PID Controller Synthesis with Specified Stability Requirement for Some Classes of MIMO Systems

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Abstract—**For certain classes of linear, time-invariant, multi-input multi-output plants, a systematic synthesis method is developed for stabilization using Proportional+Integral+Derivative (PID) controllers, where the closed-loop poles can be assigned to the left of an axis shifted to the left of the origin. For some of these plant classes, the real-parts of the closed-loop poles can be smaller than any arbitrary pre-chosen negative value. Stable and some unstable multi-input multi-input plants with transmission-zeros in the left-half complex-plane are included in these classes that admit PID-controllers with this property of small negative real-part assignability of closed-loop poles.**

Keywords: ***simultaneous stabilization and tracking, PID control, integral action, stability margin***

1 Introduction

Many practical control designs use Proportional+Integral+Derivative (PID) controllers, which are preferred due to their simplicity, integral-action property, and low-order (see e.g., [1]). Rigorous PID synthesis methods based on modern control theory have been explored recently in e.g., [6, 8, 5]. Sufficient conditions for PID stabilizability of linear, time-invariant (LTI), multi-input multi-output (MIMO) plants were given in [5] and several plant classes that admit PID-controllers were identified.

One important criterion for control design is the assignment of the closed-loop poles sufficiently far from the imaginary-axis of the complex-plane in order to have small time-constants, implying short settling times. Therefore, it is desirable for the closed-loop poles to have real-parts less than $-h$ for a pre-specified positive constant h. The goal of this paper is to identify plant classes such that closed-loop poles can be assigned to the left of an axis shifted away from the origin and to develop a synthesis procedure that explicitly describes PID controllers that achieve this performance objective.

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Since the order of PID-controllers is restricted to two, their simplicity and low-order present the constraint that only certain classes of plants can be controlled by using PID-controllers, but others may require higher order controllers, or those with poles with positive real-parts, for stabilization. In fact, strong stabilizability is a necessary condition for PID stabilizability of the plant, although it is not a sufficient condition [5]. For example, the plant with transfer-function $G(s) = \frac{s-1}{s^2-p^2}$ is not PID stabilizable for any $p > 1$ since it is not strongly stabilizable; on the other hand, the plant $G(s) = \frac{1}{(s-1)^3}$ is not PID stabilizable although it is strongly stabilizable. In addition to closed-loop stability, it is desirable to asymptotically track important test signals with zero steady-state error. If the integral constant is non-zero, PID-controllers achieve asymptotic tracking of step-input references. The integral-constant of the PID-controller can be non-zero only if the plant has no (transmission) zeros at the origin of the complex-plane. Therefore, we only consider subclasses of MIMO plants that are strongly stabilizable and have no (transmission) zeros at $s = 0$.

We study three plant classes in detail: The first class is the set of plants whose poles have real-parts less than $-h$ for a prescribed $h \geq 0$. The zeros of this class are unrestricted except that there are no zeros at $s = 0$. For this class, the objective of obtaining PID-controllers such that the closed-loop poles have real-part less than $-h$ can be achieved only for certain values of h as shown in Proposition 2.1-(i). The restriction on h is removed if the plants that have no finite zeros with real-parts larger than the given $-h$; the closed-loop poles can be assigned to the left of an axis going through this $-h$ for any chosen value of h as shown in Proposition 2.1-(ii)-(iii). The second class under consideration is the set of plants that have no zeros at infinity and whose (transmission) zeros have real-parts less than $-h$ for a prescribed $h \geq 0$. This time the pole locations are unrestricted. The third class of plants allows zeros at infinity but no other zeros that have real-parts less than $-h$ for a prescribed $h \geq 0$. For the second and third plant classes, Propositions 2.2 and 2.3 present systematic PID-controller synthesis methods, where the closed-loop poles can be pushed as far as to the left of the finite zero with the largest negative real-part.

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The paper is organized as follows: Following the introduction, the main results are presented in Section 2, starting with the problem statement and basic definitions. The three plant classes under consideration are studied under three separate subsections. Several illustrative numerical examples are given for each of the plant cases for single-input single-output (SISO) and MIMO plant transfer-functions. The choice of the free parameters can be optimized with a chosen cost function. Section 3 gives concluding remarks.

Although we discuss continuous-time systems here, all results apply also to discrete-time systems with appropriate modifications.

Notation: Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$ denote complex, real, positive real numbers. For $h \in \mathbb{R}_+ \cup \{0\}$, let $\mathcal{U}_h := \{s \in \mathbb{C} \mid \mathcal{R}e(s) \geq -h\} \cup \{\infty\}$. If $h = 0$, $\mathcal{U}_h = \mathcal{U}_0 := \{s \in \mathbb{C} \mid \mathcal{R}e(s) \geq 0\} \cup \{\infty\}$ is the extended closed right-half complex plane. Let $\mathbf{R_p}$ denote real proper rational functions of s. For $h \geq 0$, $\mathbf{S}_h \subset \mathbf{R_p}$ is the subset with no poles in $\mathcal{U}_h$. The set of matrices with entries in $\mathbf{S}_h$ is denoted by $\mathcal{M}(\mathbf{S}_h)$; ${\mathbf{S}_h}^{m\times m}$ is used instead of $\mathcal{M}(\mathbf{S}_h)$ to indicate the matrix size explicitly. A matrix $M \in \mathcal{M}(\mathbf{S}_h)$ is called $\mathbf{S}_h$-stable; $M \in \mathcal{M}(\mathbf{S}_h)$ is called $\mathbf{S}_h$-unimodular iff $M^{-1} \in \mathcal{M}(\mathbf{S}_h)$. The H_∞-norm of $M(s) \in \mathcal{M}(\mathbf{S}_h)$ is $\|M\| := \sup_{s\in\partial\mathcal{U}_h} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}_h$ is the boundary of $\mathcal{U}_h$. We drop (s) in transfer-matrices such as $G(s)$ where this causes no confusion. We use coprime factorizations over $\mathbf{S}_h$; i.e., for $G \in {\mathbf{R_p}}^{m\times m}$, $G = Y^{-1}X$ denotes a left-coprime-factorization (LCF) and $G = N_g D_g^{-1}$ denotes a right-coprime-factorization (RCF) where $X, Y, N_g, D_g \in \mathcal{M}(\mathbf{S}_h)$, $\det Y(\infty) \neq 0$, $\det D_g(\infty) \neq 0$. For MIMO transfer-functions, we refer to transmission-zeros simply as zeros; blocking-zeros are a subset of transmission-zeros. If $G \in {\mathbf{R_p}}^{m\times m}$ is full (normal) rank, then $z_o \in \mathcal{U}_h$ is called a transmission-zero of $G = Y^{-1}X$ if $\mathrm{rank}X(z_o) < m$; $z_b \in \mathcal{U}_h$ is called a blocking-zero of $G = Y^{-1}X$ if $X(z_b) = 0$ and equivalenty, $G(z_b) = 0$.

2 Main results

Consider the LTI, MIMO unity-feedback system $Sys(G, C)$ shown in Fig. 1, where $G \in {\mathbf{R_p}}^{m\times m}$ and $C \in {\mathbf{R_p}}^{m\times m}$ are the plant and controller transfer-functions. Assume that $Sys(G, C)$ is well-posed, G and C have no unstable hidden-modes, and $G \in {\mathbf{R_p}}^{m\times m}$ is full rank.

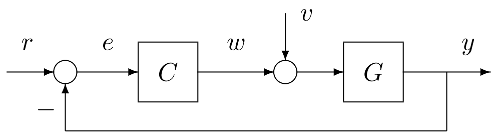

Figure 1: Unity-Feedback System $Sys(G, C)$

We consider the realizable form of proper PID-controllers given by (1), where K_p, K_i, $K_d \in \mathbb{R}^{m\times m}$ are the proportional, integral, derivative constants, respectively, and $\tau \in \mathbb{R}_+$ (see [3]):

$$C_{pid}(s) = K_p + \frac{K_i}{s} + \frac{K_d\ s}{\tau s + 1}\ . \tag{1}$$

For implementation, a (typically fast) pole is added to the derivative term so that C_{pid} in (1) is proper. The integral-action in C_{pid} is present when $K_i \neq 0$. Subsets of PID-controllers are obtained by setting one or two of the three constants equal to zero: (1) becomes a PI-controller C_{pi} when $K_d = 0$, an ID-controller C_{id} when $K_p = 0$, a PD-controller C_{pd} when $K_i = 0$, a P-controller C_p when $K_d = K_i = 0$, an I-controller C_i when $K_p = K_d = 0$, a D-controller C_d when $K_p = K_i = 0$.

