ADDITION AND SUBTRACTION THEOREMS FOR THE SINE AND THE COSINE IN MEDIEVAL INDIA

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The paper deals with the rules of finding the sines and the cosines of the sum and difference of two angles when those of the two angles are known separately. The rules, as found in the important medieval Indian works, are equivalent to the correct modern mathematical results. Indians of the said period also knew several proofs of the formulas. These proofs are based on simple algebraic and geometrical reasoning, including proportionality of sides of similar triangles and the Ptolemy's Theorem. The enunciations and derivations of the formulas presented in the paper are taken from the works of the famous authors of the period, namely, Bhāskara II (A.D. 1150), Nīlakaṇṭha Somayāji (1500), Jyeṣṭhadeva (16th century) Munīśvara (1st half of the 17th century) and Kamalākara (2nd half of the 17th century).

1. INTRODUCTION

According to Carl B. Boyer[1], the introduction of the sine function represents the chief contribution of the *Siddhāntas* (Indian astronomical works) to the history of mathematics. The Indian Sine (usually written with a capital S to distinguish it from the modern sine) of any arc in a circle is defined as the length of half the the chord of double the arc. Thus the (Indian) Sine of any arc is equal to $R \sin A$, where R is the radius (norm or *Sinus totus*) of the circle of reference and $\sin A$ is the modern sine of the angle, A, subtended at the centre by the arc. Likewise, the (Indian) Cosine function is equivalent to $R \cos A$ and similarly for the Versed Sine and its complement. The *Āryabhaṭīya* of Āryabhaṭa I (born 476 A.D.) is the earliest extant historical work of the dated type in which the Indian trigonometry is definitely used.

The modern forms of the Addition and the Subtraction Theorems for the sine and the cosine functions are :

$$\sin (A+B) = (\sin A).(\cos B)+(\cos A).(\sin B) \quad \ldots (1)$$

$$\sin (A-B) = (\sin A).(\cos B)-(\cos A).(\sin B) \quad \ldots (2)$$

$$\cos (A+B) = (\cos A).(\cos B)-(\sin A).(\sin B) \quad \ldots (3)$$

$$\cos (A-B) = (\cos A).(\cos B)+(\sin A).(\sin B) \quad \ldots (4)$$

The present paper concerns the equivalent forms of the above four Theorems for the corresponding Indian trigonometric functions. Statements as well as derivations of these formulas, as found in important Indian works, are described in it.

2. Statement of the Theorems

Bhāskara II (A.D. 1150) in his *Jyotpatti*, which is given at the end of the *Golādhyāya* part of his famous astronomical work called *Siddhānta Śiromaṇi*, states[2]

> *Cāpayoriṣṭayor-dorjye mithaḥ koṭijyakāhate* /
> *Trijyā-bhakte tayoraikyaṃ syāccāpaikyasya dorjyakā* //21//
> *Cāpāntarasya jīvā syāt tayorantarasammitā* //21½//

'The Sines of the two given arcs are crossly multiplied by (their) Cosines and (the products are) divided by the radius. Their (that is, of the quotients obtained) sum is the Sine of the sum of the arcs ; their difference is the Sine of the difference of the arcs.'

$$R \sin (A \pm B) = (R \sin A).(R \cos B)/R \pm (R \cos A).(R \sin B)/R \quad .. \quad (5), (6)$$

Thus we get the Addition Theorem (called the *Samāsa-Bhāvanā* by Bhāskara II) and the Subtraction Theorem (called the *Antara-Bhāvanā*) for the Sine.

We have some reason (*see* below and also *Section* 3) to believe that Bhāskara II was aware of the corresponding Theorems for the Cosine. According to the *Marīci* commentary (= *MC*) by Munīśvara (1638) on the *Jyotpatti*, a reason for Bhāskara's omission (*upekṣā*) of the Cosine formulas was that the following alternately shorter procedure, after having obtained $R \sin (A \pm B)$, was known[3]

$$R \cos (A \pm B) = \sqrt{R^2 - \{R \sin (A \pm B)\}^2} \quad .. \quad (7)$$

Kamalākara (1658) also mentions (or quotes *MC*) in his commentary on his own *Siddhānta Tattva-viveka* (= *STV*) that the *Ācārya* (Bhāskara) has not followed or given the Cosine Theorems because of the exactly same reason as stated in the *MC*[4].

