. Let $\rho : \widehat{\Lambda} \twoheadrightarrow \widehat{\Lambda}_0$ denote the Pontryagin dual of the inclusion map, so that we can view the pair $(0, \rho)$ as a homomorphism from $\widehat{\Lambda}$ to X. Our assumption that $\mathscr{F}$ is $\mathbf{G}$-pretempered guarantees that all of the transition maps in the filtered diagram

$$\{A^T_{\mathbf{G}_0} \otimes_{A^{T \times B\widehat{\Lambda'}}_{\mathbf{G}}} \mathscr{F}(T \times B\widehat{\Lambda'})\}_{\Lambda_0 \subseteq \Lambda' \subseteq \Lambda}$$

of Remark 6.4.11 are equivalences. We therefore obtain an equivalence

$$\begin{aligned} \overline{\Phi}(\mathscr{F})(T) &\simeq A^T_{\mathbf{G}_0} \otimes_{A^{T \times B\hat{\Lambda}_0}_{\mathbf{G}}} \mathscr{F}(T \times B\widehat{\Lambda}_0) \\ &\simeq A^T_{\mathbf{G}_0} \otimes_{\prod_\alpha A^{\mathcal{L}^\Lambda(T \times B\widehat{\Lambda_0})_\alpha}_{\mathbf{G}_0}} (\prod_\alpha \mathscr{F}(T \times B\widehat{\Lambda}_0)_\alpha) \\ &\simeq A^T_{\mathbf{G}_0} \otimes_{A^{T \times B\hat{\Lambda}_0}_{\mathbf{G}_0}} \mathscr{F}(T \times B\widehat{\Lambda}_0)_{(0,\rho)} \\ &\simeq A \otimes_{A^{B\hat{\Lambda}_0}_{\mathbf{G}_0}} \mathscr{F}(T \times B\widehat{\Lambda}_0)_{(0,\rho)}; \end{aligned}$$

here the tensor product is formed along the augmentation map $\epsilon : A^{B\widehat{\Lambda}_0}_{\mathbf{G}_0} \to A$

Proposition 6.4.13. *Let $\mathscr{F}$ be an $\underline{A}_{\mathsf{X}}$-module object of* $\operatorname{Fun}(\mathscr{T}_{/\mathsf{X}}^{\mathrm{op}}, \mathrm{Sp})$. *If $\mathscr{F}$ is a $\mathbf{G}$-pretempered local system on X, then $\overline{\Phi}(\mathscr{F})$ is a $\mathbf{G}_0$-pretempered local system on $\mathcal{L}^\Lambda(\mathsf{X})$. If $\mathbf{G}$ is oriented and $\mathscr{F}$ is a $\mathbf{G}$-tempered local system on X, then $\overline{\Phi}(\mathscr{F})$ is a $\mathbf{G}_0$-tempered local system on $\mathcal{L}^\Lambda(\mathsf{X})$.*

Proof. Assume first that $\mathscr{F}$ is a $\mathbf{G}$-pretempered local system on X. Fix an object $T \in \mathscr{T}$ equipped with a map $f : T \to \mathcal{L}^\Lambda(\mathsf{X})$, and let T' be a connected covering space of T. We wish to show that the canonical map

$$\theta : A^{T'}_{\mathbf{G}_0} \otimes_{A^T_{\mathbf{G}_0}} \Phi^{\mathrm{pre}}(\mathscr{F})(T) \to \overline{\Phi}(\mathscr{F})(T')$$

is an equivalence. Choose a representative of f by a map of orbispaces $f_0 : T \times B\widehat{\Lambda_0} \to \mathsf{X}$, where Λ_0 is a finite subgroup of Λ, and let $\rho : \widehat{\Lambda} \twoheadrightarrow \widehat{\Lambda_0}$ be the Pontryagin dual of the inclusion map. Using Example 6.4.12, we see that θ is obtained from the tautological map

$$\overline{\theta} : A_{\mathbf{G}_0}^{T' \times B\hat{\Lambda}_0} \otimes_{A_{\mathbf{G}_0}^{T \times B\hat{\Lambda}_0}} \mathscr{F}(T \times B\widehat{\Lambda}_0)_{(0,\rho)} \to \mathscr{F}(T' \times B\widehat{\Lambda}_0)_{(0,\rho)}$$

by extending scalars along the augmentation map $\epsilon : A_{\mathbf{G}_0}^{B\hat{\Lambda}_0} \to A$. It will therefore suffice to show that $\overline{\theta}$ is an equivalence, which follows from our assumption that $\mathscr{F}$ is $\mathbf{G}$-pretempered (by virtue of assertion (a) of Remark 6.4.4).

Now suppose that $\mathscr{F}$ is a $\mathbf{G}$-tempered local system on X and that $\mathbf{G}$ is oriented. We wish to show that $\overline{\Phi}(\mathscr{F})$ is a $\mathbf{G}_0$-tempered local system on $\mathscr{L}^{\Lambda}(\mathsf{X})$. We will prove this by verifying condition (B'') of Remark 5.2.8. Let $f : T \to \mathscr{L}^{\Lambda}(\mathsf{X})$ and $f_0 : T \times B\widehat{\Lambda_0} \to \mathsf{X}$ be as above, and let T_0 be a connected covering space of T for which the automorphism group $\mathrm{Aut}(T_0/T)$ is cyclic of order p, for some prime number p. We then have a fiber sequence $T_0 \to T \to BC_p$, and the ideal $I(T_0/T) \subseteq A^0_{\mathbf{G}_0}(T)$ of Notation 5.2.1 is generated by the image of the augmentation ideal $I_{C_p} \subseteq A^0_{\mathbf{G}_0}(BC_p)$. Using the description of $\overline{\Phi}(\mathscr{F})(T)$ and $\overline{\Phi}(\mathscr{F})(T_0)$ supplied by Example 6.4.12, we can identify ξ with the composition of the natural map

$$\xi' : A \otimes_{A_{\mathbf{G}_0}^{B\hat{\Lambda}_0}} \mathscr{F}(T \times B\widehat{\Lambda}_0)_{(0,\rho)} \to A \otimes_{A_{\mathbf{G}_0}^{B\hat{\Lambda}_0}} \mathscr{F}(T \times B\widehat{\Lambda}_0)^{hC_p}_{(0,\rho)}$$

with the map ξ'' which appears in the diagram of fiber sequences maps ξ' and ξ'' appearing in the diagram

$$\begin{tikzcd}
A \otimes_{A_{\mathbf{G}_0}^{B\hat{\Lambda}_0}} (\mathscr{F}(T_0 \times B\widehat{\Lambda}_0)_{(0,\rho)})_{hC_p} \arrow[r, "\sim"] \arrow[d, "\mathrm{Nm}"] & (A \otimes_{A_{\mathbf{G}_0}^{B\hat{\Lambda}_0}} \mathscr{F}(T_0 \times B\widehat{\Lambda}_0)_{(0,\rho)})_{hC_p} \arrow[d, "\mathrm{Nm}"] \\
A \otimes_{A_{\mathbf{G}_0}^{B\hat{\Lambda}_0}} \mathscr{F}(T \times B\widehat{\Lambda}_0)^{hC_p}_{(0,\rho)} \arrow[r, "\xi''"] \arrow[d] & (A \otimes_{A_{\mathbf{G}_0}^{B\hat{\Lambda}_0}} \mathscr{F}(T \times B\widehat{\Lambda}_0)_{(0,\rho)})^{hC_p} \arrow[d] \\
A \otimes_{A_{\mathbf{G}_0}^{B\hat{\Lambda}_0}} \mathscr{F}(T \times B\widehat{\Lambda}_0)^{tC_p}_{(0,\rho)} \arrow[r, "\gamma"] & (A \otimes_{A_{\mathbf{G}_0}^{B\hat{\Lambda}_0}} \mathscr{F}(T \times B\widehat{\Lambda}_0)_{(0,\rho)})^{tC_p}.
\end{tikzcd}$$

Our assumption that $\mathscr{F}$ is $\mathbf{G}$-tempered guarantees that the fiber $\mathrm{fib}(\xi')$ is $I(T_0/T)$-local (see Remark 6.4.4). It will therefore suffice to show that $\mathrm{fib}(\xi')$ is also $I(T_0/T)$-local, or equivalently that it is I_{C_p}-local when viewed as a module over $A_{\mathbf{G}_0}^{BC_p}$. Since the square on the lower right is a pullback, it induces an equivalence $\mathrm{fib}(\xi') \xrightarrow{\sim} \mathrm{fib}(\gamma)$. It will therefore suffice to show that $\mathrm{fib}(\gamma)$ is I_{C_p}-local. In fact, our assumption that $\mathbf{G}$ is oriented guarantees that the domain and codomain of γ are individually I_{C_p}-local (Proposition 4.6.8). $\square$

Notation 6.4.14 (The Functor Φ)**.** It follows from Proposition 6.4.13 that the functor $\overline{\Phi} : \mathrm{Mod}_{\underline{A}_{\mathsf{X}}} \to \mathrm{Mod}_{\underline{A}_{\mathcal{L}^{\Lambda}(\mathsf{X})}}$ of Construction 6.4.10 restricts to a functor $\Phi^{\mathrm{pre}} : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$, which is left adjoint to the functor Ψ^{pre} of Notation 6.4.7. If the $\mathbf{P}$-divisible group $\mathbf{G}$ is oriented, then Φ^{pre} restricts to a functor

$$\Phi : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})),$$

which is left adjoint to the functor $\Psi : \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ of Notation 6.4.7.

Remark 6.4.15. Suppose that $\mathbf{G}$ is oriented. Then the functor $\Phi^{\mathrm{pre}} : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$ carries the full subcategory $\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}}(\mathsf{X}) \subseteq \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ into the full subcategory $\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})) \subseteq \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$. This follows from Theorem 5.7.3, together with the fact that the right adjoint $\Psi^{\mathrm{pre}} : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X})$ carries $\mathbf{G}_0$-tempered local systems to $\mathbf{G}$-tempered local systems (Proposition 6.4.6).

Proposition 6.4.16. *The functor* $\Phi^{\mathrm{pre}} : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})$ *is fully faithful. In particular, if* $\mathbf{G}$ *is oriented, then the functor*

$$\Phi : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})$$

is fully faithful.

Proof. Let $\mathscr{F}$ be a $\mathbf{G}_0$-pretempered local system on X; we wish to show that the unit map $u : \mathscr{F} \to (\Psi^{\mathrm{pre}} \circ \Phi^{\mathrm{pre}})(\mathscr{F})$ is an equivalence. Choose an object $T \in \mathscr{T}$ equipped with a map $f : T \to \mathsf{X}$, and let $\alpha : \widehat{\Lambda} \to \pi_1(T)$ be a homomorphism; we wish to show that u induces an equivalence of $A^{T}_{\mathbf{G}_0}$-modules

$$u_{T,\alpha} : \mathscr{F}(T)_{\alpha} \to \Psi^{\mathrm{pre}}(\Phi^{\mathrm{pre}}(\mathscr{F}))(T)_{\alpha}.$$

Note that the image of α can be identified with the Pontryagin dual $\widehat{\Lambda}_0$ for some finite subgroup $\Lambda_0 \subseteq \Lambda$, and that the map $\mathcal{L}^{\Lambda}(T)_{\alpha} \hookrightarrow \mathcal{L}^{\Lambda}(T) \xrightarrow{\mathcal{L}^{\Lambda}(f)} \mathcal{L}^{\Lambda}(\mathsf{X})$ is then represented by the composition

$$\mathcal{L}^{\Lambda}(T)_{\alpha} \times B\widehat{\Lambda}_0 \xrightarrow{a} T \xrightarrow{f} \mathsf{X},$$

where a is obtained by amalgamating the homotopy equivalence $\mathcal{L}^{\Lambda}(T)_{\alpha} \simeq T$ with the inclusion map $\widehat{\Lambda}_0 \hookrightarrow \pi_1(T)$. Let $\rho : \widehat{\Lambda} \twoheadrightarrow \widehat{\Lambda}_0$ denote the Pontryagin dual of the inclusion $\Lambda_0 \hookrightarrow \Lambda$. Using the descriptions of Φ^{pre} and Ψ^{pre} supplied by Remark 6.4.5 and Example 6.4.12, we can identify $u_{T,\alpha}$ with the composite map

$$\mathscr{F}(T)_{\alpha} \to \mathscr{F}(\mathcal{L}^{\Lambda}(T)_{\alpha} \times B\widehat{\Lambda}_0)_{(0,\rho)} \to A \otimes_{A_{\mathbf{G}_0}^{B\hat{\Lambda}_0}} \mathscr{F}(\mathcal{L}^{\Lambda}(T)_{\alpha} \times B\widehat{\Lambda}_0)_{(0,\rho)}$$

determined by a. Since a has connected homotopy fibers and $\mathscr{F}$ is $\mathbf{G}$-pretempered, we can use Remark 6.4.4 to rewrite this map as

$$\begin{aligned} \mathscr{F}(T)_{\alpha} &\to A_{\mathbf{G}_0}^{\mathcal{L}^{\Lambda}(T)_{\alpha} \times B\hat{\Lambda}_0} \otimes_{A_{\mathbf{G}}^{T}} \mathscr{F}(T)_{\alpha} \\ &\to A \otimes_{A_{\mathbf{G}_0}^{B\hat{\Lambda}_0}} A_{\mathbf{G}_0}^{\mathcal{L}^{\Lambda}(T)_{\alpha} \times B\hat{\Lambda}_0} \otimes_{A_{\mathbf{G}_0}^{T}} \mathscr{F}(T)_{\alpha} \\ &\simeq A_{\mathbf{G}_0}^{T'} \otimes_{A_{\mathbf{G}_0}^{T}} \mathscr{F}(T)_{\alpha}, \end{aligned}$$

where T' denotes the homotopy fiber of the map $\mathcal{L}^{\Lambda}(T)_{\alpha} \times B\widehat{\Lambda}_0 \to B\widehat{\Lambda_0}$ given by projection onto the second factor. The desired result now follows from the observation that the composite map $T' \to \mathcal{L}^{\Lambda}(T)_{\alpha} \times B\widehat{\Lambda}_0 \xrightarrow{a} T$ is a homotopy equivalence. $\square$

Proof of Theorem 6.4.9. Suppose that $\mathbf{G}$ is oriented. Then the functors Ψ^{pre} and Ψ of Notation 6.4.7 admit left adjoints Φ^{pre} and Φ (Notation 6.4.14), which are fully faithful by virtue of Proposition 6.4.16. Moreover, since the functors Ψ^{pre} and Ψ are lax symmetric monoidal with respect to the tensor products $\overline{\otimes}$ and $\otimes$, the left adjoints Φ^{pre} and Φ inherit the structure of colax symmetric monoidal functors with respect to $\overline{\otimes}$ and $\otimes$. In particular, for every pair of $\mathbf{G}$-tempered local systems $\mathscr{F}, \mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, we have canonical maps

$$\overline{\theta} : \Phi^{\mathrm{pre}}(\mathscr{F} \overline{\otimes} \mathscr{G}) \to \Phi^{\mathrm{pre}}(\mathscr{F}) \overline{\otimes} \Phi^{\mathrm{pre}}(\mathscr{G}) \qquad \theta : \Phi(\mathscr{F} \otimes \mathscr{G}) \to \Phi(\mathscr{F}) \otimes \Phi(\mathscr{G})$$

which fit into a commutative diagram

$$\begin{tikzcd} \Phi^{\mathrm{pre}}(\mathscr{F} \overline{\otimes} \mathscr{G}) \arrow[r, "\overline{\theta}"] \arrow[d, "u"] & \Phi^{\mathrm{pre}}(\mathscr{F}) \overline{\otimes} \Phi^{\mathrm{pre}}(\mathscr{G}) \arrow[d, "v"] \\ \Phi(\mathscr{F} \otimes \mathscr{G}) \arrow[r, "\theta"] & \Phi(\mathscr{F}) \otimes \Phi(\mathscr{G}). \end{tikzcd}$$

To complete the proof, it will suffice to show that θ is an equivalence (and to prove an analogous assertion for unit objects, which we leave to the reader). From the description of Φ^{pre} supplied by Example 6.4.12, it is easy to see that the map $\overline{\theta}$ is an equivalence. It follows from the construction of the tempered tensor product on $\mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$ that the fiber $\mathrm{fib}(v)$ belongs to $\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$. Similarly, using the construction of the tempered tensor product on $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ together with Remark 6.4.15, we conclude that $\mathrm{fib}(u)$ belongs to $\mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$. It follows that the fiber $\mathrm{fib}(\theta)$ must belong to the intersection $\mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})) \cap \mathrm{LocSys}^{\mathrm{nul}}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$, and therefore vanishes (Theorem 5.7.3). $\square$

6.5 Isotropic Local Systems

Let $\mathbf{G}_0$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring, and let $\mathbf{G} = \mathbf{G}_0 \oplus \underline{\Lambda}$, where Λ is a colattice. For every orbispace X, Theorem 6.4.1 supplies a fully faithful embedding

$$\Phi : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})).$$

The goal of this section is to describe the essential image of the embedding Φ.

We begin with a few heuristic remarks. Let Y be an orbispace and let $T = BH$ be the classifying space of a finite abelian group H. Evaluation at the base point of T then determines a map

$$\mathrm{ev} : \mathrm{Map}_{\mathcal{OS}}(T, \mathsf{Y}) \to \mathrm{Map}_{\mathcal{OS}}(*, \mathsf{Y}) \simeq |\mathsf{Y}|.$$

Roughly speaking, one can think of a point f of the space $\mathsf{Y}^T \simeq \mathrm{Map}_{\mathcal{OS}}(T, \mathsf{Y})$ as consisting of a point $y = \mathrm{ev}(f)$ of the underlying space $|\mathsf{Y}|$, together with an "action" of the group H on y.

Suppose now that $\mathsf{Y} = \mathcal{L}^{\Lambda}(\mathsf{X})$, for some colattice Λ. In this case, every point $y \in |\mathsf{Y}|$ can be represented by a map of orbispaces $f_0 : B\widehat{\Lambda}_0 \to \mathsf{X}$, for some finite subgroup $\Lambda_0 \subseteq \Lambda$ (see Remark 6.4.11). For any homomorphism of finite abelian groups $u : H \to \widehat{\Lambda}_0$, we obtain the composite map

$$B(H \times \widehat{\Lambda}_0) \xrightarrow{(u,\mathrm{id})} B\widehat{\Lambda}_0 \xrightarrow{f_0} \mathsf{X}$$

then represents a map $BH \to \mathcal{L}^{\Lambda}(\mathsf{X}) = \mathsf{Y}$. This can be viewed as an action of H on the point y of a special type: roughly speaking, it is associated to the monodromy of the profinite torus $B\widehat{\Lambda}$. We now axiomatize a relative version of this condition.

Definition 6.5.1. Let X be an orbispace and let Λ be a colattice. We will say that a morphism $f : T' \to T$ be a morphism in the ∞-category $\mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}$ is *relatively* *monodromic* if the structure map $T \to \mathcal{L}^{\Lambda}$ can be represented (in the sense of Remark 6.4.11) by a composite map

$$T \times B\widehat{\Lambda}_0 \xrightarrow{g} BH \to \mathsf{X}$$

where Λ_0 is a finite subgroup of Λ, H is a finite abelian group, and the composite map

$$\pi_1(T' \times B\widehat{\Lambda}_0) \to \pi_1(T \times B\widehat{\Lambda}_0) \xrightarrow{g} \pi_1(BH) = H$$

is surjective.

Warning 6.5.2. In the situation of Definition 6.5.1, let Y denote the formal loop space $\mathcal{L}^{\Lambda}(\mathsf{X})$. The notion of relatively monodromic morphism in $\mathscr{T}_{/\mathsf{Y}}$ is not intrinsic to the orbispace Y: it depends on the presentation of Y as a formal loop space.

Example 6.5.3. Let X be an orbispace, let Λ be a colattice, and let $f : T' \to T$ be a morphism in the ∞-category $\mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}$. If f induces a surjection $\pi_1(T') \to \pi_1(T)$, then it is relatively monodromic. The converse holds if $\Lambda = 0$ is the trivial colattice (in which case $\mathcal{L}^{\Lambda}(\mathsf{X})$ can be identified with X).

Remark 6.5.4. Let X be an orbispace, let Λ be a colattice, and suppose we are given a composable pair of morphisms $T'' \xrightarrow{f} T' \xrightarrow{g} T$ in the ∞-category $\mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}$. If $(g \circ f)$ is relatively monodromic, then g and f are relatively monodromic. Conversely, if g is relatively monodromic and f induces a surjection of fundamental groups $\pi_1(T'') \to \pi_1(T)$, then $g \circ f$ is relatively monodromic.

Warning 6.5.5. In the situation of Definition 6.5.1, the collection of relatively monodromic morphisms is not necessarily closed under composition. However, one can show that it is closed under composition if the orbispace X is corporeal (see Remark 3.3.15).

Example 6.5.6. Let X be an orbispace and let Λ be a colattice. Then any morphism $f : T' \to T$ in the ∞-category $\mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}$ admits an essentially unique factorization as a composition $T' \xrightarrow{g} T_0 \xrightarrow{h} T$, where g induces an epimorphism of fundamental groups $\pi_1(T') \twoheadrightarrow \pi_1(T_0)$ and h induces a monomorphism of fundamental groups $\pi_1(T_0) \hookrightarrow \pi_1(T)$. It follows from Remark 6.5.4 that f is relatively monodromic if and only if h is relatively monodromic.

Remark 6.5.7. Let X be an orbispace and let Λ be a colattice. Using Remark 6.5.4 and Example 6.5.6, we see that every relatively monodromic morphism $T' \to T$ in $\mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}$ can be factored as a composition of relatively monodromic morphisms

$$T' \xrightarrow{f} T_0 \to T_1 \to \cdots \to T_n = T$$

where f induces a surjection $\pi_1(T') \twoheadrightarrow \pi_0(T_0)$, and each of the maps $T_i \to T_{i+1}$ exhibits T_i as a connected covering space of T_{i+1}, where the automorphism group $\mathrm{Aut}(T_i/T_{i+1})$ is cyclic of order p_i for some prime number p_i.

Definition 6.5.8. Let Λ be a colattice, let X be an orbispace, and let $\mathbf{G}_0$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. We will say that a $\mathbf{G}_0$-pretempered local system $\mathscr{G}$ on $\mathcal{L}^{\Lambda}(\mathsf{X})$ is *isotropic* if, for every relatively monodromic morphism $f : T' \to T$ in $\mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}$, the induced map

$$A^{T'}_{\mathbf{G}_0} \otimes_{A^{T}_{\mathbf{G}_0}} \mathscr{G}(T) \to \mathscr{G}(T')$$

is an equivalence.

Remark 6.5.9. In the situation of Definition 6.5.8, it suffices to verify that the map

$$A^{T'}_{\mathbf{G}_0} \otimes_{A^{T}_{\mathbf{G}_0}} \mathscr{G}(T) \to \mathscr{G}(T')$$

is an equivalence in the special case where $f : T' \to T$ is a relatively monodromic map which exhibits T' as a connected covering space of T whose automorphism group $\mathrm{Aut}(T'/T)$ is cyclic of prime order (see Remark 6.5.7).

Remark 6.5.10. Let Λ be a colattice, let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces, and let $\mathbf{G}_0$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Let $\mathscr{G}$ be a $\mathbf{G}_0$-pretempered local system on the formal loop space $\mathcal{L}^{\Lambda}(\mathsf{Y})$. If $\mathscr{G}$ is isotropic, then the pullback $\mathcal{L}^{\Lambda}(f)^*(\mathscr{G}) \in \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$ is isotropic.

Proposition 6.5.11. *Let Λ be a colattice, let X be an orbispace, and let $\mathbf{G}_0$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Let $\mathscr{G}$ be a $\mathbf{G}_0$-tempered local system on the formal loop space $\mathcal{L}^{\Lambda}(\mathsf{X})$. Then $\mathscr{G}$ is isotropic if and only if it satisfies the following condition:*

- $(*)$ *Let T be an object of $\mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}$ and let $T_0 \in \mathrm{Cov}(T)$ be a connected covering space of T such for which the map $T_0 \to T$ is relatively monodromic and $\mathrm{Aut}(T_0/T)$ is a cyclic group of order p, for some prime number p. Then $\mathrm{Aut}(T_0/T)$ acts trivially on the homotopy groups $\pi_*(\mathscr{G}(T_0)[1/p])$.*

Proof. Let T and $T_0 \in \mathrm{Cov}(T)$ be as in $(*)$, and let M denote the cofiber of the map

$$\theta : A_{\mathbf{G}_0}^{T_0} \otimes_{A_{\mathbf{G}_0}^{T}} \mathscr{G}(T) \to \mathscr{G}(T_0).$$

Since $\mathscr{G}$ is pretempered, the canonical map $M \to M[1/p]$ is an equivalence (Theorem 5.5.1). We therefore obtain another fiber sequence

$$A_{\mathbf{G}_0}^{T_0}[1/p] \otimes_{A_{\mathbf{G}_0}^{T}[1/p]} \to \mathscr{G}(T)[1/p] \to \mathscr{G}(T_0)[1/p] \to M.$$

Note that the group $\mathrm{Aut}(T_0/T)$ acts trivially on the first term (since $A_{\mathbf{G}}^{T_0}[1/p]$ is a direct factor of $A_{\mathbf{G}}^{T}[1/p]$), and that $\mathrm{Aut}(T_0/T)$ has no nonzero fixed points on $\pi_*(M)$ (Theorem 5.5.1). It follows that the induced map $\pi_*(\mathscr{G}(T_0)[1/p]) \to \pi_*(M)$ is an epimorphism, whose kernel is the subgroup of $\pi_*(\mathscr{G}(T_0)[1/p])$ which is fixed by the action of $\mathrm{Aut}(T_0/T)$. Consequently, the group $\mathrm{Aut}(T_0/T)$ acts trivially on $\pi_*(\mathscr{G}(T_0)[1/p])$ if and only if $M \simeq 0$: that is, if and only if θ is an equivalence. The desired result now follows from Remark 6.5.9. □

Corollary 6.5.12. *Let Λ be a colattice, let X be an orbispace, and let $\mathbf{G}_0$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Suppose that $\Lambda = \Lambda_{(p)}$ is p-nilpotent, for some prime number p. Then every p-nilpotent $\mathbf{G}_0$-tempered local system $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^\Lambda(\mathsf{X}))$ is isotropic.*

Proof. We verify condition $(*)$ of Proposition 6.5.11. Let T be an object of $\mathscr{T}_{/\mathcal{L}^\Lambda(\mathsf{X})}$ and let $T_0 \in \mathrm{Cov}(T)$ be a connected covering space of T such for which the map $T_0 \to T$ is relatively monodromic and $\mathrm{Aut}(T_0/T)$ is a cyclic group of order ℓ, for some prime number ℓ. Using the assumption that $\Lambda = \Lambda_{(p)}$, we deduce that $\ell = p$. Our assumption that $\mathscr{G}$ is p-nilpotent then guarantees that $\pi_*(\mathscr{G}(T_0)[1/p])$ vanishes, and therefore carries a trivial action of $\mathrm{Aut}(T_0/T)$. □

We can now state the main result of this section:

Theorem 6.5.13. *Let Λ be a colattice, let X be an orbispace, let $\mathbf{G}_0$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and set $\mathbf{G} = \mathbf{G}_0 \oplus \underline{\Lambda}$. Let $\mathscr{G}$ be a $\mathbf{G}_0$-pretempered local system on the formal loop space $\mathcal{L}^\Lambda(\mathsf{X})$. Then $\mathscr{G}$ is isotropic if and only if it belongs to the essential image of the functor*

$$\Phi^{\mathrm{pre}} : \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}_0}^{\mathrm{pre}}(\mathcal{L}^\Lambda(\mathsf{X}))$$

of Notation 6.4.14.

Variant 6.5.14. In the situation of Theorem 6.5.13, suppose that the $\mathbf{P}$-divisible group $\mathbf{G}$ is oriented. Then the functor

$$\Phi : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$$

is a fully faithful embedding, whose essential image consists of the isotropic $\mathbf{G}_0$-tempered local systems on $\mathcal{L}^{\Lambda}(\mathsf{X})$. This follows from Theorem 6.5.13: note that if a $\mathbf{G}_0$-tempered local system $\mathscr{G}$ on $\mathcal{L}(\mathsf{X})$ is isotropic, then it can be identified with $\Phi(\mathscr{F})$ (Theorem 6.5.13), where $\mathscr{F} = \Phi(\mathscr{G})$ is $\mathbf{G}$-tempered by virtue of Proposition 6.4.6.

