The Gaussian Centre and the Projection Centre of a Set of Points in $\mathbb{R}^3$

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Abstract

We define the Gaussian centre and the projection centre of a non-empty finite set of points $P \subseteq \mathbb{R}^3$. We show the two centres are equal for any $P \subseteq \mathbb{R}^3$.

1 Introduction

Finding a centre point is a fundamental problem of geometry. The Euclidean centre, or centre of the smallest enclosing sphere, provides a natural definition for the centre of a set of points. As shown in [BBKS00] and [DK04], the Euclidean centre of a set of points $P \subseteq \mathbb{R}^d$ is unstable; small perturbations at only a few points of P can result in an arbitrarily large relative change in the position of the Euclidean centre. To define a centre Υ more stable than the Euclidean centre requires, at least for some sets of points, that Υ differ from the Euclidean centre. Presumably, remaining central to P is desirable. These two factors are in opposition; high stability implies high eccentricty and vice-versa. In [DK04], the Gaussian centre of a set of points in the plane is introduced toward the objective of identifying a good centre that balances high stability with low eccentricty. The projection centre of a set of points in the plane is also defined and shown to be equivalent to the Gaussian centre.

The Gaussian centre's benefits extend beyond its definition as the centre of a set of static points. Recently, several questions of facility location have been posed within the setting of mobile facility location (e.g. [AGG02, AH01, BBKS00, Her03]). Given a set of mobile points, the fitness of a mobile facility is determined both by its eccentricity and also by the maximum velocity and continuity of its motion. As shown in [DK04], the stability of a centre is inversely related to the maximum velocity of a mobile facility, providing further motivation for the need of stability in a centre point.

The question of whether the Gaussian centre generalizes to three dimensions remained open. In this paper we define the Gaussian centre and the projection centre of a set of points $P \subseteq \mathbb{R}^3$. We show the equivalence of the two centres for any non-empty finite set $P \subseteq \mathbb{R}^3$.

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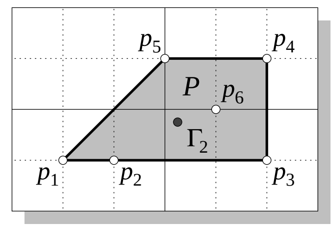

Figure 1: the 2D Gaussian centre of P, $\Gamma_2(P) = (\frac{1}{4}, -\frac{1}{4})$

2 Gaussian Centre Definition

Definition 1 *Let $H = \{x \in \mathbb{R}^d \mid h_1x_1 + \ldots + h_dx_d + h_{d+1} = 0\}$ define a $(d-1)$-dimensional hyperplane in $\mathbb{R}^d$, where $h \in \mathbb{R}^{d+1} - \{(0,\ldots,0)\}$ is fixed. Let $H^+ = \{x \in \mathbb{R}^d \mid h_1x_1 + \ldots + h_dx_d + h_{d+1} > 0\}$ and let $H^- = \{x \in \mathbb{R}^d \mid h_1x_1 + \ldots + h_dx_d + h_{d+1} < 0\}$ define the respective positive and negative half-spaces of $\mathbb{R}^d$ induced by H. A point $p \in P \subseteq \mathbb{R}^d$ is an* extreme point *of P if and only if there exists a $(d-1)$-dimensional hyperplane $H \subseteq \mathbb{R}^d$ with induced partition of $\mathbb{R}^d$, $\{H^+, H, H^-\}$, such that $H \cap P = \{p\}$, $P \subseteq H^+ \cup H$, and $P \cap H^- = \varnothing$.*

The Gaussian centre was first defined for a set of points in $\mathbb{R}^2$ [DK04]. The two-dimensional definition provides intuition for the three-dimensional case and we reproduce it here:

Definition 2 ([DK04]) *Let $P \subseteq \mathbb{R}^2$ be a non-empty finite set of points. Let $V_P \subseteq P$ be the set of extreme points of P. If $|P| \geq 2$, for every $p \in V_P$, let α_p be the interior angle formed on the convex hull boundary at p. The* Gaussian centre *of P is*

$$\Gamma_2(P) = \frac{1}{2\pi} \sum_{p \in P} w_p p \,, \tag{1}$$

where w_p is the Gaussian weight of point p given by

$$w_p = \begin{cases} 2\pi & \text{if } |P| = 1 \\ \pi - \alpha_p & \text{if } |P| \geq 2 \text{ and } p \in V_P \\ 0 & \text{if } p \in P - V_P \end{cases} . \tag{2}$$

Thus, the Gaussian weight of a point p on the convex hull of P corresponds to the turn angle at p. The greater the turn angle, the more significant the contribution p to $\Gamma_2(P)$. The turn angles of a polygon sum to 2π, hence the normalizing factor.

