Beyond odious and evil

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Abstract. In a recent post on the Seqfan list the third author proposed conjectures concerning the summatory function of odious numbers (i.e., of numbers whose sum of binary digits is odd), and of evil numbers (i.e., of numbers whose sum of binary digits is even). We prove these conjectures here. We will also study the sequences of "generalized" odious and evil numbers, and their iterations, giving in particular a characterization of the sequences of usual odious and evil numbers in terms of functional equations satisfied by their compositions.

Mathematics Subject Classification. 11A63, 11B83, 11B85, 11A07, 05A15, 68R15.

Keywords. Odious numbers, evil numbers, Thue–Morse sequence, summatory functions, iteration of sequences.

> [...] The lights hanging
> from oak beams above the readers
> light *and* illuminate every page. Each book dusted
> each day. Original jackets, no odious numbers glued to spines,
> not one decimal, Dewey or otherwise, in the entire place! [...]
> (Thomas Lux, *The Ambrosiana Library*)

1. Introduction

The purpose of this paper, whose title is reminiscent of [11], is revisiting the study of two families of integers respectively called odious and evil numbers, as well as the study of some generalizations. A natural integer is called *odious* if the sum of its binary digits is odd. A natural integer is called *evil* if the sum of its binary digits is even. This terminology was introduced by the authors of [3, 4], see [4, p. 463]; the words "odious" and "evil" were chosen because they begin respectively like "odd" and "even". Let $\mathbf{a} = (a(n))_{n \geq 0}$ denote the increasing sequence of odious numbers, and $\mathbf{b} = (b(n))_{n \geq 0}$ denote the increasing sequence

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of evil numbers. Sequences **a** and **b** are respectively A000069 and A001969 in [12], except that we let the indexes start from 0 instead of 1. Sequences **a** and **b** begin like

$$\mathbf{a} = 1\ 2\ 4\ 7\ 8\ 11\ 13\ 14\ 16\ 19\ 21\ 22\ 25\ 26\ 28\ 31\ldots$$
$$\mathbf{b} = 0\ 3\ 5\ 6\ 9\ 10\ 12\ 15\ 17\ 18\ 20\ 23\ 24\ 27\ 29\ 30\ldots$$

Remark 1. We seem to remember having read somewhere on the web (but where was it?) that mathematicians probably do not like numbers, since for them numbers are necessarily either evil or they are odd! Also note that for some authors the expression "evil numbers" has a quite different meaning. This is explained, e.g., in [15]: a number is called evil in that terminology if the first n decimal digits of its fractional part sum to 666 for some integer n. The expression "evil numbers" is also used with other meanings, inspiring in particular artists, like Fabio Mauri (see [6]).

Remark 2. There is a large literature on questions related to the study of integers that satisfy some congruence, and whose sum of digits in a given base satisfies some other congruence. The seminal papers are the papers of Fine [7] and Gel'fond [8].

The purpose of the present paper is twofold: to prove the above conjectures and to describe iterations of sequences **a** and **b** and generalizations. Namely it was noted by the second author in [12, A000069] that $a(a(n)) = 2a(n)$. We will generalize this result by proving Theorem 1 below which gives an expression of $a_{j,d}(a_{i,d}(n))$, where $(a_{j,d}(n))_{n\geq 0}$ denotes the increasing sequence of integers whose sum of d-ary digits is congruent to j modulo d. The paper is constructed as follows: first we prove formulas for the iteration of sequences of (generalized) odious and even numbers; then we compute the summatory functions of these numbers; Sect. 3 is devoted to proving Shevelev's conjectures; Sect. 5 gives a characterization of the sequences of odious and evil numbers in terms of functional equations satisfied by their compositions.

2. Iteration of the sequences of generalized odious and evil numbers

As recalled in the introduction, it was noted by the second author [12, A000069] that the increasing sequence of odious numbers **a** satisfies $a(a(n)) = 2a(n)$. We will generalize this result by proving Theorem 1 below.

We begin with a definition and a lemma.

Definition 1.

- Let $d \geq 2$ be an integer. For any integer x we let $\overline{(x)}_d$ denote the residue modulo d of x, i.e., the integer belonging to $[0, d-1]$ and congruent to x modulo d. Note that we have $x = d\lfloor \frac{x}{d} \rfloor + \overline{(x)}_d$.
- We let $s_d(n)$ denote the sum of the d-ary digits of n. We let $\mathbf{t}_d = (t_d(n))_{n \geq 0}$ denote the sequence of integers defined by $t_d(n) \equiv s_d(n) \bmod d$ and $0 \leq t_d(n) \leq d-1$, i.e., $t_d(n) = \overline{(s_d(n))}_d$.
- For $j \in [0, d-1]$, we let $\mathbf{a}_{j,d} = (a_{j,d}(n))_{n \geq 0}$ denote the increasing sequence of integers k such that $s_d(k) \equiv j \bmod d$ (i.e., $t_d(k) = j$).

Lemma 1.