Definition 2.1 *a)* $Sys(G, C)$ is said to be $\mathbf{S}_h$-stable iff the closed-loop transfer-function from (r, v) to (y, w) is in $\mathcal{M}(\mathbf{S}_h)$. *b)* C is said to $\mathbf{S}_h$-stabilize G iff C is proper and $Sys(G, C)$ is $\mathbf{S}_h$-stable. *c)* $G \in {\mathbf{R_p}}^{m\times m}$ is said to admit a PID-controller such that the closed-loop poles of $Sys(G, C)$ are in $\mathcal{U}_h$ iff there exists $C = C_{pid}$ as in (1) such that $Sys(G, C_{pid})$ is $\mathbf{S}_h$-stable. We say that G is $\mathbf{S}_h$-stabilizable by a PID-controller, and C_{pid} is an $\mathbf{S}_h$-stabilizing PID-controller. ∎

Let $G = Y^{-1}X$ be any LCF of G, $C = N_c D_c^{-1}$ be any RCF of C; for $G \in {\mathbf{R_p}}^{m\times m}$, $X, Y \in \mathcal{M}(\mathbf{S}_h)$, $\det Y(\infty) \neq 0$, and for $C \in {\mathbf{R_p}}^{n_u\times n_y}$, $N_c, D_c \in \mathcal{M}(\mathbf{S}_h)$, $\det D_c(\infty) \neq 0$. Then C $\mathbf{S}_h$-stabilizes G if and only if

$$M := Y D_c + X N_c \tag{2}$$

is $\mathbf{S}_h$-unimodular (see [4, 9]).

The problem can be described as follows: Suppose that $h \in \mathbb{R}_+$ is a given non-negative constant. Is there a PID-controller C_{pid} that stabilizes the system $Sys(G, C_{pid})$ with a guaranteed stability margin, i.e., with real parts of all closed-loop poles of the system $Sys(G, C_{pid})$ less than $-h$? For $h = 0$, the problem is the same as placing the closed-loop poles anywhere into the left-half complex-plane with arbitrary negative real parts. It is clear that this goal is not achievable for some plants. Even when it is achievable, it may be possible to place the closed-loop poles to the left of a shifted-axis that goes through $-h$ only for certain $h \in \mathbb{R}_+$.

Let $\hat{s}$ and $\hat{G}$, $\hat{C}_{pid}$ be defined as

$$\hat{s} := s + h, \ \text{ equivalently, } \ s =: \hat{s} - h\ ; \tag{3}$$

$$\hat{G}(\hat{s}) \ := \ G(\hat{s} - h)\ ; \tag{4}$$

$$\hat{C}_{pid}(\hat{s}) \ := \ C_{pid}(\hat{s} - h) := K_p + \frac{K_i}{\hat{s} - h} + \frac{K_d\,(\hat{s} - h)}{\tau(\hat{s} - h) + 1}\ . \tag{5}$$

Then $C_{pid}(s)$ $\mathbf{S}_h$-stabilizes $G(s)$ if and only if $\hat{C}_{pid}(\hat{s})$ $\mathbf{S}_0$-stabilizes $\hat{G}(\hat{s})$. For any $\alpha \in \mathbb{R}_+$, an RCF of $\hat{C}_{pid}(\hat{s})$ is given by

$$\hat{C}_{pid} = (\ \frac{(\hat{s}-h)}{\hat{s}+\alpha}\hat{C}_{pid}\ )\ (\frac{(\hat{s}-h)}{\hat{s}+\alpha}I\ )^{-1}\ . \tag{6}$$

We consider plant classes that admit PID-controllers and identify values of $h \in \mathbb{R}$ such that the closed-loop poles lie to the left of $-h$. A necessary condition for existence of PID-controllers with nonzero integral-constant K_i is that the plant $G(s)$ has no zeros (transmission-zeros or blocking-zeros) at $s = 0$ (see [5]). Therefore, all plants under consideration are assumed to be free of zeros at the origin (of the s-plane). The three specific classes under consideration are defined as follows:

1) The first class of plants, called $\mathcal{G}_{ph}$, is the set of $\mathbf{S}_h$-stable $m \times m$ plants that have no (transmission or blocking) zeros at $s = 0$; i.e., for a given $h \in \mathbb{R}_+ \cup \{0\}$, let $\mathcal{G}_{ph} \subset {\mathbf{S}_h}^{m\times m}$ be defined as

$$\mathcal{G}_{ph} := \{\ G(s) \in {\mathbf{S}_h}^{m\times m}\ |\ \det G(0) \neq 0\}. \tag{7}$$

For $G(s) \in \mathcal{G}_{ph}$, with $\hat{G}(\hat{s}) := G(\hat{s}-h)$, $\det G(0) \neq 0$ is equivalent to $\det \hat{G}(h) \neq 0$. Clearly, the plants $G \in \mathcal{G}_{ph}$ may have transmission-zeros or blocking-zeros anywhere in $\mathbb{C}$ other than $s = 0$.

2) The second class of plants, called $\mathcal{G}_{zh}$, is the set of $m \times m$ plants that have no (transmission or blocking) zeros in $\mathcal{U}_h$; i.e., for a given $h \in \mathbb{R}_+ \cup \{0\}$, let $\mathcal{G}_{zh} \subset {\mathbf{R_p}}^{m\times m}$ be defined as

$$\mathcal{G}_{zh} := \{\ G(s) \in {\mathbf{R_p}}^{m\times m}\ |\ G^{-1}(s) \in {\mathbf{S}_h}^{m\times m}\ \}. \tag{8}$$

In the SISO case, this class represents plants without zeros in $\mathcal{U}_h$ that have zero relative degree. Some plants in the set $\mathcal{G}_{zh}$ are not $\mathbf{S}_h$-stable; therefore, these plants either have poles in $\mathcal{U}_0$, or they are $\mathbf{S}_0$-stable but some poles have negative real-parts larger than the specified $-h$. Obviously, the plants in $\mathcal{G}_{zh}$ satisfy the necessary condition for existence of PID-controllers with nonzero integral-constant K_i since the fact that they have no zeros in $\mathcal{U}_h$ implies that they have no zeros at $s = 0$.

3) The third class of plants, called $\mathcal{G}_\infty$, is the set of $m \times m$ strictly-proper plants that have no (transmission or blocking) zeros in $\mathcal{U}_h$ except at infinity. For a given $h \in \mathbb{R}_+ \cup \{0\}$, let $\mathcal{G}_\infty \subset {\mathbf{R_p}}^{m\times m}$ be defined as

$$\mathcal{G}_\infty := \{\ G(s) \in {\mathbf{R_p}}^{m\times m}\ |\ \frac{1}{s+a}G^{-1}(s) \in {\mathbf{S}_h}^{m\times m} \quad \text{for any } a > h\}. \tag{9}$$

In the SISO case, this class represents plants without zeros in $\mathcal{U}_h$, that have relative degree one. Some plants in the set $\mathcal{G}_\infty$ are not $\mathbf{S}_h$-stable; therefore, these plants either have poles in $\mathcal{U}_0$, or they are $\mathbf{S}_0$-stable but some poles have negative real-parts larger than the specified $-h$. Obviously, the plants in $\mathcal{G}_\infty$ satisfy the necessary condition for existence of PID-controllers with nonzero integral-constant K_i since the fact that they have no zeros in $\mathcal{U}_h$ implies that they have no zeros at $s = 0$.

The set $\mathcal{G}_{ph} \cap \mathcal{G}_{zh}$ corresponds to $\mathbf{S}_h$-stable plants with no poles and no zeros in $\mathcal{U}_h$ (including infinity). The set $\mathcal{G}_{ph} \cap \mathcal{G}_\infty$ corresponds to $\mathbf{S}_h$-stable plants with no poles in $\mathcal{U}_h$ and no zeros in $\mathcal{U}_h$ except at infinity.

