Computational content of the Axiom of Univalence

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The Axiom of Univalence

A strengthening of Church's axiom of extensionality in simple type theory

Two forms of this axiom $10^{\alpha\beta}$, function extensionality, and 10^{o}

10^{o} states that two propositions are logical equivalent if, and only if, they are equal

10^{o} is not taken as an axiom in Church's original paper

We remark, however, on the possibility of introducing the additional axiom of extensionality, $p \equiv q \supset p = q$*, which has the effect of imposing so broad a criterion of identity between propositions that they are in consequence only two propositions, and which, in conjunction with* $10^{\alpha\beta}$*, makes possible the identification of classes with propositional functions*

Equivalent Types

Voevodsky formulated a strengthening of this axiom of extensionality, stating roughly that two *equivalent* types are equal

He also shows how to formulate *uniformely* in dependent type theory a notion of equivalence which generalizes

-logical equivalence of propositions

-bijection between sets

-equivalence of groupoids, . . .

The Axiom of Univalence

The axiom implies (in dependent type theory) that two isomorphic algebraic structures are equal (P. Aczel, T.C. and N.A. Danielsson) and that two equivalent categories are equal (M. Shulman)

But this is an *axiom*

How to justify it?

Explanation of Extensionality

Takeuti (1953) Gandy (1956)

explain the axiom of extensionality (both for propositions and for functions)

For propositions, the natural idea is to *define* equality as logical equivalence

This idea is present in Russell *The Theory of Implications*, 1906

For functions, one defines the equality by induction of the type

Can we generalize this to Dependent Type Theory?

Explanation of Extensionality

One can present this explanation of extensionality as follows

First we interpret a type as a collection with a relation

There is a natural notion of function space (logical relation) where

two functions are related if they send related input to related output

One shows that these operations only define *equivalence relations*

provided we start from equivalence relations at base types

Collection with equivalence relation = Bishop's notion of set a.k.a setoids

Explanation of Extensionality

We generalize this method

Collection with (reflexive) relation $\rightarrow$ *cubical set*

Equivalence relation $\rightarrow$ cubical set satisfying the *Kan condition*

Cubical sets

We have points, lines, square, cubes, ...

Two operations

-we can take the faces (semi-cubical sets)

-we have degeneracies: any point gives a constant line, any line gives a "constant" square (constant in one direction), and so on

The face operations are clear; the degeneracy operations are more subtle

They correspond to a generalization of the notion of *reflexive* relations

Identity Types

Using the degeneracy operations we have a natural interpretation of the identity type

Given two points a and b of a cubical set X one can define a cubical set corresponding to $Y = \mathsf{Id}_X\ a\ b$

-the points of Y are *lines* $a \rightarrow b$ in X

-the lines of Y are *squares* $a \rightarrow b$ in X where a (resp. b) point of Y, degenerate line from a (resp. b) in X

-the squares of Y are *cubes* $a \rightarrow b$ of X where a (resp. b) line of Y, degenerate square from a (resp. b) in X, and so on

Kan condition, classically

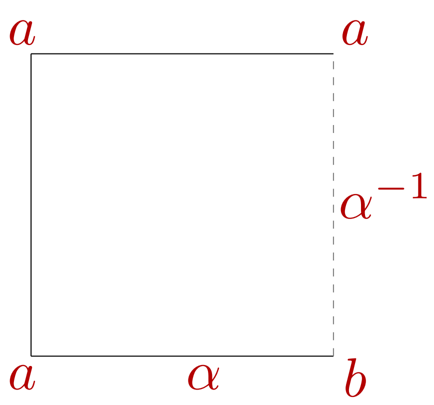
Classically, the Kan condition is simply that

any open box can be filled

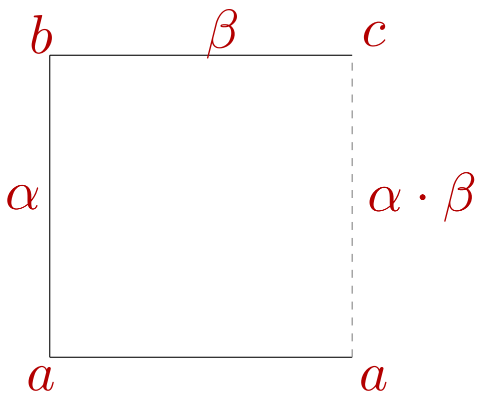
If we restrict ourselves to points, and lines we recover symmetry and transitivity (given that we already have reflexivity)

So, this notion is a remarkably simple generalization of the notion of equivalence relation (Kan, 1955)

Recovering symmetry and transitivity



symmetry



transitivity

Kan condition, classically

Considering only points and lines, we get *equivalence relations* à la Bishop

With squares, we get *groupoids*

With cubes, we get *2-groupoids*

. . .

