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The algebraic structures of complex intuitionistic fuzzy soft sets associated with groups and subgroups

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Abstract. In recent years, the theory of complex fuzzy sets has captured the attention of many researchers, and research in this area has intensified over the past five years. This paper focuses on developing the algebraic structures pertaining to groups and subgroups for the complex intuitionistic fuzzy soft set model. Besides examining some of the properties of these structures, the relationship between these structures and corresponding structures in fuzzy group theory is also examined.

1. Introduction

Uncertainty, imprecision, and vagueness are characteristics that are pervasive in problems occurring in the real world, and these features cannot be handled effectively using mathematical tools that are traditionally used to deal with uncertainties and vagueness. Some of the pioneering theories used to deal with these limitations include fuzzy set theory [1], intuitionistic fuzzy set theory [2], and soft set theory [3]. To overcome the problems that are inherent in each of these theories, researchers have chosen to combine these theories to develop new fuzzy-based hybrid models. The more well known among these include fuzzy soft sets [4], intuitionistic fuzzy soft sets [5], interval-valued fuzzy soft sets [6], interval-valued intuitionistic fuzzy soft sets [7], and vague soft sets [8]. Although all of the above-mentioned theories are able to handle the uncertainties and fuzziness that exist in the data, all of these models are not able to handle the periodicity or seasonality that exists in many real-life problems. This led to the introduction of the complex fuzzy set model in [9] and, subsequently, the development and extension of this theory.

The notion of complex sets stems from the concept of complex numbers, which is a primary concept for solving problems, especially in the field of engineering. Complex sets notion, in practice, has the ability to solve many problems that cannot be solved using traditional mathematical concepts such as number theory, probability theory, and fuzzy set theory. Examples of these instances include solving the improper integrals that are used to represent resistance in electrical engineering and also represent the phase or wave-like qualities in two-dimensional problems. This

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led to the notion of complex fuzzy sets in [9], which is an improved and extended version of ordinary fuzzy sets. Kumar & Bajaj [10], then, proposed the notion of Complex Intuitionistic Fuzzy Soft Sets (CIFSS) that combines the characteristics and advantages of complex sets, soft sets, and intuitionistic fuzzy sets in a single set. The CIFSS is parametric in nature and characterized by an amplitude term, which is equivalent to the membership and non-membership functions in an ordinary IFSS, and a phase term that represents the seasonality and/or periodicity of the elements. The novelty of CIFSS is manifested in the additional dimension of membership, which is the phase of the grade of membership. This feature gives CIFSS the added advantage of being able to represent data or information occurring repeatedly over a period of time, which is often the case with problems that are two-dimensional in nature.

Although research studies pertaining to the theory of CFSs and other complex fuzzy-based models are still in their infancy, they have been steadily gaining momentum in recent years. As of now, almost all of the work done in this area has revolved around the study of the theoretical properties of CFSs, complex fuzzy computing and modeling, complex fuzzy logic, complex fuzzy optimization and decision-making, and the application of these in solving time-periodic problems. The phase term in the structure of CFSs is the key defining feature of this model and can be used to model the seasonality and/or periodicity of time-periodic phenomena. However, this is not the only interpretation for the phase term. Instead, the phase term can be used to represent different aspects of the information, depending on the context of the scope of the problem or area that is being studied. In most of the existing literature, the phase term has been used to represent the time factor and seasonality of the problems and has been applied to multi-attribute decision-making problems in a myriad of areas including supplier selection, economics, pattern recognition, engineering, and artificial intelligence.

The phase term can also be used to accurately represent the cycles present in fuzzy algebraic structures. In the study of complex fuzzy algebraic theory, the fuzzy algebraic structures are defined in a complex fuzzy setting; therefore, the structures consist of an amplitude term and a phase term. The amplitude term is equivalent to the membership function in ordinary fuzzy sets, whereas the phase term can be used to aptly represent the cycles of the algebraic structures. For example, when dealing with fuzzy alternating groups, different cycles can be represented aptly and accurately using the phase term if the fuzzy alternating groups are defined in terms of CFSs or any complex fuzzy-based models. This would make it easier to identify different cycles and their corresponding membership functions in a systematic manner. The desire to utilize this unique ability of the phase term present in the CFS model and other complex fuzzy-based models in the study of fuzzy algebra served as the main motivation to introduce and develop the theory of complex intuitionistic fuzzy soft groups in this paper. In this regard, the notion of CIFS groups and other supporting algebraic structures for CIFSGs are introduced and developed. The lack of proper research pertaining to the algebraic theory of complex fuzzy-based models in the literature served as another motivation for the study done in this paper.

The rest of this paper is organized as follows. In Section 2, some important background information pertaining to the concepts introduced here is recapitulated. In Section 3, the algebraic structures of complex intuitionistic fuzzy subgroups and complex intuitionistic fuzzy soft groups are derived, and the properties and structural characteristics of these algebraic structures are proposed and, subsequently, verified. The relationship between the structures introduced here and corresponding concepts in fuzzy group theory and classical group theory are also examined and verified in this section. In Section 4, normal complex intuitionistic fuzzy soft groups are proposed, and the properties of this structure are discussed and verified. Concluding remarks are presented in Section 5, followed by acknowledgments and a list of references.

2. Preliminaries

In this section, we recapitulate some of the important background information pertaining to the development of the algebraic structures that will be proposed here.

2.1. Intuitionistic fuzzy sets

An Intuitionistic Fuzzy Set (IFS) [2] is an extension of the classical fuzzy set and is characterized by a membership function and a non-membership function, each of which describes the degree of belongingness and non-belongingness of the elements with respect to each attribute. The concept of IFS was then further extended by incorporating the concept of soft set to derive the concept of Intuitionistic Fuzzy Soft Set (IFSS) [5].

In all that follows, U shall be used to denote a universal set.

Definition 2.1 [2]. Let $A = \{(x, \mu_A(x), \nu_A(x)) : x \in U\}$, where both μ_A and ν_A are functions from U to $[0, 1]$, satisfying $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1$ for all $x \in U$. Then, A is called an *intuitionistic fuzzy set* on U, where μ_A is the *membership function* of A and ν_A is the *non-membership function* of A.

Define $\pi_A(x) = 1 - \mu_A - \nu_A$. Then, for each $x_0 \in U$:

(i) The value of $\mu_A(x_0)$ is called the *degree of belongingness* of x_0 to A;

(ii) The value of $\nu_A(x_0)$ is called the *degree of non-belongingness* of x_0 to A;

(iii) The value of $\pi_A(x_0)$ is called the *degree of uncertainty or indeterminacy* of x_0 to A.

Henceforth, A and B shall be used to denote two intuitionistic fuzzy sets on U, which are defined below:

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in U\},$$

$$B = \{(x, \mu_B(x), \nu_B(x)) : x \in U\}.$$

Definition 2.2 [2]. The *subset* and *equality* of A and B are defined below:

(a) $A \subseteq B$, if $\mu_A(x) \leqslant \mu_B(x)$ and $\nu_A(x) \geqslant \nu_B(x)$ for all $x \in U$;

(b) $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Definition 2.3 [2]. The *complement*, *union*, and *intersection* of A and B are defined below:

(a) $A = \{(x, \nu_A(x), \mu_A(x)) : x \in U\}$;

(b) $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}) : x \in U\}$;

(c) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}) : x \in U\}$.

Definition 2.4 [2]. The set $\{x \in U : \nu_A(x) > 0\}$ is called the *support* of A and is denoted by $\mathscr{S}_A$. Moreover:

(a) A is said to be *null* if $\mathscr{S}_A = \emptyset$; otherwise, it is said to be *non-null*;

(b) A is said to be *absolute* if $\mathscr{S}_A = U$.

2.2. Soft sets and intuitionistic fuzzy soft sets

Definition 2.5 [3]. Let E be a set of parameters. Denote $\wp(U)$ to be the power set of U, and let $\mathcal{F}$ be a function from E to $\wp(U)$. Then, the set of ordered pairs $\{(\varepsilon, \mathcal{F}(\varepsilon)) : \varepsilon \in E, \mathcal{F}(\varepsilon) \in \wp(U)\}$, denoted by $(\mathcal{F}, E)$, is called a *soft set* on U. Moreover, for each $\varepsilon_0 \in E$, $\mathcal{F}(\varepsilon_0)$ is called the *set of* ε_0*-elements* of $(\mathcal{F}, E)$, or the ε_0*-approximate elements* of $(\mathcal{F}, E)$.

