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Dedicated to the 60th Anniversary
of the Shubnikov Institute of Crystallography
of the Russian Academy of Sciences

To the 150th Anniversary of the Birth of Evgraf Stepanovich Fedorov (1853–1919)
Irregularities in the Fate of the Theory of Regularity

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There are two world-class geometers in our country—
Lobachevskii and Fedorov.
B.N. Delaunay

All will be regular, regularity underlies the world.
Woland[2]

E.S. Fedorov

Mankind has been guided by the notion of regularity since ancient times. Platonic and Archimedean bodies, Mauritanian ornaments, parquets, Kepler snowflakes, Hauy mineralogy, Galois' theory, Lobachevskii geometry, and the Mendeleev Table are examples of the exhaustive interpretation of the particular manifestations of regularity. However, the general meaning of regularity as a universal and natural law, even in human thought, was first realized by the prominent thinker, humanist, and Russian patriot Evgraf Stepanovich Fedorov (December 10, 1853–May 21, 1919). In its essence, science is unified, and, therefore, Fedorov's theory of regularity covers all knowledge accumulated by mankind throughout its entire existence. The 21st century began with the triumph of Fedorov's ideas in both the world of quarks and the global structure of the Universe and in the new school handbooks on geometry.

Fedorov's life was far from easy. He became interested in geometry in his childhood and, being only sixteen years old, started writing a book which, to a large extent, anticipated the development of geometry. He failed to enter the Medical–Surgical Academy (1874), and entered the Technological Institute, where he carefully studied *Foundations of Chemistry* by D.I. Mendeleev (1834–1907), one of the most important scientific books written in the 19th century. Fedorov combined these studies with revolutionary activity—he was a member of the organization *Land and Freedom*. It is believed that it was Italian (Fedorov's conspiratorial name) who was the excellent violinist who helped famous revolutionary P.A. Kropotkin to escape from the Peter and Paul Fortress in 1876. In 1880, Fedorov, who intuitively realized the fundamental importance of crystallography in the development of geometry, started his studies at the Mining Institute. Since then, all Fedorov's weekdays, joys, and troubles were associated with this institute. Fedorov, a Member of the Academy, died of starvation in 1919.

[1] Fedorov Prize winner, 1997.
[2] From Bulgakov's *Master and Margarita*.

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His coffin was carried high in people's hands for the whole way from the Mining Institute (Gorny) to the Smolensk cemetery [1].

FEDOROV'S *ELEMENTS*

Fedorov recollected [2] that in 1863 he came across his elder brother's handbook on elementary geometry written by Shul'gin for military schools. He looked through its first pages and the contents gave rise to such an emotion that he was carried away. A ten year old boy had got through Shulgin's planimetry in two days. Six years later, being a cadet of the St. Nicholas Military Engineering School, Fedorov started writing his first book, *The Elements of the Study of Configurations* [3].

The St. Nikolas Military Engineering School was located at the Mikhailovsky (Engineering) Castle. It seems that the spirit of the former owner of the castle, Emperor Paul I, brought him to the concepts of regularity and order, which Paul I tried to establish in Russia. The book was completed in 1879 but was published only in 1885 with the help of general of artillery and Professor of physics A.V. Gadolin (1828–1892), the author of the most progressive method of derivation of 32 crystal classes [4].

Fedorov's book was preceded by two famous treatises—Euclid's *Elements* and Newton's *Principia*. Here, the question may arise as to how a young scientist could dare to choose such a title for his first book. However, by this time (1878), Fedorov was already an organizer of an illegal socialist newspaper *Nachala* (elements, principles), which criticized the existing social system and had the aim to unite all kinds of socialists for writing a revolutionary program. In this newspaper, Fedorov supervised the column *Chronicles of the Socialist Motion in the West*. The police made all possible efforts to find its anonymous publishers. After the successful self-liquidation of the newspaper office located at Kirochnaya ulitsa (in the house of A.A. Panyutina, a landowner from the Perm district and the mother of Fedorov's future wife), Fedorov directed all his revolutionary ardor to *The Elements of the Study of Configurations*. He wrote that this extremely elegant section of elementary geometry was still almost unstudied despite the fact that the need in this theory was so urgent that many representatives of other natural sciences, and first of all mineralogists, made numerous attempts to create such a theory. However, all these attempts failed, because the authors considered only those aspects of the problem which were necessary for solving their own specific problems. As a result, their spontaneously developed theories lacked satisfactory nomenclature and integrity, whereas mathematicians, usually unaware of the results obtained in other branches of science, formulated the problem quite differently.

The distinctive feature of Fedorov's geometry, which distinguishes it from all the other geometries, is the use of the concept of regularity—a configuration composed by equivalent parts, with each of these parts being surrounded in the same way with other equivalent parts. Only such systems can possess the minimum energy [5]. Thus, the finite state of any varying system is the crystalline state, because it is only in an ideal crystal that the particles are absolutely equivalent, i.e., cannot be distinguished from one another [6]. Similar speculations brought Fedorov to the Mining Institute, from which he graduated in 1883 at the top of his class. According to the rules, he had to be sent on probation work to Germany, but he never went there, because he considered it to be humiliating to plead for something that should be granted according to the rules.

Today, Fedorov's *Elements* is considered to be one of the deepest monographs on elementary geometry (elementary in the sense fundamental and not simplest). More exactly, the largest part of *Elements* is dedicated to planimetry and not only to Euclidean but also spherical planimetry. Stereometry is considered only in the sections dealing with division of space (parallelohedra and stereohedra).

Now, consider the different sections of *Elements*. Such a consideration can also be included into all modern handbooks on geometry, including school handbooks.

Euclidean planimetry is planimetry on a Euclidean plane, i.e., conventional planimetry. The unusual aspect of the Fedorov planimetry is that it is based on regularity. It is regular division that reveals the fundamental properties of space. Fedorov called a planigon any polygon that could divide a plane in a regular way. Already the first studies of planigons gave very interesting results. Thus, it turned out that only triangles, tetra-, penta-, and hexagons can be planigons (dashed polygons in Fig. 1), that any tetragon (including nonconvex one) is a planigon, etc. Today, these results are included even in school handbooks on geometry [7]. The exhaustive theory of planigons was developed by outstanding geometer B.N. Delaunay (Delauné) (1890–1980) [8].

A division dual to the division of a plane into planigons (the apices of this division form a regular system, Fig. 1) were considered in 1916 by outstanding crystallographer A.V. Shubnikov (1887–1970) in the solution of the following problem [9]. Let each atom in the plane possess the same number of bonds with other atoms. Then, how many atomic networks are formed? Since the problem was solved by topological methods using the generalized Euler formula, it followed that two-dimensional crystallography was a purely topological science. In other words, growth of a two-dimensional crystal did not necessarily require that the bond lengths and the bond angles formed by these bonds be fixed. At the first stages of growth, they can be arbitrary. The main requirement is that these patterns could be transformed into regular patterns forming new bonds and breaking old ones. These results were generalized by Delaunay [8], who showed that all such networks can

