Multiple Solutions of H-Systems and Rellich's Conjecture

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0. Introduction

Let $\Omega = \{(x, y) \in \mathbf{R}^2;\ x^2 + y^2 < 1\}$. We look for a function $u: \bar{\Omega} \to \mathbf{R}^3$ satisfying the *H-system*

$$\text{(H)} \qquad \Delta u = 2Hu_x \wedge u_y \quad \text{on} \quad \Omega$$

together with *one* of the following conditions:

either *Dirichlet*:

$$\text{(D)} \qquad u = \gamma \quad \text{on} \quad \partial\Omega,$$

or *Plateau*:

$$\text{(P)} \qquad \begin{gathered} |u_x|^2 - |u_y|^2 = u_x \cdot u_y = 0 \quad \text{on} \quad \Omega, \\ u(\partial\Omega) = \Gamma \text{ and } u \text{ is non-decreasing on } \partial\Omega, \end{gathered}$$

where $H > 0$ is a given constant, $\gamma: \partial\Omega \to \mathbf{R}^3$ is a given function and $\Gamma \subset \mathbf{R}^3$ is a given oriented Jordan curve.

If u is a solution of (H)–(P), then $u(\bar{\Omega})$ represents a "soap bubble", that is, a surface with mean curvature H (at all points $x \in \Omega$ where $\nabla u(x) \neq 0$) spanning Γ.

Let us assume that $\gamma(\partial\Omega)$ (respectively Γ) is contained in a closed ball of radius R. It was proved by S. Hildebrandt [8] that both the Dirichlet and Plateau problems have at least one solution if $HR \leqq 1$ (this was an improvement over earlier results of Heinz [6] and Werner [19]). Moreover this result is *sharp* when Γ is a *circle*: there is no solution of (H)–(P) if $HR > 1$ (see Heinz [7]). In case Γ is a *circle* of radius R and $HR < 1$, it is easy to check that there exist two solutions of (H)–(P), namely:

1. the "small" spherical bubble B_1 of curvature H spanned by Γ,
2. the "large" spherical bubble B_2 of curvature H spanned by Γ;

see Figure 1).

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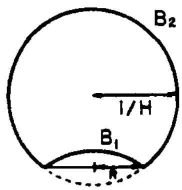

Figure 1

This observation led Rellich to conjecture that for *any curve* Γ there exist at least two solutions of (H)–(P) for H small enough (see [9]). We prove that this is indeed true for every $H>0$ with $HR<1$—see Theorem 2 (previously Steffen [13] had established this fact for some sequence $H_n \to 0$). A similar result holds for the Dirichlet problem (H)–(D) provided γ is not constant on $\partial\Omega$—see Theorem 1. If $\gamma = C$ is a constant on $\partial\Omega$, it was shown by Wente [17] that $u \equiv C$ is the *only* solution of (H)–(D).

Our approach is the following. In section 1 we consider the Dirichlet problem (H)–(D). We recall Hildebrandt's result: there exists a "small" solution $\underline{u}$ of (H)–(D) obtained by a simple minimization argument. We look for a second solution of (H)–(D) of the form $u = \underline{u} + v$ so that v satisfies

$$(0.1)\qquad \begin{aligned} \mathscr{L}v = -\Delta v + 2H(\underline{u}_x \wedge v_y + v_x \wedge \underline{u}_y) &= -2H(v_x \wedge v_y) \quad \text{on} \quad \Omega, \\ v &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

This problem has a *variational structure*:

(i) the linear operator $\mathscr{L}$ is selfadjoint and corresponds to the functional $\frac{1}{2}(\mathscr{L}v, v)$, where

$$(\mathscr{L}v, v) = \int |\nabla v|^2 + 4H \int \underline{u} \cdot v_x \wedge v_y;$$

(ii) the nonlinear term $v_x \wedge v_y$ is the derivative of the volume functional $\frac{1}{3}Q(v)$, where

$$Q(v) = \int v \cdot (v_x \wedge v_y).$$

The non-zero solutions of (0.1) are the nontrivial critical points of the functional $(\mathscr{L}v, v) + \frac{4}{3}HQ(v)$. Another view point—which we shall use—is to look for critical points of the functional $(\mathscr{L}v, v)$ on the "manifold" $Q(v) = 1$. After "stretching out" the Lagrange multiplier we obtain a non-zero solution of (0.1). In fact we prove that

$$(0.2)\qquad \inf_{\substack{v \in H_0^1(\Omega) \\ Q(v)=1}} (\mathscr{L}v, v) \quad \text{is achieved.}^1$$

[1] We denote by $H_0^1(\Omega)$ (or simply by H_0^1) the Sobolev space $H_0^1(\Omega; \mathbf{R}^3)$.

First we establish that

$$(\mathscr{L}v, v) \geqq \delta \|v\|^2_{H_0^1} \quad \text{for all} \quad v \in H_0^1 \quad \text{with} \quad \delta > 0,$$

see Lemma 3.

The major difficulty in proving (0.2) comes from the fact that $Q(v)$ is *not* continuous under weak convergence in H_0^1. To overcome this "*lack of compactness*" we use the same strategy as in [3] (see also [1]). Namely we consider the *isoperimetric inequality* (see [16])

$$\int |\nabla v|^2 \geqq S|Q(v)|^{2/3} \quad \text{for all} \quad v \in H_0^1 \tag{0.3}$$

with the best constant $S = \text{Inf}_{v \in H_0^1(\Omega), Q(v)=1} \int |\nabla v|^2$ (which is not achieved) and we prove that

$$\operatorname*{Inf}_{\substack{v \in H_0^1(\Omega) \\ Q(v)=1}} (\mathscr{L}v, v) < S. \tag{0.4}$$

(Here we use the fact that γ is not a constant; otherwise when $\gamma \equiv C$, then $\underline{u} \equiv C$ and $(\mathscr{L}v, v) = \int |\nabla v|^2$). Next, we rely on (0.4) in order to establish (0.2). At this point we use an argument which is related to a method introduced by E. Lieb [10].

In some ways problem (0.1) is reminiscent of the problem

$$\begin{cases} \mathscr{L}v = -\Delta v - \lambda v = v^p & \text{on} \quad \vartheta \subset \mathbb{R}^N, \\ v > 0 & \text{on} \quad \vartheta, \\ v = 0 & \text{on} \quad \partial\vartheta, \end{cases} \tag{0.5}$$

where ϑ is a bounded domain, $N \geqq 4$ and $p = (N+2)/(N-2)$. It is proved in [3] that (0.5) has a solution for every $0 < \lambda < \lambda_1$ (λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet data). The solutions of (0.5) correspond (after stretching) to the critical points of the functional $(\mathscr{L}v, v) = \int |\nabla v|^2 - \lambda \int v^2$ subject to the constraint $\int |v|^{p+1} = 1$. Here again the major difficulty comes from the fact that the Sobolev embedding $H^1 \subset L^{p+1}$ is continuous but *not compact.*[2] One uses the following technique (see [3]).

Here, the *Sobolev inequality*

$$\int |\nabla v|^2 \geqq S\|v\|^2_{L^{p+1}} \quad \text{for all} \quad v \in H_0^1,$$

with the best constant S, plays the role of the isoperimetric inequality (0.3). First one proves that

$$\operatorname*{Inf}_{\substack{v \in H_0^1 \\ \int |v|^{p+1}=1}} (\mathscr{L}v, v) < S \tag{0.6}$$

[2] The same difficulty occurs in Yamabe's conjecture (see [1]).

and then, using (0.6), one shows that

$$\operatorname*{Inf}_{\substack{v \in H_0^1 \\ \int |v|^{p+1} = 1}} (\mathscr{L}v, v) \text{ is achieved.}$$

In case $\lambda = 0$ (and ϑ is star-shaped) it has been proved by Pokhozaev [12] that (0.5) has *no* solution. This fact should be put in parallel with the "*non existence*" result of Wente [17] quoted above when γ is a constant.

Our results concerning (H)–(D) remind one also of the problem

$$(0.7) \qquad \begin{aligned} -\Delta u &= H(1+u)^p \quad \text{on} \quad \vartheta \subset \mathbb{R}^N, \\ u &= 0 \qquad\qquad \text{on} \quad \partial\vartheta, \end{aligned}$$

for which there exists some constant H^* such that

(a) if $0 < H < H^*$, there are *at least two* positive solutions of (0.7)—a small solution $\underline{u}$ and a large solution $\bar{u}$;

(b) if $H = H^*$, there is *exactly one* positive solution of (0.7): H^* is a turning point;

(c) if $H > H^*$, there is *no* positive solution of (0.7);

see Crandall–Rabinowitz [5] for the case where $p < (N+2)/(N-2)$ and [3] for the case $p = (N+2)/(N-2)$.

In Section 2, we deal with the Plateau problem (H)–(P). Our approach is the following. We introduce the class

$$\mathscr{C} = \{\gamma : \partial\Omega \to \mathbb{R}^3;\ \gamma(\partial\Omega) = \Gamma \text{ and } \gamma \text{ is nondecreasing}\}.$$

For each $\gamma \in \mathscr{C}$ there is a large solution $\bar{u}$ of the Dirichlet problem (H)–(D). We consider its "energy"

$$A(\gamma) = \int |\nabla \bar{u}|^2 + \tfrac{4}{3} H Q(\bar{u}).$$

Then we show that

$$\operatorname*{Inf}_{\gamma \in \mathscr{C}} A(\gamma) = A(\gamma^0) \text{ is achieved}$$

and we prove that the large solution $\bar{u}^0$ of the Dirichlet problem (H)–(D) with data γ^0 is a solution of the Plateau problem (H)–(P).

After our results were announced in [2] we learned that Struwe [15] has independently obtained some partial results in the same direction as ours. He has proved that (H)–(P) has at least two solutions, for a class of "admissible" curves Γ if $0 < H < H^*$, where H^* is some small constant which is *not explicitly* stated. Subsequently Steffen [14] was able to show that *any* Jordan curve is admissible—but again without any explicit estimate for H^*; they prove similar results for (H)–(D).

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1. The Dirichlet Problem

Let $\Omega = \{(x, y) \in \mathbf{R}^2;\ x^2 + y^2 < 1\}$. We consider the following problem: find $u \in H^1(\Omega; \mathbf{R}^3)$ satisfying

$$\Delta u = 2Hu_x \wedge u_y \quad \text{on} \quad \Omega, \tag{1}$$

$$u = \gamma \quad \text{on} \quad \partial\Omega, \tag{2}$$

where $H > 0$ is a given constant and γ is a given function on $\partial\Omega$ such that

$$\gamma \in H^{1/2}(\partial\Omega; \mathbf{R}^3) \cap L^\infty(\partial\Omega; \mathbf{R}^3). \tag{3}$$

Set

$$R = \operatorname*{Sup}_{\partial\Omega} |\gamma|.$$

Our main result is the following.

THEOREM 1. *Assume* (3),

$$HR < 1 \tag{4}$$

and

$$\gamma \text{ is not a constant on } \partial\Omega. \tag{5}$$

Then there exist at least two distinct solutions of (1)–(2).

Remark 1. It follows from Lemma A.1 (see the appendix) that every solution u of (1)–(2) lies in $L^\infty(\Omega; \mathbf{R}^3) \cap C(\Omega; \mathbf{R}^3)$; in addition, if $\gamma \in C(\partial\Omega; \mathbf{R}^3)$, then $u \in C(\bar{\Omega}; \mathbf{R}^3)$. By a result of Wente [16], extending an earlier classical theorem of Morrey [11] we know that every solution u of (1) lies in $C^\infty(\Omega; \mathbf{R}^3)$.

Remark 2. It has been known (see Hildebrandt [8]) that if $\gamma \in H^{1/2}(\partial\Omega; \mathbf{R}^3) \cap C(\partial\Omega; \mathbf{R}^3)$ and $HR \leq 1$, then there exists at least *one* solution of (1)–(2). We believe that if $\gamma(x, y) = (x, y, 0)$ (so that $R = 1$) and $H = 1$, then there exists *exactly one* solution of (1)–(2); this means that (4) is presumably sharp for such a γ.

Remark 3. Assumption (5) can not be relaxed. In case $\gamma = C$ is a constant it was proved by Wente [17] that $u = C$ is the *unique* solution of (1)–(2).

The proof of Theorem 1 is divided into four steps:

Step 1. We sketch the proof of the existence of a "small" solution $\underline{u}$ of (1)–(2) following an elegant argument due to S. Hildebrandt (see [8], [9]).

Step 2. We prove that the small solution $\underline{u}$ satisfies

$$\int |\nabla u|^2 + 4H \int \underline{u} \cdot (v_x \wedge v_y) \geqq \delta \int |\nabla v|^2 \quad \text{for all} \quad v \in H_0^1(\Omega; \mathbb{R}^3) \quad \text{for some} \quad \delta > 0.$$

Step 3. We introduce the volume integral

$$Q(v) = \int v \cdot (v_x \wedge v_y) \quad \text{for} \quad v \in H_0^1(\Omega; \mathbb{R}^3)$$

and we prove that

$$J = \operatorname*{Inf}_{\substack{v \in H_0^1 \\ Q(v)=1}} \left\{ |\nabla v|^2 + 4H \int \underline{u} \cdot (v_x \wedge v_y) \right\} < \operatorname*{Inf}_{\substack{v \in H_0^1 \\ Q(v)=1}} \int |\nabla v|^2.$$

Step 4. We prove that the infimum which defines J is achieved by some v^0 and that $\bar{u} = \underline{u} - (J/2H)v^0$ is another solution of (1)–(2).

