Elliptic Cohomology II: Orientations

April 26, 2018

Contents

Introduction

Let A be a homotopy commutative ring spectrum. We say that A is *even periodic* if the graded ring $\pi_*(A)$ is isomorphic to $\pi_0(A)[u^{\pm 1}]$, for some element $u \in \pi_2(A)$. In this case, Quillen observed that the formal spectrum $\widehat{\mathbf{G}} = \mathrm{Spf}(A^0(\mathbf{CP}^\infty))$ can be regarded as a 1-dimensional formal group over the commutative ring $R = \pi_0(A)$. The formal group $\widehat{\mathbf{G}}$ is a very powerful invariant of the ring spectrum A. In many cases, it is a complete invariant:

Theorem 0.0.1 (Landweber Exact Functor Theorem)**.** *Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional commutative formal group over R, equipped with an isomorphism of formal R-schemes $\widehat{\mathbf{G}} \simeq \mathrm{Spf}(R[[t]])$. Assume that:*

- $(*)$ *For every prime number p, let $[p] : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$ be the map given by multiplication by p, so that we can write $[p]^*(t)$ as a formal power series $\sum_{n \geqslant 0} c_n t^n$. Then $\{c_{p^n}\}_{n \geqslant 0}$ is a regular sequence in R: in other words, each c_{p^n} is a non-zero divisor in the quotient ring $R/(c_1, c_p, \ldots, c_{p^{n-1}})$.*

Then there exists an even periodic homotopy commutative ring spectrum A with $\pi_0(A) \simeq R$ and $\mathrm{Spf}(A^0(\mathbf{CP}^\infty) \simeq \widehat{\mathbf{G}}$. Moreover, the ring spectrum A is unique (up to homotopy equivalence).

Remark 0.0.2. Condition $(*)$ of Theorem 0.0.1 is known as *Landweber's criterion*; one can show that it is independent of the choice of coordinate t.

Example 0.0.3. Let $R = \mathbf{Z}$ be the ring of integers and let $\widehat{\mathbf{G}}_m$ be the formal multiplicative group over $\mathbf{Z}$, defined as the formal completion of the multiplicative group $\mathbf{G}_m = \mathrm{Spec}(\mathbf{Z}[u^{\pm 1}])$ along the identity section. Then $t = u - 1$ is a coordinate on $\widehat{\mathbf{G}}_m$ which satisfies

$$[p]^*(t) = (1+t)^p - 1 = pt + \binom{p}{2} t^2 + \cdots + pt^{p-1} + t^p$$

for each prime number p. Since $(p, 1, 0, 0, \ldots)$ is a regular sequence in $\mathbf{Z}$, the formal group $\widehat{\mathbf{G}}$ satisfies Landweber's criterion. It follows from Theorem 0.0.1 that there exists an essentially unique even periodic ring spectrum A with $\pi_0(A) \simeq \mathbf{Z}$ and $\mathrm{Spf}(A^0(\mathbf{CP}^\infty)) \simeq \widehat{\mathbf{G}}_m$.

Let KU denote the complex K-theory spectrum. Then KU is an even periodic ring spectrum with $\pi_0(\mathrm{KU}) \simeq \mathbf{Z}$ and $\mathrm{Spf}(\mathrm{KU}^0(\mathbf{CP}^\infty)) \simeq \widehat{\mathbf{G}}_m$. It follows that KU is homotopy equivalent to the ring spectrum A of Example 0.0.3. Consequently, we can regard Example 0.0.3 as providing a "purely algebraic" construction of KU, which makes no reference to the theory of complex vector bundles. However, this algebraic approach has certain limitations. In the situation of Theorem 0.0.1, the homology theory $X \mapsto A_*(X)$ can be recovered functorially from the pair $(R, \widehat{\mathbf{G}})$. However, one passes from the homology theory A_* to its representing spectrum A by means of the Brown representability theorem, rather than an explicit procedure. Consequently, the construction $(R, \widehat{\mathbf{G}}) \to A$ is functorial only up to homotopy.

Example 0.0.4. Let $\widehat{\mathbf{G}}$ be a formal group satisfying Landweber's criterion and let A be the associated even periodic ring spectrum. The two-element group $\langle \pm 1 \rangle$ acts on the formal group $\widehat{\mathbf{G}}$ and this extends to an action of $\langle \pm 1 \rangle$ on A as an object in the homotopy category of spectra hSp. When $R = \mathbf{Z}$ and $\widehat{\mathbf{G}} = \widehat{\mathbf{G}}_m$, we recover the action of $\langle \pm 1 \rangle \simeq \mathrm{Gal}(\mathbf{C} / \mathbf{R})$ on KU by complex conjugation. In this case, one can do better: the action of $\langle \pm 1 \rangle$ on KU in the homotopy category hSp can be *rectified* to an action of $\langle \pm 1 \rangle$ on the spectrum KU itself. This rectification contains useful information: for example, it allows us to recover the real K-theory spectrum KO as the homotopy fixed point spectrum $\mathrm{KU}^{h\langle \pm 1 \rangle}$. However, it cannot be obtained formally from Theorem 0.0.1.

Example 0.0.5. To every formal group $\widehat{\mathbf{G}}$ satisfying Landweber's criterion, Theorem 0.0.1 supplies a homotopy commutative ring spectrum A: that is, a commutative algebra object in the homotopy category of spectra hSp. However, in the case $R = \mathbf{Z}$ and $\widehat{\mathbf{G}} = \widehat{\mathbf{G}}_m$, we can do better: KU is an $\mathbb{E}_\infty$-ring spectrum. In other words, the multiplication on KU is commutative and associative not only up to homotopy, but up to *coherent* homotopy. The $\mathbb{E}_\infty$-structure on KU is not directly visible at the level of the homology theory $X \mapsto \mathrm{KU}_*(X)$, and therefore cannot be extracted from Theorem 0.0.1.

Examples 0.0.4 and 0.0.5 illustrate a general phenomenon: in many cases of interest, the ring spectra associated to formal groups satisfying Landweber's criterion possess additional structure which is not visible at the level of the homotopy category of spectra.

Example 0.0.6 (Lubin-Tate Spectra)**.** Let κ be a perfect field of characteristic $p > 0$ and let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group over κ of height $n < \infty$. Lubin and Tate showed that $\widehat{\mathbf{G}}_0$ admits a universal deformation $\widehat{\mathbf{G}}$ defined over a regular local ring $R_{\mathrm{LT}} \simeq W(k)[[v_1, \ldots, v_{n-1}]]$. Morava observed that the formal group $\widehat{\mathbf{G}}$ satisfies

Landweber's criterion, so that Theorem 0.0.1 supplies an even periodic ring spectrum E with $R_{\mathrm{LT}} \simeq \pi_0(E)$ and $\widehat{\mathbf{G}} \simeq \mathrm{Spf}(E^0(\mathbf{CP}^\infty))$. We will refer to E as the *Lubin-Tate spectrum of* $\widehat{\mathbf{G}}_0$ (it is also commonly known as Morava E-theory). A theorem of Goerss, Hopkins, and Miller asserts that E admits an essentially unique $\mathbb{E}_\infty$-ring structure and depends functorially on the pair $(\kappa, \widehat{\mathbf{G}}_0)$; in particular, the Lubin-Tate spectrum E carries an action of the automorphism group $\mathrm{Aut}(\widehat{\mathbf{G}}_0)$ (which is defined at the spectrum level, rather than the level of homotopy category hSp).

Example 0.0.7 (Elliptic Cohomology)**.** Let R be a commutative ring and let E be an elliptic curve over R, classified by a map $\rho : \mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{Ell}}$ where $\mathcal{M}_{\mathrm{Ell}}$ denotes the moduli stack of elliptic curves. Let $\widehat{E}$ denote the formal completion of E along its identity section, and assume that the formal group $\widehat{E}$ admits a coordinate (this condition is always satisfied locally on $\mathrm{Spec}(R)$). If the morphism ρ is flat, then the formal group $\widehat{E}$ satisfies Landweber's criterion, so that Theorem 0.0.1 supplies an even-periodic spectrum A_ρ with $R \simeq \pi_0(A_\rho)$ and $\widehat{E} \simeq \mathrm{Spf}(A^0_\rho(\mathbf{CP}^\infty))$. In the case where ρ is étale, a theorem of Goerss, Hopkins, and Miller asserts that A_ρ can be promoted to an $\mathbb{E}_\infty$-ring which depends functorially on the pair (R, E). Moreover, the construction $\rho \mapsto A_\rho$ determines a sheaf $\mathscr{O}$ on the étale site of $\mathcal{M}_{\mathrm{Ell}}$, taking values in the ∞-category of $\mathbb{E}_\infty$-ring spectra. Passing to global sections, we obtain an $\mathbb{E}_\infty$-ring $\mathrm{TMF} = \Gamma(\mathcal{M}_{\mathrm{Ell}}; \mathscr{O})$, known as the spectrum of (periodic) *topological modular forms.*

Our goal in this paper is to prove a variant of the Landweber exact functor theorem which explains the special features of Examples 0.0.3, 0.0.6, and 0.0.7. Our main result can be understood as a generalization of Example 0.0.6 to the case where κ is not a field. Suppose we are given a Noetherian $\mathbf{F}_p$-algebra R_0 and a p-divisible group $\mathbf{G}_0$ over R_0. We define a *deformation* of $\mathbf{G}_0$ to be a pullback diagram

$$\begin{tikzcd}
\mathbf{G}_0 \arrow[r] \arrow[d] & \mathbf{G}_A \arrow[d] \\
\mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(A)
\end{tikzcd}$$

where the lower horizontal map is determined by a surjection of Noetherian rings $\rho : A \to R$, $\mathbf{G}_A$ is a p-divisible group over A, and the ring A is complete with respect to $\ker(\rho)$. In this case, we also abuse terminology and say that $\mathbf{G}_A$ is a *deformation of* $\mathbf{G}_0$ which is *defined over* A. Under some mild assumptions, one can show that $\mathbf{G}_0$ admits a *universal* deformation $\mathbf{G}$, which is defined over a Noetherian ring $R^{\mathrm{cl}}_{\mathbf{G}_0}$ which we will refer to as the *classical deformation ring* of $\mathbf{G}_0$. The content of this paper is that $R^{\mathrm{cl}}_{\mathbf{G}_0}$ arises as the underlying commutative ring of an $\mathbb{E}_\infty$-ring (for a more precise formulation, see Theorem 6.0.3):

Theorem 0.0.8. *Let R_0 be a Noetherian $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a 1-dimensional p-divisible group over R_0. Assume that the Frobenius map $\varphi_{R_0} : R_0 \to R_0$ is finite and that $\mathbf{G}_0$ is nonstationary (see Definition 3.0.8). Then:*

(a) The p-divisible group $\mathbf{G}_0$ admits a universal deformation $\mathbf{G}$, defined over a Noetherian commutative ring $R^{\mathrm{cl}}_{\mathbf{G}_0}$.

(b) There exists an weakly 2-periodic $\mathbb{E}_\infty$-ring spectrum E, whose homotopy groups are concentrated in even degrees, such that $\pi_0(E) \simeq R^{\mathrm{cl}}_{\mathbf{G}_0}$ and $\mathrm{Spf}(E^0(\mathbf{CP}^\infty))$ is the identity component of the p-divisible group $\mathbf{G}$.

Moreover, the $\mathbb{E}_\infty$-ring E can be chosen to depend functorially on the pair $(R_0, \mathbf{G}_0)$.

It is possible to prove a weaker version of Theorem 0.0.8 using the Landweber exact functor theorem. Namely, one begins by constructing the universal deformation $\mathbf{G}$ (and the classical deformation ring $R^{\mathrm{cl}}_{\mathbf{G}_0}$); this is a purely algebraic problem. The p-divisible group $\mathbf{G}$ has an identity component $\mathbf{G}^\circ$, which is a 1-dimensional formal group over $R^{\mathrm{cl}}_{\mathbf{G}_0}$. If the formal group $\mathbf{G}^\circ$ admits a coordinate, then one can show that it satisfies condition $(*)$ of Theorem 0.0.1, which proves the existence of a *homotopy commutative* ring spectrum E satisfying requirement (b) of Theorem 0.0.8 (if $\mathbf{G}^\circ$ does not admit a coordinate, one instead uses a mild generalization of Theorem 0.0.1).

Our proof of Theorem 0.0.8 will proceed differently, and will not use Landweber's theorem. Instead, we will realize E as the solution to a moduli problem in the setting of $\mathbb{E}_\infty$-ring spectra. Note that an essential feature of E is that its associated formal group $\mathrm{Spf}(E^0(\mathbf{CP}^\infty))$ is the identity component of a deformation of $\mathbf{G}_0$ over the commutative ring $\pi_0(E)$. We will establish a more refined statement: the formal group $\mathrm{Spf}(E^0(\mathbf{CP}^\infty))$ can be promoted to a formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_E$ over the $\mathbb{E}_\infty$-ring E itself, which can be realized as the identity component of a p-divisible group which is a deformation of $\mathbf{G}_0$ (in a suitable sense). Moreover, we will characterize E as *universal* among $\mathbb{E}_\infty$-rings for which such a realization exists.

Let us now outline the contents of this paper. Our primary objective is to turn the idea sketched above into a precise construction of an $\mathbb{E}_\infty$-ring E, and to prove that E satisfies the requirements of Theorem 0.0.8. We begin with the observation that the notion of p-divisible group makes sense over an arbitrary $\mathbb{E}_\infty$-ring A (see Definition AV.6.5.1). Consequently, given $(R_0, \mathbf{G}_0)$ as in the statement of Theorem 0.0.8, one can consider *deformations* of $\mathbf{G}_0$ which are defined over $\mathbb{E}_\infty$-rings. In §3, we will prove that there is a universal such deformation $\mathbf{G}$, which is defined over an $\mathbb{E}_\infty$-ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ which we refer to as the *spectral deformation ring* of $\mathbf{G}_0$ (Theorem 3.0.11). This can be regarded as a generalization of part (a) of Theorem 0.0.8: the classical deformation ring $R^{\mathrm{cl}}_{\mathbf{G}_0}$ can be recovered as the ring of connected components $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$, and the classical universal deformation of $\mathbf{G}_0$ is obtained from $\mathbf{G}$ by extension of scalars along the projection map $R^{\mathrm{un}}_{\mathbf{G}_0} \to \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$ (see Corollary 3.0.13).

The spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is not yet the $\mathbb{E}_\infty$-ring we are looking for. Roughly speaking, it classifies *arbitrary* deformations of $\mathbf{G}_0$, while we would like to consider only those deformations whose underlying formal group is fixed. To make this precise, we need to extend the theory of formal groups to the setting of structured ring spectra. We will develop this extension in §1; in particular, we will define the notion of a formal group $\widehat{\mathbf{G}}$ over an arbitrary $\mathbb{E}_\infty$-ring A (Definition 1.6.1). In §2, we specialize to the case where the $\mathbb{E}_\infty$-ring A is (p)-complete, and show that the theories of formal groups and p-divisible groups are closely related. More precisely, we show that every p-divisible group over A admits an *identity component* $\mathbf{G}^\circ$, which is a formal group over A (Theorem 2.0.8), and that in many cases the passage from $\mathbf{G}$ to $\mathbf{G}^\circ$ does not lose very much information (Theorem 2.3.12).

If A is an even periodic homotopy commutative ring spectrum, then we can regard $\mathrm{Spf}(A^0(\mathbf{CP}^\infty))$ as a 1-dimensional formal group over the commutative ring $\pi_0(A)$. However, if A is an $\mathbb{E}_\infty$-ring, we can do better: in §4, we introduce a formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ defined over A itself, which we will refer to as *the Quillen formal group of* A. Most of §4 is devoted to the problem of *recognizing* the Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$. Our main result is that, for any 1-dimensional formal group $\widehat{\mathbf{G}}$ over A, choosing an equivalence $\widehat{\mathbf{G}} \simeq \widehat{\mathbf{G}}^{\mathcal{Q}}_A$ is equivalent to choosing a map from the 2-sphere S^2 to the space of A-valued points of $\widehat{\mathbf{G}}$, satisfying a certain nondegeneracy condition (Proposition 4.3.23); we will refer to such a map as an *orientation* of $\widehat{\mathbf{G}}$ (Definition 4.3.9). Given an arbitrary 1-dimensional formal group $\widehat{\mathbf{G}}$ over an $\mathbb{E}_\infty$-ring A, we show that there is a universal $\mathbb{E}_\infty$-algebra $\mathfrak{O}_{\widehat{\mathbf{G}}}$ for which $\widehat{\mathbf{G}}$ acquires an orientation after extending scalars to $\mathfrak{O}_{\widehat{\mathbf{G}}}$; we will refer to $\mathfrak{O}_{\widehat{\mathbf{G}}}$ as the *orientation classifier* of $\widehat{\mathbf{G}}$ (Definition 4.3.14).

Let us now return to the setting of Theorem 0.0.8. Let R_0 be an $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a p-divisible group over R_0 which satisfies the hypotheses of Theorem 0.0.8. Then $\mathbf{G}_0$ admits a universal deformation $\mathbf{G}$ over the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$. The identity component $\mathbf{G}^\circ$ is then 1-dimensional formal group over $R^{\mathrm{un}}_{\mathbf{G}_0}$, which has an orientation classifier that we denote by $R^{\mathrm{or}}_{\mathbf{G}_0}$. We will complete the proof by showing that $R^{\mathrm{or}}_{\mathbf{G}_0}$ satisfies the requirements of Theorem 0.0.8. Note here the contrast with the approach based on Landweber's theorem: from our construction, it is is obvious that $R^{\mathrm{or}}_{\mathbf{G}_0}$ is an $\mathbb{E}_\infty$-ring which depends functorially on the pair $(R_0, \mathbf{G}_0)$. What is not at all obvious that $R^{\mathrm{or}}_{\mathbf{G}_0}$ has the desired homotopy groups (or even that it is nonzero).

Let us first consider the case where $R_0 = \kappa$ is a perfect field of characteristic p and the p-divisible group $\mathbf{G}_0$ is connected (and can therefore be identified with the formal group $\widehat{\mathbf{G}}_0 = \mathbf{G}_0^\circ$, having some height n). In §5, we will show that the $K(n)$-localization $L_{K(n)} R^{\mathrm{or}}_{\mathbf{G}_0}$ can be identified with the Lubin-Tate spectrum E of Example 0.0.6, and therefore satisfies condition (b) of Theorem 0.0.8. Roughly speaking, this follows from Yoneda's lemma: from the construction of $R^{\mathrm{or}}_{\mathbf{G}_0}$ one can immediately extract a universal property of its $K(n)$-localization (Theorem 5.1.5), which is shared by the spectrum E. However, we give a different argument, which exploits the universal property of $L_{K(n)} R^{\mathrm{or}}_{\mathbf{G}_0}$ together with Quillen's work on complex bordism to give a direct calculation of the homotopy groups of $L_{K(n)} R^{\mathrm{or}}_{\mathbf{G}_0}$ (Theorem 5.4.1). As a byproduct, we give a new proof of the existence and uniqueness of the $\mathbb{E}_\infty$-structure on E (Theorem 5.0.2), by completely different methods than those of [12].

In §6, we return to the case of a general pair $(R_0, \mathbf{G}_0)$ satisfying the requirements of Theorem 0.0.8. By combining the results of §5 with certain localization techniques, we show that the homotopy groups of $R^{\mathrm{or}}_{\mathbf{G}_0}$ are concentrated in even degrees and that $\pi_0(R^{\mathrm{or}}_{\mathbf{G}_0})$ agrees with the classical deformation ring $R^{\mathrm{cl}}_{\mathbf{G}_0}$. This completes the proof of Theorem 0.0.8. It also shows that the $K(n)$-localization appearing in §5 is unnecessary: in the special case where $R_0 = \kappa$ and $\mathbf{G}_0$ is a connected p-divisible group of height n, the spectrum $R^{\mathrm{or}}_{\mathbf{G}_0}$ is already equivalent to the Lubin-Tate spectrum of Example 0.0.6 (and is therefore $K(n)$-local). As another application, we give a quick proof of a theorem of Snaith, which asserts that the complex K-theory spectrum KU can obtained from the suspension spectrum $\Sigma^\infty_+(\mathbf{CP}^\infty)$ by inverting the Bott class $\beta \in \pi_2(\Sigma^\infty_+(\mathbf{CP}^\infty))$ (see Theorem 6.5.1).

Ultimately, all of the theory developed in this paper was developed in order to better understand the elliptic cohomology theories of Example 0.0.7. Recall that in §AV.2 , we introduced the notion of a *strict elliptic curve* X over an $\mathbb{E}_\infty$-ring R (Definition AV.2.0.2). To such an object, one can associate a formal group $\widehat{\mathsf{X}}$ over R, which we will refer to as the *formal completion of* X. We define an *orientation* of a strict elliptic curve X to be an orientation of its formal completion $\widehat{\mathsf{X}}$: in the case where R is even periodic, this is equivalent to giving an equivalence of formal groups $\widehat{\mathbf{G}}^{\mathcal{Q}}_R \simeq \widehat{\mathsf{X}}$. In §7, we show that oriented elliptic curves are classified by a nonconnective spectral Deligne-Mumford stack, which we will denote by $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ (Proposition 7.2.10). Combining the results of §6 with the Serre-Tate theorem, we show that $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ behaves like a “2-periodic version” of the classical moduli stack of elliptic curves $\mathcal{M}_{\mathrm{Ell}}$, from which the assertions of Example 0.0.7 follow immediately (see Theorem 7.0.1).

Remark 0.0.9. To analyze the moduli stack of oriented elliptic curves $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$, one needs only Theorem 0.0.8 (and its proof) for p-divisible groups of height 2. However, one can use a similar strategy to construct “2-periodic analogues” of other moduli spaces. For example, Theorem 0.0.8 was applied by Behrens and Lawson to construct 2-periodic versions of certain (p-adically completed) Shimura stacks and thereby introduce a theory of *topological automorphic forms*; we refer the reader to [3] for more details.

Acknowledgements

During the interminable process of writing this paper, I have benefitted from discussing these ideas with many people, including (but surely not limited to) Matt Ando, Mark Behrens, Bhargav Bhatt, Lukas Brantner, Omar Antolin-Camarena Dustin Clausen, Paul Goerss, Gijs Heuts, Mike Hill, Tyler Lawson, Akhil Mathew, Haynes Miller, Jack Morava, Anatoly Preygel, Doug Ravenel, Charles Rezk, Nick Rozenblyum, Stephan Stolz, Peter Teichner, Gabriele Vezzosi, and Allen Yuan. It is my pleasure to thank them all. Most of all, I would like to thank Mike Hopkins, to whom this project owes a tremendous intellectual debt.

I also thank the National Science Foundation for supporting this work under grant number 1510417.

Notation and Terminology

Throughout this paper, we will assume that the reader is familiar with the language of higher category theory developed in [23] and [24]. We will also use some notions of spectral algebraic geometry (particularly formal spectral algebraic geometry) as developed in [25], and will refer frequently to the first paper in this series ([26]). Since we will need to refer to these texts frequently, we adopt the following conventions:

(HTT) We will indicate references to [23] using the letters HTT.

(HA) We will indicate references to [24] using the letters HA.

(SAG) We will indicate references to [25] using the letters SAG.

(AV) We will denote references to [26] using the letters AV.

For example, Theorem HTT.6.1.0.6 refers to Theorem 6.1.0.6 of [23].

For the reader's convenience, we now review some cases in which the conventions of this this paper differ from those of the texts listed above, or from the established mathematical literature.

- We will generally not distinguish between a category $\mathcal{C}$ and its nerve $\mathrm{N}(\mathcal{C})$. In particular, we regard every category $\mathcal{C}$ as an ∞-category.

- We will generally abuse terminology by not distinguishing between an abelian group M and the associated Eilenberg-MacLane spectrum: that is, we view the ordinary category of abelian groups as a full subcategory of the ∞-category Sp of spectra. Similarly, we regard the ordinary category of commutative rings as a full subcategory of the ∞-category CAlg of $\mathbb{E}_\infty$-rings.

- Let A be an $\mathbb{E}_\infty$-ring. We will refer to A-module spectra simply as *A-modules*. The collection of A-modules can be organized into a stable ∞-category which we will denote by Mod_A and refer to as *the ∞-category of A-modules*. This convention has an unfortunate feature: when A is an ordinary commutative ring, it does not reduce to the usual notion of A-module. In this case, Mod_A is not the abelian category of A-modules but is closely related to it: the homotopy category hMod_A is equivalent to the derived category $D(A)$. *Unless otherwise specified, the term "A-module" will be used to refer to an object of* Mod_A*, even when A is an ordinary commutative ring.* When we wish to consider an A-module M in the usual sense, we will say that M is a *discrete* A-module or an *ordinary* A-module.

- Unless otherwise specified, all algebraic constructions we consider in this paper should be understood in the "derived" sense. For example, if we are given discrete modules M and N over a commutative ring A, then the tensor product $M \otimes_A N$ denotes the *derived* tensor product $M \otimes^L_A N$. This may not be a discrete A-module: its homotopy groups are given by $\pi_n(M \otimes_A N) \simeq \mathrm{Tor}^A_n(M, N)$. When we wish to consider the usual tensor product of M with N over A, we will denote it by $\mathrm{Tor}^A_0(M, N)$ or by $\pi_0(M \otimes_A N)$.

- If M and N are spectra, we will denote the smash product of M with N by $M \otimes_S N$, rather than $M \wedge N$ (here S denotes the sphere spectrum). More generally, if M and N are modules over an $\mathbb{E}_\infty$-ring A, then we will denote the smash product of M with N over A by $M \otimes_A N$, rather than $M \wedge_A N$. Note that when A is an ordinary commutative ring and the modules M and N are discrete, this agrees with the preceding convention.

- If $\mathcal{C}$ is an ∞-category, we let $\mathcal{C}^{\simeq}$ denote the largest Kan complex contained in $\mathcal{C}$: that is, the ∞-category obtained from $\mathcal{C}$ by discarding all non-invertible morphisms.

- If A is an $\mathbb{E}_\infty$-ring, we let $\mathrm{Spec}(A)$ denote the nonconnective spectral Deligne-Mumford stack given by the étale spectrum of A (denoted by $\mathrm{Sp\acute{e}t}(A)$ in [25] and [26]). We will generally not distinguish between $\mathrm{Spec}(A)$ and the functor that it represents (given by the formula $B \mapsto \mathrm{Map}_{\mathrm{CAlg}}(A, B)$). We will engage in other related abuses: for example, if A is connective (or discrete), we identify $\mathrm{Spec}(A)$ with the functor $\mathrm{Map}_{\mathrm{CAlg}}(A, \bullet)$ restricted to connective (or discrete) $\mathbb{E}_\infty$-rings. In the discrete case, we will also use the notation $\mathrm{Spec}(A)$ to refer to the usual Zariski spectrum of A, regarded as an affine scheme.

Adic $\mathbb{E}_\infty$-Rings

Let R be a commutative ring and let $I \subseteq A$ be a finitely generated ideal. Then we can regard R as equipped with the *I-adic topology*, having a basis of open sets of the form $x + I^n$ where $x \in R$ and $n \geqslant 0$. We say that a topology on R is *adic* if it coincides with the I-adic topology, for some finitely generated ideal $I \subseteq R$. In this case, we will say that I is an *ideal of definition for R*.

Remark 0.0.10. Let R be a commutative ring with an adic topology. We say that an element $t \in R$ is *topologically nilpotent* if the sequence $\{t^n\}_{n \geqslant 0}$ converges to zero in the topology on R. Equivalently, t is topologically nilpotent if R admits an ideal of definition containing t.

In this paper, we will need the following variant:

Definition 0.0.11. An *adic $\mathbb{E}_\infty$-ring* is a pair (A, τ), where A is an $\mathbb{E}_\infty$-ring and τ is an adic topology on the commutative ring $\pi_0(A)$. If (A, τ) and (A', τ') are adic $\mathbb{E}_\infty$-rings, we let $\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(A, A')$ denote the summand of $\mathrm{Map}_{\mathrm{CAlg}}(A, A')$ consisting of those morphisms of $\mathbb{E}_\infty$-rings $f : A \to A'$ for which the underlying ring homomorphism $\pi_0(A) \to \pi_0(A')$ is continuous (with respect to the topologies τ and τ').

The collection of adic $\mathbb{E}_\infty$-rings forms an ∞-category $\mathrm{CAlg}^{\mathrm{ad}}$, with morphisms given by $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}}(A, A') = \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(A, A')$.

Remark 0.0.12. In the situation of Definition 0.0.11, we will generally abuse notation by identifying an adic $\mathbb{E}_\infty$-ring (A, τ) with its underlying $\mathbb{E}_\infty$-ring A; in this case, we implicitly assume that an adic topology has been specified on $\pi_0(A)$.

In this paper, we will encounter adic $\mathbb{E}_\infty$-rings for two essentially different reasons:

- If $\widehat{\mathbf{G}}$ is a formal group over an $\mathbb{E}_\infty$-ring R, then it has an "algebra of functions" $\mathscr{O}_{\widehat{\mathbf{G}}}$ that is naturally viewed as an adic $\mathbb{E}_\infty$-algebra over R (for example, when $R = \kappa$ is a field, then $\mathscr{O}_{\widehat{\mathbf{G}}}$ will isomorphic to a power series ring $\kappa[[x_1, \ldots, x_n]]$, which we will want to regard as a topological ring by endowing it with the I-adic topology for $I = (x_1, \ldots, x_n)$.

- Given a p-divisible group $\mathbf{G}_0$ defined over an $\mathbf{F}_p$-algebra R_0, the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is naturally viewed as an adic $\mathbb{E}_\infty$-ring, by endowing the underlying commutative ring $R^{\mathrm{cl}}_{\mathbf{G}_0} = \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$ with the $\ker(\rho)$-adic topology for $\rho : R^{\mathrm{cl}}_{\mathbf{G}_0} \to R_0$. Moreover, we will construct the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ as the solution to a moduli problem which is naturally defined on adic $\mathbb{E}_\infty$-rings (see Definition 3.1.4).

Suppose that R is an $\mathbb{E}_\infty$-ring and M is an R-module. If x is an element of $\pi_0(R)$, we will say that M is (x)-*complete* if the limit of the tower

$$\cdots \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M$$

vanishes. More generally, if $I \subseteq \pi_0(R)$ is a finitely generated ideal, then we say that M is I-*complete* if it is (x)-complete for each $x \in I$ (it suffices to check this condition for a set of generators for the ideal I; see Corollary SAG.7.3.3.3). We say that an adic $\mathbb{E}_\infty$-ring R is *complete* if it is I-complete for some finitely generated ideal of definition $I \subseteq \pi_0(R)$ (in this case, it is I-complete for every finitely generated ideal of definition $I \subseteq \pi_0(R)$).

Warning 0.0.13. Let R be an ordinary commutative ring and let $I \subseteq \pi_0(R)$ be a finitely generated ideal. We say that R is *classically I-adically complete* if the canonical map $\rho : R \to \varprojlim R/I^n$ is an isomorphism. In this case, R is also I-complete in the sense defined above. Conversely, if R is I-complete, then the map ρ is surjective (Corollary SAG.7.3.6.2), but ρ need not be injective. However, the intersection $\bigcap_{n \geqslant 0} I^n$ is always a nilpotent ideal in R (Corollary SAG.7.3.6.4).

We refer the reader to Chapter SAG.7 for an extensive discussion of I-completeness in the setting of $\mathbb{E}_\infty$-rings and their modules.

Nonconnective Ring Spectra

A central theme of this paper (and the prequel [26]) is that it is sensible to consider algebro-geometric objects of various flavors (formal groups, elliptic curves, p-divisible groups, finite flat group schemes, etcetera) which are defined over $\mathbb{E}_\infty$-rings, rather than merely over ordinary commutative rings. For our ultimate applications, it will be convenient to work with $\mathbb{E}_\infty$-rings which are not assumed to be connective. However, we should emphasize that allowing nonconnective ring spectra does not represent any actual gain in generality. Giving a formal group over an $\mathbb{E}_\infty$-ring R is equivalent to giving a formal group over the connective cover $\tau_{\geqslant 0}(R)$ (and similarly for elliptic curves, p-divisible groups, and so forth). The distinction between working over R and working over $\tau_{\geqslant 0}(R)$ will be completely irrelevant until §4, when we introduce the notion of an *orientation* of a formal group $\widehat{\mathbf{G}}$. This notion is fundamentally "nonconnective": a formal group over a connective $\mathbb{E}_\infty$-ring R can never be oriented (except in the trivial case $R \simeq 0$).

1 Formal Groups

Let κ be a field. A *smooth, connected, commutative formal group* over κ is an abelian group object in the category of formal κ-schemes which is isomorphic (as a formal scheme) to a formal affine space $\widehat{\mathbf{A}}^n = \mathrm{Spf}(\kappa[[t_1, \ldots, t_n]])$ for some $n \geqslant 0$. We will henceforth refer to such objects simply as *formal groups* over κ.

Our goal in this section is to introduce a more general notion of formal group, where we replace the ground field κ by an arbitrary $\mathbb{E}_\infty$-ring R. Let us assume for simplicity that R is connective. We will define a *formal group over R* to be a functor of ∞-categories $\widehat{\mathbf{G}} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$, where $\mathrm{CAlg}^{\mathrm{cn}}_R$ denotes the ∞-category of connective $\mathbb{E}_\infty$-algebras over R and $\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ denotes the ∞-category of connective $\mathbf{Z}$-module spectra (or, equivalently, the ∞-category of simplicial abelian groups). We will require this functor to have a certain representability property: namely, the underlying $\mathcal{S}$-valued functor

$$\mathrm{CAlg}^{\mathrm{cn}}_R \xrightarrow{\widehat{\mathbf{G}}} \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}} \xrightarrow{\Omega^\infty} \mathcal{S}$$

should be (representable by) a very particular type of formal R-scheme, which we will refer to as a *formal hyperplane*.

In the special case where $R = \kappa$ is a field, we define a formal hyperplane over κ to be a formal scheme over κ which has the form $\widehat{\mathbf{A}}^n = \mathrm{Spf}(\kappa[[t_1, \ldots, t_n]])$, for some

$n \geqslant 0$. When extending this definition to the case of an arbitrary $\mathbb{E}_\infty$-ring R, we encounter two (unrelated) subtleties:

(1) Even when R is an ordinary commutative ring, we do want to require every formal hyperplane over R to be isomorphic (as a formal scheme) to a formal affine space $\widehat{\mathbf{A}}^n = \mathrm{Spf}(R[[t_1, \ldots, t_n]])$: this requirement cannot be tested Zariski-locally on R. Instead, we allow objects which are isomorphic to the the formal completion of some vector bundle $\mathscr{E} \to \mathrm{Spec}(R)$ along its zero section.

(2) In the case where R is an $\mathbb{E}_\infty$-ring, the notion of a formal power series ring over R is potentially ambiguous. In general, there can be many R-algebras A with homotopy given by $\pi_*(A) \simeq \pi_*(R)[[t_1, \ldots, t_n]]$, and we would like to allow the formal spectrum of *any* such algebra to qualify as a formal hyperplane over R.

Most of this section is devoted to developing a good theory of formal hyperplanes over an arbitrary $\mathbb{E}_\infty$-ring R. We begin by observing that over a field κ, the datum of a formal hyperplane X can be encoded in (at least) three closely related ways:

(a) One can consider the *ring of functions* $\mathscr{O}_X$: this is an algebra over κ which is isomorphic to a formal power series ring $\kappa[[t_1, \ldots, t_n]]$. In particular, it is a local ring which is complete with respect to its maximal ideal $\mathfrak{m} = (t_1, \ldots, t_n)$.

(b) One can consider the *coalgebra of distributions* $C(X)$: that is, the collection of all κ-linear maps $\mathscr{O}_X \to \kappa$ which are $\mathfrak{m}$-adically continuous (that is, they annihilate some power of the maximal ideal $\mathfrak{m} \subseteq \mathscr{O}_X$).

(c) One can consider the *functor of points* $h_X : \mathrm{CAlg}_\kappa^\heartsuit \to \mathcal{S}\mathrm{et}$, which assigns to each commutative κ-algebra A the set $h_X(A) = \mathrm{Hom}_\kappa^{\mathrm{cont}}(\mathscr{O}_X, A)$ of κ-algebra homomorphisms from $\mathscr{O}_X$ to A which are $\mathfrak{m}$-adically continuous (that is, they annihilate some power of the maximal ideal $\mathfrak{m}$).

Each of these perspectives comes with advantages and disadvantages. Approach (a) is perhaps the most concrete: generally speaking, algebras are easier to work with than coalgebras (or presheaves on the category of commutative algebras). However, algebras of the form $\mathscr{O}_X$ come with some additional baggage: they should really be regarded as *topological* algebras (with respect to the $\mathfrak{m}$-adic topology). Replacing the algebra $\mathscr{O}_X$ by its continuous dual $C(X)$ is a convenient way of “remembering” this topology, and the passage from algebras to coalgebras allows us to avoid various technical complications. For example, if Y is another formal hyperplane over κ, then the coalgebra of distributions on the product $X \times Y$ is given by the tensor product $C(X \times Y) = C(X) \otimes_\kappa C(Y)$, while the algebra $\mathscr{O}_{X \times Y}$ is instead computed by a *completed* tensor product $\mathscr{O}_X \widehat{\otimes}_\kappa \mathscr{O}_Y$. Approach (c) is perhaps the most abstract, but is extremely useful (perhaps even indispensable in the homotopy-theoretic context) when we want to contemplate group structures on formal hyperplanes.

Our primary objective in this section is to show that each of these approaches remains viable when we replace the field κ by an arbitrary $\mathbb{E}_\infty$-ring R. More precisely, we will introduce an ∞-category $\mathrm{Hyp}(R)$ of formal hyperplanes over R which admits three equivalent incarnations:

(a') The ∞-category $\mathrm{Hyp}(R)$ can be identified with a full subcategory of the ∞-category $(\mathrm{CAlg}^{\mathrm{ad}}_R)^{\mathrm{op}}$ of adic $\mathbb{E}_\infty$-algebras over R.

(b') The ∞-category $\mathrm{Hyp}(R)$ can be identified with a full subcategory of the ∞-category cCAlg_R of *commutative coalgebras over* R (see Definition 1.2.1).

(c') When R is connective, the ∞-category $\mathrm{Hyp}(R)$ can be identified with a full subcategory of the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})$ of $\mathcal{S}$-valued functors on the ∞-category of connective $\mathbb{E}_\infty$-algebras over R.

We will begin by studying approach (b'). In §1.1, we review the theory of commutative coalgebras over a commutative ring R (Definition 1.1.1) and recall the construction of the *divided power coalgebra* $\Gamma^*_R(M)$ associated to a flat R-module M (Construction 1.1.11). In §1.2, we study a generalization of this theory, where we allow R to be an arbitrary $\mathbb{E}_\infty$-ring. We say that a commutative coalgebra C over R is *smooth* if it is flat as an R-module and $\pi_0(C)$ is isomorphic to the divided power coalgebra of projective $\pi_0(R)$-module of finite rank (Definition 1.2.4). The collection of smooth coalgebras over R can be organized into an ∞-category, which we will denote by $\mathrm{cCAlg}^{\mathrm{sm}}_R$; this ∞-category can be regarded as the coalgebraic incarnation of our theory of formal hyperplanes over R.

We study the relationship between approaches (a') and (b') in §1.3. For any $\mathbb{E}_\infty$-ring R, the formation of R-linear duals $C \mapsto C^\vee$ carries commutative coalgebras over R to $\mathbb{E}_\infty$-algebras over R. When C is a smooth coalgebra over R, there is a canonical topology on the commutative ring $\pi_0(C^\vee)$, which endows $C^\vee$ with the structure of an adic $\mathbb{E}_\infty$-ring (see Proposition 1.3.10). Our main result asserts that the construction $C \mapsto C^\vee$ determines a fully faithful embedding of ∞-categories

$$\theta : \mathrm{cCAlg}^{\mathrm{sm}}_R \hookrightarrow (\mathrm{CAlg}^{\mathrm{ad}}_R)^{\mathrm{op}}$$

(see Theorem 1.3.15). We will describe the essential image of this embedding in §1.4: roughly speaking, it consists of those adic $\mathbb{E}_\infty$-algebras A over R with homotopy given by $\pi_*(A) \simeq \pi_*(R)[[t_1, \ldots, t_n]]$, at least Zariski-locally on $|\operatorname{Spec}(R)|$ (see Proposition 1.4.11).

Assume now that R is connective. In §1.5, we will associate to each flat coalgebra C over R a functor $\operatorname{cSpec}(C) : \operatorname{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$, which we will refer to as the *cospectrum of* C (see Construction 1.5.4). Roughly speaking, the functor $\operatorname{cSpec}(C)$ carries an object $R' \in \operatorname{CAlg}^{\mathrm{cn}}_R$ to the space $\operatorname{GLike}(C')$ of *grouplike elements* of the coalgebra $C' = (R' \otimes_R C) \in \operatorname{cCAlg}^{\mathrm{cn}}_{R'}$. In the special case where C is smooth, we show that the cospectrum $\operatorname{cSpec}(C)$ can also be described as the formal spectrum of the adic $\mathbb{E}_\infty$-algebra $C^\vee$, whose value on $R' \in \operatorname{CAlg}^{\mathrm{cn}}_R$ is given by the space $\operatorname{Map}^{\mathrm{cont}}_{\operatorname{CAlg}_R}(C^\vee, R')$ of $\mathbb{E}_\infty$-algebra maps from $C^\vee$ to R' for which the underlying ring homomorphism $\pi_0(C^\vee) \to \pi_0(R')$ is continuous. (Proposition 1.5.8). We then define a *formal hyperplane over* R to be a functor $X : \operatorname{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ which has the form $\operatorname{cSpec}(C)$ for some $C \in \operatorname{cCAlg}^{\mathrm{sm}}_R$ (or, equivalently, which has the form $\operatorname{Spf}(A)$ where $A \in \operatorname{CAlg}^{\mathrm{ad}}_R$ belongs to the essential image of the functor θ above); see Definition 1.5.10.

In §1.6, we apply our theory of formal hyperplanes to introduce the notion of a *formal group* $\widehat{\mathbf{G}}$ over an $\mathbb{E}_\infty$-ring R (Definition 1.6.1). Much of the remainder of this paper (particularly §2) is devoted to showing that the theory of formal groups over $\mathbb{E}_\infty$-rings behaves as one might expect: for example, there is a close connection between formal groups and p-divisible groups whenever R is (p)-complete. However, there are also a few surprises: for example, we show that the formal multiplicative group $\widehat{\mathbf{G}}_m$ can be lifted to a formal group over the sphere spectrum S (Construction 1.6.16), but the formal additive group $\widehat{\mathbf{G}}_a$ cannot (Proposition 1.6.20).

1.1 Coalgebras over Commutative Rings

In this section, we review the classical theory of (flat) commutative coalgebras over a commutative ring R. In particular, we recall the construction of the *divided power coalgebra* $\Gamma^*_R(M)$ associated to a flat R-module M (Construction 1.1.11) and characterize it by a universal property (Proposition 1.1.16). We will say that a coalgebra C over R is *smooth* if it has the form $\Gamma^*_R(M)$, where M is a projective R-module of finite rank (Definition 1.1.14). We show that class of smooth coalgebras satisfies (effective) descent with respect to the étale topology (Proposition 1.1.19).

1.1.1 Flat Coalgebras

We begin by reviewing some commutative algebra.

Definition 1.1.1. Let R be a commutative ring. A *flat commutative coalgebra over* R is a triple $(C, \Delta_C, \epsilon_C)$, where C is a flat R-module, $\Delta_C : C \to C \otimes_R C$ is an R-module homomorphism (called the *comultiplication* on C) and $\epsilon_C : C \to R$ is an R-module homomorphism (called the *counit* of C) for which the diagrams

$$\begin{tikzcd} & C \arrow[dl, "\Delta_C"'] \arrow[dr, "\Delta_C"] & \\ C \otimes_R C \arrow[rr, "\sigma"] & & C \otimes_R C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "\Delta_C"] \arrow[dr, "\sim"'] & & C \otimes_R C \arrow[dl, "\epsilon_C \otimes \mathrm{id}"] \\ & R \otimes_R C & \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"] & C \otimes_R C \arrow[d, "\Delta_C \otimes \mathrm{id}"] \\ C \otimes_R C \arrow[r, "\mathrm{id} \otimes \Delta_C"] & C \otimes_R C \otimes_R C \end{tikzcd}$$

commute (here σ denotes the automorphism of $C \otimes_R C$ given by $\sigma(x \otimes y) = y \otimes x$).

Let $(C, \Delta_C, \epsilon_C)$ and $(C', \Delta_{C'}, \epsilon_{C'})$ be flat commutative coalgebras over R. A *coalgebra homomorphism* from $(C, \Delta_C, \epsilon_C)$ to $(C', \Delta_{C'}, \epsilon_{C'})$ is an R-algebra homomorphism $f : C \to C'$ satisfying $\epsilon_{C'} \circ f = \epsilon_C$ and $\Delta_{C'} \circ f = (f \otimes f) \circ \Delta_C$. We let $\mathrm{cCAlg}^\flat_R$ denote the category whose objects are flat commutative coalgebras over R and whose morphisms are coalgebra homomorphisms

Remark 1.1.2. In the situation of Definition 1.1.1, the counit $\epsilon_C : C \to R$ is uniquely determined by the comultiplication $\Delta_C : C \to C \otimes_R C$.

Warning 1.1.3. Our terminology is slightly nonstandard; most authors use the term *cocommutative coalgebra* for what we have opted to call a *commutative coalgebra*.

Remark 1.1.4. We will generally abuse terminology by identifying a flat commutative coalgebra $(C, \Delta_C, \epsilon_C)$ with its underlying R-module C; in this case, we implicitly assume that a comultiplication $\Delta_C : C \to C \otimes_R C$ and a counit map $\epsilon_C : C \to R$ have also been specified.

Remark 1.1.5. Let R be a commutative ring and let $\mathrm{Mod}^\flat_R$ be the category of flat R-modules. Then $\mathrm{Mod}^\flat_R$ is a symmetric monoidal category (with respect to the usual tensor product of R-modules $\otimes_R$), and $\mathrm{cCAlg}^\flat_R$ can be identified with the category of commutative coalgebra objects of $\mathrm{Mod}^\flat_R$.

Remark 1.1.6 (Grouplike Elements)**.** Let R be a commutative ring and let C be a flat commutative coalgebra over R. An element $\eta \in C$ is said to be *grouplike* if it satisfies the identities $\Delta_C(\eta) = \eta \otimes \eta$ and $\epsilon_C(\eta) = 1$. Equivalently, η is grouplike if multiplication by η induces a coalgebra homomorphism $R \to C$.

Remark 1.1.7 (Primitive Elements)**.** Let R be a commutative ring and let C be a flat commutative coalgebra over R. Suppose we are given a grouplike element $\eta \in C$. We let $\mathrm{Prim}_\eta(C) = \{x \in C : \Delta_C(x) = \eta \otimes x + x \otimes \eta\}$. We refer to $\mathrm{Prim}_\eta(C)$ as the *set of η-primitive elements of C*. Note that $\mathrm{Prim}_\eta(C)$ is an R-submodule of C.

Warning 1.1.8. Let R be a commutative ring. Then both the abelian category $\mathrm{Mod}_R^\heartsuit$ of discrete R-modules and the stable ∞-category Mod_R of R-module spectra admit symmetric monoidal structures. Consequently, we can consider commutative coalgebra objects in both $\mathrm{Mod}_R^\heartsuit$ (that is, coalgebras in the sense of ordinary algebra) and in the ∞-category Mod_R. However, the inclusion functor $\mathrm{Mod}_R^\heartsuit \hookrightarrow \mathrm{Mod}_R$ is only a *lax* symmetric monoidal functor, and generally does not carry coalgebra objects to coalgebra objects. For our applications, we will only be interested in studying commutative coalgebra objects of $\mathrm{Mod}_R^\heartsuit$ which *remain* commutative coalgebras when regarded as objects of the ∞-category Mod_R. For this reason, we consider only *flat* commutative coalgebras in Definition 1.1.1 (note that $\mathrm{Mod}_R^\flat$ can be regarded as a symmetric monoidal subcategory of both $\mathrm{Mod}_R^\heartsuit$ and Mod_R).

1.1.2 Divided Power Coalgebras

We now isolate an important class of commutative coalgebras.

Notation 1.1.9. Let R be a commutative ring and let M be a flat R-module. For each integer $n \geqslant 0$, the symmetric group Σ_n acts on the n-fold tensor product $M^{\otimes n}$. We let $\Gamma^n_R(M)$ denote the submodule of $M^{\otimes n}$ given by the invariants for the action of Σ_n, and we let $\mathrm{Sym}^n_R(M)$ denote the quotient of $M^{\otimes n}$ given by coinvariants for the action of Σ_n. We let $\Gamma^*_R(M)$ and $\mathrm{Sym}^*_R(M)$ denote the graded R-modules given by the formulae

$$\Gamma^*_R(M) = \bigoplus_{n \geqslant 0} \Gamma^n_R(M) \qquad \mathrm{Sym}^*_R(M) = \bigoplus_{n \geqslant 0} \mathrm{Sym}^n_R(M).$$

Warning 1.1.10. In Notation 1.1.9, the formation of invariants and coinvariants is carried out in the abelian category $\mathrm{Mod}_R^\heartsuit$ of discrete R-modules. If R is not a $\mathbf{Q}$-algebra, then this formation is not compatible with the inclusion $\mathrm{Mod}_R^\heartsuit \hookrightarrow \mathrm{Mod}_R$.

In particular, the symmetric algebra $\mathrm{Sym}^*_R(M)$ is the free commutative R-algebra generated by M in the sense of classical commutative algebra, and is generally *not* the free $\mathbb{E}_\infty$-algebra over R generated by M.

The usual multiplication on the symmetric algebra $\mathrm{Sym}^*_R(M)$ has a dual incarnation:

Construction 1.1.11. Let R be a commutative ring and let M be a flat R-module which admits a direct sum decomposition $M \simeq M' \oplus M''$. For every pair of nonnegative integers n' and n'', the evident projection map $M^{\otimes n'+n''} \to M'^{\otimes n'} \otimes_R M''^{\otimes n''}$ determines maps

$$\mu^{n',n''} : \Gamma^{n'+n''}_R(M) \to \Gamma^{n'}_R(M') \otimes_R \Gamma^{n''}_R(M'').$$

Summing over n' and n'' then yields an isomorphism of graded R-modules

$$\mu : \Gamma^*_R(M) \simeq \Gamma^*_R(M' \oplus M'') \simeq \Gamma^*_R(M') \otimes_R \Gamma^*_R(M'').$$

For any flat R-module M, the diagonal map $M \to M \oplus M$ induces a map of R-modules

$$\Delta : \Gamma^*_R(M) \to \Gamma^*_R(M \oplus M) \xrightarrow{\mu} \Gamma^*_R(M) \otimes_R \Gamma^*_R(M).$$

It is not difficult to verify that Δ exhibits $\Gamma^*_R(M)$ as a commutative coalgebra in the category of (flat) R-modules. We will refer to $\Gamma^*_R(M)$ (with this coalgebra structure) as the *divided power coalgebra of M*.

Remark 1.1.12. In the situation of Construction 1.1.11, the counit for the coalgebra $\Gamma^*_R(M)$ is the map $\epsilon : \Gamma^*_R(M) \to R$ which is the identity on $\Gamma^0_R(M) \simeq R$ and vanishes on $\Gamma^n_R(M)$ for $n > 0$.

Remark 1.1.13. In the situation of Notation 1.1.9, there are additional structures on $\mathrm{Sym}^*_R(M)$ and $\Gamma^*_R(M)$: both can be regarded as (commutative and cocommutative) Hopf algebras over R. However, in the discussion which follows, these additional structures will play no role: we will be interested only in the algebra structure on $\mathrm{Sym}^*_R(M)$ and the coalgebra structure on $\Gamma^*_R(M)$.

Definition 1.1.14. Let R be a commutative ring. A *smooth coalgebra* over R is a flat commutative coalgebra C over R which is isomorphic to $\Gamma^*_R(M)$, where M is a projective R-module of finite rank. If M has rank r, then we will say that the smooth coalgebra C *has dimension* r. We let $\mathrm{cCAlg}^{\mathrm{sm}}_R$ denote the full subcategory of $\mathrm{cCAlg}^{\flat}_R$ spanned by the smooth coalgebras over R.

Remark 1.1.15. Let R be a commutative ring. Then the collection of smooth coalgebras over R is closed under tensor products (this follows immediately from the analysis of Construction 1.1.11).

Divided power coalgebras can be characterized by the following universal property:

Proposition 1.1.16. *Let R be a commutative ring, let M be a flat R-module, and let $C \in \mathrm{cCAlg}_R^\flat$. Then composition with the projection map $\Gamma_R^*(M) \to \Gamma_R^1(M) \simeq M$ induces a monomorphism*

$$\theta : \mathrm{Hom}_{\mathrm{cCAlg}_R^\flat}(C, \Gamma_R^*(M)) \to \mathrm{Hom}_{\mathrm{Mod}_R^\flat}(C, M),$$

whose image is the collection of R-module homomorphisms $f : C \to M$ which satisfy the following condition:

- $(*)$ *For each element $x \in C$, the composite map*

$$R \xrightarrow{x} C \xrightarrow{\Delta_C^{(n)}} C^{\otimes n} \xrightarrow{f^{\otimes n}} M^{\otimes n}$$

vanishes for almost every integer $n \geqslant 0$; here $\Delta_C^{(n)}$ is the map given by iterated comultiplication on C.

Proof. We explicitly construct a (partially defined) inverse to the map θ. Let $f : C \to M$ be any R-module homomorphism. For each $n \geqslant 0$, let $f^{(n)}$ denote the composition $C \xrightarrow{\Delta_C^{(n)}} C^{\otimes n} \xrightarrow{f^{\otimes n}} M^{\otimes n}$. It follows from the cocommutativity of C that the map $f^{(n)}$ factors through the submodule $\Gamma^n(M) \subseteq M^{\otimes n}$. If f satisfies condition $(*)$, then we can define a single map $F : C \to \Gamma_R^*(M)$ by the formula $F(x) = \sum_{n \geqslant 0} f^{(n)}(x)$. We leave it to the reader to verify that $F : C \to \Gamma_R^*(M)$ is the unique coalgebra homomorphism satisfying $\theta(F) = f$. $\square$

Example 1.1.17 (Grouplike Elements of $\Gamma_R^*(M)$)**.** Let R be a commutative ring and let M be a projective module of finite rank over R. Applying Proposition 1.1.16 in the case $C = R$, we obtain a bijection

$$\{\text{Grouplike elements of } \Gamma_R^*(M)\} \simeq \{R\text{-linear maps } f : M^\vee \to \sqrt{R}\},$$

where $\sqrt{R}$ denotes the nilradical of R.

In particular, if R is reduced, then $\Gamma_R^*(M)$ has a unique grouplike element.

Warning 1.1.18. Let R be a commutative ring and let C be a smooth coalgebra over R. Then C is isomorphic to a divided power coalgebra $\Gamma^*_R(M)$, where M is a projective R-module of finite rank. However, the module M is not functorially determined by C. We can recover the module M by choosing a grouplike element $\eta \in R$: in this case, there is an isomorphism $M \simeq \mathrm{Prim}_\eta(C)$. However, the element η need not be unique if R is non-reduced.

1.1.3 Descent For Smooth Coalgebras

Let $f : R \to R'$ be a homomorphism of commutative rings. Then extension of scalars along f determines a symmetric monoidal functor $\mathrm{Mod}^\flat_R \to \mathrm{Mod}^\flat_{R'}$, and therefore determines a functor $f^* : \mathrm{cCAlg}^\flat_R \to \mathrm{cCAlg}^\flat_{R'}$. This functor carries smooth coalgebras over R to smooth coalgebras over R': note that if M is a projective R-module of finite rank, then we have a canonical isomorphism $f^*(\Gamma^*_R(M)) \simeq \Gamma^*_{R'}(R' \otimes_R M)$. We can therefore regard the construction $R \mapsto \mathrm{cCAlg}^{\mathrm{sm}}_R$ as a functor from the category of commutative rings to the 2-category of (essentially small) categories.

Proposition 1.1.19. *The construction $R \mapsto \mathrm{cCAlg}^{\mathrm{sm}}_R$ satisfies descent for the étale topology.*

The proof of Proposition 1.1.19 will require some preliminaries.

Lemma 1.1.20. *Let R be a commutative ring and let C be a flat commutative coalgebra over R. Assume that:*

(a) The coalgebra C contains a grouplike element η.

(b) There exists a faithfully flat map $R \to R'$ such that $C' = R' \otimes_R C$ is a smooth coalgebra over R'.

Then C is smooth.

Proof. Set $M = \mathrm{Prim}_\eta(C)$. Using the flatness of R' over R, we obtain an isomorphism $R' \otimes_R M \simeq \mathrm{Prim}_{\eta'}(C')$, so that M is a projective R-module of finite rank. Set $C_0 = \Gamma^*_R(M)$, so that C_0 is a smooth coalgebra over R equipped with a grouplike element $\eta_0 \in C_0$ and a canonical isomorphism $M \simeq \mathrm{Prim}_{\eta_0}(C_0)$. For every R-algebra A, let C_A and C_{0A} denote the flat commutative coalgebras over A given by $A \otimes_R C$ and $A \otimes_R C_0$, respectively. We will abuse notation by identifying η and η_0 with their images in C_A and C_{0A}, respectively. Let $\mathscr{F}(A)$ denote the set of all coalgebra isomorphisms $u : C_A \simeq C_{0A}$ for which $u(\eta) = \eta_0$ and the induced map

$$A \otimes_R M \simeq \mathrm{Prim}_\eta(C_A) \xrightarrow{u} \mathrm{Prim}_{\eta_0}(C_{0A}) \simeq A \otimes_R M$$

is the identity. Then the construction $A \mapsto \mathscr{F}(A)$ is a sheaf for the flat topology, and we wish to show that this sheaf admits a global section.

Note that $\mathscr{F}$ is a torsor for the sheaf of groups $\mathscr{G}$ which assigns to each R-algebra A is the collection of coalgebra automorphisms of C_{0A} which restrict to the identity on $\Gamma^i_R(M)$ for $i \leqslant 1$. The sheaf $\mathscr{G}$ admits a filtration by normal subgroups

$$\cdots \to \mathscr{G}_3 \to \mathscr{G}_2 \to \mathscr{G}_1 = \mathscr{G},$$

where $\mathscr{G}_n(A)$ is the subgroup of $\mathscr{G}(A)$ consisting of those automorphisms of C_{0A} which are the identity on $\Gamma^i_R(M)$ for $i \leqslant n$. Moreover, the canonical map $\mathscr{G} \to \varprojlim \mathscr{G}/\mathscr{G}_n$ is an isomorphism. It follows that $\mathscr{F}$ can be identified with the inverse limit $\varprojlim \mathscr{F}/\mathscr{G}_n$. Consequently, to prove Lemma 1.1.20, it will suffice to produce a compatible family of global sections $s_n \in (\mathscr{F}/\mathscr{G}_n)(R)$. In fact, we claim that each of the transition maps $(\mathscr{F}/\mathscr{G}_n)(R) \to (\mathscr{F}/\mathscr{G}_{n-1})(R)$ is surjective: this follows from the fact that the quotient $\mathscr{F}/\mathscr{G}_n$ is a $(\mathscr{G}_{n-1}/\mathscr{G}_n)$-torsor over $\mathscr{F}/\mathscr{G}_{n-1}$, which is automatically trivial because the sheaf $\mathscr{G}_{n-1}/\mathscr{G}_n$ is quasi-coherent (and therefore has vanishing cohomology on the affine scheme $\mathrm{Spec}\, R$, with respect to the flat topology). □

Lemma 1.1.21. *Let R be a commutative ring and let C be a flat commutative coalgebra over R. Suppose that there exists a nilpotent ideal $I \subseteq R$ such that C/IC is a smooth coalgebra over R/I. Then C is a smooth coalgebra over R.*

Proof. Without loss of generality, we may assume that $I^2 = 0$. Fix an isomorphism $\alpha : C/IC \simeq \Gamma^*_{R/I}(\overline{M})$, where $\overline{M}$ is a projective R/I-module of finite rank. Then α is classified by a map of R/I-modules $f_0 : C/IC \to \overline{M}$, where f_0 satisfies condition $(*)$ of Proposition 1.1.16. Write $\overline{M} = M/IM$, where M is a projective R-module of finite rank. We have a short exact sequence

$$\mathrm{Hom}_R(C, M) \to \mathrm{Hom}_R(C, M/IM) \to \mathrm{Ext}^1_R(C, IM),$$

and the flatness of C as an R-module supplies an isomorphism $\mathrm{Ext}^1_R(C, IM) \simeq \mathrm{Ext}^1_{R/I}(C/IC, IM)$. The existence of the isomorphism α shows that C/IC is a projective module over R/I, so the group $\mathrm{Ext}^1_{R/I}(C/IC, IM)$ vanishes. It follows that the composite map $C \to C/IC \xrightarrow{f_0} M/IM$ can be lifted to an R-module homomorphism

$f : C \to M$. Since f_0 satisfies condition $(*)$ of Proposition 1.1.16 and I is nilpotent, the map f also satisfies condition $(*)$ of Proposition 1.1.16. It follows that f extends (uniquely) to a coalgebra homomorphism $\overline{\alpha} : C \to \Gamma^*_R(M)$, whose reduction modulo I is the isomorphism α. Because I is nilpotent (and both C and $\Gamma^*_R(M)$ are flat over R), it follows that $\overline{\alpha}$ is also an isomorphism, so that C is smooth over R as desired. □

Proof of Proposition 1.1.19. It follows from the usual theory of faithfully flat descent that the construction $R \mapsto \mathrm{cCAlg}^{\flat}_R$ satisfies descent for the étale topology (in fact, it even satisfies descent for the fpqc topology). It will therefore suffice to show that if R is a commutative ring and C is a flat commutative coalgebra over R with the property that $C' = R' \otimes_R C$ is smooth for some faithfully flat étale R-algebra R', then C is itself smooth.

Assume first that R is reduced. In this case, R' is also reduced, so that C' contains a unique grouplike element η' (Example 1.1.17). Set $C'' = R' \otimes_R R' \otimes_R C$. Since $R' \otimes_R R'$ is also reduced, the commutative coalgebra C'' also has a unique grouplike element. Using the exactness of the diagram

$$0 \longrightarrow C \longrightarrow C' \rightrightarrows C'',$$

we deduce that η' is the image of a (unique) element $\eta \in C$, which is also grouplike. It now follows from Lemma 1.1.20 that C is smooth over R, as desired.

We now treat the general case. Let $J \subseteq R$ be the nilradical of R. The preceding argument shows that C/JC admits a grouplike element η_J, when regarded as a commutative coalgebra over R/J. Choose an element $\overline{\eta}_J \in C$ representing η_J. For each ideal $I \subseteq J$, let η_I denote the image of $\overline{\eta}_J$ in the quotient C/IC. Since η_J is grouplike, we can choose some finitely generated ideal $I \subseteq J$ such that η_I is grouplike. Applying Lemma 1.1.20 again, we conclude that C/IC is a smooth coalgebra over R/I. Since I is a nilpotent ideal, Lemma 1.1.21 guarantees that C is smooth over R. □

Warning 1.1.22. The analogue of Proposition 1.1.19 can fail if we replace the étale topology by some finer topology, like the fppf topology. For example, suppose that k is a field of characteristic $p > 0$ containing an element x which has no pth root. Let A be the completion of the polynomial ring $k[t]$ with respect to the maximal ideal $(t^p - x)$. Then the complete local ring A can be written as the dual $C^\vee$ for an essentially unique commutative coalgebra C over k. The coalgebra C is not smooth (since it contains no grouplike elements), but the tensor product $k' \otimes_k C$ is smooth over k', where $k' = k(\sqrt[p]{x})$.

1.2 Coalgebras over $\mathbb{E}_\infty$-Rings

In this section, we review the theory of commutative coalgebras over an arbitrary $\mathbb{E}_\infty$-ring R. In particular, we introduce an ∞-category $\mathrm{cCAlg}_R^{\mathrm{sm}}$ of *smooth* coalgebras over R (Definition 1.2.4). In §1.2.2 we study the behavior of the ∞-category $\mathrm{cCAlg}_R^{\mathrm{sm}}$ as the $\mathbb{E}_\infty$-algebra R varies and summarize some basic deformation-theoretic principles which can be used to reduce questions about (smooth) coalgebras over $\mathbb{E}_\infty$-rings to questions about smooth coalgebras over commutative rings.

1.2.1 Commutative Coalgebras

Recall that, to any symmetric monoidal ∞-category $\mathcal{C}$, we can associate an ∞-category $\mathrm{cCAlg}(\mathcal{C})$ of *commutative coalgebra objects of* $\mathcal{C}$, given by the formula $\mathrm{cCAlg}(\mathcal{C}) = \mathrm{CAlg}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ (see §AV.3.1). We now specialize this observation to the case where $\mathcal{C} = \mathrm{Mod}_R$, for some $\mathbb{E}_\infty$-ring R.

Definition 1.2.1. Let R be an $\mathbb{E}_\infty$-ring. We will denote the ∞-category $\mathrm{cCAlg}(\mathcal{C})$ by cCAlg_R and refer to its objects as *commutative coalgebras over R*. We will say that a commutative coalgebra C over R is *flat* if it is flat when regarded as an R-module. We let $\mathrm{cCAlg}_R^\flat$ denote the full subcategory of cCAlg_R spanned by the flat R-coalgebras.

Remark 1.2.2. Let R be an $\mathbb{E}_\infty$-ring and let $\mathrm{Mod}_R^\flat$ denote the full subcategory of Mod_R spanned by the flat R-algebras. Then the subcategory $\mathrm{Mod}_R^\flat \subseteq \mathrm{Mod}_R$ contains R and is closed under tensor products, and therefore inherits the structure of a symmetric monoidal ∞-category. Moreover, we can identify $\mathrm{cCAlg}_R^\flat$ with the ∞-category of commutative coalgebra objects of $\mathrm{Mod}_R^\flat$.

Remark 1.2.3. For any $\mathbb{E}_\infty$-ring R, the construction $M \mapsto \pi_0(M)$ determines symmetric monoidal functor from the ∞-category of flat modules over R to the ordinary category of flat modules over $\pi_0(R)$. Passing to commutative coalgebra objects, we obtain a functor $\pi_0 : \mathrm{cCAlg}_R^\flat \to \mathrm{cCAlg}_{\pi_0(R)}^\flat$ whose domain is the ∞-category of commutative coalgebras introduced in Definition 1.2.1, and wose codomain the ordinary category of commutative coalgebras introduced in Definition 1.1.1. If the $\mathbb{E}_\infty$-ring R is discrete, then this functor is an equivalence of ∞-categories: that is, the two notions of commutative coalgebra are the same.

Definition 1.2.4. Let R be an $\mathbb{E}_\infty$-ring and let C be a commutative coalgebra over R. We will say that C is *smooth* if it is flat (when regarded as an R-module) and the commutative coalgebra $\pi_0(C)$ is smooth over $\pi_0(R)$, in the sense of Definition 1.1.14.

We will say that a smooth coalgebra C *has dimension* r if $\pi_0(C)$ has dimension r over $\pi_0(R)$, in the sense of Definition 1.1.14.

Warning 1.2.5. Let R be an $\mathbb{E}_\infty$-ring and let C be a smooth coalgebra over R. If R is discrete, then C admits a grouplike element: that is, there exists a map of commutative coalgebras $R \to C$. However, such a map need not exist if R is not discrete.

Remark 1.2.6. Let R be an $\mathbb{E}_\infty$-ring. If C and D are smooth coalgebras over R, then the tensor product $C \otimes_R D$ is also a smooth coalgebra over R (this follows immediately from Remark 1.2.6). Note that the tensor product $C \otimes_R D$ can be identified with the Cartesian product of C and D in the ∞-category cCAlg_R.

Remark 1.2.7 (Functoriality)**.** Let $f : R \to R'$ be a morphism of $\mathbb{E}_\infty$-rings. Then extension of scalars along f determines a symmetric monoidal functor $\mathrm{Mod}_R \to \mathrm{Mod}_{R'}$. We therefore obtain a functor $f^* : \mathrm{cCAlg}_R \to \mathrm{cCAlg}_{R'}$. This functor carries flat commutative coalgebras over R to flat commutative coalgebras over R', and smooth commutative coalgebras over R to smooth commutative coalgebras over R'. Consequently, we can view the constructions $R \mapsto \mathrm{cCAlg}_R$, $R \mapsto \mathrm{cCAlg}^\flat_R$, and $R \mapsto \mathrm{cCAlg}^{\mathrm{sm}}_R$ as functors from the ∞-category CAlg of $\mathbb{E}_\infty$-rings to the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$ of (not necessarily small) ∞-categories.

1.2.2 Deformation Theory of Coalgebras

We now record a few simple observations about the behavior of cCAlg_R as the $\mathbb{E}_\infty$-ring R varies.

Proposition 1.2.8. *Let R be an $\mathbb{E}_\infty$-ring and let $\tau_{\geqslant 0}R$ be the connective cover of R. Then the extension of scalars functors*

$$\mathrm{cCAlg}^\flat_{\tau_{\geqslant 0}R} \to \mathrm{cCAlg}^\flat_R \qquad \mathrm{cCAlg}^{\mathrm{sm}}_{\tau_{\geqslant 0}R} \to \mathrm{cCAlg}^{\mathrm{sm}}_R$$

are equivalences of ∞-categories.

Proof. By virtue of Proposition HA.7.2.2.16 , the extension of scalars functor

$$\mathrm{Mod}^\flat_{\tau_{\geqslant 0}R} \to \mathrm{Mod}^\flat_R$$

is an equivalence of (symmetric monoidal) ∞-categories. The desired result now follows by passing to commutative coalgebra objects. $\square$

Remark 1.2.9. In the situation of Proposition 1.2.8, both of the inverse equivalences are given by the construction $C \mapsto \tau_{\geqslant 0} C$.

Remark 1.2.10. Let R be an $\mathbb{E}_\infty$-ring and let C be a smooth coalgebra over R. Then C is a projective R-module: that is, it is equivalent (as an R-module) to a direct summand of a coproduct of copies of R. To prove this, we may assume without loss of generality that R is connective (Proposition 1.2.8), in which case it follows from Proposition HA.7.2.2.18 , since C is flat over R and $\pi_0(C)$ is a projective module over $\pi_0(R)$.

Proposition 1.2.11 (Nilcompleteness)**.** *Let R be a connective $\mathbb{E}_\infty$-ring. Then the canonical maps*

$$\mathrm{cCAlg}^\flat_R \to \varprojlim \mathrm{cCAlg}^\flat_{\tau_{\leqslant n} R} \qquad \mathrm{cCAlg}^{\mathrm{sm}}_R \to \varprojlim \mathrm{cCAlg}^{\mathrm{sm}}_{\tau_{\leqslant n} R}$$

are equivalences of ∞-categories.

Proof. It follows from Proposition SAG.19.2.1.3 that the canonical map $\mathrm{Mod}^\flat_R \to \varprojlim \mathrm{Mod}^\flat_{\tau_{\leqslant n} R}$ is an equivalence of (symmetric monoidal) ∞-categories; the desired result now follows by passing to commutative coalgebra objects. $\square$

Proposition 1.2.12 (Clutching)**.** *Suppose we are given a pullback diagram of $\mathbb{E}_\infty$-rings*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01}. \end{tikzcd}$$

Then:

(1) *The natural map $\mathrm{cCAlg}_A \to \mathrm{cCAlg}_{A_0} \times_{\mathrm{cCAlg}_{A_{01}}} \mathrm{cCAlg}_{A_1}$ is fully faithful.*

(2) *Assume that the $\mathbb{E}_\infty$-rings A_0, A_1, and A_{01} are connective and that the ring homomorphisms $\pi_0(A_0) \to \pi_0(A_{01}) \leftarrow \pi_0(A_1)$ are surjective. Then the natural map*

$$\mathrm{cCAlg}^\flat_A \to \mathrm{cCAlg}^\flat_{A_0} \times_{\mathrm{cCAlg}^\flat_{A_{01}}} \mathrm{cCAlg}^\flat_{A_1}$$

is an equivalence of ∞-categories.

(3) *In the situation of (2), suppose that the map $\pi_0(A) \to \pi_0(A_0)$ has nilpotent kernel. Then the canonical map*

$$\mathrm{cCAlg}^{\mathrm{sm}}_A \to \mathrm{cCAlg}^{\mathrm{sm}}_{A_0} \times_{\mathrm{cCAlg}^{\mathrm{sm}}_{A_{01}}} \mathrm{cCAlg}^{\mathrm{sm}}_{A_1}$$

is an equivalence of ∞-categories.

Proof. Assertion (1) follows from Theorem SAG.16.2.0.2 , and assertion (2) follows from (1) together with Proposition SAG.16.2.3.1 . To deduce (3) from (2), it will suffice to show that if $C \in \mathrm{cCAlg}^{\flat}_A$ has the property that $A_0 \otimes_A C$ is a smooth coalgebra over A_0, then C is a smooth coalgebra over A. This follows immediately from Lemma 1.1.21. □

Proposition 1.2.13. *The construction $R \mapsto \mathrm{cCAlg}^{\mathrm{sm}}_R$ satisfies descent for the étale topology.*

Proof. Since the construction $R \mapsto \mathrm{Mod}_R$ satisfies faithfully flat descent (see Theorem SAG.D.6.3.5), the construction $R \mapsto \mathrm{cCAlg}_R$ also satisfies faithfully flat descent. It will therefore suffice to show that if C is a commutative coalgebra over R and there exists a faithfully flat étale map $R' \to R$ such that $R' \otimes_R C$ is a smooth coalgebra over R', then C is a smooth coalgebra over R. This is an immediate consequence of the corresponding assertion in the discrete case (Proposition 1.1.19). □

1.2.3 Compactness of Coalgebras

Let R be an $\mathbb{E}_\infty$-ring. Then the ∞-category cCAlg_R is presentable (Corollary AV.3.1.4). In particular, the ∞-category cCAlg_R admits small colimits, which are preserved by the forgetful functor $\mathrm{cCAlg}_R \to \mathrm{Mod}_R$ (Proposition AV.3.1.2). Since the full subcategory $\mathrm{Mod}^{\flat}_R \subseteq \mathrm{Mod}_R$ is closed under small filtered colimits, it follows that the full subcategory $\mathrm{cCAlg}^{\flat}_R \subseteq \mathrm{cCAlg}_R$ is also closed under filtered colimits. In particular, the ∞-category $\mathrm{cCAlg}^{\flat}_R$ admits small filtered colimits (which are preserved by the forgetful functor $\mathrm{cCAlg}^{\flat}_R \to \mathrm{Mod}^{\flat}_R$). It therefore makes sense to ask if an object $C \in \mathrm{cCAlg}^{\flat}_R$ is compact. We will need the following result:

Proposition 1.2.14. *Let R be a connective $\mathbb{E}_\infty$-ring which is n-truncated for some $n \gg 0$. Let C be a commutative coalgebra over R which is projective of finite rank when regarded as an R-module. Then C is a compact object of the ∞-category $\mathrm{cCAlg}^{\flat}_R$.*

Warning 1.2.15. In the situation of Proposition 1.2.14, the commutative coalgebra C need not be compact as an object of the larger ∞-category cCAlg_R. For example, let $R = \mathbf{Q}$ be the field of rational numbers and take $C = \mathbf{Q}$. For any vector space V over $\mathbf{Q}$, the direct sum $\mathbf{Q} \oplus \Sigma(V)$ can be endowed with the structure of a commutative coalgebra over $\mathbf{Q}$, depending functorially on V. One can show that the mapping space $\mathrm{Map}_{\mathrm{cCAlg}_{\mathbf{Q}}}(\mathbf{Q}, \mathbf{Q} \oplus \Sigma(V))$ is connected with fundamental group $\widehat{L}(V) = \varprojlim L(V)/L_n(V)$; here $L(V)$ denotes the free Lie algebra on V and $L_n(V) \subseteq L(V)$

denotes the nth stage of its lower central series. Note that the construction $V \mapsto \widehat{L}(V)$ does not commute with filtered colimits.

Proposition 1.2.14 is an immediate consequence of the following variant of Lemma SAG.5.2.2.6, applied to the ∞-category $\mathcal{C} = \mathrm{Mod}_R^\flat$:

Proposition 1.2.16. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category which admits filtered colimits, and suppose that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves filtered colimits. Let $C \in \mathrm{cCAlg}(\mathcal{C})$. Assume that $\mathcal{C}$ is equivalent to an n-category for some integer $n \geqslant 1$ and that C is compact object when viewed as an object of $\mathcal{C}$. Then C is a compact object of $\mathrm{cCAlg}(\mathcal{C})$.*

Proof. Since $\mathcal{C}$ is equivalent to an n-category, the forgetful functor

$$\mathrm{cCAlg}(\mathcal{C}) = \mathrm{CAlg}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Alg}_{\mathbb{E}_m}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$$

is an equivalence for $m > n$ (Corollary HA.5.1.1.7). It will therefore suffice to show that C is compact when viewed as an object of $\mathrm{Alg}_{\mathbb{E}_m}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$, for each $m \geqslant 1$. We proceed by induction on m. If $m > 1$, then Theorem HA.5.1.2.2 provides an equivalence $\mathrm{Alg}_{\mathbb{E}_m}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_{m-1}}(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}$, and our inductive hypothesis allows us to assume that C is compact when viewed as an object of $\mathcal{C}' = \mathrm{Alg}_{\mathbb{E}_{m-1}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. Note that that the forgetful functor $\mathcal{C}' \to \mathcal{C}$ is conservative and preserves filtered colimits (Corollary HA.3.2.2.4), so that the tensor product on $\mathcal{C}'$ also preserves filtered colimits separately in each variable. We may therefore replace $\mathcal{C}$ by $\mathcal{C}'$ and thereby reduce to the case $m = 1$: that is, we wish to show that $\mathcal{C}$ is compact when viewed as an object of the ∞-category $\mathrm{Alg}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ of *associative* coalgebra objects of $\mathcal{C}$. It will actually be more convenient to prove the following variant:

($*$) The coalgebra C is compact when viewed as an object of the ∞-category $\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ of *nonunital* associative coalgebra objects of $\mathcal{C}$.

Let us assume ($*$) for the moment, and show that it guarantees that C is also compact when viewed as an object of $\mathrm{Alg}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. Let $\{D_\alpha\}$ be a filtered diagram in $\mathrm{Alg}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ having a colimit D; we wish to show that the upper horizontal map in the diagram

$$\begin{tikzcd}
\varinjlim \mathrm{Map}_{\mathrm{Alg}(\mathcal{C}^{\mathrm{op}})}(D_\alpha, C) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Alg}(\mathcal{C}^{\mathrm{op}})}(D, C) \arrow[d] \\
\varinjlim \mathrm{Map}_{\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^{\mathrm{op}})}(D_\alpha, C) \arrow[r] & \mathrm{Map}_{\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^{\mathrm{op}})}(D, C)
\end{tikzcd}$$

is a homotopy equivalence. This follows from Theorem HA.5.4.3.5 (which guarantees that the preceding diagram is a pullback square) together with assumption $(*)$ (which implies that the lower horizontal map is a homotopy equivalence).

It remains to prove $(*)$. Since $\mathcal{C}$ is equivalent to an n-category, the forgetful functor $\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^{\mathrm{op}}) \to \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_k}(\mathcal{C}^{\mathrm{op}})$ is an equivalence of ∞-categories for $k \geqslant n+2$ (Corollary HA.4.1.6.17). It will therefore suffice to show that C is cocompact when viewed as an object of $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_k}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ for each $k \geqslant 1$. We proceed by induction on k. In the case $k = 1$, the forgetful functor $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_k}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{C}$ is an equivalence (Example HA.4.1.4.6) and the desired result follows from our assumption that C is compact as an object of $\mathcal{C}$. Let us therefore assume that $k \geqslant 2$. Choose a diagram $\{D_\alpha\}$ in $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_k}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ indexed by a filtered partially ordered set A, having colimit D. We wish to show that the upper horizontal map in the diagram

$$\begin{tikzcd}
\varinjlim \mathrm{Map}_{\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_k}(\mathcal{C}^{\mathrm{op}})}(D_\alpha, C) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_k}(\mathcal{C}^{\mathrm{op}})}(D, C) \arrow[d] \\
\varinjlim \mathrm{Map}_{\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_{k-1}}(\mathcal{C}^{\mathrm{op}})}(D_\alpha, C) \arrow[r] & \mathrm{Map}_{\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_{k-1}}(\mathcal{C}^{\mathrm{op}})}(D, C)
\end{tikzcd}$$

is a homotopy equivalence. Since the lower horizontal map is a homotopy equivalence by our inductive hypothesis, it will suffice to show that this diagram is a homotopy pullback square. We now show that the preceding diagram induces a homotopy equivalence after taking vertical homotopy fibers with respect to any choice of base point in $\varinjlim \mathrm{Map}_{\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_{k-1}}(\mathcal{C}^{\mathrm{op}})}(D_\alpha, C)$, which we can assume is represented by a morphism $f : D_\alpha \to C$ in $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_{k-1}}(\mathcal{C}^{\mathrm{op}})$ for some $\alpha \in A$. Using Theorem HA.4.1.6.13, we are reduced to showing that the diagram

$$\begin{tikzcd}
\varinjlim_{\beta \geqslant \alpha} \mathrm{Map}_{\mathcal{C}}(C, D_\beta^{\otimes k}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(C, D^{\otimes k}) \arrow[d] \\
\varinjlim_{\beta \geqslant \alpha} \mathrm{Map}_{\mathcal{C}}(C, D_\beta^{\otimes k})^{S^{k-3}} \arrow[r] & \mathrm{Map}_{\mathcal{C}}(C, D^{\otimes k})^{S^{k-3}}
\end{tikzcd}$$

is a homotopy pullback square, which follows from our assumptions that C is compact in $\mathcal{C}$ and that the tensor product on $\mathcal{C}$ commutes with filtered colimits. $\square$

1.3 Duality for Coalgebras

Let κ be a field and let C be a (flat) commutative coalgebra over κ, with comultiplication $\Delta_C : C \to C \otimes_\kappa C$. Then the κ-linear dual $C^\vee = \mathrm{Hom}_\kappa(C, \kappa)$ inherits the structure of a commutative algebra over κ, where the product of two linear functionals $\lambda, \mu : C \to \kappa$ is given by the composite map

$$\lambda\mu : C \xrightarrow{\Delta_C} C \otimes_\kappa C \xrightarrow{\lambda \otimes \mu} \kappa \otimes_\kappa \kappa \simeq \kappa.$$

The construction $C \mapsto C^\vee$ determines a functor from the category of commutative coalgebras over κ to the category of commutative algebras over κ. This functor is not fully faithful: in general, passing from a commutative coalgebra C to its dual space $C^\vee$ involves a loss of information. However, we can remedy the situation by equipping $C^\vee$ with a topology: namely, the topology of pointwise convergence (inherited from the product topology on the set $\prod_{x \in C} \kappa$ of *all* κ-valued functions on C). One can then show that the construction $C \mapsto C^\vee$ induces an equivalence of categories

$$\begin{array}{c} \{\text{Commutative coalgebras over } \kappa\} \\ \Big\downarrow \sim \\ \{\text{Linearly compact topological algebras over } \kappa\}; \end{array}$$

for details, we refer the reader to [6].

We will be particularly interested in the case of *smooth* coalgebras: that is, coalgebras of the form $C = \Gamma^*_\kappa(V)$, where V is a finite-dimensional vector space over κ. In this case, the dual $C^\vee$ is isomorphic to a power series ring $\kappa[[t_1, \ldots, t_n]]$ for $n = \dim_\kappa(V)$, and the topology of pointwise convergence coincides with the $\mathfrak{m}$-adic topology, where $\mathfrak{m} = (t_1, \ldots, t_n)$ is the maximal ideal of $C^\vee$. In particular, the dual $C^\vee$ is an example of an *adic ring*: that is, a commutative ring equipped with a topology which admits a finitely generated ideal of definition (see Definition 0.0.11).

In this section, we generalize the preceding discussion can be generalized to the setting of (smooth) coalgebras over an arbitrary $\mathbb{E}_\infty$-ring R. We begin in §1.3.1 by showing that for every commutative coalgebra C over R, the R-linear dual $C^\vee = \underline{\mathrm{Map}}_R(C, R)$ can be endowed with the structure of an $\mathbb{E}_\infty$-algebra over R. In §1.3.2, we specialize to the case where C is smooth over R and show that the commutative ring $\pi_0(C^\vee)$ inherits a canonical topology, which we will refer to as the *coradical topology*. The main result of this section asserts that the construction $C \mapsto C^\vee$ determines a fully faithful embedding

$$\begin{array}{c} \{\text{Smooth coalgebras over } R\} \\ \Big\downarrow \\ \{\text{Adic } \mathbb{E}_\infty\text{-algebras over } R\}. \end{array}$$

We deduce this in §1.3.4 (see Theorem 1.3.15) from a preliminary result on the compatibility of duality with base change (Proposition 1.3.13), which we prove in §1.3.3.

1.3.1 R-Linear Duality

Let R be an $\mathbb{E}_\infty$-ring. For every pair of R-modules M and N, we let $\underline{\mathrm{Map}}_R(M,N)$ denote the R-module which classifies maps from M into N (characterized by the universal mapping property $\mathrm{Map}_{\mathrm{Mod}_R}(K, \underline{\mathrm{Map}}_R(M,N)) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(K \otimes_R M, N)$). The construction $(M,N) \mapsto \underline{\mathrm{Map}}_R(M,N)$ determines a lax symmetric monoidal functor

$$\underline{\mathrm{Map}}_R : \mathrm{Mod}_R^{\mathrm{op}} \times \mathrm{Mod}_R \to \mathrm{Mod}_R .$$

Passing to commutative algebra objects, we obtain a functor $\mathrm{cCAlg}_R^{\mathrm{op}} \times \mathrm{CAlg}_R \to \mathrm{CAlg}_R$ which we will also denote by $\underline{\mathrm{Map}}_R$. We can summarize the situation more informally as follows: if C is a commutative coalgebra over R and A is an $\mathbb{E}_\infty$-algebra over R, then $\underline{\mathrm{Map}}_R(C,A)$ inherits the structure of an $\mathbb{E}_\infty$-algebra over R.

Example 1.3.1. Let R be an $\mathbb{E}_\infty$-ring, which we regard as a commutative coalgebra over itself. Then the construction $A \mapsto \underline{\mathrm{Map}}_R(R,A)$ is equivalent to the identity functor from CAlg_R to itself.

Example 1.3.2. Let R be an $\mathbb{E}_\infty$-ring, which we regard as an $\mathbb{E}_\infty$-algebra over itself. For any R-module C, we will denote $\underline{\mathrm{Map}}_R(C,R)$ by $C^\vee$ and refer to it as the *R-linear dual of C*. When C is a commutative coalgebra over R, then $C^\vee$ inherits the structure of an $\mathbb{E}_\infty$-algebra over R.

Remark 1.3.3. Let R be an $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0(R)$. Then, for any R-module M, the R-linear dual $M^\vee$ is also I-complete. In particular, if C is a coalgebra over R, then $C^\vee$ is an I-complete $\mathbb{E}_\infty$-algebra over R.

Remark 1.3.4. Let R be an $\mathbb{E}_\infty$-ring, let C be a commutative coalgebra over R, and let A be an $\mathbb{E}_\infty$-algebra over R. Then the counit map $\epsilon : C \to R$ can be regarded as a morphism of commutative coalgebras over R. By functoriality, we obtain a morphism of $\mathbb{E}_\infty$-algebras

$$A \simeq \underline{\mathrm{Map}}_R(R,A) \xrightarrow{\circ\epsilon} \underline{\mathrm{Map}}_R(C,A).$$

In other words, we can regard $\underline{\mathrm{Map}}_R(C,A)$ as an $\mathbb{E}_\infty$-algebra over A.

Remark 1.3.5 (Compatibility with Base Change)**.** Let $f : R \to R'$ be a morphism of $\mathbb{E}_\infty$-rings, let A be an $\mathbb{E}_\infty$-algebra over R' (which, by a slight abuse of notation, we also regard as an $\mathbb{E}_\infty$-algebra over R), and let C be a commutative coalgebra over R. Then $C' = R' \otimes_R C$ inherits the structure of a commutative coalgebra over R', and we have a canonical equivalence $\underline{\mathrm{Map}}_R(C, A) \simeq \underline{\mathrm{Map}}_{R'}(C', A)$ in the ∞-category CAlg_R of $\mathbb{E}_\infty$-algebras over R (or even in the ∞-category CAlg_A; see Remark 1.3.4).

In particular, taking $R' = A$, we obtain a canonical equivalence $\underline{\mathrm{Map}}_R(C, A) \simeq \underline{\mathrm{Map}}_A(A \otimes_R C, A) \simeq (A \otimes_R C)^\vee$ (here $(A \otimes_R C)^\vee$ denotes the A-linear dual of $A \otimes_R C$, which we regard as an $\mathbb{E}_\infty$-algebra over A via Example 1.3.2.

Remark 1.3.6. Let R be an $\mathbb{E}_\infty$-ring, let $\mathrm{Mod}_R^{\mathrm{perf}}$ denote the full subcategory of Mod_R spanned by the perfect R-modules, and define full subcategories $\mathrm{CAlg}_R^{\mathrm{perf}} \subseteq \mathrm{CAlg}_R$ and $\mathrm{cCAlg}_R^{\mathrm{perf}} \subseteq \mathrm{cCAlg}_R$ similarly. Then R-linear duality determines a symmetric monoidal equivalence $(\mathrm{Mod}_R^{\mathrm{perf}})^{\mathrm{op}} \simeq \mathrm{Mod}_R^{\mathrm{perf}}$. Passing to commutative algebra objects, we deduce that R-linear duality defines an equivalence $(\mathrm{cCAlg}_R^{\mathrm{perf}})^{\mathrm{op}} \to \mathrm{CAlg}_R^{\mathrm{perf}}$.

Example 1.3.7 (The Dual of Divided Power Coalgebra)**.** Let R be a commutative ring and let M be a projective R-module of finite rank, with dual $M^\vee$. Then we can identify $M^{\otimes n}$ with the set of multilinear maps $M^\vee \times \cdots \times M^\vee \to R$. Under this identification, the submodule $\Gamma_R^n(M) \subseteq M^{\otimes n}$ corresponds to the set of *symmetric* multilinear maps $M^\vee \times \cdots \times M^\vee \to R$: that is, with the R-linear dual of the symmetric power $\mathrm{Sym}_R^n(M^\vee)$. Allowing n to vary, we obtain an R-algebra isomorphism $\Gamma_R^*(M)^\vee \simeq \widehat{\mathrm{Sym}}_R^*(M^\vee)$, where $\widehat{\mathrm{Sym}}_R^*(M)$ denotes the product $\prod_{n \geqslant 0} \mathrm{Sym}_R^n(M)$. We will refer to $\widehat{\mathrm{Sym}}_R^*(M)$ as the *completed symmetric algebra of* M (note that it can be identified with the completion of $\mathrm{Sym}_R^*(M)$ with respect to the finitely generated ideal $\mathrm{Sym}_R^{>0}(M) = \bigoplus_{n>0} \mathrm{Sym}_R^n M)$.

Let R be an $\mathbb{E}_\infty$-ring and let M and N be R-modules. Then we have an evaluation map $e : M \otimes_R \underline{\mathrm{Map}}_R(M, N) \to N$ which induces a map of graded abelian groups

$$\pi_*(\underline{\mathrm{Map}}_R(M, N)) \to \mathrm{Hom}_{\pi_0(R)}(\pi_0(M), \pi_*(N)).$$

If M is a projective R-module (that is, if M appears as a direct summand of a coproduct of copies of R), then this map is an isomorphism. Using this observation, we obtain the following generalization of Example 1.3.7:

Example 1.3.8 (Duals of Smooth Coalgebras)**.** Let R be an $\mathbb{E}_\infty$-ring and let C be a smooth coalgebra over R. Fix an isomorphism $\pi_0(C) \simeq \Gamma^*_{\pi_0(R)}(M)$, where M is a projective module of finite rank over R. Since C is projective as an R-module (Remark

1.2.10), for any $\mathbb{E}_\infty$-algebra A over R we have canonical isomorphisms

$$\pi_*(\underline{\mathrm{Map}}_R(C, A)) \simeq \prod_{n \geqslant 0} (\mathrm{Sym}^n_{\pi_0(R)}(M^\vee) \otimes_{\pi_0(R)} \pi_*(A)).$$

In particular, $\pi_0(C^\vee)$ is isomorphic to the completed symmetric algebra $\widehat{\mathrm{Sym}}^*_{\pi_0(R)}(M^\vee)$.

Remark 1.3.9. It follows from Example 1.3.8 that if C is a smooth coalgebra over R, then the the construction $A \mapsto \underline{\mathrm{Map}}_R(C, A)$ commutes with passage to the connective cover and with truncation.

1.3.2 The Coradical Topology

Our next goal is to show that, if C is a smooth coalgebra over an $\mathbb{E}_\infty$-ring R, then the dual $C^\vee$ can be equipped with the structure of an adic $\mathbb{E}_\infty$-algebra over R. This is a consequence of the following:

Proposition 1.3.10. *Let R be a commutative ring and let C be a smooth coalgebra over R. Let us regard the R-linear dual $C^\vee$ as a subset of the Cartesian product $\prod_{x \in C} R$. Then the product topology on $C^\vee$ endows it with the structure of an adic commutative ring: that is, it admits a finitely generated ideal of definition $I \subseteq C^\vee$.*

Proof. Choose an isomorphism $\alpha : C \simeq \Gamma^*_R(M)$, where M is a projective R-module of finite rank. Then the R-linear dual $C^\vee$ can be identified with the completed symmetric algebra $\widehat{\mathrm{Sym}}^*_R(M^\vee)$ of Remark 1.3.7. Since every finite collection of elements of C is contained in the finitely generated submodule $\bigoplus_{0 \leqslant m \leqslant n} \Gamma^m_R(M)$ for sufficiently large n, the topology of pointwise convergence on $C^\vee$ coincides with the topology induced by the product decomposition

$$\widehat{\mathrm{Sym}}^*_R(M^\vee) \simeq \prod_{m \geqslant 0} \mathrm{Sym}^*_R(M^\vee).$$

This topology is I-adic, where I denotes the finitely generated ideal $\widehat{\mathrm{Sym}}^{>0}_R(M^\vee)$. $\square$

Definition 1.3.11. Let R be a commutative ring and let C be a smooth coalgebra over R. We will refer to the topology of Proposition 1.3.10 as the *coradical topology* on $C^\vee$.

More generally, if R is an $\mathbb{E}_\infty$-ring and C is a smooth coalgebra over R, then the isomorphism $\pi_0(C^\vee) \simeq \underline{\mathrm{Map}}_{\pi_0(R)}(\pi_0(C), \pi_0(R))$ induces a topology on the commutative ring $\pi_0(C^\vee)$, which we will also refer to as the *coradical topology*. This topology endows $C^\vee$ with the structure of an adic $\mathbb{E}_\infty$-algebra over R, in the sense of Definition 0.0.11.

Warning 1.3.12. Let C be a smooth coalgebra over a commutative ring R. Then we can choose a grouplike element $\eta \in C$, which determines an R-algebra homomorphism $\epsilon : C^\vee \to R$ having some kernel $I_\eta \subseteq C^\vee$. The proof of Proposition 1.3.10 shows that the coradical topology on $C^\vee$ coincides with the I_η-adic topology (and, in particular, that the I_η-adic topology is independent of the choice of grouplike element η). Beware that a smooth coalgebra C over an $\mathbb{E}_\infty$-ring need not admit any grouplike elements. However, the coradical topology on $\pi_0(C^\vee)$ is still well-defined (note that $\pi_0(C)$ always admits a grouplike element, when regarded as a coalgebra over $\pi_0(R)$).

1.3.3 Duality and Base Change

Let R be an $\mathbb{E}_\infty$-ring, let C be a commutative coalgebra over R, and let A be an $\mathbb{E}_\infty$-algebra over R. Then the canonical maps

$$A \simeq \underline{\mathrm{Map}}_R(R, A) \to \underline{\mathrm{Map}}_R(C, A) \leftarrow \underline{\mathrm{Map}}_R(C, R) \simeq C^\vee$$

determine a morphism of $\mathbb{E}_\infty$-rings

$$A \otimes_R C^\vee \to \underline{\mathrm{Map}}_R(C, A) \simeq (A \otimes_R C)^\vee,$$

where $(A \otimes_R C)^\vee$ denotes the A-linear dual of $A \otimes_R C$. This morphism is an equivalence if either A or C is perfect when regarded as an R-module, but not in general. Nevertheless, when C is a smooth coalgebra over R, it fails to be an equivalence in a very specific way: namely, we can identify $\underline{\mathrm{Map}}_R(C, A)$ as a completion of $A \otimes_R C^\vee$ with respect to the coradical topology of Definition 1.3.11. More generally, we have the following:

Proposition 1.3.13. *Let R be an $\mathbb{E}_\infty$-ring, let C be a smooth coalgebra over R, and let M be any R-module. Then the tautological map*

$$C^\vee \otimes_R M \to \underline{\mathrm{Map}}_R(C, M)$$

exhibits $\underline{\mathrm{Map}}_R(C, M)$ as the completion of $C^\vee \otimes_R M$ with respect to any ideal $I \subseteq \pi_0(C^\vee)$ which is a finitely generated ideal of definition for the coradical topology of Proposition 1.3.10.

The proof of Proposition 1.3.13 will require some preliminaries.

Lemma 1.3.14. *Let R be a commutative ring, let M be a projective module of finite rank over R, and let N be any discrete R-module. Then the canonical map*

$$u : \bigoplus_{n \geqslant 0} (\mathrm{Sym}^n_R(M) \otimes_R N) \to \prod_{n \geqslant 0} (\mathrm{Sym}^n_R(M) \otimes_R N)$$

exhibits the product $\prod_{n \geqslant 0} (\mathrm{Sym}^n_R(M) \otimes_R N)$ as the (derived) completion of the direct sum $\bigoplus_{n \geqslant 0} (\mathrm{Sym}^n_R(M) \otimes_R N)$ with respect to the ideal $\mathrm{Sym}^{>0}_R(M) \subseteq \mathrm{Sym}^*_R(M)$.

Proof. Choose a finitely generated subring $R_0 \subseteq R$ and a projective R_0-module M_0 of finite rank such that M is isomorphic to the tensor product $R \otimes_{R_0} M_0$. Replacing R by R_0 (and M by M_0), we can reduce to the case where the commutative ring R is Noetherian, so that the symmetric algebra $A = \mathrm{Sym}^*_R(M)$ is also Noetherian. Let $I \subseteq A$ denote the ideal $\mathrm{Sym}^{>0}_R(M)$ generated by M. For every discrete A-module $K \in \mathrm{Mod}^{\heartsuit}_A$, we let $\mathrm{Cpl}(K; I) = \varprojlim K / I^n K$ denote the classical I-adic completion of K and we let $K^{\wedge}_I$ denote the (derived) I-completion of K. It follows immediately from the definitions that the classical I-adic completion of $A \otimes_R N$ is the product $\prod_{n \geqslant 0} (\mathrm{Sym}^n_R(M) \otimes_R N)$. Consequently, Lemma 1.3.14 is equivalent to the assertion that the canonical map $(A \otimes_R N)^{\wedge}_I \to \mathrm{Cpl}((A \otimes_R N); I)$ is an equivalence. To prove this, choose a resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to N$$

of N by projective R-modules, so that $A \otimes_R P_\bullet$ is a resolution of N_A by projective A-modules. According to Corollary SAG.7.3.7.5 , the functor of I-completion is the nonabelian left derived functor of classical I-adic completion. In particular $(A \otimes_R N)^{\wedge}_I$ is represented (as an object of the ∞-category Mod_A) by the chain complex

$$\cdots \to \mathrm{Cpl}((A \otimes_R P_2); I) \to \mathrm{Cpl}((A \otimes_R P_1); I) \to \mathrm{Cpl}(A \otimes_R P_0; I).$$

We now observe that this chain complex factors as a direct product of the chain complexes

$$\cdots \to \mathrm{Sym}^n_R(M) \otimes_R P_2 \to \mathrm{Sym}^n_R(M) \otimes_R P_1 \to \mathrm{Sym}^n_R(M) \otimes_R P_0,$$

each of which is a projective resolution of $\mathrm{Sym}^n_R(M) \otimes_R N$. □

Proof of Proposition 1.3.13. We first treat the case where R is a connective $\mathbb{E}_\infty$-ring. Fix a grouplike element $\eta \in \pi_0(C)$ and let I_η be as in Warning 1.3.12. Let C be a smooth coalgebra over R. For every R-module M, let $F(M) \in \mathrm{Mod}_{C^\vee}$ denote the completion of $C^\vee \otimes_R M$ with respect to the ideal I_η. We wish to show that for every

R-module M, the canonical map $\rho_M : F(M) \to \underline{\mathrm{Map}}_R(C, M)$ is an equivalence. Note that we have a commutative diagram

$$\begin{tikzcd}
\varinjlim F(\tau_{\geqslant -n} M) \arrow[r] \arrow[d] & \varinjlim \underline{\mathrm{Map}}_R(C, \tau_{\geqslant -n} M) \arrow[d] \\
F(M) \arrow[r, "\rho_M"] & \underline{\mathrm{Map}}_R(C, M),
\end{tikzcd}$$

where the right vertical map is an equivalence (since C is projective as an R-module by Remark 1.2.10) and the left vertical map exhibits $F(M)$ as an I_η-completion of $\varinjlim F(\tau_{\geqslant -n} M)$. Note that if the upper horizontal map is an equivalence, then $\varinjlim F(\tau_{\geqslant -n} M) \simeq \underline{\mathrm{Map}}_R(C, M)$ is already I-complete, so the left vertical map is also an equivalence. It then follows that ρ_M is also an equivalence. Writing the upper horizontal map as a colimit of the maps $\rho_{\tau_{\geqslant -n} M}$, we are reduced to showing that ρ_M is an equivalence in the special case where M is $(-n)$-connective for some $n \gg 0$. Replacing M by $\Sigma^n M$, we may reduce to the case where M is connective. We will complete the proof by establishing the following connectivity estimate, for every integer k:

$(*_k)$ Let M be a connective R-module. Then the fiber $\mathrm{fib}(\rho_M)$ is k-connective.

Note that if M is connective, then $F(M)$ and $\underline{\mathrm{Map}}_R(C, M)$ are both connective, so that $(*_{-1})$ is automatically satisfied. We will complete the proof by showing that $(*_k)$ implies $(*_{k+1})$. For this, choose a free R-module P and a map $u : P \to M$ which is surjective on π_0, so that we have a fiber sequence of R-modules $M' \to P \to M$ where M' is connective. Since the functor F is exact, we have a fiber sequence $\mathrm{fib}(\rho_{M'}) \to \mathrm{fib}(\rho_P) \to \mathrm{fib}(\rho_M)$ whose first term is k-connective by virtue of our inductive hypothesis. Consequently, to show that $\mathrm{fib}(\rho_M)$ is $(k+1)$-connective, it will suffice to show that ρ_P is an equivalence.

Write P as a direct sum $\bigoplus_{\alpha \in J} R$, indexed by some set J. To show that ρ_P is an equivalence, we must show that the canonical map

$$\bigoplus_{\alpha \in J} \underline{\mathrm{Map}}_R(C, R) \to \underline{\mathrm{Map}}_R(C, \bigoplus_{\alpha \in J} R)$$

exhibits $\underline{\mathrm{Map}}_R(C, \bigoplus_{\alpha \in J} R)$ as an I_η-completion of $\bigoplus_{\alpha \in J} \underline{\mathrm{Map}}_R(C, R)$. In fact, we will prove something stronger: for every integer n, the induced map of homotopy groups

$$\phi : \pi_n \bigoplus_{\alpha \in J} \underline{\mathrm{Map}}_R(C, R) \to \pi_n \underline{\mathrm{Map}}_R(C, \bigoplus_{\alpha \in J} R)$$

exhibits $\pi_n \underline{\mathrm{Map}}_R(C, \bigoplus_{\alpha \in J} R)$ as an I_η-completion of $\pi_n \bigoplus_{\alpha \in J} \underline{\mathrm{Map}}_R(C, R)$. Set $N = \pi_n R$ and write $\pi_0(C) = \Gamma^*_{\pi_0(R)}(M^\vee)$, where $M^\vee$ is a projective module of finite rank over $\pi_0(R)$. Unwinding the definitions, we can identify ϕ with the lower horizontal map in the commutative diagram

$$\begin{tikzcd}
 & \bigoplus_{\alpha \in J} \bigoplus_{m \geqslant 0} (\mathrm{Sym}^m_R(M) \otimes_R N) \arrow[dl] \arrow[dr] & \\
\bigoplus_{\alpha \in J} \prod_{m \geqslant 0} (\mathrm{Sym}^m_R(M) \otimes_R N) \arrow[rr] & & \prod_{m \geqslant 0} \bigoplus_{\alpha \in J} (\mathrm{Sym}^m_R(M) \otimes_R N).
\end{tikzcd}$$

Without loss of generality, we may assume that I_η is the ideal $\widehat{\mathrm{Sym}}^{>0}_R(M)$, so that the right vertical map exhibits $\prod_{m \geqslant 0} \bigoplus_{\alpha \in J} (\mathrm{Sym}^m_R(M) \otimes_R N)$ as an I_η-completion of $\bigoplus_{\alpha \in J} \bigoplus_{m \geqslant 0} (\mathrm{Sym}^m_R(M) \otimes_R N)$ by vLemma 1.3.14 (applied to the R-module $\bigoplus_{\alpha \in J} N$). Consequently, to complete the proof of $(*_{k+1})$, it will suffice to show that the left vertical map induces an equivalence after I_η-completion. This is clear, since it is a direct sum of maps

$$\bigoplus_{m \geqslant 0} (\mathrm{Sym}^m_R(M) \otimes_R N) \to \prod_{m \geqslant 0} (\mathrm{Sym}^m_R(M) \otimes_R N),$$

each of which induces an equivalence of I_η-completions by virtue of Lemma 1.3.14. This completes the proof of Proposition 1.3.13 in the case where R is assumed to be connective.

We now treat the case where R is not assumed to be connective. Let R_0 denote the connective cover of R. Using Lemma 1.2.8, we can write $C = R \otimes_R R_0$, where C_0 is a smooth coalgebra over R_0. Note that we can identify $C_0^\vee$ with the connective cover of $C^\vee$, so that the ideal I_η of Proposition 1.3.10 can be viewed as an ideal of $\pi_0(C_0^\vee)$. For any R-module M, we have a commutative diagram

$$\begin{tikzcd}
C_0^\vee \otimes_{R_0} M \arrow[r] \arrow[d] & \underline{\mathrm{Map}}_{R_0}(C_0, M) \arrow[d] \\
C^\vee \otimes_R M \arrow[r] & \underline{\mathrm{Map}}_{R_0}(C, M),
\end{tikzcd}$$

where the right vertical map is an equivalence and the upper horizontal map exhibits $\underline{\mathrm{Map}}_{R_0}(C_0, M)$ as an I_η-completion of $C_0^\vee \otimes_{R_0} M$ (by the first part of the proof). To complete the proof, it will suffice to show that the left vertical map

$$C_0^\vee \otimes_{R_0} M \simeq (C_0^\vee \otimes_{R_0} R) \otimes_R M \to C^\vee \otimes_R M$$

induces an equivalence after I_η-completion. In fact, we claim that the map $C_0^\vee \otimes_{R_0} R \to C^\vee$ already induces an equivalence after I_η-completion: this follows from the first part of the proof, applied to the object $R \in \mathrm{Mod}_{R_0}$. $\square$

1.3.4 Recovering a Coalgebra from its Dual

We now show that, if C is a smooth coalgebra over an $\mathbb{E}_\infty$-ring R, then we can recover C from its R-linear dual $C^\vee$, regarded as an adic $\mathbb{E}_\infty$-algebra via the coradical topology of Definition 1.3.11. More precisely, we have the following:

Theorem 1.3.15. *Let R be an $\mathbb{E}_\infty$-ring and let $C, D \in \mathrm{cCAlg}_R^{\mathrm{sm}}$. Then R-linear duality induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{cCAlg}_R}(D, C) \to \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, D^\vee).$$

Example 1.3.16. Let R be a commutative ring and let M and N be projective R-modules of finite rank. Then we have canonical bijections

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mod}_R^\heartsuit}(\Gamma_R^*(M), N) &\simeq \mathrm{Hom}_{\mathrm{Mod}_R^\heartsuit}(N^\vee, \Gamma_R^*(M)^\vee) \\ &\simeq \mathrm{Map}_{\mathrm{CAlg}_R^\heartsuit}(\mathrm{Sym}_R^*(N^\vee), \widehat{\mathrm{Sym}}_R^*(M^\vee)).\end{aligned}$$

Note that a map $f : \Gamma_R^*(M) \to N$ satisfies condition $(*)$ of Proposition 1.1.16 if and only if its image under this bijection is an R-algebra homomorphism $\lambda : \mathrm{Sym}_R^*(N^\vee) \to \widehat{\mathrm{Sym}}_R^*(M^\vee)$ for which the composite map

$$\mathrm{Sym}_R^*(N^\vee) \xrightarrow{\lambda} \widehat{\mathrm{Sym}}_R^*(M^\vee) \to \mathrm{Sym}_R^0(M^\vee) \simeq R$$

carries each element of $N^\vee$ to a nilpotent element of R. In this case, λ extends uniquely to a continuous ring homomorphism

$$\widehat{\lambda} : \widehat{\mathrm{Sym}}_R^*(N^\vee) \to \widehat{\mathrm{Sym}}_R^*(M^\vee),$$

where we equip $\widehat{\mathrm{Sym}}_R^*(M^\vee)$ and $\widehat{\mathrm{Sym}}_R^*(N^\vee)$ with their coradical topologies.

Combining this observation with Proposition 1.1.16, we deduce that R-linear duality induces a monomorphism

$$\mathrm{Hom}_{\mathrm{cCAlg}_R^\flat}(\Gamma_R^*(M), \Gamma_R^*(N)) \to \mathrm{Hom}_{\mathrm{CAlg}_R^\heartsuit}(\widehat{\mathrm{Sym}}_R^*(N^\vee), \widehat{\mathrm{Sym}}_R^*(M^\vee)),$$

whose image is the collection of continuous R-algebra homomorphisms from $\widehat{\mathrm{Sym}}_R^*(N^\vee)$ to $\widehat{\mathrm{Sym}}_R^*(M^\vee)$.

Proof of Theorem 1.3.15. We begin with some general remarks. Let C and D be smooth coalgebras over some $\mathbb{E}_\infty$-ring R. Let R' be an $\mathbb{E}_\infty$-algebra over R, and set $C' = R' \otimes_R C$ and $D' = R' \otimes_R D$. Then C' and D' are smooth $\mathbb{E}_\infty$-algebras over R', and we have a canonical map

$$\rho_{R'} : \mathrm{Map}_{\mathrm{cCAlg}_{R'}}(D', C') \to \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{R'}}(C'^\vee, D'^\vee).$$

Using Proposition 1.3.13, we can identify the codomain of $\rho_{R'}$ with

$$\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{R'}}(R' \otimes_R C^\vee, D'^\vee) \simeq \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, \underline{\mathrm{Map}}_R(D, R')).$$

Note that this map depends functorially on R'.

To prove Theorem 1.3.15, we must show that the map ρ_R is a homotopy equivalence. We first reduce to the case where R is connective. Let $R_0 = \tau_{\geqslant 0} R$. Using Proposition 1.2.8, we can assume that $C = R \otimes_{R_0} C_0$ and $D = R \otimes_{R_0} D_0$ for some smooth coalgebras C_0 and D_0 over the connective cover $R_0 = \tau_{\geqslant 0} R$. In this case, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{cCAlg}_{R_0}}(D_0, C_0) \arrow[r] \arrow[d] & \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{R_0}}(C_0^\vee, D_0^\vee) \arrow[d] \\
\mathrm{Map}_{\mathrm{cCAlg}_R}(D, C) \arrow[r, "\rho_R"] & \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{R_0}}(C_0^\vee, \underline{\mathrm{Map}}_{R_0}(D_0, R)).
\end{tikzcd}$$

Here the left vertical map is a homotopy equivalence by virtue of Proposition 1.2.8, and the right vertical map is a homotopy equivalence because $C_0^\vee$ is connective and $D_0^\vee$ is the connective cover of $D^\vee = \underline{\mathrm{Map}}_{R_0}(D_0, R)$ (Remark 1.3.9). Consequently, to show that ρ_R is an equivalence, we can replace R by R_0 (and the coalgebras C and D by C_0 and D_0, respectively) and thereby reduce to the case where R is connective.

