The Convex Hull of Points on a Sphere is a Spanner*

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Abstract

Let S be a finite set of points on the unit-sphere $\mathbb{S}^2$. In 1987, Raghavan suggested that the convex hull of S is a Euclidean t-spanner, for some constant t. We prove that this is the case for $t = 3\pi(\pi/2 + 1)/2$. Our proof consists of generalizing the proof of Dobkin *et al.* [2] from the Euclidean Delaunay triangulation to the spherical Delaunay triangulation.

1 Introduction

Let S be a finite set of points in Euclidean space and let G be a graph with vertex set S. We denote the Euclidean distance between any two points p and q by $d(p, q)$. Let the length of any edge (p, q) in G be equal to $d(p, q)$, and define the length of a path in G to be the sum of the lengths of the edges on this path. For any two vertices a and b in G, we denote by $\delta_G(a, b)$ the minimum length of any path in G between a and b. For a real number $t \geq 1$, we say that G is a *Euclidean t-spanner* of S, if $\delta_G(a, b) \leq t \cdot d(a, b)$ for all vertices a and b. The *stretch factor* of G is the smallest value of t such that G is a Euclidean t-spanner of S. See [3] for an overview of results on Euclidean spanners.

It is well-known that the stretch factor of the Delaunay triangulation in $\mathbb{R}^2$ is bounded from above by a constant. The first proof of this fact is due to Dobkin *et al.* [2], who obtained an upper bound of $(1 + \sqrt{5})\pi/2 \approx 5.08$. The currently best known upper bound, due to Xia [4], is 1.998.

Since there is a close connection between the Delaunay triangulation in $\mathbb{R}^2$ and the convex hull in $\mathbb{R}^3$, it is natural to ask if the graph defined by the convex hull edges has a bounded stretch factor as well. It is easy to define a point set in $\mathbb{R}^3$ whose convex hull is long and skinny, resulting in an unbounded stretch factor. In 1987, Raghavan suggested, in a private communication to Dobkin *et al.* [2], that the convex hull of a finite set of points on a sphere in $\mathbb{R}^3$ has bounded stretch factor. By scaling and translating, we may assume, without loss of generality, that the points are on the unit-sphere $\mathbb{S}^2$, which is the set of all points in $\mathbb{R}^3$ that have distance 1 to the origin. In this paper, we prove that this is indeed the case:

Theorem 1 *Let S be a finite set of points on the unit-sphere $\mathbb{S}^2$. The graph defined by the convex hull edges of S is a Euclidean t-spanner of S, where*

$$t = 3\pi(\pi/2 + 1)/2.$$

We will prove this result using the well-known fact that the convex hull of a set S of points on the unit-sphere is "equal" (to be formalized in Lemma 2) to the spherical Delaunay triangulation of S. Based on this, we will show how the proof of Dobkin *et al.* [2] can be modified to show that the spherical Delaunay triangulation has bounded stretch factor (where distances are measured along the unit-sphere), resulting in a proof of Theorem 1.

2 Preliminaries

Let S be a finite set of points on the unit-sphere $\mathbb{S}^2$. We denote the convex hull of S by $CH(S)$. Let a and b be two distinct points on $\mathbb{S}^2$ and consider the plane through a, b, and the origin. The intersection of this plane with $\mathbb{S}^2$ is a *great circle* and the shorter of the two arcs on this circle connecting a and b is a *great arc*. The length of this great arc is the *spherical distance* between a and b, which we will denote by $\breve{d}(a, b)$. This distance function gives rise to the *spherical Voronoi diagram $SVD(S)$* of S and its dual, the *spherical Delaunay triangulation $SDT(S)$*; note that these graphs are entirely on the unit-sphere and each of their edges is a great arc. The following result is well-known:

Lemma 2 *Consider the graph with vertex set S that is obtained by replacing each edge (p, q) of the spherical Delaunay triangulation $SDT(S)$ by the straight-line segment between p and q. This graph is the convex hull $CH(S)$ of S.*

Let G be a graph with vertex set S, such that each of its edges (p, q) is a great arc of length $\breve{d}(p, q)$. As before, the length of a path in G is the sum of the lengths of its edges. For any two vertices a and b in G, let $\breve{\delta}_G(a, b)$ denote the minimum length of any path in G between a and b. We say that G is a *spherical t-spanner* of S, if $\breve{\delta}_G(a, b) \leq t \cdot \breve{d}(a, b)$ for all vertices a and b.

