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3D Controllable Set of Linear Time-Invariant Open-Loop Unstable Systems with Constrained Input – A Submarine Case

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Abstract—**This paper analyzes the controllable set (stability region) of a linear time-invariant open-loop unstable system with constrained input. In particular, we apply the Lyapunov descent criterion and Kuhn-Tucker Theorem to the case when the input is allowed to saturate. We demonstrate our approach by a real submarine case, and find the 3D controllable set for the case.**

Index Terms—**3D controllable set, Lyapunov descent criterion, constrained input.**

I. Introduction

The concept of controllable set in control systems was introduced by Snow [18], when he defined the controllable set as the reachable set of the system with time reversed. For linear systems, there is complete duality between *reachability* (the ability to reach any desired final state from a given initial state) and *controllability* (the ability to reach a given final state from any initial state). But this is not generally true for nonlinear systems. Determination of the reachable set under input saturation has been widely studied using an open-loop approach. See Summers [20], Summers, Wu and Sabin [21], Sabin and Summers [16], and Quinn and Summers [13]. However, none of these papers covers the controllable set of a closed-loop system.

The problem of stabilization has widely been studied for a linear system which is subject to input saturation. See Sontag and Sussmann [19], Teel [23], Yang, Sussmann and Sontag [26], Lin and Saberi [10], Sussmann, Sontag, and Yang [19], Lin, Saberi, and Teel [9], Kosut [6], and Chen and Wang [2].

Another interesting aspect of this problem was shown by Lin, Saberi [10] and Lin, Saberi and Teel [11]. They showed that global stabilization via linear state feedback requires all the eigenvalues of the open-loop system be in the closed left-half plane with those on the imaginary axis simple, while semi-global stabilization via linear state feedback requires only that all the eigenvalues of the open-loop system be in the closed left-half plane. However, the stability of an open-loop unstable system with input saturation had never been studied until Lee and Hedrick [9] studied such a case. Specifically they considered the case in which the input $u(t)$ is bounded, $\|u(t)\|_\infty \le 1$, and at least one of the eigenvalues is located

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on the open right-half plane or at least one multiple eigenvalue is located on the imaginary axis. In this case, they chose as a stable region an ellipsoid which lies in a non-saturated zone. Quinn [14] developed a method of constructing a graphical representation of the set of null controllability by linear feedback for 2-dimensional linear systems, and addressed that the algorithm to construct a 3-dimensional systems would be too complicated.

Russell and Alpigini [15] created the controllable regions by using a 3D-Julia set methodology for a real-time nonlinear system. However, neither the constraint of the input nor the open-loop instability was studied in that paper.

In this paper, we study the case of 3D controllable set (stability region) of open-loop unstable systems with constrained input for a practical submarine case [4]. We also note that, in this paper, we do not distinguish between the stability region and controllable set.

We consider a linear time-invariant continuous-time system

$$\dot{x} = Ax(t) + Bu(t) \tag{1}$$

$$u(t) = -\text{sat}(Kx(t)), \tag{2}$$

where $\text{sat}(\cdot)$ denotes the saturation function. The one-dimensional version of the saturation function is defined by

$$\text{sat}(y) = \begin{cases} 1, & \text{if } y \geq 1 \\ y, & \text{if } y \in (-1,1) \\ -1, & \text{if } y \leq -1 \end{cases} \tag{3}$$

and we componentwise extend its definition to the multi-dimensional version:

$$\text{sat}(y) = \begin{bmatrix} \text{sat}(y_1) \\ \text{sat}(y_2) \\ \vdots \\ \text{sat}(y_m) \end{bmatrix}, \quad \forall y \in \Re^m. \tag{4}$$

II. Linear Time-Invariant System with Constrained Input

Consider the linear time-invariant continuous system (1), where $A \in \Re^{n\times n}$ is a given constant matrix, $B \in \Re^{n\times m}$ is a given constant matrix, $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control vector, with $u(t) = [u_1(t),\ldots,u_m(t)]$. Assume that A is not asymptotically stable. We also assume that the above open-loop system (A,B) is linearly stabilizable. In other words, it is assumed that, without saturation, the system would be stabilizable.