- *Sequence $\mathbf{t}_d$ is the fixed point of the morphism defined on $\{0, 1, \ldots, d-1\}$ by*
  $0 \to 0\ 1\ \ldots\ d-1,\ 1 \to 1\ 2\ \ldots\ d-1\ 0, \ldots, d-1 \to d-1\ 0\ 1\ \ldots\ d-2$.
- *If α belongs to $[0, d-1]$, then, for all $n \geq 0$, we have*

$$t_d(dn + \alpha) = \overline{(t_d(n) + \alpha)}_d = \begin{cases} t_d(n) + \alpha & \text{if} \quad t_d(n) + \alpha \leq d-1 \\ t_d(n) + \alpha - d & \text{if} \quad t_d(n) + \alpha \geq d. \end{cases}$$

- *If j belongs to $[0, d-1]$, then, for all $n \geq 0$, we have*

$$d - 1 - t_d(dn + d - 1 - j) = \begin{cases} j - t_d(n) & \text{if} \quad 0 \leq t_d(n) \leq j \\ d + j - t_d(n) & \text{if} \quad j+1 \leq t_d(n) \leq d-1. \end{cases}$$

Proof. The proof of the first two items is easy and left to the reader. The last item is an easy consequence of the second item. $\qquad\square$

Now we prove a helpful proposition.

Proposition 1. *The sequence $(a_{j,d}(n))_{n \geq 0}$ satisfies*

$$a_{j,d}(n) = dn + \overline{(j - t_d(n))}_d.$$

This can also be written as

$$a_{j,d}(n) = dn + \begin{cases} j - t_d(n) & \text{if} \quad 0 \leq t_d(n) \leq j \\ d + j - t_d(n) & \text{if} \quad j+1 \leq t_d(n) \leq d-1 \end{cases}$$

$$= dn + d - 1 - t_d(dn + d - j - 1).$$

Proof. These equalities are easy consequences of Lemma 1, which implies in particular that sequence $\mathbf{t}_d$ consists of consecutive blocks taken from $(0\ 1\ \ldots\ d-1), (1\ 2\ \ldots\ d-1\ 0), \ldots, (d-1\ 0\ 1\ \ldots\ d-2)$, where the rth block begins with $t_d(r)$. $\qquad\square$

We are ready to state and prove the result of this section.

Theorem 1. *For all $n \geq 0$, for all $i, j \in [0, d-1]$, we have*

$$a_{j,d}(a_{i,d}(n)) = da_{i,d}(n) + \overline{(j-i)}_d = \begin{cases} da_{i,d}(n) + j - i & \text{if } j \geq i \\ da_{i,d}(n) + d + j - i & \text{if } j < i. \end{cases}$$

Proof. Using Proposition 1 we can write

$$a_{j,d}(a_{i,d}(n)) = da_{i,d}(n) + \overline{j - t_d(a_{i,d}(n))}_d.$$

But, from the definition of $a_{i,d}$ we have $t_d(a_{i,d}(n)) = i$. Hence

$$a_{j,d}(a_{i,d}(n)) = da_{i,d}(n) + \overline{(j-i)}_d = \begin{cases} da_{i,d}(n) + j - i & \text{if } j \geq i \\ da_{i,d}(n) + d + j - i & \text{if } j < i. \end{cases}$$

$\square$

3. Summatory function of generalized odious and evil numbers

In order to address Shevelev's conjectures recalled in the introduction, we have to study the summatory function of odious numbers. This section is devoted to studying the summatory function of generalized odious and evil numbers. The first step is the following proposition. (We keep the notation in Definition 1.)

Proposition 2. *Let a and r be integers in $[0, d-1]$. Then*

$$\sum_{\ell=0}^{r} \overline{(a - \ell)}_d = \begin{cases} a(r+1) - \dfrac{r(r+1)}{2} & \text{if } r \leq a \\ a(r+1-d) + dr - \dfrac{r(r+1)}{2} & \text{if } r > a. \end{cases}$$

This can also be written as

$$\sum_{\ell=0}^{r} \overline{(a - \ell)}_d = a(r+1) - \frac{r(r+1)}{2} + d\max\{r - a, 0\}.$$

Proof. If $r \leq a$, then

$$\sum_{\ell=0}^{r} \overline{(a - \ell)}_d = \overline{(a)}_d + \overline{(a-1)}_d + \cdots + \overline{(a-r)}_d$$

$$= a + (a - 1) + \cdots + (a - r) = a(r+1) - \frac{r(r+1)}{2}.$$

Now, if $r > a$, then

$$\sum_{\ell=0}^{r} \overline{(a - \ell)}_d = \overline{(a)}_d + \overline{(a-1)}_d + \cdots + \overline{(0)}_d + \overline{(-1)}_d + \overline{-(r-a)}_d$$

$$= a + (a - 1) + \cdots + 0 + (d - 1) + \cdots + (d - r + a)$$

$$= a(r+1-d) + dr - \frac{r(r+1)}{2}.$$

$\square$

Using Propositions 1 and 2 we obtain the following theorem.