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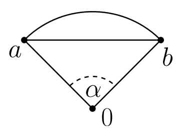
Lemma 3 *If $SDT(S)$ is a spherical t-spanner of S, then $CH(S)$ is a Euclidean $(t\pi/2)$-spanner of S.*

Proof. Let a and b be two distinct points in S, and let P be a path in $SDT(S)$ of length at most $t \cdot \breve{d}(a,b)$. Let P' be the path obtained by replacing each edge (a great arc) of P by a straight-line segment. Then, P' is a path in $CH(S)$ between a and b, and the length of P' is at most the length of P, which is at most $t \cdot \breve{d}(a,b)$.



Let α be the angle between the two vectors pointing from the origin to a and b. Then $\breve{d}(a,b) = \alpha$ and $d(a,b) = 2\sin(\alpha/2)$. It follows that

$$\breve{d}(a,b) = \frac{\alpha/2}{\sin(\alpha/2)} \cdot d(a,b).$$

Since the function $f(x) = x/\sin x$ is non-decreasing for $0 \le x \le \pi/2$, it follows that

$$\breve{d}(a,b) \le f(\pi/2) \cdot d(a,b) = (\pi/2) \cdot d(a,b).$$

□

Based on Lemma 3, Theorem 1 will follow from the following result:

Theorem 4 *Let S be a finite set of points on the unit-sphere $\mathbb{S}^2$. The spherical Delauny triangulation of S is a spherical $3(\pi/2+1)$-spanner of S.*

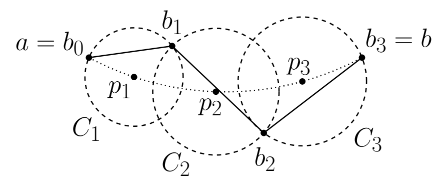
In the rest of this paper, we will prove Theorem 4.

3 Direct Paths in $SDT(S)$

Let a and b be two distinct points of S and consider the great arc on $\mathbb{S}^2$ between a and b. Let $p_1, p_2, \ldots, p_n$ be the ordered sequence of points on Voronoi region boundaries of the spherical Voronoi diagram $SVD(S)$ that are encountered when traversing this great arc from a to b. Thus, each point p_i is contained in some Voronoi edge of $SVD(S)$. Let $b_0 = a, b_1, b_2, \ldots, b_n = b$ be the ordered sequence of points of S whose Voronoi regions are visited during this traversal. Observe that, for each i with $1 \le i \le n$, p_i is on the Voronoi edge that is shared by the Voronoi regions of b_{i-1} and b_i. We call

$$a = b_0, b_1, b_2, \ldots, b_n = b$$

the *direct path* between a and b. Observe that this is a path in the spherical Delaunay triangulation $SDT(S)$.



Lemma 5 *The direct path is longitudinally monotone: Let GC be the great circle through a and b. For each i with $1 \le i \le n$, let b'_i be the point on GC whose spherical distance to b_i is minimum. Then, when traversing the great arc along GC from a to b, we visit the points $b'_1, b'_2, \ldots, b'_n$ in this order.*

Proof. We may assume without loss of generality that a and b are on the equator, have positive y-coordinates, and the x-coordinate of a is less than that of b.

Let i be an index with $1 \le i \le n$. The spherical bisector of b_{i-1} and b_i is contained in their Euclidean bisector, which is a plane that contains p_i and separates b_{i-1} from b_i. Since b_{i-1} is to the left of this plane and, thus, b_i is to its right, the x-coordinate of b_{i-1} is less than that of b_i. As a result, when traversing the great arc along G from a to b, we visit the point b'_{i-1} before b'_i. □

Consider the midpoint c of the great arc between a and b. The *spherical cap* $SC(a,b)$ is defined to be

$$SC(a,b) = \{x \in \mathbb{S}^2 : \breve{d}(c,x) \le \breve{d}(a,b)/2\}.$$

We will refer to the point c as the *pole* of the spherical cap.