Assume now that R is connective. For each $n \geqslant 0$, let us identify the truncations

$$\tau_{\leqslant n} C \simeq (\tau_{\leqslant n} R) \otimes_R C \qquad \tau_{\leqslant n} D \simeq (\tau_{\leqslant n} R) \otimes_R D$$

with smooth coalgebra over $\tau_{\leqslant n} R$. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{cCAlg}_R}(D, C) \arrow[r, "\rho_R"] \arrow[d] & \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, \underline{\mathrm{Map}}_R(D, R)) \arrow[d] \\
\varprojlim \mathrm{Map}_{\mathrm{cCAlg}_{\tau_{\leqslant n} R}}(\tau_{\leqslant n} D, \tau_{\leqslant n} C) \arrow[r] & \varprojlim \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, \underline{\mathrm{Map}}_R(D, \tau_{\leqslant n} R)).
\end{tikzcd}$$

Here the right vertical map is clearly a homotopy equivalence, the left vertical map is also a homotopy equivalence by virtue of Proposition 1.2.11, and the bottom vertical map can be identified with the limit of the tower of maps $\{\rho_{\tau_{\leqslant n} R}\}_{n \geqslant 0}$. Consequently, to show that ρ_R is a homotopy equivalence, it will suffice to show that each $\rho_{\tau_{\leqslant n} R}$ is a homotopy equivalence. We may therefore replace R by $\tau_{\leqslant n} R$ and thereby reduce to the case where R is n-truncated, for some $n \geqslant 0$.

We now proceed by induction on n. We first explain how to carry out the inductive step. Assume that $n > 0$, and set $R' = \tau_{\leqslant n-1} R$, and let $M = \pi_n R$. According to Theorem HA.7.4.1.26 , we can identify R with a square-zero extension of R' by the module $\Sigma^n M$: that is, there is a pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \pi_0(R) \arrow[d] \\ R' \arrow[r] & \pi_0(R) \oplus \Sigma^{n+1} M. \end{tikzcd}$$

Using Proposition 1.2.12, we obtain a pullback diagram

$$\begin{tikzcd} \rho_R \arrow[r] \arrow[d] & \rho_{\pi_0(R)} \arrow[d] \\ \rho_{R'} \arrow[r] & \rho_{\pi_0(R) \oplus \Sigma^{n+1} M}, \end{tikzcd}$$

in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{S})$ of morphisms in $\mathcal{S}$. Consequently, to show that ρ_R is a homotopy equivalence, it will suffice to show that $\rho_{R'}$, $\rho_{\pi_0(R)}$, and $\rho_{\pi_0(R) \oplus \Sigma^{n+1} M}$ are homotopy equivalences. The first case follows from our inductive hypothesis. To handle the other two, it will suffice to prove the following (which also establishes the base case for our induction):

- $(*)$ Let R be a commutative ring, let C and D be smooth coalgebras over R, and let A be an $\mathbb{E}_\infty$-algebra over R which is connective and truncated. Then the map ρ_A is a homotopy equivalence.

To prove $(*)$, we first invoke the smoothness of C and D to choose isomorphisms $C \simeq \Gamma^*_R(M)$ and $D \simeq \Gamma^*_R(N)$, where M and N are projective R-modules of finite rank. Note that M and N are defined over some finitely generated subring of $R_0 \subseteq R$; we may therefore replace R by R_0 and thereby reduce to the case where R is Noetherian. For each $m \geqslant 0$, set $C_m = \bigoplus_{0 \leqslant k \leqslant m} \Gamma^k_R(M)$ and $D_m = \bigoplus_{0 \leqslant k \leqslant m} \Gamma^k_R(N)$, which we regard as (non-smooth) coalgebras over R. Note that $A \otimes_R D$ can be identified with the colimit $\varinjlim A \otimes_R D_m$ (formed in the ∞-category cCAlg_A), and that $\underline{\mathrm{Map}}_R(D, A)$ can be identified with the inverse limit $\varprojlim \underline{\mathrm{Map}}_R(D_m, A)$ (formed in the ∞-category

CAlg_A). Moreover, an R-algebra map $C^\vee \to \underline{\mathrm{Map}}_R(D, A)$ is continuous (with respect to the coradical topology of Definition 1.3.11) if and only if each of the induced maps $C^\vee \to \underline{\mathrm{Map}}_R(D, A)$ is continuous (where the target is equipped with the discrete topology). We can therefore write ρ_A as the inverse limit of a tower of maps

$$\rho_m : \mathrm{Map}_{\mathrm{cCAlg}_A}(A \otimes_R D_m, A \otimes_R C) \to \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, \underline{\mathrm{Map}}_R(D_m, A)).$$

We will complete the proof by by showing that each ρ_m is an equivalence.

Let us identify $C^\vee$ with the completed symmetric algebra $\widehat{\mathrm{Sym}}^*_R(N^\vee)$, equipped with the topology determined by the ideal $I = \widehat{\mathrm{Sym}}^{>0}_R(N^\vee)$. For each $k \geqslant 0$, we can identify the quotient $C^\vee/I^{k+1}$ with the dual of the subcoalgebra $C_k \subseteq C$. We therefore have a commutative diagram

$$\begin{tikzcd}
\varinjlim_k \mathrm{Map}_{\mathrm{cCAlg}_A}(A \otimes_R D_m, A \otimes_R C_k) \arrow[r, "\widetilde{\rho}"] \arrow[d] & \varinjlim \mathrm{Map}_{\mathrm{CAlg}_R}(C_k^\vee, \underline{\mathrm{Map}}_R(D_m, A)) \arrow[d] \\
\mathrm{Map}_{\mathrm{cCAlg}_A}(A \otimes_R D_m, A \otimes_R C) \arrow[r, "\rho_m"] & \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, \underline{\mathrm{Map}}_R(D_m, A)),
\end{tikzcd}$$

where the right vertical map is a homotopy equivalence by Lemma SAG.17.3.5.7 (note that A is truncated). We claim that the left vertical map is also a homotopy equivalence. Note that we can identify $A \otimes_R C$ with the colimit of the diagram $\{A \otimes_R C_k\}_{k \geqslant 0}$ in the ∞-category $\mathrm{cCAlg}(\mathrm{Mod}^\flat_A)$ of commutative coalgebra objects of $\mathrm{Mod}^\flat_A$. To establish the desired result, it suffices to show that $A \otimes_R D_m$ is a compact object of $\mathrm{cCAlg}(\mathrm{Mod}^\flat_A)$. Since A is truncated, the ∞-category $\mathrm{Mod}^\flat_A$ is equivalent to a q-category for $q \gg 0$. Applying Proposition 1.2.16, we are reduced to showing that $A \otimes_R D_m$ is compact as an object of $\mathrm{Mod}^\flat_A$, which is clear (since D_m is a projective R-module of finite rank).

To complete the proof that ρ_m is a homotopy equivalence, it will suffice to show that the map $\widetilde{\rho}$ is a homotopy equivalence. By construction, $\widetilde{\rho}$ can be written as the colimit of maps

$$\begin{aligned}
\rho_{m,k} : \mathrm{Map}_{\mathrm{cCAlg}_A}(A \otimes_R D_m, A \otimes_R C_k) &\to \mathrm{Map}_{\mathrm{CAlg}_R}(C_k^\vee, \underline{\mathrm{Map}}_R(D_m, A)) \\
&\simeq \mathrm{Map}_{\mathrm{CAlg}_A}((A \otimes_R C_k)^\vee, (A \otimes_R D_m)^\vee).
\end{aligned}$$

Each of these maps is a homotopy equivalence by virtue of Remark 1.3.6, since $A \otimes_R C_k$ and $A \otimes_R D_m$ are perfect A-modules. $\square$

1.4 Duals of Smooth Coalgebras

Let R be an $\mathbb{E}_\infty$-ring and let $\mathrm{CAlg}_R^{\mathrm{ad}} = \mathrm{CAlg}_R \times_{\mathrm{CAlg}} \mathrm{CAlg}^{\mathrm{ad}}$ denote the ∞-category of adic $\mathbb{E}_\infty$-algebras over R. It follows from Theorem 1.3.15 that the construction $C \mapsto C^\vee$ determines a fully faithful embedding $\mathrm{cCAlg}_R^{\mathrm{op}} \to \mathrm{CAlg}_R^{\mathrm{ad}}$. We now ask to describe the essential image of this embedding:

Question 1.4.1. Let A be an adic $\mathbb{E}_\infty$-algebra over R. Under what conditions does there exist an equivalence $A \simeq C^\vee$, where C is a smooth coalgebra over R?

In the case where R is discrete, Question 1.4.1 is answered by Example 1.3.7: the duals of smooth coalgebras are precisely those adic R-algebras of the form $\widehat{\mathrm{Sym}}^*_R(M)$, where M is a projective R-module of finite rank. In particular, if R is local, then they are precisely the power series rings $R[[t_1, \ldots, t_n]]$ for $n \geqslant 0$.

Our goal in this section is to provide a complete an answer to Question 1.4.1 in general. Roughly speaking, we show that the duals of smooth coalgebras are characterized by the fact that they "resemble" power series algebras over R, at the level of homotopy groups. We make this precise in §1.4.1 by introducing an $\mathbb{E}_\infty$-algebra $R[[t_1, \ldots, t_n]]$ and establishing a weak universality principle: an $\mathbb{E}_\infty$-algebra A over R is equivalent to $R[[t_1, \ldots, t_n]]$ *as an* $\mathbb{E}_1$-*algebra* if and only if the homotopy group $\pi_*(A)$ is isomorphic to $\pi_*(R)[[t_1, \ldots, t_n]]$ (Proposition 1.4.5). In §1.4.2, we show that every such $\mathbb{E}_\infty$-algebra arises as the dual of a smooth coalgebra over R (Proposition 1.4.10). In §1.4.3, we use this result to give a complete answer to Question 1.4.1 (Proposition 1.4.11).

1.4.1 Power Series Algebras

Let R be an $\mathbb{E}_\infty$-ring. There is an essentially unique symmetric monoidal functor $\mathcal{S} \to \mathrm{Mod}_R$ which preserves small colimits, which we will denote by $X \mapsto C_*(X; R)$. For each $n \geqslant 0$, we let $R[t_1, \ldots, t_n]$ denote the R-module given by $C_*(\mathbf{Z}^n_{\geqslant 0}; R)$. Since $\mathbf{Z}^n_{\geqslant 0}$ can be regarded as a commutative algebra object of the ∞-category $\mathcal{S}$ (given by the usual addition on $\mathbf{Z}_{\geqslant 0}$), we obtain a commutative algebra structure on $R[t_1, \ldots, t_n]$: in other words, we can regard $R[t_1, \ldots, t_n]$ as an $\mathbb{E}_\infty$-algebra over R. Note that we have a canonical isomorphism of graded rings $\pi_*(R[t_1, \ldots, t_n]) \simeq (\pi_* R)[t_1, \ldots, t_n]$. In particular, if R is discrete, then $R[t_1, \ldots, t_n]$ agrees with the usual polynomial algebra on n generators over A.

Warning 1.4.2. In the situation described above, the polynomial algebra $R[t_1, \ldots, t_n]$ is generally different from the *free* $\mathbb{E}_\infty$-algebra over R generated by indeterminates

$t_1, \dots, t_n$. We denote this free algebra by $R\{t_1, \dots, t_n\}$. Its universal property yields a canonical map

$$R\{t_1, \dots, t_n\} \to R[t_1, \dots, t_n],$$

which is an equivalence if $\pi_0(R)$ is a $\mathbf{Q}$-algebra, but not in general.

Remark 1.4.3. Let R be a connective $\mathbb{E}_\infty$-ring. Then $R[t]$ is the free $\mathbb{E}_1$-algebra over R on one generator: this follows from Proposition HA.4.1.1.18 (or from the observation that $\mathbf{Z}_{\geqslant 0}$ is the free $\mathbb{E}_1$-algebra object of $\mathcal{S}$ on one generator).

Construction 1.4.4 (Power Series Algebras)**.** Let R be an $\mathbb{E}_\infty$-ring. For each $n \geqslant 0$, we let $R[[t_1, \dots, t_n]]$ denote the completion of $R[t_1, \dots, t_n]$ with respect to the ideal $(t_1, \dots, t_n) \subseteq \pi_0(R[t_1, \dots, t_n])$. It follows from Lemma 1.3.14 that the homotopy groups of $R[[t_1, \dots, t_n]]$ are given by the formula

$$\pi_*(R[[t_1, \dots, t_n]]) = \pi_*(R)[[t_1, \dots, t_n]].$$

In particular, if R is an ordinary commutative ring (regarded as a discrete $\mathbb{E}_\infty$-ring), then $R[[t_1, \dots, t_n]]$ can be identified with the usual ring of formal power series over R.

Construction 1.4.4 has a weak universal property:

Proposition 1.4.5. *Let R be an $\mathbb{E}_\infty$-ring and let A be an $\mathbb{E}_\infty$-algebra over R. The following conditions are equivalent:*

(1) *There exists an equivalence $R[[t_1, \dots, t_n]] \simeq A$ in the ∞-category* Alg_R *of $\mathbb{E}_1$-algebras over R.*

(2) *There exists an isomorphism of graded $\pi_*(R)$-algebras $\pi_*(R)[[t_1, \dots, t_n]] \simeq \pi_*(A)$.*

Warning 1.4.6. In assertion (1) of Proposition 1.4.5, we cannot replace Alg_R by the ∞-category CAlg_R of $\mathbb{E}_\infty$-algebras over R: in general, there can be many *different* $\mathbb{E}_\infty$-algebras over R which become equivalent to $R[[t_1, \dots, t_n]]$ when viewed as (augmented) $\mathbb{E}_1$-algebras (see Warning 1.6.18).

The proof depends on the following:

Lemma 1.4.7. *Let R be an $\mathbb{E}_\infty$-ring, let A be an $\mathbb{E}_\infty$-algebra over R, and let $\phi_0 : R[t_1, \dots, t_n] \to A$ be a morphism of $\mathbb{E}_1$-algebras over R which exhibits A as complete with respect to the ideal $(t_1, \dots, t_n)$ when regarded as a* left *module over $R[t_1, \dots, t_n]$. Then ϕ_0 admits an (essentially unique) extension to a map $\phi : R[[t_1, \dots, t_n]] \to A$ (again of $\mathbb{E}_1$-algebras over R).*

Proof. To simplify notation, let us assume that $n = 1$ and denote $R[t_1, \ldots, t_n]$ by $R[t]$ (the proof in the general case is the same). Let $\mathcal{C} = {}_{R[t]}\mathrm{BMod}_{R[t]}(\mathrm{Mod}_R)$ denote the ∞-category of $R[t]$-$R[t]$ bimodule objects of Mod_R, which we will equip with the monoidal structure given by relative tensor product over $R[t]$. Note that we can identify $\mathcal{C}$ with the ∞-category of modules over the tensor product $R[t] \otimes_R R[t]$. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory spanned by those objects which are $(t \otimes 1 - 1 \otimes t)$-nilpotent. We first claim that $\mathcal{C}'$ is a monoidal subcategory of $\mathcal{C}$. Since it clearly contains the unit object $R[t] \in \mathcal{C}$, it will suffice to show that for every pair of objects $M, N \in \mathcal{C}'$, the tensor product $M \otimes_{R[t]} N$ belongs to $\mathcal{C}$. Note that we can view $M \otimes_{R[t]} N$ as a module over the tensor product $R[t] \otimes_R R[t] \otimes_R R[t]$. The assumption that $M \in \mathcal{C}'$ guarantees that $M \otimes_{R[t]} N$ is $(t \otimes 1 \otimes 1 - 1 \otimes t \otimes 1)$-nilpotent, and the assumption that $N \in \mathcal{C}'$ guarantees that $M \otimes_{R[t]} N$ is $(1 \otimes t \otimes 1 - 1 \otimes 1 \otimes t)$-nilpotent. Applying Proposition SAG.7.1.1.5 , we deduce that $M \otimes_{R[t]} N$ is also $(t \otimes 1 \otimes 1 - 1 \otimes 1 \otimes t)$-nilpotent, and therefore belongs to $\mathcal{C}'$ as desired.

Let us view $\mathcal{C}'$ as an $(R[t] \otimes_R R[t])$-linear ∞-category. By construction, every $(t \otimes 1 - 1 \otimes t)$-local object of $\mathcal{C}'$ is zero. It follows that each $M \in \mathcal{C}'$ is $(t \otimes 1 - 1 \otimes t)$-complete *when viewed as an object of* $\mathcal{C}'$. Using Corollary SAG.7.3.3.3 , we deduce the following:

(∗) An object $M \in \mathcal{C}'$ is $(t \otimes 1)$-complete (when viewed as an object of $\mathcal{C}'$) if and only if it is $(1 \otimes t)$-complete (when viewed as an object of $\mathcal{C}'$).

Let $\mathcal{C}''$ denote the full subcategory of $\mathcal{C}'$ spanned by those objects which satisfy the condition of (∗). The inclusion functor $\mathcal{C}'' \hookrightarrow \mathcal{C}'$ admits a left adjoint $L : \mathcal{C}' \to \mathcal{C}''$. We now claim that L is compatible with the monoidal structure on $\mathcal{C}$ (given by tensor product over $R[t]$), in the sense of Definition HA.2.2.1.6 . In other words, we claim that if $\alpha : M \to M'$ is an L-equivalence in $\mathcal{C}'$ and is and N is an arbitrary object of $\mathcal{C}'$, then the induced maps $\beta_N : M \otimes_{R[t]} N \to M' \otimes_{R[t]} N$ and $\gamma_N : N \otimes_{R[t]} M \to N \otimes_{R[t]} N'$ are also L-equivalences. We will show that β_N is an L-equivalence; the proof for γ_N is similar. Our assumption that $N \in \mathcal{C}'$ guarantees that N can be written as a colimit of fibers $N_n = \mathrm{fib}((t \otimes 1 - 1 \otimes t)^n : N \to N)$. It will therefore suffice to show that each β_{N_n} is an L-equivalence. Proceeding by induction on n and using the fiber sequences

$$N_{n-1} \to N_n \xrightarrow{(t \otimes 1 - 1 \otimes t)^{n-1}} N_1,$$

we can reduce to the case $n = 1$. We are therefore reduced to show that β_N is an L-equivalence under the assumption that N belongs to the image of the forgetful functor $\mathrm{Mod}_{R[t]} \to \mathcal{C}' \subseteq {}_{R[t]}\mathrm{BMod}_{R[t]}(\mathrm{Mod}_R)$. Since $\mathrm{Mod}_{R[t]}$ is generated under small colimits by desuspensions of $R[t]$, we may assume that $N = R[t]$, in which case $\beta_N \simeq \alpha$ is an L-equivalence by assumption.

The $\mathbb{E}_1$-algebra maps $\phi_0 : R[t] \to A$ and $R[t] \to R[[t]]$ exhibit A and $R[[t]]$ as associative algebra objects of the monoidal ∞-category $\mathcal{C}$. The commutativity of the multiplications on A and $R[[t]]$ guarantee that they belong to the subcategory $\mathcal{C}' \subseteq \mathcal{C}$. Moreover, both A and $R[[t]]$ are $(t \otimes 1)$-complete when viewed as objects of $\mathcal{C}$, and therefore also when viewed as objects of the subcategory $\mathcal{C}' \subseteq \mathcal{C}$. Moreover, the tautological map $\rho : R[t] \to R[[t]]$ exhibits $R[[t]]$ as an L-localization of $R[t]$: to prove this, it suffices to observe that the fiber $\mathrm{fib}(\rho)$ is a $(t \otimes 1)$-local object of $\mathcal{C}'$, which is clear (since it is already a $(t \otimes 1)$-local object of $\mathcal{C}$). It follows that the map ρ_0 admits an essentially unique factorization $R[t] \to R[[t]] \xrightarrow{\phi} A$ in the ∞-category $\mathrm{Alg}(\mathcal{C}') \subseteq \mathrm{Alg}(\mathcal{C}) \simeq (\mathrm{Alg}_R)_{R[t]/}$. □

Proof of Proposition 1.4.5. Let R be an $\mathbb{E}_\infty$-ring and let A be an $\mathbb{E}_\infty$-algebra over R. Assume that there exists an isomorphism of graded $\pi_*(R)$-algebras

$$u : \pi_*(R)[[t_1, \ldots, t_n]] \simeq \pi_*(A);$$

we wish to show u can be lifted an equivalence $R[[t_1, \ldots, t_n]] \simeq A$ in the ∞-category Alg_R of $\mathbb{E}_1$-algebras over R (the converse was already noted in Construction 1.4.4). For $1 \leqslant i \leqslant n$, set $T_i = u(t_i) \in \pi_0(A)$. Using Remark 1.4.3, we see that each T_i classifies a map of $\mathbb{E}_1$-algebras $v_i : R[t_i] \to A$ for which the induced map $\pi_0(R[t_i]) \to \pi_0(A)$ carries t_i to T_i. Taking the tensor product of these maps (and composing with the multiplication on A), we obtain a map of $\mathbb{E}_1$-algebras $v : R[t_1, \ldots, t_n] \to A$, which carries each t_i to T_i. Using Lemma 1.4.7, we can factor v as a composition

$$R[t_1, \ldots, t_n] \to R[[t_1, \ldots, t_n]] \xrightarrow{\overline{v}} A.$$

Assumption (2) now guarantees that $\overline{v}$ induces an isomorphism on homotopy groups and is therefore an equivalence of $\mathbb{E}_1$-algebras over R. □

1.4.2 Duals of Standard Smooth Coalgebras

We now address a special case of Question 1.4.1.

Definition 1.4.8. Let R be a commutative ring. We will say that a coalgebra C over R is *standard smooth* if it is isomorphic to a divided power coalgebra $\Gamma^*_R(M)$, where M is a *free* R-module of finite rank.

More generally, if R is any $\mathbb{E}_\infty$-ring, then we will say that a commutative coalgebra $C \in \mathrm{cCAlg}_R$ is *standard smooth* if C is flat over R and the coalgebra $\pi_0(C)$ is standard smooth over $\pi_0(R)$.

Remark 1.4.9. Let R be an $\mathbb{E}_\infty$-ring and let C be a smooth coalgebra over R. Then C is standard smooth locally with respect to the Zariski topology on $\mathrm{Spec}(R)$. More precisely, there exists a collection of elements $a_1, \ldots, a_k \in \pi_0(R)$ which generate the unit ideal, for which each localization $C[a_i^{-1}]$ is a standard smooth coalgebra over $R[a_i^{-1}]$.

Proposition 1.4.10. *Let R be an $\mathbb{E}_\infty$-ring and let A be an adic $\mathbb{E}_\infty$-ring over R. The following conditions are equivalent:*

(a) *There exists an isomorphism $u : \pi_*(R)[[t_1, \ldots, t_n]] \simeq \pi_*(A)$, and the topology on $\pi_0(A)$ has an ideal of definition $(u(t_1), \ldots, u(t_n))$.*

(b) *There exists an equivalence $R[[t_1, \ldots, t_n]] \simeq A$ of $\mathbb{E}_1$-algebras over R which is a homeomorphism on $\pi_0(R)$ (where $\pi_0(R[[t_1, \ldots, t_n]])$ is equipped with the $(t_1, \ldots, t_n)$-adic topology).*

(c) *There exists a standard smooth coalgebra C over R and an equivalence of adic $\mathbb{E}_\infty$-algebras $A \simeq C^\vee$.*

Proof. The equivalence of (a) and (b) follows from Proposition 1.4.5, and the implication $(c) \Rightarrow (a)$ follows from Example 1.3.8. We will complete the proof by showing that $(b) \Rightarrow (c)$. Replacing R by $\tau_{\geqslant 0}R$ and A by $\tau_{\geqslant 0}A$, we can reduce to the case where R is connective. Let $f : R[[t_1, \ldots, t_n]] \to A$ be as in (2), so that f exhibits A as the completion of $R[t_1, \ldots, t_n]$ with respect to the ideal $I = (t_1, \ldots, t_n) \subseteq \pi_0(R[t_1, \ldots, t_n]))$. For every $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$, let $A_{R'}$ denote the completion of $R' \otimes_R A$ with respect to $f(I)$. Then f exhibits $A_{R'}$ as a completion of $R'[t_1, \ldots, t_n]$ with respect to $(t_1, \ldots, t_n)$, and therefore extends to an equivalence $R'[[t_1, \ldots, t_n]] \to A_{R'}$ of $\mathbb{E}_1$-algebras over R' (Lemma 1.4.7). It follows that $A_{R'}$ also satisfies condition (b) (when regarded as an adic $\mathbb{E}_\infty$-algebra over R').

Let us say that a connective R-algebra R' is *good* if $A_{R'}$ satisfies condition (c): that is, if there exists a smooth coalgebra $C_{R'} \in \mathrm{cCAlg}_{R'}^{\mathrm{sm}}$ and an equivalence of adic $\mathbb{E}_\infty$-algebras $A_{R'} \simeq C_{R'}^\vee$. Note that if $R' \to R''$ is a morphism of connective R-algebras and R' is good, then R'' is also good and we have a canonical equivalence $C_{R''} \simeq R'' \otimes_{R'} C_{R'}$.

We now prove that, for each $n \geqslant 0$, the truncation $\tau_{\leqslant n} R$ is good. The proof proceeds by induction on n, the case $n = 0$ being obvious. We now carry out the inductive step. To simplify the notation, let us replace R by $\tau_{\leqslant n} R$ and thereby reduce to the case where R is n-truncated. Invoking Theorem HA.7.4.1.26 , we obtain a pullback diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_0 \arrow[d] \\ R_1 \arrow[r] & R_{01}, \end{tikzcd}$$

with $R_0 = \pi_0(R)$, $R_1 = \tau_{\leqslant n-1} R$, and $R_{01} = \pi_0(R) \oplus \Sigma^{n+1}(\pi_n R)$. Our inductive hypothesis guarantees that R_0, R_1, and R_{01} are good, and Proposition 1.2.12 guarantees that the diagram

$$\begin{tikzcd} \mathrm{cCAlg}^{\mathrm{sm}}_R \arrow[r] \arrow[d] & \mathrm{cCAlg}^{\mathrm{sm}}_{R_0} \arrow[d] \\ \mathrm{cCAlg}^{\mathrm{sm}}_{R_1} \arrow[r] & \mathrm{cCAlg}^{\mathrm{sm}}_{R_{01}} \end{tikzcd}$$

is a pullback square. It follows that there exists a smooth coalgebra D over R equipped with compatible equivalences

$$C_{R_0} \simeq R_0 \otimes_R D \quad C_{R_{01}} \simeq R_{01} \otimes_R D \quad C_{R_1} \simeq R_1 \otimes_R D.$$

We then compute

$$\begin{aligned} D^\vee &\simeq \underline{\mathrm{Map}}_R(D, R) \\ &\simeq \underline{\mathrm{Map}}_R(D, R_0) \times_{\underline{\mathrm{Map}}_R(D, R_{01})} \underline{\mathrm{Map}}_R(D, R_1) \\ &\simeq C^\vee_{R_0} \times_{C^\vee_{R_{01}}} C^\vee_{R_1} \\ &\simeq A_{R_0} \times_{A_{R_{01}}} A_{R_1} \\ &\simeq (A \otimes_R (R_0 \times_{R_{01}} R_1))^\wedge_I \\ &\simeq (A \otimes_R R)^\wedge_I \\ &\simeq A^\wedge_I . \end{aligned}$$

Since A is I-complete, it follows that $A \simeq A^\wedge_I \simeq D^\vee$ and therefore R is good, as desired. This completes the induction.

We now return to the general case, where R is not assumed to be truncated. The preceding argument shows that each truncation $\tau_{\leqslant n} R$ is good. Set $C_n = C_{\tau_{\leqslant n} R}$, so that $\{C_n\}$ is an object of the inverse limit $\varprojlim \mathrm{cCAlg}^{\mathrm{sm}}_{\tau_{\leqslant n} R}$. Applying Proposition 1.2.11, we deduce that there exists a smooth coalgebra C over R and a compatible family of equivalences $C_n \simeq (\tau_{\leqslant n} R) \otimes_R C$. We now compute

$$\begin{aligned} C^\vee & \simeq \underline{\mathrm{Map}}_R(C, R) \\ & \simeq \varprojlim \underline{\mathrm{Map}}_R(C, \tau_{\leqslant n} R) \\ & \simeq \varprojlim C_n^\vee \\ & \simeq \varprojlim A_{\tau_{\leqslant n} R} \\ & \simeq \varprojlim \tau_{\leqslant n} A \\ & \simeq A, \end{aligned}$$

so that A satisfies condition (c) as desired. $\square$

1.4.3 Duals of Arbitrary Smooth Coalgebras

We now prove an analogue of Proposition 1.4.10 which applies to the class of *all* smooth coalgebras over an $\mathbb{E}_\infty$-ring R.

Proposition 1.4.11. *Let R be an $\mathbb{E}_\infty$-ring and let A be an adic $\mathbb{E}_\infty$-algebra over R. The following conditions are equivalent:*

(1) *There exists a smooth coalgebra C over R and an equivalence of adic $\mathbb{E}_\infty$-algebras $A \simeq C^\vee$.*

(2) *There exists an isomorphism of graded $\pi_*(R)$-algebras*

$$\pi_*(A) \simeq \prod_{n \geqslant 0} (\mathrm{Sym}^n_{\pi_0(R)}(M) \otimes_{\pi_0(R)} \pi_*(R)),$$

where M is a projective module of finite rank over $\pi_0(R)$ which generates an ideal of definition in $\pi_0(R)$.

Proof. The implication (1) $\Rightarrow$ (2) follows immediately from Example 1.3.8. Conversely, suppose that there exists an isomorphism $\pi_*(A) \simeq \prod_{n \geqslant 0} (\mathrm{Sym}^n_{\pi_0(R)}(M) \otimes_{\pi_0(R)} \pi_*(R))$ satisfying the requirements of (2), and let $I \subseteq \pi_0(A)$ be the ideal generated by M (so that I is an ideal of definition for the topology on $\pi_0(A)$). Let us say that an element $x \in \pi_0(R)$ is *good* if the localization $M[x^{-1}]$ is free as a module over the commutative ring $\pi_0(R)[x^{-1}]$. We will prove the following:

$(*)$ Let x be a good element of $\pi_0(R)$. Then there exists an equivalence

$$A[x^{-1}]^\wedge_I \simeq R[x^{-1}][[T_1, \ldots, T_n]]$$

of $\mathbb{E}_1$-algebras over R which carries $I\pi_0(A[x^{-1}]^\wedge_I)$ to the ideal $(T_1, \ldots, T_n)$.

Let us assume $(*)$ for the moment and use it to complete the proof of Proposition 1.4.11. Let $\mathcal{C} \subseteq \mathrm{CAlg}_R$ be the full subcategory spanned by those R-algebras of the form $R[x^{-1}]$, where $x \in \pi_0(R)$ is good. For each object $R' \in \mathcal{C}$, assertion $(*)$ and Proposition 1.4.10 guarantee that $(R' \otimes_R A)^\wedge_I$ has the form $C^\vee_{R'}$, where $C_{R'}$ is a standard smooth coalgebra over R'. Using Theorem 1.3.15 and the fact that smooth coalgebras satisfy descent for the Zariski topology (Proposition 1.2.13), we conclude that there is a smooth coalgebra C over R equipped with equivalences $C_{R'} \simeq R' \otimes_R C$, depending functorially on $R' \in \mathcal{C}$. Dualizing, we obtain natural equivalences $(R' \otimes_R A)^\wedge_I \simeq \underline{\mathrm{Map}}_R(C, R')$. Passing to the inverse limit over $R' \in \mathcal{C}$, we obtain equivalences

$$\begin{aligned} A &\simeq A^\wedge_I \\ &\simeq (\varprojlim_{R' \in \mathcal{C}} (R' \otimes_R A))^\wedge_I \\ &\simeq \varprojlim_{R' \in \mathcal{C}} ((R' \otimes_R A)^\wedge_I) \\ &\simeq \varprojlim_{R' \in \mathcal{C}} \underline{\mathrm{Map}}_R(C, R') \\ &\simeq \underline{\mathrm{Map}}_R(C, \varprojlim_{R' \in \mathcal{C}} R') \\ &\simeq \underline{\mathrm{Map}}_R(C, R) \\ &= C^\vee. \end{aligned}$$

It remains to prove $(*)$. Let x be a good element of $\pi_0(R)$, and choose a collection of elements $t_1, \ldots, t_n \in M$ which form a basis for $M[x^{-1}]$ as a module over $\pi_0(R)[x^{-1}]$. We will abuse notation by identifying each t_i with its image in $\pi_0(A)$. Arguing as in the proof of Proposition 1.4.5, we see that there is a map $u : R[T_1, \ldots, T_n] \to A$ of $\mathbb{E}_1$-algebras over R which carries each T_i to the element t_i. Let $J \subseteq \pi_0(R[T_1, \ldots, T_n])$ be the ideal generated by the elements of T_i, so that I and J generate the same ideal of the localization $A[x^{-1}]$. Then u induces a map of localizations $u_x : R[x^{-1}][T_1, \ldots, T_n] \to A[x^{-1}]$, and Lemma 1.4.7 guarantees that u_x extends canonically to a map of $\mathbb{E}_1$-algebras $u^\wedge_x : R[x^{-1}][[T_1, \ldots, T_n]] \to A[x^{-1}]^\wedge_I$. We will complete the proof by showing that $u^\wedge_x$ is an equivalence: that is, that u_x induces an equivalence after J-completion. In fact, we claim that this happens at the level of each individual homotopy group:

that is, for each integer k, the canonical map

$$\begin{aligned}
\pi_k(R)[x^{-1}][T_1, \ldots, T_n] &\simeq (\bigoplus_{m \geqslant 0} \operatorname{Sym}^m_{\pi_0(R)}(M) \otimes_{\pi_0(R)} \pi_k(R))[x^{-1}] \\
&\xrightarrow{v} (\prod_{m \geqslant 0} \operatorname{Sym}^m_{\pi_0(R)}(M) \otimes_{\pi_0(R)} \pi_k(R))[x^{-1}] \\
&\simeq \pi_k(A[x^{-1}])
\end{aligned}$$

induces an equivalence after J-completion. Note that the domain and codomain of v are modules over the symmetric algebra $\operatorname{Sym}^*_{\pi_0(R)}(M)[x^{-1}]$, and that $J \operatorname{Sym}^*_{\pi_0(R)}(M)[x^{-1}]$ coincides with the ideal $J' = \operatorname{Sym}^{>0}_{\pi_0(R)}(M)[x^{-1}]$. It will therefore suffice to show that v induces an equivalence after J'-completion, which follows from Lemma 1.3.14. $\square$

1.5 Formal Hyperplanes

Let R be a connective $\mathbb{E}_\infty$-ring and let $A \in \operatorname{CAlg}^{\mathrm{cn}}_R$ be a connective $\mathbb{E}_\infty$-algebra over R. We let $\operatorname{Spec}(A) : \operatorname{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ denote the functor corepresented by A, given concretely by the formula

$$\operatorname{Spec}(A)(B) = \operatorname{Map}_{\operatorname{CAlg}_R}(A, B).$$

If A is an adic $\mathbb{E}_\infty$-algebra over R, we let $\operatorname{Spf}(A) \subseteq \operatorname{Spec}(A)$ denote the subfunctor whose value on an object $B \in \operatorname{CAlg}^{\mathrm{cn}}_R$ is given by the summand

$$\operatorname{Spf}(A)(B) = \operatorname{Map}^{\mathrm{cont}}_{\operatorname{CAlg}_R}(A, B) \subseteq \operatorname{Map}_{\operatorname{CAlg}_R}(A, B)$$

spanned by those maps $f : A \to B$ for which the underlying ring homomorphism $\pi_0(A) \to \pi_0(B)$ is continuous (where we equip $\pi_0(B)$ with the discrete topology). We will refer to $\operatorname{Spec}(A)$ as the *spectrum* of A and $\operatorname{Spf}(A)$ as the *formal spectrum* of A.

Our goal in this section is to study functors $X : \operatorname{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ which have the form $\operatorname{Spf}(C^\vee)$, where C is a smooth coalgebra over R. In this case, we will see that the functor X can be defined directly in terms of C (without appealing to duality or to any notion of continuity). More precisely, it can be realized as the *cospectrum of* C (see Construction 1.5.4 and Proposition 1.5.8). We will show that the cospectrum construction determines a fully faithful embedding

$$\operatorname{cSpec} : \operatorname{cCAlg}^{\mathrm{sm}}_R \hookrightarrow \operatorname{Fun}(\operatorname{CAlg}^{\mathrm{cn}}_R, \mathcal{S})$$

(Proposition 1.5.9), and we will say that a functor $X : \operatorname{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ is a *formal hyperplane* if it belongs to the essential image of this embedding.

1.5.1 The Cospectrum of a Coalgebra

We begin with some general remarks.

Definition 1.5.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let C be a commutative coalgebra object of $\mathcal{C}$. A *grouplike element* of C is a morphism of commutative coalgebras $\mathbf{1} \to C$ (here $\mathbf{1}$ denotes the unit object of $\mathcal{C}$, which we regard as a final object of $\mathrm{cCAlg}(\mathcal{C})$). We let $\mathrm{GLike}(C)$ denote the space $\mathrm{Map}_{\mathrm{cCAlg}(\mathcal{C})}(\mathbf{1}, C)$ of grouplike elements of C.

Example 1.5.2. Let R be a commutative ring and let C be a flat commutative coalgebra over R, with comultiplication $\Delta : C \to C \otimes_R C$ and counit $\epsilon : C \to R$. Then we can identify $\mathrm{GLike}(C)$ with the subset of C consisting of those elements x which satisfy the identities

$$\Delta(x) = x \otimes x \qquad \epsilon(x) = 1.$$

In other words, the general notion of grouplike element introduced in Definition 1.5.1 reduces to the notion defined in Remark 1.1.6.

Remark 1.5.3 (Functoriality)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories and let F be a symmetric monoidal functor. Then F determines a functor $\mathrm{cCAlg}(\mathcal{C}) \to \mathrm{cCAlg}(\mathcal{D})$, which carries the unit object $\mathbf{1}_{\mathcal{C}}$ of $\mathcal{C}$ to the unit object $\mathbf{1}_{\mathcal{D}}$ of $\mathcal{D}$. It follows that, for every commutative coalgebra object $C \in \mathrm{cCAlg}(\mathcal{C})$, the functor F induces a canonical map $\mathrm{GLike}(C) \to \mathrm{GLike}(F(C))$.

Construction 1.5.4 (The Cospectrum of a Coalgebra)**.** Let R be a connective $\mathbb{E}_\infty$-ring and let C be a flat commutative coalgebra over R. For every morphism of connective $\mathbb{E}_\infty$-rings $R \to A$, the extension of scalars functor

$$\mathrm{Mod}_R \to \mathrm{Mod}_A \qquad M \mapsto A \otimes_R M$$

is symmetric monoidal, so we can regard $A \otimes_R C$ as a commutative coalgebra over R'. We let $\mathrm{cSpec}(C)(A)$ denote the space $\mathrm{GLike}(A \otimes_R C) = \mathrm{Map}_{\mathrm{cCAlg}_A}(A, A \otimes_R C)$ of grouplike elements of $A \otimes_R C$. The construction $A \mapsto \mathrm{cSpec}(C)(A)$ determines a functor from the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}_R$ of connective $\mathbb{E}_\infty$-algebras over R to the ∞-category $\mathcal{S}$ of spaces (see Remark 1.5.3). We will denote this functor by $\mathrm{cSpec}(C) : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$, and refer to it as the *cospectrum of* C.

Variant 1.5.5 (The Nonconnective Case)**.** Let R be any $\mathbb{E}_\infty$-ring and let C be a flat commutative coalgebra over R. We define the *cospectrum of* C to be the functor

$\mathrm{cSpec}(C) : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R} \to \mathcal{S}$ obtained by applying Construction 1.5.4 to the connective cover $\tau_{\geqslant 0}C$, which we regard as a flat commutative coalgebra over $\tau_{\geqslant 0}R$ (see Remark 1.2.9).

Remark 1.5.6. Let R be an $\mathbb{E}_\infty$-ring. Then the cospectrum functor

$$\mathrm{cSpec} : \mathrm{cCAlg}^{\flat}_R \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R}, \mathcal{S})$$

preserves finite products. In particular, if C and D are flat commutative coalgebras over R, then we have a canonical equivalence $\mathrm{cSpec}(C \otimes_R D) \simeq \mathrm{cSpec}(C) \times \mathrm{cSpec}(D)$.

Remark 1.5.7. The construction $C \mapsto \mathrm{cSpec}(C)$ is compatible with base change. More precisely, suppose that we are given a morphism of $\mathbb{E}_\infty$-rings $R \to R'$ and let C be a flat commutative coalgebra over R, so that $C' = R' \otimes_R C$ inherits the structure of a flat commutative coalgebra over R'. Then the functor $\mathrm{cSpec}(C')$ is equivalent to the composition of $\mathrm{cSpec}(C)$ with the forgetful functor $\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R'} \to \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R}$.

1.5.2 Comparison with $\mathrm{Spec}(C^\vee)$

Let R be a connective $\mathbb{E}_\infty$-ring and let C be a flat commutative coalgebra over R. For every connective $\mathbb{E}_\infty$-algebra A over R, we have a natural map

$$\begin{aligned}
\mathrm{cSpec}(C)(A) &= \mathrm{Map}_{\mathrm{cCAlg}_A}(A, A \otimes_R C) \\
&\to \mathrm{Map}_{\mathrm{CAlg}_A}((A \otimes_R C)^\vee, A^\vee) \\
&= \mathrm{Map}_{\mathrm{CAlg}_A}((A \otimes_R C)^\vee, A') \\
&\to \mathrm{Map}_{\mathrm{CAlg}_A}(A \otimes_R C^\vee, A) \\
&\simeq \mathrm{Map}_{\mathrm{CAlg}_R}(C^\vee, A) \\
&= \mathrm{Spec}(C^\vee)(A).
\end{aligned}$$

It is not hard to see that this map depends functorially on A, and therefore gives rise to a natural transformation of functors $\mathrm{cSpec}(C) \to \mathrm{Spec}(C^\vee)$ (here $\mathrm{Spec}(C^\vee)$ denotes the functor corepresented by $C^\vee$). This natural transformation is an equivalence when C is a finitely generated projective R-module (see Remark 1.3.6), but need not be an equivalence in general. In the case of a *smooth* coalgebra, we have the following:

Proposition 1.5.8. *Let R be a connective $\mathbb{E}_\infty$-ring and let C be a smooth coalgebra over R. Then the natural transformation $\rho : \mathrm{cSpec}(C) \to \mathrm{Spec}(C^\vee)$ induces an equivalence $\mathrm{cSpec}(C) \simeq \mathrm{Spf}(C^\vee) \subseteq \mathrm{Spec}(C^\vee)$.*

Proof. Using Lemma 1.3.13, we can replace C by $C \otimes_R A$ and thereby reduce to the case $R = A$, in which case the desired result is a special case of Theorem 1.3.15. $\square$

Proposition 1.5.9. *Let R be an $\mathbb{E}_\infty$-ring. Then the construction $C \mapsto \mathrm{cSpec}(C)$ induces a fully faithful embedding of ∞-categories*

$$\mathrm{cCAlg}_R^{\mathrm{sm}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R}, \mathcal{S}).$$

Proof. Using Proposition 1.2.8, we can reduce to the case where R is connective. Let C and D be smooth coalgebras over R; we wish to show that the canonical map

$$\rho : \mathrm{Map}_{\mathrm{cCAlg}_R}(D, C) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})}(\mathrm{cSpec}(D), \mathrm{cSpec}(C))$$

is a homotopy equivalence. Using Proposition 1.5.8 and Theorem 1.3.15, we can identify ρ with the map

$$\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, D^\vee) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})}(\mathrm{Spf}(D^\vee), \mathrm{Spf}(C^\vee)),$$

which is a homotopy equivalence by virtue of Theorem SAG.8.1.5.1 (and Corollary SAG.8.1.5.4). $\square$

1.5.3 Formal Hyperplanes

We are now ready to introduce our principal objects of interest.

Definition 1.5.10. Let R be a connective $\mathbb{E}_\infty$-ring. We will say that a functor $X : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ is a *formal hyperplane over R* if it belongs to the essential image of the fully faithful embedding $\mathrm{cSpec} : \mathrm{cCAlg}^{\mathrm{sm}}_R \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})$ of Proposition 1.5.9.

We let $\mathrm{Hyp}(R)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})$ spanned by the formal hyperplanes over R, so that the cospectrum functor determines an equivalence of ∞-categories $\mathrm{cSpec} : \mathrm{cCAlg}^{\mathrm{sm}}_R \simeq \mathrm{Hyp}(R)$. We will refer to $\mathrm{Hyp}(R)$ as *the ∞-category of formal hyperplanes over R.*

Variant 1.5.11 (The Nonconnective Case)**.** Let R be an arbitrary $\mathbb{E}_\infty$-ring. We will say that a functor $X : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R} \to \mathcal{S}$ is a *formal hyperplane over R* if it is a formal hyperplane over $\tau_{\geqslant 0}R$, in the sense of Definition 1.5.10. Equivalently, X is a formal hyperplane over R if and only if it is equivalent to the cospectrum $\mathrm{cSpec}(C)$ for some object $C \in \mathrm{cCAlg}^{\mathrm{sm}}_R$ (see Proposition 1.2.8). We let $\mathrm{Hyp}(R) = \mathrm{Hyp}(\tau_{\geqslant 0}R)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R}, \mathcal{S})$ spanned by the formal hyperplanes over R.

Notation 1.5.12 (The Ring of Functions)**.** Let R be an $\mathbb{E}_\infty$-ring and let $X : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R} \to \mathcal{S}$ be a formal hyperplane over R, so that we can write $X = \mathrm{cSpec}(C)$ for some smooth coalgebra C over R. We let $\mathscr{O}_X$ denote the R-linear dual $C^\vee$, which we regard as an adic $\mathbb{E}_\infty$-algebra over R. We will refer to $\mathscr{O}_X$ as the *$\mathbb{E}_\infty$-ring of functions of X*. Note that the construction $X \mapsto \mathscr{O}_X$ determines a fully faithful embedding $\mathrm{Hyp}(R) \hookrightarrow \mathrm{CAlg}^{\mathrm{ad}}_R$, whose essential image consists of those adic $\mathbb{E}_\infty$-algebras which satisfy the conditions described in Proposition 1.4.11. Moreover, we describe X as the formal spectrum $\mathrm{Spf}(\tau_{\geqslant 0}\,\mathscr{O}_X)$.

Remark 1.5.13. Let R be an $\mathbb{E}_\infty$-ring. Then the collection of formal hyperplanes over R is closed under finite products (when regarded as a full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R}, \mathcal{S})$). This follows immediately from Remarks 1.5.6 and 1.2.6.

Remark 1.5.14 (Functoriality)**.** Let $f : R \to R'$ be a morphism of $\mathbb{E}_\infty$-rings. If X is a formal hyperplane over R, then X determines a formal hyperplane $X_{R'}$ over R', given by composing X with the forgetful functor $\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R'} \to \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R}$. This follows immediately from Remark 1.5.7.

1.5.4 Examples from Algebraic Geometry

The following result supplies a large class of examples of formal hyperplanes:

Proposition 1.5.15. *Let R be a connective $\mathbb{E}_\infty$-ring, let $f : \mathsf{X} \to \mathrm{Spec}(R)$ be a separated fiber-smooth morphism of spectral algebraic spaces, let $s : \mathrm{Spec}(R) \to \mathsf{X}$ be a section of f, and let $\widehat{\mathsf{X}}$ denote the formal completion of X along the image of s. Then (the functor represented by) $\widehat{\mathsf{X}}$ is a formal hyperplane over R.*

We will need the following:

Lemma 1.5.16. *Let R be a connective $\mathbb{E}_\infty$-ring and let $X : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be a functor which is a sheaf for the étale topology. Suppose that there exists an étale covering $\{R \to R_\alpha\}$ such that each restriction $X|_{\mathrm{CAlg}^{\mathrm{cn}}_{R_\alpha}}$ is a formal hyperplane over R_α. Then X is a formal hyperplane over R.*

Proof. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{CAlg}^{\mathrm{cn}}_R$ spanned by those connective R-algebras A for which the structure map $R \to A$ factors through R_α, for some α. It follows from Proposition 1.2.13 the construction $A \mapsto \mathrm{Hyp}(A)$ satisfies descent for the étale topology, so that the restriction map

$$\rho : \mathrm{Hyp}(R) \to \varprojlim_{A \in \mathcal{C}} \mathrm{Hyp}(A)$$

is an equivalence of ∞-categories. Note that the construction $A \mapsto X_A$ determines an object of the codomain of ρ, which we can therefore lift to a formal hyperplane Y over R. Then $X, Y : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ are functors which agree when restricted to $\mathcal{C}$. Since both are sheaves with respect to the étale topology (Proposition 1.5.17), it follows that $X \simeq Y$ is a formal hyperplane over R. $\square$

Proof of Proposition 1.5.15. Choose an étale map $u : \mathrm{Spec}(A) \to \mathsf{X}$ whose image contains the image of s. Since the desired assertion is local with respect to the étale topology on $\mathrm{Spec}(R)$ (Lemma 1.5.16), we may assume without loss of generality that s factors as a composition

$$\mathrm{Spec}(R) \xrightarrow{\widetilde{s}} \mathrm{Spec}(A) \xrightarrow{u} \mathsf{X}.$$

The assumption that u is étale then guarantees that the formal completion of X along the image of s is equivalent to the formal completion of $\mathrm{Spec}(A)$ along the image of $\widetilde{s}$. We may therefore replace X by $\mathrm{Spec}(A)$ and thereby reduce to the case where $\mathsf{X} \simeq \mathrm{Spec}(A)$ is affine. Then A is a fiber-smooth $\mathbb{E}_\infty$-algebra over R, and s is classified by an augmentation $\epsilon : A \to R$. Let $I \subseteq \pi_0(A)$ be the kernel of of the induced map $\pi_0(A) \to \pi_0(R)$. Passing to a further localization of R, we may assume that I is generated by a regular sequence $t_1, \ldots, t_n \in \pi_0(A)$. In this case, the fiber-smoothness of A over R supplies an equivalence $\pi_*(A^\wedge_I) \simeq \pi_*(R)[[t_1, \ldots, t_n]]$, so that $\widehat{X} \simeq \mathrm{Spf}(A^\wedge_I)$ is a formal hyperplane by virtue of Proposition 1.4.10. $\square$

1.5.5 Properties of Formal Hyperplanes

We now study the deformation-theoretic features of formal hyperplanes.

Proposition 1.5.17. *Let R be a connective $\mathbb{E}_\infty$-ring and let $X : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ be a formal hyperplane over R. Then:*

(1) *The functor X is a sheaf for the étale topology.*

(2) *The functor X is nilcomplete: that is, for every connective $\mathbb{E}_\infty$-algebra A over R, the map $X(A) \to \varprojlim X(\tau_{\leqslant n} A)$ is an equivalence.*

(3) *The functor X is cohesive: that is, for every pullback diagram*

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

in $\mathrm{CAlg}^{\mathrm{cn}}_R$ *for which the maps* $\pi_0(A) \to \pi_0(B)$ *and* $\pi_0(B') \to \pi_0(B)$ *are surjective, the induced diagram*

$$\begin{tikzcd} X(A') \arrow[r] \arrow[d] & X(A) \arrow[d, "X(f)"] \\ X(B') \arrow[r, "X(g)"] & X(B) \end{tikzcd}$$

is a pullback square in $\mathcal{S}$.

Proof. All three assertions follow immediately from the description of X as a formal spectrum $\mathrm{Spf}(C^\vee)$, for some smooth coalgebra C over R. □

Remark 1.5.18. Let R be a connective $\mathbb{E}_\infty$-ring and let X be a formal hyperplane over R. Then, for every reduced $\mathbb{E}_\infty$-algebra A over R, the space $X(A)$ is contractible. To prove this, we can replace R by A and thereby reduce to the case where R is a reduced commutative ring, in which case the desired result follows from Example 1.1.17.

Proposition 1.5.19. *Let* R *be a connective* $\mathbb{E}_\infty$*-ring, and suppose we are given a natural transformation* $X \to \mathrm{Spec}(R)$ *in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ *corresponding to a formal hyperplane* $\mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$. *The map* $X \to \mathrm{Spec}(R)$ *admits a relative cotangent complex* $L_{X/\mathrm{Spec}(R)} \in \mathrm{QCoh}(X)$ *(see Definition SAG.17.2.4.2), which is connective and almost perfect.*

Proof. Since X is representable by a formal spectral Deligne-Mumford stack, the existence and connectivity of $L_{X/\mathrm{Spec}(R)}$ follow from Proposition SAG.17.2.5.1 . Fix a connective $\mathbb{E}_\infty$-ring A and a point $\eta \in X(A)$; we wish show that $\eta^* L_{X/\mathrm{Spec}(R)}$ is almost perfect as an A-module. Using Proposition SAG.2.7.3.2 we can reduce to the case where A is discrete. In this case, we can replace R by $\pi_0(R)$ and thereby assume that R is also discrete. Working locally on $|\mathrm{Spec}(R)|$, we can further assume that X is (representable by) the formal spectrum $\mathrm{Spf}(R[[t_1, \ldots, t_n]])$ (Remark 1.4.9) and is therefore obtained as the formal completion of the affine space $\mathrm{Spec}(R[t_1, \ldots, t_n])$ along its zero section. We can therefore identify $\eta^* L_{X/\mathrm{Spec}(R)}$ with the tensor product

$$A \otimes_{R[t_1,\ldots,t_n]} L_{R[t_1,\ldots,t_n]/R} \simeq A \otimes_{\mathbf{Z}[t_1,\ldots,t_n]} L_{\mathbf{Z}[t_1,\ldots,t_n]/\mathbf{Z}}.$$

It now suffices to observe that the cotangent complex $L_{\mathbf{Z}[t_1,\ldots,t_n]/\mathbf{Z}}$ is almost perfect as a module over $\mathbf{Z}[t_1, \ldots, t_n]$, since $\mathbf{Z}[t_1, \ldots, t_n]$ is almost of finite presentation over $\mathbf{Z}$ (Proposition HA.7.2.4.31). □

Corollary 1.5.20. *Let R be a connective $\mathbb{E}_\infty$-ring and let $X : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ be a formal hyperplane over R. Then X is locally almost of finite presentation over R: that is, for each $n \geqslant 0$, the functor X commutes with filtered colimits when restricted to $\tau_{\leqslant n} \mathrm{CAlg}_R^{\mathrm{cn}}$.*

Proof. By virtue of Proposition 1.5.17, Proposition 1.5.19, and Corollary SAG.17.4.2.2 , it will suffice to treat the case $n = 0$. We may therefore replace R by $\pi_0(R)$ and thereby reduce to the case where R is discrete. In this case, we can write $X = \mathrm{cSpec}(\Gamma_R^*(M))$ where M is a projective module of finite rank over R. It follows from Example 1.1.17 that the functor X is given on discrete R-algebras by the formula $X(A) = \mathrm{Hom}_R(M^\vee, \sqrt{A})$, which clearly commutes with filtered colimits (here $\sqrt{A}$ denotes the nilradical of A). □

1.6 Formal Groups

We are now ready to introduce the main objects of study in this paper.

Definition 1.6.1. Let R be a connective $\mathbb{E}_\infty$-ring. A *formal group over R* is a functor $\widehat{\mathbf{G}} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ with the following property: the composition

$$\mathrm{CAlg}_R^{\mathrm{cn}} \xrightarrow{\widehat{\mathbf{G}}} \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}} \xrightarrow{\Omega^\infty} \mathcal{S}$$

is a formal hyperplane over R (in the sense of Definition 1.5.10). We let $\mathrm{FGroup}(R)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$ spanned by the formal groups over R.

Variant 1.6.2 (The Nonconnective Case)**.** Let R be an arbitrary $\mathbb{E}_\infty$-ring. We define a *formal group over R* to be a formal group over the connective cover $\tau_{\geqslant 0}R$: that is, a functor $\widehat{\mathbf{G}} : \mathrm{CAlg}_{\tau_{\geqslant 0}R}^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ for which the composition $\Omega^\infty \circ \widehat{\mathbf{G}}$ is a formal hyperplane over R, in the sense of Variant 1.5.11. We let $\mathrm{FGroup}(R) = \mathrm{FGroup}(\tau_{\geqslant 0}R) \subseteq \mathrm{Fun}(\mathrm{CAlg}_{\tau_{\geqslant 0}R}^{\mathrm{cn}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$ denote the ∞-category of formal groups over R.

Notation 1.6.3 (The Ring of Functions)**.** Let R be an $\mathbb{E}_\infty$-ring and let

$$\widehat{\mathbf{G}} : \mathrm{CAlg}_{\tau_{\geqslant 0}R}^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$$

be a formal group over R. We will refer to the functor $X = \Omega^\infty \circ \widehat{\mathbf{G}}$ as the *underlying formal hyperplane* of $\widehat{\mathbf{G}}$. We let $\mathscr{O}_{\widehat{\mathbf{G}}}$ denote the adic $\mathbb{E}_\infty$-ring $\mathscr{O}_X$ of Notation 1.5.12; we will refer to $\mathscr{O}_{\widehat{\mathbf{G}}}$ as the $\mathbb{E}_\infty$*-ring of functions on* $\widehat{\mathbf{G}}$.

Remark 1.6.4 (Functoriality)**.** Let $f : R \to R'$ be a morphism of $\mathbb{E}_\infty$-rings. It follows from Remark 1.5.14 that if $\widehat{\mathbf{G}} : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ is a formal group over R, then the composite functor

$$\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R'} \to \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R} \xrightarrow{\widehat{\mathbf{G}}} \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$$

is a formal group over R'. We will sometimes denote this formal group by $\widehat{\mathbf{G}}_{R'}$.

1.6.1 Variations

Definition 1.6.1 emphasizes the "functor of points" perspective on formal groups, which will be most convenient to us in what follows. However, it admits various reformulations which are sometimes useful:

Remark 1.6.5 (Formal Groups as Abelian Group Objects)**.** For every ∞-category $\mathcal{C}$ which admits finite products, let $\mathrm{Ab}(\mathcal{C})$ denote the ∞-category of abelian group objects of $\mathcal{C}$ (see Definition AV.1.2.4). Let R be an $\mathbb{E}_\infty$-ring. Since the collection of formal hyperplanes is closed under finite products in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R}, \mathcal{S})$, we have a pullback diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Ab}(\mathrm{Hyp}(R)) \arrow[r] \arrow[d] & \mathrm{Ab}(\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R}, \mathcal{S})) \arrow[d] \\
\mathrm{Hyp}(R) \arrow[r] & \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{R}, \mathcal{S}).
\end{tikzcd}$$

Example AV.1.2.9 supplies an equivalence of ∞-categories $\mathrm{Ab}(\mathcal{S}) \simeq \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$, which induces an equivalence

$$\mathrm{Ab}(\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R}, \mathcal{S})) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R}, \mathrm{Ab}(\mathcal{S})) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R}, \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}).$$

Combining these observations, we obtain an equivalence of ∞-categories $\mathrm{FGroup}(R) \simeq \mathrm{Ab}(\mathrm{Hyp}(R))$.

Remark 1.6.6 (Formal Groups as Hopf Algebras)**.** Let R be an $\mathbb{E}_\infty$-ring and let $\mathrm{cCAlg}^{\mathrm{sm}}_R$ be the ∞-category of smooth coalgebras over R (Definition 1.1.14). Using the equivalence of ∞-categories $\mathrm{cSpec} : \mathrm{cCAlg}^{\mathrm{sm}}_R \simeq \mathrm{Hyp}(R)$ supplied by Proposition 1.5.9, we obtain an equivalence of ∞-categories $\mathrm{Ab}(\mathrm{cCAlg}^{\mathrm{sm}}_R) \simeq \mathrm{FGroup}(R)$. Roughly speaking, the domain of this equivalence can be viewed as an ∞-category of Hopf algebras over R, which are required to be commutative and cocommutative in a strong sense (because they are abelian group objects of $\mathrm{cCAlg}^{\mathrm{sm}}_R$, rather than merely commutative monoid objects of $\mathrm{cCAlg}^{\mathrm{sm}}_R$), and smooth when viewed as coalgebras over R.

Remark 1.6.7 (Formal Groups as Formal Schemes)**.** Let R be a connective $\mathbb{E}_\infty$-ring. For every smooth coalgebra C over R, let us regard $C^\vee$ as an adic $\mathbb{E}_\infty$-algebra over R, and let $\mathrm{Spf}(C^\vee)$ denote its formal spectrum in the sense of Construction SAG.8.1.1.10 (so that $\mathrm{Spf}(C^\vee)$ is a spectrally ringed ∞-topos). The construction $C \mapsto \mathrm{Spf}(C^\vee)$ determines a fully faithful embedding from $\mathrm{cCAlg}_R^{\mathrm{sm}}$ to the ∞-category $(\mathrm{fSpDM})_{/\mathrm{Spec}(R)}$ of formal spectral Deligne-Mumford stacks over R. Passing to abelian group objects, we obtain a fully faithful embedding $\mathrm{FGroup}(R) \hookrightarrow \mathrm{Ab}((\mathrm{fSpDM})_{/\mathrm{Spec}(R)})$.

1.6.2 Properties of Formal Groups

From Proposition 1.5.17 and Corollary 1.5.20, we immediately deduce the following:

Proposition 1.6.8. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ be a formal group over R. Then:*

(1) *The functor $\widehat{\mathbf{G}}$ is a sheaf for the étale topology.*

(2) *The functor $\widehat{\mathbf{G}}$ is nilcomplete: that is, for every connective $\mathbb{E}_\infty$-algebra A over R, the map $\widehat{\mathbf{G}}(A) \to \varprojlim \widehat{\mathbf{G}}(\tau_{\leqslant n} A)$ is an equivalence in $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$.*

(3) *The functor $\widehat{\mathbf{G}}$ is cohesive: that is, for every pullback diagram*

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

in $\mathrm{CAlg}_R^{\mathrm{cn}}$ for which the maps $\pi_0(A) \to \pi_0(B)$ and $\pi_0(B') \to \pi_0(B)$ are surjective, the induced diagram

$$\begin{tikzcd} \widehat{\mathbf{G}}(A') \arrow[r] \arrow[d] & \widehat{\mathbf{G}}(A) \arrow[d, "\widehat{\mathbf{G}}(f)"] \\ \widehat{\mathbf{G}}(B') \arrow[r, "\widehat{\mathbf{G}}(g)"] & \widehat{\mathbf{G}}(B) \end{tikzcd}$$

is a pullback square in $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$.

(4) *The functor $\widehat{\mathbf{G}}$ is locally almost of finite presentation over R: that is, it commutes with filtered colimits when restricted to $\tau_{\leqslant n} \mathrm{CAlg}_R^{\mathrm{cn}}$ for every nonnegative integer n.*

Warning 1.6.9. In the statement of Proposition 1.6.8, it is essential that all of the relevant limits are formed in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ of *connective* $\mathbf{Z}$-module spectra, rather than in the larger ∞-category $\mathrm{Mod}_{\mathbf{Z}}$ of *all* $\mathbf{Z}$-module spectra.

1.6.3 Example: The Formal Multiplicative Group

To every commutative ring R, we can associate the abelian group $\mathrm{GL}_1(R) = \{x \in R : x \text{ is invertible }\}$. The construction $R \mapsto \mathrm{GL}_1(R)$ determines a functor from the category of commutative rings to the category of abelian groups. This functor is representable by a commutative group scheme over $\mathbf{Z}$, which is typically denoted by $\mathbf{G}_m$; it can be described concretely as the spectrum of the Laurent polynomial ring $\mathbf{Z}[t^{\pm 1}]$.

The construction $R \mapsto \mathrm{GL}_1(R)$ generalizes in a natural way to the setting of $\mathbb{E}_\infty$-rings. Note that the construction $X \mapsto \Sigma^\infty_+ X$ determines a symmetric monoidal functor from the ∞-category $\mathcal{S}$ of spaces (with symmetric monoidal structure given by Cartesian product) to the ∞-category Sp (endowed with the smash product symmetric monoidal structure). It follows that the right adjoint functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ is lax symmetric monoidal, and therefore carries commutative algebra objects to commutative algebra objects. In other words, for any $\mathbb{E}_\infty$-ring R, we can regard the 0th space $\Omega^\infty R$ as an $\mathbb{E}_\infty$-space (with the $\mathbb{E}_\infty$-structure induced by the *multiplication* on R). We let $\mathrm{GL}_1(R)$ denote the summand of $\Omega^\infty R$ consisting of those connected components which are invertible when regarded as elements of the commutative ring $\pi_0(R)$. The construction $R \mapsto \mathrm{GL}_1(R)$ can be regarded as a sort of algebraic group in the setting of spectral algebraic geometry. Note that, if we neglect the $\mathbb{E}_\infty$-structure on $\mathrm{GL}_1(R)$, then it is a corepresentable functor of R: there is a canonical homotopy equivalence $\mathrm{GL}_1(R) \simeq \mathrm{Map}_{\mathrm{CAlg}}(S\{t^{\pm 1}\}, R)$, where $S\{t\}$ denotes the free $\mathbb{E}_\infty$-ring on a single generator t, and $S\{t^{\pm 1}\} = S\{t\}[t^{-1}]$ denotes the $\mathbb{E}_\infty$-ring obtained by inverting that generator.

For our purposes, the functor GL_1 suffers from two closely related defects:

- Through we can think of GL_1 as a sort of commutative group scheme in spectral algebraic geometry, it is not commutative enough: in general, $\mathrm{GL}_1(R)$ is the 0th space of a spectrum, but not of a $\mathbf{Z}$-module spectrum.

- As a spectral scheme, GL_1 is not flat over the sphere spectrum S (that is, $S\{t^{\pm 1}\}$ is not flat as an S-module).

However, these deficiencies have a common remedy.

Construction 1.6.10 (The Strict Multiplicative Group)**.** Let Lat denote the category of lattices (that is, the category of free abelian groups of finite rank). If R is an $\mathbb{E}_\infty$-ring, then the construction $(M \in \mathrm{Lat}) \mapsto \mathrm{Map}_{\mathrm{CAlg}}(\Sigma^\infty_+ M, R)$ determines a functor

$F_R : \mathrm{Lat}^{\mathrm{op}} \to \mathcal{S}$ which commutes with finite products. It follows from Remark AV.1.2.10 that there is an essentially unique connective $\mathbf{Z}$-module spectrum $\mathbf{G}_m(R)$ equipped with homotopy equivalences

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}_m(R)) \simeq \mathrm{Map}_{\mathrm{CAlg}}(\Sigma^{\infty}_{+} M, R)$$

depending functorially on M. The construction $R \mapsto \mathbf{G}_m(R)$ determines a functor

$$\mathbf{G}_m : \mathrm{CAlg} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}},$$

which we will refer to as the *strict multiplicative group*.

Remark 1.6.11. In the setting of Construction 1.6.10, we do not need to restrict our attention to lattices. For any connective $\mathbf{Z}$-module spectrum M, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}_m(R)) \simeq \mathrm{Map}_{\mathrm{CAlg}}(\Sigma^{\infty}_{+} \Omega^{\infty} M, R).$$

Remark 1.6.12 (Relationship with GL_1)**.** Let R be an $\mathbb{E}_\infty$-ring. Then there is a canonical map of $\mathbb{E}_\infty$-spaces

$$\alpha : \Omega^{\infty} \mathbf{G}_m(R) \to \mathrm{GL}_1(R).$$

Moreover, $\mathbf{G}_m(R)$ is *universal* among connective $\mathbf{Z}$-module spectra equipped with such a map. More precisely, for any connective $\mathbf{Z}$-module spectrum M, composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}_m(R)) \simeq \mathrm{Map}_{\mathrm{CMon}(\mathcal{S})}(\Omega^{\infty} M, \mathrm{GL}_1(R))$$

(this is an immediate consequence of Remark 1.6.11 and the definition of $\mathrm{GL}_1(R)$).

Put another way, the $\mathbf{Z}$-module spectrum $\mathbf{G}_m(R)$ is given by the formula

$$\mathbf{G}_m(R) = \tau_{\geqslant 0} \underline{\mathrm{Map}}_S(\mathbf{Z}, \mathrm{gl}_1(R)),$$

where $\mathrm{gl}_1(R)$ denotes the connective spectrum corresponding to the grouplike $\mathbb{E}_\infty$-space $\mathrm{GL}_1(R)$.

Remark 1.6.13 (Flatness)**.** By virtue of Remark 1.6.11, the functor $R \mapsto \Omega^{\infty} \mathbf{G}_m(R)$ is corepresented by the $\mathbb{E}_\infty$-ring $\Sigma^{\infty}_{+}(\mathbf{Z})$. This spectrum is flat over the sphere (it can be obtained from the $\mathbb{E}_\infty$-ring $S[t]$ of §1.4.1 by inverting t).

Remark 1.6.14. If R is an ordinary commutative ring, then there is essentially no difference between $\mathbf{G}_m(R)$ and $\mathrm{GL}_1(R)$: they can both be identified with the abelian group of invertible elements of R. However, they are often quite different when R is not discrete. Note that the homotopy groups of $\mathrm{GL}_1(R)$ are given by the formula

$$\pi_* \mathrm{GL}_1(R) = \begin{cases} \pi_0(R)^\times & \text{if } * = 0 \\ \pi_*(R) & \text{otherwise.} \end{cases}$$

However, the homotopy groups of $\mathbf{G}_m(R)$ are more unpredictable. For example, if $R = \mathrm{KU}^\wedge_{(p)}$ is the (p)-completed complex K-theory spectrum, then one can show that $\pi_0(\mathbf{G}_m(R))$ is isomorphic to $\mathbf{F}_p^\times$ (a cyclic group of order $(p-1)$). The canonical map

$$\pi_0(\mathbf{G}_m(R)) \to \pi_0(\mathrm{GL}_1(R)) \simeq \mathbf{Z}_p^\times$$

assigns to each invertible element $x \in \mathbf{F}_p$ its Teichmüller representative $[x] \in \mathbf{Z}_p \simeq W(\mathbf{F}_p)$.

Remark 1.6.15. Let R be an $\mathbb{E}_\infty$-ring. Then the canonical map $\tau_{\geqslant 0} R \to R$ induces equivalences

$$\mathrm{GL}_1(\tau_{\geqslant 0} R) \to \mathrm{GL}_1(R) \qquad \mathbf{G}_m(\tau_{\geqslant 0} R) \to \mathbf{G}_m(R).$$

Consequently, no information is lost by restricting the functors GL_1 and $\mathbf{G}_m$ to the full subcategory $\mathrm{CAlg}^{\mathrm{cn}} \subseteq \mathrm{CAlg}$ spanned by the connective $\mathbb{E}_\infty$-rings.

Construction 1.6.16 (The Formal Multiplicative Group)**.** Let R be a connective $\mathbb{E}_\infty$-ring. We let $\widehat{\mathbf{G}}_m(R)$ denote the fiber of the canonical map $\mathbf{G}_m(R) \to \mathbf{G}_m(R^{\mathrm{red}})$ (formed in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ of connective $\mathbf{Z}$-module spectra). The construction $R \mapsto \widehat{\mathbf{G}}_m(R)$ determines a functor

$$\widehat{\mathbf{G}}_m : \mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}},$$

which we will refer to as the *formal multiplicative group*.

Proposition 1.6.17. *The formal multiplicative group $\widehat{\mathbf{G}}_m$ is a formal group over the sphere spectrum S (in the sense of Definition 1.6.1).*

Proof. Unwinding the definitions, we see that the functor $\Omega^\infty \widehat{\mathbf{G}}_m$ is representable by the formal spectrum $\mathrm{Spf}(A)$, where A is the completion of the $\mathbb{E}_\infty$-ring $\Sigma^\infty_+(\mathbf{Z}) \simeq S[t^{\pm 1}]$ with respect to the ideal generated by $u = (t-1)$. The homotopy groups of A are given by $(\pi_* S)[[u]]$, so that $\mathrm{Spf}(A)$ is a formal hyperplane over S by virtue of Proposition 1.4.10. $\square$

Warning 1.6.18. The $\mathbb{E}_\infty$-ring A appearing in the proof of Proposition 1.6.17 is *not* equivalent to the power series algebra $S[[u]]$ of Construction 1.4.4, despite the fact that they have isomorphic homotopy rings.

Remark 1.6.19. Since the formal multiplicative group $\widehat{\mathbf{G}}_m$ is defined over the sphere spectrum S, it determines a formal group over any $\mathbb{E}_\infty$-ring R, given by the composition

$$\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R} \to \mathrm{CAlg}^{\mathrm{cn}} \xrightarrow{\widehat{\mathbf{G}}_m} \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}.$$

We will refer to this functor as *the formal multiplicative group over R*. We will generally abuse notation by writing it also as $\widehat{\mathbf{G}}_m$.

1.6.4 Non-Example: The Formal Additive Group

Let $\mathbf{G}_a$ denote the affine line $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{Z}[t])$ over $\mathrm{Spec}(\mathbf{Z})$. We regard $\mathbf{G}_a$ as a commutative group scheme over $\mathbf{Z}$, with additional law classified by the comultiplication

$$\Delta : \mathbf{Z}[t] \to \mathbf{Z}[t] \otimes_{\mathbf{Z}} \mathbf{Z}[t] \qquad t \mapsto t \otimes 1 + 1 \otimes t.$$

Then the formal completion $\widehat{\mathbf{G}}_a \simeq \mathrm{Spf}(\mathbf{Z}[[t]])$ is a formal group over $\mathbf{Z}$, which we refer to as the *formal additive group*. On ordinary commutative rings, these functors are given by

$$\mathbf{G}_a(A) = A \qquad \widehat{\mathbf{G}}_a(A) = \{x \in A : x \text{ is nilpotent}\}.$$

For any connective $\mathbb{E}_\infty$-ring R, we can contemplate analogous constructions

$$\mathbf{G} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathrm{Sp}^{\mathrm{cn}}$$

which associates to each connective $\mathbb{E}_\infty$-algebra A its underlying spectrum, and we can define its formal completion $\widehat{X} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathrm{Sp}^{\mathrm{cn}}$ by the formula $\widehat{\mathbf{G}}(A) = A \times_{\pi_0(A)} \{x \in \pi_0(A) : x \text{ is nilpotent}\}$. Beware that $\widehat{\mathbf{G}}$ is generally *not* a formal group in the sense of Definition 1.6.1, for two reasons:

- The functor $\widehat{\mathbf{G}}$ takes values in the ∞-category of connective spectra, rather than the ∞-category $\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ of connective $\mathbf{Z}$-module spectra.
- The functor $X = \Omega^\infty \circ \widehat{\mathbf{G}}$ is not a formal hyperplane over R unless R admits the structure of an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$. Note that X can be identified with the formal spectrum $\mathrm{Spf}(A)$, where A denotes the (t)-completion of the free $\mathbb{E}_\infty$-algebra $R\{t\}$. In general, $R\{t\}$ is not flat over R.

We now show that these difficulties are actually essential: it is not possible to lift the formal additive group $\widehat{\mathbf{G}}_a$ to a formal group over the sphere spectrum S. In fact, we can say more:

Proposition 1.6.20. *The formal additive group $\widehat{\mathbf{G}}_a$ cannot be lifted to $\tau_{\leqslant 1}(S)$: in other words, it does not belong to the essential image of the forgetful functor* $\mathrm{FGroup}(\tau_{\leqslant 1}S) \to \mathrm{FGroup}(\pi_0(S)) = \mathrm{FGroup}(\mathbf{Z})$.

Remark 1.6.21. Our proof of Proposition 1.6.20 will actually establish something stronger: it is not possible to lift $\widehat{\mathbf{G}}_a$ to a formal hyperplane over $\tau_{\leqslant 1}(S)$ which is equipped with a unital multiplication. We do not need to assume that this multiplication is commutative or associative, even up to homotopy.

Let us begin by reviewing some elementary facts about power operations which will be useful for proving Proposition 1.6.20.

Construction 1.6.22. Let X be an $\mathbb{E}_\infty$-space. Then every point $x \in X$ extends (in an essentially unique way) to a map of $\mathbb{E}_\infty$-spaces $u : \mathrm{Sym}(*) \to X$, where $\mathrm{Sym}(*) \simeq \amalg_{n \geqslant 0} \mathrm{B}\Sigma_n$ denotes the free $\mathbb{E}_\infty$-space generated by a point (here Σ_n denotes the symmetric group on n letters). In particular, we obtain a map $u|_{\mathrm{B}\Sigma_2} : \mathrm{B}\Sigma_2 \to X$. This map carries the base point b of $\mathrm{B}\Sigma_2$ to a point of X which we will denote by x^2, and induces a map of fundamental groups $\Sigma_2 = \pi_1(\mathrm{B}\Sigma_2, b) \to \pi_1(X, x^2)$. We will let $\mu(x) \in \pi_1(X, x^2)$ denote the image of the nontrivial element of Σ_2 under this map.

Example 1.6.23. Let E be a spectrum. Then $\Omega^\infty(E)$ can be regarded as a grouplike $\mathbb{E}_\infty$-space. Consequently, any point $x \in \Omega^\infty(E)$ determines an element $\mu(x)$ of the abelian group $\pi_1(\Omega^\infty(E), x^2) \simeq \pi_1 E$. Then $\mu(x)$ depends only on the connected component $[x] \in \pi_0(E)$ of x. Moreover, the construction $[x] \mapsto \mu(x)$ coincides with the homomorphism $\pi_0(E) \to \pi_1 E$ given by multiplication by the nontrivial element $\eta \in \pi_1(S)$.

Example 1.6.24. Let A be an $\mathbb{E}_\infty$-ring. Since the 0th space functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ is lax symmetric monoidal, the $\mathbb{E}_\infty$-structure on A determines an $\mathbb{E}_\infty$-structure on the 0th space $\Omega^\infty(A)$, which we will refer to as the *multiplicative $\mathbb{E}_\infty$-structure*. Note that this structure is difference from the additive $\mathbb{E}_\infty$-structure of Example 1.6.23 (the induced monoid structure on $\pi_0(A)$ is given by multiplication, rather than addition; in particular, the space $\Omega^\infty A$ is not grouplike with respect to the multiplicative $\mathbb{E}_\infty$-structure). Applying Construction 1.6.22 to the multiplicative $\mathbb{E}_\infty$-structure, we see that every point $x \in \Omega^\infty(A)$ determines an element $\mu(x) \in \pi_1(\Omega^\infty A, x^2) \simeq \pi_1 A$. The element $\mu(x)$ depends only on the connected component $[x] \in \pi_0(A)$ of x. We will emphasize this dependence by writing $\mu(x)$ as $\eta_m([x])$, so that we obtain a function $\eta_m : \pi_0(A) \to \pi_1(A)$.

Example 1.6.25. Let $\mathcal{F}\mathrm{in}$ denote the category of finite sets and let $\mathcal{F}\mathrm{in}^{\simeq}$ denote its underlying groupoid. We will regard $\mathcal{F}\mathrm{in}$ as equipped with the symmetric monoidal structure given by the Cartesian product. The groupoid $\mathcal{F}\mathrm{in}^{\simeq}$ inherits the structure of a symmetric monoidal ∞-category, so that the nerve $\mathrm{N}(\mathcal{F}\mathrm{in}^{\simeq})$ can be regarded as an $\mathbb{E}_\infty$-space. Let J be a point of $\mathrm{N}(\mathcal{F}\mathrm{in}^{\simeq})$, viewed as a finite set. Applying Construction 1.6.22, we obtain an element $\mu(J) \in \pi_1(\mathrm{N}(\mathcal{F}\mathrm{in}^{\simeq}), J \times J)$, which corresponds to the permutation of $J \times J$ given by $(i, j) \mapsto (j, i)$. A simple calculation shows that this permutation is even when the cardinality of J is congruent to 0 or 1 modulo 4, and is odd when the cardinality of J is congruent to 2 or 3 modulo 4.

Example 1.6.26. Let S denote the sphere spectrum. Then there is a canonical map of $\mathbb{E}_\infty$-spaces $\mathrm{N}(\mathcal{F}\mathrm{in}^{\simeq}) \to \Omega^\infty S$, where we regard $\mathrm{N}(\mathcal{F}\mathrm{in}^{\simeq})$ as equipped with the $\mathbb{E}_\infty$-structure of Example 1.6.25 and $\Omega^\infty S$ as equipped with the multiplicative $\mathbb{E}_\infty$-structure of Example 1.6.24. This map carries each finite set J into the connected component of $\Omega^\infty S$ classified by the cardinality $|J| \in \mathbf{Z} \simeq \pi_0(S)$, and carries each permutation $\sigma : J \to J$ to the element $\begin{cases} 0 \text{ if } \sigma \text{ is even } \eta \text{ if } \sigma \text{ is odd} \end{cases} \in \pi_1(S)$. Using the functoriality of Construction 1.6.22, we deduce that $\eta_m(|J|) = \begin{cases} 0 & \text{if } |J| \equiv 0, 1 \pmod 4 \\ \eta & \text{if } |J| \equiv 2, 3 \pmod 4. \end{cases}$

Lemma 1.6.27. *Let A be an $\mathbb{E}_\infty$-ring. Then, for every pair of elements $x, y \in \pi_0(A)$, we have an equality $\eta_m(x + y) = \eta_m(x) + \eta_m(y) + \eta xy$ in $\pi_1(A)$.*

Proof. Without loss of generality, we may assume that $A = S\{x, y\}$ is the free $\mathbb{E}_\infty$-ring generated by x and y. Then A is given as a spectrum by the

$$\begin{aligned} S\{x\} \otimes S\{y\} &\simeq (\bigoplus_{a \geqslant 0} \Sigma^\infty_+ \mathrm{B}\Sigma_a) \otimes (\bigoplus_{b \geqslant 0} \Sigma^\infty_+ \mathrm{B}\Sigma_b) \\ &\simeq \bigoplus_{a,b \geqslant 0} \Sigma^\infty_+ (\mathrm{B}\Sigma_a \times \mathrm{B}\Sigma_b). \end{aligned}$$

With respect to this decomposition, we can write $\eta_m(x + y) = \sum_{a,b \geqslant 0} c_{a,b}$ for some elements $c_{a,b} \in \pi_1 \Sigma^\infty_+ (\mathrm{B}\Sigma_a \times \mathrm{B}\Sigma_b)$. It follows immediately from the construction that $c_{a,b}$ vanishes for $a + b \neq 2$. Moreover, by setting x or y equal to zero, we deduce that $c_{2,0} = \eta_m(x)$ and $c_{0,2} = \eta_m(y)$. We therefore have $\eta_m(x+y) = \eta_m(x)+\eta_m(y)+c_{1,1}xy$ for some element $c_{1,1} \in \pi_1 S$. Since this equality holds in the free $\mathbb{E}_\infty$-ring on two generators, it must hold in *any* $\mathbb{E}_\infty$-ring A containing elements $x, y \in \pi_0(A)$. In particular, taking A to be the sphere spectrum and $x = y = 1$, we obtain $\eta_m(2) = \eta_m(1) + \eta_m(1) + c_{1,1}$ in $\pi_1 S$. Example 1.6.26 shows that $\eta_m(2) = \eta$ and $\eta_m(1) = 0$, so that $c_{1,1} = \eta$ as desired. $\square$

Proof of Proposition 1.6.20. Let $\widehat{\mathbf{G}}$ be a formal group over $\tau_{\leqslant 1} S$ and let $X = \Omega^\infty \widehat{\mathbf{G}}$ be the underlying formal hyperplane of $\widehat{\mathbf{G}}$. Write $X = \mathrm{Spf}(A)$ and $X \times X = \mathrm{Spf}(B)$, where A and B are adic $\mathbb{E}_\infty$-algebras over $\tau_{\leqslant 1} S$, so that the multiplication on X induces a morphism $\rho : A \to B$. Suppose that, after extension of scalars to $\pi_0(S) \simeq \mathbf{Z}$, the formal group $\widehat{\mathbf{G}}$ is equivalent to the formal additive group $\widehat{\mathbf{G}}_a$. A choice of equivalence then yields isomorphisms $\alpha : \mathbf{Z}[[t]] \to \pi_0(A)$ and $\beta : \mathbf{Z}[[t_0, t_1]] \to \pi_0(B)$ for which the diagram

$$\begin{tikzcd}
\mathbf{Z}[[t]] \arrow[r, "\alpha"] \arrow[d, "t \mapsto t_0 + t_1"] & \pi_0(A) \arrow[d, "\rho"] \\
\mathbf{Z}[[t_0, t_1]] \arrow[r, "\beta"] & \pi_0(B)
\end{tikzcd}$$

commutes. Note that multiplication by $\eta \in \pi_1(S)$ induces isomorphisms

$$\pi_0(A)/2\pi_0(A) \to \pi_1(A) \qquad \pi_0(B)/2\pi_0(B) \simeq \pi_1(B).$$

In particular, there is a unique power series $f(t) \in \mathbf{Z}[[t]]$, uniquely determined modulo 2, which satisfies the equation $\eta\alpha(f(t)) = \eta_m(\alpha(t))$. Applying the map ρ and invoking Lemma 1.6.27, we obtain

$$\begin{aligned}
\eta\beta(f(t_0 + t_1)) &= \eta_m(\beta(t_0 + t_1)) \\
&= \eta_m(\beta(t_0)) + \eta_m(\beta(t_1)) + \eta\beta(t_0)\beta(t_1) \\
&= \eta\beta(f(t_0)) + \eta\beta(f(t_1)) + \eta t_0 t_1.
\end{aligned}$$

The power series f must then satisfy the identity $f(t_0 + t_1) \equiv f(t_0) + f(t_1) + t_0 t_1 \pmod 2$. This is a contradiction, since the coefficient of $t_0 t_1$ in the power series $f(t_0 + t_1)$ is necessarily even. $\square$

2 Identity Components of p-Divisible Groups

For every commutative ring R, let $\mathbf{G}_m(R)$ denote the set of invertible elements of R, which we regard as a group under multiplication. We will view the construction $R \mapsto \mathbf{G}_m(R)$ as a functor from the category of commutative rings to the category of abelian groups. We denote this functor by $\mathbf{G}_m$ and refer to it as the *multiplicative group*; it is representable by the affine group scheme $\operatorname{Spec} \mathbf{Z}[t^{\pm 1}]$ (which we also denote by $\mathbf{G}_m$). Consider the following subfunctors of $\mathbf{G}_m$:

- The *formal multiplicative group* $\widehat{\mathbf{G}}_m \subseteq \mathbf{G}_m$ assigns to every commutative ring R the subset

$$\widehat{\mathbf{G}}_m(R) = \{x \in R : (x-1) \text{ is nilpotent}\} \subseteq \mathbf{G}_m(R).$$

  This functor is representable by the formal scheme

$$\operatorname{Spf} \mathbf{Z}[[(t-1)]] = \varinjlim \operatorname{Spec} \mathbf{Z}[t]/(t-1)^m.$$

- For each prime number p, the *p-divisible multiplicative group* $\mu_{p^\infty} \subseteq \mathbf{G}_m$ assigns to every commutative ring R the subset

$$\mu_{p^\infty}(R) = \{x \in R : x^{p^m} = 1 \text{ for } m \gg 0\} \subseteq \mathbf{G}_m(R).$$

  This functor is representable by the Ind-scheme $\varinjlim \operatorname{Spec} \mathbf{Z}[t]/(t^{p^m} - 1)$.

The functors $\widehat{\mathbf{G}}_m$ and μ_{p^∞} do not coincide (as subfunctors of $\mathbf{G}_m$): in fact, neither contains the other. However, they are closely related by the following elementary observation:

$(*)$ Let R be a commutative ring in which p is nilpotent. Then $\widehat{\mathbf{G}}_m(R) = \mu_{p^\infty}(R)$.

Roughly speaking, $(*)$ asserts that the formal group $\widehat{\mathbf{G}}_m$ and the p-divisible group μ_{p^∞} become interchangeable after p-adic completion: after extending scalars to a ring in which p is nilpotent, either can be recovered from the other.

More generally, if R is any commutative ring in which p is nilpotent, the theory of p-divisible groups over R is closely related to the theory of formal groups over R, by virtue of the following result of [27]:

Theorem 2.0.1 (Messing)**.** *Let R be a commutative ring in which p is nilpotent and let $\mathbf{G}$ be a p-divisible group over R, and define $\mathbf{G}^\circ : \mathrm{CAlg}_R^\heartsuit \to \mathrm{Mod}_{\mathbf{Z}}^\heartsuit$ by the formula*

$$\mathbf{G}^\circ(A) = \ker(\mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}})).$$

Then $\mathbf{G}^\circ$ is (representable by) a formal group over R.

In [26], we introduced the notion of a p-divisible group over an arbitrary $\mathbb{E}_\infty$-ring R. Let us recall the definition in a form which will be convenient for our purposes (see Proposition AV.6.5.8 for a slight variant):

Definition 2.0.2. Let R be a connective $\mathbb{E}_\infty$-ring. A *p-divisible group over R* is a functor $\mathbf{G} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ with the following properties:

(1) For every object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$, the $\mathbf{Z}$-module spectrum $\mathbf{G}(A)$ is p-nilpotent: that is, we have $\mathbf{G}(A)[1/p] \simeq 0$.

(2) For every finite abelian p-group M, the functor

$$(A \in \mathrm{CAlg}_R^{\mathrm{cn}}) \mapsto (\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}(A)) \in \mathcal{S})$$

is corepresentable by a finite flat R-algebra.

(3) The map $p : \mathbf{G} \to \mathbf{G}$ is locally surjective with respect to the finite flat topology. In other words, for every object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$ and every element $x \in \pi_0(\mathbf{G}(A))$, there exists a finite flat map $A \to B$ for which $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective and the image of x in $\pi_0(\mathbf{G}(B))$ is divisible by p.

Remark 2.0.3. In the situation of Definition 2.0.2, if condition (3) is satisfied, then it suffices to check condition (2) in the special case $M = \mathbf{Z}/p\,\mathbf{Z}$.

Remark 2.0.4. In the situation of Definition 2.0.2, if M is a finite abelian p-group, we let $\mathbf{G}[M]$ denote the functor given by $A \mapsto \underline{\mathrm{Map}}_{\mathbf{Z}}(M, \mathbf{G}(A))$, so that $\mathbf{G}[M]$ is a finite flat group scheme over R. In the special case $M = \mathbf{Z}/p^k\,\mathbf{Z}$, we will denote $\mathbf{G}[M]$ by $\mathbf{G}[p^k]$.

Warning 2.0.5. In the situation of Definition 2.0.2, each $\mathbf{G}[p^k]$ is a sheaf for the flat topology, and we have $\mathbf{G} \simeq \varinjlim_k \mathbf{G}[p^k]$. However, it is not clear that $\mathbf{G}$ is also a sheaf for the flat topology (or even the finite flat topology), at least when regarded as a $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$-valued functor.

We will extend Definition 2.0.2 to the nonconnective case in a purely formal way (see Remark AV.6.5.3):

Variant 2.0.6 (The Nonconnective Case)**.** Let R be an arbitrary $\mathbb{E}_\infty$-ring. We define a p-divisible group over R to be a p-divisible group over the connective cover $\tau_{\geqslant 0}R$: that is, a functor $\mathbf{G} : \mathrm{CAlg}_{\tau_{\geqslant 0}R}^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ satisfying conditions (1), (2), and (3) of Definition 2.0.2. We let $\mathrm{BT}^p(R)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_{\tau_{\geqslant 0}R}^{\mathrm{cn}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$ spanned by the p-divisible groups over R.

Remark 2.0.7 (Functoriality)**.** Let A and B be $\mathbb{E}_\infty$-rings and suppose we are given a morphism of $\mathbb{E}_\infty$-rings $f : \tau_{\geqslant 0}(A) \to \tau_{\geqslant 0}(B)$. If $\mathbf{G}$ is a p-divisible group over A, we let $\mathbf{G}_B$ denote the p-divisible group over B given by the composite functor

$$\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}B} \to \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}(A)} \xrightarrow{\mathbf{G}} \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}.$$

In this case, we will say that $\mathbf{G}_B$ is obtained from $\mathbf{G}$ by *extending scalars along* f. Note that this construction makes sense even when f does not arise from a map $A \to B$; for example, we can always extend scalars from A to $\pi_0(A)$.

We can now formulate the main result of this section:

Theorem 2.0.8. *Let R be a (p)-complete $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. Then there exists an essentially unique formal group $\mathbf{G}^\circ \in \mathrm{FGroup}(R)$ with the following property:*

$(*)$ *Let $\mathcal{E} \subseteq \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}(R)}$ denote the full subcategory spanned by those connective $\tau_{\geqslant 0}(R)$-algebras which are truncated and (p)-nilpotent. Then the functor $\mathbf{G}^\circ|_{\mathcal{E}}$ is given by the construction $A \mapsto \mathrm{fib}(\mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}}))$.*

Remark 2.0.9. To prove Theorem 2.0.8, we are free to replace R by its connective cover and thereby reduce to the case where R is connective.

Definition 2.0.10. Let R be a connective $\mathbb{E}_\infty$-ring which is (p)-complete for some prime number p, and let $\mathbf{G}$ be a p-divisible group over R. We will refer to the formal group $\mathbf{G}^\circ$ of Theorem 2.0.8 as the *identity component* of $\mathbf{G}$.

Theorem 2.0.8 is more general than Theorem 2.0.1 in three main respects:

(a) In the statement of Theorem 2.0.8, we allow R to be an $\mathbb{E}_\infty$-ring rather than an ordinary commutative ring.

(b) Theorem 2.0.1 requires that p is nilpotent in R, while Theorem 2.0.8 requires only that R is (p)-complete. However, the difference is slight, at least when R is Noetherian: in this case, one can construct the identity component $\mathbf{G}^\circ$ by amalgamating its restrictions to the subschemes $\mathrm{Spec}(R/(p^k)) \subseteq \mathrm{Spec}(R)$.

(c) Even when R is an ordinary commutative ring and p is nilpotent in $\pi_0(R)$, the content of Theorem 2.0.8 is stronger than that of Theorem 2.0.1. If $\mathbf{G}$ is a p-divisible group over R, then Theorem 2.0.1 asserts that there is a unique formal group $\mathbf{G}^\circ$ over R having the property that $\mathbf{G}^\circ(A) = \ker(\mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}}))$ when A is a discrete R-algebra. Theorem 2.0.8 asserts that this formal group $\mathbf{G}^\circ$ has a stronger property: there is a canonical equivalence $\mathbf{G}^\circ(A) \simeq \mathrm{fib}(\mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}}))$ for *every* truncated object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$.

Let us now sketch the contents of this section. Our first goal will be to show that the identity component $\mathbf{G}^\circ$ of Theorem 2.0.8 is uniquely determined by requirement $(*)$. Roughly speaking, we can think of $(*)$ as prescribing the restriction of the formal group $\mathbf{G}^\circ$ to the formal spectrum $\mathrm{Spf}(R) \subseteq \mathrm{Spec}(R)$, where we endow $\pi_0(R)$ with the p-adic topology. In §2.1, we prove more generally that if an $\mathbb{E}_\infty$-ring R is complete with respect to a finitely generated ideal $I \subseteq \pi_0(R)$, then a formal group $\widehat{\mathbf{G}}$ over $\mathrm{Spec}(R)$ is determined by its restriction to $\mathrm{Spf}(R)$ (see Theorem 2.1.1 for a precise statement).

We will give the proof of Theorem 2.0.8 in §2.2. In [27], Theorem 2.0.1 is proved by first treating the case where R is an $\mathbf{F}_p$-algebra (in which case one can exploit special features of the Frobenius and Verschiebung endomorphisms of $\mathbf{G}$), and this is extended to the general case using deformation-theoretic arguments. Our strategy will be essentially the same: we will apply deformation-theoretic arguments (in the more general setting of $\mathbb{E}_\infty$-rings) to reduce to the case where R is a discrete $\mathbf{F}_p$-algebra. The arguments of [27] (which we reproduce here, for the sake of completeness) then show that there exists a formal group $\mathbf{G}^\circ$ satisfying $\mathbf{G}^\circ(A) = \ker(\mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}}))$ whenever A is a discrete R-algebra. We then show that $\mathbf{G}^\circ$ represents the desired functor on *all* truncated $\mathbb{E}_\infty$-algebras over R by exploiting a deformation-theoretic property of the Frobenius map (Proposition 2.2.3).

Roughly speaking, the difference between a p-divisible group $\mathbf{G}$ and its identity component $\mathbf{G}^\circ$ is controlled by the values of $\mathbf{G}$ on *reduced* $\mathbf{F}_p$-algebras. In §2.3, we study the class of *connected* p-divisible groups, which are defined by the requirement that $\mathbf{G}(A) \simeq 0$ when A is reduced (see Proposition 2.3.9). We will show that the construction $\mathbf{G} \mapsto \mathbf{G}^\circ$ is fully faithful when restricted to connected p-divisible groups (Corollary 2.3.13), and study its essential image.

In §2.5, we study the class of *étale* p-divisible groups (Definition 2.5.3), which are in some sense as far as possible from being connected (Proposition 2.5.8). The theory of étale p-divisible groups offers no surprises: the datum of an étale p-divisible group over an $\mathbb{E}_\infty$-ring R is equivalent to the datum of an étale p-divisible group over the commutative ring $\pi_0(R)$ (Proposition 2.5.9). Such objects are well-understood: when the topological space $|\mathrm{Spec}(R)|$ is connected, they can be identified with free $\mathbf{Z}_p$-modules of finite rank, equipped with a continuous action of the étale fundamental group $\pi_1(\mathrm{Spec}(R), \eta)$ (where η is any geometric point of $\mathrm{Spec}(R)$). In good cases of interest, an arbitrary p-divisible group $\mathbf{G}$ can be "built" from connected and étale pieces (which can be understood in terms of formal groups and Galois representations, respectively). To make this precise, we introduce in §2.4 the notion of a *short exact sequence* of p-divisible groups

$$0 \to \mathbf{G}' \xrightarrow{f} \mathbf{G} \xrightarrow{g} \mathbf{G}'' \to 0$$

(see Definition 2.4.9). We will be particularly interested in short exact sequences where $\mathbf{G}'$ is (formally) connected and $\mathbf{G}''$ is étale. We will refer to such an exact sequence as a *connected-étale sequence* for $\mathbf{G}$ (Definition 2.5.15). In §2.5, we will show that such sequences are automatically unique (Theorem 2.5.13), and give necessary and sufficient conditions for their existence (Proposition 2.4.1).

2.1 Formal Groups over $\mathrm{Spec}(R)$ and $\mathrm{Spf}(R)$

Let R be a connective $\mathbb{E}_\infty$-ring. We can think of a formal group $\widehat{\mathbf{G}}$ over R as an abelian group object in the ∞-category formal schemes over the spectrum $\mathrm{Spec}(R)$ (see Remark 1.6.7 for a precise statement). If R is an adic $\mathbb{E}_\infty$-ring, then we can also consider the notion of a formal group over the formal spectrum $\mathrm{Spf}(R)$. Such an object can be described geometrically as a certain kind of abelian group object in the ∞-category $(\mathrm{fSpDM})_{/\mathrm{Spf}(R)}$, or more concretely as a compatible family of formal groups $\{\widehat{\mathbf{G}}_A \in \mathrm{FGroup}(A)\}$ where A ranges over connective $\mathbb{E}_\infty$-rings equipped with a map $\mathrm{Spec}(A) \to \mathrm{Spf}(R)$. We will refrain from giving a precise definition, because this turns out to be unnecessary: if R is a complete adic $\mathbb{E}_\infty$-ring, then the notions of formal group over $\mathrm{Spec}(R)$ and $\mathrm{Spf}(R)$ turn out to be equivalent. The equivalence is a consequence of the following general statement:

Theorem 2.1.1. *Let R be a connective $\mathbb{E}_\infty$-ring which is complete with respect to some finitely generated ideal $I \subseteq \pi_0(R)$ and let $\mathcal{E}_R$ denote the full subcategory of $\mathrm{CAlg}^{\mathrm{cn}}_R$ spanned by those connective $\mathbb{E}_\infty$-algebras over R which are truncated and I-nilpotent. Then the restriction functor $\widehat{\mathbf{G}} \mapsto \widehat{\mathbf{G}}|_{\mathcal{E}_R}$ determines a fully faithful embedding $\mathrm{FGroup}(R) \to \mathrm{Fun}(\mathcal{E}_R, \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}})$. The essential image of this embedding consists of those functors $\widehat{\mathbf{G}}_0 : \mathcal{E}_R \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ which satisfy the following pair of conditions:*

(1) *The composite functor*

$$\mathcal{E}_{(\pi_0 R)/I} \to \mathcal{E}_R \xrightarrow{\widehat{\mathbf{G}}_0} \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$$

can be extended to a formal group over the commutative ring $(\pi_0 R)/I$.

(2) *The functor* $\widehat{\mathbf{G}}_0$ *is cohesive: that is, for every pullback diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

in the ∞*-category* $\mathcal{E}_R$ *where the ring homomorphisms* $\pi_0(A_0) \to \pi_0(A_{01}) \leftarrow \pi_0(A_1)$ *are surjective, the induced diagram*

$$\begin{tikzcd} \widehat{\mathbf{G}}_0(A) \arrow[r] \arrow[d] & \widehat{\mathbf{G}}_0(A_0) \arrow[d] \\ \widehat{\mathbf{G}}_1(A_1) \arrow[r] & \widehat{\mathbf{G}}_0(A_{01}) \end{tikzcd}$$

is a pullback square in $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$*.*

Theorem 2.1.1 is a consequence of a more general assertion concerning formal hyperplanes.

Notation 2.1.2. Let R be a connective $\mathbb{E}_\infty$-ring. A *pointed formal hyperplane over* R is a functor $X : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}_*$ with the property that the composite functor

$$\mathrm{CAlg}_R^{\mathrm{cn}} \xrightarrow{X} \mathcal{S}_* \to \mathcal{S}$$

is a formal hyperplane over R, in the sense of Definition 1.5.10. We let $\mathrm{Hyp}_*(R)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S})$ spanned by the pointed formal hyperplanes over R.

Remark 2.1.3. Let R be a connective $\mathbb{E}_\infty$-ring. The equivalence of ∞-categories $\mathrm{cSpec} : \mathrm{cCAlg}_R^{\mathrm{sm}} \simeq \mathrm{Hyp}(R)$ induces an equivalence of ∞-categories $(\mathrm{cCAlg}_R^{\mathrm{sm}})_{R/} \simeq \mathrm{Hyp}_*(R)$, where $(\mathrm{cCAlg}_R^{\mathrm{sm}})_{R/}$ is the ∞-category of *augmented* smooth coalgebras over R (that is, the ∞-category of smooth coalgebras C over R equipped with a point $\eta \in \mathrm{GLike}(C)$).

Proposition 2.1.4. *Let* R *be a connective* $\mathbb{E}_\infty$*-ring which is* I*-complete for some finitely generated ideal* $I \subseteq \pi_0 R$ *and let* $\mathcal{E}_R \subseteq \mathrm{CAlg}_R^{\mathrm{cn}}$ *be as in the statement of Theorem 2.1.1. Then the restriction functor* $\mathrm{Hyp}_*(R) \to \mathrm{Fun}(\mathcal{E}_R, \mathcal{S}_*)$ *is fully faithful, and its essential image consists of those functors* $X_0 : \mathcal{E}_R \to \mathcal{S}_*$ *which satisfy the following pair of conditions:*

(1) *The composite functor*

$$\mathcal{E}_{(\pi_0 R)/I} \to \mathcal{E}_R \xrightarrow{X_0} \mathcal{S}_*$$

can be extended to a pointed formal hyperplane over the commutative ring $(\pi_0 R)/I$.

(2) *The functor* X_0 *is cohesive: that is, for every pullback diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

in the ∞*-category* $\mathcal{E}_R$ *where the morphisms* $\pi_0(A_0) \to \pi_0(A_{01}) \leftarrow \pi_0(A_1)$ *are surjective, the induced diagram*

$$\begin{tikzcd} X_0(A) \arrow[r] \arrow[d] & X_0(A_0) \arrow[d] \\ X_0(A_1) \arrow[r] & X_0(A_{01}) \end{tikzcd}$$

is a pullback square in $\mathcal{S}_*$.

Proof of Theorem 2.1.1 from Proposition 2.1.4. For any ∞-category $\mathcal{C}$ which admits finite products, let $\mathcal{C}_*$ denote the ∞-category of pointed objects of $\mathcal{C}$. Then the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ induces an equivalence of ∞-categories $\rho : \mathrm{Ab}(\mathcal{C}_*) \to \mathrm{Ab}(\mathcal{C})$: to see this, we observe that ρ can be identified with the forgetful functor $\mathrm{Ab}(\mathcal{C})_* \to \mathrm{Ab}(\mathcal{C})$, which is an equivalence because the ∞-category $\mathrm{Ab}(\mathcal{C})$ is pointed. Applying this observation in the case $\mathcal{C} = \mathcal{S}$, we obtain an equivalence $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}} \simeq \mathrm{Ab}(\mathcal{S}_*)$. Consequently, the equivalence of Theorem 2.1.1 can be obtained from the equivalence of Proposition 2.1.4 by passing to abelian group objects. $\square$

The rest of this section is devoted to the proof of Proposition 2.1.4. We will first prove in §2.1.2 that the restriction functor $X_0 \mapsto X_0|_{\mathcal{E}_R}$ is fully faithful (Proposition 2.1.10) using a slightly technical property of formal hyperplanes (Proposition 2.1.6). The characterization of the essential image of the restriction functor will be established in §2.1.3.

Warning 2.1.5. In the statement of Proposition 2.1.4, the restriction to *pointed* formal hyperplanes is essential. For example, if we take $R = \mathbf{Z}_p$ to be the ring of p-adic integers, I to be the principal ideal (p), and $X = \widehat{\mathbf{A}}^1 = \mathrm{Spf}(R[[t]])$ to be the formal affine line over R, then the element $p \in R$ determines a natural transformation $\mathrm{Spec}(R)|_{\mathcal{E}} \to X|_{\mathcal{E}}$ which cannot be lifted to a point of $X(R)$ (the element p is topologically nilpotent but not nilpotent in R).

2.1.1 A Finiteness Property of Formal Hyperplanes

Our proof of Proposition 2.1.4 will make use of the following:

Proposition 2.1.6. *Let R be a connective $\mathbb{E}_\infty$-ring, let $\mathcal{C} \subseteq \mathrm{CAlg}_R^{\mathrm{cn}}$ be the full subcategory spanned by those connective $\mathbb{E}_\infty$-algebras over R which are almost perfect when regarded as R-modules, and let $X : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ be a formal hyperplane over R. Then X is a left Kan extension of $X|_{\mathcal{C}}$.*

The proof will require some preliminaries.

Lemma 2.1.7. *Let R be a connective $\mathbb{E}_\infty$-ring and let M be an $R[[t]]$-module such that $M[t^{-1}] \simeq 0$. If M is almost perfect as an $R[[t]]$-module, then it is almost perfect as an R-module.*

Proof. Without loss of generality, we may assume that M is connective. We will show that, for every integer n, the module M is perfect to order n over R (in the sense of Definition SAG.2.7.0.1). For $n < 0$, this follows automatically from our assumption that M is connective (Example SAG.2.7.0.3). The proof in general proceeds by induction on n. Since M is connective and almost perfect as an $R[[t]]$-module, the abelian group $\pi_0(M)$ is finitely generated as a module over $\pi_0(R)[[t]]$. Let $x_1, \ldots, x_k$ be a set of generators for $\pi_0(M)$ as a module over $\pi_0(R)[[t]]$. Using the assumption that $M[t^{-1}] \simeq 0$, we can choose an integer m such that $t^m x_i = 0$ for $1 \leqslant i \leqslant k$. Let $R[[t]]/(t^m)$ denote the cofiber of the map $t^m : R[[t]] \to R[[t]]$. Then we can find maps $f_i : R[[t]]/(t^m) \to M$ which carry the element $1 \in \pi_0(R)[[t]]/(t^m)$ to x_i. Amalgamating these maps, we obtain a map $u : \bigoplus_{1 \leqslant i \leqslant k} R[[t]]/(t^m) \to M$ which is surjective on π_0. Note that the domain of u is perfect as an $R[[t]]$-module, so that $\mathrm{fib}(u)$ is an almost perfect $R[[t]]$-module. Applying our inductive hypothesis, we conclude that $\mathrm{fib}(u)$ is perfect to order $(n-1)$ as an R-module. Since the domain of u is perfect as an R-module, it follows that M is perfect to order n as an R-module, as desired. □

Lemma 2.1.8. *Let R be a connective $\mathbb{E}_\infty$-ring and let M be an almost perfect $R[[t_1, \ldots, t_n]]$-module such that $M[t_i^{-1}]$ vanishes for $1 \leqslant i \leqslant n$. Then M is almost perfect when regarded as an R-module.*

Proof. Apply Lemma 2.1.7 repeatedly. $\square$

Lemma 2.1.9. *Let R be a connective $\mathbb{E}_\infty$-ring, let C be a smooth coalgebra over R, and let M be an almost perfect module over $C^\vee$. Assume that M is I_η-nilpotent, where $I_\eta \subseteq \pi_0(C^\vee)$ is the ideal of Warning 1.3.12. Then M is almost perfect as an R-module.*

Proof. Using Remark 1.4.9, we can choose elements $\{x_i\}_{i \in I}$ which generate the unit ideal of $\pi_0(R)$ such that each localization $C[x_i^{-1}]$ is a standard smooth coalgebra over $R[x_i^{-1}]$. Since the condition of being almost perfect is local for the Zariski topology on R (see Proposition SAG.2.8.4.2), it will suffice to show that each localization $M[x_i^{-1}]$ is almost perfect as an $R[x_i^{-1}]$-module. Set $C_i^\vee = \underline{\mathrm{Map}}_R(C, R[x_i^{-1}])$, so that $C_i^\vee$ is the I_η-completion of $C^\vee[x_i^{-1}]$ (Proposition 1.3.13). In particular, the fiber of the map $v : C^\vee[x_i^{-1}] \to C_i^\vee$ is I-local, so the tensor product $\mathrm{fib}(v) \otimes_{C^\vee} M$ vanishes. It follows that the natural map

$$M[x_i^{-1}] \simeq C^\vee[x_i^{-1}] \otimes_{C^\vee} M \to C_i^\vee \otimes_{C^\vee} M$$

is an equivalence. We are therefore reduced to proving that each $C_i^\vee \otimes_{C^\vee} M$ is almost perfect as an $R[x_i^{-1}]$-module. We may therefore replace R by $R[x_i^{-1}]$ and M by $C_i^\vee \otimes_{C^\vee} M$ and thereby reduce to the case where the smooth coalgebra C is standard. In this case, Proposition 1.4.10 supplies an equivalence $R[[t_1, \ldots, t_n]] \to C^\vee$ of $\mathbb{E}_1$-algebras over R. Then M is almost perfect when regarded as an $R[[t_1, \ldots, t_n]]$-module and our assumption that M is I_η-nilpotent guarantees the vanishing of each localization $M[t_j^{-1}]$. Applying Lemma 2.1.8, we deduce that M is almost perfect when regarded as an R-module, as desired. $\square$

Proof of Proposition 2.1.6. Let R be a connective $\mathbb{E}_\infty$-ring and let X be a formal hyperplane over R. Write $X = \mathrm{cSpec}(C)$ for some smooth coalgebra C over R, and let $I_\eta \subseteq \pi_0(C^\vee)$ be as in Warning 1.3.12. Applying Lemma SAG.8.1.2.2 , we can choose a tower

$$\cdots \to A_3 \to A_2 \to A_1$$

of connective $\mathbb{E}_\infty$-algebras over $C^\vee$ with the following properties:

(a) For each $n > 0$, the image of I_η in $\pi_0(A_n)$ is nilpotent.

(b) Each A_n is almost perfect when regarded as a $C^\vee$-module.

(c) For every connective R-algebra B, the canonical map

$$\varinjlim \mathrm{Map}_{\mathrm{CAlg}_R}(A_n, B) \to \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, B)$$

is a homotopy equivalence.

It follows that X can be identified with the filtered colimit of the functors $\operatorname{Spec} A_n$ corepresented by the objects $A_n \in \mathrm{CAlg}_R^{\mathrm{cn}}$. Consequently, to show that X is a left Kan extension of $X|_{\mathcal{C}}$, it will suffice to show that each of the corepresentable functors $\operatorname{Spec} A_n$ is a left Kan extension of $(\operatorname{Spec} A_n)|_{\mathcal{C}}$. This follows from the observation that each A_n belongs to $\mathcal{C}$, by virtue of Lemma 2.1.9. □

2.1.2 Formal Hyperplanes over I-Complete $\mathbb{E}_\infty$-Rings

We now prove a weak version of Proposition 2.1.4:

Proposition 2.1.10. *Let R be a connective $\mathbb{E}_\infty$-ring which is complete with respect to some finitely generated ideal $I \subseteq \pi_0(R)$. Let $\mathcal{E}$ denote the full subcategory of $\mathrm{CAlg}_R^{\mathrm{cn}}$ spanned by those connective $\mathbb{E}_\infty$-algebras over R which are truncated and I-nilpotent. Then the restriction functor $X \mapsto X|_{\mathcal{E}}$ determines a fully faithful embedding $\mathrm{Hyp}_*(R) \to \mathrm{Fun}(\mathcal{E}, \mathcal{S}_*)$.*

The proof of Proposition 2.1.10 requires the following simple observation:

Lemma 2.1.11. *Let R be a connective $\mathbb{E}_\infty$-ring, let X be a pointed formal hyperplane over R, and let x be a point of $X(A)$ for some connective $\mathbb{E}_\infty$-algebra A over R. Then there exists a morphism $f : A \to B$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$ with the following properties:*

- *(i) The image of x in $X(B)$ belongs to the connected component of the base point.*
- *(ii) The $\mathbb{E}_\infty$-ring B is almost of finite presentation as an A-module.*
- *(iii) For every element $a \in \pi_0(A)$, if $f(a)$ is nilpotent in $\pi_0(B)$, then a is nilpotent in $\pi_0(A)$.*

Proof. Note that the image of x in $X(A^{\mathrm{red}})$ belongs to the identity component (since $X(A^{\mathrm{red}})$ is contractible by virtue of Remark 1.5.18). It follows from Proposition 2.1.6 that the canonical map $A \to A^{\mathrm{red}}$ factors as a composition $A \to B \to A^{\mathrm{red}}$, where B satisfies conditions (i) and (ii). Note that if $a \in \pi_0(A)$ has the property that $f(a)^k = 0$ in $\pi_0(B)$, then the image of a^k vanishes in A^{red}, so that a^k is nilpotent in $\pi_0(A)$ and therefore a is also nilpotent in $\pi_0(A)$. □

Proof of Proposition 2.1.10. Let R be a connective $\mathbb{E}_\infty$-ring and let X and Y be pointed formal hyperplanes over R; we wish to show that the restriction map

$$\rho : \mathrm{Map}_{\mathrm{Hyp}_*(R)}(X, Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{E}, \mathcal{S}_*)}(X|_{\mathcal{E}}, Y|_{\mathcal{E}})$$

is a homotopy equivalence. Let $Y^{\mathrm{cont}} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}_*$ be a right Kan extension of $Y|_{\mathcal{E}}$. The identity map $\mathrm{id} : Y|_{\mathcal{E}} \to Y|_{\mathcal{E}}$ extends uniquely to a natural transformation $u : Y \to Y^{\mathrm{cont}}$ and we can identify ρ with the map

$$\mathrm{Map}_{\mathrm{Hyp}_*(R)}(X, Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S}_*)}(X, Y^{\mathrm{cont}})$$

given by precomposition with u.