Kan condition, classically

It will be convenient to think of the Kan filling operation as a combination of

-one *composition* which produces the missing face of a given open box

-one *filling operation* which fills the closed box

Kan condition, classically

It is rather direct that $\mathsf{Id}_X\ a\ b$ satisfies the Kan condition whenever X satisfies the Kan condition

We can associate to any Kan cubical set a group $\pi_1(X,a)$ in a purely combinatorial way, and then define $\pi_2(X,a) = \pi_1(\mathsf{Id}_X\ a\ a, 1_a),\ \pi_3(X,a),\ \dots$

This seems to be the simplest way for a combinatorial definition of homotopy groups (and was essentially Kan's original approach)

Maps of Kan cubical sets

If A and B are two Kan cubical sets a *morphism* $f : A \rightarrow B$ is a map of cubical set

We don't require any commutation with the Kan filling operation

If A and B are groupoids we obtain a notion of weak functor $A \rightarrow B$

It preserves strictly identity but only compositions in a weak way

Kan condition, classically

Effectivity problem: to prove closure under exponentiation of the Kan condition requires *decidability* of degeneracy

For a related issue, a Kripke counter-model shows that the simple Kan filling condition cannot work in a constructive framework if we want to interpret type theory (M. Bezem and T.C.)

So we need to refine this condition

How? We first need to understand better the notion of degeneracy

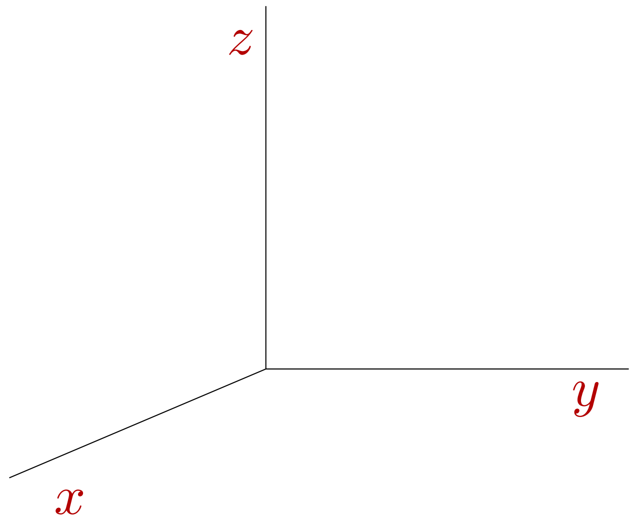
Cubical sets, reformulated

We want a simple formal representation of the (topological) intuition of points, lines, squares, cubes, ...

We give such a representation and then refine the Kan condition in this framework, in such a way that all operations become effective

Cubical sets, reformulated

We start from the trihedral picture



In a cubical set we can form lines $l(x)$ and squares $s(x, y)$ and cubes $c(x, y, z)$ using the *directions* x, y, z

Cubical sets, reformulated

Given a square $s(x, y)$ we can

-consider faces $s(0, y), s(1, y), s(x, 0), s(x, 1)$

-consider vertices $s(0, 0), s(0, 1), s(1, 0), s(1, 1)$

-build the degenerate cube $s'(x, y, z) = s(x, y)$ by adding a new variable z

This “explains” the notion of degeneracy: we add variables

In general we have a set of I-cubes for any finite set I of directions

Cubical sets, reformulated

More precisely, we introduce the following category $\mathcal{C}$

We fix a countable sets of *names* $x, y, z, \dots$ distinct from $0, 1$

A name should be thought of as an abstract notion of direction

An object of $\mathcal{C}$ is a finite set of names

A morphism $I \rightarrow J$ is a set map $I \rightarrow J \cup \{0, 1\}$ which is injective on its domain, i.e. if $i_0 \neq i_1$ and $f(i_0), f(i_1)$ in J then $f(i_0) \neq f(i_1)$

This represents a *substitution*: we can assign the value 0 or 1 or do renaming or add new variables

Cubical sets, reformulated

Definition: *a* cubical set *is a functor* $\mathcal{C} \to$ Set*.*

Definition: *If* X *is a cubical set, an* I-cube *of* X *is an element of* $X(I)$*.*