Lemma 6 *The direct path between a and b is contained in $SC(a,b)$.*

Proof. Consider the pole c of $SC(a,b)$, and let k be the index such that the points $p_1, \ldots, p_k$ are on the great arc connecting a and c, and the points $p_{k+1}, \ldots, p_n$ are on the great arc connecting c and b. If i is such that $1 \le i \le k$, then the spherical bisector of b_{i-1} and b_i is a great circle that divides $\mathbb{S}^2$ into two half-spheres. The point b_{i-1} is in one of these half-spheres, whereas both b_i and c are in the other half-sphere. It follows that $\breve{d}(c,b_i) \le \breve{d}(c,b_{i-1})$. Thus, we have

$$\breve{d}(c,b_k) \le \breve{d}(c,b_{k-1}) \le \ldots \le \breve{d}(c,b_0) = \breve{d}(c,a).$$

By a symmetric argument, we have

$$\breve{d}(c,b_{k+1}) \le \breve{d}(c,b_{k+2}) \le \ldots \le \breve{d}(c,b_n) = \breve{d}(c,b).$$

□

For each i with $1 \le i \le n$, define

$$C_i = SC(b_{i-1}, b_i).$$

This spherical cap C_i has the point p_i as its pole and does not contain any point of S in its interior. Define

$$\mathcal{C} = \bigcup_{i=1}^{n} C_i.$$

Let Π be the plane through a, b, and the origin. If the direct path between a and b is completely contained in one of the two closed halfspaces bounded by Π, then we say that this path is *one-sided.*

In Lemma 11, we will use the set $\mathcal{C}$ to prove that, if the direct path between a and b is one-sided, then its length is at most $(\pi/2) \cdot \breve{d}(a, b)$. Before we can prove this result, we need some properties of the set $\mathcal{C}$.

Lemma 7 *Let x and y be distinct points on the equator, and consider the spherical cap $SC(x, y)$. Let L be the length of the part of the boundary of this cap that is above the equator. Then $L \leq (\pi/2) \cdot \breve{d}(x, y)$.*

Proof. Consider the plane through x and y whose normal is the vector pointing from the origin to the midpoint c of the straight-line segment connecting x and y. The boundary of $SC(x, y)$ is the circle in this plane that is centered at c and has x and y on its boundary. It follows that $L = (\pi/2) \cdot d(x, y)$.

Let α be the angle between the two vectors pointing from the origin to x and y. Then $\breve{d}(x, y) = \alpha$ and

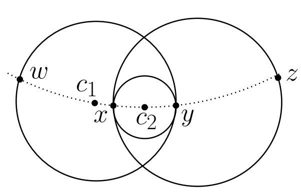
$$\begin{aligned} L &= (\pi/2) \cdot d(x, y) \\ &= \pi \cdot \sin(\alpha/2) \\ &\leq \pi \cdot \alpha/2 \\ &= (\pi/2) \cdot \breve{d}(x, y). \end{aligned}$$

□

Lemma 8 *Let w, x, y, and z be four points that appear, in this order, on a great arc. Then*

$$SC(x, y) \subseteq SC(w, y) \cap SC(x, z).$$

Proof. Let c_1 be the midpoint of the great arc between w and y, and let c_2 be the midpoint of the great arc between x and y. Thus, c_1 and c_2 are the poles of $SC(w, y)$ and $SC(x, y)$, respectively.



Since

$$\breve{d}(c_2, y) = \breve{d}(x, y)/2 \leq \breve{d}(c_1, y),$$

the point c_2 is on the great arc between c_1 and y.

Let v be an arbitrary point in $SC(x, y)$. Then,

$$\begin{aligned} \breve{d}(c_1, v) &\leq \breve{d}(c_1, c_2) + \breve{d}(c_2, v) \\ &\leq \breve{d}(c_1, c_2) + \breve{d}(c_2, y) \\ &= \breve{d}(c_1, y), \end{aligned}$$

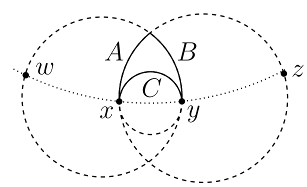
implying that v is in $SC(w, y)$. By a symmetric argument, v is in $SC(x, z)$. □

Lemma 9 *Let w, x, y, and z be four points that appear, in this order, on a great arc along the equator. Define the following:*

- *A is the part of the boundary of $SC(x, z)$ that is above the equator and inside $SC(w, y)$, and L_A is its length.*
- *B is the part of the boundary of $SC(w, y)$ that is above the equator and inside $SC(x, z)$, and L_B is its length.*
- *C is the part of the boundary of $SC(x, y)$ that is above the equator, and L_C is its length.*

Then $L_C \leq L_A + L_B$.