Let $\mathcal{E}^+ \subseteq \mathrm{Mod}^{\mathrm{cn}}_R$ be the full subcategory spanned by those connective $\mathbb{E}_\infty$-algebras over R which are I-nilpotent. Note that the functor Y^{cont} is given on objects $A \in \mathcal{E}^+$ by the formula $Y^{\mathrm{cont}}(A) = \varprojlim Y(\tau_{\leqslant n} A)$. Using Proposition 1.5.17, we see that u induces an equivalence $Y(A) \to Y^{\mathrm{cont}}(A)$ for $A \in \mathcal{E}^+$. In other words, we can also identify Y^{cont} with the right Kan extension of $Y|_{\mathcal{E}^+}$.

For each object $A \in \mathrm{CAlg}^{\mathrm{cn}}_R$, let $\mathrm{Spec}(A) : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ denote the functor corepresented by A, and let $\mathrm{Spf}(A) : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ denote the subfunctor of $\mathrm{Spec}(A)$ given by the formula

$$\mathrm{Spf}(A)(B) = \begin{cases} \mathrm{Map}_{\mathrm{CAlg}_R}(A, B) & \text{if } B \in \mathcal{E}^+ \\ \varnothing & \text{otherwise.} \end{cases}$$

Unwinding the definitions, we can identify $Y(A) \to Y^{\mathrm{cont}}(A)$ with the map of pointed space spaces

$$u_A : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})}(\mathrm{Spec}(A), Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})}(\mathrm{Spf}(A), Y)$$

given by precomposition with the inclusion $\mathrm{Spf}(A) \hookrightarrow \mathrm{Spec}(A)$. Write $Y = \mathrm{cSpec}(C)$, where C is a smooth coalgebra over R. Using Remark SAG.8.1.2.4 and Corollary SAG.8.1.5.4 , we can identify u_A with the map

$$\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, A) \to \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, A^\wedge_I);$$

here we regard $\pi_0(A)$ as equipped with the discrete topology, and $\pi_0(A^\wedge_I)$ as equipped with the I-adic topology. Note that if A is already I-complete, then this map is the inclusion of a summand.

Let $\mathcal{C} \subseteq \mathrm{Mod}^{\mathrm{cn}}_R$ be as in the statement of Proposition 2.1.6, so that X is a left Kan extension of its restriction to $\mathcal{C}$. We are therefore reduced to showing that u induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{S}_*)}(X|_{\mathcal{C}}, Y|_{\mathcal{C}}) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{S}_*)}(X|_{\mathcal{C}}, Y^{\mathrm{cont}}|_{\mathcal{C}}).$$

Our assumption that R is I-complete guarantees that each object $A \in \mathcal{C}$ is also I-complete (Proposition SAG.7.3.5.7), so that the map $Y|_{\mathcal{C}} \to Y^{\mathrm{cont}}|_{\mathcal{C}}$ is a monomorphism. It follows that θ is the inclusion of a summand. To complete the proof, it will suffice to show that every natural transformation $v : X \to Y^{\mathrm{cont}}$ carries each point $q \in X(A)$ into the summand $Y(A) \subseteq Y^{\mathrm{cont}}(A)$ for $A \in \mathcal{C}$. Choose a grouplike element $\eta \in \pi_0(C^\vee)$ and let $J_\eta \subseteq \pi_0(C^\vee)$ be the ideal of Proposition 1.3.10. Let us identify $v(q)$ with a morphism $h : C^\vee \to A^\wedge_I \simeq A$ of $\mathbb{E}_\infty$-rings over R satisfying $h(J_\eta^n) \subseteq I\pi_0(A)$. We wish to show that $f(J_\eta^m) = 0$ for $m \gg 0$. Choose a map $f : A \to B$ satisfying the requirements of Lemma 2.1.11, so that the image of q belongs to the identity component of $X(B)$. It follows by naturality that the composite map $C^\vee \xrightarrow{h} A \xrightarrow{f} B$ belongs to the identity component of $Y^{\mathrm{cont}}(B)$; in particular, it belongs to the summand $Y(B) \subseteq Y^{\mathrm{cont}}(B)$, so that $f \circ h$ annihilates some power of the ideal J_η. Since the kernel of $\pi_0(f)$ consists of nilpotent elements of $\pi_0(A)$, it follows that h also annihilates some power of J_η, so that $v(q)$ belongs to the summand $Y(A) \subseteq Y^{\mathrm{cont}}(A)$ as desired. $\square$

2.1.3 The Proof of Proposition 2.1.4

Let R be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0(R)$, let $\mathcal{E}_R \subseteq \mathrm{CAlg}_R^{\mathrm{cn}}$ denote the full subcategory spanned by those connective $\mathbb{E}_\infty$-algebras which are truncated and I-nilpotent, and let $X_0 : \mathcal{E}_R \to \mathcal{S}_*$ be a functor. We wish to show that X_0 can be extended to a (pointed) formal hyperplane $X : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}_*$ if and only if it satisfies conditions (1) and (2) of Proposition 2.1.4 (in this case, the extension X is essentially unique, by virtue of Proposition 2.1.10). The necessity of conditions (1) and (2) follows from Proposition 1.5.17. Let us therefore assume that (1) and (2) are satisfied.

Let A be a connective $\mathbb{E}_\infty$-algebra over R which is I-complete. We will say that A is *good* if the composite functor $\mathcal{E}_A \to \mathcal{E}_R \xrightarrow{X_0} \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ can be extended to a pointed formal hyperplane $X_A \in \mathrm{Hyp}_*(A)$. In this case, Proposition 2.1.10 guarantees that X_A is uniquely determined and depends functorially on A. We will prove Proposition 2.1.4 by showing that every I-complete $\mathbb{E}_\infty$-algebra over R is good (in particular, R itself is good). The proof proceeds in several steps.

(*a*) Suppose we are given a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

of connective, I-complete $\mathbb{E}_\infty$-algebras over R, where the morphisms $\pi_0(A_0) \to \pi_0(A_{01}) \leftarrow \pi_0(A_1)$ are surjective with nilpotent kernel. If A_0 and A_1 are good, then A is good. To prove this, let us suppose that $X_0|_{\mathcal{E}_{A_0}}$ and $X|_{\mathcal{E}_{A_1}}$ can be extended to pointed formal hyperplanes X_{A_0} and X_{A_1} over A_0 and A_1, respectively. It follows that $X_0|_{\mathcal{E}_{A_{01}}}$ can also be extended to a pointed formal hyperplane $X_{A_{01}}$ over A_{01}. Using Proposition 1.2.12, we can amalgamate X_{A_0} and X_{A_1} to obtain a pointed formal hyperplane $X_A \in \mathrm{Hyp}_*(A)$. For each $A' \in \mathcal{E}_A$, set

$$A'_0 = A' \otimes_A A_0 \qquad A'_{01} = A' \otimes_A A_{01} \qquad A'_1 = A' \otimes_A A_1.$$

We then have homotopy equivalences equivalences

$$\begin{aligned} X_A(A') &\simeq X_A(A'_0 \otimes_{A'_{01}} A'_1) \\ &\simeq X_A(A'_0) \times_{X_A(A'_{01})} X_A(A'_1) \\ &\simeq X_{A_0}(A'_0) \times_{X_{A_{01}}(A'_{01})} X_{A_1}(A'_1) \\ &\simeq \varprojlim_n (X_{A_0}(\tau_{\leqslant n} A'_0) \times_{X_{A_{01}}(\tau_{\leqslant n} A'_{01})} X_{A_1}(\tau_{\leqslant n} A'_1)) \\ &\simeq \varprojlim_n X_0(\tau_{\leqslant n} A'_0) \times_{X_0(\tau_{\leqslant n} A'_{01})} X_0(\tau_{\leqslant n} A'_1) \\ &\simeq \varprojlim_n X_0(\tau_{\leqslant n} A'_0 \otimes_{\tau_{\leqslant n} A'_{01}} \tau_{\leqslant n} A'_1) \\ &\simeq X_0(A') \end{aligned}$$

depending functorially on A' (here we invoke Proposition 1.5.17 together with assumption (2)). It follows that the restriction $X_0|_{\mathcal{E}_A}$ extends to the pointed formal hyperplane X_A, so that A is good.

(*b*) Let $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$ be an I-complete $\mathbb{E}_\infty$-algebra over R and let $\widetilde{A}$ be a square-zero extension of A by a connective I-complete A-module M (in the sense of Definition HA.7.4.1.6). If A is good, then $\widetilde{A}$ is also good. This is a special case of assertion (*a*).

(*c*) Let A be a discrete $\mathbb{E}_\infty$-algebra over R with $I^k R' = 0$ for some $k \gg 0$. Then R' is good. This follows by induction on k: the case $k = 1$ follows from assumption

(1), and the inductive step follows from (b) (since R'/I^kR' is a square-zeroe extension of $R'/I^{k-1}R'$ for $k > 1$ by virtue of Theorem HA.7.4.1.26).

(d) Let A be an $\mathbb{E}_\infty$-algebra over R which is I-nilpotent and k-truncated for some $k \gg 0$. Then A is good. This follows by induction on k: the case $k = 0$ follows from (c), and the inductive step follows from (b) (since each truncation $\tau_{\leqslant m}A$ is a square-zero extension of $\tau_{\leqslant m-1}A$ by virtue of Theorem HA.7.4.1.26).

(e) Let A be an $\mathbb{E}_\infty$-algebra over R which is I-complete. If each truncation $\tau_{\leqslant n}A$ is good, then A is good. To prove this, extend each restriction $X_0|_{\mathcal{E}_{\tau_{\leqslant n}A}}$ to a pointed formal hyperplane $\mathrm{cSpec}(C_n)$, where C_n is an augmented smooth coalgebra over $\tau_{\leqslant n}A$. Invoking Proposition 1.2.11, we deduce that there is an augmented smooth coalgebra C over A equipped with compatible equivalences $C_n \simeq (\tau_{\leqslant n}A) \otimes_A C$. It is then easy to check that $\mathrm{cSpec}(C)|_{\mathcal{E}_A}$ is canonically equivalent to $X_0|_{\mathcal{E}_A}$.

(f) Let A be an $\mathbb{E}_\infty$-algebra over R which is I-nilpotent. Then A is good. This follows by combining (d) and (e).

(g) Let A be a discrete $\mathbb{E}_\infty$-algebra over R which is classically I-complete: that is, the canonical map $A \to \varprojlim A/I^kA$ is an isomorphism of commutative rings. We will show that A is good. Using Lemma SAG.8.1.2.2 , we can write A as the limit of a tower

$$\cdots \to A_3 \to A_2 \to A_1 \to A_0$$

of I-nilpotent connective $\mathbb{E}_\infty$-algebras over A with the following properties: each A_n is almost perfect as an A-module, each of the transition maps is surjective on π_0, and the colimit $\varinjlim \mathrm{Map}_{\mathrm{CAlg}_A}(A_n, B)$ is contractible for every connective I-nilpotent $\mathbb{E}_\infty$-algebra B over A. Shifting this tower if necessary, we may assume that the kernel of the map $A \to \pi_0(A_0)$ is contained in some power of I. It follows from (f) that each restriction $X_0|_{\mathcal{E}_{A_n}}$ can be extended to a pointed formal hyperplane over A_n, which we can write as $\mathrm{cSpec}(C_n)$ for some smooth coalgebra C_n over A_n, equipped with augmentation given by a grouplike element $\eta_n \in \mathrm{GLike}(C_n)$. Choose a coalgebra isomorphism $\alpha_0 : \pi_0(C_0) \simeq \Gamma^*_{\pi_0(A_0)}(M_0)$, where M_0 is a projective module of finite rank over $\pi_0(A_0)$. Without loss of generality, we may assume that $\alpha_0(\eta_0) = 1$, so that α_0 is classified by a map $\beta_0 : \pi_0(C_0) \to M_0$ satisfying $\beta_0(\eta_0) = 0$ (see Proposition 1.1.16).

Since A is complete with respect to the kernel of the map $A \to \pi_0(A_0)$, we can lift M_0 to a module M which is projective of finite rank over A. Since A is I-complete, we can lift M_0 to a module M which is projective of finite rank over $\pi_0(R)$. For $n \geqslant 0$, set $M_n = \pi_0(A_n) \otimes_A M$. Using the projectivity of each $\pi_0(C_n)$ as a module over $\pi_0(R_n)$, we can extend β_0 to a compatible sequence of maps $\beta_n : \pi_0(C_n) \to M_n$. Adjusting these maps if necessary, we can further assume that $\beta_n(\eta_n) = 0$, so that β_n classifies a map of augmented coalgebras $\alpha_n : \pi_0(C_n) \to \Gamma^*_{\pi_0(A_n)}(M_n)$. Each α_n is a map of flat modules over the commutative ring $\pi_0(A_n)$ which is an isomorphism modulo the nilpotent ideal $I\pi_0(A_n)$, and is therefore an isomorphism.

For every pair of integers $k, n \geqslant 0$, the projectivity of $\Gamma^k_A(M)$ guarantees that the composite map The projectivity of $N(k)$ guarantees that the map

$$\Gamma^k_A(M) \to \Gamma^k_{\pi_0(A_n)}(M_n) \xrightarrow{\alpha_n^{-1}} \pi_0(C_n)$$

to a morphism of A-modules $\gamma_{n,k} : \Gamma^k_A(M) \to C_n$, which is well-defined up to homotopy. Moreover, each of the diagrams

$$\begin{tikzcd}
 & \Gamma^k_A(M) \arrow[dl, "\gamma_{n,k}"'] \arrow[dr, "\gamma_{n-1,k}"] & \\
C_n \arrow[rr] & & C_{n-1}
\end{tikzcd}$$

commutes up to homotopy. Choosing such a homotopy for each n, we obtain a map $\gamma_k : \Gamma^k_A(M) \to \varprojlim C_n$ (beware that the map γ_k is not quite canonical, but this will not be important in what follows).

For each $n \geqslant 0$, let $B_n = C_n^\vee = \underline{\mathrm{Map}}_{A_n}(C_n, A_n)$, and set $B = \varprojlim B_n$; we will regard B as an augmented $\mathbb{E}_\infty$-algebra over A. For fixed n, the morphisms $\{\gamma_{n,k}\}_{k \geqslant 0}$ can be amalgamated to a map $A_n \otimes_A \Gamma^*_A(M) \to C_n$, which is an equivalence (since it induces the isomorphism α_n^{-1} on π_0, and the domain and codomain are flat over A_n). We therefore obtain R_n-module equivalences

$$B_n \simeq \underline{\mathrm{Map}}_{A_n}(C_n, A_n) \simeq \underline{\mathrm{Map}}_A(\Gamma^*_A(M), A_n) \simeq \prod_{k \geqslant 0} \mathrm{Sym}^k_A(M^\vee) \otimes_A A_n.$$

Passing to the inverse limit over n, we obtain an A-module equivalence $B \simeq \prod_{k \geqslant 0} \mathrm{Sym}^k_A(M^\vee)$, which is easily seen to be a homomorphism of A-algebras. It follows that the formal spectrum $X = \mathrm{Spf}(B)$ (where we regard B as equipped with the topology given by the ideal $J = \prod_{k>0} \mathrm{Sym}^k_A(M^\vee))$ can be identified with the cospectrum of the smooth coalgebra $\Gamma^*_A(M)$, and is therefore a formal hyperplane over A. We will complete the argument by showing that the functors $X|_{\mathcal{E}_A}$ and $X_0|_{\mathcal{E}_A}$ are canonically equivalent.

By construction, we can identify $\mathcal{E}_A$ with the colimit of the filtered diagram of ∞-categories

$$\mathcal{E}_{A_0} \to \mathcal{E}_{A_1} \to \mathcal{E}_{A_2} \to \cdots .$$

It will therefore suffice to construct a compatible family of equivalences $X|_{\mathcal{E}_{A_n}} \simeq X_0|_{\mathcal{E}_{A_n}}$. Note that both sides are obtained from formal hyperplanes over A_n, which can be described as the formal spectra of $A_n \otimes_A B$ and B_n, respectively. We will complete the proof by showing that the canonical map $\rho : A_n \otimes_A B \simeq B_n$ induces an equivalence after J-completion (in fact, it is already an equivalence before J-completion, but we will not need to know this). Unwinding the definitions, we can write ρ as a composition

$$A_n \otimes_A B \xrightarrow{\rho'} \varprojlim_{m \geqslant n} A_n \otimes_A B_m \xrightarrow{\rho''} \varprojlim_{m \geqslant n} A_n \otimes_{A_m} B_m \xrightarrow{\rho'''} B_n.$$

Here the map ρ' is an equivalence because A_n is almost perfect as an A-module and each B_m is connective, and the map ρ''' induces an equivalence after J-completion because each individual map $A_n \otimes_{A_m} B_m \to R_n$ induces an equivalence after J-completion (and the J-completion functor commutes with limits). We will prove that ρ'' is an equivalence by showing that the natural map $u : \{A_n \otimes_A B_m\}_{m \geqslant n} \to \{A_n \otimes_{A_m} B_m\}_{m \geqslant n}$ is an equivalence of Pro-objects of $\mathrm{CAlg}^{\mathrm{cn}}_A$. Note that u is a pushout of the natural map $u_0 : \{A_n \otimes_A A_m\}_{m \geqslant n} \to \{A_n\}_{m \geqslant n}$. It will therefore suffice to show that u_0 is an equivalence of Pro-objects: that is, that for every object $S \in \mathrm{CAlg}^{\mathrm{cn}}_A$, the induced map

$$\begin{aligned} \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_A}(A_n, S) \quad &\to \quad \varinjlim_{m \geqslant n} \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_A}(A_n \otimes_A A_m, S) \\ &\simeq \quad \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_A}(A_n, S) \times \varinjlim_{m \geqslant n} \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_A}(A_m, S) \end{aligned}$$

is a homotopy equivalence. This is clear: if S is not I-nilpotent, then both sides are empty; if S is I-nilpotent, then the direct limit $\varinjlim_{m \geqslant n} \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_A}(A_m, S)$ is contractible.

(h) Let A be a discrete $\mathbb{E}_\infty$-ring over R which is I-complete. Then the canonical map $A \to \varprojlim A/I^k A$ is surjective (Corollary SAG.7.3.6.2) whose kernel is a square-zero ideal $J \subseteq A$. Note that the inverse limit $\varprojlim A/I^k A$ is classically I-complete (since it is I-complete and I-adically separated; see Corollary SAG.7.3.6.3), and therefore good by virtue of (g). Applying (b), we conclude that A is good.

(i) Let A be an $\mathbb{E}_\infty$-algebra over R which is I-complete and k-truncated for some $k \gg 0$. Then A is good. This follows by induction on k: the case $k = 0$ follows from (h), and the inductive step follows from (b) (since each truncation $\tau_{\leqslant m} A$ is a square-zero extension of $\tau_{\leqslant m-1} A$ by virtue of Theorem HA.7.4.1.26).

(j) Let A be any I-complete $\mathbb{E}_\infty$-algebra over R. Combining (e) and (i), we conclude that A is good.

2.2 Construction of Identity Components

Our goal in this section is to prove Theorem 2.0.8. Using the results of §2.1, we can easily reduce to the following special case:

Theorem 2.2.1. *Let R be a commutative $\mathbf{F}_p$-algebra and let $\mathbf{G}$ be a p-divisible group over R. Then there exists an essentially unique formal group $\mathbf{G}^\circ \in \mathrm{FGroup}(R)$ with the following property:*

$(*)$ *Let $\mathcal{E} \subseteq \mathrm{CAlg}_R^{\mathrm{cn}}$ denote the full subcategory spanned by the truncated objects. Then the functor $\mathbf{G}^\circ|_{\mathcal{E}}$ is given by $A \mapsto \mathrm{fib}(\mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}}))$.*

Proof of Theorem 2.0.8 from Theorem 2.2.1. Let R be a (p)-complete $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a p-divisible group over R, and let $\mathcal{E} \subseteq \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}(R)}$ be the full subcategory spanned by those objects which are truncated and I-nilpotent. Define a functor $\mathbf{G}_0^\circ : \mathcal{E} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ by the formula $\mathbf{G}_0^\circ(A) = \mathrm{fib}(\mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}}))$. We wish to show that $\mathbf{G}_0^\circ$ admits an essentially unique extension $\mathbf{G}^\circ : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}(R)} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ which is a formal group over R. Replacing R by $\tau_{\geqslant 0}(R)$, we may assume that R is connective. It will then suffice to show that $\mathbf{G}_0^\circ$ satisfies conditions (1) and (2) of Theorem 2.1.1. To verify (1), we can replace R by $\pi_0(R)/(p)$, in which case the desired result follows from Theorem 2.2.1. For (2), it suffices to observe that the functor $\mathbf{G}$ is cohesive (since it is a filtered colimit of functors $\mathbf{G}[p^k]$, each of which is representable by a finite flat group scheme over R). □

The remainder of this section is devoted to the proof of Theorem 2.2.1. We proceed as in [27], which contains a proof of the analogous fact in the setting of classical algebraic geometry. Roughly speaking, the idea is to realize $\mathbf{G}^\circ$ as the union of finite flat group schemes $\mathbf{G}[F^n]$, where each $\mathbf{G}[F^n]$ denotes the kernel of the nth power of the Frobenius map on $\mathbf{G}$ (or equivalently in $\mathbf{G}[p^n]$). Here the kernel is taken in the *ordinary* category of group schemes over R. To guarantee that the direct limit $\varinjlim \mathbf{G}[F^n]$ defines the correct functor on all (truncated and connective) $\mathbb{E}_\infty$-algebras A over R, we will exploit the heuristic idea that the difference between A and $\pi_0(A)$ consists of "nilpotent" data, which can be annihilated by applying a sufficiently large power of the Frobenius map (see Proposition 2.2.3 for a precise statement).

2.2.1 The Relative Frobenius Map

Let R be a commutative ring of characteristic p and let $\varphi_R : R \to R$ denote the Frobenius homomorphism, given by the formula $\varphi_R(x) = x^p$. For every commutative R-algebra A and every integer $n \geqslant 0$, we let A^{1/p^n} denote the R-algebra obtained from A by restricting scalars along the map $\varphi_R^n : R \to R$. In other words, if A is a commutative ring equipped with an R-algebra structure $f : R \to A$, then A^{1/p^n} denotes the same commutative ring, equipped with the R-algebra structure given by the composition $R \xrightarrow{\varphi_R^n} R \xrightarrow{f} A$.

Notation 2.2.2. Let X be a functor from the category $\mathrm{CAlg}_R^\heartsuit$ of commutative R-algebras to some other category $\mathcal{C}$ (in practice, $\mathcal{C}$ will be either the category of sets or the category of abelian groups). For each $n \geqslant 0$, we let $X^{(p^n)} : \mathrm{CAlg}_R^\heartsuit \to \mathcal{C}$ denote the functor given by the formula $X^{(p^n)}(A) = X(A^{1/p^n})$. Note that for every commutative R-algebra A, the nth power of the Frobenius map φ_A can be regarded as an R-algebra homomorphism from A to A^{1/p^n}, which induces a map $X(A) \to X(A^{1/p^n}) = X^{(p^n)}(A)$. These maps are natural in A, and therefore define a natural transformation of functors $\varphi_{X/R}^n : X \to X^{(p^n)}$ which we will refer to as the *relative Frobenius map.*

Proposition 2.2.3. *Let R be a commutative ring of characteristic p and let X be a flat R-scheme which is a local complete intersection (relative to R). Let us regard each Frobenius pullback $X^{(p^n)}$ as a spectral scheme over R, which we identify with its functor of points $X^{(p^n)} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$. Let $X^{(p^\infty)}$ denote the direct limit of the sequence*

$$X \to X^{(p)} \to X^{(p^2)} \to \cdots ,$$

formed in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S})$, where the transition maps are given by the relative Frobenius of Notation 2.2.2. Then, for every truncated $\mathbb{E}_\infty$-algebra A over R, the canonical map $X^{(p^\infty)}(A) \to X^{(p^\infty)}(A^{\mathrm{red}})$ is a homotopy equivalence.

Remark 2.2.4. In the statement of Proposition 2.2.3, the hypothesis that X is a local complete intersection over R can be eliminated: it is only important that X is flat and of finite presentation over R.

Proof of Proposition 2.2.3. Let us say that a flat R-scheme X is *good* if the map $X^{(p^\infty)}(A) \to X^{(p^\infty)}(A^{\mathrm{red}})$ is a homotopy equivalence for every truncated object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$. Equivalently, X is good if, for every truncated object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$ and every point $\eta \in X^{(p^\infty)}(A^{\mathrm{red}})$, the homotopy fiber $X^{(p^\infty)}(A) \times_{X^{(p^\infty)}(A^{\mathrm{red}})} \{\eta\}$ is contractible. This assertion is local with respect to the Zariski topology on $\mathrm{Spec}(A)$. Consequently, if we are given some open cover $\{U_\alpha\}$ of X where each U_α is good, then X is also good. We may therefore assume without loss of generality that X fits into a pullback diagram of affine R-schemes σ :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbf{A}_R^n \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & \mathbf{A}_R^m. \end{tikzcd}$$

where the vertical maps are flat. In this case, σ is also a pullback diagram of spectral schemes over R, so that the diagram of functors

$$\begin{tikzcd} X^{(p^\infty)} \arrow[r] \arrow[d] & (\mathbf{A}_R^n)^{(p^\infty)} \arrow[d] \\ (\mathrm{Spec}(R))^{(p^\infty)} \arrow[r] & (\mathbf{A}_R^m)^{(p^\infty)} \end{tikzcd}$$

is also a pullback square. Consequently, to show that X is good, it will suffice to show that every affine space $\mathbf{A}_R^k$ is good. Writing $\mathbf{A}_R^k$ as a product of affine lines, we are reduced to showing that $\mathbf{A}_R^1$ is good. Passing to an open cover again, we can reduce to showing that the punctured affine line $X = \mathrm{Spec}(R[t^{\pm 1}])$ is good. Note that for each $n \geqslant 0$, we can identify $X^{(p^n)}$ with the spectrum of the subalgebra $R[t^{\pm p^n}] \subseteq R[t^{\pm 1}]$.

Let A be a truncated $\mathbb{E}_\infty$-algebra over R and let I denote the fiber of the map $A \to A^{\mathrm{red}}$. We wish to show that the map

$$\rho : \varinjlim_n \mathrm{Map}_{\mathrm{CAlg}_R}(R[t^{\pm p^n}], A) \to \varinjlim_n \mathrm{Map}_{\mathrm{CAlg}_R}(R[t^{\pm p^n}], A^{\mathrm{red}})$$

is a homotopy equivalence. Let $\mathrm{gl}_1(A)$ denote the (connective) spectrum of units of A (see §1.6.3) and let $\mathrm{gl}_1(I)$ denote the fiber of the map $\mathrm{gl}_1(A) \to \mathrm{gl}_1(A^{\mathrm{red}})$. Unwinding the definitions, we see that ρ fits into a fiber sequence

$$\begin{array}{c} \varinjlim \mathrm{Map}_{\mathrm{Sp}}(p^n\,\mathbf{Z}, \mathrm{gl}_1(A)) \\ \downarrow{\scriptstyle\rho} \\ \varinjlim \mathrm{Map}_{\mathrm{Sp}}(p^n\,\mathbf{Z}, \mathrm{gl}_1(A^{\mathrm{red}})) \\ \downarrow \\ \varinjlim \mathrm{Map}_{\mathrm{Sp}}(p^n\,\mathbf{Z}, \Sigma^{-1}\,\mathrm{gl}_1(I)). \end{array}$$

It will therefore suffice to show that the colimit of the diagram

$$\mathrm{Map}_{\mathrm{Sp}}(\mathbf{Z}, \Sigma^{-1}\,\mathrm{gl}_1(I)) \xrightarrow{p} \mathrm{Map}_{\mathrm{Sp}}(\mathbf{Z}, \Sigma^{-1}\,\mathrm{gl}_1(I)) \xrightarrow{p} \mathrm{Map}_{\mathrm{Sp}}(\mathbf{Z}, \Sigma^{-1}\,\mathrm{gl}_1(I)) \to \cdots$$

is contractible. Since $\mathbf{Z}$ is almost perfect as a module over the sphere spectrum and $\Sigma^{-1}\,\mathrm{gl}_1(I)$ is truncated, we can identify this colimit with $\mathrm{Map}_{\mathrm{Sp}}(\mathbf{Z}, \Sigma^{-1}\,\mathrm{gl}_1(I)[p^{-1}])$. To complete the proof, it will suffice to show that $\mathrm{gl}_1(I)[p^{-1}] \simeq 0$: that is, that the action of p is locally nilpotent on each homotopy group $\pi_s\,\mathrm{gl}_1(I)$. For $s > 0$, this is obvious: the abelian group $\pi_s\,\mathrm{gl}_1(I) \simeq \pi_s A$ is annihilated by p. For $s = 0$, we can identify $\pi_s\,\mathrm{gl}_1(A)$ with the group of units in the commutative ring $\pi_0(A)$ which have the form $1 + \epsilon$, where ϵ is nilpotent. The desired result now follows from the observation that $(1+\epsilon)^{p^n} = 1 + \epsilon^{p^n} = 1$ for $n \gg 0$. $\square$

2.2.2 p-Divisible Groups in Characteristic p

In this section, we review some basic facts concerning p-divisible groups defined over commutative rings R of characteristic p which will be needed for the proof of Theorem 2.0.8. We essentially follow the presentation of Messing ([27]), with some minor deviations.

Notation 2.2.5. Let $\mathbf{G}$ be a p-divisible group over R, which we regard as a functor from $\mathrm{CAlg}_R^\heartsuit$ to the category Ab of abelian groups. For each integer $n \geqslant 0$, we let $\mathbf{G}[F^n]$ denote the kernel of the relative Frobenius map $\varphi^n_{\mathbf{G}/R} : \mathbf{G} \to \mathbf{G}^{(p^n)}$. We regard $\mathbf{G}[F^n]$ as a functor from the category $\mathrm{CAlg}_R^\heartsuit$ of commutative R-algebras to the category of abelian groups.

Proposition 2.2.6. *Let R be a commutative ring of characteristic p and let $\mathbf{G}$ be a p-divisible group over R. Then, for each $n \geqslant 0$, the functor $\mathbf{G}[F^n] : \mathrm{CAlg}_R^\heartsuit \to \mathrm{Ab}$ is (representable by) a finite flat group scheme over R.*

Proof. Let $V^n : \mathbf{G}^{(p^n)} \to \mathbf{G}$ be the n-fold Verschiebung map and let $\mathbf{G}^{(p^n)}[V^n]$ denote its kernel. Note that the compositions

$$\mathbf{G}^{(p^n)} \xrightarrow{V^n} \mathbf{G} \xrightarrow{\varphi^n_{\mathbf{G}/R}} \mathbf{G}^{(p^n)} \qquad \mathbf{G} \xrightarrow{\varphi^n_{\mathbf{G}/R}} \mathbf{G}^{(p^n)} \xrightarrow{V^n} \mathbf{G}$$

are given by multiplication by p^n on $\mathbf{G}$ and $\mathbf{G}^{(p^n)}$, respectively. It follows that we can identify $\mathbf{G}[F^n]$ and $\mathbf{G}^{(p^n)}[V^n]$ with the kernels of the maps $\varphi^n_{\mathbf{G}[p^n]/R} : \mathbf{G}[p^n] \to \mathbf{G}^{(p^n)}[p^n]$ and $V^n : \mathbf{G}^{(p^n)}[p^n] \to \mathbf{G}[p^n]$, respectively. Both of these maps are homomorphisms of finite flat group schemes over R. It follows that $\mathbf{G}[F^n]$ and $\mathbf{G}^{(p^n)}[V^n]$ are (representable by) group schemes which are finite and of finite presentation over R: that is, we have isomorphisms

$$\mathbf{G}[F^n] \simeq \mathrm{Spec}(A) \qquad \mathbf{G}^{(p^n)}[V^n] \simeq \mathrm{Spec}(B)$$

where $A, B \in \mathrm{CAlg}^{\heartsuit}_R$ are finitely generated as R-modules.

We have an exact sequence of functors

$$0 \to \mathbf{G}[F^n] \to \mathbf{G}[p^n] \xrightarrow{f} \mathbf{G}^{(p^n)}[V^n],$$

where f is induced by the relative Frobenius map $\varphi^n_{\mathbf{G}/R}$. Since $\mathbf{G}^{(p^n)}$ is a p-divisible group over R, the map $p^n : \mathbf{G}^{(p^n)} \to \mathbf{G}^{(p^n)}$ is an epimorphism of sheaves for the fppf topology, so that the relative Frobenius map $\varphi^n_{\mathbf{G}/R} : \mathbf{G} \to \mathbf{G}^{(p^n)}$ is also an epimorphism of fppf sheaves. It follows that f is an epimorphism of fppf sheaves. We can therefore choose a faithfully flat map $B \to B'$ for which the induced map $\mathrm{Spec}(B') \to \mathrm{Spec}(B) \simeq \mathbf{G}^{(p^n)}[V^n]$ factors through f. It follows that the fiber product $\mathrm{Spec}(B') \times_{\mathrm{Spec}(B)} \mathbf{G}[p^n]$ splits as a product $\mathrm{Spec}(B') \times_{\mathrm{Spec}(R)} \mathbf{G}[F^n]$ (here all products are formed in the category of schemes). Write $\mathrm{Spec}(B') \times_{\mathrm{Spec}(R)} \mathbf{G}[F^n] = \mathrm{Spec}(C)$, where $C = \mathrm{Tor}^R_0(A, B')$. Since B' is flat over B and $\mathbf{G}[p^n]$ is flat over R, we conclude that C is flat over R.

The zero section of $\mathbf{G}^{(p^n)}[V^n]$ determines an R-algebra homomorphism $\epsilon : B \to R$. Set $R' = B' \otimes_R R$, so that R' is faithfully flat over R. Note that the tautological map

$$A \otimes_R R' \to C \otimes_R R' \simeq \mathrm{Tor}^R_0(A, B' \otimes_R R')$$

admits a left inverse (given by the muliplication map $B' \otimes_R R' \to R'$). It follows that $A \otimes_R R'$ is a retract of $C \otimes_R R'$, and is therefore flat over R'. Since R' is faithfully flat over R, it follows that A is flat over R: that is, $\mathbf{G}[F^n]$ is a finite flat group scheme, as desired. □

Variant 2.2.7. Let R be a commutative ring of characteristic p and let $\mathbf{G}$ be a p-divisible group over R. Then, for each $n \geqslant 0$, the group scheme $\mathbf{G}^{(p^n)}[V^n]$ appearing in the proof of Proposition 2.2.6 is also finite flat over R. This can be seen by a slight modification of the proof of Proposition 2.2.6, or alternatively by applying Proposition 2.2.6 to the Cartier dual of $\mathbf{G}$.

Lemma 2.2.8. *Let κ be a field of characteristic p, let G be a finite flat group scheme over κ, and suppose that the relative Frobenius map $\varphi_{G/\kappa} : G \to G^{(p)}$ vanishes. Then we can write $G = \operatorname{Spec}(\kappa[x_1, \ldots, x_d]/(x_1^p, \ldots, x_d^p))$ for some $n \geqslant 0$.*

Proof. Write $G = \operatorname{Spec}(A)$, where A is a cocommutative Hopf algebra over κ. The zero section of G determines an augmentation $\epsilon : A \to \kappa$, whose kernel is an ideal $I \subseteq A$. Choose a collection of elements $y_1, \ldots, y_d \in I$ whose images form a basis for I/I^2 as a vector space over κ. The vanishing of $\varphi_{G/\kappa}$ guarantees that $y_i^p = 0$ for $1 \leqslant i \leqslant n$, so that we obtain a κ-algebra homomorphism $\alpha : \kappa[x_1, \ldots, x_d]/(x_1^p, \ldots, x_d^p) \xrightarrow{\alpha} A$ given by $\alpha(x_i) = y_i$. It follows easily by induction that the composite map

$$\kappa[x_1, \ldots, x_d]/(x_1^p, \ldots, x_d^p) \xrightarrow{\alpha} A \to A/I^k$$

is surjective for each $k \geqslant 0$. The vanishing of $\varphi_{G/\kappa}$ guarantees that G is connected: that is, the ideal I is nilpotent. It follows that α is surjective. We will complete the proof by showing that α is injective. Suppose otherwise: then we can choose some nonzero polynomial $f(x_1, \ldots, x_d)$, having degree $< p$ in each x_i, such that $f(y_1, \ldots, y_d) = 0$ in A. Write f as a sum $\sum_{n=0}^{(p-1)d} f_n(x_1, \ldots, x_d)$, where each $f_n(x_1, \ldots, x_d)$ is homogeneous of degree n. Then there exists some smallest integer k such that $f_k \neq 0$. Let us assume that f has been chosen such that k is as small as possible. Note that the equation $f(y_1, \ldots, y_d) = 0$ guarantees that $k > 0$.

Let $\Delta : A \to A \otimes_\kappa A$ be the comultiplication on A. The relations $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$ guarantee that we have $\Delta(y_i) \equiv y_i \otimes 1 + 1 \otimes y_i \pmod{I \otimes I}$. It follows that

$$f_k(y_1 \otimes 1 + 1 \otimes y_1, \ldots, y_d \otimes 1 + 1 \otimes y_d) \equiv \Delta(f(y_1, \ldots, y_k)) \equiv 0 \pmod{J},$$

where J is the ideal of $A \otimes_\kappa A$ generated by $I^a \otimes I^b$ for $a + b > k$. In particular, we have

$$\sum_{i=1}^{d} y_i \otimes \frac{\partial f_k(x_1, \ldots, x_d)}{\partial x_i}(y_1, \ldots, y_k) \equiv 0 \quad (\text{mod } A \otimes I^{k+1} + I \otimes I^k + I^2 \otimes A.)$$

Since the elements $1, y_1, \ldots, y_d$ form a basis for A/I^2, it follows that each expression $\frac{\partial f_k(x_1,\ldots,x_d)}{\partial x_i}(y_1, \ldots, y_k)$ belongs to I^k: that is, it can be written as a sum of homogeneous polynomials of degree $\geqslant k$ in the y_i. Invoking the minimality of k, we conclude that the expression $\frac{\partial f_k(x_1,\ldots,x_d)}{\partial x_i}$ vanishes for each i. Writing $f_k(x_1, \ldots, x_d)$ as a linear combination of monomials $c_{e_1,\ldots,e_d} x_1^{e_1} \cdots x_d^{e_d}$, we conclude that each e_i is divisible by p. Since f has degree $< p$ in each x_i, we must have $e_1 = e_2 = \cdots = e_d = 0$, contradicting the inequality $e_1 + \cdots + e_d = k > 0$. □

Lemma 2.2.9. *Let R be a commutative ring of characteristic p, let $\mathbf{G}$ be a p-divisible group over R, and let $\mathfrak{p} \subseteq R$ be a prime ideal. Then there exists an element $t \in R - \mathfrak{p}$ for which the group scheme $\mathbf{G}[F] \times_{\operatorname{Spec}(R)} \operatorname{Spec}(R[t^{-1}])$ is isomorphic, as an $R[t^{-1}]$-scheme, to the spectrum of a truncated polynomial ring $R[t^{-1}, x_1, \ldots, x_d]/(x_1^p, \cdots, x_d^p)$. Moreover, we can further assume that the zero section of $\mathbf{G}[F] \times_{\operatorname{Spec}(R)} \operatorname{Spec}(R[t^{-1}])$ is given by the map*

$$R[t^{-1}, x_1, \ldots, x_d]/(x_1^p, \cdots, x_d^p) \to R[t^{-1}] \qquad x_i \mapsto 0.$$

Proof. By virtue of Proposition 2.2.6, we can write $\mathbf{G}[F] = \operatorname{Spec}(A)$, where A is a finite flat R-algebra. The zero section of $\mathbf{G}$ determines an R-algebra homomorphism $\epsilon : A \to R$ with kernel $I \subseteq A$. Let $\kappa = \kappa(\mathfrak{p})$ denote the residue field of R at the prime ideal $\mathfrak{p}$, let $A_\kappa = \kappa \otimes_R A$ denote the associated fiber of A, and let $I_\kappa = \kappa \otimes_R I$ denote the augmentation ideal in A_κ. Using Lemma 2.2.8, we deduce that there exist elements $\overline{x}_1, \ldots, \overline{x}_d \in I_\kappa$ which induce an isomorphism $\rho_0 : \kappa[\overline{x}_1, \ldots, \overline{x}_d]/(\overline{x}_1^p, \ldots \overline{x}_d^p) \to A_\kappa$. Replacing R by a localization if necessary, we may assume that each $\overline{x}_i$ can be lifted to an element $x_i \in I$. Since the relative Frobenius map $\varphi_{\mathbf{G}/R}$ vanishes on $\mathbf{G}[F]$, each x_i satisfies $x_i^p = 0$, so we can lift ρ_0 to an R-algebra homomorphism $\rho : R[x_1, \ldots, x_d]/(x_1^p, \ldots, x_d^p) \to A$. Then ρ is a map between projective R-modules of finite rank which induces an isomorphism after tensoring with the field $\kappa = \kappa(\mathfrak{p})$. It follows that there exists an element $t \notin \mathfrak{p}$ such that the induced map $R[t^{-1}][x_1, \ldots, x_d]/(x_1^p, \ldots, x_d^p) \to A[t^{-1}]$ is an isomorphism. □

In the situation of Lemma 2.2.9, we have an analogous description of the group schemes $\mathbf{G}[F^n]$ for all $n \geqslant 0$:

Lemma 2.2.10. *Let R be a commutative ring of characteristic p, let $\mathbf{G}$ be a p-divisible group over R, and let $n \geqslant 2$ be an integer. Suppose that there exists an isomorphism of R-schemes*

$$\alpha : \mathbf{G}[F^{n-1}] \simeq \operatorname{Spec}(R[x_1, \ldots, x_d]/(x_1^{p^{n-1}}, \ldots, x_d^{p^{n-1}}))$$

which carries the zero section of $\mathbf{G}[F^{n-1}]$ *to the map*

$$\operatorname{Spec}(R) \to \operatorname{Spec}(R[x_1, \ldots, x_d]/(x_1^{p^{n-1}}, \ldots, x_d^{p^{n-1}})) \qquad x_i \mapsto 0.$$

Then α *extends to an isomorphism* $\mathbf{G}[F^n] \simeq \operatorname{Spec} R[x_1, \ldots, x_d]/(x_1^{p^n}, \ldots, x_d^{p^n})$.

Proof. Note that α restricts to an isomorphism of R-schemes

$$\mathbf{G}[F] \simeq \operatorname{Spec}(R[x_1, \ldots, x_d]/(x_1^p, \ldots, x_d^p)).$$

It follows that $\mathbf{G}[F]$ is a finite flat group scheme of degree p^d, and that $\mathbf{G}[F^{n-1}]$ is a finite flat group scheme of degree $p^{(n-1)d}$. We have a short exact sequence of fppf sheaves

$$0 \to \mathbf{G}[F^{n-1}] \to \mathbf{G}[F^n] \to \mathbf{G}[F]^{(p^{n-1})} \to 0,$$

which proves that $\mathbf{G}[F^n]$ is a finite flat group scheme of degree p^{nd}. Write $\mathbf{G}[F^n] = \operatorname{Spec}(A)$, and let $I \subseteq A$ be the kernel of the augmentation $A \to R$ given by the zero section of $\mathbf{G}[F^n]$. Note that the inclusion $\mathbf{G}[F^{n-1}] \hookrightarrow \mathbf{G}[F^n]$ is a closed immersion of affine schemes, so that each pullback $\alpha^*(x_i)$ can be lifted to an element $y_i \in I$. Since $\mathbf{G}[F^n]$ is annihilated by $\varphi^n_{\mathbf{G}/R}$, we have $y_i^{p^n} = 0$, so that the elements $\{y_i\}_{1 \leqslant i \leqslant d}$ determine a map of R-algebras

$$\rho : R[x_1, \ldots, x_d]/(x_1^{p^n}, \ldots, x_d^{p^n}) \to A \qquad x_i \mapsto y_i.$$

Note that the spectrum $\operatorname{Spec}(A/I^2)$ is contained in $\mathbf{G}[F^{n-1}] \subseteq \mathbf{G}[F^n]$, so the composite map

$$R[x_1, \ldots, x_d]/(x_1^{p^n}, \ldots, x_d^{p^n}) \xrightarrow{\rho} A \to A/I^2$$

is surjective. Since the ideal I is nilpotent, it follows that ρ is surjective. Because the domain and codomain of ρ are projective modules of rank p^{nd} over R, it follows that ρ is an isomorphism. $\square$

2.2.3 The Proof of Theorem 2.2.1

Let R be a commutative ring of characteristic p and let $\mathbf{G}$ be a p-divisible group over R. To prove Theorem 2.2.1, we must show that there exists a formal group $\mathbf{G}^\circ$ over R having the property that for every truncated object $A \in \operatorname{CAlg}^{\mathrm{cn}}_R$, we have a canonical equivalence $\mathbf{G}^\circ(A) \simeq \operatorname{fib}(\mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}}))$.

For each $n \geqslant 0$, let $\mathbf{G}[F^n]$ be defined as in Notation 2.2.5. Then $\mathbf{G}[F^n]$ is a finite flat group scheme over R (Proposition 2.2.6), which (by a slight abuse of notation) we will identify with the corresponding functor $\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$. Define $\mathbf{G}_0' : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ by the formula $\mathbf{G}_0(A) = \varinjlim(\mathbf{G}[F^n])(A)$. We will complete the proof by verifying the following pair of assertions:

(a) For each $A \in \mathcal{E}$, we have a fiber sequence $\mathbf{G}_0(A) \to \mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}})$.

(b) The functor $\mathbf{G}_0|_{\mathcal{E}}$ can be extended to a formal group over R.

We first prove (b). By virtue of Proposition 2.1.10, (b) is equivalent to the assertion that the functor $(\Omega^\infty \circ \mathbf{G}_0)|_{\mathcal{E}} : \mathcal{E} \to \mathcal{S}_*$ can be extended to a (pointed) formal hyperplane over R. This assertion is local with respect to the Zariski topology on $\mathrm{Spec}(R)$. Using Lemma 2.2.9, we can assume that there exists an isomorphism of R-schemes $\alpha_1 : \mathbf{G}[F] \simeq \mathrm{Spec}(R[x_1, \ldots, x_d]/(x_1^p, \ldots, x_d^p))$ which carries each x_i to a regular function on $\mathbf{G}[F]$ which vanishes along the zero section (Lemma 2.2.9). Using Lemma 2.2.10, we can extend α_1 to a compatible sequence of isomorphisms

$$\alpha_n : \mathbf{G}[F^n] \simeq \mathrm{Spec}(R[x_1, \ldots, x_d]/(x_1^{p^n}, \ldots, x_d^{p^n})).$$

Let C_n denote the R-linear dual of $R[x_1, \ldots, x_d]/(x_1^{p^n}, \ldots, x_d^{p^n})$, regarded as a commutative coalgebra over R. Note that the composite functor

$$\mathrm{CAlg}_R^{\mathrm{cn}} \xrightarrow{\mathbf{G}[F^n]} \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}} \xrightarrow{\Omega^\infty} \mathcal{S}$$

can be identified with the cospectrum $\mathrm{cSpec}(C_n)$. Consequently, if $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$ is truncated, then Proposition 1.2.14 supplies a homotopy equivalence

$$\begin{aligned}
\Omega^\infty \mathbf{G}_0(A) &\simeq \varinjlim_n \Omega^\infty \mathbf{G}[F^n](A) \\
&\simeq \varinjlim_n \mathrm{cSpec}(C_n)(A) \\
&\simeq \varinjlim_n \mathrm{Map}_{\mathrm{cCAlg}_A}(A, C_n \otimes_R A) \\
&\simeq \mathrm{Map}_{\mathrm{cCAlg}_A}(A, (\varinjlim C_n) \otimes_R A) \\
&\simeq \mathrm{cSpec}(C)(A),
\end{aligned}$$

where C denotes the colimit $\varinjlim C_n$ (formed in the category of flat coalgebras over R). We now observe that C is a smooth coalgebra over R (isomorphic to the divided power coalgebra $\Gamma_R^*(R^d)$), so that $\mathrm{cSpec}(C) : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is a formal hyperplane which extends the functor $(\Omega^\infty \circ \mathbf{G}_0)|_{\mathcal{E}} : \mathcal{E} \to \mathcal{S}$.

We now prove (a). For each $n \geqslant 0$, the proof of Proposition 2.2.6 supplies a short exact sequence of finite flat group schemes

$$0 \to \mathbf{G}[F^n] \to \mathbf{G}[p^n] \to \mathbf{G}^{(p^n)}[V^n] \to 0$$

(see Variant 2.2.7). These exact sequences are compatible n varies: more precisely, we have commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}[F^n] \arrow[r] \arrow[d] & \mathbf{G}[p^n] \arrow[r] \arrow[d] & \mathbf{G}^{(p^n)}[V^n] \arrow[r] \arrow[d, "f(n)"] & 0 \\
0 \arrow[r] & \mathbf{G}[F^{n+1}] \arrow[r] & \mathbf{G}[p^{n+1}] \arrow[r] & \mathbf{G}^{(p^{n+1})}[V^{n+1}] \arrow[r] & 0,
\end{tikzcd}$$

where $f(n)$ denotes the restriction of the relative Frobenius map $\mathbf{G}^{(p^n)} \to \mathbf{G}^{(p^{n+1})}$. Let H denote the direct limit of the sequence

$$\mathbf{G}^{(p)}[V] \xrightarrow{f(1)} \mathbf{G}^{(p^2)}[V^2] \xrightarrow{f(2)} \mathbf{G}^{(p^3)}[V^3] \to \cdots,$$

formed in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}})$. Then the exact sequences above yield a fiber sequence $\mathbf{G}_0 \to \mathbf{G} \to H$ in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}})$. Note that for any $A \in \mathrm{CAlg}^{\mathrm{cn}}_R$, the group $\mathbf{G}_0(A^{\mathrm{red}})$ vanishes; it follows that the map $\mathbf{G}(A^{\mathrm{red}}) \to H(A^{\mathrm{red}})$ is a monomorphism of abelian groups. Consequently, to prove (a), it will suffice to show the following:

- (c) Let A be a truncated $\mathbb{E}_\infty$-algebra over R. Then the canonical map $H(A) \to H(A^{\mathrm{red}})$ is an equivalence in $\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$.

Note that each of the maps $f(n) : \mathbf{G}^{(p^n)}[V^n] \to \mathbf{G}^{(p^{n+1})}[V^{n+1}]$ factors as a composition

$$\mathbf{G}^{(p^n)}[V^n] \xrightarrow{f'(n)} \mathbf{G}^{(p^{n+1})}[V^n] \hookrightarrow \mathbf{G}^{(p^{n+1})}[V^{n+1}],$$

where $f'(n)$ is the relative Frobenius map associated to the finite flat group scheme $\mathbf{G}^{(p^n)}[V^n]$. We can therefore write H as a filtered colimit $\varinjlim H_n$, where each H_n is defined as the colimit of the diagram of relative Frobenius maps

$$\mathbf{G}[V^n]^{(p^n)} \to \mathbf{G}[V^n]^{(p^{n+1})} \to \mathbf{G}[V^n]^{(p^{n+2})} \to \cdots.$$

It will therefore suffice to show that for every truncated $\mathbb{E}_\infty$-algebra A over R, the map $H_n(A) \to H_n(A^{\mathrm{red}})$ is an equivalence in $\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$, or equivalently that the map $(\Omega^\infty \circ H_n)(A) \to (\Omega^\infty \circ H_n)(A^{\mathrm{red}})$ is a homotopy equivalence of spaces. This is a special case of Proposition 2.2.3.

2.2.4 Example: The Identity Component of μ_{p^∞}

Let R be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}_m(R)$ be the strict multiplicative group of R (Construction 1.6.10). For every positive integer n, we let $\mu_n(R)$ denote the fiber of the map $n : \mathbf{G}_m(R) \to \mathbf{G}_m(R)$, formed in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ of connective $\mathbf{Z}$-module spectra. The functors $\{\mu_{p^k}\}_{k \geqslant 0}$ can be assembled into a p-divisible group:

Proposition 2.2.11. *Let $\mu_{p^\infty} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ denote the functor given by the formula*

$$\mu_{p^\infty}(R) = \mathrm{fib}(\mathbf{G}_m(R) \to \mathbf{G}_m(R)[1/p]).$$

Then μ_{p^∞} is (representable by) a p-divisible group over the sphere spectrum S.

We will refer to the p-divisible group μ_{p^∞} as the *multiplicative p-divisible group* over S.

Proof of Proposition 2.2.11. By construction, the $\mathbf{Z}$-module $\mu_{p^\infty}(R)$ is (p)-nilpotent for each R. For every finite abelian p-group M, the functor

$$R \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mu_{p^\infty}(R)) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}_m(R)) \simeq \mathrm{Map}_{\mathrm{Sp}}(M, \mathrm{GL}_1(R))$$

is representable by the suspension spectrum $\Sigma^\infty_+(M)$ (regarded as a connective $\mathbb{E}_\infty$-ring). We complete the proof by observing that if $u : M \to N$ is a monomorphism of finite abelian groups, then the induced map $\Sigma^\infty_+(M) \to \Sigma^\infty_+(N)$ is finite flat: in fact, $\Sigma^\infty_+(N)$ is a free module of finite rank over $\Sigma^\infty_+(M)$, with a basis given by any collection of coset representatives for M in N. □

Since the p-divisible group μ_{p^∞} is defined over the sphere spectrum, it determines a p-divisible group over every $\mathbb{E}_\infty$-ring R. We will abuse notation by denoting this p-divisible group also by μ_{p^∞}.

Proposition 2.2.12. *Let R be a (p)-complete $\mathbb{E}_\infty$-ring, and let us regard the p-divisible group μ_{p^∞} of Proposition 2.2.11 and the formal group $\widehat{\mathbf{G}}_m$ of Construction 1.6.16 as defined over R. Then there is a canonical equivalence of formal groups $\mu^\circ_{p^\infty} \simeq \widehat{\mathbf{G}}_m$*

Proof. Without loss of generality, we can assume that R is connective (in fact, it suffices to treat the universal case where $R = S^\wedge_{(p)}$ is the (p)-completed sphere spectrum). For every connective R-algebra A, we have a commutative diagram

$$\begin{tikzcd}
\widehat{\mathbf{G}}_m[p^\infty](A) \arrow[r] \arrow[d] & \mu_{p^\infty}(A) \arrow[r] \arrow[d] & \mu_{p^\infty}(A^{\mathrm{red}}) \arrow[d] \\
\widehat{\mathbf{G}}_m(A) \arrow[r] \arrow[d] & \mathbf{G}_m(A) \arrow[r] \arrow[d] & \mathbf{G}_m(A^{\mathrm{red}}) \arrow[d] \\
\widehat{\mathbf{G}}_m(A)[p^{-1}] \arrow[r] & \mathbf{G}_m(A)[p^{-1}] \arrow[r] & \mathbf{G}_m(A^{\mathrm{red}})[p^{-1}]
\end{tikzcd}$$

in which the rows and columns are fiber sequences. Let $\mathcal{E} \subseteq \mathrm{CAlg}_R^{\mathrm{cn}}$ be the full subcategory spanned by the connective R-algebras which are truncated and (p)-nilpotent. For $A \in \mathcal{E}$, the $\mathbf{Z}$-module spectrum $\widehat{\mathbf{G}}_m(A)[p^{-1}]$ vanishes (Lemma 2.3.24), so this diagram supplies an identification

$$\begin{aligned}
\widehat{\mathbf{G}}_m(A) &\xleftarrow{\sim} \widehat{\mathbf{G}}_m[p^\infty](A) \\
&\simeq \mathrm{fib}(\mu_{p^\infty}(A) \to \mu_{p^\infty}(A^{\mathrm{red}})) \\
&= \mu^\circ_{p^\infty}(A)
\end{aligned}$$

depending functorially on A. The desired result now follows from Theorem 2.1.1. □

2.3 Connected p-Divisible Groups

Let R be a (p)-complete $\mathbb{E}_\infty$-ring. The construction $\mathbf{G} \mapsto \mathbf{G}^\circ$ determines a functor from the ∞-category of p-divisible groups over R. In this section, we consider the following:

Question 2.3.1. Let $\widehat{\mathbf{G}}$ be a formal group over R. When can we find a p-divisible group $\mathbf{G}$ having $\widehat{\mathbf{G}}$ as its identity component?

In the case where R is a complete local Noetherian ring, Question 2.3.1 was studied by Tate in [35], who proved the following:

Theorem 2.3.2 (Tate)**.** *Let R be a complete local Noetherian ring whose residue field κ has characteristic p. Then the construction $\mathbf{G} \mapsto \mathbf{G}^\circ$ determines an equivalence between the following categories:*

(a) The category of p-divisible groups $\mathbf{G}$ over R for which the special fiber $\mathbf{G}_\kappa$ is connected (see Definition 2.3.5).

(*b*) *The category of formal groups $\widehat{\mathbf{G}}$ over R for which the map $[p] : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$ determines a finite flat map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to \mathscr{O}_{\widehat{\mathbf{G}}}$*

Remark 2.3.3. In [35], the formulation of Theorem 2.3.2 is slightly different. In particular, it is phrased in terms of the *inverse* of the identity component functor $\mathbf{G} \mapsto \mathbf{G}^\circ$ (which is defined only on those formal groups as in (*b*)).

Warning 2.3.4. In [35], Tate uses the term *connected* to refer to a p-divisible group $\mathbf{G}$ satisfying the requirement described in part (*a*) of Theorem 2.3.2. We will instead use the term *formally connected* (to emphasize the role of the formal scheme $\operatorname{Spf}(R)$), and reserve the term *connected* for a stronger condition on p-divisible groups (see Definition 2.3.5).

The purpose of this section is to establish a generalization of the equivalence of Theorem 2.3.2. Let R be a complete adic $\mathbb{E}_\infty$-ring, and suppose that p is a topologically nilpotent element of $\pi_0(R)$. Our first goal will be to introduce the notion of a *formally connected* p-divisible over R (Definition 2.3.10). We then show that the construction $\mathbf{G} \mapsto \mathbf{G}^\circ$ is fully faithful when restricted to formally connected p-divisible groups (Corollary 2.3.13). We will say that a formal group $\widehat{\mathbf{G}}$ over R is a *p-divisible formal group* if it arises as the identity component of a formally connected p-divisible group (Definition 2.3.14). We then show that the class of p-divisible formal groups admit a characterization similar to part (*b*) of Theorem 2.3.2 (see Theorems 2.3.20 and 2.3.26).

2.3.1 Connectedness and Formal Connectedness

We begin by introducing some terminology.

Definition 2.3.5. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R, so that the functor $(\Omega^\infty \circ \mathbf{G}[p]) : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ is corepresentable by a finite flat R-algebra A. We say that $\mathbf{G}$ is *connected* if the underlying map of topological spaces $|\operatorname{Spec}(A)| \to |\operatorname{Spec}(R)|$ bijective.

More generally, if $\mathbf{G}$ is a p-divisible group over an arbitrary $\mathbb{E}_\infty$-ring R, we will say that $\mathbf{G}$ is connected if it is connected when viewed as a p-divisible group over the connective cover $\tau_{\geqslant 0}(R)$.

Remark 2.3.6. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. Then $\mathbf{G}$ is connected if and only if, for each residue field κ of the commutative ring $\pi_0(R)$, the fiber $\mathbf{G}_\kappa$ is a connected p-divisible group over κ.

Remark 2.3.7. Let $\mathbf{G}$ be a p-divisible group of height $h > 0$ over an $\mathbb{E}_\infty$-ring R. If $\mathbf{G}$ is connected, then every residue field of the commutative ring $\pi_0(R)$ must have characteristic p. Consequently, the prime number p must be nilpotent in $\pi_0(R)$.

Remark 2.3.8. In the situation of Definition 2.3.5, the zero section of $\mathbf{G}$ determines an augmentation $\epsilon : A \to R$. Note that $\mathbf{G}$ is connected if and only if the closed embedding $|\operatorname{Spec}(R)| \hookrightarrow |\operatorname{Spec}(A)|$ determined by ϵ is a homeomorphism: that is, if and only if the kernel ideal $I = \ker(\pi_0(A) \to \pi_0(R))$ is locally nilpotent.

Proposition 2.3.9. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. The following conditions are equivalent:*

(a) *The p-divisible group $\mathbf{G}$ is connected.*

(b) *Let B be a reduced commutative algebra over $\pi_0(R)$. Then $\mathbf{G}(A) \simeq 0$.*

Proof. Suppose first that (a) is satisfied, and let B be a reduced commutative algebra over $\pi_0(R)$; we wish to show that $\mathbf{G}(B) = 0$. Replacing R by B, we can assume that $R = B$ is a reduced commutative ring. Suppose that there exists a nonzero element $x \in \mathbf{G}(R)$. Multiplying x by a suitable power of p, we can assume that $px = 0$; that is, x can be regarded as a nonzero R-valued point of $\mathbf{G}[p]$. Let A be as in Definition 2.3.5, so that x determines an R-algebra map $\rho : A \to R$. Let $\epsilon : A \to R$ be the augmentation of Remark 2.3.8. Assumption (a) guarantees that $\ker(\epsilon)$ is locally nilpotent. Since R is reduced, it follows that $\rho(\ker(\epsilon)) = 0$ so that ρ factors through ϵ, contradicting our assumption that $x \neq 0$.

Now suppose that (b) is satisfied. Let A be as above, and set $B = A^{\mathrm{red}}$. Then the canonical map $\rho : A \to B$ determines a p-torsion element $x \in \mathbf{G}(B)$. Assumption (b) guarantees that $x = 0$, so that ρ factors through the map $\epsilon : A \to R$. In particular, every element of the ideal $I = \ker(\pi_0(A) \xrightarrow{\rho} \pi_0(R))$ vanishes in B and is therefore nilpotent. It follows from Remark 2.3.8 that $\mathbf{G}$ is connected. $\square$

For many applications, Definition 2.3.5 is overly restrictive: note that it essentially requires the the $\mathbb{E}_\infty$-ring to be (p)-nilpotent (Remark 2.3.7). It will be therefore be useful to contemplate a weaker notion of connectedness.

Definition 2.3.10. Let R be an adic $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. We will say that $\mathbf{G}$ is *formally connected* if $\mathbf{G}_{\pi_0(R)/I}$ is a connected p-divisible group over the commutative ring $\pi_0(R)/I$, where $I \subseteq \pi_0(R)$ is a finitely generated ideal of definition.

Remark 2.3.11. Let R and $\mathbf{G}$ be as in Definition 2.3.10, and let $K \subseteq |\operatorname{Spec}(R)|$ denote the vanishing locus of a finitely generated ideal of definition $I \subseteq \pi_0(R)$. Then $\mathbf{G}$ is formally connected if and only if, for every point $x \in K$, the p-divisible group $\mathbf{G}_{\kappa(x)}$ is connected; here $\kappa(x)$ denotes the residue field of $\pi_0(R)$ at the point x. In particular, this condition depends only the closed subset $K \subseteq |\operatorname{Spec}(R)|$ (or, equivalently, on the topology of $\pi_0(R)$), and not on the choice of an ideal of definition $I \subseteq \pi_0(R)$.

2.3.2 Identity Components of Connected p-Divisible Groups

We can now state the first main result of this section.

Theorem 2.3.12. *Let R be a complete adic $\mathbb{E}_\infty$-ring, and suppose that p is topologically nilpotent in $\pi_0(R)$ (so that R is (p)-complete). Let $\mathbf{G}$ and $\mathbf{G}'$ be p-divisible groups over R, and assume that $\mathbf{G}$ is formally connected. Then the canonical map*

$$\operatorname{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}, \mathbf{G}') \to \operatorname{Map}_{\mathrm{FGroup}(R)}(\mathbf{G}^\circ, \mathbf{G}'^\circ)$$

is a homotopy equivalence.

Corollary 2.3.13. *Let R be a complete adic $\mathbb{E}_\infty$-ring and suppose that p is topologically nilpotent in $\pi_0(R)$. Let $\mathrm{BT}^p(R)^{\mathrm{fc}}$ denote the full subcategory of $\mathrm{BT}^p(R)$ spanned by the formally connected p-divisible groups over R. Then the construction $\mathbf{G} \mapsto \mathbf{G}^\circ$ induces a fully faithful functor $\mathrm{BT}^p(R)^{\mathrm{fc}} \to \mathrm{FGroup}(R)$.*

Proof of Theorem 2.3.12. Without loss of generality, we may assume that R is connective. Let $I \subseteq \pi_0(R)$ be a finitely generated ideal of definition, and let $\mathcal{E} \subseteq \mathrm{CAlg}_R^{\mathrm{cn}}$ be the full subcategory spanned by those connective $\mathbb{E}_\infty$-algebras over R which are truncated and I-nilpotent. Using Proposition 2.3.9, we deduce that the canonical map $\mathbf{G}^\circ|_{\mathcal{E}} \to \mathbf{G}|_{\mathcal{E}}$ is an equivalence. It follows from Theorem 2.1.1 that the restriction map

$$\operatorname{Map}_{\mathrm{FGroup}(R)}(\mathbf{G}^\circ, \mathbf{G}'^\circ) \to \operatorname{Map}_{\mathrm{Fun}(\mathcal{E}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(\mathbf{G}^\circ|_{\mathcal{E}}, \mathbf{G}'^\circ|_{\mathcal{E}})$$

is a homotopy equivalence. Consequently, the canonical map

$$\theta : \operatorname{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}, \mathbf{G}') \to \operatorname{Map}_{\mathrm{FGroup}(R)}(\mathbf{G}^\circ, \mathbf{G}'^\circ)$$

can be identified with the composition

$$\operatorname{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}, \mathbf{G}') \xrightarrow{\theta'} \operatorname{Map}_{\mathrm{Fun}(\mathcal{E}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(\mathbf{G}|_{\mathcal{E}}, \mathbf{G}'|_{\mathcal{E}}) \xrightarrow{\theta''} \operatorname{Map}_{\mathrm{Fun}(\mathcal{E}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(\mathbf{G}|_{\mathcal{E}}, \mathbf{G}'^\circ|_{\mathcal{E}}).$$

We will complete the proof by showing that θ' and θ'' are homotopy equivalences.

Let $H : \mathcal{E} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ denote the functor given by the formula $H(A) = \mathbf{G}'(A^{\mathrm{red}})$. We then have a fiber sequence of functors $\mathbf{G}'^{\circ}|_{\mathcal{E}} \to \mathbf{G}'|_{\mathcal{E}} \to H$, which gives a fiber sequence of mapping spaces

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Fun}(\mathcal{E},\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(\mathbf{G}|_{\mathcal{E}}, \mathbf{G}'|_{\mathcal{E}}) \arrow[d, "\theta''"] \\ \mathrm{Map}_{\mathrm{Fun}(\mathcal{E},\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(\mathbf{G}|_{\mathcal{E}}, \mathbf{G}'^{\circ}|_{\mathcal{E}}) \arrow[d] \\ \mathrm{Map}_{\mathrm{Fun}(\mathcal{E},\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(\mathbf{G}|_{\mathcal{E}}, H). \end{tikzcd}$$

Let $\mathcal{E}_0 \subseteq \mathcal{E}$ be the full subcategory spanned by those objects $A \in \mathcal{E} \subseteq \mathrm{CAlg}_R^{\mathrm{cn}}$ which are reduced. Then H is a right Kan extension of $H|_{\mathcal{E}_0}$. Moreover, our connectivity assumption on $\mathbf{G}$ guarantees that $\mathbf{G}|_{\mathcal{E}_0}$ vanishes (Proposition 2.3.9). It follows that the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathcal{E},\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(\mathbf{G}|_{\mathcal{E}}, H)$ is contractible, so that θ'' is a homotopy equivalence.

Writing $\mathbf{G}$ as a direct limit $\varinjlim_k \mathbf{G}[p^k]$, we can identify θ' with a limit of maps

$$\theta'_k : \mathrm{Map}_{\mathrm{Fun}(\mathcal{E},\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(\mathbf{G}[p^k], \mathbf{G}') \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{E},\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(\mathbf{G}[p^k]|_{\mathcal{E}}, \mathbf{G}'|_{\mathcal{E}}).$$

We will complete the proof by showing that each θ'_k is a homotopy equivalence. Unwinding the definitions, we see that θ'_k can be identified with the restriction map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{E},\mathrm{Mod}_{\mathbf{Z}/p^k\mathbf{Z}}^{\mathrm{cn}})}(\mathbf{G}[p^k], \mathbf{G}'[p^k]) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{E},\mathrm{Mod}_{\mathbf{Z}/p^k\mathbf{Z}}^{\mathrm{cn}})}(\mathbf{G}[p^k]|_{\mathcal{E}}, \mathbf{G}'[p^k]|_{\mathcal{E}}).$$

Using Lemma SAG.8.1.2.2 , we can write R as the limit of a tower

$$\cdots \to R_3 \to R_2 \to R_1$$

of connective I-nilpotent $\mathbb{E}_\infty$-algebras, having the property that the direct limit $\varinjlim \mathrm{Map}_{\mathrm{CAlg}_R}(R_n, A)$ is contractible for every I-nilpotent $\mathbb{E}_\infty$-algebra over R. Let $T : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}/p^k\mathbf{Z}}^{\mathrm{cn}}$ denote a right Kan extension of $\mathbf{G}'[p^k]|_{\mathcal{E}}$; unwinding the definitions, we see that T is given by the formula

$$T(A) = \varprojlim \mathbf{G}'[p^k](\tau_{\leqslant n}(A \otimes_R R_n)).$$

Since $\mathbf{G}'$ is a p-divisible group, the functor $\Omega^\infty \mathbf{G}'[p^k] : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is corepresentable (by a finite flat R-algebra), and therefore preserves small limits. It follows that $\mathbf{G}'[p^k]$ also preserves small limits, so that T is given by the formula $T(A) = \mathbf{G}'[p^k](\varprojlim \tau_{\leqslant n}(A \otimes_R R_n)) \simeq \mathbf{G}'[p^k](A^\wedge_I)$.

Invoking the universal property of T as a right Kan extension, we can identify θ'_k with the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{E}, \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}/p^k\mathbf{Z}})}(\mathbf{G}[p^k], \mathbf{G}'[p^k]) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{E}, \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}/p^k\mathbf{Z}})}(\mathbf{G}[p^k], T).$$

Our assumption that $\mathbf{G}$ is a p-divisible group over R guarantees that the functor $\mathbf{G}[p^k]$ is representable by a finite flat group scheme over R, and is therefore a left Kan extension of its restriction to the full subcategory $\mathrm{CAlg}^{\mathrm{ff}}_R \subseteq \mathrm{CAlg}^{\mathrm{cn}}_R$ spanned by the finite flat R-algebras. Consequently, to show that θ'_k is a homotopy equivalence, it will suffice to show that the canonical map $\mathbf{G}'[p^k](A) \to T(A) = \mathbf{G}'[p^k](A^{\wedge}_I)$ is an equivalence whenever A is finite flat over R. This is clear, since our assumption that R is I-complete guarantees that A is also I-complete (Proposition SAG.7.3.5.7). $\square$

2.3.3 p-Divisible Formal Groups

We now study the essential image of the fully faithful embedding

$$\mathrm{BT}^p(R)^{\mathrm{fc}} \hookrightarrow \mathrm{FGroup}(R) \qquad \mathbf{G} \mapsto \mathbf{G}^{\circ}$$

described in Corollary 2.3.13.

Definition 2.3.14. Let R be a complete adic $\mathbb{E}_\infty$-ring, and suppose that p is nilpotent in $\pi_0(R)$. Let $\widehat{\mathbf{G}}$ be a formal group over R. We will say that $\widehat{\mathbf{G}}$ is a *p-divisible formal group* if there exists a formally connected p-divisible group $\mathbf{G}$ and an equivalence of formal groups $\widehat{\mathbf{G}} \simeq \mathbf{G}^{\circ}$.

Remark 2.3.15. In the situation of Definition 2.3.14, the p-divisible group $\mathbf{G}$ is determined (up to equivalence) by $\widehat{\mathbf{G}}$, by virtue of Corollary 2.3.13.

Example 2.3.16. Let R be a commutative algebra over $\mathbf{F}_p$ and let $\widehat{\mathbf{G}}_a$ be the formal additive group over R. Then $\widehat{\mathbf{G}}_a$ is not p-divisible: note that the map $p : \widehat{\mathbf{G}}_a \to \widehat{\mathbf{G}}_a$ vanishes, but multiplication by p can never vanish on a nonzero p-divisible group over R.

Warning 2.3.17. In the situation of Definition 2.3.14, the condition that a formal group $\widehat{\mathbf{G}}$ is p-divisible depends on the choice of topology on $\pi_0(R)$. However, the associated p-divisible group $\mathbf{G}$ (if it exists) does not depend on the choice of topology. Suppose that there exist finitely generated ideals $I, J \subseteq \pi_0(R)$ containing p, such that R is both I-complete and J-complete, and $\widehat{\mathbf{G}}$ is a formal p-divisible group with respect to either the I-adic and J-adic topology on $\pi_0(R)$. Then we can choose p-divisible groups $\mathbf{G}_I, \mathbf{G}_J \in \mathrm{BT}^p(R)$ and equivalences $\mathbf{G}_I^\circ \simeq \widehat{\mathbf{G}} \simeq \mathbf{G}_J^\circ$, where $\mathbf{G}_I$ is formally connected for the I-adic topology and $\mathbf{G}_J$ is formally connected for the J-adic topology. It follows that $\mathbf{G}_I$ and $\mathbf{G}_J$ are both formally connected for the $(I+J)$-adic topology. Since R is $(I+J)$-complete (Corollary SAG.7.3.3.3), Theorem 2.3.12 implies that the equivalence of formal groups $\mathbf{G}_I^\circ \simeq \widehat{\mathbf{G}} \simeq \mathbf{G}_J^\circ$ can be lifted (in an essentially unique way) to an equivalence of p-divisible groups $\mathbf{G}_I \simeq \mathbf{G}_J$.

Remark 2.3.18. Let R be a complete adic $\mathbb{E}_\infty$-ring with p topologically nilpotent in $\pi_0(R)$, and let $\widehat{\mathbf{G}}$ be a formal group over R. Then $\widehat{\mathbf{G}}$ is p-divisible when regarded as a formal group over R if and only if it is p-divisible when regarded as a formal group over the connective cover $\tau_{\geqslant 0}(R)$.

Remark 2.3.19 (Functoriality)**.** Let $f : R \to R'$ be a morphism of complete adic $\mathbb{E}_\infty$-rings, where p is topologically nilpotent in $\pi_0(R)$ (hence also in $\pi_0(R')$). Let $\widehat{\mathbf{G}}$ be a p-divisible formal group over R. Then $\widehat{\mathbf{G}}_{R'}$ is a p-divisible formal group over R'.

2.3.4 A Criterion for p-Divisibility

In the case where the topology on $\pi_0(R)$ is discrete, p-divisibility can be tested by the following criterion:

Theorem 2.3.20. *Let R be a connective $\mathbb{E}_\infty$-ring and assume that p is nilpotent in $\pi_0(R)$. Let $\widehat{\mathbf{G}}$ be a formal group over R, and let $\widehat{\mathbf{G}}[p] : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ denote the fiber of the map $[p] : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$. Then $\widehat{\mathbf{G}}$ is a p-divisible formal group (with respect to the discrete topology on $\pi_0(R)$) if and only $\widehat{\mathbf{G}}[p]$ is (representable by) a finite flat group scheme over R. Moreover, if this condition is satisfied, then the pullback map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to \mathscr{O}_{\widehat{\mathbf{G}}}$ is finite flat.*

Warning 2.3.21. Let R and $\widehat{\mathbf{G}}$ be as in Theorem 2.3.20, and suppose that the pullback map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to \mathscr{O}_{\widehat{\mathbf{G}}}$ is finite flat. In this case, the functor $(\Omega^\infty \mathbf{G}[p]) : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ has the form $\mathrm{Spf}(A)$, where A is the adic $\mathbb{E}_\infty$-ring given by a pushout diagram

$$\begin{tikzcd}
\mathscr{O}_{\widehat{\mathbf{G}}} \arrow[r, "\epsilon"] \arrow[d, "{[p]^*}"] & R \arrow[d] \\
\mathscr{O}_{\widehat{\mathbf{G}}} \arrow[r] & A.
\end{tikzcd}$$

Note that A is finite flat over R. However, this is not enough to guarantee that $\widehat{\mathbf{G}}$ is a p-divisible formal group (with respect to the discrete topology on $\pi_0(R)$). To apply

Theorem 2.3.20, we need to know not only that A is finite flat over R, but also that the topology on $\pi_0(A)$ is discrete.

Example 2.3.22. Let $R = \mathbf{Z}_p$ be the ring of p-adic integers, and let $\widehat{\mathbf{G}}_m$ be the formal multiplicative group over R (this does not quite fit the paradigm of Warning 2.3.21 since p is not nilpotent in R, but is a good concrete illustration of the phenomenon). Then the pullback map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}_m} \to \mathscr{O}_{\widehat{\mathbf{G}}_m}$ can be identified with the map of power series rings

$$\mathbf{Z}_p[[t]] \to \mathbf{Z}_p[[t]] \qquad t \mapsto (t+1)^p - 1 = pt + \cdots + pt^{p-1} + t^p.$$

This map is finite flat of degree p, and the ring A appearing in Warning 2.3.21 can be identified with the polynomial ring $\mathbf{Z}_p[t]/((t+1)^p - 1)$. However, the (t)-adic topology on this ring is not discrete (since t is not nilpotent).

In the situation of Warning 2.3.21, suppose that R is a field. In this case, any finite flat R-algebra A is an Artinian ring. Consequently, if $I \subseteq A$ is an ideal for which A is I-complete (or even I-adically separated), then I is nilpotent. We therefore obtain the following:

Corollary 2.3.23. *Let κ be a field of characteristic p and let $\widehat{\mathbf{G}}$ be a formal group over κ. Then $\widehat{\mathbf{G}}$ is a p-divisible formal group (with respect to the discrete topology on κ) if and only the map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to \mathscr{O}_{\widehat{\mathbf{G}}}$ is finite flat.*

We now turn to the proof of Theorem 2.3.20.

Lemma 2.3.24. *Let R be a connective $\mathbb{E}_\infty$-ring, let $\widehat{\mathbf{G}}$ be a formal group over R, and let $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$ be truncated and (p)-nilpotent. Then $\widehat{\mathbf{G}}(A)[1/p] \simeq 0$.*

Proof. Note that the functor $\widehat{\mathbf{G}}$ is locally almost of finite presentation (Proposition 1.6.8), so we can identify $\widehat{\mathbf{G}}(A^{\mathrm{red}}) \simeq 0$ with the colimit $\varinjlim_I \widehat{\mathbf{G}}(\pi_0(A)/I)$, where I ranges over the collection of all nilpotent ideals of $\pi_0(A)$. It will therefore suffice to prove the following:

- $(*)$ Let I be a nilpotent ideal of $\pi_0(A)$. Then the canonical map $\widehat{\mathbf{G}}(A)[1/p] \to \widehat{\mathbf{G}}(\pi_0(A)/I)[1/p]$ is an equivalence.

To prove $(*)$, choose an integers k and m such that $I^k = 0$ and A is m-truncated. Then the canonical map $A \to \pi_0(A)/I$ factors as a composition of square-zero extensions

$$A = \tau_{\leqslant m} A \to \tau_{\leqslant m-1} A \to \cdots \to \tau_{\leqslant 0} A = \pi_0(A)/I^k \to \pi_0(A)/I^{k-1} \to \cdots \to \pi_0(A)/I.$$

It will therefore suffice to prove:

($*'$) Let $f : A \to B$ be a morphism in $\mathrm{CAlg}^{\mathrm{cn}}_R$ which exhibits A as a square-zero extension of B by a module $M \in \mathrm{Mod}^{\mathrm{cn}}_B$ which is truncated and satisfies $M[1/p] \simeq 0$. Then the canonical map $\widehat{\mathbf{G}}(A)[1/p] \to \widehat{\mathbf{G}}(B)[1/p]$ is an equivalence.

To prove ($*'$), we choose a pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ B \arrow[r] & B \oplus \Sigma(M). \end{tikzcd}$$

Since the functor $\widehat{\mathbf{G}}$ is cohesive (Proposition 1.6.8), the canonical map $\widehat{\mathbf{G}}(A)[1/p] \to \widehat{\mathbf{G}}(B)[1/p]$ is a pullback of the map $\widehat{\mathbf{G}}(B)[1/p] \to \widehat{\mathbf{G}}(B \oplus \Sigma(M))[1/p]$.

For the rest of the proof, we regard $B \in \mathrm{CAlg}^{\mathrm{cn}}_R$ as fixed. For every connective B-module N, the canonical maps $\widehat{\mathbf{G}}(B) \to \widehat{\mathbf{G}}(B \oplus N) \to \widehat{\mathbf{G}}(B)$ exhibit $\widehat{\mathbf{G}}(B)$ as a direct summand of $\widehat{\mathbf{G}}(B \oplus N)$, with complementary summand $F(N) \in \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$. Since $\widehat{\mathbf{G}}$ is cohesive, the functor $F : \mathrm{Mod}^{\mathrm{cn}}_B \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ is additive. In particular, the canonical map $p : F(N) \to F(N)$ can be obtained by applying the functor F to the map $p : N \to N$. Since $\widehat{\mathbf{G}}$ is locally almost of finite presentation over R (Proposition 1.6.8), the functor F commutes with filtered colimits when restricted to k-truncated objects of $\mathrm{Mod}^{\mathrm{cn}}_B$, for every integer k. In particular, if $N \in \mathrm{Mod}^{\mathrm{cn}}_B$ is truncated, we have

$$\begin{aligned} F(N)[1/p] &\simeq \varinjlim(F(N) \xrightarrow{p} F(N) \xrightarrow{p} F(N) \to \cdots) \\ &\simeq F(\varinjlim(N \xrightarrow{p} N \xrightarrow{p} N \to \cdots)) \\ &\simeq F(N[1/p]). \end{aligned}$$

In the situation of ($*'$), it follows that $F(\Sigma(M))[1/p] \simeq F(\Sigma(M)[1/p]) \simeq 0$, so that the canonical map $\widehat{\mathbf{G}}(B) \to \widehat{\mathbf{G}}(B \oplus \Sigma(M)) \simeq \widehat{\mathbf{G}}(B) \oplus \widehat{\mathbf{G}}(\Sigma(M))$ induces an equivalence $\widehat{\mathbf{G}}(B)[1/p] \simeq \widehat{\mathbf{G}}(B \oplus \Sigma(M))[1/p]$, as desired. $\square$

Proof of Theorem 2.3.20. Suppose first that $\widehat{\mathbf{G}}$ is a p-divisible formal group. Write $\widehat{\mathbf{G}} = \mathbf{G}^{\circ}$, where $\mathbf{G}$ is a connected p-divisible group over R. For each object $A \in \mathrm{CAlg}^{\mathrm{cn}}_R$, we have canonical homotopy equivalences

$$\begin{aligned} \widehat{\mathbf{G}}[p](A) &= \mathrm{fib}(p : \widehat{\mathbf{G}}(A) \to \widehat{\mathbf{G}}(A)) \\ &\simeq \varprojlim_n \mathrm{fib}(p : \widehat{\mathbf{G}}(\tau_{\leqslant n} A) \to \widehat{\mathbf{G}}(\tau_{\leqslant n} A) \\ &\simeq \varprojlim_n \mathrm{fib}(p : \mathbf{G}(\tau_{\leqslant n} A) \to \mathbf{G}(\tau_{\leqslant n} A)) \\ &= \varprojlim_n \mathbf{G}[p](\tau_{\leqslant n} A) \\ &\simeq \mathbf{G}[p](A); \end{aligned}$$

the first equivalence results from the fact that $\widehat{\mathbf{G}}$ is nilcomplete (Proposition 1.6.8), the second from the connectedness of $\mathbf{G}$, and third from the nilcompleteness of $\mathbf{G}[p]$. It follows that $\widehat{\mathbf{G}}[p]$ is equivalent to $\mathbf{G}[p]$ and is therefore (representable by) a finite flat group scheme over R.

We now prove the converse. Assume that $\widehat{\mathbf{G}}[p]$ is representable by a finite flat group scheme over R. Without loss of generality, we may assume that this finite flat group scheme has some fixed degree d. Let $\mathscr{O} = \mathscr{O}_{\widehat{\mathbf{G}}}$ denote the $\mathbb{E}_\infty$-algebra of functions on $\mathbf{G}$. The map $p : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$ induces a map $[p]^*$ from $\mathscr{O}$ to itself. To avoid confusion, we let $\mathscr{O}'$ denote the $\mathbb{E}_\infty$-algebra $\mathscr{O}$, regarded as a $\mathscr{O}$-algebra via the map $[p]^*$. Let $I \subseteq \pi_0(\mathscr{O})$ and $I' \subseteq \pi_0(\mathscr{O}')$ denote the kernels of the augmentation maps

$$\pi_0(\mathscr{O}) \to \pi_0(R) \qquad \pi_0(\mathscr{O}') \to \pi_0(R)$$

determined by the zero section of $\widehat{\mathbf{G}}$. Unwinding the definitions, we see that $\Omega^\infty \circ \widehat{\mathbf{G}}[p]$ is representable by the formal spectrum $\mathrm{Spf}(A)$, where $A = R \otimes_{\mathscr{O}} \mathscr{O}'$ is endowed with the $(I'\pi_0(A))$-adic topology. Consequently, the assumption that $\widehat{\mathbf{G}}[p]$ is finite flat of degree d guarantees both that A is finite flat of degree d over R, and that the ideal $I'\pi_0(A)$ is nilpotent. It follows that the I'-adic topology on $\pi_0(\mathscr{O}')$ coincides with the $I\pi_0(\mathscr{O}')$-topology. Since $\mathscr{O}'$ is complete with respect to the ideal I', it is also complete with respect to the ideal I (when regarded as a $\mathscr{O}$-algebra). Invoking Proposition SAG.8.3.5.7 , we deduce that the map $[p]^* : \mathscr{O} \to \mathscr{O}'$ is finite flat of degree d.

Applying the above argument repeatedly, we deduce that for each $k \geqslant 0$, the map $[p^k]^* : \mathscr{O} \to \mathscr{O}$ is finite flat of degree d^k. Moreover, the I-adic topology on $\pi_0(\mathscr{O})$ coincides with the $[p^k]^*(I)\pi_0(\mathscr{O})$-adic topology. It follows that $\widehat{\mathbf{G}}[p^k] = \mathrm{fib}(p^k : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}})$ is representable by a finite flat group scheme over R. Set $\mathbf{G} = \varinjlim \widehat{\mathbf{G}}[p^k] = \mathrm{fib}(\widehat{\mathbf{G}} \to \widehat{\mathbf{G}}[1/p])$. Using the fact that $[p]^*$ is finite flat of degree d, we see that each of the maps $p : \mathbf{G}[p^k] \to \mathbf{G}[p^{k-1}]$ is locally surjective on π_0 with respect to the finite flat topology, so that $\mathbf{G}$ is a p-divisible group over R. Note that $\mathbf{G}(A) = \mathrm{fib}(\widehat{\mathbf{G}}(A) \to \widehat{\mathbf{G}}(A)[1/p])$ vanishes when A is reduced, so that $\mathbf{G}$ is connected. We will complete the proof by showing that the canonical map $\mathbf{G}(A) \to \widehat{\mathbf{G}}(A)$ is an equivalence when A is truncated (so that $\widehat{\mathbf{G}}$ is the identity component of $\mathbf{G}$). For this, it will suffice to show that $\widehat{\mathbf{G}}(A)[1/p]$ vanishes, which is a special case of Lemma 2.3.24. □

Remark 2.3.25. In the situation of Theorem 2.3.20, suppose that $\widehat{\mathbf{G}}$ is p-divisible, so that we can write $\widehat{\mathbf{G}} \simeq \mathbf{G}^\circ$ for some formally connected p-divisible group $\mathbf{G}$ over R. Then $\mathbf{G}$ has height h if and only if the finite flat group scheme $\widehat{\mathbf{G}}[p]$ has degree p^h.

2.3.5 The Non-Discrete Case

The general case of Definition 2.3.14 can always be reduced to the case where $\pi_0(R)$ has the discrete topology, by virtue of the following:

Theorem 2.3.26. *Let R be a connective complete adic $\mathbb{E}_\infty$-ring. Assume that p is topologically nilpotent in $\pi_0(R)$ and let $J \subseteq \pi_0(R)$ be a finitely generated ideal of definition. Then a formal group $\widehat{\mathbf{G}}$ over R is p-divisible if and only if $\widehat{\mathbf{G}}_{\pi_0(R)/J}$ is a p-divisible formal group over the commutative ring $\pi_0(R)/I$, endowed with the discrete topology. Moreover, if this condition is satisfied, then the map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to \mathscr{O}_{\widehat{\mathbf{G}}}$ is finite flat.*

Proof. The "only if" direction follows from Remarks 2.3.18 and 2.3.19. To prove the converse, let us assume that $\widehat{\mathbf{G}}_{\pi_0(R)/J}$ is a p-divisible formal group; we wish to show that $\widehat{\mathbf{G}}$ is also a p-divisible formal group. Without loss of generality we may assume that R is connective. We first treat the case where the topology on $\pi_0(R)$ is discrete, so that the ideal J is nilpotent. Let $I \subseteq \pi_0(\mathscr{O})$, $I' \subseteq \pi_0(\mathscr{O}')$, and $A = R \otimes_{\mathscr{O}} \mathscr{O}'$ be as in the proof of Theorem 2.3.20. Our assumption that $\widehat{\mathbf{G}}_{\pi_0(R)/J}$ is a p-divisible formal group guarantees that $B = (\pi_0(R)/J) \otimes_R A$ is finite flat over the commutative ring $\pi_0(R)/J$, and that the ideal $I'B$ is nilpotent. Since the ideal J is nilpotent, Proposition SAG.2.7.3.2 guarantees that A is finite flat over R. Moreover, the ideal $I'\pi_0(A)$ is becomes nilpotent modulo J, and is therefore nilpotent. It follows that $\widehat{\mathbf{G}}[p] \simeq \mathrm{Spf}(A)$ is finite flat over R, so that $\widehat{\mathbf{G}}$ is a p-divisible formal group by virtue of Theorem 2.3.20.

We now treat the general case. Using Lemma SAG.8.1.2.2 , we can write R as the limit of a tower of $\mathbb{E}_\infty$-rings

$$\cdots \to R_4 \to R_3 \to R_2 \to R_1,$$

where each R_i is connective and J-nilpotent, and the direct limit $\varinjlim \mathrm{Map}_{\mathrm{CAlg}_R}(R_n, B)$ is contractible whenever $B \in \mathrm{CAlg}_R^{\mathrm{cn}}$ is J-nilpotent. If $\widehat{\mathbf{G}}_{\pi_0(R)/J}$ is a p-divisible formal group, then the first part of the proof shows that each $\widehat{\mathbf{G}}_{R_n}$ is a p-divisible formal group over R_n. We can therefore write $\widehat{\mathbf{G}}_{R_n}$ as the identity component of a connected p-divisible group $\mathbf{G}_{R_n}$. Using Proposition 3.2.2, we can arrange that the tower of p-divisible groups $\{\mathbf{G}_{R_n} \in \mathrm{BT}^p(R_n)\}$ arises from an essentially unique p-divisible group $\mathbf{G} \in \mathrm{BT}^p(R)$. Since each $\mathbf{G}_{R_n}$ is connected, the p-divisible group $\mathbf{G}$ is formally connected. Let $\mathbf{G}^\circ$ be the identity component of $\mathbf{G}$. By construction, the formal group $\mathbf{G}^\circ$ is equipped with a compatible family of equivalences $(\mathbf{G}^\circ)_{R_n} \simeq \widehat{\mathbf{G}}_{R_n}$. Using

Theorem 2.1.1, we deduce that $\mathbf{G}^\circ$ and $\widehat{\mathbf{G}}$ are equivalent, so that $\widehat{\mathbf{G}}$ is a p-divisible formal group as desired.

It remains to show that if $\widehat{\mathbf{G}}$ is a p-divisible formal group, then the map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to \mathscr{O}_{\widehat{\mathbf{G}}}$ is finite flat. Note that Lemma SAG.8.1.2.2 guarantees that we can take R_1 to be almost perfect as an R-module. Set $\widehat{\mathbf{G}}' = \widehat{\mathbf{G}}_{R_1}$, so that we can identify $\mathscr{O}_{\widehat{\mathbf{G}}'}$ with the tensor product $R_1 \otimes_R \mathscr{O}_{\widehat{\mathbf{G}}}$. It follows from Theorem 2.3.20 that the map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to \mathscr{O}_{\widehat{\mathbf{G}}}$ is finite flat after tensoring over R with R_1. The flatness of $[p]^*$ then follows from Proposition SAG.8.3.5.7 (note that $\mathscr{O}_{\widehat{\mathbf{G}}}$ is automatically J-complete, since can be described as the R-linear dual of a coalgebra over R). $\square$

We can use Theorems 2.3.20 and 2.3.26 to recover Tate's theorem:

Proof of Theorem 2.3.2. Let R be a complete local Noetherian ring whose residue field κ has characteristic p. Let us regard R as an adic commutative ring by endowing it with the $\mathfrak{m}$-adic topology, where $\mathfrak{m} \subseteq R$ is the maximal ideal. Using Theorem 2.3.26 and Corollary 2.3.23, we conclude that a formal group $\widehat{\mathbf{G}}$ over R is a p-divisible formal group if and only if the pullback map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to \mathscr{O}_{\widehat{\mathbf{G}}}$ is finite flat. The equivalence of Theorem 2.3.2 now follows from Corollary 2.3.13. $\square$

2.3.6 Pointwise Characterization of p-Divisibility

Let R be a connective $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}$ be a formal group over R. If $\widehat{\mathbf{G}}$ is a p-divisible formal group over R, then the fiber $\widehat{\mathbf{G}}_{\kappa(x)}$ is a p-divisible formal group over the residue field $\kappa(x)$ of each point $x \in |\operatorname{Spec}(R)|$. We now prove a converse:

Theorem 2.3.27. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}$ be a formal group over R. Then $\widehat{\mathbf{G}}$ is p-divisible (with respect to the discrete topology on $\pi_0(R)$) if and only if it satisfies the following pair of conditions:*

(a) *For each point $x \in |\operatorname{Spec}(R)|$, the formal group $\widehat{\mathbf{G}}_{\kappa(x)}$ is p-divisible.*

(b) *The functor $x \mapsto \operatorname{ht}(\widehat{\mathbf{G}}_{\kappa(x)})$ is a locally constant function on the topological space $|\operatorname{Spec}(R)|$.*

The proof of Theorem 2.3.27 will require some preliminaries.

Lemma 2.3.28. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}$ be a formal group over R. Then $\widehat{\mathbf{G}}$ is p-divisible (with respect to the discrete topology on $\pi_0(R)$) if and only if it satisfies the following pair of conditions:*

(a) *For each point $x \in |\operatorname{Spec}(R)|$, the formal group $\widehat{\mathbf{G}}_{\kappa(x)}$ is p-divisible.*

(b') *The functor $\widehat{\mathbf{G}}[p] : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is corepresentable by a connective $\mathbb{E}_\infty$-algebra over R.*

Proof. The necessity of condition (a) is clear, and the necessity of (b') follows from Theorem 2.3.20. Conversely, suppose that (a) and (b') are satisfied; we wish to show that $\widehat{\mathbf{G}}$ is p-divisible. Without loss of generality we may assume that R is discrete (Theorem 2.3.26) and that $\mathscr{O}_{\widehat{\mathbf{G}}} \simeq R[[x_1, \ldots, x_n]]$ is a power series algebra over R (where each x_i vanishes along the zero section of $\widehat{\mathbf{G}}$). Let A be a connective $\mathbb{E}_\infty$-algebra over R which corepresents the functor $\widehat{\mathbf{G}}[p] : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$. Then $\pi_0(A)$ is a quotient of $R[[x_1, \ldots, x_n]]/(x_1^k, \ldots, x_n^k)$ for $k \gg 0$, and is therefore finitely generated as an R-module. Since the functor $\widehat{\mathbf{G}}[p]$ is locally almost of finite presentation (Proposition 1.6.8), the $\mathbb{E}_\infty$-ring A is almost of finite presentation over R. Applying Corollary SAG.5.2.2.2 , we see that A is almost perfect as an R-module. Using (b') and the fiberwise flatness criterion (Corollary SAG.6.1.4.9), we conclude that A is flat over R. Applying Theorem 2.3.20, we conclude that the formal group $\widehat{\mathbf{G}}$ is p-divisible. $\square$

Lemma 2.3.29. *Let R be a connective $\mathbb{E}_\infty$-ring, let $\widehat{\mathbf{G}}$ be a formal group over R, and let I be the nilradical of $\pi_0(R)$. If $\widehat{\mathbf{G}}_{\pi_0(R)/I}$ is a p-divisible formal group (over the reduced commutative ring $\pi_0(R)/I$), then $\widehat{\mathbf{G}}$ is p-divisible.*

Proof. By virtue of Theorem 2.3.26, we can replace R by $\pi_0(R)$ and thereby reduce to the case where R is a commutative ring. The assertion is local with respect to the Zariski topology on $|\operatorname{Spec}(R)|$, so we may further assume that the ring of functions $\mathscr{O}_{\widehat{\mathbf{G}}}$ is isomorphic to a power series ring $R[[x_1, \ldots, x_n]]$ for some $n \geqslant 0$, where the kernel of the augmentation map $\mathscr{O}_{\widehat{\mathbf{G}}} \to R$ is the ideal $J = (x_1, \ldots, x_n)$. The map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to \mathscr{O}_{\widehat{\mathbf{G}}}$ carries each x_i to some power series $f_i(\vec{x}) \in R[[x_1, \ldots, x_n]]$ with vanishing constant term.

Set $J' = (f_1(\vec{x}), \ldots, f_n(\vec{x}))$, and let $\overline{J}$ and $\overline{J}'$ denote the images of J and J' in the ring $(R/I)[[x_1, \ldots, x_n]]$. Our hypothesis guarantees that the map of power series rings

$$(R/I)[[x_1, \ldots, x_n] \xrightarrow{x_i \mapsto \overline{f}_i(\vec{x})} (R/I)[[x_1, \ldots, x_n]]$$

is adic: that is, the ideal $\overline{J}'$ contains $\overline{J}^m$ for some integer $m \gg 0$. For every sequence of nonnegative integers $\vec{d} = (d_1, \ldots, d_n)$ with $d_1 + \cdots + d_n = m$, set $\vec{x}^{\vec{d}} = x_1^{d_1} \cdots x_n^{d_n}$, so that we have an equation of the form

$$\vec{x}^{\vec{d}} \equiv \sum_i f_i(\vec{x}) r_i^{\vec{d}}(\vec{x}) \pmod{I[[x_1, \ldots, x_n]]}$$

for some power series $r_i^{\vec{d}}(\vec{x}) \in R[[x_1, \ldots, x_n]]$. Let $I_0 \subseteq I$ be the ideal generated by all of the coefficients of monomials of degree $\leqslant m$ in the expressions $\vec{x}^{\vec{d}} - \sum_i f_i(\vec{x}) r_i^{\vec{d}}(\vec{x})$. Since I_0 is a nilpotent ideal, we can replace R by the quotient ring R/I_0 (Theorem 2.3.26) and thereby reduce to the case where $I_0 = 0$: that is, we have $J^m \subseteq J' + J^{m+1}$. It follows that for every element $g_0(\vec{x}) \in J^m$, we can find elements $g_k(\vec{x}) \in J^{m+k}$ and $h_k(\vec{x}) \in J^k J'$ satisfying $g_k(\vec{x}) = h_k(\vec{x}) + g_{k+1}(\vec{x})$. Then we can identify $g_0(\vec{x})$ with the infinite sum $\sum_{k \geqslant 0} h_k(\vec{x}) \in J'$. Allowing $g_0(\vec{x})$ to vary, we have an inclusion $J^m \subseteq J'$, so that the map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to \mathscr{O}_{\widehat{\mathbf{G}}}$ is adic and therefore the functor $\widehat{\mathbf{G}}[p]$ is corepresentable. The desired result now follows from Lemma 2.3.28. $\square$

Proof of Theorem 2.3.27. The necessity of conditions (a) and (b) is clear. To prove sufficiency, suppose that we are given a formal group $\widehat{\mathbf{G}}$ over a connective $\mathbb{E}_\infty$-ring R which satisfies (a) and (b); we wish to show that $\widehat{\mathbf{G}}$ is p-divisible. By virtue of Lemma 2.3.29, we may assume without loss of generality that R is a reduced commutative ring. Working locally on $|\operatorname{Spec}(R)|$, we may further assume that $\mathscr{O}_{\widehat{\mathbf{G}}}$ is a power series algebra $R[[t_1, \ldots, t_n]]$ and that the function $x \mapsto \operatorname{ht}(\widehat{\mathbf{G}}_{\kappa(x)})$ is constant with value h. For $1 \leqslant i \leqslant n$, let $f_i(\vec{t})$ denote the image of t_i under the pullback map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to \mathscr{O}_{\widehat{\mathbf{G}}}$. For each $k \geqslant 0$, set $A_k = R[[t_1, \ldots, t_n]]/(f_1(\vec{t}), \ldots, f_n(\vec{t}), t_1^k, \ldots, t_n^k)$, so that we have a tower of (discrete) R-algebras

$$\cdots \to A_4 \to A_3 \to A_2 \to A_1 \simeq R,$$

where the transition maps are surjective and each A_k is finitely presented as an R-module. We will complete the proof by showing that this tower is eventually constant, so that $\widehat{\mathbf{G}}$ satisfies condition (b') of Lemma 2.3.28. Note that hypothesis (a) guarantees that for every point $x \in |\operatorname{Spec}(R)|$, the tower of vector spaces

$$\cdots \to \pi_0(\kappa(x) \otimes_R A_3) \to \pi_0(\kappa(x) \otimes_R A_2) \to \pi_0(\kappa(x) \otimes_R A_1) \simeq \kappa(x)$$

is eventually constant (and of dimension p^h).

We will prove the following:

$(*)$ For every maximal ideal $\mathfrak{m}$ of R, the tower of $R_{\mathfrak{m}}$-algebras

$$\cdots \to (A_4)_{\mathfrak{m}} \to (A_3)_{\mathfrak{m}} \to (A_2)_{\mathfrak{m}} \to (A_1)_{\mathfrak{m}} \simeq R_{\mathfrak{m}},$$

is eventually constant and equivalent to an $R_{\mathfrak{m}}$-module which is free of rank p^h.

Assume $(*)$ for the moment. Let x be any closed point of $|\operatorname{Spec}(R)|$, corresponding to a maximal ideal $\mathfrak{m} \subseteq R$. It follows from $(*)$ that there exists some integer k for which

$(A_k)_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module of rank p^h, and the transition maps $(A_{k'})_{\mathfrak{m}} \to (A_k)_{\mathfrak{m}}$ are isomorphisms for $k' \geqslant k$. Choose a map of R-modules $\rho : R^{p^h} \to A_k$ which induces an isomorphism after localizing at the maximal ideal $\mathfrak{m}$. Since A_k is finitely presented as an R-module, the kernel $\ker(\rho)$ is finitely generated as an R-module. The localization $\ker(\rho)_{\mathfrak{m}}$ vanishes, so we can choose an element $u \in R - \mathfrak{m}$ which annihilates $\ker(\rho)$. We will complete the proof by arguing that the transition maps

$$\cdots \to A_{k+2}[1/u] \to A_{k+1}[1/u] \to A_k[1/u]$$

are isomorphisms, so that the tower $\{A_m\}_{m\geqslant 1}$ is eventually constant when restricted to a neighborhood of the point $x \in |\operatorname{Spec}(R)|$. To prove this, we can replace R by $R[1/u]$ and thereby reduce to the situation where the map ρ is injective. Fix an integer $k' \geqslant k$; we wish to show that the map $A_{k'} \to A_k$ is an isomorphism. For this, it will suffice to show that the induced map $\mu : (A_{k'})_{\mathfrak{n}} \to (A_k)_{\mathfrak{n}}$ is an isomorphism, for every maximal ideal $\mathfrak{n}$ of R. Enlarging k' if necessary, we may assume without loss of generality that $(A_{k'})_{\mathfrak{n}}$ is a free $R_{\mathfrak{n}}$-module of rank p^h. Suppose, for a contradiction, that $\ker(\mu) \neq 0$. Then $\ker(\mu)$ is a nonzero submodule of a free $R_{\mathfrak{n}}$-module, and therefore admits a nonzero map to $R_{\mathfrak{n}}$. Since R is reduced, we can choose a minimal prime ideal $\mathfrak{p} \subseteq \mathfrak{n}$ for which $\ker(\mu)_{\mathfrak{p}} \neq 0$. Then $R_{\mathfrak{p}}$ is a field, and we have an exact sequence of vector spaces

$$0 \to \ker(\mu)_{\mathfrak{p}} \to (A_{k'})_{\mathfrak{p}} \to (A_k)_{\mathfrak{p}} \to 0$$

which shows that $(A_k)_{\mathfrak{p}}$ is a vector space of dimension $< p^h$ over $R_{\mathfrak{p}}$. This is a contradiction, since ρ induces a monomorphism of vector spaces $R_{\mathfrak{p}}^{p^h} \hookrightarrow (A_k)_{\mathfrak{p}}$.

It remains to prove $(*)$. For this, we can replace R by the localization $R_{\mathfrak{m}}$ and thereby reduce to the case where R is a (reduced) local ring with maximal ideal $\mathfrak{m}$. Let $\kappa = R/\mathfrak{m}$ denote the residue field of R. Choose an integer k for which $\pi_0(\kappa \otimes_R A_k)$ is a vector space of dimension p^h (hence the transition maps $\pi_0(\kappa \otimes_R A_{k'}) \to \pi_0(\kappa \otimes_R A_k)$ are bijective for $k' \geqslant k$). Since the map $\varprojlim_{k' \geqslant k} A_{k'} \to \pi_0(\kappa \otimes_R A_k)$ is surjective, we can lift a basis of $\pi_0(\kappa \otimes_R A_k)$ to a map of R-modules $\nu : R^{p^h} \to \varprojlim_{k' \geqslant k} A_{k'}$. For each $k' \geqslant k$, let $\nu_{k'}$ denote the composition $R^{p^h} \xrightarrow{\nu} \varprojlim_{k' \geqslant k} A_{k'} \to A_{k'}$. It follows from Nakayama's lemma that each $\nu_{k'}$ is surjective. We will complete the proof of $(*)$ by showing that the map $\nu_{k'}$ is injective for $k' \gg k$.

For $1 \leqslant a \leqslant p^h$ and $k' \geqslant k$, let $I_{a,k'} \subseteq R$ denote the ideal given by the image of $\ker(\nu_{k'})$ along the projection map $R^{p^h} \to R$ onto the ath factor. We will complete the proof by showing that for each fixed a, the ideal $I_{a,k'}$ is zero for $k' \gg k$. Since $A_{k'}$ is finitely presented as an R-module, each of the R-modules $\ker(\nu_{k'})$ is finitely generated, so that the ideals $I_{a,k'} \subseteq R$ are finitely generated. Let $X_{a,k'} \subseteq |\operatorname{Spec}(R)|$ denote the vanishing locus of the ideal $I_{a,k'}$. Since R is reduced, it will suffice to show that $X_{a,k'} = |\operatorname{Spec}(R)|$ for $k' \gg k$. Note that each $X_{a,k'}$ is a cocompact closed subset of $|\operatorname{Spec}(R)|$, hence open with respect to the constructible topology on $|\operatorname{Spec}(R)|$. Since $|\operatorname{Spec}(R)|$ is compact with respect to the constructible topology, it will suffice to show that $\bigcup_{k' \geqslant k} X_{a,k'} = |\operatorname{Spec}(R)|$. Fix a point $y \in |\operatorname{Spec}(R)|$, corresponding to a prime ideal $\mathfrak{q} \subseteq R$; we wish to show that y belongs to $X_{a,k'}$ for $k' \gg k$. Let $\mathfrak{q}'$ be a minimal prime ideal of R contained in $\mathfrak{q}$ and let y' denote the corresponding point of $|\operatorname{Spec}(R)|$. Since each $X_{a,k'}$ is closed, it will suffice to show that y' is contained in $X_{a,k'}$ for $k' \gg k$. We may therefore replace y by y' and thereby reduce to the case where $\mathfrak{q}$ is a minimal prime ideal of R. Since R is reduced, the residue field $\kappa(y)$ coincides with the localization $R_{\mathfrak{q}}$. We can therefore choose $k' \geqslant k$ for which the localization $(A_{k'})_{\mathfrak{q}}$ is a vector space of dimension p^h over $R_{\mathfrak{q}}$. Using the exact sequence

$$0 \to \ker(\nu_{k'})_{\mathfrak{q}} \to R_{\mathfrak{q}}^{p^h} \to (A_{k'})_{\mathfrak{q}} \to 0,$$

we deduce that $\ker(\nu_{k'})_{\mathfrak{q}} = 0$, so that the ideal $I_{a,k'} R_{\mathfrak{q}}$ vanishes. It follows that the point y belongs to $X_{a,k'}$, as desired. $\square$

2.4 Exact Sequences of p-Divisible Groups

Let R be an $\mathbb{E}_\infty$-ring. In §AV.6.3, we introduced the notion of an *exact sequence* of commutative finite flat group schemes over R (Definition AV.6.3.7). In this section, we adapt this notion to the setting of p-divisible groups over R.

2.4.1 Monomorphisms and Strict Epimorphisms

We begin with some general remarks.

Proposition 2.4.1. *Let R be an $\mathbb{E}_\infty$-ring and let $f : \mathbf{G} \to \mathbf{G}'$ be a morphism of p-divisible groups over $\mathbf{G}$. The following conditions are equivalent:*

(1) *For every finite abelian p-group M, the induced map $\mathbf{G}[M] \to \mathbf{G}'[M]$ is an epimorphism of finite flat group schemes over R (in the sense of Definition AV.6.2.2).*

(2) *For each $m \geqslant 0$, the induced map $\mathbf{G}[p^m] \to \mathbf{G}'[p^m]$ is an epimorphism of finite flat group schemes over R.*

(3) *The induced map $\mathbf{G}[p] \to \mathbf{G}'[p]$ is an epimorphism of finite flat group schemes over R.*

Proof. The implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) are immediate. Assume that (3) is satisfied; we prove (2) by induction on m. Without loss of generality, we may assume that R is connective. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}})$ spanned by those functors which are sheaves with respect to the fppf topology. For $m \geqslant 2$, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{G}[p] \arrow[r] \arrow[d, "u'"] & \mathbf{G}[p^m] \arrow[r, "p"] \arrow[d, "u"] & \mathbf{G}[p^{m-1}] \arrow[d, "u''"] \\
\mathbf{G}'[p] \arrow[r] & \mathbf{G}'[p^m] \arrow[r, "p"] & \mathbf{G}'[p^{m-1}]
\end{tikzcd}$$

where the rows are cofiber sequences in $\mathcal{C}$. We therefore obtain a cofiber sequence $\mathrm{cofib}(u') \to \mathrm{cofib}(u) \to \mathrm{cofib}(u'')$ in $\mathcal{C}$. Assumption (3) guarantees that $\tau_{\leqslant 0} \mathrm{cofib}(u')$ vanishes in $\mathcal{C}$, so that multiplication by p induces an equivalence $\tau_{\leqslant 0} \mathrm{cofib}(u) \simeq \tau_{\leqslant 0} \mathrm{cofib}(u'')$. Our inductive hypothesis guarantees that $\tau_{\leqslant 0} \mathrm{cofib}(u'')$ vanishes, so that $\tau_{\leqslant 0} \mathrm{cofib}(u)$ vanishes: that is, u is an epimorphism of finite flat commutative group schemes. $\square$

Corollary 2.4.2. *Let R be an $\mathbb{E}_\infty$-ring and let $f : \mathbf{G} \to \mathbf{G}'$ be a morphism of p-divisible groups over $\mathbf{G}$. The following conditions are equivalent:*

(1) *For every finite abelian p-group M, the induced map $\mathbf{G}[M] \to \mathbf{G}'[M]$ is a monomorphism of finite flat group schemes over R (in the sense of Definition AV.6.2.2).*

(2) *For each $m \geqslant 0$, the induced map $\mathbf{G}[p^m] \to \mathbf{G}'[p^m]$ is a monomorphism of finite flat group schemes over R.*

(3) *The induced map $\mathbf{G}[p] \to \mathbf{G}'[p]$ is a monomorphism of finite flat group schemes over R.*

(4) *For every discrete $\tau_{\geqslant 0}R$-algebra A, the induced map $\mathbf{G}(A) \to \mathbf{G}'(A)$ is a monomorphism of abelian groups.*

Proof. The equivalence of (1), (2), and (3) follows by applying Proposition 2.4.1 to the Cartier dual map $f^\vee : \mathbf{G}^\vee \to \mathbf{G}'^\vee$. Let A is a discrete $\tau_{\geqslant 0}R$-algebra. If (4) is satisfied, then the map $\mathbf{G}(A) \to \mathbf{G}'(A)$ is a monomorphism of abelian groups, and therefore induces a monomorphism $u_n : \mathbf{G}[p^n](A) \to \mathbf{G}'[p^n](A)$ for each $n \geqslant 0$, so that condition (2) is satisfied by Remark AV.6.2.8. Conversely, if (2) is satisfied, then each u_n is a monomorphism; passing to the direct limit over n, we conclude that the map $u : \mathbf{G}(A) \to \mathbf{G}'(A)$ is a monomorphism of abelian groups. $\square$

Definition 2.4.3. Let R be an $\mathbb{E}_\infty$-ring and let $f : \mathbf{G} \to \mathbf{G}'$ be a morphism of p-divisible groups over R. We will say that f is an *strict epimorphism* if it satisfies the equivalent conditions of Proposition 2.4.1, and we will say that f is a *monomorphism* if it satisfies the equivalent conditions of Corollary 2.4.2.

