Journal of Applied Analysis and Computation
Volume 6, Number 3, August 2016, 865–875
Website:http://jaac-online.com/
DOI:10.11948/2016055

GLOBAL DYNAMICS ANALYSIS OF A NONLINEAR IMPULSIVE STOCHASTIC CHEMOSTAT SYSTEM IN A POLLUTED ENVIRONMENT*

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Abstract This paper intends to develop a new method to obtain the threshold of an impulsive stochastic chemostat model with saturated growth rate in a polluted environment. By using the theory of impulsive differential equations and stochastic differential equations, we obtain conditions for the extinction and the permanence of the microorganisms of the deterministic chemostat model and the stochastic chemostat model. We develop a new numerical computation method for impulsive stochastic differential system to simulate and illustrate our theoretical conclusions. The biological results show that a small stochastic disturbance can cause the microorganism to die out, that is, a permanent deterministic system can go to extinction under the white noise stochastic disturbance. The theoretical method can also be used to explore the threshold of some impulsive stochastic differential equations.

Keywords Stochastic chemostat model, impulsive stochastic differential equations, extinction, permanence in mean, saturated growth rate.

MSC(2010) 34D23, 34K20, 92D30.

1. Introduction

The chemostat is an important laboratory apparatus to investigate the growth of microorganism in a deterministic environment. Chemostat models are always used to study the continuous culture of microorganism in laboratory [2, 10, 21]. Moreover, the chemostat is also a common model of waste-treatment or fermentation process [3,4,23]. Industrial environmental pollution is a socio-ecological focus problem in the world today. The toxicant in the environment is a serious threat to the survival of the exposed biology. Consequently, it is essential to investigate the effects of toxicants on the ecological system and to obtain a theoretical threshold which governs the extinction and permanence of the biology a polluted environ-

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*This work is supported by the National Natural Science Foundation of China (11371230), the SDUST Research Fund (2014TDJH102), Shandong Provincial Natural Science Foundation, China (ZR2012AM012, ZR2015AQ001), Joint Innovative Center for Safe And Effective Mining Technology and Equipment of Coal Resources, Shandong Province, a Project of Shandong Province Higher Educational Science and Technology Program of China (J13LI05).

ment. This arouses many biologists, chemists and mathematicians's research interests [5,6,13]. Many authors have investigated the effects of toxicant on biological population. Hallam etal. initially proposed a deterministic toxicant-population model. The waste water with toxicant is always input impulsively to a ecological environment. Afterwards, population models with impulsive toxicant input were further researched [14,19,20,25]. Moreover, population system in the real world is inevitably affected by environmental noise [3,4,7,8,22,24]. Recently, some authors have introduced environmental noise into polluted population systems to study the effects of environmental noise on the persistence and extinction of single-species, see e.g. [15–17].

The bilinear growth rate means that the more microorganism individuals is, the larger number of microorganism individuals yields. As the number of microorganism individuals is large, owing to the density-dependent population growth, there is a saturation effect which makes the number of individuals constant. Comparing with bilinear growth rate, saturated growth rate may be more suitable for many cases for example, [11,20]. To the best of our knowledge, the research on stochastic chemostat model with saturated growth rate and pulsed toxicant input in a polluted environment is not too much yet. Therefore, based on a deterministic chemostat model, we shall propose a new model by taking the white noise into account. For this new system, we will investigate the influences of noise stochastic disturbance and impulsive toxicant input on system dynamics and explore the threshold which governs the extinction and permanence of the microorganism. A deterministic chemostat model with saturated growth rate and pulsed toxicant input in a polluted environment is described by the following impulsive differential equation:

$$\begin{cases} \left.\begin{aligned} &\dot{S}(t) = Q(S_0 - S(t)) - \frac{\mu S(t)x(t)}{\delta(a + x(t))}, \\ &\dot{x}(t) = \frac{\mu S(t)x(t)}{a + x(t)} - Qx(t) - rc_0(t)x(t), \\ &\dot{c}_0(t) = kc_e(t) - gc_0(t) - mc_0(t), \\ &\dot{c}_e(t) = -hc_e(t), \end{aligned}\right\} t \neq n\tau, n \in Z^+, \\ \Delta S(t) = 0, \Delta x(t) = 0, \Delta c_0(t) = 0, \Delta c_e(t) = u, t = n\tau, n \in Z^+, \end{cases} \tag{1.1}$$

where $S(t)$ represents the concentration of the unconsumed nutrient at time t, $x(t)$ represents the biomass of the population of microorganism at time t, $c_0(t)$ and $c_e(t)$ denote the concentrations of the toxicant in the organism and in the environment at time t, respectively. S_0 and Q are positive constants and denote, respectively, the concentration of the growth-limiting nutrient and the flow rate of the chemostat. μ is the maximum specific growth rate of the microorganism, δ is the yield of the microorganism $x(t)$ per unit mass of substrate, a is the so-called half-saturation constant, $r > 0$ is the rate of decrease of the intrinsic growth rate, k represents environmental toxicant uptake rate per unit mass organism, g and m are organismal net ingestion and depuration rates of toxicant, respectively, h denotes the loss rate of toxicant from the environment itself by volatilization, u is the amount of pulsed input concentration of the toxicant at each τ, and all the coefficients are positive. The function $\frac{\mu S(t)x(t)}{a+x(t)}$ represents saturated growth rate of the microorganism population.

We assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the saturated response rate, so that $\frac{\mu S(t)x(t)}{a+x(t)} \to \frac{\mu S(t)x(t)}{a+x(t)} +$

$\frac{\sigma S(t)x(t)}{a+x(t)}\dot{B}(t)$, where $B(t)$ is a standard Brownian motion with intensity $\sigma^2 > 0$. Then a stochastic version takes the following form:

$$\begin{cases} \left.\begin{array}{l} \mathrm{d}S(t) = \left(Q(S_0 - S(t)) - \dfrac{\mu S(t)x(t)}{\delta(a+x(t))}\right)\mathrm{d}t - \dfrac{\sigma S(t)x(t)}{\delta(a+x(t))}\mathrm{d}B(t), \\ \mathrm{d}x(t) = \left(\dfrac{\mu S(t)x(t)}{a+x(t)} - Qx(t) - rc_0(t)x(t)\right)\mathrm{d}t + \dfrac{\sigma S(t)x(t)}{a+x(t)}\mathrm{d}B(t), \\ \mathrm{d}c_0(t) = (kc_e(t) - gc_0(t) - mc_0(t))\mathrm{d}t, \\ \mathrm{d}c_e(t) = -hc_e(t)\mathrm{d}t, \end{array}\right\} t \neq n\tau, n \in Z^+, \\ \Delta S(t) = 0, \Delta x(t) = 0, \Delta c_0(t) = 0, \Delta c_e(t) = u, t = n\tau, n \in Z^+, \end{cases} \tag{1.2}$$

where σ is the environmental noise disturbance coefficient.

This paper will study the stochastic chemostat model with a saturated growth response rate and pulsed toxicant input in a polluted environment. The main objective of this paper is to investigate the extinction and permanence of the microorganism population and explore the threshold of the above two chemostat systems.

2. Preliminary results

In the section, we will give some notations, definitions and some lemmas which will be used for our main results. To this end, we throughout this paper assume that $S(t), x(t)$ and $c_0(t)$ are continuous at $t = nT$, and $c_e(t)$ is left continuous at $t = nT$ and $c_e(nT^+) = \lim_{t\to nT^+} c_e(t)$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets). Further assume that $B(t)$ is a scalar Brownian motion defined on the complete probability space Ω. Also let $\mathbb{R}^4_+ = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4 | z_i > 0, i = 1,2,3,4\}$. If f is an integrable function on $[0, +\infty)$, define $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(\theta)\mathrm{d}\theta$.

Definition 2.1. (i) The microorganism $x(t)$ is said to be extinctive if $\lim_{t\to+\infty} x(t) = 0$.

(ii) The species $x(t)$ is said to be permanent in the mean if there exists a positive constant λ such that $\liminf_{t\to+\infty}\langle x(t)\rangle \geq \lambda$.

Now we give some basic properties of the following subsystem of systems (1.1) and (1.2),

$$\begin{cases} \left.\begin{array}{l} \mathrm{d}c_0(t) = (kc_e(t) - gc_0(t) - mc_0(t))\mathrm{d}t, \\ \mathrm{d}c_e(t) = -hc_e(t)\mathrm{d}t, \end{array}\right\} t \neq n\tau, n \in Z^+, \\ \Delta c_0(t) = 0, \Delta c_e(t) = u, t = n\tau, n \in Z^+. \end{cases} \tag{2.1}$$