Proof. The following figure illustrates the assumptions in the lemma.

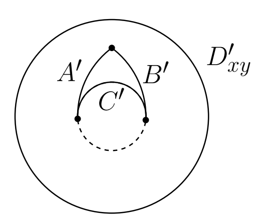


Let Π_{xy} be the plane that contains the boundary of $SC(x, y)$, let Π'_{xy} be the plane through the origin that is parallel to Π_{xy}, and let D'_{xy} be the disk in Π'_{xy} of radius 1 that is centered at the origin.

Let A', B', and C' be the orthogonal projections of A, B, and C onto Π'_{xy}, respectively. Observe that A', B', and C' are contained in D'_{xy}. Let L'_A, L'_B, and L'_C be the lengths of A', B', and C', respectively. Then $L'_A \leq L_A$, $L'_B \leq L_B$, and $L'_C = L_C$. Thus, it is sufficient to prove that

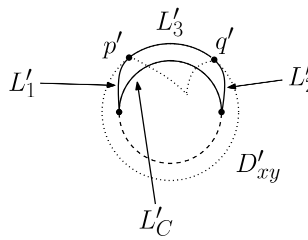
$$L'_C \leq L'_A + L'_B. \quad (1)$$

First assume that both A and B are entirely on the same side of Π'_{xy} as C.



Then, using Lemma 8, the convex curve C' is contained inside the curve obtained by concatenating A' and B'. Since these curves have the same endpoints, (1) follows from Benson [1, page 42].

Now assume that A and B are not entirely on the same side of Π'_{xy} as C. In this case, it may happen that the common endpoint of A' and B' is inside the circle through C'. Therefore, we proceed as follows.



Let p' be the intersection between A' and the boundary of D'_{xy}, and let q' be the intersection between B' and the boundary of D'_{xy}. Let L'_1 be the length of the part of A' between x's projection and p', let L'_2 be the length of the part of B' between y's projection and q', and let L'_3 be the length of the part of the boundary of D'_{xy} between p' and q'. Observe that $L'_3 = \breve{d}(p', q')$. Then, again by Benson [1, page 42],

$$L'_C \leq L'_1 + L'_2 + L'_3,$$

which, by the triangle inequality, is at most $L'_A + L'_B$. Thus, also in this case, (1) holds. □

In the next lemma, we consider the set

$$\mathcal{C} = \bigcup_{i=1}^{n} C_i$$

that was defined before.

Lemma 10 *Assume that the points a and b are on the equator. Let L be the length of the part of the boundary of $\mathcal{C}$ that is above the equator. Then*

$$L \leq (\pi/2) \cdot \breve{d}(a, b).$$

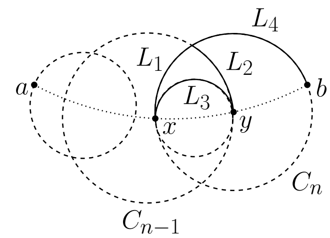
Proof. The proof is by induction on the number n of edges on the direct path between a and b. If $n = 1$, then the claim follows from Lemma 7.

Assume that $n \geq 2$. Consider the set

$$\mathcal{C}' = \bigcup_{i=1}^{n-1} C_i,$$

let L' be the length of the part of its boundary that is above the equator, and let y be the point on the equator and on the boundary of C_{n-1} whose spherical distance to b is minimum. By induction, we have

$$L' \leq (\pi/2) \cdot \breve{d}(a, y).$$



Let x be the point on the equator and on the boundary of C_n whose spherical distance to b is maximum. Define the following quantities:

- L_1 is the length of the part of the boundary of C_n that is above the equator and inside C_{n-1}.
- L_2 is the length of the part of the boundary of C_{n-1} that is above the equator and inside C_n.
- L_3 is the length of the part of the boundary of $SC(x, y)$ that is above the equator.
- L_4 is the length of the part of the boundary of C_n that is above the equator and outside C_{n-1}.