Warning 2.4.4. The terminology of Definition 2.4.3 is potentially misleading: beware that a monomorphism (respectively strict epimorphism) need not be a *categorical* monomorphism (respectively epimorphism) in the ∞-category $\mathrm{BT}^p(R)$, unless R is discrete.

Warning 2.4.5. Let R be an $\mathbb{E}_\infty$-ring and let $f : \mathbf{G} \to \mathbf{G}'$ be a morphism of p-divisible groups over R, which we view as functors from $\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R}$ to $\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$. If f is a strict epimorphism, then the induced map $\pi_0(\mathbf{G}) \to \pi_0(\mathbf{G}')$ is surjective locally for the finite flat topology (since this holds for the induced natural transformation $\pi_0(\mathbf{G}[p^n]) \to \pi_0(\mathbf{G}'[p^n])$, for each $n \geqslant 0$). However, the converse fails: for example, multiplication by p induces a finite flat surjection $\pi_0(\mathbf{G}) \to \pi_0(\mathbf{G})$, but the map $p : \mathbf{G} \to \mathbf{G}$ is never a strict epimorphism (unless $\mathbf{G} = 0$).

Remark 2.4.6. Let R be an $\mathbb{E}_\infty$-ring, let $f : \mathbf{G} \to \mathbf{G}'$ be a morphism of p-divisible groups over R, and let $f^\vee : \mathbf{G}'^\vee \to \mathbf{G}^\vee$ be its Cartier dual (Construction AV.6.6.2). Then f is a monomorphism if and only if $f^\vee$ is a strict epimorphism (this follows immediately from the corresponding assertion for commutative finite flat group schemes: see Proposition AV.6.3.4).

Remark 2.4.7. Let R be an $\mathbb{E}_\infty$-ring and let $f : \mathbf{G} \to \mathbf{G}'$ be a morphism of p-divisible groups over R. Then f is a monomorphism (strict epimorphism) if and only if, for every closed point $x \in |\operatorname{Spec}(R)|$, the induced map $\mathbf{G}_{\kappa(x)} \to \mathbf{G}'_{\kappa(x)}$ is a monomorphism (strict epimorphism) of p-divisible groups over the residue field $\kappa(x)$. For monomorphisms, this follows immediately from Nakayama's lemma; the analogous statement for strict epimorphisms follows by duality from Remark 2.4.6.

2.4.2 Exact Sequences

We can now state the main result of this section:

Proposition 2.4.8. *Let R be an $\mathbb{E}_\infty$-ring, let $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Nil}(p)}$ denote the ∞-category of connective (p)-torsion $\mathbf{Z}$-module spectra, and let $\mathcal{C} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Nil}(p)})$ denote the full subcategory spanned by those functors $X : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Nil}(p)}$ which are sheaves with respect to the finite flat topology. Suppose we are given a commutative diagram σ :*

$$\begin{tikzcd} \mathbf{G}' \arrow[r, "f"] \arrow[d] & \mathbf{G} \arrow[d, "g"] \\ 0 \arrow[r] & \mathbf{G}'' \end{tikzcd}$$

in the ∞-category $\mathcal{C}$. The following conditions are equivalent:

(1) *The functors $\mathbf{G}$ and $\mathbf{G}''$ are p-divisible groups, the map g is a strict epimorphism of p-divisible groups, and the diagram σ is a pullback square in $\mathcal{C}$.*

(2) *The functors $\mathbf{G}'$ and $\mathbf{G}$ are p-divisible groups, the map f is a monomorphism of p-divisible groups, and σ is a pushout square in $\mathcal{C}$.*

(3) *The functors $\mathbf{G}'$ and $\mathbf{G}''$ are p-divisible groups and σ is a pushout square in $\mathcal{C}$.*

Definition 2.4.9. Let R be an $\mathbb{E}_\infty$-ring. A *short exact sequence of p-divisible groups over R* is a commutative diagram

$$\begin{tikzcd} \mathbf{G}' \arrow[r, "f"] \arrow[d] & \mathbf{G} \arrow[d, "g"] \\ 0 \arrow[r] & \mathbf{G}'' \end{tikzcd}$$

in $\mathrm{BT}^p(R)$ which satisfies the equivalent conditions of Proposition 2.4.8. We will generally abuse notation by identifying a short exact sequence with the diagram

$$0 \to \mathbf{G}' \xrightarrow{f} \mathbf{G} \xrightarrow{g} \mathbf{G}'' \to 0;$$

in this case, we implicitly assume that a nullhomotopy of $g \circ f$ has also been specified.

Remark 2.4.10. Let R be an $\mathbb{E}_\infty$-ring. Using Proposition 2.4.8, we see that the following data are equivalent:

- The datum of a monomorphism $f : \mathbf{G}' \to \mathbf{G}$ of p-divisible groups over R.
- The datum of a strict epimorphism $g : \mathbf{G} \to \mathbf{G}''$ of p-divisible groups over R.

- The datum of a short exact sequence $0 \to \mathbf{G}' \xrightarrow{f} \mathbf{G} \xrightarrow{g} \mathbf{G}'' \to 0$ of p-divisible groups over R.

Proof of Proposition 2.4.8. Without loss of generality, we may assume that R is connective. We first show that (1) $\Rightarrow$ (2). Let $g : \mathbf{G} \to \mathbf{G}''$ be a strict epimorphism of p-divisible groups over R, and set $\mathbf{G}' = \mathrm{fib}(g)$ (formed in the ∞-category $\mathcal{C}$ of $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Nil}(p)}$-valued sheaves on $\mathrm{CAlg}_R^{\mathrm{cn}}$, or equivalently in the larger ∞-category $\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$). We wish to show that:

(a) The functor $\mathbf{G}'$ is a p-divisible group over R.

(b) The natural map $f : \mathbf{G}' \to \mathbf{G}$ is a monomorphism of p-divisible groups.

(c) The canonical map $\mathrm{cofib}(f) \to \mathbf{G}''$ is an equivalence (where the cofiber is formed in the ∞-category $\mathcal{C}$).

Since the ∞-category $\mathcal{C}$ is prestable, assertion (c) is equivalent to the requirement that the induced map $\pi_0(\mathbf{G}) \to \pi_0(\mathbf{G}'')$ is surjective locally with respect to the flat topology, which follows from (but is weaker than) our assumption that g is a strict epimorphism (Warning 2.4.5). Since $\mathbf{G}$ and $\mathbf{G}''$ are p-divisible, we can identify the cofibers of $p : \mathbf{G} \to \mathbf{G}$ and $p : \mathbf{G}'' \to \mathbf{G}''$ with the suspensions of $\mathbf{G}[p]$ and $\mathbf{G}''[p]$, respectively. Using (c), we obtain a cofiber sequence

$$\mathrm{cofib}(p : \mathbf{G}' \to \mathbf{G}') \to \Sigma(\mathbf{G}[p]) \to \Sigma(\mathbf{G}''[p]).$$

Consequently, the local surjectivity of the map $p : \pi_0(\mathbf{G}') \to \pi_0(\mathbf{G}')$ is equivalent to the local surjectivity of the map $\pi_0(\mathbf{G}[p]) \to \pi_0(\mathbf{G}''[p])$, and therefore follows from our assumption that g is a strict epimorphism. To complete the proofs of (a) and (b), it will suffice to show that $\mathbf{G}'[p]$ is (representable by) a commutative finite flat group scheme over R and that f induces a monomorphism $\mathbf{G}'[p] \to \mathbf{G}[p]$. This follows by applying Proposition AV.6.2.9 to the evident fiber sequence $\mathbf{G}'[p] \to \mathbf{G}[p] \to \mathbf{G}''[p]$.

We next show that (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). Let us henceforth assume that we are given a cofiber sequence

$$\mathbf{G}' \xrightarrow{f} \mathbf{G} \xrightarrow{g} \mathbf{G}''$$

in the ∞-category $\mathcal{C}$. Assume first that (2) is satisfied: that is, f is a monomorphism of p-divisible groups over R. We wish to show that $\mathbf{G}''$ is also a p-divisible group. We have a commutative diagram

$$\begin{tikzcd} \pi_0(\mathbf{G}) \arrow[r, "g"] \arrow[d, "p"] & \pi_0(\mathbf{G}'') \arrow[d, "p"] \\ \pi_0(\mathbf{G}) \arrow[r, "g"] & \pi_0(\mathbf{G}'') \end{tikzcd}$$

where the horizontal maps and the left vertical map are locally surjective with respect to the flat topology, so that the right vertical map is also locally surjective with respect to the flat topology. It follows that the diagram

$$\mathbf{G}'[p] \to \mathbf{G}[p] \to \mathbf{G}''[p]$$

is also a cofiber sequence in $\mathcal{C}$. Applying Corollary AV.6.3.5 , we deduce that $\mathbf{G}''[p]$ is (representable by) a commutative finite flat group scheme over R, so that $\mathbf{G}''$ is a p-divisible group as desired.

We now complete the proof by showing that (3) implies (1). Assume that $\mathbf{G}'$ and $\mathbf{G}''$ is a p-divisible group over R; we wish to show that $\mathbf{G}$ is also a p-divisible group and that g is a strict epimorphism. From the assumption that the maps $\pi_0(\mathbf{G}') \xrightarrow{p} \pi_0(\mathbf{G}')$ and $\pi_0(\mathbf{G}'') \xrightarrow{p} \pi_0(\mathbf{G}'')$ are locally surjective for the finite flat topology, we deduce that the map $\pi_0(\mathbf{G}) \xrightarrow{p} \pi_0(\mathbf{G})$ is also locally surjective for the finite flat topology. As above, we conclude that the diagram

$$\mathbf{G}'[p] \xrightarrow{u} \mathbf{G}[p] \xrightarrow{v} \mathbf{G}''[p]$$

is a cofiber sequence in $\mathcal{C}$. We first claim that $\mathbf{G}[p]$ is a commutative finite flat group scheme over R: that is, the functor $\Omega^\infty \mathbf{G}[p] : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is corepresentable by a finite flat R-algebra. Since we have assumed that $\Omega^\infty \mathbf{G}''[p]$ is corepresentable by a finite flat R-algebra, it will suffice to show that the natural transformation $\Omega^\infty \mathbf{G}[p] \to \Omega^\infty \mathbf{G}''[p]$ is finite flat. Because the functors $\Omega^\infty \mathbf{G}[p]$ and $\Omega^\infty \mathbf{G}''[p]$ are sheaves for the finite flat topology, this assertion can be tested locally for the finite flat topology: in particular, it suffices to show that the projection map

$$\pi : \Omega^\infty \mathbf{G}[p] \times_{\Omega^\infty \mathbf{G}''[p]} \Omega^\infty \mathbf{G}[p] \to \Omega^\infty \mathbf{G}[p]$$

is finite flat. This is clear, since π is a pullback of the projection $\Omega^\infty \mathbf{G}'[p] \to \mathrm{Spec}(R)$ (which is finite flat by virtue of our assumption that $\mathbf{G}'$ is a p-divisible group). This completes the proof that $\mathbf{G}$ is p-divisible; the assertion that f is a monomorphism of p-divisible groups is equivalent to the assertion that u is a monomorphism of finite flat commutative group schemes, which follows by applying Proposition AV.6.2.9 to the cofiber sequence $\mathbf{G}'[p] \xrightarrow{u} \mathbf{G}[p] \xrightarrow{v} \mathbf{G}''[p]$ (which is also a fiber sequence since $\mathcal{C}$ is prestable). $\square$

Remark 2.4.11. Let R be an $\mathbb{E}_\infty$-ring and suppose we are given a diagram σ :

$$\begin{tikzcd} \mathbf{G}' \arrow[r, "f"] \arrow[d] & \mathbf{G} \arrow[d, "g"] \\ 0 \arrow[r] & \mathbf{G}'' \end{tikzcd}$$

in the ∞-category $\mathrm{BT}^p(R)$. Then σ is a short exact sequence of p-divisible groups if and only if it satisfies either of the following conditions (which are *a priori* weaker than the corresponding conditions in Proposition 2.4.8, but equivalent by virtue of Remark 2.4.10):

(1′) The map g is a strict epirmorphism and σ is a pullback square in the ∞-category $\mathrm{BT}^p(R)$.

(2′) The map f is a monomorphism and σ is a pushout square in the ∞-category $\mathrm{BT}^p(R)$.

Remark 2.4.12. Let R be an $\mathbb{E}_\infty$-ring and suppose we are given a diagram σ :

$$\begin{tikzcd} \mathbf{G}' \arrow[r, "f"] \arrow[d] & \mathbf{G} \arrow[d, "g"] \\ 0 \arrow[r] & \mathbf{G}''. \end{tikzcd}$$

in the ∞-category $\mathrm{BT}^p(R)$. Then σ is a short exact sequence of p-divisible groups if and only if the induced diagram

$$\begin{tikzcd} \mathbf{G}'[p] \arrow[r] \arrow[d] & \mathbf{G}[p] \arrow[d] \\ 0 \arrow[r] & \mathbf{G}''[p] \end{tikzcd}$$

is an exact sequence of commutative finite flat group schemes, in the sense of Definition AV.6.3.7 .

Remark 2.4.13. Let R be an $\mathbb{E}_\infty$-ring and suppose we are given an exact sequence

$$0 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 0$$

of p-divisible groups over R. Then the Cartier dual sequence

$$0 \to \mathbf{G}''^{\vee} \to \mathbf{G}^{\vee} \to \mathbf{G}'^{\vee} \to 0$$

is also short exact (this follows from Remark 2.4.12 and Proposition AV.6.3.4).

Remark 2.4.14. Let R be an $\mathbb{E}_\infty$-ring and suppose we are given a diagram σ :

$$\begin{tikzcd} \mathbf{G}' \arrow[r, "f"] \arrow[d] & \mathbf{G} \arrow[d, "g"] \\ 0 \arrow[r] & \mathbf{G}''. \end{tikzcd}$$

in the ∞-category $\mathrm{BT}^p(R)$. Then σ is a short exact sequence if and only if, for every maximal ideal $\mathfrak{m} \subseteq \pi_0(R)$ with residue field $\kappa = \pi_0(R)/\mathfrak{m}$, the image of σ in $\mathrm{BT}^p(\kappa)$ is an exact sequence of p-divisible groups over κ.

2.5 The Connected-Étale Sequence

In good cases, an arbitrary p-divisible group $\mathbf{G}$ can be "built from" connected and étale pieces in an essentially unique way.

Definition 2.5.1. Let R be a complete adic $\mathbb{E}_\infty$-ring and assume that $p \in \pi_0(R)$ is topologically nilpotent. We will say that a short exact sequence of p-divisible groups

$$0 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 0$$

is a *connected-étale sequence* if $\mathbf{G}'$ is formally connected (Definition 2.3.10) and $\mathbf{G}''$ is étale (Definition 2.5.3).

In this section, we will show that connected-étale sequences are essentially unique when they exist (Theorem 2.5.13), and that existence is equivalent to the assumption that the identity component $\mathbf{G}^\circ$ is a p-divisible formal group (Proposition 2.5.17).

2.5.1 Étale p-Divisible Groups

Let R be an $\mathbb{E}_\infty$-ring. Every p-divisible group $\mathbf{G}$ over R determines a p-divisible group $\mathbf{G}_{\pi_0(R)}$ over the commutative ring $\pi_0(R)$, in the sense of classical algebraic geometry. In general, passage from $\mathbf{G}$ to $\mathbf{G}_{\pi_0(R)}$ involves a loss of information. Our goal in this section is to show that no information is lost when we restrict our attention to the class of *étale* p-divisible groups (Proposition 2.5.9).

Proposition 2.5.2. *Let R be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. The following conditions are equivalent:*

(1) *For every finite abelian p-group M, the functor*

$$\Omega^\infty \mathbf{G}[M] : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R} \to \mathcal{S} \qquad A \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}(A))$$

is corepresentable by an étale $(\tau_{\geqslant 0} R)$-algebra.

(2) *For each $n \geqslant 0$, the functor*

$$\Omega^\infty \mathbf{G}[p^n] : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R} \to \mathcal{S} \qquad A \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}/p^n\,\mathbf{Z}, \mathbf{G}(A))$$

is corepresentable by an étale $(\tau_{\geqslant 0} R)$-algebra.

(3) *The functor*

$$\Omega^\infty \mathbf{G}[p] : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R} \to \mathcal{S} \qquad A \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}/p\,\mathbf{Z}, \mathbf{G}(A))$$

is corepresentable by an étale $(\tau_{\geqslant 0} R)$-algebra.

Proof. The implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) are immediate. Assume that (3) is satisfied; we will prove that (2) is satisfied by induction on n. For $n > 0$, multiplication by p induces a natural transformation $u : \Omega^\infty \mathbf{G}[p^n] \to \Omega^\infty \mathbf{G}[p^{n-1}]$. Our inductive hypothesis guarantees that the functor $\Omega^\infty \mathbf{G}[p^{n-1}]$ is corepresentable by an étale $(\tau_{\geqslant 0} R)$-algebra. Consequently, to show that $\Omega^\infty \mathbf{G}[p^n]$ is representable by an étale $(\tau_{\geqslant 0} R)$-algebra, it will suffice to show that the natural transformation u is étale. This can be tested locally with respect to the flat topology. Since u is an effective epimorphism for the flat topology, we are reduced to showing that the projection map

$$\pi : \Omega^\infty \mathbf{G}[p^n] \times_{\Omega^\infty \mathbf{G}[p^{n-1}]} \Omega^\infty \mathbf{G}[p^n] \to \Omega^\infty \mathbf{G}[p^n]$$

is étale. This follows from (3), since π is a pullback of the projection map $\Omega^\infty \mathbf{G}[p] \to \mathrm{Spec}(\tau_{\geqslant 0} R)$. $\square$

Definition 2.5.3. Let R be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. We will say that $\mathbf{G}$ is *étale* if it satisfies the equivalent conditions of Proposition 2.5.2. We let $\mathrm{BT}^p_{\mathrm{\acute{e}t}}(R)$ denote the full subcategory of $\mathrm{BT}^p(R)$ spanned by the étale p-divisible groups over R.

Remark 2.5.4. Let R be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. Then $\mathbf{G}$ is étale if and only if, for each maximal ideal $\mathfrak{m} \subseteq \pi_0(R)$, the induced p-divisible group $\mathbf{G}_{\pi_0(R)/\mathfrak{m}}$ is étale over the residue field $\pi_0(R)/\mathfrak{m}$.

In particular, if R is complete with respect to a finitely generated ideal $I \subseteq \pi_0(R)$, then $\mathbf{G}$ is étale if and only $\widehat{\mathbf{G}}_{\pi_0(R)/I}$ is an étale p-divisible group over the commutative ring $\pi_0(R)$.

Remark 2.5.5. Let R be an $\mathbb{E}_\infty$-ring, let $\mathbf{G}$ be a p-divisible group over R, and let $R^\wedge_{(p)}$ denote the (p)-completion of R. Then $\mathbf{G}$ is étale if and only if $\mathbf{G}_{R^\wedge_{(p)}}$ is an étale p-divisible group over $R^\wedge_{(p)}$. The "only if" direction is immediate. Conversely, suppose that $\mathbf{G}_{R^\wedge_{(p)}}$ is étale; we wish to show that $\mathbf{G}$ is étale. Without loss of generality, we may assume that R is connective. By virtue of Remark 2.5.4, it will suffice to show that for every field κ and every map $f : R \to \kappa$, the p-divisible group $\mathbf{G}_\kappa$ is étale. This is automatic if κ has characteristic different from p, and follows from our hypothesis in the case where κ has characteristic p because f factors through the completion $R^\wedge_{(p)}$.

Remark 2.5.6. Let R be an $\mathbb{E}_\infty$-ring, let $\mathrm{CAlg}^{\mathrm{f\acute{e}t}}_R$ denote the ∞-category of finite étale R-algebras, and let $\mathrm{Ab}^p_{\mathrm{fin}}$ denote the category of finite abelian p-groups. Using Proposition AV.6.5.5 , we can identify $\mathrm{BT}^p_{\mathrm{\acute{e}t}}(R)$ with the full subcategory of $\mathrm{Fun}(\mathrm{Ab}^p_{\mathrm{fin}}, \mathrm{CAlg}^{\mathrm{f\acute{e}t}}_R)^{\mathrm{op}}$ spanned by those functors $F : \mathrm{Ab}^p_{\mathrm{fin}} \to \mathrm{CAlg}^{\mathrm{f\acute{e}t}}_R$ satisfying the following conditions:

- (i) The functor F preserves finite coproducts: that is, it carries direct sums to tensor products of R-algebras (in particular, the unit map $R \to F(0)$ is an equivalence).

- (ii) For every monomorphism $M' \to M$ of finite abelian p-groups, the induced map $F(M') \to F(M)$ is faithfully flat.

- (iii) For every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite abelian p-groups, the diagram

$$\begin{tikzcd} F(M') \arrow[r] \arrow[d] & F(M) \arrow[d] \\ F(0) \arrow[r] & F(M'') \end{tikzcd}$$

is a pushout square in $\mathrm{CAlg}_R^{\text{fét}}$.

Example 2.5.7 (Constant Groups)**.** Let R be an $\mathbb{E}_\infty$-ring and let M be an abelian group. We let $\underline{M} : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ denote the constant sheaf with the value M (taken with respect to the flat, étale, or Zariski topology on the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R}$; the result is the same). More concretely, the functor $\underline{M}$ is given by the formula

$$\underline{M}(A) = \{\text{Locally constant functions } |\operatorname{Spec}(A)| \to M\}.$$

If M is a direct sum of finitely many copies of $\mathbf{Q}_p / \mathbf{Z}_p$, then $\underline{M}$ is an étale p-divisible group over R (conversely, every étale p-divisible group of height h over R can be made equivalent to $\underline{\mathbf{Q}_p / \mathbf{Z}_p}^h$ after extending scalars to a faithfully flat R-algebra).

Proposition 2.5.8. *Let R be a (p)-complete $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. Then $\mathbf{G}$ is étale if and only if the formal group $\mathbf{G}^\circ$ is trivial.*

Proof. The "only if" direction is immediate from the definitions. For the converse, we can use Remark 2.5.4 to reduce to the case where $R = \kappa$ is a field of characteristic p. Without loss of generality, we may assume that κ is algebraically closed. Then the functor $\Omega^\infty \mathbf{G}[p] : \mathrm{CAlg}^{\mathrm{cn}}_\kappa \to \mathcal{S}$ is corepresentable by a finite flat κ-algebra A, which factors as a product of local Artinian rings $A \simeq \prod_{i \in I} A_i$. The identity section of $\mathbf{G}$ determines a map of κ-algebras $A \to \kappa$, which factors through A_{i_0} for some $i_0 \in I$. Then the identity map $\mathrm{id} : A_{i_0} \to A_{i_0}$ determines a p-torsion element of the abelian group $\mathbf{G}(A_{i_0})$, whose image vanishes in $A_{i_0}^{\mathrm{red}} \simeq \kappa$. If $\mathbf{G}^\circ$ is trivial, it follows that the identity map $\mathrm{id} : A_{i_0} \to A_{i_0}$ factors through κ, so that $A_{i_0} \simeq \kappa$. Using the group structure of $\mathbf{G}[p]$, we conclude that each A_i is isomorphic to κ, so that $A \simeq \prod_{i \in I} A_{i_0}$ is étale over κ. □

Proposition 2.5.9. *Let R be an $\mathbb{E}_\infty$-ring. Then the forgetful functor $\mathrm{BT}^p_{ét}(R) \to \mathrm{BT}^p_{ét}(\pi_0(R))$ is an equivalence of ∞-categories.*

Proof. Proposition HA.7.5.0.6 supplies an equivalence of ∞-categories $\mathrm{CAlg}^{\text{ét}}_R \simeq \mathrm{CAlg}^{\text{ét}}_{\pi_0(R)}$, which restricts to an equivalence $\mathrm{CAlg}^{\text{fét}}_R \simeq \mathrm{CAlg}^{\text{fét}}_{\pi_0(R)}$. The desired result now follows from the description of the ∞-category $\mathrm{BT}^p_{\text{ét}}(R)$ given in Remark 2.5.6. □

We will need the following extension of Proposition 2.5.9:

Corollary 2.5.10. *Let R be an $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0(R)$. Then the functor $\mathrm{BT}^p(R) \to \mathrm{BT}^p(\pi_0(R)/I)$ restricts to an equivalence of ∞-categories $\mathrm{BT}^p_{ét}(R) \to \mathrm{BT}^p_{ét}(\pi_0(R)/I)$.*

Proof. By virtue of Proposition 2.5.9, we may assume without loss of generality that R is discrete. The assumption that R is I-complete guarantees that the pair (R, I) is Henselian (Corollary SAG.7.3.6.5), so that extension of scalars induces an equivalence of categories $\mathrm{CAlg}_R^{\text{fét}} \to \mathrm{CAlg}_{R/I}^{\text{fét}}$ (Corollary SAG.B.3.3.7). Using the descriptions of $\mathrm{BT}^p_{\text{ét}}(R)$ and $\mathrm{BT}^p_{\text{ét}}(R/I)$ supplied by Remark 2.5.6, we are reduced to proving the following:

- $(*)$ Let $F : \mathrm{Ab}^p_{\text{fin}} \to \mathrm{CAlg}_R^{\text{fét}}$ be a functor having the property that the composite map $\mathrm{Ab}^p_{\text{fin}} \xrightarrow{F} \mathrm{CAlg}_R^{\text{fét}} \simeq \mathrm{CAlg}_{R/I}^{\text{fét}}$ satisfies conditions (i), (ii) and (iii) of Remark 2.5.6. Then F also satisfies conditions (i), (ii), and (iii) of Remark 2.5.6.

For conditions (i) and (iii), this is immediate. For condition (ii), it will suffice to prove:

- $(*')$ Let $f : A \to B$ be a morphism of finite étale R-algebras. If the induced map $A/IA \to B/IB$ is faithfully flat, then f is faithfully flat.

To prove $(*')$, we note that f is an étale morphism, and is therefore faithfully flat if the quotient $B/\mathfrak{m}B$ is nonzero for each maximal ideal $\mathfrak{m} \subseteq A$. This is clear, since $\mathfrak{m}$ automatically contains IA (the pair (A/IA) is also Henselian by Corollary SAG.B.3.3.1). $\square$

Remark 2.5.11. In the situation of Corollary 2.5.10, a p-divisible group $\mathbf{G}$ over R is étale if and only if its image in $\mathrm{BT}^p(\pi_0(R)/I)$ is étale (Remark 2.5.4). Consequently, Corollary 2.5.10 is equivalent to the statement that the projection map

$$\mathrm{BT}^p_{\text{ét}}(\pi_0(R)/I) \times_{\mathrm{BT}^p(\pi_0(R)/I)} \mathrm{BT}^p(R) \to \mathrm{BT}^p_{\text{ét}}(\pi_0(R)/I)$$

is an equivalence of ∞-categories.

We will also need the following variant of Corollary 2.5.10:

Proposition 2.5.12. *Let R be an $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0(R)$, and let $\mathbf{G}$ and $\mathbf{G}'$ be p-divisible groups over R. If $\mathbf{G}'$ is étale, then the canonical map $\mathrm{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}, \mathbf{G}') \to \mathrm{Map}_{\mathrm{BT}^p(\pi_0(R)/I)}(\mathbf{G}, \mathbf{G}')$ is a homotopy equivalence.*

Proof. Without loss of generality, we may assume that R is connective. Set $R' = \pi_0(R)/I$. Using the description of the ∞-categories $\mathrm{BT}^p(R)$ and $\mathrm{BT}^p(R')$ supplied by Proposition AV.6.5.5 , we are reduced to proving the following more concrete algebraic assertion:

($*$) .Let A and B be finite flat R-algebras, and assume that A is étale over R. Then the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}_R}(A, B) \to \mathrm{Map}_{\mathrm{CAlg}_R}(A, R' \otimes_R B) \simeq \mathrm{Map}_{\mathrm{CAlg}_{R'}}(R' \otimes_R A, R' \otimes_R B)$$

is a homotopy equivalence.

Since A is étale over R, we can use Corollary HA.7.5.4.6 to replace R by $\pi_0(R)$ and thereby reduce to the case where R is discrete. In this case, our assumption that R is I-complete guarantees that (R, I) is a Henselian pair (Corollary SAG.7.3.6.5), which immediately implies ($*$) (see Proposition SAG.B.3.2.2). $\square$

2.5.2 Uniqueness of the Connected-Étale Sequence

We now show that if a p-divisible group $\mathbf{G}$ admits a connected-étale sequence, then that sequence is essentially unique:

Theorem 2.5.13. *Let R be a complete adic $\mathbb{E}_\infty$-ring with p topologically nilpotent in $\pi_0(R)$. Let $\mathcal{C} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{BT}^p(R))$ denote the full subcategory spanned by those diagrams of p-divisible groups σ :*

$$\begin{tikzcd} \mathbf{G}' \arrow[r] \arrow[d] & \mathbf{G} \arrow[d] \\ 0 \arrow[r] & \mathbf{G}'' \end{tikzcd}$$

which are connected-étale sequences, in the sense of Definition 2.5.1. Then the construction $\sigma \mapsto \mathbf{G}$ induces a fully faithful functor $\mathcal{C} \to \mathrm{BT}^p(R)$.

Proof. Let σ_0 and σ_1 be objects of $\mathcal{C}$, which we identify with connected-étale sequences

$$0 \to \mathbf{G}_0' \to \mathbf{G}_0 \to \mathbf{G}_0'' \to 0$$

$$0 \to \mathbf{G}_1' \to \mathbf{G}_1 \to \mathbf{G}_1'' \to 0.$$

We wish to show that the canonical map

$$\rho : \mathrm{Map}_{\mathcal{C}}(\sigma_0, \sigma_1) \to \mathrm{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}_0, \mathbf{G}_1)$$

is a homotopy equivalence. Using Remark 2.4.10, we see that ρ fits into a homotopy pullback square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(\sigma_0, \sigma_1) \arrow[r, "\rho"] \arrow[d] & \mathrm{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}_0, \mathbf{G}_1) \arrow[d] \\ \mathrm{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}_0', \mathbf{G}_1') \arrow[r, "\rho'"] & \mathrm{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}_0', \mathbf{G}_1). \end{tikzcd}$$

It will therefore suffice to show that ρ' is a homotopy equivalence. Using the fiber sequence

$$\mathrm{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}'_0, \mathbf{G}'_1) \to \mathrm{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}'_0, \mathbf{G}_1) \to \mathrm{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}'_0, \mathbf{G}''_1),$$

we are reduced to proving that the mapping space $\mathrm{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}'_0, \mathbf{G}''_1)$ is contractible. Since $\mathbf{G}'_0$ is formally connected, Theorem 2.3.12 supplies a homotopy equivalence

$$\mathrm{Map}_{\mathrm{BT}^p(R)}(\mathbf{G}'_0, \mathbf{G}''_1) \simeq \mathrm{Map}_{\mathrm{FGroup}(R)}(\mathbf{G}'^{\circ}_0, \mathbf{G}''^{\circ}_1).$$

The desired contractibility now follows from the observation that $\mathbf{G}''^{\circ}_1$ vanishes (Proposition 2.5.8). $\square$

Remark 2.5.14. In the proof of Theorem 2.5.13, we did not use the full strength of our assumption that σ and σ' are connected-étale sequences. It suffices to assume that σ and σ' are short exact sequences with $\mathbf{G}'_0$ formally connected and $\mathbf{G}''_1$ étale; we do not need to know that $\mathbf{G}'_1$ is formally connected or that $\mathbf{G}''_0$ is étale.

Definition 2.5.15. Let R be a complete adic $\mathbb{E}_\infty$-ring. Assume that $p \in \pi_0(R)$ is topologically nilpotent and let $\mathbf{G}$ be a p-divisible group over R. We will say that $\mathbf{G}$ *admits a connected-étale sequence* if there is a short exact sequence of p-divisible groups

$$0 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 0,$$

where $\mathbf{G}'$ is formally connected and $\mathbf{G}''$ is étale. In this case, the short exact sequence is determined up to equivalence (and depends functorially on $\mathbf{G}$) by virtue of Theorem 2.5.13. We will refer to it as the *connected-étale sequence of* $\mathbf{G}$.

Warning 2.5.16. In the situation of Definition 2.5.15, the condition that $\mathbf{G}$ admits a connected-étale sequence depends on the topology on the commutative ring $\pi_0(R)$. However, the connected-étale sequence itself is independent of that choice (provided that it exists). To see this, let us suppose that R is complete with respect to finitely generated ideals $I, J \subseteq \pi_0(R)$, and that we are connected-étale sequences

$$0 \to \mathbf{G}'_I \to \mathbf{G} \to \mathbf{G}''_I \to 0$$

$$0 \to \mathbf{G}'_J \to \mathbf{G} \to \mathbf{G}''_J \to 0$$

with respect to the I-adic and J-adic topologies, respectively. Then both short exact sequences are connected-étale sequences with respect to the $(I + J)$-adic topology on R. Since R is $(I + J)$-complete (Corollary SAG.7.3.3.3), Theorem 2.5.13 implies that the exact sequences are equivalent.

2.5.3 Existence of Connected-Étale Sequences

We now give a criterion for the existence of a connected-étale sequence:

Proposition 2.5.17. *Let R be a complete adic $\mathbb{E}_\infty$-ring. Assume that $p \in \pi_0(R)$ is topologically nilpotent and let $\mathbf{G}$ be a p-divisible group over R. Then $\mathbf{G}$ admits a connected-étale sequence (in the sense of Definition 2.5.15) if and only if the identity component $\mathbf{G}^\circ$ is a p-divisible formal group (in the sense of Definition 2.3.14).*

The proof of Proposition 2.5.17 will make use of the following simple observation:

Lemma 2.5.18. *Let R be an $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}} : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0} R} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ be a formal group over R. If $\Omega\widehat{\mathbf{G}}$ vanishes, then $\widehat{\mathbf{G}}$ vanishes.*

Proof. Without loss of generality we may assume that $R = \kappa$ is a field, so that $\mathscr{O}_{\widehat{\mathbf{G}}}$ is isomorphic to $\kappa[[t_1, \ldots, t_d]]$ for some integer $d \geqslant 0$. Note that the functor $\Omega^{\infty+1}\widehat{\mathbf{G}} : \mathrm{CAlg}^{\mathrm{cn}}_{\kappa} \to \mathcal{S}$ is corepresentable by the tensor product $A = \kappa \otimes_{\mathscr{O}_{\widehat{\mathbf{G}}}} \kappa$. A standard calculation shows that $\pi_1(A)$ is a vector space of dimension d over κ. If $\Omega\widehat{\mathbf{G}}$ vanishes, then we must have $A \simeq \kappa$. This implies that $d = 0$, so that $\widehat{\mathbf{G}}$ vanishes as well. $\square$

Proof of Proposition 2.5.17. Suppose first that $\mathbf{G}$ admits a connected-étale sequence

$$0 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 0.$$

Since $\mathbf{G}''$ is étale, its identity component vanishes (Proposition 2.5.8). It follows that the identity component $\mathbf{G}^\circ$ is equivalent to $\mathbf{G}'^\circ$. Since $\mathbf{G}'$ is formally connected, we conclude that $\mathbf{G}^\circ$ is a p-divisible formal group.

We now prove the converse. Assume that $\mathbf{G}^\circ$ is a p-divisible formal group. Then we can choose a formally connected p-divisible group $\mathbf{G}'$ over R and an equivalence $f^\circ : \mathbf{G}'^\circ \simeq \mathbf{G}^\circ$. Using Theorem 2.3.12, we can lift f° to a morphism of p-divisible groups $f : \mathbf{G}' \to \mathbf{G}$. We first claim that f is a monomorphism of p-divisible groups. To prove this, we can replace R by its residue field at any closed point of $|\operatorname{Spec}(R)|$, and thereby reduce to the case where R is a field of characteristic p and $\mathbf{G}'$ is connected (see Remark 2.4.7). We wish to show that for each object $A \in \mathrm{CAlg}^{\heartsuit}_R$, the induced map of abelian groups $\mathbf{G}'(A) \to \mathbf{G}(A)$ is injective (see Corollary 2.4.2). This follows by inspecting the commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbf{G}'^\circ(A) \arrow[r] \arrow[d] & \mathbf{G}'(A) \arrow[r] \arrow[d] & \mathbf{G}'(A^{\mathrm{red}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{G}^\circ(A) \arrow[r] & \mathbf{G}(A) \arrow[r] & \mathbf{G}(A^{\mathrm{red}}) \arrow[r] & 0,
\end{tikzcd}$$

since the left vertical map is an isomorphism of abelian groups and $\mathbf{G}'(A^{\mathrm{red}})$ vanishes (Definition 2.3.5).

Since f is a monomorphism, we can apply Proposition 2.4.8 to construct a short exact sequence of p-divisible groups

$$0 \to \mathbf{G}' \xrightarrow{f} \mathbf{G} \to \mathbf{G}'' \to 0.$$

To complete the proof, it will suffice to show that this is a connected-étale sequence: that is, that the p-divisible group $\mathbf{G}''$ is étale. Passing to identity components, we obtain a fiber sequence

$$\mathbf{G}'^{\circ} \xrightarrow{f^{\circ}} \mathbf{G}^{\circ} \to \mathbf{G}''^{\circ},$$

so that $\Omega\mathbf{G}''$ can be identified with the fiber of the map $f^{\circ} : \mathbf{G}'^{\circ} \to \mathbf{G}^{\circ}$. Since f° is an equivalence, it follows that the functor $\Omega\mathbf{G}''^{\circ} : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ vanishes, so that $\mathbf{G}''^{\circ}$ itself vanishes (Lemma 2.5.18). Invoking Proposition 2.5.8, we conclude that $\mathbf{G}''$ is étale, as desired. □

Corollary 2.5.19. *Let R be a complete adic $\mathbb{E}_{\infty}$-ring. Assume that p is topologically nilpotent in $\pi_0(R)$ and let $I \subseteq \pi_0(R)$ be a finitely generated ideal of definition. Then a p-divisible group $\mathbf{G}$ over R admits a connected-étale sequence if and only if $\mathbf{G}_{\pi_0(R)/I}$ admit a connected-étale sequence in the category of p-divisible groups over the ordinary commutative ring $\pi_0(R)/I$ (endowed with the discrete topology).*

Proof. Combine Proposition 2.5.17 with Theorem 2.3.26. □

In some cases, the existence of a connected-étale sequence is automatic:

Proposition 2.5.20. *Let R be an $\mathbb{E}_{\infty}$-ring for which the spectrum $|\operatorname{Spec}(R)|$ contains a single point. Assume that p is nilpotent in $\pi_0(R)$. Then every p-divisible group over R admits a connected-étale sequence (with respect to the discrete topology on $\pi_0(R)$).*

Remark 2.5.21. In the situation of Proposition 2.5.20, the requirement that is p nilpotent in $\pi_0(R)$ is a matter of convention: note that we have only defined the notion of connected-étale sequence in the case where R is (p)-complete. If p were not nilpotent in $\pi_0(R)$, then the hypothesis that $|\operatorname{Spec}(R)|$ contains a single point guarantees that p is invertible in $\pi_0(R)$, so that every p-divisible group over R is étale.

Proof of Proposition 2.5.20. By virtue of Corollary 2.5.19, we can replace R by $\pi_0(R)$ and thereby reduce to the case where R is discrete (though this reduction is not actually needed in the argument). Let $\mathrm{Ab}^{p}_{\mathrm{fin}}$ denote the category of finite abelian p-groups and let $\mathrm{CAlg}_R^{\mathrm{ff}}$ denote the category of finite flat R-algebras. By virtue of Proposition AV.6.5.5 , we can identify p-divisible groups over R with functors $F : \mathrm{Ab}_{\mathrm{fin}}^p \to \mathrm{CAlg}_R^{\mathrm{ff}}$ satisfying the following conditions:

(i) The functor F preserves finite coproducts.

(ii) For every monomorphism $M' \to M$ of finite abelian p-groups, the induced map $F(M') \to F(M)$ is faithfully flat.

(iii) For every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite abelian p-groups, the induced map $F(M) \otimes_{F(M')} R \to F(M'')$ is an isomorphism.

Our assumption that $|\operatorname{Spec}(R)|$ has a single point guarantees that each of the topological spaces $|\operatorname{Spec}(F(M))|$ is a finite set with the discrete topology. Moreover, each $|\operatorname{Spec}(F(M))|$ contains a distinguished point $\mathfrak{p}_M$, given by the image of the map $|\operatorname{Spec}(R)| = |\operatorname{Spec}(F(0))| \to |\operatorname{Spec}(F(M))|$. Let $F'(M)$ denote the localization of $F(M)$ at the prime ideal $\mathfrak{p}_M$ (which is a direct factor of $F(M)$, hence finite flat over R). It is then easy to check that the construction $M \mapsto F'(M)$ satisfies conditions (i), (ii), and (iii), and therefore determines a p-divisible group $\mathbf{G}'$ over R. It follows from the construction that $\mathbf{G}'$ is connected and that the canonical map $\mathbf{G}' \hookrightarrow \mathbf{G}$ induces an isomorphism on identity components, so that $\mathbf{G}^\circ$ is a p-divisible formal group; the desired result now follows from Proposition 2.5.17. □

Corollary 2.5.22. *Let R be a complete local Noetherian $\mathbb{E}_\infty$-ring whose residue field has characteristic p. Then every p-divisible group $\mathbf{G}$ over R admits a connected-étale sequence (where we endow $\pi_0(R)$ with the $\mathfrak{m}$-adic topology, with $\mathfrak{m} \subseteq \pi_0(R)$ the maximal ideal).*

Proof. Combine Proposition 2.5.20 with Corollary 2.5.19. □

Remark 2.5.23. In the situation of Corollary 2.5.22, we can weaken the assumption that R is Noetherian: it suffices to assume that R is complete with respect to a finitely generated ideal $I \subseteq \pi_0(R)$ for which the spectrum $|\operatorname{Spec}(\pi_0(R)/I)|$ consists of a single point. This condition is satisfied in other cases of interest: for example, if R is a complete valuation ring of rank 1.

Remark 2.5.24. Let $\mathbf{G}$ be a p-divisible group over a field κ of characteristic p. It follows from Proposition 2.5.20 that $\mathbf{G}$ admits a connected-étale sequence

$$0 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 0.$$

If the field κ is perfect, then this exact sequence admits a (unique) splitting. Identifying $\mathbf{G}$ with a functor $F : \mathrm{Ab}^p_{\mathrm{fin}} \to \mathrm{CAlg}^{\mathrm{ff}}_R$ as in the proof of Proposition 2.5.20, one can argue that the functor $M \mapsto F(M)^{\mathrm{red}}$ defines a p-divisible subgroup $\mathbf{G}^{\mathrm{red}} \subseteq \mathbf{G}$ which maps isomorphically to $\mathbf{G}''$.

3 Deformations of p-Divisible Groups

Let $\widehat{\mathbf{G}}_0$ be a formal group defined over a field κ. Suppose we are given a complete local Noetherian ring A and a ring homomorphism $\rho_A : A \to \kappa$ which induces an isomorphism $A/\mathfrak{m}_A \simeq \kappa$ (where $\mathfrak{m}_A$ denotes the maximal ideal of A). A *deformation of* $\widehat{\mathbf{G}}_0$ *along* ρ_A is a pair $(\widehat{\mathbf{G}}, \alpha)$, where $\widehat{\mathbf{G}}$ is a formal group over A and α is an isomorphism of formal groups $\widehat{\mathbf{G}}_0 \simeq \widehat{\mathbf{G}}_\kappa$ (here $\widehat{\mathbf{G}}_\kappa$ denotes the formal group obtained from $\widehat{\mathbf{G}}$ by extension of scalars along ρ_A). The collection of such deformations can be organized into a category $\mathrm{Def}_{\widehat{\mathbf{G}}_0}(A, \rho_A)$, which depends functorially on A. Under some mild assumptions, Lubin and Tate showed that there exists a *universal* deformation of $\widehat{\mathbf{G}}_0$ in the following sense:

Theorem 3.0.1 (Lubin-Tate)**.** *Let κ be a perfect field of characteristic $p > 0$ and let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group of height $n < \infty$. Then there exists a complete local Noetherian ring R_{LT}, a ring homomorphism $\rho_{\mathrm{LT}} : R_{\mathrm{LT}} \to \kappa$ which induces an isomorphism $R_{\mathrm{LT}}/\mathfrak{m}_{R_{\mathrm{LT}}} \simeq \kappa$, and a deformation $(\widehat{\mathbf{G}}, \alpha)$ of $\widehat{\mathbf{G}}_0$ along ρ_{LT} with the following universal property: if A is any complete local ring equipped with a map $\rho_A : A \to \kappa$ inducing an isomorphism $A/\mathfrak{m}_A \simeq \kappa$, extension of scalars induces an equivalence*

$$\mathrm{Hom}_{/\kappa}(R_{\mathrm{LT}}, A) \simeq \mathrm{Def}_{\widehat{\mathbf{G}}_0}(A, \rho_A);$$

here $\mathrm{Hom}_{/\kappa}(R_{\mathrm{LT}}, A)$ denotes the set of ring homomorphisms $f : R_{\mathrm{LT}} \to A$ satisfying $\rho_{\mathrm{LT}} = \rho_A \circ f$ (which we regard as a category with only identity morphisms).

Remark 3.0.2. In the situation of Theorem 3.0.1, the category $\mathrm{Def}_{\widehat{\mathbf{G}}_0}(A, \rho_A)$ is always discrete: that is, given two deformations of $\widehat{\mathbf{G}}_0$ over A, there is at most one morphism between them (which is then automatically an isomorphism). This is not obvious from the definitions (and need not be true if $\widehat{\mathbf{G}}_0$ has infinite height); instead, it can be regarded as part of the content of Theorem 3.0.1.

Our goal in this section is to prove a version of Theorem 3.0.1 which is more general in three respects:

(a) We replace the formal group $\widehat{\mathbf{G}}_0$ by an arbitrary p-divisible group $\mathbf{G}_0$, which is not required to be either connected or 1-dimensional (though only the 1-dimensional case is relevant to our applications).

(b) We relax the assumption that κ is a (perfect) field: instead, we begin with an arbitrary Noetherian $\mathbf{F}_p$-algebra R_0 for which the Frobenius morphism $\varphi_{R_0} : R_0 \to R_0$ is finite, and an arbitrary p-divisible group $\mathbf{G}_0$ over R_0 which is *nonstationary* (see Definition 3.0.8; this condition is automatic in the case where R_0 is a perfect field, and is satisfied in many other cases as well).

(c) We consider a more general class of deformations of $\mathbf{G}_0$, which are defined over (connective) $\mathbb{E}_\infty$-rings rather than over ordinary commutative rings.

Before stating our main result, we need to introduce some notation.

Definition 3.0.3. Let $\mathbf{G}_0$ be a p-divisible group over a commutative ring R_0. Let A be a connective $\mathbb{E}_\infty$-ring equipped with a map $\rho_A : A \to R_0$. A *deformation of* $\mathbf{G}_0$ *along* ρ_A is a pair $(\mathbf{G}, \alpha)$, where $\mathbf{G}$ is a p-divisible group over A and $\alpha : \mathbf{G}_0 \simeq \mathbf{G}_{R_0}$ is an equivalence of p-divisible groups over R_0. The collection of all such deformations can be organized into an ∞-category $\mathrm{Def}_{\mathbf{G}_0}(A, \rho_A)$, given by the homotopy fiber product $\mathrm{BT}^p(A) \times_{\mathrm{BT}^p(R_0)} \{\mathbf{G}_0\}$.

Warning 3.0.4. Through Definition 3.0.3 makes sense for an arbitrary morphism $\rho_A : A \to R_0$, we will only be interested in the case where ρ_A induces a surjection $\epsilon_A : \pi_0(A) \to R_0$, the kernel $\ker(\epsilon_A)$ is finitely generated, and A is complete with respect to $\ker(\epsilon_A)$.

Example 3.0.5. Let $\mathbf{G}_0$ be connected p-divisible group defined over a field κ of characteristic p and let $\widehat{\mathbf{G}}_0 = \mathbf{G}_0^\circ$ denote the identity component of $\mathbf{G}_0$. Suppose that A is a complete local Noetherian ring equipped with a map $\rho_A : A \to \kappa$ inducing an isomorphism $A/\mathfrak{m}_A \simeq \kappa$. Then the ∞-category $\mathrm{Def}_{\mathbf{G}_0}(A, \rho_A)$ of Definition 3.0.3 can be identified with the groupoid $\mathrm{Def}_{\widehat{\mathbf{G}}_0}(A, \rho_A)$ appearing in the statement of Theorem 3.0.1. To prove this, it will suffice (by virtue of Theorem 2.3.12) to show that every deformation of $\widehat{\mathbf{G}}_0$ over A arises as the identity component of a deformation of $\mathbf{G}_0$, which is a special case of Theorem 2.3.26.

Example 3.0.6 (First Order Deformation)**.** Let $\mathbf{G}_0$ be a p-divisible group defined over a field κ. A *first order deformation of* $\mathbf{G}_0$ is a deformation of $\mathbf{G}_0$ along the projection map $\kappa[\epsilon]/(\epsilon^2) \to \kappa$: that is, a p-divisible group $\mathbf{G}$ over $\kappa[\epsilon]/(\epsilon^2)$ equipped with an isomorphism $\alpha : \mathbf{G}_0 \simeq \mathbf{G}_\kappa$. We will say that the first-order deformation $(\mathbf{G}, \alpha)$ is *trivial* if α can be lifted to an isomorphism of p-divisible groups $(\mathbf{G}_0)_{\kappa[\epsilon]/(\epsilon^2)} \simeq \mathbf{G}$; otherwise, we say that $(\mathbf{G}, \alpha)$ is *nontrivial*.

Construction 3.0.7. Let $\mathbf{G}$ be a p-divisible group defined over a commutative ring R. Suppose that we are given a point $x \in |\operatorname{Spec}(R)|$ and a derivation $d : R \to \kappa(x)$, where $\kappa(x)$ denotes the residue field of R at x. Then the canonical map $\beta_0 : R \to \kappa(x)$ lifts to a ring homomorphism $\beta : R \to \kappa(x)[\epsilon]/(\epsilon^2)$, given by the formula $\beta(t) = \beta_0(t) + \epsilon dt$. Let $\mathbf{G}_d$ denote the p-divisible group over $\kappa(x)[\epsilon]/(\epsilon^2)$ obtained from $\mathbf{G}$ by extending scalars along β. Then $\mathbf{G}_d$ is a first-order deformation of the p-divisible group $\mathbf{G}_{\kappa(x)}$. If $d = 0$, then $\mathbf{G}_d$ is a trivial first order deformation of $\mathbf{G}_{\kappa(x)}$.

Definition 3.0.8. Let R be a commutative ring and let $\mathbf{G}$ be a p-divisible group over R. We will say that $\mathbf{G}$ is *nonstationary* if it satisfies the following condition:

($*$) For every point $x \in |\operatorname{Spec}(R)|$ and every nonzero derivation $d : R \to \kappa(x)$, the p-divisible group $\mathbf{G}_d$ of Construction 3.0.7 is a nontrivial first order deformation of $\mathbf{G}_{\kappa(x)}$.

Remark 3.0.9. Let R be a commutative ring and let $\mathbf{G}$ be a p-divisible group over R. Then we can think of $\mathbf{G}$ as encoding a family of p-divisible groups parametrized by the affine scheme $\operatorname{Spec}(R)$. The condition that $\mathbf{G}$ is nonstationary can be understood heuristically as the requirement that this family is nonconstant along every tangent vector in $\operatorname{Spec}(R)$. Put another way, it can be understood as the requirement that $\mathbf{G}$ is classified by an unramified morphism from $\operatorname{Spec}(R)$ to the moduli stack of p-divisible groups (see Remark 3.4.4 for a more precise formulation of this heuristic).

Example 3.0.10. Let R be a commutative $\mathbf{F}_p$-algebra. Suppose that R is semiperfect: that is, the Frobenius map $\varphi_R : R \to R$ is surjective. Then, for every R-module M, every derivation $d : R \to M$ is zero (since d satisfies $d(x^p) = px^{p-1}dx = 0$). It follows that condition ($*$) of Definition 3.0.8 is vacuous: that is, every p-divisible group $\mathbf{G}$ over R is nonstationary.

We can now state our main result:

Theorem 3.0.11. *Let R_0 be a Noetherian $\mathbf{F}_p$-algebra which is F-finite (that is, the Frobenius morphism $\varphi : R_0 \to R_0$ is finite) and let $\mathbf{G}_0$ be a nonstationary p-divisible group over R_0. Then there exists a morphism of connective $\mathbb{E}_\infty$-rings $\rho : R^{\mathrm{un}}_{\mathbf{G}_0} \to R_0$ and a deformation $\mathbf{G}$ of $\mathbf{G}_0$ along ρ with the following properties:*

(a) *The $\mathbb{E}_\infty$-ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is Noetherian, the morphism ρ induces a surjection of commutative rings $\epsilon : \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0}) \to R_0$, and $R^{\mathrm{un}}_{\mathbf{G}_0}$ is complete with respect to the ideal* $\ker(\epsilon)$.

(b) *Let A be any Noetherian $\mathbb{E}_\infty$-ring equipped with a map $\rho_A : A \to R_0$ for which the underlying ring homomorphism $\epsilon_A : \pi_0(A) \to R_0$ is surjective and A is complete with respect to* $\ker(\epsilon_A)$. *Then extension of scalars induces an equivalence of ∞-categories* $\mathrm{Map}_{\mathrm{CAlg}_{/R_0}}(R^{\mathrm{un}}_{\mathbf{G}_0}, A) \to \mathrm{Def}_{\mathbf{G}_0}(A, \rho_A)$.

In the situation of Theorem 3.0.11, we will refer to $R^{\mathrm{un}}_{\mathbf{G}_0}$ as the *spectral deformation ring* of the p-divisible group $\mathbf{G}_0$, and to the p-divisible group $\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}_0})$ as the *universal deformation* of $\mathbf{G}_0$. It is clear that the pair $(R^{\mathrm{un}}_{\mathbf{G}_0}, \mathbf{G})$ is uniquely determined (up to equivalence) by the pair $(R_0, \mathbf{G}_0)$, provided that it exists: the content of Theorem 3.0.11 is that a universal deformation always exists when R_0 is F-finite and $\mathbf{G}_0$ is nonstationary (these assumptions are more or less necessary, at least for a slightly stronger version of Theorem 3.0.11: see Remarks 3.1.16 and 3.4.5).

Remark 3.0.12. The notation of Theorem 3.0.11 is intended to emphasize that the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ classifies *unoriented* deformations of $\mathbf{G}_0$ (that is, deformations as a p-divisible group, without any additional structure). In §6 we will introduce a variant $R^{\mathrm{or}}_{\mathbf{G}_0}$ which classifies *oriented* deformations of $\mathbf{G}_0$.

Note that Theorem 3.0.11 has the following concrete consequence:

Corollary 3.0.13. *Let R_0 be an F-finite Noetherian $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a nonstationary p-divisible group over R_0. Then there exists a map of commutative rings $\rho : R^{\mathrm{cl}}_{\mathbf{G}_0} \to R_0$ and a deformation $\mathbf{G}$ of $\mathbf{G}_0$ along ρ with the following properties:*

(a) *The $\mathbb{E}_\infty$-ring $R^{\mathrm{cl}}_{\mathbf{G}_0}$ is Noetherian, the morphism ρ is surjective, and $R^{\mathrm{cl}}_{\mathbf{G}_0}$ is complete with respect to the kernel ideal* $\ker(\rho)$.

(b) *Let A be any Noetherian ring equipped with a surjective ring homomorphism $\rho_A : A \to R_0$ such that A is complete with respect to* $\ker(\rho_A)$. *Then extension of scalars induces an equivalence of categories* $\mathrm{Map}_{\mathrm{CAlg}_{/R_0}}(R^{\mathrm{cl}}_{\mathbf{G}_0}, A) \to \mathrm{Def}_{\mathbf{G}_0}(A, \rho_A)$.

Proof. Let $R^{\mathrm{un}}_{\mathbf{G}_0}$ be as in Theorem 3.0.11, and set $R^{\mathrm{cl}}_{\mathbf{G}_0} = \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$. □

In the situation of Corollary 3.0.13, we will refer to $R^{\mathrm{cl}}_{\mathbf{G}_0}$ as the *classical deformation ring* of the p-divisible group $\mathbf{G}_0$.

Remark 3.0.14. In the special case where R_0 is a perfect field and the p-divisible group $\mathbf{G}_0$ is 1-dimensional, Corollary 3.0.13 reduces to Theorem 3.0.1 (see Example 3.0.5). In particular, the classical deformation ring $R^{\mathrm{cl}}_{\mathbf{G}_0}$ for the p-divisible group $\mathbf{G}_0$ can be identified with the Lubin-Tate ring R_{LT} for the identity component $\mathbf{G}_0^\circ$.

Warning 3.0.15. Corollary 3.0.13 is an immediate consequence of Theorem 3.0.11, but the reverse implication is not *a priori* obvious. The spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ generally contains more information than the classical deformation ring $R^{\mathrm{cl}}_{\mathbf{G}_0}$, because the higher homotopy groups of $R^{\mathrm{un}}_{\mathbf{G}_0}$ do not vanish. For example, in the case where $R_0 = \mathbf{F}_p$ and $\mathbf{G}_0 = \mu_{p^\infty}$, the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ can be identified with the (p)-completed sphere spectrum (Corollary 3.1.19).

Most of this section is devoted to the proof of Theorem 3.0.11. Let us begin with an outline of our strategy. We first note that the spectral deformation ring is an example of a complete adic $\mathbb{E}_\infty$-ring: that is, the underlying commutative ring $R^{\mathrm{cl}}_{\mathbf{G}_0} = \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$ is equipped with a topology having a finitely generated ideal of definition I (given by the kernel of the map $R^{\mathrm{cl}}_{\mathbf{G}_0} \to R_0$), and the $\mathbb{E}_\infty$-ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is I-complete. In §3.1 we will formulate a stronger version of Theorem 3.0.11, which characterizes $R^{\mathrm{un}}_{\mathbf{G}_0}$ among *all* complete adic $\mathbb{E}_\infty$-rings A (which we do not require to be Noetherian or equipped with a reference map $\rho_A : A \to R_0$): see Theorem 3.1.15. Roughly speaking, this characterization asserts that the formal spectrum $\mathrm{Spf}(R^{\mathrm{un}}_{\mathbf{G}_0})$ can be realized as the formal completion of the moduli stack of p-divisible groups $\mathcal{M}_{\mathrm{BT}}$ along the map $f : \mathrm{Spec}(R_0) \to \mathcal{M}_{\mathrm{BT}}$ classifying the p-divisible group $\mathbf{G}_0$ (though this heuristic should be handled with care; see Warning 3.1.9). Using a general representability theorem, we reduce the question of existence for of the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ to a statement about the relative cotangent complex $L_{\mathrm{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}}}$: namely, that it is 1-connective and almost perfect. The first condition is equivalent to our assumption that $\mathbf{G}_0$ is nonstationary (Remark 3.4.4). We verify the second by studying the cofiber sequence of R-modules

$$L_{\mathcal{M}_{\mathrm{BT}}}|_{\mathrm{Spec}(R_0)} \to L_{\mathrm{Spec}(R_0)} \to L_{\mathrm{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}}}.$$

We prove in §3.2 that the restriction $L_{\mathcal{M}_{\mathrm{BT}}}|_{\mathrm{Spec}(R_0)}$ is automatically almost perfect when R_0 is an $\mathbf{F}_p$-algebra (see Proposition 3.2.5). In §3.3, we show that $L_{\mathrm{Spec}(R_0)} = L_{R_0}$ is almost perfect under the assumption that R_0 is F-finite (Proposition 3.3.1). In §3.4, we combine these ingredients to prove Theorem 3.1.15 (and therefore also Theorem 3.0.11).

3.1 Spectral Deformation Rings

Let R_0 be a commutative ring and let $\mathbf{G}_0$ be a p-divisible group over R_0. In Definition 3.0.3, we introduced the ∞-category $\mathrm{Def}_{\mathbf{G}_0}(A, \rho_A)$ of deformations of $\mathbf{G}_0$ along a morphism of $\mathbb{E}_\infty$-rings $\rho_A : A \to R_0$. The ∞-category $\mathrm{Def}_{\mathbf{G}_0}(A, \rho_A)$ depends functorially on the pair (A, ρ_A) as an object of the overcategory $\mathrm{CAlg}^{\mathrm{cn}}_{/R_0}$. Our goal in this section is to introduce a variant of this construction, which does not require us to fix the map ρ_A in advance.

Definition 3.1.1. Let R_0 be a commutative ring and let A be an adic $\mathbb{E}_\infty$-ring. Let $\mathbf{G}_0$ be a p-divisible group over R_0 and let $\mathbf{G}$ be a p-divisible group over A. A $\mathbf{G}_0$*-tagging of* $\mathbf{G}$ is a triple (I, μ, α), where $I \subseteq \pi_0(A)$ is a finitely generated ideal of definition, $\mu : R_0 \to \pi_0(A)/I$ is a ring homomorphism, and $\alpha : (\mathbf{G}_0)_{\pi_0(A)/I} \simeq \mathbf{G}_{\pi_0(A)/I}$ is an isomorphism of p-divisible groups over the commutative ring $\pi_0(A)/I$.

We will say that a pair of $\mathbf{G}_0$-taggings (I, μ, α) and (I', μ', α') are *equivalent* if there exists a finitely generated ideal of definition $J \subseteq \pi_0(A)$ containing both I and I' for which the diagram of ring homomorphisms

$$\begin{tikzcd} R_0 \arrow[r, "\mu"] \arrow[d, "\mu'"] & \pi_0(A)/I \arrow[d] \\ \pi_0(A)/I' \arrow[r] & \pi_0(A)/J \end{tikzcd}$$

commutes, and the isomorphisms α and α' agree when restricted to $(\mathbf{G}_0)_{\pi_0(A)/J}$.

Example 3.1.2. Let R_0 be a commutative ring and let $\mathbf{G}_0$ be a p-divisible group over R_0. Then $\mathbf{G}_0$ is equipped with a tautological $\mathbf{G}_0$-tagging, given by the triple $((0), \mathrm{id}_{R_0}, \mathrm{id}_{\mathbf{G}_0})$. Here we regard R_0 as an adic commutative ring by endowing it with the discrete topology.

Remark 3.1.3. In the situation of Definition 3.1.1, let $I \subseteq \pi_0(A)$ be a finitely generated ideal of definition. Then giving a $\mathbf{G}_0$-tagging of $\mathbf{G}$ is equivalent to giving a $\mathbf{G}_0$-tagging of the p-divisible group $\mathbf{G}_{\pi_0(A)/I}$. Here we regard $\pi_0(A)/I$ as an adic commutative ring by equipping it with the discrete topology.

Definition 3.1.4. Let $\mathbf{G}_0$ be a p-divisible group defined over a commutative ring R_0 and let A be an adic $\mathbb{E}_\infty$-ring. A *deformation of* $\mathbf{G}_0$ *over* A consists of a p-divisible group $\mathbf{G}$ over A together with an equivalence class of $\mathbf{G}_0$-taggings of $\mathbf{G}$. The collection of deformations of $\mathbf{G}_0$ over A can be organized into an ∞-category $\mathrm{Def}_{\mathbf{G}_0}(A)$. More precisely, we let $\mathrm{Def}_{\mathbf{G}_0}(A)$ denote the filtered colimit

$$\varinjlim_{I} \mathrm{BT}^p(A) \times_{\mathrm{BT}^p(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I),$$

where I ranges over all finitely generated ideals of definition $I \subseteq \pi_0(A)$. Here $\mathrm{Hom}(R_0, \pi_0(A)/I)$ denotes the set of ring homomorphisms from R_0 to $\pi_0(A)/I$.

Remark 3.1.5 (Functoriality)**.** Let $\mathbf{G}_0$ be a p-divisible group defined over a commutative ring R_0 and let $f : A \to A'$ be a morphism of adic $\mathbb{E}_\infty$-rings. Then extension of scalars along f determines a functor $\mathrm{Def}_{\mathbf{G}_0}(A) \to \mathrm{Def}_{\mathbf{G}_0}(A')$.

Remark 3.1.6 (Relationship with $\mathrm{Def}_{\mathbf{G}_0}(A, \rho)$)**.** Let $\mathbf{G}_0$ be a p-divisible group defined over a commutative ring R_0 and let $\rho : A \to R_0$ be a morphism of connective $\mathbb{E}_\infty$-rings. Assume that ρ induces a surjection of commutative rings $\epsilon : \pi_0(A) \to R_0$ and that the kernel ideal $\ker(\epsilon)$ is finitely generated. Let us regard A as an adic $\mathbb{E}_\infty$-ring by endowing $\pi_0(A)$ with the $\ker(\epsilon)$-adic topology. For any p-divisible group $\mathbf{G}$ over A, the datum of a $\mathbf{G}_0$-tagging of $\mathbf{G}$ is equivalent to the datum of a $\mathbf{G}_0$-tagging of the p-divisible group $\mathbf{G}_{R_0}$ obtained by extending scalars along ρ (see Remark 3.1.3). It follows that we have a homotopy fiber sequence

$$\mathrm{Def}_{\mathbf{G}_0}(A, \rho) \to \mathrm{Def}_{\mathbf{G}_0}(A) \xrightarrow{\rho} \mathrm{Def}_{\mathbf{G}_0}(R_0),$$

where the fiber is taken over the p-divisible group $\mathbf{G}_0 \in \mathrm{Def}_{\mathbf{G}_0}(R_0)$ (equipped with the $\mathbf{G}_0$-tagging of Example 3.1.2).

Remark 3.1.7. In the situation of Definition 3.1.4, suppose that A is an adic $\mathbb{E}_\infty$-ring which admits a largest finitely generated ideal of definition $I \subseteq \pi_0(A)$ (this condition is automatically satisfied if the commutative ring $\pi_0(A)$ is Noetherian). Then we can identify $\mathrm{Def}_{\mathbf{G}_0}(A)$ with the fiber product

$$\mathrm{BT}^p(A) \times_{\mathrm{BT}^p(\pi_0(A)/I)} \mathrm{Hom}(R, \pi_0(A)/I).$$

More informally, a deformation of $\mathbf{G}_0$ over A consists of a p-divisible group $\mathbf{G}$ over A, a ring homomorphism $\mu : R \to \pi_0(A)/I$, and an isomorphism of p-divisible groups $(\mathbf{G}_0)_{\pi_0(A)/I} \simeq \mathbf{G}_{\pi_0(A)/I}$.

Remark 3.1.8. In the situation of Definition 3.1.4, let $J \subseteq R_0$ be a finitely generated nilpotent ideal, and let $\mathbf{G}_1 = (\mathbf{G}_0)_{R_0/J}$ be the p-divisible group over R_0/J obtained from $\mathbf{G}_0$ by extension of scalars. Then, for any adic $\mathbb{E}_\infty$-ring A and any p-divisible group $\mathbf{G} \in \mathrm{BT}^p(A)$, the set of equivalence classes of $\mathbf{G}_0$-taggings of $\mathbf{G}$ can be identified with the set of equivalence classes of $\mathbf{G}_1$-taggings of $\mathbf{G}$. It follows that we have canonical equivalences $\mathrm{Def}_{\mathbf{G}_0}(A) \simeq \mathrm{Def}_{\mathbf{G}_1}(A)$, depending functorially on A.

It follows that, if the ring R_0 is Noetherian, then there is no harm in replacing R_0 by its reduction R_0^{red}: this does not change the deformation functor $A \mapsto \mathrm{Def}_{\mathbf{G}_0}(A)$.

Warning 3.1.9. In the situation of Definition 3.1.4, let A be a commutative ring endowed with the discrete topology. Then the space $\mathrm{Def}_{\mathbf{G}_0}(A)$ is given by the direct limit

$$\varinjlim_I \mathrm{BT}^p(A) \times_{\mathrm{BT}^p(A/I)} \mathrm{Hom}(R_0, A/I),$$

where I ranges over all finitely generated nilpotent ideals of A. In particular, we have a canonical map

$$\mathrm{Def}_{\mathbf{G}_0}(A) \to \mathrm{BT}^p(A) \times_{\mathrm{BT}^p(A^{\mathrm{red}})} \mathrm{Hom}(R_0, A^{\mathrm{red}}),$$

which is an equivalence if the nilradical of A is finitely generated (for example, if A is Noetherian). However, it is not an equivalence in general. For example, let $R_0 = \kappa$ be a perfect field of characteristic p and set $A = \kappa[x^{1/p^\infty}]/(x-1)$. Then $(x^{1/p}, x^{1/p^2}, \ldots)$ is a compatible system of (p^n)th roots of unity in A, which we can regard as a morphism of p-divisible groups $\gamma : \underline{\mathbf{Q}_p / \mathbf{Z}_p} \to \mu_{p^\infty}$. Set $\mathbf{G} = \underline{\mathbf{Q}_p / \mathbf{Z}_p} \oplus \mu_{p^\infty}$, regarded as a p-divisible group over A. Then the matrix $\begin{pmatrix} \mathrm{id} & \gamma \\ 0 & \mathrm{id} \end{pmatrix}$ determines an automorphism of $\mathbf{G}$ which reduces to the identity after extending scalars to $A^{\mathrm{red}} \simeq \kappa$, but not after extending scalars to A/I for any finitely generated nilpotent ideal $I \subseteq A$.

Recall that an adic $\mathbb{E}_\infty$-ring A is *complete* if it is I-complete, where $I \subseteq \pi_0(A)$ is any finitely generated ideal of definition for the topology on $\pi_0(A)$ (this condition does not depend on the choice of I). We let $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ denote the full subcategory of $\mathrm{CAlg}^{\mathrm{ad}}$ spanned by the complete adic $\mathbb{E}_\infty$-rings.

Lemma 3.1.10. *Let R_0 be a commutative ring and let $\mathbf{G}_0$ be a p-divisible group over R_0. For every complete adic $\mathbb{E}_\infty$-ring A, the ∞-category $\mathrm{Def}_{\mathbf{G}_0}(A)$ is a Kan complex.*

Proof. Without loss of generality we may assume that A is connective. Let $f : \mathbf{G} \to \mathbf{G}'$ be a morphism of $\mathbf{G}_0$-tagged p-divisible groups over A; we wish to show that f is an equivalence. For this, it will suffice to show that the induced map $f[p] : \mathbf{G}[p] \to \mathbf{G}'[p]$ is an equivalence of finite flat group schemes over A. Note that $f[p]$ is an equivalence over some open subset $U \subseteq |\operatorname{Spec}(A)|$ which includes the vanishing locus of an ideal of definition $I \subseteq \pi_0(A)$. Since A is I-complete, it follows that $U = |\operatorname{Spec}(A)|$, so that f is an equivalence as desired. □

Definition 3.1.11 (Universal Deformations)**.** Let R_0 be a commutative ring and let $\mathbf{G}_0$ be a p-divisible group over R_0. Let $\mathbf{G}$ be a deformation of R_0 over a complete adic $\mathbb{E}_\infty$-ring R, in the sense of Definition 3.1.4. We will say that $\mathbf{G}$ is a *universal deformation* if, for every complete adic $\mathbb{E}_\infty$-ring A, extension of scalars induces a homotopy equivalence

$$\operatorname{Map}_{\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}}(R, A) \to \operatorname{Def}_{\mathbf{G}_0}(A).$$

Note that, if $\mathbf{G}_0$ admits a universal deformation, then the pair $(R, \mathbf{G})$ is uniquely determined up to equivalence. In this case, we will denote R by $R^{\mathrm{un}}_{\mathbf{G}_0}$ and refer to it as the *spectral deformation ring* of $\mathbf{G}$.

Remark 3.1.12. Let R_0 be a commutative ring and let $\mathbf{G}_0$ be a p-divisible group over R_0. Assume that $\mathbf{G}_0$ admits a universal deformation $\mathbf{G}$ (in the sense of Definition 3.1.11). Note that we can regard $\mathbf{G}_0$ as a deformation of itself over R_0 (where we endow R_0 with the discrete topology); see Example 3.1.2. This deformation is then classified by a map $\rho : R^{\mathrm{un}}_{\mathbf{G}_0} \to R_0$. We can then regard $\mathbf{G}$ as a deformation of $\mathbf{G}_0$ along ρ, in the sense of Definition 3.0.3.

Remark 3.1.13. Let R_0 be a commutative ring, let $\mathbf{G}_0$ be a p-divisible group over R_0, and let $\mathbf{G}_0^\vee$ denote its Cartier dual. Then Cartier duality induce an equivalence of deformation functors $\operatorname{Def}_{\mathbf{G}_0} \simeq \operatorname{Def}_{\mathbf{G}_0^\vee}$. It follows that $\mathbf{G}_0$ admits a universal deformation if and only if $\mathbf{G}_0^\vee$ admits a universal deformation; if either exists, then the spectral deformation rings $R^{\mathrm{un}}_{\mathbf{G}_0}$ and $R^{\mathrm{un}}_{\mathbf{G}_0^\vee}$ are canonically equivalent to one another.

Beware that we have now introduced two *a priori* different notions of universal deformation: one via the universal property of Theorem 3.0.11, and one via the universal property of Definition 3.1.11. However, we will see in a moment that they agree (provided that the hypotheses of Theorem 3.0.11 are satisfied).

Remark 3.1.14. Let R_0 be a Noetherian ring, let $\mathbf{G}_0$ be a p-divisible group over R_0, and suppose that $\mathbf{G}_0$ admits a universal deformation $(R^{\mathrm{un}}_{\mathbf{G}_0}, \mathbf{G})$ in the sense of Definition 3.1.11. Suppose further that we are given a map of Noetherian $\mathbb{E}_\infty$-rings $\rho_A : A \to R_0$ which induces a surjection of commutative rings $\epsilon : \pi_0(A) \to R_0$. Let us regard A as an adic $\mathbb{E}_\infty$-ring by endowing $\pi_0(A)$ with the $\ker(\epsilon)$-adic topology. We then have a commutative diagram σ :

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}}(R^{\mathrm{un}}_{\mathbf{G}_0}, A) \arrow[r, "\rho_A \circ"] \arrow[d] & \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}}(R^{\mathrm{un}}_{\mathbf{G}_0}, R_0) \arrow[d] \\
\mathrm{Def}_{\mathbf{G}_0}(A) \arrow[r] & \mathrm{Def}_{\mathbf{G}_0}(R_0).
\end{tikzcd}$$

Let $\rho : R^{\mathrm{un}}_{\mathbf{G}_0} \to R_0$ be the map of Remark 3.1.12. Note that any morphism of $\mathbb{E}_\infty$-rings $u : R^{\mathrm{un}}_{\mathbf{G}_0} \to A$ which fits into a commutative diagram

$$\begin{tikzcd}
R^{\mathrm{un}}_{\mathbf{G}_0} \arrow[rr, "u"] \arrow[dr, "\rho"'] & & A \arrow[dl, "\rho_A"] \\
& R_0 &
\end{tikzcd}$$

is automatically a morphism of adic $\mathbb{E}_\infty$-rings (since ρ annihilates an ideal of definition for $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$). Passing to homotopy fibers in the horizontal direction (and using Remark 3.1.6), we see that σ determines a comparison

$$\theta : \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}_{/R_0}}(R^{\mathrm{un}}_{\mathbf{G}_0}, A) \to \mathrm{Def}_{\mathbf{G}_0}(A, \rho).$$

If A is complete with respect to the ideal $\ker(\epsilon)$, then the vertical maps in the diagram σ are homotopy equivalences, so that θ is also a homotopy equivalence. It follows that $\mathbf{G}$ automatically satisfies condition (b) of Theorem 3.0.11.

We are now ready to formulate a refinement of Theorem 3.0.11:

Theorem 3.1.15. *Let R_0 be an F-finite Noetherian $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a nonstationary p-divisible group over R_0. Then:*

(1) *The p-divisible group $\mathbf{G}_0$ admits a universal deformation (in the sense of Definition 3.1.11): that is, the functor $\mathrm{Def}_{\mathbf{G}_0} : \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}$ is corepresentable by a complete adic $\mathbb{E}_\infty$-ring $R^{\mathrm{un}}_{\mathbf{G}_0}$.*

(2) *The spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is connective and Noetherian.*

(3) *The canonical map $\rho : R^{\mathrm{un}}_{\mathbf{G}_0} \to R_0$ (see Remark 3.1.12) induces a surjective ring homomorphism $\epsilon : \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0}) \to R_0$.*

(4) *The kernel $\ker(\epsilon)$ is an ideal of definition for $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$. In particular, the $\mathbb{E}_\infty$-ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is complete with respect to $\ker(\epsilon)$.*

We will prove Theorem 3.1.15 in §3.4.

Proof of Theorem 3.0.11 from Theorem 3.1.15. Let R_0 be an F-finite Noetherian $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a nonstationary p-divisible group over R_0. Applying Theorem 3.1.15, we deduce that $\mathbf{G}_0$ admits a universal deformation $\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}})$ in the sense of Definition 3.1.11. Applying Remark 3.1.12, we can regard $\mathbf{G}$ as a deformation of $\mathbf{G}_0$ along a map $\rho : R^{\mathrm{un}}_{\mathbf{G}_0} \to R_0$. We claim that this deformation satisfies the requirements of Theorem 3.0.11: requirement (a) follows from the statement of Theorem 3.1.15, and requirement (b) from Remark 3.1.14. □

Remark 3.1.16. Let R_0 be a Noetherian $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a p-divisible group over R_0. If conditions (1) through (4) of Theorem 3.1.15 are satisfied, then R_0 must be F-finite and $\mathbf{G}_0$ must be nonstationary; see Remark 3.4.5.

Warning 3.1.17. Let R_0 be an F-finite Noetherian $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a p-divisible group over R_0. If we do not assume that $\mathbf{G}_0$ is nonstationary, then it is possible for $\mathbf{G}_0$ to admit a universal deformation (in the sense of Definition 3.1.11) for which the map $\rho : R^{\mathrm{un}}_{\mathbf{G}_0} \to R_0$ is not surjective on π_0 (in which case $R^{\mathrm{un}}_{\mathbf{G}_0}$ does not satisfy the conclusions of Theorem 3.0.11). For example, suppose that $R_0 = R_1[\epsilon]/(\epsilon^2)$ and that $\mathbf{G}_0$ is obtained by extension of scalars from a nonstationary p-divisible group $\mathbf{G}_1$ over R_1. In this case, the deformation functors $\mathrm{Def}_{\mathbf{G}_0}$ and $\mathrm{Def}_{\mathbf{G}_1}$ coincide (Remark 3.1.8). Applying Theorem 3.1.15 to the p-divisible group $\mathbf{G}_1$, we conclude that $\mathbf{G}_0$ admits a universal deformation $\mathbf{G}$, classified by a map $\rho : R^{\mathrm{un}}_{\mathbf{G}_0} \to R_0$. However, the map ρ factors through the subring $R_1 \subseteq R_1[\epsilon]/(\epsilon^2)$, and therefore fails to be surjective.

The deformation theory of étale p-divisible groups is essentially trivial:

Proposition 3.1.18. *Let $\mathbf{G}_0$ be an étale p-divisible group over the finite field $\mathbf{F}_p$. Then:*

- (a) *The deformation functor $\mathrm{Def}_{\mathbf{G}_0} : \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}$ is given by*

$$\mathrm{Def}_{\mathbf{G}_0}(A) = \begin{cases} * & \text{if } p \text{ is topologically nilpotent in } \pi_0(A) \\ \varnothing & \text{otherwise.} \end{cases}$$

- (b) *The spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is equivalent to the (p)-completed sphere spectrum $S^{\wedge}_{(p)}$, regarded as an adic $\mathbb{E}_\infty$-ring with ideal of definition $(p) \subseteq \mathbf{Z}_p \simeq \pi_0(S^{\wedge}_{(p)})$.*

Proof. Let A be an adic $\mathbb{E}_\infty$-ring. By definition, the space $\mathrm{Def}_{\mathbf{G}_0}(A)$ is given by the fiber product

$$(\varinjlim_I \mathrm{Hom}(\mathbf{F}_p, \pi_0(A)/I)) \times_{\varinjlim_I \mathrm{BT}^p(\pi_0(A)/I)} \mathrm{BT}^p(A),$$

where I ranges over the collection of all finitely generated ideals of definition for $\pi_0(A)$. Note that the set $\mathrm{Hom}(\mathbf{F}_p, \pi_0(A)/I)$ has a unique element when I contains p and is empty otherwise. It follows that the direct limit $(\varinjlim_I \mathrm{Hom}(\mathbf{F}_p, \pi_0(A)/I))$ is contractible when there exists an ideal of definition containing p (that is, when p is topologically nilpotent in $\pi_0(A)$), and otherwise empty. In the former case, we can rewrite $\mathrm{Def}_{\mathbf{G}_0}(A)$ as the direct limit

$$\varinjlim_I \mathrm{BT}^p(A) \times_{\mathrm{BT}^p(\pi_0(A/I)} \{\mathbf{G}_0\},$$

where I ranges over the collection of finitely generated ideals of definition containing p. Corollary 2.5.10 implies that each of the fiber products $\mathrm{BT}^p(A) \times_{\mathrm{BT}^p(\pi_0(A/I)} \{\mathbf{G}_0\}$ is contractible, so that $\mathrm{Def}_{\mathbf{G}_0}(A)$ is contractible. This proves (a); assertion (b) is an immediate consequence. □

Corollary 3.1.19. *Let $\mathbf{G}_0$ denote μ_{p^∞}, regarded as a p-divisible group over $\mathbf{F}_p$. Then the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is equivalent to the p-completed sphere spectrum $S^\wedge_{(p)}$.*

Proof. Combine Remark 3.1.13 with Proposition 3.1.18. □

3.2 The Moduli Stack of p-Divisible Groups

Let p be a prime number, which we regard as fixed throughout this section.

Definition 3.2.1. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathrm{BT}^p(R)$ denote the ∞-category of p-divisible groups over R. We let $\mathcal{M}_{\mathrm{BT}}(R)$ denote the underlying Kan complex $\mathrm{BT}^p(R)^{\simeq}$ (that is, the subcategory of $\mathrm{BT}^p(R)$ whose morphisms are equivalences of p-divisible groups). The construction $R \mapsto \mathcal{M}_{\mathrm{BT}}(R)$ determines a functor $\mathcal{M}_{\mathrm{BT}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will refer to $\mathcal{M}_{\mathrm{BT}}$ as the *moduli stack of p-divisible groups.*

The following result summarizes some of the formal properties of the moduli stack $\mathcal{M}_{\mathrm{BT}}$:

Proposition 3.2.2. (1) *The functor $\mathcal{M}_{\mathrm{BT}}$ is cohesive. That is, for every pullback diagram of connective $\mathbb{E}_\infty$-rings*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

for which the underlying ring homomorphisms $\pi_0(A_0) \to \pi_0(A_{01}) \leftarrow \pi_0(A_1)$ are surjective, the diagram of spaces

$$\begin{tikzcd} \mathcal{M}_{\mathrm{BT}}(A) \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{BT}}(A_0) \arrow[d] \\ \mathcal{M}_{\mathrm{BT}}(A_1) \arrow[r] & \mathcal{M}_{\mathrm{BT}}(A_{01}) \end{tikzcd}$$

is a pullback square.

(2) *The functor $\mathcal{M}_{\mathrm{BT}}$ is nilcomplete. That is, for every connective $\mathbb{E}_\infty$-ring A, the canonical map $\mathcal{M}_{\mathrm{BT}}(A) \to \varprojlim \mathcal{M}_{\mathrm{BT}}(\tau_{\leqslant n} A)$ is a homotopy equivalence.*

(3) *The functor $\mathcal{M}_{\mathrm{BT}}$ admits a (-1)-connective cotangent complex $L_{\mathcal{M}_{\mathrm{BT}}}$ (see Definition SAG.17.2.4.2).*

(4) *Let A be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0(A)$ be a finitely generated ideal, and write the formal spectrum* $\mathrm{Spf}(A)$ *as a filtered colimit* $\varinjlim \mathrm{Spec}(A_n)$ *(see Lemma SAG.8.1.2.2) for some tower of A-algebras $\{A_n\}_{n \geqslant 1}$. If A is I-complete, then the canonical map $\mathcal{M}_{\mathrm{BT}}(A) \to \varprojlim \mathcal{M}_{\mathrm{BT}}(A_n)$ is a homotopy equivalence.*

(5) *The functor $\mathcal{M}_{\mathrm{BT}}$ satisfies descent for the fpqc topology (in particular, it satisfies descent for the étale topology).*

Remark 3.2.3. We can state assertion (4) of Proposition 3.2.2 more informally as follows: if A is a connective $\mathbb{E}_\infty$-ring which is complete with respect to an ideal I, then the datum of a p-divisible group over the spectrum $\mathrm{Spec}(A)$ is equivalent to the datum of a p-divisible group over the formal spectrum $\mathrm{Spf}(A)$.

Remark 3.2.4. It follows from assertion (4) of Proposition 3.2.2 that the functor $\mathcal{M}_{\mathrm{BT}}$ is *integrable*, in the sense of Definition SAG.17.3.4.1 .

Proof of Proposition 3.2.2. Assertions (1) and (2) follow from Proposition AV.7.1.4 and assertion (3) from Proposition AV.7.1.5 . We will prove (4); the proof of (5) is similar. For every $\mathbb{E}_\infty$-ring R, let $\mathrm{Mod}_R^{\mathrm{ff}}$ denote the full subcategory of Mod_R spanned by the finite flat R-algebras and define $\mathrm{CAlg}_R^{\mathrm{ff}} \subseteq \mathrm{CAlg}_R$ similarly. Let $\mathrm{Ab}_{\mathrm{fin}}^p$ denote the category of finite abelian p-groups. By virtue of Proposition AV.6.5.5 , we can identify $\mathcal{M}_{\mathrm{BT}}(R)$ with the full category of $\mathrm{Fun}(\mathrm{Ab}_{\mathrm{fin}}^p, \mathrm{CAlg}_R^{\mathrm{ff}})^{\simeq}$ spanned by those functors F satisfying the following conditions:

(i) The functor F preserves finite coproducts.

(ii) For every monomorphism $M' \to M$ of finite abelian p-groups, the induced map $F(M') \to F(M)$ is faithfully flat.

(iii) For every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite abelian p-groups, the diagram

$$\begin{tikzcd} F(M') \arrow[r] \arrow[d] & F(M) \arrow[d] \\ F(0) \arrow[r] & F(M'') \end{tikzcd}$$

is a pushout square in $\mathrm{CAlg}_R^{\mathrm{ff}}$.

Let A be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0(A)$, and let $\{A_n\}$ be as in assertion (4). Using Theorem SAG.8.3.4.4 and Proposition SAG.8.3.5.7 , we see that $\mathrm{Mod}_A^{\mathrm{ff}}$ can be identified with the limit $\varprojlim \mathrm{Mod}_{A_n}^{\mathrm{ff}}$. Passing to commutative algebra objects, we obtain an equivalence $u : \mathrm{CAlg}_A^{\mathrm{ff}} \simeq \varprojlim \mathrm{CAlg}_{A_n}^{\mathrm{ff}}$, hence also an equivalence $\mathrm{Fun}(\mathrm{Ab}_{\mathrm{fin}}^p, \mathrm{CAlg}_A^{\mathrm{ff}}) \simeq \varprojlim \mathrm{Fun}(\mathrm{Ab}_{\mathrm{fin}}^p, \mathrm{CAlg}_{A_n}^{\mathrm{ff}}$. To complete the proof, it will suffice to show that a functor $F : \mathrm{Ab}_{\mathrm{fin}}^p \to \mathrm{CAlg}_A^{\mathrm{ff}}$ satisfies conditions (i), (ii), and (iii) if and only if each of the composite maps $\mathrm{Ab}_{\mathrm{fin}}^p \xrightarrow{F} \mathrm{CAlg}_A^{\mathrm{ff}} \to \mathrm{CAlg}_{A_n}^{\mathrm{ff}}$ satisfies conditions (i), (ii), and (iii). The "only if" direction is clear. The converse is immediate for conditions (i) and (iii) (since u is an equivalence of ∞-categories); for condition (ii), it follows from Proposition SAG.8.3.5.7 . □

Assertion (3) of Proposition 3.2.2 admits the following refinement:

Proposition 3.2.5. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R, corresponding to a point $\eta \in \mathcal{M}_{\mathrm{BT}}(R)$. If p is nilpotent in the commutative ring $\pi_0(R)$, then the R-module $\eta^* L_{\mathcal{M}_{\mathrm{BT}}}$ is connective and almost perfect.*

Warning 3.2.6. If we do not assume that p is nilpotent in $\pi_0(R)$, then the R-module $\eta^* L_{\mathcal{M}_{\mathrm{BT}}}$ is generally neither connective nor almost perfect.

Remark 3.2.7. Let R be a connective $\mathbb{E}_\infty$-ring. Assume that p is nilpotent in $\pi_0(R)$ and that $\pi_0(R)$ is a Grothendieck ring. The product $\mathrm{Spec}(R) \times \mathcal{M}_{\mathrm{BT}}$ represents a functor $\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$, given by composing $\mathcal{M}_{\mathrm{BT}}$ with the forgetful functor $\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CAlg}^{\mathrm{cn}}$. It follows from Propositions 3.2.2 and 3.2.5 that this functor satisfies all but one of the hypotheses of Artin's representability theorem (in the form given by Theorem SAG.18.3.0.1), which provide necessary and sufficient conditions for $\mathrm{Spec}(R) \times \mathcal{M}_{\mathrm{BT}}$ to be representable by a spectral Deligne-Mumford stack which is locally almost of finite presentation over R. However, Artin's theorem does not apply because $\mathrm{Spec}(R) \times \mathcal{M}_{\mathrm{BT}}$ is not locally almost of finite presentation over R: the functor $\mathcal{M}_{\mathrm{BT}}$ does not preserve filtered colimits when restricted to discrete R-algebras (see Warning 3.1.9). Nevertheless, Proposition 3.2.5 guarantees that the relative cotangent complex of the morphism $\mathrm{Spec}(R) \times \mathcal{M}_{\mathrm{BT}} \xrightarrow{\pi} \mathrm{Spec}(R)$ is almost perfect, so that π behaves "infinitesimally" as if it were locally almost of finite presentation. The central idea in the proof of Theorem 3.1.15 is that this infinitesimal finiteness property is good enough to guarantee the representability of $\mathcal{M}_{\mathrm{BT}}$ in a formal neighborhood of any sufficiently nice R-valued point.

The proof of Proposition 3.2.5 will require some auxiliary results.

Notation 3.2.8. Let R and Λ be connective $\mathbb{E}_\infty$-rings, and suppose we are given functors $X, Y : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_\Lambda^{\mathrm{cn}}$. For every object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$, we let X_A and Y_A denote the compositions of X and Y with the forgetful functor $\mathrm{CAlg}_A^{\mathrm{cn}} \to \mathrm{CAlg}_R^{\mathrm{cn}}$. The construction $A \mapsto \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_A^{\mathrm{cn}}, \mathrm{Mod}_\Lambda^{\mathrm{cn}})}(X_A, Y_A)$ determines a functor $\mathrm{CAlg}_R^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, which is classified by a natural transformation $\underline{\mathrm{Map}}_\Lambda(X, Y) \to \mathrm{Spec}(R)$ of functors from $\mathrm{CAlg}^{\mathrm{cn}}$ to $\widehat{\mathcal{S}}$. Here $\widehat{\mathcal{S}}$ denotes the ∞-category of spaces which are not necessarily small.

Lemma 3.2.9. *Let R be a connective $\mathbb{E}_\infty$-ring and let $X, Y : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Sp}^{\mathrm{cn}}$ be functors which are (representable by) finite flat group schemes over R. Then the projection map $u : \underline{\mathrm{Map}}_S(X, Y) \to \mathrm{Spec}(R)$ admits a cotangent complex which is connective and almost perfect.*

Proof. The existence of $L_{\underline{\mathrm{Map}}_S(X,Y)/\mathrm{Spec}(R)}$ follows from Proposition AV.6.1.12 , and the assertion that it is almost perfect follows from Remark AV.6.1.13 . We will complete the proof by showing that $L_{\underline{\mathrm{Map}}_S(X,Y)/\mathrm{Spec}(R)}$ is connective. Fix a connective $\mathbb{E}_\infty$-ring

A and a point $\eta \in \underline{\mathrm{Map}}_S(X,Y)(A)$; we wish to show that $M = \eta^* L_{\underline{\mathrm{Map}}_S(X,Y)/\operatorname{Spec}(R)}$ is connective. Note that M is automatically almost connective (this is part of the definition of a cotangent complex). We may therefore replace A by $\pi_0(A)$ and thereby reduce to the case where A is discrete (Proposition SAG.2.7.3.2).

If M is not connective, then there exists some largest integer k such that $\pi_{-k} M \neq 0$. Set $N = \pi_{-k} M$. Then the identity map from N to itself supplies a nonzero element of

$$\pi_k(\mathrm{Map}_{\mathrm{Mod}_R}(M,N)) \simeq \pi_k(\mathrm{fib}(\underline{\mathrm{Map}}_S(X,Y)(A \oplus N)) \to \underline{\mathrm{Map}}_S(X,Y)(A)).$$

This is impossible, since the spaces $\underline{\mathrm{Map}}_S(X,Y)(A)$ and $\underline{\mathrm{Map}}_S(X,Y)(A \oplus N)$ are both discrete. □

Lemma 3.2.10. *Let Λ denote the commutative ring $\mathbf{Z}/N\mathbf{Z}$ for some integer $N > 0$, and let S be the sphere spectrum. Then Λ is almost perfect when viewed as a module over $\Lambda \otimes_S \Lambda$.*

Proof. It follows from Proposition HA.7.2.4.31 that Λ is almost of finite presentation when viewed as an $\mathbb{E}_\infty$-algebra over S. Consequently, the tensor product $\Lambda \otimes_S \Lambda$ is almost of finite presentation when viewed as an $\mathbb{E}_\infty$-algebra over Λ. Applying Proposition HA.7.2.4.31 again, we deduce that $\Lambda \otimes_S \Lambda$ is Noetherian. It follows that a connective module M over $\Lambda \otimes_S \Lambda$ is almost perfect if and only if each homotopy group of M is finitely generated as a module over Λ (Proposition HA.7.2.4.17). In particular, Λ is almost perfect as a module over $\Lambda \otimes_S \Lambda$. □

Lemma 3.2.11. *Let R be a connective $\mathbb{E}_\infty$-ring, let $\Lambda = \mathbf{Z}/N\mathbf{Z}$ for some $N > 0$, and let $X, Y : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_\Lambda^{\mathrm{cn}}$ be functors which are representable by finite flat group schemes over R (that is, the functors $\Omega^\infty X, \Omega^\infty Y : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ are corepresentable by finite flat R-algebras). Then the projection map $\underline{\mathrm{Map}}_\Lambda(X,Y) \to \operatorname{Spec}(R)$ admits a cotangent complex $L_{\underline{\mathrm{Map}}_\Lambda(X,Y)/\operatorname{Spec}(R)}$ which is connective and almost perfect.*

Proof. For every connective bimodule $M \in {}_\Lambda\mathrm{BMod}_\Lambda(\mathrm{Sp}^{\mathrm{cn}})$, let Z_M denote the functor $\underline{\mathrm{Map}}_\Lambda(M \otimes_\Lambda X, Y)$. Let us say that a bimodule M is *weakly good* if the projection map $Z_M \to \operatorname{Spec}(R)$ admits a connective cotangent complex $L_{Z_M/\operatorname{Spec}(R)}$, and *good* if $L_{Z_M/\operatorname{Spec}(R)}$ is almost perfect. It follows immediately from the definitions that the construction $M \mapsto Z_M$ carries colimits in ${}_\Lambda\mathrm{BMod}_\Lambda(\mathrm{Sp}^{\mathrm{cn}})$ to limits in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\operatorname{Spec}(R)}$. Consequently, the collection of weakly good objects of ${}_\Lambda\mathrm{BMod}_\Lambda(\mathrm{Sp}^{\mathrm{cn}})$ is closed under small colimits (Remark SAG.17.2.4.5). Note that the functor $Z_{\Lambda \otimes_S \Lambda}$ can be identified with $\underline{\mathrm{Map}}_S(X,Y)$, so that $\Lambda \otimes_S \Lambda$ is good (Lemma 3.2.9). Since ${}_\Lambda\mathrm{BMod}_\Lambda(\mathrm{Sp}^{\mathrm{cn}})$ is generated under small colimits by the object $\Lambda \otimes_S \Lambda$, we deduce that every bimodule $M \in {}_\Lambda\mathrm{BMod}_\Lambda(\mathrm{Sp}^{\mathrm{cn}})$ is weakly good. We wish to show that Λ is good. By virtue of Lemma 3.2.10 and Corollary SAG.17.4.2.2 , it will suffice to prove the following:

(∗) If $M \in {}_\Lambda\mathrm{BMod}_\Lambda(\mathrm{Sp}^{\mathrm{cn}})$ is almost perfect when viewed as a module over $\Lambda \otimes_S \Lambda$, then the map $Z_M \to \mathrm{Spec}(R)$ is locally almost of finite presentation.

To prove (∗), let M be a connective almost perfect module over $\Lambda \otimes_S \Lambda$, and let $n \geqslant 0$. We wish to show that the functor $A \mapsto \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_A, \mathrm{Mod}^{\mathrm{cn}}_\Lambda)}(M \otimes_\Lambda X_A, Y_A)$ commutes with filtered colimits when restricted to $\tau_{\leqslant n}\,\mathrm{CAlg}^{\mathrm{cn}}_R$. Applying Corollary SAG.2.7.2.2 , we can choose a cofiber sequence $M' \to M \to M''$ in ${}_\Lambda\mathrm{BMod}_\Lambda(\mathrm{Sp})$ where M' is $(n+1)$-connective and M'' is perfect. Let $A \in \mathrm{CAlg}^{\mathrm{cn}}_R$ be n-truncated. Since X is representable by a finite flat group scheme over R, the functor X_A is a left Kan extension of its restriction to the full subcategory $\mathcal{E} \subseteq \mathrm{CAlg}^{\mathrm{cn}}_A$ spanned by the finite flat A-algebras. Since Y is a finite flat group scheme over R, the functor Y_A takes n-truncated values on $\mathcal{E}$. It follows that the space

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_A, \mathrm{Mod}^{\mathrm{cn}}_\Lambda)}(M' \otimes_\Lambda X_A, Y_A) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{E}, \mathrm{Mod}^{\mathrm{cn}}_\Lambda)}(M' \otimes_\Lambda X_A|_{\mathcal{E}}, Y_A|_{\mathcal{E}})$$

is contractible. Using the evident pullback diagram

$$\begin{tikzcd}
\underline{\mathrm{Map}}_\Lambda(M'' \otimes_R X, Y) \arrow[r] \arrow[d] & \underline{\mathrm{Map}}_\Lambda(M \otimes_R X, Y) \arrow[d] \\
\mathrm{Spec}(R) \arrow[r] & \underline{\mathrm{Map}}_\Lambda(M' \otimes_R X, Y),
\end{tikzcd}$$

we see that the projection map $\underline{\mathrm{Map}}_\Lambda(M \otimes_R X, Y) \to \underline{\mathrm{Map}}_\Lambda(M'' \otimes_R X, Y)$ is an equivalence when restricted to $\tau_{\leqslant n}\,\mathrm{CAlg}^{\mathrm{cn}}_R$. We may therefore replace M by M'' and thereby reduce to the case where M is perfect.

The assumption that M is perfect guarantees that there exists some integer k such that M has Tor-amplitude $\leqslant k$ (when viewed as a module over $\Lambda \otimes_S \Lambda$). We proceed by induction on k. Since M is connective and perfect, the homotopy group $\pi_0(M)$ is finitely generated as a module over Λ. We can therefore choose free $(\Lambda \otimes_S \Lambda)$-module P of finite rank and a morphism $\alpha : P \to M$ which is surjective on π_0. Set $Q = \mathrm{fib}(\alpha)$, so that we have a pullback diagram

$$\begin{tikzcd}
Z_M \arrow[r] \arrow[d] & Z_P \arrow[d] \\
\mathrm{Spec}(R) \arrow[r] & Z_Q.
\end{tikzcd}$$

If $k > 0$, then Q has Tor-amplitude $< k$, and our inductive hypothesis guarantees that the map $Z_Q \to \mathrm{Spec}(R)$ is locally almost of finite presentation. If $k = 0$, then M is a projective Λ-module, so that α admits a right homotopy inverse which exhibits Z_M as a retract of Z_P. In either case, we are reduced to showing that the map $Z_P \to \mathrm{Spec}(R)$ is locally almost of finite presentation. We may therefore replace M by P and thereby reduce to the case where M is a free module of finite rank r over $\Lambda \otimes_S \Lambda$. Proceeding by induction on r, we can further reduce to the case $r = 1$: that is, to the case $M = \Lambda \otimes_S \Lambda$. In this case, we have $Z_M = \underline{\mathrm{Map}}_S(X, Y)$ and the desired result follows from Remark AV.6.1.11. $\square$

Lemma 3.2.12. *Let R be a connective $\mathbb{E}_\infty$-ring which is (p)-nilpotent, and let $\mathbf{G}$ and $\mathbf{G}'$ be p-divisible groups over R. Then the projection map $\underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}, \mathbf{G}') \to \mathrm{Spec}(R)$ admits a cotangent complex which is 1-connective and almost perfect.*

Proof. The existence of the cotangent complex $L_{\underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}, \mathbf{G}')/\mathrm{Spec}(R)}$ follows from Proposition AV.7.1.5. Fix a connective $\mathbb{E}_\infty$-ring A and a point $\eta \in \underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}, \mathbf{G}')(A)$, corresponding to a map of $\mathbb{E}_\infty$-rings $R \to A$ and a morphism of p-divisible groups $f : \mathbf{G}_A \to \mathbf{G}'_A$. We wish to show that the pullback $M = \eta^* L_{\underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}, \mathbf{G}')/\mathrm{Spec}(R)}$ is 1-connective and almost perfect. By homogeneity, we may assume without loss of generality that $f = 0$.

Choose an integer k such that $p^k = 0$ in $\pi_0(R)$, and form a fiber sequence

$$\mathbf{G}[p^k] \to \mathbf{G} \xrightarrow{p^k} \mathbf{G}.$$

Since $\mathbf{G}$ is a p-divisible group, this fiber is also a cofiber sequence of $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$-valued sheaves with respect to the fppf topology. We therefore obtain a pullback diagram

$$\begin{tikzcd}
\underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}, \mathbf{G}') \arrow[r] \arrow[d] & \underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}, \mathbf{G}') \arrow[d, "u"] \\
\mathrm{Spec}(R) \arrow[r] & \underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}[p^k], \mathbf{G}'),
\end{tikzcd}$$

whose bottom right corner can be identified with the functor

$$Z = \underline{\mathrm{Map}}_{\mathbf{Z}/p^k\mathbf{Z}}(\mathbf{G}[p^k], \mathbf{G}'[p^k]).$$

It follows from Lemma 3.2.11 that the projection map $Z \to \mathrm{Spec}(R)$ admits a cotangent complex which is connective and almost perfect. We therefore obtain a fiber sequence of A-modules $u(\eta)^* L_{Z/\mathrm{Spec}(R)} \to M \xrightarrow{q} M$, where q is induced by the endomorphism of $\underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}, \mathbf{G}')$ given by multiplication by p^k. It follows easily that q coincides with the map given by multiplication p^k on M, and is therefore nullhomotopic (since p^k vanishes in $\pi_0(A)$). The preceding fiber sequence therefore supplies a splitting $u(\eta)^* L_{Z/\operatorname{Spec}(R)} \simeq M \oplus \Sigma^{-1} M$. In particular, we see that M appears as a direct summand of $u(\eta)^* \Sigma^1 L_{Z/\operatorname{Spec}(R)}$, and is therefore 1-connective and almost perfect. □

Notation 3.2.13. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. For any connective $\mathbb{E}_\infty$-ring R, we can identify A-valued points of $\underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}, \mathbf{G})$ with pairs (u, f), where $u : R \to A$ is a map of $\mathbb{E}_\infty$-rings and $f : X_A \to X_A$ is a morphism of p-divisible groups over A. We let $\underline{\mathrm{Aut}}(\mathbf{G})(A)$ denote the summand of $\underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}, \mathbf{G})(A)$ corresponding to those pairs (u, f) where f is an equivalence of p-divisible groups. We regard the construction $A \mapsto \underline{\mathrm{Aut}}(\mathbf{G})(A)$ as a functor $\underline{\mathrm{Aut}}(\mathbf{G}) : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$.

Lemma 3.2.14. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. Then the inclusion $\underline{\mathrm{Aut}}(\mathbf{G}) \hookrightarrow \underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}, \mathbf{G})$ is an open immersion of functors (see Definition SAG.19.2.4.1).*

Proof. Suppose we are given an R'-valued point of $\underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}, \mathbf{G})$, given by a morphism of connective $\mathbb{E}_\infty$-rings $R \to R'$ and a morphism of p-divisible groups $f : \mathbf{G}_{R'} \to \mathbf{G}_{R'}$. We wish to show that $U = \operatorname{Spec}(R') \times_{\underline{\mathrm{Map}}_{\mathbf{Z}}(\mathbf{G}, \mathbf{G})} \underline{\mathrm{Aut}}(\mathbf{G})$ is an open subfunctor of $\operatorname{Spec}(R')$. Without loss of generality, we may assume that $R = R'$, so that f determines a map from $\mathbf{G}$ to itself, which induces a map $f_0 : \mathbf{G}[p] \to \mathbf{G}[p]$. The functor $\Omega^\infty \mathbf{G}[p]$ is representable by an $\mathbb{E}_\infty$-algebra $H \in \mathrm{CAlg}_R^{\mathrm{cn}}$ which is finite flat over R, so f_0 determines an R-module map $u : H \to H$. We claim that U is the open subfunctor of $\operatorname{Spec}(R)$ complementary to the support of the perfect R-module $\operatorname{cofib}(u)$. To prove this, it will suffice (by virtue of the compatibility of support with base change) to show that f is an equivalence if and only if $\operatorname{cofib}(u)$ vanishes: that is, if and only if f_0 is an equivalence. The "only if" direction is obvious. For the converse, we observe that if f_0 is an equivalence, then it follows by induction that f induces an equivalence $\mathbf{G}[p^k] \to \mathbf{G}[p^k]$ for each $k \geqslant 0$; passing to the direct limit over k, we conclude that f is an equivalence. □

Lemma 3.2.15. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. If R is (p)-nilpotent, then the projection map $\underline{\mathrm{Aut}}(\mathbf{G}) \to \operatorname{Spec}(R)$ admits a cotangent complex which is 1-connective and almost perfect.*

Proof. Combine Lemmas 3.2.12 and 3.2.14. □

Proof of Proposition 3.2.5. Let R be a connective $\mathbb{E}_\infty$-ring which is (p)-nilpotent and let $\eta \in \mathcal{M}_{\mathrm{BT}}(R)$, corresponding to a p-divisible group $\mathbf{G}$ over R. We wish to show that the R-module $\eta^* L_{\mathcal{M}_{\mathrm{BT}}}$; connective and almost perfect. Unwinding the definitions, we have a pullback diagram of functors

$$\begin{tikzcd}
\underline{\mathrm{Aut}}(\mathbf{G}) \arrow[r] \arrow[d] & \mathrm{Spec}(R) \arrow[d] \\
\mathrm{Spec}(R) \arrow[r, "(\eta,\mathrm{id})"] & \mathcal{M}_{\mathrm{BT}} \times \mathrm{Spec}(R).
\end{tikzcd}$$

Setting $Z = \mathcal{M}_{\mathrm{BT}} \times \mathrm{Spec}(R)$ and let $\delta : \mathrm{Spec}(R) \to \underline{\mathrm{Aut}}(\mathbf{G})$ denote the diagonal map (classifying the identity automorphism of $\mathbf{G}$). We then have natural equivalences

$$\begin{aligned}
\eta^* L_{\mathcal{M}_{\mathrm{BT}}} &\simeq (\eta, \mathrm{id})^* L_{Z/\mathrm{Spec}(R)} \\
&\simeq \Sigma^{-1} L_{\mathrm{Spec}(R)/Z} \\
&\simeq \Sigma^{-1} \delta^* L_{\underline{\mathrm{Aut}}(\mathbf{G})/\mathrm{Spec}(R)}.
\end{aligned}$$

The desired result now follows from the observation that $L_{\underline{\mathrm{Aut}}(\mathbf{G})/\mathrm{Spec}(R)}$ is 1-connective and almost perfect (Lemma 3.2.15). $\square$

3.3 The Cotangent Complex of an F-Finite Ring

Our primary goal in this section is to prove the following purely algebraic fact, which will be needed in our proof of Theorem 3.1.15:

Proposition 3.3.1. *Let R be an F-finite Noetherian $\mathbf{F}_p$-algebra. Then the absolute cotangent complex L_R is almost perfect as an R-module.*

Remark 3.3.2. We will prove a converse to Proposition 3.3.1 in §2.3.18 (see Theorem 3.5.1).

In the statement of Proposition 3.3.1, we use L_R to denote the cotangent complex of R in the setting of $\mathbb{E}_\infty$-rings (that is, the R-module spectrum whose homotopy groups are the *topological* André-Quillen homology groups of R). Since R is an ordinary commutative ring, we can also consider the *algebraic* cotangent complex of R: that is, the R-module spectrum whose homotopy groups are the classical André-Quillen homology groups of R. We will denote the latter by L_R^{alg}. The analogue of Proposition 3.3.1 for the algebraic cotangent complex is somewhat easier to prove. However, it formally implies Proposition 3.3.1, by virtue of the following:

Lemma 3.3.3. *Let $f : A \to B$ be a homomorphism of commutative rings. The following conditions are equivalent:*

(*a*) *The relative algebraic cotangent complex $L^{\mathrm{alg}}_{B/A}$ is almost perfect as a B-module.*

(*b*) *The relative topological cotangent complex $L_{B/A}$ is almost perfect as a B-module.*

Proof. Let B^+ be the $\mathbb{E}_1$-ring of Remark SAG.25.3.3.4 , so that $L_{B/A}$ has the structure of a left B^+-module and we have an equivalence $L^{\mathrm{alg}}_{B/A} \simeq B \otimes_{B^+} L_{B/A}$ (Remark SAG.25.3.3.7). It follows from Proposition SAG.2.7.3.1 that (a) is equivalent to the following:

(a') The relative topological cotangent complex $L_{B/A}$ is almost perfect as a left B^+-module.

To show that (a') and (b) are equivalent, it will suffice to show that B^+ is almost perfect when viewed as a left B-module. This follows from the equivalence $B^+ \simeq B \otimes_S \mathbf{Z}$ of Proposition SAG.25.3.4.2 , since $\mathbf{Z}$ is almost perfect when viewed as an S-module. □

Proposition 3.3.4. *Let R be a commutative ring. The following conditions are equivalent:*

(1) *The algebraic cotangent complex $L^{\mathrm{alg}}_{R} \simeq L^{\mathrm{alg}}_{R/\mathbf{Z}}$ is almost perfect as an R-module.*

(2) *The topological cotangent complex L_R is almost perfect as an R-module.*

(3) *The relative topological cotangent complex $L_{R/\mathbf{Z}}$ is almost perfect as an R-module.*

Proof. The equivalence of (1) and (3) is a special case of Lemma 3.3.3. We will show that (2) and (3) are equivalent. Using the fiber sequence $R \otimes_{\mathbf{Z}} L_{\mathbf{Z}} \to L_R \to L_{R/\mathbf{Z}}$, we are reduced to showing that $L_{\mathbf{Z}} = L_{\mathbf{Z}/S}$ is almost perfect as a module over $\mathbf{Z}$. This is clear, since $\mathbf{Z}$ is almost of finite presentation as an $\mathbb{E}_\infty$-algebra over the sphere spectrum (Proposition HA.7.2.4.31). □

We now specialize to the study of $\mathbf{F}_p$-algebras.

