M_r-FACTORS AND Q_r-FACTORS FOR NEAR QUASINORM ON CERTAIN SEQUENCE SPACES

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We study the multiplicativity factor and quadraticity factor for near quasinorm on certain sequence spaces of Maddox, namely, $l(p)$ and $l_\infty(p)$, where $p = (p_k)$ is a bounded sequence of positive real numbers.

1. Introduction

Let X be an algebra over a field F (R or C). A *quasinorm* on X is a function $|\cdot| : X \to R$ such that

(i) $|0| = 0$,

(ii) $|x| \geq 0$, for all $x \in X$,

(iii) $|-x| = |x|$, for all $x \in X$,

(iv) $|x + y| \leq |x| + |y|$, for all $x, y \in X$,

(v) if $t_k \in F$, $|t_k - t| \to 0$, and $x_k, x \in X$, $|x_k - x| \to 0$, then $|t_k x_k - tx| \to 0$.

If $|\cdot|$ satisfies only properties (i) to (iv), then we call $|\cdot|$ a *near quasinorm*. If the quasinorm satisfies $|x| = 0$ if and only if $x = 0$, then it is said to be *total*.

A *quasinormed linear space* (QNLS) is a pair $(X, |\cdot|)$ where $|\cdot|$ is a quasinorm on X. If $(X, |\cdot|)$ is a quasinorm space, then the map $|\cdot| : X \to R$ is continuous. For $p > 0$, a p-seminorm on X is a function $\|\cdot\| : X \to R$ satisfying

(i) $\|x\| \geq 0$, for all $x \in X$,

(ii) $\|tx\| = |t|^p \|x\|$, for all $t \in F$, for all $x \in X$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

A seminorm is called a norm if it satisfies the following condition:

(iv) $\|x\| = 0$ if and only if $x = 0$.

A p-seminormed linear space (p-semi-NLS) is a pair $(X, \|\cdot\|)$ where $\|\cdot\|$ is a seminorm on X. p-normed linear spaces (p-normed-LS) are defined similarly.

In [1, 2], *multiplicativity factors* (*or M-factors*) and *quadrativity factors* (*or Q-factors*) for seminorms on an algebra X have been introduced and studied in detail. A number $\mu > 0$ is said to be a multiplicativity factor for a seminorm S if and only if $S(xy) \leq \mu S(x)S(y)$, for all $x, y \in X$. Similarly, a number $\lambda > 0$ is said to be a quadrativity factor for S if and

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only if $S(x^2) \leq \lambda S(x)^2$, for all $x \in X$. The necessary and sufficient conditions for existence of M-factor and Q-factor for S are answered in the following results.

Theorem 1.1. *Let X be an algebra and let $S \neq 0$ be a seminorm on X. Then*

(a) *S has M-factors on X if and only if* $\operatorname{Ker} S$ *is an ideal in X and*

$$\mu_{\inf} \equiv \sup\{S(xy) : x, y \in X,\ S(x) = S(y) = 1\} < +\infty, \tag{1.1}$$

(b) *if S has M-factors on X and $\mu_{\inf} > 0$, then $\mu_{\inf}$ is the best (least) M-factor for S,*

(c) *if S has M-factors on X and $\mu_{\inf} = 0$, then μ is an M-factor for S if and only if $\mu > 0$.*

Theorem 1.2. *Let X be an algebra and let $S \neq 0$ be a seminorm on X. Then*

(a) *S has Q-factors on X if and only if* $\operatorname{Ker} S$ *is closed under squaring (i.e.,* $(\operatorname{Ker} S)^2 \subset \operatorname{Ker} S$*) and*

$$\lambda_{\inf} \equiv \sup\{S(x^2) : x \in X,\ S(x) = 1\} < +\infty, \tag{1.2}$$

(b) *if S has Q-factors on X and $\lambda_{\inf} > 0$, then $\lambda_{\inf}$ is the best (least) Q-factor for S,*