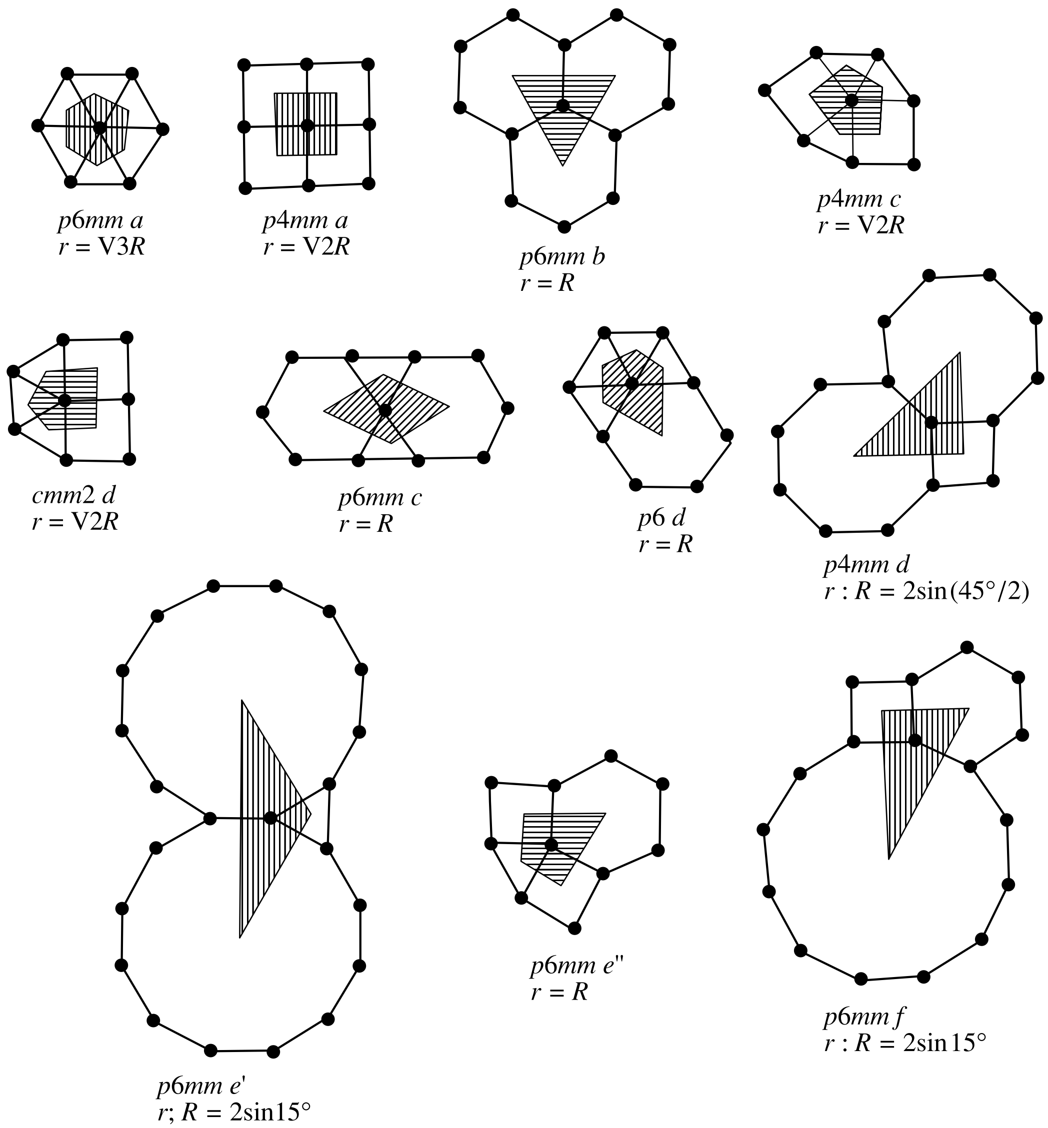


Fig. 1. 11 Kepler–Shubnikov–Delaunay networks and the corresponding Dirichlet planigons. Each infinite regular network is represented by a star of regular polygons converging at the network point. The corresponding Dirichlet planigon (composed of the points of the plane that are closer to the given lattice point than to any other point) is represented by the etched polygon. For each network, the symbol of its two-dimensional Fedorov group is indicated as well as the corresponding Wyckoff position in this group, and the ratio of the discreteness radius in this network, r, to the radius of its coverage, R.

be constructed only from regular polygons (Fig. 1) and that two-dimensional crystallography has a purely topological basis. Thus, Delaunay related the theory of planigons with the Kepler parquetry from his *World's Harmony.*

The most outstanding recent achievement of the theory of planigons is the Shtogrin theorem (the idea of this theorem was suggested by the author of the present article), according to which each regular system on a Euclidean plane is defined locally, i.e., by the same environment of any point of the system with other points of this system lying within a sphere of a fixed radius [10]. It follows from Shtogrin's theorem that long-range order is the consequence of the short-range order. The long-range order can exist only in crystal structures.

Fedorov's planigons have not been considered as yet in traditional handbooks on crystallography. In this respect, the school handbooks of geometry turned out to be more progressive [7, 11, 12].

Spherical planimetry deals with regular division of a two-dimensional sphere, i.e., a sphere's surface. All topologically nonequivalent divisions are exhausted by Platonic and Catalani bodies and two infinite series of bipyramids and deltahedra (Fig. 2). These polyhedra are called isohedra. Polyhedra dual to isohedra are called isogons. All the topologically nonequivalent polyhedra are exhausted by Platonic and Archimedean

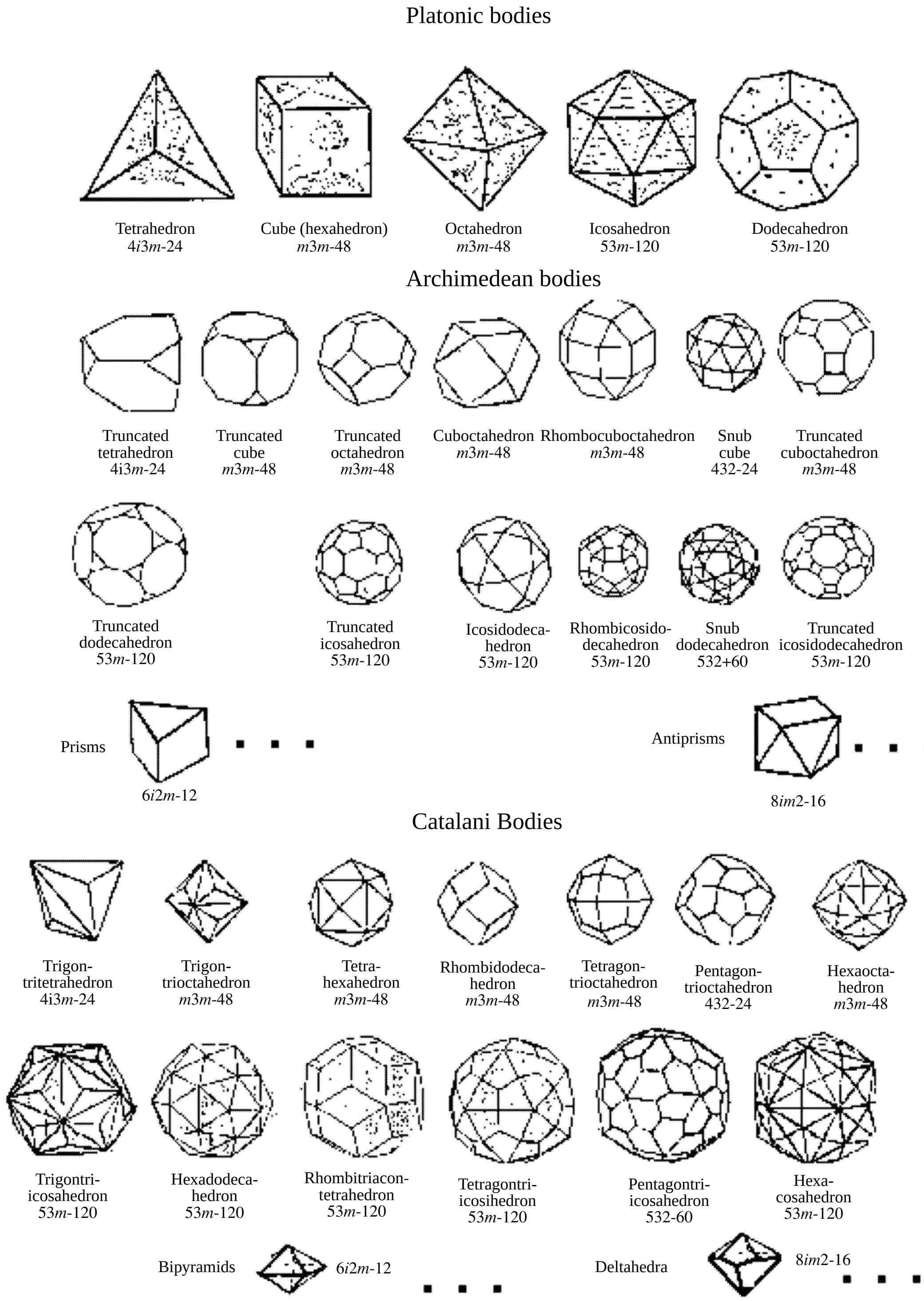


Fig. 2. Regular and semiregular convex polygons. Regular polygons: 5 Platonic bodies. Semiregular isogons: 13 Archimedean bodies and two infinite series of prisms and antiprisms. Semiregular isohedra: 13 Catalani bodies and two infinite series of bipyramids and deltahedra. For each polygon, its symmetry is indicated.

bodies and two infinite series of prisms and antiprisms (Fig. 2). In fact, isogons and isohedra form the basis of the theory of polyhedra. Therefore, the latter theory should be related not to stereometry but rather to planimetry on a sphere.