Step 1. Fix $R' > R$ such that $HR' < 1$. Let

$$K = \{v \in H^1(\Omega; \mathbb{R}^3);\ v = \gamma \text{ on } \partial\Omega \text{ and } \|v\|_{L^\infty} \leqq R'\},$$

and

$$E(v) = \int |\nabla v|^2 + \tfrac{4}{3}H \int v \cdot (v_x \wedge v_y) \quad \text{for} \quad v \in H^1 \cap L^\infty.$$

LEMMA 1. *There exists some $\underline{u} \in K$ such that*

$$E(\underline{u}) = \operatorname*{Inf}_{v \in K} E(v);$$

moreover every minimizing sequence is relatively compact in $H^1(\Omega; \mathbb{R}^3)$.

Proof: Clearly we have

$$(6) \qquad E(v) \geqq (1 - \tfrac{2}{3}H\|v\|_{L^\infty}) \int |\nabla v|^2 \geqq \tfrac{1}{3} \int |\nabla v|^2 \quad \text{for all} \quad v \in K.$$

Let (u^n) be a minimizing sequence, that is $u^n \in K$ and

$$E(u^n) = \operatorname*{Inf}_{v \in K} E(v) + o(1). \tag{7}$$

After extracting a subsequence we may assume that

$$\begin{aligned} u^n &\rightharpoonup \underline{u} \quad \text{in} \quad H^1 \text{ weakly}, \\ u^n &\rightharpoonup \underline{u} \quad \text{in} \quad L^\infty \text{ weak*}, \\ u^n &\to \underline{u} \quad \text{a.e.} \quad \text{on} \quad \Omega, \end{aligned}$$

with $\underline{u} \in K$. Set $\vartheta^n = u^n - \underline{u}$ so that $\vartheta^n \in H_0^1$ and

$$\begin{aligned} \vartheta^n &\rightharpoonup 0 \quad \text{in} \quad H^1 \text{ weakly}, \\ \vartheta^n &\rightharpoonup 0 \quad \text{in} \quad L^\infty \text{ weak*}, \\ \vartheta^n &\to 0 \quad \text{a.e.} \quad \text{on} \quad \Omega, \end{aligned}$$

and $\|\vartheta^n\|_{L^\infty} \leqq 2R'$. We have

$$E(u^n) = \int |\nabla \underline{u}|^2 + \int |\nabla \vartheta^n|^2 + o(1) + \tfrac{4}{3}H \int u^n \cdot (\underline{u}_x + \vartheta_x^n) \wedge (\underline{u}_y + \vartheta_y^n). \tag{8}$$

But

$$\tfrac{4}{3}H \left| \int u^n \cdot \vartheta_x^n \wedge \vartheta_y^n \right| \leqq \tfrac{2}{3}HR' \int |\nabla \vartheta^n|^2 \leqq \tfrac{2}{3} \int |\nabla \vartheta^n|^2. \tag{9}$$

On the other hand,

$$\int u^n \cdot \vartheta_x^n \wedge \underline{u}_y = o(1); \tag{10}$$

indeed

$$\int u^n \cdot \vartheta_x^n \wedge \underline{u}_y = -\int \vartheta_x^n \cdot u^n \wedge \underline{u}_y$$

and $\vartheta_x^n \rightharpoonup 0$ weakly in L^2, while $u^n \wedge \underline{u}_y \to \underline{u} \wedge \underline{u}_y$ strongly in L^2 (by dominated convergence). Similarly we have

$$\int u^n \cdot \underline{u}_x \wedge \vartheta_y^n = o(1). \tag{11}$$

Clearly,

$$\int u^n \cdot \underline{u}_x \wedge \underline{u}_y = \int \underline{u} \cdot \underline{u}_x \wedge \underline{u}_y + o(1) \tag{12}$$

since $u^n \rightharpoonup \underline{u}$ in L^∞ weak*. Combining (7), (8), (9), (10), (11) and (12) we find

$$E(\underset{\sim}{u})+\frac{1}{3}\int|\nabla\vartheta^n|^2\leqq \operatorname*{Inf}_{v\in K} E(v)+o(1).$$

Therefore,

$$E(\underset{\sim}{u})=\operatorname*{Inf}_{v\in K} E(v)$$

and moreover $\int|\nabla\vartheta^n|^2\to 0$; thus $u^n\to \underset{\sim}{u}$ strongly in H^1.

LEMMA 2. *Suppose $\underset{\sim}{u}\in K$ is such that*

$$E(\underset{\sim}{u})=\operatorname*{Inf}_{v\in K} E(v);$$

then $\underset{\sim}{u}$ satisfies (1) *and moreover $\|\underset{\sim}{u}\|_{L^\infty}\leqq R$.*

Proof: Let $\eta\in\mathcal{D}_+(\Omega;\mathbb{R})$ so that $(1-\varepsilon\eta)\underset{\sim}{u}\in K$ for $\varepsilon>0$ small enough and thus

$$E(\underset{\sim}{u})\leqq E((1-\varepsilon\eta)\underset{\sim}{u}).$$

It follows that

$$2\int\nabla\underset{\sim}{u}\cdot\nabla(\eta\underset{\sim}{u})+\tfrac{4}{3}H\left[\int\eta\underset{\sim}{u}\cdot\underset{\sim}{u}_x\wedge\underset{\sim}{u}_y+\int\underset{\sim}{u}\cdot(\eta\underset{\sim}{u})_x\wedge\underset{\sim}{u}_y+\int\underset{\sim}{u}\cdot\underset{\sim}{u}_x\wedge(\eta\underset{\sim}{u})_y\right]\leqq 0.$$

Using Lemma A.5 we deduce that

$$\int\nabla\underset{\sim}{u}\cdot\nabla(\eta\underset{\sim}{u})+2H\int\eta\underset{\sim}{u}\cdot\underset{\sim}{u}_x\wedge\underset{\sim}{u}_y\leqq 0,$$

that is,

$$-\tfrac{1}{2}\Delta|\underset{\sim}{u}|^2+|\nabla\underset{\sim}{u}|^2+2H\underset{\sim}{u}\cdot\underset{\sim}{u}_x\wedge\underset{\sim}{u}_y\leqq 0\quad\text{in}\quad\mathcal{D}'(\Omega).$$

Hence

$$-\Delta|\underset{\sim}{u}|^2\leqq 0\quad\text{in}\quad\mathcal{D}'(\Omega)$$

and thus, by Stampacchia's maximum principle, we conclude that

$$\operatorname*{Sup}_{\Omega}|\underset{\sim}{u}|=\operatorname*{Sup}_{\partial\Omega}|\underset{\sim}{u}|=R.$$

Finally, let $v\in\mathcal{D}(\Omega;\mathbb{R}^3)$ so that $\underset{\sim}{u}+tv\in K$ for $t\in\mathbb{R}$ with $|t|$ small enough. Then we have

$$E(\underset{\sim}{u})\leqq E(\underset{\sim}{u}+tv)$$

and consequently

$$\int\nabla\underset{\sim}{u}\cdot\nabla v+2H\int v\cdot\underset{\sim}{u}_x\wedge\underset{\sim}{u}_y=0,$$

that is, (1) holds.

Step 2. The main result of Step 2 is the following:

LEMMA 3. *Suppose $\underline{u} \in K$ satisfies*

$$E(\underline{u}) = \operatorname*{Inf}_{v \in K} E(v);$$

then there is some $\delta > 0$ such that

$$\int |\nabla v|^2 + 4H \int \underline{u} \cdot v_x \wedge v_y \geqq \delta \int |\nabla v|^2 \quad \textit{for all} \quad v \in H_0^1. \tag{13}$$

Proof: Let $v \in H_0^1 \cap L^\infty$; we have (using Lemma A.5)

$$E(\underline{u}+v) = E(\underline{u}) + E(v) + 2\int \nabla \underline{u} \cdot \nabla v + 4H \int v \cdot \underline{u}_x \wedge \underline{u}_y + 4H \int \underline{u} \cdot v_x \wedge v_y$$

and since $\underline{u}$ satisfies (1) we see that

$$E(\underline{u}+v) = E(\underline{u}) + E(v) + 4H \int \underline{u} \cdot v_x \wedge v_y \quad \text{for all} \quad v \in H_0^1 \cap L^\infty. \tag{14}$$

For $|t|$ small enough, $\underline{u} + tv \in K$, and thus we obtain

$$t^2 \int |\nabla v|^2 + \tfrac{4}{3}Ht^3 \int v \cdot v_x \wedge v_y + 4Ht^2 \int \underline{u} \cdot v_x \wedge v_y \geqq 0;$$

hence

$$\int |\nabla v|^2 + 4H \int \underline{u} \cdot v_x \wedge v_y \geqq 0 \quad \text{for all} \quad v \in H_0^1 \cap L^\infty.$$

It follows by density that

$$\int |\nabla v|^2 + 4H \int \underline{u} \cdot v_x \wedge v_y \geqq 0 \quad \text{for all} \quad v \in H_0^1. \tag{15}$$

We claim that

$$\int |\nabla v|^2 + 4H \int \underline{u} \cdot v_x \wedge v_y > 0 \quad \text{for all} \quad v \in H_0^1, \quad v \neq 0. \tag{16}$$

Indeed suppose that

$$\int |\nabla \underline{v}|^2 + 4H \int \underline{u} \cdot \underline{v}_x \wedge \underline{v}_y = 0 \quad \text{for some} \quad \underline{v} \in H_0^1; \tag{17}$$

we shall prove that $\underline{v} = 0$. Set, for v, $w \in H_0^1$,

$$B(v, w) = \int \nabla v \cdot \nabla w + 2H \int \underline{u} \cdot [(v_x \wedge w_y) + (w_x \wedge v_y)]$$

so that B is a bilinear symmetric form on H_0^1; moreover,

$$B(v, v) \geqq 0 \quad \text{for all} \quad v \in H_0^1 \qquad \text{(by (15))}$$

and

$$B(\underset{\sim}{v}, \underset{\sim}{v}) = 0.$$

It follows that $B(\underset{\sim}{v}, w) = 0$ for all $w \in H_0^1$. Using Lemma A.4 we obtain

$$\int \nabla \underset{\sim}{v} \cdot \nabla w + 2H \int w \cdot [(\underset{\sim}{u}_x \wedge \underset{\sim}{v}_y) + (\underset{\sim}{v}_x \wedge \underset{\sim}{u}_y)] = 0 \quad \text{for all} \quad w \in \mathscr{D},$$

that is

$$\begin{aligned} \Delta \underset{\sim}{v} &= 2H[(\underset{\sim}{u}_x \wedge \underset{\sim}{v}_y) + (\underset{\sim}{v}_x \wedge \underset{\sim}{u}_y)] \\ &= 2H[(\underset{\sim}{u} + \underset{\sim}{v})_x \wedge (\underset{\sim}{u} + \underset{\sim}{v})_y - \underset{\sim}{u}_x \wedge \underset{\sim}{u}_y - \underset{\sim}{v}_x \wedge \underset{\sim}{v}_y]. \end{aligned} \tag{18}$$

We rely on Lemma A.1 (or rather Remark A.1) to conclude that $\underset{\sim}{v} \in L^\infty$. Therefore, $\underset{\sim}{u} + t\underset{\sim}{v} \in K$ for $|t|$ small enough and we see, as above, that

$$t^2 \int |\nabla \underset{\sim}{v}|^2 + \tfrac{4}{3} H t^3 \int \underset{\sim}{v} \cdot \underset{\sim}{v}_x \wedge \underset{\sim}{v}_y + 4Ht^2 \int \underset{\sim}{u} \cdot \underset{\sim}{v}_x \wedge \underset{\sim}{v}_y \geqq 0.$$

It follows from (17) that

$$\int \underset{\sim}{v} \cdot \underset{\sim}{v}_x \wedge \underset{\sim}{v}_y = 0 \tag{19}$$

and thus, by (14),

$$E(\underset{\sim}{u} + t\underset{\sim}{v}) = E(\underset{\sim}{u}).$$

Applying Lemma 2 we see that for $|t|$ small enough

$$\Delta(\underset{\sim}{u} + t\underset{\sim}{v}) = 2H(\underset{\sim}{u} + t\underset{\sim}{v})_x \wedge (\underset{\sim}{u} + t\underset{\sim}{v})_y$$

and therefore

$$\underset{\sim}{v}_x \wedge \underset{\sim}{v}_y = 0.$$

Finally, we deduce from (17) that $\underset{\sim}{v} = 0$ and hence we have established (16). We turn now to the proof of (13). Assume, by contradiction, that there is a sequence (v^n) in H_0^1 such that

$$\int |\nabla v^n|^2 = 1, \tag{20}$$

and

$$\int |\nabla v^n|^2 + 4H \int \underset{\sim}{u} \cdot v_x^n \wedge v_y^n \to 0. \tag{21}$$

We may as well assume that $v^n \rightharpoonup \tilde{v}$ in H_0^1 weakly. In view of the lower semicontinuity of the function $B(v, v)$,

$$\int |\nabla \tilde{v}|^2 + 4H \int \underline{u} \cdot \tilde{v}_x \wedge \tilde{v}_y \leqq 0$$

and thus (by (16)) $\tilde{v}=0$. Hence $v^n \rightharpoonup 0$ in H_0^1 weakly. We deduce from Lemma A.9 that

$$\int \underline{u} \cdot v_x^n \wedge v_y^n \to 0. \tag{22}$$

Combining (20), (21) and (22) we obtain a contradiction.