Notation 3.3.5. Let R be a commutative algebra over $\mathbf{F}_p$. We let $R^{1/p}$ denote the same commutative ring, regarded as an R-algebra via the Frobenius morphism

$$\varphi_R : R \to R \qquad \varphi_R(x) = x^p.$$

Lemma 3.3.6. *Let $f : A \to B$ be a morphism of commutative $\mathbf{F}_p$-algebras. Then the canonical map $\rho : B^{1/p} \otimes_B L^{\mathrm{alg}}_{B/A} \to L^{\mathrm{alg}}_{B^{1/p}/A^{1/p}}$ is nullhomotopic.*

Proof. Choose a simplicial A-algebra $P_\bullet$, where each P_k is a polynomial algebra over A (possibly on infinitely many generators) and $B \simeq |P_\bullet|$. Then ρ can be identified with the geometric realization of a map of simplicial $B^{1/p}$-modules

$$\rho_\bullet : B^{1/p} \otimes_{P_\bullet} L^{\mathrm{alg}}_{P_\bullet/A} \to B^{1/p} \otimes_{P_\bullet^{1/p}} L^{\mathrm{alg}}_{P_\bullet^{1/p}/A^{1/p}}.$$

Note that each ρ_n is a map of free $B^{1/p}$-modules. It will therefore suffice to show that each ρ_n is nullhomotopic. Observe that ρ_n is obtained by extension of scalars from the map of P_n-modules $\Omega_{P_n/A} \to \Omega_{P_n^{1/p}/A^{1/p}}$ induced by the Frobenius map on P_n. This map vanishes by virtue of the calculation $d(x^p) = px^{p-1}dx = 0$ for $x \in P_n$. $\square$

Proposition 3.3.7. *Let R be a commutative $\mathbf{F}_p$-algebra. The following conditions are equivalent:*

1. *The commutative ring R satisfies the equivalent conditions of Proposition 3.3.4.*
2. *The relative algebraic cotangent complex $L^{\mathrm{alg}}_{R/\mathbf{F}_p}$ is almost perfect as an R-module.*
3. *The relative topological cotangent complex $L_{R/\mathbf{F}_p}$ is almost perfect as an R-module.*
4. *The relative algebraic cotangent complex $L^{\mathrm{alg}}_{R^{1/p}/R}$ is almost perfect as an $R^{1/p}$-module.*
5. *The relative topological cotangent complex $L_{R^{1/p}/R}$ is almost perfect as an $R^{1/p}$-module.*

Proof. Note that $L^{\mathrm{alg}}_{\mathbf{F}_p} \simeq \Sigma(\mathbf{F}_p)$ is a perfect $\mathbf{F}_p$-module. so that $R \otimes_{\mathbf{F}_p} L^{\mathrm{alg}}_{\mathbf{F}_p} \simeq \Sigma(R)$ is perfect as an R-module. Using the fiber sequence $R \otimes_{\mathbf{F}_p} L^{\mathrm{alg}}_{\mathbf{F}_p} \to L^{\mathrm{alg}}_R \to L^{\mathrm{alg}}_{R/\mathbf{F}_p}$, we deduce that (1) $\Leftrightarrow$ (2). The equivalences (2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5) follow from Lemma 3.3.3. We have a fiber sequence

$$R^{1/p} \otimes_R L^{\mathrm{alg}}_{R/\mathbf{F}_p} \xrightarrow{\alpha} L^{\mathrm{alg}}_{R^{1/p}/\mathbf{F}_p} \to L^{\mathrm{alg}}_{R^{1/p}/R}$$

where α is nullhomotopic (Lemma 3.3.6), and therefore a splitting of $R^{1/p}$-modules

$$L^{\mathrm{alg}}_{R^{1/p}/R} \simeq L^{\mathrm{alg}}_{R^{1/p}/\mathbf{F}_p} \oplus \Sigma(R^{1/p} \otimes_R L^{\mathrm{alg}}_{R/\mathbf{F}_p})$$

which shows that (2) and (4) are equivalent (note that $L^{\mathrm{alg}}_{R/\mathbf{F}_p}$ is almost perfect as an R-module if and only if $L^{\mathrm{alg}}_{R^{1/p}/\mathbf{F}_p}$ is almost perfect as an $R^{1/p}$-module). $\square$

Corollary 3.3.8. *Let R be a commutative $\mathbf{F}_p$-algebra. Suppose that the Frobenius map φ_R exhibits $R^{1/p}$ as an almost perfect R-module. Then L_R and L_R^{alg} are almost perfect as R-modules.*

Proof. If $R^{1/p}$ is almost perfect as an R-module, then it is almost of finite presentation as an $\mathbb{E}_\infty$-algebra over R (Corollary SAG.5.2.2.2). It follows that $L_{R^{1/p}/R}$ is almost perfect as an $R^{1/p}$-module, so that the desired result follows from Proposition 3.3.7. □

Proof of Proposition 3.3.1. Since R is Noetherian, the assumption that R is F-finite guarantees that $R^{1/p}$ is almost perfect as an R-module (Proposition HA.7.2.4.17). Invoking Corollary 3.3.8, we conclude that L_R and L_R^{alg} are almost perfect. □

We close this section by establishing a useful closure property for F-finite rings:

Proposition 3.3.9. *Let R be a Noetherian $\mathbf{F}_p$-algebra which is complete with respect to an ideal $I \subseteq R$. If R/I is F-finite, then R is F-finite.*

Proof. Since R is Noetherian, we can choose a finite collection of elements $x_1, \ldots, x_n$ which generate the ideal I. Let J denote the ideal $(x_1^p, \ldots, x_n^p)$, so that we have $I^{p^n} \subseteq J \subseteq I$. It follows that R/J admits a finite filtration whose successive quotients $(I^a + J)/(I^{a+1} + J)$ are finitely generated as modules over R/I. Our assumption that R/I is F-finite guarantees that $(R/I)^{1/p}$ is finitely generated as an R-module, so that $(R/J)^{1/p}$ is also finitely generated as an R-module. We can therefore choose elements $y_1, \ldots, y_m \in R$ with the following property:

($*$) For every element $t \in R$, there exists coefficients $c_{1,1}, \ldots, c_{1,m} \in R$ such that

$$t \equiv c_{1,1}^p y_1 + \cdots + c_{1,m}^p y_m \pmod{J}.$$

Applying ($*$) repeatedly, we can choose elements $c_{i,1}, \ldots, c_{i,m} \in R$ for each $i > 1$ satisfying

$$t \equiv c_{i,1}^p y_1 + \cdots + c_{i,m}^p y_m \pmod{J^i} \qquad c_{i+1,j} \equiv c_{i,j} \pmod{I^i}$$

Since R is I-complete, the sequences $\{c_{i,j}\}_{i \geqslant 1}$ converge for each $1 \leqslant j \leqslant m$ to an element $c_i \in R$, and these elements satisfy $t = c_1^p y_1 + \cdots + c_m^p y_m$. It follows that the elements $y_1, \ldots, y_m$ generate $R^{1/p}$ as an R-module, so that R is F-finite. □

3.4 The Proof of Theorem 3.1.15

We will deduce Theorem 3.1.15 from the following more general statement:

Theorem 3.4.1. *Let R_0 be a commutative ring and let $\mathbf{G}_0$ be a p-divisible group over R_0. Assume that p is nilpotent in R_0, that $\mathbf{G}_0$ is nonstationary, and that the absolute cotangent complex L_{R_0} is almost perfect as a module over R_0. Then:*

(1) *The p-divisible group $\mathbf{G}_0$ admits a universal deformation (in the sense of Definition 3.1.11): that is, the functor $\mathrm{Def}_{\mathbf{G}_0} : \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}$ is corepresentable by a complete adic $\mathbb{E}_\infty$-ring $R^{\mathrm{un}}_{\mathbf{G}_0}$.*

(2) *The spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is connective.*

(3) *The canonical map $\rho : R^{\mathrm{un}}_{\mathbf{G}_0} \to R_0$ (see Remark 3.1.12) induces a surjective ring homomorphism $\epsilon : \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0}) \to R_0$.*

(4) *The map ρ exhibits R_0 as an almost perfect module over $R^{\mathrm{un}}_{\mathbf{G}_0}$. In particular, the kernel ideal $\ker(\epsilon)$ is finitely generated.*

(5) *The kernel $\ker(\epsilon)$ is an ideal of definition for $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$. In particular, the $\mathbb{E}_\infty$-ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is complete with respect to $\ker(\epsilon)$.*

(6) *If R_0 is Noetherian, then $R^{\mathrm{un}}_{\mathbf{G}_0}$ is also Noetherian.*

Proof of Theorem 3.1.15. Combine Theorems 3.4.1 and 3.3.1. □

Remark 3.4.2. In the case where R_0 is Noetherian, Theorem 3.4.1 is not really any more general than Theorem 3.1.15. Since p is nilpotent in R_0, there is no harm in replacing R_0 by the quotient R_0/pR_0 (see Remark 3.1.8), in which case the assumption that L_{R_0} is almost perfect is equivalent to the requirement that R_0 is F-finite (Theorem 3.5.1). Nevertheless, Theorem 3.4.1 can be regarded as an improvement of Theorem 3.1.15 because it can also be applied to *non-Noetherian* rings. For example, if $\mathbf{G}_0$ is any p-divisible group defined over a perfect $\mathbf{F}_p$-algebra R_0, then $\mathbf{G}_0$ is automatically nonstationary (Example 3.0.10) and the cotangent complex L_{R_0} is almost perfect (Corollary 3.3.8), so Theorem 3.4.1 guarantees the existence of a universal deformation of $\mathbf{G}_0$.

We now turn to the proof of Theorem 3.4.1. In what follows, we will assume that the reader is familiar with the notion of the *relative de Rham space* $(X/Y)_{\mathrm{dR}}$ associated to a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ (see

Definition SAG.18.2.1.1). We will also abuse notation by identifying the ∞-category CAlg of $\mathbb{E}_\infty$-rings with a full subcategory of the ∞-category $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ of complete adic $\mathbb{E}_\infty$-rings, via the construction which associates to each $\mathbb{E}_\infty$-ring A the discrete topology on the underlying commutative ring $\pi_0(A)$.

The following statement is *almost* an immediate consequence of the definitions:

Proposition 3.4.3. *Let R_0 be a commutative ring and let $\mathbf{G}_0$ be a p-divisible group over R_0, classified by a natural transformation of functors $f : \mathrm{Spec}(R_0) \to \mathcal{M}_{\mathrm{BT}}$. Assume that p is nilpotent in R_0 and that the cotangent complex L_{R_0} is almost perfect as an R_0-module. Then the functor $\mathrm{Def}_{\mathbf{G}_0}|_{\mathrm{CAlg}^{\mathrm{cn}}}$ can be identified with the relative de Rham space $(\mathrm{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}})_{\mathrm{dR}}$.*

Proof. Set $X = (\mathrm{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}})_{\mathrm{dR}}$. By definition, X is a functor from the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-rings to the ∞-category $\mathcal{S}$ of spaces, given by the formula

$$X(A) = \varinjlim_{I \in \mathrm{Nil}(A)} \mathcal{M}_{\mathrm{BT}}(A) \times_{\mathcal{M}_{\mathrm{BT}}(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I),$$

where the colimit is taken over the collection $\mathrm{Nil}(A)$ of all nilpotent ideals in the commutative ring $\pi_0(A)$. On the other hand, the space $\mathrm{Def}_{\mathbf{G}_0}(A)$ is defined as the colimit $\varinjlim_{I \in \mathrm{Nil}_0(A)} \mathcal{M}_{\mathrm{BT}}(A) \times_{\mathcal{M}_{\mathrm{BT}}(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I)$, where $\mathrm{Nil}_0(A) \subseteq \mathrm{Nil}(A)$ is the collection of all finitely generated nilpotent ideals of $\pi_0(A)$ (these are precisely the finitely generated ideals of definition for the trivial topology on $\pi_0(A)$). We therefore have a canonical map $\theta_A : \mathrm{Def}_{\mathbf{G}_0}(A) \to X(A)$, depending functorially on A. We will complete the proof by showing that each θ_A is a homotopy equivalence.

Note that θ_A is a pullback of $\theta_{\pi_0(A)}$; we may therefore replace A by $\pi_0(A)$ and thereby reduce to the case where A is discrete (which permits us to simplify our notation). Define $F : \mathrm{Nil}(A) \to \mathcal{S}$ by the formula $F(I) = \mathcal{M}_{\mathrm{BT}}(A) \times_{\mathcal{M}_{\mathrm{BT}}(A/I)} \mathrm{Hom}(R_0, A/I)$. We will show that F is a left Kan extension of its restriction $F|_{\mathrm{Nil}_0(A)}$. To prove this, fix a nilpotent ideal $I \subseteq A$; we wish to show that the canonical map $\rho : \varinjlim_{J \subseteq I, J \in \mathrm{Nil}_0(A)} F(J) \to F(I)$ is a homotopy equivalence. Unwinding the definitions, we see that ρ fits into a pullback diagram

$$\begin{tikzcd}
\varinjlim_{J \subseteq I, J \in \mathrm{Nil}_0(A)} F(J) \arrow[r, "\rho"] \arrow[d] & F(I) \arrow[d] \\
\varinjlim_{J \subseteq I, J \in \mathrm{Nil}_0(A)} \mathrm{Hom}(R_0, A/J) \arrow[r, "\rho'"] & \mathrm{Hom}(R_0, A/I) \times_{X(A/I)} \varinjlim_{J \subseteq I, J \in \mathrm{Nil}_0(A)} X(A/J).
\end{tikzcd}$$

It will therefore suffice to show that ρ' is an equivalence. In fact, we claim that the natural transformation $\mathrm{Spec}(R_0) \to X$ is locally almost of finite presentation.

By virtue of Proposition SAG.18.2.1.13 , it will suffice to show that f is nilcomplete, infinitesimally cohesive, and admits an almost perfect cotangent complex. The first two assertions follow from Proposition AV.7.1.4 , and the third follows from the existence of a fiber sequence

$$L_{\mathcal{M}_{\mathrm{BT}}}|_{\mathrm{Spec}(R_0)} \to L_{\mathrm{Spec}(R_0)} \to L_{\mathrm{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}}};$$

note that the first term is almost perfect by virtue of Proposition 3.2.5 and the middle term is almost perfect by assumption. □

Remark 3.4.4. Let R_0 and $\mathbf{G}_0$ be as in the statement of Proposition 3.4.3. Using the fiber sequence

$$L_{\mathcal{M}_{\mathrm{BT}}}|_{\mathrm{Spec}(R_0)} \to L_{\mathrm{Spec}(R_0)} \to L_{\mathrm{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}}}$$

together with Proposition 3.2.5, we see that $L_{\mathrm{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}}}$ is connective and almost perfect. Moreover, the following conditions are equivalent:

(a) The relative cotangent complex $L_{\mathrm{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}}}$ is 1-connective.

(b) The canonical map $\beta : L_{\mathcal{M}_{\mathrm{BT}}}|_{\mathrm{Spec}(R_0)} \to L_{\mathrm{Spec}(R_0)}$ is surjective on π_0.

(c) For each residue field κ of R_0, the induced map $\beta_\kappa : L_{\mathcal{M}_{\mathrm{BT}}}|_{\mathrm{Spec}(\kappa)} \to \kappa \otimes_{R_0} L_{R_0}$ is surjective on π_0.

(d) For every residue field κ of R_0 and every nonzero map $\gamma : L_{R_0} \to \kappa$, the composite map $\gamma \circ \beta$ is also nonzero.

(e) For every residue field κ of R_0 and every derivation $d : R_0 \to \kappa$, classifying a ring homomorphism $R_0 \to \kappa[\epsilon]/(\epsilon^2)$, the p-divisible group $(\mathbf{G}_0)_d$ of Construction 3.0.7 is nontrivial as a first order deformation of $(\mathbf{G}_0)_\kappa$.

(f) The p-divisible group $\mathbf{G}_0$ is nonstationary, in the sense of Definition 3.0.8.

The implications $(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$ are immediate, while the implication $(c) \Rightarrow (b)$ follows from Nakayama's lemma (since $\pi_0 L_{R_0}$ is finitely generated as a module over R_0).

Proof of Theorem 3.4.1. Let R_0 be a commutative ring in which p is nilpotent and let $\mathbf{G}_0$ be p-divisible group over R_0, classified by a natural transformation $f : \mathrm{Spec}(R_0) \to$

$\mathcal{M}_{\mathrm{BT}}$. Using Proposition 3.2.2, we see that the functor $\mathcal{M}_{\mathrm{BT}}$ is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. If the cotangent complex L_{R_0} is almost perfect and that $\mathbf{G}_0$ is nonstationary, then the relative cotangent complex $L_{\mathrm{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}}}$ is 1-connective and almost perfect (Remark 3.4.4). By virtue of Proposition 3.4.3, we can identify the restriction $\mathrm{Def}_{\mathbf{G}_0}|_{\mathrm{CAlg}^{\mathrm{cn}}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ with the relative de Rham space $(\mathrm{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}})_{\mathrm{dR}}$. Applying Theorem SAG.18.2.3.1 , we deduce that $\mathrm{Def}_{\mathbf{G}_0}|_{\mathrm{CAlg}^{\mathrm{cn}}}$ is (representable by) formal thickening of $\mathrm{Spec}(R_0)$. It follows from Using Corollary SAG.18.2.3.3 and Proposition SAG.18.2.2.8 , we see that the functor $\mathrm{Def}_{\mathbf{G}_0}|_{\mathrm{CAlg}^{\mathrm{cn}}}$ is corepresentable by an object $R \in \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ satisfying conditions (2) through (5) (which is Noetherian if R_0 is Noetherian, by virtue of Corollary SAG.18.2.4.4). Let $h^R : \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}$ denote the functor corepresented by R; we will complete the proof by showing that the equivalence $h^R|_{\mathrm{CAlg}^{\mathrm{cn}}} \simeq \mathrm{Def}_{\mathbf{G}_0}|_{\mathrm{CAlg}^{\mathrm{cn}}}$ can be lifted (in an essentially unique way) to an equivalence $h^R \simeq \mathrm{Def}_{\mathbf{G}_0}$.

Let $\mathcal{C} \subseteq \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ denote the full subcategory spanned by those complete adic $\mathbb{E}_\infty$-rings which are connective. To complete the proof, it will suffice to prove the following:

(*i*) The functors h^R and $\mathrm{Def}_{\mathbf{G}_0}$ are left Kan extensions of the restrictions $h^R|_{\mathcal{C}}$ and $\mathrm{Def}_{\mathbf{G}_0}|_{\mathcal{C}}$, respectively.

(*ii*) The functors $h^R|_{\mathcal{C}}$ and $\mathrm{Def}_{\mathbf{G}_0}|_{\mathcal{C}}$ are right Kan extensions of the restrictions $h^R|_{\mathrm{CAlg}^{\mathrm{cn}}}$ and $\mathrm{Def}_{\mathbf{G}_0}|_{\mathrm{CAlg}^{\mathrm{cn}}}$, respectively.

Claim (*i*) is essentially tautological: it asserts that for any complete adic $\mathbb{E}_\infty$-ring A, the canonical maps

$$\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(R, \tau_{\geqslant 0}A) \to \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(R, A) \qquad \mathrm{Def}_{\mathbf{G}_0}(\tau_{\geqslant 0}A) \to \mathrm{Def}_{\mathbf{G}_0}(A)$$

are homotopy equivalences. The map on the left is a homotopy equivalence since R is connective, and the map on the right is a homotopy equivalence because the extension of scalars functor $\mathrm{BT}^p(\tau_{\geqslant 0}A) \to \mathrm{BT}^p(A)$ is an equivalence of ∞-categories (Remark AV.6.5.3). We now prove (*ii*). Fix a connective complete adic $\mathbb{E}_\infty$-ring A; we wish to show that $h^R|_{\mathcal{C}}$ and $\mathrm{Def}_{\mathbf{G}_0}|_{\mathcal{C}}$ are right Kan extensions of $h^R|_{\mathrm{CAlg}^{\mathrm{cn}}}$ and $\mathrm{Def}_{\mathbf{G}_0}|_{\mathrm{CAlg}^{\mathrm{cn}}}$ at A, respectively. Let $I \subseteq \pi_0(A)$ be a finitely generated ideal of definition. Invoking Lemma SAG.8.1.2.2 , we can write A as the limit of a tower of square-zero extensions

$$\cdots \to A_3 \to A_2 \to A_1$$

with the following property: for each $B \in \mathrm{CAlg}^{\mathrm{cn}}$, the canonical map

$$\varinjlim \mathrm{Map}_{\mathrm{CAlg}}(A_n, B) \to \mathrm{Map}_{\mathrm{CAlg}}(A, B)$$

is a homotopy equivalence onto the summand of $\mathrm{Map}_{\mathrm{CAlg}}(A, B)$ consisting of those maps $A \to B$ which annihilate some power of I. Unwinding the definitions, we wish to show that the canonical map

$$\phi : \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(R, A) \to \varprojlim \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(R, A_n) \qquad \psi : \mathrm{Def}_{\mathbf{G}_0}(A) \to \varprojlim \mathrm{Def}_{\mathbf{G}_0}(A_n).$$

Note that a morphism of $\mathbb{E}_\infty$-algebras $R \to A$ is continuous (with respect to the topologies on $\pi_0(R)$ and $\pi_0(A)$) if and only if the composite map $R \to A \to A_1$ is continuous (where we endow $\pi_0(A_1)$ with the discrete topology), so that ϕ fits into a pullback diagram

$$\begin{tikzcd}
\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(R, A) \arrow[r, "\phi"] \arrow[d] & \varprojlim \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(R, A_n) \arrow[d] \\
\mathrm{Map}_{\mathrm{CAlg}}(R, A) \arrow[r, "\phi'"] & \varprojlim \mathrm{Map}_{\mathrm{CAlg}}(R, A_n).
\end{tikzcd}$$

Our assumption that A is complete guarantees that ϕ' is a homotopy equivalence, so that ϕ is also a homotopy equivalence. Unwinding the definitions, we see that ψ fits into a pullback diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Def}_{\mathbf{G}_0}(A) \arrow[r, "\psi"] \arrow[d] & \varprojlim \mathrm{Def}_{\mathbf{G}_0}(A_n) \arrow[d] \\
\mathcal{M}_{\mathrm{BT}}(A) \arrow[r, "\psi'"] & \varprojlim \mathcal{M}_{\mathrm{BT}}(A_n).
\end{tikzcd}$$

Since ψ' is a homotopy equivalence (Proposition 3.2.2), it follows that ψ is a homotopy equivalence as desired. $\square$

Remark 3.4.5. Let R_0 be a commutative ring in which p is nilpotent and let $\mathbf{G}_0$ be a p-divisible group over R_0. Suppose that $\mathbf{G}_0$ admits a spectral deformation ring $R = R^{\mathrm{un}}_{\mathbf{G}_0}$ satisfying conditions (3) and (4) of Theorem 3.4.1 (note that condition (2) is automatically satisfied as well). Then the relative de Rham space $(\mathrm{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}})_{\mathrm{dR}}$ can be identified with the formal spectrum $\mathrm{Spf}(R)$. Using Corollary SAG.18.2.1.11 , we deduce that the relative cotangent complex $L_{\mathrm{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}}}$ can be identified with $L_{R_0/R}$, and is therefore 1-connective and almost perfect. Combining this observation with Proposition 3.2.5 and Remark 3.4.4, we conclude that $\mathbf{G}_0$ must be nonstationary and the absolute cotangent complex L_{R_0} must be almost perfect. In particular, if R_0 is a Noetherian $\mathbf{F}_p$-algebra, then it must be F-finite (Theorem 3.5.1).

3.5 A Differential Characterization of F-Finiteness

Our goal in this section is to prove the following stronger version of Proposition 3.3.1:

Theorem 3.5.1. *Let R be a Noetherian $\mathbf{F}_p$-algebra. The following conditions are equivalent:*

(1) *There exists a regular Noetherian $\mathbf{F}_p$-algebra A which is F-finite and a surjection $A \to R$.*

(2) *The $\mathbf{F}_p$-algebra R is F-finite.*

(3) *The absolute cotangent complex L_R is almost perfect as an R-module (see Proposition 3.3.7 for various equivalents of this condition).*

Remark 3.5.2. The implication (2) $\Rightarrow$ (1) of Theorem 3.5.1 is due to Gabber; see Remark 13.6 of [10].

Theorem 3.5.1 will not be needed later in this paper. However, some of the ingredients in the proof (specifically, Proposition 3.5.5 below) will be useful in §6.1.

Lemma 3.5.3. *Let $f : \kappa \to \kappa'$ be an extension of fields. Then the algebraic cotangent complex $L^{\mathrm{alg}}_{\kappa'/\kappa}$ is 1-truncated.*

Proof. Writing κ' as a filtered colimit of finitely generated subfields, we may assume that there are finitely many elements $x_1, \ldots, x_n \in \kappa'$ which generate κ' as a field extension of κ. Proceeding by induction on n, we can reduce to the case where κ' is generated by a single element x. There are two possibilities:

- The element x is transcendental over κ: that is, κ' is isomorphic to the fraction field of the polynomial ring $\kappa[x]$. In this case, $L^{\mathrm{alg}}_{\kappa'/\kappa}$ is a free κ'-module of rank 1.

- The element x is algebraic over κ, so that we can write $\kappa' = \kappa[x]/(f(x))$ for some polynomial $f(x)$. In this case, we can identify $L^{\mathrm{alg}}_{\kappa'/\kappa}$ with the cofiber of the map $\kappa' \to \kappa'$ given by multiplication by $\frac{df(t)}{dt}|_{t=x}$

$\square$

Lemma 3.5.4. *Let κ be a field and let R be a Noetherian $\mathbb{E}_\infty$-algebra over κ. Assume that the relative cotangent complex $L_{R/\kappa}$ vanishes. Then R is discrete and regular.*

Proof. Without loss of generality, we may assume that R is local. Let κ' denote the residue field of R. Using the fiber sequence

$$\kappa' \otimes_R L_{R/\kappa} \to L_{\kappa'/\kappa} \xrightarrow{\theta} L_{\kappa'/R},$$

we deduce that θ is an equivalence. It follows that $\pi_0(L_{\kappa'/\kappa})$ vanishes, so that the comparison map $L_{\kappa'/\kappa} \to L^{\mathrm{alg}}_{\kappa'/\kappa}$ induces an isomorphism $\pi_2(L_{\kappa'/\kappa}) \to \pi_2(L^{\mathrm{alg}}_{\kappa'/\kappa})$. It follows from Lemma 3.5.3 that the homotopy group $\pi_2(L^{\mathrm{alg}}_{\kappa'/\kappa})$ vanishes, so that $\pi_2(L_{\kappa'/\kappa})$ also vanishes. Invoking the fact that θ is an equivalence again, we obtain $\pi_2(L_{\kappa'/R}) \simeq 0$.

Using the fiber sequence

$$\kappa' \otimes_{\pi_0(R)} L_{\pi_0(R)/R} \to L_{\kappa'/R} \to L_{\kappa'/\pi_0(R)},$$

we obtain a short exact sequence of vector spaces

$$\pi_2(L_{\kappa'/R}) \to \pi_2(L_{\kappa'/\pi_0(R)}) \to \pi_1(\kappa' \otimes_{\pi_0(R)} L_{\pi_0(R)/R}).$$

Here the first term vanishes by the preceding argument, and the third term vanishes because the projection map $R \to \pi_0(R)$ has connected fibers. It follows that $\pi_2(L_{\kappa'/\pi_0(R)})$ also vanishes. Applying Lemma SAG.11.2.3.9 , we conclude that $\pi_0(R)$ is regular.

Since R is Noetherian, the projection map $R \to \pi_0(R)$ is almost of finite presentation, so the relative cotangent complex $L_{\pi_0(R)/R}$ is almost perfect as a $\pi_0(R)$-module. It is also 2-connective (Corollary HA.7.4.3.2). Using the fiber sequence

$$\pi_0(R) \otimes_R L_{R/\kappa} \to L_{\pi_0(R)/\kappa} \to L_{\pi_0(R)/R},$$

we conclude that $L_{\pi_0(R)/\kappa}$ is also 2-connective and almost perfect as a $\pi_0(R)$-module. Applying Lemma 3.3.3, we deduce that the algebraic cotangent complex $L^{\mathrm{alg}}_{\pi_0(R)/\kappa}$ is also 2-connective and almost perfect. We have a fiber sequence

$$\kappa' \otimes_{\pi_0(R)} L^{\mathrm{alg}}_{\pi_0(R)/\kappa} \to L^{\mathrm{alg}}_{\kappa'/\pi_0(R)} \to L^{\mathrm{alg}}_{\kappa'/\kappa},$$

where $L^{\mathrm{alg}}_{\kappa'/\pi_0(R)}$ and $L^{\mathrm{alg}}_{\kappa'/\kappa}$ are 1-truncated (the first because κ' is the quotient of $\pi_0(R)$ by a regular sequence, and the second by virtue of Lemma 3.5.3). It follows that the tensor product $\kappa' \otimes_{\pi_0(R)} L^{\mathrm{alg}}_{\pi_0(R)/\kappa}$ is both 2-connective and 1-truncated, and therefore vanishes. Since $L^{\mathrm{alg}}_{\pi_0(R)/\kappa}$ is almost perfect, it follows that $L^{\mathrm{alg}}_{\pi_0(R)/\kappa}$ itself vanishes, so that the topological cotangent complex $L_{\pi_0(R)/\kappa}$ vanishes as well (Remark SAG.25.3.3.7). the canonical map $\pi_2 L_{\kappa(x)/\pi_0(R)} \to \pi_2 L^{\mathrm{alg}}_{\kappa(x)/\pi_0(R)}$ is an isomorphism, so that $\pi_2 L_{\kappa(x)/\pi_0(R)} \simeq 0$. Applying Lemma SAG.11.2.3.9 , we deduce that the local ring $\pi_0(R)_{\mathfrak{p}}$ is regular. Returning to the fiber sequence

$$\pi_0(R) \otimes_R L_{R/\kappa} \to L_{\pi_0(R)/\kappa} \to L_{\pi_0(R)/R},$$

we conclude that $L_{\pi_0(R)/R}$ also vanishes. It follows that the projection map $R \to \pi_0(R)$ is an equivalence (Corollary HA.7.4.3.4), so that $R \simeq \pi_0(R)$ is discrete and regular as desired. □

Proposition 3.5.5. *Let $f : A \to B$ be a morphism of Noetherian $\mathbb{E}_\infty$-rings, and suppose that the relative cotangent complex $L_{B/A}$ vanishes. Then f is flat.*

Proof. By virtue of Lemma SAG.6.1.2.4 , it will suffice to show that the tensor product $\kappa \otimes_A B$ is discrete for every residue field κ of A. We may therefore replace A by κ, in which case the desired result follows from Lemma 3.5.4. □

Proposition 3.5.6. *Let $f : A \to B$ be a morphism of Noetherian $\mathbf{F}_p$-algebras. The following conditions are equivalent:*

(1) *The relative cotangent complex $L_{B/A}$ vanishes.*

(2) *The morphism f is flat and the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi_A"] & B \arrow[d, "\varphi_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is a pushout square in the category of commutative rings.

(3) *The diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi_A"] & B \arrow[d, "\varphi_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is a pushout square in the ∞-category of $\mathbb{E}_\infty$-rings.

Proof. The implication (2) ⇒ (3) is obvious. If condition (3) is satisfied, then the canonical map

$$u : B^{1/p} \otimes_B L^{\mathrm{alg}}_{B/A} \to L^{\mathrm{alg}}_{B^{1/p}/A^{1/p}}$$

is an equivalence. Since u is nullhomotopic (Lemma 3.3.6), it follows that the algebraic cotangent complex $L^{\mathrm{alg}}_{B/A}$ vanishes. Applying Proposition SAG.25.3.5.1 , we conclude that $L_{B/A}$ also vanishes; this shows that (3) implies (1).

We complete the proof by showing that (1) implies (2). Assume that $L_{B/A}$ vanishes, so that f is flat by virtue of Proposition 3.5.5. We wish to show that the relative Frobenius map $\varphi_f : A^{1/p} \otimes_A B \to B^{1/p}$ is an isomorphism. Since $A \simeq A^{1/p}$ is Noetherian, it will suffice to show that φ_f induces an isomorphism

$$\kappa^{1/p} \otimes_A B \to \kappa^{1/p} \otimes_{A^{1/p}} B^{1/p}$$

for each residue field κ of A (Lemma SAG.2.6.1.3). We may therefore replace A by κ and B by $\kappa \otimes_A B$, and thereby reduce to the case where $A = \kappa$ is a field (note that this replacement does not injure our hypothesis that B is Noetherian).

Applying Lemma 3.5.4, we deduce that B is a regular Noetherian ring. Factoring B as a product, we can assume without loss of generality that B is an integral domain. Let K denote the fraction field of B. For every subfield $\kappa' \subseteq \kappa^{1/p}$ which is a finite extension of κ, the tensor product $B_{\kappa'} = \kappa' \otimes_\kappa B$ is also Noetherian (since it is finite over B) and the relative cotangent complex $L_{B_{\kappa'}/\kappa'}$ vanishes, so Lemma 3.5.4 guarantees that $B_{\kappa'}$ is also a regular Noetherian ring. Note that $B_{\kappa'}$ is also an integral domain (since the map $|\operatorname{Spec}(B_{\kappa'})| \to |\operatorname{Spec}(B)|$ is a homeomorphism), whose fraction field can be identified with $\kappa' \otimes_\kappa K$. The regularity of $B_{\kappa'}$ guarantees that it is integrally closed in its fraction field, and can therefore be identified with the integral closure of B in $\kappa' \otimes_\kappa K$. Passing to the direct limit over κ', we conclude that $\kappa^{1/p} \otimes_\kappa B$ can be identified with the integral closure of B in $\kappa' \otimes_\kappa K$. Similarly, the regularity of B guarantees that $B^{1/p}$ is integrally closed in its fraction field $K^{1/p}$, and can therefore be identified with the integral closure of B in $K^{1/p}$. Consequently, to show that the map φ_f is an isomorphism, it will suffice to show that the natural map $\kappa^{1/p} \otimes_\kappa K \to K^{1/p}$ is an isomorphism. In other words, we can replace B by its fraction field K, and thereby reduce to the case where B is also a field.

Let Ω_κ denote the module of Kähler differentials of κ. Choose a collection of elements $\{x_i \in \kappa\}_{i \in I}$ with the property that $\{dx_i\}_{i \in I}$ is a basis for Ω_κ as a vector space over κ. Then the elements $\{x_i\}_{i \in I}$ form a *p-basis* for the field κ: that is, there is an isomorphism of κ-algebras

$$\kappa[\{t_i\}_{i \in I}]/(t_i^p - x_i) \xrightarrow{\sim} \kappa^{1/p},$$

carrying each t_i to $x_i^{1/p}$. It follows that we can identify the relative Frobenius φ_f with the map $K[\{t_i\}_{i \in I}]/(t_i^p - f(x_i)) \to K^{1/p}$ given by $t_i \mapsto f(x_i)^{1/p}$. To show that this map is an isomorphism, it suffices to show that the elements $\{f(x_i)\}_{i \in I}$ form a p-basis for the field K, or equivalently that the elements $\{df(x_i)\}$ form a basis for Ω_K as a vector space over K. This is clear: our hypothesis that the relative cotangent complex $L_{K/\kappa}$ vanishes guarantees that the natural map $K \otimes_\kappa \Omega_\kappa \to \Omega_K$ is an isomorphism. $\square$

Corollary 3.5.7. *Let $f : A \to B$ be a morphism of Noetherian $\mathbf{F}_p$-algebras. Suppose that the relative cotangent complex $L_{B/A}$ vanishes and that f induces an isomorphism of perfections $A^{1/p^\infty} \to B^{1/p^\infty}$. Then f is an isomorphism.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A^{1/p} \arrow[r] \arrow[d] & A^{1/p^2} \arrow[r] \arrow[d] & \cdots \\
B \arrow[r] & B^{1/p} \arrow[r] & B^{1/p^2} \arrow[r] & \cdots
\end{tikzcd}$$

Our assumption that $L_{B/A}$ vanishes guarantees that each square in this diagram is a pushout (Proposition 3.5.6). Passing to the colimit in the horizontal direction, we deduce that the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A^{1/p^\infty} \arrow[d] \\
B \arrow[r] & B^{1/p^\infty}
\end{tikzcd}$$

is a pushout square. Our second assumption guarantees that the right vertical map in this diagram is an isomorphism: that is, f becomes an isomorphism after extending scalars to A^{1/p^∞}. It follows that, for every residue field κ of A, the induced map $\kappa^{1/p^\infty} \to \kappa^{1/p^\infty} \otimes_A B$ is an isomorphism. Since κ^{1/p^∞} is faithfully flat over κ, we conclude that the map $\kappa \to \kappa \otimes_A B$ is an isomorphism. Because A is Noetherian, this guarantees that f is an isomorphism (Lemma SAG.2.6.1.3). $\square$

Corollary 3.5.8. *Let $f : A \to B$ be a morphism of Noetherian $\mathbb{E}_\infty$-rings. Assume that p is nilpotent in the commutative ring $\pi_0(A)$. Then f is an equivalence if and only if it satisfies the following pair of conditions:*

(a) *The relative cotangent complex $L_{B/A}$ vanishes.*

(b) *For every perfect $\mathbf{F}_p$-algebra R, composition with f induces a homotopy equivalence $\mathrm{Map}_{\mathrm{CAlg}}(B, R) \to \mathrm{Map}_{\mathrm{CAlg}}(A, R)$.*

Proof. The "only if" direction is obvious. For the converse, suppose that (a) and (b) are satisfied; we wish to show that f is an equivalence of $\mathbb{E}_\infty$-rings. Since p is nilpotent in $\pi_0(A)$, it will suffice to show that the induced map $\pi_0(A)/(p) \to (\pi_0(A)/(p)) \otimes_A B$ is an equivalence of $\mathbb{E}_\infty$-rings. We may therefore replace A by $\pi_0(A)/(p)$ and thereby reduce to the case where A is a commutative $\mathbf{F}_p$-algebra (note that this does not injure our hypothesis that B is Noetherian, because $\pi_0(A)/(p)$ is almost of finite presentation over A). In this case, hypothesis (a) guarantees that f is flat (Proposition 3.5.5), so that B is also a commutative $\mathbf{F}_p$-algebra. Condition (b) is now equivalent to the requirement that f induces an isomorphism of perfections $A^{1/p^\infty} \simeq B^{1/p^\infty}$, so that f is an equivalence by virtue of Corollary 3.5.7 $\square$

Proof of Theorem 3.5.1. The implication $(1) \Rightarrow (2)$ is obvious, and the implication $(2) \Rightarrow (3)$ follows from Proposition 3.3.1. We will complete the proof by showing that $(3) \Rightarrow (1)$. Let R be a Noetherian $\mathbf{F}_p$-algebra, and suppose that the absolute cotangent complex L_R is almost perfect. We wish to show that R can be written as the quotient of an $\mathbf{F}_p$-algebra which is regular, Noetherian, and F-finite. Our assumption that L_R is almost perfect guarantees that the module of Kähler differentials $\Omega_{R/\mathbf{F}_p} \simeq \pi_0(L_R)$ is finitely generated as an R-module. Choose elements $x_1, \ldots, x_n \in R$ with the property that their images $dx_1, \ldots, dx_n$ generate $\Omega_{R/\mathbf{F}_p}$ as an R-module. Set $P = \mathbf{F}_p[X_1, \ldots, X_n]$, so that the elements $\{x_i\}_{1 \leqslant i \leqslant n}$ determine an $\mathbf{F}_p$-algebra homomorphism $\rho : P \to R$. Then L_P is almost perfect as a P-module (this follows from Proposition 3.3.1 since P is F-finite, or more directly from the observation that P is almost of finite presentation over the sphere spectrum). Using the fiber sequence $R \otimes_P L_P \to L_R \to L_{R/P}$, we conclude that $L_{R/P}$ is almost perfect as an R-module. It is also 1-connective (by virtue of our assumption that $\Omega_{R/\mathbf{F}_p}$ is generated by the elements dx_i).

The ring homomorphism ρ determines a map of schemes $u : \mathrm{Spec}(R) \to \mathrm{Spec}(P)$. Let us regard u as a natural transformation between functors

$$\mathrm{Spec}(R), \mathrm{Spec}(P) : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S},$$

and let X denote the relative de Rham space $(\mathrm{Spec}(R)/\mathrm{Spec}(P))_{\mathrm{dR}}$ of Definition SAG.18.2.1.1 . Applying Theorem SAG.18.2.3.1 (together with Corollary SAG.18.2.3.3 and Corollary SAG.18.2.4.4), we conclude that the functor X has the form $\mathrm{Spf}(A)$, where A is a Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to an ideal $I \subseteq \pi_0(A)$ satisfying $R \simeq \pi_0(A)/I$. We will complete the proof by showing that A is discrete, regular, and F-finite.

Let us write the relative de Rham space $(\operatorname{Spec}(R^{1/p})/\operatorname{Spec}(P^{1/p}))_{\mathrm{dR}}$ as a formal spectrum $\operatorname{Spf}(A^{1/p})$ (so that $A^{1/p}$ is abstractly equivalent to A, but it will be convenient to avoid identifying them in what follows). The Frobenius morphisms on R and P induce a map $\varphi_A : A \to A^{1/p}$. Beware that we do not yet know that A is discrete, so we cannot *a priori* φ_A by declaring that it is the Frobenius map from A to itself. However, it is not difficult to see that the induced map $\pi_0(A) \to \pi_0(A^{1/p}) \simeq \pi_0(A)^{1/p}$ is the usual Frobenius map on the commutative $\mathbf{F}_p$-algebra $\pi_0(A)$. Note that we have a commutative diagram σ :

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\varphi_P"] & A \arrow[d, "\varphi_A"] \\
P^{1/p} \arrow[r] & A^{1/p},
\end{tikzcd}$$

where the left vertical map is finite flat.

We first claim that σ is a pushout square of $\mathbb{E}_\infty$-rings. Let $\theta : P^{1/p} \otimes_P A \to A^{1/p}$ be the morphism of $\mathbb{E}_\infty$-rings determined by the diagram σ; we wish to show that θ is an equivalence. Let us regard θ as a morphism of A-algebras. Note that $P^{1/p} \otimes_P A$ is a finite flat A-module, and therefore I-complete (since A is I-complete). Similarly, the $\mathbb{E}_\infty$-ring $A^{1/p}$ is $I^{1/p}$-complete, and is therefore also I-complete when regarded as an A-module via φ_A (since the ideals $I^{1/p}$ and $I\pi_0(A)^{1/p}$ generate the same topology on $\pi_0(A)^{1/p}$). Consequently, to prove that θ is an equivalence, it will suffice to show that the induced map

$$\theta_R : P^{1/p} \otimes_P R \to A^{1/p} \otimes_A R$$

is an equivalence. We now argue that θ_R satisfies criteria (a) and (b) of Corollary 3.5.8:

(a) From the description of X as a relative de Rham space, we see that the relative cotangent complex $L_{X/\operatorname{Spec}(P)}$ vanishes. It follows that the relative cotangent complex $L_{A/P}$ is I-rational: that is, it vanishes after I-completion. Similarly, the relative cotangent complex $L_{A^{1/p}/P^{1/p}}$ is $I^{1/p}$-rational, and therefore also I-rational (since $I^{1/p}$ and $I\pi_0(A)^{1/p}$ generate the same topology on the commutative ring $\pi_0(A)^{1/p}$). Let $L = L_{A^{1/p}/P^{1/p}\otimes_P A}$ denote the relative cotangent complex of the morphism θ. Using the cofiber sequence

$$A^{1/p} \otimes_A L_{A/P} \to L_{A^{1/p}/P^{1/p}} \to L,$$

we deduce that L is I-rational when regarded as an A-module. The relative cotangent complex $L_{A^{1/p}\otimes_A R/P^{1/p}\otimes_P R}$ can be identified with the tensor product

$$(A^{1/p} \otimes_A R) \otimes_{A^{1/p}} L \simeq R \otimes_A L,$$

and therefore vanishes (since it is simultaneously I-nilpotent and I-local as an A-module).

(b) Let B be a perfect $\mathbf{F}_p$-algebra; we wish to show that the the upper horizontal map in the diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{CAlg}}(A^{1/p} \otimes_A R, B) \arrow[r, "\circ\theta_R"] \arrow[d] & \mathrm{Map}_{\mathrm{CAlg}}(P^{1/p} \otimes_P R) \arrow[d] \\ \mathrm{Map}_{\mathrm{CAlg}}(A^{1/p}, B) \arrow[r, "\circ\theta"] & \mathrm{Map}_{\mathrm{CAlg}}(P^{1/p} \otimes_P A, B) \end{tikzcd}$$

is a homotopy equivalence (of discrete spaces). Since the diagram is a pullback square, it suffices to show that the lower horizontal map is a homotopy equivalence: that is, that the diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{CAlg}}(A^{1/p}, B) \arrow[r] \arrow[d, "\circ\varphi_A"] & \mathrm{Map}_{\mathrm{CAlg}}(P^{1/p}, B) \arrow[d, "\circ\varphi_P"] \\ \mathrm{Map}_{\mathrm{CAlg}}(A, B) \arrow[r] & \mathrm{Map}_{\mathrm{CAlg}}(P, B) \end{tikzcd}$$

is a pullback square. This is clear: the vertical maps are both homotopy equivalences (of discrete spaces), by virtue of our assumption that B is perfect.

Since σ is a pushout square and the Frobenius map $\varphi_P : P \to P^{1/p}$ is finite flat, it follows that the map $\varphi_A : A \to A^{1/p}$ is finite flat. In particular, $\pi_0(A)$ is F-finite. Proposition 3.3.1 now guarantees that the absolute cotangent complex $L_{\pi_0(A)}$ is almost perfect as a module over $\pi_0(A)$. Using the fiber sequence

$$\pi_0(A) \otimes_A L_A \to L_{\pi_0(A)} \to L_{\pi_0(A)/A}$$

and the observation that $\pi_0(A)$ is almost of finite presentation over A, we conclude that $\pi_0(A) \otimes_A L_A$ is almost perfect as a $\pi_0(A)$-module, so that L_A is almost perfect as an A-module (Proposition SAG.2.7.3.2). It follows that the relative cotangent complex $L_{A/P}$ is almost perfect as an A-module, and therefore I-complete (since A is I-complete). Since it is also I-local, we conclude that $L_{A/P} \simeq 0$. Applying Proposition SAG.2.7.3.2 , we see that A is flat over P, and therefore discrete. Moreover, it follows from Lemma 3.5.4 shows that the morphism $P \to A$ is geometrically regular. Since P is regular, A is also regular (alternatively, we can deduce the regularity of A from the fact that the Frobenius map $\varphi_A : A \to A$ is finite flat). $\square$

4 Orientations and Quillen's Construction

In [29], Quillen discovered a remarkable relationship between cohomology theories and formal groups. Let A be a homotopy commutative ring spectrum (that is, a commutative algebra object in the homotopy category of spectra hSp), and let $\pi_{\mathrm{ev}}(A)$ denote the commutative ring $\bigoplus_{n \in \mathbf{Z}} \pi_{2n}(A)$. If A is complex orientable, then the inverse limit $B = \varprojlim_m \bigoplus_{n \in \mathbf{Z}} A^{2n}(\mathbf{CP}^m)$ is isomorphic to a power series ring $\pi_{\mathrm{ev}}(A)[[t]]$, where t is any choice of complex orientation for A. The formal spectrum $\widehat{\mathbf{G}}^{\mathcal{Q}_{\mathrm{ev}}}_A = \mathrm{Spf}(B)$ can then be regarded as a 1-dimensional formal group over the commutative ring $\pi_{\mathrm{ev}}(A)$. This formal group is an extremely useful invariant, which determines the underlying cohomology theory A in many cases of interest (Theorem 0.0.1).

In this paper, we consider Quillen's construction only under the additional assumption that the ring spectrum A is *complex periodic* (Definition 4.1.8). In this case, the formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_{\mathrm{ev}}}_A$ above is naturally defined over the subring $\pi_0(A) \subseteq \pi_{\mathrm{ev}}(A)$. More precisely, it can be obtained from a formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_A$ over $\pi_0(A)$ by extending scalars along the inclusion map $\pi_0(A) \hookrightarrow \pi_{\mathrm{ev}}(A)$. The formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_A$ can be described as the formal spectrum $\mathrm{Spf}(A^0(\mathbf{CP}^\infty))$. We will refer to it as the *classical Quillen formal group* of A.

Our goal in this section is to study a refinement of the classical Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_A$, which is defined over the ring spectrum A itself rather than over the commutative ring $\pi_0(A)$. Assume now that A is an $\mathbb{E}_\infty$-ring (which is also complex periodic). In §4.1, we introduce a formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ which we refer to as the *Quillen formal group* of A (Construction 4.1.13). Roughly speaking, $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ can be described as the formal spectrum of the function spectrum $\mathscr{O} = C^*(\mathbf{CP}^\infty; A)$ of maps from $\mathbf{CP}^\infty$ into A (while $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_A$ is the formal spectrum of the ordinary commutative ring $\pi_0(\mathscr{O})$).

Most of this section is devoted to answering the following:

Question 4.0.1. Let A be a complex periodic $\mathbb{E}_\infty$-ring. How can we characterize $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ among the collection of all formal groups over A?

To address Question 4.0.1, we need to say a bit more about how the formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ is defined. The formal spectrum $\mathrm{Spf}(C^*(\mathbf{CP}^\infty; A))$ is only the underlying formal hyperplane of $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$; to see the group structure, we will need exploit the fact that the space $\mathbf{CP}^\infty$ is an abelian group object of the ∞-category of spaces $\mathcal{S}$. More precisely, we can realize $\mathbf{CP}^\infty \simeq K(\mathbf{Z}, 2)$ as the 0th space of the $\mathbf{Z}$-module spectrum $\Sigma^2(\mathbf{Z})$. This realization supplies a universal property of $\mathbf{CP}^\infty$: as an abelian group object of $\mathcal{S}$, it is freely generated by the 2-sphere S^2 (as a pointed space). For any abelian group object $M \in \mathrm{Ab}(\mathcal{S})$, we have a canonical homotopy equivalence $\mathrm{Map}_{\mathrm{Ab}(\mathcal{S})}(\mathbf{CP}^\infty, M) \simeq \mathrm{Map}_{\mathcal{S}_*}(S^2, M)$. This translates to a heuristic description of $\widehat{\mathbf{G}}^{Q}_A$: roughly speaking, it should be "freely generated by S^2" as an object of the ∞-category $\mathrm{FGroup}(A)$ of formal groups over A. In §4.3, we make this heuristic precise by showing that for any formal group $\widehat{\mathbf{G}}$ over A, the datum of a morphism of formal groups $\widehat{\mathbf{G}}^{Q}_A \to \widehat{\mathbf{G}}$ is equivalent to the datum of a (pointed) map $e : S^2 \to \Omega^\infty \mathbf{G}(\tau_{\geqslant 0}(A))$ (Proposition 4.3.21). We will refer to such a map as a *preorientation* of the formal group $\widehat{\mathbf{G}}$ (Definition 4.3.19).

To give a more complete answer to Question 4.0.1, we would like to address the following:

Question 4.0.2. Let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over A, and let e be a preorientation of $\widehat{\mathbf{G}}$. When is the induced map $f : \widehat{\mathbf{G}}^{Q}_A \to \widehat{\mathbf{G}}$ an equivalence of formal groups?

The classical theory of formal groups suggests a heuristic approach to answering Question 4.0.2: a map of 1-dimensional formal groups $f : \widehat{\mathbf{G}}' \to \widehat{\mathbf{G}}$ should be an equivalence if its derivative at the origin is invertible. In 4.2, we make this heuristic precise by introducing the *dualizing line* $\omega_{\widehat{\mathbf{G}}}$ of a 1-dimensional formal group over A (or, more generally, a 1-dimensional formal hyperplane over A), and showing that f is an equivalence if and only if the pullback map $f^* : \omega_{\widehat{\mathbf{G}}} \to \omega_{\widehat{\mathbf{G}}'}$ is an equivalence (Remark 4.2.5). In the special case where $\widehat{\mathbf{G}}' = \widehat{\mathbf{G}}^{Q}_A$ is the Quillen formal group of A, there is a canonical equivalence $\omega_{\widehat{\mathbf{G}}'} \simeq \Sigma^{-2}(A)$, so that a preorientation e of $\widehat{\mathbf{G}}$ induces a map

$$\beta_e : \omega_{\widehat{\mathbf{G}}} \to \Sigma^{-2}(A).$$

We will refer to β_e as the *Bott map* of the preorientation e. In the case where A is complex periodic, we can identify β_e with the "derivative" of the map $\widehat{\mathbf{G}}^{Q}_A \to \widehat{\mathbf{G}}$ corresponding to the preorientation e. However, the map β_e can be defined without assuming that A is complex periodic (and without reference to the group structure on $\widehat{\mathbf{G}}$); see Construction 4.3.7. Using this observation, we show that for any 1-dimensional formal group $\widehat{\mathbf{G}}$ over any $\mathbb{E}_\infty$-ring A, there is a universal example of a map of $\mathbb{E}_\infty$-rings $A \to A'$ for which A' is complex periodic and $\widehat{\mathbf{G}}_{A'}$ is equivalent to the Quillen formal group $\widehat{\mathbf{G}}^{Q}_{A'}$ (see Propositions 4.3.13 and 4.3.23). We will refer to A' as the *orientation classifier* of $\widehat{\mathbf{G}}$ and denote it by $\mathfrak{O}_{\widehat{\mathbf{G}}}$. The construction $\widehat{\mathbf{G}} \mapsto \mathfrak{O}_{\widehat{\mathbf{G}}}$ will play an important role in the later sections of this paper.

Quillen's construction $A \mapsto \widehat{\mathbf{G}}^{Q}_A$ provides an important supply of examples of formal groups over $\mathbb{E}_\infty$-ring spectra. We studied another class of examples in §2: if A

is (p)-complete, then we can to every p-divisible group $\mathbf{G}$ over A a formal group $\mathbf{G}^\circ$ (Definition 2.0.10). The second half of this section is devoted to the following:

Question 4.0.3. Let A be an $\mathbb{E}_\infty$-ring which is complex periodic and (p)-complete. When can we find a p-divisible group $\mathbf{G}$ over A and an equivalence of formal groups $\widehat{\mathbf{G}}^{\mathcal{Q}}_A \simeq \widehat{\mathbf{G}}^\circ$?

The theory developed in §2 provides a partial answer to Question 4.0.3: it is sufficient to assume that $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ is a p-divisible formal group over A, in the sense of Definition 2.3.14. By virtue of Theorem 2.3.26, this is equivalent to the requirement that the classical Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_A$ is p-divisible. We therefore ask the following more general question:

Question 4.0.4. Let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over a commutative ring R. When is $\widehat{\mathbf{G}}$ a p-divisible formal group?

Question 4.0.4 is addressed by the classical theory of *heights* of 1-dimensional formal groups, which we review in §4.4. Recall that if $\widehat{\mathbf{G}}$ is a formal group over a field κ of characteristic p, then $\widehat{\mathbf{G}}$ is p-divisible if and only if the map $[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to \mathscr{O}_{\widehat{\mathbf{G}}}$ is finite flat. In this case, the degree of the map $[p]^*$ is an integer of the form p^n; we refer to n as the *height* of the formal group $\widehat{\mathbf{G}}$ and write $n = \mathrm{ht}(\widehat{\mathbf{G}})$. More generally, if R is a complete adic $\mathbb{E}_\infty$-ring, then a formal group $\widehat{\mathbf{G}}$ over R is p-divisible if and only if function $x \mapsto \mathrm{ht}(\widehat{\mathbf{G}}_{\kappa(x)})$ is finite and locally constant on the topological space $|\operatorname{Spf}(R)|$ (Theorem 4.4.14).

We will be particularly interested in the case where this function takes some constant value n. Note that the function $x \mapsto \mathrm{ht}(\widehat{\mathbf{G}}_{\kappa(x)})$ is always upper-semicontinuous on $|\operatorname{Spec}(R)|$. More precisely, the each of the sets $\{x \in |\operatorname{Spec}(R)| : \mathrm{ht}(\widehat{\mathbf{G}}_{\kappa(x)}) \geqslant m\}$ can be realized as the vanishing locus of a certain ideal $\mathfrak{I}^{\widehat{\mathbf{G}}}_m$, which we will refer to as the *mth Landweber ideal* (Definition 4.4.11). Theorem 4.4.14 can then be restated as follows: the formal group $\widehat{\mathbf{G}}$ can be realized as the identity component of a formally connected p-divisible group $\mathbf{G}$ of height n if and only if the commutative ring R is complete with respect to the nth Landweber ideal $\mathfrak{I}^{\widehat{\mathbf{G}}}_n$, and the $(n+1)$st Landweber ideal $\mathfrak{I}^{\widehat{\mathbf{G}}}_{n+1}$ is equal to R. In §4.5, we specialize to the case where $\widehat{\mathbf{G}} = \widehat{\mathbf{G}}^{\mathcal{Q}_0}_A$ is the classical Quillen formal group of a complex periodic $\mathbb{E}_\infty$-ring A and show that these conditions have a homotopy-theoretic interpretation: they are equivalent to the requirement that the spectrum A is $K(n)$-local, where $K(n)$ denotes the nth Morava K-theory spectrum (Theorem 4.5.2). In this case, the general machinery of §2 guarantees that the Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ can be realized (in an essentially unique way) as the identity component of a p-divisible group $\mathbf{G}_A^Q$ of height n, which we will refer to as the *Quillen p-divisible group of A*. We give an explicit description of the p-divisible group $\mathbf{G}_A^Q$ in §4.6 (Construction 4.6.2), which is mostly independent of the formalism developed in §2.

4.1 The Quillen Formal Group

In this section, we review the notions of complex periodic ring spectrum (Definition 4.1.1) and weakly 2-periodic ring spectrum (Definition 4.1.5). We will say that an $\mathbb{E}_\infty$-ring A is *complex periodic* if it is both complex orientable and weakly 2-periodic (Definition 4.1.8). In this case, we construct a formal group $\widehat{\mathbf{G}}_A^Q$, which we will refer to as the *Quillen formal group* of A (Construction 4.1.13).

4.1.1 Complex Orientations of Ring Spectra

In this section, we briefly review the theory of complex orientations. Our presentation is terse; for a more detailed discussion, we refer the reader to [1].

Definition 4.1.1. Let $\mathbf{CP}^\infty$ denote the infinite dimensional projective space $\varinjlim \mathbf{CP}^n$, which we can also regard as the classifying space $\mathrm{BU}(1)$ for the unitary group $\mathrm{U}(1)$, or an Eilenberg-MacLane space $K(\mathbf{Z}, 2)$. Let A be an $\mathbb{E}_0$-ring: that is, a spectrum equipped with a unit map $e : S \to A$. We say that A is *complex orientable* if the map e factors as a composition

$$S \simeq \Sigma^{\infty-2}\,\mathbf{CP}^1 \to \Sigma^{\infty-2}\,\mathbf{CP}^\infty \xrightarrow{\overline{e}} A.$$

In this case, we will say that $\overline{e}$ is a *complex orientation* of A.

Example 4.1.2. Let R be an $\mathbb{E}_0$-ring, and suppose that the homotopy groups $\pi_n R$ vanish when n is odd. Then R is complex orientable. To prove this, we note that the space $\mathbf{CP}^\infty$ admits a filtration

$$\mathbf{CP}^1 \hookrightarrow \mathbf{CP}^2 \hookrightarrow \mathbf{CP}^3 \hookrightarrow \cdots.$$

To construct a complex orientation of R, it suffices to construct a compatible family of maps $e_n : \Sigma^{\infty-2}\,\mathbf{CP}^n \to R$ where e_1 is the unit map of R. Assume that $n > 1$ and that e_{n-1} has been constructed. Then the obstruction to finding the map e_n lies in $\mathrm{Ext}^1_{\mathrm{Sp}}(\mathrm{cofib}(\Sigma^{\infty-2}\,\mathbf{CP}^{n-1} \to \Sigma^{\infty-2}\,\mathbf{CP}^n), R) \simeq \pi_{2n-3}R \simeq 0$.

Remark 4.1.3. Let $\phi : A \to B$ be a map of $\mathbb{E}_0$-rings. Then the induced map $\mathrm{Map}_{\mathrm{Sp}}(\Sigma^{\infty-2}\,\mathbf{CP}^\infty, A) \to \mathrm{Map}_{\mathrm{Sp}}(\Sigma^{\infty-2}\,\mathbf{CP}^\infty, B)$ carries complex orientations of A to complex orientations of B. In particular, if A is complex-orientable, then B is also complex-orientable.

Remark 4.1.4. The spectrum $\Sigma^{\infty-2}\,\mathbf{CP}^\infty$ is connective. Consequently, if A is any spectrum, the vertical maps in the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Sp}}(\Sigma^{\infty-2}\,\mathbf{CP}^\infty, \tau_{\geqslant 0}A) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Sp}}(S, \tau_{\geqslant 0}A) \arrow[d] \\
\mathrm{Map}_{\mathrm{Sp}}(\Sigma^{\infty-2}\,\mathbf{CP}^\infty, A) \arrow[r] & \mathrm{Map}_{\mathrm{Sp}}(S, A)
\end{tikzcd}$$

are homotopy equivalences. In particular, if A is an $\mathbb{E}_0$-ring, then giving a complex orientation of A is equivalent to giving a complex orientation of its connective cover $\tau_{\geqslant 0}A$.

4.1.2 Periodic Ring Spectra

We now restrict our attention to ring spectra A having periodic homotopy groups.

Definition 4.1.5. Let A be an $\mathbb{E}_\infty$-ring. We will say that A is *weakly 2-periodic* if the suspension $\Sigma^2(A)$ is locally free of rank 1 as an A-module. Equivalently, A is weakly 2-periodic if it satisfies the following conditions:

- (a) The homotopy group $L = \pi_2(A)$ is a projective module of rank 1 over the commutative ring $\pi_0(A)$.
- (b) For every integer n, the canonical map $L \otimes_{\pi_0(A)} \pi_n(A) \to \pi_{n+2}(A)$ is an isomorphism.

Remark 4.1.6. In the situation of Definition 4.1.5, it suffices to verify condition (b) in the case $n = -2$. Moreover, condition (b) implies (a).

Remark 4.1.7. Let A be a weakly 2-periodic $\mathbb{E}_\infty$-ring. Then, for every A-module M, the canonical map $\pi_2(A) \otimes_{\pi_0(A)} \pi_*(M) \to \pi_{*+2}(M)$ is an isomorphism. In particular, if $\phi : A \to B$ is a morphism of $\mathbb{E}_\infty$-rings, then $\pi_2(B) \simeq \pi_2(A) \otimes_{\pi_0(A)} \pi_0(B)$ is an invertible module over $\pi_0(B)$, and we have isomorphisms

$$\pi_2(B) \otimes_{\pi_0(B)} \pi_*(B) \simeq \pi_2(A) \otimes_{\pi_0(A)} \pi_*(B) \simeq \pi_{*+2}(B).$$

It follows that B is also weakly 2-periodic.

Definition 4.1.8. Let A be an $\mathbb{E}_\infty$-ring. We will say that A is *complex periodic* if it is weakly 2-periodic and complex orientable.

Example 4.1.9 (Even Periodic Ring Spectra)**.** Let A be an $\mathbb{E}_\infty$-ring. We say that A is *even periodic* if the graded ring $\pi_*(A)$ is isomorphic to $\pi_0(A)[t^{\pm 1}]$, where t is an element of degree (-2). Every even periodic $\mathbb{E}_\infty$-ring is complex periodic: the weak 2-periodicity of A is obvious, and the complex orientability follows from Example 4.1.2.

Remark 4.1.10. Let $\phi : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings. If A is complex periodic, then B is also complex periodic (this follows from Remarks 4.1.3 and 4.1.7. This observation makes the class of complex periodic $\mathbb{E}_\infty$-rings more convenient to work with than the class of even periodic $\mathbb{E}_\infty$-rings.

4.1.3 The Quillen Formal Group

Let A be an $\mathbb{E}_\infty$-ring, and regard the ∞-category Mod_A as equipped with the symmetric monoidal structure given by smash product relative to A. Then there is an essentially unique symmetric monoidal functor $\mathcal{S} \to \mathrm{Mod}_A$ which preserves small colimits (where we regard the ∞-category $\mathcal{S}$ of spaces as equipped with the Cartesian symmetric monoidal structure). We will denote this functor by $X \mapsto C_*(X; A)$; here $C_*(X; A)$ is a spectrum whose homotopy groups are the A-homology groups of the space X. Note that every object $X \in \mathcal{S}$ can be regarded as a commutative coalgebra with respect to the Cartesian symmetric monoidal structure on $\mathcal{S}$ in an essentially unique way (see Corollary HA.2.4.3.10), so that $C_*(X; A)$ inherits the structure of a commutative coalgebra over A. In §4.1.4, we will prove the following:

Theorem 4.1.11. *Let A be a complex periodic $\mathbb{E}_\infty$-ring. Then $C_*(\mathbf{CP}^\infty; A)$ is a smooth coalgebra of dimension 1 over A, in the sense of Definition 1.1.14.*

Remark 4.1.12. Let A be a complex periodic $\mathbb{E}_\infty$-ring. Then, for every free abelian group M of rank $r < \infty$, the coalgebra $C_*(K(M, 2); A)$ is smooth of dimension r. To prove this, we can use the fact that the functor $X \mapsto C_*(X; A)$ is symmetric monoidal (and the fact that the class of smooth coalgebras over A is closed under tensor products; see Remark 1.2.6) to reduce to the case $r = 1$, in which case the desired result follows from Theorem 4.1.11.

Construction 4.1.13 (The Quillen Formal Group)**.** Let A be a complex periodic $\mathbb{E}_\infty$-ring and let Lat denote the category of free abelian groups of finite rank. It follows from Remark 4.1.12 that the construction

$$M \mapsto C_*(K(M^\vee, 2); A)$$

determines a functor $\mathrm{Lat}^{\mathrm{op}} \to \mathrm{cCAlg}_A^{\mathrm{sm}}$. This functor commutes with finite products, and can therefore be regarded as an abelian group object of the ∞-category $\mathrm{cCAlg}_A^{\mathrm{sm}}$. We may therefore regard the construction

$$M \mapsto \mathrm{cSpec}(C_*(K(M^\vee, 2); A))$$

as an abelian group object of the ∞-category $\mathrm{Hyp}(A)$ of formal hyperplanes over A: that is, as a formal group over A. We will denote this formal group by $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ and refer to it as the *Quillen formal group of A*.

Notation 4.1.14. Let A be a complex periodic $\mathbb{E}_\infty$-ring. We let $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ denote the image of $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ under the forgetful functor $\mathrm{FGroup}(A) \to \mathrm{FGroup}(\pi_0(A))$. We will refer to $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ as the *classical Quillen formal group* of A. Concretely, the classical Quillen formal group of A is given by the formula

$$\widehat{\mathbf{G}}_A^{\mathcal{Q}_0} = \mathrm{Spf}(A^0(\mathbf{CP}^\infty)) \simeq \mathrm{cSpec}(A_0(\mathbf{CP}^\infty)).$$

Note that this definition makes sense in somewhat greater generality: we do not need to assume that the multiplicative structure on A is $\mathbb{E}_\infty$ (it is sometimes useful to consider the formal group $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ even in cases where the multiplication on A is not homotopy commutative; for example, when A is a Morava K-theory spectrum at the prime 2).

Warning 4.1.15. In the situation of Construction 4.1.13, we can identify $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ with a formal group over the connective cover $\tau_{\geqslant 0} A$ (see Variant 1.6.2). Beware that $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ is *not* the Quillen formal group associated to $\tau_{\geqslant 0} A$: the $\mathbb{E}_\infty$-ring $\tau_{\geqslant 0} A$ is never complex periodic (except in the trivial case $A \simeq 0$), and the coalgebra $C_*(\mathbf{CP}^\infty; \tau_{\geqslant 0} A)$ is not smooth over $\tau_{\geqslant 0} A$.

Example 4.1.16 (The Case of a $\mathbf{Z}$-Algebra)**.** The cohomology ring $\mathrm{H}^*(\mathbf{CP}^\infty; \mathbf{Z})$ can be identified with the polynomial ring $\mathbf{Z}[t]$ on a generator t of degree 2. Moreover, the multiplication map $m : \mathbf{CP}^\infty \times \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ induces a pullback map

$$\mathbf{Z}[t] \simeq \mathrm{H}^*(\mathbf{CP}^\infty; \mathbf{Z}) \xrightarrow{m^*} \mathrm{H}^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty; \mathbf{Z}) \simeq \mathbf{Z}[t] \otimes_{\mathbf{Z}} \mathbf{Z}[t]$$

given by $t \mapsto t \otimes 1 + 1 \otimes t$. If A is any $\mathbb{E}_\infty$-algebra over $\mathbf{Z}$ containing an invertible element $u \in \pi_2(A)$, then A is complex periodic and the classical Quillen formal group can be described by the formula

$$\mathscr{O}_{\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}} = A^0(\mathbf{CP}^\infty) \simeq \pi_0(A)[[t']]$$

where $t' = tu$; the comultiplication on $\mathscr{O}_{\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}}$ then carries t' to $t' \otimes 1 + 1 \otimes t'$. It follows that $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ is isomorphic to the formal additive group $\widehat{\mathbf{G}}_a$ over the commutative ring $\pi_0(A)$. This isomorphism is not quite canonical: it depends on a chosen invertible element $u \in \pi_2(A)$.

4.1.4 The Proof of Theorem 4.1.11

The proof of Theorem 4.1.11 will require a few simple calculations. Choose a generator t of the cohomology group $\mathrm{H}^2(\mathbf{CP}^\infty; \mathbf{Z})$, so that $\mathrm{H}^*(\mathbf{CP}^\infty; \mathbf{Z})$ is isomorphic to the polynomial ring $\mathbf{Z}[t]$ and $\mathrm{H}^*(\mathbf{CP}^n; \mathbf{Z})$ is isomorphic to the truncated polynomial ring $\mathbf{Z}[t]/(t^{n+1})$ for each $n \geqslant 0$. Note that for each $n \geqslant 0$, there exists a cofiber sequence of pointed spaces

$$\mathbf{CP}^{n-1} \to \mathbf{CP}^n \xrightarrow{e_n} S^{2n};$$

we will assume that e_n has been normalized so that the induced map $e_n^* : \mathrm{H}^{2n}(S^{2n}; \mathbf{Z}) \to \mathrm{H}^{2n}(\mathbf{CP}^n; \mathbf{Z})$ carries the generator of $\mathrm{H}^{2n}(S^{2n}; \mathbf{Z})$ to t^n.

Lemma 4.1.17. *Let n be a nonnegative integer, let $\delta : \mathbf{CP}^\infty \to (\mathbf{CP}^\infty)^{\wedge n}$ denote the composition of the diagonal map $\mathbf{CP}^\infty \to (\mathbf{CP}^\infty)^n$ with the collapse map $(\mathbf{CP}^\infty)^n \to (\mathbf{CP}^\infty)^{\wedge n}$, let $\iota_n : \mathbf{CP}^n \to \mathbf{CP}^\infty$ be the inclusion map. Then the diagram*

$$\begin{tikzcd}
\mathbf{CP}^n \arrow[r, "e_n"] \arrow[d, "\iota_n"] & S^{2n} \arrow[d, "\iota_1^{\wedge n}"] \\
\mathbf{CP}^\infty \arrow[r, "\delta"] & (\mathbf{CP}^\infty)^{\wedge n}
\end{tikzcd}$$

commutes up to homotopy.

Proof. Note that the composite map

$$\mathbf{CP}^{n-1} \hookrightarrow \mathbf{CP}^n \xrightarrow{\iota_n} \mathbf{CP}^\infty \xrightarrow{\delta} (\mathbf{CP}^\infty)^{\wedge n}$$

is nullhomotopic, since the domain has dimension $(2n-2)$ and the codomain is $(2n)$-connective. It follows that there exists a map of pointed spaces $u : S^{2n} \to (\mathbf{CP}^\infty)^{\wedge n}$ for which the diagram

$$\begin{tikzcd} \mathbf{CP}^n \arrow[r, "e_n"] \arrow[d, "\iota_n"] & S^{2n} \arrow[d, "u"] \\ \mathbf{CP}^\infty \arrow[r, "\delta"] & (\mathbf{CP}^\infty)^{\wedge n} \end{tikzcd}$$

commutes up to homotopy. To complete the proof, it will suffice to show that u is homotopic to the nth smash power of $\iota_1 : S^2 \simeq \mathbf{CP}^1 \hookrightarrow \mathbf{CP}^\infty$. We assume $n > 0$ (otherwise, there is nothing to prove). In this case, we can identify the homotopy classes of $\iota_1^{\wedge n}$ and u with elements of $\pi_{2n}(\mathbf{CP}^\infty)^{\wedge n}$. Since $(\mathbf{CP}^\infty)^{\wedge n}$ is $(2n)$-connective, the Hurewicz map $\pi_{2n}(\mathbf{CP}^\infty)^{\wedge n} \to \mathrm{H}_{2n}((\mathbf{CP}^\infty)^{\wedge n}$ is an isomorphism. It will therefore suffice to show that $\iota_1^{\wedge n}$ and u induce the same map on the homology group H_{2n} or equivalently (since all homology groups are free abelian) that the maps

$$(\iota_1^{\wedge n})^*, u^* : \mathrm{H}^{2n}((\mathbf{CP}^\infty)^{\wedge n}; \mathbf{Z}) \to \mathrm{H}^{2n}(S^{2n}; \mathbf{Z})$$

coincide. It now suffices to observe that, after composition with the isomorphism $e_n^* : \mathrm{H}^{2n}(S^{2n}; \mathbf{Z}) \to \mathrm{H}^{2n}(\mathbf{CP}^n; \mathbf{Z})$, both of these maps carry the generator $t \otimes t \otimes \cdots \otimes t \in \mathrm{H}^{2n}((\mathbf{CP}^\infty)^{\wedge n}; \mathbf{Z})$ to the element $t^n \in \mathrm{H}^{2n}(\mathbf{CP}^n; \mathbf{Z})$. $\square$

Proof of Theorem 4.1.11. Let A be a complex periodic $\mathbb{E}_\infty$-ring set $C = C_*(\mathbf{CP}^\infty; A)$; we wish to show that C is a smooth coalgebra over A. Then the inclusion $\mathbf{CP}^0 \hookrightarrow \mathbf{CP}^\infty$ determines a coalgebra map $\rho : A \to C$. Let C_0 denote the cofiber of ρ, which we can identify with the tensor product $A \otimes \Sigma^\infty(\mathbf{CP}^\infty)$. Then a choice of complex orientation of A determines an A-module map $u : C_0 \to \Sigma^2(A)$ having the property that the composition

$$u_0 : \Sigma^2(A) \simeq A \otimes \Sigma^\infty(\mathbf{CP}^1) \to A \otimes \Sigma^\infty(\mathbf{CP}^\infty) \simeq C_0 \xrightarrow{u} \Sigma^2(A)$$

is homotopic to the identity. For each $n \geqslant 0$, let $\Delta(n) : C \to C^{\otimes n}$ be the n-ary comultiplication on C, and let β_n denote the composition

$$C \xrightarrow{\Delta(n)} C^{\otimes n} \to C_0^{\otimes n} \xrightarrow{u^{\otimes n}} \Sigma^{2n}(A).$$

We will prove the following:

$(*)$ For each $n \geqslant 0$, the composite map

$$C_*(\mathbf{CP}^n; A) \to C \xrightarrow{(\beta_0, \beta_1, \ldots, \beta_n)} A \oplus \Sigma^2(A) \oplus \cdots \oplus \Sigma^{2n}(A)$$

is an equivalence of A-modules.

The proof proceeds by induction on n, the case $n = 0$ being trivial. To carry out the inductive step, it will suffice to verify that the diagram

$$\begin{tikzcd}
C_*(\mathbf{CP}^n; A) \arrow[r] \arrow[d] & \mathrm{cofib}(C_*(\mathbf{CP}^{n-1}; A) \to C_*(\mathbf{CP}^n; A)) \arrow[d, "\sim"] \\
C \arrow[r, "\beta_n"] & \Sigma^{2n}(A)
\end{tikzcd}$$

commutes up to homotopy, which follows immediately from Lemma 4.1.17 and our assumption that u_0 is the identity.

Since A is weakly 2-periodic, assertion $(*)$ guarantees that each $C_*(\mathbf{CP}^n; A)$ is locally free of rank $(n+1)$ as an A-module. In particular, each $C_*(\mathbf{CP}^n; A)$ is a flat A-module, so that $C = \varinjlim C_*(\mathbf{CP}^n; A)$ is likewise flat over A. We complete the proof by showing that $\pi_0 C$ is a smooth coalgebra of dimension 1 over the commutative ring $\pi_0 A$. Setting $L = \pi_{-2} A$, we observe that $(*)$ supplies coalgebra isomorphisms $\pi_0 C_*(\mathbf{CP}^n; A) \simeq \bigoplus_{0 \leqslant m \leqslant n} \Gamma^m_{\pi_0(A)}(L)$ which induce, after passing to the limit over n, a coalgebra isomorphism $\pi_0(C) \simeq \Gamma^*_{\pi_0(A)}(L)$. $\square$

4.2 Dualizing Lines

Let R be a commutative ring and let X be a formal hyperplane over R, equipped with a base point $\eta \in X(R)$. To the pair (X, η), we can associate a projective R-module of finite rank, which we will denote by $T^*_{X,\eta}$ and refer to as *the cotangent space of X at the point η*. It admits several equivalent descriptions:

(a) For any (discrete) R-module M, giving an R-module homomorphism $T^*_{X,\eta} \to M$ is equivalent to giving a point $\overline{\eta} \in X(R \oplus M)$ lying over the chosen base point $\eta \in X(R)$.

(b) Writing X as the formal spectrum $\mathrm{Spf}(\mathscr{O}_X)$, we can consider the module of *continuous* Kähler differentials $\widehat{\Omega}_{\mathscr{O}_X/R}$. This is a projective $\mathscr{O}_X$-module of finite rank (playing the role of the cotangent *bundle* of X), and $T^*_{X,\eta} = R \otimes_{\mathscr{O}_X} \widehat{\Omega}_{\mathscr{O}_X/R}$ is obtained by extending scalars along the augmentaiton $\mathscr{O}_X \to R$ determined by the point η.

(c) Writing $X = \mathrm{Spf}(\mathscr{O}_X)$ as in (b), we can identify $T^*_{X,\eta}$ with the quotient I/I^2, where $I \subseteq \mathscr{O}_X$ denotes the kernel of the augmentation $\mathscr{O}_X \to R$ determined by η.

(d) Writing $X = \mathrm{cSpec}(C)$, where C is a smooth coalgebra over R, we can identify η with a grouplike element of C. Then the set of primitive elements $\mathrm{Prim}_\eta(C) = \{x \in C : \Delta_C(x) = \eta \otimes x + x \otimes \eta\}$ of Remark 1.1.7 is a projective R-module of finite rank, and $T^*_{X,\eta}$ can be identified with the R-linear dual of $\mathrm{Prim}_\eta(C)$.

More generally, suppose that X is a formal hyperplane over a connective $\mathbb{E}_\infty$-ring R, equipped with a base point $\eta \in X(R)$. Proposition 1.5.19 implies that the functor X admits a cotangent complex (relative to R). In particular, we obtain an R-module spectrum $\eta^* L_{X/\mathrm{Spec}(R)}$, which we will refer to as the *cotangent fiber of X at η*. The cotangent fiber $\eta^* L_{X/\mathrm{Spec}(R)}$ can be regarded as an analogue of the classical cotangent space $T^*_{X,\eta}$ defined above. It is characterized by the following analogue of (a): for every connective R-module spectrum M, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_R}(\eta^* L_{X/\mathrm{Spec}(R)}, M) \simeq \mathrm{fib}(X(R \oplus M) \to X(R)).$$

Moreover, it can be computed by an analogue of (b): writing $X = \mathrm{Spf}(\mathscr{O}_X)$, Proposition SAG.17.2.5.1 supplies an equivalence

$$\eta^* L_{X/\mathrm{Spec}(R)} \simeq R \otimes_{\mathscr{O}_X} L^{\wedge}_{\mathscr{O}_X/R} \simeq R \otimes_A L_{\mathscr{O}_X/R},$$

where $L_{\mathscr{O}_X/R}$ denotes the relative cotangent complex of $\mathscr{O}_X$ over R and $L^{\wedge}_{\mathscr{O}_X/R}$ its completion with respect to the kernel ideal $\ker(\pi_0(\mathscr{O}_X) \to \pi_0(R))$. However, there are some respects in which the cotangent fiber $\eta^* L_{X/\mathrm{Spec}(R)}$ is a poor replacement for the classical cotangent space $T^*_{X,\eta}$:

- If R is an ordinary commutative ring, then there is a canonical R-module isomorphism $\pi_0(\eta^* L_{X/\mathrm{Spec}(R)}) \simeq T^*_{X,\eta}$. However, unless R is a $\mathbf{Q}$-algebra, the cotangent fiber $\eta^* L_{X/\mathrm{Spec}(R)}$ is usually not equivalent to $T^*_{X,\eta}$, because it has nonvanishing homotopy groups in positive degrees.

- Unless R is a $\mathbf{Q}$-algebra, the cotangent fiber $\eta^* L_{X/\mathrm{Spec}(R)}$ is usually not projective as an R-module (though it is almost perfect over R, by virtue of Proposition 1.5.19).

In the case where X is 1-dimensional, there is a different analogue of the cotangent space $T^*_{X,\eta}$ which does not share these defects, defined instead by a homotopy-theoretic analogue of (c) (Definition 4.2.1). We will denote the resulting object by $\omega_{X,\eta}$ and refer to it as the *dualizing line* of X.

4.2.1 Construction of the Dualizing Line

Definition 4.2.1. Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional formal hyperplane over R equipped with a base point $\eta \in X(\tau_{\geqslant 0}R)$, classified by an augmentation $\epsilon : \mathscr{O}_X \to R$. We let $\mathscr{O}_X(-\eta)$ denote the fiber of ϵ, which we regard as a module over $\mathscr{O}_X$. We let $\omega_{X,\eta}$ denote the tensor product $R \otimes_{\mathscr{O}_X} \mathscr{O}_X(-\eta)$. We will refer to $\omega_{X,\eta}$ as the *dualizing line of X at the point η.*

We first show that, as the terminology suggests, the dualizing line $\omega_{X,\eta}$ of Definition 4.2.1 is an invertible R-module:

Proposition 4.2.2. *Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional formal hyperplane over R equipped with a base point $\eta \in X(\tau_{\geqslant 0}R)$. Then:*

(a) *The fiber $\mathscr{O}_X(-\eta)$ is locally free of rank 1 as a $\mathscr{O}_X$-module.*

(b) *The dualizing line $\omega_{X,\eta}$ is locally free of rank 1 as an R-module.*

Proof. Assertion (a) follows from Proposition 1.4.11, and (b) follows from (a). □

Example 4.2.3. Let R be a commutative ring and let X be a 1-dimensional formal hyperplane over R equipped with a base point $\eta \in X(R)$. Then $\omega_{X,\eta}$ can be identified with I/I^2, where $I = \mathscr{O}_X(-\eta) \subseteq \mathscr{O}_X$ is the kernel ideal of the augmentation $\mathscr{O}_X \to R$ determined by ϵ. In particular, we have a canonical R-module isomorphism $\omega_{X,\eta} \simeq T^*_{X,\eta}$.

Remark 4.2.4 (Functoriality in R)**.** Let $f : R \to R'$ be a morphism of $\mathbb{E}_\infty$-rings, let X be a 1-dimensional formal hyperplane over R with a base point $\eta \in X(R)$, let $X' = X|_{\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R'}}$ be the induced formal hyperplane over R', and let $\eta' \in X'(\tau_{\geqslant 0}R') \simeq X(\tau_{\geqslant 0}R')$ be the image of η. Then we have a canonical equivalence $\omega_{X',\eta'} \simeq R' \otimes_R \omega_{X,\eta}$.

Applying this observation to the maps $R \leftarrow \tau_{\geqslant 0}R \to \pi_0(R)$, we obtain a canonical isomorphism $\pi_0 \omega_{X,\eta} \simeq T^*_{X_0,\eta_0}$, where X_0 denotes the formal hyperplane over $\pi_0(R)$ determined by X, and $\eta_0 \in X_0(\pi_0(R))$ the image of η. In particular, we see that as an object of the homotopy category hMod_R, the dualizing line $\omega_{X,\eta}$ is canonically determined by the pair (X_0, η_0) (see Corollary HA.7.2.2.19).

Remark 4.2.5 (Functoriality in X)**.** Let R be an $\mathbb{E}_\infty$-ring, let $f : X \to X'$ be a morphism of 1-dimensional formal hyperplanes over R, and let $\eta \in X(\tau_{\geqslant 0}R)$ be a point having image $\eta' \in X'(\tau_{\geqslant 0}R)$. Then pullback along f determines a map of augmented R-algebras $\mathscr{O}_{X'} \to \mathscr{O}_X$, which in turn induces a map $f^* : \omega_{X',\eta'} \to \omega_{X,\eta}$. Note that f is an equivalence if and only if the pullback map f^* is an equivalence.

4.2.2 Comparison with and the Cotangent Fiber

Let X, R, and η be as in Definition 4.2.1, and let $m : \mathscr{O}_X \otimes_R \mathscr{O}_X \to \mathscr{O}_X$ denote the multiplication map, so that we have a fiber sequence

$$\mathrm{fib}(m) \to \mathscr{O}_X \otimes_R \mathscr{O}_X \to \mathscr{O}_X$$

in the ∞-category of $(\mathscr{O}_X \otimes_R \mathscr{O}_X)$-modules. Note that, after extending scalars along the map

$$(\epsilon \otimes \mathrm{id}) : \mathscr{O}_X \otimes_R \mathscr{O}_X \to R \otimes_R \mathscr{O}_X \simeq \mathscr{O}_X,$$

this fiber sequence reduces to $\mathscr{O}_X(-\eta) \to \mathscr{O}_X \xrightarrow{\epsilon} R$. It follows that the dualizing line $\omega_{X,\eta}$ is can be obtained from $\mathrm{fib}(m)$ by extending scalars along the map $(\epsilon \otimes \epsilon) : \mathscr{O}_X \otimes_R \mathscr{O}_X \to R$. Combining this observation with Theorem HA.7.3.5.1 , we obtain the following characterization of the dualizing line $\omega_{X,\eta}$:

Proposition 4.2.6. *Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional formal hyperplane over R equipped with a base point $\eta \in X(\tau_{\geq 0} R)$. Then, for any R-module M, we have a canonical homotopy equivalence*

$$\mathrm{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, M) \simeq \mathrm{fib}(\mathrm{Map}_{\mathrm{Alg}_R}(\mathscr{O}_X, R \oplus M) \to \mathrm{Map}_{\mathrm{Alg}_R}(\mathscr{O}_X, R));$$

here the fiber is taken over the point of $\mathrm{Map}_{\mathrm{Alg}_R}(\mathscr{O}_X, R)$ determined by η.

Remark 4.2.7 (Comparing the Dualizing Line to the Cotangent Fiber)**.** Let R, X, and η be as in Proposition 4.2.6. For every R-module M, we have a commutative diagram of mapping spaces σ :

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CAlg}_R}(\mathscr{O}_X, R \oplus M) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{CAlg}_R}(\mathscr{O}_X, R) \arrow[d] \\
\mathrm{Map}_{\mathrm{Alg}_R}(\mathscr{O}_X, R \oplus M) \arrow[r] & \mathrm{Map}_{\mathrm{Alg}_R}(\mathscr{O}_X, R).
\end{tikzcd}$$

Passing to homotopy fibers in the horizontal direction, we obtain a comparison map

$$\mathrm{Map}_{\mathrm{Mod}_R}(\eta^* L_{X/\mathrm{Spec}(R)}, M) \to \mathrm{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, M).$$

This map depends functorially on M, and is therefore given by precomposition with an R-module morphism $\rho : \omega_{X,\eta} \to \eta^* L_{X/\mathrm{Spec}(R)}$. This map has the following features:

(i) If R is connective, then ρ induces an isomorphism $\pi_0(\omega_{X,\eta}) \simeq \pi_0(\eta^* L_{X/\operatorname{Spec}(R)})$. To prove this, we can reduce to the case where R is discrete, in which case it follows from the observation that the vertical maps in the diagram σ are homotopy equivalences when M is also discrete. More concretely, this amounts to the observation that $\pi_0(\omega_{X,\eta})$ and $\pi_0(\eta^* L_{X/\operatorname{Spec}(R)})$ can be identified with the cotangent space $T^*_{X_0,\eta_0}$, where X_0 denotes the underlying formal hyperplane over the commutative ring $\pi_0(R)$ (and η_0 the image of η in $X_0(\pi_0(R))$).

(ii) If R is an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$, then ρ is an equivalence. To prove this, we can reduce to the case where R is connective, in which case it follows from (i) (since both $\omega_{X,\eta}$ and $\eta^*_{L_{X/\operatorname{Spec}(R)}}$ are locally free of rank 1 over R).

Beware that if R is a connective $\mathbb{E}_\infty$-ring which is not a $\mathbf{Q}$-algebra, then the comparison map ρ is *never* an equivalence.

4.2.3 The Linearization Map

We now give a different description of the dualizing line $\omega_{X,\eta}$, which will be more useful for our purposes.

Proposition 4.2.8. *Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional formal hyperplane over R equipped with a base point $\eta \in X(\tau_{\geqslant 0}R)$. Then we have a canonical fiber sequence of R-modules*

$$\Sigma(\omega_{X,\eta}) \to R \otimes_{\mathscr{O}_X} R \xrightarrow{m} R,$$

where m denotes the multiplication on R (regarded an algebra over $\mathscr{O}_X$ via the augmentation determined by η).

Proof. Let $\iota : R \to R \otimes_{\mathscr{O}_X} R$ denote the inclusion of the first tensor factor. Then the composition $m \circ \iota$ is the identity on R: in other words, m and ι exhibit R as a direct summand of $R \otimes_{\mathscr{O}_X} R$, whose complementary summand can be described either as the fiber of m or the cofiber of ι. We now observe that ι is obtained from the augmentation map $\epsilon : \mathscr{O}_X \to R$ by extending scalars along ϵ; we therefore have equivalences

$$\begin{aligned}
\operatorname{fib}(m) &\simeq \operatorname{cofib}(\iota) \\
&\simeq R \otimes_{\mathscr{O}_X} \operatorname{cofib}(\epsilon) \\
&\simeq \Sigma(R \otimes_{\mathscr{O}_X} \operatorname{fib}(\epsilon)) \\
&= \Sigma(R \otimes_{\mathscr{O}_X} \mathscr{O}_X(-\eta)) \\
&= \Sigma(\omega_{X,\eta}).
\end{aligned}$$

$\square$

The importance of the dualizing line in this paper arises from the following construction, which will play an important role in §4.3:

Construction 4.2.9 (The Linearization Map)**.** Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional formal hyperplane over R equipped with a base point $\eta \in X(\tau_{\geqslant 0}R)$. If R is connective and A is a connective $\mathbb{E}_\infty$-algebra over R, we obtain a canonical map

$$\begin{aligned} \Omega X(A) &\simeq \operatorname{Map}_{\mathrm{CAlg}_R}(R \otimes_{\mathscr{O}_X} R, A) \\ &\to \operatorname{Map}_{\mathrm{Mod}_R}(R \otimes_{\mathscr{O}_X} R, A) \\ &\xrightarrow{u} \operatorname{Map}_{\mathrm{Mod}_R}(\Sigma(\omega_{X,\eta}), A) \\ &\simeq \Omega \operatorname{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, A). \end{aligned}$$

where u is the map induced by the identification of Proposition 4.2.8. We will denote the composite map by

$$\mathfrak{L} : \Omega X(A) \to \Omega \operatorname{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, A)$$

and refer to it as the *linearization map* associated to the pair (X, η).

More generally, if R and A are not assumed to be connective, then we can apply the preceding construction to their connective covers to obtain a map

$$\mathfrak{L} : \Omega X(\tau_{\geqslant 0}A) \to \Omega \operatorname{Map}_{\mathrm{Mod}_{\tau_{\geqslant 0}R}}(\tau_{\geqslant 0}\omega_{X,\eta}, \tau_{\geqslant 0}A) \simeq \Omega \operatorname{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, A),$$

which we will also refer to as *the linearization map* associated to (X, η).

We now make Construction 4.2.9 a bit more concrete by describing its effects on homotopy groups.

Remark 4.2.10. In the situation of Construction 4.2.9, suppose we are given a map of pointed spaces $u : T \to \Omega X(A)$, which we can identify with a pointed map $\Sigma T \to X(A) \subseteq \operatorname{Map}_{\mathrm{CAlg}_R}(\mathscr{O}_X, A)$. This determines a commutative diagram of $\mathbb{E}_\infty$-algebras

$$\begin{tikzcd} \mathscr{O}_X \arrow[r] \arrow[d] & C^*(\Sigma T; A) \arrow[d] \\ R \arrow[r] & A, \end{tikzcd}$$

where the left vertical map is given by the base point of X and the right vertical map is evaluation at the base point of T. Passing to vertical homotopy fibers, we obtain a map of $\mathscr{O}_X$-modules $u' : \mathscr{O}_X(-\eta) \to C^*_{\mathrm{red}}(\Sigma T; A)$. Using the pullback diagram of $\mathbb{E}_\infty$-algebras

$$\begin{tikzcd}
C^*(\Sigma T; A) \arrow[r] \arrow[d] & A \arrow[d] \\
A \arrow[r] & C^*(T; A),
\end{tikzcd}$$

we see that the action of $C^*(\Sigma T; A)$ on $C^*_{\mathrm{red}}(\Sigma T; A) \simeq \Sigma^{-1} C^*_{\mathrm{red}}(T; A)$ factors through A (in two *a priori* different ways), so the action of $\mathscr{O}_X$ on $C^*_{\mathrm{red}}(\Sigma T; A)$ factors through R. This determines a factorization of u' as a composition

$$\mathscr{O}_X(-\eta) \to \omega_{X,\eta} \xrightarrow{v'} C^*_{\mathrm{red}}(\Sigma T; A),$$

and we can identify v' with a pointed map $v : T \to \Omega \operatorname{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, A)$. Unwinding the definitions, we see that v is given by the composition

$$T \xrightarrow{u} \Omega X(A) \xrightarrow{\mathfrak{L}} \Omega \operatorname{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, A),$$

where $\mathfrak{L}$ is the linearization map of Construction 4.2.9.

Example 4.2.11. In the situation of Construction 4.2.9, suppose that we are given an element of $\pi_n X(R)$ for $n > 0$, which we can identify with the homotopy class of a pointed map $u : S^{n-1} \to \Omega X(R)$. Then u determines a map of augmented $\mathbb{E}_\infty$-algebras $\mathscr{O}_X \to C^*(S^n; R)$, hence a map of augmented $\pi_0(R)$-algebras $\rho : \pi_0(\mathscr{O}_X)) \to \pi_0(R) \oplus \pi_n(R)$. If we let $\epsilon : \pi_0(\mathscr{O}_X) \to \pi_0(R)$ denote the augmentation map and $I = \ker(\epsilon)$ the augmentation ideal, then ρ is given by the formula $\rho(x) = (\epsilon(x), dx)$ for some $\pi_0(R)$-linear derivation $d : \pi_0(R) \to \pi_n(R)$, which we can identify with a $\pi_0(R)$-linear map $I/I^2 \to \pi_n(R)$. Using Remark 4.2.10, we see that the linearization map $\mathfrak{L}$ of Construction 4.2.9 carries $[u] \in \pi_n X(R)$ to d, regarded as an element of

$$\operatorname{Hom}_{\pi_0(R)}(I/I^2, \pi_n(R)) \simeq \pi_n \operatorname{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, R).$$

4.2.4 Relationship with Grothendieck Duality

We now describe another interpretation of the dualizing line $\omega_{X,\eta}$: up to a shift, it can be realized as the pullback along η of the *relative dualizing complex* of the projection map $q : X \to \operatorname{Spec}(R)$. We will regard this as a heuristic, since a general discussion of Grothendieck duality in formal spectral algebraic geometry would take us too far afield. However, we can use this heuristic to motivative a precise definition: given a reasonable notion of the relative dualizing complex of q, we should expect that its pullback along η is *inverse* to the relative dualizing complex of the morphism $\eta : \mathrm{Spec}(R) \to X$. Since η is a closed immersion, the latter dualizing complex should be realized concretely as the $\mathscr{O}_X$-linear dual of R (where we regard R as a $\mathscr{O}_X$-module via the augmentation $\mathscr{O}_X \to R$ determined by η). In the case where X is 1-dimensional, this agrees with Definition 4.2.1:

Proposition 4.2.12. *Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional formal hyperplane over R equipped with a base point $\eta \in X(\tau_{\geqslant 0}R)$. Then there is a canonical equivalence of R-modules $\underline{\mathrm{Map}}_{\mathscr{O}_X}(R, \mathscr{O}_X) \simeq \Sigma^{-1}\omega_{X,\eta}^{-1}$.*

Proof. Let $e : R \to \underline{\mathrm{Map}}_{\mathscr{O}_X}(R, R)$ be the unit map and let $f : \underline{\mathrm{Map}}_{\mathscr{O}_X}(R, \mathscr{O}_X) \to \underline{\mathrm{Map}}_{\mathscr{O}_X}(R, R)$ be the map given by postcomposition with with augmentation $\epsilon : \mathscr{O}_X \to R$. A simple calculation (using our assumption that X is 1-dimensional) shows that the map

$$(e \oplus f) : R \oplus \underline{\mathrm{Map}}_{\mathscr{O}_X}(R, \mathscr{O}_X) \to \underline{\mathrm{Map}}_{\mathscr{O}_X}(R, R)$$

is an equivalence. We can therefore identify $\underline{\mathrm{Map}}_{\mathscr{O}_X}(R, \mathscr{O}_X)$ with the cofiber of e. Unwinding the definitions, we can describe e as the R-linear dual of the multiplication map $m : R \otimes_{\mathscr{O}_X} R \to R$, so that the cofiber $\mathrm{cofib}(e)$ is the R-linear dual of the fiber $\mathrm{fib}(m)$. The desired result now follows from Proposition 4.2.8. $\square$

Remark 4.2.13 (Dualizing Lines in Higher Dimensions)**.** Let R be an $\mathbb{E}_\infty$-ring and let X be a formal hyperplane of dimension n over R, equipped with a base point $\eta \in X(\tau_{\geqslant 0}R)$. It is not difficult to show that $\underline{\mathrm{Map}}_{\mathscr{O}_X}(R, \mathscr{O}_X)$ is an invertible R-module. We can then define an R-module $\omega_{X,\eta}$ by the formula $\omega_{X,\eta} = \Sigma^{-n}\underline{\mathrm{Map}}_{\mathscr{O}_X}(R, \mathscr{O}_X)^{-1}$. It follows from Proposition 4.2.12 that, in the case $n = 1$, this agrees with the dualizing line of Definition 4.2.1. Heuristically, one can think of the R-module $\Sigma^n\omega_{X,\eta}$ as the pullback along η of the relative dualizing complex of the projection map $X \to \mathrm{Spec}(R)$. One can show that $\omega_{X,\eta}$ is always locally free of rank 1 as an R-module, and is equipped with a canonical isomorphism

$$\pi_0(\omega_{X,\eta}) \simeq \det(T^*_{X_0,\eta_0}) = \bigwedge\nolimits^n_{\pi_0(R)}(T^*_{X_0,\eta_0}),$$

where $T^*_{X_0,\eta_0}$ denotes the cotangent space to the underlying formal hyperplane X_0 over the commutative ring $\pi_0(R)$, equipped with the base point $\eta_0 \in X_0(\pi_0(R))$ determined by η.

4.2.5 The Dualizing Line of a Formal Group

Let R be an $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}$ be a formal group over R. We will say that $\widehat{\mathbf{G}}$ *has dimension n* if the underlying formal hyperplane $\Omega^\infty\widehat{\mathbf{G}}$ has dimension n.

Definition 4.2.14. Let $\widehat{\mathbf{G}}$ be a formal group of dimension 1 over an $\mathbb{E}_\infty$-ring R. If $\widehat{\mathbf{G}}$ has dimension 1, we let $\omega_{\widehat{\mathbf{G}}} \in \mathrm{Mod}_R$ denote the dualizing line $\omega_{X,\eta}$ of Definition 4.2.1, where $X = \Omega^\infty \circ \widehat{\mathbf{G}}$ is the underlying formal hyperplane of $\widehat{\mathbf{G}}$ and $\eta \in X(\tau_{\geqslant 0} R)$ is the base point (that is, the identity with respect to the group structure on $\widehat{\mathbf{G}}$). Then $\omega_{\widehat{\mathbf{G}}}$ is a locally free R-module of rank 1, which we will refer to as the *dualizing line of* $\widehat{\mathbf{G}}$.

Example 4.2.15. If $\widehat{\mathbf{G}}$ is a 1-dimensional formal group over a commutative ring R, then the dualizing line $\omega_{\widehat{\mathbf{G}}}$ is the R-linear dual of the Lie algebra $\mathrm{Lie}(\widehat{\mathbf{G}})$.

Example 4.2.16 (The Dualizing Line of $\widehat{\mathbf{G}}_m$)**.** By construction, the $\mathbb{E}_\infty$-ring of functions $\mathscr{O}_{\widehat{\mathbf{G}}_m}$ can be identified with the completion of $S[t^{\pm 1}] \simeq \Sigma^\infty_+(\mathbf{Z})$ with respect to the augmentation ideal $(t-1)$. It follows that we have a canonical equivalence

$$\begin{aligned} S \otimes_{\mathscr{O}_{\widehat{\mathbf{G}}_m}} S &\simeq S \otimes_{S[t^{\pm 1}]} S \\ &\simeq \Sigma^\infty_+(*) \otimes_{\Sigma^\infty_+(\mathbf{Z})} \Sigma^\infty_+(*) \\ &\simeq \Sigma^\infty_+(B\,\mathbf{Z}) \\ &= \Sigma^\infty_+(S^1) \end{aligned}$$

which restricts to an equivalence $\omega_{\widehat{\mathbf{G}}_m} \simeq S$, where $\omega_{\widehat{\mathbf{G}}_m}$ is the dualizing line of Definition 4.2.1. (Of course, the existence of such an equivalence is automatic, since every locally free S-module of rank 1 is equivalent to S; however, it would *a priori* be ambiguous up to a sign.)

Remark 4.2.17 (Linearization)**.** For every connective $\mathbb{E}_\infty$-ring R, Construction 4.2.9 determines a map of spaces

$$\mathfrak{L} : \Omega^{\infty+1}\widehat{\mathbf{G}}_m(R) \to \Omega^1 \mathrm{Map}_{\mathrm{Sp}}(\omega_{\widehat{\mathbf{G}}_m}, R) \simeq \Omega^{\infty+1}(R).$$

Unwinding the definitions, we see that $\mathfrak{L}$ is given by the composition

$$\begin{aligned} \Omega^{\infty+1}\widehat{\mathbf{G}}_m(R) &\simeq \Omega^{\infty+1}\mathbf{G}_m(R) \\ &\xrightarrow{\Omega\alpha} \Omega\,\mathrm{GL}_1(R) \\ &\xrightarrow{\gamma} \Omega^{\infty+1}R, \end{aligned}$$

where $\alpha : \Omega^\infty \mathbf{G}_m(R) \to \mathrm{GL}_1(R)$ is the map of Remark 1.6.12, and γ is the homotopy equivalence induced by translation by (-1).

Remark 4.2.18 (Functoriality)**.** The dualizing line $\omega_{\widehat{\mathbf{G}}}$ of Definition 4.2.14 depends functorially on R and $\widehat{\mathbf{G}}$. More precisely:

- For any morphism of $\mathbb{E}_\infty$-rings $R \to R'$, we have a canonical equivalence of R'-modules $\omega_{\widehat{\mathbf{G}}_{R'}} \simeq R' \otimes_R \omega_{\widehat{\mathbf{G}}}$.

- Every morphism $f : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}'$ of 1-dimensional formal groups over R induces a pullback map $f^* : \omega_{\widehat{\mathbf{G}}'} \to \omega_{\widehat{\mathbf{G}}}$, which is an equivalence if and only if f is an equivalence.

See Remarks 4.2.4 and 4.2.5.

Example 4.2.19. [The Quillen Formal Group] Let A be a complex periodic $\mathbb{E}_\infty$-ring. Then the Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ admits a dualizing line $\omega_{\widehat{\mathbf{G}}^{\mathcal{Q}}_A}$ (Definition 4.2.14). By virtue of Proposition 4.2.8, the dualizing line $\omega_{\widehat{\mathbf{G}}^{\mathcal{Q}}_A}$ is characterized by the existence of a fiber sequence

$$\Sigma(\omega_{\widehat{\mathbf{G}}^{\mathcal{Q}}_A}) \to A \otimes_{C^*(\mathbf{CP}^\infty; A)} A \xrightarrow{m} A,$$

where m denotes the multiplication on A (regarded as a $C^*(\mathbf{CP}^\infty; A)$-algebra by means of evaluation at the base point of $\mathbf{CP}^\infty$). Here the middle term can be identified with $C^*(S^1; A)$, and this identification carries m to the map given by evaluation at the base point of S^1. We therefore obtain an A-module equivalence $\Sigma(\omega_{\widehat{\mathbf{G}}^{\mathcal{Q}}_A}) \simeq C^*_{\mathrm{red}}(S^1; A) \simeq \Sigma^{-1} A$. Desuspending both sides, we obtain an A-module equivalence $\omega_{\widehat{\mathbf{G}}^{\mathcal{Q}}_A} \simeq \Sigma^{-2}(A)$ (note that the right hand side is locally free of rank 1 as an A-module, by virtue of our assumption that A is weakly 2-periodic).

4.3 Orientations

Let A be a complex periodic $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ be the Quillen formal group of A (Construction 4.1.13). Our goal in this section is to articulate a universal property that characterizes $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ as an object of the ∞-category $\mathrm{FGroup}(A)$ of formal groups over A: for any formal group $\widehat{\mathbf{G}}$, giving a morphism of formal groups $f : \widehat{\mathbf{G}}^{\mathcal{Q}}_A \to \widehat{\mathbf{G}}$ is equivalent to giving a *preorientation* of $\widehat{\mathbf{G}}$ (see Definition 4.3.19 and Proposition 4.3.21). Moreover, to any preorientation e we associate a *Bott map* $\beta_e : \omega_{\widehat{\mathbf{G}}} \to \Sigma^{-2}(A)$, which is invertible if and only if A is complex periodic and the associated map of formal groups $f : \widehat{\mathbf{G}}^{\mathcal{Q}}_A \to \widehat{\mathbf{G}}$ is an equivalence (Proposition 4.3.23).

4.3.1 Preorientations of Formal Hyperplanes

We begin with some general remarks.

Definition 4.3.1. Let R be an $\mathbb{E}_\infty$-ring and let $X : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}(R)} \to \mathcal{S}_*$ be a pointed formal hyperplane over R. A *preorientation* of X is a map of pointed spaces

$$e : S^2 \to X(\tau_{\geqslant 0}(R)).$$

We let $\mathrm{Pre}(X) = \Omega^2 X(\tau_{\geqslant 0}R)$ denote the space of preorientations of X. A *preoriented formal hyperplane* is a pair (X, e), where X is a pointed formal hyperplane over R and $e \in \mathrm{Pre}(X)$ is a preorientation of X.

Remark 4.3.2. To define the notion of a preoriented formal hyperplane, it is not necessary to mention base points: the data of a preoriented formal hyperplane (X, e) over R is equivalent to the data of a formal hyperplane X and a map of spaces $e : S^2 \to X(\tau_{\geqslant 0}R)$; we can then regard e as a pointed map by equipping X with the base point given by applying e to the base point of S^2. However, the slightly baroque phrasing of Definition 4.3.1 is better suited to our applications: we will be interested in studying the totality of preorientations which are compatible with a fixed base point of X (in practice, we will take X to be the underlying formal hyperplane of a formal group, with base point given by the zero section).

Remark 4.3.3. Let R be an $\mathbb{E}_\infty$-ring, and let X be a pointed formal hyperplane R, which we can also view as a pointed formal hyperplane over the connective cover $\tau_{\geqslant 0}R$ (Variant 1.5.11). The space of preorientations $\mathrm{Pre}(X)$ does not depend on whether we view $\widehat{\mathbf{G}}$ as a formal group over R or over $\tau_{\geqslant 0}(R)$. Consequently, for the purpose of studying preorientations, it is harmless to replace R by its connective cover.

Remark 4.3.4. Let X be a pointed formal hyperplane over an $\mathbb{E}_\infty$-ring R. Then the set of homotopy classes of preorientations of X can be identified with $\pi_2 X(\tau_{\geqslant 0}R)$.

Example 4.3.5. Let R be a connective $\mathbb{E}_\infty$-ring which is 1-truncated: that is, the homotopy groups $\pi_* R$ vanish for $* \notin \{0, 1\}$. Then, for any pointed formal hyperplane X over R, the space $X(R)$ is always 1-truncated. It follows that the space of preorientations $\mathrm{Pre}(X) = \Omega^2 X(R)$ is contractible. In particular, if R is an ordinary commutative ring, then every pointed formal hyperplane X admits an essentially unique preorientation.

Remark 4.3.6 (Functoriality)**.** Let R be an $\mathbb{E}_\infty$-ring and let X be a pointed formal hyperplane over R. Then, for any $\mathbb{E}_\infty$-algebra $R' \in \mathrm{CAlg}_R$, we can identify $\Omega^2 X(\tau_{\geqslant 0}(R'))$ with the space $\mathrm{Pre}(X_{R'})$, where $X_{R'}$ is the pointed formal hyperplane over R' obtained from X by extending scalars. In particular, the construction $R' \mapsto \mathrm{Pre}(X_{R'})$ depends functorially on R'.

4.3.2 Orientations of Formal Hyperplanes

Note that a pointed formal hyperplane X over an $\mathbb{E}_\infty$-ring R can be always equipped with a preorientation, by taking $e : S^2 \to X(\tau_{\geqslant 0}(R))$ to be a constant map. We now study a special class of preorientations which are, in some sense, very far from being constant.

Construction 4.3.7 (The Bott Map)**.** Let R be an $\mathbb{E}_\infty$-ring, let X be a 1-dimensional formal hyperplane over R, equipped with a base point $\eta \in X(\tau_{\geqslant 0}R)$ and associated dualizing line $\omega_{X,\eta}$ (Definition 4.2.1). Applying Construction 4.2.9, we obtain a linearization map

$$\mathfrak{L} : \Omega X(\tau_{\geqslant 0}(R) \to \Omega^1 \operatorname{Map}_{\operatorname{Mod}_R}(\omega_{X,\eta}, R).$$

Passing to loop spaces, we obtain a map

$$\operatorname{Pre}(X) \to \operatorname{Map}_{\operatorname{Mod}_R}(\omega_{X,\eta}, \Sigma^{-2}(R)).$$

For each preorientation $e \in \operatorname{Pre}(X)$, we denote its image under this map by $\beta_e : \omega_{X,\eta} \to \Sigma^{-2}(R)$. We will refer to β_e as *the Bott map of* e.

Remark 4.3.8. The terminology of Construction 4.3.7 is motivated by the consideration of a particular case, where $R = \mathrm{KU}$ is the complex K-theory spectrum, X is the underlying formal hyperplane of the formal multiplicative group $\widehat{\mathbf{G}}_m$ (see Construction 1.6.16), and

$$e : S^2 = \mathbf{CP}^1 \to \Omega^\infty \widehat{\mathbf{G}}_m(\mathrm{KU})$$

is the preorientation corresponding to the complex line bundle $\mathscr{O}(1)$ on $\mathbf{CP}^1$; in this case, β_e can be identified with the classical Bott periodicity equivalence $\mathrm{KU} \to \Sigma^{-2}(\mathrm{KU})$. We will return to this example in §6.5 (see also §4.3.6).

Definition 4.3.9. Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional formal hyperplane over R with a base point η. An *orientation* of X is a preorientation $e \in \operatorname{Pre}(X)$ for which the Bott map $\beta_e : \omega_{X,\eta} \to \Sigma^{-2}(R)$ is an equivalence. We let $\operatorname{OrDat}(X)$ denote the summand of $\operatorname{Pre}(X)$ spanned by the orientations of X.

Remark 4.3.10 (Functoriality)**.** Let $f : R \to R'$ be a morphism of $\mathbb{E}_\infty$-rings, X be a 1-dimensional formal hyperplane over R, and let $X' = X_{R'}$ be the formal hyperplane over R' obtained by extending scalars along f. Suppose we are given a base point $\eta \in X(\tau_{\geqslant 0}(R))$ having image $\eta' \in X'(\tau_{\geqslant 0}(R'))$, and a preorientation $e \in \operatorname{Pre}(X)$ having image $e' \in \mathrm{Pre}(X')$. Then the Bott map $\beta_{e'} : \omega_{X',\eta'} \to \Sigma^{-2}(R')$ can be identified with the composition

$$\omega_{X',\eta'} \simeq R' \otimes_R \omega_{X,\eta} \xrightarrow{\beta_e} R' \otimes_R \Sigma^{-2}(R) \simeq \Sigma^{-2}(R').$$

In particular, if β_e is an equivalence, then so is $\beta_{e'}$. It follows that the natural map $\mathrm{Pre}(X) \to \mathrm{Pre}(X')$ carries the summand $\mathrm{OrDat}(X) \subseteq \mathrm{Pre}(X)$ into the summand $\mathrm{OrDat}(X') \subseteq \mathrm{Pre}(X')$.

