# ON PROVING TERM REWRITING SYSTEMS ARE NOETHERIAN

Dallas Lankford

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Noetherian term rewriting systems are finding increasing applications in computer science and mathematics. These applications include[1] finding decision procedures for equational theories [Knuth&Bendix 70], [Lankford&Ballantyne 77A,B,C], [Stickel&Peterson 77], automatic deduction [Guard et al. 69], [Huet 77], [Knuth 70], [Lankford 75], [Lankford&Ballantyne 79], [Lankford&Musser 78], [Nevins 74], [Slagle 74], [Winker 75], proof of open problems in mathematics [Degano&Sirovich 79], [Guard et al. 69], [Knuth 70], [Winker 79], program verification [Boyer&Moore 75], [Courcelle 79], [Goguen&Tardo 78], [Good-London-Bledsoe 75], [Guttag-Horowitz-Musser 78], [Huet&Lang 78], [Musser 78], [Rosen 73], [Sethi 74], algebraic manipulation systems [Moses 70], and symbolic integration [Moses 70A]. We have made no attempt to include a complete bibliography of applications of Noetherian term rewriting systems, but hope we have captured the flavor of those applications in the references above. Also, not all of the term rewriting systems mentioned above are of the kind discussed in this article, but they are all similar enough so that many of the properties discussed below carry over.

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1. complete unification procedures [Fay 79]

A set T of terms is defined as follows. There is a countably infinite number of variable symbols $v_1, v_2, v_3,$ and a finite number of function symbols $f_1, \ldots, f_N$ . Associated with each function symbol $f_j$ is a non-negative integer $d_j$ called the number of arguments of[2] $f_j$ . Function symbols $f_j$ which satisfy $d_j = 0$ are called constant symbols. The set T of terms is defined by (1) variable symbols are terms, and (2) if $t_1, \ldots, t_{d_j}$ are terms, then $f_j(t_1,\ldots,t_{d_j})$ is a term. (Purists might want to add that no other strings are terms. Also, some might insist on a separate case for constant symbols, but we consider that the degenerate case (2) above.) We will take the usual liberties with notation, often writing terms in infix rather than prefix form, and using other symbols f[or] the variable and function symbols.

A term rewriting system (also called a set of rewrite rules or a set of reductions, and elsewhere called a set of simplifiers [Slagle 74]) is a finite set of expressions $L \longrightarrow R$ where L and R are terms, and each variable symbol which occurs in R also occurs in L . (The assumption about variable symbol occurrence is made because term rewriting systems without it are not Noetherian. Moreover, term rewriting systems without the variable symbol occurrence property can be transformed into a system with the property by introducing new function symbols.)

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2. sometimes called degree or arity

Term rewriting systems will be _denoted_ $\mathcal{R}$ . We say that u is an _immediate reduction_ of t relative to $\mathcal{R}$ in case there is some $L \longrightarrow R$ in $\mathcal{R}$ and some substitution $\theta$ such that u is the result of replacing one occurrence of $L\theta$ in t by $R\theta$ . (The notion of _substitution_ we have in mind is the notion introduced by [Robinson 65]. Thus substitutions are denoted by Greek letters, and are finite sets of _substitution components_, expressions of the form t/v where t is a term and v is a variable symbol.) We _denote_ that u is an immediate reduction of t by $t \longrightarrow u$ . A term rewriting system is called _Noetherian_ iff there is no infinite sequence $t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow \cdots$ of immediate reductions. (Sometimes for "is Noetherian" we use _has the finite termination property_, abbreviated _FTP_.)

In general, it is undecidable if a term rewriting system is Noetherian [Huet 77A] (also announced by [Lipton& Snyder 77]). However, potentially useful Noetherian tests are given by [Dershowitz&Manna 78], [Knuth & Bendix 70], [Lankford 75], [Manna&Ness 70], and [Plaisted 78].

The results presented in this article are a combination of the approaches of [Knuth&Bendix 70] and [Manna&Ness 70] which generalizes both of those approaches. We also show how these generalizations apply to equivalence class term rewriting systems, like [Lankford&Ballantyne 77A,B,C], [Stickel&Peterson 77].