Definition 2.6 [11]. Let E be a set of parameters. Let $(\mathcal{F}, E)$ be a soft set on U. Then, the set $\{\varepsilon \in E : \mathcal{F}(\varepsilon) \neq \emptyset\}$, denoted by $\mathscr{S}(\mathcal{F}, E)$, is called the *support* of $(\mathcal{F}, E)$. Moreover, $(\mathcal{F}, E)$ is said to be *null* if $\mathscr{S}(\mathcal{F}, E) = \emptyset$; otherwise, it is said to be *non-null*.

Definition 2.7 [5]. Let E be a set of parameters. IFS(U) denotes a collection of all intuitionistic fuzzy sets on U and let $\mathcal{F}$ be a function from E to IFS(U). Then, the set of ordered pairs $\{(\varepsilon, \mathcal{F}(\varepsilon)) : \varepsilon \in E, \mathcal{F}(\varepsilon) \in \text{IFS}(U)\}$, denoted by $(\mathcal{F}, E)$, is called an *intuitionistic fuzzy soft set* on U.

Definition 2.8 [5]. Let $(\mathcal{F}, E)$ be an intuitionistic fuzzy soft set on U. Then, the set $\{\varepsilon \in E : \mathcal{F}(\varepsilon) \neq \emptyset\}$, denoted by $\mathscr{S}(\mathcal{F}, E)$, is called the *support* of $(\mathcal{F}, E)$. Moreover, $(\mathcal{F}, E)$ is said to be *null* if $\mathscr{S}(\mathcal{F}, E) = \emptyset$; otherwise, it is said to be *non-null*.

Definition 2.9 [5]. Let $(\mathcal{F}_1, E_1)$ and $(\mathcal{F}_2, E_2)$ be two intuitionistic fuzzy soft sets on U. Then, $(\mathcal{F}_1, E_1)$ is an *intuitionistic fuzzy soft subset* of $(\mathcal{F}_2, E_2)$, denoted by $(\mathcal{F}_1, E_1)\widetilde{\subseteq}(\mathcal{F}_2, E_2)$, if:

(i) $E_1 \subseteq E_2$;

(ii) $\mathcal{F}_1(\varepsilon) \subseteq \mathcal{F}_2(\varepsilon)$ for all $\varepsilon \in \mathscr{S}(\mathcal{F}_1, E_1)$.

Remark. For each $\varepsilon \in \mathscr{S}(\mathcal{F}_1, E_1)$, $\mathcal{F}_1(\varepsilon)$ is non-null. Thus, if $(\mathcal{F}_1, E_1)\widetilde{\subseteq}(\mathcal{F}_2, E_2)$, then $\mathcal{F}_1(\varepsilon) \subseteq \mathcal{F}_2(\varepsilon)$, and we also have $\mathcal{F}_2(\varepsilon)$ being non-null, which implies $\varepsilon \in \mathscr{S}(\mathcal{F}_2, E_2)$. As a result, the condition $\mathscr{S}(\mathcal{F}_1, E_1) \subseteq \mathscr{S}(\mathcal{F}_2, E_2)$ follows.

Definition 2.10 [5]. Let $(\mathcal{F}_1, E_1)$ and $(\mathcal{F}_2, E_2)$ be two intuitionistic fuzzy soft sets on U. Define $R = E_1 \cup E_2$, $S = E_1 \cap E_2$; for all $\varepsilon \in S$, $\overline{\mathcal{H}}(\varepsilon) = \mathcal{F}_1(\varepsilon) \cup \mathcal{F}_2(\varepsilon)$ and $\overline{\mathcal{K}}(\varepsilon) = \mathcal{F}_1(\varepsilon) \cap \mathcal{F}_2(\varepsilon)$.

$$\mathcal{H}(\varepsilon) = \begin{cases} \mathcal{F}_1(\varepsilon), & \varepsilon \in E_1 - S \\ \mathcal{F}_2(\varepsilon), & \varepsilon \in E_2 - S \\ \mathcal{F}_1(\varepsilon) \cup \mathcal{F}_2(\varepsilon), & \varepsilon \in S \end{cases}$$

and:

$$\mathcal{K}(\varepsilon) = \begin{cases} \mathcal{F}_1(\varepsilon), & \varepsilon \in E_1 - S \\ \mathcal{F}_2(\varepsilon), & \varepsilon \in E_2 - S \\ \mathcal{F}_1(\varepsilon) \cap \mathcal{F}_2(\varepsilon), & \varepsilon \in S \end{cases}$$

Then:

(i) $(\mathcal{H}, R)$ is called the *union* of $(\mathcal{F}_1, E_1)$ and $(\mathcal{F}_2, E_2)$ and is denoted by $(\mathcal{H}, R) = (\mathcal{F}_1, E_1)\widetilde{\cup}(\mathcal{F}_2, E_2)$;

(ii) $(\mathcal{K}, R)$ is called the *intersection* of $(\mathcal{F}_1, E_1)$ and $(\mathcal{F}_2, E_2)$ and is denoted by $(\mathcal{K}, R) = (\mathcal{F}_1, E_1)\widetilde{\cap}(\mathcal{F}_2, E_2)$;

(iii) $(\overline{\mathcal{H}}, S)$ is called the *restricted union* of $(\mathcal{F}_1, E_1)$ and $(\mathcal{F}_2, E_2)$ and is denoted by $(\overline{\mathcal{H}}, S) = (\mathcal{F}_1, E_1)\widehat{\cup}(\mathcal{F}_2, E_2)$;

(iv) $(\overline{\mathcal{K}}, S)$ is called the *restricted intersection* of $(\mathcal{F}_1, E_1)$ and $(\mathcal{F}_2, E_2)$ and is denoted by $(\overline{\mathcal{K}}, S) = (\mathcal{F}_1, E_1)\widehat{\cap}(\mathcal{F}_2, E_2)$;

2.3. Complex fuzzy sets

In this section, an overview of the concept of Complex Fuzzy Sets (CFS) [9] and Complex Intuitionistic Fuzzy Soft Sets (CIFSS) [10] is presented. Since the introduction of CFS, attempts to improve and overcome the drawbacks that are inherent in the CFS model have led to the introduction of several complex fuzzy-based hybrid models. We refer the readers to [10,12-18] for more details on these models.

Definition 2.11 [9]. A *complex fuzzy set* A defined on a universe of discourse U is characterized by a membership function $\mu_A(x)$ that assigns a complex-valued grade of membership in A toany element $x \in U$. By definition, all values of $\mu_A(x)$ lie within the unit circle on the complex plane and are expressed by $\mu_A(x) = r_A(x)\mathrm{e}^{\mathrm{i}\omega_A(x)}$, where $\mathrm{i} = \sqrt{-1}$, $r_A(x)$ and $\omega_A(x)$ are both real-valued, $r_A(x) \in [0,1]$, and $\omega_A(x) \in (0, 2\pi]$. A complex fuzzy set A is thus of the following form:

$$\begin{aligned} A &= \{(x, \mu_A(x)) : x \in U\} \\ &= \left\{ \left(x, r_A(x)\mathrm{e}^{\mathrm{i}\omega_A(x)}\right) : x \in U \right\}. \end{aligned}$$

Henceforth, symbol i is used to denote the imaginary unit $\sqrt{-1}$, whereas symbol O_1 is used to denote $\{z \in \mathbb{C} : |z| \leqslant 1\}$. Up until Section 2.2, we have reached the concept of Intuitionistic Fuzzy Soft Sets (IFSS), involving the relations $\geqslant$, $\leqslant$, max, min on the outcomes of membership and non-membership functions of the IFSS model. Such relations are inherently defined for real numbers only. On the other hand, a CFS and all its generalizations have membership and non-membership functions that can lie anywhere in O_1. We must, therefore, generalize the concept of $\geqslant$, $\leqslant$, max, min for all complex numbers in O_1. To achieve these, the following definitions and lemmas are given.

Definition 2.12. Let $\mu = r\mathrm{e}^{\mathrm{i}\omega}$ and $\nu = \tau\mathrm{e}^{\mathrm{i}\psi}$, with $r, \tau \in [0,1]$ and $\omega, \psi \in (0, 2\pi]$. The relations $\geqslant$ and $\leqslant$ are given as follows:

(i) $\mu \geqslant \nu$, when both $r \geqslant \tau$ and $\omega \geqslant \psi$, or when $\nu = 0$;

(ii) $\mu \leqslant \nu$, when both $r \leqslant \tau$ and $\omega \leqslant \psi$, or when $\mu = 0$.