Theorem 2. *The summatory function of the sequence* $\mathbf{a}_{j,d}$ *is given by*

$$\sum_{k=0}^{N} a_{j,d}(k) = \frac{dN(N+1)}{2} + \frac{\lfloor N/d \rfloor d(d-1)}{2} + \overline{(j - t_d(\lfloor N/d \rfloor))}_d(\overline{(N)}_d + 1)$$

$$- \frac{\overline{(N)}_d(\overline{(N)}_d + 1)}{2} + d \max\left\{\overline{(N)}_d - \overline{(j - t_d(\lfloor N/d \rfloor))}_d, 0\right\}$$

Proof. We first note that, for any integer a, we have

$$\sum_{k=0}^{ad-1} \overline{(j - t_d(k))}_d = \sum_{j=0}^{a-1} \sum_{k=jd}^{(j+1)d-1} \overline{(j - t_d(k))}_d = \sum_{j=0}^{a-1} \frac{d(d-1)}{2} = \frac{ad(d-1)}{2}$$

since for each j, when k runs in $[jd, (j+1)d-1]$, $t_d(k)$ takes exactly once every value in $[0, d-1]$, thus $\overline{(j - t_d(k))}_d$ also takes exactly once every value in $[0, d-1]$.

Now, using Proposition 1, we get

$$\sum_{k=0}^{N} a_{j,d}(k) = \sum_{k=0}^{N} \left(dk + \overline{(j - t_d(k))}_d \right) = \frac{dN(N+1)}{2} + \sum_{k=0}^{N} \overline{(j - t_d(k))}_d$$

$$= \frac{dN(N+1)}{2} + \sum_{k=0}^{d\lfloor N/d \rfloor - 1} \overline{(j - t_d(k))}_d + \sum_{k=d\lfloor N/d \rfloor}^{N} \overline{(j - t_d(k))}_d$$

$$= \frac{dN(N+1)}{2} + \frac{\lfloor N/d \rfloor d(d-1)}{2} + \sum_{k=d\lfloor N/d \rfloor}^{N} \overline{(j - t_d(k))}_d.$$

But

$$\sum_{k=d\lfloor N/d \rfloor}^{N} \overline{(j - t_d(k))}_d = \sum_{\ell=0}^{N-d\lfloor N/d \rfloor} \overline{(j - t_d(\ell + d\lfloor N/d \rfloor))}_d$$

$$= \sum_{\ell=0}^{N-d\lfloor N/d \rfloor} \overline{\left(j - \overline{(t_d(\ell) + t_d(\lfloor N/d \rfloor))}_d\right)}_d$$

$$= \sum_{\ell=0}^{N-d\lfloor N/d \rfloor} \overline{(j - t_d(\lfloor N/d \rfloor) - \ell)}_d$$

$$= \sum_{\ell=0}^{N-d\lfloor N/d \rfloor} \overline{\left(\overline{(j - t_d(\lfloor N/d \rfloor))}_d - \ell\right)}_d.$$

Now, using Proposition 2 with a replaced by $\overline{(j - t_d(\lfloor N/d \rfloor))}_d$ and r replaced by $N - d\lfloor N/d \rfloor$ yields the result. $\square$

Now we give a corollary of Theorem 2 above, recalling in passing some notation of the introduction.

Corollary 1. *Let* $\mathbf{t} = (t(n))_{n\geq0}$ *be the Thue–Morse sequence defined by* $t(0) = 0$, *and for all* $n \geq 0$, $t(2n) = t(n)$, $t(2n + 1) = 1 - t(n)$. *Let* $\mathbf{a} = (a(n))_{n\geq0}$ *be the sequence of odious numbers. Let* $\mathbf{b} = (b(n))_{n\geq0}$ *be the sequence of evil numbers. Let* $S(n) = a(0) + a(1) + \cdots + a(n)$ *be the summatory function of sequence* $\mathbf{a}$. *Let* $R(n) = b(0) + b(1) + \cdots + b(n)$ *be the summatory function of sequence* $\mathbf{b}$. *Then*

- $a(n) = 2n + 1 - t(n)$
- $b(n) = 2n + t(n)$
- $S(n) = n^2 + \frac{3n}{2} + \frac{1}{2} + \frac{1+(-1)^n}{4}(1 - 2t(n)) = \begin{cases} n^2 + \frac{3n}{2} + \frac{1}{2} & \text{if } n \text{ is odd} \\ n^2 + \frac{3n}{2} + 1 - t(n) & \text{if } n \text{ is even.} \end{cases}$
- $R(n) = n^2 + \frac{3n}{2} + \frac{1}{2} + \frac{1+(-1)^n}{4}(2t(n) - 1) = \begin{cases} n^2 + \frac{3n}{2} + \frac{1}{2} & \text{if } n \text{ is odd} \\ n^2 + \frac{3n}{2} + t(n) & \text{if } n \text{ is even.} \end{cases}$

Proof. Put $d = 2$ and $j = 0, 1$ in Theorem 2. (Also see, e.g., [9, Section 8], [12, A000069] and [12, A173209].)