Remark 4.3.11. Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensionalformal hyperplane over R with a base point η. Then the dualizing line $\omega_{X,\eta}$ is locally free of rank 1 as an R-module. Consequently, the existence of an orientation of X implies that R is weakly 2-periodic. In particular, if R is nonzero and connective, then the space $\mathrm{OrDat}(X)$ is empty.

Warning 4.3.12. Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional pointed formal hyperplane over R. Then X can be identified with a pointed formal hyperplane X_0 over the connective cover $\tau_{\geqslant 0}(R)$, and giving a preorientation of X is the same as giving a preorientation of X_0. Beware, however, that giving an *orientation* of X is not the same as giving an orientation of X_0. In fact, the formal hyperplane X_0 never admits an orientation, except in the trivial case $R \simeq 0$ (Remark 4.3.11).

4.3.3 Orientation Classifiers

Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional pointed formal hyperplane over R. By virtue of Remark 4.3.10, we can regard the construction $R' \mapsto \mathrm{OrDat}(X_{R'})$ as a functor from the ∞-category CAlg_R to the ∞-category $\mathcal{S}$ of spaces (it is a subfunctor of $R' \mapsto \mathrm{Pre}(R') = \Omega^2 X(\tau_{\geqslant 0} R')$). Better yet, it is a corepresentable functor:

Proposition 4.3.13. *Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional pointed formal hyperplane over R. Then there exists an $\mathbb{E}_\infty$-algebra $\mathfrak{O}_X$ and an orientation $e \in \mathrm{OrDat}(X_{\mathfrak{O}_X})$ which is universal in the following sense: for every object $R' \in \mathrm{CAlg}_R$, evaluation on e induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}_R}(\mathfrak{O}_X, R') \to \mathrm{OrDat}(X_{R'}).$$

Definition 4.3.14. In the situation of Proposition 4.3.13, we will refer to $\mathfrak{O}_X$ as the *orientation classifier* of X.

If $\widehat{\mathbf{G}}$ is a 1-dimensional formal group over R, we let $\mathfrak{O}_{\widehat{\mathbf{G}}}$ denote the orientation classifier $\mathfrak{O}_X$, where X denotes the underlying formal hyperplane $\Omega^\infty \widehat{\mathbf{G}}$ (with base point given by the identity section of $\widehat{\mathbf{G}}$).

We first prove a version of Proposition 4.3.13 for preorientations.

Lemma 4.3.15. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be a formal hyperplane over R, equipped with a base point $\eta \in X(R)$. Then:*

(a) *The base point η is classified by a map $\mathscr{O}_X \to R$ which exhibits R as a perfect $\mathscr{O}_X$-module.*

(b) *The tensor product $R \otimes_{\mathscr{O}_X} R$ is perfect as an R-module.*

(c) *the functor $\Omega X : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is corepresentable by an object $B \in \mathrm{CAlg}_R^{\mathrm{cn}}$ which is perfect as an R-module.*

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear. We will prove (a). By virtue of Corollary SAG.8.3.5.9 , it will suffice to show that the tensor product $R \otimes_{\mathscr{O}_X} R$ is perfect as an R-module. To prove this, we can work locally on $|\operatorname{Spec}(R)|$ and thereby reduce to the case where there is an isomorphism $\pi_*(\mathscr{O}_X) \simeq \pi_*(R)[[t_1, \ldots, t_n]]$. Without loss of generality, we may assume that this isomorphism is chosen so that t_i belongs to the kernel of the augmentation $\epsilon : \pi_0(\mathscr{O}_X) \to \pi_0(R)$ determined by η. Then ϵ extends to a map of A-modules $\rho_i : \mathrm{cofib}(t_i : \mathscr{O}_X \to \mathscr{O}_X) \to R$. Tensoring these maps together, we obtain a single map $\rho : \bigotimes_{1 \leqslant i \leqslant n} \mathrm{cofib}(t_i : \mathscr{O}_X \to \mathscr{O}_X) \to R$, which is easily checked to be an equivalence. $\square$

Lemma 4.3.16. *Let R be an $\mathbb{E}_\infty$-ring and let X be a pointed formal hyperplane over R. Then the functor*

$$(R' \in \mathrm{CAlg}_R) \mapsto (\mathrm{Pre}(X_{R'}) \in \mathcal{S})$$

is corepresentable by an $\mathbb{E}_\infty$-algebra A over R. Moreover, if R is connective, then A is almost of finite presentation over R.

Proof. It follows from Lemma 4.3.15 that the functor $\Omega X : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is corepresentable by an $\mathbb{E}_\infty$-algebra B which is perfect as an R-module, and therefore almost of finite presentation if R is connective (Corollary SAG.5.2.2.2). It follows that that the functor $\Omega^2 X$ is corepresentable by $A = R \otimes_B R$, which is also almost of finite presentation if R is connective. $\square$

To deduce Proposition 4.3.13 from Lemma 4.3.16, we will need to study the procedure of "inverting" the Bott map.

Proposition 4.3.17. *Let R be an $\mathbb{E}_\infty$-ring and suppose we are given a pair of invertible objects $L, L' \in \mathrm{Mod}_R$, together with a map $u : L' \to L$.*

(a) *There exists an object* $R[u^{-1}] \in \mathrm{CAlg}_A$ *with the following universal property: for every object* $A \in \mathrm{CAlg}_R$*, the mapping space* $\mathrm{Map}_{\mathrm{CAlg}_R}(R[u^{-1}], A)$ *is contractible if* u *induces an equivalence* $u_A : A \otimes_R L' \to A \otimes_R L$*, and is otherwise empty.*

(b) *As an* R*-module,* $R[u^{-1}]$ *can be identified with the direct limit of the sequence*

$$R \xrightarrow{u} L'^{-1} \otimes_R L \xrightarrow{u} (L'^{-1})^{\otimes 2} \otimes_R L^{\otimes 2} \xrightarrow{u} (L'^{-1})^{\otimes 3} \otimes_R L^{\otimes 3} \to \cdots .$$

Remark 4.3.18. Proposition 4.3.17 is most familiar in the case $L = L' = R$; in this case, we can identify u with an element of $\pi_0(R)$ and $R[u^{-1}]$ with the associated Zariski localization of R. However, we can also apply Proposition 4.3.17 to invertible R-modules which are not locally free, like the suspension $\Sigma^n(R)$.

Proof of Proposition 4.3.17. Replacing L by the tensor product $L'^{-1} \otimes_R L$, we can reduce to the case where $L' = R$. For every R-module M, let $u_M : M \to M \otimes_R L$ denote the tensor product of id_M with u, and let $M[u^{-1}]$ denote the colimit of the diagram

$$M \xrightarrow{u_M} M \otimes_R L \xrightarrow{u_{M \otimes_R L}} M \otimes_R L \otimes_R L \to \cdots$$

Let us say that M is *u-local* if the map u_M is an equivalence. Let $\mathcal{C} \subseteq \mathrm{Mod}_R$ denote the full subcategory spanned by the u-local objects. We will deduce Proposition 4.3.17 from the following assertion:

$(*)$ For every R-module M, the canonical map $M \to M[u^{-1}]$ exhibits $M[u^{-1}]$ as a $\mathcal{C}$-localization of M. In particular, the inclusion functor $\mathcal{C} \hookrightarrow \mathrm{Mod}_R$ admits a left adjoint, given by $M \mapsto M[u^{-1}]$.

Let us assume $(*)$ for the moment. The explicit description of the localization $M[u^{-1}]$ shows that for any R-module N, we have a canonical equivalence $(N \otimes_R M)[u^{-1}] \simeq N \otimes_R M[u^{-1}]$. It follows that if $f : M \to M'$ is a morphism of R-modules which induces an equivalence $M[u^{-1}] \simeq M'[u^{-1}]$, then the induced map $(\mathrm{id}_N \otimes f) : N \otimes_R M \to N \otimes_R M'$ has the same property. In other words, the localization functor $M \mapsto M[u^{-1}]$ is compatible with the symmetric monoidal structure on Mod_R, in the sense of Definition HA.2.2.1.6 . It follows that the ∞-category $\mathcal{C}$ inherits a symmetric monoidal structure, which is characterized by the requirement that the construction $M \mapsto M[u^{-1}]$ determines a symmetric monoidal functor from Mod_R to $\mathcal{C}$ (see Proposition HA.2.2.1.9). Note that the inclusion functor $\mathcal{C} \hookrightarrow \mathrm{Mod}_R$ induces a fully faithful embedding $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathrm{Mod}_R) \simeq \mathrm{CAlg}_R$, whose essential image is spanned by the u-local $\mathbb{E}_\infty$-algebras over R. The construction $A \mapsto A[u^{-1}]$ is left adjoint to this inclusion, and therefore carries R to an initial object $R[u^{-1}] \in \mathrm{CAlg}(\mathcal{C})$. It follows immediately that $R[u^{-1}]$ has the desired universal property required by (a), and assertion $(*)$ guarantees that it has the explicit description required by (b).

It remains to prove $(*)$. We first show that, for every R-module M, the canonical map $M \to M[u^{-1}]$ induces a homotopy equivalence $\mathrm{Map}_{\mathrm{Mod}_R}(M[u^{-1}], N) \to \mathrm{Map}_{\mathrm{Mod}_R}(M, N)$ whenever N is u-local. To prove this, it will suffice to show that each of the transition maps

$$\mathrm{Map}_{\mathrm{Mod}_R}(M \otimes_R L^{\otimes n+1}, N) \to \mathrm{Map}_{\mathrm{Mod}_R}(M \otimes_R L^{\otimes n}, N)$$

is a homotopy equivalence. Replacing M by $M \otimes_R L^{\otimes n}$, we can reduce to the case $n = 0$. We wish to show that composition with u induces an equivalence of R-modules

$$v_N : \underline{\mathrm{Map}}_R(L, N) \to \underline{\mathrm{Map}}_R(R, N) \simeq N.$$

Since L is invertible, we can identify $v_N \otimes \mathrm{id}_L$ with an R-module map $u'_N : N \to N \otimes_R L$. Let $s : L \otimes_R L \to L \otimes_R L$ be the map given by exchanging the tensor factors. Then s is an equivalence from $L \otimes_R L$ to itself, and is therefore (by virtue of our assumption that L is invertible) given by multiplication by an invertible element $\epsilon \in \pi_0(R)$. Unwinding the definitions, we see that the diagram

$$\begin{tikzcd} N \arrow[rr, "\epsilon"] \arrow[dr, "u_N"'] & & N \arrow[dl, "u'_N"] \\ & N \otimes_R L & \end{tikzcd}$$

commutes. Consequently, our assumption that N is u-local guarantees that u'_N is an equivalence, as desired.

To complete the proof, it will suffice to show that for every R-module M, the R-module $M[u^{-1}]$ is u-local. For each $k \geqslant 0$, let $T^k(M)$ denote the tensor product $M \otimes_R L^{\otimes 2k}$, and define $\gamma : T^k(M) \to T^{k+1}(M)$ to be the composition

$$T^k(M) \xrightarrow{u_{T^k(M)}} T^k(M) \otimes_R L \xrightarrow{u_{T^k(M) \otimes_R L}} T^{k+1}(M).$$

Then $M[u^{-1}]$ can be identified with the colimit of the sequence

$$M = T^0(M) \xrightarrow{\gamma} T^1(M) \xrightarrow{\gamma} T^2(M) \to \cdots.$$

We will show that this colimit is u-local by verifying that the commutative diagram

$$\begin{tikzcd} T^0(M) \arrow[r, "\gamma"] \arrow[d, "u_{T^0(M)}"] & T^1(M) \arrow[r, "\gamma"] \arrow[d, "u_{T^1(M)}"] & T^2(M) \arrow[r] \arrow[d, "u_{T^2(M)}"] & \cdots \\ T^0(M) \otimes_R L \arrow[r, "\gamma \otimes \mathrm{id}"] & T^1(M) \otimes_R L \arrow[r, "\gamma \otimes \mathrm{id}"] & T^2(M) \otimes_R L \arrow[r] & \cdots \end{tikzcd}$$

becomes an equivalence after passing to the colimit in the horizontal direction. For this, it will suffice to show that for every square in this diagram, we can produce a map $\rho : T^k(M) \otimes_R L \to T^{k+1}(M)$ which makes the diagram

$$\begin{tikzcd}
T^k(M) \arrow[r, "\gamma"] \arrow[d, "u_{T^k(M)}"] & T^{k+1}(M) \arrow[d, "u_{T^{k+1}(M)}"] \\
T^k(M) \otimes_R L \arrow[r, "\gamma \otimes \mathrm{id}"] \arrow[ur, "\rho"] & T^{k+1}(M) \otimes_R L
\end{tikzcd}$$

commute up to homotopy. We accomplish this by taking ρ to be the tensor product of $\mathrm{id}_{T^k(M)}$ with the map $L \simeq L \otimes_R R \xrightarrow{\mathrm{id}_L \otimes u} L \otimes_R L$. A simple calculation then shows that the upper triangle commutes, and the lower triangle commutes up to multiplication by $\epsilon^2 \in \pi_0(R)$. To complete the proof, it will suffice to show that $\epsilon^2 = 1$, which is clear (since s^2 is homotopic to the identity). $\square$

Proof of Proposition 4.3.13. Let R be an $\mathbb{E}_\infty$-ring and let X be a pointed formal hyperplane of dimension 1 over R. By virtue of Lemma 4.3.16, the functor $R' \mapsto \mathrm{Pre}(X_{R'})$ is corepresented by an $\mathbb{E}_\infty$-algebra A over R. In particular, the formal group X_A is equipped a tautological preorientation e. Let η denote the base point of X_A and let $\beta_e : \omega_{X_A,\eta} \to \Sigma^{-2}(A)$ be the Bott map of Construction 4.3.7. We can then take $\mathfrak{O}_X$ to be the localization $A[\beta_e^{-1}]$ of Proposition 4.3.17. $\square$

4.3.4 Preorientations of Formal Groups

We now specialize to the case of interest.

Definition 4.3.19. Let R be an $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}$ be a formal group over R. A *preorientation of* $\widehat{\mathbf{G}}$ is a preorientation of the underlying pointed formal hyperplane $X = \Omega^\infty \widehat{\mathbf{G}}$. We let $\mathrm{Pre}(\widehat{\mathbf{G}}) = \mathrm{Pre}(X)$ denote the space of preorientations of $\widehat{\mathbf{G}}$.

Remark 4.3.20. Let $\widehat{\mathbf{G}}$ be a formal group over an $\mathbb{E}_\infty$-ring R. Then we have a canonical homotopy equivalence

$$\mathrm{Pre}(\widehat{\mathbf{G}}) = \mathrm{Map}_{\mathcal{S}_*}(S^2, \Omega^\infty \widehat{\mathbf{G}}(\tau_{\geqslant 0} R)) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma^2(\mathbf{Z}), \widehat{\mathbf{G}}(\tau_{\geqslant 0} R)).$$

We now establish the promised universal property of the Quillen formal group:

Proposition 4.3.21. *Let R be a complex periodic $\mathbb{E}_\infty$-ring, let $\widehat{\mathbf{G}}^{\mathcal{Q}}_R \in \mathrm{FGroup}(R)$ denote the Quillen formal group of Construction 4.1.13, and let $\widehat{\mathbf{G}}$ be any formal group over R. Then we have a canonical homotopy equivalence*

$$\mathrm{Pre}(\widehat{\mathbf{G}}) \simeq \mathrm{Map}_{\mathrm{FGroup}(R)}(\widehat{\mathbf{G}}^{\mathcal{Q}}_R, \widehat{\mathbf{G}}).$$

Proof. Let C denote the image of $\widehat{\mathbf{G}}$ under the equivalence

$$\mathrm{FGroup}(R) \simeq \mathrm{Ab}(\mathrm{Hyp}(R)) \xleftarrow{\mathrm{cSpec}} \mathrm{Ab}(\mathrm{cCAlg}_R^{\mathrm{sm}}),$$

We then have canonical homotopy equivalences

$$\begin{aligned} \mathrm{Pre}(\widehat{\mathbf{G}}) &= \mathrm{Map}_{\mathcal{S}_*}(S^2, \Omega^\infty \widehat{\mathbf{G}}(\tau_{\geqslant 0} R)) \\ &\simeq \mathrm{Map}_{\mathrm{Ab}(\mathcal{S})}(\mathbf{CP}^\infty, \mathrm{Map}_{\mathrm{cCAlg}_R}(R, C)) \\ &\simeq \mathrm{Map}_{\mathrm{Ab}(\mathrm{cCAlg}_R)}(C_*(\mathbf{CP}^\infty; R), C) \\ &\simeq \mathrm{Map}_{\mathrm{Ab}(\mathrm{Hyp}(R))}(\mathrm{cSpec}(C_*(\mathbf{CP}^\infty; R)), \mathrm{cSpec}(C)) \\ &\simeq \mathrm{Map}_{\mathrm{FGroup}(R)}(\widehat{\mathbf{G}}_R^{\mathcal{Q}}, \widehat{\mathbf{G}}). \end{aligned}$$

□

4.3.5 Orientations of Formal Groups

Let R be an $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R, equipped with a preorientation $e \in \mathrm{Pre}(\widehat{\mathbf{G}})$. We will say that is an *orientation* if it is an orientation of the underlying formal hyperplane of $\widehat{\mathbf{G}}$, in the sense of Definition 4.3.9: that is, if the Bott map

$$\beta_e : \omega_{\widehat{\mathbf{G}}} \to \Sigma^{-2}(R)$$

is an equivalence.

Example 4.3.22. Let R be a complex periodic $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}_R^{\mathcal{Q}}$ be the Quillen formal group over R (Construction 4.1.13). Then the identity map $\mathrm{id} : \widehat{\mathbf{G}}_R^{\mathcal{Q}} \simeq \widehat{\mathbf{G}}_R^{\mathcal{Q}}$ corresponds, under the homotopy equivalence of Proposition 4.3.21, to a preorientation $e \in \mathrm{Pre}(\widehat{\mathbf{G}}_R^{\mathcal{Q}})$. Unwinding the definitions, we see that the associated Bott map $\beta_e : \omega_{\widehat{\mathbf{G}}_R^{\mathcal{Q}}} \to \Sigma^{-2}(R)$ agrees with the equivalence described in Example 4.2.19. In particular, e is an orientation.

In fact, all orientations of formal groups are of the form described in Example 4.3.22:

Proposition 4.3.23. *Let R be an $\mathbb{E}_\infty$-ring, let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R, and let $e \in \mathrm{Pre}(\widehat{\mathbf{G}})$ be a preorientation of $\widehat{\mathbf{G}}$. Then e is an orientation if and only if the following condition are satisfied:*

(1) *The $\mathbb{E}_\infty$-ring R is complex periodic.*

(2) *Let* $f : \widehat{\mathbf{G}}^{\mathcal{Q}}_R \to \widehat{\mathbf{G}}$ *denote the image of* e *under the homotopy equivalence* $\mathrm{Pre}(\widehat{\mathbf{G}}) \simeq \mathrm{Map}_{\mathrm{FGroup}(R)}(\widehat{\mathbf{G}}^{\mathcal{Q}}_R, \widehat{\mathbf{G}})$ *of Proposition 4.3.21. Then* f *is an equivalence of formal groups over* R*.*

Proof. We first show that if e is an orientation of $\widehat{\mathbf{G}}$, then R is complex periodic. We have already seen that R is weakly 2-periodic (Remark 4.3.11). We will show that R is complex orientable. Let $\mathscr{O}_{\widehat{\mathbf{G}}}(-1)$ denote the fiber of the augmentation $\mathscr{O}_{\widehat{\mathbf{G}}} \to R$. Pullback along f determines a map of $\mathbb{E}_\infty$-algebras $f^* : \mathscr{O}_{\widehat{\mathbf{G}}} \to C^*(\mathbf{CP}^\infty; R)$ which fits into a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{\widehat{\mathbf{G}}}(-1) \arrow[r] \arrow[d] & \mathscr{O}_{\widehat{\mathbf{G}}} \arrow[r] \arrow[d, "f^*"] & R \arrow[d, "\mathrm{id}"] \\
C^*_{\mathrm{red}}(\mathbf{CP}^\infty; R) \arrow[r] \arrow[d] & C^*(\mathbf{CP}^\infty; R) \arrow[r] \arrow[d] & R \arrow[d, "\mathrm{id}"] \\
C^*_{\mathrm{red}}(\mathbf{CP}^1; R) \arrow[r] & C^*(\mathbf{CP}^1; R) \arrow[r] & R.
\end{tikzcd}$$

Using Remark 4.2.10, we see that the left vertical composition can be identified with the map

$$\begin{aligned}
\mathscr{O}_{\widehat{\mathbf{G}}}(-1) &\xrightarrow{u} R \otimes_{\mathscr{O}_{\widehat{\mathbf{G}}}} \mathscr{O}_{\widehat{\mathbf{G}}}(-1) \\
&= \omega_{\widehat{\mathbf{G}}} \\
&\xrightarrow{\beta_e} \Sigma^{-2}(R) \\
&\simeq C^*_{\mathrm{red}}(\mathbf{CP}^1; R).
\end{aligned}$$

The assumption that e is an orientation guarantees that the Bott map β_e is an equivalence. Since u is surjective on homotopy groups, it follows that the composite map

$$\pi_{-2}\, \mathscr{O}_{\widehat{\mathbf{G}}}(-1) \to \pi_{-2} C^*_{\mathrm{red}}(\mathbf{CP}^\infty; R) \xrightarrow{\pi}_{-2} C^*_{\mathrm{red}}(\mathbf{CP}^1; R)$$

is surjective. In particular, the canonical generator of $\pi_{-2} C^*_{\mathrm{red}}(\mathbf{CP}^1; R)$ can be lifted to an element of $\pi_{-2} C^*_{\mathrm{red}}(\mathbf{CP}^\infty; R)$, so that R is complex orientable as desired.

Let us now assume that (1) is satisfied, so that the Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_R$ is well-defined and e can be identified with a map of formal groups $f : \widehat{\mathbf{G}}^{\mathcal{Q}}_R \to \widehat{\mathbf{G}}$, which carries the tautological preorientation $e_0 \in \mathrm{Pre}(\widehat{\mathbf{G}}^{\mathcal{Q}}_R)$ to e. It follows that the map β_e factors as a composition

$$\omega_{\widehat{\mathbf{G}}} \xrightarrow{f^*} \omega_{\widehat{\mathbf{G}}^{\mathcal{Q}}_R} \xrightarrow{\beta_{e_0}} \Sigma^{-2}(R),$$

where the map β_{e_0} is an equivalence (Example 4.3.22). Consequently, the map $\beta_e : \omega_{\widehat{\mathbf{G}}} \to \Sigma^{-2}(R)$ is an equivalence if and only if f induces an equivalence of dualizing lines, which is equivalent to condition (2) (see Remark 4.2.18). $\square$

4.3.6 Example: Orientations of $\widehat{\mathbf{G}}_m$

Let $\widehat{\mathbf{G}}_m$ be the formal multiplicative group of Construction 1.6.16. In this section, we study the classification of (pre)orientations of $\widehat{\mathbf{G}}_m$.

Remark 4.3.24. Let R be an $\mathbb{E}_\infty$-ring. By definition, a preorientation of $\widehat{\mathbf{G}}_m$ (regarded as a formal group over R) is a map of pointed spaces $e : S^2 \to \Omega^\infty \widehat{\mathbf{G}}_m(\tau_{\geqslant 0} R)$. Note that we have a fiber sequence

$$\widehat{\mathbf{G}}_m(\tau_{\geqslant 0} R) \xrightarrow{u} \mathbf{G}_m(R) \to \mathbf{G}_m(\pi_0(R)^{\mathrm{red}}),$$

where the third term is discrete. It follows that that u induces a homotopy equivalence

$$\mathrm{Pre}(\widehat{\mathbf{G}}_m) = \Omega^{\infty+2} \widehat{\mathbf{G}}_m(\tau_{\geqslant 0} R) \to \Omega^{\infty+2} \mathbf{G}_m(R).$$

Put more informally, giving a preorientation of the formal multiplicative group $\widehat{\mathbf{G}}_m$ is equivalent to giving a preorientation of the strict multiplicative group $\mathbf{G}_m$.

In the situation of Remark 4.3.24, we can identify $\Omega^{\infty+2}\mathbf{G}_m(R)$ with the mapping space $\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma^2(\mathbf{Z}), \mathbf{G}_m(R))$. Combining this observation with Remark 1.6.11, we obtain the following:

Proposition 4.3.25. *Let R be an $\mathbb{E}_\infty$-ring and regard $\widehat{\mathbf{G}}_m$ as a formal group over R. Then we have a canonical homotopy equivalence*

$$\mathrm{Pre}(\widehat{\mathbf{G}}_m) \simeq \mathrm{Map}_{\mathrm{CAlg}}(\Sigma^\infty_+ \mathbf{CP}^\infty, R).$$

In §4.3.5, we associated to each preorientation e of a 1-dimensional formal group $\widehat{\mathbf{G}}$ a *Bott map* $\beta_e : \omega_{\widehat{\mathbf{G}}} \to \Sigma^{-2}(R)$. In the special case where $\widehat{\mathbf{G}} = \widehat{\mathbf{G}}_m$ is the formal multiplicative group, the dualizing line $\omega_{\widehat{\mathbf{G}}}$ can be canonically identified with R (Example 4.2.16), so that β_e is classified by an element of $\pi_2(R)$.

Proposition 4.3.26. *Let R be an $\mathbb{E}_\infty$-ring and let e be a preorientation of the formal multiplicative group $\widehat{\mathbf{G}}_m$ (regarded as a formal group over R), classified by a morphism of $\mathbb{E}_\infty$-rings $f : \Sigma^\infty_+(\mathbf{CP}^\infty) \to R$. Then the Bott map β_e of Construction 4.3.7 is given by multiplication by the element $\beta \in \pi_2(R)$ represented by the composition*

$$S^2 = \mathbf{CP}^1 \to \mathbf{CP}^\infty \to \Omega^\infty \Sigma^\infty_+(\mathbf{CP}^\infty) \xrightarrow{f} R.$$

Proof. Apply Remark 4.2.17. □

Corollary 4.3.27. *The localization $\Sigma^\infty_+(\mathbf{CP}^\infty)[\beta^{-1}]$ is an orientation classifier for the formal multiplication group $\widehat{\mathbf{G}}_m \in \mathrm{FGroup}(S)$, in the sense of Definition 4.3.14.*

4.4 Formal Groups of Dimension 1

In this section, we review some standard facts about 1-dimensional formal groups over ordinary commutative rings.

4.4.1 The Height of a Formal Group

Let p be a prime number, which we regard as fixed throughout this section.

Definition 4.4.1. Let R be a commutative ring, let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R, and let $[p] : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$ be the map given by multiplication by p. For $n \geqslant 1$, we will say that $\widehat{\mathbf{G}}$ *has height* $\geqslant n$ if $p = 0$ in R and the map $[p]$ factors through the iterated relative Frobenius map $\widehat{\mathbf{G}} \xrightarrow{\varphi_{\widehat{\mathbf{G}}^n}} \widehat{\mathbf{G}}^{(p^n)}$ (see Notation 2.2.2).

We extend this terminology to the case $n = 0$ by declaring that *all* formal groups over R have height $\geqslant 0$ (in this case, we do not require that $p = 0$ in R).

Warning 4.4.2. The terminology of Definition 4.4.1 is potentially confusing, because the condition that a 1-dimensional formal group $\widehat{\mathbf{G}}$ has height $\geqslant n$ depends on the choice of prime number p. However, the danger of confusion is slight, since a formal group which has height $\geqslant 1$ at a prime p cannot have positive height at any other prime (except in the trivial case $R \simeq 0$).

Remark 4.4.3. Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. Suppose that there exists a prime number p for which R is (p)-local and $\widehat{\mathbf{G}}$ has height $\geqslant n$, for some positive integer n. Then the prime number p is uniquely determined, except in the trivial case $R = \{0\}$: it is the unique prime number which vanishes in R.

Example 4.4.4. Let R be a commutative ring and let $\widehat{\mathbf{G}}_m$ be the formal multiplicative group over R (see Construction 1.6.16). Then the ring of functions $\mathscr{O}_{\widehat{\mathbf{G}}_m}$ is the completion of the Laurent polynomial ring $R[q^{\pm 1}]$ with respect to the ideal $(q-1)$, which we will write as a power series ring $R[[\epsilon]]$ for $\epsilon = q-1$. Under this identification, the map $p : \widehat{\mathbf{G}}_m \to \widehat{\mathbf{G}}_m$ is classified by the map of power series rings

$$[p]^* : \mathscr{O}_{\widehat{\mathbf{G}}_m} \to \mathscr{O}_{\widehat{\mathbf{G}_m}} \qquad \epsilon \mapsto (\epsilon+1)^p - 1 = p\epsilon + \binom{p}{2}\epsilon^2 + \cdots + \epsilon^p.$$

If R is an $\mathbf{F}_p$-algebra, then this map is given more simply by $\epsilon \mapsto \epsilon^p$. It follows that $\widehat{\mathbf{G}}_m$ has height $\geqslant 1$, but does not have height $\geqslant 2$ (except in the trivial case $R \simeq 0$).

4.4.2 Differentials and Frobenius Maps

To make use of Definition 4.4.1, it is useful to have a criterion for detecting when a map of formal groups factors through the Frobenius.

Proposition 4.4.5. *Let R be a commutative $\mathbf{F}_p$-algebra and let $f : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}'$ be a morphism of 1-dimensional formal groups over R. The following conditions are equivalent:*

(1) *The pullback map $f^* : \omega_{\widehat{\mathbf{G}}'} \to \omega_{\widehat{\mathbf{G}}}$ vanishes (see Remark 4.2.18).*

(2) *The morphism f factors as a composition*

$$\widehat{\mathbf{G}} \xrightarrow{\varphi_{\widehat{\mathbf{G}}}} \widehat{\mathbf{G}}^{(p)} \xrightarrow{g} \widehat{\mathbf{G}}'.$$

Moreover, if these conditions are satisfied, then the map g is uniquely determined.

Remark 4.4.6. In the statement of Proposition 4.4.5, we do not need to assume that $\widehat{\mathbf{G}}$ and $\widehat{\mathbf{G}}'$ are 1-dimensional. However, in the general case, we should replace the dualizing lines of $\widehat{\mathbf{G}}$ and $\widehat{\mathbf{G}}'$ by their cotangent spaces at the identity (see §4.2)

Proof of Proposition 4.4.5. Working locally on $\operatorname{Spec}(R)$, we can reduce to the case where $\mathscr{O}_{\widehat{\mathbf{G}}}$ is isomorphic to a power series ring $R[[T]]$. Then the Frobenius pullback $\widehat{\mathbf{G}}^{(p)}$ can be identified with the formal spectrum of the subalgebra $R[[T^p]] \subseteq R[[T]]$. It follows that the factorization described in (2) exists if and only if the pullback map $f^* : \mathscr{O}_{\widehat{\mathbf{G}}'} \to \mathscr{O}_{\widehat{\mathbf{G}}} \simeq R[[T]]$ factors through the subalgebra $R[[T^p]] \subseteq R[[T]]$, and is automatically unique if it exists.

Let $\Omega = \Omega_{\mathscr{O}_{\widehat{\mathbf{G}}}/R}$ denote the module of Kähler differentials of $\mathscr{O}_{\widehat{\mathbf{G}}} \simeq R[[T]]$ over R, so that Ω is a free $R[[T]]$-module on a single generator dT. We observe that a power series $u \in R[[T]]$ belongs to the subalgebra $R[[T^p]]$ if and only if du vanishes in Ω. Set $\Omega' = \Omega_{\mathscr{O}_{\widehat{\mathbf{G}}'}/R}$, so that f^* induces a map of $\mathscr{O}_{\widehat{\mathbf{G}}'}$-modules $f^* : \Omega' \to \Omega$ characterized by the equation $df^*(u) = f^*(du)$. Consequently, assertion (2) is equivalent to the vanishing of the pullback map $f^* : \Omega' \to \Omega$. Using the group structure on $\widehat{\mathbf{G}}$ and $\widehat{\mathbf{G}}'$, we can write $\Omega \simeq \mathscr{O}_{\widehat{\mathbf{G}}} \otimes_R \omega_{\widehat{\mathbf{G}}}$ and $\Omega' = \mathscr{O}_{\widehat{\mathbf{G}}'} \otimes_R \omega_{\widehat{\mathbf{G}}'}$, where we identify $\omega_{\widehat{\mathbf{G}}}$ and $\omega_{\widehat{\mathbf{G}}'}$ with the R-submodules of Ω and Ω' consisting of translation-invariant differentials on $\widehat{\mathbf{G}}$ and $\widehat{\mathbf{G}}'$, respectively. The equivalence of (1) and (2) now follows from the commutativity of the diagram

$$\begin{tikzcd} \omega_{\widehat{\mathbf{G}}'} \arrow[r, "f^*"] \arrow[d] & \omega_{\widehat{\mathbf{G}}} \arrow[d] \\ \Omega' \arrow[r, "f^*"] & \Omega. \end{tikzcd}$$

□

Example 4.4.7. Let R be a commutative $\mathbf{F}_p$-algebra and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. Then the map $[p] : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$ automatically satisfies condition 1) of Proposition 4.4.5 (since $\omega_{\widehat{\mathbf{G}}}$ is an R-module), and therefore factors uniquely as a composition

$$\widehat{\mathbf{G}} \xrightarrow{\varphi_{\widehat{\mathbf{G}}}} \widehat{\mathbf{G}}^{(p)} \xrightarrow{V} \widehat{\mathbf{G}}.$$

Corollary 4.4.8. *Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. Then $\widehat{\mathbf{G}}$ has height $\geqslant 1$ if and only if $p = 0$ in R.*

4.4.3 Hasse Invariants and Landweber Ideals

Construction 4.4.9 (The Hasse Invariant)**.** Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R which has height $\geqslant n$, for some nonnegative integer n. If $n > 0$, then the prime number p vanishes in R and the map $[p] : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$ factors as a composition

$$\widehat{\mathbf{G}} \xrightarrow{\varphi_{\widehat{\mathbf{G}}^n}} \widehat{\mathbf{G}}^{(p^n)} \xrightarrow{T} \widehat{\mathbf{G}}.$$

Moreover, it follows from Proposition 4.4.5 that T is uniquely determined, and therefore induces a pullback map

$$T^* : \omega_{\widehat{\mathbf{G}}} \to \omega_{\widehat{\mathbf{G}}^{(p^n)})} \simeq \omega_{\widehat{\mathbf{G}}}^{\otimes p^n},$$

which we can identify with an element $v_n \in \omega_{\widehat{\mathbf{G}}}^{\otimes(p^n-1)}$. We will refer to v_n as the *nth Hasse invariant*. By virtue of Proposition 4.4.5, it vanishes if and only if $\widehat{\mathbf{G}}$ has height $\geqslant n+1$.

This construction extends to the case $n = 0$ by setting $v_0 = p \in R \simeq \omega_{\widehat{\mathbf{G}}}^{\otimes(p^0-1)}$. which can be identified with the endomorphism of $\omega_{\widehat{\mathbf{G}}}$ induced by the map $[p] : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$; note that p vanishes in R if and only if $\widehat{\mathbf{G}}$ has height $\geqslant 1$ (Corollary 4.4.8).

Proposition 4.4.10. *Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a formal group of dimension 1 over R. Then, for each integer $n \geqslant 0$, there exists a finitely generated ideal $\mathfrak{I}_n^{\widehat{\mathbf{G}}} \subseteq R$ with the following property: a ring homomorphism $R \to R'$ annihilates $\mathfrak{I}_n^{\widehat{\mathbf{G}}}$ if and only if the formal group $\widehat{\mathbf{G}}_{R'}$ has height $\geqslant n$.*

Proof. Let m be the largest element of $\{0, \dots, n\}$ for which $\widehat{\mathbf{G}}$ has height $\geqslant m$. We proceed by descending induction on m. If $m = n$, then the formal group $\widehat{\mathbf{G}}$ itself has height $\geqslant n$ and we can take $\mathfrak{I}_n^{\widehat{\mathbf{G}}} = (0)$. Otherwise, let $v_m \in \omega_{\widehat{\mathbf{G}}}^{\otimes(p^m-1)}$ be the mth Hasse invariant of $\widehat{\mathbf{G}}$ (Construction 4.4.9), and view v_m as an R-module homomorphism $\omega_{\widehat{\mathbf{G}}}^{\otimes(1-p^m)} \to R$. The image of this map is a finitely generated ideal $J \subseteq R$, and the formal group $\widehat{\mathbf{G}}_{R/J}$ has vanishing mth Hasse invariant and therefore has height $\geqslant m+1$. Invoking our inductive hypothesis, we deduce that there exists a finitely generated ideal $I \subseteq R/J$ with the property that a ring homomorphism $R/J \to R'$ annihilates the ideal I if and only if the formal group $\widehat{\mathbf{G}}_{R'}$ has height $\geqslant n$. Let $\mathfrak{I}_n^{\widehat{\mathbf{G}}}$ denote the inverse image of I under the reduction map $R \to R/J$. This ideal has the desired universal property by construction, and is finitely generated because both I and J are finitely generated. $\square$

Definition 4.4.11. Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a formal group of dimension 1 over R. We will refer to the ideal $\mathfrak{I}_n^{\widehat{\mathbf{G}}} \subseteq R$ as the *nth Landweber ideal of* $\widehat{\mathbf{G}}$.

Remark 4.4.12. Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a formal group of dimension 1 over R, and suppose that the dualizing line $\omega_{\widehat{\mathbf{G}}}$ is trivial: that is, it is isomorphic to R as an R-module. Then, for every $n \geqslant 0$, the nth Landweber ideal $\mathfrak{I}_n^{\widehat{\mathbf{G}}}$ is generated by n elements. Roughly speaking, it is generated by the Hasse invariants $v_0 = p, v_1, v_2, \dots, v_{n-1}$ of Construction 4.4.9, modulo the caveat that each v_m is only well-defined modulo the ideal $(v_0, \dots, v_{m-1})$ (because the construction of v_m requires that the formal group $\widehat{\mathbf{G}}$ has height $\geqslant m$).

Variant 4.4.13 (Formal Groups over $\mathbb{E}_\infty$-Rings)**.** Let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over an $\mathbb{E}_\infty$-ring R. Then $\widehat{\mathbf{G}}$ determines a 1-dimensional formal group $\widehat{\mathbf{G}}_0$ over the ordinary commutative ring $\pi_0(R)$. For each $n \geqslant 0$, we set $\mathfrak{I}_n^{\widehat{\mathbf{G}}} = \mathfrak{I}_n^{\widehat{\mathbf{G}}_0} \subseteq \pi_0(R)$. We will refer to $\mathfrak{I}_n^{\widehat{\mathbf{G}}}$ as the *nth Landweber ideal* of the formal group $\widehat{\mathbf{G}}$.

4.4.4 p-Divisibility of 1-Dimensional Formal Groups

We now use the theory of heights to formulate a criterion for p-divisibility of 1-dimensional formal groups.

Theorem 4.4.14. *Let R be a complete adic $\mathbb{E}_\infty$-ring. Assume that p is topologically nilpotent in $\pi_0(R)$ and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. Fix an integer $n \geqslant 1$. The following conditions are equivalent:*

(1) *There exists a formally connected p-divisible group $\mathbf{G}$ of height n and an equivalence $\widehat{\mathbf{G}} \simeq \mathbf{G}^\circ$.*

(2) *For every point $x \in |\operatorname{Spf}(R)| \subseteq |\operatorname{Spec}(R)|$, the formal group $\widehat{\mathbf{G}}_{\kappa(x)}$ has height n (see Definition 4.4.17 below).*

Example 4.4.15. Let R be an $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. Suppose that R is complete with respect to the nth Landweber ideal $\mathfrak{I}_n^{\widehat{\mathbf{G}}}$ for some $n \geqslant 1$ (in particular, R is (p)-complete) and that $\mathfrak{I}_{n+1}^{\widehat{\mathbf{G}}} = \pi_0(R)$. Then the pair $(R, \widehat{\mathbf{G}})$ satisfies the requirements of Theorem 4.4.14, where we endow $\pi_0(R)$ with the $\mathfrak{I}_n^{\widehat{\mathbf{G}}}$-adic topology. Consequently, the formal group $\widehat{\mathbf{G}}$ can be realized as the identity component of an (essentially unique) p-divisible group $\mathbf{G}$ of height n.

Before giving the proof of Theorem 4.4.14, let us introduce some useful terminology.

Definition 4.4.16. Let R be an $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. We will say that $\widehat{\mathbf{G}}$ has height $< n$ if the nth Landweber ideal $\mathfrak{I}_n^{\widehat{\mathbf{G}}}$ is equal to $\pi_0(R)$ (see Variant 4.4.13). Equivalently, we say that $\widehat{\mathbf{G}}$ has height $< n$ if there does not exist a point $x \in |\operatorname{Spec}(R)|$ for which the fiber $\widehat{\mathbf{G}}_{\kappa(x)}$ has height $\geqslant n$, as a formal group over the residue field $\kappa(x)$.

Definition 4.4.17. Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. We will say that $\widehat{\mathbf{G}}$ has *exact height* n if it has height $\geqslant n$ (Definition 4.4.1) and height $< n+1$ (Definition 4.4.16).

Remark 4.4.18. Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. Then $\widehat{\mathbf{G}}$ has exact height n if and only if $\mathfrak{I}_n^{\widehat{\mathbf{G}}} = (0)$ and $\mathfrak{I}_{n+1}^{\widehat{\mathbf{G}}} = R$. The first condition guarantees that the Hasse invariant $v_n \in \omega_{\widehat{\mathbf{G}}}^{\otimes(p^n-1)}$ is well-defined, and the second condition is equivalent to the requirement that v_n is a generator of $\omega_{\widehat{\mathbf{G}}}^{\otimes(p^n-1)}$.

Remark 4.4.19. Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. If $n > 0$, then $\widehat{\mathbf{G}}$ has exact height n if and only if $p = 0$ in R and the map $p : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$ factors as a composition

$$\widehat{\mathbf{G}} \xrightarrow{\varphi^n_{\widehat{\mathbf{G}}}} \widehat{\mathbf{G}}^{(p^n)} \xrightarrow{\alpha} \widehat{\mathbf{G}},$$

where $\varphi^n_{\widehat{\mathbf{G}}}$ denotes the nth iterate of the relative Frobenius map and α is an isomorphism of formal groups.

Remark 4.4.20. Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. The following conditions are equivalent:

(1) The formal group $\widehat{\mathbf{G}}$ has exact height 0.

(2) The prime number p is invertible in R.

(3) The map $p : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$ is an isomorphism.

Remark 4.4.21. Let κ be a field of characteristic p and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over κ. Then one of the following possibilities occurs:

(a) There exists a unique integer $n \geqslant 1$ such that $\widehat{\mathbf{G}}$ has exact height n.

(b) The map $p : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$ vanishes.

In case (b), $\widehat{\mathbf{G}}$ is not a p-divisible formal group (in fact, one can show that $\widehat{\mathbf{G}}$ is isomorphic to the formal additive group $\widehat{\mathbf{G}}_a$).

4.4.5 The Proof of Theorem 4.4.14

We will deduce Theorem 4.4.14 from the following standard algebraic assertion:

Proposition 4.4.22. *Let R be a commutative $\mathbf{F}_p$-algebra and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R having exact height n. Then there exists a connected p-divisible group $\mathbf{G}$ over R of height n and an isomorphism $\widehat{\mathbf{G}} \simeq \mathbf{G}^{\circ}$.*

Proof. Let $\widehat{\mathbf{G}}[p]$ denote the fiber of the map $p : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$. By virtue of Theorem 2.3.20 (and Remark 2.3.25), it will suffice to show that $\Omega^{\infty}\widehat{\mathbf{G}}[p]$ is (representable by) a finite flat R-algebra of degree p^n. Using Remark 4.4.19, we deduce that $\widehat{\mathbf{G}}[p]$ can also be identified with the fiber of the iterated relative Frobenius map $\varphi^n_{\widehat{\mathbf{G}}} : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}^{(p^n)}$, from which the desired result follows immediately (note that Zariski locally on $|\operatorname{Spec}(R)|$, this can be identified with the map of formal schemes $\operatorname{Spf}(R[[t]]) \to \operatorname{Spf}(R[[t^{p^n}]])$). $\square$

Proof of Theorem 4.4.14. Let R be a complete adic $\mathbb{E}_\infty$-ring with p topologically nilpotent in $\pi_0(R)$, and let $\widehat{\mathbf{G}}$ be a formal group over R. Suppose first that condition (2) is satisfied. The condition that $\widehat{\mathbf{G}}_{\kappa(x)}$ has height $\geqslant n$ for each $x \in |\operatorname{Spf}(R)| \subseteq |\operatorname{Spec}(R)|$ guarantees that we can choose a finitely generated ideal of definition $I \subseteq \pi_0(R)$ which contains the nth Landweber ideal $\mathfrak{I}^{\widehat{\mathbf{G}}}_n$. Then $\widehat{\mathbf{G}}_{\pi_0(R)/I}$ is a formal group of height $\geqslant n$ over the commutative ring $\pi_0(R)/I$. Using the assumption that $\widehat{\mathbf{G}}_{\kappa(x)}$ has height $\leqslant n$ at each point $x \in |\operatorname{Spf}(R)|$, we deduce that $\widehat{\mathbf{G}}_{\pi_0(R)/I}$ has exact height n. Applying

Proposition 4.4.22, we see that $\widehat{\mathbf{G}}_{\pi_0(R)/I}$ can be realized as the identity component of a connected p-divisible group $\mathbf{G}'$ of height n over $\pi_0(R)/I$. In particular, $\widehat{\mathbf{G}}_{\pi_0(R)/I}$ is a p-divisible formal group over $\pi_0(R)/I$, so Theorem 2.3.26 implies that $\widehat{\mathbf{G}}$ is a p-divisible formal group over R. We can therefore identify $\widehat{\mathbf{G}}$ with the identity component $\mathbf{G}^\circ$, where $\mathbf{G}$ is a formally connected p-divisible group over R. Corollary 2.3.13 then supplies an equivalence $\mathbf{G}' \simeq \mathbf{G}_{\pi_0(R)/I}$, so that $\mathbf{G}$ must have height n at every point $x \in |\operatorname{Spf}(R)| \subseteq |\operatorname{Spec}(R)|$. Since every closed subset of $|\operatorname{Spec}(R)|$ has nonempty intersection with $|\operatorname{Spf}(R)|$, it follows that $\mathbf{G}$ is a p-divisible group of height n, which completes the proof of (1).

We now show that (1) $\Rightarrow$ (2). Assume that $\widehat{\mathbf{G}} \simeq \mathbf{G}^\circ$, where $\mathbf{G}$ is a formally connected p-divisible group of height n over R. We wish to show that, for each point $x \in |\operatorname{Spf}(R)| \subseteq |\operatorname{Spec}(R)|$, the formal group $\widehat{\mathbf{G}}_{\kappa(x)}$ has height n. Note that since multiplication by p is nonzero on the connected p-divisible group $\mathbf{G}_{\kappa(x)}$, it is also nonzero on the formal group $\widehat{\mathbf{G}}_{\kappa(x)} \simeq \mathbf{G}^\circ_{\kappa(x)}$ (Corollary 2.3.13). Consequently, there exists some positive integer m such that $\widehat{\mathbf{G}}_{\kappa(x)}$ has exact height m (Remark 4.4.21). Applying Proposition 4.4.22, we deduce that $\widehat{\mathbf{G}}_{\kappa(x)}$ is the identity component of a connected p-divisible group $\mathbf{G}'$ of height m over $\kappa(x)$. It follows from Corollary 2.3.13 that the p-divisible groups $\mathbf{G}_{\kappa(x)}$ and $\mathbf{G}'$ are isomorphic, so we must have $m = n$. $\square$

4.4.6 Descending Formal Groups Along the Frobenius

The following fact about formal groups of exact height n will be useful in §5:

Proposition 4.4.23. *Let R be commutative $\mathbf{F}_p$-algebra and let $\mathrm{FGroup}^{=n}(R)$ denote the full subcategory of $\mathrm{FGroup}(R)$ spanned by those formal groups over R which are 1-dimensional of exact height n. Then the extension of scalars functor $\mathrm{FGroup}^{=n}(R) \to \mathrm{FGroup}^{=n}(R^{1/p^\infty})$ is an equivalence of categories.*

Remark 4.4.24. In fact, one can prove a stronger assertion: for each $n > 0$, the moduli stack of formal groups of exact height n can be realized as a gerbe for a profinite group (so that $\mathrm{FGroup}^{=n}(R)$ depends only on the category of finite étale R-algebras). However, we will not need this stronger assertion.

Proof of Proposition 4.4.23. For every $\mathbf{F}_p$-algebra R, let $\mathrm{FGroup}^{\varphi^n}(R)$ denote the category whose objects are pairs $(\widehat{\mathbf{G}}, \rho)$, where $\widehat{\mathbf{G}}$ is a formal group over R and $\rho : \widehat{\mathbf{G}}^{(p^n)} \simeq \widehat{\mathbf{G}}$ is an isomorphism of formal groups; here $\widehat{\mathbf{G}}^{(p^n)}$ denotes the n-fold Frobenius pullback of $\widehat{\mathbf{G}}$; a morphism from $(\widehat{\mathbf{G}}, \rho)$ to $(\widehat{\mathbf{G}}', \rho')$ in the category $\mathrm{FGroup}^{\varphi^n}(R)$ is given by a map of formal groups $f : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}'$ for which the diagram

$$\begin{tikzcd} \widehat{\mathbf{G}}^{(p^n)} \arrow[r, "\rho"] \arrow[d, "f^{(p^n)}"] & \widehat{\mathbf{G}} \arrow[d, "f"] \\ \widehat{\mathbf{G}}'^{(p^n)} \arrow[r, "\rho'"] & \widehat{\mathbf{G}}' \end{tikzcd}$$

commutes. Note that, for each object $(\widehat{\mathbf{G}}, \rho) \in \mathrm{FGroup}^{\varphi^n}(R)$, the Frobenius pullback $(\widehat{\mathbf{G}}^{(p^n)}, \rho^{(p^n)})$ is also an object of $\mathrm{FGroup}^{\varphi^n}(R)$. Moreover, the relative Frobenius map $\varphi^n : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}^{(p^n)}$ and $\rho : \widehat{\mathbf{G}}^{(p^n)} \to \widehat{\mathbf{G}}$ can be regarded as morphisms in $\mathrm{FGroup}^{\varphi^n}(R)$. Unwinding the definitions, we can identify $\mathrm{FGroup}^{=n}(R)$ with the full subcategory of $\mathrm{FGroup}^{\varphi^n}(R)$ spanned by those objects $(\widehat{\mathbf{G}}, \rho)$ for which $\widehat{\mathbf{G}}$ is 1-dimensional and the composite map

$$\widehat{\mathbf{G}} \xrightarrow{\varphi^n} \widehat{\mathbf{G}}^{(p^n)} \xrightarrow{\rho} \widehat{\mathbf{G}}$$

coincides with multiplication by p. Consequently, to show that extension of scalars determines an equivalence of categories $\mathrm{FGroup}^{=n}(R) \to \mathrm{FGroup}^{=n}(R^{1/p^\infty})$, it will suffice to show that it determines an equivalence of categories $\mathrm{FGroup}^{\varphi^n}(R) \to \mathrm{FGroup}^{\varphi^n}(R^{1/p^\infty})$.

For every commutative $\mathbf{F}_p$-algebra R, let $\mathrm{Hyp}_*(R)$ denote the category of pointed formal hyperplanes over R. We let $\mathrm{Hyp}_*^{\varphi^n}(R)$ denote the category whose objects are pairs (X, ρ), where X is a pointed formal hyperplane over R and $\rho : X^{(p^n)} \simeq X$ is an isomorphism of pointed formal hyperplanes. Note that $\mathrm{FGroup}^{\varphi^n}(R)$ can be identified with the category of abelian group objects of $\mathrm{Hyp}_*^{\varphi^n}(R)$. Consequently, to prove Proposition 4.4.23, it will suffice to show that the extension of scalars functor $\mathrm{Hyp}_*^{\varphi^n}(R) \to \mathrm{Hyp}_*^{\varphi^n}(R^{1/p^\infty})$ is an equivalence of categories.

For every $\mathbf{F}_p$-algebra R and every integer $m \geqslant 0$, let $\mathcal{C}_m(R)$ denote the category of augmented R-algebras which have the form $\bigoplus_{0 \leqslant i \leqslant m} \mathrm{Sym}^i_R(M)$, where M is a projective R-module of finite rank, and let $\mathcal{C}_m^{\varphi^n}(R)$ denote the category of pairs (A, ρ) where $A \in \mathcal{C}_m$ and $\rho : A \simeq A^{(p^n)}$ is an isomorphism in $\mathcal{C}_m(R)$. The category of pointed formal hyperplanes $\mathrm{Hyp}_*(R)$ can be identified with the inverse limit of the tower

$$\cdots \to \mathcal{C}_m(R) \to \mathcal{C}_{m-1}(R) \to \mathcal{C}_{m-2}(R) \to \cdots \to \mathcal{C}_0(R) \simeq *,$$

and this induces an identification of $\mathrm{Hyp}_*^{\varphi^n}(R)$ with the limit of the tower $\{\mathcal{C}_m^{\varphi^n}(R)\}$. We are therefore reduced to showing that, for each $m \geqslant 0$, the extension of scalars functor $\mathcal{C}_m^{\varphi^n}(R) \to \mathcal{C}_m^{\varphi^n}(R^{1/p^\infty})$ is an equivalence of categories. However, it is easy to see that the construction $R \mapsto \mathcal{C}_m^{\varphi^n}(R)$ commutes with filtered colimits (since algebras of the form $\bigoplus_{0\leqslant i\leqslant m}\mathrm{Sym}^i_R(M)$ are finitely presented over R). Writing the perfection $R^{frm[o]--/p^\infty}$ as the direct limit of the sequence $R \xrightarrow{\varphi_R} R \xrightarrow{\varphi_R} R \xrightarrow{\varphi_R} R \to \cdots$, we are reduced to proving that the Frobenius map $\varphi_R : R \to R$ induces an equivalence of categories $\mathcal{C}^{\varphi^n}_m(R) \to \mathcal{C}^{\varphi^n}_m(R)$, which follows immediately from the definition. $\square$

Corollary 4.4.25. *Let R_0 be a perfect $\mathbf{F}_p$-algebra, let $\mathbf{G}_0$ be a connected 1-dimensional p-divisible group of exact height n over R_0, and let $\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}_0})$ be its universal deformation. Then the kernel of the augmentation map $\epsilon : \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$ is the nth Landweber ideal $\mathfrak{I}^{\mathbf{G}^\circ}_n \subseteq \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$.*

Proof. Since the formal group $\mathbf{G}^\circ_{\pi_0(R)/\ker(\epsilon)} \simeq \mathbf{G}^\circ_0$ has height $\geqslant n$, it is clear that the nth Landweber ideal $\mathfrak{I}^{\mathbf{G}^\circ}_n$ is contained in the kernel $\ker(\epsilon)$. We wish to prove the reverse inclusion. Set $R = \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})/\mathfrak{I}^{\mathbf{G}^\circ}_n$ and let $I \subseteq R$ be the image of $\ker(\epsilon)$ (that is, the kernel ideal of the natural map $R \to R_0$). Then the ideal I is finitely generated and the commutative ring R is I-complete (since $R^{\mathrm{un}}_{\mathbf{G}_0}$ is complete with respect to $\ker(\epsilon)$). We can regard the $\mathbf{F}_p$-algebra R/I^2 as a square-zero extension of R_0 by I/I^2. Since R_0 is perfect, this square-zero extension admits a unique splitting $s : R_0 \to R/I^2$. Let $\mathbf{G}'$ denote the p-divisible group over R/I^2 obtained from $\mathbf{G}_0$ by extending scalars along s. We then have a pair of p-divisible groups $\mathbf{G}'$ and $\mathbf{G}_{R/I^2}$ over the commutative ring R/I^2, and a canonical isomorphism α_0 between their restrictions to $\mathrm{Spec}(R_0)$. Since the identity components $\mathbf{G}'^\circ$ and $\mathbf{G}^\circ_{R/I^2}$ have exact height n and the projection map $R/I^2 \to R_0$ induces an equivalence of perfections, Proposition 4.4.23 guarantees that we can lift α_0 uniquely to an isomorphism of formal groups $\mathbf{G}'^\circ \simeq \mathbf{G}^\circ_{R/I^2}$, or equivalently to an isomorphism of p-divisible groups $\alpha : \mathbf{G}' \simeq \mathbf{G}_{R/I^2}$ (since both $\mathbf{G}'$ and $\mathbf{G}_{R/I^2}$ are connected). Invoking the universal property of the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$, we conclude that the canonical map $R^{\mathrm{un}}_{\mathbf{G}_0} \to R/I^2$ factors through s. In particular, s is surjective, so that the quotient I/I^2 vanishes. Since R is I-complete, the ideal I is contained in every maximal ideal $\mathfrak{m}$ of R. It follows that the quotient $I/\mathfrak{m}I$ also vanishes, so that (by Nakayama's lemma) the localization $I_\mathfrak{m}$ vanishes. Allowing $\mathfrak{m}$ to vary, we conclude that $I \simeq \ker(\epsilon)/\mathfrak{I}^{\mathbf{G}^\circ}_n$ vanishes, so that $\ker(\epsilon) = \mathfrak{I}^{\mathbf{G}^\circ}_n$ as desired. $\square$

4.5 The $K(n)$-Local Case

Throughout this section, we fix a prime number p. Let $K(n)$ denote the nth Morava K-theory (at the prime p). Our goal is to supply a purely algebraic criterion which can be used to determine when a complex periodic ring spectrum A is $K(n)$-local. First, let us introduce some terminology.

Definition 4.5.1. Let A be a (p)-local $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}}$ be a formal group of dimension 1 over A. Then $\widehat{\mathbf{G}}$ determines a 1-dimensional formal group over the commutative ring $\pi_0 A$, which we will denote by $\widehat{\mathbf{G}}_{\pi_0 A}$. For each $n \geqslant 0$, we set $\mathfrak{I}_n^{\widehat{\mathbf{G}}} = \mathfrak{I}_n^{\widehat{\mathbf{G}}_{\pi_0 A}}$. We will refer to $\mathfrak{I}_n^{\widehat{\mathbf{G}}}$ as the *nth Landweber ideal of* $\widehat{\mathbf{G}}$.

In the special case where A is complex periodic and $\widehat{\mathbf{G}} = \widehat{\mathbf{G}}_A^{\mathcal{Q}}$ is the Quillen formal group, we will denote the Landweber ideal $\mathfrak{I}_n^{\widehat{\mathbf{G}}}$ simply by $\mathfrak{I}_n^A$, and refer to it as the *nth Landweber ideal of A*. Let $\widehat{\mathbf{G}}_A^{\mathcal{Q}_n}$ denote the formal group obtained from $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ by extending scalars along the quotient map $\pi_0(A) \to \pi_0(A)/\mathfrak{I}_n^A$. Then $\widehat{\mathbf{G}}_A^{\mathcal{Q}_n}$ has height $\geqslant n$, and its dualizing line is given by $\pi_2(A)/\mathfrak{I}_n^A \pi_2(A)$ (see Example 4.2.19). We let v_n denote the nth Hasse invariant of $\widehat{\mathbf{G}}_A^{\mathcal{Q}_n}$, which we regard as an element of $\pi_{2p^n-2}(A)/\mathfrak{I}_n^A \pi_{2p^n-2}(A)$. Then the ideal $\mathfrak{I}_{n+1}^A$ is generated by $\mathfrak{I}_n^A$ together with $\overline{v}_n \pi_{2-2p^n}(A)$, where $\overline{v}_n \in \pi_{2p^n-2}(A)$ is any lift of v_n.

We can now state our main result:

Theorem 4.5.2. *Let A be a (p)-local complex periodic $\mathbb{E}_\infty$-ring and let n be a positive integer. Then A is $K(n)$-local if and only if the following conditions are satisfied:*

- (a) *The $\mathbb{E}_\infty$-ring A is complete with respect to the nth Landweber ideal $\mathfrak{I}_n^A \subseteq \pi_0(A)$.*
- (b) *The $(n+1)$st Landweber ideal $\mathfrak{I}_{n+1}^A$ is equal to $\pi_0(A)$. In other words, the formal group $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ has height $\leqslant n$, in the sense of Definition 4.4.16.*

Let A be a (p)-local complex periodic $\mathbb{E}_\infty$-ring. Hypothesis (a) of Theorem 4.5.2 is equivalent to the requirement that each homotopy group $\pi_m(A)$ is $\mathfrak{I}_n^A$-complete, when regarded as a discrete module over $\pi_0(A)$ (Theorem SAG.7.3.4.1). Since A is weakly 2-periodic, it suffices to verify this condition for $m \in \{0, 1\}$. In particular, we obtain the following:

Corollary 4.5.3. *Let A be an even periodic (p)-local $\mathbb{E}_\infty$-ring and let n be a positive integer. Then A is $K(n)$-local if and only if the commutative ring $\pi_0(A)$ is $\mathfrak{I}_n^A$-complete and the formal group $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ has height $\leqslant n$.*

4.5.1 $K(n)$-Locality of Modules

We will deduce Theorem 4.5.2 from the following more general assertion:

Proposition 4.5.4. *Let A be a (p)-local complex periodic $\mathbb{E}_\infty$-ring, let M be an A-module, and let n be a positive integer. Then M is $K(n)$-local if and only if the following conditions are satisfied:*

(a) *The module M is complete with respect to the nth Landweber ideal $\mathfrak{I}_n^A \subseteq \pi_0(A)$.*

(b) *Let $v_n \in \pi_{2p^n-2}(A)/\mathfrak{I}_n^A \pi_{2p^n-2}(A)$ be as in Definition 4.5.1 and let $\overline{v}_n \in \pi_{2p^n-2}(A)$ be any lift of v_n. Then multiplication by $\overline{v}_n$ induces an equivalence from $\Sigma^{2p^n-2}M$ to M (note that if condition (a) is satisfied, then this condition is independent of the choice of $\overline{v}_n$).*

Proof of Theorem 4.5.2 from Proposition 4.5.4. Suppose first that A is $K(n)$-local, and let $\overline{v}_n \in \pi_{2p^n-2}(A)$ be a lift of $v_n \in \pi_{2p^n-2}(A)/\mathfrak{I}_n^A \pi_{2p^n-2}(A)$. It follows from Proposition 4.5.4 that A is complete with respect to $\mathfrak{I}_n^A$ and that $\overline{v}_n$ is invertible in $\pi_*(A)$. Since $\mathfrak{I}_{n+1}^A$ contains the image of the map $\pi_{2-2p^n}A \xrightarrow{\overline{v}_n} \pi_0 A$, it follows that $\mathfrak{I}_{n+1}^A = \pi_0(A)$.

Now suppose that conditions (a) and (b) of Theorem 4.5.2 are satisfied. Then $\mathfrak{I}_{n+1}^A = \pi_0(A)$, so that $\pi_0(A)$ is generated by $\mathfrak{I}_n^A$ together with the image of the map $\overline{v}_n : \pi_{2-2p^n}(A) \to \pi_0(A)$. It follows that there exists an element $x \in \pi_{2-2p^n}(A)$ such that $\overline{v}_n x \equiv 1 \mod \mathfrak{I}_n^A$. Condition ($a$) implies that $\pi_0(A)$ is $\mathfrak{I}_n^A$-complete, so that $\overline{v}_n x$ is invertible in $\pi_0(A)$ (Corollary SAG.7.3.4.9). Consequently, $\overline{v}_n$ is an invertible element of the graded ring $\pi_*(A)$, so that A is $K(n)$-local by virtue of Proposition 4.5.4. □

4.5.2 The Proof of Proposition 4.5.4

We need the following simple observation:

Lemma 4.5.5. *Let A be a weakly 2-periodic $\mathbb{E}_\infty$-ring, let $x \in \pi_{2m}(A)$, and let $I \subseteq \pi_0(A)$ be an ideal which contains the image of the map $\pi_{-2m}(A) \to \pi_0(A)$ given by multiplication by x. Let M be an A-module. If x induces an equivalence $\Sigma^{2m}M \to M$, then M is I-local.*

Proof. The assertion is local with respect to the Zariski topology on A. We may therefore assume that $\pi_{-2}(A)$ contains an element t which is invertible in $\pi_*(A)$. In this case, multiplication by the element $t^m x \in I \subseteq \pi_0 A$ induces an equivalence from M to itself, so that M is $(t^m x)$-local and therefore I-local. □

Proof of Proposition 4.5.4. Let A be a (p)-local complex periodic $\mathbb{E}_\infty$-ring and let M be an A-module. Suppose first that M is $K(n)$-local; we will show that conditions (a) and (b) of Proposition 4.5.4 are satisfied. We begin by proving (a). By virtue of Corollary SAG.7.3.3.3 , it will suffice to show that M is (t)-complete, for each element $t \in \mathfrak{I}_n^A$. Let N be an $A[1/t]$-module; we will show that the mapping space

$\mathrm{Map}_{\mathrm{Mod}_A}(N, M)$ vanishes. Since M is $K(n)$-local, it will suffice to show that $K(n) \otimes_S N$ vanishes. Note that $K(n) \otimes_S N$ can be viewed as a module over the ring spectrum $K(n) \otimes_S A[1/t]$. We are therefore reduced to showing that $B = K(n) \otimes_S A[1/t]$ vanishes. This is clear, since the classical Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_B$ has height n (since it is obtained by extension of scalars from $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{K(n)}$) and also height $< n$ (since it is obtained by extension of scalars from $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{A[1/t]}$. (Beware that $K(n)$ is not an $\mathbb{E}_\infty$-ring, or even homotopy commutative at the prime $p = 2$; however, the classical Quillen formal group of $K(n)$ is still well-defined.)

We now prove (b). Let $f : \Sigma^{2p^n - 2} M \to M$ be the map induced by $\overline{v}_n$; we wish to prove that f is an equivalence. Since f is a map between $K(n)$-local spectra, it will suffice to show that the induced map $\Sigma^{2p^n-2} M \otimes_S K(n) \to M \otimes_S K(n)$ is an equivalence. This is clear, since the image of $\overline{v}_n$ is invertible in $\pi_*(A \otimes_S K(n))$ (by virtue of the fact that the classical Quillen formal group of $A \otimes_S K(n)$ has height exactly n).

Now suppose that conditions (a) and (b) of Proposition 4.5.4 are satisfied; we wish to show that the module M is $K(n)$-local. For $0 \leqslant m \leqslant n$, let $\overline{v}_m \in \pi_{2(p^m-1)}(A)$ be a lift of the mth Hasse invariant $v_m \in \pi_{2(p^m-1)}(A)/\mathfrak{I}^A_m \pi_{2(p^m-1)}(A)$. Using condition (b), we see that M can be regarded as a module over the localization $A[v_n^{-1}]$; we may therefore replace A by $A[v_n^{-1}]$ and thereby reduce to the case where v_n is invertible in $\pi_*(A)$. We will prove the following more general result for $0 \leqslant m \leqslant n$:

$(*_m)$ Let N be a perfect A-module which is $\mathfrak{I}^A_m$-nilpotent. Then the tensor product $M \otimes_A N$ is $K(n)$-local.

Note that assertion $(*_0)$ implies that $M \simeq M \otimes_A A$ is $K(n)$-local. We will prove $(*_m)$ by descending induction on m. Let us first carry out the inductive step. Assume that $m < n$ and that $(*_{m+1})$ is satisfied; we will prove that $(*_m)$ is also satisfied. Let N be a perfect A-module which is $\mathfrak{I}^A_m$-nilpotent. Since M is $\mathfrak{I}^A_n$-complete and N is perfect, the tensor product $M \otimes_A N$ is also $\mathfrak{I}^A_n$-complete. It follows that we can identify $M \otimes_A N$ with the homotopy limit of the tower

$$\{M \otimes_A \mathrm{cofib}(v_m^k : \Sigma^{2(p^m-1)k} N \to N)\}_{k \geqslant 0}.$$

Since each cofiber $\mathrm{cofib}(v_m^k : \Sigma^{2(p^m-1)k} N \to N)$ is $\mathfrak{I}^A_{m+1}$-nilpotent, our inductive hypothesis guarantees that the tensor product $M \otimes_A \mathrm{cofib}(v_m^k : \Sigma^{2(p^m-1)k} N \to N)$ is $K(n)$-local. Passing to the limit, we deduce that $M \otimes_A N$ is $K(n)$-local.

To complete the proof, it will suffice to show that assertion $(*_n)$ holds. Let N be a perfect A-module which is $\mathfrak{I}^A_n$ nilpotent and let X be a $K(n)$-local spectrum; we wish to show that the mapping space $\mathrm{Map}_{\mathrm{Sp}}(X, M \otimes_A N)$ is contractible. For this, it will suffice to show that the smash product $X \otimes_S N^\vee$ vanishes, where $N^\vee$ denotes the A-linear dual of N. Let MP denote the periodic complex bordism spectrum (see Construction 5.3.9). Then the smash product $A \otimes_S \mathrm{MP}$ is faithfully flat over A (Theorem 5.3.13 and Proposition 5.3.12). It will therefore suffice to show that the smash product

$$X \otimes_S N^\vee \otimes_A (A \otimes_S \mathrm{MP}) \simeq X \otimes_S N^\vee \otimes_S \mathrm{MP}$$

vanishes.

Let u be an invertible element in $\pi_2(\mathrm{MP})$. For $0 \leqslant m \leqslant n$, choose $w_m \in \pi_0(\mathrm{MP})$ so that $u^{p^m-1}w_m$ represents the mth Hasse invariant in $\pi_{2(p^m-1)}(\mathrm{MP})/\mathfrak{I}_m^{\mathrm{MP}}\pi_{2(p^m-1)}(\mathrm{MP})$. Then the elements $(w_0, w_1, \ldots, w_{n-1})$ generate the nth Landweber ideal $\mathfrak{I}_n^{\mathrm{MP}}$. Note that $\mathfrak{I}_n^A$ and $\mathfrak{I}_n^{\mathrm{MP}}$ generate the same ideal $\pi_0(A \otimes_S \mathrm{MP})$. Since N is perfect and $\mathfrak{I}_n^A$-nilpotent, it follows that $N^\vee \otimes_S \mathrm{MP}$ is a perfect module over $A \otimes_S \mathrm{MP}$ which is $\mathfrak{I}_n^{\mathrm{MP}}$-nilpotent. Since v_n is invertible in $\pi_*(A)$, the image w_n in $\pi_0(A \otimes_S \mathrm{MP})$ is invertible modulo the ideal $\mathfrak{I}_n^{\mathrm{MP}}$, and therefore acts invertibly on $N^\vee \otimes_S \mathrm{MP}$. We can therefore identify $N^\vee \otimes_S \mathrm{MP}$ with the localization $N^\vee \otimes_S \mathrm{MP}[w_n^{-1}]$.

By construction, the $(A \otimes_S \mathrm{MP})$-module $N^\vee \otimes_S \mathrm{MP}$ is a retract of the smash product

$$N^\vee \otimes_S \mathrm{cofib}(w_0^k : \mathrm{MP} \to \mathrm{MP}) \otimes_{\mathrm{MP}} \cdots \otimes_{\mathrm{MP}} \mathrm{cofib}(w_{n-1}^k : \mathrm{MP} \to \mathrm{MP}).$$

for $k \gg 0$. It will therefore suffice to show that each of the smash products

$$X \otimes_S N^\vee \otimes_S \mathrm{cofib}(w_0^k : \mathrm{MP} \to \mathrm{MP}) \otimes_{\mathrm{MP}} \cdots \otimes_{\mathrm{MP}} \mathrm{cofib}(w_{n-1}^k : \mathrm{MP} \to \mathrm{MP})$$

vanishes. Note that such a smash product admits a filtration by k^n copies of the smash product

$$X \otimes_S N^\vee \otimes_S (\mathrm{cofib}(w_0 : \mathrm{MP} \to \mathrm{MP}) \otimes_{\mathrm{MP}} \cdots \otimes_{\mathrm{MP}} \mathrm{cofib}(w_{n-1}^k : \mathrm{MP} \to \mathrm{MP}))[w_n^{-1}].$$

We complete the proof by observing that the spectrum

$$X \otimes_S (\mathrm{cofib}(w_0 : \mathrm{MP} \to \mathrm{MP}) \otimes_{\mathrm{MP}} \cdots \otimes_{\mathrm{MP}} \mathrm{cofib}(w_{n-1}^k : \mathrm{MP} \to \mathrm{MP}))[w_n^{-1}]$$

vanishes, since X is $K(n)$-acyclic and the smash product $\mathrm{MP}[w_n^{-1}] \otimes_{\mathrm{MP}} \mathrm{cofib}(w_0 : \mathrm{MP} \to \mathrm{MP}) \otimes_{\mathrm{MP}} \cdots \otimes_{\mathrm{MP}} \mathrm{cofib}(w_{n-1}^k : \mathrm{MP} \to \mathrm{MP})$ can be written as a sum of copies of $K(n)$. □

4.6 The Quillen p-Divisible Group

Let A be a complex periodic $\mathbb{E}_\infty$-ring, and suppose that A is $K(n)$-local for some $n > 0$. Then A is complete with respect to the nth Landweber ideal $\mathfrak{I}_n^A$, and the $(n+1)$st Landweber ideal $\mathfrak{I}_{n+1}^A$ is equal to A (Theorem 4.5.2). Let us regard A as a complete adic $\mathbb{E}_\infty$-ring by endowing $\pi_0(A)$ with the $\mathfrak{I}_n^A$-adic topology. Applying Theorems 4.4.14 and 2.3.26, we deduce that $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ is a p-divisible formal group: that is, it can be realized as the identity component of an (essentially unique) formally connected p-divisible group over A. Our goal in this section is to give an explicit construction of this p-divisible group, which we will denote by $\mathbf{G}_A^{\mathcal{Q}}$ and refer to as the *Quillen p-divisible group* of A. The details of this construction will not be needed elsewhere in this paper (but will be used in a sequel, where we discuss applications of these ideas to transchromatic character theory).

4.6.1 The Construction of $\mathbf{G}_A^{\mathcal{Q}}$

For the purposes of constructing $\mathbf{G}_A^{\mathcal{Q}}$, it will be useful to temporarily abandon the conventions of Definition 2.0.2 and return to the definition of p-divisible groups given in [26]: namely, as certain p-torsion objects of the ∞-category $\mathrm{SpDM}_A^{\mathrm{nc}}$ of nonconnective spectral Deligne-Mumford stacks over A.

Notation 4.6.1. Let $\mathrm{Ab}_{\mathrm{fin}}^p$ denote the category of finite abelian p-groups. For each $M \in \mathrm{Ab}_{\mathrm{fin}}^p$, we let M^* denote the Pontryagin dual of M, given by

$$M^* = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}).$$

Let A be an $\mathbb{E}_\infty$-ring. For every space X, we let $C^*(X;A)$ denote the function spectrum of (unpointed) maps from X into A. Then $C^*(X;A)$ is an $\mathbb{E}_\infty$-algebra over A, so we can regard the spectrum $\mathrm{Spec}(C^*(X;A))$ as an object of $\mathrm{SpDM}_A^{\mathrm{nc}}$.

Construction 4.6.2 (The Quillen p-Divisible Group)**.** Let A be an $\mathbb{E}_\infty$-ring. For every finite abelian p-group M, we let $\mathbf{G}_A^{\mathcal{Q}}[M]$ denote the spectrum $\mathrm{Spec}(C^*(BM^*;A))$. We regard the construction $M \mapsto \mathbf{G}_A^{\mathcal{Q}}[M]$ as a functor

$$\mathbf{G}_A^{\mathcal{Q}} : (\mathrm{Ab}_{\mathrm{fin}}^p)^{\mathrm{op}} \to \mathrm{SpDM}_A^{\mathrm{nc}}.$$

Theorem 4.6.3. *Let A be a complex periodic $\mathbb{E}_\infty$-ring which is $K(n)$-local for some $n > 0$. Then the functor $M \mapsto \mathbf{G}_A^{\mathcal{Q}}[M]$ of Construction 4.6.2 is a p-divisible group of height n over A, in the sense of Proposition AV.6.5.5. In other words:*

(1) *For every pair of finite abelian p-groups M and M', the canonical map*

$$C^*(BM; A) \otimes_A C^*(BM'; A) \to C^*(B(M \times M'); A)$$

is an equivalence of $\mathbb{E}_\infty$-algebras over A.

(2) *For every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite abelian p-groups, the diagram of $\mathbb{E}_\infty$-algebras*

$$\begin{tikzcd} C^*(BM''; A) \arrow[r] \arrow[d] & C^*(BM; A) \arrow[d] \\ A \arrow[r] & C^*(BM'; A) \end{tikzcd}$$

is a pushout square in CAlg_A.

(3) *For each $m > 0$, the map $C^*(B\,\mathbf{Z}/p^{m-1}\,\mathbf{Z}; A) \to C^*(B\,\mathbf{Z}/p^m\,\mathbf{Z}; A)$ is finite flat of degree p^n.*

In particular, each $C^*(BM; A)$ is finite flat of degree $|M|^n$ as an A-module.

Definition 4.6.4. Let A be a complex periodic $\mathbb{E}_\infty$-ring which is $K(n)$-local for some $n > 0$. We will refer to the p-divisible group $\mathbf{G}_A^{\mathcal{Q}}$ of Theorem 4.6.3 as the *Quillen p-divisible group over A*.

4.6.2 Universal Flatness

We will need some preliminaries.

Definition 4.6.5. Let A be an $\mathbb{E}_\infty$-ring and let $f : X \to Y$ be a map of spaces. We will say that f is *universally A-flat* if the following conditions are satisfied:

(a) The pullback map $C^*(Y; A) \to C^*(X; A)$ is finite flat.

(b) For every point $y \in Y$ with homotopy fiber $X_y = X \times_Y \{y\}$, the diagram of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} C^*(Y; A) \arrow[r] \arrow[d] & C^*(\{y\}; A) \arrow[d] \\ C^*(X; A) \arrow[r] & C^*(X_y; A) \end{tikzcd}$$

is a pushout square; in particular, $C^*(X_y; A)$ is finite flat over A.

If, in addition, the map $C^*(Y;A) \to C^*(X;A)$ has degree d, then we will say that f is *universally A-flat of degree d*.

Proposition 4.6.6. *Let A be an $\mathbb{E}_\infty$-ring and suppose we are given a pullback diagram of spaces*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

If f is universally A-flat (of degree d), then f' is universally A-flat (of degree d) and the diagram of $\mathbb{E}_\infty$-rings σ :

$$\begin{tikzcd} C^*(Y;A) \arrow[r] \arrow[d] & C^*(Y';A) \arrow[d] \\ C^*(X;A) \arrow[r] & C^*(X';A) \end{tikzcd}$$

is a pushout square.

Proof. We first show that σ is a pushout square: that is, it induces an equivalence of $\mathbb{E}_\infty$-rings

$$\theta_{Y'} : C^*(X;A) \otimes_{C^*(Y;A)} C^*(Y';A) \to C^*(X';A).$$

Observe that since $C^*(X;A)$ is finite flat over $C^*(Y;A)$, the construction $Y' \mapsto \theta_{Y'}$ carries colimits in $\mathcal{S}_{/Y}$ to limits in the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{CAlg})$. Since the ∞-category $\mathcal{S}_{/Y'}$ is generated under small colimits by maps $\{y\} \hookrightarrow Y$ for $y \in Y$, we can reduce to the case where Y' is a point, in which case the desired result follows from condition (b) of Definition 4.6.5.

Since σ is a pushout square, the map $C^*(Y';A) \to C^*(X';A)$ is finite flat (and has degree d if $C^*(Y;A) \to C^*(X;A)$ has degree d). To complete the proof, it will suffice to show that f' also satisfies condition (b) of Definition 4.6.5: that is, for every point $y' \in Y'$, the right square in the diagram

$$\begin{tikzcd} C^*(Y;A) \arrow[r] \arrow[d] & C^*(Y';A) \arrow[r] \arrow[d] & C^*(\{y'\};A) \arrow[d] \\ C^*(X;A) \arrow[r] & C^*(X';A) \arrow[r] & C^*(X'_{y'};A) \end{tikzcd}$$

is a pushout diagram of $\mathbb{E}_\infty$-rings. This is clear, since the left square and the outer rectangle are pushout diagrams. $\square$

The main ingredient in the proof of Theorem 4.6.3 is the following:

Proposition 4.6.7. *Let A be a complex periodic $\mathbb{E}_\infty$-ring which is $K(n)$-local for some integer $n > 0$. Then the pth power map $p : \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ is universally A-flat of degree p^n.*

We will give the proof of Proposition 4.6.7 in §4.6.3.

Corollary 4.6.8. *Let A be a complex periodic $\mathbb{E}_\infty$-ring which is $K(n)$-local for some integer $n > 0$ and let $f : M \to N$ be a surjection of finite abelian p-groups. Then the induced map of classifying spaces $BM \to BN$ is universally A-flat of degree $|\ker(f)|^n$.*

Proof. Proceeding by induction on the order of $\ker(f)$, we can reduce to the case where $\ker(f)$ has order p. Let $\mathrm{U}(1)$ denote the circle group, so that there exists an embedding $\alpha_0 : \ker(f) \to \mathrm{U}(1)$. Since $\mathrm{U}(1)$ is injective as an abelian group, we can extend α_0 to a group homomorphism $\alpha : M \to \mathrm{U}(1)$. The map α fits into a pullback diagram of abelian groups

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\alpha"] & N \arrow[d] \\ \mathrm{U}(1) \arrow[r, "p"] & \mathrm{U}(1). \end{tikzcd}$$

Passing to classifying spaces, we obtain a homotopy pullback square

$$\begin{tikzcd} BM \arrow[r] \arrow[d] & BN \arrow[d] \\ \mathbf{CP}^\infty \arrow[r, "p"] & \mathbf{CP}^\infty . \end{tikzcd}$$

By virtue of Proposition 4.6.6, we are reduced to showing that the map $p : \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ is universally A-flat of degree p^n, which follows from Proposition 4.6.7. $\square$

Corollary 4.6.9. *Let A be a complex periodic $\mathbb{E}_\infty$-ring which is $K(n)$-local for some $n > 0$, and suppose we are given a pullback diagram of finite abelian p-groups*

$$\begin{tikzcd} M' \arrow[r] \arrow[d] & M \arrow[d] \\ N' \arrow[r] & N \end{tikzcd}$$

where the vertical maps are surjective. Then the diagram σ :

$$\begin{tikzcd} C^*(BM'; A) & C^*(BM; A) \arrow[l] \\ C^*(BN'; A) \arrow[u] & C^*(BN; A) \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout square in CAlg_A.

Proof. Combine Corollary 4.6.8 with Proposition 4.6.6. $\square$

Proof of Theorem 4.6.3. Let A be a complex periodic $\mathbb{E}_\infty$-ring which is $K(n)$-local for some $n > 0$. We wish to show that the construction

$$M \mapsto \mathbf{G}_A^{\mathcal{Q}}[M] = \mathrm{Spec}(C^*(BM^*; A))$$

satisfies conditions (1), (2), and (3) of Theorem 4.6.3. Conditions (1) and (2) are special cases of Corollary 4.6.9, and condition (3) follows from Corollary 4.6.8. $\square$

4.6.3 The Proof of Proposition 4.6.7

Let A be a complex periodic $\mathbb{E}_\infty$-ring; we wish to show that the map $p : \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ is universally A-flat of degree p^n. Conditions (a) and (b) of Definition 4.6.5 follow immediately from the following results:

Proposition 4.6.10. *Let A be an $\mathbb{E}_\infty$-ring which is complex periodic and $K(n)$-local for some integer $n \geqslant 1$. Then the map $p : \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ induces map of $\mathbb{E}_\infty$-rings $\phi : C^*(\mathbf{CP}^\infty; A) \to C^*(\mathbf{CP}^\infty; A)$ which is finite flat of rank p^n.*

Proposition 4.6.11. *Let A be a complex orientable $\mathbb{E}_\infty$-ring. For every fiber sequence of spaces $\widetilde{X} \to X \xrightarrow{\phi} \mathbf{CP}^\infty$, the canonical map*

$$C^*(X; A) \otimes_{C^*(\mathbf{CP}^\infty; A)} A \to C^*(\widetilde{X}; A)$$

is an equivalence.

Let us begin with the proof of Proposition 4.6.11, which is slightly easier (note that it does not require the hypothesis that A is $K(n)$-local).

Remark 4.6.12. Let A be an $\mathbb{E}_\infty$-ring, and let $\overline{e}$ be a complex orientation of A. We can regard $\overline{e}$ as an element of

$$\pi_{-2}\,\mathrm{fib}(C^*(\mathbf{CP}^\infty; A) \to C^*(*; A)).$$

Given any commutative diagram of spaces σ :

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & \mathbf{CP}^\infty, \end{tikzcd}$$

the pullback of $\overline{e}$ determines an element $\xi_\sigma \in \pi_{-2} \operatorname{fib}(C^*(X;A) \to C^*(\widetilde{X};A))$. For every space Y, let $\underline{A}_Y \in \operatorname{Fun}(Y;\operatorname{Mod}_A)$ denote the constant local system (of A-module spectra) on Y with value A. Multiplication by ξ_σ then determines a map

$$\Sigma^{-2}\underline{A}_X \to \operatorname{fib}(\underline{A}_X \to \pi_* \underline{A}_{\widetilde{X}}).$$

We claim that if σ is a pullback diagram, then this map is an equivalence: in other words, we obtain a fiber sequence

$$\Sigma^{-2}\underline{A}_X \to \underline{A}_X \to \pi_* \underline{A}_{\widetilde{X}}$$

of local systems on X. It suffices to verify this stalkwise: that is, we can assume that X is a point. In this case, we wish to show that ξ_σ determines an equivalence of spectra

$$\Sigma^{-2}A \to \operatorname{fib}(C^*(*;A) \to C^*(\mathrm{U}(1);A)).$$

Let σ' denote the rightmost square in the diagram of spaces

$$\begin{tikzcd}
\mathrm{U}(1) \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & S^2 \arrow[r] & \mathbf{CP}^\infty .
\end{tikzcd}$$

Here the left square is a pushout, and therefire induces an equivalence

$$\operatorname{fib}(C^*(S^2;A) \to C^*(*;A)) \to \operatorname{fib}(C^*(*;A) \to C^*(\mathrm{U}(1);A)).$$

We are therefore reduced to proving that $\xi_{\sigma'}$ induces an equivalence

$$\Sigma^{-2}A \to \operatorname{fib}(C^*(S^2;A) \to C^*(*;A)).$$

This follows immediately from our assumption that $\overline{e}$ is a complex orientation.

Remark 4.6.13. Let A be an $\mathbb{E}_\infty$-ring equipped with a complex orientation $\overline{e}$, and suppose we are given a fiber sequence of spaces $\widetilde{X} \to X \to \mathbf{CP}^\infty$. Then $\overline{e}$ determines an element $\xi \in A^2(X)$. Using Remark 4.6.12, we deduce the existence of a fiber sequence of A-module spectra

$$\Sigma^{-2}C^*(X;A) \xrightarrow{\xi} C^*(X;A) \to C^*(\widetilde{X};A).$$

Proof of Proposition 4.6.11. For every space Y, let $\underline{A}_Y$ denote the constant local system (of A-module spectra) on Y taking the value A. For every local system of A-module spectra $\mathscr{F}$ on $\mathbf{CP}^\infty$, we have a canonical map

$$e_{\mathscr{F}} : C^*(X; A) \otimes_{C^*(\mathbf{CP}^\infty; A)} C^*(\mathbf{CP}^\infty; \mathscr{F}) \to C^*(X; \phi^* \mathscr{F}).$$

Let $\psi : * \to \mathbf{CP}^\infty$ be the inclusion of the base point. Unwinding the definitions, we wish to prove that $e_{\mathscr{F}}$ is an equivalence when $\mathscr{F} = \psi_* \underline{A}_*$. According to Remark 4.6.13, a choice of complex orientation of A gives a fiber sequence of local systems

$$\Sigma^{-2} \underline{A}_{\mathbf{CP}^\infty} \to \underline{A}_{\mathbf{CP}^\infty} \to \psi_* \underline{A}_*.$$

We are therefore reduced to proving that $e_{\mathscr{F}}$ is an equivalence when $\mathscr{F} = \underline{A}_{\mathbf{CP}^\infty}$, which follows immediately from the definitions. $\square$

We now turn to the proof of Proposition 4.6.10. We will need a few algebraic preliminaries.

Notation 4.6.14. Let R be a commutative ring and let ω be an invertible R-module. For every R-module spectrum M, we let $M[[\omega]]$ denote the product $\prod_{n \geqslant 0} \omega^{\otimes n} \otimes_R M$. In the special case where ω is the free R-module on a generator x and M is discrete, the R-module $M[[\omega]]$ is also discrete; we identify elements of $M[[\omega]]$ with formal sums $\sum_{n \geqslant 0} c_n x^n$, where $c_n \in M$.

Let ω' be another invertible R-module, and suppose we are given a map $f : \omega' \to R[[\omega]]$, given by a family of R-module homomorphisms $f_n : \omega' \to \omega^{\otimes n}$. If $f_0 = 0$, then f induces a "composition map" $\lambda_M : M[[\omega']] \to M[[\omega]]$, given informally by the formula $\lambda_M(g(x')) = (g \circ f)(x)$. More formally, λ_M is given by the collection of maps

$$\lambda_{M,n} : M[[\omega']] \to \omega^n \otimes_R M,$$

where $\lambda_{M,n} = \sum_{k \geqslant 0} \sum_{n = n_1 + \cdots + n_k} \lambda_{M, n_1, \ldots, n_k}$, with $\lambda_{M, n_1, \ldots, n_k}$ given by the composite map

$$M[[\omega']] \to \omega'^{\otimes k} \otimes_R M \xrightarrow{f_{n_1} \otimes \cdots \otimes f_{n_k}} \omega^{\otimes n} \otimes_R M.$$

Lemma 4.6.15. *Let R be a commutative ring, $I \subseteq R$ a finitely generated ideal, $n > 0$ a positive integer Let ω and ω' be invertible R-modules, and let $f = \{f_m\}_{m \geqslant 0}$ be a map from ω' to $R[[\omega]]$. Suppose that $f_n : \omega' \to \omega^{\otimes n}$ is an isomorphism, $f_0 = 0$, and $f_m(\omega') \subseteq I \omega^{\otimes m}$ for $0 < m < n$. If M is an I-complete R-module, then the map $\lambda_M : M[[\omega']] \to M[[\omega]]$ of Notation 4.6.14 induces an isomorphism*

$$\theta_M : (\bigoplus_{0 \leqslant m < n} \omega^{\otimes m}) \otimes_R M[[\omega']] \to M[[\omega]].$$

Proof. Note that the maps λ_M and θ_M are well-defined for any R-module spectrum M; however, the construction is compatible with passage to homotopy groups, so we may assume without loss of generality that M is discrete. Choose a finite set $x_1, \ldots, x_a$ of generators for the ideal I. We proceed by induction on a. Suppose $a > 0$. For each $b \geqslant 0$, let M_b denote the cofiber of the map $x_1^b : M \to M$. Since M is I-complete, it can be identified with the limit of the tower of R-module spectra

$$\cdots \to M_2 \to M_1 \to M_0 \simeq 0.$$

It follows that θ_M is a limit of the tower of morphisms $\{\theta_{M_b}\}_{b \geqslant 0}$. To prove that θ_M is an equivalence, it suffices to show that each θ_{M_b} is an equivalence. Since M_b can be written as a successive extension of b copies of M_1, we are reduced to proving that θ_{M_1} is an equivalence. Equivalently, we must show that θ_N is an equivalence for $N = \pi_0 M_1 = \operatorname{coker}(x_1 : M \to M)$ and $N = \pi_1 M_1 \simeq \ker(x_1 : M \to M)$. Replacing M by N, we are reduced to the case where M is annihilated by x_1. We may then replace R by $R/(x_1)$ and I by its image in $R/(x_1)$, which is generated by the images of the elements $x_2, x_3, \ldots, x_a \in R$. The desired result then follows from the inductive hypothesis.

It remains to handle the case $a = 0$: that is, where I is the zero ideal in R. Consider the filtration

$$M[[\omega]] = F^0 M[[\omega]] \supseteq F^1 M[[\omega]] \supseteq \cdots$$

where $F^d M[[\omega]] = \prod_{i \geqslant d} (\omega^{\otimes i} \otimes_R M)$. Let $X = (\bigoplus_{0 \leqslant m < n} \omega^{\otimes m}) \otimes_R M[[\omega']]$, and define a filtration $X = F^0 X \supseteq F^1 X \supseteq \cdots$ so that

$$F^d X = \bigoplus_{0 \leqslant m < n} (\omega^{\otimes m} \otimes_R \prod_{m + in \geqslant d} (\omega'^{\otimes i} \otimes_R M)).$$

Then θ_M is an inverse limit of maps

$$\theta_d : X / F^d X \to M[[\omega]] / F^d M[[\omega]].$$

It will therefore suffice to show that each of these maps is an isomorphism. Proceeding by induction on d, we are reduced to showing that each of the maps of successive quotients

$$F^d X / F^{d+1} X \to F^d M[[\omega]] / F^{d+1} M[[\omega]]$$

is an isomorphism. This follows immediately from our assumption that $f_n : \omega' \to \omega^{\otimes n}$ is an isomorphism. $\square$

Proof of Proposition 4.6.10. Let A be a complex periodic $\mathbb{E}_\infty$-ring which is $K(n)$-local for some $n > 0$. Let $R = \pi_0(A)$ and set $\omega = \pi_2(A)$. A choice of complex orientation of A gives an equivalence $C^*(\mathbf{CP}^\infty; A) \simeq \prod_{m \geqslant 0} \Sigma^{-2m} A$, hence isomorphisms of homotopy groups $\pi_m C^*(\mathbf{CP}^\infty; A) \simeq (\pi_m A)[[\omega]]$. In particular, we can identify $\pi_0 C^*(\mathbf{CP}^\infty; A)$ with $R[[\omega]]$. The map $p : \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ induces a ring homomorphism from $R[[\omega]]$ to itself, which is determined by $f : \omega \to R[[\omega]]$ given by a family of maps $\{f_m : \omega \to \omega^{\otimes m}\}$. Let $\mathfrak{I}^A_n \subseteq R$ be the nth Landweber ideal of A (Definition 4.5.1). Since the image of $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ in $\mathrm{FGroup}(\pi_0(A)/\mathfrak{I}^A_n)$ has height $\geqslant n$, the composite map $\omega \xrightarrow{f} R[[\omega]] \to (R/\mathfrak{I}^A_n)[[\omega]]$ factors through the subalgebra $(R/\mathfrak{I}^A_n)[[\omega^{p^n}]]$. It follows that $f_m(\omega) \subseteq \mathfrak{I}^A_n \omega^{\otimes m}$ for $m < p^n$. Moreover, the element $f_{p^n} \in \omega^{\otimes (p^n - 1)} \simeq \pi_{2p^n - 2}(A)$ is a lift of the Hasse invariant $v_n \in \pi_{2p^n-2}(A)/\mathfrak{I}^A_n \pi_{2p^n-2}(A)$.

Let $B = B' = C^*(\mathbf{CP}^\infty; A)$, and regard ϕ as a map from B to B'. The product decomposition $B' \simeq \prod_{m \geqslant 0} \Sigma^{-2m} A$ gives a collection of A-module maps $\Sigma^{-2m} A \to B'$, and therefore a collection of B-module maps $\Sigma^{-2m} B \to B'$. Since A is $K(n)$-local, Theorem 4.5.2 implies that the homotopy groups of A are $\mathfrak{I}^A_n$-complete and that f_{p^n} is an isomorphism. Lemma 4.6.15 guarantees that the map $\bigoplus_{0 \leqslant m < p^n} \Sigma^{-2m} B \to B'$ is an equivalence of B-modules, so that ϕ is finite flat of rank p^n. $\square$

4.6.4 The Identity Component of $\mathbf{G}^{\mathcal{Q}}_A$

We now show that the Quillen p-divisible group $\mathbf{G}^{\mathcal{Q}}_A$ bears the promised relationship to the Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$:

Theorem 4.6.16. *Let A be a complex periodic $\mathbb{E}_\infty$-ring which is $K(n)$-local for some $n > 0$, let $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ denote the Quillen formal group of A (Construction 4.1.13), and let $\mathbf{G}^{\mathcal{Q}}_A$ denote the Quillen p-divisible group of A (Definition 4.6.4). Then there is a canonical equivalence $\alpha : \widehat{\mathbf{G}}^{\mathcal{Q}}_A \simeq \mathbf{G}^{\mathcal{Q}\circ}_A$ of formal groups over A: that is, $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ can be realized as the identity component of $\mathbf{G}^{\mathcal{Q}}_A$.*

The proof of Theorem 4.6.16 will require an auxiliary construction.

Notation 4.6.17. Let $\mathrm{Lat}(p)$ denote the category of abelian groups which are either free $\mathbf{Z}$-modules of finite rank or finite abelian p-groups (see Notation AV.6.4.5). Note that each object $M \in \mathrm{Lat}(p)$ can be regarded as a perfect $\mathbf{Z}$-module spectrum, and therefore admits a dual $M^\vee$ in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$. The construction

$$M \mapsto \Omega^{\infty-2} M^\vee$$

determines a functor of ∞-categories $(\mathrm{Lat}(p))^{\mathrm{op}} \to \mathcal{S}$. This functor carries an object $M \in \mathrm{Lat}(p)$ to the classifying space BM^*, where $M^* = \mathrm{Hom}(M, \mathrm{U}(1))$ is the Pontryagin dual of M in the category of locally compact abelian groups. More concretely:

- If M is a finite abelian p-group, then $\Omega^{\infty-2}M^\vee$ is the classifying space BM^* of the finite abelian p-group $M^* = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$.

- If M is a free abelian group of rank r, then $\Omega^{\infty-2}M^\vee \simeq K(M^\vee, 2)$ is the Eilenberg-MacLane space associated to free abelian group $M^\vee$; noncanonically, it is homotopy equivalent to a product of r copies of $\mathbf{CP}^\infty$.

Construction 4.6.18. Let A be a $K(n)$-local complex periodic $\mathbb{E}_\infty$-ring. For each object $M \in \mathrm{Lat}(p)$, we let $\mathscr{O}_M$ denote the $\mathbb{E}_\infty$-algebra over A given by $C^*(\Omega^{\infty-2}M^\vee; A)$. Then:

- If M is a finite abelian p-group, then $\mathscr{O}_M = C^*(BM^*; A)$ is a finite flat A-algebra of degree $|M|^n$ (Corollary 4.6.8).

- If M is a free abelian group, then $\mathscr{O}_M$ is the dual of $C_*(K(M^\vee, 2); A)$, which is a smooth coalgebra over A by virtue of Remark 4.1.12.

We let $\mathscr{O}_M^{\geqslant 0}$ denote the connective cover of $\mathscr{O}_M$, which we regard as an $\mathbb{E}_\infty$-algebra over $\tau_{\geqslant 0}A$. Note that the base point of $\Omega^{\infty-2}M$ determines a ring homomorphism $\epsilon_M : \pi_0 \mathscr{O}_M \to \pi_0 A$; we will denote the kernel of this homomorphism by $J_M \subseteq \pi_0 \mathscr{O}_M \simeq \pi_0 \mathscr{O}_M^{\geqslant 0}$. Note that the ideal J_M is always finitely generated. We will regard $\mathscr{O}_M^{\geqslant 0}$ as an adic $\mathbb{E}_\infty$-ring by equipping $\pi_0 \mathscr{O}_M^{\geqslant 0}$ with the J_M-adic topology. Then the construction $M \mapsto \mathrm{Spf}(\mathscr{O}_M^{\geqslant 0})$ determines a functor

$$X : (\mathrm{Lat}(p))^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}A}, \mathcal{S}).$$

Remark 4.6.19. Let $\mathrm{Lat} \subseteq \mathrm{Lat}(p)$ be the full subcategory spanned by the free abelian groups, and let X be as in Construction 4.6.18. By construction, the restriction $X|_{\mathrm{Lat}^{\mathrm{op}}}$ agrees with the Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$, regarded as an abelian group object of the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}A}, \mathcal{S})$.

Remark 4.6.20. Let X be as in Construction 4.6.18. For every finite abelian p-group M, we can identify $X(M) = \mathrm{Spf}(\tau_{\geqslant 0}C^*(BM^*; A))$ with the subfunctor of $\mathrm{Spec}(\tau_{\geqslant 0}C^*(BM^*; A)) \simeq \mathbf{G}^{\mathcal{Q}}_A[M]$, whose value on an $B \in \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}A}$ is the summand of $\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}^{\mathcal{Q}}_A(B))$ consisting of those maps $\mathrm{Spec}(B) \to \mathbf{G}^{\mathcal{Q}}_A[M]$ which factor through the zero section at the level of topological spaces: in other words, the summand given by the fiber of the map $\mathbf{G}_A^{\mathcal{Q}}[M](B) \to \mathbf{G}_A^{\mathcal{Q}}[M](B^{\mathrm{red}})$. It follows that the restriction of X to the subcategory $(\mathrm{Ab}_{\mathrm{fin}}^p)^{\mathrm{op}} \subseteq (\mathrm{Lat}(p))^{\mathrm{op}}$ is a p-torsion object of the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}A}, \mathcal{S})$ in the sense of Proposition AV.6.5.5 . In particular, we can identify $X|_{(\mathrm{Ab}_{\mathrm{fin}}^p)^{\mathrm{op}}}$ with the functor $\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}A} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ given by $B \mapsto \mathrm{fib}(\mathbf{G}_A^{\mathcal{Q}}(B) \to \mathbf{G}_A^{\mathcal{Q}}(B^{\mathrm{red}}))$.

We will deduce Theorem 4.6.16 from the following:

Proposition 4.6.21. *Let A be a $K(n)$-local complex periodic $\mathbb{E}_\infty$-ring and let $X : (\mathrm{Lat}(p))^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}A}, \mathcal{S})$ be the functor of Construction 4.6.18. Then X is a right Kan extension of $X|_{\mathrm{Lat}^{\mathrm{op}}}$.*

We defer the proof of Proposition 4.6.21 until the end of this section.

Corollary 4.6.22. *Let A be a $K(n)$-local complex periodic $\mathbb{E}_\infty$-ring. Then, for every object $B \in \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}A}$, we have a canonical equivalence*

$$\mathrm{fib}(\widehat{\mathbf{G}}_A^{\mathcal{Q}}(B) \to \widehat{\mathbf{G}}_A^{\mathcal{Q}}(B)[1/p]) \simeq \mathrm{fib}(\mathbf{G}_A^{\mathcal{Q}}(B) \to \mathbf{G}_A^{\mathcal{Q}}(B^{\mathrm{red}})).$$

Proof. Combining Remarks 4.6.19 and 4.6.20 with Proposition 4.6.21, we deduce that the functor $B \mapsto \mathrm{fib}(\mathbf{G}_A^{\mathcal{Q}}(B) \to \mathbf{G}_A^{\mathcal{Q}}(B^{\mathrm{red}}))$ can be identified with the functor $\widehat{\mathbf{G}}_A^{\mathcal{Q}}[p^\infty]$ of Construction AV.6.4.7 . $\square$

Proof of Theorem 4.6.16. Let A be a $K(n)$-local complex periodic $\mathbb{E}_\infty$-ring and let $\mathcal{E} \subseteq \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}A}$ be the full subcategory spanned by those connective $\mathbb{E}_\infty$-algebras over $\tau_{\geqslant 0}A$ which are truncated and (p)-nilpotent. It follows from Lemma 2.3.24 that for $B \in \mathcal{E}$, the localization $\widehat{\mathbf{G}}_A^{\mathcal{Q}}(B)[1/p]$ vanishes. Consequently, Corollary 4.6.22 supplies an equivalence

$$\widehat{\mathbf{G}}_A^{\mathcal{Q}}(B) \simeq \mathrm{fib}(\mathbf{G}_A^{\mathcal{Q}}(B) \to \mathbf{G}_A^{\mathcal{Q}}(B^{\mathrm{red}})) = (\mathbf{G}_A^{\mathcal{Q}})^\circ(B),$$

depending functorially on $B \in \mathcal{E}$. Applying Proposition 2.1.1, we obtain an equivalence of formal groups $\widehat{\mathbf{G}}_A^{\mathcal{Q}} \simeq (\mathbf{G}_A^{\mathcal{Q}})^\circ$. $\square$

We now turn to the proof of Proposition 4.6.21. By virtue of Remark AV.6.4.8 , it will suffice to establish the following:

Proposition 4.6.23. *Let A be a $K(n)$-local complex periodic $\mathbb{E}_\infty$-ring and let M be a finite abelian p-group. Then there exists a short exact sequence $0 \to \Lambda' \to \Lambda \to M \to 0$, where Λ is a free abelian group of finite rank, for which the diagram*

$$\begin{tikzcd} X(M) \arrow[r] \arrow[d] & X(\Lambda) \arrow[d] \\ X(0) \arrow[r] & X(\Lambda') \end{tikzcd}$$

is a pullback square in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}A}, \mathcal{S})$.

Proof. Choose a direct sum decomposition $M \simeq \mathbf{Z}/p^{e_1}\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p^{e_r}\mathbf{Z}$. We can then take $\Lambda = \mathbf{Z}^r$ and $\Lambda' = p^{e_1}\mathbf{Z} \oplus \cdots \oplus p^{e_r}\mathbf{Z}$. It follows from Remark 4.6.19 that the functor $X|_{\mathrm{Lat}^{\mathrm{op}}}$ commutes with finite products, and from Remark 4.6.20 that the functor $X|_{(\mathrm{Ab}^p_{\mathrm{fin}})^{\mathrm{op}}}$ commutes with finite products. We may therefore reduce to the case $r = 1$. In this case, we wish to show that the pullback diagram of spaces σ :

$$\begin{tikzcd} B\,\mathbf{Z}/p^e\,\mathbf{Z} \arrow[r] \arrow[d] & \mathbf{CP}^\infty \arrow[d, "p^e"] \\ * \arrow[r] & \mathbf{CP}^\infty \end{tikzcd}$$

induces a pullback diagram of functors τ :

$$\begin{tikzcd} \mathrm{Spf}(\tau_{\geqslant 0}C^*(B\,\mathbf{Z}/p^e\,\mathbf{Z}; A)) \arrow[r] \arrow[d] & \mathrm{Spf}(\tau_{\geqslant 0}C^*(\mathbf{CP}^\infty; A)) \arrow[d] \\ \mathrm{Spf}(\tau_{\geqslant 0}A) \arrow[r] & \mathrm{Spf}(\tau_{\geqslant 0}C^*(\mathbf{CP}^\infty; A)). \end{tikzcd}$$

It follows from iterated application of Proposition 4.6.7 that the diagram σ determines a pushout square of $\mathbb{E}_\infty$-rings σ' :

$$\begin{tikzcd} C^*(B\,\mathbf{Z}/p^e\,\mathbf{Z}; A) & C^*(\mathbf{CP}^\infty; A) \arrow[l] \\ A \arrow[u] & C^*(\mathbf{CP}^\infty; A) \arrow[l] \arrow[u] \end{tikzcd}$$

in which the vertical maps are finite flat of degree p^{ne}. It follows that σ' induces a pushout square of connective covers σ'':

$$\begin{tikzcd} \tau_{\geqslant 0}C^*(B\,\mathbf{Z}/p^e\,\mathbf{Z}; A) & \tau_{\geqslant 0}C^*(\mathbf{CP}^\infty; A) \arrow[l] \\ \tau_{\geqslant 0}A \arrow[u] & \tau_{\geqslant 0}C^*(\mathbf{CP}^\infty; A). \arrow[l] \arrow[u] \end{tikzcd}$$

Since the lower horizontal map is surjective on π_0, it follows that the upper horizontal map is also surjective on π_0; consequently, the ideal $J_{\mathbf{Z}/p^e\mathbf{Z}} \subseteq \pi_0 C^*(B\,\mathbf{Z}/p^e\,\mathbf{Z}; A)$ appearing in Construction 4.6.18 is generated by the image of $J_{\mathbf{Z}} \subseteq \pi_0 C^*(\mathbf{CP}^\infty; A)$. It follows that σ'' is also a pushout diagram in the ∞-category of adic $\mathbb{E}_\infty$-rings, so that $\tau \simeq \mathrm{Spf}(\sigma'')$ is a pullback diagram of (functors represented by) formal spectral Deligne-Mumford stacks. □

5 Lubin-Tate Spectra

Let κ be a perfect field of characteristic and let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group of height $n < \infty$ over κ. Then $\widehat{\mathbf{G}}_0$ admits a universal deformation $\widehat{\mathbf{G}}$ defined over a complete local Noetherian ring R_{LT} (see Theorem 3.0.1). Morava observed that $\widehat{\mathbf{G}}$ can be realized as the classical Quillen formal group of an even periodic ring spectrum E, and Goerss-Hopkins-Miller proved that E admits an essentially unique $\mathbb{E}_\infty$-ring structure. We begin with a brief review of their work, which will require a bit of terminology.

Notation 5.0.1. We define a category $\mathcal{FG}$ as follows:

- The objects of $\mathcal{FG}$ are pairs $(R, \widehat{\mathbf{G}})$, where R is a commutative ring and $\widehat{\mathbf{G}}$ is a 1-dimensional formal group over R.
- A morphism from $(R, \widehat{\mathbf{G}})$ to $(R', \widehat{\mathbf{G}}')$ in the category $\mathcal{FG}$ is a pair (f, α), where $f : R \to R'$ is a ring homomorphism and $\alpha : \widehat{\mathbf{G}}' \simeq \widehat{\mathbf{G}}_{R'}$ is an isomorphism of formal groups over R'.

If A is a complex periodic $\mathbb{E}_\infty$-ring and $\mathfrak{I}_n^A$ denotes the nth Landweber ideal of the classical Quillen formal group $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$, we let $\widehat{\mathbf{G}}_A^{\mathcal{Q}_n}$ denote the formal group obtained from $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ by extension of scalars along the surjection $\pi_0(A) \to \pi_0(A)/\mathfrak{I}_n^A$ (so that $\widehat{\mathbf{G}}_A^{\mathcal{Q}_n}$ has height $\geqslant n$).

Theorem 5.0.2 (Goerss-Hopkins-Miller)**.** *Let κ be a perfect field of characteristic $p > 0$ and let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group of height $n < \infty$ over κ. Then there exists a complex periodic $\mathbb{E}_\infty$-ring E and an isomorphism*

$$\alpha : (\kappa, \widehat{\mathbf{G}}_0) \simeq (\pi_0(E)/\mathfrak{I}_n^E, \widehat{\mathbf{G}}_E^{\mathcal{Q}_n})$$

in the category $\mathcal{FG}$ with the following features:

(*i*) *The $\mathbb{E}_\infty$-ring E is even periodic and the composite map*

$$(\pi_0(E), \widehat{\mathbf{G}}_E^{\mathcal{Q}_0}) \to (\pi_0(E)/\mathfrak{I}_n^E, \widehat{\mathbf{G}}_E^{\mathcal{Q}_n}) \xrightarrow{\alpha^{-1}} (\kappa, \widehat{\mathbf{G}}_0)$$

exhibits the classical Quillen formal group $\widehat{\mathbf{G}}_E^{\mathcal{Q}_0}$ as a universal deformation of $\widehat{\mathbf{G}}_0$ (in the sense of Theorem 3.0.1). In particular, $\pi_0(E)$ can be identified with the Lubin-Tate ring R_{LT} of $\widehat{\mathbf{G}}_0$.

(*ii*) *The $\mathbb{E}_\infty$-ring E is $K(n)$-local. Moreover, for every complex periodic $K(n)$-local $\mathbb{E}_\infty$-ring A, composition with α induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}}(E, A) \simeq \mathrm{Hom}_{\mathcal{FG}}((\kappa, \widehat{\mathbf{G}}_0), (\pi_0(A)/\mathfrak{I}_n^A, \widehat{\mathbf{G}}_A^{\mathcal{Q}_n})).$$

In particular, the mapping space $\mathrm{Map}_{\mathrm{CAlg}}(E, A)$ is discrete.