In three dimensions, the Gaussian centre of $P \subseteq \mathbb{R}^3$ is again defined as a normalized weighted mean. This time, however, a point $p \in P$ is adjacent to two or more faces; the Gaussian weight of p is defined in terms of the angles formed at the faces that meet at p.

Definition 3 *Let $P \subseteq \mathbb{R}^3$ be a non-empty finite set of points. Let $V_P \subseteq P$ be the set of extreme points of P. If $|P| \geq 2$, for every $p \in V_P$, let F_p be the set of faces that meet at p. For every face $f_j \in F_p$, let $\theta_{p,j}$ be the interior plane angle on f_j at p. The* Gaussian centre *of P is*

$$\Gamma(P) = \frac{1}{4\pi} \sum_{p \in P} w_p p \ , \tag{3}$$

where w_p is the three-dimensional Gaussian weight of point p given by

$$w_p = \begin{cases} 4\pi & \text{if } |P| = 1 \\ 2\pi - \sum_{f_j \in F_p} \theta_{p,j} & \text{if } |P| \geq 2 \text{ and } p \in V_P \\ 0 & \text{if } p \in P - V_P \end{cases} . \tag{4}$$

The sum of the plane angles at a point p ranges from 2π (when p is coplanar with its neighbours) and approaches a limit of 0 (when the neighbours of p approach a single point). The three-dimensional Gaussian weights of any polyhedron sum to 4π. Thus, in three dimensions we normalize by $1/4\pi$.

The Gaussian centre is invariant under many affine transformations. Specifically, it is easy to show that $\Gamma(f(P)) = f(\Gamma(P))$ where $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is any translational, uniform scaling, or rotational transformation function and $P \subseteq \mathbb{R}^3$ is any non-empty finite set of points.

When vertices are coplanar, a set of points P in three dimensions reduces to the two-dimensional case.

Lemma 1 *Let $P \subseteq \mathbb{R}^3$ be a non-empty finite set of coplanar points. The three-dimensional Gaussian centre of P matches the two-dimensional Gaussian centre of P.*

Proof sketch. Let $\Gamma_2(P)$ and $\Gamma(P)$ be the respective two- and three-dimensional Gaussian centres of P.

$$\Gamma(P) = \frac{1}{4\pi} \sum_i (2\pi - 2\theta_i) p = \frac{1}{2\pi} \sum_i (\pi - \theta_i) p = \Gamma_2(P) \ . \ \square \tag{5}$$

3 Projection Centre Definition

The projection centre was first defined in two dimensions [DK04]. Again, we introduce the three-dimensional definition by first presenting its two-dimensional analogue.

Let l_θ be the line through the origin parallel to the unit vector $u_\theta = (\cos\theta, \sin\theta)$. Expressed in slope-intercept form, l_θ is $y = \tan\theta \ x$. Given a set of points $P \subseteq \mathbb{R}^2$, let P_θ be the projection of P onto l_θ. See Figure 2A. That is,

$$P_\theta = \{ u_\theta \langle p, u_\theta \rangle \mid p \in P \} \ , \tag{6}$$

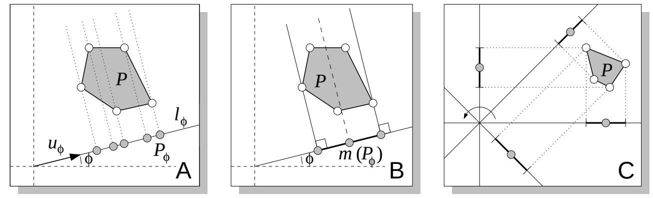

Figure 2: defining the projection centre $\Lambda_2(P)$

where $\langle x, y \rangle$ denotes the inner product of x and y. The midpoint of P_θ is

$$m(P_\theta) = \frac{1}{2} \left(\min_{p \in P_\theta} p + \max_{q \in P_\theta} q \right) \ , \tag{7}$$

where max and min return the extrema along line l_θ. See Figure 2B.

