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Zero-coupon yield curves: technical documentation

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This volume was originally prepared following a meeting on estimation of zero-coupon yield curves held at the Bank for International Settlements in June 1996, and the papers are technical in character. This volume is a revised version with updated papers from the reporting central banks.

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Introduction

Following a meeting on the estimation of zero-coupon yield curves held at the BIS in June 1996, participating central banks have since been reporting their estimates to the Bank for International Settlements. The BIS Data Bank Services provide access to these data, which consist of either spot rates for selected terms to maturity or represent estimated parameters from which spot and forward rates can be derived. In the case estimated parameters are reported, the Data Bank Services provides, in addition to the parameters also the generated spot rates.

The purpose of this document is to facilitate the use of these data. It provides information on the reporting central banks' approaches to the estimation of the zero-coupon yield curves and the data transmitted to the BIS Data Bank. In most cases, the contributing central banks adopted the so-called Nelson and Siegel approach or the Svensson extension thereof. A brief overview of the relevant estimation techniques and the associated mathematics is provided below. General issues concerning the estimation of yield curves are discussed in Section 1. Sections 2 and 3 document the term structure of interest rate data available from the BIS. The final section provides examples of estimated parameter and selected spot and forward rates derived thereof. A list of contacts at central banks can be found after the references. The remainder of this document consists of brief notes provided by the reporting central banks on approaches they have taken to estimate the yield curves.

Since the last release of this manual in March 1999 there have been four major changes: Switzerland started to report their estimates of the yield curve to the BIS in August 2002. Furthermore, Sweden began to use a new estimation method in 2001, the United Kingdom since September 2002 and Canada since January 2005. These changes are included in Tables 1 and 2.

1. Zero-coupon yield curve estimation techniques

The estimation of a zero-coupon yield curve is based on an assumed functional relationship between either par yields, spot rates, forward rates or discount factors on the one hand and maturities on the other. Discount factors are the quantities used at a given point in time to obtain the present value of future cash flows. A discount function $d_{t,m}$ is the collection of discount factors at time t for all maturities m. Spot rates $s_{t,m}$, the yields earned on bonds which pay no coupon, are related to discount factors according to:

$$d_{t,m} = \exp\left(-s_{t,m}\ m\right) \text{ and } s_{t,m} = -\frac{1}{m}\log d_{t,m} \qquad (1)$$

Because spot interest rates depend on the time horizon, it is natural to define the forward rates $f_{t,m}$ as the instantaneous rates which, when compounded continuously up to the time to maturity, yield the spot rates (instantaneous forward rates are, thus, rates for which the difference between settlement time and maturity time approaches zero):

$$s_{t,m} = -\frac{1}{m}\int_0^m f(u)du \qquad (2)$$

or, equivalently:

$$d_{t,m} = \exp\left[-\int_0^m f(u)du\right] \qquad (3)$$

These relations can be inverted to express forward rates directly as a function of discount factors or spot rates:

$$f_{t,m} = s_{t,m} + m\dot{s}_{t,m} \text{ and } f_{t,m} = -\frac{\dot{d}_{t,m}}{d_{t,m}} \tag{4}$$

where dots stand for derivatives with respect to time to maturity.

However, the general absence of available pure discount bonds that can be used to compute zero-coupon interest rates presents a problem to practitioners. In other words, zero coupon rates are rarely directly observable in financial markets. Attempting to extract zero-coupon rates from the prices of those risk-free coupon-bearing instruments which are observable, namely government bonds, various models and numerical techniques have been developed. Such models can broadly be categorised into ***parametric*** and ***spline-based approaches***, where a different trade-off between the flexibility to represent shapes generally associated with the yield curve (*goodness-of-fit*) and the *smoothness* characterizes the different approaches. These main modelling approaches are now briefly discussed below.

Parametric Models

The underlying principle of parametric models, also referred to as function-based models, is the specification of a single-piece function that is defined over the entire maturity domain. Whilst the various approaches in this class of models advocate different choices of this function, they all share the general approach that the model parameters are determined through the minimisation of the squared deviations of theoretical prices from observed prices.

The Nelson and Siegel method

The method developed by Nelson and Siegel (1987) attempts to estimate these relationships by fitting for a point in time t a discount function to bond price data by assuming explicitly the following function form for the instantaneous forward rates:

$$f_{t,m} = \beta_{t,0} + \beta_{t,1} \exp\left(\frac{-m}{\tau_{t,1}}\right) + \beta_{t,2} \frac{m}{\tau_{t,1}} \exp\left(\frac{-m}{\tau_{t,1}}\right) \tag{5}$$

In this equation m denotes time to maturity, t the time index and $\beta_{t,0}$, $\beta_{t,1}$, $\beta_{t,2}$ and $\tau_{t,1}$ are parameters to be estimated.[1] The zero-coupon or spot interest rate curve s_m can be derived by integrating the forward rate curve:

$$s_m = \beta_0 + \beta_1\left(1 - \exp\left(-\frac{m}{\tau_1}\right)\right)\left(\frac{m}{\tau_1}\right)^{-1} + \beta_2\left(\left(1 - \exp\left(-\frac{m}{\tau_1}\right)\right)\left(\frac{m}{\tau_1}\right)^{-1} - \exp\left(-\frac{m}{\tau_1}\right)\right) \tag{6}$$

which is equivalent to:

$$s_m = \beta_0 + (\beta_1 + \beta_2)\frac{\tau_1}{m}\left(1 - \exp\left(-\frac{m}{\tau_1}\right)\right) - \beta_2 \exp\left(-\frac{m}{\tau_1}\right) \tag{7}$$

For long maturities, spot and forward rates approach asymptotically the value β_0 which must be positive. ($\beta_0 + \beta_1$) determines the starting value of the curve at maturity zero; β_1 thus represents the deviation from the asymptote β_0. In addition, ($\beta_0 + \beta_1$) must also be positive. The remaining two parameters β_2 and τ_1 are responsible for the “hump”. The hump’s magnitude is given by the absolute size of β_2 while its direction is given by the sign: a negative sign indicates a U-shape and a positive sign a hump. τ_1, which again must be positive, determines the position of the hump.

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[1] To simplify the notation, the time index t is dropped below.

The Svensson method

To improve the flexibility of the curves and the fit, Svensson (1994) extended Nelson and Siegel's function by adding a further term that allows for a second "hump". The extra precision is achieved at the cost of adding two more parameters, β_3 and τ_2, which have to be estimated. The instantaneous forward rates curve thus becomes:

$$f_m = \beta_0 + \beta_1 \exp\left(\frac{-m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(\frac{-m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(\frac{-m}{\tau_2}\right) \tag{8}$$

with β_3 and τ_2 having the same characteristics as β_2 and τ_1 discussed above. Again, to derive the spot rates curve the instantaneous forward rates curve is integrated:

$$s_m = \beta_0 + \beta_1\left(1 - \exp\left(-\frac{m}{\tau_1}\right)\right)\left(-\frac{m}{\tau_1}\right)^{-1} + \beta_2\left(\left(1 - \exp\left(-\frac{m}{\tau_1}\right)\right)\left(\frac{m}{\tau_1}\right)^{-1} - \exp\left(-\frac{m}{\tau_1}\right)\right)$$

$$+\beta_3\left(\left(1 - \exp\left(-\frac{m}{\tau_2}\right)\right)\left(\frac{m}{\tau_2}\right)^{-1} - \exp\left(-\frac{m}{\tau_2}\right)\right) \tag{9}$$

For zero-coupon bonds, spot rates can be derived directly from observed prices. For coupon-bearing bonds usually their "yield to maturity" or "par yield" only is quoted. The yield to maturity is its internal rate of return, that is the constant interest rate r_k that sets its present value equal to its price:

$$P_k = \sum_{i=1}^{n} \frac{CF_i}{(1+r_k)^{t_i}} \tag{10}$$

where P_k is the price of bond k which generates n cash-flows CF at periods t_j (i = 1, 2, .. , n). These cash flows consist of the coupon payments and the final repayment of the principal or face value. Yields to maturity on coupon bonds of the same maturity with different coupon payments are not identical. In particular, the yield to maturity on a coupon-bearing bond differs from the yield to maturity - or spot rate - of a zero-coupon bond of the same maturity. Nevertheless, if the cash flow structure of a bond trading at the market ("at par") is known, it is possible to derive from estimated spot rates uniquely the coupon bond's theoretical yield to maturity, ie the rate the bond would require in order to trade at its face value ("at par"). Drawing on the spot rates $s_{t,m}$, the price equation can be expressed as:

$$P_k = \frac{C}{(1+s_{t,1})} + \frac{C}{(1+s_{t,2})^2} + \ldots + \frac{C}{(1+s_{t,m})^m} + \frac{V}{(1+s_{t,m})^m} \tag{11}$$

where C represents the coupon payments and V the repayment of the principal. The yield to maturity of a coupon-bearing bond is therefore an average of the spot rates which, in general, varies with the term to maturity.

To derive the term structure of interest rates, the discount function is estimated by applying a (constrained) non-linear optimisation procedure to data observed on a trade day. More important than the choice of a particular optimisation method (eg maximum likelihood, non-linear least squares, generalised method of moments) is the decision whether the (sum of squared) yield or price errors should be minimised. If one is primarily interested in interest rates, it suggests itself to minimise the deviation between estimated and observed yields. In this case the estimation proceeds in two stages: first, the discount function $d_{t,m}$ is used to compute estimated prices and, secondly, estimated yields to maturity are calculated by solving the following equation for each coupon-bearing bond k:

$$P_k = \sum_{i=1}^{m} C \exp(-r_k i) + V \exp(-r_k m) \tag{12}$$

At both stages, the starting point is from pre-selected values for the relevant parameters and to run through an iterative process until convergence is achieved. It is computationally easier to minimise price errors than yield errors, as this only requires finding a solution for the first stage. Unfortunately, minimising price errors can lead to large yield errors for financial instruments with relatively short remaining term to maturity. Considering how yield, price and term to maturity of a bond are related, it is not surprising to observe this heteroscedasticity problem: drawing on the concept of duration,[2] the elasticity of the price with respect to one plus the yield is equal to the duration of the bond. A given change in the yield corresponds to a small/large change in the price of a bond with a short/long term to maturity or duration. Fitting prices to each bond, given an equal weight irrespective of its duration, leads to over-fitting of the long-term bond prices at the expense of the short-term prices. One approach to correct for this problem is to weight the price error of each bond by a value derived from the inverse of its duration.

Other factors can also contribute to fairly large yield errors at the short end of the term structure. For instance, the trading volume of a bond can decrease considerably when it approaches its maturity date. The quoted price for such a bond may not accurately reflect the price at which trading would take place. For such reasons it may be appropriate to exclude price data of bonds close to expiration when fitting term structures.

Spline-based Models

Rather than specifying a single functional form over the entire maturity range, spline-based methods fit the yield curve by relying on a piecewise polynomial, the spline function[3], where the individual segments are joined smoothly at the so-called knot points. Over a closed interval, a given continuous function can be approximated by selecting an arbitrary polynomial, where the goodness-of-fit increases with the order of the polynomial. Higher-order polynomials, however, quite frequently display insufficient smoothing properties. This problem can be avoided by relying on a piecewise polynomial whereby the higher-order polynomial is approximated by a sequence of lower-order polynomials.

Consequently, spline functions are generally based on lower-order polynomials (mostly quadratic or cubic). A cubic spline, for instance, is a piecewise cubic polynomial restricted at the knot points such that their levels and first two derivatives are identical. One parameter corresponds to each knot in the spline.

The "smoothing splines" method

This method developed by Fisher, Nychka and Zervos (1995) represents an extension of the more traditional cubic spline techniques (eg Vasicek and Fong (1982)). In the case of "smoothing splines" the number of parameters to be estimated is not fixed in advance. Instead, one starts from a model which is initially over-parameterised. Allowing for a large number of knot points guarantees sufficient flexibility for curvature throughout the spline. The optimal number of knot points is then determined by minimizing the ratio of a goodness-of-fit measure to the number of parameters. This approach penalizes for the presence of parameters which do not contribute significantly to the fit. It is not convenient to draw on the (varying number of) parameters in disseminating yield curve information.

There is a broad range of spline-based models which use this "smoothing method" pioneered by Fisher et al. The main difference among the various approaches simply lies in the extent to and fashion by which the smoothing criteria are applied to obtain a better fix. The "variable penalty roughness" (VRP) approach recently implemented by the Bank of England allows the "roughness" parameter to vary with the maturity, permitting more curvature at the short end.[4]

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2 Recall that the duration of a zero-coupon bond is equal to its maturity. Assuming a flat yield curve, the sensitivity of a zero-coupon bond to a change in the term structure should be directly proportional to its maturity. A change in the interest rate divided by one minus the interest rate of 1% corresponds to a change in the price of 1% of a bond with a maturity of one year and of 10% of a 10-year bond.

3 Spline functions, such as basis or B-splines, are used in the context of yield curve estimation. At times there exists some confusion among practitioners between spline functions and spline-based interpolation. While the former technique uses polynomials in order to approximate (unknown) functions, the latter is simply a specific method to interpolate between two data points.

4 See James and Webber (2000) for a comprehensive overview and comparison of the various approaches and Anderson and Sleath (1999) for a detailed description of the VRP approach.

Generally, the estimation method largely depends on intended use of data: no-arbitrage pricing and valuation of fixed-income and derivative instruments vs information extraction for investment analytical and monetary policy purposes. One of the main advantages of spline-base techniques over parametric forms, such as the Svensson method, is that, rather than specifying a single functional form to describe spot rates, they fit a curve to the data that is composed of many segments, with the constraint that the overall curve is continuous and smooth.[5]

2. Provision of information on the term structure of interest rates

The term structure of interest rates, defined as the functional relationship between term to maturity and the spot interest rate of zero-coupon bonds, consists of an infinite number of points. In many respects forward interest rates are more interesting than spot rates, as implied by the spot rate curve or vice versa, as the former can pertain information about expected future time paths of spot rates. At any point along the maturity spectrum there exists an infinite number of forward rates which differ in terms of their investment horizon. The *instantaneous* forward rate represents just a special case, the one for which the investment horizon approaches zero.