2.1 Plants with no poles in $\mathcal{U}_h$

We start our investigation by considering the $\mathbf{S}_h$-stable plant class $\mathcal{G}_{ph}$ described in (7). In Proposition 2.1-(i), we obtain a sufficient condition on h for existence of PID-controllers that $\mathbf{S}_h$-stabilize the plant $G \in \mathcal{G}_{ph}$ such that none of the closed-loop poles are in $\mathcal{U}_h$. We propose a systematic PID-controller synthesis procedure, where the controller parameters are explicitly defined. In Proposition 2.1-(ii), we consider the subclass $\mathcal{G}_{ph} \cap \mathcal{G}_{zh}$ of $\mathcal{G}_{ph}$, where the plants have no (transmission or blocking) zeros in $\mathcal{U}_h$, i.e., $G \in \mathcal{G}_{ph}$ such that $G^{-1} \in \mathcal{M}(\mathbf{S}_h)$. For these $\mathbf{S}_h$-unimodular plants, there exist stabilizing PID-controllers such that none of the closed-loop poles are in $\mathcal{U}_h$ *for any* $h \in \mathbb{R}_+$. In Proposition 2.1-(iii), we consider the subclass $\mathcal{G}_{ph} \cap \mathcal{G}_\infty$, where the plants are strictly-proper and have no finite (transmission or blocking) zeros in $\mathcal{U}_h$, i.e., $G \in \mathcal{G}_{ph}$ such that $\frac{1}{s+a}G^{-1} \in \mathcal{M}(\mathbf{S}_h)$ for any $a > h$. For these plants, there exist stabilizing PID-controllers such that none of the closed-loop poles are in $\mathcal{U}_h$ *for any* choice of $h \in \mathbb{R}_+$. Proposition 2.1-(ii) and (iii) indicate that PID-controllers can be designed so that the closed-loop poles have negative real-parts less than any $-h$ if the open-loop poles *and* (finite) zeros are not in $\mathcal{U}_h$. A methodology leading to explicit design parameter choices is proposed for each special case.

Proposition 2.1 *(PID controller synthesis for* $\mathbf{S}_h$*-stable plants):*
Let $h \in \mathbb{R}_+$ and $G(s) \in \mathcal{G}_{ph}$ be given.
i) ($\mathbf{S}_h$*-stable plants with zeros in* $\mathcal{U}_h$): Define $\Theta(\hat{s})$ as

$$\Theta(\hat{s}) := \hat{G}(\hat{s})(\hat{K}_p + \frac{\hat{K}_d(\hat{s}-h)}{\tau(\hat{s}-h)+1}) + \frac{\hat{G}(\hat{s})G(0)^{-1} - I}{\hat{s}-h}\ . \tag{10}$$

If the given $h \in \mathbb{R}_+$ satisfies

$$h < \frac{1}{2}\ \|\ \Theta(\hat{s})\ \|^{-1}, \tag{11}$$

for some $\hat{K}_p \in \mathbb{R}^{m\times m}$, $\hat{K}_d \in \mathbb{R}^{m\times m}$ and $\tau < 1/h$, then there exists an $\mathbf{S}_h$-stabilizing PID-controller. Furthermore, C_{pid} can be designed as follows: Choose any $\hat{K}_p \in \mathbb{R}^{m\times m}$, $\hat{K}_d \in \mathbb{R}^{m\times m}$, $\tau \in \mathbb{R}_+$ satisfying $\tau < 1/h$. Let $K_p = (\alpha+h)\hat{K}_p$, $K_d = (\alpha+h)\hat{K}_d$, $K_i = (\alpha+h)G(0)^{-1} = (\alpha+h)\hat{G}(h)^{-1}$, where $\alpha \in \mathbb{R}_+$ satisfies

$$h < \alpha < \|\ \Theta(\hat{s})\ \|^{-1} - h\ . \tag{12}$$

Then an $\mathbf{S}_h$-stabilizing PID-controller C_{pid} is given by

$$C_{pid} = (\alpha+h)\hat{K}_p + \frac{(\alpha+h)G(0)^{-1}}{s} + \frac{(\alpha+h)\hat{K}_d\, s}{\tau s+1} \ . \quad (13)$$

ii) ($\mathbf{S}_h$*-stable plants with no zeros in* $\mathcal{U}_h$): Let $G(s) \in \mathcal{G}_{ph} \cap \mathcal{G}_{zh}$. Then there exists an $\mathbf{S}_h$-stabilizing PID-controller. Furthermore, C_{pid} can be designed as follows: Choose any nonsingular $\hat{K}_p \in \mathbb{R}^{m\times m}$, any $K_d \in \mathbb{R}^{m\times m}$, and $\tau \in \mathbb{R}_+$ satisfying $\tau < 1/h$. Choose any $g \in \mathbb{R}_+$ satisfying

$$g > 2h \ . \quad (14)$$

Define $\widetilde{\Phi}(\hat{s})$ as

$$\widetilde{\Phi}(\hat{s}) := \hat{K}_p^{-1} \ [\ \hat{G}^{-1}(\hat{s}) + \frac{K_d(\hat{s}-h)}{\tau(\hat{s}-h)+1}\ ] \ . \quad (15)$$

Let $K_p = \tilde{\beta}\hat{K}_p$, $K_i = g\,\tilde{\beta}\hat{K}_p$, where $\tilde{\beta} \in \mathbb{R}_+$ satisfies

$$\tilde{\beta} > \|\ \widetilde{\Phi}(\hat{s})\ \| \ . \quad (16)$$

Then an $\mathbf{S}_h$-stabilizing PID-controller C_{pid} is given by

$$C_{pid} = \tilde{\beta}\hat{K}_p + \frac{g\,\tilde{\beta}\,\hat{K}_p}{s} + \frac{K_d\ s}{\tau s+1} \ . \quad (17)$$

iii) ($\mathbf{S}_h$*-stable strictly-proper plants*): Let $G(s) \in \mathcal{G}_{ph} \cap \mathcal{G}_{\infty}$. Then there exists an $\mathbf{S}_h$-stabilizing PID-controller. Furthermore, C_{pid} can be designed as follows: Let $Y(\infty)^{-1} := sG(s)|_{s\to\infty}$. Choose any $K_d \in \mathbb{R}^{m\times m}$, and $\tau \in \mathbb{R}_+$ satisfying $\tau < 1/h$. Choose any $g \in \mathbb{R}_+$ satisfying

$$g > h \ . \quad (18)$$

Define $\tilde{\Psi}(\hat{s})$ as

$$\tilde{\Psi}(\hat{s}) := \ [\ \hat{G}^{-1}(\hat{s}) + \frac{K_d(\hat{s}-h)}{\tau(\hat{s}-h)+1}\ ]\frac{(\hat{s}-h)}{(\hat{s}-h+g)}Y(\infty)^{-1} - \hat{s}\,I \ . \quad (19)$$

Let $K_p = \tilde{\delta}Y(\infty)$, $K_i = g\,\tilde{\delta}\,Y(\infty)$, where $\tilde{\delta} \in \mathbb{R}_+$ satisfies

$$\tilde{\delta} > \|\ \tilde{\Psi}(\hat{s})\ \| \ . \quad (20)$$

Then an $\mathbf{S}_h$-stabilizing PID-controller C_{pid} is given by

$$C_{pid} = \tilde{\delta}\,Y(\infty) + \frac{g\,\tilde{\delta}\,Y(\infty)}{s} + \frac{K_d\ s}{\tau s+1} \ . \quad (21)$$

■

Remark: Condition (11) is obviously satisfied if $h = 0$, i.e., $\mathcal{U}_h = \mathcal{U}_0$; therefore, there exists a PID-controller C_{pid} of the form in (1) that stabilizes a given stable plant G, where the closed-loop poles of the system $Sys(G, C_{pid})$ may be anywhere in the open left-half complex-plane. ■

Proof of Proposition 2.1: **i)** Substitute $\hat{s} = s + h$ as in (3), (4), (5). With $\hat{G}(\hat{s}) = I^{-1}\hat{G}$, write $\hat{C}_{pid}(\hat{s})$ as in (6). By (2), $\hat{C}_{pid}$ in (13) stabilizes $\hat{G}(\hat{s})$ if and only if $\hat{M}(\hat{s})$ is $\mathbf{S}_0$-unimodular:

$$\begin{aligned}\hat{M}(\hat{s}) &= \frac{(\hat{s}-h)}{\hat{s}+\alpha}I + \hat{G}(\hat{s})\frac{(\hat{s}-h)}{\hat{s}+\alpha}\hat{C}_{pid} \\ &= I - \frac{(\alpha+h)}{\hat{s}+\alpha}I + \frac{(\hat{s}-h)}{\hat{s}+\alpha}\hat{G}(\hat{s})\hat{C}_{pid} \\ &= I + \frac{(\alpha+h)(\hat{s}-h)}{\hat{s}+\alpha}\,\Theta(\hat{s}) \ . \quad (22)\end{aligned}$$