Proof of Theorem 6.5.13. Assume first that we can write $\mathscr{G} = \Phi^{\mathrm{pre}}(\mathscr{F})$, for some $\mathbf{G}$-pretempered local system $\mathscr{F}$ on X. We wish to show that $\mathscr{G}$ is isotropic. Fix a relatively monodromic morphism $g : T' \to T$ in $\mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}$. Then the structural map $f : T \to \mathsf{X}$ is represented by a map $f_0 : T \times B\widehat{\Lambda}_0 \to \mathsf{X}$ which factors as a composition $T \times B\widehat{\Lambda}_0 \xrightarrow{f_0'} \overline{T} \to \mathsf{X}$ for some object $\overline{T} \in \mathscr{T}$ for which the composition $T' \times B\widehat{\Lambda}_0 \to T \times B\widehat{\Lambda}_0 \xrightarrow{f_0'} \overline{T}$ has connected homotopy fibers. We wish to show that the canonical map $\theta : A_{\mathbf{G}_0}^{T'} \otimes_{A_{\mathbf{G}_0}^{T}} \mathscr{G}(T) \to \mathscr{G}(T')$ is an equivalence. Let $\rho : \widehat{\Lambda} \twoheadrightarrow \widehat{\Lambda}_0$ denote the Pontryagin dual of the inclusion map. Using the description of Φ^{pre} supplied by Example 6.4.12, we see that θ can be obtained from a map

$$\overline{\theta} : A_{\mathbf{G}_0}^{T' \times B\widehat{\Lambda}_0} \otimes_{A_{\mathbf{G}_0}^{T \times B\widehat{\Lambda}_0}} \mathscr{F}(T \times B\widehat{\Lambda}_0)_{(0,\rho)} \to \mathscr{F}(T' \times B\widehat{\Lambda}_0)_{(0,\rho)}$$

by extending scalars along the augmentation map $\epsilon : A_{\mathbf{G}_0}^{B\widehat{\Lambda}_0} \to A$. It will therefore suffice to show that $\overline{\theta}$ is an equivalence. This follows from our assumption that $\mathscr{F}$ is $\mathbf{G}$-pretempered (and Remark 6.4.4), which allows us to identify both sides with the tensor product $A_{\mathbf{G}_0}^{T' \times B\widehat{\Lambda}_0} \otimes_{A_{\mathbf{G}_0}^{\overline{T}}} \mathscr{F}(\overline{T})_{\alpha}$; here α denotes the composite homomorphism

$$\widehat{\Lambda} \xrightarrow{\rho} \widehat{\Lambda_0} \hookrightarrow \pi_1(T \times B\widehat{\Lambda}_0) \xrightarrow{\pi_1(f_0')} \pi_1(\overline{T}).$$

We now prove the converse. Let $\mathscr{G}$ be a $\mathbf{G}_0$-pretempered local system on the formal loop space $\mathcal{L}^{\Lambda}(\mathsf{X})$ which is isotropic, in the sense of Definition 6.5.8. We wish to show that the counit map $v : \Phi^{\mathrm{pre}}(\Psi^{\mathrm{pre}}(\mathscr{G})) \to \mathscr{G}$ is an equivalence. Fix an object T' in $\mathscr{T}$ and a map of orbispaces $f' : T' \to \mathcal{L}^{\Lambda}(\mathsf{X})$, which we may assume is represented by $f_0' : T' \times B\widehat{\Lambda}_0 \to \mathsf{X}$ for some finite subgroup $\Lambda_0 \subseteq \Lambda$ (Remark 6.4.11). Let $\rho : \widehat{\Lambda} \twoheadrightarrow \widehat{\Lambda}_0$ denote the Pontryagin dual of the inclusion, and set $T = \mathcal{L}(T' \times B\widehat{\Lambda_0})_{(0,\rho)}$, so that the map f' factors as a composition $T' \xrightarrow{g} T \xrightarrow{f} \mathcal{L}^{\Lambda}(\mathsf{X})$ where f is given by the restriction of the map $\mathcal{L}(f_0') : \mathcal{L}^{\Lambda}(T' \times B\widehat{\Lambda}_0) \to \mathcal{L}^{\Lambda}(\mathsf{X})$. Using the descriptions of

Φ^{pre} and Ψ^{pre} supplied by Remark 6.4.5 and Example 6.4.12, we see that the map $\Phi^{\mathrm{pre}}(\Psi^{\mathrm{pre}}(\mathscr{G}))(T') \to \mathscr{G}(T')$ determined by v can be identified with the composition

$$\begin{aligned} A \otimes_{A_{\mathbf{G}_0}^{B\widehat{\Lambda}_0}} \mathscr{G}(\mathcal{L}^{\Lambda}(T' \times B\widehat{\Lambda}_0)_{(0,\rho)}) &\simeq A_{\mathbf{G}_0}^{T'} \otimes_{A_{\mathbf{G}_0}^{T}} \mathscr{G}(T) \\ &\xrightarrow{\gamma} \mathscr{G}(T') \end{aligned}$$

where γ is obtained by applying the functor $\mathscr{G}$ to the morphism $g : T' \to T$ in the ∞-category $\mathscr{T}_{/\mathcal{L}^{\Lambda}(\mathsf{X})}$. Note that the evaluation map $\mathrm{ev} : \mathcal{L}^{\Lambda}(T' \times B\widehat{\Lambda}_0)_{(0,\rho)} \times B\widehat{\Lambda} \to T' \times B\widehat{\Lambda}_0$ factors as a composition

$$\mathcal{L}^{\Lambda}(T' \times B\widehat{\Lambda}_0)_{(0,\rho)} \times B\widehat{\Lambda} \xrightarrow{\mathrm{id} \times B\rho} \mathcal{L}^{\Lambda}(T' \times B\widehat{\Lambda}_0)_{(0,\rho)} \times B\widehat{\Lambda}_0 \xrightarrow{e} T' \times B\widehat{\Lambda}_0,$$

so the map f is represented by the composition

$$T \times B\widehat{\Lambda}_0 = \mathcal{L}^{\Lambda}(T' \times B\widehat{\Lambda}_0)_{(0,\rho)} \times B\widehat{\Lambda}_0 \xrightarrow{e} T' \times B\widehat{\Lambda}_0 \xrightarrow{f_0'} \mathsf{X}.$$

The composition $T' \times B\widehat{\Lambda}_0 \xrightarrow{g \times \mathrm{id}} T \times B\widehat{\Lambda}_0 \xrightarrow{e} T' \times B\widehat{\Lambda}_0$ is homotopic to the identity map, and therefore has connected homotopy fibers. It follows that g is relatively monodromic, so that γ is an equivalence by virtue of our assumption that $\mathscr{G}$ is isotropic. □

Corollary 6.5.15. *Let Λ be a colattice, let $\mathbf{G}_0$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and set $\mathbf{G} = \mathbf{G}_0 \oplus \underline{\Lambda}$. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces, and let $\mathscr{G}$ be a $\mathbf{G}_0$-tempered local system on the formal loop space $\mathcal{L}^{\Lambda}(\mathsf{Y})$. If $\mathscr{G}$ belongs to the essential image of the functor $\Phi : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \hookrightarrow \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{Y}))$, then the pullback $\mathcal{L}^{\Lambda}(f)^*(\mathscr{G})$ belongs to the essential image of the functor $\Phi : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \hookrightarrow \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$.*

Proof. Combine Variant 6.5.14 with Remark 6.5.10. □

Corollary 6.5.16. *Let p be a prime number, let $\Lambda \simeq (\mathbf{Q}_p / \mathbf{Z}_p)^n$ be a p-nilpotent colattice, let X be an orbispace, let $\mathbf{G}_0$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and set $\mathbf{G} = \mathbf{G}_0 \oplus \underline{\Lambda}$. Then every p-nilpotent object of $\mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X}))$ belongs to the essential image of the embedding*

$$\Phi : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(\mathsf{X})).$$

Proof. Combine Variant 6.5.14 with Corollary 6.5.12. □

7 Ambidexterity for Tempered Local Systems

Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. In §5.2, we associated to each map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$ a pullback functor $f^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ (Remark 5.2.9). If $\mathscr{F}$ is a $\mathbf{G}$-local system on Y, then the pullback $f^* \mathscr{F}$ is given concretely by the formula

$$(f^* \mathscr{F})(T \xrightarrow{\eta} \mathsf{X}) = \mathscr{F}(T \xrightarrow{f \circ \eta} \mathsf{Y});$$

in particular, it preserves small limits and colimits (since these are computed levelwise). It follows from the adjoint functor theorem (Corollary HTT.5.5.2.9) that the pullback functor f^* admits both left and right adjoints.

Notation 7.0.1 (Direct Image Functors)**.** Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring and let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces. We let $f_! : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ denote a left adjoint to the pullback functor $f^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, and we let $f_* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ denote a right adjoint to the pullback functor $f^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$.

Our goal in this section is to prove Theorem 1.1.21, which asserts that if $f : X \to Y$ is a map of π-finite spaces, then there is a canonical equivalence $\mathrm{Nm}_f : f_! \simeq f_*$ which we call the *norm map* of f (see Theorem 7.2.10 for a more precise statement). In [6], we proved an analogous assertion for (ordinary) local systems with values in the ∞-category $\mathrm{Sp}_{K(n)}$ of $K(n)$-local spectra. Let us begin by recalling some of the main steps in the argument given in [6]:

(a) Let $f : X \to Y$ be a map of π-finite spaces, and let $f_!, f_* : \mathrm{LocSys}_{\mathrm{Sp}_{K(n)}}(X) \to \mathrm{LocSys}_{\mathrm{Sp}_{K(n)}}(Y)$ denote the left and right adjoint of the pullback functor $f^* : \mathrm{LocSys}_{\mathrm{Sp}_{K(n)}}(Y) \to \mathrm{LocSys}_{\mathrm{Sp}_{K(n)}}(X)$. Then there exists an integer $m \gg 0$ for which the homotopy fibers of f are m-truncated. The norm equivalence of [6] was constructed by a recursive procedure: more precisely, the norm map $\mathrm{Nm}_f : f_! \to f_*$ was constructed using the *inverse norm map* $\mathrm{Nm}_\delta^{-1} : \delta_* \to \delta_!$ associated to the relative diagonal $\delta : X \to X \times_Y X$ (which we can assume to have been previously constructed, since the homotopy fibers of δ are $(m-1)$-truncated). The difficulty is then to show that the map Nm_f is invertible.

(b) Let p denote the residue characteristic of the Morava K-theory $K(n)$. Then the ∞-category $\mathrm{Sp}_{K(n)}$ is p-local, in the sense that the multiplication $\ell : M \to M$ is an equivalence for every $K(n)$-local spectrum and every prime number $\ell \neq p$.

Combining this observation with formal arguments, we can reduce to the case where the spaces X and Y are connected and p-finite.

(c) Any map of connected p-finite spaces $f : X \to Y$ can be factored as a composition

$$X = X(0) \to X(1) \to \cdots \to X(t) = Y,$$

where each of the maps $X(i) \to X(i+1)$ is equivalent to a principal fibration whose fiber is an Eilenberg-MacLane space $K(\mathbf{F}_p, d)$. We can therefore assume without loss of generality that the map f fits into a fiber sequeence

$$X \xrightarrow{f} Y \to K(\mathbf{F}_p, d+1).$$

(d) Using the fact that the norm transformation $\mathrm{Nm}_f : f_! \to f_*$ can be computed fiberwise, one can reduce to the case where $Y = *$ consists of a single point, so that $X = K(\mathbf{F}_p, d)$ is an Eilenberg-MacLane space.

(e) Let $\underline{E} \in \mathrm{LocSys}_{\mathrm{Sp}_{K(n)}}(X)$ be the constant local system $\underline{E}$ associated to a Lubin-Tate spectrum E. In this case, the norm map $\mathrm{Nm}_f : f_!(\mathscr{F}) \to f_*(\mathscr{F})$ can be identified with a bilinear form $\mathrm{AForm}(f) : E[X] \otimes_E E[X] \to E$ on the E-module spectrum $E[X] = L_{K(n)}(E \otimes_S \Sigma^\infty_+(X))$. Using formal arguments, one can reduce to proving that this bilinear form is nondegenerate (that is, it exhibits $[X]$ as a self-dual object of the ∞-category of $K(n)$-local E-modules).

(f) In the case where $X = K(\mathbf{F}_p, d)$, the homotopy groups of $E[X]$ can be computed explicitly (by a mild extension fo the work of Ravenel-Wilson on the $K(n)$-homology of Eilenberg-MacLane spaces). In particular, one can show that $E[X]$ is a projective E-module of finite rank, and the nondegeneracy of the bilinear form b can be verified by an algebraic calculation.

Our proof of Theorem 1.1.21 will loosely follow the same approach. We begin in §7.1 by giving a concrete description of the direct image functor

$$f_* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$$

associated to a map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$. Using this description, we show that both the functors $f_!$ and f_* of Notation 7.0.1 satisfy a Beck-Chevalley condition for pullback diagrams of orbispaces (Theorem 7.1.6 and Corollary 7.1.7). In §7.2, we carry out an analogue of (a) by using the Beck-Chevalley construction to produce a norm map $\mathrm{Nm}_f : f_! \to f_*$, under the assumption that we have already constructed an invertible norm map $\mathrm{Nm}_\delta : \delta_! \simeq \delta_*$ for the relative diagonal $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ (see Notation 7.2.3).

Recall that, for any orbispace X, the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ can be equipped with a symmetric monoidal structure given by the *tempered tensor product* of studied in §5.8. In §7.3, we combine the results of §6 to show that the functor $f_! : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ always satisfies a projection formula with respect to the tempered tensor product (Theorem 7.3.1). Using this, we carry out an analogue of step (e): assuming that the norm transformation $\mathrm{Nm}_f : f_! \to f_*$ has been constructed, we show that it is an equivalence if and only if a certain map $\mathrm{AForm}(f) : [X/Y] \otimes [X/Y] \to \underline{A}_Y$ is a duality pairing in the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ (Proposition 7.3.15).

Choose a prime number p. In §7.5, we will prove that the norm map $\mathrm{Nm}_f : f_! \to f_*$ is an equivalence in the special case where $\mathsf{X} = X^{(-)}$ and $\mathsf{Y} = Y^{(-)}$ are representable by p-finite spaces X and Y, respectively (Theorem 7.5.1). In this case, we can proceed as in (c) to reduce to the case where the map f fits into a fiber sequence $X \xrightarrow{f} Y \to K(\mathbf{F}_p, d+1)$. The essential case is where $Y = *$ is a single point, so that $X = K(\mathbf{F}_p, d)$ is an Eilenberg-MacLane space. In this case, the calculations of §4 show that the tempered function spectrum $A^X_{\mathbf{G}}$ is a projective A-module of finite rank, which can be described explicitly in terms of the arithmetic of the p-divisible group $\mathbf{G}_{(p)}$ (Theorem 4.4.16). The map $\mathrm{AForm}(f) : [X/Y] \otimes [X/Y] \to \underline{A}_Y$ can then be identified with a bilinear form on the A-linear dual $(A^X_{\mathbf{G}})^\vee$, whose nondegeneracy can be verified by an explicit calculation as in (f); see Proposition 7.5.2. Beware that the analogue of step (d) is somewhat nontrivial in our case: a tempered local system $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(Y)$ is generally not determined by its restriction to the *points* of Y (the forgetful functor $\mathrm{LocSys}_{\mathbf{G}}(Y) \to \mathrm{LocSys}_A(Y)$ of Variant 5.1.15 is usually not conservative). Consequently, the reduction to the essential case $Y = *$ will require considerably more effort than the analogous reduction in [6].

In §7.7, we show that the norm map $\mathrm{Nm}_f : f_! \to f_*$ is an equivalence for a general map of π-finite spaces $f : X \to Y$. In the special case where $\mathbf{G} = \mathbf{G}_{(p)}$ is a p-divisible group for some fixed prime number p, this is a straightforward consequence of the analogous assertion for p-finite spaces. It is possible to reduce to the case $\mathbf{G} = \mathbf{G}_{(p)}$ by combining the categorified character theory of §6.4 with descent arguments (Proposition 6.2.6). However, we will adopt a different approach, which is instead based on tempered versions of the celebrated *induction theorems* of Artin and Brauer. In §7.4, we associate to a *transfer map* $\mathrm{tr}_{X/Y} : A^*_{\mathbf{G}}(X) \to A^*_{\mathbf{G}}(Y)$ to each map $f : X \to Y$ of π-finite spaces. In the special case where $A = \mathrm{KU}$ is the complex K-theory spectrum,

$\mathbf{G} = \mu_{\mathbf{P}^\infty}$ is the multiplicative $\mathbf{P}$-divisible group, and $f : BH \to BG$ is the covering map associated to an inclusion of finite groups $H \hookrightarrow G$, the map $\mathrm{tr}_{X/Y}$ recovers the induction homomorphism $\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ of representation theory (Proposition 7.6.7). The classical Artin (Brauer) induction theorem asserts that $\mathrm{Rep}(G)$ is generated rationally (integrally) by elements of the form $\mathrm{Ind}_H^G(x)$, where H is a cyclic (elementary) subgroup of G. In §7.6, we prove analogues of both of these theorems for the tempered cohomology theory of π-finite spaces (Theorems 7.6.3 and 7.6.5), which we apply in §7.7 to reduce the study of the norm map $\mathrm{Nm}_f : f_! \to f_*$ to the case where f is a map between nilpotent π-finite spaces (and therefore factors as a product of maps between p-finite spaces, for various primes p).

We conclude in §7.8 and §7.9 by describing some applications to the theory of tempered local systems on π-finite spaces:

- Let X be a π-finite space. Then the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(X)$ is compactly generated (Corollary 5.3.3). Moreover, an object $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(X)$ is compact if and only if it is dualizable (Proposition 7.8.8).

- Let $f : X \to Y$ be a map of π-finite spaces. Then the functors $f_! \simeq f_*$ and f^* carry compact objects to compact objects (Proposition 7.8.5).

- Let X be a π-finite space. Then $\mathrm{LocSys}_{\mathbf{G}}(X)$ is proper when viewed as an A-linear ∞-category. That is, for every pair of compact objects $\mathscr{F}, \mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(X)$, the mapping spectrum $\underline{\mathrm{Map}}(\mathscr{F}, \mathscr{G})$ is a perfect A-module (the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(X)$ is generally not smooth, but satisfies a weaker "p-adic smoothness" property for each prime number p: see Warning 7.9.12).

- Let X be a π-finite space and let $\mathscr{F}$ be a $\mathbf{G}$-tempered local system on X. If $\mathscr{F}$ is dualizable, then $\mathscr{F}(T)$ is a perfect A-module for each $T \in \mathscr{T}_{/\mathsf{X}}$ (Proposition 7.9.1). The converse holds if $\mathscr{F}$ is p-nilpotent, for any prime number p (Theorem 7.9.2).

- Let X and Y be π-finite spaces. Then external tensor product

  $$\boxtimes : \mathrm{LocSys}_{\mathbf{G}}(X) \times \mathrm{LocSys}_{\mathbf{G}}(Y) \to \mathrm{LocSys}_{\mathbf{G}}(X \times Y)$$

  induces fully faithful embedding of ∞-categories

  $$\lambda : \mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y) \hookrightarrow \mathrm{LocSys}_{\mathbf{G}}(X \times Y)$$

  (Corollary 7.8.12). If $\mathbf{G}$ is a p-divisible group, then the essential image of λ includes all p-nilpotent objects of $\mathrm{LocSys}_{\mathbf{G}}(X \times Y)$ (Proposition 7.8.13).

Warning 7.0.2. The exposition in this section has been arranged in a somewhat circular fashion:

- In §7.2, we define the notion of a $\upsilon_{\mathbf{G}}$-*ambidextrous* morphism of orbispaces $f : \mathsf{X} \to \mathsf{Y}$ (Notation 7.2.3) and state (but do not yet prove) that every truncated relatively π-finite morphism $f : \mathsf{X} \to \mathsf{Y}$ is $\upsilon_{\mathbf{G}}$-ambidextrous (Theorem 7.2.10).
- In §7.4, we associate a transfer map $\mathrm{tr}_{\mathsf{X}/\mathsf{Y}} : A^*_{\mathbf{G}}(\mathsf{X}) \to A^*_{\mathbf{G}}(\mathsf{Y})$ to each relatively π-finite morphism $f : \mathsf{X} \to \mathsf{Y}$. The definition of this transfer map $\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}$ depends on Theorem 7.2.10.
- In §7.7 we give a proof of Theorem 7.2.10 which exploits the existence of the transfer maps $\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}$ and their basic properties (Theorem 7.2.10 also depends on Theorem 7.5.1, which we will prove using transfer maps).

However, the circularity is only apparent: to prove Theorem 7.2.10 for an n-truncated morphism $f : \mathsf{X} \to \mathsf{Y}$, we will only make use of transfer maps $\mathrm{tr}_{\mathsf{X}'/\mathsf{Y}'}$ associated to $(n-1)$-truncated morphisms $f' : \mathsf{X}' \to \mathsf{Y}'$, which can be constructed assuming that Theorem 7.2.10 holds for $(n-1)$-truncated morphisms of orbispaces.

7.1 Direct Images of Tempered Local Systems

Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Our goal in this section is to give an explicit description of the direct image functor $f_* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ associated to a map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$.

Construction 7.1.1 (The Direct Image Functor)**.** Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces, so that composition with f determines a functor of ∞-categories $(\circ f) : \mathscr{T}_{/\mathsf{X}} \to \mathscr{T}_{/\mathsf{Y}}$. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $\mathscr{F}$ be an $\underline{A}_{\mathsf{X}}$-module object of $\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}, \mathrm{Sp})$ and let $q : \mathrm{Mod}(\mathrm{Sp}) \to \mathrm{CAlg}$ be the fibration of Construction 5.1.8, so that we can identify $\mathscr{F}$ with a functor $\mathscr{T}^{\mathrm{op}}_{/\mathsf{X}} \to \mathrm{Mod}(\mathrm{Sp})$ such that $q \circ \mathscr{F} = \underline{A}_{\mathsf{X}}$. We let $f_* \mathscr{F}$ denote a q-right Kan extension of $\mathscr{F}$ along the functor $(\circ f) : \mathscr{T}_{/\mathsf{X}} \to \mathscr{T}_{/\mathsf{Y}}$, which we view as an $\underline{A}_{\mathsf{Y}}$-module object of the ∞-category $\mathrm{Fun}(\mathscr{T}^{\mathrm{op}}_{/\mathsf{Y}}, \mathrm{Sp})$. We refer to $f_* \mathscr{F}$ as the *direct image of* $\mathscr{F}$ *along* f. Concretely, it is given by the formula

$$(f_* \mathscr{F})(T) = \varprojlim_{\overline{T} \in \mathscr{T}^{\mathrm{op}}_{/T \times_{\mathsf{Y}} \mathsf{X}}} \mathscr{F}(\overline{T}).$$

Note that the construction $\mathscr{F} \mapsto f_* \mathscr{F}$ determines a functor $f_* : \mathrm{Mod}_{\underline{A}_{\mathsf{X}}} \to \mathrm{Mod}_{\underline{A}_{\mathsf{Y}}}$, which is right adjoint to the restriction functor $f^* : \mathrm{Mod}_{\underline{A}_{\mathsf{Y}}} \to \mathrm{Mod}_{\underline{A}_{\mathsf{X}}}$.

Proposition 7.1.2. *Let $\mathbf{G}$ be a preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring and let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces. Then:*

1. *If $\mathscr{F}$ is a $\mathbf{G}$-pretempered local system on X, then the direct image $f_* \mathscr{F}$ is a $\mathbf{G}$-pretempered local system on Y.*
2. *If $\mathscr{F}$ is a $\mathbf{G}$-tempered local system on X, then the direct image $f_* \mathscr{F}$ is a $\mathbf{G}$-tempered local system on Y.*

Notation 7.1.3. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces. By virtue of Proposition 7.1.2, the direct image functor f_* of Construction 7.1.1 restricts to functors

$$\mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{Y}) \qquad \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}).$$

We will abuse notation by denoting both of these functors by f_*. Note that they are right adjoint to the pullback functors

$$f^* : \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}^{\mathrm{pre}}_{\mathbf{G}}(\mathsf{X}) \qquad f^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}).$$

of Remarks 5.1.6 and 5.2.9.

Example 7.1.4 (The Global Sections Functor)**.** Let $\mathbf{G}$ be an preoriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $\mathscr{F}$ be a $\mathbf{G}$-tempered local system on an orbispace X. We let $\Gamma(\mathsf{X}; \mathscr{F})$ denote the image of $\mathscr{F}$ under the functor

$$\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \xrightarrow{q_*} \mathrm{LocSys}_{\mathbf{G}}(*) \simeq \mathrm{Mod}_A,$$

where $q : \mathsf{X} \to *$ denotes the projection map from X to a point. We will refer to the construction $\mathscr{F} \mapsto \Gamma(\mathsf{X}; \mathscr{F})$ as the *tempered global sections functor*. Concretely, it is given by the formula

$$\Gamma(\mathsf{X}; \mathscr{F}) = \varprojlim_{T \in \mathscr{T}^{\mathrm{op}}_{/\mathsf{X}}} \mathscr{F}(T).$$

In the special case where $\mathsf{X} = X^{(-)}$ is the orbispace represented by a space X, we will denote the A-module $\Gamma(\mathsf{X}; \mathscr{F})$ by $\Gamma(X; \mathscr{F})$.

Example 7.1.5 (Tempered Cohomology)**.** Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. For every orbispace X, the tempered function spectrum $A^{\mathsf{X}}_{\mathbf{G}}$ can be identified with $\Gamma(\mathsf{X}; \underline{A}_{\mathsf{X}})$, where $\underline{A}_{\mathsf{X}}$ is the trivial $\mathbf{G}$-tempered local system of Notation 5.1.2.

Proof of Proposition 7.1.2. Suppose first that $\mathscr{F}$ is a $\mathbf{G}$-pretempered local system on X; we will show that the direct image $f_* \mathscr{F}$ is a $\mathbf{G}$-pretempered local system on Y. Fix a morphism $T' \to T$ in $\mathscr{T}_{/\mathsf{Y}}$ with connected homotopy fibers; we wish to show that the canonical map

$$\theta : A_{\mathbf{G}}^{T'} \otimes_{A_{\mathbf{G}}^{T}} (f_* \mathscr{F})(T) \to (f_* \mathscr{F})(T')$$

is an equivalence. Unwinding the definitions, we see that this map factors as a composition

$$\begin{aligned} A_{\mathbf{G}}^{T'} \otimes_{A_{\mathbf{G}}^{T}} \Big(\varprojlim_{\overline{T} \in \mathscr{T}^{\mathrm{op}}_{/T \times_{\mathsf{Y}} \mathsf{X}}} \mathscr{F}(\overline{T})\Big) &\xrightarrow{\sim} \varprojlim_{\overline{T} \in \mathscr{T}^{\mathrm{op}}_{/T \times_{\mathsf{Y}} \mathsf{X}}} \big(A_{\mathbf{G}}^{T'} \otimes_{A_{\mathbf{G}}^{T}} \mathscr{F}(\overline{T})\big) \\ &\xrightarrow{\sim} \varprojlim_{\overline{T} \in \mathscr{T}^{\mathrm{op}}_{/T \times_{\mathsf{Y}} \mathsf{X}}} \mathscr{F}(T' \times_T \overline{T}) \\ &\xleftarrow{\sim} \varprojlim_{\overline{T}' \in \mathscr{T}^{\mathrm{op}}_{/T' \times_{\mathsf{Y}} \mathsf{X}}} \mathscr{F}(\overline{T}') \end{aligned}$$

where the first map is an equivalence because $A_{\mathbf{G}}^{T'}$ is a projective $A_{\mathbf{G}}^{T}$-module of finite rank, the second is an equivalence by virtue of our assumption that $\mathscr{F}$ is $\mathbf{G}$-pretempered, and the third map is supplied by the left cofinality of the functor

$$\rho : \mathscr{T}_{/T \times_{\mathsf{Y}} \mathsf{X}} \to \mathscr{T}_{/T' \times_{\mathsf{Y}} \mathsf{X}} \qquad \overline{T} \mapsto T' \times_T \overline{T}$$

(which follows from the observation that ρ is right adjoint to the forgetful functor). This completes the proof of (1).

Now suppose that $\mathscr{F}$ is $\mathbf{G}$-tempered; we wish to show that the direct image $f_* \mathscr{F}$ is also $\mathbf{G}$-tempered. Choose an object $T \in \mathscr{T}_{/\mathsf{Y}}$ and a connected covering space $T_0 \in \mathrm{Cov}(T)$; we wish to show that the fiber of the canonical map $\alpha : (f_* \mathscr{F})(T) \to (f_* \mathscr{F})(T_0)^{\mathrm{hAut}(T_0/T)}$ is $I(T_0/T)$-local. Let $\mathcal{C}$ denote the ∞-category $\mathscr{T}_{/T \times_{\mathsf{Y}} \mathsf{X}}$, and let $\mathcal{D} \subseteq \mathcal{C}$ denote the full subcategory spanned by those objects $\overline{T}$ for which the map $\overline{T} \to T$ factors through T_0. Unwinding the definitions, we see that α can be identified with the restriction map

$$\varprojlim_{\overline{T} \in \mathcal{C}^{\mathrm{op}}} \mathscr{F}(\overline{T}) \to \varprojlim_{\overline{T} \in \mathcal{D}^{\mathrm{op}}} \mathscr{F}(\overline{T}),$$

and can therefore be written as a limit of maps

$$\alpha_{\overline{T}} : \mathscr{F}(\overline{T}) \to \varprojlim_{\overline{T}' \in \mathcal{D}^{\mathrm{op}}_{/\overline{T}}} \mathscr{F}(\overline{T}').$$

For each $\overline{T} \in \mathcal{C}$, let $\overline{T}_0$ denote any connected component of the fiber product $\overline{T} \times_T T_0$. Then $\alpha_{\overline{T}}$ coincides with the canonical map $\mathscr{F}(\overline{T}) \to \mathscr{F}(\overline{T}_0)^{\mathrm{hAut}(\overline{T}_0/\overline{T})}$. Note that the ideal $I(\overline{T}_0/\overline{T}) \subseteq A^0_{\mathbf{G}}(\overline{T})$ is generated by the image of the ideal $I(T_0/T) \subseteq A^0_{\mathbf{G}}(T)$.Our assumption that $\mathscr{F}$ is $\mathbf{G}$-tempered guarantees that the fiber $\mathrm{fib}(\alpha_{\overline{T}})$ is $I(\overline{T}_0/\overline{T})$-local when viewed as a module over $A^{\overline{T}}_{\mathbf{G}}$, hence also $I(T_0/T)$-local when viewed as a module over $A^T_{\mathbf{G}}$. Since the collection of $I(T_0/T)$-local $A^T_{\mathbf{G}}$-modules is closed under limits, it follows that $\mathrm{fib}(\alpha)$ is also $I(T_0/T)$-local. □

Theorem 7.1.6 (The Beck-Chevalley Condition)**.** *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group and let* σ :

$$\begin{tikzcd}
\mathsf{X}' \arrow[r, "f'"] \arrow[d, "g'"] & \mathsf{Y}' \arrow[d, "g"] \\
\mathsf{X} \arrow[r, "f"] & \mathsf{Y}
\end{tikzcd}$$

be a pullback diagram of orbispaces. Then the associated diagram of pullback functors

$$\begin{tikzcd}
\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \arrow[r, "f^*"] \arrow[d, "g^*"] & \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \arrow[d, "g'^*"] \\
\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}') \arrow[r, "f'^*"] & \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}')
\end{tikzcd}$$

is right adjointable. In other words, the canonical equivalence $f'^*g^* \simeq g'^*f^*$ *induces a natural transformation* $g^*f_* \to f'_*g'^*$ *which is also an equivalence.*

Proof. Let $\mathscr{F}$ be a $\mathbf{G}$-tempered local system on X; we wish to show that the Beck-Chevalley map $g^*f_*\mathscr{F} \to f'_*g'^*\mathscr{F}$ is an equivalence of $\mathbf{G}$-tempered local systems on Y'. This follows from the description of the direct image supplied by Construction 7.1.1: when evaluated on an object $T \in \mathscr{T}_{/\mathsf{Y}'}$, both sides can be identified with the limit $\varprojlim_{\overline{T}} \mathscr{F}(\overline{T})$, indexed by the opposite of the ∞-category $\mathscr{T}_{/T\times_{\mathsf{Y}'}\mathsf{X}'} \simeq \mathscr{T}_{/T\times_{\mathsf{Y}}\mathsf{X}}$. □

Passing to left adjoints (and exchanging the roles of X and Y'), we obtain the following formal consequence of Theorem 7.1.6:

Corollary 7.1.7. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group and let* σ :

$$\begin{tikzcd}
\mathsf{X}' \arrow[r, "f'"] \arrow[d, "g'"] & \mathsf{Y}' \arrow[d, "g"] \\
\mathsf{X} \arrow[r, "f"] & \mathsf{Y}
\end{tikzcd}$$

be a pullback diagram of orbispaces. Then the associated diagram of pullback functors

$$\begin{tikzcd}
\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \arrow[r, "f^*"] \arrow[d, "g^*"] & \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \arrow[d, "g'^*"] \\
\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}') \arrow[r, "f'^*"] & \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}')
\end{tikzcd}$$

is left adjointable: that is, the canonical equivalence $g'^* f^* \simeq f'^* g^*$ *induces a natural transformation* $f'_! g'^* \to g^* f_!$ *which is also an equivalence.*

7.2 The Tempered Ambidexterity Theorem

Let $\mathcal{X}$ be an ∞-category which admits pullbacks and let $\upsilon : \mathcal{C} \to \mathcal{X}$ be a functor of ∞-categories. Recall that υ is said to be a *Beck-Chevalley fibration* if the following conditions are satisfied (see Definition Ambi.4.1.3):

(1) The map υ is both a Cartesian fibration and a coCartesian fibration. In particular, every object $X \in \mathcal{X}$ determines an ∞-category $\mathcal{C}_X = \mathcal{C} \times_{\mathcal{X}} \{X\}$, and every morphism $f : X \to Y$ in $\mathcal{C}$ determines an adjunction

$$\begin{tikzcd}
\mathcal{C}_X \arrow[r, shift left, "f_!"] & \mathcal{C}_Y \arrow[l, shift left, "f^*"]
\end{tikzcd}.$$

(2) For every pullback square

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

in the ∞-category $\mathcal{X}$, the Beck-Chevalley transformation $f'_! g'^* \to g^* f_!$ is an equivalence of functors from $\mathcal{C}_{Y'}$ to $\mathcal{C}_X$.