Remark 4. There is a *unique* element $\underline{u} \in K$ such that

$$E(\underline{u}) = \operatorname*{Inf}_{v \in K} E(v).$$

Indeed suppose $\underline{\underline{u}}$ is another such element. Recall that (see (14)) for all $v \in H_0^1 \cap L^\infty$ we have

$$E(\underline{u}+v) = E(\underline{u}) + E(v) + 4H \int \underline{u} \cdot v_x \wedge v_y, \tag{23}$$

$$E(\underline{\underline{u}}-v) = E(\underline{\underline{u}}) + E(-v) + 4H \int \underline{\underline{u}} \cdot v_x \wedge v_y. \tag{24}$$

Choosing $v = \underline{\underline{u}} - \underline{u}$ and subtracting we obtain

$$\int v \cdot v_x \wedge v_y = 0.$$

Going back to (23) we deduce that

$$\int |\nabla v|^2 + 4H \int \underline{u} \cdot v_x \wedge v_y = 0$$

and thus (by Lemma 3) $v=0$. Throughout the paper we shall say that $\underline{u}$ is *the small solution* of Problem (1)–(2).

Step 3. We set $Q(v) = \int v \cdot v_x \wedge v_y$ for $v \in H_0^1 \cap L^\infty$ and we recall the isoperimetric inequality

$$|Q(v)|^{2/3} \leqq C \int |\nabla v|^2 \quad \text{for all} \quad v \in H_0^1 \cap L^\infty \tag{25}$$

(see Lemma A.3). The best constant in (25) is given by

LEMMA 4. *We have*

$$|Q(v)|^{2/3} \leqq \frac{1}{S} \int |\nabla v|^2 \quad \textit{for all} \quad v \in H_0^1 \cap L^\infty,$$

where $S = (32\pi)^{1/3}$ *provides the best constant.*

For the proof of Lemma 4 we refer to Wente [16]; the argument in [16] relies essentially on some inequality due to Bononcini.

Remark 5. We may extend by continuity the function Q to $H_0^1(\Omega)$ (see Lemma A.10) and we still have

$$|Q(v)|^{2/3} \leqq \frac{1}{S} \int |\nabla v|^2 \quad \text{for all} \quad v \in H_0^1(\Omega).$$

However the best constant is *not* achieved in $H_0^1(\Omega)$; otherwise we would obtain a non-zero function $v \in H_0^1(\Omega)$ which satisfies $\Delta v = v_x \wedge v_y$ on Ω—a contradiction with Wente [17] (see also Remark 3). On the other hand, we may consider

$$Q(\varphi) = \int_{\mathbb{R}^2} \varphi \cdot \varphi_x \wedge \varphi_y \quad \text{defined} \quad \text{for} \quad \varphi \in H^1(\mathbb{R}^2; \mathbb{R}^3) \cap L^\infty(\mathbb{R}^2; \mathbb{R}^3).$$

We deduce from Lemma 4 (by stretching variables) that

$$|Q(\varphi)|^{2/3} \leqq \frac{1}{S} \int_{\mathbb{R}^2} |\nabla \varphi|^2, \quad \text{for all} \quad \varphi \in \mathcal{D}(\mathbb{R}^2; \mathbb{R}^3),$$

and by density we obtain

$$(26) \quad |Q(\varphi)|^{2/3} \leqq \frac{1}{S} \int_{\mathbb{R}^2} |\nabla \varphi|^2, \quad \text{for all} \quad \varphi \in L^\infty(\mathbb{R}^2; \mathbb{R}^3) \quad \text{with} \quad \varphi_x, \varphi_y \in L^2(\mathbb{R}^2; \mathbb{R}^3).$$

For later purpose, it is important to observe that the best constant in (26) is achieved when

$$\varphi(x, y) = (1 + r^2)^{-1}(x, y, 1) \quad \text{with} \quad r^2 = x^2 + y^2,$$

or more generally when

$$\varphi(x, y) = \varphi_\varepsilon(x, y) = (\varepsilon^2 + r^2)^{-1}(x, y, \varepsilon) \quad \text{with} \quad \varepsilon \in \mathbb{R}, \ \varepsilon \neq 0.$$

A similar phenomenon occurs with the best constants of Sobolev inequalities (see [3]).

The main result of Step 3 is the following.

LEMMA 5. *Assume u is a given function such that* $u \in C^2(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ *and u is not constant on* Ω. *Set*

$$J = \operatorname*{Inf}_{\substack{v \in H_0^1 \cap L^\infty \\ Q(v) \neq 0}} \left\{ \frac{\displaystyle\int |\nabla v|^2 + 4H \int u \cdot v_x \wedge v_y}{|Q(v)|^{2/3}} \right\};$$

then

$$J < S.$$

Proof: Fix a point $(x_0, y_0) \in \Omega$ such that $\nabla u(x_0, y_0) \neq 0$. Set $\vec{a} = u_x(x_0, y_0)$ and $\vec{b} = u_y(x_0, y_0)$. We choose an orthonormal basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$ in $\mathbb{R}^3$ having the same orientation as the canonical basis of $\mathbb{R}^3$ and such that

(27) $$\vec{a} \cdot \vec{\imath} + \vec{b} \cdot \vec{\jmath} < 0$$

(for example, if $\vec{a} \neq 0$ we take $\vec{\imath} = -\vec{a}/|\vec{a}|$ and then $\vec{\jmath}$ such that $|\vec{\jmath}| = 1$, $\vec{\imath} \cdot \vec{\jmath} = 0$, $\vec{b} \cdot \vec{\jmath} = 0$). Next we use a technique inspired by [3] (see also [1]). We set

$$v^\varepsilon = \zeta \varphi^\varepsilon \quad \text{for} \quad \varepsilon > 0,$$

where $\zeta \in \mathscr{D}(\Omega; \mathbb{R})$ is a fixed function such that $\zeta \equiv 1$ near (x_0, y_0) and

$$\varphi^\varepsilon(x, y) = f_\varepsilon(r)(x - x_0, y - y_0, \varepsilon)$$

with $f_\varepsilon(r) = (\varepsilon^2 + r^2)^{-1}$ and $r^2 = (x - x_0)^2 + (y - y_0)^2$ (φ^ε is written with respect to the basis $\vec{\imath}, \vec{\jmath}, \vec{k}$).

We consider

$$R(v^\varepsilon) = \frac{\displaystyle\int |\nabla v^\varepsilon|^2 + 4H \int u \cdot v_x^\varepsilon \wedge v_y^\varepsilon}{|Q(v^\varepsilon)|^{2/3}}$$

and we shall establish that

(28) $$R(v^\varepsilon) = S + SH(\vec{a} \cdot \vec{\imath} + \vec{b} \cdot \vec{\jmath})\varepsilon + O(\varepsilon^2 |\log \varepsilon|) \quad \text{as} \quad \varepsilon \to 0.$$

The conclusion of Lemma 5 follows from (28) by choosing ε small enough. We have

$$v_x^\varepsilon = \zeta_x \varphi^\varepsilon + \zeta \varphi_x^\varepsilon, \qquad v_y^\varepsilon = \zeta_y \varphi^\varepsilon + \zeta \varphi_y^\varepsilon$$

and thus

$$\int |\nabla v^\varepsilon|^2 = \int \zeta^2 |\nabla \varphi^\varepsilon|^2 + O(1) = \int |\nabla \varphi^\varepsilon|^2 + \int (\zeta^2 - 1)|\nabla \varphi^\varepsilon|^2 + O(1)$$
$$= \int |\nabla \varphi^\varepsilon|^2 + O(1).$$

On the other hand, it is easy to check that

(29) $$|\nabla \varphi^\varepsilon|^2 = 2 f_\varepsilon^2$$

and

$$\int_\Omega f_\varepsilon^2 = \int_{\mathbf{R}^2} f_\varepsilon^2 + O(1) = \frac{\pi}{\varepsilon^2} + O(1). \tag{30}$$

Therefore we obtain

$$\int |\nabla v^\varepsilon|^2 = \frac{2\pi}{\varepsilon^2} + O(1). \tag{31}$$

Next we have

$$v^\varepsilon \cdot (v_x^\varepsilon \wedge v_y^\varepsilon) = \zeta^3 \varphi^\varepsilon \cdot (\varphi_x^\varepsilon \wedge \varphi_y^\varepsilon) = \varepsilon \zeta^3 f_\varepsilon^3 = \varepsilon f_\varepsilon^3 + \varepsilon(\zeta^3 - 1) f_\varepsilon^3$$

and

$$\int_\Omega f_\varepsilon^3 = \int_{\mathbf{R}^2} f_\varepsilon^3 + O(1) = \frac{\pi}{2\varepsilon^4} + O(1).$$

Hence

$$Q(v^\varepsilon) = \frac{\pi}{2\varepsilon^3} + O(\varepsilon)$$

and thus

$$|Q(v^\varepsilon)|^{2/3} = (\tfrac{1}{2}\pi)^{2/3} \frac{1}{\varepsilon^2} (1 + O(\varepsilon^4)). \tag{32}$$

Finally we write

$$u(x, y) = u(x_0, y_0) + \vec{a}(x - x_0) + \vec{b}(y - y_0) + O(r^2)$$

and thus

$$\int u \cdot (v_x^\varepsilon \wedge v_y^\varepsilon) = I + II + III,$$

where

$$I = \int u(x_0, y_0) \cdot (v_x^\varepsilon \wedge v_y^\varepsilon),$$

$$II = \int [\vec{a}(x - x_0) + \vec{b}(y - y_0)] \cdot (v_x^\varepsilon \wedge v_y^\varepsilon),$$

$$III = \int O(r^2) \cdot (v_x^\varepsilon \wedge v_y^\varepsilon).$$

From Lemma A.5 we deduce that

$$I = 0. \tag{33}$$

We shall verify (see below) that

$$II=(\vec{a}\cdot\vec{\imath}+\vec{b}\cdot\vec{\jmath})\frac{\pi}{2\varepsilon}+O(1) \tag{34}$$

and

$$III=O(|\log\varepsilon|) \tag{35}$$

which imply that

$$\int u\cdot(v_x^\varepsilon\wedge v_y^\varepsilon)=(\vec{a}\cdot\vec{\imath}+\vec{b}\cdot\vec{\jmath})\frac{\pi}{2\varepsilon}+O(|\log\varepsilon|). \tag{36}$$

Combining (31), (32) and (36) we obtain (28).

Proof of (34): By Lemma A.5,

$$\begin{aligned}II&=\frac{1}{2}\int v^\varepsilon\cdot[(\vec{a}\wedge v_y^\varepsilon)+(v_x^\varepsilon\wedge\vec{b})]\\&=\frac{1}{2}\int\zeta^2\varphi^\varepsilon\cdot[(\vec{a}\wedge\varphi_y^\varepsilon)+(\varphi_x^\varepsilon\wedge\vec{b})]\\&=\frac{1}{2}\int\zeta^2\varphi^\varepsilon f_\varepsilon\cdot[\vec{a}\wedge\vec{\jmath}+\vec{\imath}\wedge\vec{b}]\\&=\frac{1}{2}\int\zeta^2 f_\varepsilon^2[(a_1+b_2)\varepsilon-a_3(x-x_0)-b_3(y-y_0)].\end{aligned}$$

From (30) we deduce that

$$\frac{1}{2}\int\zeta^2 f_\varepsilon^2(a_1+b_2)\varepsilon=(a_1+b_2)\frac{\pi}{2\varepsilon}+O(\varepsilon).$$

On the other hand,

$$\frac{1}{2}\int_\Omega\zeta^2 f_\varepsilon^2[a_3(x-x_0)+b_3(y-y_0)]=\frac{1}{2}\int_B f_\varepsilon^2[a_3(x-x_0)+b_3(y-y_0)]+O(1),$$

where B denotes a small ball centered at (x_0, y_0), and then

$$\int_B f_\varepsilon^2(x-x_0)=\int_B f_\varepsilon^2(y-y_0)=0.$$

Proof of (35): Using (29) we obtain

$$\begin{aligned}|III|&\le C\int r^2|\nabla v^\varepsilon|^2\le C\int r^2|\nabla\varphi^\varepsilon|^2+O(1)\\&=2C\int r^2 f_\varepsilon^2+O(1)\le 2C\int f_\varepsilon+O(1)=O(|\log\varepsilon|).\end{aligned}$$

Step 4. We consider the function Q defined on H_0^1 to be the continuous extension of $Q(v)=\int v\cdot v_x\wedge v_y$ $(v\in H_0^1\cap L^\infty)$—see Lemma A.10. We set

$$J=\operatorname*{Inf}_{\substack{v\in H_0^1\\ Q(v)=1}}\left\{\int|\nabla v|^2+4H\int \underline{u}\cdot v_x\wedge v_y\right\}. \tag{37}$$