A cubical set X is a presheaf on the category $\mathcal{C}^{opp}$

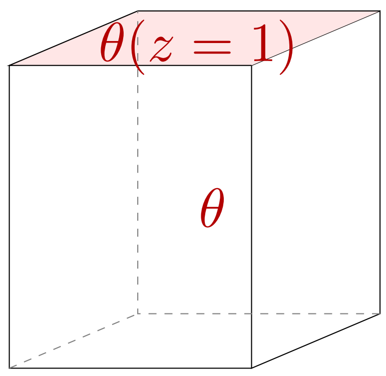
Via Yoneda, the object I can be thought of as a cubical set

We may think of this cubical set as a formal version of $[0,1]^I$

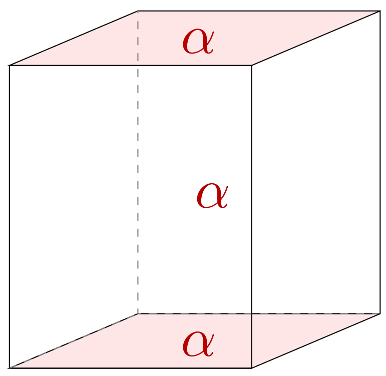
An I-cube is then a formal version of a map $[0,1]^I \to X$

Cf. singular cubical complexes

Cubical sets, reformulated



face map



degeneracy map

Example: the unit interval

We have two points a and b and a line $l(x) : a \rightarrow_x b$

a, b and $l(x)$ are primitive objects (constructors)

In each direction x we have three lines

$a(x) = a : a \rightarrow a \qquad b(y) = b : b \rightarrow b \qquad l(x) : a \rightarrow b$

In each directions x, y we have four squares

$a(x, y) = a, \qquad b(x, y) = b, \qquad u(x, y) = l(x), \qquad v(x, y) = l(y)$

and so on

The unit interval is a cubical set, which does not satisfy the Kan condition

Kan filling conditions, reformulated

Let $(u, \vec{u})$ be an open box, e.g.

(u_0^x, u_0^y, u_1^y) with $u_0^x(y = i) \ = \ u_i^y(x = 0)$

We have

one *composition operation* $X^+(u, \vec{u})$ which closes the box and

one *filling operation* $X \uparrow (u, \vec{u})$ which fills the box

Kan filling conditions, reformulated

$$X \uparrow (u_0^x, u_0^y, u_1^y)(x = 0) \;=\; u_0^x$$

$$X \uparrow (u_0^x, u_0^y, u_1^y)(y = 0) \;=\; u_0^y$$

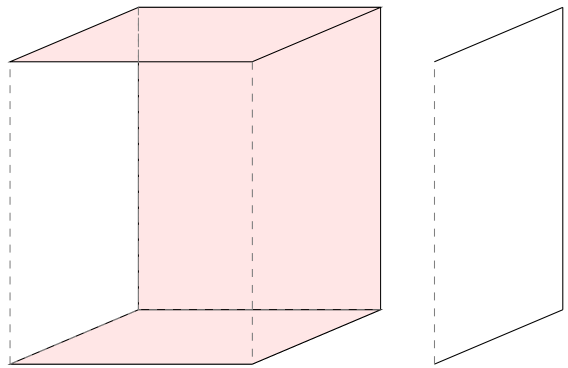
$$X \uparrow (u_0^x, u_0^y, u_1^y)(y = 1) \;=\; u_1^y$$

$$X \uparrow (u_0^x, u_0^y, u_1^y)(x = 1) \;=\; X^+(u_0^x, u_0^y, u_1^y)$$

We *allow* $u, \vec{u}$ to depend on more variables than only x and y and we *require* that the composition and filling operations commute with renaming and addition of variables

Kan filling conditions, reformulated

Geometrically addition of variables corresponds to new filling operations



In this example we have added the y direction

The faces of this filling should be the filling of the faces

The degeneracy of a filling should be the filling of the degeneracy

Kan filling conditions, reformulated

We define a *Kan cubical set* to be a cubical set with composition and filling operations that commute with renaming and addition of variables

This extra condition is natural for this notion of cubical sets

Adding this extra condition *solves* the effectivity problem

We get a(n effective) model of dependent type theory, interpreting a type by a Kan cubical set

Kan filling conditions, reformulated

Classically, from any simple Kan filling operations, we can get other filling operations that satisfy the stronger uniformity condition

In order to define these new operations, we have to rely on the decidability of degeneracy

Universe

What should be an equality $A \rightarrow B$ between two types A and B?

