The Homotopy Theory of n-Fold Categories

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January 8th, 2009

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Motivation

When do we consider two categories A and B the same?

Two *different* possibilities:

1. If there is a functor $F\colon A \longrightarrow B$ such that $NF\colon NA \longrightarrow NB$ is a weak homotopy equivalence. (Thomason 1980)
2. If there is a fully faithful and essentially surjective functor $F\colon A \longrightarrow B$. (Joyal–Tierney 1991)

2) $\Rightarrow$ 1)

Motivation: 2-categories vs. Double Categories

- A **2-category** is like an ordinary category except a 2-category has *Hom-categories*.
  Example: **Top**.

- A **double category** is like an ordinary category except a double category has a *category of objects* and a *category of morphisms*.
  Example: Bimodules.

- Recent examples show 2-categories are not enough, we need double categories.

Motivation: Why consider model structures on **DblCat** and **nFoldCat**?

Model categories have found great utility in comparing notions of $(\infty, 1)$-category.

Theorem (Bergner, Joyal–Tierney, Rezk, Toën,...) The following model categories are Quillen equivalent: simplicial categories, Segal categories, complete Segal spaces, and quasicategories.

So we can expect model structures to also be of use in an investigation of iterated internalizations.

Double Categories

Definition (Ehresmann 1963)

A double category $\mathbb{D}$ *is an internal category* $(\mathbb{D}_0, \mathbb{D}_1)$ *in* **Cat**.

Double Categories

Definition (Ehresmann 1963)

A double category $\mathbb{D}$ consists of
a set of objects,
a set of horizontal morphisms,
a set of vertical morphisms, and
a set of squares with source and target as follows

$$A \xrightarrow{f} B \qquad \begin{array}{c} A \\ {\scriptstyle j}\downarrow \\ C \end{array} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

and compositions and units that satisfy the usual axioms and the interchange law.

Examples of Double Categories

1. Any 2-category is a double category with trivial vertical morphisms.
2. Compact closed 1-manifolds, 2-cobordisms, diffeomorphisms of 1-manifolds, diffeomorphisms of 2-cobordisms compatible with boundary diffeomorphisms.
3. Rings, bimodules, ring maps, and twisted maps.
4. Topological spaces, parametrized spectra, continuous maps, and squares like in 3.

Bisimplicial Nerve of a Double Category

$$N\colon \mathbf{DblCat} \longrightarrow [\Delta^{op} \times \Delta^{op}, \mathbf{Set}]$$

$$(N\mathbb{D})_{j,k} = j \times k - \text{matrices of composable squares in } \mathbb{D}$$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{11}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{12}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{13}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{14}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{15}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{16}"] & \cdot \arrow[d] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{21}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{22}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{23}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{24}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{25}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{26}"] & \cdot \arrow[d] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{31}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{32}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{33}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{34}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{35}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{36}"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

N admits a left adjoint c called *double categorification*.

Next Topic

Model Structures for Higher Categories in Low Dimensions

Model Categories

A *model category* is a complete and cocomplete category $\mathbf{C}$ equipped with three subcategories:

1. weak equivalences
2. fibrations
3. cofibrations

which satisfy various axioms.

Notably: given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\text{cofibration } i"'] & X \arrow[d, "p \text{ fibration}"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

in which at least one of i or p is a weak equivalence, then there exists a lift $h\colon B \dashrightarrow X$.

Example The category **Top** with π_*-isomorphisms and Serre fibrations is a model category.

Model Structures on **Cat**

Theorem (Thomason 1980)

There is a model structure on **Cat** *such that*

- F *is a weak equivalence if and only if* Ex^2NF *is so.*
- F *is a fibration if and only if* Ex^2NF *is so.*

Theorem (Joyal–Tierney 1991)

There is a model structure on **Cat** *such that*

- F *is a weak equivalence if and only if* F *is an equivalence of categories.*
- F *is a fibration if and only if* F *is an isofibration.*

Model Structures on **2-Cat**

Theorem (Worytkiewicz–Hess–Parent–Tonks 2007)

There is a model structure on **2-Cat** *such that*

- F *is a weak equivalence if and only if* $Ex^2 N_2 F$ *is so.*
- F *is a fibration if and only if* $Ex^2 N_2 F$ *is so.*

Theorem (Lack 2004)

There is a model structure on **2-Cat** *such that*

- F *is a weak equivalence if and only if* F *is a biequivalence of 2-categories.*
- F *is a fibration if and only if* F *is an equivfibration.*

Model Structures on **DblCat**

Theorem (Fiore–Paoli–Pronk, AGT, 2008)

There exist model structures on **DblCat** *for each of the following types of weak equivalences.*

- F *is a weak equivalence if and only if* F *is fully faithful and "essentially surjective."*
- F *is a weak equivalence if and only if* F *is a weak equivalence of double categories as algebras in* $\mathrm{Cat}(\mathbf{Graph})$.
- F *is a weak equivalence if and only if* $N_h F$ *is a weak equivalence in* $[\Delta^{op}, \mathbf{Cat}]$.