Lemma 2.1. ([14]) System (2.1) has a unique positive τ-periodic solution $(c_0^*(t), c_e^*(t))^T$ and for each solution $(c_0(t), c_e(t))^T$ of (2.1), $c_0(t) \to c_0^*(t), c_e(t) \to c_e^*(t)$ as $t \to +\infty$. Moreover, $c_0(t) > c_0^*(t), c_e(t) > c_e^*(t)$ for all $t \geq 0$ if $c_0(0) > c_0^*(0), c_e(0) > c_e^*(0)$, where

$$\begin{cases} c_0^*(t) = c_0^*(0)e^{-(g+m)(t-n\tau)} + \dfrac{ku\left(e^{-(g+m)(t-n\tau)} - e^{-h(t-n\tau)}\right)}{(h-g-m)\left(1-e^{-h\tau}\right)}, \\ c_e^*(t) = \dfrac{ue^{-h(t-n\tau)}}{1-e^{-h\tau}}, \\ c_0^*(0) = \dfrac{ku\left(e^{-(g+m)\tau} - e^{-h\tau}\right)}{(h-g-m)\left(1-e^{-(g+m)\tau}\right)\left(1-e^{-h\tau}\right)}, \\ c_e^*(0) = \dfrac{u}{1-e^{-h\tau}}, \end{cases} \tag{2.2}$$

for $t \in (n\tau, (n+1)\tau]$ and $n \in Z^+$.

Lemma 2.2. For any positive solution $(S(t), x(t), c_0(t), c_e(t))$ of system (1.1) or (1.2) with initial value $(S(0), x(0), c_0(0), c_e(0^+)) \in \mathbb{R}_+^4$, we have

$$\limsup_{t\to+\infty} S(t) \le S_0, \limsup_{t\to+\infty} x(t) \le \delta S_0, \lim_{t\to+\infty} \langle c_0(t)\rangle = \frac{ku}{h(g+m)\tau} \triangleq \overline{c_0}.$$

Proof. From the first two equations of system (1.1) or (1.2), we have

$$\frac{\mathrm{d}\left(S(t)+\frac{1}{\delta}x(t)\right)}{\mathrm{d}t} \le Q\left[S_0 - \left(S(t)+\frac{1}{\delta}x(t)\right)\right].$$

This implies that $\lim_{t\to+\infty}\left(S(t)+\frac{1}{\delta}x(t)\right) \le S_0$, then $\limsup_{t\to+\infty} S(t) \le S_0, \limsup_{t\to+\infty} x(t) \le \delta S_0$. By Lemma 2.1, we have

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t c_0(s)\mathrm{d}s = \lim_{t\to+\infty}\frac{1}{t}\int_0^t c_0^*(s)\mathrm{d}s = \frac{1}{\tau}\int_0^\tau c_0^*(t)\mathrm{d}t = \frac{ku}{h(g+m)\tau}.$$

This completes the proof of Lemma 2.2. □

Let

$$\mathcal{R} = \frac{\mu S_0}{a(Q+r\bar{c}_0)}.$$

Then we can prove

Lemma 2.3. For system (1.1), we have

(a) if $\mathcal{R} < 1$, then the microorganism goes to extinction and the system has a unique stable 'microorganism-extinction' periodic solution $(S_0, 0, c_0^*(t), c_e^*(t))$; and

(b) if $\mathcal{R} > 1$, then the microorganism of the system is permanent.

Proof. By Lemma 2.1, we can see that system (1.1) has a unique 'microorganism-extinction' periodic solution $(S_0, 0, c_0^*(t), c_e^*(t))$. The stability of the periodic solution $(S_0, 0, c_0^*(t), c_e^*(t))$ is determined by the eigenvalues of

$$M = \begin{pmatrix} \exp(-Q\tau) & * & 0 & 0 \\ 0 & \exp\left(\int_0^\tau \frac{\mu S_0}{a} - Q - rc_0^*(t)\mathrm{d}t\right) & 0 & 0 \\ 0 & 0 & \exp(-(g+m)\tau) & k \\ 0 & 0 & 0 & \exp(-h\tau) \end{pmatrix},$$

which are

$$\lambda_1 = \exp(-Q\tau) < 1, \quad \lambda_2 = \exp\left(\int_0^\tau \frac{\mu S_0}{a} - Q - rc_0^*(t)\mathrm{d}t\right),$$
$$\lambda_3 = \exp(-(g+m)\tau) < 1, \quad \lambda_4 = \exp(-h\tau) < 1.$$

Then according to Floquet theory [1], $(S_0, 0, c_0^*(t), c_e^*(t))$ is stable if $\lambda_2 < 1$, i.e., $\mathcal{R} < 1$, which gives the first conclusion.