The projection centre is defined as the normalized average midpoint over all projections of P onto lines l_θ. See Figure 2C.

Definition 4 ([DK04]) *Let $P \subseteq \mathbb{R}^2$ be a non-empty bounded set of points. The* projection centre *of P is*

$$\Lambda_2(P) = \frac{2}{\pi} \int_0^\pi m(P_\theta) \ d\theta \ , \tag{8}$$

where $m(P_\theta)$ is the midpoint of the projection of P onto the line $y = \tan\theta \ x$.

The factor of 2 is necessary since $\frac{1}{\pi}\int_0^\pi u_\theta \langle p, u\theta \rangle \ d\theta = p/2$. The factor of $1/\pi$ normalizes for the range of integration.

In three dimensions, we express the projection centre in terms of spherical coordinates. Let $l_{\theta,\phi}$ be the line through the origin parallel to the unit vector $u_{\theta,\phi} = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. Let $P_{\theta,\phi}$ and $m(P_{\theta,\phi})$ be the natural generalizations of P_θ and $m(P_\theta)$ to spherical coordinates in $\mathbb{R}^3$, respectively.

The projection centre is defined as the normalized average midpoint over all projections of P onto lines $l_{\theta,\phi}$.

Definition 5 *Let $P \subseteq \mathbb{R}^3$ be a non-empty bounded set of points. The* projection centre *of P is*

$$\Lambda(P) = \frac{3}{2\pi} \int_0^\pi \int_0^\pi \sin\phi \cdot m(P_{\theta,\phi}) \ d\phi \ d\theta \ , \tag{9}$$

where $m(P_{\theta,\phi})$ is the midpoint of the projection of P onto the line through the origin parallel to $u_{\theta,\phi} = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$.

Note the factor $\sin\phi$ to account for non-uniform density using spherical coordinates, the normalizing factor $1/2\pi$ since $\int_0^\pi \int_0^\pi \sin\phi \ d\phi \ d\theta = 2\pi$, and the factor 3 since $\frac{1}{2\pi}\int_0^\pi \int_0^\pi \sin\phi \cdot u_{\theta,\phi}\langle p, u\phi \rangle \ d\phi \ d\theta = p/3$.

Similarly to the Gaussian centre, the projection centre is invariant under many affine transformations. Specifically, it is easy to show that $\Lambda(f(P)) = f(\Lambda(P))$ where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is any translational, uniform scaling, or rotational transformation function and $P \subseteq \mathbb{R}^3$ is any non-empty finite set of points.

Lemma 2 *Let $P \subseteq \mathbb{R}^3$ be a non-empty bounded set of coplanar points. The three-dimensional projection centre of P matches the two-dimensional projection centre of P.*

Proof sketch. Let $\Lambda_2(P)$ and $\Lambda(P)$ be the respective two- and three-dimensional projection centres of P. Since Λ is invariant under rotation and translation, assume P is coplanar with the xy-plane. For any $p \in P$, $p = (x, y, 0)$ and

$$\int_0^\pi \sin\phi \cdot u_{\theta,\phi}\langle (x,y,0), u_{\theta,\phi}\rangle \, d\phi = \frac{4}{3} u_\theta \langle (x,y), u_\theta\rangle \ . \quad (10)$$

Given θ, if p is an extreme point of $P_{\theta,\phi}$ for some $\phi \neq 0 \mod \pi$, then p is an extreme point of $P_{\theta,\phi}$ for any ϕ. Therefore,

$$\begin{aligned} \Lambda(P) &= \frac{3}{2\pi}\int_0^\pi \int_0^\pi \sin\phi \cdot m(P_{\theta,\phi})\, d\phi\, d\theta && (11)\\ &= \frac{3}{2\pi}\int_0^\pi \frac{4}{3} m(P_{\theta,\pi/2})\, d\theta && (12)\\ &= \frac{2}{\pi}\int_0^\pi m(P_{\theta,\pi/2})\, d\theta && (13)\\ &= \frac{2}{\pi}\int_0^\pi m(P_\theta)\, d\theta && (14)\\ &= \Lambda_2(P) \ . \ \square && (15) \end{aligned}$$