(c) *if S has Q-factors on X and $\lambda_{\inf} = 0$, then λ is a Q-factor for S if and only if $\lambda > 0$.*

If S is a norm, then $\operatorname{Ker} S = \{0\}$. If in addition X is finite-dimensional, then a simple compactness argument shows that $\mu_{\inf}$ is finite. Therefore, by Theorem 1.1, norms on finite-dimensional algebras always have M-factors. If S is a seminorm on a finite-dimensional algebra X, then S has M-factors on X if and only if $\operatorname{Ker} S$ is a (two-sided) ideal in X. In [1, 2] several examples of seminorms having M-factors and Q-factors are given. In [3], scalar multiplicativity factors for near quasinorms on certain sequence spaces of Maddox are studied. Motivated by these results we define *M_r-factors* and *Q_r-factors* for a near quasinorm q on an algebra X as follows.

A number $\mu > 0$ is an *M_r-factor* for q if and only if $q(txy) \leq \mu |t|^r q(x) q(y)$, there exists $r > 0$, for all $t \in F$, for all $x, y \in X$.

A number $\lambda > 0$ is a *Q_r-factor* for q if and only if $q(tx^2) \leq \lambda |t|^r q(x)^2$, there exists $r > 0$, for all $t \in F$, for all $x \in X$.

Let

$$\begin{aligned} \mu_{\inf} &\equiv \sup\left\{\frac{q(txy)}{|t|^r q(x) q(y)} : t \in F - \{0\},\ x, y \in X - \operatorname{Ker} q\right\}, \\ \lambda_{\inf} &\equiv \sup\left\{\frac{q(tx^2)}{|t|^r q(x)^2} : t \in F - \{0\},\ x \in X - \operatorname{Ker} q\right\}. \end{aligned} \tag{1.3}$$

2. M_r-factors and Q_r-factors for near quasinorms

In this section, we will prove the following theorems.

Theorem 2.1. *Let X be an algebra over a field F ($F = C$ or R). Let q be a near quasinorm on X. Then*

(a) *q has M_r-factors on X if and only if* $\operatorname{Ker} q$ *is a (two-sided) ideal in X and $\mu_{\inf} < +\infty$,*

(b) *if q has M_r-factors on X and $\mu_{\inf} > 0$, then $\mu_{\inf}$ is the best (least) M_r-factor for q,*

(c) *if q has M_r-factors on X and $\mu_{\inf} = 0$, then μ is an M_r-factor for q if and only if $\mu > 0$.*

Theorem 2.2. *Let X be an algebra over a field F ($F = C$ or R). Let q be a near quasinorm on X. Then*

(a) *q has Q_r-factors on X if and only if* $\operatorname{Ker} q$ *is closed under squaring (i.e., $x^2 \in \operatorname{Ker} q$, for all $x \in \operatorname{Ker} q$) and $\lambda_{\inf} < +\infty$,*

(b) *if q has Q_r-factors on X and $\lambda_{\inf} > 0$, then $\lambda_{\inf}$ is the best (least) Q_r-factors for q,*

(c) *if q has Q_r-factors on X and $\lambda_{\inf} = 0$, then λ is a Q_r-factors for q if and only if $\lambda > 0$.*