Note that properties (*i*) and (*ii*) of Theorem 5.0.2 have a different character. We can regard (*i*) as a description of the homotopy groups of the spectrum E: they are given by the formula

$$\pi_m(E) \simeq \begin{cases} \omega_{\widehat{\mathbf{G}}}^{\otimes k} & \text{if } m = 2k \text{ is even} \\ 0 & \text{if } m \text{ is odd,} \end{cases}$$

where $\widehat{\mathbf{G}}$ is the universal deformation of $\widehat{\mathbf{G}}_0$. Roughly speaking, this gives information about the structure of maps *to* the spectrum E. By contrast, property (*ii*) gives information about maps *from* E to other $K(n)$-local $\mathbb{E}_\infty$-rings A: they are given by the formula

$$\mathrm{Map}_{\mathrm{CAlg}}(E, A) \simeq \begin{cases} \mathrm{Hom}_{\mathcal{FG}}((\kappa, \widehat{\mathbf{G}}_0), (\pi_0(A)/\mathfrak{I}_n^A, \widehat{\mathbf{G}}_A^{\mathcal{Q}_n})) & \text{for } A \text{ complex periodic} \\ \varnothing & \text{otherwise.} \end{cases}$$

Once we know that there exists an $\mathbb{E}_\infty$-ring E (and an isomorphism α) having both of these properties, either one can be taken as a characterization of E:

Proposition 5.0.3. *Let κ be a perfect field of characteristic $p > 0$ and let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group of height $n < \infty$ over κ. Suppose that E is a $K(n)$-local complex periodic $\mathbb{E}_\infty$-ring and that we are given a map*

$$\alpha : (\kappa, \widehat{\mathbf{G}}_0) \to (\pi_0(E)/\mathfrak{I}_n^E, \widehat{\mathbf{G}}_E^{\mathcal{Q}_n})$$

which satisfies condition (ii) of Theorem 5.0.2. Then α is an isomorphism which also satisfies condition (i).

Proof. By virtue of Theorem 5.0.2, we can choose a complex periodic $\mathbb{E}_\infty$-ring E' and an isomorphism of formal groups

$$\alpha' : (\kappa, \widehat{\mathbf{G}}_0) \simeq (\pi_0(E')/\mathfrak{I}_n^{E'}, \widehat{\mathbf{G}}_{E'}^{\mathcal{Q}_n})$$

which satisfies conditions (i) and (ii) of Theorem 5.0.2. Since the map α satisfies condition (ii) of Theorem 5.0.2, there is an essentially unique morphism of $\mathbb{E}_\infty$-rings $f : E \to E'$ for which α' factors as a composition

$$(\kappa, \widehat{\mathbf{G}}_0) \xrightarrow{\alpha} (\pi_0(E)/\mathfrak{I}_n^{E}, \widehat{\mathbf{G}}_{E}^{\mathcal{Q}_n}) \xrightarrow{f} (\pi_0(E')/\mathfrak{I}_n^{E'}, \widehat{\mathbf{G}}_{E'}^{\mathcal{Q}_n}).$$

For any complex periodic $K(n)$-local $\mathbb{E}_\infty$-ring A, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CAlg}}(E', A) \arrow[rr, "\circ f"] \arrow[dr] & & \mathrm{Map}_{\mathrm{CAlg}}(E, A) \arrow[dl] \\
& \mathrm{Hom}_{\mathcal{FG}}((\kappa, \widehat{\mathbf{G}}_0), (\pi_0(A)/\mathfrak{I}_n^{A}, \widehat{\mathbf{G}}_{A}^{\mathcal{Q}_n})) &
\end{tikzcd}$$

where the vertical maps are homotopy equivalences. It follows that the upper horizontal map is also a homotopy equivalence. Applying Yoneda's lemma, we deduce that f is a homotopy equivalence. It follows that α is an isomorphism which exhibits E as a Lubin-Tate spectrum of $\widehat{\mathbf{G}}_0$. $\square$

Proposition 5.0.4. *Let κ be a perfect field of characteristic $p > 0$ and let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group of height $n < \infty$ over κ. Suppose that E is an even periodic $\mathbb{E}_\infty$-ring and that we are given a map $K(n)$-local complex periodic $\mathbb{E}_\infty$-ring and that we are given a map of formal groups*

$$\beta : (\pi_0(E)/\mathfrak{I}_n^{E}, \widehat{\mathbf{G}}_{E}^{\mathcal{Q}_n}) \to (\kappa, \widehat{\mathbf{G}}_0)$$

for which the composite map

$$(\pi_0(E), \widehat{\mathbf{G}}_{E}^{\mathcal{Q}_0}) \to (\pi_0(E)/\mathfrak{I}_n^{E}, \widehat{\mathbf{G}}_{E}^{\mathcal{Q}_n}) \xrightarrow{\beta} (\kappa, \widehat{\mathbf{G}}_0)$$

exhibits the classical Quillen formal group $\widehat{\mathbf{G}}_{E}^{\mathcal{Q}_0}$ as a universal deformation of $\widehat{\mathbf{G}}_0$. Then β is an isomorphism, and the inverse map $\alpha = \beta^{-1}$ satisfies condition (ii) of Theorem 5.0.2.

Proof. If β exhibits $\widehat{\mathbf{G}}_{E}^{\mathcal{Q}_0}$ is a universal deformation of $\widehat{\mathbf{G}}_0$, then we can identify $\pi_0(E)$ with the Lubin-Tate ring R_{LT} of $\widehat{\mathbf{G}}_0$. Using Corollary 4.4.25, we see that the nth

Landweber ideal $\mathfrak{I}_n^E$ is the maximal ideal of $\pi_0(E)$, so that β is an isomorphism. Note also that E is complete with respect to $\mathfrak{I}_n^E$ and $\mathfrak{I}_{n+1}^E$ vanishes, so that the $\mathbb{E}_\infty$-ring E is $K(n)$-local (Theorem 4.5.2).

By virtue of Theorem 5.0.2, we can choose a complex periodic $\mathbb{E}_\infty$-ring E' and an isomorphism of formal groups

$$\alpha' : (\kappa, \widehat{\mathbf{G}}_0) \simeq (\pi_0(E')/\mathfrak{I}_n^{E'}, \widehat{\mathbf{G}}_{E'}^{\mathcal{Q}_n})$$

which satisfies conditions (i) and (ii) of Theorem 5.0.2. Using (ii), we see that there is an essentially unique morphism of $\mathbb{E}_\infty$-rings $g : E' \to E$ for which the composite map

$$(\kappa, \widehat{\mathbf{G}}_0) \xrightarrow{\alpha'} (\pi_0(E')/\mathfrak{I}_n^{E'}, \widehat{\mathbf{G}}_{E'}^{\mathcal{Q}_n}) \xrightarrow{g} (\pi_0(E)/\mathfrak{I}_n^{E}, \widehat{\mathbf{G}}_{E}^{\mathcal{Q}_n})$$

is equal to $\alpha = \beta^{-1}$. We then have a commutative diagram of formal groups

$$\begin{tikzcd}
(\pi_0(E'), \widehat{\mathbf{G}}_{E'}^{\mathcal{Q}_0}) \arrow[rr, "g"] \arrow[dr] & & (\pi_0(E), \widehat{\mathbf{G}}_{E}^{\mathcal{Q}_0}) \arrow[dl] \\
& (\kappa, \widehat{\mathbf{G}}_0), &
\end{tikzcd}$$

where the vertical maps both exhibit their domains as universal deformations of $\widehat{\mathbf{G}}_0$. It follows that g induces an isomorphism of commutative rings $\pi_0(E') \to \pi_0(E)$. Since the $\mathbb{E}_\infty$-rings E and E' are both even periodic, we conclude that g is an equivalence. It follows that α also satisfies condition (ii) of Theorem 5.0.2. $\square$

Remark 5.0.5 (Uniqueness)**.** Let κ be a perfect field of characteristic $p > 0$ and let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group of height $n < \infty$ over κ. It follows immediately from the definition that if there exists a complex periodic $\mathbb{E}_\infty$-ring E satisfying conclusion (ii) of Theorem 5.0.2, then E is unique up to equivalence (as an object of the ∞-category CAlg of $\mathbb{E}_\infty$-rings). We will emphasize this uniqueness by referring to E as the *Lubin-Tate spectrum of* $\widehat{\mathbf{G}}_0$. It follows from Proposition 5.0.4 that the Lubin-Tate spectrum E is also characterized up to equivalence by conclusion (i) of Theorem 5.0.2. In particular, if E' is any other $\mathbb{E}_\infty$-ring and there exists an isomorphism $\gamma : E \simeq E'$ in the ordinary category CAlg(hSp) of homotopy commutative ring spectra, then γ can be promoted (in an essentially unique way) to an equivalence $E \simeq E'$ in the ∞-category CAlg = CAlg(Sp) of $\mathbb{E}_\infty$-rings. We can informally summarize the situation by saying that the $\mathbb{E}_\infty$-structure on the ring spectrum E is unique (more precisely, it is uniquely determined by the underlying homotopy commutative multiplication on E).

Remark 5.0.6 (Maps Between Lubin-Tate Spectra)**.** Let κ and κ' be perfect fields of characteristic p, and let $\widehat{\mathbf{G}}_0$ and $\widehat{\mathbf{G}}_0'$ be 1-dimensional formal groups of the same height n over κ and κ', respectively. Let E and E' denote Lubin-Tate spectra for $\widehat{\mathbf{G}}_0$ and $\widehat{\mathbf{G}}_0'$, respectively. Then we have a canonical homotopy equivalences

$$\begin{aligned}\mathrm{Map}_{\mathrm{CAlg}}(E, E') &\simeq \mathrm{Hom}_{\mathcal{FG}}((\kappa, \widehat{\mathbf{G}}_0), (\pi_0(E')/\mathfrak{I}_n^{E'}, \widehat{\mathbf{G}}_{E'}^{\mathcal{Q}_n})) \\ &\simeq \mathrm{Hom}_{\mathcal{FG}}((\kappa, \widehat{\mathbf{G}}_0), (\kappa', \widehat{\mathbf{G}}_0')).\end{aligned}$$

The first homotopy equivalence arises by invoking part (ii) of Theorem 5.0.2 for the Lubin-Tate spectrum E, and the second arises from invoking part (i) of Theorem 5.0.2 for the Lubin-Tate spectrum E'.

Remark 5.0.7 (Functoriality)**.** Let n be a positive integer, and let $\mathcal{C} \subseteq \mathcal{FG}$ be the full subcategory spanned by those pairs $(\kappa, \widehat{\mathbf{G}}_0)$, where κ is a perfect field and $\widehat{\mathbf{G}}_0$ is a 1-dimensional formal group of height n over κ. It follows from Remark 5.0.6 that there is a fully faithful functor $\mathcal{C} \to \mathrm{CAlg}$, which carries a formal group $(\kappa, \widehat{\mathbf{G}}_0)$ to the associated Lubin-Tate spectrum E. We will say that an $\mathbb{E}_\infty$-ring E is a *Lubin-Tate spectrum of height* n if it belongs to the essential image of this embedding. Note $\mathcal{C}$ is an ordinary category: it follows that the full subcategory of CAlg spanned by the Lubin-Tate spectra of height n is also (equivalent to) an ordinary category.

Remark 5.0.8 (The Morava Stabilizer Group)**.** Let κ be a perfect field of characteristic p, let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group over κ of height $n < \infty$, and let $\Gamma = \mathrm{Aut}(\kappa, \widehat{\mathbf{G}}_0)$ be the automorphism group of $(\kappa, \widehat{\mathbf{G}})$ in the category $\mathcal{FG}$: the objects of Γ are given by pairs (σ, α), where σ is an automorphism of the field κ and $\alpha : \widehat{\mathbf{G}}_0 \to \sigma^* \widehat{\mathbf{G}}_0$ is an isomorphism of formal groups over κ. It follows from Remark 5.0.7 that the action of Γ on $(\kappa, \widehat{\mathbf{G}}_0)$ can be lifted, in an essentially unique way, to an action of Γ on the associated Lubin-Tate spectrum E, as an object of the ∞-category CAlg. More precisely, there is a functor of ∞-categories $B\Gamma \to \mathrm{CAlg}$ which carries the base point of $B\Gamma$ to the Lubin-Tate spectrum E; here $B\Gamma$ denotes the classifying space of the group Γ.

Remark 5.0.9. Let κ be a perfect field of characteristic p, let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group over κ of height $n < \infty$, and let E be the associated Lubin-Tate spectrum. It follows from Theorem 5.0.2 that for any $K(n)$-local $\mathbb{E}_\infty$-ring A, the mapping space $\mathrm{Map}_{\mathrm{CAlg}}(E, A)$ is discrete.

A proof of Theorem 5.0.2 was given by Goerss and Hopkins in [12] (modulo the slight caveat that condition (ii) is stated only in the case where A is also a Lubin-Tate spectrum). Let us give a brief summary of their approach. Fix a perfect field κ of characteristic p and a 1-dimensional formal group $\widehat{\mathbf{G}}_0$ of height n over κ. Let R_{LT} be the Lubin-Tate ring of $\widehat{\mathbf{G}}_0$ and let $\widehat{\mathbf{G}} \in \mathrm{FGroup}(R_{\mathrm{LT}})$ be the universal deformation (in the sense of Theorem 3.0.1). Then the formal group $\widehat{\mathbf{G}}$ satisfies Landweber's criterion, so we can apply the Landweber exact functor theorem (Theorem 0.0.1) to construct an even periodic ring spectrum E equipped with isomorphisms

$$\pi_0(E) \simeq R_{\mathrm{LT}} \qquad \widehat{\mathbf{G}}_E^{\mathcal{Q}_0} \simeq \widehat{\mathbf{G}}.$$

By construction, the ring spectrum E satisfies requirement (i) of Theorem 5.0.2, with the caveat that the ring structure on E is *a priori* only commutative up to homotopy, rather than $\mathbb{E}_\infty$ as required by Theorem 5.0.2. To remedy this, Goerss and Hopkins study the fiber product

$$\mathcal{M}(E) = \mathrm{CAlg}(\mathrm{Sp}) \times_{\mathrm{CAlg}(\mathrm{hSp})} \{E\},$$

which can be viewed as a "moduli space" of $\mathbb{E}_\infty$-structures on E which are compatible with the homotopy commutative multiplication supplied by Landweber's theorem. Using techniques of obstruction theory, they introduce a spectral sequence converging to the homotopy groups of the moduli space $\mathcal{M}$. By studying the geometry of the formal group $\widehat{\mathbf{G}}_0$, they show that this spectral sequence vanishes identically at the second page, so that the moduli space $\mathcal{M}$ is contractible. This provides an essentially unique $\mathbb{E}_\infty$-structure on the spectrum E (as in Remark 5.0.5). Moreover, if A is any $K(n)$-local $\mathbb{E}_\infty$-ring equipped with a map of homotopy commutative ring spectra $f : E \to A$, then the same methods can be applied to prove the contractibility of the moduli space

$$\mathcal{M}(f) = \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(E, A) \times_{\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{hSp})}(E,A)} \{f\}$$

of $\mathbb{E}_\infty$-structures on f. This reduces the proof of assertion (ii) of Theorem 5.0.2 to the calculation of the set $\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{hSp})}(E, A)$, which can be computed using Landweber's methods.

Our goal in this section is to present a new proof of Theorem 5.0.2. Our strategy can be summarized as follows. Writing $\widehat{\mathbf{G}}_0$ as the identity component of a connected p-divisible group $\mathbf{G}_0$ over κ, we let $R^{\mathrm{un}}_{\mathbf{G}_0}$ denote the spectral deformation ring of $\mathbf{G}_0$ and $\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}_0})$ its universal deformation. Let $R^{\mathrm{or}}_{\mathbf{G}_0} = \mathfrak{O}_{\mathbf{G}^\circ}$ be an orientation classifier for the identity component $\mathbf{G}^\circ$ (Definition 4.3.14). In §5.1, we prove directly that the $K(n)$-localization $E = L_{K(n)} R^{\mathrm{or}}_{\mathbf{G}_0}$ satisfies condition (ii) of Theorem 5.0.2: this follows more or less formally from the relevant universal properties. We will prove (i) by computing the homotopy groups of E (Theorem 5.4.1). Our calculation appeals directly to Quillen's work on the homotopy of the complex bordism spectrum and its relationship with formal group laws, which we review in §5.3. We will also need a general algebraic result about the deformation theory of relatively perfect ring extensions, which we explain in §5.2.

The present approach to Theorem 5.0.2 differs from that of [12] in several ways:

- In the approach of [12], one begins with a homotopy commutative ring spectrum E satisfying property (i) of Theorem 5.0.2, and must work hard to show that E has an $\mathbb{E}_\infty$-ring structure satisfying property (ii). In our approach, the spectrum $L_{K(n)} R^{\mathrm{or}}_{\mathbf{G}_0}$ already has an $\mathbb{E}_\infty$-structure, and property (ii) follows easily from the definitions. We will instead need to do some work to verify condition (i): it is not obvious from the definitions that $L_{K(n)} R^{\mathrm{or}}_{\mathbf{G}_0}$ has the expected homotopy groups, or even that it is nonzero.

- Theorem 5.0.2 is a variant of an older result of Hopkins and Miller, which asserts that the Lubin-Tate spectrum E can be characterized by analogues of properties (i) and (ii) in the ∞-category $\mathrm{Alg}(\mathrm{Sp})$ of associative ring spectra. From the obstruction-theoretic perspective, this requires less elaborate machinery and is considerably easier to prove (see [31] for a nice account). By contrast, the proof of Theorem 5.0.2 given in this section is specific to the commutative case: it requires us to contemplate p-divisible and formal groups defined over a ring spectrum R; this is sensible only when R is commutative.

- In the approach of [12], the underlying spectrum E is constructed using the Landweber exact functor theorem (Theorem 0.0.1). To ensure that the formal group $\widehat{\mathbf{G}}$ satisfies Landweber's criterion, one needs to know something about the Lubin-Tate ring R_{LT}: namely, that it is isomorphic to a power series ring $W(\kappa)[[u_1, \ldots, u_{n-1}]]$ whose generators u_i can be chosen to represent the Hasse invariants of the formal group $\widehat{\mathbf{G}}$. Our approach does not appeal to Landweber's theorem, and the structure of the Lubin-Tate ring emerges as a consequence of our method (see Corollary 5.4.3; note however that our proof uses Lazard's work on the classification of formal group laws, which can also be used to describe R_{LT} directly).

- In some sense, both proofs of Theorem 5.0.2 involve a kind of deformation theory, in the sense that they produce an $\mathbb{E}_\infty$-ring as a "limit" of partial approximations. However, these approximations have very different flavors. Roughly speaking, the strategy of [12] is to produce the Lubin-Tate spectrum E by starting with the graded ring $E_*E = \pi_*(E \otimes_S E)$, viewed as a comodule over itself, and successively adding information about homology cooperations and Massey products of higher and higher orders. In our approach, the Lubin-Tate spectrum $E \simeq L_{K(n)} R^{\mathrm{or}}_{\mathbf{G}_0}$ is built from the the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$, which is ultimately constructed (in the proof of Theorem SAG.18.2.3.1) as the limit of a sequence of square-zero extensions

$$\cdots \to A_3 \to A_2 \to A_1 \to A_0 \simeq \kappa$$

which parametrize successively "larger" families of deformations of $\mathbf{G}_0$.

5.1 Construction of Lubin-Tate Spectra

For later applications, it will be convenient to consider a slightly more general version of Theorem 5.0.2, where we begin with a perfect ring in place of a perfect field (see Remark 5.1.6 below).

Construction 5.1.1. Let R_0 be a perfect $\mathbf{F}_p$-algebra and let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group of exact height n over R_0 (Definition 4.4.17). Write $\widehat{\mathbf{G}}_0$ as the identity component of a connected p-divisible group $\mathbf{G}_0$ over R_0 (Proposition 4.4.22). Let $R^{\mathrm{un}}_{\mathbf{G}_0}$ denote the spectral deformation ring of $\mathbf{G}_0$ and let $\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}_0})$ be its universal deformation (Theorem 3.4.1 and Remark 3.4.2). Let $R^{\mathrm{or}}_{\mathbf{G}_0}$ denote the orientation classifier for the formal group $\mathbf{G}^\circ$ (Definition 4.3.14). We let $E(\widehat{\mathbf{G}}_0)$ denote the $K(n)$-localization $L_{K(n)} R^{\mathrm{or}}_{\mathbf{G}_0}$. We will refer to $E(\widehat{\mathbf{G}}_0)$ as the *Lubin-Tate spectrum of* $\widehat{\mathbf{G}}_0$.

Remark 5.1.2. We will later show that when R_0 is a perfect field, the final step in Construction 5.1.1 is unnecessary: the orientation classifier $R^{\mathrm{or}}_{\mathbf{G}_0}$ is already $K(n)$-local. See Corollary 6.0.6.

Remark 5.1.3. In the situation of Construction 5.1.1, the Lubin-Tate spectrum $E(\widehat{\mathbf{G}}_0)$ is $K(n)$-local (by construction) and complex periodic (by Proposition 4.3.23, since there exists an oriented formal group over $E(\widehat{\mathbf{G}}_0)$).

We now justify the terminology of Construction 5.1.1 by showing that the Lubin-Tate spectrum $E(\widehat{\mathbf{G}}_0)$ satisfies property (ii) of Theorem 5.0.2.

Construction 5.1.4. In the situation of Construction 5.1.1, we have a canonical surjective map $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0}) \to R_0$, whose kernel is the nth Landweber ideal $\mathfrak{I}^{\mathbf{G}^\circ}_n$ (Corollary

4.4.25). By construction, the formal group $\mathbf{G}^{\circ}$ acquires an orientation after extending scalars to $R^{\mathrm{or}}_{\mathbf{G}_0}$, and therefore also after extending scalars to the Lubin-Tate spectrum $E = E(\widehat{\mathbf{G}}_0)$. This orientation determines an equivalence of formal groups $\mathbf{G}^{\circ}_E \simeq \widehat{\mathbf{G}}^{\mathcal{Q}}_E$ over E, hence an isomorphism of formal groups $\mathbf{G}^{\circ}_{\pi_0(E)/\mathfrak{I}^E_n} \simeq \widehat{\mathbf{G}}^{\mathcal{Q}_n}_E$ which we can identify with a morphism

$$\alpha : (R_0, \widehat{\mathbf{G}}_0) \to (\pi_0(E)/\mathfrak{I}^E_n, \widehat{\mathbf{G}}^{\mathcal{Q}_n}_E)$$

in the category $\mathcal{FG}$ of Notation 5.0.1.

Theorem 5.1.5. *Let R_0 be a perfect $\mathbf{F}_p$-algebra, let $\widehat{\mathbf{G}}_0$ be a formal group of exact height n over R_0, and let $E = E(\widehat{\mathbf{G}}_0)$ be the Lubin-Tate spectrum of Construction 5.1.1. Then, for every complex periodic $K(n)$-local $\mathbb{E}_\infty$-ring A, composition with the map α of Construction 5.1.4 induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}}(E, A) \to \mathrm{Hom}_{\mathcal{FG}}((R_0, \widehat{\mathbf{G}}_0), (\pi_0(A)/\mathfrak{I}^A_n, \widehat{\mathbf{G}}^{\mathcal{Q}_n}_A).$$

Proof. Let us regard A as an adic $\mathbb{E}_\infty$-ring by endowing $\pi_0(A)$ with the $\mathfrak{I}^A_n$-adic topology. Let $\mathbf{G}^{\mathcal{Q}}_A$ be the Quillen p-divisible group of A (Definition 4.6.4), so that $\mathbf{G}^{\mathcal{Q}}_A$ is a formally connected p-divisible group over A whose identity component is the Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ (note that the existence of such a p-divisible group follows formally from Theorem 4.4.14; for the present purposes, it is not necessary to know that $\mathbf{G}^{\mathcal{Q}}_A$ has the explicit description of Construction 4.6.2). If $\mathbf{G}'$ is any other formally connected p-divisible group over A, then the space $\mathrm{OrDat}(\mathbf{G}'^{\circ})$ of orientations of the identity component $\mathbf{G}'^{\circ}$ can be identified with the space of equivalences of $\mathbf{G}'^{\circ}$ with $\widehat{\mathbf{G}}^{\mathcal{Q}}_A$ (Proposition 4.3.23), or equivalently with the space of equivalences between $\mathbf{G}'$ and $\mathbf{G}^{\mathcal{Q}}_A$ (Corollary 2.3.13).

Note that any morphism of $\mathbb{E}_\infty$-rings $E \to A$ automatically carries $\mathfrak{I}^E_n = \mathfrak{I}^{\mathbf{G}^{\circ}}_n \pi_0(E)$ into $\mathfrak{I}^A_n$, so that the composite map $R^{\mathrm{un}}_{\mathbf{G}} \to E \to A$ is a morphism of adic $\mathbb{E}_\infty$-rings. Invoking the definition of the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$, we can identify the mapping space $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}}(R^{\mathrm{un}}_{\mathbf{G}_0}, A)$ with the classifying space

$$\mathrm{Def}_{\mathbf{G}_0}(A) = \varinjlim_I \mathrm{BT}^p(A) \times_{\mathrm{BT}^p(\pi_0(A)/I)} \mathrm{Hom}(R_0, I)$$

of $\mathbf{G}_0$-tagged p-divisible groups over A (here the direct limit is taken over all finitely generated ideals of definition in $\pi_0(A)$. Under this identification, the mapping space $\mathrm{Map}_{\mathrm{CAlg}}(E, A) \simeq \mathrm{Map}_{\mathrm{CAlg}}(R^{\mathrm{or}}_{\mathbf{G}_0}, A)$ corresponds to the classifying space

$$\varinjlim_I \mathrm{BT}^p_{\mathrm{or}}(A) \times_{\mathrm{BT}^p(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I)$$

of *oriented* p-divisible groups over A equipped with a $\mathbf{G}_0$-tagging. Since $\mathbf{G}_0$ is connected, any $\mathbf{G}_0$-tagged p-divisible group $\mathbf{G}'$ over A is formally connected. We may therefore replace $\mathrm{BT}^p_{\mathrm{or}}(A)$ with the full subcategory spanned by the formally connected oriented p-divisible groups over A, which is equivalent to the singleton $\{\mathbf{G}^{\mathcal{Q}}_A\}$. Using Corollary 2.3.13 again, we can identify $\mathrm{Map}_{\mathrm{CAlg}}(E, A)$ with the direct limit

$$\varinjlim_I \{\widehat{\mathbf{G}}^{\mathcal{Q}}_A\} \times_{\mathrm{FGroup}(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I),$$

again taken over all finitely generated ideals of definition $I \subseteq \pi_0(A)$.

Note that if we are given finitely generated ideals of definition $I \subseteq J \subseteq \pi_0(A)$, then the quotient J/I is a nilpotent ideal of $\pi_0(A)/I$; our assumption that R_0 is perfect guarantees that the reduction map $\mathrm{Hom}(R_0, \pi_0(A)/I) \to \mathrm{Hom}(R_0, \pi_0(A)/J)$ is bijective. Moreover, if we are given a ring homomorphism $R_0 \to \pi_0(A)/I$ and I contains the nth Landweber ideal $\mathfrak{I}^A_n$, then $(\widehat{\mathbf{G}}^{\mathcal{Q}}_A)_{\pi_0(A)/I}$ and $(\widehat{\mathbf{G}}_0)_{\pi_0(A)/I}$ are 1-dimensional formal groups of exact height n over the commutative $\mathbf{F}_p$-algebra $\pi_0(A)/I$. Consequently, the set of isomorphisms between $(\widehat{\mathbf{G}}^{\mathcal{Q}}_A)_{\pi_0(A)/I}$ and $(\widehat{\mathbf{G}}_0)_{\pi_0(A)/I}$ is in bijection with the set of isomorphisms between $(\widehat{\mathbf{G}}^{\mathcal{Q}}_A)_{\pi_0(A)/J}$ and $(\widehat{\mathbf{G}}_0)_{\pi_0(A)/J}$ (since $\pi_0(A)/I$ and $\pi_0(A)/J$ have the same perfection; see Proposition 4.4.23). It follows that the diagram of spaces $\{\{\widehat{\mathbf{G}}^{\mathcal{Q}}_A\} \times_{\mathrm{FGroup}(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I)\}$ is constant when restricted to finitely generated ideals of definition which contain the nth Landweber ideal, so that we can identify $\mathrm{Map}_{\mathrm{CAlg}}(E, A)$ with the single space

$$\{\widehat{\mathbf{G}}^{\mathcal{Q}}_(A)\} \times_{\mathrm{FGroup}(\pi_0(A)/\mathfrak{I}^A_n)} \mathrm{Hom}(R_0, \pi_0(A)/\mathfrak{I}^A_n) \simeq \mathrm{Hom}_{\mathcal{FG}}((R_0, \widehat{\mathbf{G}}_0), (\pi_0(A)/\mathfrak{I}^A_n, \widehat{\mathbf{G}}^{\mathcal{Q}_n}_A)).$$

By construction, this identification is given by composition with the map α of Construction 5.1.4. $\square$

Remark 5.1.6 (Smash Products of Lubin-Tate Spectra)**.** Let R_0 and R_1 be commutative rings, and let $\widehat{\mathbf{G}}_0$ and $\widehat{\mathbf{G}}_1$ be formal groups over R_0 and R_1, respectively. Then the objects $(R_0, \widehat{\mathbf{G}}_0)$ and $(R_1, \widehat{\mathbf{G}}_1)$ have a coproduct $(R_{01}, \widehat{\mathbf{G}}_{01})$ in the category $\mathcal{FG}$. Moreover, if R_0 and R_1 are perfect $\mathbf{F}_p$-algebras and $\widehat{\mathbf{G}}_0$ and $\widehat{\mathbf{G}}_1$ are of exact height n, then R_{01} is also a perfect $\mathbf{F}_p$-algebra (this follows immediately from Proposition 4.4.23), and $\widehat{\mathbf{G}}_{01}$ also has exact height n. By construction, there are maps of Lubin-Tate spectra

$$E(\widehat{\mathbf{G}}_0) \to E(\widehat{\mathbf{G}}_{01}) \leftarrow E(\widehat{\mathbf{G}}_1).$$

It follows from the universal property of Theorem 5.1.5 that these maps exhibit $E(\widehat{\mathbf{G}}_{01})$ as a coproduct of $E(\widehat{\mathbf{G}}_0)$ and $E(\widehat{\mathbf{G}}_1)$ in the ∞-category $\mathrm{CAlg}(\mathrm{Sp}_{K(n)})$ of $K(n)$-local

$\mathbb{E}_\infty$-rings. In other words, we can identify $E(\widehat{\mathbf{G}}_{01})$ with the $K(n)$-local smash product $L_{K(n)}(E(\widehat{\mathbf{G}}_0) \otimes_S E(\widehat{\mathbf{G}}_1))$. Beware that if R_0 and R_1 are perfect fields, then R_{01} will *never* be a field; this is one motivation for not restrict our attention to fields in the setting of Construction 5.1.1.

5.2 Thickenings of Relatively Perfect Morphisms

Let κ be a perfect field of characteristic $p > 0$. The theory of Witt vectors provides a "lift" of κ to characteristic zero. More precisely, it allows us to write κ as a quotient $W(\kappa)/pW(\kappa)$, where $W(\kappa)$ is a (p)-complete commutative ring in which p is not a zero-divisor. The ring $W(\kappa)$ is uniquely determined by κ (up to canonical isomorphism); we refer to it as *the ring of Witt vectors of* κ. There are many concrete constructions of the ring $W(\kappa)$: for example, we can identify the elements of $W(\kappa)$ with formal expressions

$$[a_0] + [a_1]p + [a_2]p^2 + [a_3]p^3 + \cdots$$

where the coefficients a_i are elements of κ; the addition and multiplication on $W(\kappa)$ can then be given explicitly by certain expressions called *Witt polynomials* (see [33] for a nice exposition). However, one can also prove the existence of $W(\kappa)$ by abstract arguments. The assumption that κ is perfect guarantees that the relative cotangent complex $L_{\kappa/\mathbf{F}_p}$ vanishes (Proposition 3.5.6). It follows from deformation theory that κ can be lifted uniquely to a flat $(\mathbf{Z}/p^n\mathbf{Z})$-algebra for each $n \geqslant 0$, and we can recover the ring of Witt vectors $W(\kappa)$ by taking the inverse limit over n. One advantage of the abstract approach is that it remains sensible in the setting of ring spectra, where it is no longer feasible to carry out constructions by writing formulas. For example, one can show that every perfect field κ can be lifted to an $\mathbb{E}_\infty$-ring $W^+(\kappa)$ which is flat over the sphere spectrum, from which the usual ring of Witt vectors can be recovered by the formula $W(\kappa) \simeq \pi_0(W^+(\kappa))$ (see Example 5.2.7 below). Our goal in this case is to describe a generalization of this construction which will be useful in our analysis of Construction 5.1.1.

Definition 5.2.1. Let A be an connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0(A)$ be finitely generated ideal, and set $A_0 = \pi_0(A)/I$. Suppose we are given a commutative diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A_0 \arrow[r, "f_0"] & B_0, \end{tikzcd}$$

where B_0 is an ordinary commutative ring. We will say that σ *exhibits f as an A-thickening of f_0* if the following conditions are satisfied:

(a) The $\mathbb{E}_\infty$-ring B is I-complete.

(b) The diagram σ induces an isomorphism of commutative rings $\pi_0(B)/I\pi_0(B) \to B_0$.

(c) Let R be any connective $\mathbb{E}_\infty$-algebra over A which is I-complete. Then canonical map

$$\mathrm{Map}_{\mathrm{CAlg}_A}(B, R) \simeq \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{A_0}}(B_0, \pi_0(R)/I\pi_0(R))$$

is a homotopy equivalence. In particular, the mapping space $\mathrm{Map}_{\mathrm{CAlg}_A}(B, R)$ is discrete.

Remark 5.2.2 (Uniqueness)**.** Let A be an connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0(A)$ be finitely generated ideal, and set $A_0 = \pi_0(A)/I$. Suppose we are given a homomorphism of commutative rings $f_0 : A_0 \to B_0$. It follows immediately from the definitions that, if there exists a diagram σ :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A_0 \arrow[r, "f_0"] & B_0, \end{tikzcd}$$

which exhibits f as an A-thickening of f_0, then the morphism f (and the diagram σ) is uniquely determined up to equivalence.

Remark 5.2.3. In Definition 5.2.1, it is not necessary to assume that A is I-complete. However, it is harmless to add this assumption; the notion of A-thickening does not change if we replace A by its I-completion $A^\wedge_I$.

Remark 5.2.4 (Functorality)**.** Suppose we are given a commutative diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"] \arrow[d] & B' \arrow[d] \\ A_0 \arrow[r, "f_0"] & B_0. \end{tikzcd}$$

Assume that A_0 and B_0 are ordinary commutative rings and that the left vertical maps induce surjective ring homomorphism $\pi_0(A) \to \pi_0(A') \to A_0$ whose composition has kernel $I \subseteq \pi_0(A)$. Suppose that the outer rectangle exhibits f as an A-thickening of f_0, and that the upper square exhibits B' as an I-completion of $B \otimes_A A'$. Then the lower square exhibits f' as an A'-thickening of f_0.

Our goal is to prove the following existence result for thickenings:

Theorem 5.2.5. *Let A be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0(A)$ be a finitely generated ideal, and set $A_0 = \pi_0(A)/I$. Suppose that A_0 is an $\mathbf{F}_p$-algebra which is almost perfect as an A-module and that the Frobenius map $\varphi_{A_0} : A_0 \to A_0$ is flat. Let $f : A_0 \to B_0$ be a morphism of commutative $\mathbf{F}_p$-algebras which is relatively perfect: that is, the diagram of commutative rings*

$$\begin{tikzcd} A_0 \arrow[r, "f"] \arrow[d, "\varphi_{A_0}"] & B_0 \arrow[d, "\varphi_{B_0}"] \\ A_0 \arrow[r, "f_0"] & B_0 \end{tikzcd}$$

is a pushout square (so that φ_{B_0} is also flat). Then there exists a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A_0 \arrow[r, "f_0"] & B_0, \end{tikzcd}$$

which exhibits f as an A-thickening of f_0. Moreover, σ is a pushout square.

Example 5.2.6 (Classical Witt Vectors)**.** In the statement of Theorem 5.2.5, take A to be the ring $\mathbf{Z}$ of integers (or the ring $\mathbf{Z}_p$ of p-adic integers), and I to be the ideal (p). Then $A_0 = A/I$ is the finite field $\mathbf{F}_p$, and a morphism $f_0 : A_0 \to B_0$ is relatively perfect if and only if B_0 is a perfect $\mathbf{F}_p$-algebra. If this condition is satisfied, then Theorem 5.2.5 allows us to lift B_0 to a flat $\mathbf{Z}$-algebra B, which is complete with respect to the ideal (p) and for which the quotient B/pB is isomorphic to B_0. This $\mathbf{Z}$-algebra is the ring of Witt vectors $W(B_0)$.

Example 5.2.7 (Spherical Witt Vectors)**.** In the statement of Theorem 5.2.5, take A to be the sphere spectrum S of integers (or its (p)-completion $S^{\wedge}_{(p)}$), and I to be the ideal (p). Then $A_0 = A/I$ is the finite field $\mathbf{F}_p$, and a morphism $f_0 : A_0 \to B_0$ is relatively perfect if and only if B_0 is a perfect $\mathbf{F}_p$-algebra. If this condition is satisfied, then Theorem 5.2.5 allows us to lift B_0 to a flat S-algebra B, which is complete with respect to the ideal (p) and for which the tensor product $\mathbf{F}_p \otimes_S B \simeq \pi_0(B)/p\pi_0(B)$ is isomorphic to B_0. This is the $\mathbb{E}_\infty$-ring $W^+(B_0)$ of "spherical" Witt vectors alluded to in the discussion preceding Definition 5.2.1.

5.2.1 Existence of Thickenings

The proof of Theorem 5.2.5 will require the following variant of Proposition 3.5.6:

Lemma 5.2.8. *Let $f_0 : A_0 \to B_0$ be a morphism of commutative $\mathbf{F}_p$-algebras. Assume that the Frobenius map $\varphi_{A_0} : A_0 \to A_0$ is flat and that f_0 is relatively perfect. Then the relative cotangent complex L_{B_0/A_0} vanishes.*

Proof. The assumption that φ_{A_0} is flat guarantees that the pushout diagram

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "\varphi_{A_0}"] & B_0 \arrow[d, "f_0"] \\ A_0^{1/p} \arrow[r] & B_0^{1/p} \end{tikzcd}$$

is also a pushout diagram of $\mathbb{E}_\infty$-rings, and therefore induces an equivalence of algebraic cotangent complexes

$$B_0^{1/p} \otimes_{B_0} L^{\mathrm{alg}}_{B_0/A_0} \to L^{\mathrm{alg}}_{B_0^{1/p}/A_0^{1/p}}.$$

Since this map is nullhomotopic (Lemma 3.3.6), we conclude that the algebraic cotangent complex $L^{\mathrm{alg}}_{B_0/A_0}$ vanishes. Applying Proposition SAG.25.3.5.1 , we deduce that L_{B_0/A_0} also vanishes. $\square$

Proof of Theorem 5.2.5. Let A be an $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0(A)$ be a finitely generated ideal for which $A_0 = \pi_0(A)/I$ is almost perfect as an A-module. Then A_0 is almost of finite presentation as an $\mathbb{E}_\infty$-algebra over A (Corollary SAG.5.2.2.2), so the relative cotangent complex $L_{A_0/A}$ is almost perfect. It is also 1-connective (since A_0 is a quotient of $\pi_0(A)$). Suppose further that A_0 is an $\mathbf{F}_p$-algebra for which the Frobenius map φ_{A_0} is flat, and let $f_0 : A_0 \to B_0$ be a morphism of $\mathbf{F}_p$-algebras which is relatively perfect. Then the relative cotangent complex L_{B_0/A_0} vanishes (Lemma 5.2.8). Using the fiber sequence

$$B_0 \otimes_{A_0} L_{A_0/A} \to L_{B_0/A} \to L_{B_0/A_0},$$

we deduce that $L_{B_0/A}$ is 1-connective and almost perfect.

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the relative de Rham space of the map $\mathrm{Spec}(B_0) \to \mathrm{Spec}(A)$, in the sense of Definition SAG.18.2.1.1 . Concretely, the functor X is given by the formula

$$X(R) = \varinjlim_J \mathrm{Map}_{\mathrm{CAlg}}(A, R) \times_{\mathrm{Map}_{\mathrm{CAlg}}(A, \pi_0(R)/J)} \mathrm{Map}_{\mathrm{CAlg}}(B_0, \pi_0(R)/J),$$

where the direct limit is taken over all nilpotent ideals $J \subseteq \pi_0(R)$. By construction, we have a commutative diagram of functors τ :

$$\begin{tikzcd} \mathrm{Spec}(A) & X \arrow[l] \\ \mathrm{Spec}(A_0) \arrow[u] & \mathrm{Spec}(B_0) \arrow[l] \arrow[u] \end{tikzcd}$$

We claim that τ is a pullback square. To prove this, we must show that for every connective $\mathbb{E}_\infty$-ring R, the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}}(B_0, R) \to \varinjlim_J \mathrm{Map}_{\mathrm{CAlg}}(A_0, R) \times_{\mathrm{Map}_{\mathrm{CAlg}}(A_0, \pi_0(R)/J)} \mathrm{Map}_{\mathrm{CAlg}}(B_0, \pi_0(R)/J)$$

is a homotopy equivalence. In fact, we prove a stronger assertion: for every nilpotent ideal $J \subseteq \pi_0(R)$, the individual map

$$\mathrm{Map}_{\mathrm{CAlg}}(B_0, R) \to \mathrm{Map}_{\mathrm{CAlg}}(A_0, R) \times_{\mathrm{Map}_{\mathrm{CAlg}}(A_0, \pi_0(R)/J)} \mathrm{Map}_{\mathrm{CAlg}}(B_0, \pi_0(R)/J)$$

is a homotopy equivalence. Writing R as an inverse limit of its truncations, we may assume that R is m-truncated. In this case, the projection map $R \to \pi_0(R)/J$ can be written as a composition of finitely many square-zero extensions. We are therefore reduced to showing that if R is a square-zero extension of a connective $\mathbb{E}_\infty$-ring R' (by a connective R'-module), then the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}}(B_0, R) \to \mathrm{Map}_{\mathrm{CAlg}}(A_0, R) \times_{\mathrm{Map}_{\mathrm{CAlg}}(A_0, R')} \mathrm{Map}_{\mathrm{CAlg}}(B_0, R')$$

is a homotopy equivalence, which follows immediately from the vanishing of L_{B_0/A_0}.

Note that X is nilcomplete, infinitesimally cohesive, and admits a cotangent complex given by $L_{\mathrm{Spec}(A)}|_X$ (see Corollary SAG.18.2.1.11). In particular, we see that that $L_{\mathrm{Spec}(B_0)/X} \simeq L_{B_0/A}$ is 1-connective and almost perfect. Applying Theorem SAG.18.2.3.2 (together with Proposition SAG.18.2.2.8), we conclude that X is (representable by) the formal spectrum $\mathrm{Spf}(B)$, where B is a connective $\mathbb{E}_\infty$-ring equipped with a surjective map $\epsilon : \pi_0(B) \to B_0$, which is complete with respect to the (finitely generated) ideal $I' = \ker(\epsilon)$, and that B_0 is almost perfect as a B-module. Note that the pullback diagram τ determines a pushout diagram σ :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A_0 \arrow[r, "f_0"] & B_0 \end{tikzcd}$$

of adic $\mathbb{E}_\infty$-rings, where we endow $\pi_0(B)$ with the I'-adic topology and the other rings with the discrete topology. In other words, we can identify B_0 with the I'-completion of the tensor product $B \otimes_A A_0$. However, since A_0 is almost perfect as an A-module, the tensor product $B \otimes_A A_0$ is almost perfect as a B-module and is therefore already I'-complete (Proposition SAG.7.3.5.7). It follows that σ is also a pushout diagram of $\mathbb{E}_\infty$-rings. In particular, it exhibits B_0 as the quotient $\pi_0(B)/I\pi_0(B)$, so that the ideal I' coincides with $I\pi_0(B)$.

We now complete the proof by showing that σ exhibits f as an A-thickening of f_0. Conditions (a) and (b) of Definition 5.2.1 have already been verified. To prove (c), suppose that we are given a connective $\mathbb{E}_\infty$-algebra R over A which is I-complete. We wish to show that the canonical map

$$\theta_R : \mathrm{Map}_{\mathrm{CAlg}_A}(B, R) \simeq \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{A_0}}(B_0, \pi_0(R)/I\pi_0(R))$$

is a homotopy equivalence. Applying Lemma SAG.8.1.2.2 , we can write R as the limit of a tower

$$\cdots \to R_3 \to R_2 \to R_1$$

for which each quotient $\pi_0(R_m)/I\pi_0(R_m)$ is isomorphic to $\pi_0(R)/I\pi_0(R)$, and each R_m is I-nilpotent. It follows that θ_R can be identified with the limit of maps $\{\theta_{R_m}\}_{m \geqslant 1}$. It will therefore suffice to show that θ_R is a homotopy equivalence under the additional assumption that R is I-nilpotent. In this case, any morphism of A-algebras $B \to R$ automatically annihilates some power of I', so we can identify $\mathrm{Map}_{\mathrm{CAlg}_A}(B, R)$ with the fiber

$$\mathrm{fib}(X(R) \to \mathrm{Map}_{\mathrm{CAlg}}(A, R)) \simeq \varinjlim_J \mathrm{Map}_{\mathrm{CAlg}_A}(B_0, \pi_0(R)/J)$$

where the colimit is taken over all nilpotent ideals $J \subseteq \pi_0(R)$. Restrict our attention to the cofinal subset of ideals which contain $I\pi_0(A)$, we can rewrite this fiber as the colimit of the diagram of sets $\{\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{A_0}}(B_0, \pi_0(R)/J)\}$. We complete the proof by observing that our hypothesis that f_0 is relatively perfect guarantees that this diagram constant with value $\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{A_0}}(B_0, \pi_0(R)/I\pi_0(R))$. □

5.2.2 Recognition of Thickenings

In the situation of Theorem 5.2.5, we have a detection principle for A-thickenings:

Proposition 5.2.9. *Let A be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0(A)$ be a finitely generated ideal, and set $A_0 = \pi_0(A)/I$. Suppose that A_0 is an $\mathbf{F}_p$-algebra which is almost perfect as an A-module and that the Frobenius map $\varphi_{A_0} : A_0 \to A_0$ is flat. Suppose we are given a commutative diagram of connective $\mathbb{E}_\infty$-rings σ :*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A_0 \arrow[r, "f_0"] & B_0, \end{tikzcd}$$

where f_0 is a relatively perfect morphism of commutative $\mathbf{F}_p$-algebras. Then σ exhibits f as an A-thickening of f_0 if and only if the following conditions are satisfied:

- (i) *The $\mathbb{E}_\infty$-ring B is I-complete (as an A-module).*
- (ii) *The diagram σ is a pushout square.*

Proof. The necessity of condition (i) is part of the definition of an A-thickening, and the necessity of condition (ii) follows from Theorem 5.2.5. We will show that they are also sufficient. Assume that σ satisfies conditions (i) and (ii). It follows from condition (ii) that σ induces an isomorphism of commutative rings $\gamma : \pi_0(B)/I\pi_0(B) \to B_0$. Using Theorem 5.2.5, we can choose a diagram σ' :

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d] & B' \arrow[d] \\ A_0 \arrow[r, "f_0"] & B_0 \end{tikzcd}$$

which exhibits f' as an A-thickening of f_0. Applying condition (i), we deduce that the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}_A}(B', B) \to \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{A_0}}(B_0, \pi_0(B)/I\pi_0(B))$$

is a homotopy equivalence. In particular, we can lift γ^{-1} to a morphism of $g : B' \to B$, which fits into a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g"] \arrow[d] & B \arrow[d] \\ A_0 \arrow[r, "f_0"] & B_0 \arrow[r, "\mathrm{id}"] & B_0. \end{tikzcd}$$

The left square in this diagram is a pushout (by Theorem 5.2.5) and the outer rectangle is a pushout (by hypothesis (ii)), so the right square is also a pushout: that is, the morphism g becomes an equivalence after tensoring over B' with $B_0 = \pi_0(B')/I\pi_0(B')$. Since B and B' are both connective and I-complete, it follows that g is an equivalence, so that σ also exhibits f as an A-thickening of f_0. $\square$

Remark 5.2.10. In the situation of Theorem 5.2.5, suppose that B_0 is Noetherian. Then B is also Noetherian. This follows from Corollary SAG.18.2.4.4 .

Remark 5.2.11. In the situation of Theorem 5.2.5, suppose that A is I-complete, that A_0 and B_0 are Noetherian, and that $f_0 : A_0 \to B_0$ is flat. Then A and B are Noetherian (Corollary SAG.18.2.4.4), so the map $f : A \to B$ is also flat (Lemma 6.1.9). In particular, if A is discrete, then B is also discrete.

5.2.3 Regular Sequences

The following observation will be useful for helping to compute with A-thickenings:

Proposition 5.2.12. *Let A be a commutative ring and let $I \subseteq A$ be an ideal which is generated by a regular sequence $x_1, x_2, \ldots, x_n \in A$. Suppose that $A_0 = A/I$ is an $\mathbf{F}_p$-algebra for which the Frobenius map $\varphi_{A_0} : A_0 \to A_0$ is flat. Let $f_0 : A_0 \to B_0$ be a relatively perfect morphism of commutative $\mathbf{F}_p$-algebras, and let σ :*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A_0 \arrow[r, "f_0"] & B_0 \end{tikzcd}$$

be a diagram which exhibits f as an A-thickening of f_0. Then B is an ordinary commutative ring and the sequence $f(x_1), f(x_2), \ldots, f(x_n)$ is regular in B.

Warning 5.2.13. In the statement of Proposition 5.2.12 (and in the situation we will apply it), we do not assume that A is Noetherian. Consequently, notion of regular sequence must be handled with some care (in general, it can depend on the order of the sequence). We interpret regularity as the statement that, for each $1 \leqslant i \leqslant n$, the element x_i is not a zero divisor in the quotient ring $A/(x_1, \ldots, x_{i-1})$.

Proof of Proposition 5.2.12. We proceed by induction on n. Set $A' = A/(x_1)$, and extend σ to a commutative diagram of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
A' \arrow[r, "f'"] \arrow[d] & B' \arrow[d] \\
A_0 \arrow[r, "f_0"] & B_0
\end{tikzcd}$$

where the upper square is a pushout. Since B is I-complete, the $\mathbb{E}_\infty$-ring $B' \simeq \mathrm{cofib}(x_1 : B \to B)$ is also I-complete. Applying Remark 5.2.4, we deduce that the bottom square exhibits f' as an A'-thickening of f_0. Applying our inductive hypothesis to the regular sequence $x_2, \ldots, x_n \in A'$, we deduce that B' is discrete and that $x_2, x_3, \ldots, x_n$ is a regular sequence in B'. For every integer k, the cofiber $\mathrm{cofib}(x_1^k : B \to B)$ can be written as an extension of finitely many copies of B' (in the ∞-category of B-modules), and is therefore also discrete. Since B is I-complete, it is (x_1)-complete, and can therefore be identified with the limit of the tower $\{\mathrm{cofib}(x_1^k : B \to B)\}_{k \geqslant 0}$ of discrete B-modules with surjective transition maps. It follows that B is discrete. Since $B' = \mathrm{cofib}(x_1 : B \to B)$ is discrete, we conclude that x_1 is not a zero-divisor in B, so that the regularity of the sequence $x_2, \ldots, x_n$ in B' implies the regularity of $x_1, \ldots, x_n$ in B. $\square$

5.3 Review: Complex Bordism and the Lazard Ring

In this section, we recall (without proofs) some foundational results about the Lazard ring and its relationship with complex bordism.

5.3.1 Formal Group Laws

Definition 5.3.1. Let R be a commutative ring and let $\widehat{\mathbf{A}}^1_R = \mathrm{Spf}(R[[t]])$ be the formal affine line over R, which we regard as a pointed formal hyperplane (with base point given by the vanishing locus of t). A *formal group law* over R is an abelian group structure on $\widehat{\mathbf{A}}^1_R$ which is compatible with this choice of base point: that is, a map of pointed formal hyperplanes

$$m : \widehat{\mathbf{A}}^1_R \times_{\mathrm{Spec}(R)} \widehat{\mathbf{A}}^1_R \to \widehat{\mathbf{A}}^1_R.$$

which endows $\widehat{\mathbf{A}}^1_R$ with the structure of an abelian group object of $\mathrm{Hyp}_*(R)$. We let $\mathcal{FGL}(R)$ denote the set of all formal group laws over R.

Remark 5.3.2. Concretely, the datum of a formal group law over R can be described as a formal power series $f(u,v) \in R[[u,v]]$ satisfying the identities

$$f(u,0) = u \qquad f(u,v) = f(v,u) \qquad f(u, f(v,w)) = f(f(u,v), w).$$

The construction $R \mapsto \mathcal{FGL}(R)$ determines a functor from the category of commutative rings to the category of sets. This functor was studied by Lazard in [22].

Theorem 5.3.3 (Lazard)**.** *There exists a commutative ring L and a formal group law $f_{\mathrm{uni}}(u,v) \in \mathcal{FGL}(L)$ with the following universal property: for every commutative ring R, evaluation on $f_{\mathrm{uni}}(u,v)$ induces a bijection* $\mathrm{Hom}(L,R) \to \mathcal{FGL}(R)$*. Moreover:*

- (a) *There is an isomorphism of commutative rings $L \simeq \mathbf{Z}[c_1, c_2, \ldots]$.*
- (b) *Let $\widehat{\mathbf{G}}_{\mathrm{uni}}$ denote the formal group over L determined by the formal group law f_{uni}. Then the generators $\{c_i\}$ of part (a) can be chosen so that, for every prime number p and every integer $n \geqslant 0$, the sequence $p, c_{p-1}, c_{p^2-1}, \ldots c_{p^{n-1}-1}$ generates the nth Landweber ideal $\mathfrak{I}_n^{\widehat{\mathbf{G}}_{\mathrm{uni}}}$ (Definition 4.4.11).*

We will refer to the commutative ring L of Theorem 5.3.3 as the *Lazard ring*, to the formal group law $f_{\mathrm{uni}}(u,v)$ as the *universal formal group law*, and to the underlying formal group $\widehat{\mathbf{G}}_{\mathrm{uni}}$ as the *universal coordinatized formal group*. For our applications, we will not need the full strength of Theorem 5.3.3. However, we will need the following weaker statement:

Corollary 5.3.4. *Let L be the Lazard ring. Fix a prime number p. Then there exists a regular sequence $p = w_0, w_1, w_2, \ldots$ in the ring L with the following property: for every $n \geqslant 0$, the nth Landweber ideal of the universal coordinatized formal group $\widehat{\mathbf{G}}_{\mathrm{uni}}$ is given by $(w_0, w_1, \ldots, w_{n-1})$. Moreover, for $n > 0$, the Frobenius morphism $\varphi : L/(w_0, \ldots, w_{n-1}) \to L/(w_0, \ldots, w_{n-1})$ is flat.*

5.3.2 Coordinates on Formal Groups

Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. A *coordinate* on $\widehat{\mathbf{G}}$ is an element of the augmentation ideal $t \in \mathscr{O}_{\widehat{\mathbf{G}}}(-e)$ whose image is a generator for the dualizing line $\omega_{\widehat{\mathbf{G}}} = \mathscr{O}_{\widehat{\mathbf{G}}}(-e)/\mathscr{O}_{\widehat{\mathbf{G}}}(-e)^2$. Equivalently, a coordinate on $\widehat{\mathbf{G}}$ is an isomorphism of formal R-schemes $\widehat{\mathbf{G}} \simeq \mathrm{Spf}(R[[t]])$ which is compatible with base points (that is, it carries the zero section of $\widehat{\mathbf{G}}$ to the vanishing locus of t).

Definition 5.3.5. A *coordinatized formal group* over R is a pair $(\widehat{\mathbf{G}}, t)$, where $\widehat{\mathbf{G}}$ is a 1-dimensional formal group over R and t is a coordinate on $\widehat{\mathbf{G}}$.

Remark 5.3.6. Let R be a commutative ring. For any formal group law $f(u,v)$ over R, the corresponding formal group (with underlying formal scheme $\mathrm{Spf}(R[[t]])$) comes with a canonical coordinate t. This construction determines a bijection

$$\mathcal{FGL}(R) \simeq \{\text{Coordinatized formal groups over } R\}/\text{isomorphism}.$$

In particular, every coordinatized formal group $(\widehat{\mathbf{G}}, t)$ over R determines a formal group law over R, classified by a homomorphism from the Lazard ring L into R.

Example 5.3.7 (Formal Group Laws from Topology)**.** Let A be a complex periodic $\mathbb{E}_\infty$-ring and let $\widehat{\mathbf{G}} = \widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ be the classical Quillen formal group of A. Suppose we are given an element $t \in \mathscr{O}_{\widehat{\mathbf{G}}}(-e) \simeq A^0_{\mathrm{red}}(\mathbf{CP}^\infty)$. Then:

- The element t is a coordinate on $\widehat{\mathbf{G}}$ if and only if its image $u \in A^0_{\mathrm{red}}(\mathbf{CP}^1) \simeq \pi_2(A)$ is an invertible element of $\pi_*(A)$.
- If u is invertible, then $u^{-1}t \in A^2_{\mathrm{red}}(\mathbf{CP}^\infty)$ is a complex orientation of A.

Conversely, if u is an element of $\pi_2(A)$ which is invertible in $\pi_*(A)$ and $e \in A^2_{\mathrm{red}}(\mathbf{CP}^\infty)$ is a complex orientation of A, then the product $ue \in A^0_{\mathrm{red}}(\mathbf{CP}^\infty)$ is a coordinate on $\widehat{\mathbf{G}}$. We therefore obtain a canonical bijection

$$\{\text{Coordinates on } \widehat{\mathbf{G}}\} \simeq \{\text{Complex Orientations of } A\} \times \{\text{Units in } \pi_2(A)\}.$$

Example 5.3.8. Let A be an $\mathbb{E}_\infty$-ring and suppose we are given a map of spectra $f : \Sigma^\infty(\mathbf{CP}^\infty) \to A$. The homotopy class of the restriction $f|_{\Sigma^\infty(\mathbf{CP}^1)}$ determines an element $u \in \pi_2(A)$. If u is invertible, then the composite map

$$\Sigma^{\infty-2}(\mathbf{CP}^\infty \xrightarrow{\Sigma^{-2}(f)} \Sigma^{-2}(A) \xrightarrow{u^{-1}} A$$

is a complex orientation of A. It follows that A is complex periodic and that the homotopy class $[f] \in A^0_{\mathrm{red}}(\mathbf{CP}^\infty)$ is a coordinate on the classical Quillen formal group $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$.

If u is not invertible in $\pi_2(A)$, then we can remedy the situation by replacing A by the localization $A[u^{-1}]$ (see Proposition 4.3.17); it follows that $A[u^{-1}]$ is complex periodic and that f determines a coordinate on the Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{A[u^{-1}]}$.

5.3.3 Periodic Complex Bordism

We now consider an important special case of Example 5.3.8.

Construction 5.3.9 (Periodic Complex Bordism)**.** Let $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$ denote the topologically enriched category whose objects are finite-dimensional complex vector spaces and whose morphisms are isomorphisms. Let $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})$ denote the nerve of $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$ (in the topologically enriched sense; see Definition HTT.1.1.5.5). We regard $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$ as a symmetric monoidal category with respect to the operation of direct sum, so that $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})$ inherits the structure of an $\mathbb{E}_\infty$-space.

For every finite-dimensional complex vector space V, we let V^c denote the one-point compactification of V. The construction $V \mapsto V^c$ determines a symmetric monoidal functor $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq}) \to \mathcal{S}_*$, where we endow $\mathcal{S}_*$ with the symmetric monoidal structure given by the smash product of pointed spaces. It follows that the colimit $Y = \varinjlim_{V \in \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})} V^c$ can be regarded as a commutative algebra object of $\mathcal{S}$ (with respect to the smash product). Note that, as a pointed space, Y can be described as an infinite wedge

$$\bigvee_{n \geqslant 0} \mathrm{BU}(n)/\mathrm{BU}(n-1),$$

where $\mathrm{BU}(n)$ denotes the classifying space of the unitary group $\mathrm{U}(n)$ (and we agree to the convention that $\mathrm{BU}(-1)$ is empty, so that the quotient $\mathrm{BU}(0)/\mathrm{BU}(-1)$ is the pointed space S^0). It follows that we can regard the suspension spectrum $\Sigma^\infty(Y)$ as a connective $\mathbb{E}_\infty$-ring. By construction, the inclusion $\mathbf{CP}^\infty \simeq \mathrm{BU}(1)/\mathrm{BU}(0) \hookrightarrow Y$ induces a map of spectra $f : \Sigma^\infty(\mathbf{CP}^\infty) \to \Sigma^\infty(Y)$. Let $u \in \pi_2(\Sigma^\infty(Y))$ be the homotopy class of $f|_{\Sigma^\infty(\mathbf{CP}^1)}$. We let MP denote the localization $\Sigma^\infty(Y)[u^{-1}]$ (see Proposition 4.3.17). We will refer to MP as the *periodic complex bordism spectrum*. By construction, the spectrum MP is complex periodic and the homotopy class of the composite map

$$\Sigma^\infty(\mathbf{CP}^\infty) \xrightarrow{f} \Sigma^\infty(Y) \to \mathrm{MP}$$

is classified by an element $t \in \mathrm{MP}^0_{\mathrm{red}}(\mathbf{CP}^\infty)$ which determines a coordinate on the classical Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathrm{MP}}$ (see Example 5.3.8).

The following result is essentially proven in [29] (though it is stated there for a non-periodic version of complex bordism):

Theorem 5.3.10 (Quillen)**.** *Let L be the Lazard ring, let* MP *be the periodic complex bordism spectrum, let $t \in \mathrm{MP}^0_{\mathrm{red}}(\mathbf{CP}^\infty)$ be as in Construction 5.3.9, and let $\rho : L \to \pi_0(\mathrm{MP})$ be the map which classifying the coordinatized formal group $(\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathrm{MP}}, t)$ (in the* *sense of Remark 5.3.6). Then ρ is an isomorphism. Moreover, the homotopy groups of* MP *are concentrated in even degrees.*

5.3.4 Homology of the Complex Bordism Spectrum

We will need a second universal property of the periodic complex bordism spectrum. To state it, we first need a bit of terminology.

Notation 5.3.11. Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. We let $\mathrm{Coord}(\widehat{\mathbf{G}}) : \mathrm{CAlg}_R^{\heartsuit} \to \mathcal{S}\mathrm{et}$ denote the functor given by

$$\mathrm{Coord}(\widehat{\mathbf{G}})(A) = \{\text{Coordinates on } \widehat{\mathbf{G}}_A\}$$

Proposition 5.3.12. *Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R. Then the functor $\mathrm{Coord}(\widehat{\mathbf{G}})$ is (representable by) an affine R-scheme which is faithfully flat over R.*

Proof. Since every 1-dimensional formal group admits a coordinate Zariski-locally, we can identify $\mathrm{Coord}(\widehat{\mathbf{G}})$ with a torsor (in the Zariski topology) for the group-valued functor $G = \mathrm{Aut}_0(\widehat{\mathbf{A}}^1)$, given by

$$\begin{aligned} G(A) &= \{f(t) \in A[[t]] : f(0) = 0 \text{ and } f \text{ is invertible } \} \\ &= \{a_1 t + a_2 t^2 + \cdots : a_1 \in R^{\times}, a_2, a_3, \ldots \in R\} \end{aligned}$$

It now suffices to observe that $G \simeq \mathrm{Spec}(R[a_1^{\pm 1}, a_2, a_3, \ldots])$ is a flat affine group scheme over R. $\square$

Let A be any complex periodic $\mathbb{E}_\infty$-ring and let B denote the smash product $A \otimes_S \mathrm{MP}$. Over the commutative ring $\pi_0(B)$, we have canonical isomorphisms of formal groups

$$(\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathrm{MP}})_{\pi_0(B)} \simeq \widehat{\mathbf{G}}^{\mathcal{Q}_0}_B \simeq (\widehat{\mathbf{G}}^{\mathcal{Q}_0}_A)_{\pi_0(B)}.$$

This isomorphism carries the canonical coordinate on $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathrm{MP}} \simeq \widehat{\mathbf{G}}_{\mathrm{uni}}$ to a coordinate on the formal group $(\widehat{\mathbf{G}}^{\mathcal{Q}_0}_A)_{\pi_0(B)}$, and therefore induces a morphism of schemes. $\mathrm{Spec}(\pi_0(B)) \to \mathrm{Coord}(\widehat{\mathbf{G}}^{\mathcal{Q}_0}_A)$.

Theorem 5.3.13. *Let A be a complex periodic $\mathbb{E}_\infty$-ring. Then $A \otimes_S \mathrm{MP}$ is flat over A, and the preceding construction induces an isomorphism $\mathrm{Spec}(\pi_0(A \otimes_S \mathrm{MP})) \to \mathrm{Coord}(\widehat{\mathbf{G}}^{\mathcal{Q}_0}_A)$.*

Remark 5.3.14. We can paraphrase Theorems 5.3.10 and 5.3.13 as follows:

- The classical Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathrm{MP}}$ is universal among coordinatized formal groups.

- For any complex periodic $\mathbb{E}_\infty$-ring A, the classical Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{A \otimes_S \mathrm{MP}}$ is universal among coordinatized formal groups which can be obtained from $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{A}$ by extending scalars.

For a proof of Theorem 5.3.13, we refer the reader to [1] (though it is stated there in a slightly different form, for the non-periodic version of complex bordism).

5.4 Homotopy Groups of Lubin-Tate Spectra

Our goal in this section is compute the homotopy groups of the Lubin-Tate spectra introduced in Construction 5.1.1. Our main result can be stated as follows:

Theorem 5.4.1. *Let R_0 be a perfect $\mathbf{F}_p$-algebra, let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group of exact height n over R_0, and let $E = E(\widehat{\mathbf{G}}_0)$ be the Lubin-Tate spectrum of Construction 5.1.1. Then:*

(a) *The map α of Construction 5.1.4 induces an isomorphism of commutative rings $R_0 \to \pi_0(E)/\mathfrak{I}^E_n$.*

(b) *The homotopy groups of E are concentrated in even degrees.*

(c) *Choose any sequence of elements $\{\overline{v}_m \in \pi_{2(p^m-1)}(E)\}_{0 \leq m < n}$ such that the image of each $\overline{v}_m$ in $\pi_*(E)/\mathfrak{I}^E_m \pi_*(E)$ is the mth Hasse invariant v_m (Construction 4.4.9). Then $p = \overline{v}_0, \overline{v}_1, \ldots, \overline{v}_{n-1}$ is a regular sequence in $\pi_*(E)$.*

Theorem 5.4.1 can be stated a bit more simply in the case where the dualizing line $\omega_{\widehat{\mathbf{G}}_0}$ is trivial. In this case, we can lift any trivialization of $\omega_{\widehat{\mathbf{G}}_0}$ to an element of $e \in \pi_2(E)$ which is invertible in $\pi_*(E)$.

Corollary 5.4.2. *Let R_0, $\widehat{\mathbf{G}}_0$, and E be as in Theorem 5.4.1, and suppose that there exists an element $e \in \pi_2(E)$ which is invertible in $\pi_*(E)$. Choose elements $\overline{v}_m \in \pi_{2(p^m-1)}(R)$ representing the Hasse invariants $v_m \in \pi_{2(p^m-1)}(R)/\mathfrak{I}^R_m \pi_{2(p^m-1)}(R)$, and set $u_m = \overline{v}_m / e^{p^m-1} \in \pi_0(R)$. Then we have a canonical isomorphisms*

$$\pi_0(R) \simeq W(R_0)[[u_1, \ldots, u_{m-1}]] \qquad \pi_*(R) \simeq W(R_0)[[u_1, \ldots, u_{m-1}]][e^{\pm 1}].$$

From Corollary 5.4.2, we can recover the structure theory of classical Lubin-Tate rings:

Corollary 5.4.3 (Lubin-Tate)**.** *Let κ be a perfect field of characteristic $p > 0$, let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group of height $n < \infty$ over κ, and let R_{LT} denote the associated Lubin-Tate ring. Then R_{LT} is (noncanonically) isomorphic to a power series ring $W(\kappa)[[u_1, \ldots, u_{n-1}]]$. In particular, it is a regular local ring of Krull dimension n.*

Proof. Let $\widehat{\mathbf{G}} \in \mathrm{FGroup}(R_{\mathrm{LT}})$ be the universal deformation of $\widehat{\mathbf{G}}_0$ in the classical sense of Theorem 3.0.1. Let $\omega_{\widehat{\mathbf{G}}}$ be the dualizing line of the formal group $\widehat{\mathbf{G}}$ (Definition 4.2.14). Since R_{LT} is a local ring, we can choose an R_{LT}-module generator $e \in \omega_{\widehat{\mathbf{G}}}$. For each $m \geqslant 0$, let $\mathfrak{I}^{\widehat{\mathbf{G}}}_m$ denote the mth Landweber ideal (Definition 4.4.11) and let $v_m \in \omega_{\widehat{\mathbf{G}}}^{\otimes(p^m-1)}/\mathfrak{I}^{\widehat{\mathbf{G}}}_m \omega_{\widehat{\mathbf{G}}}^{\otimes(p^m-1)}$ be the mth Hasse invariant (Construction 4.4.9). Then each v_m can be lifted to an element of $\omega_{\widehat{\mathbf{G}}}^{\otimes(p^m-1)}$, which we can write as $u_m e^{p^m-1}$ for some element $u_m \in R_{\mathrm{LT}}$. Note that the elements $p = u_0, u_1, \ldots, u_{n-1}$ generate the nth Landweber ideal $\mathfrak{I}^{\widehat{\mathbf{G}}}_n$, which is also the maximal ideal of R_{LT} by virtue of Corollary 4.4.25. It follows that the sequence $u_1, \ldots, u_{n-1}$ determines a surjection of complete local Noetherian rings $\rho : W(\kappa)[[u_1, \ldots, u_{n-1}]] \to R_{\mathrm{LT}}$. To prove Corollary 5.4.3, we must show that ρ is also injective. Write $\widehat{\mathbf{G}}_0$ as the identity component of a connected p-divisible group $\mathbf{G}_0$, and let $E = E(\widehat{\mathbf{G}}_0)$ be its Lubin-Tate spectrum. Using Corollary 5.4.2, we see that the composite map

$$W(\kappa)[[u_1, \ldots, u_{n-1}]] \xrightarrow{\rho} R_{\mathrm{LT}} \simeq \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0}) \to \pi_0(E)$$

is an isomorphism. It follows that ρ is injective, as desired. □

Proof of Theorem 5.0.2. Let κ be a perfect field of characteristic $p > 0$, let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group of height $n < \infty$ over κ, let $E = E(\widehat{\mathbf{G}}_0)$ be the Lubin-Tate spectrum of Construction 5.1.1, and let

$$\alpha : (\kappa, \widehat{\mathbf{G}}_0) \to (\pi_0(E)/\mathfrak{I}^E_n, \widehat{\mathbf{G}}^{\mathcal{Q}_n}_E)$$

be the morphism in $\mathcal{FG}$ described in Construction 5.1.4. It follows from Theorem 5.4.1 that α is an isomorphism. We claim that α has properties (i) and (ii) of Theorem 5.0.2. Property (ii) was established in Theorem 5.1.5. To prove (i), we first observe that $\pi_0(E)$ is a complete local Noetherian ring with residue field κ (Corollary 5.4.2), so that α is classified by a map of commutative rings $\mu : R_{\mathrm{LT}} \to \pi_0(E)$. Let $\rho : W(\kappa)[[u_1, \ldots, u_{n-1}]] \to R_{\mathrm{LT}}$ be the isomorphism appearing in the proof of Corollary 5.4.3. Corollary 5.4.2 implies that the composition $\mu \circ \rho$ is an isomorphism, so that μ is also an isomorphism. It follows that α exhibits the classical Quillen formal group $\widehat{\mathbf{G}}_E^{\mathcal{Q}_0}$ as a universal deformation of $\widehat{\mathbf{G}}_0$ in the sense of Theorem 3.0.1. It will therefore suffice to show that the homotopy groups of E are concentrated in even degrees, which follows from Theorem 5.4.1. $\square$

5.4.1 The Proof of Theorem 5.4.1

For the remainder of this section, we fix a perfect $\mathbf{F}_p$-algebra R_0 and a formal group $\widehat{\mathbf{G}}_0$ of exact height n over R_0. Let L denote the Lazard ring, and let $I_n = \mathfrak{I}_n^{\widehat{\mathbf{G}}^{\mathrm{uni}}}$ denote the nth Landweber ideal of the universal coordinatized formal group. By virtue of Proposition 5.3.12, the functor

$$\mathrm{Coord}(\widehat{\mathbf{G}}_0) : \mathrm{CAlg}_{R_0}^{\heartsuit} \to \mathcal{S}\mathrm{et}$$

is representable by a faithfully flat R_0-algebra, which we will denote by $\widetilde{R}_0$. Let t denote the tautological coordinate on the formal group $(\widehat{\mathbf{G}}_0)_{\widetilde{R}_0}$, so that the coordinatized formal group $((\widehat{\mathbf{G}}_0)_{\widetilde{R}_0}, t)$ is classified by a ring homomorphism $L \to \widetilde{R}_0$. Since $\widehat{\mathbf{G}}_0$ has exact height n, the formal group $(\widehat{\mathbf{G}}_0)_{\widetilde{R}_0}$ also has exact height n. It follows that the map $L \to \widetilde{R}_0$ annihilates the ideal Landweber $I_n \subseteq L$, and therefore induces a ring homomorphism $\rho_0 : L/I_n \to \widetilde{R}_0$.

Lemma 5.4.4. *The homomorphism $\rho_0 : L/I_n \to \widetilde{R}_0$ is a relatively perfect morphism of $\mathbf{F}_p$-algebras.*

Proof. For every commutative ring A, let $\mathcal{FGL}^{\geqslant n}(A)$ denote the subset of $\mathcal{FGL}(A)$ consisting of those formal group laws over A for which the associated formal group has height $\geqslant n$, and let $\mathcal{FGL}^{=n}(A)$ denote the subset consisting of those formal groups which have exact height n. By construction, the commutative ring L/I_n corepresents the functor $\mathcal{FGL}^{\geqslant n}$. Suppose that A is R_0-algebra, and let $A^{(p)}$ denote the R_0-algebra obtained from A by extending scalars along the Frobenius isomorphism $\varphi_{R_0} : R_0 \to R_0$. The statement of Lemma 5.4.4 is equivalent to the assertion that the outer rectangle in the diagram

$$\begin{tikzcd}
\mathrm{Coord}(\widehat{\mathbf{G}}_0)(A^{(p)}) \arrow[r] \arrow[d] & \mathcal{FGL}^{=n}(A^{(p)}) \arrow[r] \arrow[d] & \mathcal{FGL}^{\geqslant n}(A^{(p)}) \arrow[d] \\
\mathrm{Coord}(\widehat{\mathbf{G}}_0)(A) \arrow[r] & \mathcal{FGL}^{=n}(A) \arrow[r] & \mathcal{FGL}^{\geqslant n}(A)
\end{tikzcd}$$

is a pullback square of sets, where the vertical maps are induced by the relative Frobenius map $A^{(p)} \to A$. Note that the right square is automatically a pullback (since

$\mathcal{FGL}^{=n}$ is an open subfunctor of $\mathcal{FGL}^{\geqslant n}$). It will therefore suffice to show that the square on the left is a pullback. Unwinding the definitions, we must show that if $(\widehat{\mathbf{G}}', t)$ is a coordinatized formal group of exact height n over $A^{(p)}$, then any isomorphism $\widehat{\mathbf{G}}'_A \simeq (\widehat{\mathbf{G}}_0)_A$ of formal groups over A can be lifted uniquely to an isomorphism $\widehat{\mathbf{G}}' \simeq (\widehat{\mathbf{G}}_0)_{A^{(p)}}$ of formal groups over $A^{(p)}$. This follows from Proposition 4.4.23, since the relative Frobenius map $A^{(p)} \to A$ induces an isomorphism of perfections. $\square$

Lemma 5.4.5. *Let* MP *denote the periodic complex bordism spectrum. Then the canonical map*

$$\tau_{\geqslant 0}(\mathrm{MP}) \to \pi_0(\mathrm{MP}) \simeq L \to L/I_n$$

exhibits L/I_n as a perfect module over $\tau_{\geqslant 0}(\mathrm{MP})$.

Proof. Let $u \in \pi_2(\mathrm{MP})$ be as in Construction 5.3.9, so that u is invertible in $\pi_*(\mathrm{MP})$. Theorem 5.3.10 implies that the homotopy groups of MP are concentrated in even degrees, so that multiplication by u determines a cofiber sequence

$$\Sigma^2(\tau_{\geqslant 0}(\mathrm{MP})) \to \tau_{\geqslant 0}(\mathrm{MP}) \to L.$$

It follows that L is perfect as a module over $\tau_{\geqslant 0}(\mathrm{MP})$. Since I_n is generated by a regular sequence in L (Corollary 5.3.4), the quotient L/I_n is perfect as a module over L and therefore also as a module over $\tau_{\geqslant 0}(\mathrm{MP})$. $\square$

Construction 5.4.6. Recall that the quotient ring L/I_n is an $\mathbf{F}_p$-algebra for which the Frobenius map $\varphi_{L/I_n} : L/I_n \to L/I_n$ is flat. Applying Theorem 5.3.3 to the map $\tau_{\geqslant 0}(\mathrm{MP}) \to L/I_n$ and the relatively perfect morphism $\rho : L/I_n \to \widetilde{R}_0$, we obtain a pushout diagram of $\mathbb{E}_\infty$-rings σ :

$$\begin{tikzcd} \tau_{\geqslant 0}(\mathrm{MP}) \arrow[r, "\rho"] \arrow[d] & \widetilde{A} \arrow[d] \\ L/I_n \arrow[r, "\rho_0"] & \widetilde{R}_0 \end{tikzcd}$$

which exhibits ρ as a $\tau_{\geqslant 0}(\mathrm{MP})$-thickening of ρ_0, in the sense of Definition 5.2.1.

Lemma 5.4.7. *Let $\widetilde{A}$ be as in Construction 5.4.6. Then:*

(1) *The element $u \in \pi_2(\mathrm{MP})$ induces an isomorphism of rings $\pi_*(R) \simeq \pi_0(\widetilde{A})[u]$.*

(2) *Any regular sequence $w_0, w_1, \ldots, w_{n-1}$ generating the ideal $I \subseteq L$ remains regular in $\pi_0(\widetilde{A})$.*

Proof. Expand the commutative square σ of Construction 5.4.6 to a diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{tikzcd}
\tau_{\geqslant 0}(\mathrm{MP}) \arrow[r, "\rho"] \arrow[d] & \widetilde{A} \arrow[d] \\
L \arrow[r, "\rho'"] \arrow[d] & \widetilde{A}_0 \arrow[d] \\
L/I_n \arrow[r, "\rho_0"] & \widetilde{R}_0
\end{tikzcd}$$

where the upper square is a pushout. Applying Remark 5.2.4, we deduce that the lower square exhibit ρ' as an L-thickening of ρ_0. It follows from Proposition 5.2.12 that $\widetilde{A}_0$ is discrete and that $w_0, \ldots, w_{n-1}$ is a regular sequence in $\widetilde{A}_0$. By construction, we have a cofiber sequence

$$\Sigma^2 \widetilde{A} \xrightarrow{u} \widetilde{A} \to \widetilde{A}_0.$$

The associated long exact sequence of homotopy groups shows that the map $\pi_0(\widetilde{A}) \to \widetilde{A}_0$ is an isomorphism and that multiplication by u induces isomorphisms $\pi_{k-2}(\widetilde{A}) \simeq \pi_k(\widetilde{A})$ for $k > 0$. It follows that $\pi_*(\widetilde{A})$ can be identified with the polynomial ring $\widetilde{A}_0[u]$, which immediately implies (1) and (2). $\square$

Notation 5.4.8. Let A denote the localization $\widetilde{A}[u^{-1}]$ (where we abuse notation by identifying the element $u \in \pi_2(\mathrm{MP})$ with its image in $\pi_2(\widetilde{A})$). Note that the map $\rho : \tau_{\geqslant 0}(\mathrm{MP}) \to A$ extends to a map of localizations

$$\overline{\rho} : \mathrm{MP} \simeq (\tau_{\geqslant 0}(\mathrm{MP}))[u^{-1}] \to \widetilde{A}[u^{-1}] = A.$$

Lemma 5.4.9. *The $\mathbb{E}_\infty$-ring A is even periodic and $K(n)$-local. Moreover, the canonical map $\widetilde{A} \to A$ exhibits $\widetilde{A}$ as a connective cover of A.*

Proof. It follows from Lemma 5.4.7 that the homotopy ring $\pi_*(A)$ can be identified with the Laurent polynomial ring $\pi_0(\widetilde{A})[u^{\pm 1}]$. This proves that A is even periodic, and that $\widetilde{A}$ is a connective cover of A. We will prove that A is $K(n)$-local by applying the criterion of Theorem 4.5.2. Note that the nth Landweber ideal $\mathfrak{I}_n^A$ is given by $I_n \pi_0(A)$. By construction, $\widetilde{A}$ is I_n-complete and therefore the commutative ring $\pi_0(\widetilde{A}$ is I_n-complete (Theorem SAG.7.3.4.1). Since each homotopy group of A is isomorphic to $\pi_0(\widetilde{A})$, it follows that A is also I_n-complete (Theorem SAG.7.3.4.1). Choose an element $w \in L$ such that $I_{n+1} = (w) + I_n$. To complete the proof that A is $K(n)$-local, it will suffice to show that the image of w is invertible in $\pi_0(\widetilde{A})$. Because $\pi_0(\widetilde{A})$ is

I-complete, it will suffice to show that w has invertible image in $\pi_0(\widetilde{A})/I_n\pi_0(\widetilde{A}) \simeq \widetilde{R}_0$. This follows from our assumption that the the formal group $\widehat{\mathbf{G}}_0$ has exact height n. $\square$

Using the commutativity of the diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & \pi_0(A) \arrow[d] \\ L/I_n \arrow[r, "\rho_0"] & \widetilde{R}_0, \end{tikzcd}$$

and the definition of ρ_0, we obtain a canonical map

$$\gamma_0 : (R_0, \widehat{\mathbf{G}}_0) \to (\widetilde{R}_0, \widehat{\mathbf{G}}_A^{\mathcal{Q}_n})$$

in the category $\mathcal{FG}$. Since A is $K(n)$-local, Theorem 5.1.5 ensures that γ_0 admits an essentially unique lift to a morphism of $\mathbb{E}_\infty$-rings $\gamma : E \to A$. We will deduce Theorem 5.4.1 from the following:

Proposition 5.4.10. *The maps $\overline{\rho} : \mathrm{MP} \to A$ and $\gamma : E \to A$ exhibit A as the I_n-completion of the smash product $\mathrm{MP} \otimes_S E$.*

Proof. Since the Quillen formal group $\widehat{\mathbf{G}}_E^{\mathcal{Q}_0}$ has height $< n+1$, we can identify the I_n-completion of $\mathrm{MP} \otimes_S E$ with its $K(n)$-localization (Theorem 4.5.2). It will therefore suffice to show that, for every $K(n)$-local $\mathbb{E}_\infty$-ring B, the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}}(A, B) \to \mathrm{Map}_{\mathrm{CAlg}}(\mathrm{MP} \otimes_S E, B) \simeq \mathrm{Map}_{\mathrm{CAlg}}(\mathrm{MP}, B) \times \mathrm{Map}_{\mathrm{CAlg}}(E, B)$$

is a homotopy equivalence. Equivalently, we must show that if B is a $K(n)$-local $\mathbb{E}_\infty$-algebra over MP, then composition with γ induces a homotopy equivalence $\theta : \mathrm{Map}_{\mathrm{CAlg}_{\mathrm{MP}}}(A, B) \to \mathrm{Map}_{\mathrm{CAlg}}(E, B)$. Set $B_0 = \pi_0(B)/I_n\pi_0(B)$, so that the map $\mathrm{MP} \to B$ determines a coordinate t on $\widehat{\mathbf{G}}_B^{\mathcal{Q}_n}$. Using Theorem 5.1.5, we can identify $\mathrm{Map}_{\mathrm{CAlg}}(E, B)$ with the set $\mathrm{Hom}_{\mathcal{FG}}((R_0, \widehat{\mathbf{G}}_0), (B_0, \widehat{\mathbf{G}}_B^{\mathcal{Q}_n}))$. Since the restriction map $\mathrm{Map}_{\mathrm{CAlg}_{\mathrm{MP}}}(A, B) \to \mathrm{Map}_{\mathrm{CAlg}_{\tau_{\geqslant 0}\mathrm{MP}}}(\widetilde{A}, \tau_{\geqslant 0}B)$ is a homotopy equivalence, we can invoke the definition ρ as a $\tau_{\geqslant 0}(\mathrm{MP})$ lift of ρ_0 to obtain a homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathrm{MP}}}(A, B) \simeq \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{L/I_n}}(\widetilde{R}_0, B_0).$$

Under these identifications, θ corresponds to a map θ' which fits into a commutative diagram of sets

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{L/I_n}}(\widetilde{R}_0, B_0) \arrow[rr, "\theta'"] \arrow[dr] & & \mathrm{Hom}_{\mathcal{FG}}((R_0, \widehat{\mathbf{G}}_0), (B_0, \widehat{\mathbf{G}}_B^{\mathcal{Q}_n})) \arrow[dl] \\ & \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}}(R_0, B_0). & \end{tikzcd}$$

To show that θ' is bijective, it will suffice to prove bijectivity after restricting to the fiber over any ring homomorphism $g : R_0 \to B_0$. This follows from an examination of the pullback square

$$\begin{tikzcd}
\mathrm{Coord}(\widehat{\mathbf{G}}_0)(B_0) \arrow[r] \arrow[d] & \{\widehat{\mathbf{G}}_0\} \arrow[d] \\
\mathcal{FGL}(B_0) \arrow[r] & \mathrm{FGroup}(B_0)
\end{tikzcd}$$

by taking fibers over (the isomorphism class of) the coordinatized formal group $(\widehat{\mathbf{G}}_B^{\mathcal{Q}_n}, t) \in \mathcal{FGL}(B_0)$. □

Proof of Theorem 5.4.1. Let $\{\overline{v}_m \in \pi_{2(p^m-1)}(E)\}_{0 \leqslant m < n}$ be as in the statement of Theorem 5.4.1. For each $0 \leqslant m < n$, let $\mathrm{cofib}(\overline{v}_m)$ denote the cofiber of the map $\overline{v}_m : \Sigma^{2(p^m-1)}E \to E$. For $0 \leqslant m \leqslant n$, we let $E(m)$ denote the "Koszul complex" given by the tensor product $\bigotimes_{i<m} \mathrm{cofib}(\overline{v}_m)$ (where the tensor product is formed in the ∞-category Mod_E), and set $A(m) = E(m) \otimes_E A$. We first prove the following:

(i_m) The canonical map $A \to A(m)$ induces an isomorphism $\pi_*(A)/I_m\pi_*(A) \simeq \pi_*(A(m))$.

We prove (i_m) by induction on m, the case $m = 0$ being trivial. Let us therefore suppose that $(*_m)$ is known for some $m < n$; we will prove $(*_{m+1})$. Using Corollary 5.3.4, we see that the ideal $I_{m+1} \subseteq L$ is generated by a regular sequence $w_0, w_1, \ldots, w_m \in L$ for which the subsequence $w_0, w_1, \ldots, w_{m-1}$ generates I_m. Let us abuse notation by identifying each w_i with its image in $\pi_0(A)$ under the map $\overline{\rho}$, and each $\overline{v}_i$ with its image in $\pi_{2(p^i-1)}(A)$ under the map γ. Note that $u^{1-p^m}\overline{v}_m$ and w_m are both non-zero divisors of $\pi_0(A)/I_m\pi_0(A)$ which generate the ideal $I_{m+1}\pi_0(A)/I_m\pi_0(A)$, so their images differ by a unit in $\pi_0(A)/I_m\pi_0(A)$. Lemma 5.4.7 implies that multiplication by w_m is injective on $\pi_*(A)/I_m\pi_*(A)$, so multiplication by $\overline{v}_m$ must also be injective. We therefore have a short exact sequence

$$0 \to \pi_{*-2(p^m-1)}(A(m)) \xrightarrow{\overline{v}_m} \pi_*(A(m)) \to \pi_*(A(m+1)) \to 0,$$

from which assertion (i_{m+1}) follows.

By virtue of Lemma 5.4.9, we can identify the quotient $\pi_*(A)/I_n\pi_*(A)$ with $\widetilde{R}_0[u^{\pm 1}]$. We can therefore reformulate $(*_n)$ as follows:

(ii) There is a canonical isomorphism of $\pi_*(A)$-modules

$$\widetilde{R}_0[u^{\pm 1}] \simeq \pi_*(A(n))$$

which carries $1 \in \widetilde{R}_0$ to the homotopy class of the natural map $A \to A(n)$.

It follows from Proposition 5.4.10 that each $A(m)$ can be identified with the completion of the smash product $\mathrm{MP} \otimes_S E(m)$ with respect to the ideal $I_n \subseteq \pi_0(\mathrm{MP})$, or equivalently with respect to the ideal $\mathfrak{I}_n^E \subseteq \pi_0(E)$ (since both generate the Landweber ideal $\mathfrak{I}_n^A \subseteq \pi_0(A)$). Note that the smash product $\mathrm{MP} \otimes_S E(n)$ is already $\mathfrak{I}_n^E$-complete (since it is annihilated by a power of each $\overline{v}_i$). Combining this observation with (ii), we obtain the following:

(iii) There is a canonical isomorphism of $\pi_*(\mathrm{MP} \otimes_S E)$-modules

$$\widetilde{R}_0[u^{\pm 1}] \simeq \pi_*(\mathrm{MP} \otimes_S E(n))$$

which carries $1 \in \widetilde{R}_0$ to the homotopy class of the natural map $\mathrm{MP} \otimes_S E \to \mathrm{MP} \otimes_S E(n)$.

By virtue of Theorem 5.3.13, the smash product $\mathrm{MP} \otimes_S E$ is flat over E. Since $(*'')$ implies that the homotopy groups of

$$\mathrm{MP} \otimes_S E(n) \simeq (\mathrm{MP} \otimes_S E) \otimes_E E(n)$$

are concentrated in even degrees, it follows that the homotopy groups of $E(n)$ are concentrated in even degrees. We now prove the following more general assertion for $0 \leqslant m \leqslant n$

(iv_m) The homotopy groups of $E(m)$ are concentrated in even degrees.

We proceed by descending induction on m; the case $m = n$ has already been treated above. Let us therefore assume that $m < n$ and that the homotopy groups of $E(m+1)$ are concentrated in even degrees. Using the cofiber sequence

$$\Sigma^{2(p^m-1)} E(m) \xrightarrow{\overline{v}_m} E(m) \to E(m+1),$$

we conclude that the map $\overline{v}_m : \pi_*(E(m)) \to \pi_*(E(m))$ is surjective on homotopy groups in odd degrees. Consequently, if $\pi_*(E(m))$ were nonzero some odd degree, then the inverse limit of the tower

$$\cdots \to \Sigma^{4(p^m-1)} E(m) \xrightarrow{\overline{v}_m} \Sigma^{2(p^m-1)} E(m) \xrightarrow{\overline{v}_m} E(m)$$

would have nonzero homotopy. This is impossible, since $E(m)$ is complete with respect to the ideal $\mathfrak{I}_n^E$ (because it is a perfect E-module and E is $\mathfrak{I}_n^E$-complete).

Applying (iv_m) in the case $m = 0$, we deduce that the homotopy groups of $E \simeq E(0)$ are concentrated in even degrees, which proves assertion (b) of Theorem 5.4.1. Moreover, it follows from (iv_m) that each of the cofiber sequences

$$\Sigma^{2(p^m-1)} E(m) \xrightarrow{\overline{v}_m} E(m) \to E(m+1)$$

induces a short exact sequence of graded abelian groups

$$0 \to \pi_{*-2(p^m-1)}(E(m)) \xrightarrow{\overline{v}_m} \pi_*(E(m)) \to \pi_*(E(m+1)) \to 0.$$

It follows that the sequence $\overline{v}_0, \overline{v}_1, \ldots, \overline{v}_{n-1}$ is regular in $\pi_*(E)$ (which proves (c)) and that the canonical map $E \to E(n)$ induces an isomorphism

$$\pi_*(E)/(\overline{v}_0, \ldots, \overline{v}_{n-1}) = \pi_*(E)/\mathfrak{I}_n^E \pi_*(E) \to \pi_*(E(n)).$$

To complete the proof of Theorem 5.4.1, we must show that the canonical map $\psi : R_0 \to \pi_0(E)/\mathfrak{I}_n^E$ is an isomorphism. Since $\widetilde{R}_0$ is faithfully flat over R_0 (Proposition 5.3.12), it will suffice to show that ψ induces an isomorphism

$$\widetilde{\psi} : \widetilde{R}_0 \to \widetilde{R}_0 \otimes_{R_0} (\pi_0(E)/\mathfrak{I}_n^E) \simeq \widetilde{R}_0 \otimes_{R_0} \pi_0(E(n)).$$

Using Theorem 5.3.13, we can identify $\widetilde{R}_0 \otimes_{R_0} \pi_0(E(n))$ with $\pi_0(\mathrm{MP} \otimes_S E(n))$, so that the desired result follows from (iii). $\square$

6 Oriented Deformation Rings

Let R_0 be a commutative $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a p-divisible group over R_0. In §3, we studied conditions which guarantee that $\mathbf{G}_0$ admits a universal deformation, defined over an $\mathbb{E}_\infty$-ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ which we refer to as the *spectral deformation ring* of $\mathbf{G}_0$. In this section, we study a variant of $R^{\mathrm{un}}_{\mathbf{G}_0}$, which classifies *oriented* deformations of $\mathbf{G}_0$.

Construction 6.0.1. Let R_0 be a Noetherian $\mathbf{F}_p$-algebra which is F-finite, let $\mathbf{G}_0$ be a nonstationary p-divisible group over R_0, and let $\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}_0})$ be a universal deformation of $\mathbf{G}_0$ (see Theorem 3.0.11). We let $R^{\mathrm{or}}_{\mathbf{G}_0}$ denote an orientation classifier for the underlying formal group $\mathbf{G}^\circ$ (Definition 4.3.14). We will refer to $R^{\mathrm{or}}_{\mathbf{G}_0}$ as the *oriented deformation ring of* $\mathbf{G}_0$.

Remark 6.0.2. In the situation of Theorem 6.0.3, let $\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}_0})$ denote the universal deformation of $\mathbf{G}_0$. Then the formal group $\mathbf{G}^\circ$ acquires an orientation after extending scalars to $R^{\mathrm{or}}_{\mathbf{G}_0}$. It follows from Proposition 4.3.23 that the oriented deformation ring $R^{\mathrm{or}}_{\mathbf{G}_0}$ is complex periodic.

In the situation of Construction 6.0.1, we have a canonical map of $\mathbb{E}_\infty$-rings $R^{\mathrm{un}}_{\mathbf{G}_0} \to R^{\mathrm{or}}_{\mathbf{G}_0}$, which induces a ring homomorphism $R^{\mathrm{cl}}_{\mathbf{G}_0} \to \pi_0(R^{\mathrm{or}}_{\mathbf{G}_0})$.

Theorem 6.0.3. *Let R_0 be a commutative $\mathbf{F}_p$-algebra which is Noetherian and F-finite and let $\mathbf{G}_0$ be a 1-dimensional nonstationary p-divisible group over R_0, with classical deformation ring $R^{\mathrm{cl}}_{\mathbf{G}_0}$ and oriented deformation ring $R^{\mathrm{or}}_{\mathbf{G}_0}$. Then:*

(a) *The canonical map $u : R^{\mathrm{cl}}_{\mathbf{G}_0} \to \pi_0(R^{\mathrm{or}}_{\mathbf{G}_0})$ is an isomorphism of commutative rings.*

(b) *The homotopy groups of $R^{\mathrm{or}}_{\mathbf{G}_0}$ are concentrated in even degrees.*

Note that Theorem 6.0.3 can be regarded as a more precise version of the main result claimed in the introduction to this paper:

Proof of Theorem 0.0.8 from Theorem 6.0.3. Let $(R_0, \mathbf{G}_0)$ be as in the statement of Theorem 6.0.3, let $\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}})$ denote the universal deformation of $\mathbf{G}_0$, and set $E = R^{\mathrm{or}}_{\mathbf{G}_0}$. By construction, the formal group $\mathbf{G}^\circ$ acquires an orientation after extending scalars from $R^{\mathrm{un}}_{\mathbf{G}_0}$ to E. It follows that E is complex periodic and that the $\mathbf{G}^\circ_E$ can be identified with the Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_E$. It follows from Theorem 6.0.3 that the classical Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_E$ agrees with the identity component of the formal group $\mathbf{G}_{\mathrm{cl}}$ obtained from $\mathbf{G}$ by extending scalars along the projection map $R^{\mathrm{un}}_{\mathbf{G}_0} \to \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$, which is the classical universal deformation of $\mathbf{G}_0$ (see Corollary 3.0.13). $\square$

Remark 6.0.4. In the situation above, the homotopy groups of $R^{\mathrm{or}}_{\mathbf{G}_0}$ are given by

$$\pi_*(R^{\mathrm{or}}_{\mathbf{G}_0}) \simeq \begin{cases} \omega^{\otimes k} & \text{if } * = 2k \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

where ω denotes the dualizing line of the formal group $\mathbf{G}^\circ_{\mathrm{cl}}$.

Let us describe some other consequences of Theorem 6.0.3. Recall first that the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is equipped with the structure of an *adic* $\mathbb{E}_\infty$-ring.

Corollary 6.0.5. *Let R_0 be a commutative $\mathbf{F}_p$-algebra which is Noetherian and F-finite, let $\mathbf{G}_0$ be a 1-dimensional nonstationary p-divisible group over R_0, and let $I \subseteq \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$ be a finitely generated ideal of definition. Then the oriented deformation ring $R^{\mathrm{or}}_{\mathbf{G}_0}$ is I-complete.*

Proof. By virtue of Theorem SAG.7.3.4.1 , it will suffice to show that each homotopy group $\pi_k(R^{\mathrm{or}}_{\mathbf{G}_0})$ is I-complete. If k is odd, then $\pi_k(R^{\mathrm{or}}_{\mathbf{G}_0})$ vanishes (Theorem 6.4.6) and there is nothing to prove. If k is even, then $\pi_k(R^{\mathrm{or}}_{\mathbf{G}_0})$ is an invertible module over $\pi_0(R^{\mathrm{or}}_{\mathbf{G}_0})$. It will therefore suffice to show that $\pi_0(R^{\mathrm{or}}_{\mathbf{G}_0})$ is I-complete. Invoking Theorem 6.4.6, we see that the unit map $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0}) \to \pi_0(R^{\mathrm{or}}_{\mathbf{G}_0})$ is an isomorphism. We are therefore reduced to showing that $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$ is I-complete, which follows from the I-completeness of $R^{\mathrm{un}}_{\mathbf{G}_0}$ (Theorem SAG.7.3.4.1). □

Corollary 6.0.6. *Let κ be a perfect field of characteristic p, let $\mathbf{G}_0$ be a connected 1-dimensional p-divisible group over κ, and let $E = E(\mathbf{G}_0^{\circ})$ denote the associated Lubin-Tate spectrum. Then the oriented deformation ring $R^{\mathrm{or}}_{\mathbf{G}_0}$ is equivalent to E.*

Proof. Recall that E was defined as the $K(n)$-localization of the oriented deformation ring $A = R^{\mathrm{or}}_{\mathbf{G}_0}$ (Construction 5.1.1). It will therefore suffice to show that A is already $K(n)$-local. By virtue of Theorem 4.5.2, it will suffice to show that A is complete with respect to the nth Landweber ideal $\mathfrak{I}^A_n$, and that $\mathfrak{I}^A_{n+1} = \pi_0(A)$. The second assertion is obvious, and the first follows from from Theorem 6.0.3, since the nth Landweber ideal defines the topology on $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$ (Corollary 4.4.25). □

Remark 6.0.7 (The Universal Property of $R^{\mathrm{or}}_{\mathbf{G}_0}$)**.** In the situation of Theorem 6.0.3, let us regard the oriented deformation ring $R^{\mathrm{or}}_{\mathbf{G}_0}$ as a complete adic $\mathbb{E}_\infty$-ring by endowing $\pi_0(R^{\mathrm{or}}_{\mathbf{G}_0})$ with the topology determined by the isomorphism $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0}) \simeq \pi_0(R^{\mathrm{or}}_{\mathbf{G}_0})$. Let A be an arbitrary adic $\mathbb{E}_\infty$-ring. Unwinding the definitions, we can identify the mapping space $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}}(R^{\mathrm{or}}_{\mathbf{G}_0}, A)$ with a space $\mathrm{Def}^{\mathrm{or}}_{\mathbf{G}_0}(A)$ classifying triples $(\mathbf{G}, \alpha, e)$ where $\mathbf{G}$ is a p-divisible group, α is an equivalence class of $\mathbf{G}_0$-taggings of A (see Definition 3.1.1), and e is an orientation of the identity component $\mathbf{G}^{\circ}$. Note that the existence of α guarantees that p is topologically nilpotent in A and that the p-divisible group $\mathbf{G}$ is 1-dimensional.

Let us now briefly outline our strategy for proving Theorem 6.0.3. Note that the statement of Theorem 6.0.3 refers to a p-divisible group $\mathbf{G}_0$ over a commutative ring R_0, which we can think of as a family of p-divisible groups parametrized by the affine scheme $\mathrm{Spec}(R_0)$. For every point $x \in |\mathrm{Spec}(R)|$, let $\overline{\kappa(x)}$ denote an algebraic closure of the residue field $\kappa(x)$, so that $\mathbf{G}_0$ determines a p-divisible group $\mathbf{G}'_0$ over the field $\overline{\kappa(x)}$ by extension of scalars. In §6.1, we will show that the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}'_0}$ is flat as a $\mathbb{E}_\infty$-algebra over the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ (Theorem 6.1.2). Using this observation, we will reduce the proof of Theorem 6.0.3 to the case where R_0 is an algebraically closed field. In this case, the p-divisible group $\mathbf{G}_0$ admits a connected-étale sequence

$$0 \to \mathbf{G}'_0 \to \mathbf{G}_0 \to \mathbf{G}''_0 \to 0.$$

In §6.2, we will show that the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ can be regarded as a flat $\mathbb{E}_\infty$-algebra over $R^{\mathrm{un}}_{\mathbf{G}'_0}$. This will allow us to replace $\mathbf{G}_0$ by $\mathbf{G}'_0$, and thereby further reduce to the situation where $\mathbf{G}_0$ is a connected p-divisible group of some height $n > 0$. In this case, we can identify $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$ with the Lubin-Tate ring R_{LT} of the formal group $\mathbf{G}^\circ_0$. Roughly speaking, the idea is then to argue that the description of Theorem 6.0.3 must be correct because it becomes correct after completing at the closed point of $|\operatorname{Spec}(R_{\mathrm{LT}})|$ (by virtue of Theorem 5.4.1) and also after deleting the closed point of $|\operatorname{Spec}(R_{\mathrm{LT}})|$ (by repeating the above analysis to reduce to a statement about p-divisible groups of smaller height). To carry out the details of this argument, we will need to show that the higher homotopy groups of the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ are rationally trivial, which we prove in §6.3 using a similar strategy (see Theorem 6.3.1).

6.1 Flatness of Comparison Maps

In §3.1, we defined the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ of a nonstationary p-divisible group $\mathbf{G}_0$ over an F-finite $\mathbf{F}_p$-algebra R_0 (Definition 3.1.11). Our goal in this section is to study the behavior of spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ as we vary the commutative ring R_0. We begin with a simple observation:

Lemma 6.1.1. *Let $f : R_0 \to R'_0$ be a morphism of commutative rings in which p is nilpotent, and assume that the absolute cotangent complexes L_{R_0} and $L_{R'_0}$ are almost perfect. Let $\mathbf{G}_0$ be a nonstationary p-divisible group over R_0, and let $\mathbf{G}'_0 = (\mathbf{G}_0)_{R'_0}$ denote the p-divisible group obtained from $\mathbf{G}_0$ by extending scalars along f. Then $\mathbf{G}'_0$ is nonstationary if and only if the module of Kähler differentials $\Omega_{R'_0/R_0}$ vanishes.*

Proof. Since $L_{\operatorname{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}}}$ is 1-connective (Remark 3.4.4), the fiber sequence

$$R' \otimes_R L_{\operatorname{Spec}(R_0)/\mathcal{M}_{\mathrm{BT}}} \to L_{\operatorname{Spec}(R'_0)/\mathcal{M}_{\mathrm{BT}}} \to L_{R'_0/R_0}$$

induces an isomorphism $\pi_0 L_{\mathrm{Spec}(R'_0)/\mathcal{M}_{\mathrm{BT}}} \simeq \pi_0 L_{R'_0/R_0} = \Omega_{R'_0/R_0}$. It follows that $L_{\mathrm{Spec}(R'_0)/\mathcal{M}_{\mathrm{BT}}}$ is 1-connective if and only if $\Omega_{R'_0/R_0}$ vanishes, so the desired result follows from Remark 3.4.4. $\square$

In the situation of Lemma 6.1.1, composition with f induces a natural transformation of deformation functors $u : \mathrm{Def}_{\mathbf{G}'_0} \to \mathrm{Def}_{\mathbf{G}_0}$. If $\Omega_{R'_0/R_0}$ vanishes, then Theorem 3.4.1 and Lemma 6.1.1 imply that the functors $\mathrm{Def}_{\mathbf{G}_0}$ and $\mathrm{Def}_{\mathbf{G}_0}$ are corepresented by spectral deformation rings $R^{\mathrm{un}}_{\mathbf{G}_0}$ and $R^{\mathrm{un}}_{\mathbf{G}'_0}$, respectively. The natural transformation u is then classified by a map $R^{\mathrm{un}}_{\mathbf{G}_0} \to R^{\mathrm{un}}_{\mathbf{G}'_0}$. Our goal in this section is to prove the following:

Theorem 6.1.2. *Let $f : R_0 \to R'_0$ be a morphism of F-finite Noetherian $\mathbf{F}_p$-algebras. Let $\mathbf{G}_0$ be a nonstationary p-divisible group over R_0, let $\mathbf{G}'_0$ denote its extension of scalars to R'_0, and assume that $\Omega_{R'_0/R_0}$ vanishes (so that $\mathbf{G}'_0$ is also nonstationary). Then the induced map of spectral deformation rings $R^{\mathrm{un}}_{\mathbf{G}_0} \to R^{\mathrm{un}}_{\mathbf{G}'_0}$ is flat.*

6.1.1 The Case of a Closed Immersion

Let us first describe an important special case of Theorem 6.1.2, in which the conclusion follows immediately from the definitions:

Example 6.1.3. Let R_0 be a commutative ring and let $\mathbf{G}_0$ be a p-divisible group over R_0. Suppose we are given an ideal $I \subseteq R_0$, and let $\mathbf{G}'_0 = (\mathbf{G}_0)_{R_0/I}$ be the induced p-divisible group over R_0/I. For any complete adic $\mathbb{E}_\infty$-ring A, the we have a pullback diagram of spaces

$$\begin{tikzcd}
\mathrm{Def}_{\mathbf{G}'_0}(A) \arrow[r] \arrow[d] & \mathrm{Def}_{\mathbf{G}_0}(A) \arrow[d] \\
\varinjlim_J \mathrm{Hom}(R_0/I, \pi_0(A)/J) \arrow[r] & \varinjlim_J \mathrm{Hom}(R_0, \pi_0(A)/J).
\end{tikzcd}$$

Let us now suppose that the hypotheses of Theorem 3.4.1 are satisfied, so that $\mathbf{G}_0$ admits a spectral deformation ring $R = R^{\mathrm{un}}_{\mathbf{G}_0}$. Assume further that the ideal I is finitely generated, so that the inverse image of I is a finitely generated ideal $\overline{I} \subseteq \pi_0(R)$. The above description shows that, for every complete adic $\mathbb{E}_\infty$-ring A, we can identify $\mathrm{Def}_{\mathbf{G}'_0}(A)$ with the summand of $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}}(R, A) \simeq \mathrm{Def}_{\mathbf{G}_0}(A)$ spanned by those maps $R \to A$ for which the underlying ring homomorphism $u : \pi_0(R) \to \pi_0(A)$ is continuous when we equip $\pi_0(R)$ with the $\overline{I}$-adic topology. It follows that the functor $\mathrm{Def}_{\mathbf{G}'_0}$ is corepresented by the $\overline{I}$-completion $R^{\wedge}_{\overline{I}}$ (regarded as an adic $\mathbb{E}_\infty$-ring with the $\overline{I}$-adic topology).

In the special case where R_0 is an F-finite Noetherian $\mathbf{F}_p$-algebra, the spectral deformation ring R is Noetherian (Theorem 3.4.1), so that $R^{\wedge}_{I}$ is flat over R by virtue of Corollary SAG.7.3.6.9 .

Example 6.1.4. Let R_0 be an F-finite Noetherian $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a nonstationary p-divisible group over R_0. Let $I \subseteq R_0$ be an ideal, and set $\mathbf{G}'_0 = (\mathbf{G}_0)_{R_0/I}$. If R_0 is I-complete, then the induced map of spectral deformation rings $R^{\mathrm{un}}_{\mathbf{G}_0} \to R^{\mathrm{un}}_{\mathbf{G}'_0}$ is an equivalence at the level of $\mathbb{E}_\infty$-rings (this follows from Example 6.1.3 and Remark SAG.7.3.6.8). Beware that it usually not not an equivalence of *adic* $\mathbb{E}_\infty$-rings (this happens if and only if the ideal I is nilpotent): that is, the commutative rings $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$ and $\pi_0(R^{\mathrm{un}}_{\mathbf{G}'_0})$ are equipped with different topologies.

The following result supplies some examples of situations in which Example 6.1.4 can be applied:

Proposition 6.1.5. *Let R_0 be an F-finite Noetherian $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a nonstationary p-divisible group over R_0. Set $R = R^{\mathrm{un}}_{\mathbf{G}_0}$ and let $\mathbf{G} \in \mathrm{BT}^p(R)$ be the universal deformation of $\mathbf{G}_0$. Then:*

(a) *The commutative ring $R_1 = \pi_0(R)/(p)$ is Noetherian and F-finite.*

(b) *The p-divisible group $\mathbf{G}_{R_1} \in \mathrm{BT}^p(R_1)$ is nonstationary.*

(c) *The natural map of spectral deformation rings $R^{\mathrm{un}}_{\mathbf{G}_1} \to R^{\mathrm{un}}_{\mathbf{G_0}} = R$ is an equivalence in the ∞-category of $\mathbb{E}_\infty$-rings (though generally not as a morphism of adic $\mathbb{E}_\infty$-rings).*

Proof. The natural map $R \to R_0$ extends to a surjective ring homomorphism $\rho : R_1 \to R_0$, and Theorem 3.4.1 guarantees that R_1 is complete with respect to the ideal $I = \ker(\rho)$. Assertion (a) follows from Proposition 3.3.9. To prove (b), we note that the relative cotangent complex $L_{\mathrm{Spec}(R_1)/\mathcal{M}_{\mathrm{BT}}}$ is almost perfect as an R_1-module; we wish to show that $L_{\mathrm{Spec}(R_1)/\mathcal{M}_{\mathrm{BT}}}$ is 1-connective (Remark 3.4.4). By Nakayama's lemma, it will suffice to show that the tensor product $\kappa \otimes_{R_1} L_{\mathrm{Spec}(R_1)/\mathcal{M}_{\mathrm{BT}}}$ is 1-connective, whenever κ is the residue field of R_1 at a maximal ideal $\mathfrak{m} \subseteq R_1$. Equivalently, we wish to show that every map of R_1-modules $u : L_{\mathrm{Spec}(R_1)/\mathcal{M}_{\mathrm{BT}}} \to \kappa$ is nullhomotopic.

Let $\rho : R_1 \to \kappa$ be the canonical quotient map. Then u determines a ring homomorphism

$$\overline{\rho} : R_1 \to \kappa[\epsilon]/(\epsilon^2) \qquad \overline{\rho}(x) = \rho(x) + \epsilon dx$$

having the property that the p-divisible group $(\mathbf{G}_1)_d$ is trivial as a first-order deformation of $(\mathbf{G}_1)_\kappa$. Since R_1 is I-complete, the maximal ideal $\mathfrak{m}$ contains I. It follows that the p-divisible group $(\mathbf{G}_1)_d$ admits a $\mathbf{G}_0$-tagging, in the sense of Definition 3.1.1; it is therefore classified by a map of adic $\mathbb{E}_\infty$-rings $\rho' : R \to \kappa[\epsilon]/(\epsilon^2)$. It follows immediately that ρ' is obtained by composing $\overline{\rho}$ with the canonical map $q : R \to R_1$. On the other hand, the fact that $(\mathbf{G}_1)_d$ is a *trivial* first order deformation of $(\mathbf{G}_1)_\kappa$ guarantees that ρ' factors through the subalgebra $\kappa \subseteq \kappa[\epsilon]/(\epsilon^2)$. Since the map q is surjective on π_0, we conclude that $\overline{\rho}$ also factors through κ: that is, the derivation $d : R_1 \to \kappa$ vanishes. This proves that u is nullhomotopic, as desired. $\square$

6.1.2 The General Case

We now turn to the proof of Theorem 6.1.2. Our strategy is to reduce to a general statement about $\mathbb{E}_\infty$-rings (Proposition 6.1.8).