Unless otherwise specified, we refer to the three-dimensional definitions of Γ and Λ. Although $\Gamma(P)$ and $\Lambda(P)$ are defined in terms of a finite set of points $P \subseteq \mathbb{R}^3$, since only extreme points of P affect the positions of $\Gamma(P)$ and $\Lambda(P)$, each definition also applies to the polyhedron induced by the convex hull of P, $Q = CH(P)$. Thus, $\Gamma(Q)$ and $\Lambda(Q)$ are well defined for any convex polyhedron Q.

4 Convex Decomposition

We give an outline of proofs that for both the Gaussian centre and the projection centre, when a convex polyhedron is partitioned by a plane into two convex polyhedra, the relationships between the centres of the two components are identical. The analogous two-dimensional proofs are given in [DK04].

Let $\overline{A}$ denote the closure of set A. Lemma 3 first derives the relationship between the Gaussian centres followed by Lemma 5 which derives the relationship between the projection centres.

Lemma 3 *Let $P \subseteq \mathbb{R}^3$ be a convex polyhedral region. Let h be a plane that intersects P. Let h^+ and h^- be the half-spaces induced by h. Let $\Gamma(Q)$ be the Gaussian centre of Q. The Gaussian centres of the components of the decomposition of P induced by h are related by*

$$\Gamma(P) = \Gamma(\overline{P\cap h^+}) + \Gamma(\overline{P \cap h^-}) - \Gamma(P\cap h) \ . \quad (16)$$

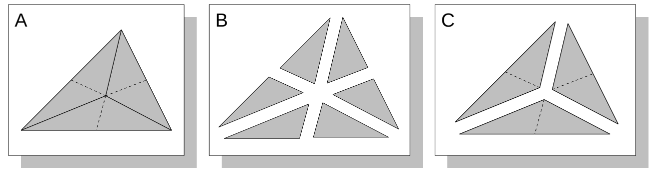

Figure 3: illustrations supporting Corollaries 4 and 6

Proof sketch. Let w_p, w_p^+, w_p^-, and w_p^h denote the Gaussian weights of a point p in polyhedra P, $\overline{P\cap h^+}$, $\overline{P\cap h^-}$, or $P \cap h$, respectively.

$$\begin{aligned} \Gamma(P) &= \frac{1}{4\pi}\sum_{p\in P} w_p p && (17)\\ &= \frac{1}{4\pi}\left[\sum_{p\in P\cap h^+} w_p p + \sum_{p \in P\cap h^-} w_p p + \sum_{p\in P\cap h} w_p p\right] && (18)\\ &= \frac{1}{4\pi}\left[\sum_{p\in P\cap h^+} w_p^+ p + \sum_{p\in P\cap h^-} w_p^- p \right.\\ &\quad \left. + \sum_{p\in P\cap h}(w_p^+ + w_p^- + w_p^h)p\right] && (19)\\ &= \frac{1}{4\pi}\left[\sum_{p\in \overline{P\cap h^+}} w_p^+ p + \sum_{p\in\overline{P\cap h^-}} w_p^- p - \sum_{p\in P\cap h} w_p^h p\right] && (20)\\ &= \Gamma(\overline{P\cap h^+}) + \Gamma(\overline{P\cap h^-}) - \Gamma(P\cap h) \ . && (21) \end{aligned}$$

$\square$

Note that w_p^A is defined respectively in terms of the faces adjacent to p in polyhedron A.