In (22), $\Theta(\hat{s}) \in \mathcal{M}(\mathbf{S}_0)$ since $\frac{\hat{G}(\hat{s})G(0)^{-1}-I}{\hat{s}-h} = \frac{\hat{G}(\hat{s})\hat{G}(h)^{-1}-I}{\hat{s}-h} \in \mathcal{M}(\mathbf{S}_0)$. If (11) and (12) hold, then $h < \alpha$ and $\alpha + h < \|\Theta(\hat{s})\|^{-1}$ imply

$$\begin{aligned}\|\frac{(\alpha+h)(\hat{s}-h)}{\hat{s}+\alpha}\,\Theta(\hat{s})\,\| &\le (\alpha+h)\|\frac{\hat{s}-h}{\hat{s}+\alpha}\|\ \|\,\Theta(\hat{s})\,\| \\ &= (\alpha+h)\ \|\,\Theta(\hat{s})\,\| < 1;\end{aligned}$$

hence, $\hat{M}(\hat{s})$ in (22) is $\mathbf{S}_0$-unimodular by the "small-gain theorem" (see e.g., [9]). Therefore, $\hat{C}_{pid}(\hat{s})$ stabilizes $\hat{G}(\hat{s})$; hence, C_{pid} is an $\mathbf{S}_h$-stabilizing controller for G.

ii) Write the controller $C_{pid}(s)$ given in (17) as

$$\begin{aligned}C_{pid}(s) &= (\frac{s}{s+g}C_{pid})(\frac{s\,I}{s+g})^{-1} \\ &= (\tilde{\beta}\hat{K}_p + \frac{K_d\,s}{(\tau\,s+1)}\frac{s}{(s+g)}\,)(\frac{s\,I}{s+g}\,)^{-1}. \quad (23)\end{aligned}$$

Substitute $\hat{s} = s+h$ into (23) to obtain an RCF of $\hat{C}_{pid}(\hat{s})$ as in (6), with $\alpha = g - h$. Then

$$\begin{aligned}\hat{C}_{pid}(\hat{s}) = (\,\tilde{\beta}\hat{K}_p &+ \frac{K_d\,(\hat{s}-h)}{(\tau(\hat{s}-h)+1)} \\ &\frac{(\hat{s}-h)}{(\hat{s}-h+g)}\,)(\frac{(\hat{s}-h)}{\hat{s}-h+g}I)^{-1} \ , \quad (24)\end{aligned}$$

where $(1 - \tau h) \in \mathbb{R}_+$ and $(g - h) \in \mathbb{R}_+$ by assumption. Since $G(s) \in \mathcal{G}_{ph} \cap \mathcal{G}_{zh}$ implies $G^{-1}(s) \in \mathcal{M}(\mathbf{S}_h)$, we have $\hat{G}^{-1}(\hat{s}) \in \mathcal{M}(\mathbf{S}_0)$. By (2), $\hat{C}_{pid}(\hat{s})$ in (17) stabilizes $\hat{G}(\hat{s})$ if and only if $\tilde{M}_\beta(\hat{s})$ is $\mathbf{S}_0$-unimodular:

$$\begin{aligned}\tilde{M}_\beta(\hat{s}) &= \frac{(\hat{s}-h)}{\hat{s}-h+g}I + \hat{G}(\hat{s})\frac{(\hat{s}-h)}{\hat{s}-h+g}\hat{C}_{pid} \\ &= \hat{G}(\hat{s})\,\tilde{\beta}\,\hat{K}_p + [\ I + \hat{G}(\hat{s})\frac{K_d\,(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}\ ]\frac{(\hat{s}-h)}{(\hat{s}-h+g)} \\ &= \tilde{\beta}\,\hat{G}(\hat{s})\hat{K}_p(\,I \\ &\quad + \frac{1}{\tilde{\beta}}\hat{K}_p^{-1}\ [\ \hat{G}^{-1}(\hat{s}) + \frac{K_d(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}]\frac{(\hat{s}-h)}{(\hat{s}-h+g)}) \\ &= \tilde{\beta}\hat{G}(\hat{s})\hat{K}_p(I + \frac{1}{\tilde{\beta}}\widetilde{\Phi}(\hat{s})\frac{(\hat{s}-h)}{(\hat{s}-h+g)}), \quad (25)\end{aligned}$$

where $\hat{K}_p$ is nonsingular and $G^{-1}(s) \in \mathcal{M}(\mathbf{S}_h)$ implies $\hat{G}^{-1}(\hat{s}) \in \mathcal{M}(\mathbf{S}_0)$ by assumption. If (14) holds, then $2h < g$ and $\tilde{\beta} > \|\widetilde{\Phi}(\hat{s})\|$ imply

$$\|\frac{1}{\tilde{\beta}}\ \widetilde{\Phi}(\hat{s})\ \frac{(\hat{s}-h)}{(\hat{s}-h+g)}\| \ \leq\ \frac{1}{\tilde{\beta}}\ \|\ \widetilde{\Phi}(\hat{s})\ \|\ \|\frac{(\hat{s}-h)}{(\hat{s}-h+g)}\| = \frac{1}{\tilde{\beta}}\ \|\ \widetilde{\Phi}(\hat{s})\ \| <\ 1;$$

hence, $\tilde{M}_\beta(\hat{s})$ in (25) is $\mathbf{S}_0$-unimodular. Therefore, $\hat{C}_{pid}(\hat{s})$ an $\mathbf{S}_0$-stabilizing controller for $\hat{G}(\hat{s})$; hence, C_{pid} is an $\mathbf{S}_h$-stabilizing controller for G.

iii) Substitute $\hat{s} = s+h$ as in (3), (4), (5). Then an LCF of $\hat{G}(\hat{s})$ is

$$\hat{G}(\hat{s}) = \hat{Y}^{-1}\hat{X} := I^{-1}\hat{G}(\hat{s})\ .$$

Write the controller $C_{pid}(s)$ given in (21) as

$$\begin{aligned} C_{pid}(s) &= (\frac{s}{s+g}C_{pid})(\frac{s\,I}{s+g})^{-1} \\ &= (\tilde{\delta}Y(\infty) + \frac{K_d\,s}{(\tau s+1)}\frac{s}{(s+g)})(\frac{s\,I}{s+g}I)^{-1}. \quad (26) \end{aligned}$$

Substitute $\hat{s} = s+h$ into (26) to obtain an RCF of $\hat{C}_{pid}(\hat{s})$ as in (6), with $\alpha = g-h$. Then

$$\hat{C}_{pid}(\hat{s}) = (\ \tilde{\delta}\,Y(\infty) + \frac{K_d\,(\hat{s}-h)}{(\tau(\hat{s}-h)+1)} \frac{(\hat{s}-h)}{(\hat{s}-h+g)}\ )(\frac{(\hat{s}-h)}{\hat{s}-h+g}I\ )^{-1}\ , \quad (27)$$

where $(1-\tau h) \in \mathbb{R}_+$ and $(g-h) \in \mathbb{R}_+$ by assumption. Since $G(s) \in \mathcal{G}_{ph} \cap \mathcal{G}_\infty$ implies $(s+a)G(s) \in \mathcal{M}(\mathbf{S}_h)$ and $\frac{1}{(s+a)}G^{-1}(s) \in \mathcal{M}(\mathbf{S}_h)$ for $a > h$, we have $(\hat{s}-h+a)\hat{G}(\hat{s}) \in \mathcal{M}(\mathbf{S}_0)$ and $\frac{1}{(\hat{s}-h+a)}\hat{G}^{-1}(\hat{s}) \in \mathcal{M}(\mathbf{S}_0)$; similarly, $(\hat{s}+\tilde{\delta})\hat{G}(\hat{s}) \in \mathcal{M}(\mathbf{S}_0)$ and $\frac{1}{(\hat{s}+\tilde{\delta})}\hat{G}^{-1}(\hat{s}) \in \mathcal{M}(\mathbf{S}_0)$. By (2), $\hat{C}_{pid}(\hat{s})$ in (26) stabilizes $\hat{G}(\hat{s})$ if and only if $\tilde{M}_\delta(\hat{s})$ is $\mathbf{S}_0$-unimodular:

