No Lebesgue Needed

James Grimmelmann

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1 Introduction: A Contest Problem

A version of the following problem appeared on on of the math contests when I was in high school:

> A sequence $k_1, k_2, k_3 \dots$ of random numbers is drawn uniformly from the interval $[0, 1]$. On average, how many numbers in the sequence are needed to make the sum of the numbers drawn so far exceed 1?

I couldn't figure it out at the time. One of the math teachers explained that I shouldn't feel bad about it; the solution requires Lebesgue integration.

Well, he was wrong. The solution does not require Lebesgue integration—which I still don't know how to do. This problem can be cracked using ordinary high-school calculus.

2 Setting Up a Solution

Solving the problem requires three things: confidence that it can be done, some care in setting up an integral, and the willingness to generalize a little. Start with the generalization. Define $f(x)$ to be the average number of draws needed such that the sum exceeds x. Thus, the problem as stated is equivalent to asking for the value of $f(1)$.

Armed with this definition, we need to figure out a way to compute $f(x)$ in general. There are a lot of plausible ways to proceed, many of which end up requiring fancy-pants integration, fancy-pants probability theory, or both. But inverting the problem provides an easier route. Instead of starting with a sum of 0 and repeatedly adding random numbers between 0 and 1 until we get x, think of us as starting with a value of x and repeatedly subtracting random numbers until we get below 0.

That is, define $g(x)$ to be the average number of random draws from $[0, 1]$ needed such that $x - k_1 - k_2 - \dots$ is less than zero. It should be apparent after a moment's thought that $f(x) = g(x)$, because if the sum of the first n numbers is k_n, then $0 + k_n > x$ if and only if $x - k_n < 0$.

This observation lets us set up an equation for $g(x)$ in terms of values of $g(t)$ for $t < x$. For $x < 0$, we don't need any more draws; we're already where we need to be, and so $g(x) = 0$. For $x \geq 0$, subtracting a random $k \in [0, 1]$ gives us a new starting point somewhere random in $[x - 1, x]$. We've used one random draw to get down to that new starting point, so the total number of draws is 1 plus however many draws it will take from the new starting point, that is the average of $g(x)$ on $[x - 1, x]$. To summarize:

$$g(x) = \begin{cases} 0 & \text{if } x < 0; \\ 1 + \int_{x-1}^{x} g(t)dt & \text{if } x \geq 0; \end{cases}$$

Let's simplify matters by specifying that from now on, we're only interested in values of $g(x)$ for $x \in [0, 1]$. That means we can look excusively to the second line of that defintion. Let's copy it out:

$$g(x) = 1 + \int_{x-1}^{x} g(t)dt$$

From here, armed with the knowledge that $0 \leq x \leq 1$, this integral equation is solvable for the function $g(x)$ using only basic first-year calculus techniques. Now might be a good time to pause and try them before reading on.

3 Completing the Solution

Since x is greater than 0 but less than 1, this means that 0 falls somewhere in $[x - 1, x]$. We can split the integral into two halves, one half for values of $x \leq 0$ and one half for values of $x \geq 0$.

$$g(x) = 1 + \int_{x-1}^{0} g(t)dt + \int_{0}^{x} g(t)dt$$

The first half simplifies immediately, since we know that $g(x) = 0$ for all $x < 0$. That yields:

$$\begin{aligned} g(x) &= 1 + \int_{x-1}^{0} 0dt + \int_{0}^{x} g(t)dt \\ &= 1 + 0 + \int_{0}^{x} g(t)dt \\ &= 1 + \int_{0}^{x} g(t)dt \end{aligned}$$

(Note that while $g(x)$ isn't continuous at 0, where it leaps from 0 to just above 1, this discontinuity at a single point doesn't stop us from taking its integral.) A solution may already be looming in mind, but let's push this one all the way through, just to be sure. Differentiatig both sides yields:

$$\frac{d}{dx} g(x) = \frac{d}{dx} \int_{0}^{x} g(t)dt$$

The right-hand side now simplifies by the fundamental theorem of calculus, giving:

$$\frac{d}{dx}g(x) = g(x)$$

I dont know abut you, but I only know of one function satisfying this condition. We can conclude that, as long as $x \in [0, 1]$, we have:

$$g(x) = e^x$$

Thus, since we were looking for $f(1)$, and since $f = g$, our answer pops right out:

$$f(1) = g(1) = e^1 = e$$

Thus, it should take an average number or e random draws on $[0, 1]$ to make the sum of the draws greater than 1.