For each function symbol $f_1, \ldots, f_N$ let $F_1, \ldots, F_N$ be functions from the positive integers to the positive integers such that

(1) the number of arguments of each $F_j$ is the same as the number of arguments of the corresponding $f_j$,

(2) $F_j(x_1,\ldots,y,\ldots,x_{d_j}) < F_j(x_1,\ldots,z,\ldots,x_{d_j})$ , when $y < z$ ,

and let $\|\cdot\|$ be the function defined on all terms by

(3) $\|v_j\|$ is some fixed positive integer for all $j$ ,

(4) $\|f_j(t_1,\ldots,t_{d_j})\| = F_j(\|t_1\|,\ldots,\|t_{d_j}\|)$ .

__Lemma 1__ If $\mathcal{R}$ is a term rewriting system and $\|L\theta\| > \|R\theta\|$ for all substitutions $\theta$ and all $L \longrightarrow R$ in $\mathcal{R}$ , then $\mathcal{R}$ is Noetherian.[3]

__Proof__ See [Lankford 75A].

To use Lemma 1, functions $F_1, \ldots, F_N$ are specified satisfying (1) and (2), a positive integer is specified for (3), and then a proof of

(5) $\|L\theta\| > \|R\theta\|$ for all substitutions $\theta$ and all $L \longrightarrow R$ in $\mathcal{R}$

is attempted. For example, if $\mathcal{R}$ consists of the single rewrite rule $-(x + y) \longrightarrow (-x) + (-y)$ , then $\|v_j\| = 1$ , $\|x + y\| = \|x\| + \|y\| + 1$ , and $\|-x\| = 3\|x\| + 1$ show that $\mathcal{R}$ is Noetherian. It should be

3. a similar (equivalent) lemma is stated by [Manna&Ness 70]

noticed that when the $F_j$ are polynomials, deciding condition (5) amounts to solving Diophantine inequalities. For the example given, the inequality is $3x + 3y + 4 > 3x + 3y + 3$.

If the $F_j$ are restricted to polynomials, then some Noetherian rewriting systems will not be found [Stickel 76]. Moreover, if we defer the selection of the positive integer in condition (3), then with polynomial $F_j$ condition (5) can be replaced by

(6) $\exists r \forall x_1 \ldots \forall x_n \; x_1 \geq r \wedge \ldots \wedge x_n > r \Longrightarrow \overline{\| L \|} > \overline{\| R \|}$ ) where $\overline{\| L \|}$ and $\overline{\| R \|}$ are obtained by replacing $\| v_{j_1} \|, \ldots, \| v_{j_n} \|$ in $\| L \|$ and $\| R \|$ by $x_1, \ldots, x_n$ respectively.

For the sentences of condition (6) to faithfully capture Lemma 1, they must be considered to be sentences over the integers. Unfortunately, methods used by [Davis 73] can be used to show that there is no algorithm to decide sentences of the form of condition (6). (A proof of this fact was given by Martin Davis at Oberwolfach in January 1976.) Still, a weaker realization of condition (6) can be obtained by considering the sentences to be interpreted over the reals, in which case decision methods of elementary algebra, [Cohen 69], [Collins 75], [Seidenberg 54], [Tarski 51], can be applied. A further weakening has been given by [Lankford 76].

For the remainder of the article, let the $F_j$ be restricted to polynomials. When the polynomials are all linear and satisfy the form

$$(7) \quad F_j(x_1,\ldots,x_{d_j}) = x_1 + \ldots + x_{d_j} + w_j$$

then the approach of [Knuth&Bendix 70] is a generalization of Lemma 1 as follows.

Lemma 2 Let the $w_j$ be non-negative integers, let $w_j > 0$ when $d_j = 0$,[4] let $f_{j_1} \gg \ldots \gg f_{j_N}$, let $d_j = 1$ imply $w_j > 0$ (except when $j = j_1$, in which case $w_N$ may be 0), and let $t > u$ iff (i) $\forall\theta \; \|t\theta\| > \|u\theta\|$ or (ii) $\forall\theta \; \|t\theta\| = \|u\theta\|$ and (ii)(a) the leading function symbol of $t \gg$ the leading function symbol of $u$ or (ii)(b) the leading function symbols of $t$ and $u$ are the same, $t \equiv f(t_1,\ldots,t_n)$ and $u \equiv f(u_1,\ldots,u_n)$, and $t_1 = u_1$, $\ldots$, $t_k = u_k$, and $t_{k+1} > u_{k+1}$. If $L > R$, for each $L \longrightarrow R$ in $\mathcal{R}$, then $\mathcal{R}$ is Noetherian.