Proceeding from the consideration above, school geometry [7] should rather be called planimetry and should be complemented with the elementary data on a Lobachevskii plane (sum of the angles of a triangle is less than 180°, each regular polygon can regularly

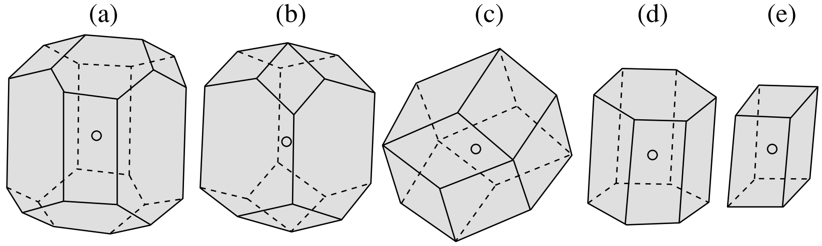


Fig. 3. Fedorov bodies: 5 combinatorially different parallelohedra for the points of three-dimensional lattices. (a) General 14-face polygon, (b) Fedorov dodecahedron, (c) parallelogrammatical dodecahedron, (d) hexagonal prism, (e) parallelepiped.

divide the Lobachevskii plane). This is already used in practice when nesting blanks in the shoe industry [13]: soles are sewn from regular hexagons, heels, from pentagons, and shin, from heptagons.

STEREOMETRY

Delaunay was the first to compare Fedorov with Plato and Archimedes [14], because it was Fedorov who composed the complete list of all the combinatorially different polyhedra that can fill the space being in parallel positions, the so-called Fedorov parallelohedra. Delaunay called these five polyhedra the Fedorov bodies (Fig. 3). Four of these bodies (the first, third, fourth and fifth ones) have been known since ancient times. The second parallelohedron can be justly called the Fedorov dodecahedron.

The division of the Euclidean space into parallelohedra (in the case of a plane, into Fedorov parallelogons) is one of its basic properties. This property is inherent only in spaces with the zero curvature. Other spaces of a constant curvature (it is only in these spaces that matter can be crystallized, i.e., form absolutely equivalent and indistinguishable particles) cannot be divided into parallelohedra. A Euclidean observer cannot move rectilinearly in these spaces. It should also be indicated that many other mathematicians arrived at parallelohedra, but their derivation gave rise to serious difficulties even for advanced mathematicians [15, 16]. Despite this, Delaunay included parallelohedra in his school book on geometrical problems [17].

The fifth chapter of *Elements*, dedicated to nonconvex polyhedra, has still not received due attention despite the fact that Russian mathematicians recently reported important results obtained in the theory of nonconvex polyhedra [18].

Elements is a versatile work, the best handbook on regularity necessary not only for mathematicians and natural scientists but also for any educated person. I was surprised to see Newton's "Principia" and Fedorov's *Elements* side by side at the honorary place in the personal library of Marutaev, a well-known musician. It shows that people with humanitarian education also realize the necessity of a mathematical picture of the world.

REGULARITY OF ATOMIC AND NUCLEAR ORBITALS

After completion of *Elements* (1869–1879), Fedorov presented to D.I. Mendeleev (1834–1907) his new manuscript (1880) in which he first stated his new idea—to consider the Periodic Law in terms of the theory of regularity. At that time, he published only the abstract of this work [19]. The manuscript of the complete work was found in Mendeleev's Archives many years later [20] and was published only in 1955 [21].

Fedorov writes [21] that the human brain always seeks regularity in everything, which is quite understandable, because a man can be oriented in his search for an appropriate work only considering regularly grouped materials and only if this regularity does not give rise to any doubt, so that he can be satisfied and become a master of this new field.

To explain the sequence of atomic weights of elements in the Mendeleev Table, Fedorov put forward the hypothesis of a planetary structure of an atom. Fedorov writes [21] that the atomic surface is the most important factor providing the occurrence of a chemical reaction. This signifies that small bodies forming an atom are not arranged continuously but, similar to planets, are spaced from one another by sufficiently large distances. Thus, at the very beginning of his scientific carrier (1880), Fedorov came to the concept of divisibility of an atom. Fedorov had an inclination for physics. He wrote a large manuscript on the theory of electricity but refused to publish it without its experimental verification. However, fate seemed to be against it—Fedorov had no chance to use equipment of any physical laboratory and, gradually, he left physics. Only at the beginning of the new 19th century did he realized that the theory he developed a quarter of century before was, in essence, the theory of an electron.

The theory of atomic orbitals fully confirmed his conjecture—the structure of an atom is regular! The equivalent charges on a sphere could be stable only if they form a regular system. All the possible configurations (within an accuracy of the combinatorial equivalence) are exhausted by the Platonic and Archimedean bodies and two infinite series of prisms and antiprisms (Fig. 2) [22]. With due regard for quantum constraints, each electronic level (*s*, *p*, *d*, and *f*) can be represented

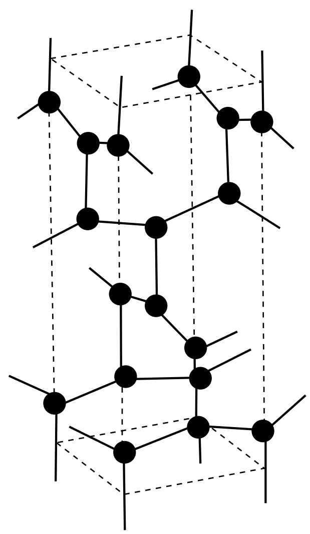


Fig. 4. Bravais parallelepiped of the hypothetical carbon structure described by the Fedorov group *I*4*r*/*amd*.

by two centrosymmetric simplexes of the four-dimensional space—the *s* level, by a segment; the *p* level, by a regular octahedron; the *d* level, by a pentagonal antiprism; and the *f* level, by a regular heptagonal antiprism.

It should be indicated that the atoms at the *d* and *f* levels have the symmetry forbidden for crystal structures in the Euclidean space. This results in partial destruction of the regularity of these levels in atoms that form a crystal structure with the Euclidean metrics [23]. The crystals consisting of such atoms have no defects only in the spherical, Lobachevskii, and Euclidean spaces with the dimension not exceeding three [24]. Therefore, the crystals with *d* and *f* elements in the Euclidean three-dimensional space should always contain defects (e.g., quasicrystals).

Only regular systems can be stable. Fersman, who dreamed of an institute of crystallography still in 1932 [25], wrote that everything which is not crystalline cannot be stable and should gradually be transformed into crystals. A crystal is such an ideal state of matter, a deep internal order to which nature strives… [26]. It should also be noted that any defect in a crystal is influenced by the gradient of the crystalline field and, therefore, such a defect would either be pushed out from the crystal or form a regular system with other similar defects [27]. In this way, the crystal "cures" itself.

The representation of the orbitals of chemical elements in the Mendeleev Table corresponds to their lowest energy level. The concrete physical conditions give rise to the corresponding displacements of electrons to other admissible levels [28]. Possibly, under certain extreme conditions, all the electrons of a carbon atom can fill the 2*p* level. Then, an extremely strong carbon modification is formed with multiple bonds [29] (Fig. 4). Possibly, the solid part of the earth's core consists of this carbon modification and is a fractal penetrating the iron–nickel melt [30].

It should also be indicated that all the regular features of the atomic shell are regularly reflected into the atomic nucleus, which should also have a regular structure. This is confirmed by the empirically determined magic numbers of protons and neutrons in the most stable nuclei—2, 8, 20, 50, 82, and 126. However, the nuclei are characterized not by spherical but by the hyperbolic regularity, and the heavier the nucleus, the more pronounced the absolute curvature of the corresponding Lobachevskii space. For example, the nuclei of the most stable isotopes of the inert gases can possess a diamond-like structure (because of the regular arrangement of the α particles constituting these nuclei [29]). The nuclei in which the regularity in the arrangement of α particles is violated are less stable. It is the solution of the problem of the atomic-nucleus structure which is based on the three theories developed by three Russian scientists—Lobachevskii (1792–1859), Mendeleev, and Fedorov—whose works predetermined the development of the world's science in the 21st century. It is most probable that the problem of low-temperature nuclear synthesis would also be solved via dissymmetrization of atomic nuclei.

FEDOROV GROUPS

It was fated that, instead of working with Mendeleev, Fedorov worked for ten years in the Geological Committee, where he composed geological maps of northwest Russia. As usual, Fedorov found brilliant solutions to routine geological problems, e.g., he developed a universal theodolite method in mineralogy and petrography (1893). He also designed a special device, the so-called Fedorov's stage, which allowed him to study by the latter method the optical properties at any point along any direction of a thin section of a rock.

The most famous Fedorov's work is associated with regularity in crystal structures. Fedorov was the first to derive 230 discrete groups of motion of the Euclidean space with a finite independent domain (1890) [31]. In Russia, these groups are justly called Fedorov groups. The story of their derivation was initiated by outstanding French mathematician Jordan (1838–1922) and German physicist Sohncke (1842–1897), a follower of Neumann (1798–1859), a prominent expert in crystal physics from Köenigsberg. Jordan in his memories entitled *On Groups of Motion* first stated that the discovery made by Galois (1811–1832) can also be interpreted as the discovery of groups of motion. At that time, Jordan had two students, Lie (1842–1899) and Klein (1849–1925), who "divided" the theory of groups into two parts—continuous groups were studied by Lie, and discrete groups, by Klein. These two theories were developed along different lines, so that today they can hardly be unified.