Proof of Theorem 2.1. (a) Suppose that q has an M_r-factor μ on X. Clearly, $\operatorname{Ker} q$ is a subspace of X. Now take any $x \in \operatorname{Ker} q$ and $y \in X$. Then $q(xy) \le \mu q(x)q(y) = 0$ which implies that $xy \in \operatorname{Ker} q$. Similarly, $yx \in \operatorname{Ker} q$, so $\operatorname{Ker} q$ is a (two-sided) ideal in X. Now for $t \in F - \{0\}$ and $x, y \in X - \operatorname{Ker} q$, we have $q(txy) \le \mu|t|^r q(x)q(y)$ or $q(txy)/|t|^r q(x)q(y) \le \mu$ which implies that $\mu_{\inf} \le \mu < +\infty$. Conversely, suppose that $\operatorname{Ker} q$ is a (two-sided) ideal in X and $\mu_{\inf} < +\infty$. If $t = 0$, $x \in \operatorname{Ker} q$, or $y \in \operatorname{Ker} q$, then $txy \in \operatorname{Ker} q$, so $0 = q(txy) = \mu_{\inf}|t|^r q(x)q(y)$. If $t \ne 0$ and $x, y \notin \operatorname{Ker} q$, then $q(txy)/|t|^r q(x)q(y) \le \mu_{\inf}$ or $q(txy) \le \mu_{\inf}|t|^r q(x)q(y)$. Therefore, $q(txy) \le \mu_{\inf}|t|^r q(x)q(y)$, for all $t \in F$ and for all $x, y \in X$ which implies that q has M_r-factors on X.

(b) Let μ be an M_r-factor for q on X and $\mu_{\inf} > 0$. Then $q(txy) \le \mu|t|^r q(x)q(y)$ for all $t \in F$ and for all $x, y \in X$. Therefore, $q(txy)/|t|^r q(x)q(y) \le \mu$, for all $t \in F - \{0\}$ and for all $x, y \in \operatorname{Ker} q$, so $\mu_{\inf} \le \mu$.

(c) This part follows directly from definition of $\mu_{\inf}$ and M_r-factors for q on X. □

Proof of Theorem 2.2. The proof of this theorem is a simple modification of the proof of Theorem 2.1 and will be omitted. □

3. M_r-factors and Q_r-factors for near quasinorm on certain sequence spaces of Maddox

Let $p = (p_k)$ be a bounded sequence of positive real numbers. The sequence spaces of Maddox $l_\infty(p)$ and $l(p)$ are defined as follows:

$$\begin{aligned} l_\infty(p) &= \Big\{(x_k) : x_k \in C,\ \sup_k |x_k|^{p_k} < \infty\Big\}, \\ l(p) &= \Big\{(x_k) : x_k \in C,\ \sum_k |x_k|^{p_k} < \infty\Big\}. \end{aligned} \tag{3.1}$$

With the usual multiplication (i.e., $(x_k)(y_k) = (x_k y_k)$), both $l_\infty(p)$ and $l(p)$ are algebras over C. We define near quasinorms q_1 on $l_\infty(p)$ and q_2 on $l(p)$ as follows:

$$\begin{aligned} q_1((x_k)) &= \sup_k |x_k|^{p_k/M}, \quad (x_k) \in l_\infty(p), \\ q_2((x_k)) &= \Big(\sum_k |x_k|^{p_k}\Big)^{1/M}, \quad (x_k) \in l(p), \end{aligned} \tag{3.2}$$

where $M = \max\{1, \sup_k p_k\}$. We observe that q_1 and q_2 may or may not be quasinorms. For example, when $(p_k) = (1/k)$, then q_1 is a near quasinorm but not a quasinorm; if $(p_k) = (1 - 1/(k+1))$, then q_1 is a quasinorm.

In this section, we give necessary and sufficient conditions for sequence spaces $l_\infty(p)$ and $l(p)$ to have M_r-factors and Q_r-factors.

Theorem 3.1. *Let $p = (p_k)$ and let M be defined as above. Then the following are equivalent.*

(a) *$p_0 = p_k = p_{k+1}$ for all $k \geq 0$ where p_0 is a positive real number.*
(b) *q_1 has M_r-factors on $l_\infty(p)$.*
(c) *q_1 has Q_r-factors on $l_\infty(p)$.*
(d) *q_1 is a p_0/M-seminorm on $l_\infty(p)$.*

Theorem 3.2. *Let $p = (p_k)$ and let M be defined as above. Then the following are equivalent.*

(a) *$p_0 = p_k = p_{k+1}$ for all $k \geq 0$ where p_0 is a positive real number.*
(b) *q_2 has M_r-factors on $l(p)$.*
(c) *q_2 has Q_r-factors on $l(p)$.*
(d) *q_2 is a p_0/M-seminorm on $l(p)$.*

Proof of Theorem 3.1. (a)⇒(b) If $p_0 = p_k = p_{k+1}$ for all $k \geq 1$, then

$$q_1(txy) = \sup_k |txy|^{p_k/M} = \sup_k |txy|^{p_0/M} \leq |t|^{p_0/M} q_1(x) q_1(y) \tag{3.3}$$

for all $x, y \in l_\infty(p)$, so q_1 has an M_r-factor on $l_\infty(p)$.