Proof See [Knuth&Bendix 70].

We now turn our attention to the primary purpose of this article, which is to generalize Lemma 2 to include all polynomials, not just those satisfying condition (7). Our solution is also a generalization of Lemma 1.

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4. We have already assumed that constants are interpreted by positive integers, but restate that assumption to show the connection between our approach and Knuth & Bendix.

Lemma 3 Let the $F_j$ be polynomials with non-negative integer coefficients;[5] let $f_{j_1} \gg \cdots \gg f_{j_N}$ ($j_k$ is a bijection between the first $N$ positive integers), let $d_j = 1$ imply $F_j(0) > 0$ (except when $j = j_1$, in which case $F_j(0)$ may be $0$), and let $t > u$ iff (i) $\forall\theta\ \|t\theta\| > \|u\theta\|$ or (ii) $\forall\theta\ \|t\theta\| = \|u\theta\|$ and (ii)(a) the leading function symbol of $t \gg$ the leading function symbol of $u$ or (ii)(b) the leading function symbols of $t$ and $u$ are the same, $t \equiv f(t_1,\ldots,t_n)$ and $u \equiv f(u_1,\ldots,u_n)$, and $t_1 = u_1, \ldots, t_k = u_k$, and $t_{k+1} > u_{k+1}$.

If $d_j = 0$ implies $F_j \geq 2$, and $L > R$ for each $L \longrightarrow R$ in $\mathcal{R}$, then $\mathcal{R}$ is Noetherian.

Proof First notice that without the assumption that constant symbols are interpreted as 2 or greater, Lemma 3 does not hold. This can be seen by letting $\mathcal{R}$ consist of $c \longrightarrow f(c,c)$, ordering $c \gg f$, and letting $F_c = 1$ and $F_f(x,y) = xy$. (There may be alternate hypotheses, but we have not examined the situation.)

Our proof follows the same form as the proof of Lemma 2 given by [Knuth&Bendix 70]. A ground term (called a pure word by [Knuth&Bendix 70]) is a term with no occurrences of variable symbols. We will show that the set of all ground terms is well ordered by the relation $>$. Then it follows easily

5. satisfying conditions (1) and (2)

that $\mathcal{R}$ is Noetherian. (Suppose $t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow \ldots$ Then, because of the assumption that each variable symbol that occurs in R also occurs in L, it follows that there is a ground sequence $g_1 > g_2 > g_3 > \ldots$ (gotten by assigning the variable symbols of $t_j$ a constant symbol c ).)

It can be shown by a somewhat tedious case analysis that

(8) $t > u > v$ implies $t > v$, and

(9) exactly one of the following holds:

$t > u$, $t = u$, $t < u$,

for any ground terms t, u, and v.

It remains to be shown that

(10) $g_1 > g_2 > g_3 > \ldots$ cannot happen for ground terms $g_j$.

For any ground term g and any non-negative integer j let n(j,g) <u>denote</u> the number of occurrences of function symbols of degree j. (There is a typo in [Knuth&Bendix 70] relating to this point.) It can be shown by induction that

(11) $n(0,g) + n(1,g) + n(2,g) + \ldots =$
$1 + 0n(0,g) + 1n(1,g) + 2n(2,g) + \ldots$,

and because of the assumption that constant symbols are assigned integers 2 or greater, it follows that

(12) $\|g\| > n(0,g)$ for any ground term g.

Since condition (11) holds, it also follows that

(13) $\|g\| > n(k,g)$ when $k \geq 2$.

If $d_j = 1$ implies $F_j(0) > 0$, then it follows that

(14) $\|g\| > n(1,g)$ ,

and so the set $\{g_1, g_2, g_3, \ldots\}$ would be finite, violating conditions (8) and (9).