Construction 7.2.1. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Then the construction $\mathsf{X} \mapsto \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ determines a functor of ∞-categories

$$\mathrm{LocSys}_{\mathbf{G}}(\bullet) : \mathcal{OS}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty.$$

We let $\upsilon_{\mathbf{G}} : \mathrm{TotSys}_{\mathbf{G}} \to \mathcal{OS}$ be a Cartesian fibration which is classified by the functor $\mathrm{LocSys}_{\mathbf{G}}$. The ∞-category $\mathrm{TotSys}_{\mathbf{G}}$ can be described more informally as follows:

- The objects of $\mathrm{LocSys}_{\mathbf{G}}$ are pairs $(\mathsf{X}, \mathscr{F})$, where X is an orbispace and $\mathscr{F}$ is a $\mathbf{G}$-tempered local system on X.

- A morphism from $(\mathsf{X}, \mathscr{F})$ to $(\mathsf{Y}, \mathscr{G})$ in $\mathrm{LocSys}_{\mathbf{G}}$ is given by a map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$ together with a map $\mathscr{F} \to f^* \mathscr{G}$ of $\mathbf{G}$-tempered local systems on X (or equivalently a map $f_! \mathscr{F} \to \mathscr{G}$ of $\mathbf{G}$-tempered local systems on Y).

Corollary 7.1.7 can now be restated as follows:

Proposition 7.2.2. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. Then the map* $\upsilon_{\mathbf{G}} : \mathrm{LocSys}_{\mathbf{G}} \to \mathcal{OS}$ *of Construction 7.2.1 is a Beck-Chevalley fibration.*

We now apply the general formalism of §Ambi.4.1 to the Beck-Chevalley fibration $\upsilon_{\mathbf{G}} : \mathrm{LocSys}_{\mathbf{G}} \to \mathcal{OS}$. For the reader's convenience, we include a brief summary.

Notation 7.2.3. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. For every map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$, we let

$$\phi_f : f_! \circ f^* \to \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})} \qquad \psi_f : \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})} \to f^* \circ f_!$$

denote a compatible counit and unit for the adjunction between the functors $f_! : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$.

Applying Construction Ambi.4.1.8 to the Beck-Chevalley fibration $\upsilon_{\mathbf{G}} : \mathrm{LocSys}_{\mathbf{G}} \to \mathcal{OS}$, we obtain the following data:

- A collection of orbispace maps $f : \mathsf{X} \to \mathsf{Y}$ which we refer to as *weakly* $\upsilon_{\mathbf{G}}$*-ambidextrous maps*, together with natural transformations $\nu_f : f^* \circ f_! \to \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}}(\mathsf{X})$ when f is weakly υ-ambidextrous.

- A smaller collection of orbispace maps $f : \mathsf{X} \to \mathsf{Y}$ which we refer to as $\upsilon_{\mathbf{G}}$*-ambidextrous maps*, for which ν_f is the counit of an adjunction (which exhibits $f_!$ as the *right* adjoint of f^*); in this case, we let $\mu_f : \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})} \to f_! \circ f^*$ denote a compatible unit for the adjunction.

This data is uniquely determined (up to homotopy) by the following requirements:

- Every equivalence of orbispaces $f : \mathsf{X} \to \mathsf{Y}$ is $\upsilon_{\mathbf{G}}$-ambidextrous. Moreover, the morphisms

  $$\mu_f : \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})} \to f_! \circ f^* \qquad \nu_f : f^* \circ f_! \to \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}$$

  are homotopy inverses to ϕ_f and ψ_f, respectively.

- A map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$ is weakly $\upsilon_{\mathbf{G}}$-ambidextrous if and only if the relative diagonal $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ is $\upsilon_{\mathbf{G}}$-ambidextrous. In this case, the natural transformation ν_f is given by the composition

  $$f^* f_! \simeq \pi_{0!} \pi_1^* \xrightarrow{\mu_\delta} \pi_{0!} \delta_! \delta^* \pi_1^* \simeq \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})} .$$

  Here the first map is the inverse of the Beck-Chevalley transformation associated to the pullback diagram

  $$\begin{tikzcd} \mathsf{X} \times_{\mathsf{Y}} \mathsf{X} \arrow[r, "\pi_0"] \arrow[d, "\pi_1"] & \mathsf{X} \arrow[d, "f"] \\ \mathsf{X} \arrow[r, "f"] & \mathsf{Y}. \end{tikzcd}$$

- A map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$ is $\upsilon_{\mathbf{G}}$-ambidextrous if and only if, for every pullback diagram

  $$\begin{tikzcd} \mathsf{X}' \arrow[r] \arrow[d, "f'"] & \mathsf{X} \arrow[d, "f"] \\ \mathsf{Y}' \arrow[r] & \mathsf{Y}, \end{tikzcd}$$

  the map f' is weakly $\upsilon_{\mathbf{G}}$-ambidextrous and the natural transformation $\nu_{f'} : f'^* \circ f'_! \to \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}}(\mathsf{X}')$ is the counit of an adjunction.

- Every $\upsilon_{\mathbf{G}}$-ambidextrous map $f : \mathsf{X} \to \mathsf{Y}$ is n-truncated for some $n \gg 0$ (so that the preceding properties supply a recursive algorithm for "computing" the natural transformations μ_f and ν_f).

If a map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$ is weakly υ-ambidextrous, then the natural transformation $\nu_f : f^* \circ f_! \to \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}}(\mathsf{X})$ can be identified with a natural transformation $\mathrm{Nm}_f : f_! \to f_*$ between the functors $f_!, f_* : \mathrm{LocSys}_{\mathbf{G}}(X) \to \mathrm{LocSys}_{\mathbf{G}}(Y)$. We will refer to Nm_f as the *norm map* associated to f. Note that ν_f is the counit of an adjunction if and only if the form map $\mathrm{Nm}_f : f_!(\mathscr{F}) \to f_*(\mathscr{F})$ is an equivalence, for every $\mathbf{G}$-tempered local system $\mathscr{F}$ on X.

We now describe a source of examples of $\upsilon_{\mathbf{G}}$-ambidextrous morphisms of orbispaces.

Definition 7.2.4. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces. We will say that f is *relatively π-finite* if, for every object $T \in \mathscr{T}_{/\mathsf{Y}}$, the orbispace $T^{(-)} \times_{\mathsf{Y}} \mathsf{X}$ is (representable by) a π-finite space.

Example 7.2.5. Every equivalence of orbispaces is relatively π-finite.

Remark 7.2.6. Suppose we are given a pullback diagram of orbispaces

$$\begin{tikzcd}
\mathsf{X}' \arrow[r] \arrow[d, "f'"] & \mathsf{X} \arrow[d, "f"] \\
\mathsf{Y}' \arrow[r] & \mathsf{Y}.
\end{tikzcd}$$

If f is relatively π-finite, then so is f'.

Proposition 7.2.7. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces, and suppose that $\mathsf{Y} = Y^{(-)}$ is representable by a π-finite space Y. Then f is relatively π-finite if and only if X is representable by a π-finite space X.*

Proof. Suppose first that X is representable by a π-finite space X. Then, for any object $T \in \mathscr{T}_{/\mathsf{Y}}$, the fiber product of orbispaces $T^{(-)} \times_{\mathsf{Y}} \mathsf{X}$ is representable by the π-finite space $T \times_Y X$, so f is relatively π-finite.

For the converse, suppose that f is relatively π-finite and let $X = |\mathsf{X}|$ be the underlying space of X (Notation 3.1.5). For each point $y \in Y$, the fiber $X \times_Y \{y\}$ underlies the orbispace $\mathsf{X} \times_{\mathsf{Y}} \{y\}$ and is therefore π-finite by virtue of our assumption on f. Since Y is π-finite, it follows that X is π-finite. We will complete the proof by showing that the canonical map $\mathsf{X} \to X^{(-)}$ is an equivalence of orbispaces. Let T be an object of $\mathscr{T}$; we wish to show that upper horizontal map in the diagram σ :

$$\begin{tikzcd}
\mathsf{X}^T \arrow[r] \arrow[d] & X^T \arrow[d] \\
\mathsf{Y}^T \arrow[r] & Y^T
\end{tikzcd}$$

is a homotopy equivalence. Since the lower horizontal map is a homotopy equivalence by assumption, it will suffice to show that σ is a pullback square. In other words, it will suffice to show that for every map of orbispaces $\eta : T^{(-)} \to \mathsf{Y}$, the diagram σ induces a homotopy equivalence $\mathsf{X}^T \times_{\mathsf{Y}^T} \{\eta\} \to X^T \times_{Y^T} \{\eta\}$. To prove this, we can replace f by the projection map $\mathsf{X} \times_{\mathsf{Y}} T^{(-)} \to T^{(-)}$, in which case the representability of X is automatic from our assumption that f is relatively π-finite. $\square$

Corollary 7.2.8. *Let $f : \mathsf{X} \to \mathsf{Y}$ and $g : \mathsf{Y} \to \mathsf{Z}$ be maps of orbispaces. If f and g are relatively π-finite, then $(g \circ f) : \mathsf{X} \to \mathsf{Z}$ is relatively π-finite.*

Proof. Fix an object $T \in \mathscr{T}_{/\mathsf{Z}}$. We wish to show that the fiber product $T^{(-)} \times_{\mathsf{Z}} \mathsf{X}$ is representable by a π-finite space. This follows by applying Proposition 7.2.7 to the map

$$f_T : T^{(-)} \times_{\mathsf{Z}} \mathsf{X} \to T^{(-)} \times_{\mathsf{Z}} \mathsf{Y};$$

note that f_T is relatively π-finite (by virtue of our assumption on f and Remark 7.2.6) and the codomain of f_T is representable by a π-finite space (by virtue of our assumption on g). $\square$

Remark 7.2.9. Let $f : \mathsf{X} \to \mathsf{Y}$ be a relatively π-finite map of orbispaces and let $n \geqslant -2$ be an integer. Then the following conditions are equivalent:

(a) The map f is n-truncated (as a morphism in the ∞-category $\mathcal{OS}$). In other words, for every orbispace Z, the induced map $\mathrm{Map}_{\mathcal{OS}}(\mathsf{Z}, \mathsf{X}) \to \mathrm{Map}_{\mathcal{OS}}(\mathsf{Z}, \mathsf{Y})$ has n-truncated homotopy fibers.

(b) For each object $T \in \mathscr{T}_{/\mathsf{Y}}$, the fiber product $T^{(-)} \times_{\mathsf{Y}} \mathsf{X}$ is representable by a π-finite space Z for which the projection map $Z \to T$ has n-truncated homotopy fibers.

(c) For each point $y \in |\mathsf{Y}|$, the fiber $\mathsf{X}_y = \{y\} \times_{\mathsf{Y}} \mathsf{X}$ is (repesentable by) an n-truncated π-finite space.

In particular, if the underlying space $|\mathsf{Y}|$ is connected, then f is n-truncated for some $n \gg 0$.

We can now state the main result of this paper; the proof will be given in §7.7.

Theorem 7.2.10 (Tempered Ambidexterity). *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* $f : \mathsf{X} \to \mathsf{Y}$ *be a map of orbispaces which is relatively* π*-finite and* n*-truncated for some* $n \gg 0$. *Then* f *is* $\upsilon_{\mathbf{G}}$*-ambidextrous.*

Remark 7.2.11. In the statement of Theorem 7.2.10, the requirement that f is relatively n-truncated is essentially a technicality. Any relatively π-finite map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$ can be realized as a coproduct of relatively π-finite maps $\{f_i : \mathsf{X}_i \to \mathsf{Y}_i\}_{i \in I}$, where the underlying spaces $|\mathsf{Y}_i|$ are connected. Then each f_i is relatively n-truncated for some integer n (which might depend on i), so Theorem 7.2.10 supplies norm equivalences $\mathrm{Nm}_{f_i} : f_{i!} \simeq f_{i*}$. Taking the product of these equivalences as i varies, we obtain an equivalence $\mathrm{Nm}_f : f_! \simeq f_*$ of functors $f_!, f_* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$.

7.3 Projection Formulas

Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. For every map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$, the pullback functor $f^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is symmetric monoidal with respect to the tempered tensor product (Proposition 5.8.13). In particular, for every pair of objects $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$, $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, we obtain a comparison map

$$\begin{aligned} f_!(f^* \mathscr{F} \otimes \mathscr{G}) &\rightarrow f_!(f^* \mathscr{F} \otimes f^* f_! \mathscr{G}) \\ &\simeq f_! f^*(\mathscr{F} \otimes f_! \mathscr{G}) \\ &\rightarrow \mathscr{F} \otimes f_! \mathscr{G}, \end{aligned}$$

which we will denote by $\beta_{\mathscr{F},\mathscr{G}}$ and refer to as the *projection morphism* from $f_!(f^* \mathscr{F} \otimes \mathscr{G})$ to $\mathscr{F} \otimes f_! \mathscr{G}$. The main result of this section can be stated as follows:

Theorem 7.3.1 (Projection Formula for $f_!$)**.** *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* $f : \mathsf{X} \to \mathsf{Y}$ *be a morphism of orbispaces. Then, for every pair of objects* $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ *and* $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$*, the projection morphism*

$$\beta_{\mathscr{F},\mathscr{G}} : f_!(f^* \mathscr{F} \otimes \mathscr{G}) \to \mathscr{F} \otimes f_! \mathscr{G}$$

is an equivalence in $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$*.*

We will give the proof of Theorem 7.3.1 at the end of this section. First, let us describe some of its consequences.

Construction 7.3.2. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces. We let $[\mathsf{X}/\mathsf{Y}]$ denote the $\mathbf{G}$-tempered local system $f_!(\underline{A}_{\mathsf{Y}}) \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$. In the special case where $\mathsf{X} = X^{(-)}$ and $\mathsf{Y} = Y^{(-)}$ are representable by spaces X and Y, we will denote $[\mathsf{X}/\mathsf{Y}]$ simply by $[X/Y]$.

For any $\mathbf{G}$-tempered local system $\mathscr{F}$ on Y, Theorem 7.3.1 supplies a canonical equivalence

$$(f_! \circ f^*)(\mathscr{F}) \simeq f_!(f^*(\mathscr{F}) \otimes \underline{A}_{\mathsf{X}}) \to \mathscr{F} \otimes f_!(\underline{A}_{\mathsf{X}}) = \mathscr{F} \otimes [\mathsf{X}/\mathsf{Y}].$$

Remark 7.3.3 (Compatibility with Pullback)**.** Every commutative diagram of orbispaces σ :

$$\begin{tikzcd} \mathsf{X}' \arrow[r, "g'"] \arrow[d, "f'"] & \mathsf{X} \arrow[d, "f"] \\ \mathsf{Y}' \arrow[r, "g"] & \mathsf{Y} \end{tikzcd}$$

determines a comparison map

$$[\mathsf{X}'/\mathsf{Y}'] = f'_! g'^*(\underline{A}_{\mathsf{X}}) \to g^* f_!(\underline{A}_{\mathsf{X}}) = g^*[\mathsf{X}/\mathsf{Y}].$$

This map is an equivalence when σ is a pullback square (Corollary 7.1.7).

Remark 7.3.4. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. For every orbispace X, we will abuse notation by identifying $[\mathsf{X}/*]$ with its image under the equivalence of ∞-categories $\mathrm{LocSys}_{\mathbf{G}}(*) \simeq \mathrm{Mod}_A$. Note that if B is an $\mathbb{E}_\infty$-algebra over A, then we have a canonical equivalence $\underline{\mathrm{Map}}_A([\mathsf{X}/*], B) \simeq B^{\mathsf{X}}_{\mathbf{G}}$. In particular, we can view $[\mathsf{X}/*]$ as an A-linear predual of the tempered function spectrum $A^{\mathsf{X}}_{\mathbf{G}}$. Concretely, one can show that it is given by the formula

$$[\mathsf{X}/*] = \varinjlim_{T \in \mathscr{T}_{/\mathsf{X}}} (A^T_{\mathbf{G}})^\vee,$$

where $(A^T_{\mathbf{G}})^\vee$ denotes the A-linear dual of the tempered function spectrum $A^T_{\mathbf{G}}$.

In the special case where $\mathsf{X} = K(\mathbf{F}_p, m)$ is (representable by) an Eilenberg-MacLane space, Theorem 4.4.16 guarantees that the functor $B \mapsto \underline{\mathrm{Map}}_A([\mathsf{X}/*], B) \simeq B^{\mathsf{X}}_{\mathbf{G}}$ commutes with filtered colimits. Restricting our attention to A-algebras of the form $B = A \oplus M$, we conclude that the functor $M \mapsto \underline{\mathrm{Map}}_{\mathrm{Mod}_A}([\mathsf{X}/*], M)$ also commutes with filtered colimits: that is, $[\mathsf{X}/*]$ is perfect as an A-module spectrum. In this case, the double duality map

$$[\mathsf{X}/*] \to [\mathsf{X}/*]^{\vee\vee} \simeq (A^{\mathsf{X}}_{\mathbf{G}})^\vee$$

is an equivalence: that is, $[\mathsf{X}/*]$ can be identified with the A-linear dual of the tempered function spectrum $A^{\mathsf{X}}_{\mathbf{G}}$. In particular, the homotopy groups $\pi_*[\mathsf{X}/*]$ can be identified with the $\mathbf{G}$-tempered homology groups $A^{\mathbf{G}}_*(\mathsf{X})$ of Notation 4.4.13.

For any map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$, the pullback functor $f^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ is symmetric monoidal (with respect to the tempered tensor products on both sides) and can therefore be regarded as a $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$-linear functor (where $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ acts on $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ via the functor f^*). It follows from Theorem 7.3.1 that the left adjoint $f_! : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ inherits the structure of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$-linear functor, Combining Theorem 7.3.1 with Remark HA.7.3.2.9 , we obtain the following:

Corollary 7.3.5. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* $f : \mathsf{X} \to \mathsf{Y}$ *be a map of orbispaces. Then the functor* $f_! : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ *can be regarded as a* $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$*-linear functor, and the unit and counit maps*

$$\phi_f : f_! f^* \to \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})} \qquad \psi_f : \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})} \to f^* f_!$$

can be regarded as $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$*-linear natural transformations.*

Remark 7.3.6. In the situation of Corollary 7.3.5, the functor $f_! f^*$ is a $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$-linear object of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$, and is therefore given by tensor product with the object $[\mathsf{X}/\mathsf{Y}] = f_! f^*(\underline{A}_{\mathsf{Y}})$ introduced in Construction 7.3.2. The $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$-linear natural transformation $\phi_f : f_! f^* \to \mathrm{id}$ can then be identified with a morphism $\epsilon : [\mathsf{X}/\mathsf{Y}] \to \underline{A}_{\mathsf{Y}}$, which is given by evaluating ϕ_f on the $\mathbf{G}$-tempered local system $\underline{A}_{\mathsf{Y}}$.

Combining Corollary 7.3.5 with a simple inductive argument, we obtain the following:

Corollary 7.3.7. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. Then:*

- (a) *If* $f : \mathsf{X} \to \mathsf{Y}$ *is a weakly* $\upsilon_{\mathbf{G}}$*-ambidextrous map of orbispaces, then* $\nu_f : f^* \circ f_! \to \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})}$ *has the structure of a* $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$*-linear natural transformation.*
- (b) *If* $f : \mathsf{X} \to \mathsf{Y}$ *is a* υ*-ambidextrous map of orbispaces, then* $\mu_f : \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})} \to f_! \circ f^*$ *has the structure of a* $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$*-linear natural transformation.*

Warning 7.3.8. The statement of Corollary 7.3.7 is somewhat imprecise: what we really mean (and will make use henceforth) is that the natural transformations ν_f and μ_f (when defined) have *canonical* promotions to $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$-linear natural transformations, which can be obtained by a suitable refinement of the ambidexterity constructions of §7.2.

Variant 7.3.9. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces. Then we can regard the pullback $f^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ as a symmetric monoidal functor from $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ to $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$. Then, for every pair of objects $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$, $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, we obtain a canonical map

$$\begin{aligned} \mathscr{F} \otimes f_*(\mathscr{G}) &\to f_* f^*(\mathscr{F} \otimes f_*(\mathscr{G})) \\ &\simeq f_*(f^*(\mathscr{F}) \otimes f^* f_*(\mathscr{G})) \\ &\to f_*(f^*(\mathscr{F}) \otimes \mathscr{G}). \end{aligned}$$

Theorem 7.3.10 (Projection Formula for f_*)**.** *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* $f : \mathsf{X} \to \mathsf{Y}$ *be a morphism of orbispaces which is* $\upsilon_{\mathbf{G}}$*-ambidextrous. Then, for every pair of objects* $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$*,* $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$*, the preceding construction induces an equivalence* $\mathscr{F} \otimes f_* \mathscr{G} \to f_*(f^* \mathscr{F} \otimes \mathscr{G})$ *of* $\mathbf{G}$*-tempered local systems on* Y*.*

Proof. Theorem 7.3.10 is equivalent to the assertion that f^* admits a $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$-linear right adjoint: that is, that we can find a $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$-linear functor $g : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ and together with compatible $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$-linear natural transformations

$$u : \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})} \to g \circ f^* \qquad v : f^* \circ g \to \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})} .$$

If f is $\upsilon_{\mathbf{G}}$-ambidextrous, this follows from Corollary 7.3.7 (we can take $g = f_!$, $u = \mu_f$, and $v = \nu_f$). □

Remark 7.3.11. It follows from Theorem 7.3.10 that if $f : \mathsf{X} \to \mathsf{Y}$ is $\upsilon_{\mathbf{G}}$-ambidextrous, then $f_* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ has the structure of a $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$-linear functor. The proof gives a more precise description of this struture: it is given by transporting the $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$-linearity of the functor $f_!$ (supplied by Theorem 7.3.1) along the norm equivalence $\mathrm{Nm}_f : f_! \xrightarrow{\sim} f_*$ supplied by our assumption that f is $\upsilon_{\mathbf{G}}$-ambidextrous.

Assuming Theorem 7.2.10, we can now deduce Theorem 4.7.1, which was stated without proof in §4.7.

Corollary 7.3.12. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* X *be a* π*-finite space. Then, for every map of* $\mathbb{E}_\infty$*-rings* $A \to B$*, the canonical map* $A^X_{\mathbf{G}} \to B^X_{\mathbf{G}}$ *extends to an equivalence* $\rho : B \otimes_A A^X_{\mathbf{G}} \to B^X_{\mathbf{G}}$ *of* $\mathbb{E}_\infty$*-algebras over* B*.*

Proof. It follows from Theorem 7.2.10 that the projection map $f : X \to *$ is $\upsilon_{\mathbf{G}}$-ambidextrous. The desired result now follows by applying the projection formula of Theorem 7.3.10 in the special case where $\mathscr{F} = B$ and $\mathscr{G} = \underline{A}_X$. □

Construction 7.3.13 (The Ambidexterity Form)**.** Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and let $f : \mathsf{X} \to \mathsf{Y}$ be a weakly $\upsilon_{\mathbf{G}}$-ambidextrous map of orbispaces. Then ν_f induces a $\mathrm{LocSys}_{\mathbf{G}}(Y)$-linear natural transformation

$$(f_! \circ f^*) \circ (f_! \circ f^*) = f_! \circ (f^* \circ f_!) \circ f^* \xrightarrow{\nu_f} f_! \circ \mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})} \circ f_* = f_! \circ f^*,$$

which we can identify with a map $m : [\mathsf{X}/\mathsf{Y}] \otimes [\mathsf{X}/\mathsf{Y}] \to [\mathsf{X}/\mathsf{Y}]$ of $\mathbf{G}$-tempered local systems on Y. We let $\mathrm{AForm}(f)$ denote the composition

$$[\mathsf{X}/\mathsf{Y}] \otimes [\mathsf{X}/\mathsf{Y}] \xrightarrow{m} [\mathsf{X}/\mathsf{Y}] \xrightarrow{\epsilon} \underline{A}_{\mathsf{Y}}.$$

We will refer to $\mathrm{AForm}(f)$ as the *ambidexterity form of* f.

Remark 7.3.14. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and suppose we are given a pullback diagram of orbispaces

$$\begin{tikzcd}
\mathsf{X}' \arrow[r] \arrow[d, "f'"] & \mathsf{X} \arrow[d, "f"] \\
\mathsf{Y}' \arrow[r, "g"] & \mathsf{Y}
\end{tikzcd}$$

where f and f' are weakly $\upsilon_{\mathbf{G}}$-ambidextrous. Then the ambidexterity form $\mathrm{AForm}(f') : [\mathsf{X}'/\mathsf{Y}'] \otimes [\mathsf{X}'/\mathsf{Y}'] \to \underline{A}_{\mathsf{Y}'}$ can be identified with the image under the pullback g^* of the ambidexterity form $\mathrm{AForm}(f) : [\mathsf{X}/\mathsf{Y}] \otimes [\mathsf{X}/\mathsf{Y}] \to \underline{A}_{\mathsf{Y}}$. In particular, if $\mathrm{AForm}(f)$ is a duality datum (in the symmetric monoidal ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$), then $\mathrm{AForm}(f')$ is a duality datum (in the symmetric monoidal ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}')$).

Proposition 7.3.15. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces which is weakly $\upsilon_{\mathbf{G}}$-ambidextrous. The following conditions are equivalent:*

- (a) *The map $\nu_f : f^* \circ f_! \to \mathrm{id}$ of Notation 7.2.3 is the counit of an adjunction.*
- (b) *For every object $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, the norm map $\mathrm{Nm}_f : f_!(\mathscr{F}) \to f_*(\mathscr{F})$ of Notation 7.2.3 is an equivalence.*
- (c) *The ambidexterity form $\mathrm{AForm}(f) : [\mathsf{X}/\mathsf{Y}] \otimes [\mathsf{X}/\mathsf{Y}] \to \underline{A}_{\mathsf{Y}}$ of Construction 7.3.13 is a duality datum: that is, it exhibits $[\mathsf{X}/\mathsf{Y}]$ as a self-dual object of the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$).*
- (d) *The map f is $\upsilon_{\mathbf{G}}$-ambidextrous.*

Proof. The equivalence $(a) \Leftrightarrow (b)$ is a tautology, and the equivalence $(a) \Leftrightarrow (c)$ follows from Proposition Ambi.5.1.8 . The implication $(d) \Rightarrow (a)$ is clear. The converse follows from the observation that if the morphism $f : \mathsf{X} \to \mathsf{Y}$ satisfies condition (c), then any pullback of f also satisfies condition (c) (Remark 7.3.14). □

Corollary 7.3.16. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces which is n-truncated for some $n \gg 0$. Then f is $\upsilon_{\mathbf{G}}$-ambidextrous if and only, for every $T \in \mathscr{T}_{/\mathsf{Y}}$, the pullback diagram of orbispaces*

$$\begin{tikzcd}
\mathsf{X}_T \arrow[r] \arrow[d, "f_T"] & \mathsf{X} \arrow[d, "f"] \\
T^{(-)} \arrow[r] & \mathsf{Y}
\end{tikzcd}$$

exhibits f_T as a $\upsilon_{\mathbf{G}}$-ambidextrous morphism of orbispaces.

Proof. The "only if" direction is clear, since the collection of $\upsilon_{\mathbf{G}}$-ambidextrous morphisms is closed under pullback. To prove the reverse direction, we proceed by induction on n. Using the inductive hypothesis, we can assume without loss of generality that f is weakly $\upsilon_{\mathbf{G}}$-ambidextrous. In this case, the desired result follows from Proposition 7.3.15 and Remark 7.3.14 (since $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ can be identified with the limit of the diagram of symmetric monoidal ∞-categories $\{\mathrm{LocSys}_{\mathbf{G}}(T)\}_{T \in \mathscr{T}^{\mathrm{op}}_{/\mathsf{Y}}}$). □

Proof of Theorem 7.3.1. Since a morphism of $\mathbf{G}$-tempered local systems is an equivalence if and only if it is an equivalence after localization at every prime, we may assume without loss of generality that the $\mathbb{E}_\infty$-ring A is p-local, for some prime number p. We will prove the following assertion:

$(*_n)$ For every oriented $\mathbf{P}$-divisible group over a p-local $\mathbb{E}_\infty$-ring A, every map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$, and every pair of objects $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ and $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, the projection morphism

$$\beta_{\mathscr{F},\mathscr{G}} : f_!(f^* \mathscr{F} \otimes \mathscr{G}) \to \mathscr{F} \otimes f_! \mathscr{G}$$

becomes an equivalence after $E(n)$-localization.

Note that Theorem 7.3.1 follows from $(*_n)$ for $n \gg 0$ (it suffices to take n to be any upper bound for the height of the p-divisible group $\mathbf{G}_{(p)}$). We will prove $(*_n)$ by induction on n. For the remainder of the proof, we regard n as fixed and assume that, if $n > 0$, then $(*_{n-1})$ holds. Note that, to prove that the projection map $\beta_{\mathscr{F},\mathscr{G}}$ is an equivalence, it will suffice to show that it becomes an equivalence after extending scalars to the localization $A_{\mathfrak{m}}$, for every maximal ideal $\mathfrak{m} \subseteq \pi_0(A)$. We may therefore assume that A is local. It follows that, for every prime number ℓ, the ℓ-divisible group $\mathbf{G}_{(\ell)}$ has some fixed height h_ℓ. By virtue of $(*_{n-1})$, it will suffice to show that $L_{K(n)}(\beta_{\mathscr{F},\mathscr{G}})$ is an equivalence in the ∞-category $\mathrm{LocSys}^{K(n)}_{\mathbf{G}}(\mathsf{Y})$. To prove this, we can extend scalars to the $K(n)$-localization $L_{K(n)}(A)$, and thereby reduce to the case where A is $K(n)$-local (beware that this replacement will generally injure our hypothesis that A is local). In this case, our hypothesis that $\mathbf{G}$ is oriented guarantees the existence of a connected-étale sequence of p-divisible groups

$$0 \to \mathbf{G}^{\mathcal{Q}}_A \xrightarrow{e} \mathbf{G}_{(p)} \to \mathbf{G}' \to 0,$$

where $\mathbf{G}^{\mathcal{Q}}_A$ denotes the Quillen p-divisible group of A and $\mathbf{G}'$ is an étale p-divisible group of height $h_p - n$ (Proposition 2.5.6). Set $\Lambda = (\mathbf{Q}_p / \mathbf{Z}_p)^{h_p - n} \oplus \bigoplus_{\ell \neq p} (\mathbf{Q}_\ell / \mathbf{Z}_\ell)^{h_\ell}$, and let $B = \mathrm{Split}_\Lambda(e)$ be the splitting algebra of the monomorphism e (Definition

2.7.12). Then B is faithfully flat over A (Proposition 2.7.15). It will therefore suffice to show that $\beta_{\mathscr{F},\mathscr{G}}$ becomes an equivalence after extending scalars from A to B. Replacing A by B (which might injure our hypothesis that A is $K(n)$-local), we are reduced to the problem of showing that $\beta_{\mathscr{F},\mathscr{G}}$ is an equivalence in the special case where $\mathbf{G}$ splits as a direct sum $\mathbf{G}_0 \oplus \underline{\Lambda}$, where $\mathbf{G}_0$ is a p-divisible group of height n. Replacing $\mathbf{G}$ by $\mathbf{G}_0$ (and f by the morphism $\mathcal{L}^{\Lambda}(f) : \mathcal{L}^{\Lambda}(\mathsf{X}) \to \mathcal{L}^{\Lambda}(\mathsf{X})$), we can reduce to the case where $\mathbf{G}$ is a p-divisible group of height n. Invoking our inductive hypothesis again, we are reduced to showing that $L_{K(n)}(\beta_{\mathscr{F},\mathscr{G}})$ is an equivalence in the ∞-category $\mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{Y})$. In this case, Theorem 6.3.1 supplies equivalences

$$\mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{X}) \simeq \mathrm{Fun}(|\mathsf{X}|, \mathrm{Mod}_A^{K(n)}) \qquad \mathrm{LocSys}_{\mathbf{G}}^{K(n)}(\mathsf{Y}) \simeq \mathrm{Fun}(|\mathsf{Y}|, \mathrm{Mod}_A^{K(n)}).$$

Under these equivalences, the pullback functor f^* can be identified with the functor $U : \mathrm{Fun}(|\mathsf{Y}|, \mathrm{Mod}_A^{K(n)}) \to \mathrm{Fun}(|\mathsf{X}|, \mathrm{Mod}_A^{K(n)})$ induced by composition with $|f| : |\mathsf{X}| \to |\mathsf{Y}|$, and the functor $L_{K(n)} f_!$ with its left adjoint $V : \mathrm{Fun}(|\mathsf{X}|, \mathrm{Mod}_A^{K(n)}) \to \mathrm{Fun}(|\mathsf{Y}|, \mathrm{Mod}_A^{K(n)})$, given by left Kan extension along the map of spaces $|\mathsf{X}| \to |\mathsf{Y}|$. Let $\mathscr{F}'$ and $\mathscr{G}'$ be the images of $L_{K(n)}\,\mathscr{F}$ and $L_{K(n)}\,\mathscr{G}$ in the ∞-categories $\mathrm{Fun}(|\mathsf{Y}|, \mathrm{Mod}_A^{K(n)})$ and $\mathrm{Fun}(|\mathsf{X}|, \mathrm{Mod}_A^{K(n)})$, respectively. Then the evaluation of $L_{K(n)}\beta_{\mathscr{F},\mathscr{G}}$ at a point $y \in |\mathsf{Y}|$ can be identified with the natural map

$$\varinjlim_{x \in |\mathsf{X}|_y} (\mathscr{F}'(y) \widehat{\otimes} \mathscr{G}'(x)) \to \mathscr{F}'(y) \widehat{\otimes} \varinjlim_{x \in |\mathsf{X}|_y} \mathscr{G}'(x),$$

where $|\mathsf{X}|_y$ denotes the homotopy fiber of the map $|\mathsf{X}| \to |\mathsf{Y}|$ over the point y. Since the tensor product $\widehat{\otimes}$ on $\mathrm{Mod}_A^{K(n)}$ preserves small colimits in each variable, we conclude that $L_{K(n)}\beta_{\mathscr{F},\mathscr{G}}$ is an equivalence. □

7.4 Transfer Maps in Tempered Cohomology

For every preoriented $\mathbf{P}$-divisible group $\mathbf{G}$ over an $\mathbb{E}_\infty$-ring A, the formation of tempered cohomology groups $A^*_{\mathbf{G}}(\bullet)$ of Construction 4.0.5 can be regarded as a contravariant functor from (the homotopy category of) the category of orbispaces to the category of graded-commutative rings. However, when $\mathbf{G}$ is oriented, then Theorem 7.2.10 supplies a much richer structure: tempered cohomology is also *covariantly* functorial for relatively π-finite maps of orbispaces with π-finite fibers.