Next let us prove the permanence of system (1.1) when $\mathcal{R} > 1$. Integrating from 0 to t and dividing by t on both sides of the first two equations of (1.1) yields

$$\epsilon(t) \triangleq \delta\frac{S(t)-S(0)}{t} + \frac{x(t)-x(0)}{t} \ge \delta Q S_0 - \delta Q\langle S(t)\rangle - (Q + rc_0^*(0))\langle x(t)\rangle,$$

then we get

$$\langle S(t)\rangle \ge S_0 - \left(\frac{1}{\delta} + \frac{rc_0^*(0)}{\delta Q}\right)\langle x(t)\rangle - \frac{\epsilon(t)}{\delta Q}. \tag{2.3}$$

Define $V(t) = a \ln x(t) + x(t)$. It is easy to check $V(t)$ is bounded. Then we have

$$\begin{aligned} D^+V(t) &= \mu S(t) - a(Q + rc_0(t)) - Qx(t) - rc_0(t)x(t) \\ &\geq \mu S(t) - a(Q + rc_0(t)) - (Q + rc_0^*(0))x(t). \end{aligned} \tag{2.4}$$

Integrating from 0 to t and dividing by t on both sides of (2.4) yields

$$\begin{aligned} \frac{V(t)}{t} - \frac{V(0)}{t} &\geq \mu\langle S(t)\rangle - a(Q + r\langle c_0(t)\rangle) - (Q + rc_0^*(0))\langle x(t)\rangle \\ &\geq \mu S_0 - a(Q + r\langle c_0(t)\rangle) - \left[\mu\left(\frac{1}{\delta} + \frac{rc_0^*(0)}{\delta Q}\right) + (Q + rc_0^*(0))\right]\langle x(t)\rangle - \frac{\mu\epsilon(t)}{\delta Q} \\ &= a(Q + r\langle c_0(t)\rangle)\left[\frac{\mu S_0}{a(Q + r\langle c_0(t)\rangle)} - 1\right] \\ &\quad - \left[\mu\left(\frac{1}{\delta} + \frac{rc_0^*(0)}{\delta Q}\right) + (Q + rc_0^*(0))\right]\langle x(t)\rangle - \frac{\mu\epsilon(t)}{\delta Q}. \end{aligned} \tag{2.5}$$

Noticing that $0 < S \leq S_0$ and $0 < x(t) \leq \delta S_0$, then we obtain $\lim_{t\to+\infty} \frac{V(t)}{t} = 0$ and $\lim_{t\to+\infty} \epsilon(t) = 0$. Finally, taking the inferior limit of both sides of (2.5) leads to

$$\liminf_{t\to+\infty}\langle x(t)\rangle \geq \frac{a\delta Q(Q + r\overline{c_0})}{(\mu + \delta Q)(Q + rc_0^*(0))}(\mathcal{R} - 1) > 0.$$

This completes the proof.

□

3. Main results

3.1. Extinction

In this section, we explore the condition for the extinction of the microorganism, which implies microculture failed. Let

$$\mathcal{R}^* = \frac{\mu S_0}{a(Q + r\overline{c}_0)} - \frac{\sigma^2 S_0^2}{2a^2(Q + r\overline{c}_0)} = \mathcal{R} - \frac{\sigma^2 S_0^2}{2a^2(Q + r\overline{c}_0)}, \tag{3.1}$$

be the threshold of the deterministic system (1.1), where $\overline{c}_0 \triangleq \frac{ku}{h(g+m)\tau}$, $\mathcal{R} = \frac{\mu S_0}{a(Q + r\overline{c}_0)}$. Then we obtain the following theorem.

Theorem 3.1. Let $(S(t), x(t), c_0(t), c_e(t))$ be the solution of system (1.2) with initial value $(S(0), x(0), c_0(0), c_e(0^+)) \in \mathbb{R}_+^4$. Then if one of the following holds

(i) $\sigma > \frac{\mu}{\sqrt{2(Q + r\overline{c_0})}}$, or

(ii) $\mathcal{R}^* < 1$ and $\sigma \leq \sqrt{\frac{a\mu}{S_0}}$,

the microorganism goes to extinction almost surely, i.e. $\lim_{t\to+\infty} x(t) = 0$, moreover, $\lim_{t\to+\infty} S(t) = S_0$, $\lim_{t\to+\infty} c_0(t) = c_0^*(t)$, $\lim_{t\to+\infty} c_e(t) = c_e^*(t)$.

Proof. Applying Itô's formula to system (1.2) yields

$$\mathrm{d}\ln x(t) = \left(\frac{\mu S(t)}{a + x(t)} - Q - rc_0(t) - \frac{\sigma^2 S^2(t)}{2(a + x(t))^2}\right)\mathrm{d}t + \frac{\sigma S(t)}{a + x(t)}\mathrm{d}B(t). \tag{3.2}$$

It then gives two cases to discuss.