LEMMA 6. *The infimum which defines J (see (37)) is achieved, i.e., there exists some $v^0\in H_0^1$ such that*

$$Q(v^0)=1 \quad \textit{and} \quad J=\int|\nabla v^0|^2+4H\int \underline{u}\cdot v_x^0\wedge v_y^0.$$

Proof: It follows from Lemma 5 that $J<S$ (recall that $\underline{u}\in C^2(\Omega;\mathbf{R}^3)$—see Remark 1—and moreover $\underline{u}$ is not constant on Ω because of assumption (5)). Let (v^n) be a minimizing sequence, that is,

$$Q(v^n)=1 \tag{38}$$

and

$$\int|\nabla v^n|^2+4H\int \underline{u}\cdot v_x^n\wedge v_y^n=J+o(1). \tag{39}$$

We deduce from Lemma 3 and (39) that (v^n) remains bounded in H_0^1. We may assume, modulo a subsequence, that $v^n\rightharpoonup v^0$ in H_0^1 weakly. In order to pass to the limit we use an argument which is related to a method introduced by E. Lieb [10]. Set

$$w^n=v^n-v^0$$

so that $w^n\rightharpoonup 0$ in H_0^1 weakly. Thus we have $Q(v^0+w^n)=1$ and using Lemma A.12 we obtain

$$Q(v^0)+Q(w^n)=1+o(1). \tag{40}$$

On the other hand, we know (see Lemma A.9) that

$$\int \underline{u}\cdot v_x^n\wedge v_y^n\to\int \underline{u}\cdot v_x^0\wedge v_y^0$$

and therefore we deduce from (39) that

$$\int|\nabla v^0|^2+\int|\nabla w^n|^2+4H\int \underline{u}\cdot v_x^0\wedge v_y^0=J+o(1). \tag{41}$$

From the definition of J (see (37)) we have

$$\int|\nabla v^0|^2+4H\int \underline{u}\cdot v_x^0\wedge v_y^0\geqq J|Q(v^0)|^{2/3}$$

which, together with (41), implies that

$$J|Q(v^0)|^{2/3}+\int|\nabla w^n|^2\leqq J+o(1). \tag{42}$$

From (40) it follows that

$$1\leqq|Q(v^0)|^{2/3}+|Q(w^n)|^{2/3}+o(1). \tag{43}$$

Combining (42) and (43) we are led to

$$\begin{aligned}\int|\nabla w^n|^2&\leqq J|Q(w^n)|^{2/3}+o(1)\\&\leqq\frac{J}{S}\int|\nabla w^n|^2+o(1)\end{aligned}$$

(by Lemma 4). Since $J<S$, we conclude that $\int|\nabla w^n|^2\to 0$ and hence $v^n\to v^0$ in H_0^1 strongly. We complete the proof of Lemma 6 by passing to the limit in (38) and (39).

We conclude the proof of Theorem 1 using

LEMMA 7. *Set*

$$\bar{u}=\underline{u}-\frac{J}{2H}v^0$$

(J and v^0 have been defined in Lemma 6). Then $\underline{u}$ is a solution of (1)–(2) and, moreover,

$$E(\bar{u})=E(\underline{u})+\frac{J^3}{12H^2},$$

Remark 6. Clearly, $\bar{u}\neq\underline{u}$ since $J>0$ (note that, by Lemmas 3 and 4, $J\geqq\delta S$).

Proof: Let $w\in H_0^1\cap L^\infty$; set

$$\varphi=\frac{1}{\mu}(v^0+tw)\quad\text{with}\quad t\in\mathbf{R}\quad\text{and}\quad\mu=|Q(v^0+tw)|^{1/3}.$$

Note that $Q(v^0+tw)\to 1$ as $t\to 0$ and thus $Q(\varphi)=1$ for $|t|$ small enough. From the definition of J (see (37)) we have

$$\begin{aligned}J&=\int|\nabla v^0|^2+4H\int\underline{u}\cdot(v_x^0\wedge v_y^0)\leqq\int|\nabla\varphi|^2+4H\int\underline{u}\cdot(\varphi_x\wedge\varphi_y)\\&=\frac{1}{\mu^2}\Big[\int|\nabla v^0|^2+2t\int\nabla v^0\cdot\nabla w+4H\int\underline{u}\cdot(v_x^0\wedge v_y^0)\\&\quad+4Ht\int\underline{u}\cdot[(v_x^0\wedge w_y)+(w_x\wedge v_y^0)]+O(t^2)\Big].\end{aligned}$$

On the other hand (see Lemma A.11), we have

$$\mu^3 = |Q(v^0) + 3tR(w, v^0) + O(t^2)| = 1 + 3tR(w, v^0) + O(t^2)$$

and thus

$$\frac{1}{\mu^2} = 1 - 2tR(w, v^0) + O(t^2).$$

Finally we obtain

$$\int \nabla v^0 \cdot \nabla w + 2H \int \underline{u} \cdot [(v_x^0 \wedge w_y) + (w_x \wedge v_y^0)] - J \int w \cdot (v_x^0 \wedge v_y^0) = 0.$$

Using Lemma A.4 we deduce that

$$\int \nabla v^0 \cdot \nabla w + 2H \int w \cdot [(\underline{u}_x \wedge v_y^0) + (v_x^0 \wedge \underline{u}_y)] - J \int w \cdot (v_x^0 \wedge v_y^0) = 0,$$

that is,

$$\Delta v^0 = 2H[(\underline{u}_x \wedge v_y^0) + (v_x^0 \wedge \underline{u}_y)] - J(v_x^0 \wedge v_y^0).$$

Hence we find

$$\begin{aligned}\Delta \bar{u} = \Delta \underline{u} - \frac{J}{2H}\Delta v^0 &= 2H\underline{u}_x \wedge \underline{u}_y - J[(\underline{u}_x \wedge v_y^0) + (v_x^0 \wedge \underline{u}_y)] + \frac{J^2}{2H}(v_x^0 \wedge v_y^0) \\ &= 2H\bar{u}_x \wedge \bar{u}_y.\end{aligned}$$

It follows that $\bar{u} \in L^\infty$ (see Remark 1) and thus $v^0 \in L^\infty$. In conclusion we have (see (14))

$$\begin{aligned}E(\bar{u}) = E\left(\underline{u} - \frac{J}{2H}v^0\right) &= E(\bar{u}) + E\left(-\frac{J}{2H}v^0\right) + \frac{J^2}{H}\int \underline{u} \cdot (v_x^0 \wedge v_y^0) \\ &= E(\underline{u}) + \frac{J^2}{4H^2}\int |\nabla v^0|^2 - \frac{J^3}{6H^2} + \frac{J^2}{H}\int \underline{u} \cdot (v_x^0 \wedge v_y^0) \\ &= E(\underline{u}) + \frac{J^3}{4H^2} - \frac{J^3}{6H^2} = E(\underline{u}) + \frac{J^3}{12H^2}.\end{aligned}$$

2. The Plateau Problem

Let $\Gamma \subset \mathbb{R}^3$ be a closed Jordan curve; more precisely we assume that $\Gamma = \alpha(\partial\Omega)$, where $\alpha: \partial\Omega \to \mathbb{R}^3$ is one-to-one and

(44) $$\alpha \in C(\partial\Omega; \mathbb{R}^3) \cap H^{1/2}(\partial\Omega; \mathbb{R}^3).$$

We set

$$R = \operatorname*{Max}_{\partial\Omega} |\alpha|.$$

DEFINITION. We say that a continuous mapping $\eta:\partial\Omega\to\partial\Omega$ is *non-decreasing* if there is a continuous non-decreasing function $f:[0,2\pi]\to\mathbb{R}$ such that

$$f(2\pi)-f(0)=2\pi \quad \text{and} \quad \eta(e^{i\vartheta})=e^{if(\vartheta)} \quad \text{for all} \quad \vartheta\in[0,2\pi].$$

We consider the following problem: find $u\in H^1(\Omega;\mathbb{R}^3)\cap C(\bar\Omega;\mathbb{R}^3)$ satisfying

$$\Delta u=2Hu_x\wedge u_y \quad \text{on} \quad \Omega, \tag{45}$$

$$|u_x|^2-|u_y|^2=u_x\cdot u_y=0 \quad \text{on} \quad \Omega, \tag{46}$$

$$u(\partial\Omega)=\Gamma \quad \text{and} \quad \alpha^{-1}\circ u \quad \text{is non-decreasing} \quad \text{on} \quad \partial\Omega, \tag{47}$$

where $H>0$ is a given constant.

Our main result is the following.

THEOREM 2. *Assume* (44) *and*

$$HR<1. \tag{48}$$

Then there exist at least two distinct solutions[3] *of* (45)–(46)–(47).

Remark 7. It has been known (see Hildebrandt [8]) that if (44) is satisfied and $HR\leqq 1$, then there exists at least *one* solution of (45)–(46)–(47). We believe that if Γ is a circle of radius R and $H=1/R$, then there exists *exactly one*[3] solution of (45)–(46)–(47); this means that assumption (48) is presumably sharp for the circle.

We shall use the following notation:

$$\mathscr{C}=\left\{\gamma\,\middle|\,\begin{array}{l}\gamma\in C(\partial\Omega;\mathbb{R}^3)\cap H^{1/2}(\partial\Omega;\mathbb{R}^3),\ \gamma(\partial\Omega)=\Gamma,\\ \alpha^{-1}\circ\gamma \text{ is non-decreasing and } \alpha^{-1}\circ\gamma \text{ leaves}\\ \text{invariant the 3 points } e^{i\vartheta} \text{ with } \vartheta=0,\ \vartheta=\pm\frac{2}{3}\pi\end{array}\right\}.$$

Clearly $\mathscr{C}\neq\varnothing$ since $\alpha\in\mathscr{C}$.

Remark 8. For the proof of Theorem 2 we shall not need the assumption that $\alpha\in H^{1/2}(\partial\Omega;\mathbb{R}^3)$ but only the fact that $\mathscr{C}\neq\varnothing$.

As before we fix $R'>R$ with $HR'<1$. Consider

$$\operatorname{Inf}\{E(v)\,|\,v\in H^1(\Omega;\mathbb{R}^3),\ \|v\|_{L^\infty}\leqq R' \text{ and } v_{|\partial\Omega}\in\mathscr{C}\}.$$

It may be shown (see Hildebrandt [8], [9]) that the infimum is achieved by some function $\underline{u}_p$ which is a "*small*" *solution of the Plateau problem* (45)–(46)–(47).

In order to prove the existence of *a second solution of the Plateau problem* we proceed as follows. For each $\gamma\in\mathscr{C}$ we have a "small" solution $\underline{u}_\gamma$ of the

[3] We say that two solutions are distinct if one cannot pass from one to the other by a conformal diffeomorphism.

Dirichlet problem (1)–(2) which is uniquely defined (see Lemmas 1, 2 and Remark 4). We set

$$J_\gamma = \inf_{\substack{\varphi \in H_0^1 \\ Q(\varphi)=1}} \left\{ \int |\nabla \varphi|^2 + 4H \int \underline{u}_\gamma \cdot \varphi_x \wedge \varphi_y \right\}; \tag{49}$$

this infimum is achieved by some function φ^γ (not necessarily unique) and then

$$\bar{u}^\gamma = \underline{u}_\gamma - \frac{J_\gamma}{2H} \varphi^\gamma$$

provides a second solution for the Dirichlet problem (1)–(2) such that

$$E(\bar{u}^\gamma) = E(\underline{u}_\gamma) + \frac{J_\gamma^3}{12H^2}$$

(see Lemma 6 and 7). We set

$$A(\gamma) = E(\underline{u}_\gamma) + \frac{J_\gamma^3}{12H^2}; \tag{50}$$

notice that $A(\gamma)$ is uniquely defined although $\bar{u}^\gamma$ is not!

The proof of Theorem 2 is divided into two steps.

Step 1. We show that

$$\inf_{\gamma \in \mathscr{C}} A(\gamma)$$

is achieved by some $\gamma^0 \in \mathscr{C}$.