Remark. The usual definition of $\geqslant$ and $\leqslant$ at the real interval $[0,1]$ is a special case of this definition. However, there remain pairs of elements of O_1 such that neither $\geqslant$ nor $\leqslant$ can be established between them, such as $0.1\mathrm{e}^{3\mathrm{i}}$ and $0.4\mathrm{e}^{2\mathrm{i}}$, because $0.1 < 0.4$, but $3 > 2$. Nonetheless, $0 \leqslant \mu \leqslant 1$ still holds for all $\mu \in \mathrm{O}_1$.

Definition 2.13. Let $S = \{\mu_n : n \in V\} \subseteq \mathrm{O}_1$. Then, $\max S$ and $\min S$ are as defined below:

(i) (a) $\max S \geqslant \mu_n$ for all $n \in V$;

(b) If $\xi \in \mathrm{O}_1$ is such that $\xi \geqslant \mu_n$ for all $n \in V$, then $\xi \geqslant \max S$;

(ii) (a) $\min S \leqslant \mu_n$ for all $n \in V$;

(b) If $\zeta \in \mathrm{O}_1$ is such that $\zeta \leqslant \mu_n$ for all $n \in V$, then $\zeta \leqslant \max S$.

Remark. Unlike subsets of $\mathbb{R}$, $\max S$ and $\min S$ may not be in S, even if S is finite. For example, if $S_0 = \{0.1\mathrm{e}^{3\mathrm{i}}, 0.4\mathrm{e}^{2\mathrm{i}}\}$, then $\max S_0 = 0.4\mathrm{e}^{3\mathrm{i}}$ and $\min S_0 = 0.1\mathrm{e}^{2\mathrm{i}}$.

Definition 2.14. Let $\mu = r\mathrm{e}^{\mathrm{i}\omega}$, with $r \in [0,1]$ and $\omega \in (0, 2\pi]$. The *complement* of μ, denoted by $1 \sim \mu$, is defined as $1 \sim \mu = (1-r)\mathrm{e}^{\mathrm{i}\omega'}$, where:

$$\omega' = \begin{cases} 2\pi - \omega, & \omega < 2\pi \\ \omega, & \omega = 2\pi \end{cases}$$

Remark. If $\mu \in [0,1]$, then $1 \sim \mu = 1 - \mu$.

Lemma 2.1. *For all $\mu \in O_1$, $1 \sim (1 \sim \mu) = \mu$.*

Remark. Let $\mu = r\mathrm{e}^{\mathrm{i}\omega}$, with $r \in [0,1]$ and $\omega \in (0, 2\pi]$. Then, $|\mu| = r$.

Lemma 2.2. *For all $\mu \in O_1$, $|1 \sim \mu| = 1 - |\mu|$.*

2.4. Complex intuitionistic fuzzy soft sets

The object of study in this paper is the CIFSS model [10], which is an adaptation of the original CFS model [9]. It is a hybrid composed of complex fuzzy sets, intuitionistic fuzzy sets, and soft sets characterized by membership and non-membership functions that represent the degree of belongingness and non-belongingness of the elements with respect to the attributes that are under consideration.

Definition 2.15 [10]. Let E be a set of parameters, CIFS(U) be the collection of all complex intuitionistic fuzzy sets on U, and $\widetilde{\mathcal{F}}$ be a function from E to CIFS(U). Then, the set of ordered pairs $\{(\varepsilon, \widetilde{\mathcal{F}}(\varepsilon)) : \varepsilon \in E, \widetilde{\mathcal{F}}(\varepsilon) \in \mathrm{CIFS}(U)\}$, denoted by $(\widetilde{\mathcal{F}}, E)$, is called a *Complex Intuitionistic Fuzzy Soft Set* (CIFSS) on U. Note that, for each $\varepsilon \in E$:

$$\begin{aligned} \widetilde{\mathcal{F}}(\varepsilon) &= \left\{ \left(x, \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x)\right) : x \in U \right\} \\ &= \left\{ \left(x, r_{\widetilde{\mathcal{F}}(\varepsilon)}(x)\mathrm{e}^{\mathrm{i}\omega_{\widetilde{\mathcal{F}}(\varepsilon)}(x)}, \tau_{\widetilde{\mathcal{F}}(\varepsilon)}(x)\mathrm{e}^{\mathrm{i}\psi_{\widetilde{\mathcal{F}}(\varepsilon)}(x)}\right) : x \in U \right\}. \end{aligned}$$

In all that follows, let CIFSS(U) denote the collection of all complex intuitionistic fuzzy soft sets on a universe U. Furthermore, we write $(\widetilde{\mathcal{F}}, E) \in \mathrm{CIFSS}(U)$ to denote that $(\widetilde{\mathcal{F}}, E)$ is a complex intuitionistic fuzzy soft set on U.

Definition 2.16. Let $(\widetilde{\mathcal{F}}, E) \in \mathrm{CIFSS}(U)$. Then, the set $\{\varepsilon \in E : \widetilde{\mathcal{F}}(\varepsilon)$ is non-null$\}$, denoted by $\mathscr{S}(\widetilde{\mathcal{F}}, E)$, and is called the *support* of $(\widetilde{\mathcal{F}}, E)$. Moreover, $(\widetilde{\mathcal{F}}, E)$ is said to be *null* if $\mathscr{S}(\widetilde{\mathcal{F}}, E) = \emptyset$; otherwise, it is said to be *non-null*.

Definition 2.17 [10]. Let $(\widetilde{\mathcal{F}}_1, E_1), (\widetilde{\mathcal{F}}_2, E_2) \in \mathrm{CIFSS}(U)$. Then, $(\widetilde{\mathcal{F}}_1, E_1)$ is a *complex intuitionistic fuzzy soft subset* of $(\widetilde{\mathcal{F}}_2, E_2)$, denoted as $(\widetilde{\mathcal{F}}_1, E_1)\widetilde{\subseteq}(\widetilde{\mathcal{F}}_2, E_2)$, if:

(i) $E_1 \subseteq E_2$;

(ii) $\widetilde{\mathcal{F}}_1(\varepsilon) \subseteq \widetilde{\mathcal{F}}_2(\varepsilon)$ for all $\varepsilon \in \mathscr{S}(\widetilde{\mathcal{F}}_1, E_1)$.

Remark. When $(\widetilde{\mathcal{F}}_1, E_1)\widetilde{\subseteq}(\widetilde{\mathcal{F}}_2, E_2)$ for each $\varepsilon \in \mathscr{S}(\widetilde{\mathcal{F}}_1, E_1)$, $\widetilde{\mathcal{F}}_1(\varepsilon)$ is non-null. Since $\widetilde{\mathcal{F}}_1(\varepsilon) \subseteq \widetilde{\mathcal{F}}_2(\varepsilon)$, we also have $\widetilde{\mathcal{F}}_2(\varepsilon)$ being non-null, which implies $\varepsilon \in \mathscr{S}(\widetilde{\mathcal{F}}_2, E_2)$. As a result, the condition $\mathscr{S}(\widetilde{\mathcal{F}}_1, E_1) \subseteq \mathscr{S}(\widetilde{\mathcal{F}}_2, E_2)$ follows.

Definition 2.18 [10]. Let $(\widetilde{\mathcal{F}}, E) \in \mathrm{CIFSS}(U)$. Then, the *complement* of $(\widetilde{\mathcal{F}}, E)$, denoted by $(\widetilde{\mathcal{F}}, E)^c$, is defined as $(\widetilde{\mathcal{F}}, E)^c = (\widetilde{\mathcal{F}}^c, \neg E)$, where $\widetilde{\mathcal{F}}^c$ is a function from $\neg E$ to $\mathrm{CIFS}(U)$ given by:

$$\widetilde{\mathcal{F}}^c(\neg\varepsilon) = \left\{\left(x, \nu_{\widetilde{\mathcal{F}}(\neg\neg\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\neg\neg\varepsilon)}(x)\right) : x \in U\right\}$$
$$= \left\{\left(x, \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x)\right) : x \in U\right\},$$

for all $\neg\varepsilon \in \neg E$.

Remark. Definition 2.18 can be restated as follows.

Let $(\widetilde{\mathcal{F}}, E) \in \mathrm{CIFSS}(U)$. Define $\widetilde{\mathcal{T}}$ as a function from E to $\mathrm{CIFS}(U)$, where $\widetilde{\mathcal{T}}(\varepsilon)$ is the complement of $\widetilde{\mathcal{F}}(\varepsilon)$ for all $\varepsilon \in E$. Then, $(\widetilde{\mathcal{T}}, E)$ is called the *complement* of $(\widetilde{\mathcal{F}}, E)$ and this can be denoted as $(\widetilde{\mathcal{T}}, E) = (\widetilde{\mathcal{F}}, E)^c$.