Adjunction:

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "c"] \arrow[r, phantom, "\perp"] & \mathbf{Cat} \arrow[l, bend left, "N"] \end{tikzcd}$$

cX is the free category on the graph (X_0, X_1) modulo the relation below.

$g \circ f \sim h$ whenever X has a 2-simplex

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, bend left=20, "\sigma"] & & \cdot \end{tikzcd}$$

The unit component $\partial\Delta[3] \longrightarrow Nc(\partial\Delta[3])$ is **not** a weak equivalence.

Thomason Structure on **Cat** continued

The unit and counit of the adjunction

$$\begin{tikzcd} \mathbf{SSet} \arrow[rr, bend left, "\mathrm{Sd}^2"] \arrow[rr, phantom, "\perp"] & & \mathbf{SSet} \arrow[ll, bend left, "\mathrm{Ex}^2"] \arrow[rr, bend left, "c"] \arrow[rr, phantom, "\perp"] & & \mathbf{Cat} \arrow[ll, bend left, "N"] \end{tikzcd}$$

are weak equivalences (Fritsch–Latch 1979, Thomason). So the Thomason model structure on **Cat** is Quillen equivalent to **SSet** and also **Top**.

n-fold Categories

Definition

An n-fold category is an internal category in **(n-1)FoldCat**.

Example

A double category is a 2-fold category.

We have a fully faithful n-fold nerve.

$$N\colon\ \mathbf{nFoldCat} \longrightarrow \mathbf{SSet^n}$$

$$(N\mathbb{D})_{j_1,\ldots,j_n} = \mathbf{nFoldCat}([j_1] \boxtimes \cdots \boxtimes [j_n], \mathbb{D}).$$

Adjunction:

$$\begin{tikzcd} \mathbf{SSet^n} \arrow[rr, bend left, "c"] \arrow[rr, phantom, "\perp"] && \mathbf{nFoldCat} \arrow[ll, bend left, "N"] \end{tikzcd}$$

The n-fold Grothendieck Construction

If $Y\colon (\Delta^{\mathrm{op}})^{\times n} \longrightarrow \mathbf{Set}$, then the *n-fold Grothendieck construction on* Y is the n-fold category $\Delta^{\boxtimes n}/Y$ with

$$\text{Objects } = \{(y, \overline{k}) | \overline{k} \in \Delta^{\times n}, y \in Y_{\overline{k}}\}$$

and n-cubes $(y, \overline{k}) \longrightarrow (z, \overline{\ell})$ are morphisms $\overline{f}\colon \overline{k} \longrightarrow \overline{\ell}$ in $\Delta^{\times n}$ such that

$$\overline{f}^*(z) = y.$$

This is the n-fold category of multisimplices of Y.

Main Theorem 1: The *n*-fold Grothendieck Construction is Homotopy Inverse to the *n*-fold Nerve

(n=1 case was Quillen, Illusie, Waldhausen, Joyal–Tierney)
Theorem (Fiore–Paoli 2008)
The n-fold Grothendieck construction is a homotopy inverse to n-fold nerve. In other words, there are natural weak equivalences

$$N(\Delta^{\boxtimes n}/Y) \longrightarrow Y$$

$$\Delta^{\boxtimes n}/N(\mathbb{D}) \longrightarrow \mathbb{D}\ .$$

Diagonal

$\mathbf{SSet} = [\Delta^{\mathrm{op}}, \mathbf{Set}]$=simplicial sets
$\mathbf{SSet^n} = [(\Delta^{\mathrm{op}})^{\times n}, \mathbf{Set}]$=multisimplicial sets
The diagonal functor

$$\delta\colon \Delta \longrightarrow \Delta^n$$

$$[m] \longmapsto ([m], \ldots, [m])$$

induces $\delta^*\colon \mathbf{SSet^n} \longrightarrow \mathbf{SSet}$ by precomposition.
Adjunction:

$$\begin{tikzcd} \mathbf{SSet} \arrow[rr, bend left, "\delta_!"] \arrow[rr, phantom, "\perp"] & & \mathbf{SSet^n} \arrow[ll, bend left, "\delta^*"] \end{tikzcd}$$

Main Theorem 2: Thomason Structure on **nFoldCat**

Theorem (Fiore–Paoli 2008)

There is a cofibrantly generated model structure on **nFoldCat** *such that*

- *F is a weak equivalence if and only if $Ex^2\delta^* NF$ is so.*
- *F is a fibration if and only if $Ex^2\delta^* NF$ is so.*

Further, the adjunction

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathrm{Ex}^2"] \arrow[r, bend left, "\delta_!"] \arrow[r, phantom, "\perp"] & \mathbf{SSet^n} \arrow[l, bend left, "\delta^*"] \arrow[r, bend left, "c"] \arrow[r, phantom, "\perp"] & \mathbf{nFoldCat} \arrow[l, bend left, "N"] \end{tikzcd}$$

is a Quillen equivalence.