Jordan described in his memoirs [32] 174 groups of motion. Zohncke singled out from these groups the discrete groups directly related to the arrangement of atoms in crystal structures and found that the list of these groups was not complete. In 1874, Zohncke derived infinite regular systems of points on a Euclidean plane [33]. In 1869, he published the list of discrete groups of motion with the finite independent domain in the Euclidean space which consisted only of the first-order symmetry transformations [34]. However, he mistakenly included into this list one group twice. This mistake was revealed by his post graduate student Arthur Schönflies (1853–1928), who established that the groups can also be derived by using second-order transformations. Schönflies started their derivation and published intermediate results obtained. Fedorov paid attention to these publications and decided to complete the derivation of such groups started still in *Elements*. He sent his results to Schönflies and indicated some inaccuracies in Schönflies' derivation. In turn, Schönflies made the same. From this moment on, they entered into a lively correspondence, which concluded with the derivation of 230 groups by both scientists. Fedorov completed his derivation somewhat earlier [31]. This derivation had become a landmark in the development of natural sciences. Thus, finally, Mankind rigorously established that crystals are regular atomic formations which, by definition, should be described by the Fedorov groups.

The Schönflies monograph [35] received wide recognition in Europe and since then the Schönflies notation has been widely used. Even Russian crystallographers use this notation although it is important only for crystal classes. Germans did not forget Fedorov either. In 1896, Fedorov, an unknown laboratory assistant from the Geological Committee, was elected a Corresponding Member of the Bavarian Academy of Sciences. Klein was going to address the Russian Tzar and to ask him to make Fedorov also a member of the Russian Academy. Only Fedorov's resolute protest prevented Klein from doing so. Fedorov's colleagues working with him in Krasnotyr'insk (where Fedorov rather successfully prospected new copper deposits) could not believe the fact that they worked with a member of a German academy.

Neither Fedorov nor Schönflies made use of the lattice classification suggested by Bravais, who had made the first steps in group–theoretical crystallography. The results obtained by Bravais can be considered as the derivation of all the Fedorov groups possessed by lattices [36]. Altogether, there are 14 such groups. Another interpretation of the Bravais results is the derivation of all the different groups of the integral automorphisms of positive quadratic forms (arithmetic holohedry [36]), which seems to be the deepest meaning of the above classification. The best lattice classification should be based on the Fedorov theory of parallelohedra [37]. Delaunay completed this classification and made it extremely elegant by dividing all the lattices into 24 kinds [38]. This classification is the most appropriate for solving a number of applied problems (the unique choice of the main frame of reference in the lattice, the rigorous description of ideal habits of crystals according to Wulff, combinatorial–symmetric classification of the first Brillouin zones [39], etc.). It should be noted that modern handbooks on crystallography consider the types of the Bravais lattices insufficiently rigorously [40].

Fedorov's classification of all the space groups is much deeper than Schönflies' classification. Fedorov divided all the groups into symmorphic (whose crystal class is the stabilizer of the Fedorov group), hemisymmorhic (in which the axial hemihedry is the maximum stabilizer of the Fedorov group), and asymmorphic (all the remaining groups). This classification of groups considerably facilitated their derivation. Also, it turned out that this classification has a rather deep mathematical sense: there is a one-to-one correspondence between the symmorphic and finite groups of integral matrices. It is not accidental that D.K. Faddeev, a well-known expert in algebra, used Fedorov's classification as the basis for the table of representations of the Fedorov groups [41]. Faddeev's classification is more natural for crystallography than the classification suggested in [42], which is confirmed by [43]. We believe that it is necessary to publish a new edition of Faddeev's tables which would be based on the modern crystallographic nomenclature of the Fedorov groups [44]. The innovations introduced into the nomenclature in [45] seem to be excessive. The nomenclature of the Fedorov groups convenient for computer work is given in [46]; it is also useful for making compact tables of these groups (Table 1) [47].

Fedorov also derived regular systems purely algebraically. This derivation was then repeated by the mathematician Bogomolov [48]. The most widespread purely algebraic method of derivation of the Fedorov groups was suggested by Zassenhaus [49]. The method was used to derive all the four-dimensional Fedorov groups [50]. Geometrization of this algorithm made in [51] resulted in the compact analytical representation of the vector systems–the complete set of vectors in any crystal structure (Galiulin–Delone formula [52]).

However, not all of Fedorov's contemporaries realized the meaning of the Fedorov groups, the convenient classification of regular point systems following from these groups, and the crystal structures composed of such systems. Thus, Vernadsky (1843–1945) in his lectures on physical crystallography delivered at the Physics Faculty of Moscow State University in 1908 stated that crystallography can confine itself only to 32 crystal classes [53]. Fedorov was also criticized by Wulff (1863–1925) [54]. As a result, the Department of Crystallography of the Physics Faculty of Moscow State University made a much more modest contribution to the development of crystallography than the Department of Crystallography of the Mining Institute

Table 1. 219 abstractly different Fedorov groups

Pi		P1+.										
P2/m	P2s/m	P2.	Pm.									
P2/b	P2s/b	P2s+.	Pb+.									
I2/m		I2.	Im.									
I2/b			Ib+.									
Pmmm"	Pmma	P222	Pmm2.	Pmc2s.								
Pccm	Pmna	P222s	Pcc2.	Pmn2s.								
Pban	Pbam	P2s2s2	Pma2.	Pca2s+.								
Pnnn	Pmmn	P2s2s2s+	Pnc2.	Pna2s+.								
	Pnnm		Pnn2.									
	Pcca		Pba2.									
	Pnna											
	Pbcm											
	Pbcn											
	Pccn											
	Pnma											
	Pbca											
Cmmm'	Cmcm	C222	Cmm2.	Cmc2s.								
Cccm	Cmca	C222s	Ccc2.									
Cmma			C2mm.									
Ccca			C2mb.									
			C2cm.									
			C2cb.									
Immm'	Imma	I222	Imm2.									
Ibam	Ibca	I2s2s2s	Ima2.									
			Iba2.									
Fmmm'		F222	Fmm2.									
Fddd			Fdd2									
P4/mmm"	P4/mbm'	P422	P4mm.	P4smc.	P4i2m	P4i2sm	P4/m	P4s/m	P4.		P4i	
P4/mcc	P4/nmm	P422s	P4cc.	P4snm.	P4i2c	P4i2sc	P4/n	P4s/n	P4s.			
P4/nbm	P4/mnc	P4s22	P4bm.	P4scm.					P4r+*			
P4/nnc	P4s/mmc'	P4s22s	P4nc.	P4sbc.	P4im2							
	P4s/mcm'	P4r22*	P4ic2									
	P4s/nnm	P4r22s*	P4ib2									
	P4s/mnm'	P4in2										
	P4s/nmc											
	P4/ncc											
	P4s/nbc											
	P4s/ncm											
	P4s/mbc											
I4/mmm'	P4r/amd	I422	I4mm.	I4rmd	I4i2m	I4i2d	I4/m	I4r/a	I4.		I4i	
I4/mcm'	I4r/acd	I4r22	I4cm.	I4rcd	I4im2				I4r			
					I4ic2							
P6/mmm"	P6s/mcm'	P622	P6mm.	P6smc.	P6i2m'		P6/m	P6s/m	P6.		P6i	
P6/mcc	P6s/mmc'	P6s22	P6cc.	P6scm.	P6i2c				P6s.			
		P6rr22*			P6im2"				P6rr*			
		P6r22*			P6ic2				P6r*+			
							P3im1		P321	P3m1.	P3i	P3.
							P3ic1		P3r21*	P3c1.		P3r*+
							P3i1m		P312	P31m.		
							P3i1c		P3r12*	P31c.		
							R3im		R32	R3m	R3i	R3
							R3ic			R3c		
Pm3m"	Pm3n'	P432			P4i3m'		Pm3'	Pa3	P23			
Pn3n	Pn3m'	P4s32			P4i3n'		Pn3		P2s3			
		P4r32*										
Im3m'	Ia3d	I432			I4i3m'	I4i3d	Im3	Ia3	I23			
		I4r32							I2s3			
Fm3m"	Fd3m'	F432			F4i3m"	Fm3'	F23					
Fm3c'	Fd3c	F4r32			F4i3c	Fd3						

$219_{sym} = 73_{sym} + 54_{hemisym} + 2_{asym}$
(230 = 73 + 54 + 103)
* – enantiomorphic (11)
\+ – uniform (10)
. – degenerate (53)
" – caleidoscopic (7)
' – Molnar (22)

32 CRYSTAL CLASSES OF MEROHEDRY

i	1								
2/m	2	m							
mmm	222	mm2							
4/mm	422	4mm	4i2m	4/m	4		4i		
6/mmm	622	6mm	6i2m	6/m	6		6i		
				3im	32	3m		3i	3
m3m	432		4i3m	m3	23				

Note: *s* is a twofold screw axis, *r* is a right-handed screw axis, and *i* is an inversion axis.

founded by Fedorov. The outstanding results of the Physics Faculty of Moscow State University were obtained by noncrystallographers. Thus, Vlasov (1908–1975) predicted the existence of long-range order in plasma [55] and bending of space by a growing crystal [56]. Ivanenko (1904–1994), together with Heisenberg, predicted the proton–neutron model of a nucleus (1932) and, together with Pomeranchuk, the synchrotron radiation (1945), formulated the problem of regularity of the global structure of the Universe (1994) [57]. A high level of crystallography at the Geology and Chemistry Faculties of Moscow State University was achieved by Fedorov's followers Bokii (1909–2001) [58], Belov (1891–1992) [59], Popov (1905–1963) [60], and Livinskaya (1920–1994) [61]. High level of crystallographic studies at the Moscow Institute of Steel and Alloys is also associated with the Leningrad school and its representative Shaskol'skaya (1913–1983) [62]. An important role in the development of Russian crystallography was also played by *Crystallographic University* created by Shubnikov, Bokii, and Shaskol'skaya [58].