Corollary 4 *Let $P \subseteq \mathbb{R}^3$ be a convex polyhedral region such that $P = P_1 \cup \ldots \cup P_k$ where $P_1, \ldots, P_k$ forms a partition of P such that each P_i is also a convex polyhedral region. Let $f_1, \ldots, f_n$ be the faces that define the decomposition of P. The Gaussian centres of the components of the decomposition of P are related by*

$$\Gamma(P) = \sum_{i=1}^k \Gamma(P_i) - \sum_{j=1}^n \Gamma(f_j) \ . \quad (22)$$

Proof sketch. Let $f_1, \ldots, f_n$ be the faces that define the decomposition of P. Let $h_1, \ldots, h_n$ be the planes such that face f_i lies in plane h_i. We further decompose P by the planes $h_1, \ldots, h_n$. See Figure 3. The result follows by recursive applications of Lemmas 1 through 3 and Theorem 7 to decompose P and subsequently reconstruct each $P_1, \ldots, P_k$. $\square$

Lemma 5 *Let $P \subseteq \mathbb{R}^3$ be a convex polyhedral region. Let h be a plane that intersects P. Let h^+ and h^- be the half-spaces induced by h. Let $\Lambda(Q)$ be the projection centre of*

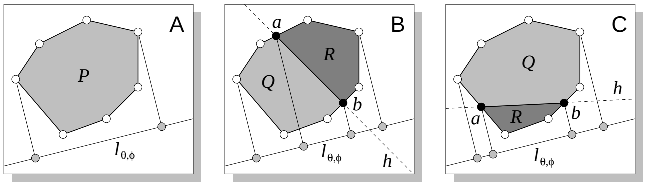

Figure 4: illustrations supporting Lemma 5

Q. The projection centres of the components of the decomposition of P induced by h are related by

$$\Lambda(P) = \Lambda(\overline{P \cap h^+}) + \Lambda(\overline{P \cap h^-}) - \Lambda(P \cap h) \ . \qquad (23)$$

Proof sketch. Let $l_{\theta,\phi}$ be the line through the origin parallel to $u_{\theta,\phi} = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$. Let $Q = \overline{P \cap h^+}$, $R = \overline{P \cap h^-}$, and $L = h \cap P$. Let $A_{\theta,\phi}$ be the projection of set A onto line $l_{\theta,\phi}$. Let $m(A_{\theta,\phi})$ be the midpoint of $A_{\theta,\phi}$. By examination of the two possible cases (Figure 4),

$$m(P_{\theta,\phi}) = m(Q_{\theta,\phi}) + m(R_{\theta,\phi}) - m(L_{\theta,\phi}) \ . \qquad (24)$$

The relationship follows:

$$\begin{aligned} \Lambda(P) &= \frac{3}{2\pi}\int_0^{\pi}\int_0^{\pi} \sin\phi \cdot m(P_{\phi,\theta}) \, d\phi \, d\theta & (25) \\ &= \frac{3}{2\pi}\int_0^{\pi}\int_0^{\pi} \sin\phi [m(Q_{\theta,\phi}) \\ &\quad + m(R_{\theta,\phi}) - m(L_{\theta,\phi})] \, d\phi \, d\theta & (26) \\ &= \frac{3}{2\pi}\int_0^{\pi}\int_0^{\pi} \sin\phi \cdot m(Q_{\theta,\phi}) \, d\phi \, d\theta \\ &\quad + \frac{3}{2\pi}\int_0^{\pi}\int_0^{\pi} \sin\phi \cdot m(R_{\theta,\phi}) \, d\phi \, d\theta \\ &\quad - \frac{3}{2\pi}\int_0^{\pi}\int_0^{\pi} \sin\phi \cdot m(L_{\theta,\phi}) \, d\phi \, d\theta & (27) \\ &= \Lambda(Q) + \Lambda(R) - \Lambda(L) \ . \ \Box & (28) \end{aligned}$$

Corollary 6 *Let $P \subseteq \mathbb{R}^3$ be a convex polyhedral region such that $P = P_1 \cup \ldots \cup P_k$ where $P_1, \ldots, P_k$ forms a partition of P such that each P_i is also a convex polyhedral region. Let $f_1, \ldots, f_n$ be the faces that define the decomposition of P. The projection centres of the components of the decomposition of P are related by*

$$\Lambda(P) = \sum_{i=1}^{k} \Lambda(P_i) - \sum_{j=1}^{n} \Lambda(f_j) \ . \qquad (29)$$

Proof sketch. The proof is analogous to the proof of Corollary 4.

