The Geometry and Topology of Orientifolds II

Dan Freed
University of Texas at Austin

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Ongoing joint work with Jacques Distler and Greg Moore

And there are simply too many slides, that's all. Just cut a few and it will be perfect. (Emperor Joseph II)

We mock the thing we are to be. (Mel Brooks)

SUMMARY

- There are new "abelian" objects in differential geometry which are *local*, so can serve as fields in the sense of physics. In our work: twistings of K-theory and its cousins, twisted spin structures and spinor fields, twisted differential KR-objects, ...
- Underlying topological objects lie in a twisted cohomology theory.
- Two theories: worldsheet (short distance, fundamental, 2d) and spacetime (long distance, effective, 10d).
- In the foundational theory of orientifolds we are proving two theorems which are *topological*:
    - Ramond-Ramond charge due to gravitational orientifold background (localization in equivariant KO-theory, KO Wu class)
    - anomaly cancellation on the worldsheet (exotic notion of orientation)

  Proofs: new variations on old themes in K-theory and index theory.
- Most intricate matching we know between topological features in a short distance theory and its long distance approximation.

TWISTINGS OF KR-THEORY

There are many approaches to twistings of K-theory: Donovan-Karoubi, Rosenberg, Atiyah-Segal, Bouwknegt-Carey-Mathai-Murray-Stevenson, etc. We adapt F.-Hopkins-Teleman (arXiv:0711.1906) to KR-theory.

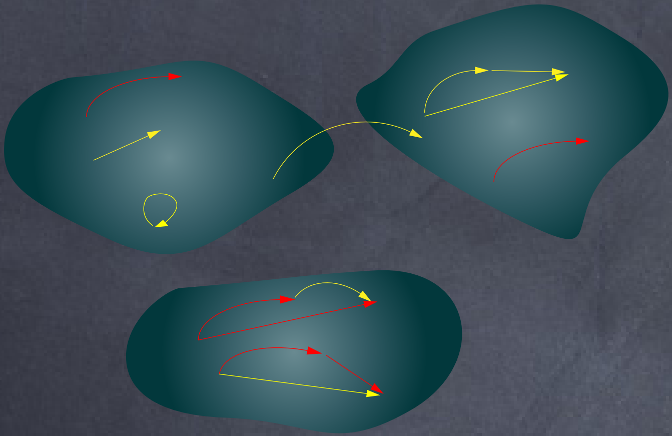
Let X be a local quotient groupoid in the sense that locally it is isomorphic to $S/\!/G$ for S a nice space (e.g. manifold) and G a compact Lie group. We write

$$X : \quad X_0 \underset{p_0}{\overset{p_1}{\leftleftarrows}} X_1$$

Specify a double cover $\pi\colon X_w \to X$ by a homomorphism $\phi\colon X_1 \to \mathbb{Z}/2\mathbb{Z}$. Then X_w is represented by the groupoid

$$X_w : \quad X_0 \underset{p_0}{\overset{p_1}{\leftleftarrows}} X_1'$$

where $X_1' = \{(a \xrightarrow{f} b) \in X_1 : \phi(f) = 0\}$ is the kernel of ϕ. It is classified by $w \in H^1(X; \mathbb{Z}/2\mathbb{Z})$ (cohomology of geometric realization).



Pictured is the groupoid X. Yellow arrows f satisfy $\phi(f) = 0$; red arrows f satisfy $\phi(f) = 1$. The groupoid X_w has only the yellow arrows.

Extend the groupoid to a simplicial space by fiber products:

$$X : \quad X_0 \leftleftarrows X_1 \Lleftarrow X_2 \leftleftarrows\!\leftleftarrows X_3 \ \cdots$$

For V is a complex vector space, $\phi \in \mathbb{Z}/2\mathbb{Z}$, set

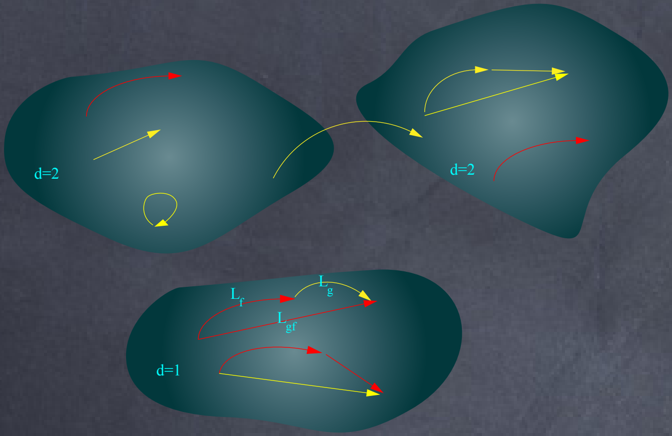
$$^{\phi}V = \begin{cases} V, & \phi = 0; \\ \overline{V}, & \phi = 1. \end{cases}$$