Note that it is also possible to give summatory functions of polynomial expressions for sequences like $\mathbf{a}$ and $\mathbf{b}$. For example, we can prove the following result.   □

Corollary 2. *We have the relations*

$$\sum_{k=0}^{b(n)} a(k) = b(n)^2 + b(n) + n + 1$$

$$\sum_{k=0}^{a(n)} b(k) = a(n)^2 + a(n) + n + 1$$

$$\sum_{k=0}^{a(n)} a(k) = a(n)^2 + 2a(n) - n$$

$$\sum_{k=0}^{b(n)} b(k) = b(n)^2 + 2b(n) - n^2.$$

We have the relations

$$\sum_{a(k)\leq n} a(k) = \begin{cases} \frac{n^2}{4} - \frac{n}{4} + nt(n) & \text{if } n \equiv 0 \bmod 4 \\ \frac{n^2}{4} + \frac{n}{4} - \frac{1}{2} + t(n) & \text{if } n \equiv 1 \bmod 4 \\ \frac{n^2}{4} - \frac{n}{4} - \frac{1}{2} + (n+1)t(n) & \text{if } n \equiv 2 \bmod 4 \\ \frac{n^2}{4} + \frac{n}{4} & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

$$\sum_{b(k)\le n} b(k) = \begin{cases} \frac{n^2}{4} + \frac{3n}{4} - nt(n) & \text{if } n \equiv 0 \bmod 4 \\ \frac{n^2}{4} + \frac{n}{4} + \frac{1}{2} - t(n) & \text{if } n \equiv 1 \bmod 4 \\ \frac{n^2}{4} + \frac{3n}{4} + \frac{1}{2} - (n+1)t(n) & \text{if } n \equiv 2 \bmod 4 \\ \frac{n^2}{4} + \frac{n}{4} & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

Proof. Left to the reader. Hint for the last two formulas: note that

$$\begin{aligned} \{k; a(k) \le n\} &= \{k; 2k+1-t(k) \le n\} \\ &= \{k; t(k) = 1, \ k \le \lfloor \tfrac{n}{2} \rfloor\} \cup \{k; \ t(k) = 0, \ k \le \lfloor \tfrac{n-1}{2} \rfloor\} \\ &= \{0, 1, 2, \dots, \lfloor \tfrac{n-1}{2} \rfloor\} \cup \Theta_n \end{aligned}$$

where

$$\Theta_n = \begin{cases} \emptyset & \text{if } n \text{ is odd, or if } n \text{ is even and } t(n) = 0 \\ \{\lfloor \tfrac{n}{2} \rfloor\} & \text{if } n \text{ is even and } t(n) = 1. \end{cases}$$

Thus

$$\sum_{a(k)\le n} a(k) = \sum_{k \le \lfloor \frac{n-1}{2} \rfloor} a(k) + \frac{1 + (-1)^n}{2} t(n) a(\lfloor n/2 \rfloor).$$

$\square$

4. Proof of Shevelev's conjecture

Before stating Shevelev's conjectures, we need a definition and a lemma.

Definition 2. If x and y are two integers, we write $x <_4 y$ (resp. $x \le_4 y$) if the residues modulo 4 of x and y, denoted by $\overline{x}$ and $\overline{y}$, belonging to $\{0, 1, 2, 3\}$ and considered as natural integers, satisfy $\overline{x} < \overline{y}$ (resp. $\overline{x} \le \overline{y}$).

Example 1. For example $17 <_4 6$ because $17 \equiv 1 \bmod 4$, $6 \equiv 2 \bmod 4$ and $1 < 2$.

Lemma 2. *Let* **a** *and* **t** *be as above the increasing sequence of odious numbers and the Thue–Morse sequence. Then, if n and m are both odd or both even, then $a(n) <_4 a(m)$ (resp. $a(n) \le_4 a(m)$) if and only if $t(m) < t(n)$ (resp. $t(m) \le t(n)$).*

Proof. Since $a(n) = 2n+1-t(n)$, we see that $\overline{a(n)} = 1 - t(n)$ if n is even, and that $\overline{a(n)} = 3 - t(n)$ if n is odd. The statement in the lemma follows. $\square$

Theorem 3 (Shevelev's conjecture). *Let $S(n)$ be the summatory function of odious numbers, i.e., $S(n) = a(0) + a(1) + \cdots + a(n)$. We have, for $n \ge 2$,*

- *if $a(n-1) <_4 a(n+1)$ and $a(n) \le_4 a(n+2)$, then $S(n) = \dfrac{a(n)a(n+1)}{4}$*

- *if $a(n-1) >_4 a(n+1)$ and $a(n) \geq_4 a(n+2)$, then $S(n) = \dfrac{a(n)a(n+1)}{4} + \dfrac{1}{2}$*
- *if $a(n-1) <_4 a(n+1)$ and $a(n) >_4 a(n+2)$, then $S(n) = \dfrac{a(n)(a(n+1)-1)}{4}$*
- *if $a(n-1) >_4 a(n+1)$ and $a(n) <_4 a(n+2)$, then $S(n) = \dfrac{(a(n)+1)a(n+1)}{4}.$*

Proof.