Note that, for each $\varepsilon \in E$, $\widetilde{\mathcal{T}}(\varepsilon) = \{(x, \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x)) : x \in U\}$ in line with Definition 2.3.

Definition 2.19 [10]. Let $(\widetilde{\mathcal{F}}_1, E_1), (\widetilde{\mathcal{F}}_2, E_2) \in \mathrm{CIFSS}(U)$. Define $R = E_1 \cup E_2$, $S = E_1 \cap E_2$; and for all $\varepsilon \in S$, $\overline{\mathcal{H}}(\varepsilon) = \widetilde{\mathcal{F}}_1(\varepsilon) \cup \widetilde{\mathcal{F}}_2(\varepsilon)$ and $\overline{\mathcal{K}}(\varepsilon) = \widetilde{\mathcal{F}}_1(\varepsilon) \cap \widetilde{\mathcal{F}}_2(\varepsilon)$.

$$\mathcal{H}(\varepsilon) = \begin{cases} \widetilde{\mathcal{F}}_1(\varepsilon), & \varepsilon \in E_1 - S \\ \widetilde{\mathcal{F}}_2(\varepsilon), & \varepsilon \in E_2 - S \\ \widetilde{\mathcal{F}}_1(\varepsilon) \cup \widetilde{\mathcal{F}}_2(\varepsilon), & \varepsilon \in S \end{cases}$$

and:

$$\mathcal{K}(\varepsilon) = \begin{cases} \widetilde{\mathcal{F}}_1(\varepsilon), & \varepsilon \in E_1 - S \\ \widetilde{\mathcal{F}}_2(\varepsilon), & \varepsilon \in E_2 - S \\ \widetilde{\mathcal{F}}_1(\varepsilon) \cap \widetilde{\mathcal{F}}_2(\varepsilon), & \varepsilon \in S \end{cases}$$

Then:

(i) $(\mathcal{H}, R)$ is called the *union* of $(\widetilde{\mathcal{F}}_1, E_1)$ and $(\widetilde{\mathcal{F}}_2, E_2)$ and is denoted by $(\mathcal{H}, R) = (\widetilde{\mathcal{F}}_1, E_1)\widetilde{\cup}(\widetilde{\mathcal{F}}_2, E_2)$;

(ii) $(\mathcal{K}, R)$ is called the *intersection* of $(\widetilde{\mathcal{F}}_1, E_1)$ and $(\widetilde{\mathcal{F}}_2, E_2)$ and is denoted by $(\mathcal{K}, R) = (\widetilde{\mathcal{F}}_1, E_1)\widetilde{\cap}(\widetilde{\mathcal{F}}_2, E_2)$;

(iii) $(\overline{\mathcal{H}}, S)$ is called the *restricted union* of $(\widetilde{\mathcal{F}}_1, E_1)$ and $(\widetilde{\mathcal{F}}_2, E_2)$ and is denoted by $(\overline{\mathcal{H}}, S) = (\widetilde{\mathcal{F}}_1, E_1)\widehat{\cup}(\widetilde{\mathcal{F}}_2, E_2)$;

(iv) $(\overline{\mathcal{K}}, S)$ is called the *restricted intersection* of $(\widetilde{\mathcal{F}}_1, E_1)$ and $(\widetilde{\mathcal{F}}_2, E_2)$ and is denoted by $(\overline{\mathcal{K}}, S) = (\widetilde{\mathcal{F}}_1, E_1)\widehat{\cap}(\widetilde{\mathcal{F}}_2, E_2)$.

We now define two new operations for the CIFSS model, namely the (α, β)-level set and the characteristic set of a CIFSS, and provide some properties of these operations. The formal definitions of these operations and the properties of these operations are given below.

Definition 2.20. Let $(\widetilde{\mathcal{F}}, E) \in \mathrm{CIFSS}(U)$, and $\alpha, \beta \in \mathrm{O}_1$. The *$(\alpha, \beta)$-level set* of $(\widetilde{\mathcal{F}}, E)$, denoted by $(\widetilde{\mathcal{F}}, E)_{(\alpha,\beta)}$, is a soft set on U defined below:

$$\left(\widetilde{\mathcal{F}}, E\right)_{(\alpha,\beta)} = \left\{\left(\varepsilon, \widetilde{\mathcal{F}}_{(\alpha,\beta)}(\varepsilon)\right) : \varepsilon \in E, \widetilde{\mathcal{F}}_{(\alpha,\beta)}(\varepsilon) \in \wp(U)\right\},$$

where $\widetilde{\mathcal{F}}_{(\alpha,\beta)}(\varepsilon) = \{x \in U : \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x) \geqslant \alpha, \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x) \leqslant \beta\}$ for all $\varepsilon \in E$.

If $\alpha = \beta$, then $(\widetilde{\mathcal{F}}, E)_{(\alpha,\alpha)}$ is called the *α-level set* of $(\widetilde{\mathcal{F}}, E)$, denoted by $(\widetilde{\mathcal{F}}, E)_\alpha$, and defined as:

$$\left(\widetilde{\mathcal{F}}, E\right)_\alpha = \left\{\left(\varepsilon, \widetilde{\mathcal{F}}_\alpha(\varepsilon)\right) : \varepsilon \in E, \widetilde{\mathcal{F}}_\alpha(\varepsilon) \in \wp(U)\right\},$$

where $\widetilde{\mathcal{F}}_\alpha(\varepsilon) = \{x \in U : \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x) \geqslant \alpha, \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x) \leqslant \alpha\}$ for all $\varepsilon \in E$.

Remark. Note that since $\widetilde{\mathcal{F}}_{(\alpha,\beta)}(\varepsilon) \in \wp(U)$ for all $\varepsilon \in E$, we have:

$$\left(\widetilde{\mathcal{F}}, E\right)_{(\alpha,\beta)} = \left\{\left(\varepsilon, \widetilde{\mathcal{F}}_{(\alpha,\beta)}(\varepsilon)\right) : \varepsilon \in E\right\}.$$

Definition 2.21. Let $(\widetilde{\mathcal{F}}, E) \in \mathrm{CIFSS}(U)$ and S be a non-null proper subset of U. If $\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)} : \varepsilon \in E\} = \mu_0$ and $\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)} : \varepsilon \in E\} = \nu_0$, in which:

$$\mu_0(x) = \begin{cases} r\mathrm{e}^{\mathrm{i}\omega}, & x \in S \\ 1 \sim r\mathrm{e}^{\mathrm{i}\omega}, & x \in U - S \end{cases}$$

and:

$$\nu_0(x) = \begin{cases} \tau\mathrm{e}^{\mathrm{i}\psi}, & x \in S \\ 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}, & x \in U - S \end{cases}$$

where $\omega + \psi \in \{2\pi, 4\pi\}$ and $r\mathrm{e}^{\mathrm{i}\omega} \geqslant \tau\mathrm{e}^{\mathrm{i}\psi}$, and then $(\widetilde{\mathcal{F}}, E)$ is said to be *characteristic over S*.

Remark. Consider the particular case where $U = \{p,q\}$ and $\mu_0(x) = \nu_0(x) = \frac{1}{2}$ for all $x \in U$.

Note that $r\mathrm{e}^{\mathrm{i}\omega} = \tau\mathrm{e}^{\mathrm{i}\psi} = r\mathrm{e}^{\mathrm{i}\omega} = \tau\mathrm{e}^{\mathrm{i}\psi} = \frac{1}{2} \in \mathrm{O}_1$, and $\omega+\psi = 4\pi$, $r = \tau$, and $\omega = \psi$. However, S can be either $\{p\}$ or $\{q\}$. We now have an example of $(\widetilde{\mathcal{F}}, E)$ as characteristic over *more than one* non-null proper subset of U.

Proposition 2.1. *Let* $(\widetilde{\mathcal{F}}, E) \in \mathrm{CIFSS}(U)$, *and* $(\widetilde{\mathcal{F}}, E)$ *be characteristic over* S, *in which:*

$$\mu_0(x) = \begin{cases} r\mathrm{e}^{\mathrm{i}\omega}, & x \in S \\ 1 \sim r\mathrm{e}^{\mathrm{i}\omega}, & x \in U - S \end{cases}$$

and:

$$\nu_0(x) = \begin{cases} \tau\mathrm{e}^{\mathrm{i}\psi}, & x \in S \\ 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}, & x \in U - S \end{cases}$$

are the membership and non-membership functions of $\widetilde{\mathcal{F}}(\varepsilon)$, *respectively. Then:*

(i) $r + \tau = 1$;

(ii) $r\mathrm{e}^{\mathrm{i}\omega} \geqslant 1 \sim r\mathrm{e}^{\mathrm{i}\omega}$ *and* $\tau\mathrm{e}^{\mathrm{i}\psi} \leqslant 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}$.