Construction 7.4.1 (The Transfer Map)**.** Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of orbispaces which is relatively

π-finite (Definition 7.2.4) and let $\mathrm{Nm}_f : f_! \xrightarrow{\sim} f_*$ be the isomorphism of Notation 7.2.3 (see Remark 7.2.11). We let $\mathrm{tr}_{\mathsf{X}/\mathsf{Y}} : A^{\mathsf{X}}_{\mathbf{G}} \to A^{\mathsf{Y}}_{\mathbf{G}}$ be the map given by the composition

$$A^{\mathsf{X}}_{\mathbf{G}} = \Gamma(\mathsf{X}; \underline{A}_{\mathsf{X}}) \simeq \Gamma(Y; f_* \underline{A}_{\mathsf{X}}) \xrightarrow{\mathrm{Nm}_f^{-1}} \Gamma(Y; f_! \underline{A}_{\mathsf{X}}) \to \Gamma(Y; \underline{A}_{\mathsf{Y}}) = A^{\mathsf{Y}}_{\mathbf{G}}.$$

We will refer to $\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}$ as the *transfer map associated to f*. Passing to homotopy groups, we obtain a map of tempered cohomology groups $A^*_{\mathbf{G}}(\mathsf{X}) \to A^*_{\mathbf{G}}(\mathsf{Y})$, which we will also denote by $\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}$ and refer to as the *transfer map*.

In the special case where $\mathsf{X} = X^{(-)}$ and $\mathsf{Y} = Y^{(-)}$ are represented by spaces X and Y, respectively, we will denote the transfer map $\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}$ by $\mathrm{tr}_{X/Y}$.

Warning 7.4.2. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of orbispaces which is relatively π-finite. The transfer map $\mathrm{tr}_{\mathsf{X}/\mathsf{Y}} : A^{\mathsf{X}}_{\mathbf{G}} \to A^{\mathsf{Y}}_{\mathbf{G}}$ is not a morphism of ring spectra. However, it is a morphism of $A^{\mathsf{Y}}_{\mathbf{G}}$. In particular, at the level of tempered cohomology rings, we have the projection formula

$$\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}((f^* u) \cdot v) = u \cdot \mathrm{tr}_{\mathsf{X}/\mathsf{Y}}(v).$$

Warning 7.4.3. To define the transfer maps of Construction 7.4.1 in complete generality, we need the full strength of Theorem 7.2.10, which asserts that every truncated relatively π-finite morphism of orbispaces is ambidextrous with respect to the Beck-Chevalley fibration $v : \mathrm{TotSys}_{\mathbf{G}} \to \mathcal{OS}$ of §7.2. However, to construct the transfer map $\mathrm{tr}_{\mathsf{X}/\mathsf{Y}} : A^{\mathsf{X}}_{\mathbf{G}} \to A^{\mathsf{Y}}_{\mathbf{G}}$ for a *particular* map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$, we only need to know that f is v-ambidextrous. Our proof of Theorem 7.2.10 will make use of this observation: to show that every n-truncated relatively π-finite morphism $f : \mathsf{X} \to \mathsf{Y}$ is v-ambidextrous, we will use transfer maps associated to $(n-1)$-truncated relatively π-finite morphisms of orbispaces.

We now summarize some of the basic formal properties of Construction 7.4.1.

Proposition 7.4.4 (Push-Pull). *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Suppose we are given a pullback diagram of orbispaces*

$$\begin{tikzcd}
\mathsf{X}' \arrow[r, "f'"] \arrow[d, "g'"] & \mathsf{Y}' \arrow[d, "g"] \\
\mathsf{X} \arrow[r, "f"] & \mathsf{Y},
\end{tikzcd}$$

where the map f (and therefore f') is relatively π-finite. Then the diagram of tempered function spectra

$$\begin{tikzcd} A_{\mathbf{G}}^{\mathsf{X}'} \arrow[r, "\mathrm{tr}_{\mathsf{X}'/\mathsf{Y}'}"] & A_{\mathbf{G}}^{\mathsf{Y}'} \\ A_{\mathbf{G}}^{\mathsf{X}} \arrow[u, "g'^*"] \arrow[r, "\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}"] & A_{\mathbf{G}}^{\mathsf{Y}} \arrow[u, "g^*"'] \end{tikzcd}$$

commutes up to homotopy. In particular, the diagram of graded abelian groups

$$\begin{tikzcd} A_{\mathbf{G}}^{*}(\mathsf{X}') \arrow[r, "\mathrm{tr}_{\mathsf{X}'/\mathsf{Y}'}"] & A_{\mathbf{G}}^{*}(\mathsf{Y}') \\ A_{\mathbf{G}}^{*}(\mathsf{X}) \arrow[u, "g'^*"] \arrow[r, "\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}"] & A_{\mathbf{G}}^{*}(\mathsf{Y}) \arrow[u, "g^*"'] \end{tikzcd}$$

Proof. Decomposing Y as a disjoint union if necessary, we can assume that f is truncated. In this case, the desired result follows from the compatibility of norm maps with pullback (Remark Ambi.4.2.3). □

Proposition 7.4.5. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and suppose we are given relatively π-finite morphisms of orbispaces $\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z}$. Then the transfer map $\mathrm{tr}_{\mathsf{X}/\mathsf{Z}} : A_{\mathbf{G}}^{\mathsf{X}} \to A_{\mathbf{G}}^{\mathsf{Z}}$ is homotopic to the composition $\mathrm{tr}_{\mathsf{Y}/\mathsf{Z}} \circ \mathrm{tr}_{\mathsf{X}/\mathsf{Y}}$. In particular, we have a commutative diagram of graded abelian groups*

$$\begin{tikzcd} & A_{\mathbf{G}}^{*}(\mathsf{Y}) \arrow[dr, "\mathrm{tr}_{\mathsf{Y}/\mathsf{Z}}"] & \\ A_{\mathbf{G}}^{*}(\mathsf{X}) \arrow[ur, "\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}"] \arrow[rr, "\mathrm{tr}_{\mathsf{X}/\mathsf{Z}}"] & & A_{\mathbf{G}}^{*}(\mathsf{Z}). \end{tikzcd}$$

Proof. Decomposing Z into connected components if necessary, we may assume that f and g are truncated. In this case, the desired result follows from the compatibility of norm maps with composition (Remark Ambi.4.2.4). □

Remark 7.4.6 (Functoriality for Correspondences)**.** Define a category $\mathcal{C}$ as follows:

- The objects of $\mathcal{C}$ are orbispaces.
- For orbispaces X and Y, $\mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y})$ is the set of equivalence classes of diagrams

  $$\mathsf{X} \leftarrow \mathsf{M} \xrightarrow{f} \mathsf{Y},$$

  where f is relatively π-finite.

- Given morphisms $\mathsf{X} \leftarrow \mathsf{M} \rightarrow \mathsf{Y}$ and $\mathsf{Y} \leftarrow \mathsf{N} \rightarrow \mathsf{Z}$ of $\mathcal{C}$, their composition is given by (the equivalence class of) the diagram
$$\mathsf{X} \leftarrow \mathsf{M} \times_{\mathsf{Y}} \mathsf{N} \rightarrow \mathsf{Z}.$$

Every oriented $\mathbf{P}$-divisible group $\mathbf{G}$ over an $\mathbb{E}_\infty$-ring A then determines a functor $\mathcal{C} \rightarrow \mathrm{hMod}_A$, which carries each orbispace X to the tempered function spectrum $A^{\mathsf{X}}_{\mathbf{G}}$, and each correspondence $\mathsf{X} \xleftarrow{f} \mathsf{M} \xrightarrow{g} \mathsf{Y}$ to the composite map

$$A^{\mathsf{X}}_{\mathbf{G}} \xrightarrow{f^*} A^{\mathsf{M}}_{\mathbf{G}} \xrightarrow{\mathrm{tr}_{\mathsf{M}/\mathsf{Y}}} A^{\mathsf{Y}}_{\mathbf{G}}.$$

The compatibility of this construction with composition is precisely the content of Propositions 7.4.4 and 7.4.5. In particular, the construction $\mathsf{X} \mapsto A^*_{\mathbf{G}}(\mathsf{X})$ determines a functor from $\mathcal{C}$ to the category of graded abelian groups (or graded modules over $\pi_{-*}(A)$).

Remark 7.4.7. The category $\mathcal{C}$ appearing in Remark 7.4.6 can be identified with the homotopy category of an ∞-category $\overline{\mathcal{C}}$ (where the morphism spaces $\mathrm{Map}_{\overline{\mathcal{C}}}(\mathsf{X}, \mathsf{Y})$ can be identified with the summand of the Kan complex $\mathcal{OS}^{\simeq}_{/\mathsf{X}\times\mathsf{Y}}$ spanned by those diagrams $\mathsf{X} \leftarrow \mathsf{M} \xrightarrow{g} \mathsf{Y}$ where g is relatively π-finite. Using a more elaborate version of the ambidexterity formalism of §Ambi.4 , one can upgrade Remark 7.4.6 to obtain a functor of ∞-categories $A^{\bullet}_{\mathbf{G}} : \overline{\mathcal{C}} \rightarrow \mathrm{Mod}_A$. We will return to this point in a future work.

Remark 7.4.8 (Change of Ring)**.** Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $f : \mathsf{X} \rightarrow \mathsf{Y}$ be a relatively π-finite morphism of orbispaces. Then, for every $\mathbb{E}_\infty$-algebra B over A, the diagram

$$\begin{tikzcd} A^{\mathsf{X}}_{\mathbf{G}} \arrow[r, "\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}"] \arrow[d] & A^{\mathsf{Y}}_{\mathbf{G}} \arrow[d] \\ B^{\mathsf{X}}_{\mathbf{G}} \arrow[r, "\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}"] & B^{\mathsf{Y}}_{\mathbf{G}} \end{tikzcd}$$

commutes (up to homotopy) in the ∞-category of $A^{Y}_{\mathbf{G}}$-modules. In particular, we obtain a commutative diagram of graded abelian groups

$$\begin{tikzcd} A^{*}_{\mathbf{G}}(\mathsf{X}) \arrow[r, "\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}"] \arrow[d] & A^{*}_{\mathbf{G}}(\mathsf{Y}) \arrow[d] \\ B^{*}_{\mathbf{G}}(\mathsf{X}) \arrow[r, "\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}"] & B^{*}_{\mathbf{G}}(\mathsf{Y}) \end{tikzcd}$$

Remark 7.4.9 (Compatibility with Character Maps)**.** Let $\mathbf{G}_0$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let Λ be a colattice, and let $\mathbf{G} = \mathbf{G}_0 \oplus \underline{\Lambda}$ (which we also regard as an oriented $\mathbf{P}$-divisible group over A). Then, for any map of orbispaces $f : \mathsf{X} \to \mathsf{Y}$, the diagram of A-modules

$$\begin{tikzcd} A_{\mathbf{G}}^{\mathsf{X}} \arrow[r, "\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}"] \arrow[d, "\sim"', "\chi"] & A_{\mathbf{G}}^{\mathsf{Y}} \arrow[d, "\sim"', "\chi"] \\ A_{\mathbf{G}_0}^{\mathcal{L}^\Lambda(\mathsf{X})} \arrow[r, "\mathrm{tr}_{\mathcal{L}^\Lambda(\mathsf{X})/\mathcal{L}^\Lambda(\mathsf{Y})}"] & A_{\mathbf{G}_0}^{\mathcal{L}^\Lambda(\mathsf{Y})} \end{tikzcd}$$

commutes (up to homotopy), where the horizontal maps are the transfer morphisms of Construction 7.4.1, and the vertical maps are the character equivalences of Notation 4.3.3. In particular, we have a commutative diagram of graded abelian groups

$$\begin{tikzcd} A_{\mathbf{G}}^*(\mathsf{X}) \arrow[r, "\mathrm{tr}_{\mathsf{X}/\mathsf{Y}}"] \arrow[d, "\sim"', "\chi"] & A_{\mathbf{G}}^*(\mathsf{Y}) \arrow[d, "\sim"', "\chi"] \\ A_{\mathbf{G}_0}^*(\mathcal{L}^\Lambda(\mathsf{X})) \arrow[r, "\mathrm{tr}_{\mathcal{L}^\Lambda(\mathsf{X})/\mathcal{L}^\Lambda(\mathsf{Y})}"] & A_{\mathbf{G}_0}^*(\mathcal{L}^\Lambda(\mathsf{Y})). \end{tikzcd}$$

We now describe the behavior of transfers in the "rational" case.

Definition 7.4.10. For every π-finite space X define the *mass* of X to be the rational number

$$\mathrm{Mass}(X) = \sum_{x} \prod_{n>0} |\pi_n(X,x)|^{(-1)^n}.$$

where the sum is taken over a set of representatives for the set $\pi_0(X)$ of connected components of X. Note that if S is a set of prime numbers and X is S-finite (Definition 1.1.25), then $\mathrm{Mass}(X)$ belongs to the subring $\mathbf{Z}[S^{-1}] \subseteq \mathbf{Q}$.

Proposition 7.4.11. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. Let* S *be a set of prime numbers with the property that, for each* $p \in S$*, the* p*-divisible group* $\mathbf{G}_{(p)}$ *vanishes (so that* p *is invertible in* $\pi_0(A)$*, by virtue of Remark 2.6.14). Then, for every connected connected* S*-finite space* X*, the unit map* $A \to A_{\mathbf{G}}^X$ *is an equivalence.*

Proof. Let $\mathscr{T}_{(S)}$ denote the full subcategory of $\mathscr{T}$ spanned by those objects of the form BH, where every prime divisor of H belongs to S. For each $T \in \mathscr{T}_{(S)}$, the unit map $A \to A^T_{\mathbf{G}}$ is an equivalence (since $\mathbf{G}_{(p)}$ vanishes for $p \in S$) and the unit map $A \to A^T$ is an equivalence (since every element of S is invertible in $\pi_0(A)$). It follows that for $T \in \mathscr{T}_{(S)}$, the Atiyah-Segal comparison map $A^T_{\mathbf{G}} \to A^T$ is an equivalence. Let $\mathcal{C}$ be the full subcategory of $\mathcal{OS}$ spanned by those orbispaces X for which the Atiyah-Segal comparison map $A^{\mathsf{X}}_{\mathbf{G}} \to A^{|\mathsf{X}|}$. Then $\mathcal{C}$ contains $\mathscr{T}_{(S)}$ and is closed under small colimits, and therefore contains every orbispace $\mathsf{X} : \mathscr{T}^{\mathrm{op}} \to \mathcal{S}$ which is a left Kan extension of its restriction to $\mathscr{T}^{\mathrm{op}}_{(S)}$. It follows that $\mathcal{C}$ contains the representable orbispace $X^{(-)}$ whenever X is S-finite. We are therefore reduced to showing that the unit map $A \to A^X$ is an equivalence, which is clear (since every element of S is invertible in $\pi_0(A)$). □

Proposition 7.4.12. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*. Let* S *be a set of prime numbers with the property that, for each* $p \in S$*, the* p*-divisible group* $\mathbf{G}_{(p)}$ *vanishes. Let* $f : X \to Y$ *be a map of* S*-finite spaces, which decompose into connected components*

$$X = \coprod_{i \in I} X_i \qquad Y = \coprod_{j \in J} Y_j.$$

Then the diagram

$$\begin{tikzcd}
\prod_{i \in I} A \arrow[r, "M"] \arrow[d, "\sim"] & \prod_{j \in J} A \arrow[d, "\sim"] \\
A^X_{\mathbf{G}} \arrow[r, "\mathrm{tr}_{X/Y}"] & A^Y_{\mathbf{G}}
\end{tikzcd}$$

commutes, where the vertical maps are the equivalences supplied by Proposition 7.4.11 and M *is given by the matrix of rational numbers*

$$M_{ij} = \mathrm{Mass}(\mathrm{fib}(X_i \to Y_j)) = \frac{\mathrm{Mass}(X_i)}{\mathrm{Mass}(Y_j)}.$$

Remark 7.4.13. In the situation of Proposition 7.4.12, each of the prime numbers $p \in S$ is invertible in the commutative ring $\pi_0(A)$, so we can view the rational numbers $M_{ij} \in \mathbf{Z}[S^{-1}]$ as elements of $\pi_0(A)$.

Proof of Proposition 7.4.12. Choose an integer n such that the homotopy fibers of f are n-truncated. We proceed by induction on n. Using Proposition 7.4.4, we can reduce to the case where $Y = \{y\}$ consists of a single point. We may also assume without loss of generality that X is connected. If $n = 0$, then f is a homotopy equivalence and there is nothing to prove. Assume therefore that $n > 0$ and choose a point $x \in X$, so that the inclusion $\{x\} \hookrightarrow X$ has $(n-1)$-truncated homotopy fibers. We then have a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "N"] & A \arrow[r, "M"] \arrow[d] & A \arrow[d] \\
A_{\mathbf{G}}^{\{x\}} \arrow[r, "\mathrm{tr}_{\{x\}/X}"] & A_{\mathbf{G}}^{X} \arrow[r, "\mathrm{tr}_{X/\{y\}}"] & A_{\mathbf{G}}^{\{}y\},
\end{tikzcd}$$

where the vertical maps are the unit morphisms (which are equivalences by virtue of Proposition 7.4.11), for some elements M and N of the commutative ring $\pi_0(A)$. The commutativity of the diagram shows that $M \cdot N = 1$, and our inductive hypothesis implies that $N = \mathrm{Mass}(\Omega(X))$. It follows that

$$M = \frac{1}{N} = \frac{1}{\mathrm{Mass}(\Omega(X))} = \mathrm{Mass}(X).$$

□

By combining Remarks 7.4.8, 7.4.9, and Proposition 7.4.12, we obtain (at least in principle) a complete recipe for computing the *rationalized* transfer map

$$\mathrm{tr}_{X/Y} : \mathbf{Q} \otimes A_{\mathbf{G}}^{*}(X) \to \mathbf{Q} \otimes A_{\mathbf{G}}^{*}(Y),$$

where X and Y are π-finite spaces. Using Remark 7.4.8, we can reduce to the case where A is an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$ and $\mathbf{G}$ is the constant $\mathbf{P}$-divisible group associated to a colattice Λ. We can then use Remark 7.4.9 to reduce to the case where $\mathbf{G} = 0$ (at the cost of replacing X and Y by the mapping spaces $X^{B\widehat{\Lambda}}$ and $Y^{B\widehat{\Lambda}}$), in which case the transfer map is given by the formula of Proposition 7.4.12. For some illustrations of this principle, see the proofs of Propositions 7.5.2 and 7.6.7.

7.5 Tempered Ambidexterity for p-Finite Spaces

We now prove a weak form of Theorem 7.2.10.

Theorem 7.5.1. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let $f : X \to Y$ be a map of p-finite spaces, for some prime number p. Then f is $\upsilon_{\mathbf{G}}$-ambidextrous.*

The proof of Theorem 7.5.1 will require some preliminaries. We begin by carrying out the essential step.

Proposition 7.5.2. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*, let* p *be a prime number, and let* $X = K(\mathbf{F}_p, m)$ *be an Eilenberg-MacLane space for some* $m > 0$*. Assume that the projection map* $f : X \to *$ *is weakly* $\upsilon_{\mathbf{G}}$*-ambidextrous. Then* f *is* $\upsilon_{\mathbf{G}}$*-ambidextrous.*

Proof. Without loss of generality, we may assume that $m > 0$ and that the p-divisible group $\mathbf{G}_{(p)}$ has some fixed height n. Let $X = K(\mathbf{F}_p, m)$, and assume that the projection map $f : X \to *$ is weakly $\upsilon_{\mathbf{G}}$-ambidextrous. We wish to show that f is $\upsilon_{\mathbf{G}}$-ambidextrous. By virtue of Proposition 7.3.15, it will suffice to show that the ambidexterity form $\mathrm{AForm}(f) : [X/*] \otimes_A [X/*] \to A$ is a duality datum: that is, that it exhibits $[X/*]$ as a self-dual object of the ∞-category Mod_A. Remark 7.3.4 implies that $[X/*]$ is a projective A-module of finite rank, and that $\pi_0[X/*]$ can be identified with the $\mathbf{G}$-tempered homology ring $A_0^{\mathbf{G}}(X)$ of Notation 4.4.13. It will therefore suffice to show that the ambidexterity form $\mathrm{AForm}(f)$ induces a perfect pairing

$$A_0^{\mathbf{G}}(X) \otimes_{\pi_0(A)} A_0^{\mathbf{G}}(X) \to \pi_0(A),$$

or equivalently that the dual map

$$\pi_0(\mathrm{AForm}(f)^\vee) : \pi_0(A) \to A^0_{\mathbf{G}}(X) \otimes_{\pi_0(A)} A^0_{\mathbf{G}}(X)$$

is a duality datum in the ordinary category $\mathrm{Mod}^\heartsuit_{\pi_0(A)}$. Unwinding the definitions, we see that this map carries the element $1 \in \pi_0(A)$ to

$$\mathrm{tr}_{X/X \times X}(1) \in A^0_{\mathbf{G}}(X \times X) \simeq A^0_{\mathbf{G}}(X) \otimes_{\pi_0(A)} A^0_{\mathbf{G}}(X),$$

where $\mathrm{tr}_{X/X\times X}$ denotes the transfer map of Construction 7.4.1 (which is well-defined by virtue of our assumption that f is weakly $\upsilon_{\mathbf{G}}$-ambidextrous).

Let $R = \pi_0(A)^{\mathrm{red}}$ denote the quotient of $\pi_0(A)$ by its nilradical. Set $B = R \otimes_{\pi_0(A)} A^0_{\mathbf{G}}(X)$, and let e denote the image of $\mathrm{tr}_{X/X\times X}(1)$ in the tensor product $B \otimes_R B$. Then B is a projective R-module of finite rank. We will complete the proof by showing that e induces an isomorphism from B to its R-linear dual. Note that the existence of the oriented $\mathbf{P}$-divisible group $\mathbf{G}$ guarantees that the tensor product $A \otimes_S \mathbf{F}_\ell$ vanishes for every prime number ℓ (see Remark 2.5.11). Applying the May nilpotence conjecture (Theorem 2 of [14]), we deduce that every torsion element of $\pi_0(A)$ is nilpotent. Consequently, the commutative ring R is torsion-free.

Using Theorem 4.4.16 (and Remark 4.4.18), we see that the spectrum $\mathrm{Spec}(A^0_{\mathbf{G}}(X))$ is a truncated p-divisible group over $\pi_0(A)$ of level 1, height $\binom{n}{m}$, and dimension $d = \binom{n-1}{m-1}$. It follows that $\mathrm{Spec}(B)$ is a truncated p-divisible group over R of level 1 and dimension d. If $m > n$, then $B \simeq 0$ and there is nothing to prove. Otherwise, Proposition Ambi.5.2.2 implies that the trace map $\mathrm{tr} : B \to R$ is divisible by p^d, and the pairing

$$(x, y) \mapsto \frac{\mathrm{tr}(xy)}{p^d}$$

determines a perfect pairing of B with itself (in the category of R-modules). We will complete the proof by showing that the dual pairing is given by e. To prove this, we are free to replace A by the localization $A[\frac{1}{p}]$, and thereby reduce to the case where $\mathbf{G}_{(p)}$ is an étale p-divisible group. Replacing A by a faithfully flat extension, we may further assume that $\mathbf{G}_{(p)} \simeq (\mathbf{Q}_p / \mathbf{Z}_p)^n$ is a constant p-divisible group. Writing $\mathbf{G} = \mathbf{G}_0 \oplus (\mathbf{Q}_p / \mathbf{Z}_p)^n$ with $\mathbf{G}_{0(p)} \simeq 0$, Theorem 4.3.2 supplies an isomorphism

$$A^0_{\mathbf{G}}(X) \simeq A^0_{\mathbf{G}_0}(X^{B\,\mathbf{Z}^n_p}) \simeq A^0(X^{B\,\mathbf{Z}^n_p}) \simeq \prod_{\alpha : B\,\mathbf{Z}^n_p \to X} \pi_0(A),$$

where the product is taken over the (finite) collection of all homotopy classes of maps $\alpha : B\,\mathbf{Z}^n_p \to X$. Using Remark 7.4.9 and Proposition 7.4.12, we see that this isomorphism carries e to the matrix of rational numbers $\{e_{\alpha,\beta}\}$ given by $e_{\alpha,\beta} = \mathrm{Mass}(\{\alpha\} \times_{X^{B\,\mathbf{Z}^n_p}} \{\beta\})$. The desired equality now follows from the observation that $e_{\alpha,\beta}$ vanishes when α and β belong to different connected components of $X^{B\,\mathbf{Z}^n_p}$, and is otherwise given by

$$\begin{aligned}
\mathrm{Mass}(K(\mathbf{F}_p, m-1)^{B\,\mathbf{Z}^n_p}) &= \prod_{i \geqslant 0} |\pi_i(K(\mathbf{F}_p, m-1)^{B\,\mathbf{Z}^n_p})|^{(-1)^i} \\
&= \prod_{i \geqslant 0} |\mathrm{H}^{m-1-i}(B\,\mathbf{Z}^n_p; \mathbf{F}_p)|^{(-1)^i} \\
&= \prod_{i \geqslant 0} p^{(-1)^i \binom{n}{m-1-i}} \\
&= p^{\sum_{i \geqslant 0} (-1)^i \binom{n}{m-1-i}} \\
&= p^{\sum_{i \geqslant 0} (-1)^i \binom{n-1}{m-1-i} + (-1)^i \binom{n-1}{m-2-i}} \\
&= p^d.
\end{aligned}$$

□

We now consider some cases where ambidexterity is easy to verify.

Proposition 7.5.3. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* $f : X \to Y$ *be a map of spaces which exhibits* X *as a summand of* Y*. Then* f *is* $\upsilon_{\mathbf{G}}$*-ambidextrous.*

Proof. By virtue of Corollary 7.3.16, we may assume without loss of generality that $Y \in \mathscr{T}$ is the classifying space of a finite abelian group. In this case, either the map f is a homotopy equivalence or the space X is empty. In the former case there is nothing to prove, and in the latter case we have that $[X/Y] \simeq 0$, so that f is $\upsilon_{\mathbf{G}}$-ambidextrous by virtue of Proposition 7.3.15. □

Proposition 7.5.4. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* $f : \mathsf{X} \to \mathsf{Y}$ *be a map of orbispaces. Suppose that* X *decomposes as a finite disjoint union* $\coprod_{i \in I} \mathsf{X}_i$*. If each* $f_i = f|_{\mathsf{X}_i}$ *is* $\upsilon_{\mathbf{G}}$*-ambidextrous, then* f *is* $\upsilon_{\mathbf{G}}$*-ambidextrous.*

Proof. The map f factors as a composition

$$\mathsf{X} = \coprod_{i \in I} \mathsf{X}_i \xrightarrow{\coprod_{i \in I} f_i} \coprod_{i \in I} \mathsf{Y} \xrightarrow{g} \mathsf{Y},$$

where g is given by the identity on each factor. Note that for every pullback diagram

$$\begin{tikzcd}
\mathsf{Z} \arrow[r] \arrow[d, "f'"] & \coprod_{i \in I} \mathsf{X}_i \arrow[d, "\coprod_{i \in I} f_i"] \\
T^{(-)} \arrow[r] & \coprod_{i \in I} \mathsf{Y}
\end{tikzcd}$$

for $T \in \mathscr{T}$, the map f' is a pullback of some f_i and is therefore $\upsilon_{\mathbf{G}}$-ambidextrous by assumption. Applying Corollary 7.3.16, we deduce that f' is $\upsilon_{\mathbf{G}}$-ambidextrous. It will therefore suffice to show that g is $\upsilon_{\mathbf{G}}$-ambidextrous. Note that g is weakly $\upsilon_{\mathbf{G}}$-ambidextrous by virtue of Proposition 7.5.3. It will therefore suffice to show that for every pullback diagram

$$\begin{tikzcd}
\coprod_{i \in I} \mathsf{Y}' \arrow[r] \arrow[d, "g'"] & \coprod_{i \in I} \mathsf{Y} \arrow[d] \\
\mathsf{Y}' \arrow[r] & \mathsf{Y},
\end{tikzcd}$$

the norm map $\mathrm{Nm}_{g'} : g'_! \to g'_*$ is an equivalence. This follows immediately from the additivity of the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}')$. □

Note that for every orbispace X, we can regard the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ as an $A^{\mathsf{X}}_{\mathbf{G}}$-linear ∞-category. If $f : \mathsf{X} \to \mathsf{Y}$ is a map of orbispaces, then it induces an $A^{\mathsf{Y}}_{\mathbf{G}}$-linear functor $f^* : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$, which we can identify with an $A^{\mathsf{X}}_{\mathbf{G}}$-linear functor $A^{\mathsf{X}}_{\mathbf{G}} \otimes_{A^{\mathsf{Y}}_{\mathbf{G}}} \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$.