(i) Proof of the first assertion. From Lemma 2 the conditions $a(n-1) <_4 a(n+1)$ and $a(n) \leq_4 a(n+2)$ are equivalent to $t(n+1) < t(n-1)$ and $t(n+2) \leq t(n)$. Thus $t(n+1) = 0$, $t(n-1) = 1$, and either $t(n) = 1$ or $t(n) = t(n+2) = 0$. But $t(n) = t(n+2) = 0$ is impossible because this would imply $((t(n), t(n+1), t(n+2)) = (0,0,0)$ and the Thue–Morse sequence does not contain any cube (see, e.g., [2] and the references therein). Thus $t(n) = 1$, $t(n-1) = 1$, $t(n+1) = 0$. Furthermore $t(n) = t(n-1)(=1)$ implies that n must be even (if $n = 2k+1$, $t(n) = 1-t(k)$ and $t(n-1) = t(k)$). So, using Corollary 1, $S(n) = n^2 + \frac{3n}{2} + 1 - t(n) = n^2 + \frac{3n}{2}$. On the other hand $a(n)a(n+1) = (2n+1-t(n))(2n+3-t(n+1)) = 2n(2n+3) = 4n^2 + 6n = 4(n^2 + \frac{3n}{2})$.

(ii) Proof of the second assertion. From Lemma 2 the conditions $a(n-1) >_4 a(n+1)$ and $a(n) \geq_4 a(n+2)$ are equivalent to $t(n+1) > t(n-1)$ and $t(n+2) \geq t(n)$. Hence $t(n-1) = 0$, $t(n+1) = 1$, and either $t(n) = t(n+2) = 1$ or $t(n) = 0$. But we cannot have $t(n) = t(n+2) = 1$, because this would give $(t(n), t(n+1), t(n+2)) = (1,1,1)$ and this would give a cube in the Thue–Morse sequence. Thus $t(n-1) = 0$, $t(n) = 0$, $t(n+1) = 1$. As previously $t(n-1) = t(n)(=0)$ implies that n must be even. So, using Corollary 1, $S(n) = n^2 + \frac{3n}{2} + 1 - t(n) = n^2 + \frac{3n}{2} + 1$. On the other hand $a(n)a(n+1) + 2 = (2n+1-t(n))(2n+3-t(n+1)) + 2 = (2n+1)(2n+2) + 2 = 4n^2 + 6n + 4 = 4(n^2 + \frac{3n}{2} + 1)$.

(iii) Proof of the third assertion. From Lemma 2 the conditions $a(n-1) <_4 a(n+1)$ and $a(n) >_4 a(n+2)$ are equivalent to $t(n+1) < t(n-1)$ and $t(n+2) > t(n)$. Thus $t(n-1) = 1$, $t(n) = 0$, $t(n+1) = 0$, $t(n+2) = 1$. Since $t(n) = t(n+1)(=0)$, n must be odd. So, using Corollary 1, $S(n) = n^2 + \frac{3n}{2} + \frac{1}{2}$. On the other hand $a(n)(a(n+1)-1) = (2n+1-t(n))(2n+2-t(n+1)) = (2n+1)(2n+2) = 4(n^2 + \frac{3n}{2} + \frac{1}{2})$.

(iv) Proof of the fourth assertion. From Lemma 2 the conditions $a(n-1) >_4 a(n+1)$ and $a(n) <_4 a(n+2)$ are equivalent to $t(n+1) > t(n-1)$ and $t(n+2) < t(n)$. Hence $t(n-1) = 0$, $t(n) = 1$, $t(n+1) = 1$, $t(n+2) = 0$. Since $t(n) = t(n+1)(=1)$, n must be odd. So, using Corollary 1, $S(n) = n^2 + \frac{3n}{2} + \frac{1}{2}$. On the other hand $(a(n)+1)a(n+1) = (2n+2-t(n))(2n+3-t(n+1)) = (2n+1)(2n+2) = 4(n^2 + \frac{3n}{2} + \frac{1}{2})$.

$\square$

5. Functional equations for sequences a and b

Several studies about iterating increasing sequences of integers can be found in the literature (see, e.g., [1, 5, 10, 13] and references therein, in particular parts of Ref. [14] that we discovered thanks to [13]).

With the previous notation, the increasing sequences of odious and evil numbers satisfy $a(n) = a_{1,2}(n)$ and $b(n) = a_{0,2}(n)$. We thus have the following relations.