The situation with the 17 two-dimensional Fedorov groups was quite different. They were derived by Fedorov in 1891 [63], although, in fact, all these groups can be found in Medieval Mauritanian ornaments [64]. Arabs decorated their mosques with such ornaments; they symbolized for Moslems infinite regular paths to Allah [65]. Unfortunately, there is still no handbook on two-dimensional crystallography, which seems to be a considerable gap in education, because many crystallographic problems can readily be understood in the two-dimensional case. As a result, two-dimensional crystallography is less used in practice than three-dimensional crystallography.

Table 2 lists the Bravais parallelograms for the 17 two-dimensional Fedorov groups. The independent domain of the group is hatched. Latin letters indicate the Wyckoff positions corresponding to this group. The symbol of the general Wyckoff position is given along with the symbols of the special positions, which are indicated at the corresponding symmetry elements (mirror planes and rotation axes). Table 2 has six columns. The first two columns correspond to holohedry and contain both symmorphic and nonsymmorphic groups in the order of a decrease in holohedry (along the vertical); the third, fourth, and fifth columns correspond to hemihedry (axial, symmorphic, and nonsymmorphic (*pb*) groups); the sixth column corresponds to tetartohedry (*p*, the sixth row).

To the merit of school teaching [7], which, in some instances, is better than teaching in modern higher schools, the elements of two-dimensional crystallography are considered in modern school handbooks. The level of understanding of crystallography and its relation to other sciences, and, first of all, to mathematics, physics, chemistry, and biology, is determined by realization of the meaning of the Fedorov groups.

PERFECTIONISM

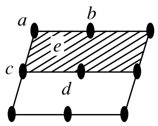
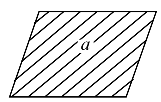
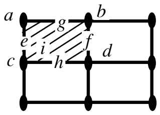
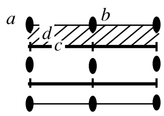
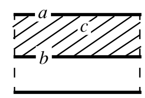
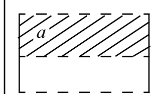
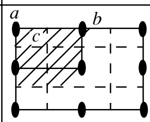
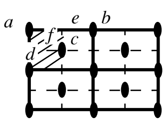
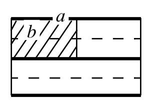
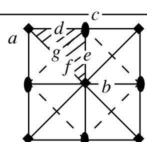
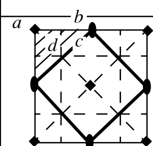
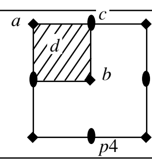
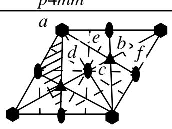
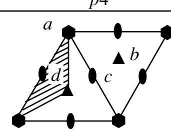
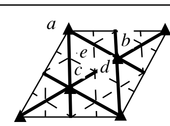
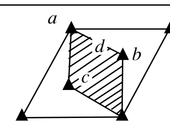
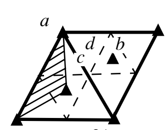
Fedorov started his philosophical work on perfectionism in 1872 and continued writing it for many years [66]. His wife, Ludmila Vasil'evna Fedorova, recollects the time before her marriage [1]: "… He told us his theory of perfectionism, which I then rewrote for him. Unfortunately, it was published with considerable censure gaps in those places where he mentioned Germans as perfectionists and predicted their future failure." The term perfectionism was coined by Fedorov and signifies the strive for perfection. Fedorov shows the universal nature of the main laws of evolution, which describe the development of the most diverse phenomena. Using the laws established in natural sciences (physics, in the broad sense), he considered the specific features of biology, psychology, and sociology. Fedorov believes that evolution can never be ended with the attainment of perfection, it can only strive for perfection. The most elegant and perfect elements, which are formed in the process of evolution, unavoidably disappear and make space for new even more perfect and harmonic elements. Perfection and harmony are attained only at the moments of their disappearance. When life is in full swing, only its unstable forms can develop. Life deals only with unstable forms.

Now, we draw your attention to the fact that double helix of DNA is associated with the action of a tenfold axis [67]. It is this axis that "protects" DNA from crystallization in the Euclidean space in a way similar to the *d* shell of an atom having the shape of a pentagonal antiprism (Fig. 5), which prevents the growth of an ideal crystal [23]. The crystal structure is uniquely reconstructed from its nucleus. It can have no mutations, so necessary for life. Thus, crystals signify death. I heard about this Fedorov concept from Alan MacKay, an English crystallographer, the founder of the theory of quasicrystals [68], who in turn referred to his teacher John Bernal (1901–1971). Bernal, the founder of protein crystallography planned to state his original position in understanding symmetry (the addition of a fivefold axis [69]) at the 7th International Congress of Crystallographers in Moscow in 1966, but he could not do it because of his illness. Instead, the Congress was addressed by Shubnikov, who, in Fedorov's spirit, called the crystallographers to keep the banner of pure crystallography [70].

The Fedorov groups form the main criterion separating crystal structures from all the other atomic formations, cannot be generalized. In the mathematical sense, Shubnikov's black and white groups and Belov's color groups are the subgroups of the Fedorov groups and, in fact, are the mathematical interpretations of these groups.

The latter studies of global crystal formation [71] allow one to emphasize the above thought of Fedorov and to state that only crystals signify depth. Other systems, e.g., quasicrystals, cannot be uniquely reconstructed and they have no long-range order. Therefore,

Table 2. Two-dimensional Fedorov groups

Holohedry		Hemihedry			Tetartohedry
nonsymmorphic	symmorphic	axial	planar symmorphic	planar nonsymmorphic	
 *p*2		 *p*1			
 *pmm*2	 *pma*2		 *pm*	 *pb*	
	 *pba*2				
 *cmm*2			 *cm*		
 *p*4*mm*	 *p*4*bm*	 *p*4			
 *p*6*mm*		 *p*6	 *p*3*m*1		 *p*3
			 *p*31*m*		

they possess the ability (although only limited) to accommodate. One can reveal some primitive elements of life in these systems. A crystal with defects also has some primitive elements of life, because defects are always subjected to the action of a force—an electric field gradient [27, 72]. Formation of twins (considered in one of the first of Fedorov's articles [73]), OD structures [74] (whose theory is consistent with the theory of twinning (Fig. 6) and unique local continuation [75]), and the Penrose-like model of a quasicrystal [76] are some examples of the attempts of matter to avoid the attainment of any stable state. In all the occasions, these attempts are stopped by Pauling's approximants, ideal crystal structures which, within the experimental accuracy, can be assigned noncrystallographic symmetry.

The simplest way of introducing mutations into a crystal is *twinning*, formation of a set of crystal structures related by symmetry transformations (twinning operations) not contained in the Fedorov group that describes the crystal structure. The atomic structure at the contact surface is another polymorphic modification of the structure (diamond–lonsdaleite, sphalerite–wurtzite, calcite–aragonite, pyrite–marcasite, etc.). Therefore, the study of the twinning laws is the most promising way of searching for new phases of a substance. Any plane of a crystal structure can play the part of a twinning plane. The axial polysynthetic twins are formed due to multiple reflection in such planes. The surface separating individuals in a twin is called a *twin boundary* (contact surface). If the intergrowth surface is a plane dividing the twin structure into two parts, the twinned structures formed are called *contact twins*; otherwise, they are called *penetration twins*. The contact surface can have fractal indices.