Step 2. We prove that $\bar{u}^{\gamma^0}$ is a solution of the Plateau problem (45)–(46)–(47); moreover, $\bar{u}^{\gamma^0} \neq \underline{u}_p$ since

$$E(\bar{u}^{\gamma^0}) = E(\underline{u}_{\gamma^0}) + \frac{J_{\gamma^0}^3}{12H^2} \geqq E(\underline{u}_p) + \frac{J_{\gamma^0}^3}{12H^2} > E(\underline{u}_p).$$

Step 1. We shall need a technical lemma dealing with the dependence of the mapping $\gamma \mapsto \underline{u}_\gamma$ under uniform convergence of the γ's. Suppose (γ^n) is a sequence such that

$$\gamma^n \in H^{1/2}(\partial\Omega; \mathbb{R}^3) \cap L^\infty(\partial\Omega; \mathbb{R}^3), \qquad \|\gamma^n\|_{L^\infty} \leqq R,$$

and let $\gamma \in H^{1/2}(\partial\Omega; \mathbb{R}^3) \cap L^\infty(\partial\Omega; \mathbb{R}^3)$ with $\|\gamma\|_{L^\infty} \leqq R$. Let $\underline{u}^n$ (respectively $\underline{u}$) denote the "small" solution of the Dirichlet problem (1)–(2) corresponding to the boundary data γ^n (respectively γ). Assume

$$\|\gamma^n - \gamma\|_{L^\infty(\partial\Omega)} \to 0 \quad \text{and} \quad \|\gamma^n\|_{H^{1/2}(\partial\Omega)} \leqq C \tag{51}$$

so that $\gamma^n \rightharpoonup \gamma$ in $H^{1/2}$ weakly. Let $h^n \in H^1(\Omega; \mathbb{R}^3)$ denote the solution of the problem

$$\Delta h^n = 0 \quad \text{on} \quad \Omega,$$
$$h^n = \gamma^n - \gamma \quad \text{on} \quad \partial\Omega.$$

By the maximum principle we have

$$\|h^n\|_{L^\infty(\Omega)} \leqq \|\gamma^n - \gamma\|_{L^\infty(\partial\Omega)} \to 0; \tag{52}$$

moreover,

$$h^n \rightharpoonup 0 \quad \text{in} \quad H^1 \quad \text{weakly}. \tag{53}$$

LEMMA 8. *Assume* (51); *then we have*

$$\|\underline{u}^n - h^n - \underline{u}\|_{H_0^1} \to 0 \tag{54}$$

and

$$E(\underline{u}) \leqq \underline{\lim}\, E(\underline{u}^n). \tag{55}$$

Proof: Set

$$K^n = \{v \in H^1(\Omega; \mathbb{R}^3);\ v = \gamma^n \text{ on } \partial\Omega \text{ and } \|v\|_{L^\infty} \leqq R'\},$$
$$K = \{v \in H^1(\Omega; \mathbb{R}^3);\ v = \gamma \text{ on } \partial\Omega \text{ and } \|v\|_{L^\infty} \leqq R'\}.$$

Since $\|\underline{u}\|_{L^\infty} \leqq R$ (see Lemma 2) it follows that $\underline{u} + h^n \in K^n$ for n large enough; therefore we have

$$E(\underline{u}^n) \leqq E(\underline{u} + h^n) \quad \text{for all} \quad n \text{ (large enough)}. \tag{56}$$

Set $v^n = \underline{u}^n - h^n$, so that $v^n = \gamma$ on $\partial\Omega$; moreover,

$$\|v^n\|_{L^\infty} \leqq \|\underline{u}^n\|_{L^\infty} + \|h^n\|_{L^\infty} \leqq R + \|h^n\|_{L^\infty}$$

and thus $v^n \in K$ for all n (large enough). We claim that

$$E(v^n) \leqq E(\underline{u}) + o(1); \tag{57}$$

this means that (v^n) is a minimizing sequence for E on K and, therefore, (see Lemma 1 and Remark 4), $v^n \to \underline{u}$ in H^1 strongly, i.e., (54) holds.

Proof of (57): Recall that (see (6))

$$\frac{1}{3}\int |\nabla \underline{u}^n|^2 \leqq E(\underline{u}^n); \tag{58}$$

since $E(\underline{u} + h^n)$ remains bounded, it follows from (56) and (58) that $\int |\nabla \underline{u}^n|^2$ remains bounded. Consequently,

$$\|\underline{u}^n\|_{H^1} \leqq C, \qquad \|v^n\|_{H^1} \leqq C. \tag{59}$$

On the other hand, we have

$$E(\underline{u}^n)=\int|\nabla\underline{u}^n|^2+\tfrac{4}{3}H\int\underline{u}^n\cdot\underline{u}_x^n\wedge\underline{u}_y^n, \tag{60}$$

$$E(\underline{u}+h^n)=\int|\nabla(\underline{u}+h^n)|^2+\tfrac{4}{3}H\int(\underline{u}+h^n)\cdot(\underline{u}_x+h_x^n)\wedge(\underline{u}_y+h_y^n), \tag{61}$$

$$\int|\nabla\underline{u}^n|^2=\int|\nabla(v^n+h^n)|^2=\int|\nabla h^n|^2+\int|\nabla v^n|^2+2\int\nabla v^n\cdot\nabla h^n. \tag{62}$$

Let us check that

$$\int\underline{u}^n\cdot(\underline{u}_x^n\wedge\underline{u}_y^n)=\int v^n\cdot(v_x^n\wedge v_y^n)+\int v^n\cdot(h_x^n\wedge h_y^n)+o(1); \tag{63}$$

indeed we have

$$\int\underline{u}^n\cdot(\underline{u}_x^n\wedge\underline{u}_y^n)=\int v^n\cdot(v_x^n\wedge v_y^n)+\int v^n\cdot(h_x^n\wedge h_y^n)+I+II$$

with

$$I=\int v^n\cdot[(v_x^n\wedge h_y^n)+(h_x^n\wedge v_y^n)]=o(1),\qquad\text{by Lemma A.7,}$$

and

$$II=\int h^n\cdot(\underline{u}_x^n\wedge\underline{u}_y^n)=o(1),\qquad\text{by (52) and (59).}$$

Combining (56), (60), (61), (62) and (63) we obtain

$$E(v^n)\leqq E(\underline{u})=2\int\nabla(\underline{u}-v^n)\cdot\nabla h^n+\tfrac{4}{3}H\int(\underline{u}-v^n)\cdot h_x^n\wedge h_y^n+o(1).$$

Finally we observe that

$$\int\nabla(\underline{u}-v^n)\nabla h^n=0,$$

since $\underline{u}-v^n=0$ on $\partial\Omega$ and $\Delta h^n=0$; moreover,

$$\int(\underline{u}-v^n)\cdot(h_x^n\wedge h_y^n)=o(1),\qquad\text{by Lemma A.6.}$$

This concludes the proof of (57) (and thus (54)).

Proof of (55): Combining (62) and (63) we have

$$E(\underline{u}^n)=E(v^n)+\int|\nabla h^n|^2+\tfrac{4}{3}H\int v^n\cdot(h_x^n\wedge h_y^n)+o(1)$$

and thus

$$E(\underline{u}^n) \geqq E(\underline{u}) + \frac{1}{3}\int |\nabla h^n|^2 + o(1) \geqq E(\underline{u}) + o(1)$$

which implies (55).

We recall a well-known compactness result (see [4], Lemma 3.2, page 103).

LEMMA 9. *Let (γ^n) be a sequence in $\mathscr{C}$ such that $\|\gamma^n\|_{H^{1/2}}$ remains bounded. Then there exist a subsequence (γ^{n_k}) and some $\gamma^0 \in \mathscr{C}$ such that*

$$\|\gamma^{n_k} - \gamma^0\|_{L^\infty(\partial\Omega)} \to 0.$$

The main existence result of Step 1 is the following.

LEMMA 10. *There exists some $\gamma^0 \in \mathscr{C}$ such that*

$$A(\gamma^0) = \operatorname*{Inf}_{\gamma \in \mathscr{C}} A(\gamma).$$

Proof: Let (γ^n) be a minimizing sequence for A, i.e.,

$$\gamma^n \in \mathscr{C} \quad \text{and} \quad A(\gamma^n) = \operatorname*{Inf}_{\gamma \in \mathscr{C}} A(\gamma) + o(1). \tag{64}$$

Let $\underline{u}^n$ denote the small solution of the Dirichlet problem (1)–(2) corresponding to the boundary data γ^n. We have

$$\frac{1}{3}\int |\nabla \underline{u}^n|^2 \leqq E(\underline{u}^n) \leqq A(\gamma^n) \leqq C.$$

Since $\|\underline{u}^n\|_{L^\infty} \leqq R$, it follows that

$$\|\underline{u}^n\|_{H^1} \leqq C$$

and in particular

$$\|\gamma^n\|_{H^{1/2}} \leqq C.$$

Using Lemma 9 we may assume that

$$\|\gamma^n - \gamma^0\|_{L^\infty} \to 0$$

for some $\gamma^0 \in \mathscr{C}$.

We denote by $\underline{u}^0$ the small solution of the Dirichlet problem (1)–(2) corresponding to the boundary data γ^0. Set

$$J_n = \operatorname*{Inf}_{\substack{\varphi \in H_0^1 \\ Q(\varphi)=1}} \left\{ \int |\nabla \varphi|^2 + 4H \int \underline{u}^n \cdot \varphi_x \wedge \varphi_y \right\},$$

$$J_0 = \operatorname*{Inf}_{\substack{\varphi \in H_0^1 \\ Q(\varphi)=1}} \left\{ \int |\nabla \varphi|^2 + 4H \int \underline{u}^0 \cdot \varphi_x \wedge \varphi_y \right\}.$$

Clearly we have

$$J_n \leqq C \tag{65}$$

(use any fixed $\varphi \in \mathcal{D}$ with $Q(\varphi)=1$). It follows from Lemma 8 that

$$E(\underline{u}^0) \leqq E(\underline{u}^n) + o(1). \tag{66}$$

On the other hand, we deduce from Lemma 6 that there exists some $\varphi^n \in H_0^1$ such that

$$Q(\varphi^n)=1 \quad \text{and} \quad J_n = \int |\nabla \varphi^n|^2 + 4H \int \underline{u}^n \cdot \varphi_x^n \wedge \varphi_y^n. \tag{67}$$

With the notations of Lemma 8 we have

$$\begin{aligned}
&\int |\nabla \varphi^n|^2 + 4H \int \underline{u}^0 \cdot \varphi_x^n \wedge \varphi_y^n \\
&\quad = J_n + 4H \int (\underline{u}^0 - \underline{u}^n + h^n) \cdot \varphi_x^n \wedge \varphi_y^n - 4H \int h^n \cdot \varphi_x^n \wedge \varphi_y^n \\
&\quad \leqq J_n + C \|\underline{u}^0 - \underline{u}^n + h^n\|_{H_0^1} \cdot \int |\nabla \varphi^n|^2 + 2H \|h^n\|_{L^\infty} \int |\nabla \varphi^n|^2
\end{aligned}$$

(here we have used Lemma A.3). We deduce from Lemma 8 that

$$\int |\nabla \varphi^n|^2 + 4H \int \underline{u}^0 \cdot \varphi_x^n \wedge \varphi_y^n \leqq J_n + o(1) \int |\nabla \varphi^n|^2. \tag{68}$$

We recall (see Lemma 3) that there exists some $\delta > 0$ such that

$$\int |\nabla \varphi^n|^2 + 4H \int \underline{u}^0 \cdot \varphi_x^n \wedge \varphi_y^n \geqq \delta \int |\nabla \varphi^n|^2 \quad \text{for all} \quad n. \tag{69}$$

Combining (65), (68) and (69) we see that

$$\int |\nabla \varphi^n|^2 \leqq C. \tag{70}$$

From the definition of J_0 we have

$$J_0 \leqq \int |\nabla \varphi^n|^2 + 4H \int \underline{u}^0 \cdot \varphi_x^n \wedge \varphi_y^n. \tag{71}$$

Relations (68), (70) and (71) together yield

$$J_0 \leqq J_n + o(1). \tag{72}$$

Finally from (66) and (72) it follows that

$$A(\gamma^0)=E(\underline{u}^0)+J_0^3/12H^2 \leqq E(\underline{u}^n)+J_n^3/12H^2+o(1)=A(\gamma^n)+o(1).$$

We conclude using (64) that $A(\gamma^0)=\operatorname{Inf}_{\gamma\in\mathscr{C}} A(\gamma)$.

Step 2. We start with some technical facts.

LEMMA 11. *Let $\gamma\in\mathscr{C}$ and let $\underline{u}$ be the small solution of the Dirichlet problem* (1)–(2). *Then*

$$\operatorname*{Sup}_{t\geqq 0} E(\underline{u}+tv)\geqq A(\gamma) \quad \textit{for all} \quad v\in H_0^1\cap L^\infty, \qquad v\not\equiv 0.$$

Proof: Recall that

$$E(\underline{u}+tv)=E(\underline{u})+t^2\left[\int\!\!\int |\nabla v|^2+4H\int \underline{u}\cdot v_x\wedge v_y\right]+\tfrac{4}{3}Ht^3Q(v) \tag{73}$$

(see (14)). Therefore,

$$\operatorname*{Sup}_{t\geqq 0} E(\underline{u}+tv)=+\infty \quad \text{provided} \quad Q(v)\geqq 0 \quad \text{and} \quad v\not\equiv 0.$$

We assume now that $Q(v)<0$; a simple computation leads to

$$\begin{aligned}\operatorname*{Sup}_{t\geqq 0} E(\underline{u}+tv)&=E(\underline{u})+\frac{1}{12H^2}\,\frac{\left[\int\!\!\int |\nabla v|^2+4H\int \underline{u}\cdot v_x\wedge v_y\right]^3}{|Q(v)|^2}\\&\geqq E(\underline{u})+\frac{J^3}{12H^2}=A(\gamma).\end{aligned}$$

The next lemma is a slight generalization of Lemma 11.

LEMMA 12. *Let $\gamma\in\mathscr{C}$ and let $\underline{u}$ be the small solution of the Dirichlet problem* (1)–(2). *Then*

$$\operatorname*{Sup}_{t\geqq 0} E(\underline{u}+\varphi+tv)\geqq A(\gamma) \quad \textit{for all} \quad \varphi\in H_0^1\cap L^\infty \quad \textit{with} \quad \|\varphi\|_{H_0^1}\leqq\frac{C\delta}{H},$$
$$\textit{for all} \quad v\in H_0^1\cap L^\infty, \qquad v\not\equiv 0,$$

where C is some universal constant and $\delta>0$ appears in Lemma 3.