Corollary 3.
- (i) $a(a(n)) = 2a(n)$
- (ii) $b(b(n)) = 2b(n)$
- (iii) $a(b(n)) = 2b(n) + 1$
- (iv) $b(a(n)) = 2a(n) + 1$
- (v) $a(a(n)) = b(a(n)) - 1$
- (vi) $b(b(n)) = a(b(n)) - 1$
- (vii) $a(n) - b(n) = 1 - 2t(n)$ *(in particular $a(n) - b(n)$ takes only the values ± 1)*
- (viii) $a(b(n)) - b(a(n)) = 4t(n) - 2$ *(in particular $a(b(n)) - b(a(n))$ takes only the values ± 2).*

Proof. The first four relations are Theorem 1 for the case $d = 2$. Relations (v) and (vi) are easy consequences of relations (i)–(iv). The last two relations are consequences of the expressions of $a(n)$ and $b(n)$ given in Corollary 1 and of the properties $t(2n) = t(n)$ and $t(2n + 1) = 1 - t(n)$. $\square$

One might ask which set of relations among relations (i)–(vi) suffices to characterize sequences **a** and **b**. The next three theorems yield three answers to the question.

Theorem 4. *Suppose that the two sets X and Y form a partition of the non-negative integers. Let $\mathbf{x} = (x(n))_{n \geq 0}$ be the increasing sequence of the elements of X, and let $\mathbf{y} = (y(n))_{n \geq 0}$ be the increasing sequence of the elements of Y. Suppose that $\mathbf{x}$ and $\mathbf{y}$ satisfy the following relations*

- $x(x(n)) = 2x(n)$ *for all $n \geq 0$*
- $y(y(n)) = 2y(n)$ *for all $n \geq 0$*
- $|x(n) - y(n)| = 1$ *for all $n \geq 0$.*

Then, either $\mathbf{x} = \mathbf{a}$ and $\mathbf{y} = \mathbf{b}$, or $\mathbf{x} = \mathbf{b}$ and $\mathbf{y} = \mathbf{a}$. In particular the sequence $(x(n) - y(n))_{n \geq 0}$ must be equal to $(1 - 2t(n))_{n \geq 0}$ or to $(2t(n) - 1)_{n \geq 0}$.

Proof. We must have that $\{0, 1\} = \{x(0), y(0)\}$. Without loss of generality we may suppose that $x(0) = 1$ thus $y(0) = 0$. We thus want to prove that $\mathbf{x} = \mathbf{a}$ and $\mathbf{y} = \mathbf{b}$. We will prove by induction on n that $\{2n, 2n + 1\} = \{x(n), y(n)\}$. The property is true for $n = 0$; suppose it is true for n and let us look at

$\{2n + 2, 2n + 3\}$. Either there exists k such that $2n + 2 = x(k)$ or there exists k such that $2n + 2 = y(k)$ (X and Y form a partition of the integers).

If $2n + 2 = x(k)$ we have necessarily $2n + 3 = y(k)$ (since $|x(k) - y(k)| = 1$). Furthermore $k \geq n + 1$ (since $\mathbf{x}$ and $\mathbf{y}$ are increasing). If we had $k \geq n + 2$ the values $2n + 2$, $2n + 3$ would not be in the range of $\mathbf{x}$ nor in the range of $\mathbf{y}$, hence $k = n + 1$.

If $2n+2 = y(k)$, the same reasoning shows that $2n+3 = x(k)$, and $k = n+1$.

We thus have $\{2n + 1, 2n + 3\} = \{x(n + 1), y(n + 1)\}$ and the induction is proven. Now, define the sequence $(\alpha(n))_{n \geq 0}$ by $x(n) = 2n + 1 - \alpha(n)$. This implies of course $\alpha(0) = 0$ and $y(n) = 2n + \alpha(n)$. We then note that, for any integer m, we have, by applying the formula $x(n) = 2n + 1 - \alpha(n)$ with $n = x(m)$, on the one hand $x(x(m)) = 2x(m) + 1 - \alpha(x(m))$, and on the other hand $x(x(m)) = 2x(m)$. Thus $\alpha(x(m)) = 1$. In the same way we have for any integer m, using the relation $y(n) = 2n + \alpha(n)$ for $n = y(m)$, that $y(y(m)) = 2y(m) + \alpha(y(m))$, while $y(y(m)) = 2y(m)$. Thus $\alpha(y(m)) = 0$. Since X and Y form a partition of the integers this gives

$$n \in X \Leftrightarrow \alpha(n) = 1 \quad \text{and} \quad n \in Y \Leftrightarrow \alpha(n) = 0.$$

Now we prove that $\alpha(n) = t_2(n)$, i.e., that the sequence $(\alpha(n))_{n \geq 0}$ is the Thue–Morse sequence beginning with 0. It suffices to prove that, for all $m \geq 0$, we have $\alpha(2m) = \alpha(m)$ and $\alpha(2m + 1) = 1 - \alpha(m)$.

If m belongs to X, then there exists a k such that $m = x(k)$. We have just seen that $\alpha(m) = 1$. We have $x(2m) = 4m + 1 - \alpha(2m)$. But

$$x(2m) = x(2x(k)) = x(xx(k)) = xx(x(k)) = 2xx(k) = 4x(k) = 4m.$$

Hence $\alpha(2m) = 1 = \alpha(m)$. Now, since we thus have that $2m$ belongs to X, we must have $2m + 1$ belongs to Y, hence $\alpha(2m + 1) = 0$.