$$\begin{aligned} \tilde{M}_\delta(\hat{s}) &= \frac{(\hat{s}-h)}{(\hat{s}-h+g)}I + \hat{G}(\hat{s})\ \frac{(\hat{s}-h)}{(\hat{s}-h+g)}\hat{C}_{pid} \\ &= (\hat{s}+\tilde{\delta})\hat{G}(\hat{s})(\ \frac{1}{(\hat{s}+\tilde{\delta})}\hat{G}^{-1}(\hat{s})\ \frac{(\hat{s}-h)}{(\hat{s}-h+g)}I \\ &\quad + \frac{1}{(\hat{s}+\tilde{\delta})}I\frac{(\hat{s}-h)}{(\hat{s}-h+g)}\hat{C}_{pid}) \\ &= (\hat{s}+\tilde{\delta})\hat{G}(\hat{s})(\ \frac{\tilde{\delta}Y(\infty)}{(\hat{s}+\tilde{\delta})} \\ &\quad + \frac{1}{(\hat{s}+\tilde{\delta})}[\hat{G}^{-1}(\hat{s}) + \frac{K_d(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}]\frac{(\hat{s}-h)}{(\hat{s}-h+g)}) \\ &= (\hat{s}+\tilde{\delta})\hat{G}(\hat{s})\ (\ I + \frac{1}{(\hat{s}+\tilde{\delta})}[\frac{(\hat{s}-h)}{(\hat{s}-h+g)}\hat{G}^{-1}(\hat{s})Y(\infty)^{-1} \\ &\quad - \hat{s}I + \frac{K_d(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}\frac{(\hat{s}-h)}{(\hat{s}-h+g)}Y(\infty)^{-1}]\ )Y(\infty) \\ &= (\hat{s}+\tilde{\delta})\ \hat{G}(\hat{s})\ (\ I + \frac{1}{(\hat{s}+\delta)}\ \widetilde{\Psi}(\hat{s})\ )Y(\infty)\ . \quad (28) \end{aligned}$$

Then $\widetilde{\Psi}(\hat{s}) \in \mathcal{M}(\mathbf{S}_0)$ and therefore $M_\delta(\hat{s})$ in (28) is $\mathbf{S}_0$-unimodular since $\delta > \|\widetilde{\Psi}(\hat{s})\|$ implies

$$\|\frac{1}{(\hat{s}+\tilde{\delta})}\ \widetilde{\Psi}(\hat{s})\ \| \leq \|\ \frac{1}{(\hat{s}+\tilde{\delta})}\ \|\ \|\ \widetilde{\Psi}(\hat{s})\ \| = \frac{1}{\tilde{\delta}}\ \|\ \widetilde{\Psi}(\hat{s})\ \| <\ 1.$$

Therefore, $\hat{C}_{pid}(\hat{s})$ an $\mathbf{S}_0$-stabilizing controller for $\hat{G}(\hat{s})$; hence, C_{pid} is an $\mathbf{S}_h$-stabilizing controller for G. ∎

The systematic PID-controller design method of Proposition 2.1 is illustrated by the following examples. Given $h \in \mathbb{R}_+$ and $G \in \mathcal{G}_h$, define

$$\rho\ :=\ max\{x|p = x + jy,\ \ where\ p\ is\ a\ pole\ of\ G(s)\}; \quad (29)$$

then $-h > \rho$ since $G \in \mathcal{G}_h$. We also define

$$\gamma(\hat{K}_p, \hat{K}_d) := \|\ \Theta(\hat{s})\ \|^{-1}\ . \quad (30)$$

Example 2.1 Consider the same plant transfer-function as that of Example 3.2 in [2].

$$G(s) = \frac{(s-5)(s^2+8s+32)}{(s+2)(s+8)(s^2+12s+40)}\ . \quad (31)$$

By (29), $\rho = -2$. Suppose that $h = 1$ and we choose $\hat{K}_p = -2.5$, $\hat{K}_d = -0.3$, $\tau = 0.05$. We compute $\gamma = 2.9 > 2h = 2$, and set $\alpha = 0.5\gamma$. The closed-loop poles are $\{-2.52 \pm j0.94, -3.44 \pm j2.22, -4.57 \pm j15.20\}$, which all have real-parts less than $-h = -1$.

Note that γ is a function of $(\hat{K}_p, \hat{K}_d)$ for a given h. The constant contour of $\gamma(\hat{K}_p, \hat{K}_d)$ is shown in Fig. 2, where the solid line represents $\gamma = 2h$ as the upper-bound for

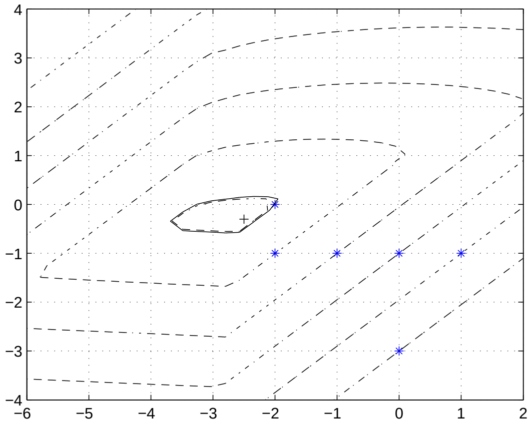

Figure 2: Contour of $\gamma(\hat{K}_p, \hat{K}_d)$ for Example 2.1

condition (11). The point $\hat{K}_p = -2.5$, $\hat{K}_d = -0.3$ is marked by "+", which is inside the region of the solid line.

From the list of the values evaluated for the contours in Table 1, we can see that there exists a maximum γ^* for the given h. The curve $\gamma^*(h)$ is plotted in Fig. 3 as the solid line, where the dashed line represents $\gamma = 2h$. We can see from Fig. 3 that there exists an absolute value $h_{max} < -\rho$ such that the sufficient condition will not be satisfied when $h > h_{max}$.

Table 1: Evaluated points for contours in Example 2.1

x	-2	-2	-1	0	0	1
y	-1	0	-1	-3	-1	-1
γ	0.80	2.09	0.50	0.23	0.37	0.29

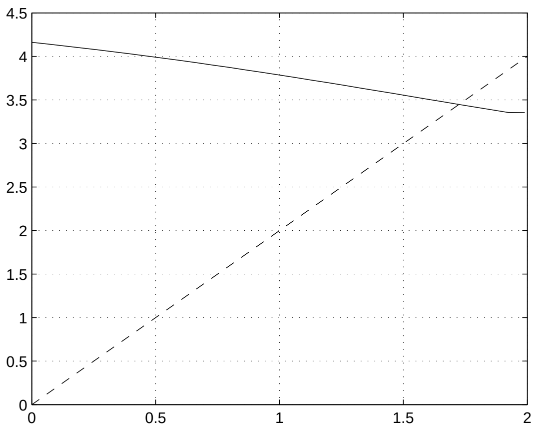

Figure 3: Plot of $\gamma^*(h)$ for Example 2.1

∎

Example 2.2 Consider the transfer-function

$$G(s) = \frac{(s+5)(s^2+8s+32)}{(s+2)(s+3)(s^2+5s+40)}\ , \tag{32}$$

which belongs to the set of $\mathbf{S}_h$-stable strictly-proper plants with no zeros in $\mathcal{U}_h$ for $h < 2$. According to Proposition 2.1(iii), for any $h < 2$, there exists a PID controller such that the closed loop transfer function is $\mathbf{S}_h$-stable. However we cannot use the procedure in Proposition 2.1(i) to achieve this for all $h < 2$ as seen in the $\gamma^*(h)$ curve shown in Fig. 4. For those h's on the right side of the intersection point, Proposition 2.1(i) is not applicable.

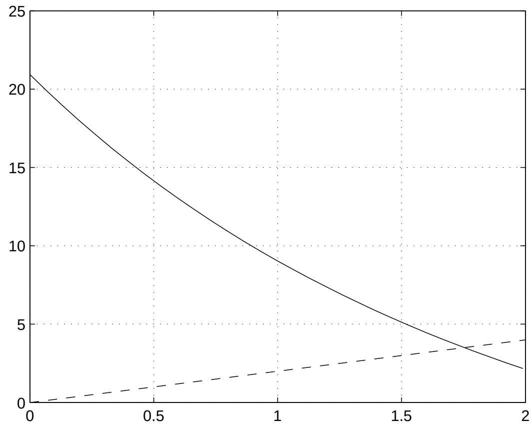

Figure 4: Plot of $\gamma^*(h)$ for Example 2.2

To use Proposition 2.1(iii) for $h = 1.99$, we can choose $\tau = 0.05$ since $0.05 < 1/1.99$. Choose arbitrarily $K_d = 2$ and $g = 4 > h$. With $Y(\infty) = 1$, we can compute $\|\tilde{\Psi}(\hat{s})\| = 31.01$ and simply choose $\tilde{\delta} = 32.01$. The closed-loop poles are $\{-3.49 \pm j3.04, -4.26, -5.29 \pm j5.29, -80.20\}$, which all have real-parts less than $-h = -1.99$. ∎