In the late Āryabhaṭa School the Addition-Subtraction Theorem for the Sine was known as the *Jīveparaspara-Nyāya* and is attributed to the famous Mādhava of Saṅgamagrāma (*circa* 1340-1425) who is also referred as *Golavid* ('Master of spherics)[5]. The *Tantra-Saṃgraha* (= *TS*), composed by Nīlakaṇṭha Somayāji (A.D. 1500), gives Mādhava's rule in Chapter II as[6]

> *Jīve paraspara-nijetara-maurvikābhyā-Mabhyasya vistṛtidalena vibhajyamāne* //
> *Anyonya-yogavirahānuguṇe bhavetāṃ Yadvā svalambakṛtibheda-padīkṛte dve* //16 //

'The Sines (of two arcs) reciprocally multiplied by the Cosines and divided by the radius, when added to and subtracted from each other, become the Sines of the sum and difference of the arcs (respectively). Or (we get the same results when the mutual addition and subtraction is performed with) the two (positive) square-roots of the (two) differences of their own (that is, of the two Sines themselves) and *lamba* squares.'

So that the first part of the rule gives the formulas (5) and (6), while the second part contains the alternate formulas

$$R \sin (A \pm B) = \sqrt{(R \sin A)^2 - (lamba)^2} \pm \sqrt{(R \sin B)^2 - (lamba)^2} \qquad .. \quad (8),(9)$$

Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya* ($=$ *NAB*) and Śaṅkara Vāriar (A.D. 1556) in his commentary ($=$ *TSC*)[8] on the above rule explain that the *lamba* involved is to be calculated from the relation

$$lamba = (R \sin A).\,(R \sin B)/R \qquad .. \quad (10)$$

Thus it will be noticed that the forms (8) and (9) are mathematically equivalent to the formulas (5) and (6).

An important point to note is that the *TSC* (pp. 22-23) makes it clear that the word *jīve* (which we have translated as Sines) can be also taken to mean Cosines. But in such a case the phrase *nijetaramaurvikās* ('the other chords') should be taken to mean, as the *TSC* points out, the corresponding Sines. In other words we can get the same Addition and Subtraction Theorems for the Sine, if we interchange "sin" and "cos" with each other in the right hand sides of (5) and (6). Following this interpretation the forms (8) and (9) can also be expressed as

$$R \sin (A \pm B) = \sqrt{(R \cos A)^2 - (lamba)^2} \pm \sqrt{(R \cos B)^2 - (lamba)^2} \qquad .. \quad (11),\ (12)$$

where the *lamba* will now be given by

$$lamba = (R \cos A).\,(R \cos B)/R \qquad .. \quad (13)$$

This interpretation also leads to the same Addition and Subtraction Theorems for the Sine.

For a geometrical interpretation of the quantity *lamba*, see *Section* 4 below.

Almost the same Sanskrit text of Mādhava's rule is also found quoted in the *NAB* (part I, p. 58) where it is explicitly mentioned to be '*Mādhava-nirmitaṃ padyam*' that is, a stanza composed by Mādhava. In this connection the *NAB* (part I, p. 60) also mentions the variant readings :

vistṛtiguṇena and *vistṛtidalena*

For the Addition and Subtraction Theorems of the Cosine function, we may quote the *Siddhānta Sārvabhauma* ($=$ *SSB*, 1646 A.D.), II, 57 which says[3]

Yadaṃśajyayorghāta-hīnā'dhikā ca Svakoṭijyayorāhatis-trijakāptā |
Tadaṃśaikyaviśleṣahīnābhranandāṃ-śayorjye sta (57||

'The product of the Sines of the degrees (of two arcs) subtracted from or added to the product of their Cosines, and (the results) divided by the radius, become the Sines of the sum or difference of the degrees diminished from the ninety degrees.' That is,

$$R \sin (90^\circ - \overline{A \pm B}) = (R \cos A\,.\,R \cos B \mp R \sin A\,.R \sin B)/R \qquad .. \quad (14),\ (15)$$

which are equivalent to the Theorems (3) and (4).

The *STV* (A.D. 1658), III, 68-69 (p. 111) puts all the four Theorems side by side in the following words clearly.