If m belongs to Y, then there exists a k such that $m = y(k)$. Thus $\alpha(m) = 0$. We have $y(2m) = 4m + \alpha(2m)$. But

$$y(2m) = y(2y(k)) = y(yy(k)) = yy(y(k)) = 2yy(k) = 4y(k) = 4m.$$

Hence $\alpha(2m) = 0$. Now, since we thus have that $2m$ belongs to Y, we must have $2m + 1$ belongs to X, hence $\alpha(2m + 1) = 1$.

Finally we thus have that $(\alpha(n))_{n \geq 0} = (t_2(n))_{n \geq 0}$, and then $\mathbf{x} = \mathbf{a}$ and $\mathbf{y} = \mathbf{b}$. $\square$

The next two theorems can be seen as variations on Theorem 4.

Theorem 5. *Let* $\mathbf{x} = (x(n))_{n \geq 0}$ *and* $\mathbf{y} = (y(n))_{n \geq 0}$ *be increasing integer sequences such that* $\{x(n), \ n \geq 0\} \cup \{y(n), \ n \geq 0\} = \mathbb{N}$ *satisfying* $x(0) = 1$, $y(0) = 0$, *and*

$$\forall n \geq 0, \ x(x(n)) = y(x(n)) - 1 \quad \text{and} \quad y(y(n)) = x(y(n)) - 1.$$

Then $\mathbf{x}$ *and* $\mathbf{y}$ *are respectively equal to* $\mathbf{a}$ *and* $\mathbf{b}$ *the sequences of odious and evil numbers.*

Proof. Let $X = \{x(n), n \in \mathbb{N}\}$ and $Y = \{y(n), \ n \in \mathbb{N}\}$. The condition on $x(x(n))$ and $y(y(n))$ can be written as follows

if $m \in X$, then $x(m) = y(m) - 1$; if $m \in Y$, then $y(m) = x(m) - 1$.

This implies in particular that $X \cap Y = \emptyset$, thus X and Y form a partition of the integers. Now let 1_X be the characteristic function of X (i.e., $1_X(n) = 1$ if and only if n belongs to X). Thus $1 - 1_X$ is the characteristic function of Y. We will prove by induction on n that, for all $n \geq 0$,

$$x(n) = 2n + 1 - 1_X(n), \quad y(n) = 2n + 1_X(n).$$

The property is true for $n = 0$ since $x(0) = 1$ and $y(0) = 0$. If it is true up to n, we first have $\{x(n), y(n)\} = \{2n, 2n + 1\}$.

- If $n + 1$ belongs to X, then on the one hand $x(n + 1) = y(n + 1) - 1$. Since $x(n+1) > \max\{2n, 2n+1\}$, this gives $x(n+1) \geq 2n+2$ and $y(n+1) = x(n+1)+1 \geq 2n+3$. But X and Y form a partition of the integers, thus $x(n+1)$ must be equal to $2n+2$ (otherwise $2n+2$ is missed both by x and by y), and $y(n+1) = x(n+1)+1 = 2n+3$. This gives $x(n+1) = 2(n+1)+1-1_X(n+1)$ and $y(n + 1) = 2(n + 1) + 1_X(n + 1)$.
- If $n + 1$ belongs to Y, then on the one hand $y(n + 1) = x(n + 1) - 1$. Since $y(n+1) > \max\{2n, 2n+1\}$, this gives $y(n+1) \geq 2n+2$ and $x(n+1) = y(n+1)+1 \geq 2n+3$. But X and Y form a partition of the integers, thus $y(n+1)$ must be equal to $2n+2$ (otherwise $2n+2$ is missed both by y and by x), and $x(n+1) = y(n+1)+1 = 2n+3$. This gives $y(n+1) = 2(n+1)+1_X(n+1)$ and $x(n + 1) = 2(n + 1) + 1 - 1_X(n + 1)$.

We then note that $x(n) = 2n + 1 - 1_X(n)$ for all n, implies that $x(x(n)) = 2x(n) + 1 - 1_X(x(n)) = 2n$ for all n. Similarly $y(n) = 2n + 1_X(n)$ for all n implies that $y(y(n)) = 2y(n)$ for all n. But we have seen that according to m being in X or Y, we have $y(m) - x(m) = \pm 1$, i.e., $|x(m) - y(m)| = 1$. We can then conclude using Theorem 4. $\square$

Theorem 6. *Let* $\mathbf{x} = (x(n))_{n \geq 0}$ *and* $\mathbf{y} = (y(n))_{n \geq 0}$ *be two sequences of integers defined by* $x(0) = 1$, $y(0) = 0$, *and for each* $n \geq 1$, $x(n)$ *and* $y(n)$ *are the smallest integers that did not occur before (i.e., that do not belong to* $\{x(k), \ k \leq n-1\} \cup \{y(k), \ k \leq n-1\}$), *with the conditions that for all* $n \geq 0$
- $x(x(n))$ *and* $y(y(n))$ *are even,*
- $x(y(n))$ *and* $y(x(n))$ *are odd.*