Case (i)

Integrating with respective to t from 0 to t on both sides of (3.2) leads to

$$\begin{aligned}\ln x(t) &= -\frac{\sigma^2}{2}\int_0^t\left(\frac{\mu}{\sigma^2}-\frac{S(t)}{a+x(t)}\right)^2\mathrm{d}t - Qt - r\int_0^t c_0(\theta)\mathrm{d}\theta + \frac{\mu^2}{2\sigma^2}t + M(t) + \ln x(0)\\ &\le -Qt - r\int_0^t c_0(\theta)\mathrm{d}\theta + \frac{\mu^2}{2\sigma^2}t + M(t) + \ln x(0),\end{aligned}\tag{3.3}$$

where $M(t) = \int_0^t \frac{\sigma S(\theta)}{a+x(\theta)}\mathrm{d}B(\theta)$ is a local continuous martingale with $M(0) = 0$ and its quadratic variation given by

$$\langle M(t), M(t)\rangle = \int_0^t \frac{\sigma^2 S^2(\theta)}{(a+x(\theta))^2}\mathrm{d}\theta.$$

Noticing that $0 < \frac{S}{a+x} \le \frac{S_0}{a}$ gives

$$\limsup_{t\to+\infty}\frac{\langle M(t), M(t)\rangle}{t} \le \frac{\sigma^2 S_0^2}{a^2} < \infty,\ \text{a.s.}$$

Then by the strong law of large numbers for martingales [9,18], we have

$$\lim_{t\to+\infty}\frac{M(t)}{t} = 0,\ \text{a.s.}$$

Dividing both sides of (3.3) by t, one obtains

$$\frac{\ln x(t)}{t} \le -\left(Q + r\langle c_0(t)\rangle - \frac{\mu^2}{2\sigma^2}\right) + \frac{M(t)}{t} + \frac{\ln x(0)}{t}.\tag{3.4}$$

Since $\sigma > \frac{\mu}{\sqrt{2(Q+r\overline{c_0})}}$ implies $-\left(Q + r\langle c_0(t)\rangle - \frac{\mu^2}{2\sigma^2}\right) < 0$, taking the limit superior of both sides of (3.4) gives

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{t} \le -\left(Q + r\langle c_0(t)\rangle - \frac{\mu^2}{2\sigma^2}\right) < 0,$$

which means $\lim_{t\to+\infty} x(t) = 0,$ a.s.

Case (ii)

Integrating this from 0 to t and dividing by t on both sides of (3.2) yields

$$\begin{aligned}\frac{\ln x(t)}{t} &= \frac{1}{t}\int_0^t\left(\frac{\mu S(\theta)}{a+x(\theta)} - Q - rc_0(\theta) - \frac{\sigma^2 S^2(\theta)}{2(a+x(\theta))^2}\right)\mathrm{d}\theta + \frac{M(t)}{t} + \frac{\ln x(0)}{t}\\ &\le \left(\frac{\mu S_0}{a} - (Q + r\langle c_0(t)\rangle) - \frac{\sigma^2 S_0^2}{2a^2}\right) + \frac{M(t)}{t} + \frac{\ln x(0)}{t}\\ &= (Q + r\langle c_0(t)\rangle)\left(\frac{\mu S_0}{a(Q + r\langle c_0(t)\rangle)} - \frac{\sigma^2 S_0^2}{2a^2(Q + r\langle c_0(t)\rangle)} - 1\right) + \frac{M(t)}{t} + \frac{\ln x(0)}{t}.\end{aligned}\tag{3.5}$$

Taking the limit superior of both sides of (3.5) leads to

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{t} \le (Q + r\overline{c_0})(\mathcal{R}^* - 1) < 0,$$

which implies $\lim_{t\to+\infty} x(t) = 0,$ a.s.

Finally, since the limit system of (1.2) is

$$\begin{cases} \left.\begin{aligned} &\mathrm{d}S(t) = [Q(S_0 - S(t))]\,\mathrm{d}t,\\ &\mathrm{d}c_0(t) = [kc_e(t) - gc_0(t) - mc_0(t)]\mathrm{d}t,\\ &\mathrm{d}c_e(t) = -hc_e(t)\mathrm{d}t, \end{aligned}\right\} t \neq n\tau, n \in Z^+,\\ \Delta S(t) = 0, \Delta c_0(t) = 0, \Delta c_e(t) = u, t = n\tau, n \in Z^+, \end{cases} \tag{3.6}$$

by Lemma 2.1, it is clear that $\lim_{t\to+\infty} S(t) = S_0$, $\lim_{t\to+\infty} c_0(t) = c_0^*(t)$, $\lim_{t\to+\infty} c_e(t) = c_e^*(t)$.