(b)⇒(a) Assume that q_1 has M_r-factors on $l_\infty(p)$. This implies that

$$\mu_{\inf} = \sup\left\{ \frac{q_1(txy)}{|t|^r q_1(x) q_1(y)} : t \in F - \{0\},\ x, y \in X - \operatorname{Ker} q_1 \right\} < +\infty. \tag{3.4}$$

We shall show that $r = \sup_k p_k/M = \inf_k p_k/M$ which implies that $p_k = p_{k+1}$ for all $k \geq 1$. To this end we observe that

$$\begin{aligned} \mu_{\inf} &= \sup\left\{ \frac{q_1(txy)}{|t|^r q_1(x) q_1(y)} : t \in F - \{0\},\ x, y \in X - \operatorname{Ker} q_1 \right\} \\ &\geq \sup\left\{ \frac{q_1(txy)}{|t|^r q_1(x) q_1(y)} : t \in F - \{0\},\ x, y = (1,1,1,\ldots) \right\} \\ &\geq \sup\left\{ \frac{\sup_k |t|^{p_k/M}}{|t|^r} : t \in F,\ |t| \geq 1 \right\} = \sup\left\{ |t|^{\sup_k p_k/M} : t \in F,\ |t| \geq 1 \right\} \end{aligned} \tag{3.5}$$

so that

$$\mu_{\inf} \geq \sup\left\{ \frac{|t|^{\sup_k p_k/M}}{|t|^r} : t \in F,\ |t| \geq 1 \right\}. \tag{3.6}$$

If $r < \sup_k p_k/M$, then $\mu_{\inf} = +\infty$ which is a contradiction. Therefore, $r \geq \sup_k p_k/M$. Similarly, we can show that $r \leq \inf_k p_k/M$ from which it follows that $r = \sup_k p_k/M = \inf_k p_k/M$ and the proof is complete.

(a)⇒(c) The same proof as (a)⇒(b).

(c)⇒(a) The same proof as (b)⇒(a).

(d)⇒(b) This is obvious.

(b)⇒(d) Assume that q_1 has M_r-factors. Then, by (a), $p_0 = p_k = p_{k+1}$ for all $k \geq 0$ where p_0 is a positive real number. Moreover, we have

$$q_1(txy) = \sup_k \left| t \cdot (x_k)(y_k) \right|^{p_0/M} = |t|^{p_0/M} \sup_k \left| x_k y_k \right|^{p_0/M} = |t|^{p_0/M} q_1(xy) \tag{3.7}$$

for all $x = (x_k)$, $y = (y_k) \in l_\infty(p)$ and all $t \in F$. Putting $y = (1, 1, 1\ldots)$ we see that

$$q_1(tx) = |t|^{p_0/M} q_1(x) \tag{3.8}$$

and the proof is complete. □

Proof of Theorem 3.2. The proof is almost the same as in Theorem 3.1 and will be omitted. □

Remark 3.3. If the algebra X has an identity element x_0 for multiplication and $q \neq 0$ is a near-quasinorm on X which has an M_r-factor on X, then we obtain $q(x_0) > 0$, $\mu_{\inf} \geq 1/q(x_0)$ and

$$\frac{1}{q(x_0)\mu_{\inf}} |t|^r q(xy) \leq q(txy) \leq \mu_{\inf} |t|^r q(x) q(y) \tag{3.9}$$

for all $x, y \in X$ and all $t \in F$.

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