Mithaḥ koṭijyakā-nighnyau trijyāpte cāpayorjyake /
Tayoryogāntare syātāṃ cāpayogāntarajyake //68//

Dorjyayoḥ koṭimaurvyośca ghātau trijyod-dhṛtau tayoḥ /
Viyogayogau jīve staścāpaikyāntara-koṭije /69//

'Multiply the Sines of the two arcs crossly by the Cosines and divide (separately) by the radius. Their (that is, of the two quotients obtained) sum and difference are the Sines of the sum and difference of the arcs (respectively). The products of the Sines and the Cosines are (each) divided by the radius. Their (that is, of the two quotient just obtained) difference and sum are (respectively) the Cosines of the, sum and difference of the arcs.'. That is,

$$R \sin (A \pm B) = (R \sin A).\ (R \cos B)/R \pm (R \sin B).\ (R \cos A)/R$$

$$R \cos (A \pm B) = (R \sin A).\ (R \sin B)/R \mp (R \cos A)\ (R \cos B)/R$$

Immediately after the statement of the above Theorems, the author, Kamalākara, in the next two verses (*STV*, III, 70-71), says

Evamānayanaṃ cakre pūrvaṃ Svīyaśiromaṇau /
Bhāvanābhyāmatispaṣṭaṃ samyagāryo'pi a bhāskaraḥ //70//
Tasya cānayanasyāryaiḥ siddhāntajñaiḥ puroditā /
Vāsanā bahubhih svasvabuddhivaicitryataḥ sphuṭā //71//

'Such a computation, which is quite evident from the two *bhāvanās* (see the next *Section*), was given earlier also by the highly respected Bhāskara in his (*Siddhānta-*) *Śiromaṇi*.* And many accurate proofs of that computation have been given previously by the respected astronomers according to the manifoldness of their intelligence.'

Below we outline the various derivations as found in some Indian works and which indicate the ways through which Indians understood the rationales of the Theorems.

3. METHOD BASED ON THE THEORY OF INDETERMINATE ANALYSIS

The second degree indeterminate equation

$$Nx^2 + k = y^2 \qquad \ldots \quad (16)$$

*Alternately, the first verse may be translated thus : 'This was very clearly computed earlier by the respected Bhāskara also through the *bhāvanās* in his *Siddhānta-Śiromaṇi*.'

is called *varga-prakṛti* ('square-nature) in the Sanskrit works. In connection with its solution the following two Lemmas, referred to as Brahmagupta's (A.D. 628) Lemmas by Datta and Singh[10]. have been quite popular in Indian mathematics since the early days.

Lemma I : If x_1, y_1 is a solution of (16) and x_2, y_2 that of

$$Nx^2+g = y^2 \qquad .. \quad (17)$$

then (*Samāsa-bhāvanā*)

$$x = x_1\, y_2+y_1\, x_2,\ y = y_1y_2+Nx_1\, x_2$$

is a solution of the equation

$$Nx^2+kg = y^2 \qquad .. \quad (18)$$

Lemma II : (*Antara-bhāvanā*)

$$x = x_1y_2-y_1x_2,\ y = y_1\, y_2\, -Nx_1\, x_2$$

is also a solution of (18).

An elaborate discussion of the subject including references to Sanskrit works. translations, proofs, and terminology is available and need not to be reproduced here[11]. Dr. Shukla's paper on Jayadeva (not later than 1073) is an additional noteworthy publication in this connection[12].

Now, as indicated by the terminology used by Bhāskara II and clearly explained by his great mathematical commentator, Munīśvara, it is evident that Bhāskara II arrived at the truth of the Addition and Subtraction Theorems for the Sine (and Cosine) possibly by applying the above Lemmas as follows.

As explained in the *MC* (pp. 150-51) on *Jyotpatti* 21-25, if we compare the equation (16) with the relation

$$-(R \sin Q)^2 +R^2 = (R \cos Q)^2 \qquad .. \quad (19)$$

we see that we can take

guṇaka (multiplier) $N = -1$

kṣepaka (interpolator) $k = R^2$

$$x = R \sin Q$$

$$y = R \cos Q$$

Hence, on identifying

$$k = g = R^2$$

$$x_1 = R \sin A,\ y_1 = R \cos A$$

$$x_2 = R \sin B,\ y_2 = R \cos B,$$

we at once see, from Lemma I, that

$$x = (R \sin A).\ (R \cos B)+(R \cos A).\ (R \sin B) \qquad .. \quad (20)$$

and

$$y = (R \cos A).\ (R \cos B)-(R \sin A).\ (R \sin B) \qquad .. \quad (21)$$

is a solution of the equation

$$-x^2+R^4 = y^2$$

that is, (x/R) and (y/R) will be a solution of (16), since

$$-(x/R)^2 + R^2 = (y/R)^2 \quad .. \quad (22)$$

Thus comparing (19) and (22), we see, from (20) and (21), that

$$\{(R \sin A).(R \cos B)+(R \cos A).(R \sin B)\}/R$$

and

$$\{(R \cos A).(R \cos B)-(R \sin A).(R \sin B)\}/R$$

will represent some sort of additive solutions for the Sine and Cosine functions respectively. The above were taken to represent R sin $(A+B)$ and R cos $(A+B)$ respectively. From mathematical point of view, there is a lacuna in such an identification without further justification[13].