□

3.2. Permanence in mean

Theorem 3.2. If $\mathcal{R}^* > 1$, then for any initial value $(S(0), x(0),\ c_0(0), c_e(0^+)) \in \mathbb{R}_+^4$, system (1.2) is permanent in the mean; moreover, the solution $(S(t), x(t), c_0(t), c_e(t))$ of system (1.2) satisfies

$$\liminf_{t\to+\infty}\langle x(t)\rangle \geq \frac{a\delta Q(Q + r\overline{c_0})}{(\mu + \delta Q)(Q + rc_0^*(0))}(\mathcal{R}^* - 1). \tag{3.7}$$

Proof. Integrating from 0 to t and dividing by t on both sides of the first two equations of (1.2) yields

$$\varepsilon(t) \triangleq \delta\frac{S(t) - S(0)}{t} + \frac{x(t) - x(0)}{t} \geq \delta QS_0 - \delta Q\langle S(t)\rangle - (Q + rc_0^*(0))\langle x(t)\rangle,$$

then one can get

$$\langle S(t)\rangle \geq S_0 - \left(\frac{1}{\delta} + \frac{rc_0^*(0)}{\delta Q}\right)\langle x(t)\rangle - \frac{\varepsilon(t)}{\delta Q}. \tag{3.8}$$

Applying Itô's formula gives

$$\begin{aligned} &\mathrm{d}\left(a\ln x(t) + x(t)\right)\\ = &\left[\mu S(t) - a(Q + rc_0(t)) - Qx(t) - rc_0(t)x(t) - \frac{a\sigma^2S^2(t)}{2(a + x(t))^2}\right]\mathrm{d}t + \sigma S(t)\mathrm{d}B(t)\\ \geq &\left[\mu S(t) - a(Q + rc_0(t)) - (Q + rc_0^*(0))x(t) - \frac{\sigma^2S_0^2}{2a}\right]\mathrm{d}t + \sigma S(t)\mathrm{d}B(t), \end{aligned} \tag{3.9}$$

from which one can get

$$\begin{aligned} &\frac{a\left(\ln x(t) - \ln x(0)\right)}{t} + \frac{x(t) - x(0)}{t}\\ \geq\ &\mu\langle S(t)\rangle - a(Q + r\langle c_0(t)\rangle) - (Q + rc_0^*(0))\langle x(t)\rangle - \frac{\sigma^2S_0^2}{2a} + \frac{M(t)}{t}\\ \geq\ &\mu S_0 - a(Q + r\langle c_0(t)\rangle) - \frac{\sigma^2S_0^2}{2a} - \left[\mu\left(\frac{1}{\delta} + \frac{rc_0^*(0)}{\delta Q}\right)\right.\\ &\left.+(Q + rc_0^*(0))\right]\langle x(t)\rangle - \frac{\mu\varepsilon(t)}{\delta Q} + \frac{M(t)}{t}\\ =\ &a(Q + r\langle c_0(t)\rangle)\left[\frac{\mu S_0}{a(Q + r\langle c_0(t)\rangle)} - \frac{\sigma^2S_0^2}{2a^2(Q + r\langle c_0(t)\rangle)} - 1\right]\\ &-\left[\mu\left(\frac{1}{\delta} + \frac{rc_0^*(0)}{\delta Q}\right) + (Q + rc_0^*(0))\right]\langle x(t)\rangle - \frac{\mu\varepsilon(t)}{\delta Q} + \frac{M(t)}{t}, \end{aligned} \tag{3.10}$$

where $M(t) = \int_0^t \sigma S(\theta) \mathrm{d}B(\theta)$. The inequality (3.10) can be rewritten as

$$\begin{aligned}
\langle x(t)\rangle \geq{}& \frac{1}{\Delta}\Bigg[a(Q+r\langle c_0(t)\rangle)\left(\frac{\mu S_0}{a(Q+r\langle c_0(t)\rangle)} - \frac{\sigma^2 S_0^2}{2a^2(Q+r\langle c_0(t)\rangle)} - 1\right) \\
&- \frac{\mu\varepsilon(t)}{\delta Q} + \frac{M(t)}{t} - \left(\frac{a\left(\ln x(t) - \ln x(0)\right)}{t} + \frac{x(t)-x(0)}{t}\right)\Bigg] \\
\geq{}& \begin{cases} \dfrac{1}{\Delta}\Bigg[a(Q+r\langle c_0(t)\rangle)\left(\dfrac{\mu S_0}{a(Q+r\langle c_0(t)\rangle)} - \dfrac{\sigma^2 S_0^2}{2a^2(Q+r\langle c_0(t)\rangle)} - 1\right) \\ \qquad - \dfrac{\mu\varepsilon(t)}{\delta Q} + \dfrac{M(t)}{t} + \dfrac{a\ln x(0)}{t} - \dfrac{x(t)-x(0)}{t}\Bigg], 0 < x(t) < 1, \\ \dfrac{1}{\Delta}\Bigg[a(Q+r\langle c_0(t)\rangle)\left(\dfrac{\mu S_0}{a(Q+r\langle c_0(t)\rangle)} - \dfrac{\sigma^2 S_0^2}{2a^2(Q+r\langle c_0(t)\rangle)} - 1\right) \\ \qquad - \dfrac{\mu\varepsilon(t)}{\delta Q} + \dfrac{M(t)}{t} - \dfrac{a\left(\ln x(t) - \ln x(0)\right)}{t} - \dfrac{x(t)-x(0)}{t}\Bigg], 1 \leq x(t), \end{cases}
\end{aligned} \tag{3.11}$$

where $\Delta = \frac{(Q+rc_0^*(0))(\mu+\delta Q)}{\delta Q}$. Note that $0 < S \leq S_0$, then

$$\limsup_{t\to+\infty} \frac{\langle M(t), M(t)\rangle}{t} \leq \sigma^2 S_0^2 < \infty, \text{ a.s.}$$