Hence there exists at least one matrix K such that

$$\dot{x}(t) = Ax(t) - BKx(t)$$

is asymptotically stable. Actually it is possible to select the location of the system eigenvalues (i.e., the eigenvalues of A-BK) arbitrarily. Hence we assume that matrix K has been selected so as to place the system eigenvalues in the desired location. Since $\tilde{A} = A - BK$ is Hurwitz, for every positive definite matrix $\tilde{Q}$, there exists an unique $P \in \Re^{n\times n}$ satisfying

$$\tilde{A}^T P + P\tilde{A} = -\tilde{Q}, \tag{5}$$

and $P > 0$.

Now we introduce saturation (2) in the system, where $\text{sat}(\cdot)$ denotes the saturation function defined in (4).

We denote the i-th row of matrix K by $k_i, i = 1,\cdots,m$:

$$K = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix}.$$

We partition $\Re^n$ into 3^m sections based on $k_i\xi, i = 1,\cdots,m$, being

(1) unsaturated: $k_i\xi \in (1,-1)$;

(2) positively saturated: $k_i\xi \leq -1$;

(3) negatively saturated: $k_i\xi \geq 1$.

Define *s* to be a function such that

$$s:\{1,2,\cdots,m\} \to \{-1,0,1\}$$

and observe that there are 3^m possible functions. Let *S* be the collection of all possible *s* functions. Define the regions where the *i*-th control u_i is unsaturated, positively saturated, and negatively saturated, respectively, as follows,

(1) $R_{i0}(K) = \{\xi \in \Re^n : |k_i\xi| < 1\};$

(2) $R_{i+}(K) = \{\xi \in \Re^n : k_i\xi \leq -1\};$

(3) $R_{i-}(K) = \{\xi \in \Re^n : k_i\xi \geq 1\}.$

And, finally, define $R(s,K)$ as the intersection of all possible regions for different control u_i, i.e.,

$$R(s,K) = \bigcap_{i=1}^{m} R_{i,s(i)}(K),\ s \in S.$$

Observe that the set of these intersections. $R(s,K),\ s \in S$, partition $\Re^n$ into 3^m regions.

We now construct the Lyapunov function *V*. Let $V:\Re^n \to \Re$ be defined by

$$V(\xi) = \xi^T P\xi, \tag{6}$$

where *P* is a given real, positive definite, symmetric $n\times n$ matrix such that $\tilde{Q} = -(\tilde{A}^T P + P\tilde{A}) > 0$. Then $V(\cdot)$ is a quadratic, continuously differentiable function. Note that the contours, $\Theta(c) = \{\xi \in \Re^n \mid V(\xi) = c\}$, with different values of *c* represent a nested family of ellipses for $n = 2$, of ellipsoids for $n = 3$, or of hyperellipsoids, for $n > 3$, all centered at $\xi = 0$.

For now, we consider the case of a single input: $m = 1$. Define

$$f(\xi) = A\xi - B\text{sat}(K\xi) \tag{7}$$

$$= \begin{cases} (A-BK)\xi, & \text{if } \xi \in H_0 = \{\xi \in \Re^n \mid |K\xi| < 1\} \\ A\xi - B, & \text{if } \xi \in H_- = \{\xi \in \Re^n \mid K\xi \geq 1 \\ A\xi + B, & \text{if } \xi \in H_+ = \{\xi \in \Re^n \mid K\xi < -1 \end{cases} \tag{8}$$

Define $\tilde{V}(t) = V(x(t))$. Taking derivative of $\tilde{V}(t)$ along the trajectory $x(t)$, we obtain the following three cases:

(i) $x(t) \in H_0$: unsaturated case, i.e., $u(t) = -Kx(t)$

$$\begin{aligned} \frac{d}{dt}\tilde{V}(t) &= x(t)^T((A-BK)^T P + P(A-BK))x(t) \\ &= -x(t)^T \tilde{Q}x(t), \end{aligned} \tag{9}$$

where

$$\tilde{Q} \overset{\Delta}{=} -[(A-BK)^T P + P(A-BK)] = -(\tilde{A}^T P + PA). \tag{10}$$

(ii) $x(t) \in H_+$: positively saturated case, i.e., $u(t) = 1$

$$\begin{aligned} \frac{d}{dt}\tilde{V}(t) &= x(t)^T(A^T P + PA)x(t) + B^T Px(t) + x(t)^T PB \\ &= -x(t)^T Qx(t) + B^T Px(t) + x(t)^T PB, \end{aligned} \tag{11}$$

where

$$Q \overset{\Delta}{=} -(A^T P + PA). \tag{12}$$

(iii) $x(t) \in H_-$: negatively saturated case, i.e., $u(t) = -1$

$$\begin{aligned} \frac{d}{dt}\tilde{V}(t) &= x(t)^T(A^T P + PA)x(t) - B^T Px(t) - x(t)^T PB \\ &= -x(t)^T Qx(t) - B^T Px(t) - x(t)^T PB, \end{aligned} \tag{13}$$

where *Q* is defined as in (12).

Inspired by the right-hand sides (9), (11) and (13) for $\frac{d}{dt}\tilde{V}(t)$, we have

$$g(\xi)=\begin{cases}-\xi^t\tilde{Q}\xi & \text{if } \xi\in H_0,\\ -\xi^T Q\xi+B^T P\xi+\xi^T PB & \text{if } \xi\in H_+,\\ -\xi^T Q\xi-B^T P\xi-\xi^T PB & \text{if } \xi\in H_-.\end{cases}$$

(14)

Observe that

$$\frac{d}{dt}\tilde{V}(t)=g(x(t)).$$

In order to satisfy the Lyapunov descent condition $g(\xi)<0$ for a given ξ, we require that for each $\xi\neq 0$, there exists at least one control value ν satisfying $\|\nu\|_\infty\leq 1$ and

$$g(\xi)=-\xi^T Q\xi\pm 2\xi^T PB\nu<0.$$

Then the state space $\Re^n$ can be divided into the following regions:

(a) $R_0=\{\xi\in\Re^n \mid \xi^T\tilde{Q}\xi>0\}$. If $\xi\in R_0$, then $g(\xi)<0.$

(b) $R_+=\{\xi\in\Re^n \mid 2\xi^T PB<\xi^T Q\xi\leq 0\}$. If $\xi\in R_+$, then set $\nu=1$ so that $g(\xi)<0.$

(c) $R_-=\{\xi\in\Re^n \mid 2\xi^T PB>-\xi^T Q\xi\geq 0\}$. If $\xi\in R_-$, then set $\nu=-1$ so that $g(\xi)<0.$

(d) $\Re^n-\{R_-\cup R_0\cup R_{+\}}\}$. If $\xi\in\Re^n-\{R_-\cup R_0\cup R_{+\}}\}$, then it is not possible to find $\nu\in[-1,1]$ such that $g(\xi)<0.$

We now seek to find the maximal level set $L_P(c^*)=\{\xi\in\Re^n \mid V(\xi)=\xi^T P\xi\leq c^*\}$ which is contained in the union of the regions (a), (b) and (c), i.e.,

$$c^*=\max\{c \mid L_P(c)\subset R_0\cup R_+\cup R_-\}.$$

In other words, we seek to obtain the maximal inner ellipsoidal approximation $L_p(c^*)$ of the controllable set Ω. Note that we are interested in finding the maximal set which is contained not only in the unsaturated region R_0, but also in the saturated region $R_+\cup R_-$.