Proof.

(i) Note that μ_0 and ν_0 are the membership and the non-membership functions of $\widetilde{\mathcal{F}}(\varepsilon)$, which is a complex intuitionistic fuzzy set. Then, the condition $0 \leqslant |\mu_0(x)| + |\mu_0(x)| \leqslant 1$ holds for all $x \in U$. We now have both $0 \leqslant r+\tau \leqslant 1$ and $0 \leqslant |1 \sim r\mathrm{e}^{\mathrm{i}\omega}| + |1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}| \leqslant 1$ by Definition 2.21. From Lemma 2.2, it follows that:

$$|1 \sim r\mathrm{e}^{\mathrm{i}\omega}| + |1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}| = (1-r) + (1-\tau)$$

$$= 2 - (r+\tau),$$

causing $0 \leqslant 2-(r+\tau) \leqslant 1$; therefore, $1 \leqslant r+\tau \leqslant 2$, which implies $r + \tau = 1$;

(ii) As $(\widetilde{\mathcal{F}}, E)$ is characteristic over S, $\omega+\psi \in \{2\pi, 4\pi\}$ and $r\mathrm{e}^{\mathrm{i}\omega} \geqslant \tau\mathrm{e}^{\mathrm{i}\psi}$. Since $r \geqslant \tau$ and $r + \tau = 1$, it follows that $r \geqslant \frac{1}{2}$ and $\tau \leqslant \frac{1}{2}$. Thus, we have $1-r \leqslant \frac{1}{2}$ and $1-\tau \geqslant \frac{1}{2}$. These further imply that $r \geqslant 1-r$ and $\tau \leqslant 1-\tau$.

Now, suppose that $\omega + \psi = 2\pi$. Since $\omega \geqslant \psi$, it follows that $\omega \geqslant \pi$ and $\psi \leqslant \pi$. We now have $1-\omega \leqslant \pi$ and $1-\psi \geqslant \pi$, implying that $\omega \geqslant 2\pi-\omega$ and $\psi \leqslant 2\pi-\psi$. In addition, note that both $\omega, \psi < 2\pi$. By Definition 2.14, we have $r\mathrm{e}^{\mathrm{i}\omega} \geqslant (1-r)\mathrm{e}^{\mathrm{i}(2\pi-\omega)} = 1 \sim r\mathrm{e}^{\mathrm{i}\omega}$ and $\tau\mathrm{e}^{\mathrm{i}\psi} \leqslant (1-\tau)\mathrm{e}^{\mathrm{i}(2\pi-\psi)} = 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}$.

On the other hand, if $\omega + \psi = 4\pi$, then $\omega = \psi = 2\pi$. Then, by Definition 2.14, we have $r\mathrm{e}^{\mathrm{i}\omega} \geqslant (1-r)\mathrm{e}^{\mathrm{i}\omega} = 1 \sim r\mathrm{e}^{\mathrm{i}\omega}$ and $\tau\mathrm{e}^{\mathrm{i}\psi} \leqslant (1-\tau)\mathrm{e}^{\mathrm{i}\psi} = 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}$. □

3. Complex intuitionistic fuzzy soft groups

The study of soft algebra and fuzzy soft algebra was initiated by Aktas & Cagman [19] and Aygunoglu & Aygun [20], respectively. Other researchers such as Feng et al. [11], Acar et al. [21], Inan and Ozturk [22], and Ghosh et al. [23] also contributed to the development of these areas. Besides, many more advanced algebraic structures pertaining to groups, rings, and hemirings of fuzzy soft sets have been introduced in the literature. Some of the latest works include the introduction of soft fuzzy rough rings and ideals by Zhu [24], I-fuzzy soft groups by Vimala et al. [25], soft union set characterizations of hemirings by Zhan et al. [26], and neutrosophic normal soft groups by Bera and Mahapatra [27]. Yamak et al. [28], Leoreanu-Fotea et al. [29], and Selvachandran and Salleh [30-34], on the other hand, were responsible for introducing the algebraic structures of soft hypergroupoids, fuzzy soft hypergroups as well as soft hyperrings, fuzzy soft hyperrings, vague soft hypergroups, and hyperrings, respectively. Khan et al. [35] proposed the notion of soft interior hyperideals of ordered semihypergroups, whereas Ma et al. [36] studied the concept of rough soft hyperrings.

Research in the area of complex fuzzy algebra is still in its infancy. The study of the complex fuzzy algebraic theory was initiated by Al-Husban et al. [37,38] through the introduction of the algebraic structures of complex fuzzy subrings and complex fuzzy rings in [37,38], respectively. Al-Husban and Salleh [39], then, defined the notion of a complex fuzzy group, which is defined in a complex fuzzy space, instead of an ordinary universe of discourse. Alsarahead and Ahmad [40,41], then, proposed the structures of complex fuzzy subgroups and complex fuzzy soft groups in [40,41], respectively. To the best of our knowledge, these are the only published works in this area of research at present.

The aim of this section is to establish the novel concept of Complex Intuitionistic Fuzzy Soft groups (CIFS-groups) in the Rosenfelds sense (i.e., using the concept of a fuzzy subgroup of a group defined by Rosenfeld [42]). The properties and structural characteristics of the proposed algebraic structures are examined and, subsequently, verified.

Henceforth, symbol G will be used to denote a group.

Definition 3.1 [19]. Let $(\mathcal{F}, E)$ be a non-null soft set on G. Then, $(\mathcal{F}, E)$ is said to be a *soft group* on G if $\mathcal{F}(\varepsilon) \leqslant G$ for all $\varepsilon \in \mathscr{S}(\mathcal{F}, E)$.

Remark. As in classical group theory, a null set cannot be a group, and a null soft set on G is not a soft group on G.

Now, define the notion of a complex intuitionistic fuzzy subgroup of group G and, then, use it to define the notion of a complex intuitionistic fuzzy soft group of a group G.

Definition 3.2. Let $M = (x, \mu_M(x), \nu_M(x)) : x \in G$ be a complex intuitionistic fuzzy set on G. Then, M is said to be a Complex Intuitionistic Fuzzy subgroup (CIF-subgroup) of G, if the following conditions hold for all $x, y \in G$:

(i) $\mu_M(xy) \geqslant \min\{\mu_M(x), \mu_M(y)\}$;

(ii) $\nu_M(xy) \leqslant \max\{\nu_M(x), \nu_M(y)\}$;

(iii) $\mu_M(x^{-1}) \geqslant \mu_M(x)$;

(iv) $\nu_M(x^{-1}) \leqslant \nu_M(x)$.

Moreover, let M and N be two complex intuitionistic fuzzy subgroups of G with $M \subseteq N$. In this case, M is said to be a complex intuitionistic fuzzy subgroup of N.

Definition 3.3. Let $(\widetilde{\mathcal{F}}, E) \in$ CIFSS(G). Then, $(\widetilde{\mathcal{F}}, E)$ is said to be a *Complex Intuitionistic Fuzzy Soft group* (*CIFS-group*) on G if $\widetilde{\mathcal{F}}(\varepsilon)$ is a complex intuitionistic fuzzy subgroup of G for all $\varepsilon \in \mathscr{S}(\widetilde{\mathcal{F}}, E)$.

In all that follows, CIFSG(G) denotes the collection of all complex intuitionistic fuzzy soft groups on a group G, and $(\widetilde{\mathcal{F}}, E) \in$ CIFSG(G) denotes that $(\widetilde{\mathcal{F}}, E)$ is a complex intuitionistic fuzzy soft group on G.