Definition: A *twisting* of $KR(X_w)$ is a triple $\tau = (d, L, \theta)$ consisting of a locally constant function $d\colon X_0 \to \mathbb{Z}$, a $\mathbb{Z}/2\mathbb{Z}$-graded complex line bundle $L \to X_1$, and for $(a \xrightarrow{f} b \xrightarrow{g} c) \in X_2$ an isomorphism

$$\theta\colon {}^{\phi(f)}L_g \otimes L_f \xrightarrow{\cong} L_{gf}.$$

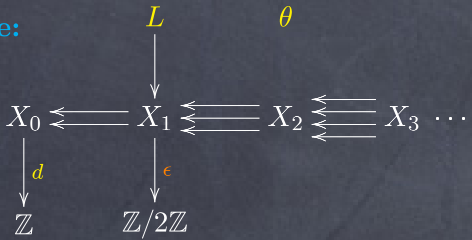
There are consistency conditions for d on X_1 and for θ on X_3.

Warning: In general, we replace X by a locally equivalent groupoid.



The degree d is the same on components of X_0 connected by an arrow. There is an isomorphism $\theta\colon \overline{L_g} \otimes L_f \to L_{gf}$ for the labeled arrows.

Another picture:



We define a (higher) groupoid of twistings and commutative composition law. Isomorphism classes of twistings of $KR(X_w)$ are classified by

$$H^0(X;\mathbb{Z}) \times H^1(X;\mathbb{Z}/2\mathbb{Z}) \times H^{w+3}(X;\mathbb{Z}),$$

$$d \qquad\qquad\qquad \epsilon \qquad\qquad\qquad (L,\theta)$$

where the last factor is cohomology in a local system. This is an isomorphism of sets but *not* of abelian groups.

Key point: We can realize twistings as objects in a cohomology theory. Special case: involution on X_w acts trivially—so twistings of $KO(X)$—twistings classified by Postnikov truncation $ko < 0 \cdots 2 >$ of connective ko with homotopy groups $\pi_{\{0,1,2\}} = \{\mathbb{Z}\,,\ \mathbb{Z}/2\mathbb{Z}\,,\ \mathbb{Z}/2\mathbb{Z}\}$.

An object in twisted $KR^q(X)$ may be represented by a pair (E, ψ), where $E \to X_0$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford$_q$-module and for each $(a \xrightarrow{f} b) \in X_1$ we have an isomorphism

$$\psi : {}^{\phi(f)}(L_f \otimes E_a) \xrightarrow{\cong} E_b$$

There is a consistency condition on X_2.

Warning: In general we need to use a more sophisticated model in which E has infinite rank and an odd skew-adjoint Fredholm operator.

Definition: A *differential twisting* of $KR(X_w)$ is a quintet $\check{\tau} = (d, L, \theta, \nabla, B)$ where $\tau = (d, L, \theta)$ is a twisting, ∇ is a covariant derivative on L, and $B \in \Omega^2(X_0)$ satisfies

$$(-1)^{\phi} p_1^* B - p_0^* B = \frac{i}{2\pi} \operatorname{curv}(\nabla) \qquad \text{on } X_1.$$

The 3-form $H = dB$ is a global *twisted* form: $(-1)^{\phi} p_1^* H = p_0^* H$. It is the *curvature* of $\check{\tau}$. (Ungraded version in Schreiber-Schweigert-Waldorf).

Remarks:

- We could continue and give a finite dimensional model for objects in the twisted differential KR-theory $\widecheck{KR}^{\check{\tau}}(X_w)$. We have not developed an infinite dimensional model along these lines.
- Because these objects have cohomological significance, we can give topological models. For the differential objects we can give models following Hopkins-Singer. Can develop products, pushforwards, etc.
- Other models of differential objects in ordinary cohomology and K-theory are being developed. (Deligne, Simons-Sullivan, Bunke-Kreck-Schick-Schroeder-Wiethaup, ...)
- There is not yet a general *equivariant* theory of differential objects. There is some work for ordinary cohomology (Gomi) and for finite group actions in K-theory (Szabo-Valentino, Ortiz).

We leave this general discussion to return to orientifolds, where the foregoing provides an explicit model of the B-field. We formulate everything in a model-independent manner.