Similarly, by using Lemma II, the expansions of $R\sin(A-B)$ and $R\cos(A-B)$ were identified.

Such a derivation undoubtedly supports the view that Bhāskara must have been aware of the Addition and Subtraction Theorems for the Cosine, although he did not state them.

The *SSB*, II, 58-59 (pp. 144-45), whose author is same as that of *MC*, also gives the same derivation of the Theorems (also see *STVC*, pp. 112-13).

4. Geometrical Derivation as given in the *NAB*

While explaining Mādhava's Sanskrit stanza (*Jīveparaspara* etc.) and the implied rules, which we have already mentioned above, the *NAB* (part I, p. 59) says :

> *Tatrādyapādatrayātmakamekaṃ*
> *vākyam. Carmaḥ pādo vākyāntaramiti vibhāgaḥ.*
> *Tatrādye vākye trairāśikena*
> *tadānayanaṃ pradarśyate. Anyasmin*
> *bhujākoṭikarṇa dvārā vargamūla*
> *parikalpanayā.*

'The first three lines (of the stanza) form one rule (or method). The last line represents another rule. This is the break-up. We demonstrate the derivation of the first rule by (applying) the Rule of Three (that is, the proportionality of sides in the similar triangles). The other (rule) will follow from the relation between the base, upright and hypotenuse (or Sine, Cosine and radius) by extracting the square-root'.

The geometrical demonstration given in the *NAB* (part I, pp. 58-61) may be substantially outlined as follows :

In Fig. 1 (East direction is upwards).

arc $EP = A$

arc $PQ =$ arc $PG = B$ (being less than A).

So that

$$\text{arc } EQ = A+B$$
$$\text{arc } EG = A-B$$

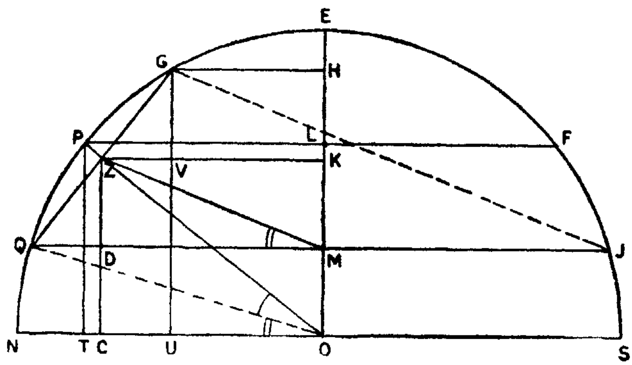

Fig. 1

Here OP and QG intersect at Z and

$$PL = R \sin A = TO$$
$$OL = R \cos A$$
$$QZ = R \sin B = ZG$$
$$OZ = R \cos B = OP-PZ$$
$$QM = R \sin (A+B), \text{ which is required to be found out.}$$

In order to find QM, we determine its two portions. QD and DM, made by the line ZC (drawn westwards from Z), separately and add them. Now to find the southern portion DM (which is equal to KZ) we have, from the similar right triangles OZK and OPL,

$$ZK/OZ = PL/OP$$

or

$$DM/(R \cos B) = (R \sin A)/R$$

giving

$$DM = (R \sin A).\ (R \cos B)/R \qquad \text{.. (23)}$$

Again, to find the northern portion DQ, we have, from the similar right triangles DQZ and OLP

$$DQ/ZQ = OL/OP$$

or

$$DQ/(R \sin B) = (R \cos A)/R$$

giving

$$DQ = (R \cos A).\ (R \sin B)/R \qquad \text{.. (24)}$$

By adding (23) and (24) we get QM which represents $R \sin (A+B)$. Thus is proved the Addition Theorem for the Sine.

For proving the Subtraction Theorem, drop perpendicular GU from G on ON. It divides ZK, which is equal to DM given by (23), into two portions VZ

and VK. The northern portion VZ is equal to DQ given by (24) because the hypotenuse ZG is equal to the hypotenuse ZQ. Hence the southern portion

$$VK = ZK - DQ$$

or

$$GH = DM - DQ$$

That is,

$$R \sin (A-B) = (R \sin A).\ (R \cos B)/R - (R \cos A).\ (R \sin B)/R$$

the required Subtraction Theorem.