Example 2.3 Consider the quadruple-tank apparatus in [7], which consists of four interconnected water tanks and two pumps. The output variables are the water levels of the two lower tanks, and they are controlled by the currents that are manipulating two pumps. The transfer-matrix of the linearized model at some operating point is

$$G = \begin{bmatrix} \frac{3.7b_1}{62s+1} & \frac{3.7(1-b_2)}{(23s+1)(62s+1)} \\ \frac{4.7(1-b_1)}{(30s+1)(90s+1)} & \frac{4.7b_2}{90s+1} \end{bmatrix} \in {\mathbf{S}_0}^{2\times 2}. \tag{33}$$

One of the two transmission-zeros of the linearized system dynamics can be moved between the positive and negative real-axis by changing a valve. The adjustable transmission-zeros depends on parameters b_1 and b_2 (the proportions of water flow into the tanks adjusted by tv valves). For the values of b_1, b_2 chosen as $b_1 = 0.$ and $b_2 = 0.34$, the plant G has transmission-zeros $z_1 = 0.0229 > 0$ and $z_2 = -0.0997$. By (29) $\rho = -1/90$ -0.0111. Suppose that $h = 0.004$. Choose $\tau = 0.05$, ar

$$\hat{K}_p = \begin{bmatrix} -22.61 & 37.61 \\ 72.14 & -43.96 \end{bmatrix} , \quad \hat{K}_d = \begin{bmatrix} 5.28 & 6.21 \\ 6.53 & 7.84 \end{bmatrix} \quad (3$$

We can compute $\gamma = 0.0099 > 2h = 0.008$, and set α 0.5γ. The maximum of the real-parts of the closed pol can now be computed as -0.0059, which is less than $-h$ -0.004. Thus the requirement is fulfilled.

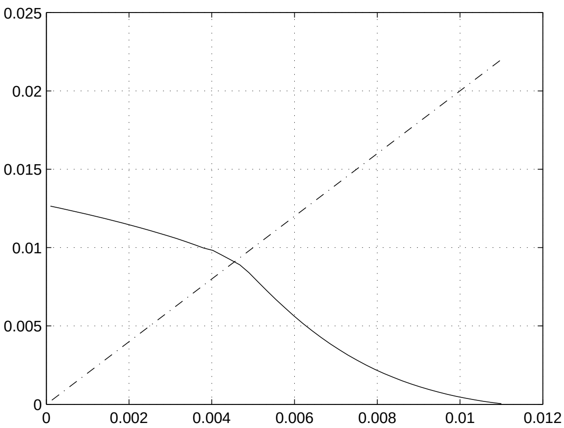

Figure 5: Plot of $\gamma(h)$ for Example 2.3

For a given $(\hat{K}_p, \hat{K}_d)$, γ can be uniquely determined. For the given value of $(\hat{K}_p, \hat{K}_d)$ in (34), the curve of $\gamma(h)$ is given in Fig. 5 as the solid line and the dash-dotted line represents the straight line $2h$. The intersection point thus decides the maximum value h_{max} such that condition (11) is violated, which clearly indicates that the choice of $h = 0.004$ is feasible. ■

Example 2.4 The PID-synthesis procedure based on Proposition 2.1 involves free parameter choices. Consider the same transfer-function as in (31) of Example 2.1. Let $h = 1$, choose $\tau = 0.05$, and set $\alpha = 0.5\gamma$ as before. If we choose ($\hat{K}_p = 2.5$, $\hat{K}_d = 0.2$), then the the dotted line in Fig. 6 shows the closed-loop step response. However, if we choose ($\hat{K}_p = 2$, $\hat{K}_d = -0.1$), then we obtain a completely different step response as shown by the dash-dotted line. It is natural to ask then if the free parameters can be chosen optimally in some sense. Consider a prototype second order model plant, with $\zeta = 0.7$ and $\omega_n = 6$; i.e.,

$$T_m = \frac{\omega_n^2}{s^2 + 2\zeta\omega s + \omega_n^2} \ . \qquad (35)$$

We want the actual step response $s_o(t)$ to be as close as possible to the closed-loop step response $s_m(t)$ using the

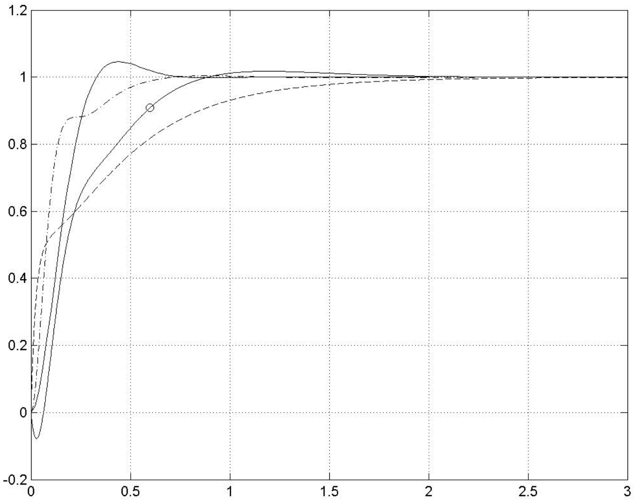

Figure 6: Step responses for Example 2.4

model plant T_m. The solid line shows the step response using T_m. Consider the cost function

$$error = \frac{1}{3}\int_0^3 (s_m(t) - s_o(t))^2 dt, \qquad (36)$$

where $s_o(t)$ is the step response for any choice of ($\hat{K}_p$, $\hat{K}_d$). By plotting the contour of the error in terms of ($\hat{K}_p$, $\hat{K}_d$), we find the global minimum of the error occurring at ($\hat{K}_p = 1.47, \hat{K}_d = -0.15$). The step response for this choice of $(\hat{K}_p, \hat{K}_d)$ is shown by the solid line marked with a circle, which is closer to the model step response than the other two.

■

2.2 Plants with no zeros in $\mathcal{U}_h$

Consider the class $\mathcal{G}_{zh}$ of $m \times m$ plants with no (transmission or blocking) zeros in $\mathcal{U}_h$ as described in (8). The plants in $\mathcal{G}_{zh}$ obviously have no zeros at $s = 0$ since they have no zeros in $\mathcal{U}_h$. The plants $G \in \mathcal{G}_{zh}$ may not be $\mathbf{S}_h$-stable but $G^{-1} \in \mathcal{M}(\mathbf{S}_h)$; an LCF of $G(s)$ is

$$G = Y^{-1}X = (G^{-1})^{-1}I \ . \qquad (37)$$

The plants in $\mathcal{G}_{zh}$ are obviously strongly stabilizable, and they admit $\mathbf{S}_0$-stabilizing PID-controllers (see [5]). Proposition 2.2 shows that these plants also admit $\mathbf{S}_h$-stabilizing PID-controllers for any pre-specified $h \in \mathbb{R}_+$, and proposes a systematic PID-controller synthesis procedure.

Proposition 2.2 *(PID controller synthesis for plants with no zeros in $\mathcal{U}_h$):*
Let $G \in \mathcal{G}_{zh}$. Then there exists an $\mathbf{S}_h$-stabilizing PID-controller C_{pid}. Furthermore, C_{pid} can be designed as

follows: Choose any nonsingular $\hat{K}_p \in \mathbb{R}^{m\times m}$. Choose any $K_d \in \mathbb{R}^{m\times m}$, and $\tau \in \mathbb{R}_+$ satisfying $\tau < 1/h$. Choose any $g \in \mathbb{R}_+$ satisfying

$$g > 2h \ . \tag{38}$$

Define $\Phi(\hat{s})$ as

$$\Phi(\hat{s}) := \hat{K}_p^{-1} \ [\ \hat{G}^{-1}(\hat{s}) + \frac{K_d(\hat{s}-h)}{\tau(\hat{s}-h)+1}\ ] \ . \tag{39}$$

Let $K_p = \beta\hat{K}_p$, $K_i = g\,\beta\hat{K}_p$, where $\beta \in \mathbb{R}_+$ satisfies

$$\beta > \|\ \Phi(\hat{s})\ \| \ . \tag{40}$$

Then an $\mathbf{S}_h$-stabilizing PID-controller C_{pid} is given by

$$C_{pid} = \beta\hat{K}_p + \frac{g\,\beta\,\hat{K}_p}{s} + \frac{K_d\ s}{\tau s+1} \ . \tag{41}$$