Example 3.1. Consider the case where G is the symmetric group of order 3, that is, $G = S_3 = \{1, (12), (23), (13), (123), (132)\}$. Next, consider a set of parameters $E = \{a, b\}$. Herein, $\mu_1 = 0.4\mathrm{e}^{\mathrm{i}}$, $\mu_2 = 0.4\mathrm{e}^{2\mathrm{i}}$, $\mu_3 = 0.7\mathrm{e}^{2\mathrm{i}}$; as well as $\nu_1 = 0.2\mathrm{e}^{4\mathrm{i}}$, $\nu_2 = 0.1\mathrm{e}^{4\mathrm{i}}$, $\nu_3 = 0.1\mathrm{e}^{3\mathrm{i}}$ are defined. Note that $\mu_1 \leqslant \mu_2 \leqslant \mu_3$ and $\nu_1 \geqslant \nu_2 \geqslant \nu_3$. Now, two CIFSSs of G are considered, which are defined as follows:

(i) $(\widetilde{\mathcal{F}}, E) = \{\widetilde{\mathcal{F}}(a), \widetilde{\mathcal{F}}(b)\}$, where:

$$\widetilde{\mathcal{F}}(a) = \left\{ \begin{array}{l} (1,\mu_3,\nu_3), ((12), \mu_1, \nu_1), \\ \quad ((13), \mu_1, \nu_1), ((23), \mu_1, \nu_1), \\ \quad ((123), \mu_2, \nu_2), ((132), \mu_2, \nu_2) \end{array} \right\},$$

and:

$$\widetilde{\mathcal{F}}(b) = \left\{ \begin{array}{l} (1,\mu_3,\nu_3), ((12), \mu_2, \nu_2), \\ \quad ((13), \mu_1, \nu_1), ((23), \mu_1, \nu_1), \\ \quad ((123), \mu_1, \nu_1), ((132), \mu_1, \nu_1) \end{array} \right\},$$

(ii) $(\widetilde{\mathcal{G}}, E) = \{\widetilde{\mathcal{G}}(a), \widetilde{\mathcal{G}}(b)\}$, where:

$$\widetilde{\mathcal{G}}(a) = \widetilde{\mathcal{F}}(a),$$

and:

$$\widetilde{\mathcal{G}}(b) = \left\{ \begin{array}{l} (1,\mu_1,\nu_1), ((12), \mu_2, \nu_2), \\ \quad ((13), \mu_2, \nu_2), ((23), \mu_1, \nu_1), \\ \quad ((123), \mu_1, \nu_1), ((132), \mu_1, \nu_1) \end{array} \right\}.$$

Accordingly, it can be verified that $(\widetilde{\mathcal{F}}, E) \in$ CIFSS(G), whereas $(\widetilde{\mathcal{G}}, E) \notin$ CIFSS(G).

Definition 3.4. Let $(\widetilde{\mathcal{F}}_1, E_1), (\widetilde{\mathcal{F}}_2, E_2) \in$ CIFSS(G). Then, $(\widetilde{\mathcal{F}}_1, E_1)$ is said to be a *Complex Intuitionistic Fuzzy Soft subgroup* (*CIFS-subgroup*) of $(\widetilde{\mathcal{F}}_2, E_2)$ if the following conditions are satisfied:

(i) $E_1 \subseteq E_2$;

(ii) For all $\varepsilon \in E_1$, $\widetilde{\mathcal{F}}_1(\varepsilon)$ is a complex intuitionistic fuzzy subgroup of $(\widetilde{\mathcal{F}}_2 \varepsilon)$.

Proposition 3.1. *Let $(\widetilde{\mathcal{F}}, E) \in$ CIFSS(G) and 1_G be the identity element of G. Then, the following results hold for all $\varepsilon \in E$ and for all $x \in G$:*

(i) $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x^{-1}) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x)$ *and* $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x^{-1}) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x)$,

(ii) $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(1_G) \geqslant \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x)$ *and* $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(1_G) \leqslant \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x)$.

Proof. Let $\varepsilon \in E$ and $x \in G$. By Definition 3.3, $\widetilde{\mathcal{F}}(\varepsilon)$ is a CIFS-subgroup of G, which enables us to utilize Definition 3.2 for proving both (i) and (ii):

(i) Both $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x^{-1}) \geqslant \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x)$ and $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x^{-1}) \leqslant \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x)$ directly follow from Definition 3.2. Since $x \in G$, we also have $x^{-1} \in G$. Thus, it follows that:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)} \left(\left(x^{-1} \right)^{-1} \right) \geqslant \mu_{\widetilde{\mathcal{F}}(\varepsilon)} \left(x^{-1} \right),$$

and:

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)} \left(\left(x^{-1} \right)^{-1} \right) \leqslant \nu_{\widetilde{\mathcal{F}}(\varepsilon)} \left(x^{-1} \right),$$

due to Definition 3.2.

(ii) Note that $1_G = xx^{-1}$; thus, the conditions $\mu_{\widetilde{\mathcal{F}}(\varepsilon)} (1_G) \geqslant \min\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x^{-1})\}$, and $\nu_{\widetilde{\mathcal{F}}(\varepsilon)} (1_G) \leqslant \max\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x^{-1})\}$, follow from Definition 3.2. By (i), we have:

$$\min \left\{ \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)} \left(x^{-1} \right) \right\}$$

$$= \min \left\{ \mu_{\widetilde{\mathcal{F}}(\varepsilon) M}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x) \right\} = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x),$$

and:

$$\max\left\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(x^{-1}\right)\right\}$$

$$= \max\left\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x)\right\} = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x).$$

This completes the proof. □

Proposition 3.2. *Let $(\widetilde{\mathcal{F}}, E) \in$ CIFSS(G). Then, $(\widetilde{\mathcal{F}}, E) \in$ CIFSG(G) if and only if the following conditions are satisfied for all $\varepsilon \in E$ and for all $x, y \in G$:*

(i) $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy^{-1}) \geqslant \min\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\}$;

(ii) $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy^{-1}) \leqslant \max\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\}$.

Proof. $(\Rightarrow)$ Suppose that $(\widetilde{\mathcal{F}}, E) \in$ CIFSG(G).

Let $\varepsilon \in E$ and $x, y \in G$. Then, $y^{-1} \in G$ too, and based on Definition 3.3, $\widetilde{\mathcal{F}}(\varepsilon)$ is a CIF-subgroup of G. The conditions:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy^{-1}) \geqslant \min\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y^{-1})\},$$

and:

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy^{-1}) \leqslant \max\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y^{-1})\},$$

directly follow from Definition 3.2. Similarly, we have $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y^{-1}) \geqslant \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)$ and $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y^{-1}) \leqslant \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)$, which implies that:

$$\min\left\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(y^{-1}\right)\right\}$$

$$\geqslant \min\left\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\},$$

and:

$$\max\left\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(y^{-1}\right)\right\}$$

$$\leqslant \max\left\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\}.$$

Thus, conditions (i) and (ii) now hold.

$(\Leftarrow)$ Suppose that conditions (i) and (ii) are satisfied for all $\varepsilon \in E$ and for all $x, y \in G$.

By considering the case $x = y$, we have:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(1_G) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(yy^{-1}\right)$$

$$\geqslant \min\left\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\} = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y),$$

and:

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(1_G) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(xx^{-1}\right)$$

$$\leqslant \max\left\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x)\right\} = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x).$$

These imply that:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(y^{-1}\right) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(1_G y^{-1}\right)$$

$$\geqslant \min\left\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(1_G), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\} = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y),$$

and:

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(y^{-1}\right) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(1_G y^{-1}\right)$$

$$\leqslant \max\left\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(1_G), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\} = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y).$$

By considering $y^{-1} \in G$, it follows that:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy) \geqslant \min\left\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(y^{-1}\right)\right\}$$

$$\geqslant \min\left\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\},$$

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy) \leqslant \max\left\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(y^{-1}\right)\right\}$$

$$\leqslant \max\left\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\}.$$

Thus, $\widetilde{\mathcal{F}}(\varepsilon)$ is proved to be a CIF-subgroup of G for all $\varepsilon \in E$; hence, it follows that $(\widetilde{\mathcal{F}}, E) \in$ CIFSG(G). □

Proposition 3.3. *Let $(\widetilde{\mathcal{F}}, E) \in$ CIFSG(G), and $\alpha, \beta \in O_1$. If $(\widetilde{\mathcal{F}}, E)_{(\alpha,\beta)}$ is non-null, then it is a soft group of G.*

Proof. The proof is straightforward. □

Proposition 3.4. *Let S be a non-null subset of G and $(\widetilde{\mathcal{F}}, E) \in$ CIFSS(G), where $(\widetilde{\mathcal{F}}, E)$ is characteristic over S. IF $(\widetilde{\mathcal{F}}, E) \in$ CIFSG(G), then S is a classical subgroup of G.*

Proof. Let $x, y \in S$. Then, by Definition 2.21, $\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)} : \varepsilon \in E\} = \mu_0$ and $\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)} : \varepsilon \in E\} = \nu_0$, in which there exist $\alpha, \beta \in \mathrm{O}_1$, with $\alpha \geqslant \beta$, such that $\mu_0(x) = \mu_0(y) = \alpha$ and $\nu_0(x) = \nu_0(y) = \beta$. Thus, it follows that $x, y \in \widetilde{\mathcal{F}}_{(\alpha,\beta)}(\varepsilon)$ for all $\varepsilon \in E$, which further implies that $\widetilde{\mathcal{F}}_{(\alpha,\beta)}(\varepsilon)$ is not empty for all $\varepsilon \in E$. Therefore, we have $\mathscr{S}(\widetilde{\mathcal{F}}, E)_{(\alpha,\beta)} = E$. Since $\mathscr{S}(\widetilde{\mathcal{F}}, E)_{(\alpha,\beta)}$ is not empty, $(\widetilde{\mathcal{F}}, E)_{(\alpha,\beta)}$ is non-null, and $(\widetilde{\mathcal{F}}, E)_{(\alpha,\beta)}$ is, therefore, a soft group of G.