By Lemma 9, we have $L_3 \leq L_1 + L_2$. It follows that

$$\begin{aligned} L &= L' + L_4 - L_2 \\ &= L' + (L_1 + L_4) - (L_1 + L_2) \\ &\leq (\pi/2) \cdot \breve{d}(a, y) + (L_1 + L_4) - L_3. \end{aligned}$$

Define the following two angles:

- α is the angle between the two vectors pointing from the origin to x and y.
- β is the angle between the two vectors pointing from the origin to y and b.

Observe that

$$L_1 + L_4 = (\pi/2) \cdot d(x, b) = \pi \sin((\alpha + \beta)/2)$$

and

$$L_3 = (\pi/2) \cdot d(x, y) = \pi \sin(\alpha/2).$$

Using the identity

$$\sin\gamma - \sin\delta = 2\sin((\gamma - \delta)/2)\cos((\gamma + \delta)/2),$$

it follows that

$$\begin{aligned} L_1 + L_4 - L_3 &= 2\pi \sin(\beta/4)\cos((2\alpha + \beta)/4) \\ &\leq 2\pi \sin(\beta/4) \\ &\leq 2\pi(\beta/4) \\ &= (\pi/2) \cdot \breve{d}(y, b). \end{aligned}$$

We conclude that

$$\begin{aligned} L &\leq (\pi/2) \cdot \breve{d}(a, y) + (\pi/2) \cdot \breve{d}(y, b) \\ &= (\pi/2) \cdot \breve{d}(a, b). \end{aligned}$$

□

Lemma 11 *If the direct path between a and b is one-sided, then its length is at most $(\pi/2) \cdot \breve{d}(a,b)$.*

Proof. Since each edge of the direct path between a and b is a great arc, the triangle inequality implies that the length of this path is at most the quantity L in Lemma 10. □

4 Constructing a Short Path in $SDT(S)$

Consider again two distinct points a and b of S, together with their direct path

$$P = (a = b_0, b_1, b_2, \ldots, b_n = b).$$

In this section, we define a path Q in $SDT(S)$ between a and b. In Section 5, we will prove that the length of Q is at most $3(\pi/2+1) \cdot \breve{d}(a,b)$.

We assume, without loss of generality, that a and b are on the equator; thus the plane Π through a, b, and the origin is the plane with equation $z = 0$.

We partition the direct path P into subpaths $P_1, P_2, \ldots, P_m$, where each subpath P_k is

- either of *type 1*, i.e., P_k is a maximal subpath of P that is completely on or above Π,
- or of *type 2*, i.e., P_k is a subpath $b_i, b_{i+1}, \ldots, b_j$ with $j \geq i+2$, where both b_i and b_j are on or above Π and all points $b_{i+1}, \ldots, b_{j-1}$ are below Π.

For example, in the figure in the beginning of Section 3, $m = 2$, $P_1 = (b_0, b_1)$, and $P_2 = (b_1, b_2, b_3)$.

In the rest of this section, we will use the subpaths $P_1, P_2, \ldots, P_m$ to define paths $Q_1, Q_2, \ldots, Q_m$. The final path will be the concatenation of the latter paths.

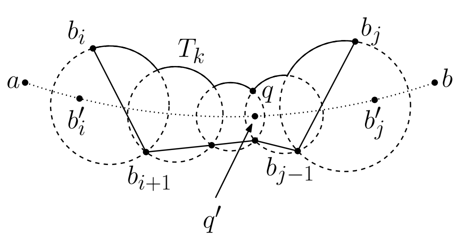
Let k be an integer with $1 \leq k \leq m$. If the subpath P_k is of type 1, then we define $Q_k = P_k$.

Assume that $P_k = (b_i, b_{i+1}, \ldots, b_j)$ is of type 2. Let b'_i and b'_j be the points on the equator whose spherical distances to b_i and b_j are minimum, respectively, and let

$$w = \breve{d}(b'_i, b'_j).$$

Let T_k be the part of the boundary of $\mathcal{C}$ that is above Π and that connects b_i and b_j. Let q be a point on T_k whose spherical distance to the equator is minimum, let q' be the point on the equator whose spherical distance to q is minimum, and let

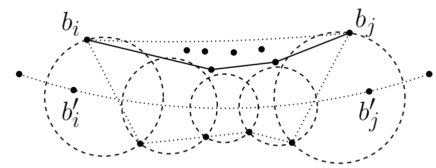
$$h = \breve{d}(q, q').$$



If $h \leq w/4$, then we define $Q_k = P_k$.