Published information on term structure of interest rates usually consists of selected spot rates at discrete points along the maturity spectrum. Occasionally, these spot rates are complemented by a selection of specific forward interest rates.

Such limitations would be mitigated if information on the term structure of interest rates could be presented in terms of algebraic expressions from which spot and forward rates can be derived. This is straightforward for parsimonious approaches such as Nelson and Siegel and Svensson discerned above, for which spot and instantaneous forward rates can be calculated using the estimated parameters (β's and τ's).

Further information can be useful in interpreting the curves such as statistics on the quality of the fit, details on the debt instruments used in the estimations, and if and what kind of efforts were made to prevent that specific premia, eg tax premia, distort estimation results. Some of this information can be found below and in the notes provided by the central banks.

Comparability of central banks' term structures of interest rates

To estimate the term structure of interest rates, most central banks reporting data have adopted either the Nelson and Siegel or the extended version suggested by Svensson. Exceptions are Canada, Japan, (in part) Sweden[6], the United Kingdom, and the United States which all apply variants of the "smoothing splines" method.

Government bonds data are used in the estimations since they carry no default risk. Occasionally, central banks complement this information by drawing on money market interest rates or swap rates. Clearly, financial markets differ considerably in terms of the number of securities actively traded and their turnover, the variety of financial instruments and specific institutional features. Such differences can give rise to a variety of premia which should be taken into consideration in the estimation process but in practice this is difficult to do.

Premia induced by tax regulations are notoriously difficult to deal with. One could attempt to remove tax-premia from the observed prices/yields before they are used in estimations. In other instances it may be preferable to simply exclude instruments with distorted prices/yields from the data set. In cases where it is expected that tax distortions have only a minor impact on the estimation results the best approach may be to ignore this problem altogether. Occasionally, central banks prefer modifying the estimation approach instead of adjusting the data to deal with specific problems (see Table 1).

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5 For example, at the long end of the yield curve, the Svensson model is constrained to converge to a constant level, directly implying that the unbiased expectation hypothesis holds.

6 Although Sveriges Riksbank adopted the "smoothing splines" method in 2001, they still only report their Svensson method estimates to the BIS Data Bank.

Not all central banks estimate the term structure for the full maturity spectrum for which debt instruments are available. Although this concerns both ends of the curves, the short-end is usually more difficult to deal with than the long-end. In modelling the short-end of the term structure, the approaches taken by monetary authorities differ considerably. On the data side this concerns mostly the choice of the types of short-term instruments regarded to be the most suitable and the minimum remaining term to maturity allowed in the estimation. On the modelling side, it is this part of the term structure on which the decision for either Nelson and Siegel's "one-hump" or Svensson's "two-hump" model may have the greatest impact. The requirement of a minimum remaining term to maturity for a bond to be included in the estimations influence the fit of the very short-end of the curve. Considering the difficulties to consistently achieve a good fit for this part of the curve helps to explain why some central banks regard the short-end of their curves as less reliable than the rest. Across the board, the interval from one to 10 years is hardly controversial.

Table 1

The term structure of interest rates - estimation details

Central bank	Estimation method	Minimised error	Shortest maturity in estimation	Adjustments for tax distortions	Relevant maturity spectrum
Belgium	Svensson or Nelson-Siegel	Weighted prices	Treasury certificates: > few days Bonds: > one year	No	Couple of days to 16 years
Canada	Merrill Lynch Exponential Spline	Weighted prices	Bills: 1 to 12 months Bonds: > 12 months	Effectively by excluding bonds	3 months to 30 years
Finland	Nelson-Siegel	Weighted prices	≥ 1 day	No	1 to 12 years
France	Svensson or Nelson-Siegel	Weighted prices	Treasury bills: all Treasury Notes: : ≥ 1 month Bonds: : ≥ 1 year	No	Up to 10 years
Germany	Svensson	Yields	> 3 months	No	1 to 10 years
Italy	Nelson-Siegel	Weighted prices	Money market rates: O/N and Libor rates from 1 to 12 months Bonds: > 1 year	No	Up to 30 years Up to 10 years (before February 2002)
Japan	Smoothing splines	Prices	≥ 1 day	Effectively by price adjustments for bills	1 to 10 years
Norway	Svensson	Yields	Money market rates: > 30 days Bonds: > 2 years	No	Up to 10 years
Spain	Svensson Nelson-Siegel (before 1995)	Weighted prices Prices	≥ 1 day ≥ 1 day	Yes No	Up to 10 years Up to 10 years
Sweden	Smoothing splines and Svensson	Yields	≥ 1 day	No	Up to 10 years
Switzerland	Svensson	Yields	Money market rates: ≥ 1 day Bonds: ≥ 1 year	No	1 to 30 years

Table 1 cont

The term structure of interest rates - estimation details

Central bank	Estimation method	Minimised error	Shortest maturity in estimation	Adjustments for tax distortions	Relevant maturity spectrum
United Kingdom[1]	VRP (government nominal)	Yields	1 week (GC repo yield)	No	Up to around 30 years
	VRP (government real/implied inflation)	Yields	1.4 years	No	Up to around 30 years
	VRP (bank liability curve)	Yields	1 week	No	Up to around 30 years
United States	Smoothing splines (two curves)	Bills: weighted prices	–	No	Up to 1 year
		Bonds: prices	≥ 30 days	No	1 to 10 years

[1] The United Kingdom used the Svensson method between January 1982 and April 1998.

3. Zero-coupon yield curves available from the BIS

Table 2 provides an overview of the term structure information available from the BIS Data Bank. Most central banks estimate term structures at a daily frequency. With the exception of the United Kingdom, central banks which use Nelson and Siegel-related models report estimated parameters to the BIS Data Bank. Moreover, Germany and Switzerland provide both estimated parameters and spot rates from the estimated term structures. Canada, the United States and Japan, which use the smoothing splines approach, provide a selection of spot rates. With the exception of France, Italy and Spain, the central banks report their data in percentage notation. Specific information on the retrieval of term structure of interest rates data from the BIS Data Bank can be obtained from BIS Data Bank Services.

Table 2

The structure of interest rates available from the BIS Data Bank[1]

Central bank	Method[2]	Estimates available since	Frequency	Available yield series	Maturity interval	Parameters[3]	Parameter notation[4]
Belgium	SV-NS	1 Sep 1997	Daily	0 to 10 years	3 months	6	Per cent
Canada	SV	23 Jun 1998 to 15 Oct 2003	Daily	1 to 10 years	3 months	6	Per cent
	SS	1 Jan 1986	Daily	3 months to 30 years	3 months	na	na
Finland	NS	3 Nov 1997	Weekly; daily from 4 Jan 1999	1 to 10 years	3 months	4	Per cent
France[5]	SV-NS	3 Jan 1992 to 1 Jun 2004	Weekly	0 to 10 years	3 months	6	Decimal
Germany[6]	SV	7 Aug 1997	Daily	1 to 10 years	3 months	6	Per cent
	SV	Jan 1973	Monthly	na	—	6	Per cent
	SV	28 Aug 1997	Daily	1 to 10 years	1 year	6	Per cent
	SV	Jan 1973	Monthly	1 to 10 years	1 year	6	Per cent
Italy	NS	1 Jan 1996	Daily	0 to 10 years	3 months	4	Decimal
Japan[5]	SS	29 Jul 1998 to 19 Apr 2000	Weekly	1 to 10 years	1 year	na	na
Norway	SV	21 Jan 1998	Once a month	0 to 10 years	3 months	6	Per cent
Spain	NS	2 Jan 1991 to 30 Dec 1994	Daily	na	—	4	Decimal
	SV	2 Jan 1995	Daily	1 to 10 years	3 months	6	Decimal
Sweden	SV	9 Dec 1992 to 1 Mar 1999	Weekly	0 to 10 years	3 months	6	Per cent
	SV	2 Mar 1999	Daily	0 to 10 years	3 months	6	Per cent
Switzerland[7]	SV	4 Jan 1988	Daily	1 to 10 years	3 months	6	Per cent
	SV	4 Jan 1998	Daily	1 to 10, 15, 20, 30 years	1 year	6	Per cent
	SV	Jan 1988	Monthly	1 to 10, 15, 20, 30 years	1 year	6	Per cent
United Kingdom	SV	4 Jan 1982 to 30 Apr 1998	Daily	2 to 10 years	6 months	na	na
	SV	Jan 1982 to April 1998	Monthly	2 to 10 years	6 months	na	na
	VRP	4 Jan 1982	Daily	5, 10 years[8]	—	na	na
	VRP	Jan 1982	Monthly	5, 10 years[8]	—	na	na
	VRP	15 Jan 1985	Daily	20 years[8]	—	na	na
	VRP	Jan 1985	Monthly	20 years[8]	—	na	na
United States	SS	14 Jun 1961	Daily	0 to 10 years	6 months	na	na
	SV	01 Dec 1987	Daily	0 to 10 years	3 months	na	na

[1] As of August 2005. [2] NS = Nelson-Siegel, SV = extended Nelson-Siegel (Svensson), SS = smoothing splines, VRP = variable roughness penalty. [3] Where there is an indication of a parameter there is also a BIS generated yield available on the BIS Data Bank. Moreover, "na" means that the country is transmitting estimated yields and not parameters. [4] Estimated parameters define spot and forward interest rates expressed in either decimal notation or per cent. [5] The yield curve is currently not estimated. [6] BIS generated yields are available at three months interval in addition to the yearly yields reported by the Bundesbank. [7] BIS generated yields are available at three months interval in addition to the yearly yields reported by the Swiss National Bank. [8] The nominal and real yields as well as the implied inflation term structure are calculated for the corresponding maturities.

4. Spot interest rates and forward rates derived from estimation parameters

Spot interest rates and *instantaneous* forward rates can be derived directly from the equations for the Nelson-Siegel and Svensson approaches presented above: replace the parameters of the equations - β_0, β_1, β_2 and τ_1 in the Nelson and Siegel and β_0, β_1, β_2, τ_1, β_3 and τ_2 in the Svensson case - by their estimated values and evaluate the equations at terms to maturity *m* for which the spot or forward rates have to be derived (eg $m = 1$ for one year to maturity). Table 3 provides examples of estimated parameters and a selection of corresponding points on the term structures. For the calculation of spot and instantaneous forward rates, it is partly relevant if the term structure was estimated either in decimal or percentage notation; the only difference is that the β-parameters are rescaled by a factor of 100. Clearly, such rescaling has no impact on the location of the humps as determined by the τ-parameters. By setting $\beta_3 = 0$ and τ_2 to an arbitrary non-zero value (eg $\tau_2 = 1$), the Svensson equations can be used to derive spot and forward rates of term structures estimated by the Nelson and Siegel approach. Thus it is sufficient to implement just the two Svensson equations to derive the spot and instantaneous forward rates for both approaches.

Table 3

Spot interest rates and instantaneous forward rates derived from estimation parameters

Estimation parameters	**Svensson (in percentage notation)**		**Nelson and Siegel (in decimal notation)**		**Nelson and Siegel (in percentage notation)**	
β_0	5.82		0.0769		7.69	
β_1	–2.55		–0.0413		–4.13	
β_2	–0.87		–0.0244		–2.44	
τ_1	3.90		0.0202		2.02	
β_3	0.45		–		–	
τ_2	0.44		–		–	
Term to maturity	**Spot rate (%)**	**Forward rate (%)**	**Spot rate**	**Forward rate**	**Spot rate (%)**	**Forward rate (%)**
0	3.27	3.27	0.0356	0.00356	3.56	3.56
1 year	3.61	3.78	0.0400	0.0444	4.00	4.44
15 months	3.65	3.84	0.0411	0.0465	4.11	4.65
18 months	3.69	3.91	0.0421	0.0486	4.21	4.86
21 months	3.72	3.98	0.0432	0.0506	4.32	5.06
2 years	3.76	4.05	0.0443	0.0526	4.43	5.26
5 years	4.17	4.80	0.0546	0.0683	5.46	6.83
10 years	4.68	5.45	0.0639	0.0758	6.39	7.58
∞	5.82	5.82	0.0769	0.0769	7.69	7.69

The calculation of forward rates with non-instantaneous term to maturity is slightly more complicated. It is important to notice that in those cases where the term structure parameters are readily available, forward rates can be derived for any desired term to maturity. To compute such *implied* forward rates, evaluate the spot rate equations at discreetly chosen term to maturity intervals, then calculate the implied forward rates recursively from the shortest to the longest term to maturity.

References

Anderson, N, F Breedon, M Deacon, A Derry and G Murthy (1996): "Estimating and interpreting the yield curve", John Wesley and Sons.

Anderson, N, F Breedon, M Deacon, A Derry, G Murthy and J Sleath (1999): "New estimates of the UK real and nominal yield curves", Bank of England *Quarterly Bulletin*, November, pp 384-392.

Cooper, N and J Steeley (1996): "G7 yield curves". Bank of England, *Quarterly Bulletin*, May, pp 199-208.

Deutsche Bundesbank (1995): "The market of German Federal Securities", Frankfurt am Main, July.

Fama, E F and R R Bliss: "The information in long-maturity forward rates". *American Economic Review*, LXXVII, pp 680-92.

Fisher, M, D Nychka and D Zervos (1995): "Fitting the term structure of interest rates with smoothing splines", Board of Governors of the Federal Reserve System, *Finance and Economics Discussion Series*, 95-1.

McCulloch, J H (1971): "Measuring the term structure of interest rates", *Journal of Business*, 44, pp 19-31.

McCulloch, J H (1975): "The tax adjusted yield curve", *Journal of Finance*, 90, pp 811-30.

Estrella, A and F S Mishkin (1997): "The predictive power of the term structure of interest rates in Europe and the United States: Implications for the European Central Bank", *European Economic Review*, 41, pp 1375-401.

James, J and Nick Webber (2000): "Interest Rate Modelling", John Wiley and Sons Ldt.

Nelson, C R and A F Siegel (1987): "Parsimonious modeling of yield curves", *Journal of Business*, 60, pp 473-89.

