How My Sister Discovered Integration by Parts

James Grimmelmann

February 11, 2007

1 Introduction: My Cool Sister

My sister is a senior in high school. She's taking AP calculus, and last week, she did something so clever in class that other kids accused her of being the daughter of the textbook author. In a nutshell, she discovered integration by parts on her own. This little pamphlet is a tribute to her and an attempt to explain what was neat about her insight.

2 My Sister's Technique

Shortly after teaching them about antidifferentiaion and the fundamental theorem of calculus, my sister's calculus teacher gave the class the following integral:

$$\int_0^4 xe^x dx = ?$$

The point of the exercise was to show the class that it can be very difficult to find an antiderivative for even a fairly simple function. After flailing for a bit trying to integrate it, the class was supposed to pull out their calculators and take a numerical approximation. It's a fair enough point, but there was one thing that he forgot to take into account: it *is* possible to find the indefinite integral of xe^x. You just need to be a little clever.

Her insight was that e^x is a perfectly easy function to integrate, and that the difference between the easy e^x and the "hard" xe^x is only a single factor of x. Perhaps if the x could be made to go away temporarily, there'd be something integrable she could work with. Given that goal, she started doing something else very smart: performing various manipulations on xe^x to see what she could come up with. Thus she sarted with differentiation, and applied the chain rule:

$$\begin{aligned}\frac{d}{dx}(xe^x) &= \left(\frac{d}{dx}x\right)e^x + x\left(\frac{d}{dx}e^x\right)\\ &= e^x + xe^x\end{aligned}$$

Well, isn't that interesting! It turns out that xe^x is almost its own derivative; it just spits out an extra e^x term. Since she already knew how to integrate e^x, this mean that a little rearrangement would make a nice simple integrable expression:

$$xe^x = \left(\frac{d}{dx}xe^x\right) - e^x$$
$$\int xe^x dx = \int \left(\left(\frac{d}{dx}xe^x\right) - e^x\right) dx$$
$$\int xe^x dx = \int \frac{d}{dx}xe^x dx - \int e^x dx$$

The right-hand side here consists of two integrals. The first involves the integral with respect to x of a derivative with respect to x; it's therefore just xe^x (plus or minus a constant) by the fundamental theorem of calculus. The second is the integral of the magical function e^x, which is both its own derivative and its own integral. That means:

$$\int xe^x dx = xe^x - e^x + C$$

And there you go. An "impossible" indefinite integral, integrated. Since the actual problem being posed in class involved the definite integral from 0 to 4, she simply evaluated the integral in the usual way:

$$\begin{aligned}\int_0^4 xe^x dx &= [xe^x - e^x]_0^4 \\ &= (4e^4 - e^4) - (0e^0 - e^0) \\ &= 4e^4 - e^4 - 0 + 1 \\ &= 3e^4 + 1\end{aligned}$$

3 Integration by Parts

Let's take a step back and examine the coolness here. It really involves two tricks. First, my sister saw that a messy function could be broken down into the product of two simpler ones. Second, she differentiated to make integration simpler, even though that might have seemed like a step in the wrong direction. In fact, these two insights, when combined in this way, make my favorite integration technique: *integration by parts*. Products can be differentiated cleanly using the chain rule, and this kind of clean differentiation can sometimes lead to much simpler integrands.

Let's run through what she did, this time more abstractly. She started with:

$$xe^x$$

She realized that x and e^x were individually functions with nice properties. Let's call $f(x) = x$ and $g(x) = e^x$. That meant that she had:

$$fg$$

She differentiated using the chain rule:

$$(fg)' = f'g + fg'$$

From here, it was a simple rearrangement to:

$$fg' = (fg)' - f'g$$

And integrating both sides produced:

$$\begin{aligned}\int fg'dx &= \int (fg)' - f'gdx \\ &= fg - \int f'gdx\end{aligned}$$

This equation is remarkable. It says that if you can break an unknown integrand into two functions, one (f) that's easy to differentiate and one (g') that's easy to integrate, you can try a different, and possibly easier integration of the products. Instead of trying to figure out the integral of fg', you can try for the integral of $f'g$. Why might that be easier? Well, here, where $f(x) = x$, that makes $f'(x) = 1$. And since $g'(x) = e^x = g(x)$, things there are easy to work with, too. Thus, instead of integrating the puzzler $fg' = xe^x$, you only have to integrate the much simpler $f'g = e^x$. Thus the name "integration by parts"—this approach breaks a product up into simpler parts. It's incredibly useful.

In this case, she had:

$$\begin{aligned}f(x) &= x \\ f'(x) &= 1 \\ g'(x) &= e^x \\ g(x) &= e^x\end{aligned}$$

Thus, the equation above becomes:

$$\begin{aligned}\int xe^x dx &= xe^x - \int 1e^x dx \\ &= xe^x - e^x\end{aligned}$$

She got there by playing around with fg and then used the fact that in her case $g = g'$ (so that $fg = fg'$) to take advantage of an easy integration when she saw one. More commonly, one starts with an ugly-looking fg' and tries to find a good f and a good g' to decompose it cleanly. There's an art to choosing a good f and g', one that partly involves some rules of thumb and partly depends on instinct. When one gets to differential equations, there are more advanced techniques that use this same basic idea of using a transformation to make an ungainly integrand more user-friendly.

I never really had a knack for it. But it seems that my sister may.