Proposition 7.5.5. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A *and let* $f : T' \to T$ *be a morphism in* $\mathscr{T}$ *with connected homotopy fibers. Then the natural map*

$$\rho : A^{T'}_{\mathbf{G}} \otimes_{A^T_{\mathbf{G}}} \mathrm{LocSys}_{\mathbf{G}}(T) \to \mathrm{LocSys}_{\mathbf{G}}(T')$$

is fully faithful.

Proof. For every pair of objects $\mathscr{F}, \mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(T)$, the functor ρ induces a map

$$A^{T'}_{\mathbf{G}} \otimes_{A^T_{\mathbf{G}}} \underline{\mathrm{Map}}(\mathscr{F}, \mathscr{G}) \to \underline{\mathrm{Map}}(f^*(\mathscr{F}), f^*(\mathscr{G})).$$

By virtue of Corollary 5.3.3, it will suffice to show that this map is an equivalence in the special case when $\mathscr{F}$ is compact. Without loss of generality, we may assume that $\mathscr{F} = [T_0/T]$, for some connected covering space $T_0 \in \mathrm{Cov}(T)$. Then $T'_0 = T_0 \times_T T'$ is a connected covering space of T', and we can identify $f^*(\mathscr{F})$ with the object $[T'_0/T']$ (Remark 7.3.3). Unwinding the definitions, we are reduced to showing that the canonical map

$$A^{T'}_{\mathbf{G}} \otimes_{A^T_{\mathbf{G}}} \mathscr{G}(T_0) \to \mathscr{G}(T'_0)$$

is an equivalence. This follows from our assumption that $\mathscr{G}$ is $\mathbf{G}$-pretempered, since the diagram of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} A^T_{\mathbf{G}} \arrow[r] \arrow[d] & A^{T'}_{\mathbf{G}} \arrow[d] \\ A^{T_0}_{\mathbf{G}} \arrow[r] & A^{T'_0}_{\mathbf{G}} \end{tikzcd}$$

is a pushout square. □

Corollary 7.5.6. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*, let* $f : T' \to T$ *be a morphism in* $\mathscr{T}$ *with connected homotopy fibers, and let* $b : \mathscr{F} \otimes \mathscr{G} \to \underline{A}_T$ *be a morphism of* $\mathbf{G}$*-tempered local systems on* T*. Then* b *is a duality datum in the* ∞*-category* $\mathrm{LocSys}_{\mathbf{G}}(T)$ *if and only if the pullback map*

$$f^*(b) : f^*(\mathscr{F}) \otimes f^*(\mathscr{G}) \to f^*(\underline{A}_T) \simeq \underline{A}_{T'}$$

is a duality datum in the ∞*-category* $\mathrm{LocSys}_{\mathbf{G}}(T')$*.*

Proof. It is clear that if b is a duality datum, then so is $f^*(b)$. Conversely, assume that $f^*(b)$ is a duality datum. Set $B^0 = A^{T'}_{\mathbf{G}}$ and let $B^\bullet$ denote the cosimplicial $A^T_{\mathbf{G}}$-algebra given by the tensor powers of B^0. Since B^0 is a faithfully flat $A^T_{\mathbf{G}}$-algebra and the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(T)$ is compactly generated, we can identify $\mathrm{LocSys}_{\mathbf{G}}(T)$ with the totalization of the cosimplicial symmetric monoidal ∞-category $B^{\bullet} \otimes_{A^T_{\mathbf{G}}} \mathrm{LocSys}_{\mathbf{G}}(T)$ (see Corollary SAG.D.7.7.7). It will therefore suffice to show that for each $k \geqslant 0$, the image of b in the ∞-category $B^k \otimes_{A^T_{\mathbf{G}}} \mathrm{LocSys}_{\mathbf{G}}(T)$ is a duality datum. Without loss of generality we may assume that $k = 0$. Set $\mathcal{C} = B^0 \otimes_{A^T_{\mathbf{G}}} \mathrm{LocSys}^{(}_{\mathbf{G}} T)$ and suppose we are given a pair of objects $\mathscr{H}, \mathscr{H}' \in \mathcal{C}$; we wish to show that the composite map

$$\mathrm{Map}_{\mathcal{C}}(\mathscr{H}, \mathscr{G} \otimes \mathscr{H}') \to \mathrm{Map}_{\mathcal{C}}(\mathscr{F} \otimes \mathscr{H}, \mathscr{F} \otimes \mathscr{G} \otimes \mathscr{H}') \xrightarrow{b} \mathrm{Map}_{\mathcal{C}}(\mathscr{F} \otimes \mathscr{H}, \mathscr{H}')$$

is an equivalence, and that an analogous statement holds with the roles of $\mathscr{F}$ and $\mathscr{G}$ reversed. By virtue of Proposition 7.5.5, we can identify $\mathcal{C}$ with a full subcategory of $\mathrm{LocSys}_{\mathbf{G}}(T')$, so that the desired result follows from our assumption that $f^*(b)$ is a duality datum. $\square$

Corollary 7.5.7. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let $g : T' \to T$ be a morphism in $\mathscr{T}$ with connected homotopy fibers, and suppose we are given a pullback diagram of spaces*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
T' \arrow[r, "g"] & T.
\end{tikzcd}$$

If f is weakly $\upsilon_{\mathbf{G}}$-ambidextrous and f' is $\upsilon_{\mathbf{G}}$-ambidextrous, then f is $\upsilon_{\mathbf{G}}$-ambidextrous.

Proof. Combine Proposition 7.3.15, Remark 7.3.14, and Corollary 7.5.6. $\square$

Proposition 7.5.8. *Let $\mathbf{G}$ be a p-divisible group of dimension $\leqslant 1$ over a commutative ring R, let V be a finite-dimensional vector space over $\mathbf{F}_p$, let $\mathbf{G}[V]$ be the finite flat group scheme over R classifying maps from V into $\mathbf{G}$, and let $U \subseteq \mathbf{G}[V]$ be the open subset whose κ-valued points are given by* injective *maps $V \to \mathbf{G}(\kappa)$, for every field κ. Let $\mathrm{Alt}^{(m)}_{\mathbf{G}[p]}$ denote the R-scheme of Construction Ambi.3.2.11 and let $\mathbf{D}(\mathrm{Alt}^{(m)}_{\mathbf{G}[p]})$ denote its Cartier dual. Let η be a nonzero element of the exterior power $\bigwedge^m(V)$, so that η induces a map*

$$\phi : \mathbf{G}[V] \to \mathbf{D}(\mathrm{Alt}^{(m)}_{\mathbf{G}[p]})$$

of finite flat group schemes over R. Then $\phi(U)$ does not intersect the zero section of $\mathbf{D}(\mathrm{Alt}^{(m)}_{\mathbf{G}[p]})$.

Proof. Without loss of generality, we may assume that R is an algebraically closed field. In this case, the p-divisible group $\mathbf{G}$ fits into a (canonically split) exact sequence

$$0 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 0$$

where $\mathbf{G}'$ is connected and $\mathbf{G}''$ is étale. We have a commutative diagram of R-schemes

$$\begin{tikzcd}
\mathbf{G}[V] \arrow[r] \arrow[d] & \mathbf{D}(\mathrm{Alt}^{(m)}_{\mathbf{G}[p]}) \arrow[d] \\
\mathbf{G}''[V] \arrow[r] & \mathbf{D}(\mathrm{Alt}^{(m)}_{\mathbf{G}''[p]})
\end{tikzcd}$$

where the vertical maps are homeomorphisms. We may therefore replace $\mathbf{G}$ by $\mathbf{G}''$ and thereby reduce to the case where $\mathbf{G}$ is étale. Since R is an algebraically closed field, $\mathbf{G}[p]$ is a constant group scheme associated to a vector space W of finite dimension over $\mathbf{F}_p$. Then $\mathbf{D}(\mathrm{Alt}^{(m)}_{\mathbf{G}[p]})$ is also a constant group scheme, associated to the $\mathbf{F}_p$-vector space $\bigwedge^m W$. We are therefore reduced to verifying the following elementary fact of linear algebra: every injective map of $\mathbf{F}_p$-vector spaces $V \to W$ induces an injection of exterior powers $\bigwedge^m V \to \bigwedge^m W$. □

Proposition 7.5.9. *Let* $\mathbf{G}$ *be an oriented* $\mathbf{P}$*-divisible group over an* $\mathbb{E}_\infty$*-ring* A*, let* p *be a prime number, and suppose we are given a fiber sequence of spaces* $X \xrightarrow{f} Y \xrightarrow{\eta} K(\mathbf{F}_p, m)$*. Then* f *is* $\upsilon_{\mathbf{G}}$*-ambidextrous.*

Proof. Given a fiber sequence of spaces $X \xrightarrow{f} Y \xrightarrow{\eta} K(\mathbf{F}_p, m)$, we will say that η is *good* if f is $\upsilon_{\mathbf{G}}$-ambidextrous. We wish to show that every map $\eta : Y \to K(\mathbf{F}_p, m)$ is good. The proof proceeds by induction on m. In the case $m = 0$, f is the inclusion of a summand and the desired result follows from Proposition 7.5.3. For $m > 0$, we observe that the relative diagonal $\delta : X \to X \times_Y X$ fits into a fiber sequence

$$X \xrightarrow{\delta} X \times_Y X \to K(\mathbf{F}_p, m-1),$$

and is therefore $\upsilon_{\mathbf{G}}$-ambidextrous by virtue of our inductive hypothesis. Note that if a map $\eta' : Y' \to K(\mathbf{F}_p, m)$ factors as a composition $Y' \to Y \xrightarrow{\eta} K(\mathbf{F}_p, m)$, then we have a commutative diagram of fiber sequences

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[r, "\eta'"] \arrow[d] & K(\mathbf{F}_p, m) \arrow[d, equal] \\
X \arrow[r] & Y \arrow[r, "\eta"] & K(\mathbf{F}_p, m)
\end{tikzcd}$$

where the left square is a pullback. Applying Corollaries 7.3.16 and 7.5.7, we deduce the following:

(a) If the map $\eta : Y \to K(\mathbf{F}_p, m)$ is good, then so is any composite map $Y' \to Y \xrightarrow{\eta} K(\mathbf{F}_p, m)$.

(b) To show that a map $\eta : Y \to K(\mathbf{F}_p, m)$ is good, it will suffice to show that any composition $T \to Y \xrightarrow{\eta} K(\mathbf{F}_p, m)$ is good for $T \in \mathscr{T}$.

(c) If $g : T' \to T$ is a morphism in $\mathscr{T}$ with connected homotopy fibers some composite map $T' \xrightarrow{g} T \xrightarrow{\eta} K(\mathbf{F}_p, m)$ is good, then η is good.

We must show that every morphism $\eta : Y \to K(\mathbf{F}_p, m)$ is good. By virtue of (b), it will suffice to prove this in the special case where $Y \simeq BM$ is the classifying space of a finite abelian group M. In this case, the map η factors through the localization $BM_{(p)}$, so we can use (a) to reduce further to the case where M is a finite abelian p-group. Let $k = \dim_{\mathbf{F}_p}(M/pM)$ denote the minimal number of generators of M. Our proof will proceed by induction on k.

Let us abuse notation by identifying η with its homotopy class, regarded as an element of the cohomology group $\mathbf{H}^m(Y; \mathbf{F}_p)$. Choose a surjection of abelian groups $\mathbf{Z}^k \to M$ and let $u : B\,\mathbf{Z}^k \to BM = Y$ be the induced map of classifying spaces. For $1 \leqslant i \leqslant k$, let $\alpha_i \in \mathrm{H}^1(B(\mathbf{Z}/p\,\mathbf{Z})^k; \mathbf{F}_p)$ denote the cohomology class corresponding to the homomorphism $(\mathbf{Z}/p\,\mathbf{Z})^k \to \mathbf{F}_p$ given by projection onto the ith factor. In what follows, we will abuse notation by identifying each α_i with its images under the natural maps

$$\mathrm{H}^1(B(\mathbf{Z}\,p\,\mathbf{Z})^k; \mathbf{F}_p) \to \mathrm{H}^1(B(\mathbf{Z}/p^t\,\mathbf{Z})^k; \mathbf{F}_p) \to \mathrm{H}^1(B\,\mathbf{Z}^k; \mathbf{F}_p).$$

A standard calculation shows that the cohomology ring $\mathrm{H}^*(B\,\mathbf{Z}^k; \mathbf{F}_p)$ is an exterior algebra on the classes $\{\alpha_i\}_{1 \leqslant i \leqslant k}$. In particular, we can write

$$u^*(\eta) = \sum_{\vec{i}} c_{\vec{i}}(\alpha_{i_1} \cup \cdots \cup \alpha_{i_m}) \in \bigwedge_{\mathbf{F}_p}^{m} \mathrm{H}^1(B\Lambda; \mathbf{F}_p) \simeq \mathrm{H}^m(B\Lambda; \mathbf{F}_p),$$

where $\vec{i}$ ranges over all sequences $0 < i_1 < \cdots < i_m \leqslant k$ and each coefficient $c_{\vec{i}}$ is an element of $\mathbf{F}_p$. For $t \gg 0$, the map u factors as a composition

$$B\,\mathbf{Z}^k \to B(\mathbf{Z}/p^t\,\mathbf{Z})^k \xrightarrow{u_t} Y$$

and the equality

$$u_t^*(\eta) = \sum_{\vec{i}} c_{\vec{i}}(\alpha_{i_1} \cup \cdots \cup \alpha_{i_m})$$

holds in the ring $\mathrm{H}^*(B(\mathbf{Z}/p^t\mathbf{Z})^k;\mathbf{F}_p)$. By virtue of (b), it will suffice to show that the composite map

$$B(\mathbf{Z}/p^t\mathbf{Z})^k \xrightarrow{u_t} Y \xrightarrow{\eta} K(\mathbf{F}_p, m)$$

is good. By construction, this map also factors as a composition

$$B(\mathbf{Z}/p^t\mathbf{Z})^k \to B(\mathbf{Z}/p\mathbf{Z})^k \xrightarrow{\eta'} K(\mathbf{F}_p, m),$$

where η' represents the cohomology class $\sum_{\vec{i}} c_{\vec{i}}(\alpha_{i_1} \cup \cdots \cup \alpha_{i_m}) \in \mathrm{H}^m(B(\mathbf{Z}/p\mathbf{Z})^k;\mathbf{F}_p)$. Applying (a), we can replace η by η' and thereby reduce to the case where Y has the form $B(\mathbf{Z}/p\mathbf{Z})^k$.

If each of the coefficients $c_{\vec{i}}$ vanishes, then the map $\eta : Y \to K(\mathbf{F}_p, m)$ is nullhomotopic. In this case, f is a pullback of the projection map $K(\mathbf{F}_p, m-1) \to *$, which is $\upsilon_{\mathbf{G}}$-ambidextrous by virtue of Proposition 7.5.2. We may therefore assume that some coefficient $c_{\vec{i}}$ is nonzero. Let $\mathrm{AForm}(f) : [X/Y] \otimes [X/Y] \to \underline{A}_Y$ be the ambidexterity form of f (Construction 7.3.13); we wish to show that $\mathrm{AForm}(f)$ is a duality pairing (Proposition 7.3.15). To prove this, it will suffice to show that for every pair of $\mathbf{G}$-tempered local systems $\mathscr{F}, \mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(Y)$, the composite map

$$\begin{aligned} \theta_{\mathscr{F},\mathscr{G}} : \underline{\mathrm{Map}}(\mathscr{F}, [X/Y] \otimes \mathscr{G}) &\to \underline{\mathrm{Map}}([X/Y] \otimes \mathscr{F}, [X/Y] \otimes \otimes [X/Y] \otimes \mathscr{G}) \\ &\xrightarrow{\mathrm{AForm}(f)} \underline{\mathrm{Map}}([X/Y] \otimes \mathscr{F}, \mathscr{G}) \end{aligned}$$

is an equivalence of $A^Y_{\mathbf{G}}$-modules (and that a similar assertion holds for the composition of $\mathrm{AForm}(f)$ with the automorphism of $[X/Y] \otimes [X/Y]$ given by exchanging the two factors, though this is actually unnecessary: one can show that the ambidexterity form of f is symmetric).

Let $\mathcal{C} \subseteq \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ denote the full subcategory spanned by those $\mathbf{G}$-tempered local systems $\mathscr{F}$ for which the map $\theta_{\mathscr{F},\mathscr{G}}$ is an equivalence of spectra. Since the construction $\mathscr{F} \mapsto \theta_{\mathscr{F},\mathscr{G}}$ carries colimits in $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ to limits in $\mathrm{Fun}(\Delta^1, \mathrm{Sp})$, the ∞-category $\mathcal{C}$ is presentable and closed under small colimits in $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$. Let $\mathcal{C}^{\perp}$ be the full subcategory of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ spanned by those objects $\mathscr{F}$ for which the spectrum $\underline{\mathrm{Map}}(\mathscr{F}_0, \mathscr{F})$ vanishes for each object $\mathscr{F}_0 \in \mathcal{C}$. Then every object $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ fits into an essentially unique fiber sequence $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$, where $\mathscr{F}' \in \mathcal{C}$ and $\mathscr{F}'' \in \mathcal{C}^{\perp}$. It will therefore suffice to show that $\mathcal{C}^{\perp}$ contains only zero objects of $\mathrm{LocSys}_{\mathbf{G}}(Y)$.

Fix an object $\mathscr{F} \in \mathcal{C}^{\perp}$; we will complete the proof by showing that $\mathscr{F}$ belongs to $\mathcal{C}$ (in which case it follows that $\mathscr{F} \simeq 0$). Note that if $Y_0 \in \mathrm{Cov}(Y)$ is a connected covering space of Y which the covering map $Y_0 \to Y$ is not an isomorphism, then fundamental group $\pi_1(Y_0$ is an $\mathbf{F}_p$-vector space of dimension $< k$, and our inductive hypothesis (together with Remark 7.3.14) guarantees that $[Y_0/Y]$ belongs to $\mathcal{C}$. It follows that the spectrum $\underline{\mathrm{Map}}([Y_0/Y], \mathscr{F}) \simeq \mathscr{F}(Y_0)$ vanishes. Since $\mathscr{F}$ is $\mathbf{G}$-tempered, the canonical map $\mathscr{F}(Y) \to \mathscr{F}(Y_0)^{\mathrm{hAut}(Y_0/Y)}$ exhibits $\mathscr{F}(Y_0)^{\mathrm{hAut}(Y_0/Y)} \simeq 0$ as the $I(Y_0/Y)$-completion of $\mathscr{F}(Y)$. It follows that $\mathscr{F}(Y)$ is $I(Y_0/Y)$-local when viewed as an $A^Y_{\mathbf{G}}$-module.

Let $\overline{J} \subseteq A^0_{\mathbf{G}}(K(\mathbf{F}_p, m))$ denote the kernel of the augmentation map

$$\epsilon : A^0_{\mathbf{G}}(K(\mathbf{F}_p, m)) \to \pi_0(A)$$

(given by pullback along the inclusion of the base point to $K(\mathbf{F}_p, m)$), and let $J \subseteq A^0_{\mathbf{G}}(Y)$ denote the ideal generated by the image of $\overline{J}$ under the pullback map $\eta^* : A^0_{\mathbf{G}}(K(\mathbf{F}_p, m)) \to A^0_{\mathbf{G}}(Y)$. It follows from Proposition 7.5.8 and Theorem 4.4.16 that the vanishing locus of J is contained in the union of the vanishing loci of the ideals $I(Y_0/Y)$, where Y_0 is a connected covering space of Y which is not equivalent to Y. The preceding argument then shows that $\mathscr{F}$ is J-local when viewed as an object fo the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ (where we view $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ as an $A^Y_{\mathbf{G}}$-linear ∞-category). Using the commutativity of the diagram

$$\begin{tikzcd}
A^0_{\mathbf{G}}(K(\mathbf{F},m)) \arrow[r, "\epsilon"] \arrow[d, "\eta^*"] & \pi_0(A) \arrow[d] \\
A^0_{\mathbf{G}}(Y) \arrow[r, "f^*"] & A^0_{\mathbf{G}}(X),
\end{tikzcd}$$

we see that J is annihilated by the pullback map f^*. In particular, for each element $x \in J$, multiplication by x induces a nullhomotopic map from $[X/Y]$ to itself. It follows that the tensor product $[X/Y] \otimes \mathscr{F}$ is simultaneously J-nilpotent and J-local, and therefore vanishes. Similarly, multiplication by each $x \in J$ induces a nullhomotopic map from $[X/Y] \otimes \mathscr{G}$ to itself, so that $[X/Y] \otimes \mathscr{G}$ is J-complete. Since $\mathscr{F}$ is J-local, the spectrum $\underline{\mathrm{Map}}(\mathscr{F}, [X/Y] \otimes \mathscr{G})$ vanishes. It follows that the domain and codomain of $\theta_{\mathscr{F},\mathscr{G}}$ are both trivial, so that $\theta_{\mathscr{F},\mathscr{G}}$ is a homotopy equivalence and $\mathscr{F}$ belongs to $\mathcal{C}$, as desired. □

Proof of Theorem 7.5.1. Let $f : X \to Y$ be a map of p-finite spaces; we wish to show that f is $\upsilon_{\mathbf{G}}$-ambidextrous. Factoring f as a composition (using the Postnikov tower of X as an object of $\mathcal{S}_{/Y}$), we can assume that there exists some integer $n \geqslant -1$ for which the homotopy fibers of f are n-truncated and n-connective. By virtue of Corollary 7.3.16, we can also assume that Y is the classifying space of a finite abelian p-group. We now consider several cases:

- If $n = -1$, then the desired result follows from Proposition 7.5.3.

- If $n = 0$, then X is a covering space of Y. Using Proposition 7.5.4, we can assume that X is a connected covering space of Y. Then f induces a monomorphism of fundamental groups $\pi_1(X) \to \pi_1(Y)$. Proceeding by induction, we can reduce to the case where the quotient group $\pi_1(Y)/\pi_1(X)$ is cyclic of order p. In this case, X is the fiber of a map $Y \to K(\mathbf{F}_p, 1)$, so the desired result follows from Proposition 7.5.9.

- Suppose that $n \geqslant 1$, so that the homotopy fiber of f has the form $K(G, n)$ for some finite p-group G (which is abelian for $n \geqslant 2$). Proceeding by induction on the order of the group G, we can reduce to the case where G is cyclic of order p (so that the fundamental group $\pi_1(X)$ automatically acts trivially on G). Then X is the homotopy fiber of a map $Y \to K(\mathbf{F}_p, n+1)$, and the desired result again follows from Proposition 7.5.9.

□

7.6 Induction Theorems

Let G be a finite group and let $H \subseteq G$ be a subgroup. If V is a finite-dimensional complex representation of H, then the tensor product $\mathbf{C}[G] \otimes_{\mathbf{C}[H]} V$ is a finite-dimensional complex representation of G, which we denote by $\mathrm{Ind}_H^G(V)$ and refer to as the *induced representation*. The construction $V \mapsto \mathrm{Ind}_H^G(V)$ determines a homomorphism of abelian groups

$$\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G) \qquad [V] \mapsto [\mathrm{Ind}_H^G(V)],$$

which we refer to as the *induction homomorphism*. The celebrated *induction theorems* of Artin and Brauer assert that every representation of G can be expressed as a linear combination of representations induced from special kinds of subgroups of G.

Theorem 7.6.1 (Artin Induction Theorem)**.** *Let G be a finite group and let* $\mathrm{Rep}(G)$ *denote its representation ring. Then the localization* $\mathrm{Rep}(G)[\frac{1}{|G|}]$ *is generated, as a module over* $\mathbf{Z}[\frac{1}{|G|}]$*, by the images of the induction maps*

$$\mathrm{Ind}_H^G : \mathrm{Rep}(H)[\frac{1}{|G|}] \to \mathrm{Rep}(G)[\frac{1}{|G|}],$$

where H ranges over the collection of cyclic subgroups of G.

Theorem 7.6.2 (Brauer Induction Theorem)**.** *Let G be a finite group. Then the representation ring* $\mathrm{Rep}(G)$ *is generated, as an abelian group, by the images of the induction maps*

$$\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G),$$

where H ranges over subgroups of G which factor as a product $C \times P$, where C is cyclic and P is a p-group (for some prime number p).

Our goal in this section is to prove analogues of Theorems 7.6.1 and 7.6.2 in the setting of tempered cohomology. Let $\vec{h} = \{h_p\}_{p \in \mathbf{P}}$ be a collection of nonnegative integers. Recall that a $\mathbf{P}$-divisible group $\mathbf{G}$ has *height* $\leqslant \vec{h}$ if each summand $\mathbf{G}_{(p)}$ is a p-divisible group of height $\leqslant h_p$, and that $\mathscr{T}(\leqslant \vec{h}) \subseteq \mathscr{T}$ denotes the full subcategory spanned by those objects of the form BH, where each p-local component $H_{(p)}$ can be generated by $\leqslant h_p$ elements (Notation 5.6.1).

Theorem 7.6.3 (Tempered Artin Induction Theorem)**.** *Let $\vec{h} = \{h_p\}_{p \in \mathbf{P}}$ be a collection of nonnegative integers, let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group of height $\leqslant \vec{h}$ over an $\mathbb{E}_\infty$-ring A, and let X be a π-finite space. Assume that each homotopy group $\pi_n(X, x)$ has order invertible in the commutative ring $\pi_0(A)$. Then the tempered cohomology ring $A^0_{\mathbf{G}}(X)$ is generated (as an abelian group) by the images of transfer maps*

$$\mathrm{tr}_{T/X} : A^0_{\mathbf{G}}(T) \to A^0_{\mathbf{G}}(X)$$

where T belongs to $\mathscr{T}(\leqslant \vec{h})$.

Remark 7.6.4. In the situation of Theorem 7.6.3, suppose that $X = BG$ is the classifying space of a finite group G. In that case, every map of classifying spaces $f : BH \to BG$ factors as a composition $BH \xrightarrow{f'} BH_0 \xrightarrow{f''} BG$, where f' induces a surjection on fundamental groups and f'' induces an injection on fundamental groups. Using Proposition 7.4.5, we see that the image of the transfer map $\mathrm{tr}_{BH/BG}$ is contained in the image of the transfer map $\mathrm{tr}_{BH_0/BG}$. Consequently, $A^0_{\mathbf{G}}(BG)$ can also be generated by the images of transfer maps $\mathrm{tr}_{BH/BG}$, where H ranges over abelian *subgroups* of G (having the property that each $H_{(p)}$ can be generated by at most h_p elements).

Theorem 7.6.5 (Tempered Brauer Induction Theorem)**.** *Let $\vec{h} = \{h_p\}_{p \in \mathbf{P}}$ be a collection of nonnegative integers, let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group of height $\leqslant \vec{h}$ over an $\mathbb{E}_\infty$-ring A, and let X be a π-finite space. Then the tempered cohomology ring $A^0_{\mathbf{G}}(X)$ is generated, as an abelian group, by the images of the transfer maps*

$$\mathrm{tr}_{Y/X} : A^0_{\mathbf{G}}(Y) \to A^0_{\mathbf{G}}(X),$$

where Y ranges over π-finite spaces (equipped with map to X) having the following property:

- (∗) *For some prime number p, the space Y factors as a product $T \times P$, where $T \in \mathscr{T}(\leqslant \vec{h})$ and P is a connected p-finite space.*

Remark 7.6.6. In the situation of Theorem 7.6.5, suppose that the π-finite space X is n-truncated for some $n \geqslant 1$. Any map of spaces $f : Z \to X$ factors as a composition $Z \xrightarrow{f'} Y \xrightarrow{f''} X$, where the homotopy fibers of f'' are $(n-1)$-truncated and the homotopy fibers of f' are n-connective. It follows that, for any base point $z \in Z$, we have isomorphisms

$$\pi_m(Y, g(z)) \simeq \begin{cases} \pi_m(Z, z) & \text{if } m < n \\ \operatorname{im}(\pi_n(Z, z) \to \pi_n(X, f(z)) & \text{if } m = n \\ 0 & \text{if } m > n \ . \end{cases}$$

If Z satisfies condition (∗) of Theorem 7.6.5, then so does Y, and the image of the transfer map $\mathrm{tr}_{Z/X}$ is contained in the image of $\mathrm{tr}_{Y/X}$ (Proposition 7.4.5). It follows that the tempered cohomology ring $A^0_{\mathbf{G}}(X)$ is generated by the images of the transfer maps $\mathrm{tr}_{Y/X}$ associated to $(n-1)$-truncated maps $Y \to X$ which satisfy condition (∗).

We now show that Theorems 7.6.1 and 7.6.2 can be deduced from their tempered counterparts. First, we need to relate the transfers of §7.4 with the classical induction maps.

Proposition 7.6.7. *Let G be a finite group and let $H \subseteq G$ be a subgroup. Let $\mathbf{G} = \mu_{\mathbf{P}^\infty}$, regarded as an oriented $\mathbf{P}$-divisible group over the complex K-theory spectrum* KU *(Construction 2.8.6). Then the diagram of abelian groups*

$$\begin{tikzcd}
\mathrm{KU}^0_{\mathbf{G}}(BH) \arrow[r, "\mathrm{tr}_{BH/BG}"] \arrow[d, "\sim"] & \mathrm{KU}^0_{\mathbf{G}}(BG) \arrow[d, "\sim"] \\
\mathrm{Rep}(H) \arrow[r, "\mathrm{Ind}_H^G"] & \mathrm{Rep}(G)
\end{tikzcd}$$

commutes, where the vertical maps are the isomorphisms supplied by Example 4.1.4.