5 Equivalence of Γ and Λ in $\mathbb{R}^3$

Theorem 7 ([DK04]) *Given any non-empty finite set of points $P \subseteq \mathbb{R}^2$, the location of its Gaussian centre, $\Gamma_2(P)$, and its projection centre, $\Lambda_2(P)$, are equal. That is,*

$$\forall P \subseteq \mathbb{R}^2, \ \Gamma_2(P) = \Lambda_2(P) \ . \qquad (30)$$

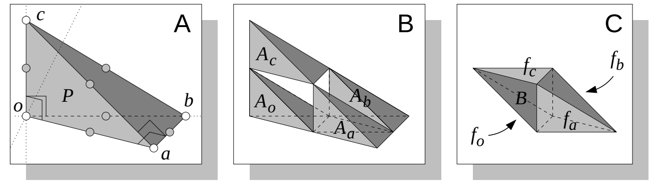

Figure 5: illustrations supporting Lemma 9

In this section we show Theorem 7 extends to any non-empty finite set $P \subseteq \mathbb{R}^3$.

Definition 6 *A set of points $P \subseteq \mathbb{R}^3$ is xyz-*symmetric *if there exists some $q \in \mathbb{R}^3$ such that for all p, $p \in P \Leftrightarrow (q - p) \in P$.*

Lemma 8 *For any xyz-symmetric non-empty finite set of points $P \subseteq \mathbb{R}^3$, $\Gamma(P) = \Lambda(P)$.*

Proof sketch. Let $P \subseteq \mathbb{R}^3$ be any xyz-symmetric non-empty finite set of points. Since Γ and Λ are invariant under translation, assume $q = (0, 0, 0)$.

Given θ and ϕ, let p_1 be an extreme point of $P_{\theta,\phi}$. By symmetry, $p_2 = -p_1$ is also an extreme point of $P_{\theta,\phi}$. Therefore $m(P_{\theta,\phi}) = \frac{1}{2}(p_1 + p_2) = (0, 0, 0)$ and $\Lambda(P) = \frac{3}{2\pi}\int_0^{\pi}\int_0^{\pi} \sin\phi \cdot m(P_{\theta,\phi}) \, d\phi \, d\theta = (0, 0, 0)$.

Furthermore, for any extreme point p_1 of P, $p_2 = -p_1$ is also an extreme point of P. If $q_1, \ldots, q_k$ are the neighbours of p_1 on the convex hull of P, then $r_1, \ldots, r_k$ are the neighbours of p_2 on the convex hull where $r_i = -q_i$. Therefore, the Gaussian weights of p_1 and p_2 are equal and $w_{p_1}p_1 + w_{p_2}p_2 = (0, 0, 0)$. Therefore, $\Gamma(P) = \frac{1}{2\pi}\sum_{p \in P} w_p p = (0, 0, 0)$. $\Box$

Definition 7 *If $P = \{o, a, b, c\}$ is a tetrahedron such that $\angle aoc = \angle boc = \angle oab = \angle cab = \frac{\pi}{2}$ then P is a* right-angle tetrahedron.

Lemma 9 *If P is a right-angle tetrahedron then $\Gamma(P) = \Lambda(P)$.*

Proof sketch. Assume $P = \{o, a, b, c\}$ is a right-angle tetrahedron. By the invariance of Γ and Λ under translation and rotation, assume $o = (0, 0, 0)$, a lies on the positive xy-plane, b lies on the positive y-axis, and c lies on the positive z-axis. See Figure 5A. Let $m_1, \ldots, m_6$ be the midpoints of the edges of P. Let $e_1, \ldots, e_{12}$ be the twelve edges induced by $m_1, \ldots, m_6$ that lie parallel to some edge of P. These new edges define a decomposition of P into four tetrahedra, A_o, A_a, A_b, and A_c, each isomorphic to P, plus one xyz-symmetric octahedron B. See Figures 5B and 5C. Let f_i be the face of intersection between tetrahedron A_i and octahedron B. Let $v_i \in \mathbb{R}^3$ be the translation vector such that for all $p, p \in A_o \Leftrightarrow (p + v_i) \in A_i$. Note that $p \in A_o \Leftrightarrow 2p \in P$. The following equation follows from Lemmas 1 through 9,