Seppälä, J and P Viertiö (1996): "The term structure of interest rates: estimation and interpretation", Bank of Finland, *Discussion Paper*, No 19.

Schich, S T (1996): "Alternative specifications of the German term structure and its information content regarding inflation", Deutsche Bundesbank, Economic Research Group, *Discussion Paper*, No 8, October.

Schich, S T (1997): "Estimating the German term structure", Deutsche Bundesbank, Economic Research Group, *Discussion Paper*, No 4.

Svensson, L E O (1994): "Estimating and interpreting forward interest rates: Sweden 1992-4", *CEPR Discussion Paper Series*, No 1051, October (also: NBER Working Paper Series, No 4871).

Ricart, R and P Sicsic (1995): "Estimating the term structure of interest rates from French data", Banque de France, *Bulletin Digest*, No 22, October.

Vasicek, O and G Fong (1982): "Term structure estimation using exponential splines", *Journal of Finance*, 38, pp 339-48.

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Technical note on the estimation procedure for the Belgian yield curve

Michel Dombrecht and Raf Wouters[1]

The purpose of this note is to document the methodology and data used for the construction of the zero coupon yield curve that is daily estimated by the National Bank of Belgium. The yield curve is based on the functional form proposed by Nelson and Siegel (1985) and extended by Svensson (1994).

Theoretical model

The following functional form is used to represent the zero coupon yield curve:

$$r(m)=\beta_0+\beta_1*\left[\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}}\right]+\beta_2*\left[\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}}-\exp\left(-\frac{m}{\tau_1}\right)\right]+\beta_3*\left[\frac{1-\exp\left(-\frac{m}{\tau_2}\right)}{\frac{m}{\tau_2}}-\exp\left(-\frac{m}{\tau_2}\right)\right] \quad (1)$$

The zero coupon yield r depends on the maturity of the bond (m) and the parameters β_0, β_1, β_2, β_3, τ_1 and τ_2. This function is used to define the discount factor $d(m)$:

$$d(m) = \exp\left(-\frac{r(m,\beta,\tau)}{100}m\right) \quad (2)$$

Each bond price can then be approximated by the discounted sum of the coupon payments and final capital:

$$P^e(m) = \sum_{i=1}^{m}\exp\left(-\frac{r(m,\beta,\tau)}{100}i\right) * Coupon + \exp\left(-\frac{r(m,\beta,\tau)}{100}m\right) * 100 \quad (3)$$

The parameters β_0, β_1, β_2, β_3, τ_1 and τ_2 are estimated by minimising the sum of squared bond price errors weighted by ($1/\Phi$):

$$\underset{\beta_0,\beta_1,\beta_2,\tau_1,\tau_2}{Min}\sum_{j=1}^{n}\left\{\left[P_j-P_j^e(\beta_0,\beta_1,\beta_2,\tau_1,\tau_2)\right]/\Phi_j\right\}^2 \quad (4)$$

where Φ equals the duration $*$ price$/(1+$yield to maturity$)$ of the bond.

Application and data

The daily estimation is based on the market price of Treasury certificates and linear bonds: all outstanding Treasury certificates (with a maturity between days and one year) and all linear bonds or OLO's in Belgian francs with a maturity longer than one year (excluding line 239) are included in the sample. This means that some 45 prices are considered, of which 18 bond prices and 27 Treasury

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certificates. This sample is adjusted over time according to the information from specialists in the bond market.

The market prices of the bonds are corrected for the accrued interest calculated as a proportion of the coupon payment. There is no correction for the deviation between the day of trade (t) and the day of settlement ($t + 3$ for bonds and $t + 2$ for the Treasury certificates).

The estimation programme starts by estimating the parameters β_0, β_1, β_2 and τ_1 with fixed $\beta_3 = 0$ and $\tau_2 = 1$. Then the programme checks whether the estimation result improves by adding β_3 and τ_2. If these coefficients are not significant, the simple Nelson-Siegel formula is retained; otherwise, the extended Svensson formula is used.

References

Nelson, C R and A F Siegel (1985): "Parsimonious modeling of yield curves for US Treasury bills", *NBER Working Paper Series*, no 1594.

Svensson, L E O (1994): "Estimating and interpreting forward interest rates: Sweden 1992-4", *NBER Working Paper Series*, no 4871.

Ricart, R and P Sicsic (1995): "Estimating the term structure of interest rates from French data", Bank of France, *Bulletin Digest*, no 22, October.

A technical note on the Merrill Lynch Exponential Spline model as applied to the Canadian term structure

David Bolder, Scott Gusba, and David Stréliski[1]

The purpose of this note is to describe the methodology used by the Bank of Canada to construct the Government of Canada yield curve. We generate zero coupon curves daily, for maturities from 0.25 to 30.00 years, by applying an estimation method based on the Merrill Lynch Exponential Spline (MLES) model to a selection of Government of Canada Treasury bill and bond prices.[2]

1. Data

The two fundamental types of Canadian dollar-denominated marketable securities issued by the government of Canada are Treasury bills and Canada bonds. Treasury bills, which do not pay periodic interest but rather are issued at a discount and mature at their par value, are currently issued at three-, six- and 12-month maturities. Government of Canada bonds pay a fixed semi-annual interest rate and have a fixed maturity date. Issuance involves maturities across the yield curve with original terms of maturity at issuance of two, five, 10 and 30 years.[3] Each issue is reopened several times to improve liquidity and achieve "benchmark status".[4] Canada bonds are currently issued on a quarterly "competitive yield" auction rotation with each maturity typically auctioned once per quarter.[5] In the interests of promoting liquidity, Canada has set targets for the total amount of issuance to achieve "benchmark status"; currently, these targets are CAD 7 billion to 10 billion for each maturity.

2. Data filtering

Our goal is to select only those bonds that are indicative of the current market yields. As a result, we use a system of filters to omit bonds which create distortions in the estimation of the yield curve.

- To avoid potential price distortions when large deviations from par exist, bonds that trade at a premium or a discount of more than 500 basis points from their coupon are excluded.[6]
- Bonds with less than CAD 500 million outstanding are excluded in order to include only those bonds with the requisite degree of liquidity. This amount was chosen in a fairly arbitrary manner to ensure a reasonable number of bonds in the sample.
- Canada benchmark bonds are the most actively traded Canada bonds in the marketplace and it is thus essential that the information contained in these bonds be incorporated into the yield curve.[7]

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2 For a more detailed description of the approach, see Bolder and Gusba (2002), or Bolder, Johnson and Metzler (2004).

3 Canada eliminated three-year bond issues in early 1997; the final three-year issue was 15 January.

4 A "benchmark" bond is analogous to an "on-the-run" US Treasury security in that it is the most actively traded security for a given maturity.

5 Government of Canada bond yields are quoted on an actual/actual day count basis net of accrued interest. The accrued interest, however, is by market convention calculated on an actual/365 day count basis.

6 This value of 500 basis points is intended to reflect a threshold at which the tax effect of a discount or premium is not believed to have an economic impact.

- Additional "discretionary" filtering of bonds is possible. It should be noted, however, that the inclusion or exclusion of a bond is based on judgment and would occur after investigating the underlying reason for a problematic (or unusual) bond quote.

3. The model

The Bank of Canada uses the Merrill Lynch Exponential Spline (MLES) model, developed by Li et al.[8] The MLES model is a parametric model which specifies a functional form for the discount function, $d(t)$, as

$$d(t) = \sum_{k=1}^{9} z_k e^{-k\alpha t} \tag{1}$$

where z_k (k = 1,...,9) and α are the parameters to be estimated.

Once a functional form for the discount function has been specified, a zero coupon interest rate function is derived. The zero coupon curve, z(t), is given by

$$z(t) = -(\ln(d(t))/t \tag{2}$$

4. The estimation

The basic process of determining the optimal parameters for the discount function which best fits the bond data is outlined as follows:

- The sample of Government of Canada bond and Treasury bills is selected and the timing and magnitude of their cashflows are determined.
- The estimation involves 9 linear parameters (the z_k), and one non-linear parameter (α). The optimization normally takes less than one minute to complete. Maximum likelihood estimation is used. As a result of the fact that the discount function is a linear function of 9 of the 10 parameters, the majority of the maximum likelihood computations can be carried out as matrix multiplications, which are computationally efficient.
- Price residuals are calculated using theoretical Government of Canada security prices and the actual price data and inversely weighted by (modified) duration. The calculation of estimated prices is straightforward as the discount function permits us to discount any cashflow occurring throughout the term to maturity spectrum. The weighting on the i-th bond (w_i) is given as follows:

  $$w_i = 1/D_i$$

  where D_i is the (modified) duration of the i-th bond.[9]

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7 As previously discussed, the new issues may require two or more reopenings to attain "benchmark status". As a result, the decision as to whether or not a bond is a benchmark is occasionally a matter of judgment.

8 Li et al (2001)

9 This is consistent with the Bliss approach. For a complete explanation of the justification for weighting price errors, see Bliss (1996).

References

Bliss, R R Jr (1996): "Testing term structure estimation methods", Federal Reserve Bank of Atlanta, *Working Paper*, no 96-12a, November.

Bolder, D, G Johnson and A Metzler (2004): "An Empirical Analysis of the Canadian Term Structure of Zero-Coupon Interest Rates", Bank of Canada Working Paper No 2004-48.

Bolder, D and S Gusba (2002): "Exponentials, Polynomials, and Fourier Series: More Yield Curve Modelling at the Bank of Canada", Bank of Canada Working Paper No 2002-29.

Li, B, E DeWetering, G Lucas, R Brenner and A Shapiro (2001): "Merrill Lynch Exponential Spline Model". Merrill Lynch Working Paper.

Notes on the estimation for the Finnish term structure

Lauri Kajanoja and Antti Ripatti[1]

1. Nelson and Siegel method as applied at the Bank of Finland

The daily term structure of interest rates for Finland is estimated using the methods developed by Nelson and Siegel (1987).[2] Given the parameter vector, β, and maturity, m, the *instantaneous forward rate* is defined as follows:

$$f(m,\beta) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) \tag{1}$$

The corresponding spot rate (zero coupon interest rate) is:

$$s\,(m,\beta) = \beta_0 + (\beta_1 + \beta_2)\,\frac{1-\exp\,(-m/\tau_1)}{m/\tau_1} - \beta_2 \exp\left(-\frac{m}{\tau_1}\right) \tag{2}$$

The parameters β_0 (labelled as BETA0 in the database), β_1 (BETA1), β_2 (BETA2), and τ_1 (TAU1) are estimated using the following assumptions:

- The estimation is based on the minimisation of the yield errors (based on the maximum likelihood method assuming that yield errors follow normal distribution).
- The spot curve is usually but not always forced to pass the overnight rate. When it is, the instantaneous forward rate with zero maturity corresponds to the overnight rate.
- The data consist of the following instruments: Eonia (pre-1999: Finnish overnight rate), one-, three-, six- and 12-month money market (Euribor interbank offered rate (actual/360), %, daily fixing) rates (pre-1999: Helibor), and a variety (four to seven different bonds) of government benchmark bonds (average of primary dealers' bids/offers at 1 pm). The data are from the Bank of Finland database. No tax corrections are made.

When the estimated parameters are used to compute spot or forward rates using the above formulas, the following applies: *time to maturity is expressed in years*; the size of the parameters is as given. The results are expressed as annualised rates.

2. Metadata

BETA0 Nelson-Siegel parameter beta 0; estimate based on the minimisation of the yield errors; original data from O/N up to 12 years of maturity.

BETA1 Nelson-Siegel parameter beta 1; estimate based on the minimisation of the yield errors; original data from O/N up to 12 years of maturity.

BETA2 Nelson-Siegel parameter beta 2; estimate based on the minimisation of the yield errors; original data from O/N up to 12 years of maturity.

TAU1 Nelson-Siegel parameter tau 1; estimate based on the minimisation of the yield errors; original data from O/N up to 12 years of maturity.

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2 C Nelson and A F Siegel, "Parsimonious modeling of yield curves", *Journal of Business*, 60, 1987, pp 473-89.

Estimating the term structure of interest rates from French data

Roland Ricart, Pierre Sicsic and Eric Jondeau[1]

The data used

The data used for estimating zero coupon yield curves cover three categories of government issues:

- French franc-denominated OAT bonds (Obligations Assimilables du Trésor) with maturities at issue ranging between seven and 30 years, which have been the main instrument used for financing the government since the mid-1980s.[2]
- Treasury notes, or BTANs (Bons du Trésor à taux fixe et intérêts ANnuels), with maturities of two to five years, which are used for medium-term financing.
- Treasury bills, or BTFs (Bons du Trésor à taux Fixe et intérêts précomptés), which are issued with maturities up to one year, offering a wide choice of maturities at the time of issue. The prices quoted are those of each Friday.

OATs are issued through a process of assimilation: they are often issued with the same characteristics as existing OATs (ie the same coupon and maturity). At the first coupon date, all the new issues are pooled with the earlier releases.

OATs are no longer issued with maturities of less than a year. With the latter category, the liquidity tended to diminish, which can lead to abrupt price swings. Indeed, market operators make their decision on the basis of yield to maturity, and a slight variation in the latter has a very strong impact on the price of assets with only a short time remaining to maturity. A comparable phenomenon occurred in the case of BTANs, leading the Treasury to stop issuing them with maturities of less than one month. The prices and yield to maturity of BTFs were calculated to make them consistent with data on OATs and BTANs, whose yields are based on a 365-day year.

For all securities, coupons are paid once a year and are subject to taxation. Households are liable to a withholding tax of 18.1% on income. For the business sector, the same rate applies as with taxes on profits (34%). For non-residents, the tax rate depends on the bilateral agreement with the country concerned.

Some notes on the estimations

In selecting data for the estimations, the following rules apply.

Concerning OATs, only the most liquid of the fixed rate and French franc-denominated issues (except strips) are used. For liquidity reasons, the following issues are excluded: OATs with a maturity of less than one year, BTANs of less than one month and BTFs of less than one week. In estimating the zero coupon yield curves, tax effects are not taken into account.

The estimation goes back to January 1992. The prices or yield to maturity quoted are those of each Friday. For OAT data, the prices used correspond to the last price; for BTAN data, the price is the average between the bid and ask prices quoted; for BTF data, the yield is the average between the bid and ask yields quoted.