Again, since

$$(R \sin A).\ (R \cos B)/R = (R \sin A).\ \{\sqrt{R^2-(R \sin B)^2}\}/R$$
$$= \sqrt{(R \sin A)^2 - \{(R \sin A).\ (R \sin B)/R\}^2}$$
$$= \sqrt{(R \sin A)^2 - (lamba)^2}$$

we can easily get the forms (8) and (9) from the forms (5) and (6) mathematically (see *NAB*, part I, pp. 86-87).

The *NAB* (part I, p. 87-88) has also given some further geometrical interpretations and computations which we now indicate. In Fig. 1, $R \sin (A+B)$, that is, QM, is the base of the triangle ZQM. The second (or smaller) Sine, $R \sin B$, that is, QZ is the left side. The greater Sine, $R \sin A$, is the right side ZM. (How ?)

The foot of the perpendicular (*lamba*), D, divides the base into two segments (*ābādhās*) DQ and DM which have been already found out. So that the *lamba*, given by (10), can be easily identified with the length ZD, the altitude of the triangle ZQM (this follows from $ZD^2 = ZQ^2 - DQ^2$). Then, from (see *NAB* part I, p. 88)

$$\sqrt{DM^2+ZD^2} = ZM,$$

we get, using (10) and (23),

$$ZM = R \sin A.$$

5. A Proof Based on Ptolemy's Theorem

Jyeṣṭhadeva (circa 1500-1610)[14] wrote *Yuktibhāṣā* (= *YB*) in Malayalam. Part I of the work presents an elaborate and systematic exposition of the rationale of the mathematical formulas[15].

YB, (pp. 206-208 and 212-13) explains Mādhava's rules concerning the Addition and Subtraction Theorems for the Sine more or less on the same lines as given in the *NAB*. However, the *YB* (pp. 237-38) also indicates a proof of the Addition Theorem for the Sine by applying the so-called Ptolemy's Theorem, namely :

'In a cyclic quadrilateral the sum of the products of the opposite sides is equal to the product of the diagonals'.

Of course, before indicating this use of the Ptolemy's Theorem, the *YB* (pp. 228-36) has given a proof of it. According to Kaye[16], a proof of the Ptolemy's Theorem was also given by a commentator (Pṛthūdaka ?, ninth century) of Brahmagupta (A.D. 628), the famous Indian mathematician who knew the correct expressions (which immediately yield the Ptolemy's Theorem on multiplication) for the diagonals of a cyclic quadrilateral[17].

The proof indicated in the *YB* and as explained by its editors (pp. 237-39) may be outlined as follows :

In Fig. 2

$$\text{arc } PE = A$$
$$\text{arc } QP = \text{arc } QG = B$$

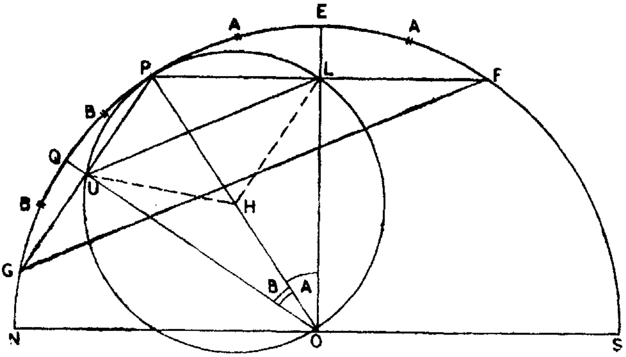

Fig. 2

The radius OQ intersects PG in U. Thus PL and OL are the Sine and the Cosine of A and PU and OU those of B. From the cyclic quadrilateral $LPUO$, we have, by applying the rule of *bhujā-pratibhujā etc.*

(that is, the Ptolemy's Theorem),

$$PL.\,OU+OL.\,PU = LU.\,OP$$

or

$$(R \sin A).\,(R \cos B)+(R \cos A).\,(R \sin B) = LU.R \qquad .. \quad (25)$$

The relation (25) will establish the Addition theorem for the Sine provided we are able to identify that LU represents the Sine of $(A+B)$. For seeing this, it may be noted that LU is the full chord of the arc $(LP+PU)$ in the circle which circumscribes the quadrilateral in question and whose radius is $R/2$ (as the centre of this smaller circle will be at H, the middle point of the radius OP equal to R). Thus

$$LU = 2\,(R/2) \sin \{(2A+2B)/2\}$$
$$= R \sin (A+B)$$

We can also prove this by observing that LU is parallel to and half of the side FG in the triangle PFG. But FG itself is the full chord of the arc GPF in the bigger circle, so that

$$FG = 2R \sin \{(2A+2B)/2\}$$

6. A Geometrical Proof Quoted in the *MC* (1638)

The *MC* (pp. 154-55) contains a geometrical proof, ascribed to others (*kecid*), which is only slightly different fromt that found in the *NAB* (see *Section* 4). It may be outlined as follows :

Firstly, the *MC* asks us to draw a figure similar to Fig. 1 which may be referred now. In the triangle ZQM, the base QM is the desired Sine of the combined arc $(A+B)$. The smaller side QZ is R sin B. The distance between Z and M, that is, the larger (lateral) side ZM, is equal to R sin A evidently (*pratyakṣa-pramāṇāvagatā* ?) In order to know the base QM, its two segments QD and DM should be found out.