NSNS SUPERSTRING BACKGROUND

The (Neveu-Schwarz)^2 =NSNS fields are relevant for both the worldsheet (2d) and spacetime (10d) theories. As in Jacques' lecture we have the following concise

Definition: An *NSNS superstring background* consists of:

(i) a 10-dimensional smooth orbifold X together with Riemannian metric and real-valued scalar (dilaton) field;

(ii) a double cover $\pi\colon X_w \to X$ (orientation double cover);

(iii) a differential twisting $\check{\beta}$ of $KR(X_w)$ (B-field);

(iv) and a *twisted spin structure* $\kappa\colon \Re(\beta) \to \tau^{KO}(TX - 2)$.

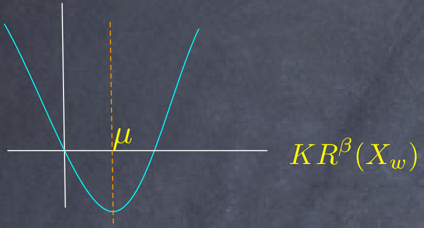
- An orbifold (in the sense of Satake) is presented by a local quotient groupoid which is locally $S/\!/\,\Gamma$ with Γ *finite*.
- We do not have time today to explicate κ, an isomorphism of twistings of $KO(X)$.
- This compact and precise definition is one of our main offerings.

THEOREM 1: RR BACKGROUND CHARGE

The (Ramond)^2 =RR current on spacetime is *self-dual*. Its definition requires an extra topological datum: a quadratic form. We fix an NSNS superstring background.

Definition: An *RR current* is an object $\check{j}$ in $\widecheck{KR}^{\check{\beta}}(X_w)$. The quadratic form of the self-dual structure is displayed on the next slide.

A quadratic form has an axis of symmetry, so defines a *center* μ in its domain. Here the domain is the group of topological equivalence classes of currents, or *charges*. (Sign: The RR background charge is $-\mu$.)



Recall $KO^0_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{pt}) \cong RO(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}[\epsilon]/(\epsilon^2 - 1)$, where ϵ is the sign representation. The quadratic form is complicated to describe (Hopkins-Singer); one manifestation is on a 12-manifold M with orientation double-cover M_w and twisted spin structure.

$$
\begin{array}{ccc}
 & KR^{\beta}(M_w) & & j \\
 & \downarrow & & \downarrow \\
KO^{\Re(\beta)}_{\mathbb{Z}/2\mathbb{Z}}(M_w) \xrightarrow[\cong]{\kappa} & KO^{\tau^{KO}(TM-4)}_{\mathbb{Z}/2\mathbb{Z}}(M_w) & & \kappa\bar{j}j \\
 & \downarrow \pi_*^{M_w} & & \downarrow \\
 & KO^{-4}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{pt}) \cong \mathbb{Z} \times \mathbb{Z}\epsilon & & \pi_*^{M_w}(\kappa\bar{j}j) \\
 & \downarrow & & \downarrow \\
 & \mathbb{Z} & & \epsilon\text{-component } \pi_*^{M_w}(\kappa\bar{j}j)
\end{array}
$$

Theorem (in progress): In the NSNS superstring background assume X_w is a *manifold*, let $i\colon F \hookrightarrow X_w$ be the fixed point set of the involution, and ν its normal bundle. After inverting 2 the center is

$$\mu = \frac{1}{2}\, i_* \left(\frac{\kappa^{-1}\ \Xi(F)}{\psi^{-1}\big(\kappa^{-1}\phi\,\mathrm{Euler}(\nu)\big)} \right) \in KR[1/2]^{\beta}(X_w).$$

- $i_*\colon KR[1/2]^{i^*\beta - \tau^{KO}(\nu)}(F) \longrightarrow KR[1/2]^{\beta}(X_w)$.
- We invert the multiplicative set $S = \{(1-\epsilon)^n\}_{n\in\mathbb{Z}^{>0}} \subset RO(\mathbb{Z}/2\mathbb{Z})$ and apply a localization theorem à la Atiyah-Segal in twisted $\mathbb{Z}/2\mathbb{Z}$-equivariant KO-theory. Here $\phi\,\mathrm{Euler}(\nu)$ is the image of the Euler class of the normal bundle after inverting S.
- ψ is a twisted version of the Adams squaring operation.
- $\Xi(F)$ is KO-analog of the Wu class: "commutator" of ψ and Thom.
- Passing to rational cohomology we recover the physicists' formula with the modified Hirzebruch L-genus, as in Jacques' lecture.