Proof. Define a $\mathbf{C}$-linear map

$$\mathrm{Ind}_H^G : \{\text{Class functions } H \to \mathbf{C}\} \to \{\text{Class functions } G \to \mathbf{C}\}$$

by the formula

$$\mathrm{Ind}_H^G(\chi)(g) = \frac{1}{|H|} \sum_{s \in G, sgs^{-1} \in H} \chi(sgs^{-1}),$$

and consider the diagram

$$\begin{tikzcd}
\mathrm{KU}^0_{\mathbf{G}}(BH) \arrow[r, "\mathrm{tr}_{BH/BG}"] \arrow[d, "\sim"] & \mathrm{KU}^0_{\mathbf{G}}(BG) \arrow[d, "\sim"] \\
\mathrm{Rep}(H) \arrow[r, "\mathrm{Ind}_H^G"] \arrow[d, "{[V] \mapsto \chi_V}"] & \mathrm{Rep}(G) \arrow[d, "{[V] \mapsto \chi_V}"] \\
\{\text{Class functions } H \to \mathbf{C}\} \arrow[r, "\mathrm{Ind}_H^G"] & \{\text{Class functions } G \to \mathbf{C}\}.
\end{tikzcd}$$

A standard elementary calculation shows that the lower square commutes. Moreover, the lower vertical maps are injective. Consequently, it will suffice to show that the outer rectangle commutes. Let $\mathrm{KU}_{\mathbf{C}} = \mathbf{C} \otimes_S \mathrm{KU}$ denote the complexification of the complex K-theory spectrum. Then, over the ring spectrum $\mathrm{KU}_{\mathbf{C}}$, we have an isomorphism of $\mathbf{P}$-divisible groups

$$\underline{\mathbf{Q}/\mathbf{Z}} \simeq \mathbf{G}_{\mathrm{KU}_{\mathbf{C}}} \qquad \alpha \mapsto \exp(2\pi i \alpha),$$

so that Theorem 4.3.2 supplies isomorphisms

$$(\mathrm{KU}_{\mathbf{C}})^0_{\mathbf{G}}(BG) \simeq \mathrm{KU}^0_{\mathbf{C}}(\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(BG)) \simeq \{\text{Class functions } G \to \mathbf{C}\}$$

$$(\mathrm{KU}_{\mathbf{C}})^0_{\mathbf{G}}(BH) \simeq \mathrm{KU}^0_{\mathbf{C}}(\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(BH)) \simeq \{\text{Class functions } H \to \mathbf{C}\}.$$

By virtue of Example 4.3.9, we are reduced to verifying the commutativity of the outer rectangle in the diagram

$$\begin{tikzcd}
\mathrm{KU}^0_{\mathbf{G}}(BH) \arrow[r, "\mathrm{tr}_{BH/BG}"] \arrow[d, "\sim"] & \mathrm{KU}^0_{\mathbf{G}}(BG) \arrow[d, "\sim"] \\
(\mathrm{KU}_{\mathbf{C}})^0_{\mathbf{G}}(BH) \arrow[r, "\mathrm{tr}_{BH/BG}"] \arrow[d, "\sim"] & (\mathrm{KU}_{\mathbf{C}})^0_{\mathbf{G}}(BG) \arrow[d, "\sim"] \\
\mathrm{KU}^0_{\mathbf{C}}(\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(BH)) \arrow[r, "\mathrm{tr}_{\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(BH)/\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(BG)}"] \arrow[d, "\sim"] & \mathrm{KU}^0_{\mathbf{C}}(\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(BG)) \arrow[d, "\sim"] \\
\{\text{Class functions } H \to \mathbf{C}\} \arrow[r, "\mathrm{Ind}_H^G"] & \{\text{Class functions } G \to \mathbf{C}\}.
\end{tikzcd}$$

In fact, the entire diagram commutes: for the upper square this follows from Remark 7.4.8, for the middle square it follows from Remark 7.4.9, and for the lower square it follows from Proposition 7.4.12. □

Proof of Theorems 7.6.1 and 7.6.2 from Theorems 7.6.3 and 7.6.5. We give an argument that Theorem 7.6.5 implies Theorem 7.6.2; the proof that Theorem 7.6.3 implies Theorem 7.6.1 is similar. Let KU denote the complex K-theory spectrum and let $\mathbf{G} = \mu_{\mathbf{P}^\infty}$ be the multiplicative $\mathbf{P}$-divisible group over KU, endowed with the orientation of Construction 2.8.6. Let G be a finite group, so that $X = BG$ is a π-finite space. Then Theorem 7.6.5 implies that the tempered cohomology ring $\mathrm{KU}^0_{\mathbf{G}}(X)$ is generated by the images of the maps $\mathrm{tr}_{Y/X} : \mathrm{KU}^0_{\mathbf{G}}(Y) \to \mathrm{KU}^0_{\mathbf{G}}(X)$, where $f : Y \to X$ is a map satisfying condition $(*)$ of Theorem 7.6.5. Moreover, since X is 1-truncated, we may assume without loss of generality that the map $f : Y \to X$ is 0-truncated (Remark 7.6.6), so that we can identify Y with the classifying space of a subgroup $H \subseteq G$. In this case, $(*)$ guarantees that H factors as a product of a cyclic group and a p-group for some prime number p. Combining this observation with Proposition 7.6.7, we deduce that the representation ring $\mathrm{Rep}(G)$ is generated by the images of the induction maps $\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$, where H ranges over subgroups of G which are products of cyclic groups with p-groups. □

Theorem 7.6.5 has a local version:

Theorem 7.6.8. *Let p be a fixed prime number, let $\vec{h} = \{h_\ell\}_{\ell \in \mathbf{P}}$ be a collection of nonnegative integers, let $\mathbf{G}$ be a $\mathbf{P}$-divisible group of height $\leqslant \vec{h}$ over a p-local $\mathbb{E}_\infty$-ring A, and let X be a π-finite space. Then the tempered cohomology ring $A^0_{\mathbf{G}}(X)$ is generated, as an abelian group, by the images of the transfer maps*

$$\mathrm{tr}_{Y/X} : A^0_{\mathbf{G}}(Y) \to A^0_{\mathbf{G}}(X),$$

where Y ranges over π-finite spaces of the form $T \times P$, where $T \in \mathscr{T}(\leqslant \vec{h})$ and P is a connected p-finite space.

Proof of Theorem 7.6.5 from Theorem 7.6.8. Let $\mathbf{G}$ be a $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let X be a π-finite space, and let $I \subseteq A^0_{\mathbf{G}}(X)$ be the subgroup generated by the images of the transfer maps $\mathrm{tr}_{Y/X} : A^0_{\mathbf{G}}(Y) \to A^0_{\mathbf{G}}(X)$, where $Y \to X$ satisfies condition $(*)$ of Theorem 7.6.8. It follows from the projection formula of Warning 7.4.2 that I is an ideal. Consequently, to show that I coincides with $A^0_{\mathbf{G}}(X)$, it will suffice to show agreement after localizing at every prime number p. By virtue of Corollary 4.7.3, we can replace A by the localization $A_{(p)}$ and thereby reduce to the case where A is p-local, in which case the desired result follows immediately from Theorem 7.6.8. $\square$

Warning 7.6.9. In our deduction of Theorem 7.6.5 from Theorem 7.6.8, we invoked the fact that the formation of tempered cohomology of π-finite spaces is compatible with faithfully flat base change (Corollary 4.7.3). To prove Theorems 7.6.3 and 7.6.8, we will *not* use this fact (despite the fact that it would simplify our argument somewhat). This is actually important to the overall logic of §7: to prove that the tempered function spectrum $A^X_{\mathbf{G}}$ of an n-truncated, π-finite space X is compatible with base change, we use the fact that the projection map $X \to *$ is υ-ambidextrous (Theorem 7.2.10), whose proof will make use of Theorem 7.6.3 and Theorem 7.6.8 (applied to the same π-finite space X), but will not use Theorem 7.6.5.

Proof of Theorem 7.6.3. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Let S be the (finite) collection of all prime numbers which divide the order of some homotopy group $\pi_n(X, x)$, and assume that each $p \in S$ is invertible in the commutative ring $\pi_0(A)$. Without loss of generality, we may assume that for each $p \in S$, the p-divisible group $\mathbf{G}_{(p)}$ has some fixed height h_p. Let Λ be the colattice $\bigoplus_{p \in S}(\mathbf{Q}_p / \mathbf{Z}_p)^{h_p}$ and let $\widehat{\Lambda} \simeq \prod_{p \in S} \mathbf{Z}_p^{h_p}$ denote its Pontryagin dual. Let I be the set of all homotopy classes of maps $B\widehat{\Lambda} \to X$, and choose a representative $f_i : B\widehat{\Lambda} \to X$ for each homotopy class $i \in I$. By virtue of Proposition 3.4.7, we can choose finite subgroups $M_i \subseteq \Lambda$ such that each of the maps f_i factors as a composition

$$B\widehat{\Lambda} \to B\widehat{M_i} \xrightarrow{g_i} X.$$

By construction, the finite group $(\widehat{M_i})_{(p)}$ can be generated by $\leqslant h_p$ elements for each $p \in S$ (and vanishes for $p \notin S$). We will complete the proof by showing that the transfer maps $\mathrm{tr}_{B\widehat{M}_i/X}$ induce a surjection

$$\bigoplus_{i \in I} A^0_{\mathbf{G}}(B\widehat{M_i}) \to A^0_{\mathbf{G}}(X).$$

Let $\mathbf{G}'$ denote the sum $\bigoplus_{p \in S} \mathbf{G}_{(p)}$ and let $C = \mathrm{Split}_{\Lambda}(\mathbf{G}')$ be a splitting algebra for $\mathbf{G}'$ (Definition 2.7.7). Then C is a direct limit of finite étale A-algebras, and there is an isomorphism $\rho : \underline{\Lambda} \to \mathbf{G}'_C$ of $\mathbf{P}$-divisible groups over C. The restriction of ρ to each $\underline{M}_i$ is then classified by a map of A-algebras $u_i : A^{B\widehat{M}_i}_{\mathbf{G}} \to C$. We can then factor the unit map $A \to C$ as a composition $A \to B \to C$, where B is a finite étale A-algebra (of nonzero degree), C is faithfully flat over B, and each of the maps u_i

factors through some map of A-algebras $A_{\mathbf{G}}^{B\widehat{M}_i} \to B$, which we can identify with a map of B-algebras $v_i : B_{\mathbf{G}}^{B\widehat{M}_i} \to B$. Since $B_{\mathbf{G}}^{B\widehat{M}_i}$ is an étale B-algebra, this map decomposes the commutative ring $B^0_{\mathbf{G}}(B\widehat{M}_i)$ as a Cartesian product of $\pi_0(B)$ with some auxiliary commutative ring R_i (so that v_i is given by projection onto the first factor). Let $\xi_i \in B^0_{\mathbf{G}}(B\widehat{M}_i)$ be the element which corresponds to the pair $(1, 0)$ under this product decomposition. We will prove the following:

(a) The sum $\sum_{i \in I} \mathrm{tr}_{B\widehat{M}_i/X}(\xi_i)$ is an invertible element of the tempered cohomology ring $B^0_{\mathbf{G}}(X)$.

Note that if (a) is satisfied, then the transfer map $\bigoplus_{i \in I} B^0_{\mathbf{G}}(B\widehat{M}_i) \to B^0_{\mathbf{G}}(X)$ is surjective (since its image is automatically an ideal). Since B is finite flat (and faithfully flat) over A, it will then follow from Remark 4.7.4 that the transfer map $\bigoplus_{i \in I} A^0_{\mathbf{G}}(B\widehat{M}_i) \to A^0_{\mathbf{G}}(X)$ is also surjective, completing the proof of Theorem 7.6.3.

Let $C^\bullet$ denote the cosimplicial B-algebra given by the iterated tensor powers of C over B. Since C is faithfully flat over B, the canonical map $B \to \mathrm{Tot}(C^\bullet)$ is an equivalence. It then follows from Lemma 4.2.11 that the map of tempered function spectra $B^X_{\mathbf{G}} \to \mathrm{Tot}(C^{\bullet X}_{\mathbf{G}})$ is also an equivalence. Consequently, to show that the element $\sum_{i \in I} \mathrm{tr}_{B\widehat{M}_i/X}(\xi_i)$ is an invertible element of the tempered cohomology ring $B^0_{\mathbf{G}}(X) = \pi_0(B^X_{\mathbf{G}})$ is invertible, it will suffice to show that its image in $C^0_{\mathbf{G}}(X) = \pi_0(C^X_{\mathbf{G}})$ is invertible. Set $\mathbf{G}_0 = \bigoplus_{p \notin S} \mathbf{G}_{(p)}$, so that the $\mathbf{P}$-divisible group $\mathbf{G}_C$ splits as a direct sum $\mathbf{G}_{0C} \oplus \underline{\Lambda}$. We then have a commutative diagram

$$\begin{tikzcd}
\bigoplus_{i \in I} C^0_{\mathbf{G}}(B\widehat{M}_i) \arrow[r, "\mathrm{tr}_{B\widehat{M}_i/X}"] \arrow[d, "\sim"] & C^0_{\mathbf{G}}(X) \arrow[d, "\sim"] \\
\bigoplus_{i \in I} C^0_{\mathbf{G}_0}(B\widehat{M}_i^{B\hat{\Lambda}}) \arrow[r, "\mathrm{tr}_{B\widehat{M}_i^{B\hat{\Lambda}}/X^{B\hat{\Lambda}}}"] & C^0_{\mathbf{G}_0}(X^{B\hat{\Lambda}}),
\end{tikzcd}$$

where the vertical maps are the character isomorphisms supplied by Theorem 4.3.2. Note that the mapping space $Z = X^{B\hat{\Lambda}}$ splits as a disjoint union of connected S-finite spaces $\coprod_{i \in I} Z_i$, so that the tempered cohomology ring $C^0_{\mathbf{G}_0}(X^{B\hat{\Lambda}})$ can be identified with $\prod_{i \in I} \pi_0(C)$ (Proposition 7.4.11). Using Proposition 7.4.12, we see that the image of $\sum_{i \in I} \mathrm{tr}_{B\widehat{M}_i/X}(\xi_i)$ under this identification is given by the tuple of rational numbers $\{\frac{\mathrm{Mass}(B\widehat{M}_i)}{\mathrm{Mass}(Z_i)}\}_{i \in I}$, each of which is invertible in $\pi_0(C)$. □

Our proof of Theorem 7.6.8 will use a similar strategy. However, it is somewhat more complicated, because we cannot explicitly describe the tempered cohomology rings which appear. We will need a few preliminary remarks. Recall that if G is a finite p-group acting on a finite set X, then the fixed point set $X^G = \{x \in X : (\forall g \in G)[x^g = x]\}$ satisfies $|X^G| \equiv |X| \pmod{p}$. We will need an analogous fact for π-finite spaces:

Lemma 7.6.10. *Let X be a π-finite space, and let p be a prime number which does not divide the order of any homotopy group of X. Let G be a finite p-group acting on X. Then:*

1. *The homotopy fixed point set X^{hG} is also a π-finite space, whose homotopy groups have order not divisible by p.*
2. *If X is connected, then X^{hG} is connected.*
3. *We have $\mathrm{Mass}(X^{hG}) \equiv \mathrm{Mass}(X) \pmod{p}$ in the commutative ring $\mathbf{Z}_{(p)}$.*

Proof. Decomposing X as a disjoint union, we may assume without loss of generality that $\pi_0(X)$ consists of a single orbit of G. If X is not connected, then G has no fixed points on the set $\pi_0(X)$ and therefore the homotopy fixed point space X^{hG} is empty. On the other hand, the mass $\mathrm{Mass}(X)$ is the product of $|\pi_0(X)|$ with the mass of any connected component of X, and is therefore divisible by p (in the ring $\mathbf{Z}_{(p)}$). We may therefore assume without loss of generality that X is connected. Since X is π-finite, there exists an integer $n \gg 0$ for which X is n-truncated. We proceed by induction on n. If $n = 0$, then X is contractible and the result is clear. To carry out the inductive step, assume that $n > 0$ and let $Y = \tau_{\leqslant n-1}(X)$ be the $(n-1)$-truncation of X. Then Y inherits an action of G, and our inductive hypothesis guarantees that Y^{hG} is a connected π-finite space satisfying $\mathrm{Mass}(Y^{hG}) \equiv \mathrm{Mass}(Y) \pmod{p}$. Fix a base point $y \in Y^{hG}$. Then G acts on the homotopy fiber $F = \{y\} \times_Y X$, and we have a homotopy fiber sequence

$$F^{hG} \to X^{hG} \to Y^{hG},$$

which yields an equality

$$\begin{aligned}
\mathrm{Mass}(X^{hG}) &= \mathrm{Mass}(Y^{hG}) \cdot \mathrm{Mass}(F^{hG}) \\
&\equiv \mathrm{Mass}(Y) \cdot \mathrm{Mass}(F^{hG}) \\
&= \frac{\mathrm{Mass}(X)}{\mathrm{Mass}(F)} \cdot \mathrm{Mass}(F^{hG}) \\
&= \mathrm{Mass}(X) \cdot \frac{\mathrm{Mass}(F^{hG})}{\mathrm{Mass}(F)}.
\end{aligned}$$

We may therefore replace X by F and thereby reduce to the case where $X \simeq K(M, n)$ is the Eilenberg-MacLane space associated to a finite group M whose order is not divisible by p.

Suppose now that $n \geqslant 2$, so that the group H is abelian. In this case, the action of G on $X = K(M, n)$ is classified by an action of G on the group M together with a k-invariant $\eta \in \mathrm{H}^{n+1}(G; M)$. Since G is a finite p-group and M has order relatively prime to p, the invariant η automatically vanishes. It follows that the homotopy fixed point space X^{hG} is nonempty, and its homotopy groups (for any choice of base point) are given by

$$\pi_*(X^{hG}) = \mathrm{H}^{n-*}(G; M) \simeq \begin{cases} M^G & \text{if } * = n \\ 0 & \text{otherwise.} \end{cases}$$

Assertions (1) and (2) are now immediate, and (3) follows from the identity $|M^G| = |M|$.

It remains to treat the case $n = 1$. In this case, the action of G on X is encoded by a homotopy fiber sequence

$$X \to X_{hG} \xrightarrow{u} BG.$$

Choose a point $x \in X_{hG}$ lying over the base point of BG and set $\widetilde{G} = \pi_1(X_{hG}, x)$, so that we have an exact sequence of finite groups $0 \to M \to \widetilde{G} \xrightarrow{\varphi} G \to 0$. Note that the space of *pointed* sections of the map u can be identified with the set of sections of φ in the category of groups, or equivalently with the set of all p-Sylow subgroups of $\widetilde{G}$ (by identifying a section of φ with its image in $\widetilde{G}$). We therefore obtain a homotopy equivalence

$$\begin{aligned} X^{hG} &\simeq \{\text{Unpointed sections of } u\} \\ &\simeq \{\text{Pointed sections of } u\}_{hM} \\ &\simeq \{p\text{-Sylow subgroups of } \widetilde{G}\}_{hM}. \end{aligned}$$

It follows immediately that X^{hG} is a π-finite space whose homotopy groups are equivalent to subgroups of M (and therefore not divisible by p). Note that the collection of p-Sylow subgroups of G form a single orbit under the action of $\widetilde{G}$ (by Sylow's theorem), hence also under the action of M (since $\widetilde{G}$ is generated by M together with any choice of p-Sylow subgroup $P \subseteq \widetilde{G}$); this proves that X^{hG} is connected. The congruence

$$\begin{aligned}
\mathrm{Mass}(X^{hG}) &= \frac{|\{p\text{-Sylow subgroups of } \widetilde{G}\}|}{|M|} \\
&= \mathrm{Mass}(X) \cdot |\{p\text{-Sylow subgroups of } \widetilde{G}\}| \\
&\equiv \mathrm{Mass}(X) \pmod{p}
\end{aligned}$$

also follows from Sylow's theorem (which guarantees that the number of p-Sylow subgroups of $\widetilde{G}$ is congruent to 1 modulo p). ☐

Variant 7.6.11. Let X be a π-finite space, let p be a prime number which does not divide the order of any homotopy group of X, and suppose that X is equipped with an action of $G = \mathbf{Z}_p^n$ for some nonnegative integer n. Then X^{hG} is a π-finite space, whose homotopy groups have order not divisible by p, and we have $\mathrm{Mass}(X^{hG}) \equiv \mathrm{Mass}(X) \pmod{p}$ in the commutative ring $\mathbf{Z}_{(p)}$.

Proof. Let $\mathrm{Aut}(X) \subseteq X^X$ denote the subspace consisting of homotopy equivalences from X to itself, and let $\mathrm{BAut}(X)$ denote its classifying space. Then $\mathrm{BAut}(X)$ is a π-finite space, and the action of G on X is classified by a pointed map $f : BG \to \mathrm{BAut}(X)$. It follows from Proposition 3.4.7 that the map f is homotopic to a composition $BG \to B(G/G_0) \to \mathrm{BAut}(X)$ for some subgroup $G_0 \subseteq G$ of finite index. Then $G_0 \simeq \mathbf{Z}_p^n$ acts trivially on X, so the homotopy fixed point space X^{hG_0} is equivalent to the mapping space $\mathrm{Fun}(BG_0, X) \simeq X$ (by virtue of our assumption that the homotopy groups of X have order relatively prime to p). The desired result now follows by applying Lemma 7.6.10 to the residual action of the finite p-group G/G_0 on X^{hG_0}. ☐

We will also need the notion of a *p-Sylow map* between π-finite spaces (see [17] for a general discussion).

Definition 7.6.12. Let X be a connected π-finite space and let p be a prime number. We say that a map of spaces $f : Y \to X$ is *p-Sylow* if Y is connected and, for each integer $m \geqslant 1$, the induced map of homotopy groups $\pi_m(Y, y) \to \pi_m(X, f(y))$ induces an isomorphism from $\pi_m(Y, y)$ to a p-Sylow subgroup of $\pi_m(X, f(y))$; here $y \in Y$ is any choice of base point.

Example 7.6.13. Let G be a finite group. Then a map of spaces $Y \to BG$ is p-Sylow if and only if it induces a homotopy equivalence of Y with a connected covering space of BG whose fundamental group is a p-Sylow subgroup $P \subseteq G$.

Remark 7.6.14. Let X be a connected π-finite space and let $f : Y \to X$ be a p-Sylow map. Then Y is a connected p-finite space. Moreover, the homotopy fiber $\mathrm{fib}(f)$ is a π-finite space whose homotopy groups have order relatively prime to p, and the mass

$$\mathrm{Mass}(\mathrm{fib}(f)) = \frac{\mathrm{Mass}(Y)}{\mathrm{Mass}(\mathrm{X})} = \prod_{n>0} (\frac{|\pi_n(Y)|}{|\pi_n(X)|})^{(-1)^n} \in \mathbf{Z}_{(p)}$$

is not divisible by p.

Lemma 7.6.15. *Let X be a connected π-finite space and let p be a prime number. Then there exists a p-Sylow map $f : Y \to X$.*

Proof. Note that X is n-truncated for some $n \gg 0$. We proceed by induction on n. If $n = 1$, the desired result follows from Sylow's theorem (Example 7.6.13). For $n > 1$, let X' denote the truncation $\tau_{\leqslant n-1}(X)$. Our inductive hypothesis then guarantees that there exists a p-Sylow map $Y' \to X'$. Replacing X by the fiber product $X \times_{X'} Y'$, we can reduce to the case where X' is p-finite. The construction $x \mapsto \pi_n(X, x)$ determines a local system of finite abelian groups on X. Since $n > 1$, this is the pullback of a local system $\mathcal{L}$ of finite abelian groups on X'. Write $\mathcal{L}$ as a direct sum $\mathcal{L}_+ \oplus \mathcal{L}_-$, where $\mathcal{L}_+$ is a local system of finite abelian p-groups on X' and $\mathcal{L}_-$ is a local system of finite abelian groups of order relatively prime to p. It follows from obstruction theory that the map $X \to X'$ is classified by a k-invariant $\eta \in \mathrm{H}^{n+1}(X'; \mathcal{L}) \simeq \mathrm{H}^{n+1}(X'; \mathcal{L}_+) \oplus \mathrm{H}^{n+1}(X'; \mathcal{L}_-)$. Since X' is p-finite, the cohomology group $\mathrm{H}^{n+1}(X'; \mathcal{L}_-)$ vanishes. It follows that η is the image of a cohomology class $\eta_+ \in \mathrm{H}^{n+1}(X'; \mathcal{L}_+)$, which is the k-invariant associated to a map $Y \to X'$. By construction, this space is equipped with a p-Sylow map $Y \to X$. □

Remark 7.6.16 (Uniqueness of p-Sylow Maps)**.** Let X be a π-finite space and let p be a prime number. One can show that the p-Sylow map $Y \to X$ of Lemma 7.6.15 is unique up to homotopy equivalence. However, it is not unique up to a contractible space of choices. More precisely, let $\mathcal{C} \subset \mathcal{S}_{/X}$ be the full subcategory spanned by the p-Sylow maps. By refining the argument of Lemma 7.6.15, one can show that $\mathcal{C}$ is a connected π-finite space, whose homotopy groups have order relatively prime to p (moreover, if X is n-truncated, then $\mathcal{C}$ is also n-truncated).

Lemma 7.6.17. *Let p be a prime number and let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over a p-local $\mathbb{E}_\infty$-ring A. Let $f : Y \to X$ be a p-Sylow map of connected π-finite spaces. Assume that, for every prime number $\ell \neq p$ which divides the order of some homotopy group of X, the ℓ-divisible group $\mathbf{G}_{(\ell)}$ vanishes. Then the transfer map $\mathrm{tr}_{Y/X} : A^0_{\mathbf{G}}(Y) \to A^0_{\mathbf{G}}(X)$ carries 1 to an invertible element of $A^0_{\mathbf{G}}(X)$.*

Proof. Let us say that an object $B \in \mathrm{CAlg}_A$ is *good* if the image of $\mathrm{tr}_{Y/X}(1)$ in the tempered cohomology ring $B^0_{\mathbf{G}}(X) \simeq \pi_0(B^X_{\mathbf{G}})$ is invertible. Using Lemma 4.2.11, we see that the collection of good $\mathbb{E}_\infty$-algebras over A is closed under limits. We wish to prove that A is good.

Without loss of generality, we may assume that the p-divisible group $\mathbf{G}_{(p)}$ has some fixed height h. Then A is $E(h)$-local. We will complete the proof by showing that every $E(n)$-local A-algebra is good, for any $n \geqslant 0$. Our proof proceeds by induction on n. For $n > 0$, we have a pullback diagram of A-algebras

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & L_{K(n)}(B) \arrow[d] \\
L_{E(n-1)}(B) \arrow[r] & L_{E(n-1)}(L_{K(n)}(B))
\end{tikzcd}$$

where the bottom left and right corners are good by virtue of our inductive hypothesis. We may therefore replace A by $L_{K(n)}(B)$ and thereby reduce to the case where A is $K(n)$-local. In this case, our orientation of $\mathbf{G}$ supplies an exact sequence of p-divisible groups

$$0 \to \mathbf{G}^{\mathcal{Q}}_A \xrightarrow{i} \mathbf{G}_{(p)} \to \mathbf{G}' \to 0,$$

where $\mathbf{G}'$ is an étale p-divisible group of height $h - n$ (Proposition 2.5.6). Set $\Lambda = (\mathbf{Q}_p / \mathbf{Z}_p)^{h-n}$ and let $C = \mathrm{Split}_\Lambda(i)$ be a splitting algebra of f (Definition 2.7.12). Then C is a faithfully flat A-algebra (Proposition 2.7.15), so A can be realized as the totalization $\mathrm{Tot}(C^\bullet)$ of the cosimplicial A-algebra $C^\bullet$ given by the iterated tensor powers of C over A. It will therefore suffice to show that C is good. Using our inductive hypothesis again, we can replace A by $L_{K(n)}(C)$ and thereby reduce to the case where A is $K(n)$-local and the p-divisible group $\mathbf{G}_{(p)}$ splits as a direct sum $\mathbf{G}_0 \oplus \underline{\Lambda}$, where $\mathbf{G}_0 = \mathbf{G}^{\mathcal{Q}}_A$ is the Quillen p-divisible group of A. In this case, Remark 7.4.9 supplies a commutative diagram

$$\begin{tikzcd}
A^0_{\mathbf{G}}(Y) \arrow[r, "\mathrm{tr}_{Y/X}"] \arrow[d] & A^0_{\mathbf{G}}(X) \arrow[d] \\
A^0_{\mathbf{G}_0}(Y^{B\widehat{\Lambda}}) \arrow[r, "\mathrm{tr}_{Y^{B\widehat{\Lambda}}/X^{B\widehat{\Lambda}}}"] & A^0_{\mathbf{G}_0}(X^{B\widehat{\Lambda}}),
\end{tikzcd}$$

where the vertical maps are the character isomorphisms of Theorem 4.3.2. Using Lemma 4.4.17 and Theorem 4.2.5, we deduce that Atiyah-Segal comparison map

$$A^0_{\mathbf{G}_0}(X^{B\widehat{\Lambda}}) \to A^0(X^{B\widehat{\Lambda}})$$

is an isomorphism. Consequently, an element of the tempered cohomology ring $A^0_{\mathbf{G}_0}(X^{B\widehat{\Lambda}})$ is invertible if and only if it is invertible when evaluated at any point u of the mapping space $X^{B\widehat{\Lambda}}$, which we can represent by a map of spaces $u : B\widehat{\Lambda} \to X$. Using Remark 7.4.8, we are reduced to showing that the transfer map associated to the projection $Y^{B\widehat{\Lambda}} \times_{X^{B\hat{\Lambda}}} \{u\} \to \{u\}$ carries the identity element $1 \in A^0_{\mathbf{G}_0}(Y^{B\widehat{\Lambda}} \times_{X^{B\hat{\Lambda}}} \{u\})$ to an invertible element in $A^0_{\mathbf{G}_0}(\{u\}) \simeq \pi_0(A)$. By virtue of Proposition 7.4.12, this is equivalent to the assertion that the mass of the π-finite space $Z = Y^{B\widehat{\Lambda}} \times_{X^{B\hat{\Lambda}}} \{u\}$ is an invertible element of the commutative ring $\pi_0(A)$. Note that Z can be identified with the homotopy fixed point space for an action of $\widehat{\Lambda}$ on the homotopy fiber $F = \mathrm{fib}(X \to Y)$. Applying Variant 7.6.11 and Remark 7.6.14, we deduce that $\mathrm{Mass}(Z) \equiv \mathrm{Mass}(F) \pmod p$ is an invertible element of the local ring $\mathbf{Z}_{(p)}$, and therefore also invertible in the commutative ring $\pi_0(A)$. □

Proof of Theorem 7.6.8. We proceed as in the proof of Theorem 7.6.3, with a few modifications. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over a p-local $\mathbb{E}_\infty$-ring A, let X be a π-finite space, and let S be the (finite) collection of all prime numbers other than p which divide the order of some homotopy group $\pi_n(X, x)$. Without loss of generality, we may assume that for each $\ell \in S$, the ℓ-divisible group $\mathbf{G}_{(\ell)}$ has some fixed height h_ℓ. Let Λ be the colattice $\bigoplus_{\ell \in S} (\mathbf{Q}_\ell / \mathbf{Z}_\ell)^{h_\ell}$ and let $\widehat{\Lambda} \simeq \prod_{\ell \in S} \mathbf{Z}_\ell^{h_\ell}$ denote its Pontryagin dual. Let Z denote the mapping space $X^{B\widehat{\Lambda}}$. Then Z is also a π-finite space, which decomposes into connected components $\coprod_{i \in I} Z_i$ where I denotes the finite set $\pi_0(Z) = \mathrm{Hom}_{\mathrm{h}\mathcal{S}}(B\widehat{\Lambda}, X)$. For each $i \in I$, we have an evaluation map

$$\mathrm{ev}_i : B\widehat{\Lambda} \times Z_i \to X,$$

which we can identify with a map $e_i : B\widehat{\Lambda} \to X^{Z_i}$. Since X^{Z_i} is also a π-finite space, each of the maps e_i factors as a composition $B\widehat{\Lambda} \to B\widehat{M_i} \xrightarrow{e_i^\circ} X^{Z_i}$ for some finite subgroup $M_i \subseteq \Lambda$ (Proposition 3.4.7). It follows that the evaluation maps ev_i admit a corresponding factorization as

$$B\widehat{\Lambda} \times Z_i \to B\widehat{M_i} \times Z_i \xrightarrow{\mathrm{ev}_i^\circ} X.$$

For each $i \in I$, choose a p-Sylow map $Y_i \to Z_i$ (Lemma 7.6.15). Let g_i denote the composite map $Y_i \times B\widehat{M_i} \to Z_i \times B\widehat{M_i} \xrightarrow{\mathrm{ev}_i^\circ} X$. We will complete the proof by showing that the transfer maps $\mathrm{tr}_{(B\widehat{M}_i \times Y_i)/X}$ induce a surjection

$$\bigoplus_{i \in I} A^0_{\mathbf{G}}(B\widehat{M_i} \times Y_i) \to A^0_{\mathbf{G}}(X).$$

Let $\mathbf{G}'$ denote the sum $\bigoplus_{\ell \in S} \mathbf{G}_{(\ell)}$ and let $C = \mathrm{Split}_{\Lambda}(\mathbf{G}')$ be a splitting algebra for $\mathbf{G}'$ (Definition 2.7.7). Then C is a direct limit of finite étale A-algebras, and there is an isomorphism $\rho : \underline{\Lambda} \to \mathbf{G}'_C$ of $\mathbf{P}$-divisible groups over C. The restriction of ρ to each $\underline{M_i}$ is then classified by a map of A-algebras $u_i : A_{\mathbf{G}}^{B\widehat{M}_i} \to C$. We can then factor the unit map $A \to C$ as a composition $A \to B \to C$, where B is a finite étale A-algebra (of nonzero degree), C is faithfully flat over B, and each of the maps u_i factors through some map of A-algebras $A_{\mathbf{G}}^{B\widehat{M}_i} \to B$, which we can identify with a map of B-algebras $v_i : B_{\mathbf{G}}^{B\widehat{M}_i} \to B$. Since $B_{\mathbf{G}}^{B\widehat{M}_i}$ is an étale B-algebra, this map decomposes the commutative ring $B^0_{\mathbf{G}}(B\widehat{M}_i)$ as a Cartesian product of $\pi_0(B)$ with some auxiliary commutative ring R_i (so that v_i is given by projection onto the first factor). Let $\xi_i \in B^0_{\mathbf{G}}(B\widehat{M}_i)$ be the element which corresponds to the pair $(1, 0)$ under this product decomposition. We will prove the following:

(a) The sum $\sum_{i \in I} \mathrm{tr}_{(B\widehat{M}_i \times Y_i)/X}(\xi_i)$ is an invertible element of the tempered cohomology ring $B^0_{\mathbf{G}}(X)$.