The case when $d_{j_1} = 1$ and $F_{j_1}(0) = 0$ is like the case in Lemma 2, i.e., like the proof in [Knuth&Bendix 70], and so is not given here.[6] This completes the proof of Lemma 3.

<u>Example 1</u> Consider the complete set of reductions for group theory derived by [Knuth&Bendix 70].

KB1. $x1 \longrightarrow x$

KB2. $1x \longrightarrow x$

KB3. $x(x^{-1}) \longrightarrow 1$

KB4. $(x^{-1})x \longrightarrow 1$

KB5. $(xy)z \longrightarrow x(yz)$

KB6. $1^{-1} \longrightarrow 1$

KB7. $(x^{-1})^{-1} \longrightarrow x$

KB8. $(xy)^{-1} \longrightarrow (y^{-1})(x^{-1})$

KB9. $x((x^{-1})y) \longrightarrow y$

KB10. $(x^{-1})(xy) \longrightarrow y$

They show that the only ordering via Lemma 2 which shows that KB1-KB10 is Noetherian is $w_{multiplication} = 0$ and $w_{-1} = 0$. With Lemma 1 and Lemma 3 there are other solutions. With Lemma 1, the following example due to Gérard Huet suffices

(15) $F_{\cdot}(x,y) = x(1 + 2y)$ , $F_{-1}(x) = x^2$ , $F_1 = 2$ , and $\|\text{variable}\| = 2$ .

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6. for that case, to the definition of $t > u$ add $f_{j_1}(\ldots(f_{j_1}(v))\ldots) > v$

With Lemma 3, the following example suffices

(16) $F_{\bullet}(x,y) = x + y + 1$ , $F_{-1}(x) = 3x + 1$ ,

$F_1 = 2$ , and $\|\text{variable}\| = 1$.

In case (16) it seems that we may allow $F_1 = 1$ , which suggests a variation of Lemma 3 when the polynomials are linear.

Here we should add a few remarks about solving the Diophantine sentences that result from Lemma 3. In general, as we have said, those sentences (at least the class of all sentences of that form, the form of condition (6)) are undecidable. However, if the polynomials are of small degree, and the rewrite rules are sufficiently simple in structure, then the Diophantine sentences can often be decided. Sometimes it is just a matter of deciding $j > k$ where $j$ and $k$ are particular positive integers. And in other cases, it is often a matter of deciding if a polynomial is "eventually positive" by deciding if each of the first partials with respect to a variable occuring in the polynomial is "eventually positive," see [Lankford 76]. This amounts to constructing the partial derivative tree of a polynomial and checking that the tips are positive integers (modulo some additional mumbling). Improvements in this basic approach can be had by checking that polynomials in one variable have leading coefficient positive.

Now we turn our attention to extending these results to equivalence class term rewriting systems, a la [Lankford& Ballantyne 77A,B,C], [Stickel&Peterson 77]. Let $\mathcal{E}$ be a finite set of equations $\{p_1 = q_1 , \ldots , p_M = q_M\}$ and let $\mathcal{R}$ be a finite set of $\mathcal{E}$-equivalence class rewrite rules, $\approx[L_1] \longrightarrow \approx[R_1] , \ldots , \approx[L_N] \longrightarrow \approx[R_N]$ , where $t \approx u$ iff $\vdash_{\mathcal{E}} t = u$ . We denote $\approx[t]$ by $[t]$ where no confusion is possible ( = everywhere). The concept of immediate reduction is the obvious "jacking up" of the ordinary concept, see [Lankford&Ballantyne 77B] and delete the requirement they make for all equivalence classes to be finite, and the concept of Noetherian is like before.

<u>Lemma 4</u> Lemma 1 goes through with the additional requirement that $\|p_j\theta\| = \|q_j\theta\|$ for $j = 1, \ldots , M$, and all $\theta$ .

<u>Proof</u> Like the proof of Lemma 1.

<u>Example 2</u> Let us consider the the complete set of C + A reductions for ring theory derived by [Lankford&Ballantyne 77C] (and independently by [Stickel&Peterson 77]).