By the strong law of large numbers for martingales [9, 18], we have $\lim_{t\to+\infty} \frac{M(t)}{t} = 0$. According to Lemma 2.2, one sees that $x(t) \leq \delta S_0$. Thus one has $\lim_{t\to+\infty} \frac{x(t)}{t} = 0, \lim_{t\to+\infty} \frac{\ln x(t)}{t} = 0$ and $\lim_{t\to+\infty} \varepsilon(t) = 0$. Taking the inferior limit of both sides of (3.11), one can derive that

$$\begin{aligned}
\liminf_{t\to+\infty} \langle x(t)\rangle &\geq \frac{a(Q+r\overline{c_0})}{\Delta}\left[\frac{\mu S_0}{a(Q+r\overline{c_0})} - \frac{\sigma^2 S_0^2}{2a^2(Q+r\overline{c_0})} - 1\right] \\
&= \frac{a\delta Q(Q+r\overline{c_0})}{(\mu+\delta Q)(Q+rc_0^*(0))}(\mathcal{R}^* - 1) > 0.
\end{aligned}$$

This completes the proof of Theorem 3.2. □

4. Simulation and Conclusion

In this section, our numerical method for impulsive stochastic differential equations is adapted from the Euler Maruyama (EM) method in [12]. In our simulations for systems (1.1) and (1.2), we set

$$\begin{aligned}
&S_0 = 4, Q = 0.4, \delta = 0.5, a = 10, \mu = 1.25, r = 0.1, \\
&k = 1, g = 0.8, h = 1, u = 0.1, \tau = 1,
\end{aligned}$$

and take parameter values $\sigma = 0$ or $\sigma = 1.5$ to investigate the effect of stochastic disturbance on the dynamics of stochastic system. Fig.1 shows that the persistent microorganism of a deterministic system maybe go to extinction under the white noise stochastic disturbance, thus the simulation is consistent with the theoretical results of Lemma 2.3 and Theorem 3.1. If $\mathcal{R}^* = \mathcal{R} - \frac{\sigma^2 S_0^2}{2a^2(Q+r\overline{c})} < 1 < \mathcal{R}$, then a persistent deterministic system becomes extinct due to the white noise disturbance. Therefore, the white noise stochastic effect is disadvantage for the persistence of system.

Keeping all parameters unchanged as in Fig.1, except u, the pulsed input concentration of the toxicant. When it is large, $u = 1$ say, we have $\mathcal{R}^* = 0.928 < 1$. Thus, the microorganism x goes to extinction, please see Fig.2 (a). Conversely, when it is small, say $u = 0.1$, we have $\mathcal{R}^* = 1.1811 > 1$. Thus, the microorganism x is persistent, see Fig.2 (b). This supports our theoretical results obtained in Theorem 3.1 and Theorem 3.2 as well.