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[2] French franc-denominated Treasury bonds were the main instrument used for financing the government until 1985.

Two specifications are used for the interpolations: the original Nelson-Siegel function and the augmented function as proposed by Svensson. The parameters of each function are obtained for each observation date by minimising the weighted sum of the square of the errors on the prices of all of the securities, using a non-linear estimation method. The weights are the interest rate sensitivity factors of prices. In fact, this function can be seen as an approximate criterion defined on the yields to maturity. This method, strictly based on the yields to maturity, would make the estimation process longer, because a system of non-linear equations has to be solved for each iteration. Thus, a criterion obtained by taking an approximation of the estimated interest rate on the basis of a first-order Taylor approximation is substituted for this function. The function is then minimised and can be interpreted as a criterion established on the weighted prices. The latter represents the derivative of the price with regard to the yields to maturity or, in other words, the interest rate sensitivity of prices.[3]

A constraint is imposed on the parameters, so that the estimated curve goes through the shortest-term interest rate available at each observation date.

In view of the number of coefficients and the high degree of non-linearity of the function to be optimised, the parameters of the "augmented" Nelson-Siegel relationship are obtained in two stages.[4] This cuts down the estimation time and thus reduces the risk of false convergence. At first, the basic function proposed by Nelson and Siegel is estimated, using as the initial coefficients values that suit all of the possible configurations of the term structure of interest rates.[5] After convergence, the results are used as the initial values for estimating the "augmented" Nelson-Siegel function. The two parameters that are specific to the augmented part of the function, which are not available in the first step, are initialised with 0 and 1 for the extra β and τ respectively. This procedure makes it possible to start the second step with values that can be assumed to be close to the real parameters of the model. After making the estimates, the term structure of interest rates found is checked to see if it justifies the use of the "augmented" relationship rather than the basic Nelson-Siegel relationship.

The selection between the basic and the augmented Nelson-Siegel functions is based on the Fisher test (at the 5% significance level). Confidence intervals based on the data method are also estimated.

The estimated zero coupon yield curves are published in Section 4 of the Bank of France's *Bulletin Digest*.

References

Ricart, Roland and Pierre Sicsic (1995): "Estimating the term structure of interest rates from French data", Bank of France, *Bulletin Digest*, no 22, October.

3 $\underset{\tilde{\alpha}}{Min} \sum_{i=1}^{n} \left\{ \left(P_i(t,m) - P_i(t,m,\tilde{\alpha})\right) / \Phi_i \right\}^2$, where $P(t,m)$ is the market value of a bond, m the time to maturity, n the number of issues, α the parameters vector, and $\Phi_i = \frac{\partial P_i(t,m)}{\partial r_i(t,m)} = c \sum_{i=1}^{[m]+1} \frac{-(m-[m]+i-1)}{(1+r_i(t,m))^{m-[m]+i}} - \frac{100m}{(1+r_i(t,m))^{m+1}}$, where c is the coupon of a bond, expressed as a percentage of its par value.

4 The estimates are made using Gauss software.

5 The initial coefficients, which are constant for all estimate dates, were obtained by testing various possibilities in order to come up with the smallest possible number of non-convergence points over the period 1992-94.

The data for estimating the German term structure of interest rates

Sebastian T Schich[1]

The choice of securities used in constructing the yield curve from the prices of government debt instruments is important since it affects the estimates. A decision has to be taken on the trade-off between "homogeneity" and the availability of sufficient observations in each range of the maturity spectrum. There is no objective criterion available for determining the optimal choice of data. The following paragraphs describe an attempt to find a compromise solution to these problems.[2]

The available set of data comprises end-month observations of the officially quoted prices ("amtlich festgestellte Kassakurse"), remaining maturities and coupons of a total of 523 listed public debt securities from September 1972 to February 1996. They include bonds issued by the Federal Republic of Germany (Anleihen der Bundesrepublik Deutschland), bonds issued by the Federal Republic of Germany - "German Unity" Fund (Anleihen der Bundesrepublik Deutschland - Fonds "Deutsche Einheit"), bonds issued by the Federal Republic of Germany - ERP Special Fund (Anleihen der Bundesrepublik Deutschland - ERP-Sondervermögen), bonds issued by the Treuhand agency (Anleihen der Treuhandanstalt), bonds issued by the German Federal Railways (Anleihen der Deutschen Bundesbahn), bonds issued by the German Federal Post Office (Anleihen der Deutschen Bundespost), five-year special federal bonds (Bundesobligationen), five-year special Treuhand agency bonds (Treuhandobligationen), special bonds issued by the German Federal Post Office (Postobligationen), Treasury notes issued by the German Federal Railways (Schatzanweisungen der Deutschen Bundesbahn), Treasury notes issued by the German Federal Post Office (Schatzanweisungen der Deutschen Bundespost), and Federal Treasury notes (Schatzanweisungen des Bundes).[3]

The vast bulk of available securities have a fixed maturity and an annual coupon. There are a few bonds with semiannual coupons and special terms, such as debtor right of notice and sinking funds. The differing coupon payment frequencies (annual, semiannual) are taken into account in the calculation of yields. Bonds with semiannual coupon payments were issued until the end of December 1970; they matured not later than December 1980. The debtor right of notice gives the issuer the right to redeem (or call) loans prematurely after expiry of a fixed (minimum) maturity; therefore these bonds are referred to as callable bonds. Such bonds were issued until September 1973 and were traded until November 1988. Bonds with a sinking fund may be redeemed prematurely and in part after a fixed (minimum) maturity. They were issued until December 1972 and traded until December 1984.

In order to obtain a more homogeneous set of data, bonds with special terms and those issued by the German Federal Railways and the German Federal Post Office were eliminated from the original set.[4] The yields of these debt securities are characterised by additional premia compared to debt securities on standard terms issued by the Federal Republic of Germany. For example, the price of a bond with a debtor right of notice can be interpreted as the price of a standard bond minus the price of a call option on that bond. Since this call option has a positive value as long as the volatility of interest rates is positive, the price of the bond with the debtor right of notice is lower and its yield higher than that of a standard bond. As for bonds issued by the German Federal Railways and the German Federal Post

[1] Deutsche Bundesbank, currently on sabbatical in the Money and Finance Division of the OECD's Economics Department.

[2] No data are available for May 1982. The May 1982 term structure estimates are proxied by the arithmetic average of the estimates for April and June 1982.

[3] For information on individual securities issued after 1984, see Deutsche Bundesbank (1995), pp 81-8.

[4] Another possible source of variation among the yields is the fact that, since the introduction of the Deutsche Terminbörse (DTB), it has been possible to deliver some bonds and Special Federal Notes under bund futures contracts. In particular, bonds with a remaining time to maturity of between eight and a half and 10 years and Special Federal Notes with a remaining time to maturity of between three and a half and five years are candidates for delivery under futures contracts. However, a close inspection of the data did not reveal significant differences due to these characteristics.

Office, they have a rating disadvantage compared to bonds issued by the Federal Republic of Germany because the perceived default risk is marginally higher.[5] In practice, the bonds of the former carry a spread with respect to the bonds of the latter, and this spread varies over time.

The final data set comprises (standard) bonds issued by the Federal Republic of Germany (170 issues), five-year special federal bonds (116 issues), and Federal Treasury notes (17 issues), making altogether 303 issues for the period September 1972-February 1996. A list of the individual securities, as of end-December 1996, is contained in the appendix to Schich (1997).[6] The debt securities available for each month vary considerably over time, especially until the mid-1980s. For example, only a few observations are available at the beginning of the 1970s, the smallest set being September 1972 with just 15 observations. The number of debt securities available grows sharply during the 1970s, increasing (almost) monotonically to more than 80 observations in 1983. During the rest of our sample period, the number of observations available varies between 80 and almost 100.

The observations are in general spaced equally over the maturity range from zero to 10 years. Nevertheless, there are a few gaps in the maturity spectrum at the beginning of the 1970s. Although there are no bonds with a short original time to maturity, the short end of the yield curve is generally well represented by medium- and long-term issues with small residual maturities.

This leads on to the question of the maturity spectrum used. We adopt the Bank of England approach and consider all bonds with a remaining time to maturity above three months. The yields of bonds with residual maturities below three months are excluded because they appear to be significantly influenced by their low liquidity and may therefore not be very reliable indicators of market expectations. Bonds with maturities between three months and one year appear to be more liquid. Including these bonds is at variance with the Bundesbank's former practice of excluding bonds with a residual time to maturity below one year. Although this exclusion would improve the overall fit in terms of the deviations between observed and estimated yields, we do not adopt that strategy here because it implies very imprecise estimates for the one-year yields. Since observations of exactly one year and slightly higher than one year are regularly missing, the estimate of the one-year rate essentially becomes an out-of-sample forecast. This forecast turned out to be often not very reliable. For example, the parametric approach adopted here could produce a "spoon effect", whereby the curve flips up at the short end when observations are sparse, thus resulting in unrealistically high estimates for the one-year rate. As the one-year rate is of special concern to policymakers and is also one of the frequently cited interest rates in reports on the capital market, these properties are particularly undesirable. Thus, bonds with a remaining time to maturity of between three months and one year are included.

Another issue is whether or not the three bonds at the very long end of the maturity spectrum should be included. There is a case for leaving them out because not all of them appear to be very actively traded. However, when the curve is very steep, the observations at the long end help to tie down the 10-year estimates. We follow the practice employed in the past at the Bundesbank and include the long-term bonds as well.

Reducing the sample to the 303 issues improved the fit of the estimates, measured as the deviation between observed and estimated yields. The extent of improvement varied over time and amounted to just 1 basis point on average. It should be noted that the reduction of the sample also rendered convergence of the estimates more difficult. Nevertheless, convergence was achieved in all periods. Thus, the sample of 303 issues seems to offer a good compromise between homogeneity and efficiency in estimation.

5 This is supported by simple statistical tests. Regressing separately for various dates the yields of the final set of securities, on the one hand, and of the omitted securities, on the other, on the coupons and maturities, the null hypothesis of equality of the estimated coefficients (Wald test) can be rejected, with the coefficients obtained from the omitted securities being generally higher.

6 Clearly, in estimating its (spot) yield curve, the Deutsche Bundesbank continuously updates the list of securities.

References

Deutsche Bundesbank (1995): *The market of German federal securities*, Frankfurt am Main, July.

Schich, Sebastian T (1996): "Alternative specifications of the German term structure and its information content regarding inflation", Deutsche Bundesbank, Economic Research Group, *Discussion paper*, no 8, October.

——— (1997): "Estimating the German term structure", Deutsche Bundesbank, Economic Research Group, *Discussion paper*, no 4, October.

Technical note on the estimation of forward and zero coupon yield curves as applied to Italian euromarket rates

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1. Estimation of the nominal yield curve: data and methodology

The nominal yield curve is estimated from Libor and swap rates, with maturity dates of one to 12 months for Libor rates and two to 10 years for swap yields, downloaded daily from Reuters. Rates are quotes in the London market provided by the British Bankers' Association and Intercapital Brokers respectively.[1] The underlying assumption is that the price (par value) of these securities equals the present values of their future cash flows (ie coupon payments and final redemption payment at maturity).

At the Bank of Italy, we have a fairly long tradition of estimating zero coupon rate yield curves and have experimented with several methodologies and models. In the middle of the 1980s, we started zero coupon yield curve estimation by using the CIR (1985) one-factor model for the short rate, estimated on a cross section of government bond prices (Barone and Cesari (1986)[2]); before that, a cubic splines interpolation was in place as a routine device to gauge the term structure of interest rates. The CIR model application was later updated (Barone et al (1989)) and then the CIR model extended to a two-factor model for the short rate (Majnoni (1993)), along the lines of Longstaff and Schwartz (1992). Drudi and Violi (1997) have tried to efficiently combine cross-section and time series information in estimating parameters for a two-factor model of the term structure, in which a stochastic central tendency rate is introduced as a second factor determining the shape of the yield curve.

More recently, we have been considering the Nelson-Siegel approach, as a viable alternative to the general equilibrium model-based yield curve estimation, because of its relatively low implementation and running cost in building a forward yield curve on a daily basis.

2. Functional specification of the discount function: Nelson-Siegel vs Svensson approach

Forward rates and yield to maturity are estimated using the methodology suggested in Nelson and Siegel (1987), subsequently extended in Svensson (1994). The modelling strategy is based on the following functional form for the discount function:

$d(\tau) = \exp\,(-y(\tau)\tau)$

with

$$y(\tau) \equiv \beta_0 + \beta_1 \left[\frac{1-\exp\left(-\frac{\tau}{\tau_1}\right)}{\left(\frac{\tau}{\tau_1}\right)} \right] + \beta_2 \left[\frac{1-\exp\left(-\frac{\tau}{\tau_1}\right)}{\left(\frac{\tau}{\tau_1}\right)} - \exp\left(-\frac{\tau}{\tau_1}\right) \right] + \beta_3 \left[\frac{1-\exp\left(-\frac{\tau}{\tau_2}\right)}{\left(\frac{\tau}{\tau_1}\right)} - \exp\left(-\frac{\tau}{\tau_2}\right) \right] \tag{1}$$

where τ represents time to maturity, $y(\tau)$ the yield to maturity and vectors (β_0, β_1, β_2, β_3, τ_1, τ_2) the parameters to be estimated, with (β_0, τ_1, τ_2) > 0.

1 Reuters RIC pages: FRBD/H and ICAQ/T respectively.

2 See also Barone et al (1991).

The spot yield function, $y(\tau)$, and forward rate function, $f(\tau)$, are related by the equation:

$$y(\tau) = \int_0^{\tau} \frac{f(s)}{\tau} ds \tag{2}$$

Replacing (1) into (2) and differentiating, one obtains the closed-form expression for the forward yield curve:

$$f(\tau) = \beta_0 + \left(\beta_1 + \beta_2 \frac{\tau}{\tau_1}\right)\left[\exp\left(-\frac{\tau}{\tau_1}\right)\right] + \beta_3 \frac{\tau}{\tau_2}\left[\exp\left(-\frac{\tau}{\tau_2}\right)\right] \tag{3}$$

where β_0 represents the (instantaneous) asymptotic rate and $(\beta_0 + \beta_1)$ the instantaneous spot rate. Restricting β_3 equal to zero in (3), one obtains the Nelson-Siegel (1987) forward rate function. This function is consistent with a forward rate process fulfilling a second-order differential equation with two identical roots. Such a restriction limits to only one local minimum (or maximum) the maturity profile, according to the sign of β_2. When β_3 differs from zero, eg Svensson extension, more than one local maximum or minimum is allowed, hence increasing flexibility in fitting the yield curves.