We now give a lemma.

Lemma 1 Suppose $P\in\Re^{n\times n}$ is positive definite, i.e., $P>0$, satisfying (5) with $\tilde{Q}>0$. Define

$$\begin{aligned}R_{g0}&=\{\xi\in H_0 : g_0(\xi)\leq 0\},\\ R_{g+}&=\{\xi\in H_+ : g_+(\xi)\leq 0\},\\ R_{g-}&=\{\xi\in H_- : g_-(\xi)\leq 0\}.\end{aligned}$$

Then, $R_g=R_{g+}\cup H_0\cup R_{g-}$.

Proof It follows from $\Re^n=H_+\cup H_0\cup H_-$ that

$$\begin{aligned}R_g&=R_g\cap(H_+\cup H_0\cup H_-)\\ &=(R_g\cap H_+)\cup(R_g\cap H_0)\cup(R_g\cap H_-)\\ &=R_{g+}\cup R_{g0}\cup R_{g-}.\end{aligned}$$

$P>0$ is selected so that $\tilde{Q}=-(\tilde{A}^T P+P\tilde{A})>0$. Hence

$$g_0(\xi)=-\xi^T\tilde{Q}\xi\leq 0 \text{ for any } \xi\in H_0$$

and

$$R_{g0}=\{\xi\in\Re^n : g_0(\xi)\leq 0\}=H_0.$$

So,

$$R_g=R_{g+}\cup H_0\cup R_{g-}. \blacksquare$$

Recall that we want to find the maximal level set *L(r)* that fits in the descent region R_g. The optimization problem can be simplified as

$$
\begin{aligned}
c^* = \ & \min && V(\xi) = \xi^T P \xi \\
& \text{subject to} && g_-(\xi) = -\xi^T Q \xi - B^T P \xi - \xi^T PB \geq 0, \\
& \text{and} && K\xi \geq 1.
\end{aligned}
$$

Applying the Kuhn-Tucker Theorem, the desired value of the level r^* can be obtained as

$$r^* = \xi^{*T} P \xi^*$$

III. Practical Example: A Submarine Case

We now apply the two techniques to the following real case from [4], except that our objective is to find the 3D controllable sets for the model instead of their objective of finding the optimal control.

Example: Consider the following linearized model of a submarine obtained from Kockumation AB, Malmö, Sweden:

$$\dot{x} = Ax + bu$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -0.005 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 0 \\ 0.005 \end{bmatrix},$$

with the control constraint

$$|\, u \,| \leq 0.005.$$

Here we note that the eigenvalues of the open-loop system are found as 0, 0, -0.005. Since there are two open-loop zero eigenvalues for the system, the system is open-loop unstable. Suppose the desired eigenvalues of the closed-loop system are -0.0039, -0.0026 ±0.0021*i* as in [4]. Therefore, by the technique of eigenvalues placement, the feedback *K* is found as

$$K = [8.7126 \times 10^{-6} \quad 6.29 \times 10^{-3} \quad 8.2 \times 10^{-1}]$$

We choose $\tilde{Q}$ as

$$\tilde{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the eigenvalues of *P* are found as

$$\lambda_1 = 4.3051 \times 10^{11},$$

$$\lambda_2 = 2.0608 \times 10^{6},$$

$$\lambda_1 = 5.4946 \times 10^{1}.$$

Hence *P* is positive definite.

We now apply the two techniques:

Technique 1. Maximize the stable region inside the linear unsaturated region.

To find the stable region proposed by Lee and Hedrick [9], we formulate the following minimization problem:

$$
\begin{aligned}
& \min && V(\xi) = \xi^{\mathrm{T}} P \xi \\
& \text{subject to} && |\, K\xi \,| \geq 0.005.
\end{aligned}
$$

The above optimization problem yields the level $r_1^* = 1.0679\text{e} + 007$.

