

The Homotopy Theory of n -Fold Categories

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When do we consider two categories A and B the same?

Two *different* possibilities:

- 1 If there is a functor $F: A \longrightarrow B$ such that $NF: NA \longrightarrow NB$ is a weak homotopy equivalence. (Thomason 1980)
- 2 If there is a fully faithful and essentially surjective functor $F: A \longrightarrow B$. (Joyal–Tierney 1991)

2) \Rightarrow 1)

Motivation: 2-categories vs. Double Categories

- A **2-category** is like an ordinary category except a 2-category has *Hom-categories*.
Example: **Top**.
- A **double category** is like an ordinary category except a double category has a *category of objects* and a *category of morphisms*.
Example: Bimodules.
- Recent examples show 2-categories are not enough, we need double categories.

Motivation: Why consider model structures on **DbICat** and **nFoldCat**?

Model categories have found great utility in comparing notions of $(\infty, 1)$ -category.

Theorem (Bergner, Joyal–Tierney, Rezk, Toën,...) The following model categories are Quillen equivalent: simplicial categories, Segal categories, complete Segal spaces, and quasicategories.

So we can expect model structures to also be of use in an investigation of iterated internalizations.

Definition (Ehresmann 1963)

*A double category \mathbb{D} is an internal category $(\mathbb{D}_0, \mathbb{D}_1)$ in **Cat**.*

Double Categories

Definition (Ehresmann 1963)

A double category \mathbb{D} consists of
a set of objects,
a set of horizontal morphisms,
a set of vertical morphisms, and
a set of squares with source and target as follows

$$\begin{array}{ccc} A \xrightarrow{f} B & & A \xrightarrow{f} B \\ & & \downarrow j \quad \alpha \quad \downarrow k \\ & & C \xrightarrow{g} D \end{array}$$

and compositions and units that satisfy the usual axioms and the interchange law.

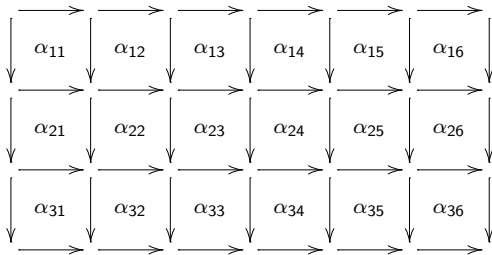
Examples of Double Categories

- 1 Any 2-category is a double category with trivial vertical morphisms.
- 2 Compact closed 1-manifolds, 2-cobordisms, diffeomorphisms of 1-manifolds, diffeomorphisms of 2-cobordisms compatible with boundary diffeomorphisms.
- 3 Rings, bimodules, ring maps, and twisted maps.
- 4 Topological spaces, parametrized spectra, continuous maps, and squares like in 3.

Bisimplicial Nerve of a Double Category

$$N: \mathbf{DbICat} \longrightarrow [\Delta^{\text{op}} \times \Delta^{\text{op}}, \mathbf{Set}]$$

$(N\mathbb{D})_{j,k} = j \times k$ – matrices of
composable squares in \mathbb{D}



N admits a left adjoint c called *double categorification*.

Model Structures for Higher Categories in Low Dimensions

Model Categories

A *model category* is a complete and cocomplete category \mathbf{C} equipped with three subcategories:

1. weak equivalences
2. fibrations
3. cofibrations

which satisfy various axioms.

Notably: given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \text{cofibration } i \downarrow & \nearrow h & \downarrow p \text{ fibration} \\ B & \longrightarrow & Y \end{array}$$

in which at least one of i or p is a weak equivalence, then there exists a lift $h: B \dashrightarrow X$.

Example The category **Top** with π_* -isomorphisms and Serre fibrations is a model category.

Model Structures on **Cat**

Theorem (Thomason 1980)

*There is a model structure on **Cat** such that*

- *F is a weak equivalence if and only if Ex^2NF is so.*
- *F is a fibration if and only if Ex^2NF is so.*

Theorem (Joyal–Tierney 1991)

*There is a model structure on **Cat** such that*

- *F is a weak equivalence if and only if F is an equivalence of categories.*
- *F is a fibration if and only if F is an isofibration.*

Model Structures on **2-Cat**

Theorem (Worytkiewicz–Hess–Parent–Tonks 2007)

*There is a model structure on **2-Cat** such that*

- *F is a weak equivalence if and only if Ex^2N_2F is so.*
- *F is a fibration if and only if Ex^2N_2F is so.*

Theorem (Lack 2004)

*There is a model structure on **2-Cat** such that*

- *F is a weak equivalence if and only if F is a biequivalence of 2-categories.*
- *F is a fibration if and only if F is an equivfibration.*

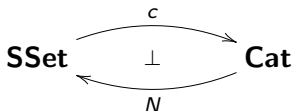
Theorem (Fiore–Paoli–Pronk, AGT, 2008)

*There exist model structures on **DbICat** for each of the following types of weak equivalences.*

- *F is a weak equivalence if and only if F is fully faithful and “essentially surjective.”*
- *F is a weak equivalence if and only if F is a weak equivalence of double categories as algebras in $\text{Cat}(\mathbf{Graph})$.*
- *F is a weak equivalence if and only if $N_h F$ is a weak equivalence in $[\Delta^{op}, \mathbf{Cat}]$.*

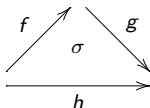
Thomason Structure on \mathbf{Cat}

Adjunction:



cX is the free category on the graph (X_0, X_1) modulo the relation below.