Then $\mathbf{x} = \mathbf{a}$ *the sequence of odious numbers, and* $\mathbf{y} = \mathbf{b}$ *the sequence of evil numbers.*

Proof. The hypothesis "the smallest numbers that did not occur before" implies that x and y do not miss any integer. In other words, defining $X = \{x(n), \ n \geq 0\}$ and $Y = \{y(n), \ n \geq 0\}$, we have $X \cup Y = \mathbb{N}$. On the other hand the intersection of X and Y is empty: if n belongs both to X and Y, then there exist k, ℓ with $n = x(k) = y(\ell)$. But then $x(n) = x(x(k))$ is even,

while $x(n) = x(y(\ell))$ should be odd, which is impossible. Thus X and Y form a partition of the integers. We will prove as above that, letting 1_X denote the characteristic function of X,

$$x(n) = 2n + 1 - 1_X(n) \quad \text{and} \quad y(n) = 2n + 1_X(n).$$

The property is true for $n = 0$. Suppose that it is true up to n, which implies in particular that $\{x(n), y(n)\} = \{2n, 2n + 1\}$.

- If $n+1$ belongs to X, i.e., $n+1 = x(k)$ for some k, then $x(n+1) = x(x(k))$ must be even, while $y(n + 1) = y(x(k))$ must be odd. All integer values being taken, this implies that $x(n + 1) = 2n + 2$ and $y(n + 1) = 2n + 3$. This can also be written as $x(n + 1) = 2(n + 1) + 1 - 1_X(n + 1)$ and $y(n+1) = 2(n+1) + 1_X(n+1)$.
- If $n+1$ belongs to Y, i.e., $n+1 = y(k)$ for some k, then $x(n+1) = x(y(k))$ must be odd, while $y(n + 1) = y(y(k))$ must be even. All integer values being taken, this implies that $y(n + 1) = 2n + 2$ and $x(n + 1) = 2n + 3$. This can also be written as $y(n+1) = 2(n+1) + 1_X(n+1)$ and $x(n+1) = 2(n+1) + 1 - 1_X(n+1)$.

Since for all n we clearly have $|x(n) - y(n)| = 1$ we conclude as in Theorem 5. □

6. Conclusion

We would like to add that all the functional equations given above for the sequences of odious and evil numbers can be translated in terms of characterizations of the Thue–Morse sequence. Furthermore analogous results can be proven for the sequences $\mathbf{a}_{d,j}$.

Acknowledgements

References

[1] Allouche, J.-P., Rampersad, N., Shallit, J.: On integer sequences whose first iterates are linear. Aequationes Math. **69**, 114–127 (2005)
[2] Allouche, J.-P., Shallit, J.: The ubiquitous Prouhet–Thue–Morse sequence. In: Sequences and their Applications (Singapore, 1998). Springer Ser. Discrete Math. Theor. Comput. Sci., pp. 1–16. Springer, London (1999)
[3] Berlekamp, E., Conway, J., Guy, R.: Winning Ways for Your Mathematical Plays, vols. 1 and 2. Academic Press, London (1982)
[4] Berlekamp, E., Conway, J., Guy, R.: Winning Ways for Your Mathematical Plays, vol. 3, 2nd edn. A. K. Peters, Natick (2003)

[5] Cloitre, B., Sloane, N.J.A., Vandermast, M.J.: Numerical analogues of Aronson's sequence. J. Integer Seq. **6** (2003) Art. 03.2.2

[6] Emmer, M.: Numeri malefici (evil numbers): homage to Fabio Mauri. In: Emmer, M. (ed.) Imagine Math 2: Between Culture and Mathematics, pp. 139–150. Springer, Italia (2013)

[7] Fine, N.J.: The distribution of the sum of digits (mod p), Bull. Am. Math. Soc. **71**, 651–652 (1965)

[8] Gel'fond, A.O.: Sur les nombres qui ont des propriétés additives et multiplicatives données. Acta Arith. **13**, 259–265 (1967/1968)

[9] Guy, R.K.: Impartial games. In: Combinatorial Games, vol. 29. MSRI Publications (1995). http://library.msri.org/books/Book29/files/imp. Accessed 18 Feb 2015

[10] Laohakosol, V., Yuttanan, B.: Iterates of increasing sequences of positive integers. Aequationes Math. **87**, 89–103 (2014)

[11] Nietzche, F.W.: Beyond Good and Evil: prelude to a Philosophy of the Future, Edited by Rolf-Peter Horstmann, Edited and translated by Judith Norman. Cambridge Texts in the History of Philosophy, Cambridge (2002)

[12] On-Line Encyclopedia of Integer Sequences, available electronically at the URL http://oeis.org

[13] Sarkaria, K.S.: Roots of translations. Aequationes Math. **75**, 304–307 (2008)

[14] Targoński, G.: Topics in iteration theory, In: Studia Mathematica: Skript, vol. 6. Vandenhoeck & Ruprecht, Göttingen (1981)

[15] Wolfram Mathworld. http://mathworld.wolfram.com/EvilNumber.html. Accessed 18 Feb 2015

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