Now the *MC* finds QD exactly in the same manner as *NAB* (see the derivation of the relation (24)). Similarly, from the similar right triangles DMZ and OZQ, we have

$$DM/MZ = OZ/OQ$$

or

$$DM/(R \sin A) = (R \cos B)/R$$

which gives the bigger segment DM and hence their sum $(QD+DM)$ proves the Addition Theorem for the Sine.

After this, the *MC* also indicates the method for proving the Subtraction Theorem for the Sine.

We note that, in proving the Addition Theorem above, the *MC* does not give any theoretical details to demostrate that the length ZM is equal to R sin A. One way of proving this could be by noting that ZM is parallel to and half of the side GJ in the triangle QGJ ; and GJ is itself the full chord of the arc GEJ which is easily seen to be equal to $2A$, so that $GJ = 2R \sin A$.

Alternately, we can see that a circle, of radius $R/2$, drawn on OQ as the diameter will pass through the points Q, Z, M and O and ZM will be a full chord (subtending angle $2A$ at the centre) of this smaller circle. So that we have

$$ZM = 2(R/2) \sin A$$

Once the flank sides of the triangle ZQM are thus identified, the perpendicular ZD could also be obtained directly by using a well-known geometrical rule equivalent to[18]

$$\text{perp.} = \frac{\text{product of flank sides}}{\text{twice the circum-radius}}$$

giving

$$ZD = ZQ.ZM/2.\,(R/2)$$
$$= (R \sin B).\,(R \sin A)/R$$

Thus, knowing ZQ, ZM and ZD, we can easily get the segments QD and DM and hence the required length QM. This provides an alternate and independent rationale of the Addition Theorem for the Sine in the form (8).

7. Proofs Found in *STVC*

We have already mentioned the observation of *STV*, III, 71 that several proofs of these Theorems were given by the previous writers. One set of derivations as given in the *STVC* (pp. 125-29) may be briefly outlined as follows :

In Fig. 3, arcs EP and EQ are equal to A and B respectively. Other constructions are obvious from the figure. It can be easily seen that

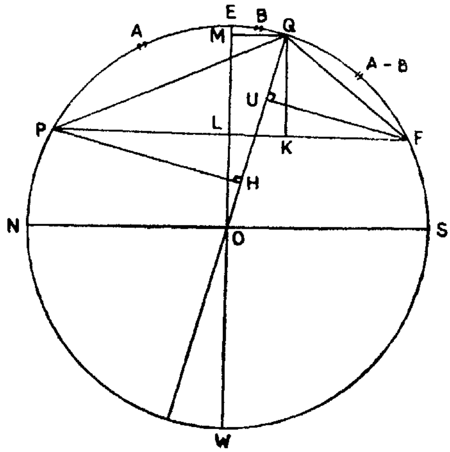

Fig. 3

$$PK = PL+MQ = R \sin A+R \sin B$$

$$QK = OM-OL = R \cos B-R \cos A$$

Therefore.

$$PQ^2 = (R \sin A+R \sin B)^2+(R \cos B-R \cos A)^2$$

$$= 2R^2+2R \sin A.\ R \sin B-2R \cos A.\ R \cos B \quad .. \quad (26)$$

But PQ is the full chord of the arc $(A+B)$, so that

$$PQ/2 = R \sin \{(A+B)/2\} \quad .. \quad (27)$$

Now from a rule given in the *Jyotpatti*, 10 (p. 282), which the *STVC* (p. 126) quotes, we have

$$R \sin\ (A+B)/2 = \sqrt{(R/2).\ R \text{ vers } (A+B)} \quad .. \quad (28)$$

That is,

$$R \text{ vers } (A+B) = (2/R).\ \{R \sin (A+B)/2\}^2$$

$$= (1/2R).\ PQ^2, \text{ by } (27)$$

Using (26), we easily get

$$R \text{ vers } (A+B) = R+(R \sin A.\ R \sin B-R \cos A.\ R \cos B)/R$$

from which the required expression for $R \cos (A+B)$ follows, since

$$R \cos (A+B) = R-R \text{ vers } (A+B).$$

Here it may be pointed out that the *STVC* (p. 126) also states that PQ, which we have found above from the triangle PQK, is also the hypotenuse for the right angled triangle PQH (PH being perpendicular to the radius OQ). Incidently, this gives an alternate procedure for proving the Addition Theorem for the Cosine. For, we have