Composition (contact) plane in twins may either coincide with the twinning plane (*mirror twin*) or not coincide with it. If a twinning axis lies in the composition plane, the twins are called *parallel*. If the twinning axis is perpendicular to the composition plane, the twins are called *normal*. If the composition planes are parallel to one another, the twins are called polysynthetic; otherwise, they are called cyclic. A twin is rational if the twinning operation in the Bravais reference system is written by rational numerals. These twins are ideal crystals, because they are described by the Fedorov group, which is a subgroup of the initial group.

The most widespread type of twins are *merohedral twins* in which the twinning operations are the symmetry operations of the holohedry of the individual that are not contained in its crystal class. In this case the corresponding groups of parallel translations of individuals are the same. Sometimes, the twin symmetry can be higher than the symmetry of an individual. These are the mimetic twins. The processes of twinning in crystals can be considered as elementary events of life.

What is life? Fedorov wrote [66]: "Thorough consideration of the conditions of development always shows that evolution is not a continuous upward band, it is similar to branching observed in crystallization from solutions. Not all the branches of the crystalline substance propagate uniformly, the situation is quite different. Almost all the branches disappear one after another, i.e., stop growing immediately after the formation of more favorable conditions for crystallization. The vital branches are those which, because of the conditions of the solution drying, would continuously maintain the highest growth rate, and these are always the most miserable shapeless crystallizing (but not crystallized) masses. To some extent, everybody can observe this phenomenon in water freezing on a window. Of course, as soon as a delicate flakelike mass of growth figures is brought into contact with a saturated solution with an introduced well-shaped little crystal, the whole flakelike mass disappears at an amazing rate, and, instead, the introduced crystal starts growing. This fact is even more emblematic of the general law of development: delicate unstable growth figures are emblematic of motion, life, and eternal and continuous changes, whereas a crystal is emblematic of death, equilibrium, and immobility. No doubt, death is stronger than life, and the attainment of the constant conditions of mobility indicates the moment of death and beginning of perfect crystallization."

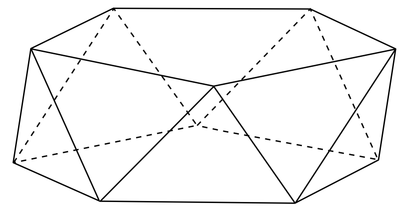


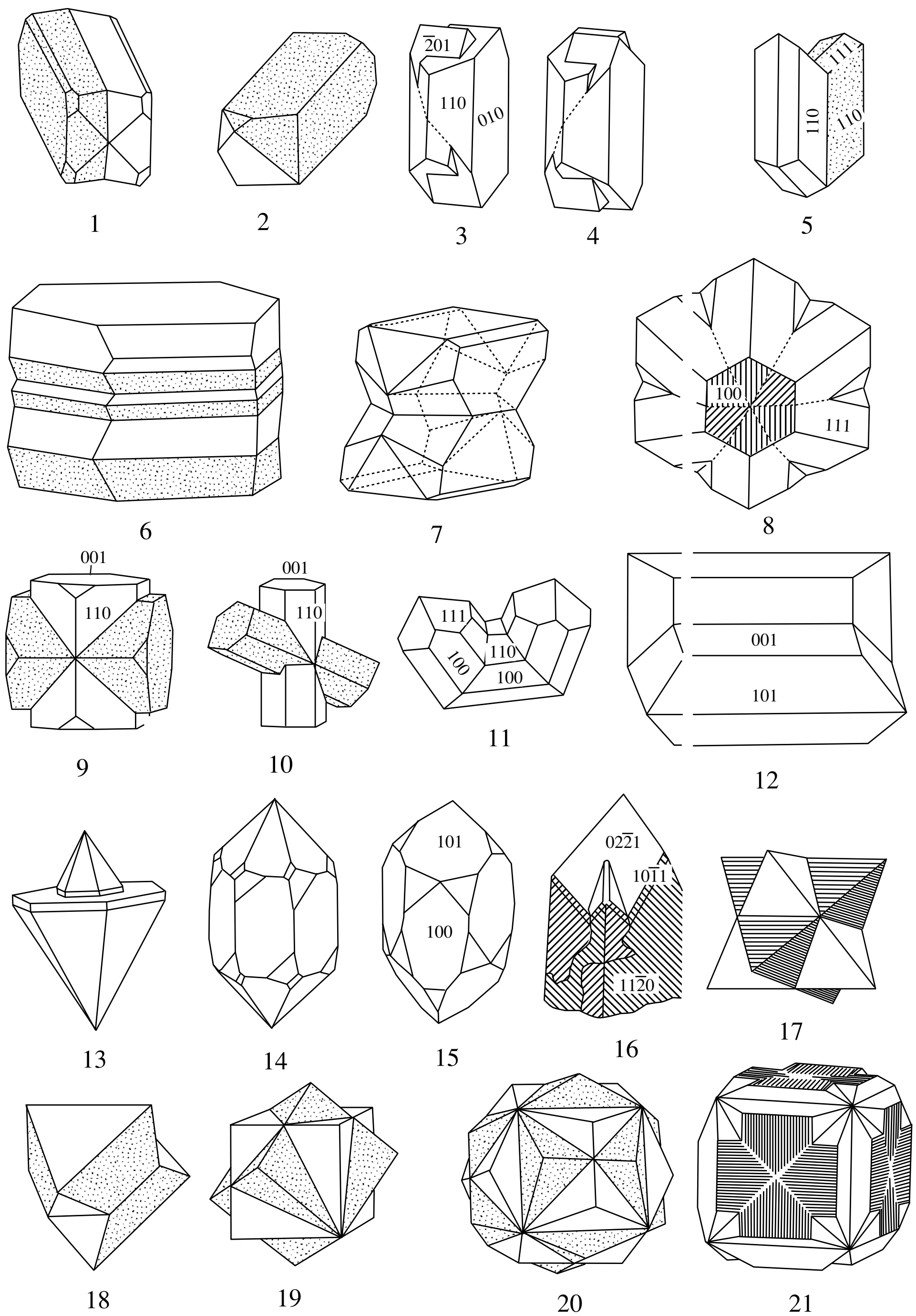
Fig. 5. Semiregular pentagonal antiprism.

Living matter cannot be stable. It seems that Nature created Man in the search for new ways of its own further development. Man differs from animals because of his strive for intellectual activity as necessary for him as food. Therefore, he must pay for being able to perform such an activity [77] in the way he pays for bread and meat. This is exactly what Fedorov himself did. Fedorov organized the Department of Crystallography at the Mining Institute with his own money despite the fact that at that time he was a Director, and a progressive one. "…Progress cannot be based on binding or unbinding hands of individual citizens, providing the most favorable conditions for an individual and the removal of all the obstacles in the development of the already accumulated forces" [66]. Fedorov was not appointed a Director by the order of the administration, he was selected by students. From 1895 to 1905, Fedorov was a Professor of geology at the St. Peter Academy at Petrovskoe–Razumovskoe (Now Timiryazev Agricultural Academy in Moscow). For ten years of his professorship, he was never invited to deliver a lecture on crystallography at Moscow University. Instead, once a week, the Moscow–St. Petersburg train stopped at the Petrovsko-Razumovskoe station to take a sole passenger, Professor Fedorov, who went to St. Petersburg to deliver one of his regular lectures on crystallography at the Mining Institute.

Up to the very end of his life, Fedorov was a Russian patriot. Because of the war with Germany, he refused to publish in German his work "The Realm of Crystals" [78], where he stated that goniometric data allows one to characterize crystals and draw some conclusions on their atomic structure. This work is still disputed by crystallographers throughout the world. English crystallographer Thomas Barker visited Fedorov to master this method. However, without Fedorov, this work had not been concluded.

UNITY OF SCIENCE BASED ON REGULARITY

Fedorov always stated that science is unified. This thought is also shared by modern scientists [79, 80]. And science can be unified only based on regularity. The most successful step in understanding the importance of such a science was made by Fedorov. The future of science lies in its unity. As an example of the



fruitful influence of such unity, we mention here the proof of the famous Fermat theorem by the mathematician Faddeev, who invoked for this almost all the branches of modern mathematics [81]. The solution was found by considering two theories—those of elliptic functions and modulated forms [82]. Both these theories are associated with the finite groups of integral matrices, i.e., lead to crystallography. Attempts of finding purely crystallographic (i.e., regularity-based) proof of this theorem were also undertaken by crystallographers [83].

It should also be noted that the Fedorov groups (discrete groups with finite independent domains) exist in all the spaces of constant curvatures [84–88], in particular, in Lobachevskii spaces. Fedorov deeply respected Lobachevskii. He wrote that Lobachevskii destroyed the artificial obstacle between mathematics and natural sciences by proving that geometry is based not on the indisputable truth but on the truth, which requires its experimental verification and confirmation [89].

This attitude to mathematics formulated still in the 19th century turned out to be extremely important not only for modern mathematics but even for teaching of

Fig. 6. Crystal twinning. Type of twins most often encountered and the symbols of twinning operations and composition planes.