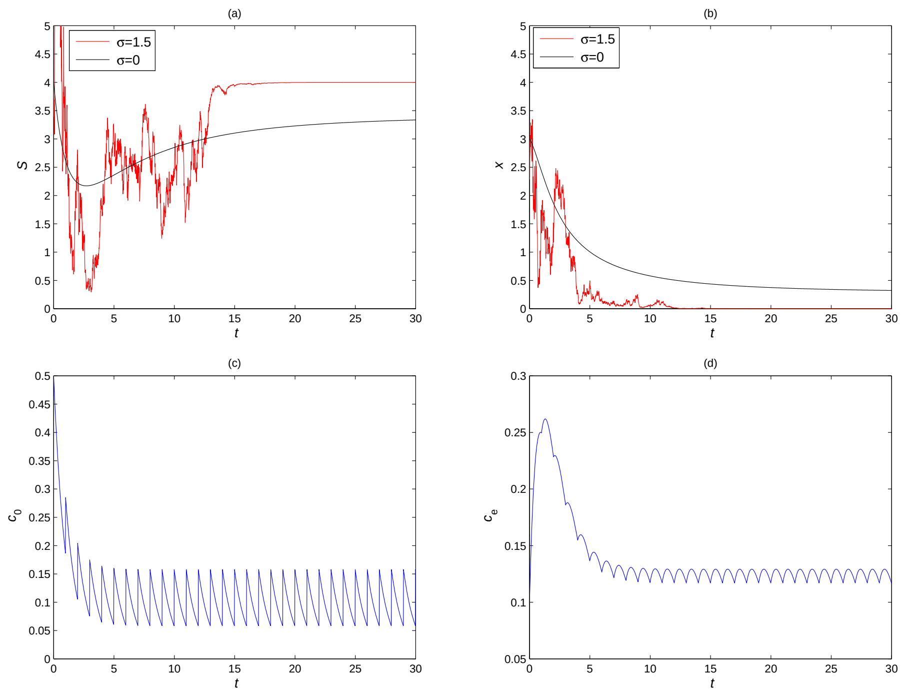

Figure 1. Numerical simulation of the paths $S(t), x(t), c_0(t), c_e(t)$ for deterministic chemostat system (1.1) and stochastic chemostat system (1.2), where black curves and red curves in (a) and (b) represent the deterministic system and the stochastic systems, respectively, (c) and (d) represent the concentrations of the toxicant in the organism and in the environment. Black curves (deterministic system): $\sigma = 0, \mathcal{R} = 1.2121 > 1$; red curves (stochastic system): $\sigma = 1.5, \mathcal{R}^* = 0.9212 < 1$.

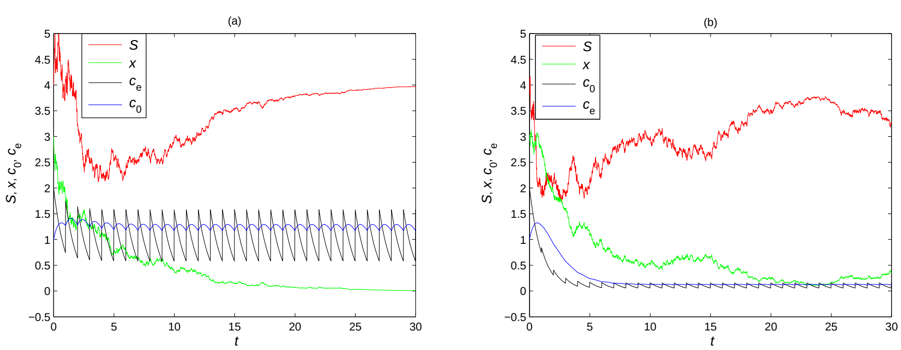

Figure 2. Numerical simulation of the paths $S(t), x(t), c_0(t), c_e(t)$ for the chemostat stochastic system (1.2), where (a) $u = 1, \mathcal{R}^* = 0.928 < 1$; (b) $u = 0.1, \mathcal{R}^* = 1.1811 > 1$.

It is clear from the sample paths plots that the concentration of the unconsumed nutrient will stabilize at an equilibrium state when the microorganism tends to extinction eventually as demonstrated in Figs. 1(a) and 1(b). However, the concentration of the unconsumed nutrient displays much more vibrations when the microorganism is persistence in random environments, see demonstration in Figs. 2(b). To see the differences between a random and a deterministic environment, we also plot the component-wise sample paths of the microorganism in Fig. 1(b). Note that in the stochastic environment, the population size of the microorganism approaches zero very quickly.

This paper explores an impulsive stochastic chemostat model with saturated growth in a polluted environment. The threshold of the impulsive stochastic system which governs the extinction and permanence of the microorganism is obtained. From Lemma 2.3, Theorems 3.1 and 3.2, we can see that, there is a significant difference compared with the threshold of system (1.1), that is, the conditions for the microorganism to become extinct in the stochastic system (1.2) are weaker than in the corresponding deterministic model (1.1). When $\mathcal{R}^* = \mathcal{R} - \frac{\sigma^2 S_0^2}{2a^2(Q+r\overline{c_0})} < 1 < \mathcal{R}$, a permanent deterministic system can go to extinction under the white noise stochastic disturbance. Therefore, the biological results show the white noise stochastic disturbance is disadvantage for the permanence of system. Our theoretical conclusions are validated and illustrated by the above numerical simulations. The theoretical method can also be used to explore the threshold of some impulsive stochastic differential equations.

Some interesting questions deserve further investigation. One could study more realistic but more complex models, for example, impulsive stochastic systems with Lévy jumps. The motivation is that the population may suffer sudden-environmental shocks, e.g., severe weather, earthquakes, floods, epidemics and so on. Moreover, it is interesting to investigate the effects of impulsive and stochastic perturbations on the probability of extinction of certain population. Also it is interesting to study adaptive dynamics of stochastic evolutionary model, and we leave these for future work.

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