$$PK^2+QK^2 = PQ^2 = PH^2+QH^2$$

or

$$(R \sin A+R \sin B)^2+(R \cos B-R \cos A)^2 = \{R \sin (A+B)\}^2 + \{R-R \cos (A+B)\}^2$$

or

$$2R^2+2R \sin A \,.\, R \sin B-2R \cos A \,.\, R \cos B = 2R^2-2R \,.\, R \cos (A+B)$$

giving the required expansion of $R \cos (A+B)$.

Anyway, after getting the expression for $R \cos (A+B)$, the *STVC* (pp. 127-28) derives the corresponding expression for $R \sin (A+B)$ by using the relation

$$\{R \sin (A+B)\}^2 = R^2- \{R \cos (A+B)\}^2$$

Again, in the same figure the arc QF represents $(A-B)$. Also we have

$$FQ^2 = FK^2+QK^2 = (PL-QM)^2+(OM-OL)^2$$

or

$$\{2R \sin (A-B)/2\}^2 = (R \sin A-R \sin B)^2+(R \cos B-R \cos A)^2 \qquad (29)$$

If we proceed as we did in the case of proving $R \cos (A+B)$ above, we easily get the desired expression for $R \cos (A-B)$. Alternately we get the same expansion by starting with the relation

$$FK^2+QK^2 = FU^2+QU^2$$

and proceeding as before.

Finally, the corresponding Subtraction Theorem for the Sine can be derived from that for the Cosine.

It is interesting to note that an equivalent of the identity (29) already occurs in the *Jyotpatti*, 13, (p. 282). Thus Bhāskara II's familiarity with the relation (29) and that implied in (28) (where $A-B$ should be used for $A+B$), was enough to derive the Subtraction Theorems by this method (if he wanted to do so).

Another proof given in the *STVC* (pp. 130-35) may be briefly outlined as follows: In Fig. 4

$$\text{arc } EP = \text{arc } EF = 2A$$
$$\text{arc } EQ = 2B$$

It is important to note that the *STVC* says that the radius of the circle drawn is $R/2$ where R is the *Sinus totus*, so that the full chords EP, EQ, etc. will themselves behave as the Sines. That is, we have

$$EP = 2\,(R/2) \sin A = R \sin A$$
$$EQ = R \sin B$$
$$PW = R \cos A$$
$$QW = R \cos B$$

etc., and, of course

$$EW = R$$

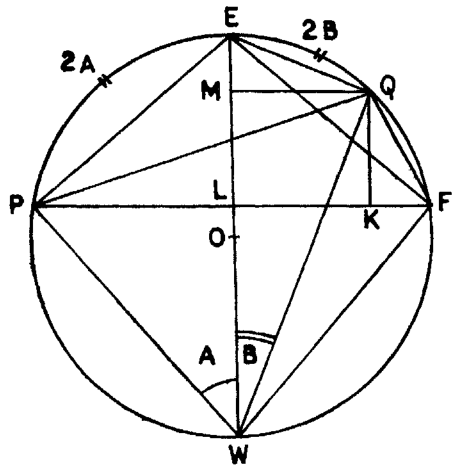

Fig. 4

Now, by the methods of finding the altitude and segments of the base in a triangle, we have

$$\text{segment } EM = (R \sin B)^2/R$$
$$\text{segment } WM = (R \cos B)^2/R$$
$$\text{perpendicular } QM = (R \sin B).(R \cos B)/R$$
$$\text{segment } EL = (R \sin A)^2/R$$
$$\text{segment } WL = (R \cos A)^2/R$$
$$\text{perp } PL = \text{perp } LF = (R \sin A).(R \cos A)/R$$

(Of course, all these results also follow from similar right triangles in the figure.) Now we have

$$PQ^2 = PK^2 + QK^2$$
$$= (PL+QM)^2 + (EL-EM)^2$$

On substituting from the above expressions, simplifying, and on taking the square-roots we easily get the required expression for $R \sin (A+B)$ represented by PQ.

Again we have

$$FQ^2 = FK^2 + KQ^2$$
$$= (PL-QM)^2 + (EL-EM)^2$$

Thus, following the same procedure, we get the required expression for $R \sin (A-B)$ represented by QF.