$\mathcal{E}1.$ $x + y = y + x$

$\mathcal{E}2.$ $(x + y) + z = x + (y + z)$

$\mathcal{E}3.$ $xy = yx$

$\mathcal{E}4.$ $(xy)z = x(yz)$

$\mathcal{R}1.$ $[x + 0] \longrightarrow [x]$

$\mathcal{R}2.$ $[x + (-x)] \longrightarrow [0]$

$\mathcal{R}3.$ $[-0] \longrightarrow [0]$

$\mathcal{R}4.$ $[-(-x)] \longrightarrow [x]$

$\mathcal{R}5.$ $[-(x + y)] \longrightarrow [(-x) + (-y)]$

$\mathcal{R}6.$ $[x1] \longrightarrow [x]$

$\mathcal{R}7.$ $[x0] \longrightarrow [0]$

$\mathcal{R}8.$ $[x(-y)] \longrightarrow [-(xy)]$

$\mathcal{R}9.$ $[x(y + z)] \longrightarrow [(xy) + (xz)]$

With Lemma 4, the following example suffices

$$(17) \quad \|\text{variable}\| = 2 \,,\quad F_0 = 1 \,,\ F_1 = 2 \,,$$
$$F_-(x) = 3x + 1 \,,\ F_+(x,y) = x + y + 1 \,,$$
$$F_\bullet(x,y) = xy \,,$$

to show that $\mathcal{R}1$- $\mathcal{R}9$ is Noetherian.

We may relax the requirement in Lemma 4 that $\|L\theta\| > \|R\theta\|$ for all $[L] \longrightarrow [R]$ in $\mathcal{R}$ and all $\theta$, to the case that some members of $\mathcal{R}$ satisfy the requirement, while the remainder of the members of $\mathcal{R}$ are Noetherian and satisfy $\|L\theta\| = \|R\theta\|$ for all $\theta$.

<u>Example 3</u> Consider the following example from [Degano&Sirovich 79].

$\mathcal{E}1.$ $x + y = y + x$

$\mathcal{E}2.$ $(x + y) + z = x + (y + z)$

$\mathcal{E}3.$ $xy = yx$

$\mathcal{E}4.$ $(xy)z = x(yz)$

$\mathcal{R}1.$ $[x + 0] \longrightarrow [x]$

$\mathcal{R}2.$ $[s(x) + y] \longrightarrow [s(x + y)]$

$\mathcal{R}3.$ $[x0] \longrightarrow [0]$

$\mathcal{R}4.$ $[s(x)y] \longrightarrow [(xy) + y]$

$\mathcal{R}5.$ $[x(y + z)] \longrightarrow [(xy) + (xz)]$

First $\mathcal{R}2$ is shown to be Noetherian (without the presence of the other members of $\mathcal{R}$ ) by

(18) $F_s(x) = x + 1$ , $F_+(x,y) = F_.(x,y) = xy$ .

Then the entire set is shown Noetherian by

(19) $\|\text{variable}\| = 2$ , $F_0 = 1$ , $F_+(x,y) = x + y + 1$

$F_.(x,y) = xy$ , $F_s(x) = x + 2$ .

When condition (19) is applied to $\mathcal{R}1$- $\mathcal{R}5$, all except $\mathcal{R}2$ satisfy the inequality, while $\mathcal{R}2$ satisfies the equality and is Noetherian.

When the equations of $\mathcal{E}$ are commutative-associative pairs, the polynomials which satisfy the equality condition of Lemma 4 are very limited, at least in the linear and second degree cases. The only polynomials that we have found have one of the following three forms

(20) $c$ , $cxy$ , $c + x + y + dxy$ .

We conjecture that these are the only forms that such polynomials can have.[7]

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7. G. Huet claims to have found others

## CONCLUDING REMARKS

Many researchers who derive complete (and incomplete) sets of reductions use human interaction to order the rewrite rules. It is only ex post facto that they prove the rewrite rule system is Noetherian, if at all. Not proving finite termination is very risky, especially for sets of rewrite rules claimed to be Church-Rosser since the Knuth and Bendix Church-Rosser algorithm demands that the rewrite rules be Noetherian.

The methods of this article and appendix provide a modest, systematic approach to proving finite termination for some term rewriting systems by some polynomial norms. Of course, the approach is still more an art than a science, and likely to remain that way because of undecidable problems on all sides. However, it still seems possible to extend these results in a number of useful directions. We are especially interested in efficiently deciding sentences of the form of condition (6) over the real numbers.