Estimation requires prior specification of a price, P_i, for the i-th security, obtained by discounting the cash flow profile, $\{C^j\}_i$, for a given time to maturity, $\{\tau^i\}$. This is carried out on a daily sample of n securities whose price is modelled as the sum of their discounted cash flows:

$$P_i(b) = \sum_{j=1}^{k_i} C_i^j d(\tau_i^j; b)$$

$$b \equiv (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2) \tag{4}$$

$$\forall i = 1, \ldots, n$$

where k_i stands for the time to maturity for the i-th security.

The econometric implementation leads to the introduction of a pricing error process, ε_i:

$$P_i^* = P_i(b) + \varepsilon_i$$

$$\forall i = 1, \ldots, n \tag{5}$$

where P^* indicates the market price of the security and ε_i is assumed to be a white noise process. The objective function minimises the squared deviation between the actual and the theoretical price, weighted by a value related to the inverse of its duration, Φ_i:

$$\underset{b}{MIN} \sum_{i=1}^{n} \varepsilon_i^2 \Phi_i^*$$

$$\Phi_i^* \equiv 1/\Phi_i \Big/ \sum_{i=1}^{n} 1/\Phi_i \tag{6}$$

$$\Phi_i \equiv \frac{\partial P_i^*}{\partial y} \frac{1+y}{P_i^*}$$

Criterion (6) is implemented by means of a non-linear least squares algorithm (TSP command LSQ) to derive the parameters' estimates. The Nelson-Siegel parsimonious parametrisation has been preferred to Svensson's extended version for practical reasons. Often, the Svensson extension seems to be less robust at the shortest end of the yield curve. In our experience, the Svensson approach offers little, if any, practical advantage in improving the precision of the estimates, in the terms of both pricing errors and information criteria (for instance, Akaike or Schwarz-Bayes). With the Nelson-Siegel specification, simulated yield curves normally show average pricing errors of some 4-5 basis points, equivalent to 1-2 basis points in terms of yield to maturity. Parameter significance tests, with the covariance matrix corrected for heteroskedasticity, are almost always passed. In comparing daily pricing errors over time across maturities, we have found some evidence of autocorrelated residuals, pointing to regression residuals which are not always "white". In addition, larger, duration-adjusted, pricing errors often seem to show up more often at the shorter end of the curve; the Svensson extension does not provide a remedy for these latter shortcomings.

References

Barone, E and R Cesari (1986): "Rischio e rendimento dei titoli a tasso fisso e a tasso variabile in un modello stocastico univariato", Bank of Italy, *Temi di Discussione*, no 73.

Barone, E, D Cuoco and E Zautzik (1989): "La struttura dei rendimenti per scadenza secondo il modello di Cox, Ingersoll e Ross: una verifica empirica", Bank of Italy, *Temi di Discussione*, no 128.

——— (1991): "Term structure estimation using the Cox, Ingersoll and Ross model: the case of Italian Treasury bonds", *Journal of Fixed Income*, 1, pp 87-95.

Drudi, F M and R Violi (1997): "The term structure of interest rates, expectations and risk premia: evidence for the eurolira", Bank of Italy, *Temi di Discussione*, no 311.

Longstaff, F A and E Schwartz (1992): "Interest rate volatility and the term structure: a two factor general equilibrium model", *Journal of Finance*, 47, pp 1259-82.

Majnoni, G (1993): *An empirical evaluation of one vs two factor model of the term structure of interest rates: the Longstaff and Schwartz and the CIR model*, Bank of Italy, Research Department, mimeo.

Nelson, C R and A F Siegel (1987): "Parsimonious modeling of yield curves", *Journal of Business*, 60, pp 473-89.

Svensson, L E O (1994): "Estimating and interpreting forward interest rates: Sweden 1992-4", International Monetary Fund, *IMF Working Paper*, 1994/114, Washington DC.

A technical note on the estimation of the zero coupon yield and forward rate curves of Japanese government securities

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This note covers data description and estimation techniques for estimating zero coupon yield curves and implied forward rate curves using Japanese government securities. The Bank of Japan estimates these curves based on the method developed by Fisher et al (1995).

1. Data description

To estimate the yield curves of risk-free fixed income assets, the following four types of Japanese government securities are used: 10-year and 20-year government bonds (hereinafter 10-yr JGBs and 20-yr JGBs, respectively) and three- and six-month Treasury bills (hereinafter 3m TBs and 6m TBs, respectively). The first two are fixed income bonds with semiannual coupon payments while the latter two are discount securities. In each case, the data required for estimation are: ID number, quote date, redemption date, coupon rate (zero for TBs), and price.

Table 1

Selected features of each security

	10-yr JGBs	**20-yr JGBs**	**3m TBs**	**6m TBs**
Frequency of issuance	Monthly	Quarterly	Monthly	Monthly
Number[1] of issues outstanding in secondary markets	Around 85	Around 35	3	6
Price data for estimation purposes	Small-lot trades,[2] price as of 3 pm on TSE[3]		Inter-dealer price as of 5 pm	

[1] Varies over time. [2] Orders of more than JPY 1 million but less than JPY 10 million, par value. [3] 3 pm is the closing time of the Tokyo Stock Exchange (TSE).

Input data set details:

1. The data set includes all JGBs listed on the TSE and TBs outstanding in the inter-dealer market with one exception: due to the existence of a "redemption fee",[1] the data for 10-yr JGBs with a remaining maturity of less than half a year are excluded.[2]

2. TB prices are adjusted for withholding tax levied at issuance and repaid at redemption according to the following formula:

   TB price for input data = quoted price ∗ 100 / (100 + withholding tax),
   where withholding tax = (100 – average issuance price) ∗ 0.18 and 0.18 is the withholding tax rate.

[1] The redemption fee (currently JPY 0.09) is gradually incorporated into the JGB price as remaining maturity becomes shorter, especially when it becomes less than half a year.

[2] Since the shortest remaining maturity of 20-yr JGBs is currently around nine years, the "redemption fee" problem applies only to 10-yr JGBs - at least for now.

2. Estimation techniques

Interpolating forward rate curves by smoothing splines

Following Fisher et al (1995), zero coupon yield and forward rate curves are extracted by smoothing the spline with the roughness penalty selected according to generalised cross validation (GCV).

First, the cash flow of each security i is decomposed into the scheduled coupon and principal payments $(c_{1i}, c_{2i}, ..., c_{mi})$ and the number of days to each payment $(t_{1i}, t_{2i}, ..., t_{mi})$ is calculated. m is the number of remaining coupon payments until maturity.

Let $\delta(t_{ji})$ be the discount factor at time t_{ji}. Also let $\tilde{\delta}(t_i)$ and c_i be the column vector of discount factors and payments for security i, respectively. Then, the price of security i can be written as $p_i = c_i' \tilde{\delta}(t_i)$.[3] This decomposition is effected for all securities in the data set $(i = 1, 2, ..., n)$, and P is defined as $(p_1, p_2, ..., p_n)'$.

Instantaneous forward rate curves are expressed by linear combination of cubic B-splines as below:

$$f(t) \equiv (\phi_1(t), \phi_2(t), ..., \phi_\kappa(t))(\beta_1, \beta_2, ..., \beta_\kappa)' \equiv \phi(t)\beta \qquad (1)$$

where $(\phi_1(t), \phi_2(t), ..., \phi_\kappa(t))$ is a cubic B-spline basis, $(\beta_1, \beta_2, ..., \beta_\kappa)'$ is a column vector of coefficients, and κ is the number of knot points plus 2.

By definition, the discount factor can be written with the above forward rates as:

$$\delta_s(t, \beta) = \exp\left(-\int_0^T \phi(s)\beta ds\right) \qquad (2)$$

where T is the largest t_{ji} for all j's and i's.

Let $\Pi(\beta)$ be the vector of prices of securities based on the above interpolating forward rates such that:

$$\Pi(\beta) \equiv (\pi_1(\beta), \pi_2(\beta), ..., \pi_m(\beta)) \qquad (3)$$

where $\pi_i(\beta) \equiv c_i' \tilde{\delta}_s(t_i, \beta)$ and $\tilde{\delta}_s(t_i, \beta) \equiv \left(\delta_s(t_{1i}, \beta), \delta_s(t_{2i}, \beta), ..., \delta_s(t_{mi}, \beta)\right)'$

The smoothing spline minimises the following problem for a given λ (stated below) with respect to β:

$$\min_\beta \left[(P - \Pi(\beta))'(P - \Pi(\beta)) + \lambda \int_0^T f''(t)^2 dt \right] \qquad (4)$$

The first term of this expression is the sum of the residuals of squares and the second term defines the roughness penalty. λ, a constant, is a weighting parameter of the roughness penalty. The bigger λ becomes, the smoother the estimated forward rate curves look. In the smoothing splines, the number of effective knots is determined automatically to secure a certain degree of smoothness and goodness of fit; at the same time, the minimiser β (denoted as $\beta^*(\lambda)$) is derived once the value of λ is set.

In order to set λ, we have to refer to the shape of the yield curve and the size of residual terms, which is inevitably a subjective operation. GCV works to choose λ in a more objective way. Once the "tuning parameter" (θ) is set by discretion, GCV selects λ under a constant criterion for each estimation time.

We choose the value of λ as the minimiser of the GCV value (γ):

$$\min_\lambda \gamma(\lambda) = \frac{\left(P - \Pi(\beta^*(\lambda))\right)' \left(P - \Pi(\beta^*(\lambda))\right)}{\left(n - \theta tr(A(\lambda))\right)^2} \qquad (5)$$

where $A(\lambda) \equiv X(\beta^*(\lambda)) \left(X(\beta^*(\lambda))' X(\beta^*(\lambda)) + \lambda H\right)^{-1} X(\beta^*(\lambda))'$

(= measure of the effective number of parameters)

[3] (′) denotes the transpose.

$$X(\beta^*(\lambda)) \equiv \left.\frac{\partial \Pi(\beta)}{\partial \beta'}\right|_{\beta=\beta^*(\lambda)} \quad \text{and} \quad H \equiv \int_0^T \phi''(t)'\phi''(t)dt$$

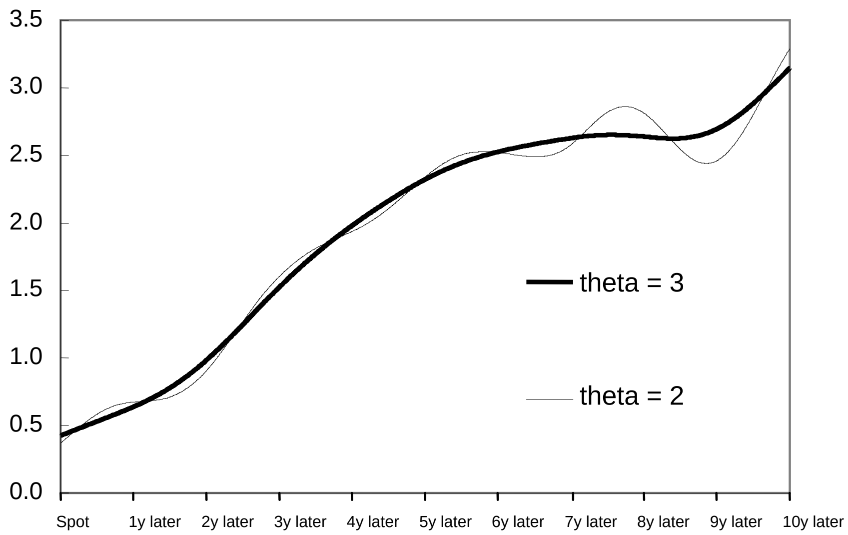
As θ gets bigger, the forward rate curves appear smoother at the expense of goodness of fit. In the original paper, θ is set to be 2 for US data. However, we found it is not always enough for the data set we examined.

Graph 1 shows instantaneous forward rate curves as of 24 December 1997 estimated using the program written by Fisher and Zervos (1996).[4] With $\theta = 2$ the estimated forward rate curve (solid curve) seems too rough, especially between five and 10 years, while with $\theta = 3$ it looks more reasonable (thick curve).[5] After examining samples between April 1997 and January 1998, we decided to set θ (at least provisionally) to be 3.

Graph 1

Estimated instantaneous forward rates with different tuning parameters

As of 24 December 1997; in percentages



Market conventions with respect to Japanese government securities

Some market conventions peculiar to Japan make the estimation procedure and the interpretation of output complicated. We adjusted the original program (see subsections 1 and 2 below).

1. Initial and final coupon payments

Coupon payments are semiannual for JGBs. The actual number of days to the next coupon payment varies, for the two reasons stated below. However, the amount of each coupon payment is fixed at half the coupon rate regardless of the actual number of days, except for initial and final payments.

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4 The estimation procedure described in this paper is implemented using this program with some adjustments (described in the next section). It runs on *Mathematica* version 2.2.

5 In either case, we obtain a humped region between five and 10 years, which implies that JGB prices with a remaining maturity of seven to 10 years are somewhat overvalued. This phenomenon might be attributable to the fact that these JGBs are eligible for settlement as JGB futures.

- Coupon payment and redemption dates for JGBs are basically the 20th of the month, becoming the nearest following business day if the 20th is not a business day.[6] Only the final payment takes account of such a shift in the redemption date.
- 10-yr JGBs are currently issued every month but redemption months are grouped by the rule[7] shown in Table 2. The month of initial coupon payment is determined accordingly. Thus, the number of months from issuance to initial coupon payment may be seven or eight, rather than six, which means an additional payment to the initial payment. In practice, the initial coupon payment is calculated based on the number of days between the issuance day and the 20th of the initial coupon payment month (including the issuance day).