Assume that $h > w/4$. Let S' be the set of points p in S such that

- p is on or above Π,
- p is on or below the plane through b_i, b_j, and the origin, and
- p', i.e., the point on the equator whose spherical distance to p is minimum, is on the great arc connecting b'_i and b'_j.



Consider the "lower" part H of the spherical convex hull of S'; this is the path of solid edges in the figure above. If $S' = \{b_i, b_j\}$, then H consists of the edge (b_i, b_j). Otherwise, H consists of the hull edges that are not equal to (b_i, b_j). Observe that H is a path on $\mathbb{S}^2$ between b_i and b_j, all of whose edges are great arcs. For each such edge on H, take the direct path in $SDT(S)$ between their endpoints, and define Q_k to be the concatenation of all these direct paths.

Having defined a path Q_k in $SDT(S)$ for each integer k with $1 \leq k \leq m$, we define

$$Q = Q_1 Q_2 \cdots Q_m.$$

5 Bounding the Length of the Path Q

Let k be an integer with $1 \leq k \leq m$, and consider the subpath P_k of the previous section. We write this subpath as

$$P_k = (b_i, b_{i+1}, \ldots, b_j).$$

Recall that T_k is the part of the boundary of $\mathcal{C}$ that is above the plane Π and that connects b_i and b_j. Let L_k be the length of T_k. As before, we denote by b'_i and b'_j the points on the equator whose spherical distances to b_i and b_j are minimum, respectively. We will prove that the length of the path Q_k is at most

$$3\left(L_k + \breve{d}(b'_i, b'_j)\right). \quad (2)$$

By Lemma 5, this will imply that the length of the path $Q = Q_1 Q_2 \cdots Q_m$ is at most

$$3\left(\sum_{k=1}^{m} L_k + \breve{d}(a,b)\right).$$

Since $\sum_{k=1}^{m} L_k$ is equal to the quantity L in Lemma 10, it will follow that the length of Q is at most

$$3\,(\pi/2+1) \cdot \breve{d}(a,b),$$

thus completing the proof of Theorem 4.

If P_k is of type 1, then the length of Q_k (which is equal to P_k) is at most L_k and, thus, the inequality in (2) holds.

Assume that P_k is of type 2 and $h \leq w/4$. The length of Q_k (which is equal to P_k) is at most

$$L_k + 2 \cdot \breve{d}(b_i, b'_i) + 2 \cdot \breve{d}(b_j, b'_j).$$

The point q splits T_k into two parts. We denote the part connecting b_i and q by T'_k, and the part connecting q and b_j by T''_k. Let L'_k and L''_k denote the lengths of T'_k and T''_k, respectively.

Let a_i be the point on the great arc connecting b_i and b'_i such that $\breve{d}(a_i, b'_i) = h$. Then we have

$$\begin{aligned}
\breve{d}(b_i, b'_i) &= \breve{d}(b_i, a_i) + \breve{d}(a_i, b'_i) \\
&= \breve{d}(b_i, a_i) + h \\
&\leq \left(L'_k + \breve{d}(q, a_i)\right) + h \\
&\leq L'_k + \breve{d}(b'_i, q') + w/4.
\end{aligned}$$

By a symmetric argument, we have

$$\breve{d}(b_j, b'_j) \leq L''_k + \breve{d}(b'_j, q') + w/4.$$

Thus, the length of Q_k is at most

$$\begin{aligned}
& L_k + 2\left(L'_k + \breve{d}(b'_i, q') + w/4\right) \\
& +2\left(L''_k + \breve{d}(b'_j, q') + w/4\right) \\
& = \ 3\left(L_k + \breve{d}(b'_i, b'_j)\right)
\end{aligned}$$

and, therefore, the inequality in (2) holds.

It remains to consider the case when P_k is of type 2 and $h > w/4$.