Lemma 6.1.6. *Let $f : R \to R'$ be a morphism of (p)-complete Noetherian $\mathbb{E}_\infty$-rings, and assume that the commutative rings $\pi_0(R)/(p)$ and $\pi_0(R')/(p)$ are F-finite. Then the (p)-completion of the relative cotangent complex $L_{R'/R}$ is almost perfect as an R'-module.*

Proof. By virtue of Lemma SAG.8.1.2.3 , it will suffice to show that for every morphism of connective $\mathbb{E}_\infty$-rings $u : R' \to A$, where A is (p)-nilpotent, the tensor product $A \otimes_{R'} L_{R'/R}$ is almost perfect. By virtue of Proposition SAG.2.7.3.2 , we can assume that A is a commutative $\mathbf{F}_p$-algebra. Without loss of generality, we can assume that $A = \pi_0(R')/(p)$. In this case, the map u factors through the relative tensor product $(\pi_0(R)/(p)) \otimes_R R'$. We may therefore replace R by $\pi_0(R)/(p)$ and thereby reduce to the case where R is a commutative $\mathbf{F}_p$-algebra and $A = \pi_0(R')$. Using the fiber sequence

$$A \otimes_{R'} L_{R'/R} \to L_{A/R} \to L_{A/R'},$$

we are reduced to showing that the relative cotangent complexes $L_{A/R}$ and $L_{A/R'}$ are almost perfect as modules over A. For the relative cotangent complex $L_{A/R}$, this follows from Theorem 3.3.1 (since R and A are Noetherian and F-finite). For the relative cotangent complex $L_{A/R'}$, we observe that $A \simeq \pi_0(R')$ is almost of finite presentation over R', since R' is Noetherian (Proposition HA.7.2.4.31). $\square$

Lemma 6.1.7. *In the situation of Theorem 6.1.2, the relative cotangent complex $L_{R^{\mathrm{un}}_{\mathbf{G}'_0}/R^{\mathrm{un}}_{\mathbf{G}_0}}$ is (p)-rational: that is, it vanishes after completion at (p).*

Proof. To simplify notation, set $R = R^{\mathrm{un}}_{\mathbf{G}_0}$ and $R' = R^{\mathrm{un}}_{\mathbf{G}'_0}$. Then R' is an adic $\mathbb{E}_\infty$-ring; let $I \subseteq \pi_0(R')$ be a finitely generated ideal of definition. Then I contains a power of p. Consequently, for any R'-module M, the canonical map from M to its (p)-completion $M^\wedge_{(p)}$ induces an equivalence of I-completions $M^\wedge_I \to (M^\wedge_{(p)})^\wedge_I$. In the special case $M = L_{R'/R}$, the (p)-completion $M^\wedge_{(p)}$ is almost perfect over R' (Proposition 6.1.5 and Lemma 6.1.6), and therefore already I-complete (since R' is I-complete; see Proposition SAG.7.3.5.7). It follows that the (p)-completion of $L_{R'/R}$ can be identified with the I-completion of $L_{R'/R}$, which can be identified with the relative cotangent complex of the map of formal spectra $\mathrm{Spf}(R') \to \mathrm{Spf}(R)$ (see Example SAG.17.1.2.9). It will therefore suffice to show that $L_{\mathrm{Spf}(R')/\mathrm{Spf}(R)}$ vanishes. Using the fiber sequence

$$L_{\mathrm{Spf}(R)/\mathcal{M}_{\mathrm{BT}}}|_{\mathrm{Spf}(R')} \to L_{\mathrm{Spf}(R')/\mathcal{M}_{\mathrm{BT}}} \to L_{\mathrm{Spf}(R')/\mathrm{Spf}(R)},$$

we are reduced to proving that the relative cotangent complexes $L_{\mathrm{Spf}(R)/\mathcal{M}_{\mathrm{BT}}}$ and $L_{\mathrm{Spf}(R')/\mathcal{M}_{\mathrm{BT}}}$ vanish. This follows the description of $\mathrm{Spf}(R) \simeq \mathrm{Def}_{\mathbf{G}_0}$ and $\mathrm{Spf}(R') \simeq \mathrm{Def}_{\mathbf{G}'_0}$ as relative de Rham spaces (Proposition 3.4.3 and Corollary SAG.18.2.1.11). □

By virtue of Proposition 6.1.5 and Lemma 6.1.7, Theorem 6.1.2 is a consequence of the following algebraic assertion:

Proposition 6.1.8. *Let $u : R \to R'$ be a morphism of (p)-complete Noetherian $\mathbb{E}_\infty$-rings. If the relative cotangent complex $L_{R'/R}$ is (p)-rational, then u is flat.*

We will need the following general fact:

Lemma 6.1.9. *Let $f : R \to R'$ be a morphism of Noetherian $\mathbb{E}_\infty$-rings. Assume that R is complete with respect to an ideal $I \subseteq \pi_0(R)$ and that R' is complete with respect to $I\pi_0(R')$. The following conditions are equivalent:*

(a) *The morphism f is flat.*

(b) *The induced map $\pi_0(R)/I \to (\pi_0(R)/I) \otimes_R R'$ is flat.*

Proof. The implication $(a) \Rightarrow (b)$ is clear. Assume that (b) is satisfied; we wish to prove that f is flat. Proceeding by induction on the number of generators of I, we can reduce to the case where $I = (x)$ is a principal ideal. Let M be a discrete R-module; we wish to show that $M \otimes_R R'$ is also discrete. Writing M as a filtered colimit of finitely generated R-modules, we can assume that M is finitely generated as a module over $\pi_0(R)$. We then have an exact sequence $0 \to M' \to M \to M'' \to 0$, where the action of x on M' is locally nilpotent and M'' has no x-torsion. We will complete the proof by showing that $M' \otimes_R R'$ and $M'' \otimes_R R'$ are discrete.

We first show that $M' \otimes_R R'$ is discrete. Since R is Noetherian, M' is finitely generated as a module over $\pi_0(R)$, and therefore annihilated by x^k for $k \gg 0$. We can therefore write M' as a successive extension of finitely many discrete R-modules, each of which is annihilated by x. Consequently, it will suffice to show that the tensor product $N \otimes_R R'$ is discrete when N is a discrete R-module annihilated by x. In this case, N admits the structure of a module over $\pi_0(R)/I$, and the desired result follows from assumption (b).

We now show that $M'' \otimes_R R'$ is discrete. Note that M'' is almost perfect as an R-module (Proposition HA.7.2.4.17), so that $M'' \otimes_R R'$ is also finitely generated as an R', and therefore (x)-complete (Proposition SAG.7.3.5.7). It follows that $M'' \otimes_R R'$ can be written as the limit of the tower $\{(M''/x^j M'') \otimes_R R'\}_{j \geqslant 0}$. Consequently, to show that $M'' \otimes_R R'$ is discrete, it will suffice to show that each $(M''/x^j M'') \otimes_R R'$ is discrete, and that each of the transition maps $(M''/x^{j+1} M'') \otimes_R R' \to (M''/x^j M'') \otimes_R R'$ is surjective. Using the fiber sequence

$$(M''/xM'') \otimes_R R' \xrightarrow{x^j} (M''/x^{j+1} M'') \otimes_R R' \to (M''/x^j M'') \otimes_R R',$$

we are reduced to showing that the tensor product $(M''/xM'') \otimes_R R'$ is discrete, which follows immediately from (b). $\square$

Proof of Proposition 6.1.8. Let $u : R \to R'$ be as in the statement of Proposition 6.1.8; we wish to show that u is flat. By virtue of Lemma 6.1.9, we can replace R by $\pi_0(R)/(p)$ and R' by the tensor product $(\pi_0(R)/(p)) \otimes_R R'$ and thereby reduce to the case where R is a commutative $\mathbf{F}_p$-algebra. In this case, our assumption guarantees that the relative cotangent complex $L_{R'/R}$ vanishes, and the desired result follows from Proposition 3.5.5. $\square$

6.2 Serre-Tate Parameters

Let R_0 be a commutative ring and suppose we are given a short exact sequence

$$0 \to \mathbf{G}'_0 \to \mathbf{G}_0 \to \mathbf{G}''_0 \to 0$$

of p-divisible groups over R_0. In this section, we will study the relationship between the deformation theories of $\mathbf{G}'_0$ and $\mathbf{G}_0$, under the assumption that $\mathbf{G}''_0$ is étale.

6.2.1 Deformations of Short Exact Sequences

We begin by introducing some terminology.

Notation 6.2.1. Let R be an $\mathbb{E}_\infty$-ring. We let $\mathcal{M}^{\mathrm{ex}}_{\mathrm{BT}}(R)$ denote the subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{BT}^p(R))$ whose objects are short exact sequences of p-divisible groups

$$\begin{tikzcd} \mathbf{G}' \arrow[r] \arrow[d] & \mathbf{G} \arrow[d] \\ 0 \arrow[r] & \mathbf{G}'' \end{tikzcd}$$

in the sense of Definition 2.4.9, and whose morphisms are equivalences.

Suppose we are given a commutative ring R_0 and a short exact sequence σ_0 :

$$\begin{tikzcd} \mathbf{G}'_0 \arrow[r] \arrow[d] & \mathbf{G}_0 \arrow[d] \\ 0 \arrow[r] & \mathbf{G}''_0 \end{tikzcd}$$

of p-divisible groups over R_0. For every complete adic $\mathbb{E}_\infty$-ring A, we let $\mathbf{B}_{\sigma_0}(A)$ denote the direct limit

$$\varinjlim_I (\mathcal{M}^{\mathrm{ex}}_{\mathrm{BT}}(A) \times_{\mathcal{M}^{\mathrm{ex}}_{\mathrm{BT}}(\pi_0(A)/I)} \mathrm{Hom}(R, \pi_0(A)/I)),$$

where I ranges over all finitely generated ideals of definition for $\pi_0(A)$.

Proposition 6.2.2. *Let R_0 be a commutative ring and let σ_0 :*

$$\begin{tikzcd} \mathbf{G}'_0 \arrow[r] \arrow[d] & \mathbf{G}_0 \arrow[d] \\ 0 \arrow[r] & \mathbf{G}''_0 \end{tikzcd}$$

be a short exact sequence of p-divisible groups over R_0. Assume that $\mathbf{G}''_0$ is étale. Then the construction

$$(0 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 0) \mapsto \mathbf{G}$$

induces an equivalence of deformation functors $\mathrm{Def}_{\sigma_0} \to \mathrm{Def}_{\mathbf{G}_0}$

Proof. Combine Proposition 2.4.8, Remark 2.5.11, and Proposition 3.1.18. □

Remark 6.2.3. Proposition 6.2.2 can be stated more informally as follows: if a p-divisible group $\mathbf{G}_0$ fits into an exact sequence

$$0 \to \mathbf{G}'_0 \to \mathbf{G}_0 \to \mathbf{G}''_0 \to 0$$

where $\mathbf{G}''_0$ is étale, then giving a deformation of $\mathbf{G}_0$ is equivalent to giving a deformation of the entire short exact sequence.

Note that the construction

$$(0 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 0) \mapsto \mathbf{G}'$$

determines another natural transformation of deformation functors $\mathrm{Def}_{\sigma_0} \to \mathrm{Def}_{\mathbf{G}'_0}$. Composing this natural transformation with the inverse of the equivalence $\mathrm{Def}_{\sigma_0} \to \mathrm{Def}_{\mathbf{G}_0}$, we obtain a forgetful functor $\mathrm{Def}_{\mathbf{G}_0} \to \mathrm{Def}_{\mathbf{G}'_0}$ (under the assumption that $\mathbf{G}''_0$ is étale).

6.2.2 The Main Theorem

The main result of this section can be stated as follows:

Theorem 6.2.4. *Let R_0 be a Noetherian $\mathbf{F}_p$-algebra which is F-finite, and suppose we are given a short exact sequence of p-divisible groups*

$$0 \to \mathbf{G}'_0 \to \mathbf{G}_0 \to \mathbf{G}''_0 \to 0$$

where $\mathbf{G}'_0$ is nonstationary and $\mathbf{G}''_0$ is étale. Then:

(*a*) *The p-divisible group $\mathbf{G}_0$ is also nonstationary.*

(*b*) *The natural transformation of deformation functors $\mathrm{Def}_{\mathbf{G}_0} \to \mathrm{Def}_{\mathbf{G}'_0}$ induces a flat morphism of spectral deformation rings $R^{\mathrm{un}}_{\mathbf{G}'_0} \to R^{\mathrm{un}}_{\mathbf{G}_0}$.*

Corollary 6.2.5. *Let R_0 be a Noetherian $\mathbf{F}_p$-algebra which is F-finite and let $\mathbf{G}_0$ be a nonstationary p-divisible group over R_0. Assume that $\mathbf{G}_0$ is ordinary: that is, for every residue field κ of R_0, the p-divisible group $(\mathbf{G}_0)_\kappa$ fits into a short exact sequence*

$$0 \to \mathbf{G}' \to (\mathbf{G}_0)_\kappa \to \mathbf{G}'' \to 0,$$

where $\mathbf{G}''$ is étale and the Cartier dual of $\mathbf{G}'$ is étale. Then the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is flat over the sphere spectrum S.

Proof. It will suffice to show that, for each maximal ideal $\mathfrak{m} \subseteq \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$, the localization $(R^{\mathrm{un}}_{\mathbf{G}_0})_{\mathfrak{m}}$ is flat over S. Note that $\mathfrak{m}$ is the inverse image of a maximal ideal $\mathfrak{m}_0 \subseteq R_0$. Let κ denote the residue field $R_0/\mathfrak{m}_0 \simeq \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})/\mathfrak{m}$, and let R' denote the spectral deformation ring of $(\mathbf{G}_0)_\kappa$. It follows from Theorem 6.1.2 that the natural map $R^{\mathrm{un}}_{\mathbf{G}_0} \to R'$ is flat, and the quotient $\pi_0(R')/\mathfrak{m}\pi_0(R')$ is nonzero. It will therefore suffice to show that R' is flat over S. We may therefore replace R_0 by κ, and thereby reduce to the case where there exists a short exact sequence

$$0 \to \mathbf{G}'_0 \to \mathbf{G}_0 \to \mathbf{G}''_0 \to 0,$$

where $\mathbf{G}''_0$ and the Cartier dual of $\mathbf{G}'_0$ are étale. By virtue of Theorem 6.2.4, we can replace $\mathbf{G}_0$ by $\mathbf{G}'_0$ and thereby reduce to the case where $\mathbf{G}_0$ is Cartier dual to an étale p-divisible group. Using Remark 3.1.13 we can replace $\mathbf{G}_0$ by its Cartier dual, and thereby reduce to the case where $\mathbf{G}_0$ is étale. Applying Theorem 6.2.4 again, we can reduce to the case where $\mathbf{G}_0 = 0$. In this case, the p-divisible group $\mathbf{G}_0$ is defined over $\mathbf{F}_p$, so we can use Theorem 6.1.2 to reduce to the case $R_0 = \mathbf{F}_p$. The desired result now follows from Proposition 3.1.18. □

6.2.3 Extensions of p-Divisible Groups

We begin by introducing some terminology which will be useful in the proof of Theorem 6.2.4.

Notation 6.2.6. Let R be an $\mathbb{E}_\infty$-ring. We note that the construction $(\mathbf{G}' \to \mathbf{G} \to \mathbf{G}'') \mapsto (\mathbf{G}', \mathbf{G}'')$ determines a map of spaces $\mathcal{M}^{\mathrm{ex}}_{\mathrm{BT}}(R) \to \mathrm{BT}^p(R)^{\simeq} \times \mathrm{BT}^p(R)^{\simeq}$. Given a pair of p-divisible groups $\mathbf{G}', \mathbf{G}'' \in \mathrm{BT}^p(R)$, we let $\mathrm{Ext}(\mathbf{G}'', \mathbf{G}')$ denote the fiber product

$$\mathcal{M}^{\mathrm{ex}}_{\mathrm{BT}}(R) \times_{\mathrm{BT}^p(R)^{\simeq} \times \mathrm{BT}^p(R)^{\simeq}} \{(\mathbf{G}', \mathbf{G}'')\}.$$

We will refer to $\mathrm{Ext}(\mathbf{G}'', \mathbf{G}')$ as the *space of extensions of* $\mathbf{G}''$ *by* $\mathbf{G}'$.

Notation 6.2.7. Let R be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. For each integer $k \geqslant 0$, we let $B\mathbf{G}[p^k] : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ denote the sheafification of the functor $A \mapsto \Sigma \mathbf{G}[p^k]$ with respect to the finite flat topology on $\mathrm{CAlg}^{\mathrm{cn}}_R$. Note that the 0th space $\Omega^\infty B\mathbf{G}[p^k](A)$ can be identified with the classifying space for $\mathbf{G}[p^k]$-torsors on $\mathrm{Spec}(A)$ (with respect to the finite flat topology). Note that since $\mathbf{G}[p^k]$ is finite flat commutative group scheme over R, every $\mathbf{G}[p^k]$-torsor in the fpqc topology is also a $\mathbf{G}[p^k]$-torsor in the finite flat topology (since it is split by itself); consequently, each of the functors $B\mathbf{G}[p^k]$ satisfies descent for the fpqc topology.

We define $B\mathbf{G} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ by the formula $B\mathbf{G}(A) = \varinjlim B\mathbf{G}[p^k](A)$. Alternatively, we can describe $B\mathbf{G}$ as the suspension of $\mathbf{G}$ in the prestable ∞-category $\mathcal{C} = \mathcal{S}\mathrm{hv}_{\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Nil}(p)}}(\mathrm{CAlg}_R^{\mathrm{cn}})$ of $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Nil}(p)}$-valued sheaves with respect to the finite flat topology on $\mathrm{CAlg}_R^{\mathrm{cn}}$.

Lemma 6.2.8. *Let R be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. Then the functor $B\mathbf{G} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ is cohesive.*

Proof. Writing $B\mathbf{G}$ as a filtered colimit of the functors $B\mathbf{G}[p^k]$, we are reduced to proving that each $B\mathbf{G}[p^k]$ is cohesive. Since the functor $\Omega^\infty : \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}} \to \mathcal{S}$ is conservative and preserves limits, it will suffice to show that the composite functor $(\Omega^\infty \circ B\mathbf{G}[p^k]) : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is cohesive. We now observe that this functor is an example of a *geometric stack* in the sense of Definition SAG.9.3.0.1 , and therefore cohesive by virtue of Example SAG.17.3.1.3 . $\square$

Construction 6.2.9. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R, which we regard as an object of the ∞-category $\mathcal{E} = \mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$. Note that the formula $\mathbf{G}(A) \simeq \varinjlim_{k \geqslant 0} \mathbf{G}[p^k](A)$ shows that $\mathbf{G}$ is $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$-valued sheaf with respect to the Zariski topology on $\mathrm{CAlg}_R^{\mathrm{cn}}$, so we have an equivalence

$$\mathrm{Map}_{\mathcal{E}}(\underline{\mathbf{Z}}, \mathbf{G}) \simeq \Omega^\infty \mathbf{G}(R),$$

where $\underline{\mathbf{Z}}$ denotes the constant sheaf introduced in Example 2.5.7. We let $\rho_{\mathbf{G}}$ denote the composite map

$$\begin{aligned} \Omega^\infty \mathbf{G}(R) &\xleftarrow{\sim} \mathrm{Map}_{\mathcal{E}}(\underline{\mathbf{Z}}, \mathbf{G}) \\ &= \mathrm{Map}_{\mathcal{E}}(\underline{\mathbf{Z}}, \Omega B\mathbf{G}) \\ &\simeq \mathrm{Map}_{\mathcal{E}}(\Sigma\underline{\mathbf{Z}}, B\mathbf{G}) \\ &\xrightarrow{\delta} \mathrm{Map}_{\mathcal{E}}(\underline{\mathbf{Q}_p / \mathbf{Z}_p}, B\mathbf{G}) \\ &\simeq \mathrm{Ext}(\mathbf{Q}_p / \mathbf{Z}_p, \mathbf{G}), \end{aligned}$$

where δ is given by composition with the boundary map $\underline{\mathbf{Q}_p / \mathbf{Z}_p} \to \Sigma\underline{\mathbf{Z}}$ associated to the short exact sequence of abelian groups

$$0 \to \mathbf{Z} \to \mathbf{Z}[1/p] \to \mathbf{Q}_p / \mathbf{Z}_p \to 0.$$

This map carries each point $q \in \Omega^\infty \mathbf{G}(R)$ to an extension of $\underline{\mathbf{Q}_p / \mathbf{Z}_p}$ by $\mathbf{G}$, which we will write as a short exact sequence of p-divisible groups

$$0 \to \mathbf{G} \to \mathbf{G}_q \to \underline{\mathbf{Q}_p / \mathbf{Z}_p} \to 0.$$

Remark 6.2.10. In the situation of Construction 6.2.9, we have a commutative diagram of fiber sequences

$$\begin{tikzcd} \underline{\mathbf{Z}} \arrow[r] \arrow[d, "q"] & \underline{\mathbf{Z}[1/p]} \arrow[r] \arrow[d] & \underline{\mathbf{Q}_p/\mathbf{Z}_p} \arrow[d, "\mathrm{id}"] \\ \mathbf{G} \arrow[r] & \mathbf{G}_q \arrow[r] & \underline{\mathbf{Q}_p/\mathbf{Z}_p}. \end{tikzcd}$$

Beware that the square on the left of this diagram is not quite a pushout square. However, it exhibits $\mathbf{G}_q$ as the $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn},\mathrm{Nil}(p)}$-valued sheaf on $\mathrm{CAlg}_R^{\mathrm{cn}}$ obtained by sheafifying the pushout

$$\underline{\mathbf{Z}[1/p]} \amalg_{\underline{\mathbf{Z}}} \mathbf{G}$$

with respect to the finite flat topology (beware that sheafification is necessary, unless the image of q in $\pi_0 \mathbf{G}(R)$ is divisible by arbitrarily high powers of p).

Remark 6.2.11. In the situation of Construction 6.2.9, $\rho_{\mathbf{G}'}$ is described as a composition of maps, all but one of which is a homotopy equivalence. It follows that $\rho_{\mathbf{G}}$ can be identified with map δ in the fiber sequence

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{E}}(\Sigma \underline{\mathbf{Z}}, B\mathbf{G}) \arrow[d, "\delta"] \\ \mathrm{Map}_{\mathcal{E}}(\underline{\mathbf{Q}_p/\mathbf{Z}_p}, B\mathbf{G}) \arrow[d] \\ \mathrm{Map}_{\mathcal{E}}(\underline{\mathbf{Z}[1/p]}, B\mathbf{G}). \end{tikzcd}$$

Using the observation that $B\mathbf{G}$ is a sheaf for the Zariski topology (since it is a filtered direct limit of functors $B\mathbf{G}[p^k]$, each of which is a sheaf for the fpqc topology), we can identify the third term in this fiber sequence with $\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}[1/p], B\mathbf{G}(R))$. It follows that It follows that $\rho_{\mathbf{G}}$ fits into a canonical fiber sequence

$$\Omega^\infty \mathbf{G}(R) \xrightarrow{\rho_{\mathbf{G}}} \mathrm{Ext}(\underline{\mathbf{Q}_p/\mathbf{Z}_p}, \mathbf{G}) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}[1/p], B\mathbf{G}(R)).$$

6.2.4 Classification of Extensions

Our next goal is to show that, under some mild assumptions, all extensions of the constant p-divisible group $\underline{\mathbf{Q}_p/\mathbf{Z}_p}$ can be obtained from Construction 6.2.9.

Proposition 6.2.12. *Let R be a connective $\mathbb{E}_\infty$-ring, and let $\mathbf{G}$ be a p-divisible group over R. Assume that R is truncated and that p is nilpotent in $\pi_0(R)$. Then, for every nilpotent ideal $I \subseteq \pi_0(R)$, the canonical map*

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}[1/p], B\mathbf{G}(R)) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}[1/p], B\mathbf{G}(\pi_0(R)/I))$$

is a homotopy equivalence.

Proof. We proceed as in the proof of Lemma 2.3.24. Choose an integers k and m such that $I^k = 0$ and R is m-truncated. Then the canonical map $R \to \pi_0(R)/I$ factors as a composition of square-zero extensions

$$R = \tau_{\leqslant m} R \to \tau_{\leqslant m-1} R \to \cdots \to \tau_{\leqslant 0} R = \pi_0(R)/I^k \to \pi_0(R)/I^{k-1} \to \cdots \to \pi_0(R)/I.$$

It will therefore suffice to prove:

- $(*)$ Let $f : A \to B$ be a morphism in $\mathrm{CAlg}_R^{\mathrm{cn}}$ which exhibits A as a square-zero extension of B by $\Sigma^k(M)$, where M is a discrete B-module. Then the canonical map $\rho : \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}[1/p], B\mathbf{G}(A)) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}[1/p], B\mathbf{G}(B))$ is a homotopy equivalence.

To prove $(*)$, we choose a pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ B \arrow[r] & B \oplus \Sigma^{k+1}(M). \end{tikzcd}$$

Since the functor $B\mathbf{G}$ is cohesive (Lemma 6.2.8), the map ρ is a pullback the map $\rho' : \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}[1/p], B\mathbf{G}(B)) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}[1/p], B\mathbf{G}(B \oplus \Sigma^{k+1}(M)))$. We show that ρ' is a homotopy equivalence.

For the rest of the proof, we regard $B \in \mathrm{CAlg}_R^{\mathrm{cn}}$ as fixed. For every connective B-module N, the canonical maps $B\mathbf{G}(B) \to B\mathbf{G}(B \oplus N) \to B\mathbf{G}(B)$ exhibit $B\mathbf{G}(B)$ as a direct summand of $B\mathbf{G}(B \oplus N)$. We will denote the auxiliary summand by $F(N) \in \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$. To show that ρ' is a homotopy equivalence, it will suffice to show that the mapping space $\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}[1/p], F(\Sigma^{k+1} M))$ is contractible. Note that this mapping space can be realized as the 0th space of the $\mathbf{Z}$-module spectrum given by the limit of the tower

$$\cdots \xrightarrow{p} F(\Sigma^{k+1}(M)) \xrightarrow{p} F(\Sigma^{k+1} M) \xrightarrow{p} F(\Sigma^{k+1} M)$$

where the transition maps are given by multiplication by p. Lemma 6.2.8 guarantees that the functor $B\mathbf{G}$ is cohesive (Lemma 6.2.8), so that the functor F is additive. In particular, multiplication by p on $F(\Sigma^{k+1}M)$ can be obtained by applying the functor F to the map $p : \Sigma^{k+1}M \to \Sigma^{k+1}M$. Our assumption that p is nilpotent in $\pi_0(R)$ guarantees that M is annihilated by p^i for $i \gg 0$. It follows that the preceding tower vanishes as a Pro-object of $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$, so that its inverse limit vanishes. □

Corollary 6.2.13. *Let R be a connective $\mathbb{E}_\infty$-ring, and let $\mathbf{G}$ be a p-divisible group over R. Assume that R is truncated and that p is nilpotent in $\pi_0(R)$. Then, for every nilpotent ideal $I \subseteq \pi_0(R)$, the diagram*

$$\begin{tikzcd} \Omega^\infty \mathbf{G}(R) \arrow[r, "\rho_{\mathbf{G}}"] \arrow[d] & \mathrm{Ext}(\underline{\mathbf{Q}_p / \mathbf{Z}_p}, \mathbf{G}) \arrow[d] \\ \Omega^\infty \mathbf{G}(\pi_0(R)/I) \arrow[r] & \mathrm{Ext}(\underline{\mathbf{Q}_p / \mathbf{Z}_p}, \mathbf{G}_{\pi_0(R)/I}) \end{tikzcd}$$

is a pullback square.

In the situation of Corollary 6.2.13, we can identify $\mathbf{G}^\circ(R)$ with the fiber product $\mathbf{G}(R) \times_{\mathbf{G}(\pi_0(R)/I)} \mathbf{G}^\circ(\pi_0(R)/I)$. We therefore obtain:

Corollary 6.2.14. *Let R be a connective $\mathbb{E}_\infty$-ring, and let $\mathbf{G}$ be a p-divisible group over R. Assume that R is truncated and that p is nilpotent in $\pi_0(R)$. Then, for every nilpotent ideal $I \subseteq \pi_0(R)$, we have a canonical pullback square*

$$\begin{tikzcd} \Omega^\infty \mathbf{G}^\circ(R) \arrow[r] \arrow[d] & \mathrm{Ext}(\underline{\mathbf{Q}_p / \mathbf{Z}_p}, \mathbf{G}) \arrow[d] \\ \Omega^\infty \mathbf{G}^\circ(\pi_0(R)/I) \arrow[r] & \mathrm{Ext}(\underline{\mathbf{Q}_p / \mathbf{Z}_p}, \mathbf{G}_{\pi_0(R)/I}) \end{tikzcd}$$

is a pullback square.

In the situation of Corollary 6.2.14, the functor $\Omega^\infty \mathbf{G}^\circ$ is locally almost of finite presentation (Corollary 1.5.20), so the direct limit $\varinjlim_I \Omega^\infty \mathbf{G}^\circ(\pi_0(R)/I) \simeq \Omega^\infty \mathbf{G}^\circ(R^{\mathrm{red}})$ is contractible (here we take I to range over the the collection of all nilpotent ideals in $\pi_0(R)$). We therefore obtain:

Corollary 6.2.15. *Let R be a connective $\mathbb{E}_\infty$-ring, and let $\mathbf{G}$ be a p-divisible group over R. Assume that R is truncated and that p is nilpotent in $\pi_0(R)$. Then we have a canonical fiber sequence*

$$\Omega^\infty \mathbf{G}^\circ(R) \to \mathrm{Ext}(\underline{\mathbf{Q}_p / \mathbf{Z}_p}, \mathbf{G}) \to \varinjlim_I \mathrm{Ext}(\underline{\mathbf{Q}_p / \mathbf{Z}_p}, \mathbf{G}_{\pi_0(R)/I}),$$

where the colimit is taken over all nilpotent ideals $I \subseteq \pi_0(R)$.

6.2.5 The Proof of Theorem 6.2.4

We will deduce Theorem 6.2.4 from a more precise statement in the special case where $\mathbf{G}_0$ splits as a direct sum $\mathbf{G}'_0 \oplus \underline{\mathbf{Q}_p / \mathbf{Z}_p}$.

Theorem 6.2.16. *Let R_0 be a commutative ring for which p is nilpotent and absolute cotangent complex L_{R_0} is almost perfect, and let $\mathbf{G}'_0$ be a nonstationary p-divisible group over R_0 and set $\mathbf{G}_0 = \mathbf{G}'_0 \oplus \underline{\mathbf{Q}_p / \mathbf{Z}_p}$. Let $\mathbf{G}^\circ$ denote the identity component of $\mathbf{G}$ and let $X = \Omega^\infty \mathbf{G}^\circ$ be its underlying formal hyperplane, which we view as an (affine) formal spectral Deligne-Mumford stack over $R^{\mathrm{un}}_{\mathbf{G}_0}$. Then there is a canonical pullback diagram of (affine) formal spectral Deligne-Mumford stacks*

$$\begin{tikzcd}
\mathrm{Spf}(R^{\mathrm{un}}_{\mathbf{G}_0}) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spf}(R^{\mathrm{un}}_{\mathbf{G}'_0}) \arrow[r] & \mathrm{Spec}(R^{\mathrm{un}}_{\mathbf{G}'_0}).
\end{tikzcd}$$

In particular, $R^{\mathrm{un}}_{\mathbf{G}_0}$ is equivalent to $\Gamma(X; \mathscr{O}_X)$ as an $\mathbb{E}_\infty$-ring (though not as an adic $\mathbb{E}_\infty$-ring).

Proof. To simplify notation, set $R = R^{\mathrm{un}}_{\mathbf{G}'_0}$. Let A be a connective $\mathbb{E}_\infty$-ring and suppose we are given a point $\eta \in \mathrm{Spf}(R)(A) \simeq \mathrm{Def}_{\mathbf{G}'_0}(A)$, corresponding to a p-divisible group $\mathbf{G}'$ over A equipped with a $\mathbf{G}'_0$-tagging (here we regard A as an adic $\mathbb{E}_\infty$-ring by endowing $\pi_0(A)$ with the discrete topology). We wish to construct a homotopy equivalence $X(A) \simeq \mathrm{Def}_{\mathbf{G}_0}(A) \times_{\mathrm{Def}_{\mathbf{G}'_0}(A)} \{\eta\}$, depending functorially on the pair (A, η). Without loss of generality, we may assume that A is truncated (since both sides are nilcomplete when regarded as functors of A). In this case, the desired homotopy equivalence is supplied by Corollary 6.2.15. $\square$

Proof of Theorem 6.2.4. Let R_0 be a Noetherian $\mathbf{F}_p$-algebra which is F-finite, and suppose we are given a short exact sequence of p-divisible groups

$$0 \to \mathbf{G}'_0 \to \mathbf{G}_0 \to \mathbf{G}''_0 \to 0$$

where $\mathbf{G}'_0$ is nonstationary and $\mathbf{G}''_0$ is étale. We first claim that $\mathbf{G}_0$ is also nonstationary. Let x be a point of $|\mathrm{Spec}(R_0)|$ and let $d : R_0 \to \kappa(x)$ be a nonzero derivation. Then d determines first-order deformations $\mathbf{G}_d$ and $\mathbf{G}'_d$ of $\mathbf{G}_{0\kappa(x)}$ and $\mathbf{G}'_{0\kappa(x)}$, respectively.

Suppose, for a contradiction, that $\mathbf{G}_d$ is a trivial first order deformation. Since $\mathbf{G}'_d$ is obtained from $\mathbf{G}_d$ by applying the construction of Remark 6.2.3, it follows that $\mathbf{G}'_d$ is also a trivial first-order deformation, contradicting our assumption that $\mathbf{G}'_0$ is nonstationary.

We wish to show that the map of spectral deformation rings $R^{\mathrm{un}}_{\mathbf{G}'_0} \to R^{\mathrm{un}}_{\mathbf{G}_0}$ is flat. Fix a maximal ideal $\mathfrak{m} \subseteq \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$; we will show that the induced map $R^{\mathrm{un}}_{\mathbf{G}'_0} \to (R^{\mathrm{un}}_{\mathbf{G}_0})_{\mathfrak{m}}$ is flat. The maximality of $\mathfrak{m}$ guarantees that it the inverse image of a maximal ideal $\mathfrak{m}_0 \subseteq R_0$. Let κ be an algebraic closure of the residue field $R_0/\mathfrak{m}_0$. Extending scalars to κ, we obtain a short exact sequence $0 \to \mathbf{G}'_{0\kappa} \to \mathbf{G}_{0\kappa} \to \mathbf{G}''_{0\kappa} \to 0$ in $\mathrm{BT}^p(\kappa)$. Applying the constructions of §6.1, we obtain a commutative diagram of spectral deformation rings:

$$\begin{tikzcd} R^{\mathrm{un}}_{\mathbf{G}'_0} \arrow[r] \arrow[d] & R^{\mathrm{un}}_{\mathbf{G}'_{0\kappa}} \arrow[d] \\ R^{\mathrm{un}}_{\mathbf{G}_0} \arrow[r] & R^{\mathrm{un}}_{\mathbf{G}_{0\kappa}} \end{tikzcd}$$

where the horizontal maps are flat (Theorem 6.1.2) and $\mathfrak{m}$ is the inverse image of the maximal ideal of $R^{\mathrm{un}}_{\mathbf{G}_{0\kappa}}$. It will therefore suffice to show that right vertical map is flat: that is, we can replace R_0 by κ and thereby reduce to the case where $R_0 = \kappa$ is an algebraically closed field. In this case, the étale p-divisible group $\mathbf{G}''_0$ is isomorphic to $\underline{\mathbf{Q}_p/\mathbf{Z}_p}^r$ for some $r \geqslant 0$, and the short exact sequence

$$0 \to \mathbf{G}'_0 \to \mathbf{G}_0 \to \mathbf{G}''_0 \to 0$$

splits (Remark 2.5.24). Proceeding by induction on r, we can reduce to the case $r = 1$, so that we have a direct sum decomposition $\mathbf{G}_0 \simeq \mathbf{G}'_0 \oplus \underline{\mathbf{Q}_p/\mathbf{Z}_p}$. Set $R = R^{\mathrm{un}}_{\mathbf{G}'_0}$ and let $\mathbf{G}' \in \mathrm{BT}^p(R)$ be the universal deformation of $\mathbf{G}'_0$. Then R is a local Noetherian $\mathbb{E}_\infty$-ring with residue field κ, so the formal hyperplane $X = \Omega^\infty \mathbf{G}'^\circ$ can be written as $\mathrm{cSpec}(C)$, where C is a standard smooth coalgebra over R. It follows from Theorem 6.2.16 that the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ can be identified with the R-linear dual $C^\vee$. It follows from Proposition 1.4.10 that the homotopy ring $\pi_*(R^{\mathrm{un}}_{\mathbf{G}_0})$ is isomorphic to $\pi_*(R)[[t_1, \ldots, t_n]]$, where n is the dimension of $\mathbf{G}'_0$. Since R is Noetherian, it follows immediately that $R^{\mathrm{un}}_{\mathbf{G}_0}$ is flat over R. $\square$

6.3 Rational Homotopy Groups of $R^{\mathrm{un}}_{\mathbf{G}_0}$

Our goal in this section is to prove the following:

Theorem 6.3.1. *Let R_0 be a F-finite Noetherian $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a nonstationary p-divisible group over R_0 with spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$. If the identity component $\mathbf{G}_0^\circ$ is 1-dimensional, then $R^{\mathrm{un}}_{\mathbf{G}_0}[p^{-1}]$ is a discrete $\mathbb{E}_\infty$-ring.*

Our proof of Theorem 6.3.1 will proceed by induction on the height n of the p-divisible group $\mathbf{G}_0$. When $n = 1$, the desired result follows from Corollary 6.2.5. To handle the general case, we will use Theorem 6.1.2 to reduce the proof of Theorem 6.3.1 to the case where $R_0 = \kappa$ is an algebraically closed field, and Theorem 6.2.4 to reduce to the case where the p-divisible group $\mathbf{G}_0$ is connected. Our inductive hypothesis then guarantees that any nonzero higher homotopy groups of $R^{\mathrm{un}}_{\mathbf{G}_0}[p^{-1}]$ must be concentrated "near" the closed point of $|\operatorname{Spec}(R^{\mathrm{un}}_{\mathbf{G}_0})|$ (in a suitable sense). For $n \geqslant 2$, we use a symmetry argument to show that this is impossible.

Remark 6.3.2. It seems likely that Theorem 6.3.1 is true even without the assumption that $\mathbf{G}_0^\circ$ is 1-dimensional. However, this would require a different proof.

We now carry out the details. We will prove the following special case of Theorem 6.3.1, using induction on n:

($*_n$) Let R_0 be a F-finite Noetherian $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a nonstationary p-divisible group of height n over R_0, with spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$. If the identity component $\mathbf{G}_0^\circ$ is 1-dimensional, then $R^{\mathrm{un}}_{\mathbf{G}_0}[p^{-1}]$ is a discrete $\mathbb{E}_\infty$-ring.

Note that assertion ($*_n$) is vacuous in the case $n = 0$, and the case $n = 1$ follows immediately from Corollary 6.2.5 (since the higher homotopy groups of the sphere spectrum S are torsion). Throughout this section, we will fix an integer $n \geqslant 2$, and assume that assertion ($*_m$) holds for $m < n$; our goal is to prove ($*_n$). The main step is to prove the following:

Lemma 6.3.3. *Let κ be an algebraically closed field of characteristic p, let $\mathbf{G}_0$ be a connected p-divisible group of height n over κ, let R_{LT} be the (classical) Lubin-Tate ring of $\mathbf{G}_0$. Let $\widetilde{\mathbf{G}}_0 \in \mathrm{BT}^p(R_{\mathrm{LT}})$ be the universal deformation of $\mathbf{G}_0$ in the sense of ordinary commutative algebra, and let $\mathbf{G}_1 \in \mathrm{BT}^p(R_{\mathrm{LT}}/(p))$ be the p-divisible group obtained from $\widetilde{\mathbf{G}}_0$ by reduction modulo p. Then:*

(a) *The $\mathbf{F}_p$-algebra $R_{\mathrm{LT}}/(p)$ is Noetherian and F-finite.*

(b) *The p-divisible group $\mathbf{G}_1$ is nonstationary, and therefore admits a spectral deformation ring $R = R^{\mathrm{un}}_{\mathbf{G}_1}$ and a universal deformation $\mathbf{G} \in \mathrm{BT}^p(R)$.*

(c) *The canonical map $R \to R^{\mathrm{un}}_{\mathbf{G}_0}$ is an equivalence of $\mathbb{E}_\infty$-rings (so that $\mathbf{G}$ can also be identified with the universal deformation of $\mathbf{G}_0$).*

(d) *The localization $R[p^{-1}]$ is discrete.*

Proof. Assertions (a), (b), and (c) follow from Proposition 6.1.5. We will prove (d). Fix an integer $k > 0$; we wish to show that the abelian group $\pi_k(R)[p^{-1}]$ vanishes. Since R is Noetherian, we can regard $M = \pi_k(R)$ as a finitely generated module over the commutative ring $\pi_0(R)$ (which is isomorphic to the Lubin-Tate ring R_{LT}, by virtue of (3)). The support of M is a Zariski-closed subset $K \subseteq |\operatorname{Spec}(R_{\mathrm{LT}})|$. Let $V \simeq |\operatorname{Spec}(R_{\mathrm{LT}}/(p))| \subseteq |\operatorname{Spec}(R_{\mathrm{LT}})|$ be the vanishing locus of p; we wish to show that $K \subseteq V$. Suppose otherwise. Then there exists some irreducible component $K' \subseteq K$ which is not contained in V. Note that the Krull dimension of K' must be positive (since V contains the closed point of $|\operatorname{Spec}(R_{\mathrm{LT}})|$. Let $\mathfrak{p} \subseteq R_{\mathrm{LT}}$ be the prime ideal corresponding to the generic point of K', and set $A = R_{\mathrm{LT}}/\mathfrak{p}$. Then A can be regarded as an algebra over the Witt vectors $W(\kappa)$, so that A/pA has the structure of a vector space over κ. We distinguish two cases:

(a) The dimension $\dim_\kappa(A/pA)$ is finite. Since A is (p)-complete, it follows that A is finitely generated as a $W(\kappa)$-module, so that $A[p^{-1}]$ is a finite extension field of $W(\kappa)[p^{-1}]$. Let $\Gamma = \operatorname{Aut}(\mathbf{G}_0)$ be the group of automorphisms of $\mathbf{G}_0$ in the category of p-divisible groups over κ. Then Γ acts on the spectral deformation ring R. This determines an action of Γ on the affine scheme $|\operatorname{Spec}(R_{\mathrm{LT}})|$ which preserves the closed subset $K \subseteq |\operatorname{Spec}(R_{\mathrm{LT}})|$. In particular, Γ acts on the set of irreducible components of K; let $\Gamma_0 \subseteq \Gamma$ be the subgroup which fixes the irreducible component $K' \subseteq K$. Then the action of Γ_0 on the Lubin-Tate ring R_{LT} fixes the prime ideal $\mathfrak{p}$, and therefore induces an action of Γ_0 on $A = R_{\mathrm{LT}}$. This action is trivial on the residue field κ, and therefore also on the subring $W(\kappa) \subseteq A$. Since the Galois group $\operatorname{Gal}(A[p^{-1}]/W(\kappa)[p^{-1}])$ is finite, there exists a finite-index subgroup $\Gamma_1 \subseteq \Gamma_0$ which acts trivially on A. We then obtain an action of Γ_1 on the p-divisible group $(\overline{\mathbf{G}}_0)_A$ in the category $\mathrm{BT}^p(A)$. This action is automatically faithful (since it is already faithful on the p-divisible group $\mathbf{G}_0$), and therefore restricts to a faithful action of Γ_1 on the identity component $(\overline{\mathbf{G}}_0^\circ)_A$. Since A is an integral domain, this induces a faithful action of Γ_1 on $\overline{\mathbf{G}}_0^\circ)_{A[p^{-1}]}$, which is a 1-dimensional formal group over a field $A[p^{-1}]$ of characteristic zero. It follows that Γ_1 acts faithfully on the Lie algebra of $\overline{\mathbf{G}}_0^\circ)_{A[p^{-1}]}$: that is, we have a monomorphism of groups $\Gamma_1 \hookrightarrow A[p^{-1}]^\times$. Since the group $A[p^{-1}]^{\times}$ is abelian, the group Γ_1 must also be abelian. In particular, the group $\Gamma = \mathrm{Aut}(\mathbf{G}_0)$ contains an abelian subgroup of finite index. However, the structure of Γ is well-understood: it is isomorphic to the group of units $\mathcal{O}_D^{\times}$, where D is a central division algebra over $\mathbf{Q}_p$ of Hasse invariant $1/n$, and $\mathcal{O}_D \subseteq D$ is its ring of integers. For $n \geqslant 2$, this is a contradiction: $\mathcal{O}_D^{\times}$ does not contain abelian subgroups of finite index.

(*b*) The dimension $\dim_{\kappa}(A/pA)$ is infinite. In this case, the ring A/pA is not Artinian. It follows that the intersection $V \cap K'$ contains a non-closed point $x \in |\operatorname{Spec}(R_{\mathrm{LT}})|$. Let $\kappa(x)$ denote the residue field of R_{LT} at the point x and let $R(x)$ denote the spectral deformation ring of $(\mathbf{G}_1)_{\kappa(x)}$. According to Corollary 4.4.25, the maximal ideal of the Lubin-Tate ring R_{LT} coincides with the nth Landweber ideal of the formal group $\widetilde{\mathbf{G}}_0^{\circ}$. Since x is not the closed point of $|\operatorname{Spec}(R_{\mathrm{LT}})|$, we conclude that the formal group $(\widetilde{\mathbf{G}}_0)^{\circ}_{\kappa(x)} \simeq (\mathbf{G}_1)^{\circ}_{\kappa(x)}$ has height $< n$. We can therefore choose an exact sequence of p-divisible groups

$$0 \to \mathbf{G}_1' \to (\mathbf{G}_1')_{\kappa(x)} \to \mathbf{G}_2'' \to 0$$

over $\kappa(x)$, where $\mathbf{G}_2''$ is étale and $\mathbf{G}_1'$ has height $< n$. Our inductive hypothesis then guarantees that the $\mathbb{E}_{\infty}$-ring $R^{\mathrm{un}}_{\mathbf{G}_1'}[p^{-1}]$ is discrete, and Theorem 6.2.4 supplies a flat map of spectral deformation rings $R^{\mathrm{un}}_{\mathbf{G}_1'} \to R(x)$. It follows that $R_x[p^{-1}]$ is also discrete. Since R_x is Noetherian, it follows that the homotopy group $\pi_k(R_x)$ is annihilated by p^a for some $a \gg 0$. On the other hand, Proposition 6.1.8 supplies a flat map of spectral deformation rings $u : R \to R(x)$. Let $\mathfrak{q}$ be the prime ideal of R_{LT} corresponding to the point x, so that u induces a faithfully flat map of local $\mathbb{E}_{\infty}$-rings $R_{\mathfrak{q}} \to R(x)$. Then we can identify $\pi_k(R_x)$ with the (non-derived) tensor product $\pi_k(R_{\mathfrak{q}}) \otimes_{\pi_0(R_{\mathfrak{q}})} \pi_0(R(x))$. Using the faithful flatness of $\pi_0(R(x))$ over $\pi_0(R_{\mathfrak{q}})$, we deduce that $\pi_k(R_{\mathfrak{q}}) \simeq M_{\mathfrak{q}}$ is annihilated by p^a. It follows that the localization of M at the generic point of K' is also annihilated by p^a and therefore vanishes (since $K' \nsubseteq V$), contradicting the definition of K'.

$\square$

Proof of Assertion $(*_n)$. Let R_0 be a F-finite Noetherian $\mathbf{F}_p$-algebra and let $\mathbf{G}_0$ be a 1-dimensional nonstationary p-divisible group over R_0, with spectral deformation ring $R = R^{\mathrm{un}}_{\mathbf{G}_0}$. Let $k > 0$ and set $M = \pi_k(R)$. Since R is Noetherian, the module M is finitely generated over $\pi_0(R)$; we wish to show that M is annihilated by some power of p. Equivalently, we wish to show that the localization $M_{\mathfrak{q}}$ vanishes, for any prime ideal $\mathfrak{q} \subseteq \pi_0(R)$ which does not contain p. Let $\mathfrak{m} \subseteq \pi_0(R)$ be a maximal ideal containing $\mathfrak{q}$. Then $\mathfrak{m}$ can be written as the inverse image of a maximal ideal $\mathfrak{m}_0 \subseteq R_0$. Let κ denote the residue field $R_0/\mathfrak{m}_0 \simeq \pi_0(R)/\mathfrak{m}$ and let $\overline{\kappa}$ be an algebraic closure of κ. Let R' denote the spectral deformation ring of the p-divisible group $(\mathbf{G}_0)_{\overline{\kappa}}$. Then R' is flat over R (Proposition 6.1.8), and therefore faithfully flat over the localization $R_{\mathfrak{m}}$. Consequently, to show that $M_{\mathfrak{q}}$ vanishes, it will suffice to show that the tensor product

$$M_{\mathfrak{q}} \otimes_{\pi_0(R_{\mathfrak{m}})} \pi_0(R') \simeq \pi_k(R')_{\mathfrak{q}}.$$

Since $\pi_k(R')_{\mathfrak{q}}$ is a localization of $\pi_k(R')[p^{-1}]$, we are reduced to showing that that the localization $\pi_k(R')[p^{-1}]$ vanishes. We may therefore replace $(R_0, \mathbf{G}_0)$ by $(\overline{\kappa}, (\mathbf{G}_0)_{\overline{\kappa}})$, and thereby reduce to the case where $R_0 = \overline{\kappa}$ is an algebraically closed field. In this case, the p-divisible group $\mathbf{G}_0$ admits a connected-étale sequence

$$0 \to \mathbf{G}_0' \to \mathbf{G}_0 \to \mathbf{G}_0'' \to 0,$$

(Proposition 2.5.20). If $\mathbf{G}_0'' = 0$, then $\mathbf{G}_0$ is connected and the desired result follows from Lemma 6.3.3. We may therefore assume that $\mathbf{G}_0''$ is nonzero, so that $\mathbf{G}_0'$ has height $h < n$. In this case, our inductive hypothesis $(*_h)$ guarantees that the localization $R^{\mathrm{un}}_{\mathbf{G}_0'}[p^{-1}]$ is discrete. Since R is flat over $R^{\mathrm{un}}_{\mathbf{G}_0'}$ (Theorem 6.2.4), it follows that $R[p^{-1}]$ is also discrete. $\square$

6.4 Proof of the Main Theorem

Let R_0 be a Noetherian $\mathbf{F}_p$-algebra which is F-finite, and let $\mathbf{G}_0$ be a nonstationary p-divisible group over R. Our goal in this section is to prove Theorem 6.0.3, which describes the homotopy groups of the oriented deformation ring $R^{\mathrm{or}}_{\mathbf{G}_0}$. By definition, $R^{\mathrm{or}}_{\mathbf{G}_0}$ is the orientation classifier of the formal group $\mathbf{G}^\circ$, where $\mathbf{G}$ denotes the universal deformation of $\mathbf{G}_0$. We begin by introducing the notion of a *balanced* formal group (Definition 6.4.1), and framing Theorem 6.0.3 as the statement that the formal group $\mathbf{G}^\circ$ is balanced. The class of balanced formal groups has good closure properties, which will allow us to use the results of §6.1 and §6.2 to reduce to the case where R_0 is a perfect field and the p-divisible group $\mathbf{G}_0$ is connected. In this case, we prove the desired result by combining Theorem 5.4.1 with an induction on the height of $\mathbf{G}_0$ (much like in the proof of Theorem 6.3.1).

6.4.1 Balanced Formal Hyperplanes

We begin by introducing some notations and auxiliary notions which will be useful in the proof of Theorem 6.0.3.

Definition 6.4.1. Let R be an connective $\mathbb{E}_\infty$-ring, let X be a 1-dimensional pointed formal hyperplane over R, and let $\mathfrak{O}_X$ denote the orientation classifier of X (Definition 4.3.14). We will say that X is *balanced* if the following conditions are satisfied:

- (a) The unit map $R \to \mathfrak{O}_X$ induces an isomorphism of commutative rings $\pi_0(R) \to \pi_0(\mathfrak{O}_X)$.
- (b) The homotopy groups of $\mathfrak{O}_X$ are concentrated in even degrees.

We will say that a 1-dimensional formal group $\widehat{\mathbf{G}}$ is *balanced* if the underlying pointed formal hyperplane $X = \Omega^\infty \widehat{\mathbf{G}}$ is balanced.

Remark 6.4.2. Let $f : R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings, let X be a 1-dimensional pointed formal hyperplane over R, and let X' be the pointed formal hyperplane over R' obtained by extending scalars along f. Let $\mathfrak{O}_X$ and $\mathfrak{O}_{X'}$ denote the orientation classifiers of X and X', respectively. It follows immediately from the definitions that we have a pushout diagram of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] & R' \arrow[d] \\ \mathfrak{O}_X \arrow[r] & \mathfrak{O}_{X'}. \end{tikzcd}$$

If f is flat, then the homotopy groups of $\mathfrak{O}_{X'}$ are given by

$$\pi_*(\mathfrak{O}_{X'}) \simeq \pi_*(\mathfrak{O}_X) \otimes_{\pi_0(R)} \pi_0(R').$$

In particular:

- (i) If f is flat and X is balanced, then X' is balanced.
- (ii) If f is faithfully flat and X' is balanced, then X is balanced.

Remark 6.4.3. Let R be a connective $\mathbb{E}_\infty$-ring and let X be a 1-dimensional pointed formal hyperplane over R. For each prime ideal $\mathfrak{p} \subseteq \pi_0(R)$, let $X_\mathfrak{p}$ denote the associated pointed formal hyperplane over the localization $R_\mathfrak{p}$. If X is balanced, then each localization $X_\mathfrak{p}$ is balanced (this is a special case of Remark 6.4.2). Conversely, if $X_\mathfrak{m}$ is balanced for each maximal ideal $\mathfrak{m} \subseteq \pi_0(R)$, then X is balanced.

6.4.2 Balanced Formal Hyperplanes over Q

When working over the rational numbers, it is easy to find balanced formal hyperplanes:

Proposition 6.4.4. *Let R be an ordinary commutative algebra over $\mathbf{Q}$ and let X be a 1-dimensional pointed formal hyperplane over R. Then X is balanced.*

Remark 6.4.5. Theorem 6.4.4 has a converse: if R is an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$ and there exists a pointed formal hyperplane over R which is balanced, then R must be discrete.

Proof of Proposition 6.4.4. The assertion can be tested Zariski-locally on $|\operatorname{Spec}(R)|$, so we may assume without loss of generality that $X = \operatorname{Spf}(R[[t]])$ is the underlying formal hyperplane of the formal multiplicative group $\widehat{\mathbf{G}}_m$. In this case, the $\mathbb{E}_\infty$-ring $C_*(\mathbf{CP}^\infty; R)[\beta^{-1}]$ is an orientation classifier for $\widehat{\mathbf{G}}$, where β denotes the canonical generator of the second homology group $\mathrm{H}_2(\mathbf{CP}^\infty; R)$ (in fact, an analogous statement is true over the sphere; see Corollary 4.3.27). We now conclude by observing that the homology ring $\mathrm{H}_*(\mathbf{CP}^\infty; R)$ is isomorphic to a divided power algebra on β, and therefore also to a polynomial algebra on β (since we have assumed that R is a $\mathbf{Q}$-algebra). $\square$

The proof of Proposition 6.4.4 suggests that it is unreasonable to hope for a formal group over an ordinary commutative ring R to be balanced, except in the case where R is a $\mathbf{Q}$-algebra.

6.4.3 The Proof of Theorem 6.0.3

We first note that Theorem 6.0.3 can be restated as follows:

Theorem 6.4.6. *Let R_0 be an F-finite Noetherian $\mathbf{F}_p$-algebra, let $\mathbf{G}_0$ be a nonstationary p-divisible group of dimension 1 over R_0, and let $\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}_0})$ be its universal deformation. Then the identity component $\mathbf{G}^\circ$ is a balanced formal group over $R^{\mathrm{un}}_{\mathbf{G}_0}$.*

We begin by proving Theorem 6.4.6 in the Lubin-Tate case:

Theorem 6.4.7. *Let κ be a perfect field of characteristic p, let $\mathbf{G}_0$ be a connected p-divisible group of height n over κ, and let $\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}_0})$ be its universal preoriented deformation. Then the identity component $\mathbf{G}^\circ$ is a balanced formal group over $R^{\mathrm{un}}_{\mathbf{G}_0}$.*

Remark 6.4.8. In the situation of Theorem 6.4.7, let $R_{\rm LT}$ denote the Lubin-Tate ring of the formal group $\mathbf{G}_0^{\circ}$. Theorem 6.4.7 asserts that the canonical map

$$R_{\rm LT} \simeq \pi_0(R^{\rm un}_{\mathbf{G}_0}) \to \pi_0(R^{\rm or}_{\mathbf{G}_0})$$

is an isomorphism of commutative rings, and that the homotopy groups of $R^{\rm or}_{\mathbf{G}_0}$ are concentrated in even degrees. Note that Theorem 5.4.1 (and Corollary 5.4.3) imply that the analogous statements holds if we replace $R^{\rm or}_{\mathbf{G}_0}$ by the Lubin-Tate spectrum $E = L_{K(n)} R^{\rm or}_{\mathbf{G}_0}$. Consequently, we can restate Theorem 6.4.7 as follows:

($*$) The canonical map $R^{\rm or}_{\mathbf{G}_0} \to L_{K(n)} R^{\rm or}_{\mathbf{G}_0}$ is an equivalence: that is, the oriented deformation ring $R^{\rm or}_{\mathbf{G}_0}$ is $K(n)$-local.

This is a special case of Corollary 6.0.6, but of course it would be circular to invoke Corollary 6.0.6 here (since it depends on Theorem 6.0.3, which has not yet been proved).

Proof of Theorem 6.4.7. We proceed by induction on the height n of $\mathbf{G}_0$. Let $R_{\rm LT} \simeq \pi_0(R^{\rm un}_{\mathbf{G_0}})$ denote the Lubin-Tate ring of the formal group $\mathbf{G}_0^{\circ}$. For each prime ideal $\mathfrak{p} \subseteq R_{\rm LT}$, let $\mathbf{G}^{\circ}_{\mathfrak{p}}$ denote the formal group over $(R^{\rm un}_{\mathbf{G}_0})_{\mathfrak{p}}$ obtained from $\mathbf{G}^{\circ}$ by extending scalars. We first prove the following:

(a) If $\mathfrak{p}$ is a non-maximal prime ideal of $R_{\rm LT}$, then the formal group $\mathbf{G}^{\circ}_{\mathfrak{p}}$ is balanced.

To prove (a), let κ' denote the fraction field of $R_{\rm LT}/\mathfrak{p}$. If κ' has characteristic zero, then the $\mathbb{E}_\infty$-ring $(R^{\rm un}_{\mathbf{G}_0})_{\mathfrak{p}}$ is a discrete $\mathbf{Q}$-algebra (Theorem 6.3.1), so the desired result is a special case of Theorem 6.4.4. Let us therefore assume that κ' has characteristic p. Let $\mathbf{G}_1$ denote the p-divisible group $\mathbf{G}_{R_{\rm LT}/(p)}$. Using Proposition 6.1.5, we see that $R_{\rm LT}/(p)$ is an F-finite Noetherian $\mathbf{F}_p$-algebra, that $\mathbf{G}_1$ is nonstationary, and that the canonical map $R^{\rm un}_{\mathbf{G}_1} \to R^{\rm un}_{\mathbf{G}_0}$ is an equivalence of $\mathbb{E}_\infty$-rings (though not of adic $\mathbb{E}_\infty$-rings; the topology on $\pi_0(R^{\rm pre}_{\mathbf{G}_1})$ is defined by the ideal (p)). It will therefore suffice to show that $\mathbf{G}^{\circ}_{\mathfrak{p}}$ is balanced when viewed as a formal group over $R^{\rm un}_{\mathbf{G}_1}$. Let $\overline{\kappa}'$ be an algebraic closure of κ' and set $\mathbf{G}_2 = \mathbf{G}_{\overline{\kappa}'} \in \mathrm{BT}^p(\overline{\kappa}')$. Then the map $R_{\rm LT} \to \overline{\kappa'}$ lifts to a flat map of spectral deformation rings $\rho : R^{\rm un}_{\mathbf{G}_1} \to R^{\rm un}_{\mathbf{G}_2}$ (Theorem 6.1.2). Note that the preimage of the maximal ideal of $\pi_0(R^{\rm un}_{\mathbf{G}_2})$ is the prime ideal $\mathfrak{p} \subseteq \pi_0(R^{\rm un}_{\mathbf{G}_1})$, so that ρ induce a faithfully flat map $(R^{\rm un}_{\mathbf{G}_1})_{\mathfrak{p}} \to R^{\rm un}_{\mathbf{G}_2}$. Using Remark 6.4.2, we are reduced to showing that $\mathbf{G}^{\circ}_{R^{\rm un}_{\mathbf{G}_2}}$ is a balanced formal group over $R^{\rm un}_{\mathbf{G}_2}$. Note that the p-divisible group $\mathbf{G}_2$ admit a connected-étale sequence

$$0 \to \mathbf{G}'_2 \xrightarrow{i_0} \mathbf{G}_2 \to \mathbf{G}''_2 \to 0.$$

Let $R^{\mathrm{un}}_{\mathbf{G}'_2}$ be the spectral deformation ring of $\mathbf{G}'_2$ and let $\mathbf{G}' \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}'_2})$ be its universal deformation. Then we have a comparison map $u : R^{\mathrm{un}}_{\mathbf{G}'_2} \to R^{\mathrm{un}}_{\mathbf{G}_2}$, which is essentially characterized by the requirement that i_0 can be lifted to a monomorphism

$$\mathbf{G}'_{R^{\mathrm{un}}_{\mathbf{G}_2}} \to \mathbf{G}_{R^{\mathrm{un}}_{\mathbf{G}_2}}$$

of p-divisible groups over $R^{\mathrm{un}}_{\mathbf{G}_2}$. In particular, the formal group $\mathbf{G}^{\circ}_{R^{\mathrm{un}}_{\mathbf{G}_2}}$ can be obtained from the formal group $\mathbf{G}'^{\circ}$ by extension of scalars along u. Since u is flat (Theorem 6.2.4), it will suffice to show that the formal group $\mathbf{G}'^{\circ}$ is balanced (Remark 6.4.2). This follows from our inductive hypothesis, since we have assumed that $\mathfrak{p}$ is not the maximal ideal of R_{LT} and therefore the formal group $\mathbf{G}^{\circ}_2$ has height $< n$ (Corollary 4.4.25).

Let $E = L_{K(n)} R^{\mathrm{or}}_{\mathbf{G}_0}$ be the Lubin-Tate spectrum associated to the formal group $\mathbf{G}^{\circ}_0$. Combining (a) with Remark 6.4.8, we obtain the following:

(b) If $\mathfrak{p}$ is a non-maximal prime ideal of R_{LT}, then the canonical map $(R^{\mathrm{or}}_{\mathbf{G}_0})_{\mathfrak{p}} \to E_{\mathfrak{p}}$ is an equivalence of $\mathbb{E}_{\infty}$-rings.

Let $\mathfrak{m} \subseteq R_{\mathrm{LT}}$ be the maximal ideal. Let K denote the fiber of the natural map $f : R^{\mathrm{or}}_{\mathbf{G}_0} \to E$, which we view as a module over the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$. It follows from (b) that the localization $K_{\mathfrak{p}}$ vanishes for every non-maximal prime ideal $\mathfrak{p} \subseteq R_{\mathrm{LT}}$. Consequently, the module K is $\mathfrak{m}$-nilpotent: that is, every element of $\pi_*(K)$ is annihilated by some power of $\mathfrak{m}$. By construction, the map f exhibits E as the $\mathfrak{m}$-completion of A. In particular, the morphism f becomes an equivalence after $\mathfrak{m}$-completion, so that K vanishes after $\mathfrak{m}$-completion. On the other hand, $\mathfrak{m}$-completion induces an equivalence from the ∞-category of $\mathfrak{m}$-nilpotent $R^{\mathrm{un}}_{\mathbf{G}_0}$-modules to the ∞-category of $\mathfrak{m}$-complete $R^{\mathrm{un}}_{\mathbf{G}_0}$-modules (Proposition SAG.7.3.1.7). It follows that K itself must vanish, so that f is an equivalence. Invoking Remark 6.4.8, we deduce that $\mathbf{G}^{\circ}$ is balanced. $\square$

Proof of Theorem 6.4.6. Let R_0 be an F-finite Noetherian $\mathbf{F}_p$-algebra, let $\mathbf{G}_0$ be a nonstationary p-divisible group of dimension 1 over R_0, and let $\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}_0})$ be its universal deformation. We wish to show that the identity component $\mathbf{G}^{\circ}$ is a balanced formal group over $R^{\mathrm{un}}_{\mathbf{G}_0}$. By virtue of Remark 6.4.3, it will suffice to show that $\mathbf{G}^{\circ}_{\mathfrak{m}}$ is a balanced formal group over $(R^{\mathrm{un}}_{\mathbf{G}_0})_{\mathfrak{m}}$, for every maximal ideal $\mathfrak{m} \subseteq \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$. Note that since $R^{\mathrm{un}}_{\mathbf{G}_0}$ is complete with respect to the kernel of the map $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0}) \to R_0$, we can write $\mathfrak{m}$ as the inverse image of a maximal ideal $\mathfrak{m}_0 \subseteq R_0$.

Let κ be any perfect extension field of $R_0/\mathfrak{m}_0$, and let $\mathbf{G}_1 = (\mathbf{G}_0)_{\kappa}$ be the p-divisible group obtained from $\mathbf{G}_0$ by extending scalars to κ. Using Theorem 6.1.2, we obtain a flat map of spectral deformation rings $\rho : R^{\mathrm{un}}_{\mathbf{G}_0} \to R^{\mathrm{un}}_{\mathbf{G}_1}$. Moreover, the inverse image under ρ of the maximal ideal of $R^{\mathrm{un}}_{\mathbf{G}_1}$ is $\mathfrak{m}$, so that ρ induces a faithfully flat map $(R^{\mathrm{un}}_{\mathbf{G}_0})_{\mathfrak{m}} \to R^{\mathrm{un}}_{\mathbf{G}_1}$. By virtue of Remark 6.4.2, it will suffice to show that the formal group $\mathbf{G}^{\circ}_{R^{\mathrm{un}}_{\mathbf{G}_1}}$ is balanced. We may therefore replace $(R_0, \mathbf{G}_0)$ by $(\kappa, \mathbf{G}_1)$ and thereby reduce to proving Theorem 6.4.6 in the special case where $R_0 = \kappa$ is a perfect field of characteristic p. In this case, the p-divisible group $\mathbf{G}_0$ admits a connected-étale sequence

$$0 \to \mathbf{G}'_0 \to \mathbf{G}_0 \to \mathbf{G}''_0 \to 0.$$

Let $R^{\mathrm{un}}_{\mathbf{G}'_0}$ be the spectral deformation ring of $\mathbf{G}'_0$ and let $\mathbf{G}' \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}'_0})$ be its universal deformation. As in the proof of Theorem 6.4.7, we observe that the formal group $\mathbf{G}^{\circ}$ can be obtained from $\mathbf{G}'^{\circ}$ by extending scalars along a comparison map $u : R^{\mathrm{un}}_{\mathbf{G}'_0} \to R^{\mathrm{un}}_{\mathbf{G}_0}$. Since u is flat (Theorem 6.2.4), we are reduced to proving that the formal group $\mathbf{G}'^{\circ}$ is balanced (Remark 6.4.2), which follows from Theorem 6.4.7. $\square$

6.5 Application: Snaith's Theorem

Let $\mathrm{Vect}^{\simeq}_{\mathbf{C}}$ denote the category whose objects are finite-dimensional complex vector spaces and whose morphisms are isomorphisms, and let $\mathrm{N}(\mathrm{Vect}^{\simeq}_{\mathbf{C}})$ denote its nerve as a topologically enriched category (as in Construction 5.3.9). Then $\mathrm{N}(\mathrm{Vect}^{\simeq}_{\mathbf{C}})$ is a Kan complex, which can be identified with the disjoint union of classifying spaces $\mathrm{BU}(n)$ as n ranges over all nonnegative integers.

The formation of direct sums of complex vector spaces determines an $\mathbb{E}_\infty$-structure on the space $\mathrm{N}(\mathrm{Vect}^{\simeq}_{\mathbf{C}})$, which we will refer to as the *additive* $\mathbb{E}_\infty$*-structure*. The group completion of $\mathrm{N}(\mathrm{Vect}^{\simeq}_{\mathbf{C}})$ (with respect to the additive $\mathbb{E}_\infty$-structure) is the 0th space of a connective spectrum, which we denote by ku and refer to as the *connective complex* K*-theory spectrum*.

There is a second symmetric monoidal structure on the category $\mathrm{Vect}^{\simeq}_{\mathbf{C}}$, given by tensor products of complex vector spaces. This symmetric monoidal structure induces a different $\mathbb{E}_\infty$-structure on the space $\mathrm{N}(\mathrm{Vect}^{\simeq}_{\mathbf{C}})$, which we refer to as the *multiplicative* $\mathbb{E}_\infty$*-structure*. Because the tensor product of complex vector spaces distributes over direct sums, the multiplicative $\mathbb{E}_\infty$-structure distributes over the additive $\mathbb{E}_\infty$-structure. More precisely, it endows $\mathrm{N}(\mathrm{Vect}^{\simeq}_{\mathbf{C}})$ with the structure of a commutative algebra object of the ∞-category $\mathrm{CMon}(\mathcal{S})$ of $\mathbb{E}_\infty$-spaces, where we regard $\mathrm{CMon}(\mathcal{S})$ as equipped with the symmetric monoidal structure given by the smash product of $\mathbb{E}_\infty$-spaces (see Proposition AV.3.6.1). Put more informally, $\mathrm{N}(\mathrm{Vect}^{\simeq}_{\mathbf{C}})$ is an $\mathbb{E}_\infty$-semiring space, with addition given by direct sum of complex vector spaces and multiplication given by the tensor product. It follows that the connective complex K-theory spectrum ku inherits the structure of an $\mathbb{E}_\infty$-ring. Moreover, the tautological map $\xi : \mathrm{N}(\mathrm{Vect}^{\simeq}_{\mathbf{C}}) \to \Omega^\infty \mathrm{ku}$ can be regarded as a map of $\mathbb{E}_\infty$-spaces, where we endow both sides with the multiplicative $\mathbb{E}_\infty$-structure.

Let us identify $\mathbf{CP}^\infty \simeq \mathrm{BU}(1)$ with the summand of $\mathrm{N}(\mathrm{Vect}^{\simeq}_{\mathbf{C}})$ spanned by the complex vector spaces of dimension 1. Then the multiplicative $\mathbb{E}_\infty$-structure on $\mathrm{N}(\mathrm{Vect}^{\simeq}_{\mathbf{C}})$ restricts to an $\mathbb{E}_\infty$-structure on $\mathbf{CP}^\infty$, which is essentially unique (since $\mathbf{CP}^\infty \simeq K(\mathbf{Z}, 2)$ is an Eilenberg-MacLane space). Note that the map ξ carries $\mathbf{CP}^\infty$ into the identity component of $\Omega^\infty \mathrm{ku}$, and therefore induces a map of $\mathbb{E}_\infty$-spaces $\mathbf{CP}^\infty \to \mathrm{GL}_1(\mathrm{ku})$ which classifies a morphism of $\mathbb{E}_\infty$-rings $\rho : \Sigma^\infty_+(\mathbf{CP}^\infty) \to \mathrm{ku}$.

The composite map

$$S^2 \simeq \mathbf{CP}^1 \hookrightarrow \mathbf{CP}^\infty \xrightarrow{\rightharpoonup} \Omega^\infty \Sigma^\infty_+(\mathbf{CP}^\infty)$$

determines an element $\beta \in \pi_2(\Sigma^\infty_+ \mathbf{CP}^\infty)$, which we will refer to as the *Bott element*. We will generally abuse notation by identifying β with its image under the map $\pi_2(\Sigma^\infty_+(\mathbf{CP}^\infty) \xrightarrow{\rho} \pi_2(\mathrm{ku})$. Inverting β on both sides (see Proposition 4.3.17), we obtain a morphism of $\mathbb{E}_\infty$-rings $\Sigma^\infty_+(\mathbf{CP}^\infty)[\beta^{-1}] \to \mathrm{ku}[\beta^{-1}]$. We denote the localization $\mathrm{ku}[\beta^{-1}]$ by KU and refer to it as the *periodic complex K-theory spectrum*. The following result was proved in [34]:

Theorem 6.5.1 (Snaith)**.** *The map $\Sigma^\infty_+(\mathbf{CP}^\infty)[\beta^{-1}] \to \mathrm{KU}$ is an equivalence of $\mathbb{E}_\infty$-rings.*

Our goal in this section is to show that Theorem 6.5.1 is a formal consequence of Theorem 6.4.6, together with the classical Bott periodicity theorem. Our starting point is the following observation:

Proposition 6.5.2. *The formal multiplicative group $\widehat{\mathbf{G}}_m$ is a balanced formal group over the sphere spectrum S.*

Proof. By virtue of Remark 6.4.3, it will suffice to show that $\widehat{\mathbf{G}}_m$ is balanced when viewed as a formal group over the p-local sphere $S_{(p)}$, for every prime number p. Let $S^\wedge_{(p)}$ denote the (p)-completed sphere spectrum. Then the natural map $S_{(p)} \to S^\wedge_{(p)}$ is faithfully flat. By virtue of Remark 6.4.2, it will suffice to show that $\widehat{\mathbf{G}}_m$ is balanced when viewed as a formal group over $S^\wedge_{(p)}$. This is a special case of Theorem 6.4.6, since we can identify $S^\wedge_{(p)}$ with the spectral deformation ring of the p-divisible group $\mathbf{G}_0 = \mu_{p^\infty}$ over $\mathbf{F}_p$ (Corollary 3.1.19), and $\widehat{\mathbf{G}}_m$ with the identity component of its universal deformation (Proposition 2.2.12). $\square$

Corollary 6.5.3. *The Bott element* $\beta \in \pi_2(\Sigma^\infty_+(\mathbf{CP}^\infty))$ *induces an isomorphism of graded rings*

$$\mathbf{Z}[\beta^{\pm 1}] \to \pi_*(\Sigma^\infty_+(\mathbf{CP}^\infty)[\beta^{-1}]).$$

Proof. Combine Proposition 6.5.2 with Corollary 4.3.27. □

Proof of Theorem 6.5.1. Let us abuse notation by not distinguishing between the Bott element $\beta \in \pi_2(\Sigma^\infty_+(\mathbf{CP}^\infty)[\beta^{-1}])$ and its image in $\pi_2(\mathrm{ku})$. It follows from Bott periodicity that the element β induces an isomorphism of graded rings $\mathbf{Z}[\beta] \to \pi_*(\mathrm{ku})$, hence an isomorphism of localizations $\mathbf{Z}[\beta^{\pm 1}] \to \pi_*(\mathrm{ku}[\beta^{-1}]) = \pi_*(\mathrm{KU})$. We have a commutative diagram

$$\begin{tikzcd} & \mathbf{Z}[\beta^{\pm 1}] \arrow[dl] \arrow[dr] & \\ \pi_*(\Sigma^\infty_+(\mathbf{CP}^\infty)[\beta^{-1}]) \arrow[rr] & & \pi_*(\mathrm{KU}) \end{tikzcd}$$

where the vertical maps are isomorphisms (Theorem 6.5.3), so that the lower horizontal map is an isomorphism as well. □

7 Elliptic Cohomology

For every commutative ring R, let $\mathrm{Ell}(R)$ denote the category of elliptic curves over R and let $\mathrm{Ell}(R)^{\simeq}$ denote its underlying groupoid. The construction $R \mapsto \mathrm{Ell}(R)^{\simeq}$ is (representable by) a Deligne-Mumford stack $\mathcal{M}_{\mathrm{Ell}}$, which we will refer to as *the moduli stack of elliptic curves*. The étale topos of $\mathcal{M}_{\mathrm{Ell}}$ can be identified with the category of $\mathcal{S}$et-valued sheaves $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathcal{U})$, where the category $\mathcal{U}$ is defined as follows:

- The objects of $\mathcal{U}$ are pairs (R, E), where R is a commutative ring and E is an elliptic curve over R which is classified by an étale map $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{Ell}}$.
- A morphism from (R, E) to (R', E') in the category $\mathcal{U}$ is given by a pullback diagram of schemes σ :

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & E' \arrow[d] \\ \mathrm{Spec}(R) \arrow[r, "f_0"] & \mathrm{Spec}(R'), \end{tikzcd}$$

having the property that f carries the zero section of E into the zero section of E' (in other words, f induces an isomorphism $E \to \mathrm{Spec}(R) \times_{\mathrm{Spec}(R')} E'$ in the category $\mathrm{Ell}(R)$).

We regard the category $\mathcal{U}$ as equipped with the Grothendieck topology given by étale coverings: a collection of morphisms $\{(R_\alpha, E_\alpha) \to (R, E)\}$ in $\mathcal{E}$ is a covering if the underlying map of schemes $\amalg \mathrm{Spec}(R_\alpha) \to \mathrm{Spec}(R)$ is surjective (note that it is automatically étale, by virtue of (a)). Note that the structure sheaf $\mathscr{O}_{\mathcal{M}_{\mathrm{Ell}}}$ of the moduli stack of elliptic curves $\mathcal{M}_{\mathrm{Ell}}$ can be viewed as a sheaf of commutative rings on the category $\mathcal{U}$, given concretely by the formula $\mathscr{O}_{\mathcal{M}_{\mathrm{Ell}}}(R, E) = R$.

For any commutative ring R and any elliptic curve $E \in \mathrm{Ell}(R)$, we can construct a 1-dimensional formal group $\widehat{E}$ by formally completing E along its identity section. If the classifying map $f : \mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{Ell}}$ is flat, one can show that the formal group $\widehat{E}$ is Landweber-exact: that is, it satisfies the hypothesis of Theorem 0.0.1 if the formal group $\widehat{E}$ admits a coordinate, and a suitable generalization otherwise. It follows that there is an essential unique even periodic ring spectrum A_E equipped with isomorphisms $R \simeq \pi_0(A_E)$ and $\widehat{E} \simeq \mathrm{Spf}(A_E^0(\mathbf{CP}^\infty))$. The flatness hypothesis is automatically satisfied when f is étale: that is, when the pair (R, E) is an object of the category $\mathcal{U}$. The construction $(R, E) \mapsto A_E$ determines a functor

$$\mathscr{O}^h_{\mathcal{M}_{\mathrm{Ell}}} : \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{hSp}),$$

which we can view as a presheaf on $\mathcal{U}$ taking values in the homotopy category $\mathrm{CAlg}(\mathrm{hSp})$ of homotopy commutative ring spectra. This presheaf can be regarded as refinement of the structure sheaf $\mathscr{O}_{\mathcal{M}_{\mathrm{Ell}}}$: they are related by the formula $\mathscr{O}_{\mathcal{M}_{\mathrm{Ell}}} = \pi_0(\mathscr{O}^h_{\mathcal{M}_{\mathrm{Ell}}})$.

The category $\mathrm{CAlg}(\mathrm{hSp})$ is not well-behaved from a categorical point of view: for example, it has very few limits and colimits. Consequently, there is no good theory of $\mathrm{CAlg}(\mathrm{hSp})$-valued sheaves, so it is not really sensible to ask if the presheaf $\mathscr{O}^h_{\mathcal{M}_{\mathrm{Ell}}}$ is a sheaf. However, one can remedy the situation by replacing the ordinary category $\mathrm{CAlg}(\mathrm{hSp})$ of homotopy commutative ring spectra by the ∞-category $\mathrm{CAlg}(\mathrm{Sp})$ of $\mathbb{E}_\infty$-rings. Our goal in this section is to prove the following:

Theorem 7.0.1 (Goerss-Hopkins-Miller). *The functor $\mathscr{O}^h_{\mathcal{M}_{\mathrm{Ell}}} : \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{hSp})$ can be promoted to a functor $\mathscr{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}} : \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Sp}) = \mathrm{CAlg}$. Moreover, $\mathscr{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}$ is a CAlg-valued sheaf (with respect to the étale topology on the category $\mathcal{U}$).*

Remark 7.0.2. The work of Goerss-Hopkins-Miller actually proves something slightly stronger. Let Z denote the fiber product

$$\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Sp})) \times_{\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{hSp}))} \{\mathscr{O}^h_{\mathcal{M}_{\mathrm{Ell}}}\},$$

which we can think of as a classifying space for all lifts of $\mathscr{O}^h_{\mathcal{M}_{\mathrm{Ell}}}$ to a sheaf of $\mathbb{E}_\infty$-rings. One can show that the space Z is connected, so that the functor $\mathscr{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}$ of Theorem 7.0.1 exists and is unique up to homotopy (beware, however, that Z is not contractible). The connectedness of Z does not follow from our methods. Instead, our arguments will produce a contractible space Z' with a map $Z' \to Z$. In other words, we will construct a sheaf $\mathscr{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}$ which is *canonical* (up to contractible ambiguity, not just up to homotopy), but not obviously unique.

Definition 7.0.3 (Topological Modular Forms)**.** We let TMF denote the $\mathbb{E}_\infty$-ring of global sections of the sheaf $\mathscr{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}$, given concretely by the formula

$$\mathrm{TMF} = \varprojlim_{(R,E)\in\mathcal{U}} \mathscr{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}(R, E).$$

We will refer to TMF as the *periodic spectrum of topological modular forms*.

We now sketch our approach to Theorem 7.0.1. Let $\mathcal{X} = \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{U})$ denote the ∞-topos of $\mathcal{S}$-valued sheaves on $\mathcal{U}$. Given a sheaf $\mathscr{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}$ as in Theorem 7.0.1, we can view the pair $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}} = (\mathcal{X}, \mathscr{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}})$ as a *nonconnective spectral Deligne-Mumford stack* in the sense of Definition SAG.1.4.4.2 . Our strategy is to show that $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ arises naturally as the solution to a moduli problem in spectral algebraic geometry, just as $\mathcal{M}_{\mathrm{Ell}}$ arises naturally in classical algebraic geometry as a classifying object for elliptic curves.

In §AV.2 , we introduced the notion of a *strict elliptic curve* over an arbitrary $\mathbb{E}_\infty$-ring R (Definition AV.2.0.2). The strict elliptic curves over R form an ∞-category $\mathrm{Ell}^s(R)$, which agrees with $\mathrm{Ell}(R)$ when R is an ordinary commutative ring. Moreover, we proved that the construction $R \mapsto \mathrm{Ell}^s(R)^{\simeq}$ is (representable by) a spectral Deligne-Mumford stack $\mathcal{M}^s_{\mathrm{Ell}}$, which we will refer to as *the moduli stack of strict elliptic curves*. In §7.1, we show that every strict elliptic curve X of an $\mathbb{E}_\infty$-ring R admits a formal completion $\widehat{\mathsf{X}}$, which is a (1-dimensional) formal group over R. We use this observation in §7.2 to construct a variant $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ of $\mathcal{M}^s_{\mathrm{Ell}}$ which classifies *oriented* elliptic curves: that is, strict elliptic curves X together with an orientation of the formal group $\widehat{\mathsf{X}}$ (in the sense of Definition 4.3.9). The structure sheaf of $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ determines a sheaf $\mathscr{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}$ on $\mathcal{U}$ with values in the ∞-category $\mathrm{CAlg} = \mathrm{CAlg}(\mathrm{Sp})$ of $\mathbb{E}_\infty$-rings. To prove Theorem 7.0.1, it will suffice to show that the underlying presheaf of homotopy commutative ring spectra agrees with $\mathscr{O}^h_{\mathcal{M}_{\mathrm{Ell}}}$. In §7.3, we reduce this to the problem of showing that the formal group of the universal elliptic curve on $\mathcal{M}^s_{\mathrm{Ell}}$ is *balanced*, in the sense of Definition 6.4.1. We prove this in §7.4 by applying a version of the Serre-Tate theorem (Theorem AV.7.0.1) to reduce to an analogous statement for p-divisible groups, which follows from Theorem 6.4.6.

7.1 The Formal Group of a Strict Abelian Variety

We begin with some general remarks.

Construction 7.1.1 (Formal Completion)**.** Let R be an $\mathbb{E}_\infty$-ring and let X be a strict abelian variety over R (see Definition AV.1.5.1), which we identify with its functor of points $\mathsf{X} : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$. We define a new functor $\widehat{\mathsf{X}} : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ by the formula $\widehat{\mathsf{X}}(A) = \mathrm{fib}(\mathsf{X}(A) \to \mathsf{X}(A^{\mathrm{red}}))$, where the fiber is formed in the ∞-category $\mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$. We will refer to $\widehat{\mathsf{X}}$ as the *formal completion of* X.

Proposition 7.1.2. *Let R be an $\mathbb{E}_\infty$-ring and let X be a strict abelian variety over R. Then the formal completion $\widehat{\mathsf{X}}$ of Construction 7.1.1 is a formal group over R (in the sense of Definition 1.6.1 and Variant 1.6.2).*

The proof of Proposition 7.1.2 is based on the following completely elementary observation:

Lemma 7.1.3. *Let X be a strict abelian variety over a connective $\mathbb{E}_\infty$-ring R, and suppose we are given an A-valued point $\eta : \mathrm{Spec}(A) \to \mathsf{X}$ for some $A \in \mathrm{CAlg}^{\mathrm{cn}}_R$. The following conditions are equivalent:*

- (i) *The homotopy class of η belongs to the kernel of the map $\pi_0 X(A) \to \pi_0 X(A^{\mathrm{red}})$, where $X : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ denotes the functor represented by X.*
- (ii) *The underlying map of topological spaces $|\mathrm{Spec}(A)| \to |\mathsf{X}|$ factors through the closed subset $K \subseteq |\mathsf{X}|$ given by the image of the zero section $s : \mathrm{Spec}(R) \to \mathsf{X}$.*

Proof. Form a pullback diagram

$$\begin{tikzcd} \mathsf{Y} \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d, "\eta"] \\ \mathrm{Spec}(R) \arrow[r, "s"] & \mathsf{X}. \end{tikzcd}$$

Since X is separated, s is a closed immersion, so the upper horizontal map is also a closed immersion. We can therefore write $\mathsf{Y} = \operatorname{Spec}(B)$, where B is some $\mathbb{E}_\infty$-algebra over A for which the map $\phi : \pi_0 A \to \pi_0 B$ is surjective. Condition (ii) is satisfied if and only if the map of topological spaces $|\operatorname{Spec}(B)| \to |\operatorname{Spec}(A)|$ is surjective: that is, if and only if the kernel $\ker(\phi)$ is contained in the nilradical of $\pi_0 A$. This is equivalent to the requirement that the tautological map $A \to A^{\mathrm{red}}$ factors through ϕ, which is a reformulation of condition (i). $\square$

Proof of Proposition 7.1.2. Without loss of generality we can assume that R is connective. For each $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$, we can identify $\Omega^\infty \widehat{\mathsf{X}}(A)$ with the summand of $\Omega^\infty \mathsf{X}(A)$ spanned by those maps $\eta : \operatorname{Spec}(A) \to \mathsf{X}$ whose restriction to $\operatorname{Spec}(A^{\mathrm{red}})$ is nullhomotopic. Using Lemma 7.1.3, we see that this is equivalent to the condition that the underlying map of topological spaces $|\operatorname{Spec}(A)| \to |\mathsf{X}|$ factors through the zero section. We can therefore identify $\Omega^\infty \widehat{\mathsf{X}}$ with the (functor represented by) the formal completion of X along its zero section, which is a formal hyperplane by Proposition 1.5.15. $\square$

Remark 7.1.4. If X is a strict abelian variety of dimension d over an $\mathbb{E}_\infty$-ring R, then the formal completion $\widehat{\mathsf{X}}$ is a formal group of dimension d over R. In particular, if X is a strict elliptic curve over R, then the formal group $\widehat{\mathsf{X}}$ has dimension 1.

7.2 Oriented Elliptic Curves

We now adapt our theory of (pre)orientations to the setting of elliptic curves.

Definition 7.2.1. Let R be an $\mathbb{E}_\infty$-ring and let X be a strict elliptic curve over R. A *preorientation of* X is a pointed map $e : S^2 \to \Omega^\infty \mathsf{X}(\tau_{\geq 0} R)$, or equivalently a point of the space $\Omega^{\infty+2} \mathsf{X}(\tau_{\geq 0} R)$. We will denote the space $\Omega^{\infty+2} \mathsf{X}(\tau_{\geq 0} R)$ by $\operatorname{Pre}(\mathsf{X})$ and refer to it as the *space of preorientations of* X.

Remark 7.2.2. Let X be a strict elliptic curve over an $\mathbb{E}_\infty$-ring R and let $\widehat{\mathsf{X}}$ denote its formal completion (Construction 7.1.1). Then the canonical map

$$\widehat{\mathsf{X}}(\tau_{\geq 0} R) \to \mathsf{X}(\tau_{\geq 0} R)$$

induces an equivalence $\Omega \widehat{\mathsf{X}}(\tau_{\geq 0} R) \to \Omega \mathsf{X}(\tau_{\geq 0} R)$, and therefore a homotopy equivalence

$$\operatorname{Pre}(\widehat{\mathsf{X}}) = \Omega^{\infty+2} \widehat{\mathsf{X}}(\tau_{\geq 0} R) \to \Omega^{\infty+2} \mathsf{X}(\tau_{\geq 0} R) = \operatorname{Pre}(\mathsf{X}).$$

In other words, giving a preorientation of X is equivalent to giving a preorientation of the formal group $\widehat{\mathsf{X}}$, in the sense of Definition 4.3.19.

Remark 7.2.3. Let R be a complex periodic $\mathbb{E}_\infty$-ring and let X be a strict elliptic curve over R. Combining Remark 7.2.2 with Proposition 4.3.21, we see that the space $\mathrm{Pre}(\mathsf{X})$ of preorientations of X can be identified with the space $\mathrm{Map}_{\mathrm{FGroup}(R)}(\widehat{\mathbf{G}}_R^{\mathcal{Q}}, \widehat{\mathsf{X}})$ of maps from the Quillen formal group $\widehat{\mathbf{G}}_R^{\mathcal{Q}}$ to the formal completion $\widehat{\mathsf{X}}$. Equivalently, we can identify $\mathrm{Pre}(\mathsf{X})$ with the space

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R}, \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}})}(\widehat{\mathbf{G}}_R^{\mathcal{Q}}, \mathsf{X}),$$

where we identify X with its functor of points $\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$.

Notation 7.2.4. Let R be an $\mathbb{E}_\infty$-ring. The construction $\mathsf{X} \mapsto \mathrm{Pre}(\mathsf{X})$ determines a functor $\mathrm{Ell}^s(R) \to \mathcal{S}$, which classifies a left fibration of ∞-categories $\mathrm{Ell}^{\mathrm{pre}}(R) \to \mathrm{Ell}^s(R)$. The objects of $\mathrm{Ell}^{\mathrm{pre}}(R)$ can be identified with pairs (X, e), where X is a strict elliptic curve over R and e is a preorientation of X. We will refer to such pairs as *preoriented elliptic curves over* R, and to $\mathrm{Ell}^{\mathrm{pre}}(R)$ as *the* ∞*-category of preoriented elliptic curves over* R.

In what follows, we let $\mathcal{M}^s_{\mathrm{Ell}}$ denote the moduli stack of strict elliptic curves (see Theorem AV.2.0.3).

Proposition 7.2.5. *The functor $R \mapsto \mathrm{Ell}^{\mathrm{pre}}(R)^{\simeq}$ is (representable by) a spectral Deligne-Mumford stack $\mathcal{M}^{\mathrm{pre}}_{\mathrm{Ell}}$. Moreover, the canonical map $\mathcal{M}^{\mathrm{pre}}_{\mathrm{Ell}} \to \mathcal{M}^s_{\mathrm{Ell}}$ is affine, locally almost of finite presentation, and induces an equivalence of the underlying classical Deligne-Mumford stacks (in particular, it is a closed immersion).*

Remark 7.2.6. We will refer to the spectral Deligne-Mumford stack $\mathcal{M}^{\mathrm{pre}}_{\mathrm{Ell}}$ as *the moduli stack of preoriented elliptic curves.*

Proof of Proposition 7.2.5. Fix a connective $\mathbb{E}_\infty$-ring R and a map $\mathrm{Spec}(R) \to \mathcal{M}^s_{\mathrm{Ell}}$, classifying strict elliptic curve X over R. We wish to show that the construction $(A \in \mathrm{CAlg}_R) \mapsto \mathrm{Pre}(\mathsf{X}_A)$ is corepresentable by an $\mathbb{E}_\infty$-algebra $R' \in \mathrm{CAlg}_R$ which is locally almost of finite presentation over R and for which the underlying map $\pi_0(R) \to \pi_0(R')$ is an isomorphism of commutative rings. This follows immediately from Remarks 7.2.2 and Lemma 4.3.16. □

Definition 7.2.7. Let R be an $\mathbb{E}_\infty$-ring and let X be a strict elliptic curve over R. We will say that a preorientation $e \in \Omega^{\infty+2}\mathsf{X}(\tau_{\geqslant 0}R)$ is an *orientation of* X if its image under the homotopy equivalence $\mathrm{Pre}(\mathsf{X}) \simeq \mathrm{Pre}(\widehat{\mathsf{X}})$ (Remark 7.2.2) is an orientation of the formal group $\widehat{\mathsf{X}}$, in the sense of Definition 4.3.9. We let $\mathrm{OrDat}(\mathsf{X})$ denote the summand of $\mathrm{Pre}(\mathsf{X})$ spanned by the orientations of X.

Remark 7.2.8. Let R be an $\mathbb{E}_\infty$-ring and let X be a strict elliptic curve over R. If R is complex periodic, then giving an orientation of X is equivalent to giving an equivalence of formal groups $\widehat{\mathbf{G}}^{\mathcal{Q}}_R \simeq \widehat{\mathsf{X}}$. If R is not complex periodic, then the space of orientations $\mathrm{OrDat}(\mathsf{X})$ is empty. This follows immediately from Remark 7.2.2 and Proposition 4.3.23.

Definition 7.2.9. Let R be an $\mathbb{E}_\infty$-ring and let $\mathrm{Ell}^{\mathrm{pre}}(R)$ denote the ∞-category of preoriented elliptic curves over R (Notation 7.2.4). We let $\mathrm{Ell}^{\mathrm{or}}(R)$ denote the full subcategory of $\mathrm{Ell}^{\mathrm{pre}}(R)$ spanned by those pairs (X, e) where e is an orientation of X. We will refer to such pairs as *oriented elliptic curves over* R, and to $\mathrm{Ell}^{\mathrm{or}}(R)$ as *the* ∞*-category of oriented elliptic curves over* R.

Proposition 7.2.10. *The functor $R \mapsto \mathrm{Ell}^{\mathrm{or}}(R)^{\simeq}$ is (representable by) a nonconnective spectral Deligne-Mumford stack $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$. Moreover, the canonical map $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}} \to \mathcal{M}^{\mathrm{pre}}_{\mathrm{Ell}}$ is affine.*

Proof. Let R be any $\mathbb{E}_\infty$-ring and let $f : \mathrm{Spec}(R) \to \mathcal{M}^{\mathrm{pre}}_{\mathrm{Ell}}$ be a map, classifying a preoriented elliptic curve (X, e) over R. Then e determines a Bott map $\beta_e : \omega_{\widehat{\mathsf{X}}} \to \Sigma^{-2}(R)$ (see Construction 4.3.7), which we can identify with a map $R \to \Sigma^{-2}(\omega_{\widehat{\mathsf{X}}})$. We now observe that the fiber product $\mathcal{M}^{\mathrm{pre}}_{\mathrm{Ell}} \times_{\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}} \mathrm{Spec}(R)$ can be identified with $\mathrm{Spec}(R[\beta_e^{-1}])$, where $R[\beta_e^{-1}]$ denotes the localization of Proposition 4.3.17. □

We wil refer to the nonconnective spectral Deligne-Mumford stack $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ as the *moduli stack of oriented elliptic curves*.

7.3 The Structure of $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$

Our next goal is to analyze the structure of the moduli stack $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ and show that it satisfies the demands of Theorem 7.0.1. This is essentially an immediate consequence of the following statement which we will prove in §7.4:

Theorem 7.3.1. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be a strict elliptic curve over R which is classified by an étale map $\mathrm{Spec}(R) \to \mathcal{M}^{s}_{\mathrm{Ell}}$. Then the formal completion $\widehat{\mathsf{X}}$ is a balanced formal group over R (in the sense of Definition 6.4.1).*

Proof of Theorem 7.0.1 from Theorem 7.3.1. Let $\mathcal{M}^{s}_{\mathrm{Ell}}$ denote the moduli stack of strict elliptic curves. We view $\mathcal{M}^{s}_{\mathrm{Ell}}$ as a spectral Deligne-Mumford stack, with underlying ∞-topos $\mathcal{X}$ and structure sheaf $\mathscr{O}_{\mathcal{M}^{s}_{\mathrm{Ell}}}$. Note that underlying 0-truncated spectral Deligne-Mumford stack $(\mathcal{X}, \pi_0(\mathscr{O}_{\mathcal{M}^{s}_{\mathrm{Ell}}}))$ can be identified with the classical moduli stack of elliptic curves $\mathcal{M}_{\mathrm{Ell}}$. In particular, the category $\mathcal{U}$ appearing in the statement of Theorem 7.0.1 can be identified with the full subcategory of $\mathcal{X}$ spanned by the affine objects (see Corollary SAG.1.4.7.3).

Let $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ denote the moduli stack of oriented elliptic curves, so that we have a map of nonconnective spectral Deligne-Mumford stacks $\phi : \mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}} \to \mathcal{M}^{s}_{\mathrm{Ell}}$. Then the direct image $\phi_* \mathscr{O}_{\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, which determines a functor $\mathscr{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}} : \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}$ which is a sheaf for the étale topology on $\mathcal{U}$. We will complete the proof by showing the underlying presheaf of commutative ring spectra coincides with the one that can be extracted from Landweber's theorem. Fix an object $U \in \mathcal{U}$, given by a commutative ring $R = \mathscr{O}_{\mathcal{M}}(U)$ together with an étale map $f : \mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{Ell}}$ classifying an elliptic curve X over R. Set $R' = \mathscr{O}^{s}_{\mathcal{M}}(U)$, so that R' is a connective $\mathbb{E}_\infty$-ring with $R \simeq \pi_0(R')$, equipped with an étale map $f' : \mathrm{Spec}(R') \to \mathcal{M}^{s}_{\mathrm{Ell}}$ classifying a lift of X to a strict elliptic curve $\mathsf{X}' \in \mathrm{Ell}^{s}(R')$. Set $A = \mathscr{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}(U)$. By construction, A is an $\mathbb{E}_\infty$-algebra over R' which classifies orientations of the strict elliptic curve X', or equivalently of its formal completion $\widehat{\mathsf{X}}'$. In particular, A is complex periodic and its Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_{A}$ can be identified with $\widehat{\mathsf{X}}'_{A}$. In particular, the classical Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{A}$ is obtained from $\widehat{\mathsf{X}}$ by extending scalars along the unit map $R \simeq \pi_0(R') \xrightarrow{u} \pi_0(A)$. To complete the proof, it will suffice to show that u is an isomorphism of commutative rings and that the homotopy groups of A are concentrated in even degrees, which is exactly the content of Theorem 7.3.1. $\square$

Remark 7.3.2. The proof of Theorem 7.0.1 shows that the affine morphism $\phi : \mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}} \to \mathcal{M}^{s}_{\mathrm{Ell}}$ has the property that the unit map

$$\mathscr{O}_{\mathcal{M}^{s}_{\mathrm{Ell}}} \to \phi_* \mathscr{O}_{\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}}$$

induces an isomorphism $\pi_0(\mathscr{O}_{\mathcal{M}^{s}_{\mathrm{Ell}}}) \simeq \pi_0(\mathscr{O}_{\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}})$. It follows that ϕ induces an equivalence of the underlying ∞-topoi. In other words, the moduli stack $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ of oriented elliptic curves has the same underlying étale topos (or ∞-topos) as the classical moduli stack of elliptic curves. Moreover, their structure sheaves are related by the formula

$$\pi_*(\mathscr{O}_{\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}}) \simeq \begin{cases} \omega^{\otimes k} & \text{if } * = 2k \text{ is even} \\ 0 \text{ otherwise,} & \end{cases}$$

where ω denotes the line bundle on $\mathcal{M}_{\mathrm{Ell}}$ which associates to each elliptic curve X over a commutative ring R its dualizing line $\omega_{\widehat{\mathsf{X}}}$ (or, equivalently, the R-module of invariant differentials on X).

7.4 The Proof of Theorem 7.3.1

Let R be an $\mathbb{E}_\infty$-ring and let X be a strict abelian variety over R (Definition AV.1.5.1). From X, we can extract a formal group $\widehat{X}$ over R by the process of formal completion along the identity section (see Proposition 7.1.2 and Remark SAG.1.2.5.3). On the other hand, for every prime number p, we can extract a p-divisible group $X[p^\infty]$ (Proposition AV.6.7.1). These are related as follows:

Proposition 7.4.1. *Let R be a (p)-complete $\mathbb{E}_\infty$-ring and let X be a strict abelian variety over R. Then the formal completion $\widehat{X}$ is canonically equivalent to the identity component of $X[p^\infty]$.*

Proof. We proceed as in the proof of Proposition 2.2.12. Without loss of generality, we may assume that R is connective. For each object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$, we have a commutative diagram

$$\begin{tikzcd}
\widehat{X}[p^\infty](A) \arrow[r] \arrow[d] & X[p^\infty](A) \arrow[r] \arrow[d] & X[p^\infty](A^{\mathrm{red}}) \arrow[d] \\
\widehat{X}(A) \arrow[r] \arrow[d] & X(A) \arrow[r] \arrow[d] & X(A^{\mathrm{red}}) \arrow[d] \\
\widehat{X}(A)[p^{-1}] \arrow[r] & X(A)[p^{-1}] \arrow[r] & X(A^{\mathrm{red}})[p^{-1}].
\end{tikzcd}$$

in which the rows and columns are fiber sequences. Let $\mathcal{E} \subseteq \mathrm{CAlg}_R^{\mathrm{cn}}$ be the full subcategory spanned by the connective R-algebras which are truncated and (p)-nilpotent. For $A \in \mathcal{E}$, the $\mathbf{Z}$-module spectrum $\widehat{X}(A)[p^{-1}]$ vanishes (Lemma 2.3.24), so this diagram supplies an identification

$$\begin{aligned}
\widehat{X}(A) &\xleftarrow{\sim} \widehat{X}[p^\infty](A) \\
&\simeq \mathrm{fib}(X[p^\infty](A) \to X[p^\infty](A^{\mathrm{red}})) \\
&= X[p^\infty]^\circ(A)
\end{aligned}$$

depending functorially on A. The desired result now follows from Theorem 2.1.1. □

Proposition 7.4.2. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be a strict elliptic curve over R which is classified by an étale morphism $f : \mathrm{Spec}(R) \to \mathcal{M}^s_{\mathrm{Ell}}$. Let $\mathfrak{m}$ be any maximal ideal of R. Then:*

(a) *The residue field $\kappa = \pi_0(A)/\mathfrak{m}$ is finite.*

(b) *Let $\widehat{R} = R^{\wedge}_{\mathfrak{m}}$ denote the completion of R with respect to $\mathfrak{m}$ and let p be the characteristic of the field κ. Then the p-divisible group $\mathsf{X}[p^\infty]_{\widehat{R}}$ is a universal deformation of $\mathbf{G}_0 = \mathsf{X}[p^\infty]_\kappa$, in the sense of Theorem 3.0.11: in other words, we can identify $\widehat{R}$ with the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$.*

Proof. Since $\mathcal{M}^s_{\mathrm{Ell}}$ is locally almost of finite presentation over the sphere spectrum (Theorem AV.2.4.1), the commutative ring $\pi_0(R)$ is finitely generated over $\mathbf{Z}$ (in fact, it is even a smooth $\mathbf{Z}$-algebra of relative dimension 1). In particular, the quotient $\pi_0(R)/\mathfrak{m}$ is a field which is finitely generated as a $\mathbf{Z}$-algebra, and is therefore finite. This proves (a). We now prove (b). Note that $\widehat{R}$ is a complete local Noetherian $\mathbb{E}_\infty$-ring with residue field κ, so that the deformation $\mathsf{X}[p^\infty]_{\widehat{R}}$ of $\mathbf{G}_0$ is classified by a map $f : R^{\mathrm{un}}_{\mathbf{G}_0} \to \widehat{R}$ which is the identity on residue fields. We will show that it is an equivalence by arguing that, for every complete local Noetherian $\mathbb{E}_\infty$-ring A equipped with a map $\rho : A \to \kappa$ which exhibits κ as the residue field of A, composition with f induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(\widehat{R}, A) \to \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R^{\mathrm{un}}_{\mathbf{G}_0}, A).$$

Writing A as an inverse limit, we can reduce to the case where A is truncated and $\pi_0(A)$ is Artinian. In this case, there exists a finite sequence of maps

$$A = A_m \to A_{m-1} \to \cdots \to A_0 = \kappa,$$

each of which exhibits A_{i+1} as a square-zero extension of A_i by an almost perfect A_i-module. It will therefore suffice to prove the following:

(∗) Let $\rho : A \to \kappa$ be as above and let $\widetilde{A}$ be a square-zero extension of A by a connective A-module M which is almost perfect over A. Then the diagram σ :

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(\widehat{R}, \widetilde{A}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R^{\mathrm{un}}_{\mathbf{G}_0}, \widetilde{A}) \arrow[d] \\ \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(\widehat{R}, A) \arrow[r] & \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R^{\mathrm{un}}_{\mathbf{G}_0}, A) \end{tikzcd}$$

is a pullback square.

Invoking the universal property of $R^{\mathrm{un}}_{\mathbf{G}_0}$ (in the form articulated in Theorem 3.0.11) and the description of $\widehat{R}$ as a completion of R, we can identify σ with the outer rectangle in the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R, \widetilde{A}) \arrow[r] \arrow[d] & \mathrm{Ell}^s(\widetilde{A}) \times_{\mathrm{Ell}^s(\kappa)} \{\mathsf{X}_\kappa\} \arrow[r] \arrow[d] & \mathrm{Def}_{\mathbf{G}_0}(\widetilde{A}; \rho) \arrow[d] \\
\mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R, A) \arrow[r] & \mathrm{Ell}^s(A) \times_{\mathrm{Ell}^s(\kappa)} \{\mathsf{X}_\kappa\} \arrow[r] & \mathrm{Def}_{\mathbf{G}_0}(A; \rho).
\end{tikzcd}$$

Our assumption that f is étale guarantees that the left square in this diagram is a pullback, while the Serre-Tate theorem (Theorem AV.7.0.1) guarantees that the right square is a pullback. □

Proof of Theorem 7.3.1. Let $f : \mathrm{Spec}(R) \to \mathcal{M}^s_{\mathrm{Ell}}$ be an étale morphism classifying a strict elliptic curve X over R. We wish to show that the formal completion $\widehat{\mathsf{X}}$ is balanced. By virtue of Remark 6.4.3, it will suffice to show that for every maximal ideal $\mathfrak{m} \subseteq \pi_0(R)$, the formal group $\widehat{\mathsf{X}}_{\mathfrak{m}}$ is balanced as a formal group over the localization $R_{\mathfrak{m}}$. Let p be the characteristic of the residue field $\kappa = \pi_0(R)/\mathfrak{m}$ and let $\widehat{R}$ denote the $\mathfrak{m}$-completion of R. Then $\widehat{R}$ is faithfully flat over $R_{\mathfrak{m}}$. By virtue of Remark 6.4.2, it will suffice to show that $\widehat{\mathsf{X}}_{\widehat{R}}$ is a balanced formal group over $\widehat{R}$. Since $\widehat{R}$ is p-complete, we can identify $\widehat{\mathsf{X}}_{\widehat{R}}$ with the identity component of the p-divisible group $\mathsf{X}[p^\infty]_{\widehat{R}}$. The desired result now follows from Theorem 6.4.6, since the p-divisible group $\mathsf{X}[p^\infty]_{\widehat{R}}$ is a universal deformation of $\mathsf{X}[p^\infty]_\kappa$ (Proposition 7.4.2). □

References

[1] Adams, F. *Stable homotopy and generalized homology.* Chicago Lectures in Mathematics, 1974.

[2] Bauer, T. *Convergence of the Eilenberg-Moore spectral sequence for generalized cohomology theories.* Available for download at http://arxiv.org/abs/0803.3798.

[3] Behrens, M. and T. Lawson. *Topological automorphic forms.* Memoirs of the American Mathematical Society, 2010, Volume 204, Number 958.

[4] Buchstaber, V. and A. Lazarev. *Dieudonné modules and p-divisible groups associated with Morava K-theory of Eilenberg-MacLane spaces.* Algebr. Geom. Topol. 7 (2007), 529-564.

[5] Demazure, M. *Lectures on p-divisible groups.* Lecture notes in Mathematics 302 (1972), Berlin, New York: Springer-Verlag.

[6] Dieudonné, J. *Introduction to the theory of formal groups.* Dekker, New York 1973.

[7] Devinatz, E., Hopkins, M., and J. Smith. *Nilpotence and stable homotopy theory. I*, Annals of Mathematics, Second Series, 128 (2), 1988, 207Ð241.

[8] Hopkins, M. and J. Smith. *Nilpotence and Stable Homotopy Theory II.* Annals of Mathematics Second Series, Vol. 148, No. 1 (Jul., 1998), pp. 1-49.

[9] Demazure, M. and P. Gabriel. *Introduction to Algebraic Geometry and Algebraic Groups.* North-Holland Publishing Company, 1980.

[10] Gabber, O. *Notes on some t-structures.* In: Geometric Aspects of Dwork Theory, Volume II, Walter de Gruyter, 2004.

[11] Goerss, P. *Hopf rings, Dieudonné modules, and $E_*\Omega^2S^3$.* Homotopy invariant algebraic structures (Baltimore, MD, 1998), 115Ð174, Contemp. Math., 239, Amer. Math. Soc., Providence, RI, 1999.

[12] Goerss, P. and M. Hopkins. *Moduli Spaces of Commutative Ring Spectra.*

[13] Grothendieck, A. *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : III. Étude cohomologique des faisceaux cohérents, Première partie.* Publications Mathématiques de l'IHÈS, 11 (1961), p. 5-167.

[14] Grothendieck, A. *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : IV. Étude locale des schŐmas et des morphismes de schémas, Troisième partie.* Publications Mathématiques de l'IHÈS, 28 (1966), p. 5-255.

[15] Hazewinkel, M. *Formal groups and applications.* Pure and Applied Mathematics, vol. 78, Academic Press Inc., New York, 1978.

[16] Hedayatzadeh, M. *Exterior Powers of Barsotti-Tate Groups.* Available for download as arXiv:1009.2460v4.

[17] Hopkins, M. and J. Lurie. *Ambidexterity in $K(n)$-Local Stable Homotopy Theory.*

[18] Hovey, M. and H. Sadofsky. *Tate cohomology lowers chromatic Bousfield classes.* Proceedings of the AMS 124, 1996, 3579-3585.

[19] Hovey, M. and H. Sadofsky. *Invertible spectra in the $E(n)$-local stable homotopy category.* Journal of the London Mathematical Society, Volume 60, 1999, 284-302.

[20] Illusie, L. *Déformations de groups de Barsotti-Tate (d'apreès A. Grothendieck)*, Astérisque (1985), no. 127, 151-198.

[21] Johnson, D. and S. Wilson. *The Brown-Peterson homology of elementary p-groups.* Amer. J. Math. 107 (1985), no. 2, 427Ð453.

[22] Lazard, M. *Sur les groupes de Lie formels à un param ́etre.* Bulletin de la Socieété Mathématique de France, 83: 251-274.

[23] Lurie, J. *Higher Topos Theory.* Available for download at http://www.math.harvard.edu/ lurie/ .

[24] Lurie, J. *Higher Algebra.* Available for download.

[25] Lurie, J. *Spectral Algebraic Geometry.* Unfinished manuscript.

[26] Lurie, J. *Elliptic Cohomology I: Spectral Abelian Varieties.*

[27] Messing, W. *The crystals associated to Barsotti-Tate groups with applications to abelian schemes.* Lecture Notes in Mathematics, 264 (1972). Berlin, New York: Springer-Verlag,

[28] Peterson, E. *The Morava E-Theory of Eilenberg-MacLane Spaces.* Available for download as arXiv:1109.5989v2.

[29] Quillen, D. *On the formal group laws of unoriented and complex cobordism theory.* Bull. Amer. Math. Soc. 75 (1969), no. 6, 1293–1298.

[30] Ravenel, D. and S. Wilson. *The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture.* Amer. J. Math. 102 (1980), no. 4, 691Ð748.

[31] Rezk, C. *Notes on the Hopkins-Miller Theorem*

[32] Sati, H., and C. Westerland. *Morara K-theory and E-theory.* Available as arXiv:1109.3867 .

[33] Serre, J. P. *Local Fields.* Springer Verlag, New York, 1979.

[34] Snaith, V. *Localized stable homotopy of some classifying spaces.* Math. Proc. Cambridge Philos. Soc. 89 (1981), no. 2, 325Ð330.

[35] Tate, J. *p-Divisible groups.* Proc. Conf. Local Fields (Dribergen, 1966), 158-183.