Table 2

Issuance month, redemption month and initial coupon payment month for 10-yr JGBs

Issuance month	Redemption month	Month for initial coupon payment
Apr, May, Jun	Jun	Dec
Jul, Aug, Sep	Sep	Mar
Oct, Nov, Dec	Dec	Jun
Jan, Feb, Mar	Mar	Sep

Note: This rule has been in effect since 1987.

Similarly, there is a rule, as shown in Table 3, regarding 20-yr JGBs; initial and final coupon payments are treated accordingly.

Table 3

Issuance month, redemption month and initial coupon payment month for 20-yr JGBs

Issuance month	Redemption month	Month for initial coupon payment
Apr	Sep	Sep
Jul	Sep	Mar
Oct	Mar	Mar
Jan	Mar	Sep

Note: This rule has been in effect since 1996.

6 The date of issuance varies irregularly.

7 This rule is not official, but de facto.

2. Accrued interest

The price data do not include accrued interest, which is calculated using the following formula[8] and added to the price.

Accrued interest = N * coupon rate / 365, where N is the number of days from the last coupon payment to settlement.

3. Business days from the quote day to the settlement day

For both JGBs and TBs, the number of business days from the quote day to the settlement day is now three. Our estimation procedure treats the settlement day as if it were the quote day. Thus, for example, estimated zero coupon rates are, strictly speaking, forward rates whose delivery day is three business days after the quote day. However, since this interval (three business days) is quite short and an appropriate short-term risk-free rate does not exist in the Japanese market, we decided to report the estimated rates as of the quote day without any adjustment.

References

Fisher, M, D Nychka and D Zervos (1995): "Fitting the term structure of interest rates with smoothing splines", Board of Governors of the Federal Reserve System, *Working Paper*, no 95-1.

Fisher, M and D Zervos (1996): "Yield curves", in *Computational economics and finance*, H R Varian (ed), Springer-Verlag New York, Inc, Chapter 12.

Muranaga, J (1997): *Generating yield curves of Japanese government bonds using spline functions* (in Japanese), Institute for Monetary and Economic Studies, Bank of Japan, mimeo.

[8] Because of the special treatment for initial payment, N in the formula is replaced with the number of days from issuance to settlement (including the issuance day) for the first payment.

Estimation of spot and forward rates from daily observations

Øyvind Eitrheim[1]

Introduction

Spot and forward interest rates are calculated from daily observations of the yield to maturity on Norwegian government bonds and their coupon payments for bonds with maturities in the range of two to 10 years, and four money market rates on one-, three-, six- and 12-month holdings respectively.

We use money market rates instead of Treasury bill rates since the secondary market for the latter is much less liquid in Norway.

Details of the estimation procedure

We consider two variants of parametric forward interest rate functions $f(m,\beta)$ proposed by Nelson and Siegel (1987) and Svensson (1994), where m denotes the remaining maturity and β the parameter vector to be estimated. The corresponding spot interest rate function can be written as the average of the instantaneous forward rates with settlement between 0 and m:

$$s(m,\beta) = \frac{1}{m}\int_{\tau=0}^{m} f(\tau,\beta)\,d\tau \tag{1)}$$

For a given trading date, let there be n bonds (c_j, m_j, y_j, p_j), $j = 1, ...,n$ represented by their coupons c_j, remaining maturity m_j and observed yield to maturity y_j. p_j denotes the observed price of a given bond. For bonds with annual coupon payments, we index the coupon payments by the sequence τ_{jk}, $k = 1, ..., K_j$, where K_j denotes the number of coupon payments for bond j. Allowing remaining maturity m_j to be non-integer, we define:

$$\tau_{jk} = m_j - [m_j] + k - 1 \tag{2}$$

$$K_j = [m_j] + 1 \tag{3}$$

where $[m_j]$ denotes the highest integer lower than m_j. The estimated price of a coupon bond $P_j(\beta)$ can be written as the sum of prices of a sequence of zero coupon (discount) bonds related to each coupon payment and the face value of the bond (normalised to 1), each priced with the discount function:

$$d(m_j,\beta) = \exp\left[-\frac{s(m_j,\beta)}{100} m_j\right] \tag{4}$$

Hence :

$$P_j(\beta) = \sum_{k=1}^{K_j} c_j d(\tau_{jk},\beta) + d(\tau_{jK},\beta), \quad j = 1, ...,n \tag{5}$$

We note that we can characterise each bond either by the observed triplet (c_j, m_j, p_j) or by the triplet (c_j, m_j, y_j) replacing the price p_j of the bond with the bond's yield to maturity y_j. From the coupons c_j, $j = 1, ..., n$ and the indexed sequence of payments $\tau_{jk}(m_j)$, k = 1, ..., K_j, we can then use the present value function and estimate a corresponding price P_j of bond j:

[1] Central Bank of Norway, Research Department.

$$P_j\left(c_j, m_j, y_j\right) = \sum_{k=1}^{K_j} \frac{c_j}{\left(1+\frac{y_j}{100}\right)^{\tau_{jk}(m_j)}} + \frac{1}{\left(1+\frac{y_j}{100}\right)^{\tau_j K_j(m_j)}}, \quad j = 1, \ldots, n \tag{6}$$

Alternatively, when we know the observed price p_j on bond j, we can estimate the yield to maturity (Y_j) by solving for Y_j in:

$$p_j = \sum_{k=1}^{k_j} \frac{c_j}{\left(1+\frac{Y_j}{100}\right)^{\tau_{jk}(m_j)}} + \frac{1}{\left(1+\frac{Y_j}{100}\right)^{\tau_j K_j(m_j)}}, \quad j = 1, \ldots, n \tag{7}$$

using a standard Newton-Raphson algorithm.

Likewise, this relationship between Y_j and P_j can be used in the parametric case when we derive the discount function from the forward interest rate function $f(m,\beta)$, hence the estimated yield to maturity for bond j denoted $Y_j(\beta)$ can then be computed from the present value function:

$$P_j(\beta) = \sum_{k=1}^{K_j} \frac{c_j}{\left(1+\frac{Y_j(\beta)}{100}\right)^{\tau_{jk}}} + \frac{1}{\left(1+\frac{Y_j(\beta)}{100}\right)^{\tau_j K_j}}, \quad j = 1, \ldots, n \tag{8}$$

also using a standard Newton-Raphson algorithm.

The observed yield to maturity y_j is assumed to differ from the estimated yield to maturity $Y_j(\beta)$ by a normally distributed error term $\varepsilon_j \sim \mathrm{Niid}(0,\sigma_\varepsilon), \forall j$:

$$y_j = Y_j(\beta) + \varepsilon_j, \quad j = 1, \ldots, n \tag{9}$$

We use the method proposed in Svensson (1994) and estimate the following forward rate function, with parameters $\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$. This relationship is also denoted as the extended Nelson-Siegel forward rate function:

$$f\left(m,\beta\right) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(-\frac{m}{\tau_2}\right) \tag{10}$$

It can be shown that the corresponding spot interest rate function can be expressed as:

$$s\left(m,\beta\right) = \beta_0 + \beta_1 \frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} + \beta_2 \left[\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} - \exp\left(-\frac{m}{\tau_1}\right)\right] + \beta_3 \left[\frac{1-\exp\left(-\frac{m}{\tau_2}\right)}{\frac{m}{\tau_2}} - \exp\left(-\frac{m}{\tau_2}\right)\right] \tag{11}$$

The parameters in the forward rate function β are estimated by solving the following maximum likelihood estimation problem:

$$\hat{\beta} : \underset{\beta}{Max} \left[-\frac{n}{2} \ln\left(2\pi\sigma_\varepsilon\right) - \frac{1}{2} \sum_{j=1}^{n} \left(\frac{y_j - Y_j(\beta)}{\sigma_\varepsilon}\right)^2\right] \tag{12}$$

inserting the following MLE : $\hat{\sigma}_\varepsilon = \sqrt{\frac{1}{n} \sum_{j=1}^{n} (y_j - Y_j(\beta))^2}$ for σ_ε. An alternative to minimising the sum of squared yield errors would be to minimise the sum of squared price errors $\sum_{j=1}^{n} \left(p_j - P_j(\beta)\right)^2$ instead.

As pointed out by Svensson (1994), however, minimising price errors sometimes results in fairly large yield errors for bonds and bills with short maturities while minimising yield errors gives a substantially better fit for short maturities, and the two procedures seem to perform equally well for long maturities. This is because prices are very insensitive to yields for short maturities.

On the other hand, minimising yield errors entails, as we have seen, an extra Newton-Raphson iteration where we solve for the yield Y_j in the price function. This could potentially cause some convergence problems to occur at certain data points.

The continually compounded spot and forward interest rates which are derived from the equations above for a given $\hat{\beta}$ are finally transformed into annually compounded interest rates, ie:

$$
\begin{aligned}
s_a(m,\hat{\beta}) &= 100\left(\exp\left[\frac{s(m,\hat{\beta})}{100}\right]-1\right) \\
f_a(m,\hat{\beta}) &= 100\left(\exp\left[\frac{f(m,\hat{\beta})}{100}\right]-1\right)
\end{aligned}
\qquad (13)
$$

References

Nelson, C R and A F Siegel (1987): "Parsimonious modeling of yield curves", *Journal of Business*, 60, pp 473-89.

Svensson, L E O (1994): "Estimating and interpreting forward interest rates: Sweden 1992-4", International Monetary Fund, *IMF Working Paper*, 1994/114.

Notes on the estimation procedure for the Spanish term structure

Soledad Núñez[1]

Data

At present, about 34 securities are used in the estimation. These securities are distributed as follows:

- 22-26 coupon bonds with a residual maturity between one and 30 years,
- four Treasury bills with a residual maturity of three, six, nine and 12 months,
- four bond repos with a maturity of one, seven, 15 and 30 days.

Bond repos (called *operaciones simultáneas* in Spain) are used for the very short-run maturity observations because the Treasury bill secondary market is rather illiquid for maturities less than three months, while the repo market is very active for such maturities. These repos are "American-style", that is, the buyer (the party who acquires the bond) is considered the owner of it so that he/she is free to sell it or make a reverse repo with the corresponding bond. Thus repos may be considered as Treasury bill equivalents (assuming no credit risk). The observed repo rate for maturity $m, r(m)$ is converted into a price as follows:

$$P = 100\,(1 - r(m)\,m / 360)$$

Estimation methods

The estimation goes back to January 1991. However, the estimation method used for the period January 1991-December 1994 is different from the one used since January 1995.

For the period January 1991-December 1994, the estimation method used is from Nelson and Siegel (1987), that is, the following equation is assumed for the instantaneous forward rate:

$$f_0(h) = \beta_0 + \beta_1 \exp\left(-\frac{h}{\tau_1}\right) + \beta_2 \frac{h}{\tau_1} \exp\left(-\frac{h}{\tau_1}\right) \tag{1}$$

where h is the horizon and $b = (\beta_0, \beta_1, \beta_2, \tau_1)$ is the parameter vector to be estimated. β_0 represents the instantaneous asymptotic rate and $(\beta_0 + \beta_1)$ the instantaneous spot rate. This equation is consistent with a forward rate process fulfilling a second-order differential equation with two identical roots. This functional form allows for only one local maximum or minimum along the maturity profile, according to the sign of β_2.

The spot rate function for maturity $m, r(m)$ and the forward rate function, $f_0(m)$, are related by:

$$r(m) = \int_0^m \frac{f(s)}{m}\, ds \tag{2}$$

Hence:

$$r(m) = \beta_0 + (\beta_0 + \beta_1)\, \frac{\tau_1}{m} \left(1 - \exp\left(-\frac{m}{\tau_1}\right)\right) - \beta_2 \exp\left(-\frac{m}{\tau_1}\right) \tag{3}$$

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[1] Bank of Spain, Research Department.

Replacing $r(m)$ in the discount function, $d(m)$:

$$d(m) = \exp(-mr(m)) \tag{4}$$

one obtains $d(m)$ as a function of $b = (\beta_0, \beta_1, \beta_2, \tau_1)$.

Therefore the price of the i-th bond is expressed as:

$$P_t^i(b) + cc_t^i = C^i \sum_{j=1}^{j=v^i} d_t\left(m_j^i; b\right) + 100 d_t\left(m_{v^i}; b\right) \tag{5}$$

$i = 1,....,n$

where t is the day of observation, n is the number of observations, v^i is the number of payments for the i-th bond, and cc_t^i is the accrued interest, calculated as:

$$cc = coupon\ \frac{actual\ date - last\ payment\ date}{next\ payment\ date - last\ payment\ date}$$

C^i = coupon

$b = (\beta_0, \beta_1, \beta_2, \tau_1)$ is the parameter vector to be estimated.

The estimation is made by using a non-linear least squares procedure. More precisely, the algorithm used is from Marquardt and runs with SAS.

Once the parameter vector, $b = (\beta_0, \beta_1, \beta_2, \tau_1)$, is estimated, instantaneous forward rates are obtained from equation (1), and zero coupon rates from equation (3). The estimation criterion is to minimise the squared price error. Observed prices used in the estimation correspond to the median of prices traded during the day of observation. Tax effects are not taken into account in the estimation.

Since January 1995, the estimation method used has been from Svensson (1994). Svensson's method is identical to Nelson and Siegel's, but adding to the instantaneous forward rate function the term $\beta_3(h/\tau_2)\exp(-h/\tau_2)$, where now the parameter vector to be estimated is $b = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$. Here, β_0 also represents the instantaneous asymptotic rate and $(\beta_0 + \beta_1)$ the instantaneous spot rate. This functional form allows for more than one local maximum or minimum along the maturity profile. Thus, Svensson's method is more flexible than Nelson and Siegel's.