Figure 1 shows the 3D ellipsoidal controllable set approximated by Technique 1 without showing the linear unsaturated region. Figure 2 shows the 3D linear unsaturated region and the controllable set fit inside the linear unsaturated region. Figure 3 and 4 show another view of 3D linear unsaturated region and the controllable set. It is clear that the ellipsoid is bounded by the linear unsaturated region and tangent to the linear unsaturated region.

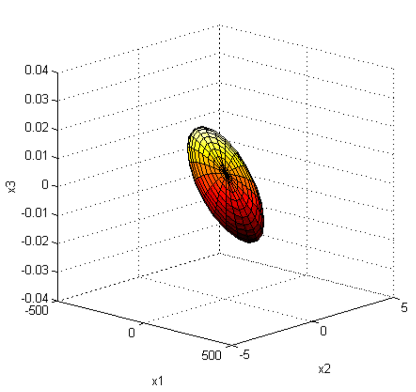

Figure 1. 3D ellipsoidal controllable set approximated by Technique 1

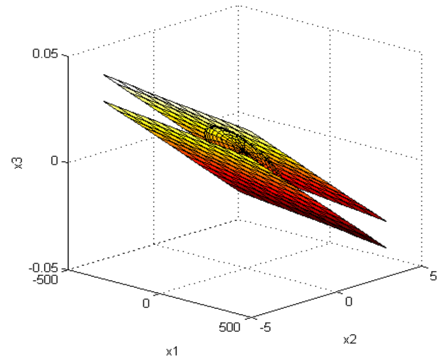

Figure 2. Linear unsaturated region and 3D controllable set approximated by technique 1

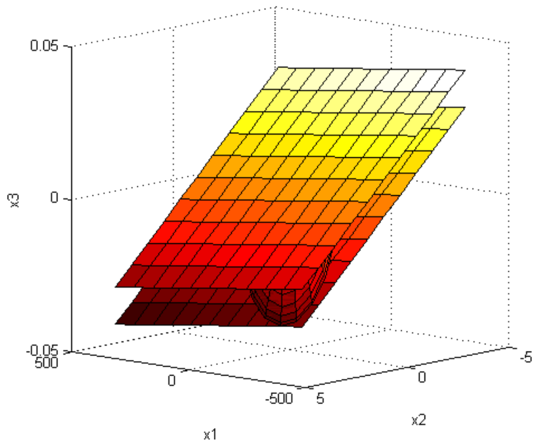

Figure 3 Another view of 3D ellipsoidal controllable set approximated by Technique 1

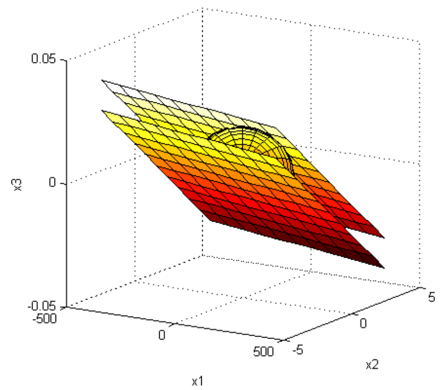

Figure 4 Another view of 3D ellipsoidal controllable set approximated by Technique 1

Technique 2. Maximize the stable region under Lyapunov descent criterion.

To find the stable region under Lyapunov descent criterion, we formulate the following minimization problem:

$$\begin{aligned} &\min && V(\xi) = \xi^T P \xi \\ &\text{subject to} && g_+(\xi) = \xi^T (A^T P + PA)\xi - 2\times 0.005 \times B^T P \xi \geq 0, \\ &&& K\xi \leq -0.005. \end{aligned}$$

The above optimization problem yields the level $r_2^* = 1.1115\mathrm{e}+007$, which is about 4% more than that of the level found from Technique 1.