Proof: We distinguish two cases:
(a) $Q(v)\geqq 0$,
(b) $Q(v)<0$.

Case (a). The leading terms in the expansion of $E(\underline{u}+\varphi+tv)$ are

$$\tfrac{4}{3}Ht^3Q(v)+t^2\left[\int|\nabla v|^2+4H\int\underline{u}\cdot v_x\wedge v_y+4H\int\varphi\cdot v_x\wedge v_y\right].$$

If $Q(v)>0$, we have $\operatorname{Sup}_{t\geqq 0}E(\underline{u}+\varphi+tv)=+\infty$; if $Q(v)=0$, we still have $\operatorname{Sup}_{t\geqq 0}E(\underline{u}+\varphi+tv)=+\infty$ provided $\|\varphi\|_{H_0^1}\leqq C\delta/H$, (we use here Lemma A.3).

Case (b). We may assume that

(74) $$\varphi+\alpha v\neq 0 \quad \text{for all} \quad \alpha\geqq 0;$$

otherwise if $\varphi=-\alpha_0 v$ for some $\alpha_0\geqq 0$ we have

$$\operatorname*{Sup}_{t\geqq 0}E(\underline{u}+\varphi+tv)=\operatorname*{Sup}_{t\geqq 0}E(\underline{u}+(t-\alpha_0)v)\geqq\operatorname*{Sup}_{s\geqq 0}E(\underline{u}+sv)\geqq A(\gamma),$$

by Lemma 11. For each $\alpha\geqq 0$ we know that

(75) $$\operatorname*{Sup}_{t\geqq 0}E(\underline{u}+t(\varphi+\alpha v))\geqq A(\gamma)$$

(by Lemma 11 and (74)). Using (73) we see that the supremum in (75) is achieved when $t=t(\alpha)$ with

$$t(\alpha)=\begin{cases}+\infty & \text{if} \quad Q(\varphi+\alpha v)\geqq 0,\\ -\dfrac{1}{2H}\dfrac{T(\varphi+\alpha v)}{Q(\varphi+\alpha v)} & \text{if} \quad Q(\varphi+\alpha v)<0,\end{cases}$$

where $T(w)=\int|\nabla w|^2+4H\int\underline{u}\cdot w_x\wedge w_y$. Clearly the function $\alpha\mapsto t(\alpha)$ is continuous from $[0,+\infty)$ into $(0,+\infty]$. Moreover, we have

(76) $$\lim_{\alpha\to+\infty}t(\alpha)=0$$

(note that $Q(\varphi+\alpha v)<0$ for α large enough). On the other hand, we have

$$t(0)=\begin{cases}+\infty & \text{if} \quad Q(\varphi)\geqq 0,\\ \dfrac{T(\varphi)}{2H|Q(\varphi)|}\geqq\dfrac{\delta S^{3/2}}{2H}\dfrac{1}{\left(\int|\nabla\varphi|^2\right)^{1/2}} & \text{if} \quad Q(\varphi)<0.\end{cases}$$

(here we have used Lemmas 3 and 4).

In both cases,

(77) $$t(0)\geqq 1 \quad \text{provided} \quad \|\varphi\|_{H_0^1}\leqq C\frac{\delta}{H}.$$

We deduce from (76) and (77) that there exists some $\alpha_0 \geqq 0$ such that $t(\alpha_0)=1$; using (75) we obtain

$$E(\underline{u}+\varphi+\alpha_0 v) \geqq A(\gamma)$$

and the conclusion of Lemma 12 follows.

We recall now some variational techniques which are well known in the study of Plateau problems (see [4], pp. 107–115). Consider a family (r_ε) of perturbations of the identity depending on a parameter $\varepsilon \geqq 0$, ε small enough. More precisely, we assume that

(78) for each $\varepsilon \geqq 0$, $r_\varepsilon = \bar{\Omega} \to \bar{\Omega}$ is a smooth diffeomorphism,

(79) $\begin{cases} r_\varepsilon : \partial\Omega \to \partial\Omega \text{ is non-decreasing and leaves invariant the 3 points } e^{i\vartheta} \\ \text{with } \vartheta = 0,\ \vartheta = \pm\frac{2}{3}\pi, \end{cases}$

(80) $$r_0 = Id \quad \text{and} \quad r_\varepsilon \underset{\varepsilon \to 0}{\to} Id \text{ uniformly on } \bar{\Omega}.$$

We denote by R^ε the operator

$$R^\varepsilon w = w \circ r^\varepsilon,$$

where w is a function, $w = \bar{\Omega} \to \mathbb{R}^3$ (respectively $w : \partial\Omega \to \mathbb{R}^3$). It is well known that the volume integral Q is invariant under orientation preserving diffeomorphisms, i.e.,

$$Q(R^\varepsilon w) = Q(w) \quad \text{for all} \quad w \in H^1 \cap L^\infty.$$

The Dirichlet integral is not invariant under diffeomorphism but we shall assume that

(81) $$\left| \int |\nabla R^\varepsilon w|^2 - \int |\nabla w|^2 \right| \leqq C\varepsilon \int |\nabla w|^2 \quad \text{for all} \quad w \in H^1$$

which clearly implies

(82) $$\left| \int \nabla R^\varepsilon v \cdot \nabla R^\varepsilon w - \int \nabla v \cdot \nabla w \right| \leqq C\varepsilon \|\nabla v\|_{L^2} \|\nabla w\|_{L^2} \quad \text{for all} \quad v, w \in H^1.$$

It follows from (80) and (81) that

(83) $$R^\varepsilon w \underset{\varepsilon \to 0}{\to} w \textit{ strongly} \text{ in } H^1 \cap L^\infty \quad \text{if} \quad w \in H^1(\Omega) \cap C(\bar{\Omega}).$$

In practice, we obtain a family (r_ε) in the following way. Fix a function $\alpha \in C^\infty(\bar{\Omega}; \mathbb{R})$ and consider the mapping $q_\varepsilon : \bar{\Omega} \to \bar{\Omega}$ defined (in complex notations) by

$$q_\varepsilon(z) = z\, e^{i\varepsilon\alpha(z)}.$$

If ε is small enough, q_ε verifies (78) and (80). In order to satisfy (79) we introduce the (unique) homographic transformation $p_\varepsilon = \bar{\Omega} \to \bar{\Omega}$ such that $q_\varepsilon \circ p_\varepsilon$ leaves invariant the 3 points $e^{i\vartheta}$ with $\vartheta = 0$, $\vartheta = \pm\frac{2}{3}\pi$. Then $r_\varepsilon = q_\varepsilon \circ p_\varepsilon$ satisfies (78), (79) and (80). Moreover, (81) holds; for the verification of (81) we refer to [4], page 109, formula (3.16), which gives a precise expansion of $\int |\nabla R^\varepsilon w|^2 = \int |\nabla Q^\varepsilon w|^2$ as $\varepsilon \to 0$.

LEMMA 13. *Let $\gamma \in \mathscr{C}$; let $\underline{u}$ be the small solution of* (1), (2) *corresponding to the Dirichlet data γ and let $\bar{u}$ be a large solution of* (1), (2)*—as given by Lemma 7—corresponding to the same Dirichlet data γ. Then*

$$\text{(84)} \qquad \sup_{t \geq 0} E(R^\varepsilon \underline{u} + tR^\varepsilon(\bar{u} - \underline{u})) \leq A(\gamma) + \int |\nabla R^\varepsilon \bar{u}|^2 - \int |\nabla \bar{u}|^2 + O(\varepsilon^2).$$

Proof: We have

$$\begin{aligned}
&E(R^\varepsilon \underline{u} + tR^\varepsilon(\bar{u} - \underline{u})) \\
&= \int |\nabla R^\varepsilon \bar{u} + (1-t)\nabla(R^\varepsilon \underline{u} - R^\varepsilon \bar{u})|^2 + \tfrac{4}{3}HQ(\underline{u} + t(\bar{u} - \underline{u})) \\
&= \int |\nabla R^\varepsilon \bar{u}|^2 + 2(1-t)\int \nabla R^\varepsilon \bar{u} \cdot \nabla(R^\varepsilon \underline{u} - R^\varepsilon \bar{u}) + (1-t)^2 \int |\nabla(R^\varepsilon \underline{u} - R^\varepsilon \bar{u})|^2 \\
&\quad + \tfrac{4}{3}HQ(\underline{u} + t(\bar{u} - \underline{u})) \\
&\leq \left[\int |\nabla R^\varepsilon \bar{u}|^2 - \int |\nabla \bar{u}|^2\right] + \left[\int |\nabla \bar{u}|^2 + 2(1-t)\int \nabla \bar{u} \cdot \nabla(\underline{u} - \bar{u}) \right.\\
&\qquad \left. + (1-t)^2 \int |\nabla(\underline{u} - \bar{u})|^2\right] \\
&\qquad + C|1-t|\varepsilon + C(1-t)^2\varepsilon + \tfrac{4}{3}HQ(\underline{u} + t(\bar{u} - \underline{u})) \\
&= \left[\int |\nabla R^\varepsilon \bar{u}|^2 - \int |\nabla \bar{u}|^2\right] + E(\underline{u} + t(\bar{u} - \underline{u})) + C|1-t|\varepsilon + C(1-t)^2\varepsilon,
\end{aligned}$$

where C depends only on $\underline{u}$ and $\bar{u}$ (here we use (81) and (82)). On the other hand, a direct computation (based on (14) and Lemmas 6, 7) shows that

$$E(\underline{u} + t(\bar{u} - \underline{u})) = A(\gamma) - \frac{J^2}{12H^2}(t-1)^2(2t+1) \quad \text{for all} \quad t.$$

Therefore we obtain

$$E(R^\varepsilon \underline{u}+tR^\varepsilon(\bar{u}-\underline{u}))$$

$$\leqq\left[\int|\nabla R^\varepsilon \bar{u}|^2-\int|\nabla \bar{u}|^2\right]+A(\gamma)-\frac{J^3}{12H^2}(t-1)^2+C\varepsilon|t-1|+C\varepsilon(t-1)^2$$

$$\leqq\left[\int|\nabla R^\varepsilon \bar{u}|^2-\int|\nabla \bar{u}|^2\right]+A(\gamma)+C\varepsilon^2 \quad \text{for } \varepsilon \text{ small enough.}$$

We are now in a position to conclude the proof.

Proof of Theorem 2: Let $\gamma^0 \in \mathscr{C}$ be such that

$$A(\gamma^0)=\operatorname*{Inf}_{\gamma\in\mathscr{C}} A(\gamma)$$

(see Lemma 10). Let $\underline{u}^0$ (respectively $\bar{u}^0$) be the small solution (respectively a large solution) of (1)–(2) corresponding to the Dirichlet data γ^0. We claim that $\bar{u}^0$ is a solution of the Plateau problem (45)–(46)–(47). We already know that $\bar{u}^0$ verifies (45) and (47). We shall establish that $\bar{u}^0$ satisfies

$$\int|\nabla R^\varepsilon \bar{u}^0|^2-\int|\nabla \bar{u}^0|^2 \geqq -C\varepsilon^2; \tag{85}$$

then one can deduce (46) from (85) by a standard argument involving the expansion of $\int|\nabla R^\varepsilon \bar{u}_0|^2$ (see [4], pages 107–115). Set $\gamma^\varepsilon=R^\varepsilon\gamma^0$, so that $\gamma^\varepsilon\in\mathscr{C}$ and let $\underline{u}^\varepsilon$ be the small solution of (1)–(2) with Dirichlet data γ^ε. We know (see (83)) that

$$\gamma^\varepsilon \to \gamma^0 \quad \text{strongly in} \quad H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega)$$

and

$$R^\varepsilon \underline{u}^0 \to \underline{u}^0 \quad \text{strongly in} \quad H^1(\Omega)\cap L^\infty(\Omega).$$

It follows from Lemma 8 that

$$\underline{u}^\varepsilon \to \underline{u}^0 \quad \text{strongly in} \quad H^1(\Omega). \tag{86}$$

We deduce from Lemma 3 that there exists a $\delta_0>0$ such that

$$\int|\nabla v|^2+4H\int \underline{u}^0\cdot v_x\wedge v_y \geqq \delta_0\int|\nabla v|^2 \quad \text{for all} \quad v\in H_0^1, \tag{87}$$

which implies that

$$\int|\nabla v|^2+4H\int \underline{u}^\varepsilon\cdot v_x\wedge v_y \geqq \tfrac{1}{2}\delta_0\int|\nabla v|^2 \quad \text{for all} \quad v\in H_0^1 \tag{88}$$

provided ε is small enough (use (86), (87) and Lemma A.8). Applying Lemma 12 we see that

$$(89)\quad A(\gamma^\varepsilon) \leqq \operatorname*{Sup}_{t\geqq 0} E(\underline{u}^\varepsilon + \varphi + tv) \quad \text{for all} \quad \varphi \in H_0^1 \cap L^\infty \text{ with } \|\varphi\|_{H_0^1} \leqq \frac{C\delta_0}{2H},$$
$$\text{for all} \quad v \in H_0^1 \cap L^\infty,\ v \not\equiv 0.$$

Choosing, in (89),

$$\varphi = R^\varepsilon \underline{u}^0 - \underline{u}^\varepsilon \quad \text{and} \quad v = R^\varepsilon(\bar{u}^0 - \underline{u}^0)$$

we obtain

$$(90)\qquad A(\gamma^\varepsilon) \leqq \operatorname*{Sup}_{t\geqq 0} E(R^\varepsilon \underline{u}^0 + tR^\varepsilon(\bar{u}^0 - \underline{u}^0)).$$

On the other hand, we have (by definition of γ^0)

$$(91)\qquad A(\gamma^0) \leqq A(\gamma^\varepsilon)$$

and, by Lemma 13,

$$(92)\quad \operatorname*{Sup}_{t\geqq 0} E(R^\varepsilon \underline{u}^0 + tR^\varepsilon(\bar{u} - \underline{u}^0)) \leqq A(\gamma^0) + \int |\nabla R^\varepsilon \bar{u}^0|^2 - \int |\nabla \bar{u}^0|^2 + O(\varepsilon^2).$$

Combining (90), (91) and (92) we obtain (85).