No.	Twin	Twinning operation and composition plane
1	Manebach twin of orthoclase $KAlSi_3O_8$	*m*(001)/(001)
2	Baveno twin of orthoclase	*m*(021)/(021)
3	Right Carlsbad twin of orthoclase	2[001]/(010)
4	Left Carlsbad twin of orthoclase	–
5	Swallow tail (gypsum $CaSO_4 \cdot 2H_2O$)	*m*(100)/(100)
6	Polysynthetic plagioclase twin	(001)/(001)
7	Kalomine twin $Zn_4(Si_2O_7)(OH)_2 \cdot H_2O$	*m*(001)/(001)
8	Pseudohexagonal antigrowth twins of chrysoberyl $BeAl_2O_4$	–
9	Staurolite $Fe[OH]_2 \cdot 2Al_2SiO_5$	*m*(032)/(032)
10	Staurolit	*m*(232)/(232)
11	Cassiterite type (rutile TiO_2)	*m*(101)/(101)
12	Aragonite law (agaronite $CaCO_3$)	*m*(110)/(110)
13	ZnO twin	*m*(001)/(001)
14	Dophin'e twin of quartz SiO_2	6[001]/(101)
15	Brazilian twin of quartz	*m*(110)/(110)
16	Twin along pinacoid (Iceland spar)	6[001]/(001)
17	Diamond twin	–
18	Diamond twin along octahedron	*m*(111)/(111)
19	Spinel twins with respect to octahedron	*m*(111)/(111)
20	Iron cross (pyrite FeS_2)	*m*(110)/(100)
21	Maltese cross (pyrite FeS_2)	*m*(110)/(100)

mathematics in school. According to Arnol'd [90], mathematics is a part of physics and, similar to physics, is an experimental science. In the well-known article *Mathematics and Natural Sciences* (1930), Gilbert wrote that geometry is a part of physics. However, there is also an opinion that mathematics and physics have nothing in common. This brought the conclusion that geometry may be excluded from all the mathematical courses. And indeed, such an attempt has already been made in Russia [90, p. 11]. It is timely to compare the negative attitude of Chebyshev (1821–1894), one of the most prominent mathematicians of the 19th century, to Fedorov's *Elements*, who wrote that the modern science has no interest in such a geometry, with the first epigraph to the present article and Delaunay's words that "Geometry is a difficult science in which one has to think at every step." As follows from Fedorov's and Delaunay's works, one cannot state modern geometry and mathematics, in general, without invocation of regularity, i.e., crystallography.

CONCLUSIONS

To E.S. Fedorov

Deep in beautiful vials,
Similar to a sculptor–magician,
Colorless dense solutions
Create for us beautiful crystals.

Based on the vague entangling
Of thoughts, expectations, and dreams,
The human brain endlessly sculptures
Visions of fantastic creations.

The World of ethereal ideas
Is close to the mineral realm,
Shining like crystal faces
The dreams are instilled in our hearts.

January 2, 1919

N.A. Morozov "Crystals. Star Songs"

Books 1, 2. Moscow 1920–1921

These verses addressed to Fedorov, written on January 2, 1919, by N.A. Morozov (1854–1946), imprisoned for 23 years (1881–1905) in the Schliesselburg Fortress, provoke deep thinking. What is the crystal civilization? What contribution did crystallography make to World's history? What is Future in the light of the opposition of Life and Crystal? How deeply is Fedorov understood by our contemporaries? Mankind should always be grateful to Fedorova (1851–1936) for the memoirs about her husband [1], which, in fact, should be regarded as a literary masterpiece. Probably, Fedorov would have never become the Fedorov we know without the understanding and constant support of his wife. We are lacking Bokii, who brought us as closer to Fedorov. Just imagine that, being a child, Bokii sat on Fedorov's knee! We should also be grateful to Nina Georgievna Furmanova, Bokii's daughter, who made a precious gift to us all— the reprints of the majority of Fedorov's articles collected by her grandfather and father and two volumes of the "Fundamentals of Differential and Integral Calculus" written by Fedorov [91, 92] still never mentioned in any of his bibliographies. The penetration into the crystallographic meaning of Fedorov's ideas was also facilitated by Shafranovskiĭ (1907–1994) and Frank-Kamenetskiĭ (1912–1994) [93].

The modern tendency of integration of various sciences made the restoration of the Fedorov Institute (created by Fedorov's brightest student, Boldyrev (1883–1946)) quite timely [94]. This Institute should be an international organization performing the studies in all the natural and humanitarian sciences under the UNESCO supervision.

ACKNOWLEDGMENTS

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REFERENCES

1. L. V. Fedorova, *Our Workdays, Sorrows, and Joys. Memoirs Scientific Heritage* (Nauka, Moscow, 1992), Vol. 20.
2. E. S. Fedorov, *The Symmetry of Real Systems of Configurations. Imperial Petersburg Academy of Sciences. Scientific Heritage* (Nauka, Moscow, 1991), Vol. 16, p. 168.
3. E. S. Fedorov, *The Elements of the Study of Configurations. Transactions of the St. Petersburg Mineralogical Society* (1885), Part 21.
4. A. V. Gadolin, *Derivation of All Crystallographic Systems and Their Subsystems from One Common Origin* (Akad. Nauk SSSR, Moscow, 1954).
5. R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1966; Mir, Moscow, 1977), Vol. 7.
6. R. V. Galiulin, Priroda (Moscow), No. 12, 20 (1991).
7. I. M. Smirnova and V. A. Smirnov, *Geometry. Handbook for the 7–9 Grades of Primary Schools* (Prosveshchenie, Moscow, 2001).
8. B. N. Delone, Izv. Akad. Nauk SSSR, Ser. Mat., No. 23, 365 (1959).
9. A. V. Shubnikov, Izv. Imp. Akad. Nauk, Ser. 6 **10** (9), 757 (1916).
10. R. V. Galiulin, Kvant, No. 11, 10 (1983).
11. I. M. Smirnova, *In the World of Polyhedra* (Prosveshchenie, Moscow, 1995).
12. I. M. Smirnova, *Geometry* (Prosveshchenie, Moscow, 1997).
13. A. N. Golovanova and R. V. Galiulin, in *Proceedings of International Conference on Mineralogy and Life* (Syktyvkar, 2000), p. 172.
14. B. N. Delone, Tr. Inst. Istor. Estestvozn. Tekh. Akad. Nauk SSSR **10**, 5 (1956).
15. P. N. Antonyuk, in *Proceedings of VII All-Russian School on Modern Problems of Gas Dynamics* (Moscow, 1991), p. 87.
16. M. Sh. Yakupov, in *Proceedings of International Conference on Geometrization of Physics* (Kazan, 1992), p. 1.
17. B. Delone and O. Zhitomirskiĭ, *Book on Geometrical Problems* (GITTL, Moscow, 1952).
18. I. Kh. Sabitov, Priroda (Moscow), No. 4, 19 (2000).
19. E. S. Fedorov, Zh. Russ. Fiz.–Khim. O–va **13** (4), 244 (1881).
20. S. A. Shchukarev and R. B. Dobrotin, *Crystallography* (Leningr. Gorn. Inst., Leningrad, 1955), No. 3, p. 81.
21. E. S. Fedorov, *Crystallography* (Leningr. Gorn. Inst., Leningrad, 1955), No. 3, p. 85.
22. R. V. Galiulin, Zh. Vychisl. Mat. Mat. Fiz. **43** (6), 791 (2003).
23. T. F. Veremĕichik and R. V. Galiulin, in *Proceedings of International Conference on Crystal Genesis and Mineralogy* (St. Petersburg, 2001), p. 409.
24. R. V. Galiulin, in *Proceedings of International Conference on Carbon, Syktyvkar, 2003* (in press).
25. A. E. Fersman, Nauka Rabot. (Petersburg), No. 1, 13 (1920).
26. A. E. Fersman, Tr. Inst. im. M.V. Lomonosova, Ser. Obshch., No. 1, 39 (1932).
27. T. F. Veremĕichik and R. V. Galiulin, Neorg. Mater. **38** (9), 1110 (2002).
28. A. A. Marakushev, *Periodic System of Extremum States of Chemical Elements* (Nauka, Moscow, 1987).
29. R. V. Galiulin, in *Proceedings of 2nd International Conference on Carbon: Fundamental Problems of Science, Materials Science, Technology* (Mosk. Gos. Univ., Moscow, 2003).
30. R. V. Galiulin, Vestn. Nizhegorod. Univ. im. N.I. Lobachevskogo, Fiz. Tverd. Tela, No. 1 (4), 152 (2001).
31. E. S. Fedorov, Zap. Imp. S.-Peterb. Mineral. O–va, No. 28 (2), 1 (1891).
32. C. Jordan, Ann. Math. Ser. 2 **2**, 167 (1868/1869); Ann. Math. Ser. 2 **2**, 322 (1868/1869).
33. L. Sohncke, J. Reine Angew. Math. **77**, 48 (1874).
34. L. Sohncke, *Entwickelung Einer Theorie der Kristallstruktur* (Tuebner, Leipzig, 1869).
35. A. Schönflies, *Kristallsystem und Kristallstruktur* (Teubner, Leipzig, 1891).
36. R. V. Galiulin and S. S. Ryshkov, *Problems in Crystallology* (Mosk. Gos. Univ., Moscow, 1971), p. 290.
37. E. S. Fedorov, Abh. K. Bauer Akad. Wiss. **11**, 465 (1899).
38. B. N. Delone, *Science and Mankind* (Znanie, Moscow, 1981), p. 160.
39. R. V. Galiulin, Kristallografiya **29** (4), 638 (1984) [Sov. Phys. Crystallogr. **29**, 378 (1984)].
40. B. K. Vaĭnshteĭn and R. V. Galiulin, in *Physical Encyclopedia* (1988), Vol. 1, p. 226.
41. D. K. Faddeev, *Tables of the Main Unitary Representations of the Fedorov Groups* (Akad. Nauk SSSR, Moscow, 1961; Pergamon Press, Oxford, 1964).
42. O. V. Kovalev, *Irreducible Representations of Space Groups* (Akad. Nauk Ukr. SSR, Kiev, 1961).