Take $\varepsilon_0 \in E$. Based on Definition 3.1, it follows that $\widetilde{\mathcal{F}}_{(\alpha,\beta)}(\varepsilon_0)$ is a subgroup of G; therefore, we have $xy^{-1} \in \widetilde{\mathcal{F}}_{(\alpha,\beta)}(\varepsilon_0)$. Then, by Definition 2.20, $\mu_{\widetilde{\mathcal{F}}(\varepsilon_0)}(xy^{-1}) \geqslant \alpha$ and $\nu_{\widetilde{\mathcal{F}}(\varepsilon_0)}(xy^{-1}) \leqslant \beta$.

Recall that $\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)} : \varepsilon \in E\} = \mu_0$ and $\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)} : \varepsilon \in E\} = \nu_0$. As a result, $\mu_0(xy^{-1}) \geqslant \alpha$ and $\nu_0(xy^{-1}) \leqslant \beta$.

We now show that both $\mu_0(xy^{-1}) = \alpha$ and $\nu_0(xy^{-1}) = \beta$, which in turn implies that $xy^{-1} \in S$. We write $\alpha = r\mathrm{e}^{\mathrm{i}\omega}$ and $\beta = \tau\mathrm{e}^{\mathrm{i}\psi}$, for some $r, \tau \in [0,1]$ and $\omega, \psi \in (0, 2\pi]$. Then, $\omega + \psi \in \{2\pi, 4\pi\}$ follows because of Definition 2.21. Furthermore, recall that $r + \tau = 1$ (and, thus, $|\alpha| + |\beta| = 1$) due to Proposition 2.1.

(a) By Definition 2.21, it is either $\mu_0(xy^{-1}) = \alpha$ or $\mu_0(xy^{-1}) = 1 \sim \alpha$. Suppose that $\mu_0(xy^{-1}) = 1 \sim \alpha$; then, $1 \sim \alpha \geqslant \alpha$. Since $1 \sim \alpha = (1-r)\mathrm{e}^{\mathrm{i}\omega'} \geqslant r\mathrm{e}^{\mathrm{i}\omega} = \alpha$, it follows that $|\beta| = 1 - |\alpha| = |1 \sim \alpha| \geqslant |\alpha| \geqslant |\beta|$. This further implies that $|1 \sim \alpha| = |\alpha|$ and, therefore, $1 - r = r$. Therefore, we now have $1 \sim \alpha = r\mathrm{e}^{\mathrm{i}\omega'}$, and there are two cases to consider:
   (i) If $\omega = 2\pi$, then $\omega' = \omega$ and, therefore, $1 \sim \alpha = r\mathrm{e}^{\mathrm{i}\omega'} = r\mathrm{e}^{\mathrm{i}\omega} = \alpha$. Hence, $\mu_0(xy^{-1}) = \alpha$ follows;
   (ii) If $\omega < 2\pi$, we have both $\omega' = (2\pi - \omega)$ and $\omega + \psi = 2\pi$. Note that $\omega \geqslant \psi$ because of $\alpha \geqslant \beta$. As a result, $\omega \geqslant \pi$ follows. On the other hand, since $1 \sim \alpha = r\mathrm{e}^{\mathrm{i}(2\pi-\omega)} \geqslant r\mathrm{e}^{\mathrm{i}\omega} = \alpha$, we have $(2\pi - \omega) \geqslant \omega$, which implies that $\pi \geqslant \omega$. Thus, we now have $\omega = \pi$, resulting in $\omega' = \pi = \omega$ and, therefore, $1 \sim \alpha = r\mathrm{e}^{\mathrm{i}\omega'} = r\mathrm{e}^{\mathrm{i}\omega} = \alpha$. Hence, $\mu_0(xy^{-1}) = \alpha$ again follows.

(b) Based on Definition 2.21, it is either $\nu_0(xy^{-1}) = \beta$ or $\nu_0(xy^{-1}) = 1 \sim \beta$. Suppose that $\nu_0(xy^{-1}) = 1 \sim \beta$; then, $1 \sim \beta \leqslant \beta$. Since $1 \sim \beta = (1-\tau)\mathrm{e}^{\mathrm{i}\psi'} \leqslant \tau\mathrm{e}^{\mathrm{i}\psi} = \beta$, it follows that $|\alpha| = 1 - |\beta| = |1 \sim \beta| \leqslant |\beta| \leqslant |\alpha|$. This further implies that $|1 \sim \beta| = |\beta|$ and, therefore, $1 - \tau = \tau$. Thus, we now have $1 \sim \beta = \tau\mathrm{e}^{\mathrm{i}\psi'}$; there are two cases to consider:
   (i) If $\psi = 2\pi$, then $\psi' = \psi$ and, therefore, $1 \sim \beta = \tau\mathrm{e}^{\mathrm{i}\psi'} = \tau\mathrm{e}^{\mathrm{i}\psi} = \beta$. Hence, $\nu_0(xy^{-1}) = \beta$ follows;
   (ii) If $\psi < 2\pi$, we have both $\psi' = (2\pi - \psi)$ and $\omega + \psi = 2\pi$. Note that $\omega \leqslant \psi$ because of $\beta \leqslant \alpha$. As a result, $\psi \leqslant \pi$ follows. On the other hand, since $1 \sim \beta = \tau\mathrm{e}^{\mathrm{i}(2\pi-\psi)} \geqslant \tau\mathrm{e}^{\mathrm{i}\psi} = \beta$, $(2\pi-\psi) \leqslant \psi$ which implies that $\pi \leqslant \psi$. Thus, we now have $\psi = \pi$, resulting in $\psi' = \pi = \psi$ and, therefore, $1 \sim \beta = \tau\mathrm{e}^{\mathrm{i}\psi'} = \tau\mathrm{e}^{\mathrm{i}\psi} = \beta$. Hence, $\nu_0(xy^{-1}) = \beta$ again follows.

Therefore, we obtain $xy^{-1} \in S$ whenever $x, y \in S$. As such, it can be concluded that S is a classical subgroup of G. □

Theorem 3.1. *Let S be a non-null subset of G and $(\widetilde{\mathcal{F}}, E) \in$ CIFSS(G), where $(\widetilde{\mathcal{F}}, E)$ is characteristic over S. Then, $(\widetilde{\mathcal{F}}, E) \in$ CIFSG(G) if and only if S is a classical subgroup of G.*

Proof. In Proposition 3.4, it has already been proved that S is a classical subgroup of G whenever $(\widetilde{\mathcal{F}}, E) \in$ CIFSG(G). Therefore, it suffices to prove that $(\widetilde{\mathcal{F}}, E) \in$ CIFSG(G) whenever S is a classical subgroup of G.

Since $(\widetilde{\mathcal{F}}, E) \in$ CIFSS(G) and $(\widetilde{\mathcal{F}}, E)$ is characteristic over S, by Definition 2.21, we have $\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)} : \varepsilon \in E\} = \mu_0$ and $\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)} : \varepsilon \in E\} = \nu_0$, in which:

$$\mu_0(x) = \begin{cases} r\mathrm{e}^{\mathrm{i}\omega}, & x \in S \\ 1 \sim r\mathrm{e}^{\mathrm{i}\omega}, & x \in U - S \end{cases}$$

and:

$$\nu_0(x) = \begin{cases} \tau\mathrm{e}^{\mathrm{i}\psi}, & x \in S \\ 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}, & x \in U - S \end{cases}$$

with $\omega + \psi \in \{2\pi, 4\pi\}$ and $r\mathrm{e}^{\mathrm{i}\omega} \geqslant \tau\mathrm{e}^{\mathrm{i}\psi}$.