Note that if (a) is satisfied, then the transfer map $\bigoplus_{i \in I} B^0_{\mathbf{G}}(B\widehat{M}_i) \to B^0_{\mathbf{G}}(X)$ is surjective (since its image is automatically an ideal). Since B is finite flat (and faithfully flat) over A, it will then follow from Remark 4.7.4 that the transfer map $\bigoplus_{i \in I} A^0_{\mathbf{G}}(B\widehat{M}_i \times Y_i) \to A^0_{\mathbf{G}}(X)$ is also surjective, completing the proof of Theorem 7.6.8.

Let $C^\bullet$ denote the cosimplicial B-algebra given by the iterated tensor powers of C over B. Since C is faithfully flat over B, the canonical map $B \to \mathrm{Tot}(C^\bullet)$ is an equivalence. It then follows from Lemma 4.2.11 that the map of tempered function spectra $B^X_{\mathbf{G}} \to \mathrm{Tot}(C^{\bullet X}_{\mathbf{G}})$ is also an equivalence. Consequently, to show that the element $\sum_{i \in I} \mathrm{tr}_{(B\widehat{M}_i \times Y_i)/X}(\xi_i)$ is an invertible element of the tempered cohomology ring $B^0_{\mathbf{G}}(X) = \pi_0(B^X_{\mathbf{G}})$ is invertible, it will suffice to show that its image in $C^0_{\mathbf{G}}(X) = \pi_0(C^X_{\mathbf{G}})$ is invertible. Set $\mathbf{G}_0 = \bigoplus_{\ell \notin S} \mathbf{G}_{(\ell)}$, so that the $\mathbf{P}$-divisible group $\mathbf{G}_C$ splits as a direct sum $\mathbf{G}_{0C} \oplus \underline{\Lambda}$. We then have a commutative diagram

$$\begin{tikzcd}
\bigoplus_{i \in I} C^0_{\mathbf{G}}(B\widehat{M}_i \times Y_i) \arrow[r, "\mathrm{tr}_{B\widehat{M}_i/X}"] \arrow[d, "\chi"] & C^0_{\mathbf{G}}(X) \arrow[d, "\chi"] \\
\bigoplus_{i \in I} C^0_{\mathbf{G}_0}((B\widehat{M}_i \times Y_i)^{B\widehat{\Lambda}}) \arrow[r] & C^0_{\mathbf{G}'}(X^{B\widehat{\Lambda}}),
\end{tikzcd}$$

where the vertical maps are the character isomorphisms supplied by Theorem 4.3.2. Since each Y_i is p-finite, the mapping spaces $(B\widehat{M}_i \times Y_i)^{B\widehat{\Lambda}}$ can be identified with a disjoint union

$$\coprod_{\alpha \in \mathrm{Hom}(\widehat{\Lambda}, M_i)} (B\widehat{M_i} \times Y_i)$$

By construction, the image of ξ under the character map can be identified with the element of the tempered cohomology ring

$$C^0_{\mathbf{G}_0}((B\widehat{M_i} \times Y_i)^{B\widehat{\Lambda}}) \simeq \prod_{\alpha \in \mathrm{Hom}(\widehat{\Lambda}, M_i)} C^0_{\mathbf{G}_0}(B\widehat{M_i} \times Y_i)$$

which takes the value 1 on the connected component corresponding to the homomorphism $\widehat{\Lambda} \to \widehat{M_i}$ which is Pontryagin dual to the inclusion map, and 0 on all other connected components. We are therefore reduced to showing that each of the transfer maps

$$\mathrm{tr}_{Y_i \times B\widehat{M_i}/Z_i} : C^0_{\mathbf{G}_0}(B\widehat{M_i} \times Y_i) \to C^0_{\mathbf{G}_0}(Z_i)$$

carries 1 to an invertible element of the tempered cohomology ring $C^0_{\mathbf{G}_0}(Z_i)$. In fact, we claim that $\mathrm{tr}_{(B\widehat{M_i} \times Y_i)/Z_i}(1) = \frac{\mathrm{tr}_{Y_i/Z_i}(1)}{|M_i|}$ (which will imply the desired result, by virtue of Lemma 7.6.17). Using the functoriality of the transfer (Proposition 7.4.5), we are reduced to verifying the identity $\mathrm{tr}_{Y_i/(B\widehat{M_i} \times Y_i)}(1) = |M_i|$ in the tempered cohomology ring $C^0_{\mathbf{G}_0}(B\widehat{M_i} \times Y_i)$. Using the push-pull identity of Proposition 7.4.4, we are reduced to showing that transfer along the base point inclusion $\{e\} \hookrightarrow B\widehat{M_i}$ satisfies $\mathrm{tr}_{\{e\}/B\widehat{M_i}}(1) = |M_i|$, which is a special case of Proposition 7.4.12. □

7.7 Proof of Tempered Ambidexterity

Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, which we regard as fixed throughout this section. Our goal in this section is to prove Theorem 7.2.10, which asserts that every n-truncated relatively π-finite morphism of orbispaces $f : \mathsf{X} \to \mathsf{Y}$ is $\upsilon_{\mathbf{G}}$-ambidextrous. Our proof will proceed by induction on n. The case $n = -1$ follows from Proposition 7.5.3 (and Corollary 7.3.16). To carry out the inductive step, we will prove the following:

Proposition 7.7.1. *Let n be a nonnegative integer, and let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of orbispaces which is relatively π-finite and n-truncated. Assume that every $(n-1)$-truncated, relatively π-finite morphism of orbispaces is $\upsilon_{\mathbf{G}}$-ambidextrous. Then f is $\upsilon_{\mathbf{G}}$-ambidextrous.*

The proof will make use of the following:

Lemma 7.7.2. *Let $f : \mathsf{X}' \to \mathsf{X}$ be a $\upsilon_{\mathbf{G}}$-ambidextrous morphism of orbispaces. Suppose that the transfer map $\mathrm{tr}_{\mathsf{X}'/\mathsf{X}} : A^0_{\mathbf{G}}(\mathsf{X}') \to A^0_{\mathbf{G}}(\mathsf{X})$ is surjective. Then every $\mathbf{G}$-tempered local system $\mathscr{F}$ on X can be realized as a direct summand of $f_!(\mathscr{G})$, for some $\mathbf{G}$-tempered local system $\mathscr{G}$ on X'.*

Proof. By virtue of the projection formula (Theorem 7.3.1), it will suffice to treat the case $\mathscr{F} = \underline{A}_{\mathsf{X}}$. Let $u : \underline{A}_{\mathsf{X}'} \to f_*(\underline{A}_{\mathsf{X}'})$ and $v : f_!(\underline{A}_{\mathsf{X}}) \to \underline{A}_{\mathsf{X}'}$ be the unit and counit maps. For any element $t \in A^0_{\mathbf{G}}(\mathsf{X}')$, the composite map

$$\underline{A}_{\mathsf{X}} \xrightarrow{u} f_*(\underline{A}_{\mathsf{X}'}) \xrightarrow{f_*(t)} f_*(\underline{A}_{\mathsf{X}'}) \xrightarrow{\mathrm{Nm}_f^{-1}} f_!(\underline{A}_{\mathsf{X}'}) \xrightarrow{v} \underline{A}_{\mathsf{X}}$$

is given by multiplication by the element $\mathrm{tr}_{\mathsf{X}'/\mathsf{X}}(t) \in A^0_{\mathbf{G}}(\mathsf{X})$. Choosing t such that $\mathrm{tr}_{\mathsf{X}'/\mathsf{X}}(t) = 1$, we see that this diagram exhibits $\underline{A}_{\mathsf{X}'}$ as a retract of $f_!(\underline{A}_{\mathsf{X}})$. ☐

Proof of Proposition 7.7.1. Let $f : \mathsf{X} \to \mathsf{Y}$ be a map of orbispaces which is n-truncated and relatively π-finite; we wish to show that f is $\upsilon_{\mathbf{G}}$-ambidextrous. By virtue of Corollary 7.3.16, we may assume without loss of generality that $\mathsf{Y} = Y^{(-)}$ where $Y \in \mathscr{T}$ is the classifying space of a finite abelian group. Our assumption that f is relatively π-finite then implies that X is representable by a π-finite space X. Using Proposition 7.5.3, we can assume that X is connected. If $n = 0$, then X is a connected covering space of $Y \in \mathscr{T}$, and is therefore also the classifying space of a finite abelian group. In this case, we can identify f with a finite product of maps $f_{(p)} : X_{(p)} \to Y_{(p)}$ between p-finite spaces, so that the desired result follows from Theorem 7.5.1. We will therefore assume that $n > 0$, so that the space X is n-truncated. Note that the relative diagonal map $\delta : X \to X \times_Y X$ is $(n-1)$-truncated, and is therefore $\upsilon_{\mathbf{G}}$-ambidextrous by virtue of our inductive hypothesis. It follows that f is weakly $\upsilon_{\mathbf{G}}$-ambidextrous. In particular, for every $\mathbf{G}$-tempered local system $\mathscr{F}$ on X, we can associate a norm map $\mathrm{Nm}_f : f_!(\mathscr{F}) \to f_*(\mathscr{F})$ (Notation 7.2.3).

Let $\mathrm{AForm}(f) : [X/Y] \otimes [X/Y] \to \underline{A}_Y$ denote the ambidexterity form of Construction 7.3.13. Then f is $\upsilon_{\mathbf{G}}$-ambidextrous if and only if $\mathrm{AForm}(f)$ is a duality datum in the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(Y)$ (Proposition 7.3.15). Let S be the (finite) set of all prime numbers which divide the order of some homotopy group of X, and let N be the product of all the numbers which belong to S. Then $A[\frac{1}{N}]$ and $\{A_{(p)}\}_{p \in S}$ comprise a faithfully flat covering of A. By virtue of Proposition 6.2.6, the ambidexterity form $\mathrm{AForm}(f)$ is a duality datum in $\mathrm{LocSys}_{\mathbf{G}}(Y)$ if and only if its image is a duality datum in each of the symmetric monoidal ∞-categories

$$A[\frac{1}{N}] \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y) \qquad A_{(p)} \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y).$$

It will therefore suffice to show that f is $\upsilon_{\mathbf{G}}$-ambidextrous after extending scalars from A to the localizations $A[\frac{1}{N}]$ and $A_{(p)}$ for $p \in S$. We now break into two cases:

- Suppose that $A = A[\frac{1}{N}]$: that is, every prime number $p \in S$ is invertible in A. To complete the proof, it will suffice to show that the norm map $\mathrm{Nm}_f : f_!(\mathscr{F}) \to f_*(\mathscr{F})$ is an equivalence for every $\mathbf{G}$-tempered local system $\mathscr{F}$ on X. Using Theorem 7.6.3, we can choose a map of spaces $g : X' \to X$, where X' is a finite disjoint union of objects of $\mathscr{T}$ and the transfer map $\mathrm{tr}_{X'/X} : A^0_{\mathbf{G}}(X') \to A^0_{\mathbf{G}}(X)$ is surjective. Moreover, we can assume that the map g is $(n-1)$-truncated (this is automatic for $n \geqslant 2$, and for $n = 1$ it follows from Remark 7.6.4). Invoking Lemma 7.7.2, we deduce that $\mathscr{F}$ can be written as a direct summand of $g_!(\mathscr{G})$, for some object $\mathscr{G}$ on $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$. It we are therefore reduced to showing that the norm map $\mathrm{Nm}_f : f_!(g_!\mathscr{G}) \to f_*(g_!(\mathscr{G}))$ is an equivalence. By assumption, every $(n-1)$-truncated morphism of π-finite spaces is $\upsilon_{\mathbf{G}}$-ambidextrous. In particular, we have a norm equivalence $\mathrm{Nm}_g : g_!(\mathscr{G}) \to g_*(\mathscr{G})$. Moreover, the composition
$$f_!(g_!(\mathscr{G})) \xrightarrow{\mathrm{Nm}_f} f_*(g_!(\mathscr{G})) \xrightarrow{f_*(\mathrm{Nm}_g)} f_*(g_*(\mathscr{G}))$$
is given by the norm map $\mathrm{Nm}_{g \circ f}$ associated the composition $(g \circ f) : X' \to Y$ (Remark Ambi.4.2.4). It will therefore suffice to show that the composite map $g \circ f$ is $\upsilon_{\mathbf{G}}$-ambidextrous. Writing X' as a union of connected components $\coprod_{i \in I} X'_i$, we are reduced to showing that each of the composite maps $h_i : X'_i \hookrightarrow X' \xrightarrow{g} X \xrightarrow{f} Y$ is $\upsilon_{\mathbf{G}}$-ambidextrous (Proposition 7.5.3). This is clear, since h_i is a map between classifying spaces of finite abelian groups, and therefore factors as a finite product of maps $(X'_i)_{(p)} \to Y_{(p)}$ between p-finite spaces (each of which is $\upsilon_{\mathbf{G}}$-ambidextrous by virtue of Theorem 7.5.1).

- Suppose that the $\mathbb{E}_\infty$-ring A is p-local, for some prime number p. As before, we will complete the proof by showing that the norm map $\mathrm{Nm}_f : f_!(\mathscr{F}) \to f_*(\mathscr{F})$ is an equivalence for every $\mathbf{G}$-tempered local system $\mathscr{F}$ on X. Using Theorem 7.6.8, we can choose a map of spaces $g : X' = \coprod_{i \in I} X'_i \to X$, where each X'_i is a product of an object of $\mathscr{T}$ with a p-finite space, and the transfer map $\mathrm{tr}_{X'/X} : A^0_{\mathbf{G}}(X') \to A^0_{\mathbf{G}}(X)$ is surjective. By virtue of Remark 7.6.6, we can assume without loss of generality that g is $(n-1)$-truncated. Lemma 7.7.2 implies that $\mathscr{F}$ can be written as a direct summand of $g_!(\mathscr{G})$, for some object $\mathscr{G}$ on $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$. It we are therefore reduced to showing that the norm map $\mathrm{Nm}_f : f_!(g_!\mathscr{G}) \to f_*(g_!(\mathscr{G}))$ is an equivalence. As above, we note that g is

$\upsilon_{\mathbf{G}}$-ambidextrous and that the composition

$$f_!(g_!(\mathscr{G})) \xrightarrow{\mathrm{Nm}_f} f_*(g_!(\mathscr{G})) \xrightarrow{f_*(\mathrm{Nm}_g)} f_*(g_*(\mathscr{G}))$$

can be identified with the norm map $\mathrm{Nm}_{g \circ f} : (g \circ f)_!(\mathscr{G}) \to (g \circ f)_*(\mathscr{G})$. It will therefore suffice to show that $g \circ f$ is $\upsilon_{\mathbf{G}}$-ambidextrous. By virtue of Proposition 7.5.3, we are reduced to showing that each of the composite maps $g \circ f$ is $\upsilon_{\mathbf{G}}$-ambidextrous. Writing X' as a union of connected components $\coprod_{i \in I} X'_i$, we are reduced to showing that each of the composite maps $h_i : X'_i \hookrightarrow X' \xrightarrow{g} X \xrightarrow{f} Y$ is $\upsilon_{\mathbf{G}}$-ambidextrous. This again follows from Theorem 7.5.1, since h_i can be written as a finite product of maps between ℓ-finite spaces (which are classifying spaces of finite abelian ℓ-groups for $\ell \neq p$).

□

7.8 Applications of Tempered Ambidexterity

Our goal in this section is to summarize some of the consequences of tempered ambidexterity. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Then Theorem 7.2.10 immediately implies the following:

Proposition 7.8.1. *Let $f : X \to Y$ be a map of π-finite spaces. Then the functors $f_!, f_* : \mathrm{LocSys}_{\mathbf{G}}(X) \to \mathrm{LocSys}_{\mathbf{G}}(Y)$ are equivalent.*

Corollary 7.8.2. *Let $f : X \to Y$ be a map of π-finite spaces. Then the functor $f_* : \mathrm{LocSys}_{\mathbf{G}}(X) \to \mathrm{LocSys}_{\mathbf{G}}(Y)$ preserves small colimits, and the functor $f_! : \mathrm{LocSys}_{\mathbf{G}}(X) \to \mathrm{LocSys}_{\mathbf{G}}(Y)$ preserves small limits.*

Fix a prime number p. It follows immediately from the definition that for every map of spaces $f : X \to Y$, the pullback functor $f^* : \mathrm{LocSys}_{\mathbf{G}}(Y) \to \mathrm{LocSys}_{\mathbf{G}}(X)$ carries $\mathrm{LocSys}_{\mathbf{G}}^{K(m)}(Y)$ into $\mathrm{LocSys}_{\mathbf{G}}^{K(m)}(X)$ for every nonnegative integer m. Since the collection of $K(m)$-local spectra is closed under the formation of limits, the description of f_* supplied by Construction 7.1.1 shows that the functor f_* carries $\mathrm{LocSys}_{\mathbf{G}}^{K(m)}(X)$ into $\mathrm{LocSys}_{\mathbf{G}}^{K(m)}(Y)$. In particular, for each object $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(X)$, the canonical map $f_* \mathscr{F} \to f_*(L_{K(m)} \mathscr{F})$ factors through $L_{K(m)} f_* \mathscr{F}$.

Corollary 7.8.3. *Let $f : X \to Y$ be a map of π-finite spaces. Then, for every object $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(X)$ and every integer $m \geqslant 0$, the canonical map $L_{K(m)}(f_* \mathscr{F}) \to f_*(L_{K(m)} \mathscr{F})$ is an equivalence in $\mathrm{LocSys}_{\mathbf{G}}(Y)$.*

Proof. We wish to prove that the map $f_* \mathscr{F} \to f_*(L_{K(m)} \mathscr{F})$ is a $K(m)$-equivalence. Let $\mathscr{F}'$ denote the fiber of the canonical map $\mathscr{F} \to L_{K(m)} \mathscr{F}$; we wish to prove that $f_* \mathscr{F}'$ is $K(m)$-acyclic: that is, that the mapping space $\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(Y)}(f_* \mathscr{F}', \mathscr{G})$ is contractible for every object $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}^{K(m)}(Y)$. Using Proposition 7.8.1, we obtain a homotopy equivalence

$$\mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(Y)}(f_* \mathscr{F}', \mathscr{G}) \simeq \mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(X)}(\mathscr{F}', f^* \mathscr{G}),$$

so that the desired result follows from the $K(m)$-acyclicity of $\mathscr{F}'$ (since the pullback $f^* \mathscr{G}$ is $K(m)$-local). □

Remark 7.8.4. Using Corollary 7.8.3, we can deduce Theorem 7.2.10 from Theorem Ambi.5.2.1 . However, we do not know a direct proof of Corollary 7.8.3.

Proposition 7.8.5. *Let $f : X \to Y$ be a map of π-finite spaces. Then the functors $f^* : \mathrm{LocSys}_{\mathbf{G}}(Y) \to \mathrm{LocSys}_{\mathbf{G}}(X)$ and $f_* : \mathrm{LocSys}_{\mathbf{G}}(X) \to \mathrm{LocSys}_{\mathbf{G}}(Y)$ preserve compact objects.*

Proof. Proposition 7.8.1 supplies an equivalence $f_* \simeq f_!$; it will therefore suffice to show that f^* and $f_!$ preserve compact objects. By virtue of Proposition HTT.5.5.7.2 , it will suffice to prove that the right adjoint functors f_* and f^* preserve filtered colimits. In the second case this is obvious, and in the first case it follows from Corollary 7.8.3. □

If $\mathscr{F}$ is a $\mathbf{G}$-tempered local system on a space X, then the A-module $\Gamma(X; \mathscr{F})$ is given by the direct image $q_* \mathscr{F}$, where q is the projection map from X to a point (and we identify $\mathrm{LocSys}_{\mathbf{G}}(*)$ with the ∞-category Mod_A). We therefore have the following consequence of Corollary 7.8.2 and Proposition 7.8.5:

Corollary 7.8.6. *Let X be a π-finite space. Then the tempered global sections functor*

$$\Gamma(X; \bullet) : \mathrm{LocSys}_{\mathbf{G}}(X) \to \mathrm{Mod}_A \qquad \mathscr{F} \mapsto \Gamma(X; \mathscr{F})$$

commutes with small colimits, and carries compact objects of $\mathrm{LocSys}_{\mathbf{G}}(X)$ to compact objects of Mod_A.

Notation 7.8.7. For every space X, we view $\mathrm{LocSys}_{\mathbf{G}}(X)$ as an A-linear ∞-category. For every pair of objects $\mathscr{F}, \mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(X)$, we write $\underline{\mathrm{Map}}(\mathscr{G}, \mathscr{F})$ for the A-module classifying morphisms from $\mathscr{F}$ to $\mathscr{G}$ (so that we have canonical homotopy equivalences $\mathrm{Map}_{\mathrm{Mod}_A}(M, \underline{\mathrm{Map}}(\mathscr{G}, \mathscr{F})) \simeq \mathrm{Map}_{\mathrm{LocSys}_{\mathbf{G}}(X)}(M \otimes_A \mathscr{G}, \mathscr{F})$, depending functorially on $M \in \mathrm{Mod}_A$).

Proposition 7.8.8. *Let X be a π-finite space and let $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(X)$. The following conditions are equivalent:*

1. *The object $\mathscr{F}$ is a compact object of* $\mathrm{LocSys}_{\mathbf{G}}(X)$
2. *The object $\mathscr{F}$ is dualizable (with respect to the tensor product introduced in §5.8.*

Proof. If $\mathscr{F}$ is dualizable, then the functor

$$\begin{aligned} \mathscr{G} & \mapsto \underline{\mathrm{Map}}(\mathscr{F}, \mathscr{G}) \\ & \simeq \underline{\mathrm{Map}}(\underline{A}_X, \mathscr{F}^\vee \otimes \mathscr{G}) \\ & \simeq \Gamma(X; \mathscr{F}^\vee \otimes \mathscr{G}) \end{aligned}$$

commutes with small colimits, since the functors $\mathscr{G} \mapsto \mathscr{F}^\vee \otimes \mathscr{G}$ and $\Gamma(X; \bullet)$ commute with small colimits (Corollary 7.8.6). This shows that (2) ⇒ (1).

We now prove that (1) ⇒ (2). Let $\mathcal{C} \subseteq \mathrm{LocSys}_{\mathbf{G}}(X)$ be the full subcategory spanned by the dualizable objects. Then $\mathcal{C}$ is a stable subcategory of $\mathrm{LocSys}_{\mathbf{G}}(X)$, and the first part of the proof shows that every object of $\mathcal{C}$ is compact in $\mathrm{LocSys}_{\mathbf{G}}$. Applying Proposition HTT.5.3.5.11 , we obtain a fully faithful embedding $f : \mathrm{Ind}(\mathcal{C}) \to \mathrm{LocSys}_{\mathbf{G}}(X)$ which preserves filtered colimits. We will show that f is an equivalence of ∞-categories. It will then follow that every compact object of $\mathrm{LocSys}_{\mathbf{G}}(X)$ is a retract of an object of $\mathcal{C}$; since $\mathcal{C}$ is closed under retracts, the implication (1) ⇒ (2) follows.

Using Corollary HTT.5.5.2.9 , we see that f admits a right adjoint g. To prove that f is an equivalence of ∞-categories, it will suffice to show that g is conservative. Fix an object $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(X)$ such that $g(\mathscr{F}) \simeq 0$; we wish to show that $\mathscr{F} \simeq 0$. Choose any object $T \in \mathscr{T}_{/X}$. Then we have an equivalence $\mathscr{F}(T) \simeq \underline{\mathrm{Map}}(f_! \underline{A}_T, \mathscr{F}) = \underline{\mathrm{Map}}([T/X], \mathscr{F})$. This spectrum vanishes, since $[T/X] = f_! \underline{A}_T$ is a self-dual object of $\mathrm{LocSys}_{\mathbf{G}}(X)$. ☐

Corollary 7.8.9. *Let X be a π-finite space. Then $\mathrm{LocSys}_{\mathbf{G}}(X)$ is a proper A-linear ∞-category. That is, for every pair of compact objects $\mathscr{F}, \mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(X)$, the A-module $\underline{\mathrm{Map}}(\mathscr{F}, \mathscr{G})$ is perfect.*

Proof. Since $\mathscr{F}$ and $\mathscr{G}$ are compact, they are dualizable (Proposition 7.8.8). We then have a equivalences

$$\underline{\mathrm{Map}}(\mathscr{F}, \mathscr{G}) \simeq \underline{\mathrm{Map}}(\underline{A}_X, \mathscr{F}^\vee \otimes \mathscr{G}) \simeq \Gamma(X; \mathscr{F}^\vee \otimes \mathscr{G}).$$

We now observe that the tensor product $\mathscr{F}^\vee \otimes \mathscr{G}$ is dualizable and therefore compact, and the functor $\Gamma(X; \bullet)$ preserves compact objects by virtue of Corollary 7.8.6. ☐

Recall that if X and Y are π-finite spaces, then Corollary 4.7.11 guarantees that the multiplication map $A^X_{\mathbf{G}} \otimes_A A^Y_{\mathbf{G}} \to A^{X \times Y}_{\mathbf{G}}$ is an equivalence of A-modules. We now establish a relative version of this result.

Notation 7.8.10 (External Tensor Products)**.** Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let X and Y be spaces, and let $q_X : X \times Y \to X$ and $q_Y : X \times Y \to Y$ denote the projection maps. Then the pullback functors

$$\mathrm{LocSys}_{\mathbf{G}}(X) \xrightarrow{q_X^*} \mathrm{LocSys}_{\mathbf{G}}(X \times Y) \xleftarrow{q_Y^*} \mathrm{LocSys}_{\mathbf{G}}(Y)$$

determine an A-linear functor

$$\lambda : \mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y) \to \mathrm{LocSys}_{\mathbf{G}}(X \times Y).$$

In particular, we obtain an external tensor product functor

$$\boxtimes : \mathrm{LocSys}_{\mathbf{G}}(X) \times \mathrm{LocSys}_{\mathbf{G}}(Y) \to \mathrm{LocSys}_{\mathbf{G}}(X \times Y),$$

which is A-linear separately in each variable, given concretely by the formula

$$\mathscr{F} \boxtimes \mathscr{G} = q_X^* \mathscr{F} \otimes q_Y^* \mathscr{G}.$$

Proposition 7.8.11 (Künneth Formula)**.** *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Let X and Y be π-finite spaces. Then, for every pair of objects $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(X)$ and $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(Y)$, the canonical map*

$$\theta : \Gamma(X; \mathscr{F}) \otimes_A \Gamma(Y; \mathscr{G}) \to \Gamma(X \times Y; \mathscr{F} \boxtimes \mathscr{G})$$

is an equivalence in Mod_A.

Proof. Form a pullback diagram of spaces

$$\begin{tikzcd}
X \times Y \arrow[r, "q_X"] \arrow[d, "q_Y"] & X \arrow[d, "p_Y"] \\
Y \arrow[r, "p_X"] & *.
\end{tikzcd}$$

Using Theorem 7.3.10 (and Theorem 7.1.6), we see that θ factors as a composition of equivalences

$$\begin{aligned}
\Gamma(X; \mathscr{F}) \otimes_A \Gamma(Y; \mathscr{G}) &= (p_{Y*} \mathscr{F}) \otimes_A (p_{X*} \mathscr{G}) \\
&\xrightarrow{\sim} p_{Y*}(\mathscr{F} \otimes p_Y^* p_{X*} \mathscr{G}) \\
&\xrightarrow{\sim} p_{Y*}(\mathscr{F} \otimes q_{X*} q_Y^* \mathscr{G}) \\
&\xrightarrow{\sim} p_{Y*}(q_{X*}(q_X^* \mathscr{F} \otimes q_Y^* \mathscr{G})) \\
&= \Gamma(X \times Y; \mathscr{F} \boxtimes \mathscr{G}).
\end{aligned}$$

□

Corollary 7.8.12. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Let X and Y be π-finite spaces. Then the A-linear functor*

$$\lambda : \mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y) \to \mathrm{LocSys}_{\mathbf{G}}(X \times Y)$$

of Notation 7.8.10 is fully faithful.

Proof. For every pair of objects $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(X)$ and $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(Y)$, we let $\mathscr{F} \boxtimes \mathscr{G}$ denote the image of $(\mathscr{F}, \mathscr{G})$ in the tensor product $\mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y)$. We also define $\mathscr{F} \boxtimes \mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(X \times Y)$ as in Notation 7.8.10, so that $\mathscr{F} \boxtimes \mathscr{G} \simeq \lambda(\mathscr{F} \boxtimes \mathscr{G})$.

Since $\mathrm{LocSys}_{\mathbf{G}}(X)$ and $\mathrm{LocSys}_{\mathbf{G}}(Y)$ are compactly generated A-linear ∞-categories, the tensor product $\mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y)$ is also compactly generated. To prove that λ is fully faithful, it will suffice to prove the following (see Proposition HTT.5.3.5.11):

(a) The functor λ carries compact objects of $\mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y)$ to compact objects of $\mathrm{LocSys}_{\mathbf{G}}(X \times Y)$.

(b) The functor λ is fully faithful when restricted to compact objects.

Let $\mathcal{C}$ denote the full subcategory of $\mathrm{LocSys}_{\mathbf{G}}(X) \otimes_{\mathrm{Mod}_A} \mathrm{LocSys}_{\mathbf{G}}(Y)$ spanned by the compact objects, and let $\mathcal{C}_0$ denote the full subcategory of $\mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y)$ spanned by objects of the form $\mathscr{F} \boxtimes \mathscr{G}$, where $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(X)$ and $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(Y)$ are compact. Then $\mathcal{C}$ is generated by $\mathcal{C}_0$ under colimits and retracts. Consequently, to prove (a), it will suffice to show that $\lambda(\mathscr{F} \boxtimes \mathscr{G}) \simeq \mathscr{F} \boxtimes \mathscr{G}$ is a compact object of $\mathrm{LocSys}_{\mathbf{G}}(X \times Y)$ whenever $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(X)$ and $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(Y)$ are compact. This follows immediately from Corollary 7.8.8, since it is clear that $\mathscr{F} \boxtimes \mathscr{G}$ is dualizable whenever $\mathscr{F}$ and $\mathscr{G}$ are dualizable.