43. M. I. Aroyo and H. Wondratschek, Z. Kristallogr. **210**, 243 (1995).
44. *International Tables for X-ray Crystallography* (Kynoch, Birmingham, 1952), Vol. 1.
45. *International Tables for Crystallography*, Vol. A: *Space-Group Symmetry*, Ed. by T. Hahn (Reidel, Dordrecht, 1983).
46. R. V. Galiulin, Kristallografiya **40** (3), 461 (1995) [Crystallogr. Rep. **40**, 421 (1995)].
47. S. A. Bogomolov, *Derivation of Regular Systems by the Fedorov Method* (KUBUCh, Leningrad, 1932), Part 1; (ONTI, Leningrad, 1934), Part 2.
48. R. V. Galiulin, Kristallografiya **32** (3), 769 (1987) [Sov. Phys. Crystallogr. **32**, 452 (1987)].
49. H. Zassenhaus, Commun. Math. Helv. **21**, 117 (1948).
50. H. Brown, R. Bulow, J. Neubusser, H. Wondratschek, and H. Zassenhaus, *Crystallographic Groups of Four-Dimensional Space* (Wiley, New York, 1978).
51. R. V. Galiulin, *Matrix–Vector Method of Deviation of the Fedorov Group*, Available from VINITI, Moscow, No. 1094-69.
52. R. V. Galiulin and B. N. Delone, Dokl. Akad. Nauk SSSR **210** (1), 85 (1973) [Sov. Phys. Dokl. **18**, 285 (1973)].
53. V. I. Vernadsky, *Selected Works. Crystallography* (Nauka, Moscow, 1988), p. 66.
54. Y. V. Wulff, Z. Kristallogr. **45**, 433 (1908).
55. A. A. Vlasov, Vestn. Mosk. Univ., Nos. 3–4, 63 (1946).
56. L. S. Kuz'menkov, *Prosesses of Real Crystal Formation* (Nauka, Moscow, 1977), p. 221.
57. D. D. Ivanenko and R. V. Galiulin, in *Proceedings of 17th Workshop Dedicated to the 140th Birth Anniversary of Henri Poincare, Protvino, 1995*, p. 180.
58. G. B. Bokiĭ, *Problems of Crystallology* (GEOS, Moscow, 1999), p. 31.
59. N. V. Belov, *Structural Crystallography* (Akad. Nauk SSSR, Moscow, 1951).
60. G. M. Popov and I. I. Shafranovskiĭ, *Crystallography* (Vysshaya Shkola, Moscow, 1972).
61. Yu. G. Zagal'skaya, G. P. Litvinskaya, R. V. Galiulin, and V. S. Kovalenko, *Problems of Crystallology* (Mosk. Gos. Univ., Moscow, 1971), p. 284.
62. M. P. Shaskol'skaya, *Crystallography* (Vysshaya Shkola, Moscow, 1984).
63. E. S. Fedorov, Zap. Imper. S.-Peterb. Mineral. O–va, No. 28 (2), 345 (1891).
64. N. V. Belov, Kristallografiya **1** (5), 612 (1956) [Sov. Phys. Crystallogr. **1**, 482 (1956)].
65. Kh. S. Mamedov, Comput. Math. Appl. **128**, 511 (1986).
66. E. S. Fedorov, Izv. S.-Peterb. Biol. Lab. **8** (1), 25 (1906); Izv. S.-Peterb. Biol. Lab. **8** (2), 9 (1906).
67. M. D. Frank-Kamenetskiĭ, *The Most Important Molecule* (Nauka, Moscow, 1983).
68. A. L. Mackay, Kristallografiya **26** (5), 910 (1981) [Sov. Phys. Crystallogr. **26**, 517 (1981)].
69. J. D. Bernal and S. H. Carlise, Kristallografiya **13** (5), 927 (1968) [Sov. Phys. Crystallogr. **13**, 811 (1968)].
70. R. V. Galiulin, Usp. Fiz. Nauk **172**, 229 (2002).
71. R. V. Galiulin, in *Proceedings of Scientific Seminar on Nontraditional Problems of Geology* (Moscow, 2002), p. 52.
72. O. V. Konovalov and R. V. Galiulin, Kristallografiya **34** (3), 731 (1989) [Sov. Phys. Crystallogr. **34**, 435 (1989)].
73. E. S. Fedorov, Zap. S.-Peterb. Mineral. O–va, Ser. 2, No. 17, 381 (1882).
74. K. Dornberger-Schiff, *Lehrgang über OD-Strukturen* (Akademie, Berlin, 1966; Mosk. Gos. Univ., Moscow, 1966).
75. R. V. Galiulin, Kristallografiya **43** (2), 366 (1998) [Crystallogr. Rep. **43**, 332 (1998)].
76. D. Gratia, Usp. Fiz. Nauk **156** (2), 347 (1988).
77. V. M. Vil'chek, *Algorithms of History* (Prometeĭ, Moscow, 1989).
78. E. S. Fedorov, *The Realm of Crystals. Tables for Crystallochemical Analysis* (Zap. Akad. Nauk Fiz.–Mat. Otd., 1920), Vol. 36.
79. V. I. Arnol'd, Izv. Ross. Akad. Nauk, Ser. Yaz. Lit. **56** (2), 63 (1997).
80. V. I. Arnol'd, Usp. Fiz. Nauk **169** (12), 1311 (1999) [Phys. Usp. **42**, 1205 (1999)].
81. L. D. Faddeev, Chto Novogo Nauke Tekh., No. 2 (4), 112 (2003).
82. S. Singh, *Fermat's Enigma: the Epic Quest to Solve the World's Greatest Mathematical Problem* (Anchor Books, New York, 1998; Mosk. Tsentr Neprer. Mat. Obraz., Moscow, 2000).
83. I. A. Sheremet'ev, Kristallografiya **46** (2), 199 (2001) [Crystallogr. Rep. **46**, 161 (2001)].
84. V. V. Nikulin and I. R. Shafarevich, *Geometries and Groups* (Nauka, Moscow, 1983).
85. S. V. Rudnev, Comput. Math. Appl. **16** (5–8), 597 (1988).
86. R. V. Galiulin, Sci. Spectra **14**, 54 (1998).
87. R. V. Galiulin, Usp. Fiz. Nauk **172** (2), 229 (2002).
88. R. V. Galiulin and T. F. Veremeĭchik, in *Proceedings of International Conference Dedicated to M.P. Shaskol'skaya, Moscow, 2003* (in press).
89. E. S. Fedorov, Zap. S.-Peterb. Mineral. O–va, Ser. 2, No. 30, 455 (1893).
90. V. I. Arnol'd, *What is Mathematics?* (Mosk. Tsentr Neprer. Mat. Obraz., Moscow, 2002).
91. E. S. Fedorov, *Fundamentals of Differential and Integral Calculus* (Imper. Akad. Nauk, St. Petersburg, 1903).
92. E. S. Fedorov, *Fundamentals of Differential and Integral Calculus, Drawings* (Imper. Akad. Nauk, St. Petersburg, 1903).
93. E. S. Fedorov, *Scientific Heritage* (Nauka, Leningrad, 1991), Vol. 16.
94. I. I. Shafranovskiĭ, *Crystallography in the USSR, 1917–1991* (Nauka, St. Petersburg, 1996).

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