■

Proof of Proposition 2.2: Substitute $\hat{s} = s + h$ as in (3), (4), (5). Then an LCF of $\hat{G}(\hat{s})$ is $\hat{G}(\hat{s}) = \hat{Y}^{-1}\hat{X} := (\ \hat{G}^{-1}(\hat{s})\ )^{-1} I$. Write the controller $C_{pid}(s)$ given in (41) as

$$\begin{aligned} C_{pid}(s) &= (\frac{s}{s+g}C_{pid})(\frac{s\,I}{s+g})^{-1} \\ &= (\beta\hat{K}_p + \frac{K_d\,s}{(\tau\,s+1)}\frac{s}{(s+g)})(\frac{s\,I}{s+g})^{-1}. \end{aligned} \tag{42}$$

Substitute $\hat{s} = s+h$ into (42) to obtain an RCF of $\hat{C}_{pid}(\hat{s})$ as in (6), with $\alpha = g - h$. Then

$$\begin{aligned} \hat{C}_{pid}(\hat{s}) = (\ \beta\hat{K}_p &+ \frac{K_d\,(\hat{s}-h)}{(\tau(\hat{s}-h)+1)} \\ &\frac{(\hat{s}-h)}{(\hat{s}-h+g)}\ )\ (\frac{(\hat{s}-h)}{\hat{s}-h+g}I\ )^{-1} \ , \end{aligned} \tag{43}$$

where $(1-\tau h) \in \mathbb{R}_+$ and $(g-h) \in \mathbb{R}_+$ by assumption. By (2), $\hat{C}_{pid}(\hat{s})$ in (42) stabilizes $\hat{G}(\hat{s})$ if and only if $M_\beta(\hat{s})$ is $\mathbf{S}_0$-unimodular:

$$\begin{aligned} M_\beta(\hat{s}) &= \hat{Y}(\hat{s})\ \frac{(\hat{s}-h)}{\hat{s}-h+g}I + \hat{X}(\hat{s})\ \frac{(\hat{s}-h)}{\hat{s}-h+g}\hat{C}_{pid} \\ &= \hat{G}^{-1}(\hat{s})\ \frac{(\hat{s}-h)}{\hat{s}-h+g}I + \ \frac{(\hat{s}-h)}{\hat{s}-h+g}\hat{C}_{pid} \\ &= \beta\,\hat{K}_p + [\ \hat{G}^{-1}(\hat{s}) + \frac{K_d\,(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}\ ]\frac{(\hat{s}-h)}{(\hat{s}-h+g)} \\ &= \beta\,\hat{K}_p\ (I + \frac{1}{\beta}\hat{K}_p^{-1}\ [\ \hat{G}^{-1}(\hat{s}) \\ &\quad + \frac{K_d\,(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}\ ]\frac{(\hat{s}-h)}{(\hat{s}-h+g)}\ ) \\ &= \beta\,\hat{K}_p\ (I + \frac{1}{\beta}\ \Phi(\hat{s})\ \frac{(\hat{s}-h)}{(\hat{s}-h+g)}\ ) \ , \end{aligned} \tag{44}$$

where $\hat{K}_p$ is unimodular and $G^{-1}(s) \in \mathcal{M}(\mathbf{S}_h)$ by assumption. If (38) holds, then $2h < g$ and $\beta > \|\Phi(\hat{s})\|$ imply

$$\begin{aligned} \|\frac{1}{\beta}\ \Phi(\hat{s})\ \frac{(\hat{s}-h)}{(\hat{s}-h+g)}\| \ &\le\ \frac{1}{\beta}\ \|\ \Phi(\hat{s})\ \|\ \|\frac{(\hat{s}-h)}{(\hat{s}-h+g)}\| \\ &= \frac{1}{\beta}\ \|\ \Phi(\hat{s})\ \| < 1; \end{aligned}$$

hence, $M_\beta(\hat{s})$ in (44) is $\mathbf{S}_0$-unimodular. Therefore, $\hat{C}_{pid}(\hat{s})$ an $\mathbf{S}_0$-stabilizing controller for $\hat{G}(\hat{s})$; hence, C_{pid} is an $\mathbf{S}_h$-stabilizing controller for G. ■

Example 2.5 Consider the MIMO system

$$G = \begin{bmatrix} \frac{(s+2)(s+3)}{(s-4)(s-8)} & 0 \\ \frac{(s+1)(s+5)}{(s+6)(s+7)} & \frac{(s+4)(s+8)}{s^2-6s+12} \end{bmatrix} \in {\mathbf{S}_0}^{2\times 2} \ , \tag{45}$$

which has no (transmission) zeros larger than -2. Thus we can choose $h = 1.99$. Since $0.05 < 1/1.99$, $\tau = 0.05$ can be selected. Let $g = 5$ to fulfill the requirement $g > 2h$. Let us arbitrarily choose

$$\hat{K}_p = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \ , \quad K_d = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \ . \tag{46}$$

We can calculate the norm $\|\ \Phi(\hat{s})\ \| = 163.8$ from (40). By arbitrarily choosing $\beta = 164.8$, the maximum of the real-parts of the closed poles can now be computed as -2.3561, which is less than $-h = -1.99$. Thus the requirement is fulfilled. ■

2.3 Plants with no finite $\mathcal{U}$-zeros

Consider the class $\mathcal{G}_\infty$ of $m \times m$ strictly-proper plants that have no other (transmission or blocking) zeros in $\mathcal{U}_h$ as described in (9). Since the plants in $\mathcal{G}_\infty$ have no zeros in $\mathcal{U}_h$ other than the one at $s = \infty$, they obviously have no zeros at $s = 0$. The plants $G \in \mathcal{G}_{zh}$ are not all $\mathbf{S}_h$-stable but $\frac{1}{s+a}G^{-1} \in \mathcal{M}(\mathbf{S}_h)$ for any $a > h$; an LCF of $G(s)$ is

$$G = Y^{-1}X = (\frac{1}{s+a}G^{-1})^{-1}(\ \frac{1}{s+a}I\ ) \ ; \tag{47}$$

in (47), $G(\infty) = 0$, and $Y(\infty)^{-1} = (s+a)G(s)|_{s\to\infty} = s\,G(s)|_{s\to\infty}$. The plants in $\mathcal{G}_\infty$ are strongly stabilizable, and they admit $\mathbf{S}_0$-stabilizing PID-controllers [5]). Proposition 2.3 shows that these plants also admit $\mathbf{S}_h$-stabilizing PID-controllers for any pre-specified $h \in \mathbb{R}_+$, and proposes a systematic PID-controller synthesis procedure.

Proposition 2.3 *(PID controller synthesis for strictly-proper plants):*
Let $G \in \mathcal{G}_\infty$. Then there exists an $\mathbf{S}_h$-stabilizing PID-controller, and C_{pid} can be designed as follows: Let

$Y(\infty)^{-1} := s\,G(s)|_{s\to\infty}$. Choose any $K_d \in \mathbb{R}^{m\times m}$, and $\tau \in \mathbb{R}_+$ satisfying $\tau < 1/h$. Choose any $g \in \mathbb{R}_+$ satisfying

$$g > h\ . \tag{48}$$

Define $\Psi(\hat{s})$ as

$$\Psi(\hat{s}) := [\ \hat{G}^{-1}(\hat{s}) + \frac{K_d(\hat{s}-h)}{\tau(\hat{s}-h)+1}\ ]\frac{(\hat{s}-h)}{(\hat{s}-h+g)}Y(\infty)^{-1} - \hat{s}I. \tag{49}$$

Let $K_p = \delta Y(\infty)$, $K_i = g\,\delta Y(\infty)$, where $\delta \in \mathbb{R}_+$ satisfies

$$\delta > \|\ \Psi(\hat{s})\ \|\ . \tag{50}$$

Then an $\mathbf{S}_h$-stabilizing PID-controller C_{pid} is given by

$$C_{pid} = \delta\,Y(\infty) + \frac{g\,\delta\,Y(\infty)}{s} + \frac{K_d\ s}{\tau s+1}\ . \tag{51}$$

■

Proof of Proposition 2.3: Substitute $\hat{s} = s + h$ as in (3), (4), (5). Then an LCF of $\hat{G}(\hat{s})$ is

$$\hat{G}(\hat{s}) = \hat{Y}^{-1}\hat{X} := (\ \frac{1}{\hat{s}-h+a}\hat{G}^{-1}(\hat{s})\ )^{-1}(\ \frac{1}{\hat{s}-h+a}I\ ).$$

Write the controller $C_{pid}(s)$ given in (51) as

$$\begin{aligned} C_{pid}(s) &= (\frac{s}{s+g}C_{pid})(\frac{s\,I}{s+g})^{-1} \\ &= (\delta Y(\infty) + \frac{K_d\,s}{(\tau\,s+1)}\frac{s}{(s+g)})(\frac{s\,I}{s+g})^{-1}. \end{aligned} \tag{52}$$