To prove (b), it will suffice to show that for every pair of objects $C, C' \in \mathcal{C}$, the canonical map

$$\theta : \underline{\mathrm{Map}}_{\mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y)}(C, C') \to \underline{\mathrm{Map}}(\lambda(C), \lambda(C'))$$

is an equivalence of A-modules If we regard C' as fixed, then the collection of those objects $C \in \mathcal{C}$ for which θ is an equivalence is closed under retracts and finite colimits; we may therefore assume without loss of generality that C has the form $\mathscr{F} \boxtimes \mathscr{G}$, where $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(X)$ and $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(Y)$ are compact. By a similar argument, we may suppose that $C' = \mathscr{F}' \boxtimes \mathscr{G}'$ where $\mathscr{F}' \in \mathrm{LocSys}_{\mathbf{G}}(X)$ and $\mathscr{G}' \in \mathrm{LocSys}_{\mathbf{G}}(Y)$. In this case, the θ can be identified with the canonical map

$$\Gamma(X; \mathscr{F}^\vee \otimes \mathscr{F}') \otimes_A \Gamma(Y; \mathscr{G}^\vee \otimes \mathscr{G}') \to \Gamma(X \times Y; (\mathscr{F}^\vee \otimes \mathscr{F}') \boxtimes (\mathscr{G}^\vee \otimes \mathscr{G}')),$$

which is an equivalence by virtue of Proposition 7.8.11. $\square$

In the situation of Corollary 7.8.12, the embedding

$$\lambda : \mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y) \to \mathrm{LocSys}_{\mathbf{G}}(X \times Y)$$

is generally not essentially surjective. However, we have the following partial result:

Proposition 7.8.13. *Let p be a prime number and let $\mathbf{G}$ be an oriented p-divisible group over an $\mathbb{E}_\infty$-ring A. Let X and Y be π-finite spaces, and let*

$$\lambda : \mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y) \hookrightarrow \mathrm{LocSys}_{\mathbf{G}}(X \times Y)$$

be the fully faithful embedding of Corollary 7.8.12. Then the essential image of λ includes all p-nilpotent objects of $\mathrm{LocSys}_{\mathbf{G}}(X \times Y)$.

Proof. Let $\mathscr{F}$ be a p-nilpotent object of the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(X \times Y)$; we wish to show that $\mathscr{F}$ belongs to the essential image of λ. Let $\lambda^R : \mathrm{LocSys}_{\mathbf{G}}(X \times Y) \to \mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y)$ be a right adjoint to λ. Then we have a canonical fiber sequence $\mathscr{F}' \to \mathscr{F} \xrightarrow{u} \mathscr{F}''$ where $\mathscr{F}' \simeq (\lambda \circ \lambda^R)(\mathscr{F})$ belongs to the essential image of λ and $\mathscr{F}''$ is annihilated by the functor λ^R. We will complete the proof by showing that u is nullhomotopic, so that $\mathscr{F}$ is equivalent to a direct summand of $\mathscr{F}'$ and therefore also belongs to the essential image of λ. By virtue of our assumption that $\mathscr{F}$ is p-nilpotent, it will suffice to show that the multiplication map $p : \mathscr{F}'' \to \mathscr{F}''$ is an equivalence. Assume otherwise. Then we can choose an object $T \in \mathscr{T}_{/\mathsf{X} \times \mathsf{Y}}$ such that for which the map $p : \mathscr{F}''(T) \to \mathscr{F}''(T)$ is not an equivalence. Choose a connected covering space $T_0 \in \mathrm{Cov}(T)$ for which the fundamental group of T_0 is the p-local summand of the fundamental group of $\pi_1(T)$. Our assumption that $\mathbf{G}$ is a p-divisible group then guarantees that the pullback map $A^T_{\mathbf{G}} \to A^{T_0}_{\mathbf{G}}$ is an equivalence, so that $I(T_0/T)$ is the zero ideal of $A^0_{\mathbf{G}}(T)$. Invoking the fact that $\mathscr{F}''$ is a tempered local system, we see that the canonical map $\mathscr{F}''(T) \to \mathscr{F}''(T_0)^{\mathrm{hAut}(T_0/T)}$ is an equivalence. It follows that the map $p : \mathscr{F}''(T_0) \to \mathscr{F}''(T_0)$ is not an equivalence: in other words, the cofiber $\mathscr{F}''/p\mathscr{F}''$ does not vanish on T_0. is nonzero. Set $f_0 = f|_{T_0}$ and $g_0 = g|_{T_0}$, and regard the product $T_0 \times T_0$ as an object of $\mathscr{T}_{/\mathsf{X} \times \mathsf{Y}}$ via the product map $f_0 \times g_0 : T_0 \times T_0 \to \mathsf{X} \times \mathsf{Y}$. Using Corollary 5.5.5, we deduce that $(\mathscr{F}''/p\mathscr{F}'')(T_0)$ can be identified with the tensor product $A^{T_0}_{\mathbf{G}} \times_{A_{\mathbf{G}}{T_0 \times T_0}} (\mathscr{F}''/p\mathscr{F}'')(T_0 \times T_0)$. It follows that the spectrum $\mathscr{F}''(T_0 \times T_0)$ does not vanish, contradicting our assumption that $\mathscr{F}''$ is annihilated by the functor λ^R. $\square$

Remark 7.8.14. Assume that $\mathbf{G}$ is an oriented p-divisible group (for some prime number p) and let X and Y be π-finite spaces. Proposition 7.8.13 is equivalent to the assertion that the embedding

$$\lambda : \mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y) \hookrightarrow \mathrm{LocSys}_{\mathbf{G}}(X \times Y)$$

becomes an equivalence after extending scalars along the p-completion functor $\mathrm{Mod}_A \to \mathrm{Mod}_A^{\mathrm{Cpl}(p)}$. More precisely, λ induces an equivalence of ∞-categories

$$\widehat{\lambda} : \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Cpl}(p)}(X) \otimes_{\mathrm{Mod}_A^{\mathrm{Cpl}(p)}} \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Cpl}(p)}(Y) \simeq \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Cpl}(p)}(X \times Y).$$

7.9 Dualizability of Tempered Local Systems

Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. For any π-finite space X, the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(X)$ is compactly generated ∞-category (Corollary 5.3.3), whose compact objects are the dualizable tempered local systems on X (Proposition 7.8.8). In this section, we study the condition of dualizability in more detail.

Proposition 7.9.1. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and let $\mathscr{F}$ be a $\mathbf{G}$-tempered local system on an orbispace X. If $\mathscr{F}$ is a dualizable (as an object of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$), then $\mathscr{F}(T)$ is a perfect A-module for each object $T \in \mathscr{T}_{/\mathsf{X}}$.*

Proof. Let T be an object of $\mathscr{T}$ equipped with a map $f : T \to \mathsf{X}$. Then $f^*(\mathscr{F})$ is a dualizable object of $\mathrm{LocSys}_{\mathbf{G}}(T)$ and therefore compact as an object of $\mathrm{LocSys}_{\mathbf{G}}(T)$ (Proposition 7.8.8). Applying Corollary 7.8.6, we conclude that $\mathscr{F}(T) \simeq \Gamma(T; f^*(\mathscr{F}))$ is a perfect A-module. □

Proposition 7.9.1 has a partial converse.

Theorem 7.9.2. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let $\mathscr{F}$ be a $\mathbf{G}$-tempered local system on an orbispace X, and let p be a prime number. The following conditions are equivalent:*

(1) *For each object $T \in \mathscr{T}_{/\mathsf{X}}$, the cofiber of the map $p : \mathscr{F}(T) \to \mathscr{F}(T)$ is perfect when regarded as an A-module.*

(2) *The cofiber $\mathscr{F}/p\mathscr{F} = \mathrm{cofib}(p : \mathscr{F} \to \mathscr{F})$ is dualizable when regarded as an object of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ (with respect to the tempered tensor product of Construction 5.8.7).*

Remark 7.9.3. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A and let p be a prime number. Let X be an orbispace and let $\mathrm{LocSys}^{\mathrm{Cpl}(p)}_{\mathbf{G}}(\mathsf{X})$ denote the full subcategory of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ spanned by the p-complete tempered local systems. Then the symmetric monoidal structure on tempered local systems (Construction 5.8.7) induces a symmetric monoidal structure

$$\widehat{\otimes} : \mathrm{LocSys}^{\mathrm{Cpl}(p)}_{\mathbf{G}}(\mathsf{X}) \times \mathrm{LocSys}^{\mathrm{Cpl}(p)}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}^{\mathrm{Cpl}(p)}_{\mathbf{G}}(\mathsf{X}) \qquad \mathscr{F} \widehat{\otimes} \mathscr{G} = (\mathscr{F} \otimes \mathscr{G})^{\wedge}_{(p)}.$$

Using Theorem 7.9.2, we see that the following conditions on p-complete tempered local system $\mathscr{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ are equivalent:

(1) For each object $T \in \mathscr{T}_{/\mathsf{X}}$, the spectrum $\mathscr{F}(T)$ is dualizable as an object of the ∞-category $\mathrm{Mod}^{\mathrm{Cpl}(p)}_{A}$ (with respect to the completed tensor product).

(2) The tempered local system $\mathscr{F}$ is a dualizable object of $\mathrm{LocSys}^{\mathrm{Cpl}(p)}_{\mathbf{G}}(\mathsf{X})$.

Remark 7.9.4. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, let p be a prime number, and let $\mathscr{F}$ be a $\mathbf{G}$-tempered local system on an orbispace X which satisfies the equivalent conditions of Remark 7.9.3. Then, for every object $T \in \mathscr{T}_{/\mathsf{X}}$, the spectrum $\mathscr{F}(T)$ is dualizable as an object of the ∞-category $\mathrm{Mod}^{\mathrm{Cpl}(p)}_{A^T_{\mathbf{G}}}$ (with respect to the completed tensor product). When $\pi_1(T)$ is a p-group, this follows from Corollary 5.5.5 (and the general case follows from a similar argument).

Example 7.9.5. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over a p-complete $\mathbb{E}_\infty$-ring A and let $f : T' \to T$ be any morphism in $\mathscr{T}$. It follows from Proposition 7.8.5 that the direct image $f_*(\underline{A}_{T'})$ is dualizable as an object of the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(T)$, and therefore also with respect to the completed tensor product on the subcategory $\mathrm{LocSys}^{\mathrm{Cpl}(p)}_{\mathbf{G}}(T)$. Applying Remark 7.9.4, we deduce that the tempered function spectrum $A^{T'}_{\mathbf{G}}$ is dualizable as an object of the ∞-category of p-complete modules over $A^T_{\mathbf{G}}$. In other words, the cofiber $A^{T'}_{\mathbf{G}}/pA^{T'}_{\mathbf{G}} \simeq \mathrm{cofib}(p : A^{T'}_{\mathbf{G}} \to A^{T'}_{\mathbf{G}})$ is a perfect $A^T_{\mathbf{G}}$-module. Beware that $A^{T'}_{\mathbf{G}}$ itself is usually not perfect as an $A^T_{\mathbf{G}}$-module (unless the map $f : T' \to T$ has connected homotopy fibers, in which case $A^{T'}_{\mathbf{G}}$ is a projective module of finite rank over $A^T_{\mathbf{G}}$).

Remark 7.9.6. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A, and let $\mathscr{F}$ be a $\mathbf{G}$-tempered local system on an orbispace X. Suppose that, for every object $T \in \mathscr{T}_{/\mathsf{X}}$, the spectrum $\mathscr{F}(T)$ is perfect as an A-module. Using Remark 7.9.3), we conclude that for every prime number p, the completion $\mathscr{F}^{\wedge}_{(p)}$ is dualizable with respect to the completed tensor product on the ∞-category $\mathrm{LocSys}^{\mathrm{Cpl}(p)}_{\mathbf{G}}(\mathsf{X})$. One can also show (by a much easier argument) that the rationalization $\mathbf{Q} \otimes_S \mathscr{F}$ is dualizable as an object of the symmetric monoidal ∞-category $\mathbf{Q} \otimes_A \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$. However, it does not follow formally that $\mathscr{F}$ is a dualizable as an object of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ (it unlikely that this is true in general: see Warning 7.9.12).

Our proof of Theorem 7.9.2 will make use of some auxiliary constructions which may be of independent interest.

Construction 7.9.7 (Integral Transforms)**.** Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_\infty$-ring A. Let X and Y be orbispaces, and let

$$\pi_{\mathsf{X}} : \mathsf{X} \times \mathsf{Y} \to \mathsf{X} \qquad \pi_{\mathsf{Y}} : \mathsf{X} \times \mathsf{Y} \to \mathsf{Y}$$

denote the projection maps. Let $\mathscr{K}$ be a $\mathbf{G}$-tempered local system on the product $\mathsf{X} \times \mathsf{Y}$. We let $T_{\mathscr{K}} : \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X}) \to \mathrm{LocSys}_{\mathbf{G}}(\mathsf{Y})$ denote the functor given by the formula

$$T_{\mathscr{K}}(\mathscr{F}) = \pi_{\mathsf{Y}!}(\mathscr{K} \otimes \pi_{\mathsf{X}}^* \mathscr{F}).$$

We refer to $T_{\mathscr{K}}$ as the *integral transform* associated to the $\mathbf{G}$-tempered local system $\mathscr{K}$.

Example 7.9.8 (The Functor $f_!$ as an Integral Transform)**.** Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of orbispaces, let $\Gamma(f) : \mathsf{X} \to \mathsf{X} \times \mathsf{Y}$ denote its graph, and set $\mathscr{K} = \Gamma(f)_!(\underline{A}_{\mathsf{X}}) \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X} \times \mathsf{Y})$. Then the integral transform $T_{\mathscr{K}}$ is given by the construction

$$\begin{aligned} T_{\mathscr{K}}(\mathscr{F}) &= \pi_{\mathsf{Y}!}(\mathscr{K} \otimes \pi_{\mathsf{X}}^*(\mathscr{F})) \\ &\simeq \pi_{\mathsf{Y}!}(\Gamma(f)_!(\underline{A}_{\mathsf{X}}) \otimes \pi_{\mathsf{X}}^*(\mathscr{F})) \\ &\simeq \pi_{\mathsf{Y}!}(\Gamma(f)_!(\underline{A}_{\mathsf{X}} \otimes \Gamma(f)^* \pi_{\mathsf{X}}^*(\mathscr{F})) \\ &\simeq (\pi_{\mathsf{Y}} \circ \Gamma(f))_!((\pi_{\mathsf{X}} \circ \Gamma(f))^*(\mathscr{F})) \\ &\simeq f_!(\mathscr{F}). \end{aligned}$$

where the second equivalence is provided by the projection formula of Theorem 7.3.1.

Example 7.9.9 (The Functor f^* as an Integral Transform)**.** Let $f : \mathsf{Y} \to \mathsf{X}$ be a morphism of orbispaces, let $\Gamma(f) : \mathsf{Y} \to \mathsf{X} \times \mathsf{Y}$ denote its graph, and set $\mathscr{K} = \Gamma(f)_!(\underline{A}_{\mathsf{X}}) \in \mathrm{LocSys}_{\mathbf{G}}(\mathsf{X} \times \mathsf{Y})$. Then the integral transform $T_{\mathscr{K}}$ is given by the construction

$$\begin{aligned}
T_{\mathscr{K}}(\mathscr{F}) &= \pi_{\mathsf{Y}!}(\mathscr{K} \otimes \pi_{\mathsf{X}}^{*}(\mathscr{F})) \\
&\simeq \pi_{\mathsf{Y}!}(\Gamma(f)_{!}(\underline{A}_{\mathsf{Y}}) \otimes \pi_{\mathsf{X}}^{*}(\mathscr{F})) \\
&\simeq \pi_{\mathsf{Y}!}(\Gamma(f)_{!}(\underline{A}_{\mathsf{Y}} \otimes \Gamma(f)^{*}\pi_{\mathsf{X}}^{*}(\mathscr{F})) \\
&\simeq (\pi_{\mathsf{Y}} \circ \Gamma(f))_{!}((\pi_{\mathsf{X}} \circ \Gamma(f))^{*}(\mathscr{F})) \\
&\simeq f^{*}(\mathscr{F})
\end{aligned}$$

where the second equivalence is provided by the projection formula of Theorem 7.3.1.

Example 7.9.10. Let X and Y be π-finite spaces, and consider the pullback diagram

$$\begin{tikzcd}
X \times Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \arrow[d, "q"] \\
Y \arrow[r, "q'"] & *.
\end{tikzcd}$$

Suppose we are given tempered local systems $\mathscr{G} \in \mathrm{LocSys}_{\mathbf{G}}(X)$ and $\mathscr{H} \in \mathrm{LocSys}_{\mathbf{G}}(Y)$, where $\mathscr{G}$ is dualizable. Set $\mathscr{K} = \mathscr{G}^{\vee} \boxtimes \mathscr{H} \in \mathrm{LocSys}_{\mathbf{G}}(X \times Y)$. The the integral transform $T_{\mathscr{K}}$ is given by the construction

$$\begin{aligned}
T_{\mathscr{K}}(\mathscr{F}) &= \pi_{Y!}(\mathscr{K} \otimes \pi_{X}^{*}(\mathscr{F})) \\
&\simeq \pi_{Y!}(\pi_{X}^{*}(\mathscr{G}^{\vee}) \otimes \pi_{Y}^{*}(\mathscr{H}) \otimes \pi_{X}^{*}(\mathscr{F})) \\
&\simeq \pi_{Y!}(\pi_{X}^{*}(\mathscr{G}^{\vee} \otimes \mathscr{F})) \otimes \mathscr{H} \\
&\simeq q'_{*}q_{!}(\mathscr{G}^{\vee} \otimes \mathscr{F}) \otimes \mathscr{H} \\
&\simeq q'^{*}q_{*}(\mathscr{G}^{\vee} \otimes \mathscr{F}) \otimes \mathscr{H} \\
&\simeq \underline{\mathrm{Hom}}(\mathscr{G}, \mathscr{F}) \otimes_{A} \mathscr{H}\,.
\end{aligned}$$

Here the second equivalence is given by the projection formula of Theorem 7.3.1, the third by the Beck-Chevalley property of Corollary 7.1.7, and the fourth from ambidexterity for the projection map $q : X \to *$ (Proposition 7.8.1).

Remark 7.9.11. Let $\mathbf{G}$ be an oriented $\mathbf{P}$-divisible group over an $\mathbb{E}_{\infty}$-ring A, and let X and Y be π-finite spaces. It follows from Proposition 7.8.8 that $\mathrm{LocSys}_{\mathbf{G}}(X)$ is canonically self-dual as an A-linear ∞-category. Consequently, we can identify the tensor product $\mathrm{LocSys}_{\mathbf{G}}(X) \otimes_{A} \mathrm{LocSys}_{\mathbf{G}}(Y)$ with the ∞-category of colimit-preserving A-linear functors from $\mathrm{LocSys}_{\mathbf{G}}(X)$ to $\mathrm{LocSys}_{\mathbf{G}}(Y)$. Under this identification, the formation of integral transforms $\mathscr{K} \mapsto T_{\mathscr{K}}$ corresponds to a functor

$$\lambda^{R} : \mathrm{LocSys}_{\mathbf{G}}(X \times Y) \to \mathrm{LocSys}_{\mathbf{G}}(X) \otimes_{A} \mathrm{LocSys}_{\mathbf{G}}(Y).$$

Unwinding the definitions, we see that λ^R can be identified with the right adjoint of the functor

$$\lambda : \mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(Y) \to \mathrm{LocSys}_{\mathbf{G}}(X \times Y).$$

classifying the external tensor product of $\mathbf{G}$-tempered local systems (Notation 7.8.10). It follows from Corollary 7.8.12 that the functor λ^R is essentially surjective: in other words, every colimit-preserving A-linear functor from $\mathrm{LocSys}_{\mathbf{G}}(X)$ to $\mathrm{LocSys}_{\mathbf{G}}(Y)$ is equivalent to the integral transform $T_{\mathscr{K}}$ for some $\mathbf{G}$-tempered local system $\mathscr{K} \in \mathrm{LocSys}_{\mathbf{G}}(X \times Y)$. Beware that $\mathscr{K}$ is not unique: it can be chosen canonically by demanding that it belongs to the essential image of the functor λ, but this choice might not be desirable (see Warning 7.9.12).

Proof of Theorem 7.9.2. Let $\mathscr{F}$ be a $\mathbf{G}$-tempered local system on an orbispace X with the property that, for every object $T \in \mathscr{T}_{/\mathsf{X}}$, the cofiber $\mathscr{F}(T)/p\mathscr{F}(T) = \mathrm{cofib}(p : \mathscr{F}(T) \to \mathscr{F}(T))$ is a perfect A-module. We wish to show that $\mathscr{F}/p\mathscr{F}$ is a dualizable object of $\mathrm{LocSys}_{\mathbf{G}}(\mathsf{X})$ (the converse follows from Proposition 7.9.1). By virtue of Remark 5.2.11, we may assume that $\mathsf{X} = X^{(-)}$ is representable by an object $X \in \mathscr{T}$. Without loss of generality, we can replace A by $A_{(p)}$ and thereby reduce to the case where A is p-local. Let S be the collection of all prime numbers $\ell \neq p$ which divide the order of the fundamental group $\pi_1(X)$. Note that replacing $\mathbf{G}$ by the oriented $\mathbf{P}$-divisible group $\mathbf{G}' = \mathbf{G}_{(p)} \oplus \bigoplus_{\ell \in S} \mathbf{G}_{(\ell)}$ does not change the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(X)$ (see Proposition 5.4.2). We may therefore replace $\mathbf{G}$ by $\mathbf{G}'$ and thereby reduce to the case where the ℓ-divisible groups $\mathbf{G}_{(\ell)}$ vanish for $\ell \notin S \cup \{p\}$. By virtue of Proposition 6.2.6, it will suffice to test the dualizability of $\mathscr{F}/p\mathscr{F}$ after faithfully flat base change. We may therefore assume without loss of generality that the $\mathbf{P}$-divisible group $\mathbf{G}$ splits as a direct sum $\mathbf{G}_0 \oplus \underline{\Lambda}$, where $\mathbf{G}_0 = \mathbf{G}_{(p)}$ is a p-divisible group and $\underline{\Lambda}$ is the constant $\mathbf{P}$-divisible group associated to a colattice with $\Lambda_{(p)} \simeq 0$. In this case, Theorem 6.4.1 supplies a fully faithful symmetric monoidal embedding

$$\Phi : \mathrm{LocSys}_{\mathbf{G}}(X) \to \mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(X)),$$

whose essential image is spanned by the isotropic objects of $\mathrm{LocSys}_{\mathbf{G}_0}(\mathcal{L}^{\Lambda}(X))$ (Theorem 6.5.13). It is not difficult to see that if the isotropic $\mathbf{G}_0$-tempered local system $\Phi(\mathscr{F}/p\mathscr{F})$ is dualizable, then the dual $\Phi(\mathscr{F}\,p\mathscr{F})^{\vee}$ is also isotropic and can therefore be written as $\Phi(\mathscr{G})$, where $\mathscr{G}$ is a dual of $\mathscr{F}/p\mathscr{F}$ in the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(X)$. We may therefore replace $\mathscr{F}$ by $\Phi(\mathscr{F})$, X by $\mathcal{L}^{\Lambda}(X)$, and $\mathbf{G}$ by $\mathbf{G}_0$, thereby reducing to the case where $\mathbf{G}$ is a p-divisible group and X is a π-finite space (which might no longer belong to $\mathscr{T}$).

Let $\delta : X \to X \times X$ be the diagonal map and set $\mathscr{K}_0 = \delta_! \underline{A}_X$. Then $T_{\mathscr{K}_0}$ is the identity functor (see Example 7.9.8 or 7.9.9), so we can identify $\mathscr{F}/p\mathscr{F}$ with the $\mathbf{G}$-tempered local system $T_{\mathscr{K}_0/p\mathscr{K}_0}(\mathscr{F})$. It follows from the above analysis that $\mathscr{K}_0/p\mathscr{K}_0$ belongs to $\mathcal{C}_0$. Let us say that an object $\mathscr{K} \in \mathrm{LocSys}_{\mathbf{G}}(X \times X)$ is *good* if the integral transform $T_{\mathscr{K}}(\mathscr{F})$ is a dualizable object of $\mathrm{LocSys}_{\mathbf{G}}(X)$. It will therefore suffice to show that $\mathscr{K}_0/p\mathscr{K}_0$ is good. In fact, we will prove something stronger: every compact p-nilpotent object of $\mathrm{LocSys}_{\mathbf{G}}(X \times X)$ is good.

Given a pair of objects $T, T' \in \mathscr{T}_{/X}$, let $\mathscr{K}_{T,T'}$ denote the external tensor product $[T/X] \boxtimes [T'/X]$, which we view as a $\mathbf{G}$-tempered local system on $X \times X$ Let $\mathcal{C}_0$ be the smallest stable subcategory of $\mathrm{LocSys}_{\mathbf{G}}(X \times X)$ which contains the objects $\mathscr{K}_{T,T'}/p\mathscr{K}_{T,T'}$ and is closed under retracts, let $\mathcal{C}_1 \subseteq \mathrm{LocSys}_{\mathbf{G}}(X \times X)$ be the subcategory generated by $\mathcal{C}_0$ under small colimits, and let $\mathcal{C}_2 \subseteq \mathrm{LocSys}_{\mathbf{G}}(X \times X)$ be the smallest subcategory which contains the objects $\mathscr{K}_{T,T'}$ and is closed under shifts and small colimits. Then we have inclusions

$$\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathrm{LocSys}_{\mathbf{G}}(X \times X)$$

with the following properties:

- The ∞-category $\mathcal{C}_2$ is the essential image of the functor λ appearing in Corollary 7.8.12. Consequently, $\mathcal{C}_2$ contains all p-nilpotent objects of $\mathrm{LocSys}_{\mathbf{G}}(X \times X)$ (Proposition 7.8.13).
- For every object $\mathscr{G} \in \mathcal{C}_2$, the fiber of the rationalization map $\mathscr{G} \to \mathscr{G}[p^{-1}]$ belongs to $\mathcal{C}_1$. Consequently, $\mathcal{C}_1$ also contains all p-nilpotent objects of $\mathrm{LocSys}_{\mathbf{G}}(X \times X)$.
- The ∞-category $\mathcal{C}_1$ is equivalent to $\mathrm{Ind}(\mathcal{C}_0)$. Consequently, if a compact object of $\mathrm{LocSys}_{\mathbf{G}}(X \times X)$ is contained in $\mathcal{C}_1$, then it is also contained in $\mathcal{C}_0$.

We will complete the proof by showing that every object of $\mathcal{C}_0$ is good. Since the collection of good objects of $\mathrm{LocSys}_{\mathbf{G}}(X \times X)$ is closed under shifts, suspensions, and retracts, it will suffice to show that each of the $\mathbf{G}$-tempered local systems $\mathscr{K}_{T,T'}/p\mathscr{K}_{T,T'}$ is good. Using the self-duality of $[T/X]$ and Example 7.9.10, we obtain an equivalence

$$T_{\mathscr{K}_{T,T'}/p\mathscr{K}_{T,T'}}(\mathscr{F}) \simeq (\mathscr{F}(T)/p\mathscr{F}(T)) \otimes_A [T'/X],$$

which is dualizable by virtue of our assumption that $\mathscr{F}(T)/p\mathscr{F}(T)$ is perfect as an A-module (together with the dualizability of the object $[T'/X]$). □

Warning 7.9.12. For every π-finite space X, the ∞-category of tempered local systems $\mathrm{LocSys}_{\mathbf{G}}(X)$ is a proper A-linear ∞-category (Corollary 7.8.9). However, it is usually not a *smooth* A-linear ∞-category: that is, the identity functor $\mathrm{id}_{\mathrm{LocSys}_{\mathbf{G}}(X)} : \mathrm{LocSys}_{\mathbf{G}}(X) \to \mathrm{LocSys}_{\mathbf{G}}(X)$ need not be compact as an object of the ∞-category $\mathrm{End}_A(\mathrm{LocSys}_{\mathbf{G}}(X))$ of A-linear endofunctors of $\mathrm{LocSys}_{\mathbf{G}}(X)$. In essence, this is due to the failure of the embedding

$$\lambda : \mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(X) \to \mathrm{LocSys}_{\mathbf{G}}(X \times X)$$

of Corollary 7.8.12 to be an equivalence of ∞-categories. By virtue of Remark 7.9.11, the smoothness of $\mathrm{LocSys}_{\mathbf{G}}(X)$ is equivalent to the compactness of $\lambda^R(\delta_! \underline{A}_X)$, where $\delta : X \to X \times X$ is the diagonal map and λ^R denotes the right adjoint of the functor

$$\lambda : \mathrm{LocSys}_{\mathbf{G}}(X) \otimes_A \mathrm{LocSys}_{\mathbf{G}}(X) \to \mathrm{LocSys}_{\mathbf{G}}(X \times X).$$

The $\mathbf{G}$-tempered local system $\delta_! \underline{A}_X$ is compact when viewed as an object of the ∞-category $\mathrm{LocSys}_{\mathbf{G}}(X \times X)$ but usually does not belong to the essential image of the functor λ, so that $\lambda^R(\delta_! \underline{A}_X)$ need not be compact.

Note that when $\mathbf{G}$ is a p-divisible group, then the functor λ induces an equivalence on p-nilpotent objects. One can use this to show that the ∞-category $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{Cpl}(p)}(X)$ is "p-adically smooth": that is, it is fully dualizable when viewed as an object of the symmetric monoidal $(\infty, 2)$-category of $\mathrm{Mod}_A^{\mathrm{Cpl}(p)}$-linear ∞-categories (this smoothness was implicitly used in our proof of Theorem 7.9.2).

References

[1] M. F. Atiyah. Characters and cohomology of finite groups. *Inst. Hautes Études Sci. Publ. Math.*, (9):23–64, 1961.

[2] A. M. Bohmann. Global orthogonal spectra. *Homology Homotopy Appl.*, 16(1):313–332, 2014.

[3] D. Gepner and A. Henriques. Homotopy theory of orbispaces.

[4] J. P. C. Greenlees and J. P. May. Localization and completion theorems for MU-module spectra. *Ann. of Math. (2)*, 146(3):509–544, 1997.

[5] M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594, 2000.

[6] M. J. Hopkins and J. Lurie. Ambidexterity in $K(n)$-local stable homotopy theory.

[7] A. Körschgen. A comparison of two models of orbispaces. *Homology Homotopy Appl.*, 20(1):329–358, 2018.

[8] J. Lurie. Elliptic cohomology I: Spectral abelian varieties.

[9] J. Lurie. Elliptic cohomology II: Orientations.

[10] J. Lurie. Elliptic cohomology IV: Equivariant elliptic cohomology.

[11] J. Lurie. *Higher Algebra.*

[12] J. Lurie. *Spectral Algebraic Geometry.*

[13] J. Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[14] A. Mathew, N. Naumann, and J. Noel. On a nilpotence conjecture of J. P. May. *J. Topol.*, 8(4):917–932, 2015.

[15] J. P. May. *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. With contributions by M. Cole, G. Comezaña, S.

Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner.

[16] H. Miller. The Sullivan conjecture on maps from classifying spaces. *Ann. of Math. (2)*, 120(1):39–87, 1984.

[17] M. Prasma and T. M. Schlank. Sylow theorems for ∞-groups. *Topology Appl.*, 222:121–138, 2017.

[18] S. Schwede. Orbispaces, orthogonal spaces, and the universal compact lie group.

[19] S. Schwede. *Global homotopy theory*, volume 34 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2018.

[20] G. Segal. Equivariant K-theory. *Inst. Hautes Études Sci. Publ. Math.*, (34):129–151, 1968.

[21] N. Stapleton. Transchromatic generalized character maps. *Algebr. Geom. Topol.*, 13(1):171–203, 2013.

[22] N. Stapleton. Transchromatic twisted character maps. *J. Homotopy Relat. Struct.*, 10(1):29–61, 2015.