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## APPENDIX TO "ON PROVING TERM REWRITING SYSTEMS ARE NOETHERIAN"

One of the bothersome things about the approaches to proving that term rewriting systems are Noetherian by the polynomial methods discussed in this article is that there is no general way to specify the polynomials a priori. We suspect that in general it is undecidable whether there exist polynomials such that a finite set of equations can be arranged into a set of rewrite rules which can be shown Noetherian by those polynomials. However, when the polynomials have the form of condition (7),

$$F_f(x_1,\ldots,x_{d_f}) = x_1 + \ldots + x_{d_f} + w_f ,$$

then it is decidable whether there is a choice of the $w_f$'s , an ordering $f_{j_1} \gg \ldots \gg f_{j_N}$ , and a choice for $\|\text{variable}\|$ which can show $\mathcal{R}^1$ is Noetherian.

This can be seen as follows. There are finitely many choices for $f_{j_1} \gg \ldots \gg f_{j_N}$ , so in the following discussion, the permutation $j_k$ of $\{1, \ldots, N\}$ is assumed fixed. In addition, for $n$ equations, there are $2^n$ potential sets $\mathcal{R}$ , so we may assume $\mathcal{R}$ is fixed. Because $\mathcal{R}$ is finite, there is a finite conjunction of inequalities to be decided. Thus we are faced with deciding

$$\begin{aligned}\exists w_1 \ldots \exists w_N \exists r \ \bigwedge_k \Big[ \forall x_{k1} \ldots \forall x_{kn} \Big\{ \big( \overline{x_{k1}} > r \wedge \ldots \wedge \overline{x_{kn}} > r \big) \\ \Longrightarrow \overline{\|L_k\|} > \overline{\|R_k\|} \Big\} \\ \vee \big( \overline{\|L_k\|} = \overline{\|R_k\|} \wedge LFS(L_k) \gg LFS(R_k) \big) \\ \bigvee_j \big( \overline{\|L_k\|} = \overline{\|R_k\|} \wedge t_1 = u_1 \wedge \cdots \wedge \\ t_j = u_j \wedge t_{j+1} > u_{j+1} \big) \Big] ^{**}\end{aligned}$$

where $LFS(t)$ is the leading function symbol of $t$ .

Since the negation of the above formula is equivalent to a universal Presburger formula, it is decidable by algorithms like [Bledsoe 74] , or [Shostak 77] . (** with the obvious hypotheses about the $w_{\text{constant}}$'s and $w_{\text{unary function}}$'s )

### APPENDIX REFERENCES

[Bledsoe 74] W. Bledsoe The SUP-INF method in Presburger arithmetic. Univ. of Texas, Math. Dept., Automatic Theorem Proving Project, Austin, Texas, 78712, report ATP-18, Dec. 1974.

[Shostak 77] R. Shostak On the SUP-INF method for proving Presburger formulas. JACM 24, 4 (Oct. 1977), 529-543.

For the ten group reductions, Example 1, there are six orderings of the function symbols. One of those orderings and the associated Presburger formula is given below.

$-1 \gg . \gg 1$

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$$\exists x \exists y \exists z \; ( \; z \geq 1 \\ \wedge \; x \geq 0 \\ \wedge \; y + z > 0 \\ \wedge \; x > 0 \vee x = 0 \\ \wedge \; 2x > 0 \vee 2x = 0 \\ \wedge \; 0 > x \vee x = 0 \; )$$

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We should remark that the method of the appendix works for any linear polynomial with positive $c_j$

$$F_f(x_1,\ldots,x_n) = c_1x_1 + \ldots + c_nx_n + w_f$$

when the $c_j$ are specified _a priori_ and the $w_f$ are left unspecified. If the $c_j$ are left unspecified, then we are up against Hilbert's 10th Problem, or so it seems.

_Postscript_ 10/19/1979 I now have some doubts about some of what is said in this appendix. That may only be an anxiety attack, but I suspect it is because of my incomplete understanding of some concepts from logic. Until now, time has not permitted me to carefully reexamine the appendix.

Bledsoe's Presburger algorithm established the above example.