Again, the estimation is made by non-linear least squares using the Marquardt algorithm. The estimation criterion is to minimise the squared price error weighted by the inverse of duration, that is:

$$\underset{b}{Min} \sum_{i=1}^{n} \left\{ \left(P_t^i - P_t^i\left(\hat{b}\right)\right) \left(\frac{1}{D_t^i}\right) \right\}^2 \tag{6}$$

where: $D_t^i = \dfrac{\partial P_t^i}{\partial y_t^i} \dfrac{(1 + y_t^i)}{P_t^i}$

y_t^i = yield to maturity

P_t^i = observed price

$P_t^i\left(\hat{b}\right)$ = estimated price

The prices used in the estimation correspond to the average between the bid and ask prices quoted at 4 pm. These prices correspond to a settlement date of $t + j$ days, where t is the trading date.[2] For the estimation, prices are valued at t and obtained from the following expression:

$$P_{t+j}^i + cc_{t+j} = \left(P_t^i + cc_t\right) \left(1 + r_t(j)\ \frac{j}{360}\right) - C^i \left(1 + r_t\left(t + j - t_1\right)\ \frac{t + j - t_1}{360}\right) \tag{7}$$

[2] Up to 30 November 1997, j was seven days. Since then, j has been three days.

where: t = observation day

P_{t+j} = average of bid-ask quoted price

cc_t = accrued interest at t

$r(j)$ = simple interest rate for j days

t_1 = date of next coupon payment

C^i = coupon if $t < t_1 < t + j$, 0 otherwise

Tax effects are not taken into account in the estimation. Since July 1997, an additional parameter (Y), which measures the differences in prices between non-stripped and stripped bonds (the last are traded at a lower yield), is estimated. In order to take this feature into account in the estimation, it is assumed that for the stripped bonds:

$$r^s(m) = r(m) + \gamma \tag{8}$$

where $r(m)$ is the spot rate of equation (3) and γ is the new parameter to be estimated. Therefore, the price of the i-th bond is expressed as:

$$P_t^i(b) + cc_t^i = C^i \sum_{j=1}^{v^i} d_t\left(m_j^i; b\right) + 100 d_t\left(m_{v^i} : b\right) \tag{9}$$

$i = 1,....,n$

$$\text{where} \begin{cases} (\beta_1, \beta_2, \beta_3, \tau_1, \tau_2) \text{ if } i \text{ is stripped} \\ (\beta_1, \beta_2, \beta_3, \tau_1, \tau_2, \gamma) \text{ if } i \text{ is not stripped} \end{cases}$$

Summary of characteristics of estimation

	January 1991-December 1994	**January 1995 onwards**
Frequency of estimation	Daily	Daily
Prices used in the estimation	Median of traded prices	Average of bid-ask quoted prices
Short-run securities	Bond repos for 1, 7, 15 and 30 days Treasury bills for 3, 6, 9 and 12 months	Bond repos for 1, 7, 15 and 30 days Treasury bills for 3, 6, 9 and 12 months
Long-run securities	Treasury bonds	Treasury bonds
Estimation method	Nelson and Siegel	Svensson
Minimisation criterion	Price error	Price error weighted by the inverse of duration
Regression procedure	Non-linear least squares	Non-linear least squares
Algorithm used	Marquardt	Marquardt
Econometric package	SAS	SAS

References

Ayuso, J and S Núñez (1996): "The yield curve as a monetary policy indicator", in *Monetary policy and inflation in Spain*, Bank of Spain.

Nelson, C R and A F Siegel (1987): "Parsimonious modeling of yield curves", *Journal of Business*, 60, pp 473-89.

Núñez, S (1995): "Estimación de la estructura temporal de los tipos de interés en España: elección entre métodos alternativos", Bank of Spain, *Documento de Trabajo*, 9522.

Svensson, L E O (1994): "Estimating and interpreting forward interest rates: Sweden 1992-4", International Monetary Fund, *IMF Working Paper*, 1994/114.

The estimation of forward interest rates and zero coupon yields at the Riksbank

Hans Dillén and Carl Fredrik Pettersson

1. Overview

Since 2001 the forward interest rates reported in the inflation report have been calculated by means of the smoothing splines method. Earlier forward interest rates were calculated by the extended Nelson-Siegel method (or the Svensson method) and we continue to calculate Nelson Siegel parameters for the sake of continuity and to have a back-up method. Moreover, the extended Nelson Siegel model has also been used in macroeconomic studies. The quantitative difference between the extended Nelson Siegel and the smoothing splines are normally very small. Recently parameters of the original Nelson Siegel (1987) model have been estimated on monthly basis (from 1996 and onwards) in order to do analysis along the lines of Diebold and Li (2003) and Diebold, Rudebusch, and Auroba (2003).

2. Data

The data we use are benchmark government bonds from 2-10 years. In the short end, T-Bills with maturities (closest to) 3, 6, 9, and 12 months are used, in addition to the repo rate.

3. Description of the methods

3.1 Smoothing splines

When fitting the zero coupon bond (ZCB) curve, we do so in yield-space, that is we minimize the weighted sum of two terms: the squared deviations of the fitted yields from the quoted yields, and a penalty for roughness, which is the integral of the squared second derivatives of the ZCB curve. The weight on the penalty term is determined by the variable roughness penalty (VRP) method, described in Waggoner (1997). It follows the original method in Fisher, Nychka, and Zervos (1995) closely, but with different penalty weights for the shorter yield maturities in order to allow a more flexible specification of the short end of the curve. The penalty weights in the different segments are determined once and for all by a trial-and-error approach. Occasionally, we reinvestigate the choice of optimal penalty weights. With the ZCB curve at hand, we then calculate the forward curve.

3.2 Extended Nelson Siegel

The chosen objective in this estimation procedure is to minimise the sum of squared yield errors. The functional form for the forward rate curve at date t to be estimated is:

$$f(t;s) = \beta_0 + \left(\beta_1 + \beta_2 \frac{s}{\tau_1}\right) \exp\left(-\frac{s}{\tau_1}\right) + \beta_3 \frac{s}{\tau_2} \exp\left(-\frac{s}{\tau_2}\right) \tag{1}$$

where s is time to settlement.

In the estimations, we impose the restriction that $\beta_0 + \beta_1$ is equal to the official repo rate, in order to facilitate the interpretation of implied forward rates as expected future repo rates. The original Nelson Siegel model is obtained by the restriction $\beta_3 = 0$.

4. Reporting routines to BIS

The Riksbank reports the daily data on the six parameters in the extended Nelson Siegel model once a week (Mondays). The data set from which these parameters are estimated normally consists of 9 benchmark bonds and 4 treasury bills (see paragraph 2). The yields used in the computations are the average yields from bid and ask yields (close yields).

References

Diebold, F and C Li (2003): “Forecasting the term structure of government bond yields”, NBER Working Paper.

Diebold, F, G Rudebusch and S Auroba (2003): “The macro economy and the yield curve: a nonstructural analysis”, CFS Working Paper 31.

Fisher, M, D Nychka and D Zervos (1995): “Fitting the term structure of interest rates with smoothing splines”, Board of Governors of the Federal Reserve System, Working Paper 1.

Nelson, C R, and A F Siegel (1987): Parsimonious modeling of yield curves”, Journal of Business 60, 473-89.

Waggoner, D (1997): “Spline methods for extracting interest rate curves from coupon bond prices”, Federal Reserve Bank of Atlanta Working Paper 10.

A technical note on the Svensson model as applied to the Swiss term structure

Robert Müller[1]

The purpose of this note is to describe the methodology used by the Swiss National Bank to construct the Swiss government zero coupon curve by applying an estimation based on the Svensson model.

1. Data and data selection

Spot and forward rates are estimated based on daily observations of the yield to maturity on Swiss government bonds and their coupon payments for bonds with maturities ranging from one to 50 years, and four money market rates on one-, three-, six- and 12-month holdings respectively. Callable bonds and bonds with a residual maturity of less than one year are excluded from the estimation. At the moment (August 2003), the estimation is done with 18 bonds and the money market rates.

2. The model

The Swiss National Bank uses a model developed by Charles Nelson and Andrew Siegel in 1987 and extended by Svensson. Nelson and Siegel assume that the instantaneous forward rate is the solution to a second-order differential equation with two equal roots. Let $f(t, t+m)$ denote the instantaneous forward rate with time to settlement m, for a given trade date t. Then the Svensson forward rate function can be written as:

$$f(t, t+m, b) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(-\frac{m}{\tau_2}\right) \tag{1}$$

where b = (β_0, β_1, β_2, β_3, τ_1, τ_2) is a vector of parameters (β_0, τ_1 *and* τ_2 must be positive). The Svensson method is identical to Nelson and Siegel's, but adds the term $\beta_3 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_2}\right)$ to the instantaneous forward rate function. In contrast to the Nelson-Siegel approach, this functional form allows for more than one local extremum along the maturity profile. This can be useful in improving the fit of yield curves.

The spot rate can be derived by integrating the forward rate and then dividing the result by the remaining time to maturity. It is given by:

$$i(t, t+m, b) = \beta_0 + \beta_1 \frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} + \beta_2 \left(\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} - \exp\left(-\frac{m}{\tau_1}\right) \right) + \beta_3 \left(\frac{1-\exp\left(-\frac{m}{\tau_2}\right)}{\frac{m}{\tau_2}} - \exp\left(-\frac{m}{\tau_2}\right) \right) \tag{2}$$

[1] Swiss National Bank, Statistics Section.

The discount function is related to the spot rate by:

$$d(t,t{+}m,b) = \exp\left(-\frac{i(t,t+m,b)}{100}\,m\right) \tag{3}$$

This discount function is used to compute the estimated (theoretical) bond price. We can define the price of a coupon bond, similarly to the traditional valuation of an investment, as the sum of discounted future coupon payments c and the present value of the face value paid after m years. It follows that the price of the bond $P(t,t+m)$ on trade day t can then be approximated by:

$$P(t,t+m,b) = \sum_{k=1}^{m} cd(t,t+k,b) + 100d(t,t+m,b) \tag{4}$$

For coupon bonds, yields to maturity are often quoted. The yield to maturity is the internal rate of return for the coupon bond that makes the present value of the coupon payments and the face value equal to the price of the bond. Thus, the price of the coupon bond can be written as a function of the yield to maturity $y(t,t+m)$:

$$P(t,t+m) = \sum_{k=1}^{m} c\,\exp\left(-\frac{y\,(t,t+m)}{100}\,k\right) + 100\,\exp\left(-\frac{y\,(t,t+m)}{100}\,m\right) \tag{5}$$

3. The estimation

The discount function is estimated for each trade date by minimising either the sum of squared price errors or the sum of squared yield errors. We choose the parameter so as to minimise yield-to-maturity errors. Minimising price errors sometimes results in fairly large yield errors for bonds and money market rates for short maturities. This is because yields are very sensitive to prices for short maturities.

The estimation is done with the restriction that the forward rate curve (and hence the spot rate curve) should start at the left end (from the overnight rate). This means that the term $(\beta_0 + \beta_1)$ equals the overnight rate.

The optimisation is performed using a numerical non-linear optimisation to maximise a log-likelihood function subject to the constraint on the parameter β_1 (= *overnight rate* – β_0). First, we use the Simplex algorithm to compute starting values and then the Berndt, Hall, Hall and Hausmann (BHHH) algorithm to estimate the final parameters. The optimisation procedure consists of the following steps for both numerical algorithms:

Initialising the parameter $b_t = (\beta_{0t}, \beta_{1t}, \beta_{2t}, \beta_{3t}, \tau_{1t}, \tau_{2t})$ for $t = 1$.

Calculating those spot rates $i(t,t+m,b_t)$ which are necessary in order to calculate the discount functions (equation (2)).

Computing the discount factors $d(t,t+m,b_t)$ for each bond (equation (3)).

Computing the estimated (theoretical) prices for the N different bonds (equation (4)).

Calculating the estimated (theoretical) yields to maturity for all the bonds (equation (5)) using the Newton-Raphson numerical algorithm.

Computing the function $\sum_{i=1}^{N}(y_{it}(t,t+m_i) - \hat{y}_{it}(t,t+m_i,b_t))^2$ (sum of squared yield errors) using first the Simplex algorithm and then the BHHH algorithm in order to determine a new b_{t+1}.

Examining the convergence condition:

$(b_{t+1} - b_t)'(b_{t+1} - b_t) < \alpha$ for an $\alpha > 0$

If the condition does not hold, go back to step 2 with b_{t+1} as a new vector instead of b_t.

The optimisation procedure alternates the parameters of *b* and with them the spot rates so as to minimise the sum of the (squared) differences between observed and calculated yields to maturity. The 95% confidence intervals are computed using the delta method.[2]

References

Nelson, C R and A F Siegel (1987): "Parsimonious modeling of yield curves", *Journal of Business*, 60, pp 473-89.

Svensson, L E O (1994): "Estimating and interpreting forward interest rates: Sweden 1992-4", *NBER Working Paper Series*, no 4871, September.

Table 1

Term structure of interest rates - estimation details

Estimation method	Minimised error	Shortest maturity in estimation	Adjustments for tax distortions	Relevant maturity spectrum
Svensson	Yields	Money market rates: ≥ 1 day bonds: ≥ 1 year	No	1 to 30 years

Table 2

Term structure of interest rates -
availability from the EASY database

Method	Estimates available since	Frequency	Spot rates	Forward rates	Parameters	Notation
Svensson	4 January 1988 5 January 1998	Weekly Daily	1 to 10 years 15 years 20 years 30 years (since 5 January 1998)	1 to 10 years 15 years 20 years 30 years	6	Percentages

[2] The delta method implies that, for the purpose of computing confidence intervals for the instantaneous forward rate, the estimated forward rate $f(m;b)$ for a given time to settlement m is considered to be distributed as a normal variable with mean $f(m;\hat{b})$ and covariance $\frac{\partial f(m,\hat{b})'}{\partial b}\hat{\Sigma}\frac{\partial f(m,\hat{b})}{\partial b}$, where $\hat{b}$, $\hat{\Sigma}$ and $\frac{\partial f(m,\hat{b})}{\partial b}$ denote the estimates of the parameter vector *b*, its covariance matrix and the column vector of partial derivatives for the parameters, respectively. When the BHHH algorithm is used, a natural estimator for the asymptotic covariance matrix for $\hat{\beta}$ is $\hat{\Sigma}_{\hat{\beta}} = \left[\sum_{i=1}^{N}\left(\frac{\partial L_n}{\partial b}\right)\left(\frac{\partial L_n}{\partial b'}\right)\right]^{-1}\Bigg|_{\hat{b},\hat{\sigma}^2}$, where L is the log-likelihood function and $i = 1, 2, ..., N$ the number of securities.