Appendix

We collect here a number of technical facts. Most of these facts are well known to the experts and have been used in various forms since the pioneering work of Wente [16]. As before, $\Omega = \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 < 1\}$.

Throughout this appendix we deal with functions defined on Ω with values in $\mathbb{R}^3$, except in Lemmas A.1 *and* A.2 *where the functions are real-valued.*

LEMMA A.1. *Assume $u, v \in H^1(\Omega)$ and let $\varphi \in W_0^{1,1}(\Omega)$ be the unique solution of*

$$(\text{A.1})\qquad \begin{cases} -\Delta\varphi = u_x v_y - u_y v_x & \text{on} \quad \Omega, \\ \varphi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then $\varphi \in C(\bar{\Omega}) \cap H_0^1(\Omega)$ and

$$\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^2} \leqq C\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}.$$

Proof: We follow essentially an argument due to Wente [18]. Assume first that $u, v \in \mathscr{D}(\mathbb{R}^2)$ and set

$$\psi = E * (u_x v_y - u_y v_x),$$

where $E(x, y) = (1/2\pi)\log(1/r)$, $r = (x^2+y^2)^{1/2}$, is the fundamental solution of $-\Delta$. Then

$$-\Delta\psi = u_x v_y - u_y v_x. \tag{A.2}$$

In polar coordinates we have

$$u_x v_y - u_y v_x = \frac{1}{r}(u_r v_\vartheta - u_\vartheta v_r).$$

Thus

$$\begin{aligned}
\psi(0) &= \frac{1}{2\pi}\iint \left(\log\frac{1}{r}\right)(u_r v_\vartheta - u_\vartheta v_r)\,dr\,d\vartheta \\
&= \frac{1}{2\pi}\iint \left(\log\frac{1}{r}\right)[(uv_\vartheta)_r - (uv_r)_\vartheta]\,dr\,d\vartheta \\
&= \frac{1}{2\pi}\iint \left(\log\frac{1}{r}\right)(uv_\vartheta)_r\,dr\,d\vartheta \\
&= \frac{1}{2\pi}\iint \frac{1}{r}(uv_\vartheta)\,dr\,d\vartheta.
\end{aligned}$$

However,

$$\int_0^{2\pi} uv_\vartheta\,d\vartheta = \int_0^{2\pi} (u-\bar{u})v_\vartheta\,d\vartheta, \quad \text{where} \quad \bar{u}(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\sigma)\,d\sigma$$

and thus

$$\left|\int_0^{2\pi} uv_\vartheta\,d\vartheta\right| \leqq \|u-\bar{u}\|_{L^2(0,2\pi)}\|v_\vartheta\|_{L^2(0,2\pi)} \leqq \|u_\vartheta\|_{L^2(0,2\pi)}\|v_\vartheta\|_{L^2(0,2\pi)}.$$

Finally we obtain

$$\begin{aligned}
|\psi(0)| &\leqq \frac{1}{2\pi}\int_0^\infty \|u_\vartheta\|_{L^2(0,2\pi)}\|v_\vartheta\|_{L^2(0,2\pi)}\frac{1}{r}\,dr \\
&\leqq \frac{1}{2\pi}\left(\int_0^\infty \|u_\vartheta\|^2_{L^2(0,2\pi)}\frac{1}{r}\,dr\right)^{1/2}\left(\int_0^\infty \|v_\vartheta\|^2_{L^2(0,2\pi)}\frac{1}{r}\,dr\right)^{1/2} \\
&\leqq \frac{1}{2\pi}\|\nabla u\|_{L^2(\mathbf{R}^2)}\|\nabla v\|_{L^2(\mathbf{R}^2)}.
\end{aligned}$$

Similarly we have

$$\|\psi\|_{L^\infty(\mathbf{R}^2)} \leqq \frac{1}{2\pi}\|\nabla u\|_{L^2(\mathbf{R}^2)}\|\nabla v\|_{L^2(\mathbf{R}^2)}.$$

Moreover, from (A.1) and (A.2) we obtain

$$\Delta(\varphi-\psi) = 0 \quad \text{on} \quad \Omega$$

and, by the maximum principle,

$$\|\varphi-\psi\|_{L^\infty(\Omega)}\leqq\|\varphi-\psi\|_{L^\infty(\partial\Omega)}=\|\psi\|_{L^\infty(\partial\Omega)}.$$

Hence

$$\|\varphi\|_{L^\infty(\Omega)}\leqq 2\|\psi\|_{L^\infty(\Omega)}\leqq\frac{1}{\pi}\|\nabla u\|_{L^2(\mathbb{R}^2)}\|\nabla v\|_{L^2(\mathbb{R}^2)},$$

and, multiplying (A.1) through by φ, we obtain

$$\int_\Omega|\nabla\varphi|^2\leqq\|\varphi\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}\leqq\frac{1}{\pi}\|\nabla u\|^2_{L^2(\mathbb{R}^2)}\|\nabla v\|^2_{L^2(\mathbb{R}^2)}.$$

In the general case where u, $v\in H^1(\Omega)$, we can find $\tilde u$, $\tilde v\in H^1(\mathbb{R}^2)$ extending u, v with

$$\|\tilde u\|_{H^1(\mathbb{R}^2)}\leqq C\|u\|_{H^1(\Omega)},\qquad \|\tilde v\|_{H^1(\mathbb{R}^2)}\leqq C\|v\|_{H^1(\Omega)}.$$

A standard density argument shows that $\varphi\in C(\bar\Omega)\cap H^1_0(\Omega)$ and that

$$\begin{aligned}\|\varphi\|_{L^\infty(\Omega)}+\|\nabla\varphi\|_{L^2(\Omega)}&\leqq C\|\nabla\tilde u\|_{L^2(\mathbb{R}^2)}\|\nabla\tilde v\|_{L^2(\mathbb{R}^2)}\\&\leqq C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.\end{aligned}$$

Finally, we note that φ is unchanged if we replace u by $u-\bar u$, where $\bar u=(1/|\Omega|)\int_\Omega u$ (similarly for v) and then use Poincaré's inequality.

Remark A.1. Assume u_i, $v_i\in H^1(\Omega)$, $1\leqq i\leqq k$, and let $\varphi\in W^{1,1}_0(\Omega)$ be the unique solution of

$$\begin{cases}-\Delta\varphi=\sum(u_{ix}v_{iy}-u_{iy}v_{ix}) & \text{on}\quad\Omega,\\ \quad\varphi=0 & \text{on}\quad\partial\Omega.\end{cases}$$

Then $\varphi\in C(\bar\Omega)\cap H^1_0(\Omega)$.

LEMMA A.2. *Assume u, $v\in H^1(\Omega)$ and $w\in\mathcal{D}(\Omega)$. Then*

$$\left|\int_\Omega(u_xv_y-u_yv_x)w\right|\leqq C\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}\|\nabla w\|_{L^2}.$$

Proof: Let φ be the solution of (A.1). We have

$$\begin{aligned}\left|\int_\Omega(u_xv_y-u_yv_x)w\right|=\left|\int_\Omega\nabla\varphi\cdot\nabla w\right|&\leqq\|\nabla\varphi\|_{L^2}\|\nabla w\|_{L^2}\\&\leqq C\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}\|\nabla w\|_{L^2},\end{aligned}$$

by Lemma A.1.

From now on *all functions are defined on Ω with values in $\mathbb{R}^3$.*

LEMMA A.3. *Assume* $u \in H^1(\Omega)$ *and* $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$. *Then*

$$\left| \int_\Omega w \cdot (u_x \wedge u_y) \right| \leqq C \|\nabla w\|_{L^2} \|\nabla u\|_{L^2}^2.$$

Proof: When $w \in \mathcal{D}(\Omega)$, the conclusion follows from Lemma A.2 used on each component. In the general case where $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we may choose a sequence $w^n \in \mathcal{D}(\Omega)$ such that $w^n \to w$ in $H^1(\Omega)$, $w^n \to w$ a.e. and $\|w^n\|_{L^\infty} \leqq C$; then we use dominated convergence.

LEMMA A.4. *Assume*

$$\text{(A.3)} \qquad u \in H^1(\Omega) \cap L^\infty(\Omega), \quad v \in H^1(\Omega) \cap L^\infty(\Omega), \quad w \in H^1(\Omega).$$

In addition assume

$$\textit{either} \quad u \wedge v = 0 \quad \textit{on} \quad \partial\Omega \quad \textit{or} \quad w = 0 \quad \textit{on} \quad \partial\Omega.[4]$$

Then

$$\int_\Omega u \cdot [(v_x \wedge w_y) + (w_x \wedge v_y)] = \int_\Omega v \cdot [(u_x \wedge w_y) + (w_x \wedge u_y)].$$

The same conclusion holds if instead of (A.3) *we assume*

$$\text{(A.3')} \qquad u \in C^1(\bar{\Omega}), \quad v \in H^1(\Omega), \quad w \in H^1(\Omega).$$

Proof: (a) Assume $w \in H_0^1(\Omega)$ and let $w^n \in \mathcal{D}(\Omega)$ be such that $w^n \to w$ in $H^1(\Omega)$. We have

$$(v_x \wedge w_y^n) + (w_x^n \wedge v_y) = (v \wedge w_y^n)_x + (w_x^n \wedge v)_y$$

and thus

$$\int_\Omega u \cdot [(v_x \wedge w_y^n) + (w_x^n \wedge v_y)] = -\int_\Omega u_x \cdot (v \wedge w_y^n) + u_y \cdot (w_x^n \wedge v)$$

$$= \int_\Omega v \cdot [(u_x \wedge w_y^n) + (w_x^n \wedge u_y)].$$

The conclusion follows easily as $n \to \infty$ both in case (A.3) and in case (A.3').

(b) Assume $u \wedge v = 0$ in $\partial\Omega$ and let $w^n \in C^\infty(\bar{\Omega})$ be such that $w^n \to w$ in $H^1(\Omega)$. We have

$$\int_\Omega u \cdot [(v_x \wedge w_y^n) + (w_x^n \wedge v_y)] = \int_\Omega v \cdot [(u_x \wedge w_y^n) + (w_x^n \wedge u_y)]$$

$$+ \int_{\partial\Omega} (u \wedge v) \cdot [w_y^n \cos(\nu, x) - w_x^n \cos(\nu, y)],$$

[4] Note that $(u \wedge v) \in H^1(\Omega)$ and therefore $u \wedge v$ has a trace on $\partial\Omega$.

where ν denotes the outward normal to Ω. The conclusion follows as $n \to \infty$, both under (A.3) and (A.3′).

LEMMA A.5. *Assume*

$$u \in H^1(\Omega) \cap L^\infty(\Omega), \qquad v \in H^1(\Omega) \cap L^\infty(\Omega), \tag{A.4}$$

$$u \wedge v = 0 \quad on \quad \partial\Omega. \tag{A.5}$$

Then

$$2 \int_\Omega u \cdot (v_x \wedge v_y) = \int_\Omega v \cdot [(u_x \wedge v_y) + (v_x \wedge u_y)].$$

The same conclusion holds if instead of (A.4) *we assume*

$$u \in C^1(\bar{\Omega}) \quad and \quad v \in H^1(\Omega). \tag{A.4′}$$

Proof: Use Lemma A.4 with $w = v$.