Substitute $\hat{s} = s+h$ into (52) to obtain an RCF of $\hat{C}_{pid}(\hat{s})$ as in (6), with $\alpha = g-h$. Then

$$\begin{aligned} \hat{C}_{pid}(\hat{s}) = (\delta\,Y(\infty) &+ \frac{K_d\,(\hat{s}-h)}{(\tau(\hat{s}-h)+1)} \\ &\frac{(\hat{s}-h)}{(\hat{s}-h+g)})(\frac{(\hat{s}-h)}{\hat{s}-h+g}I)^{-1}\ , \end{aligned} \tag{53}$$

where $(1-\tau h) \in \mathbb{R}_+$ and $(g-h) \in \mathbb{R}_+$ by assumption. By (2), $\hat{C}_{pid}(\hat{s})$ in (52) stabilizes $\hat{G}(\hat{s})$ if and only if $M_\delta(\hat{s})$ is $\mathbf{S}_0$-unimodular:

$$\begin{aligned} M_\delta(\hat{s}) &= \hat{Y}(\hat{s})\,\frac{(\hat{s}-h)}{(\hat{s}-h+g)}I + \hat{X}(\hat{s})\,\frac{(\hat{s}-h)}{(\hat{s}-h+g)}\hat{C}_{pid} \\ &= \frac{1}{(\hat{s}-h+a)}\hat{G}^{-1}(\hat{s})\,\frac{(\hat{s}-h)}{(\hat{s}-h+g)}I \\ &\quad + \frac{1}{(\hat{s}-h+a)}I\frac{(\hat{s}-h)}{(\hat{s}-h+g)}\hat{C}_{pid} \\ &= \frac{1}{(\hat{s}-h+a)}\delta\,Y(\infty) + \frac{1}{(\hat{s}-h+a)}[\ \hat{G}^{-1}(\hat{s}) \\ &\quad + \frac{K_d\,(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}\ ]\frac{(\hat{s}-h)}{(\hat{s}-h+g)} \\ &= \frac{(\hat{s}+\delta)}{(\hat{s}-h+a)}(\frac{\delta\,I}{\hat{s}+\delta} + \frac{1}{(\hat{s}+\delta)}[\hat{G}^{-1}(\hat{s}) \\ &\quad + \frac{K_d(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}]\frac{(\hat{s}-h)Y(\infty)^{-1}}{(\hat{s}-h+g)})Y(\infty) \\ &= \frac{(\hat{s}+\delta)}{(\hat{s}-h+a)}(\,I + \frac{1}{(\hat{s}+\delta)}\ \Psi(\hat{s})\ )Y(\infty). \end{aligned} \tag{54}$$

Then $\Psi(\hat{s}) \in \mathcal{M}(\mathbf{S}_0)$ since

$$\begin{aligned} \Psi(\hat{s}) &= \hat{G}^{-1}(\hat{s})\frac{(\hat{s}-h)}{(\hat{s}-h+g)}Y(\infty)^{-1} - \hat{s}I \\ &\quad + \frac{K_d\,(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}\frac{(\hat{s}-h)}{(\hat{s}-h+g)}Y(\infty)^{-1} \\ &= (\hat{s}-h+a)\hat{Y}(\hat{s})\frac{(\hat{s}-h)}{(\hat{s}-h+g)}Y(\infty)^{-1} - \hat{s}I \\ &\quad + \frac{K_d\,(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}\frac{(\hat{s}-h)}{(\hat{s}-h+g)}Y(\infty)^{-1} \\ &= \hat{s}[\frac{(\hat{s}-h)}{(\hat{s}-h+g)}\hat{Y}(\hat{s})Y(\infty)^{-1} - I] \\ &\quad + [(a-h)\hat{Y}(\hat{s}) + \frac{K_d\,(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}\ ]\frac{(\hat{s}-h)Y(\infty)^{-1}}{(\hat{s}-h+g)}, \end{aligned}$$

and $\hat{Y}(\infty) = Y(\infty)$ implies $[\ \frac{(\hat{s}-h)}{(\hat{s}-h+g)}\hat{Y}(\hat{s})Y(\infty)^{-1} - I\ ]$ is strictly-proper. Therefore $M_\delta(\hat{s})$ in (54) is $\mathbf{S}_0$-unimodular since $\delta > \|\Psi(\hat{s})\|$ implies

$$\begin{aligned} \|\frac{1}{(\hat{s}+\delta)}\ \Psi(\hat{s})\ \| \ &\le\ \|\ \frac{1}{(\hat{s}+\delta)}\ \|\,\|\,\Psi(\hat{s})\,\| \\ &= \frac{1}{\delta}\ \|\,\Psi(\hat{s})\,\| < 1. \end{aligned}$$

Therefore, $\hat{C}_{pid}(\hat{s})$ an $\mathbf{S}_0$-stabilizing controller for $\hat{G}(\hat{s})$; hence, C_{pid} is an $\mathbf{S}_h$-stabilizing controller for G. ■

Example 2.6 Consider a similar transfer-function as in Example 2.2 by changing all its stable poles into unstable ones:

$$G(s) = \frac{(s+5)(s^2+8s+32)}{(s-2)(s-3)(s^2-5s+40)}\ . \tag{55}$$

This transfer-function is an unstable strictly-proper plant with no zeros in $\mathcal{U}_h$ for $h < 4$. According to Proposition 2.3, there exists a PID controller for any $h < 4$ such that the closed loop transfer function is $\mathbf{S}_h$-stable.

Let us consider $h = 2.5$. We can choose $\tau = 0.05$ since $0.05 < 1/2.5$. Choose arbitrarily $K_d = 2$ and $g = 5 = 2h > h$. With $Y(\infty) = 1$, we can compute the norm $\| \Psi(\hat{s}) \| = 52.58$ and simply choose $\delta = 60$. The closed-loop poles are $\{-2.605 \pm j3.827, -2.899, -9.859 \pm j9.524, -82.173\}$, which all have real-parts less than $-h = -2.5$.

To demonstrate a PID controller exists for all $h < 4$, let us choose $h = 3.99$. We can choose $\tau = 0.05$ as before since $0.05 < 1/2.5$. Choose arbitrarily $K_d = 2$ and $g = 8 > h$. With $Y(\infty) = 1$, we can compute the norm $\| \Psi(\hat{s}) \| = 13905.36$ and simply choose $\delta = 14000$. The closed-loop poles are $\{-3.99007 \pm j3.99997, -4.96577, 8.10888, -19.90415, -14009.04107\}$, which all have real-parts less than $-h = -3.99$. Even though the goal is achieved, δ is now very large since the value of h approaches its limit. ■

Example 2.7 Consider the MIMO system

$$G = \begin{bmatrix} \frac{2(s+3)}{(s-4)(s-8)} & \frac{1}{(s+20)} \\ \frac{(s+5)}{(s+6)(s+7)} & \frac{(s+4)}{s^2-6s+12} \end{bmatrix} \in \mathbf{S}_0{}^{2\times 2} , \qquad (56)$$

which has no (transmission) zeros larger than -1.39. Thus we can choose $h = 1$. Since $0.05 < 1$, $\tau = 0.05$ can be selected. Let $g = 2$ to fulfill the requirement $g > h$. Let us arbitrarily choose

$$K_d = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} . \qquad (57)$$

We can compute the norm $\| \Phi(\hat{s}) \| = 94.77$ from (49). By arbitrarily choosing $\delta = 96$, the maximum of the real-parts of the closed poles can now be computed as -1.25, which is less than $-h = -1$. Thus the requirement is fulfilled. ■

3 Conclusions

For several important classes of LTI MIMO plants, systematic synthesis procedures were developed so that that closed-loop system is stabilized using a PID-controller that places the closed-loop poles in the left-half complex-plane to the left of the plant zero with the largest negative real-part. The plants under consideration are either stable, or unstable with restrictions on the location of the zeros. For the unstable plant case, only one zero at infinity is allowed, which in the SISO plant case means that the relative degree is no more than one. The proposed synthesis methods allow freedom in the choice of parameters. Illustrative examples were given, including one that demonstrates how this freedom can be used to improve an SISO system's performance. Extending the optimal parameter selection to MIMO systems would be a challenging goal. Future directions of this work would explore extending the design method to a wider class of unstable plants, perhaps with more zeros at infinity. Optimal parameter selections for the MIMO case will also be explored.

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