Yield curve estimation at the Bank of England

Matthew Hurd[1]

The Bank of England estimates yield curves for the United Kingdom on a daily basis. Three basic curves are produced: a nominal and a real government liability curve and a nominal commercial bank liability curve. These curves are published daily on the Bank's website (www.bankofengland.co.uk). This note provides a brief description of the curves we produce and the methods used to derive them.

Government liability curve

The government liability curve is based on yields on UK government bonds (gilts) and yields in the general collateral repo market. The nominal yield curves are derived from UK gilt prices and General Collateral (GC) repo rates. The real yield curves are derived from UK index-linked bond prices. Using the Fisher relationship,[2] we are also able to estimate a term structure of inflation expectations for the United Kingdom.

Estimates for the nominal curve are available from 2 January 1979. Estimates for the real yield curve and implied inflation term structure are available from 2 January 1985. Depending on the range of available bonds, we aim to publish estimates of both the spot rate and the instantaneous forward rate out to a maturity of about 25 to 30 years.

GC repo rates are used to estimate the nominal curve down to a maturity of one week. By generating synthetic zero coupon bonds from the GC repo rates, we improve the estimation of the short end of the nominal curve, where gilts tend to be less liquid.

Commercial bank liability curve

The commercial bank liability curve[3] is based on sterling interbank rates (Libor) and on yields on instruments linked to Libor, specifically short sterling futures, forward rate agreements and Libor-based interest rate swaps. These commercial bank liability curves are nominal only.

Estimates of the commercial bank liability curve are available out to a maturity of 10 years from November 1990 to July 1997, out to a maturity of 15 years from July 1997, and out to a maturity of at least 25 years from January 1999.

Estimation

The technique currently used by the Bank to estimate its yield curves is the variable roughness penalty (VRP) method, which replaced the approach by Svensson.[4] The VRP methodology is based on the

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1 Bank of England, Monetary Analysis, Monetary Instruments and Markets Division.

2 The Fisher relationship means that an implied inflation term structure can be calculated as the difference between nominal and real yields.

3 For further details, see Brooke et al (2000).

4 Estimates of the nominal yield curve were first published by the Bank in the November 1999 *Inflation Report* and are discussed in an article by Anderson and Sleath in the November 1999 *Bank of England Quarterly Bulletin*.

spline-based technique proposed by Waggoner (1997). This method was considered superior to the Svensson method and other estimation techniques, based upon the criteria of smoothness, flexibility and stability.[5] An important innovation of this technique is that the degree of smoothing is a function of maturity, and in particular that the curve is more flexible at the short end (where the curve is likely to exhibit the greatest curvature) than at the long end (where expectations are likely to be more smooth).

The spline-based technique models forward rates as a piecewise cubic polynomial, with the segments joined at "knot points". The coefficients of the individual polynomials are restricted so that both the curve and its first derivatives are continuous at all maturities, and knot points are placed at the time of maturity of each bond. Although the cubic spline is more flexible than parametric forms, an unconstrained spline would be far too flexible to generate the type of yield curves necessary for the purpose of monetary policy. In order to control the trade-off between smoothness of the curve and goodness of fit, a roughness penalty is included to penalise excessive curvature of the forward curve. The size of this roughness penalty is determined by a time-invariant function, $\lambda(m)$, which varies with horizon m.

Formally, the VRP method minimises the objective function X_s:

$$X_s = X_p + \int_0^M \lambda_t\,(m)\,[f''(m)]^2\,dm \tag{1}$$

where*:*

$$X_p = \sum_{i=1}^{N} \left[\frac{P_i - \Pi_i(\beta)}{D_i}\right]^2$$

P_i is the observed price of the i-th bond, D_i is its modified duration, $\Pi_i(\beta)$ is the fitted price, m is the maturity, $f''(m)$ is the second derivative of the fitted forward curve, M is the maturity of the longest bond and β is the vector of spline parameters.

From the above objective function, it can be seen that the VRP technique minimises the sum of the squared bond price residuals (X_p)[6] subject to a penalty for curvature. In addition, the bond prices are weighted according to the inverse modified duration.

The optimisation procedure for the VRP technique has two steps. First, the parameters of the smoothing function, $\lambda(m)$, are optimised and then, holding these parameters constant, we estimate the spline parameters on a daily basis.

The parameters of the smoothing function were chosen to maximise out-of-sample goodness of fit[7] over the period 1 May 1996 to 31 December 1998. The function, $\lambda(m)$, is defined to be a continuous function of three parameters:

$$\log \lambda(m) = L - (L - S)\,\exp\left(-\frac{m}{\mu}\right) \tag{2}$$

with L, S and μ the parameters maximised over the sample period.

5 A more in-depth comparison of the VRP method with other techniques can be found in Anderson and Sleath (2001).

6 Minimising the squared bond price residuals weighted by duration is a first-order approximation to minimising yields. Proceeding in this way has the advantage of not needing to numerically calculate yields at each step of the optimisation.

7 Following Waggoner, the main criterion for choosing the parameters was to maximise the out-of-sample goodness of fit averaged over the sample period. It was found that many combinations of these parameters gave similar measures of goodness of fit. We therefore opted for the set of parameters that led to the highest level of smoothing among these combinations.

References

Anderson, N and J Sleath (1999): "New estimates of the UK real and nominal yield curves", Bank of *England Quarterly Bulletin*, November, pp 384-92.

——— (2001): "New estimates of the UK real and nominal yield curves", *Bank of England Working Paper*, no 126.

Brooke, M, N Cooper and C Scholtes (2000): "Inferring market interest rate expectations from money market rates", *Bank of England Quarterly Bulletin*, November, pp 392-402.

Waggoner, D (1997): "Spline methods for extracting interest rate curves from coupon bond prices", Federal Reserve Bank of Atlanta, *Working Paper series*, 97-10.

In addition to these references, a note entitled "Notes on the Bank of England yield curves" is available from our website (www.bankofengland.co.uk). This note accompanies the estimates we provide and is aimed at explaining the terminology and concepts behind our yield curves.

Fitting the US term structure at the Federal Reserve Board

Mark Fisher[1]

1. Some notes on the US bond market

Treasury bills are discount securities with maturities of three and six months and one year. The three- and six-month bills are auctioned weekly and the year bill is auctioned monthly.

Treasury notes and *Treasury bonds* are coupon securities issued with various original terms to maturity in various auction cycles. Currently, the Treasury is issuing coupon securities with maturities of two, five, 10 and 30 years. The two-year note is auctioned monthly; the five-, 10- and 30-year notes are auctioned quarterly.

Reopenings. Not all auctions result in a new security. For bills, three-month bill auctions are always reopenings of outstanding six-month bills, and every fourth six-month bill auction is the reopening of an outstanding year bill. Coupon bonds are occasionally reopened. This occurs most often for the 30-year bond, but also for the 10-year note.

The bond price quotes are for *regular delivery*, for which the *settlement date* is the next business day ($t+1$). (Note that the yields on the FRBNY quote sheets are skip-day yields ($t+2$) computed from regular-delivery price quotes.)

Accrued interest. Notes and bonds pay coupons semiannually. Accrued interest is calculated as follows:

$$a = \left(\frac{c}{2}\right)\left(\frac{d}{b}\right) \tag{1}$$

where a is the accrued interest *per dollar* of face value,

b is the number of days in the coupon period (the basis),

c is the annual coupon rate, and

d is the number of days from issue or last coupon to settlement.

Tax treatment. Coupon payments and Treasury bill capital gains are subject to income tax as ordinary income. Capital gains on coupon securities are subject to income tax as capital gains, which have often been taxed at a lower rate than ordinary income.

Special features. In the past, the Treasury has issued notes and bonds with special features.

- Callable bonds are typically callable at par in the last five years of their life.
- Flower bonds are low-coupon bonds (typically) that can be redeemed at par for the payment of estate taxes.

Repo market. Government securities dealers finance their positions in the repurchase agreement market. They borrow funds (to finance their long positions) using the securities they own outright as collateral (so-called repos), and they take in securities (to deliver on their short positions) as collateral on loans they make (so-called reverses). When the aggregate short positions are large, dealers that are short may have to pay a premium to those who have possession of the collateral to acquire the specific collateral needed. The premium is paid by lending funds at less than the risk-free rate, in which case the security is said to be *on special* in the repo market. The (expected) difference between

[1] The views expressed herein are the author's and do not necessarily reflect those of the Board of Governors of the Federal Reserve System.

the risk-free rate and the special rate (over the life of the security) is capitalised into the security's price, pushing its yield below the yield curve.

The STRIPs (Separate Trading of Registered Interest and Principal) market provides direct observations on zero coupon securities at quarterly intervals out to 30 years. Ten-year notes and 30-year bonds are eligible for stripping (as well as some older 20-year bonds). Market participants strip (decompose) whole coupon bonds into a series of individual *coupon strips* and a final *principal strip*. While coupon strips with the same maturity date are fungible, principal strips are not - the coupon bond cannot be reconstituted without the principal strip. As a result, *principal* strips often trade at a premium relative to coupon strips.

When-issued trading is a forward market for a security that has yet to be issued. It starts when the Treasury announces the amount to be auctioned (about one week before the auction) and continues until the security is issued (about one week after the auction).

Futures market. There are a number of futures contracts for which various Treasury securities are deliverable. At times, the demand for an individual Treasury security may be affected by its deliverability.

2. Fitting the term structure

We have described in detail our technique for fitting the term structure in Fisher et al (1995). Here is a very brief summary:

- We fit a cubic spline to the forward rate curve.
- We minimise the weighted sum of two terms:
  - the sum of squared deviations of the fitted bond prices from the observed bond prices, and
  - a penalty for non-linearity - the integral of the squared second derivative of the forward rate curve.
- The weighting of the two terms is determined by minimising the generalised cross-validation (GCV) criterion:
  - the ratio of (i) a quasi-out-of-sample goodness-of-fit measure to (ii) the effective number of parameters.
- This can be thought of as a signal extraction technique.
  - The greater the forward rate signal in the bond prices, the more the forward rate curve may deviate from linearity.
  - The "amount" of signal extracted can be controlled by adjusting the trade-off between the two factors in the GCV criterion ratio.

Treasury bills vs notes and bonds. Currently, we fit separate yield curves for bills and coupon securities. We do not consider this to be an entirely satisfactory situation. The practice began because we were fitting the yield curve to measure deviations from the yield curve largely for coupon securities, and we noticed that coupon securities with less than one year to maturity were priced measurably differently from bills. This difference may be driven by liquidity, taxes, or other effects. We have estimated both curves daily starting in December 1987.

Coupon curve:

- We make no adjustments for tax effects.
- We exclude:
  (a) Treasury bills.
  (b) Notes and bonds with less than 30 days to maturity. We found that the prices of coupon securities very near maturity did not behave well.

(c) Callable bonds and flower bonds. The price of these securities reflects special features not captured by the discount function. As a result, we currently have no bonds in the estimation with maturities in the range of 10 to 18 years.

(d) The two most recently issued securities of each original term to maturity. We exclude these securities even though they are the most actively traded. The reason is that these securities are most likely to be on special in the repo market. Thus the price of these securities probably reflects special features not captured by the discount function.

Bill curve. There are three differences in the settings we use for the bill curve relative to the coupon curve.

(a) No bills are dropped. (There are about 32 bills outstanding each day.)

(b) We weight the observations by the inverse of the maturity.

(c) The GCV ratio has been adjusted to extract more signal.

Bills and coupons. We have a project under way to use the CRSP[2] daily bond file to estimate a curve daily starting in June 1961 that incorporates both bills and coupon securities. Given the limited number of long-term securities outstanding in the 1960s and 1970s, this curve will rely to some extent on callable bonds and flower bonds.

Linearised Nelson-Siegel. Prior to developing the smoothing spline approach to fitting the term structure using market prices for bonds, a simpler approach was developed, which could be applied directly to the *constant maturity yields*. This technique uses a linear approximation to the relationship between yields and prices.

STRIPs quotes. In principle, the STRIPs market is an excellent source of information. Currently, we have only limited historical data and no current data.

3. Modelling the term premia

It has been well documented that the expectations hypothesis does not hold: forward rates do not equal expected future spot rates. One reflection of the failure of the expectations hypothesis is that the slope of the yield curve is not an unbiased predictor of future changes in yields.[3] This failure results from the presence of time-varying term premia. Therefore, to extract the path of the expected short rate implicit in the term structure, we need to model these random term premia.

The approach we have undertaken is to fit multifactor models of the term structure that are based on absence-of-arbitrage conditions. (These models should not be confused with the so-called arbitrage-free models that exactly match a given term structure by fitting, for example, a deterministic time trend.) There is a class of numerically tractable models (the exponential-affine class, which includes the multifactor Cox, Ingersoll and Ross model, among many others) that has the potential to capture many of the stylised facts of the term structure, including the Campbell-Shiller regression results.

For this project, we take the zero coupon rates estimated with our smoothing spline techniques as "data". Thus, it is of prime importance that the cross-sectional and time series properties of the zero coupon rates not be obscured by the method used to obtain them. In particular, since intuition regarding the shape of the yield curve that relies on the expectations hypothesis may well be quite wrong, it is important not to impose an overly restrictive functional form that may obscure important relationships.

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2 Center for Research in Security Prices.

3 See, for example, the regression results in Campbell and Shiller (1991).

References

Campbell, J and R Shiller (1991): “Yield spreads and interest rate movements: a bird’s eye view”, *Review of Economic Studies*, 58, pp 495-514.

Fisher, M (1996): *Fitting and interpreting the US yield curve at the Federal Reserve Board*.

Fisher, M, D Nychka and D Zervos (1995): “Fitting the term structure of interest rates with smoothing splines”, Board of Governors of the Federal Reserve System, *Federal Reserve Board Working Paper* 95-1.