However, before closing this article, it may not be out of place to mention that by using the Ptolemy's Theorem in Fig. 4, we get the expressions for PQ and QF almost in one step. For, Ptolemy's Theorem applied to the quadrilateral $EPWQ$ yields

$$EP.QW + PW.EQ = PQ.EW$$

which gives the desired PQ ; and Ptolemy's Theorem applied to the quadrilateral $EQFW$ yields

$$EQ.FW + QF.EW = EF.QW$$

which gives the desired QF.

Acknowledgements

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References and Notes

[1] Boyer, Carl B. : *A History of Mathematics*. Willey, New York, 1968, p. 232.

[2] The *Siddhānta Śiromaṇi* (with the author's own commentary called *Vāsanābhāṣya*) edited by Bapu Deva Sastri, Chowkhamba Sanskrit Series Office. Benares (Varanasi). 1929, p. 283. (Kashi Sanskrit Series No. 72).

The *Jyotpatti*, consisting of 25 Sanskrit verses, given towards the end of the above edition, may be regarded either as Chapter XIV of Or as an Appendix to the *Golādhyāya* part of the work.

[3] *Golādhyāya* (with *Vāsanābhāṣya* and *Marīci*) edited by D. V. Apte. Anandarama Sanskrit Series No. 122, Part I, Poona, 1943, p. 152.

The *MC* on *Jyotpatti* is found here in Chapter V itself, instead of at the end of the work. All page references to *MC* are as per this edition.

[4] *The Siddhānta Tattva-viveka* (with the author's own commentary on it) edited by S. Dvivedi. Benares Sanskrit Series No. 1, Fasciculus I. Braj Bhusan Das & Co., Benares (Varanasi). 1924, p. 113. All page references to *STV* and its commentary are as per this edition.

[5] Sarma, K. V. : *A History of the Kerala School of Hindu Astronomy* (in *Perspective*), Vishveshvaranand Institute, Hoshiarpur. 1972, p. 51. (Vishveshvaranand Indological Series No. 55).

[6] The *Tantrasaṃgraha* (with the commentary of Śaṅkara Vāriar) edited by S. K. Pillai, Trivandrum Sanskrit Series No. 188, Trivandrum, 1956, p. 22.

[7] The *Āryabhaṭīya* with the *Bhāṣya* (gloss) of Nīlakaṇṭha Somasutvan, edited by K. Sambasiva Sastri, Part I (*Gaṇitapāda*), Trivandrum Sanskrit Series No. 1, Trivandrum, 1930, p. 87.

[8] *TS*, cited above, p. 23 and Sarma, *op. cit.*, p. 58.

[9] The *Siddhānta Sārvabhauma* (with the author's own commentary) edited by Muralidhara Thakkura, Saraswati Bhavana Texts No. 41, Part I, Benares, 1932, p. 143.

[10] Datta, B. B. and A. N. Singh, *History of Hindu Mathematics. A Source Book*. Single volume edition, Asia Publishing House, Bombay, 1962, Part II, p. 146.

[11] *Ibid.*, pp. 141-172.

[12] Shukla, K. S., "Ācārya Jayadeva, the mathematician." *Gaṇita* 5. No. 1 (June 1954), pp. 1-20.

[13] If we compare equation (16) with the identity

$$\tan^2 Q + 1 = \sec^2 Q,$$

we see that ($\tan A$, $\sec A$) and ($\tan B$, $\sec B$) are solutions of the equation

$$x^2 + 1 = y^2$$

whose *samāsa-bhāvanā* solution, therefore, will be given by

$x = \tan A . \sec B + \sec A . \tan B$

$y = \sec A . \sec B + \tan A . \tan B$.

But here x and y do not represent $\tan (A+B)$ and $\sec (A+B)$.

[14] Sarma, K. V., *op. cit.*, pp. 59-60.

[15] *Yuktibhāṣā Part I* (in Malayalam) edited with notes by Ramavarma Maru Thampuran and A. R. Akhileswar Aiyer, Mangalodayam Press, Trichur, 1948.

[16] Kaye, G. R., "Indian Mathematics." *Isis*, Vol. 2 (1919), pp. 340-41.

[17] *Ibid.*, p. 339 and Boyer, *op. cit.*, p. 243.

[18] *Brāhmasphuṭa Siddhānta* of Brahmagupta (A.D. 628), XII, 27 (New Delhi edition, 1966, Vol. III, p. 834); *Siddhānta-śekhara* of Śrīpati (1039), XIII, 31 (Calcutta edition, 1947, Part II, p. 48); *YB*, p. 231.