Lemma 12 *For each edge (x, y) of the lower part of the spherical convex hull of the set S', the direct path in $SDT(S)$ between x and y is one-sided.*

Proof. The proof uses Lemma 6 and is a straighforward generalization of the proof of Lemma 4 in Dobkin *et al.* [2]. □

Let Σ denote the sum of the lengths of the edges of the lower spherical convex hull H of the set S'. Then, by Lemmas 11 and 12, the length of the path Q_k is at most $(\pi/2)\Sigma$.

Since each edge of H is a great arc, it follows from Lemma 13 in the appendix that $\Sigma \leq L_k$. Thus, the inequality in (2) holds.

6 Concluding Remarks

We have shown that the spherical Delaunay triangulation $SDT(S)$ of a finite set S of points on the unit-sphere $\mathbb{S}^2$ is a spherical t-spanner of S, for $t = 3(\pi/2 + 1)$. We proved this result by modifying the proof of Dobkin *et al.* [2] for the Euclidean Delaunay triangulation in $\mathbb{R}^2$.

By "straightening" the edges of $SDT(S)$, we obtain the convex hull $CH(S)$ of S (see Lemma 2), implying that $CH(S)$ is a Euclidean $(t\pi/2)$-spanner of S (see Lemma 3). We leave as an open problem to decide if the proof technique of Dobkin *et al.* can be used directly on $CH(S)$.

We also leave as an open problem to improve our upper bound on the stretch factor of the convex hull of points on the unit-sphere.

References

[1] R. Benson. *Euclidean Geometry and Convexity.* McGraw-Hill, New York, 1966.

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[3] G. Narasimhan and M. Smid. *Geometric Spanner Networks.* Cambridge University Press, Cambridge, UK, 2007.

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Appendix

Lemma 13 *Let p and q be two distinct points on $\mathbb{S}^2$, and let H and R be curves on $\mathbb{S}^2$ between p and q. Assume that*

- *p, q, H, and R are on or above the equator,*
- *p and q are not contained in a great circle through the north and south poles,*
- *both H and R are longitudinally monotone,*
- *H is on or below the plane through p, q, and the origin,*
- *H consists of a finite number of great arcs,*
- *H is spherically convex,*
- *and for each vertex x of H, the great arc between x and the south pole intersects R.*

Then the length of H is at most the length of R.

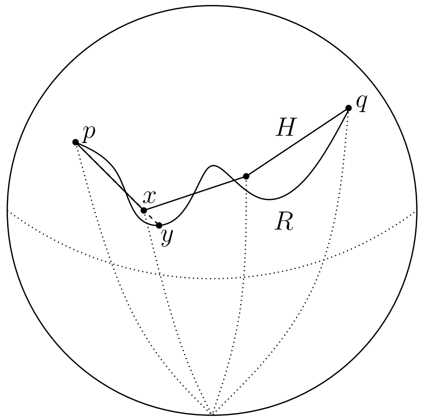
Proof. For any two points x and y on H, we denote by Σ_H^{xy} the length of the portion of the curve H between x and y. We define Σ_R^{xy} similarly with respect to the curve R. Using this notation, the lemma states that

$$\Sigma_H^{pq} \leq \Sigma_R^{pq}.$$

The proof is by induction on the number of great arcs on H. To prove the base case, assume that H consists of one single arc. Since this is a great arc, we have

$$\Sigma_H^{pq} = \breve{d}(p, q) \leq \Sigma_R^{pq}.$$

Now assume that H consists of at least two great arcs. Consider the first great arc (p, x) of H. Starting at x, walk along the great circle through this arc, in the opposite direction of p, and stop as soon as a point, say y, on R is encountered. (Observe that this point y exists.)



Let H' be the portion of H between x and q, and let R' be the curve obtained by concatening the great arc between x and y, and the portion of R between y and q. Since H' and R' satisfy the assumptions in the lemma and the number of great arcs on H' is one less than the number of great arcs on H, it follows by induction that

$$\Sigma_H^{xq} \leq \check{d}(x, y) + \Sigma_R^{yq}.$$

It follows that

$$\begin{aligned} \Sigma_H^{pq} &= \check{d}(p, x) + \Sigma_H^{xq} \\ &\leq \check{d}(p, x) + \check{d}(x, y) + \Sigma_R^{yq} \\ &= \check{d}(p, y) + \Sigma_R^{yq} \\ &\leq \Sigma_R^{py} + \Sigma_R^{yq} \\ &= \Sigma_R^{pq}. \end{aligned}$$

□