We require to have a "heterogeneous" notion of lines $a \rightarrow b$ where a point of A and b point of B

and a notion of squares $p \rightarrow q$ where p line of A and q line of B, $\dots$

This makes sense if A and B are only cubical sets and this represents a *relation* between A and B

In order to get an *equality* $A \rightarrow B$ we furthermore require that this heterogeneous notion of line, square, cube $\dots$ also satisfies the Kan condition

(Equivalently, this is a Kan fibration over the unit interval)

Example for Bishop sets

What is an equality $A \to B$?

The Kan condition tells us that if $a \to b,\ a \to a',\ b \to b'$ then $a' \to b'$

but also that if $a \to b,\ a' \to b',\ b \to b'$ then $a \to a'$ and so on

All these conditions tell us exactly that this equality is the

graph of an isomorphism

between the (Bishop) sets A and B

This is the essence of *univalence*

An equality between two types is the graph of an equivalence

Example for (weak) groupoids

In this case, an equality $A \to B$ correspond to the “graph” of an equivalence between A and B (seen as categories)

Universe

We have defined a notion of lines $A \to B$ between types

Similarly we can define a notion of squares, cubes, $\ldots$ between types

This defines a large cubical set

We have a natural notion of composition, which generalizes the notion of composition of relations, and preserve the Kan filling property

It can be shown that this large cubical set *satisfies the Kan condition*

Kan completion

If X is a cubical set, we want to "complete" it to a Kan cubical set

For this we add operations $X^{+}, X\uparrow, X^{-}, X\downarrow$ in a *free* way, i.e. considering these operations as *constructors*

The uniformity condition defines what should be the degeneracies of these elements

We get in this way a new cubical set

By construction, this cubical set satisfies the Kan condition

Defining $\mathsf{S}^1,\ \mathsf{S}^2,\dots$

The same idea can be used to define S^1

We add one point a and one line $l : a \to a$ as *constructors*

We close it by the composition and filling operations

This satisfies the required induction principle of S^1

Defining inh X

Similarly we define inh X for any Kan cubical set X

This is a *proposition* stating that X is inhabited

We add a constructor $\alpha_x(a_0, a_1)$ connecting formally along the direction x any two I-cubes a_0 and a_1 (with x not in I)

$$\alpha_x(a_0, a_1)(x = 0) \;=\; a_0 \qquad \alpha_x(a_0, a_1)(x = 1) \;=\; a_1$$

We define degeneracy of these new elements by commutation with substitution

This satisfies the required induction principle of inh X: if Y is a proposition and $X \to Y$ then we have inh $X \to Y$

Axiom of Description

$(\exists x : A)B \quad = \quad \mathsf{inh}\ (\Sigma x : A)B$

If $(\Sigma x : A)B$ is a proposition we have

$(\exists x : A)B \rightarrow (\Sigma x : A)B$

This is a generalization of the *axiom of description*

If A set, B proposition and $(\exists! x : A)B$ then $(\Sigma x : A)B$ is a proposition

Both Russell and Church use the symbol ι to represent this as an operation

We have just given a justification of this axiom, different from Russell's one

What is a Kan Fibration

We want to represent the notion of family of types $A \vdash B$

For each I-cube α of A we have a corresponding set $B\alpha$ of cubes above α

If v in $B\alpha$ and $f : I \to J$ is a substitution then vf is in $B\alpha f$

We say that $A \vdash B$ is a *Kan fibration* if we have composition operations $B\alpha^+,\ B\alpha^-$ and filling operations $B\alpha\uparrow,\ B\alpha\downarrow$ above α

What is a Kan Fibration

In particular if α is a line connecting α_0 and α_1 we get a transfer map

$$B\alpha_0 \rightarrow B\alpha_1$$

which expresses Leibniz's law of substitution

Because of the uniformity condition this defines a map of cubical set

Without the uniformity condition, this cannot be done effectively

What is a Kan Fibration

This generalizes the notion of family of sets in Bishop's framework

Cf. Exercice 3.2 in Bishop's book

In the first edition, only families over discrete sets are considered while the Bishop-Bridges edition presents a more general definition, due to F. Richman

E.g. Id_X defines a Kan fibration over $X \times X$ whenever X is a Kan cubical set