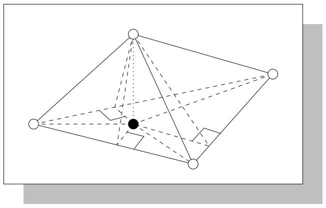

Figure 6: decomposition of a tetrahedron into right-angle tetrahedra

Corollaries 4 and 6, and Theorem 7.

$$
\begin{aligned}
\Gamma(P) &= \Gamma(A_o \cup A_a \cup A_b \cup A_c \cup B) && (31)\\
&= \Gamma(A_o) + \Gamma(A_a) + \Gamma(A_b) + \Gamma(A_c) + \Gamma(B) \\
&\quad -\Gamma(f_o) - \Gamma(f_a) - \Gamma(f_b) - \Gamma(f_c) && (32)\\
&= 4\Gamma(A_o) + v_a + v_b + v_c + \Gamma(B) \\
&\quad -\Gamma(f_o) - \Gamma(f_a) - \Gamma(f_b) - \Gamma(f_c) && (33)\\
&= 4\left[\frac{1}{2}\Gamma(P)\right] + v_a + v_b + v_c + \Gamma(B) \\
&\quad -\Gamma(f_o) - \Gamma(f_a) - \Gamma(f_b) - \Gamma(f_c) && (34)\\
&= -v_a - v_b - v_c - \Gamma(B) \\
&\quad +\Gamma(f_o) + \Gamma(f_a) + \Gamma(f_b) + \Gamma(f_c) && (35)\\
&= -v_a - v_b - v_c - \Lambda(B) \\
&\quad +\Lambda(f_o) + \Lambda(f_a) + \Lambda(f_b) + \Lambda(f_c) && (36)\\
&= 4\left[\frac{1}{2}\Lambda(P)\right] + v_a + v_b + v_c + \Lambda(B) \\
&\quad -\Lambda(f_o) - \Lambda(f_a) - \Lambda(f_b) - \Lambda(f_c) && (37)\\
&= 4\Lambda(A_o) + v_a + v_b + v_c + \Lambda(B) \\
&\quad -\Lambda(f_o) - \Lambda(f_a) - \Lambda(f_b) - \Lambda(f_c) && (38)\\
&= \Lambda(A_o) + \Lambda(A_a) + \Lambda(A_b) + \Lambda(A_c) + \Lambda(B) \\
&\quad -\Lambda(f_o) - \Lambda(f_a) - \Lambda(f_b) - \Lambda(f_c) && (39)\\
&= \Lambda(A_o \cup A_a \cup A_b \cup A_c \cup B) && (40)\\
&= \Lambda(P)\ \Box && (41)
\end{aligned}
$$

Theorem 10 *Given any non-empty finite set of points $P \subseteq \mathbb{R}^3$, the location of its Gaussian centre, $\Gamma(P)$, and its projection centre, $\Lambda(P)$, are equal. That is,*

$$\forall P \subseteq \mathbb{R}^3,\ \Gamma(P) = \Lambda(P)\ . \qquad (42)$$

Proof sketch. By induction on $|V_P|$. Let V_P be the set of extreme points of P. In the base case, $|V_P| \leq 3$. When $|V_P| \leq 3$, the points of P must be coplanar and $\Gamma(P) = \Lambda(P)$ by Lemmas 1 and 2 and Theorem 7. If $|V_P| \geq 4$, tetrahedralize P (may require the addition of new vertices). If a tetrahedron T does not have any vertex that lies on a line perpendicular to the opposite face, then add a point p at the centre of the insphere of T and decompose T further into the four tetrahedra induced by p. Each of these can be decomposed into right-angle tetrahedra. See Figure 6. By Lemma 9, the Gaussian centre and projection centre of any right-angle tetrahedron match. P can be reconstituted from these right-angle tetrahedra and by Corollaries 4 and 6, $\Gamma(P) = \Lambda(P)$. $\Box$

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