LEMMA A.6. *Assume* (u^n) *and* (v^n) *are sequences such that*

$$u^n \in H^1(\Omega) \cap L^\infty(\Omega), \quad v^n \in H^1(\Omega) \cap L^\infty(\Omega), \quad u^n \wedge v^n = 0 \quad on \quad \partial\Omega,$$
$$\|u^n\|_{H^1} \leqq C, \quad \|v^n\|_{H^1} \leqq C, \quad \|v^n\|_{L^\infty} \to 0.$$

Then

$$\int_\Omega u^n \cdot (v_x^n \wedge v_y^n) \to 0.$$

Proof: By Lemma A.5 we have

$$\int_\Omega u^n \cdot (v_x^n \wedge v_y^n) = \frac{1}{2} \int_\Omega v^n \cdot [(u_x^n \wedge v_y^n) + (v_x^n \wedge u_y^n)] \to 0.$$

LEMMA A.7. *Assume* (u^n) *and* (v^n) *are sequences such that*

$$u^n \in H^1(\Omega) \cap L^\infty(\Omega), \qquad v^n \in H^1(\Omega) \cap L^\infty(\Omega),$$
$$\|u^n\|_{H^1} \leqq C, \quad \|v^n\|_{H^1} \leqq C, \quad \|v^n\|_{L^\infty} \to 0,$$

$u^n = \gamma$ *on* $\partial\Omega$ *for some fixed function* $\gamma \in H^1(\Omega) \cap L^\infty(\Omega)$. *Then*

$$\omega_n = \int_\Omega u^n \cdot [(u_x^n \wedge v_y^n) + (v_x^n \wedge u_y^n)] \to 0.$$

Proof. Set $\varphi^n = u^n - \gamma$ so that $\varphi^n \in H_0^1(\Omega)$ and $\|\varphi^n\|_{H^1} \leqq C$. We have

$$\omega_n = \int_\Omega (\varphi^n + \gamma) \cdot [(\varphi_x^n + \gamma_x) \wedge v_y^n + v_x^n \wedge (\varphi_y^n + \gamma_y)]$$

$$= \int_\Omega \varphi^n \cdot [(\varphi_x^n \wedge v_y^n) + (v_x^n \wedge \varphi_y^n)]$$

$$+ \int_\Omega \varphi^n \cdot [(\gamma_x \wedge v_y^n) + (v_x^n \wedge \gamma_y)]$$

$$+ \int_\Omega \gamma \cdot [(\varphi_x^n \wedge v_y^n) + (v_x^n \wedge \varphi_y^n)]$$

$$+ \int_\Omega \gamma \cdot [(\gamma_x \wedge v_y^n) + (v_x^n \wedge \gamma_y)].$$

Using Lemma A.4 we obtain

$$\omega_n = 2 \int_\Omega v^n \cdot [(\varphi_x^n \wedge \varphi_y^n) + (\gamma_x \wedge \varphi_y^n) + (\varphi_x^n \wedge \gamma_y)]$$

$$+ \int_\Omega \gamma \cdot [(\gamma_x \wedge v_y^n) + (v_x^n \wedge \gamma_y)].$$

The conclusion follows since $\|v^n\|_{L^\infty} \to 0$ and $v^n \rightharpoonup 0$ in $H^1(\Omega)$ weakly.

LEMMA A.8. *Assume*

$$u \in H^1(\Omega) \cap L^\infty(\Omega) \quad \textit{and} \quad v \in H_0^1(\Omega);$$

then

$$\left| \int_\Omega u \cdot (v_x \wedge v_y) \right| \leqq C \|\nabla u\|_{L^2} \|\nabla v\|_{L^2}^2.$$

Proof: Assume first that $v \in \mathcal{D}(\Omega)$. By Lemma A.5 we have

$$\int_\Omega u \cdot (v_x \wedge v_y) = \frac{1}{2} \int_\Omega v \cdot [(u_x \wedge v_y) + (v_x \wedge u_y)]$$

$$= \frac{1}{2} \int_\Omega v \cdot [(u+v)_x \wedge (u+v)_y - u_x \wedge u_y - v_x \wedge v_y].$$

We deduce from Lemma A.3 that

$$\left| \int_\Omega u \cdot (v_x \wedge v_y) \right| \leqq C \|\nabla v\|_{L^2} (\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2).$$

Replacing v by λv with $\lambda = \|\nabla u\|_{L^2} / \|\nabla v\|_{L^2}$ we obtain

$$\left| \int_\Omega u \cdot (v_x \wedge v_y) \right| \leqq C \|\nabla u\|_{L^2} \|\nabla v\|_{L^2}^2 \quad \text{for all} \quad v \in \mathcal{D}(\Omega).$$

For the general case where $v \in H_0^1(\Omega)$ we argue by density.

LEMMA 1.9. *Assume $u \in H^1(\Omega) \cap L^\infty(\Omega)$ and let (v^n) be a sequence such that $v^n \in H_0^1(\Omega)$ and*

$$v^n \rightharpoonup v \quad in \quad H_0^1(\Omega) \quad weakly.$$

Then

$$\int_\Omega u \cdot (v_x^n \wedge v_y^n) \to \int_\Omega u \cdot (v_x \wedge v_y).$$

Proof: Clearly it suffices to consider the case where $v = 0$. Given $\varepsilon > 0$ we fix $\tilde{u} \in C^1(\bar{\Omega})$ such that $\|u - \tilde{u}\|_{H^1} < \varepsilon$. By Lemma A.8 we have

$$\left| \int_\Omega u \cdot (v_x^n \wedge v_y^n) - \int_\Omega \tilde{u} \cdot (v_x^n \wedge v_y^n) \right| \leqq C\varepsilon.$$

On the other hand (see Lemma A.5), we have

$$\int_\Omega \tilde{u} \cdot (v_x^n \wedge v_y^n) = \frac{1}{2} \int v^n \cdot [(\tilde{u}_x \wedge v_y^n) + (v_x^n \wedge \tilde{u}_y)] \to 0$$

since $v^n \to 0$ in $L^2(\Omega)$ strongly. Thus

$$\limsup_{n \to \infty} \left| \int_\Omega u \cdot (v_x^n \wedge v_y^n) \right| \leqq C\varepsilon$$

and hence

$$\int_\Omega u \cdot (v_x^n \wedge v_y^n) \to 0.$$

LEMMA A.10. *There is a unique continuous map*

$$R: H_0^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$$

such that

$$R(u, v) = \int_\Omega u \cdot (v_x \wedge v_y) \quad \textit{for all} \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad \textit{and for all} \quad v \in H^1(\Omega).$$

Moreover,

$$|R(u, v)| \leqq C \|\nabla u\|_{L^2} \|\nabla v\|_{L^2}^2 \quad \textit{for all} \quad u \in H_0^1(\Omega), \quad \textit{and for all} \quad v \in H^1(\Omega).$$

Proof: Fix $v \in H^1(\Omega)$ and consider the mapping

$$u \in \mathscr{D}(\Omega) \mapsto \int_\Omega u \cdot (v_x \wedge v_y) \in \mathbb{R}.$$

Using Lemma A.3 we may extend it by continuity to H_0^1 and we denote it by $R(u, v)$. In particular we have

$$|R(u, v)| \leq C\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}^2 \quad \text{for all} \quad u \in H_0^1(\Omega) \quad \text{and for all} \quad v \in H^1(\Omega).$$

On the other hand, in case $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ we choose a sequence (u^n) in $\mathscr{D}(\Omega)$ such that $u^n \to u$ in $H_0^1(\Omega)$, $u^n \to u$ a.e. and $\|u^n\|_{L^\infty} \leq C$. We have

$$R(u^n, v) = \int_\Omega u^n \cdot (v_x \wedge v_y);$$

clearly $R(u^n, v) \to R(u, v)$, while $\int_\Omega u^n \cdot (v_x \wedge v_y) \to \int_\Omega u \cdot (v_x \wedge v_y)$

by dominated convergence.

Hence

$$R(u, v) = \int_\Omega u \cdot (v_x \wedge v_y) \quad \text{for all} \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega) \quad \text{and for all} \quad v \in H^1(\Omega).$$

We check now that R is continuous on $H_0^1(\Omega) \times H^1(\Omega)$. Let $u^n \to u$ in $H_0^1(\Omega)$, $v^n \to v$ in $H^1(\Omega)$ strongly. We have

$$\begin{aligned} |R(u^n, v^n) - R(u, v)| &\leq |R(u^n - u, v^n)| + |R(u, v^n) - R(u, v)| \\ &\leq C\|\nabla(u^n - u)\|_{L^2}\|\nabla v^n\|_{L^2}^2 + |R(u, v^n) - R(u, v)|. \end{aligned}$$

Given $\varepsilon > 0$ we fix $\tilde{u} \in \mathscr{D}$ such that $\|u - \tilde{u}\|_{H^1} < \varepsilon$. Clearly,

$$\int_\Omega \tilde{u} \cdot (v_x^n \wedge v_y^n) \to \int_\Omega \tilde{u} \cdot (v_x \wedge v_y).$$

Hence

$$\begin{aligned} |R(u, v^n) - R(u, v)| &\leq |R(u - \tilde{u}, v^n)| + |R(\tilde{u}, v^n) - R(\tilde{u}, v)| + |R(\tilde{u} - u, v)| \\ &\leq C\varepsilon + o(1). \end{aligned}$$

Therefore, $|R(u^n, v^n) - R(u, v)| \to 0$.

Definition. We set

$$Q(v) = R(v, v) \quad \text{for} \quad v \in H_0^1(\Omega)$$

so that Q is continuous on $H_0^1(\Omega)$, $|Q(v)| \leq C\|\nabla v\|_{L^2}^3$ for all $v \in H_0^1(\Omega)$ and

$$Q(v) = \int v \cdot (v_x \wedge v_y) \quad \text{for all} \quad v \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

LEMMA A.11. *We have*

$$Q(v+w)=Q(v)+Q(w)+3R(v,w)+3R(w,v) \quad \text{for all} \quad v\in H_0^1(\Omega) \quad \text{and for all} \quad w\in H_0^1(\Omega).$$

Proof: By continuity it suffices to consider the case where $v, w\in \mathcal{D}(\Omega)$ and then use Lemma A.5.

LEMMA A.12. *Assume $v\in H_0^1(\Omega)$ and let (w^n) be a sequence in $H_0^1(\Omega)$ such that*

$$w^n \rightharpoonup 0 \quad in \quad H_0^1(\Omega) \quad weakly.$$

Then

$$|Q(v+w^n)-Q(v)-Q(w^n)|\to 0.$$

Proof: In view of Lemma A.11 it suffices to verify that $R(v, w_n)\to 0$ and $R(w^n, v)\to 0$. The second point is clear since for fixed v the mapping $w\mapsto R(w, v)$ is a continuous linear form on $H_0^1(\Omega)$. We check now that $R(v, w_n)\to 0$. Given $\varepsilon>0$ we fix $\tilde{v}\in\mathcal{D}(\Omega)$ such that $\|v-\tilde{v}\|_{H_0^1}<\varepsilon$. We have (by Lemma A.5)

$$\int \tilde{v}\cdot(w_x^n\wedge w_y^n)=\frac{1}{2}\int_\Omega w^n\cdot[(\tilde{v}_x\wedge w_y^n)+(w_x^n\wedge\tilde{v}_y)]\to 0$$

since $w^n\to 0$ in $L^2(\Omega)$ strongly. Finally we have

$$|R(v, w_n)\leqq|R(v-\tilde{v}, w_n)|+|R(\tilde{v}, w_n)|\leqq C\varepsilon+o(1)$$

and thus $R(v, w_n)\to 0$.

Bibliography

[1] Aubin, Th., *Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures et Appl. 55, 1976, pp. 269–296.

[2] Brezis, H., and Coron, J. M., *Sur la conjecture de Rellich pour les surfaces à courbure moyenne prescrite*, C.R. Acad. Sc. Paris 295, 1982, pp. 615–618.

[3] Brezis, H., and Nirenberg, L., *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. 36, 1983, pp. 437–478.

[4] Courant, R., *Dirichlet's Principle*, Interscience, New York, 1950.

[5] Crandall, M., and Rabinowitz, P., *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Archive Rat. Mech. Anal. 58, 1975, pp. 207–218.

[6] Heinz, E., *Über die Existenz einer Flähce konstanter mittlerer Krümmung bei vorgegebener Berandung*, Math. Ann. 127, 1954, 258–287.

[7] Heinz, E., *On the non-existence of a surface of constant mean curvature with finite area and prescribed rectifiable boundary*, Archive Rat. Mech. Anal. 35, 1969, pp. 249–252.

[8] Hildebrandt, S., *On the Plateau problem for surfaces of constant mean curvature*, Comm. Pure Appl. Math. 23, 1970, pp. 97–114.

[9] Hildebrandt, S., *Nonlinear elliptic systems and harmonics mappings*, in Proc. 1980 Beijing Symp. Diff. Geom. and Diff. Eq., Science Press, Beijing and Gordon and Breach, New York, 1982.

[10] Lieb, E., *Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities*, Ann. of Math. 118, 1983, pp. 349–374.

[11] Morrey, C., *Multiple Integrals in the Calculus of Variations*, Springer Verlag, New York, 1966.

[12] Pokhozaev, S., *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR 165, 1965, pp. 33–36. (In Russian.)

[13] Steffen, K., *Flächen konstanter mittlerer Krümmung mit vorgegebenem Volumen oder Flächeninhalt*, Archive Rat. Mech. Anal. 49, 1972, pp. 99–128.

[14] Steffen, K., *On the uniqueness of surfaces with prescribed mean curvature spanning a given contour*, to appear.

[15] Struwe, M., *Nonuniqueness in the Plateau problem for surfaces of constant mean curvature*, to appear.

[16] Wente, H., *An existence theorem for surfaces of constant mean curvature*, J. Math. Anal. Appl. 26, 1969, pp. 318–344.

[17] Wente, H., *The differential equation* $\Delta x = 2H(x_u \wedge x_v)$ *with vanishing boundary values*, Proc. A.M.S. 50, 1975, pp. 131–137.

[18] Wente, H., *Large solutions to the volume constrained Plateau problem*, Archive Rat. Mech. Anal. 75, 1980, pp. 59–77.

[19] Werner, H., *Das Problem von Douglas für Flächen konstanter mittlerer Krümmung*, Math. Ann. 133, 1957, pp. 303–319.

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