THEOREM 2: WORLDSHEET ANOMALY CANCELLATION

To specify a field theory we give a domain category of manifolds, fields, and an action. For the 2d worldsheet theory the fields are contained in

Definition: A *worldsheet* consists of

(i) a compact smooth 2-manifold Σ (possibly with boundary) with Riemannian structure;

(ii) a spin structure α on the orientation double cover $\hat{\pi}\colon \widehat{\Sigma} \to \Sigma$ whose underlying orientation is that of $\widehat{\Sigma}$ (notation: $\hat{w}$ for $\widehat{\Sigma} \to \Sigma$);

(iii) a smooth map $\phi\colon \Sigma \to X$;

(iv) an isomorphism $\phi^* w \to \hat{w}$, or equivalently a lift of ϕ to an equivariant map $\widehat{\Sigma} \to X_w$;

(v) a positive chirality spinor field ψ on $\widehat{\Sigma}$ with coefficients in $\hat{\pi}^*\phi^*(TX)$;

(vi) and a negative chirality spinor field χ on $\widehat{\Sigma}$ with coefficients in $T^*\widehat{\Sigma}$ (the gravitino).

We focus on two factors in the effective action after integrating out the fermionic fields:

$$\operatorname{pfaff} D_{\widehat{\Sigma},\alpha}\big(\hat{\pi}^*\phi^*(TX)-2\big)\cdot \exp\Big(i\int_\Sigma \check{\zeta}\cdot\phi^*\check{\beta}\Big).$$

The first factor is the pfaffian of a (real) Dirac operator on the orientation double cover $\widehat{\Sigma}$. The second factor is the integral of the B-field over the worldsheet.

Work over a parameter space S of worldsheets. Then the $\{\text{first}, \text{second}\}$ factor is a section of a flat hermitian line bundle $\{L_\psi, L_B\}\to S$. The first is the standard pfaffian line bundle with its Quillen metric and Bismut-F. covariant derivative. We discuss the second below.

Theorem (in progress): There is a canonical geometric trivialization of the tensor product

$$L_\psi\otimes L_B\longrightarrow S$$

which is constructed from the twisted spin structure κ on spacetime X.

$$\operatorname{pfaff} D_{\hat{\Sigma},\alpha}\big(\hat{\pi}^*\phi^*(TX) - 2\big) \cdot \exp\Big(i\int_\Sigma \check{\zeta}\cdot\phi^*\check{\beta}\Big) : S \longrightarrow L_\psi \otimes L_B$$

- $\hat{\Sigma}$ has an orientation-reversing isometry which preserves all data except the spin structure α. The line bundle $L_\psi \to S$ can be computed in terms of a torsion class which measures the nonequivariance of α. Variation of Atiyah-Patodi-Singer.

- We implicitly project the pullback $\phi^*\check{\beta}$ of the B-field modulo the Bott periodicity action. This lands in a certain *multiplicative* cohomology theory R which is the Postnikov section $ko\langle 0\dots 4\rangle$, more precisely in $\check{R}^{\hat{w}-1}(\Sigma)$. Sadly, our data does not include an orientation on Σ which would allow us to integrate $\phi^*\check{\beta}$. This is the genesis of the mysterious $\check{\zeta}$. We explain by analogy on next slide.

- Denote the trivialization in the theorem as $\mathbf{1}$. Then

$$\frac{\operatorname{pfaff} D_{\hat{\Sigma},\alpha}\big(\hat{\pi}^*\phi^*(TX) - 2\big) \cdot \exp\Big(i\int_\Sigma \check{\zeta}\cdot\phi^*\check{\beta}\Big)}{\mathbf{1}} : S \longrightarrow \mathbb{C}$$

is a function on S which is part of the “quantum integrand”.

Let M be a smooth compact n-manifold. Integration is defined as

$$\int_M : \Omega^{\hat{w}+n}(M) \longrightarrow \mathbb{R}$$

with domain the space of *densities*. A density pulls back to an n-form on the orientation double cover $\widehat{M} \to M$, odd under the involution.

An orientation in ordinary cohomology is a section of $\widehat{M} \to M$. Then let $o \in \Omega^{\hat{w}}(M)$ be the function on $\widehat{M}$ which is 1 on the image. (Alternative: o is an iso of twistings $0 \to \hat{w}$ of real cohomology.) Integration on forms is now defined:

$$\Omega^n(M) \longrightarrow \mathbb{R}$$
$$\omega \longmapsto \int_M o\,\omega$$

In $\exp\left(i \int_\Sigma \check{\zeta} \cdot \phi^* \check{\beta}\right)$ the "orientation data" is an object $\check{\zeta}$ which is a trivialization of an object $\check{\epsilon} \in \check{R}^{\tau^{KO}(T\Sigma)-\hat{w}-1}(\Sigma)$. It is closely related to the class which measures the nonequivariance of the spin structure α on $\widehat{\Sigma}$. The details involve explicit models with Clifford modules. . .

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