Figure 5 shows the Lyapunov controllable set for approximated by Technique 2 without showing the linear unsaturated region. Figure 6 shows the linear unsaturated region and the Lyapunov controllable set. Figure 7 and 8 show the linear unsaturated region and the Lyapunov controllable set. We note that a portion of the ellipsoid breaking through the linear unsaturated region indicates that indeed the Lyapunov controllable set extends beyond the linear unsaturated region, i.e., the Lyapunov controllable set is larger than the controllable set found inside the linear unsaturated region through Technique 2.

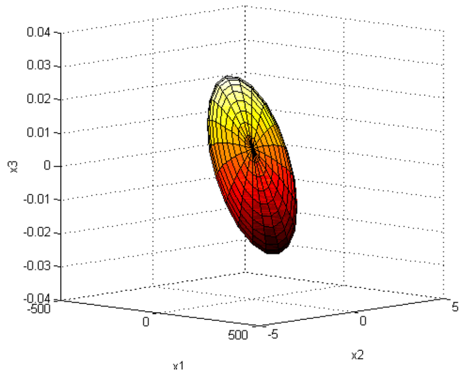

Figure 5. 3D ellipsoidal controllable set approximated by Technique 2

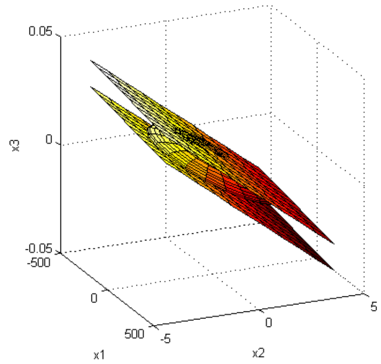

Figure 6. Linear unsaturated region and 3D controllable set approximated by technique 2

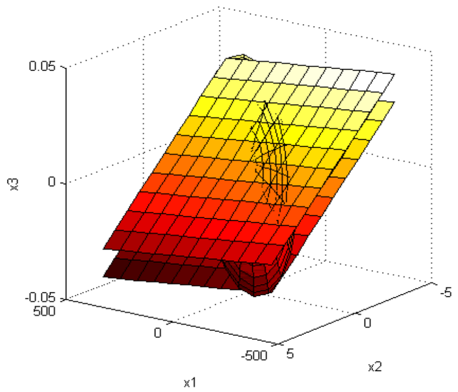

Figure 7 Another view of 3D ellipsoidal controllable set approximated by Technique Figure 8 Another view of 3D ellipsoidal controllable set approximated by Technique 2

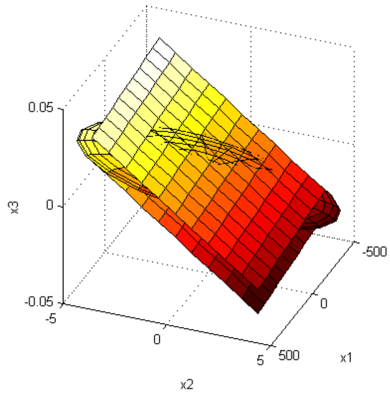

Figure 8 Another view of 3D ellipsoidal controllable set approximated by Technique Figure 8 Another view of 3D ellipsoidal controllable set approximated by Technique

IV. Conclusion

In this paper, we study the case of 3D controllable set for open-loop unstable systems with constrained input. Inspired by the result that the stability region of the system can be approximated by an ellipsoid inside the linear unsaturated region, we showed that the asymptotic stability region of the system can actually be expanded to the areas in which the control is saturated. In particular, we first separated the entire state space into three areas, namely, unsaturated area, positively saturated, and negatively saturated area; then we applied the Lyapunov descent criterion and Kuhn-Tucker Theorem to the problem to the case when the input is allowed to saturate. We demonstrated our approach by a real submarine case, and found the 3D controllable set for the case. It is shown that the controllable set found by our approach is actually superior to that of set found inside the linear unsaturated region, and thus, yield a better approximation.

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