$g \circ f \sim h$ whenever X has a 2-simplex



The unit component $\partial\Delta[3] \longrightarrow Nc(\partial\Delta[3])$ is **not** a weak equivalence.

Thomason Structure on **Cat** continued

The unit and counit of the adjunction

$$\begin{array}{ccc} & \xrightarrow{\text{Sd}^2} & \\ \mathbf{SSet} & \perp & \mathbf{SSet} & \xrightarrow{c} & \mathbf{Cat} \\ & \xleftarrow{\text{Ex}^2} & & \xleftarrow{N} & \end{array}$$

are weak equivalences (Fritsch–Latch 1979, Thomason). So the Thomason model structure on **Cat** is Quillen equivalent to **SSet** and also **Top**.

n -fold Categories

Definition

An n -fold category is an internal category in $(n-1)\mathbf{FoldCat}$.

Example

A double category is a 2-fold category.

We have a fully faithful n -fold nerve.

$$N: \mathbf{nFoldCat} \longrightarrow \mathbf{SSet}^n$$

$$(N\mathbb{D})_{j_1, \dots, j_n} = \mathbf{nFoldCat}([j_1] \boxtimes \cdots \boxtimes [j_n], \mathbb{D}).$$

Adjunction:

$$\begin{array}{ccc} & \xrightarrow{c} & \\ \mathbf{SSet}^n & \perp & \mathbf{nFoldCat} \\ & \xleftarrow{N} & \end{array}$$

The n -fold Grothendieck Construction

If $Y: (\Delta^{\text{op}})^{\times n} \longrightarrow \mathbf{Set}$, then the n -fold Grothendieck construction on Y is the n -fold category $\Delta^{\boxtimes n}/Y$ with

$$\text{Objects} = \{(y, \bar{k}) \mid \bar{k} \in \Delta^{\times n}, y \in Y_{\bar{k}}\}$$

and n -cubes $(y, \bar{k}) \longrightarrow (z, \bar{\ell})$ are morphisms $\bar{f}: \bar{k} \longrightarrow \bar{\ell}$ in $\Delta^{\times n}$ such that

$$\bar{f}^*(z) = y.$$

This is the n -fold category of multisimplices of Y .

Main Theorem 1: The n -fold Grothendieck Construction is Homotopy Inverse to the n -fold Nerve

($n=1$ case was Quillen, Illusie, Waldhausen, Joyal–Tierney)

Theorem (Fiore–Paoli 2008)

The n -fold Grothendieck construction is a homotopy inverse to n -fold nerve. In other words, there are natural weak equivalences

$$N(\Delta^{\boxtimes n}/Y) \longrightarrow Y$$

$$\Delta^{\boxtimes n}/N(\mathbb{D}) \longrightarrow \mathbb{D} .$$

SSet = $[\Delta^{\text{op}}, \mathbf{Set}]$ = simplicial sets

SSetⁿ = $[(\Delta^{\text{op}})^{\times n}, \mathbf{Set}]$ = multisimplicial sets

The diagonal functor

$$\delta: \Delta \longrightarrow \Delta^n$$

$$[m] \longmapsto ([m], \dots, [m])$$

induces $\delta^*: \mathbf{SSet}^n \longrightarrow \mathbf{SSet}$ by precomposition.

Adjunction:

$$\begin{array}{ccc} & \delta_! & \\ \mathbf{SSet} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathbf{SSet}^n \\ & \delta^* & \end{array}$$

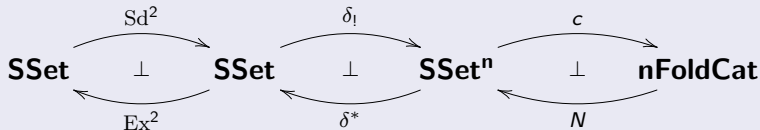
Main Theorem 2: Thomason Structure on **nFoldCat**

Theorem (Fiore–Paoli 2008)

There is a cofibrantly generated model structure on **nFoldCat** such that

- F is a weak equivalence if and only if $Ex^2\delta^*NF$ is so.
- F is a fibration if and only if $Ex^2\delta^*NF$ is so.

Further, the adjunction



is a Quillen equivalence.