Now, let $\varepsilon \in E$ and $x, y \in G$. Then both: $\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy^{-1})\} \subseteq \{r\mathrm{e}^{\mathrm{i}\omega}, 1 \sim r\mathrm{e}^{\mathrm{i}\omega}\}$ and $\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy^{-1})\} \subseteq \{\tau\mathrm{e}^{\mathrm{i}\psi}, 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}\}$. Furthermore, by Proposition 2.1, $r\mathrm{e}^{\mathrm{i}\omega} \geqslant 1 \sim r\mathrm{e}^{\mathrm{i}\omega}$ and $\tau\mathrm{e}^{\mathrm{i}\psi} \leqslant 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}$, which imply that $\min\{r\mathrm{e}^{\mathrm{i}\omega}, 1 \sim r\mathrm{e}^{\mathrm{i}\omega}\} = 1 \sim r\mathrm{e}^{\mathrm{i}\omega}$ and $\max\{\tau\mathrm{e}^{\mathrm{i}\psi}, 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}\} = 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}$, respectively.

Without loss of generality, suppose that $x \in G - S$. Then, $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x) = 1 \sim r\mathrm{e}^{\mathrm{i}\omega}$ and $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x) = 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}$ which causes $\min\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\} = 1 \sim r\mathrm{e}^{\mathrm{i}\omega}$ and $\max\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\} = 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}$, respectively. Since $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy^{-1}) \in \{r\mathrm{e}^{\mathrm{i}\omega}, 1 \sim r\mathrm{e}^{\mathrm{i}\omega}\}$ and $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy^{-1}) \in \{\tau\mathrm{e}^{\mathrm{i}\psi}, 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}\}$, we conclude that:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(xy^{-1}\right) \geqslant \min\left\{r\mathrm{e}^{\mathrm{i}\omega}, 1 \sim r\mathrm{e}^{\mathrm{i}\omega}\right\} = 1 \sim r\mathrm{e}^{\mathrm{i}\omega} = \min\left\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\},$$

and:

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(xy^{-1}\right) \leqslant \max\left\{\tau\mathrm{e}^{\mathrm{i}\psi}, 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi}\right\} = 1 \sim \tau\mathrm{e}^{\mathrm{i}\psi} = \max\left\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\}.$$

Now, let $x, y \in S$. Since S is a classical subgroup of G, $xy^{-1} \in S$; therefore, it follows that:

$$\left\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(xy^{-1}\right)\right\} \subseteq \left\{r\mathrm{e}^{\mathrm{i}\omega}\right\},$$

and:

$$\left\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(xy^{-1}\right)\right\} \subseteq \left\{\tau\mathrm{e}^{\mathrm{i}\psi}\right\}.$$

As a result, we have:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy^{-1}) = r\mathrm{e}^{\mathrm{i}\omega} = \min\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\},$$

and:

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy^{-1}) = \tau e^{i\psi} = \max\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\}.$$

Hence, the conditions:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(xy^{-1}\right) \geqslant \min\left\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\},$$

and:

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(xy^{-1}\right) \leqslant \max\left\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\},$$

are shown to be satisfied for all $\varepsilon \in E$ and $x, y \in G$. This proves that $(\widetilde{\mathcal{F}}, E) \in \text{CIFSG}(G)$.

Theorem 3.2. *Let $(\widetilde{\mathcal{F}}, E) \in \text{CIFSS}(G)$, where $(\widetilde{\mathcal{F}}, E)$ is non-null. Then, the following statements are equivalent:*

(i) $(\widetilde{\mathcal{F}}, E) \in \text{CIFSG}(G)$;

(ii) *For all $\alpha, \beta \in O_1$, either $(\widetilde{\mathcal{F}}, E)_{(\alpha,\beta)}$ is null, or $(\widetilde{\mathcal{F}}, E)_{(\alpha,\beta)}$ is a soft group of G.*

Proof.

(i)⇒(ii) Take any arbitrary $\alpha, \beta \in O_1$. By Proposition 3.3, if $(\widetilde{\mathcal{F}}, E)_{(\alpha,\beta)}$ is non-null, then it is a soft group of G. Thus, statement (ii) is proved true;

(ii)⇒(i) Let $\varepsilon \in E$ and $x, y \in G$, and note that $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y) \in O_1$.

Take $\alpha = \min\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\}$ and $\beta = \max\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\}$. Then, we have $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y) \geqslant \alpha$ and $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y) \leqslant \beta$, which means that $x, y \in \widetilde{\mathcal{F}}_{(\alpha,\beta)}(\varepsilon)$. This implies that $(\widetilde{\mathcal{F}}, E)_{(\alpha,\beta)}$ is not null; therefore, it is a soft group of G. Thus, we now have $\widetilde{\mathcal{F}}_{(\alpha,\beta)}(\varepsilon) \leqslant G$ and, therefore, $xy^{-1} \in \widetilde{\mathcal{F}}_{(\alpha,\beta)}(\varepsilon)$, which in turn implies that:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(xy^{-1}\right) \geqslant \alpha = \min\left\{\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\},$$

and:

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}\left(xy^{-1}\right) \leqslant \beta = \max\left\{\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)\right\}.$$

Hence, by Proposition 3.2, statement (i) now follows. □

Theorem 3.3. *Let $(\widetilde{\mathcal{F}}_1, E_1), (\widetilde{\mathcal{F}}_2, E_2) \in \text{CIFSG}(G)$. Then, $(\widetilde{\mathcal{F}}_1, E_1) \widetilde{\cap} (\widetilde{\mathcal{F}}_2, E_2) \in \text{CIFSG}(G)$ too.*

Proof. The proof is straightforward by Definition 2.19 and is, therefore, omitted. □

Remark. This property also holds for the restricted intersection operation between CIFSSs.

Definition 3.5. Let U_1, U_2 be two universal sets, $\varphi : U_1 \to U_2$ be a function, E, B be two sets of parameters: $(\widetilde{\mathcal{T}}, E) \in \text{CIFSS}(U_1)$ and $(\widetilde{\mathcal{F}}, B) \in \text{CIFSS}(U_2)$. Define $(\varphi(\widetilde{\mathcal{T}}), E) \in \text{CIFSS}(U_2)$ and $(\varphi^{-1}(\widetilde{\mathcal{F}}), B) \in \text{CIFSS}(U_1)$ as follows:

(i) $(\varphi(\widetilde{\mathcal{T}}), E)$ is such that for all $y \in U_2$ and $\varepsilon \in E$:

$$\mu_{\varphi(\widetilde{\mathcal{T}})(\varepsilon)}(y) = \max\left\{\left\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(u) : u \in U_1, \varphi(u) = y\right\} \cup \{0\}\right\},$$

and:

$$\nu_{\varphi(\widetilde{\mathcal{T}})(\varepsilon)}(y) = \min\left\{\left\{\nu_{\widetilde{\mathcal{T}}(\varepsilon)}(u) : u \in U_1, \varphi(u) = y\right\} \cup \{1\}\right\}.$$

(ii) $(\varphi^{-1}(\widetilde{\mathcal{F}}), B)$ is such that, for all $x \in U_1$ and $s \in B$, $\mu_{\varphi^{-1}(\widetilde{\mathcal{F}})(s)}(x) = \mu_{\widetilde{\mathcal{F}}(s)}(\varphi(x))$ and $\nu_{\varphi^{-1}(\widetilde{\mathcal{F}})(s)}(x) = \nu_{\widetilde{\mathcal{F}}(s)}(\varphi(x))$.

Theorem 3.4. *Let $\varphi : G \to G'$ be a surjective group homomorphism. Let $(\widetilde{\mathcal{T}}, E) \in \text{CIFSG}(G)$ and $(\widetilde{\mathcal{F}}, B) \in \text{CIFSG}(G')$. Then:*

(i) $(\varphi(\widetilde{\mathcal{T}}), E) \in \text{CIFSG}(G')$ *provided that:*

$$\max\left\{\min\left\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(p), \mu_{\widetilde{\mathcal{T}}(\varepsilon)}(q)\right\} : \begin{matrix} p, q \in G \\ \varphi(p) = x, \varphi(q) = y \end{matrix}\right\} \geqslant \min\left\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{T}}(\varepsilon)}(y)\right\}$$

and:

$$\min\left\{\max\left\{\nu_{\widetilde{\mathcal{T}}(\varepsilon)}(p), \nu_{\widetilde{\mathcal{T}}(\varepsilon)}(q)\right\} : \begin{matrix} p, q \in G \\ \varphi(p) = x, \varphi(q) = y \end{matrix}\right\} \leqslant \max\left\{\nu_{\widetilde{\mathcal{T}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{T}}(\varepsilon)}(y)\right\},$$

for all $x, y \in G'$;

(ii) $(\varphi^{-1}(\widetilde{\mathcal{F}}), B) \in \text{CIFSG}(G)$.

Proof.

(i) Let $x, y \in G'$ and $\varepsilon \in E$. Then, by Definition 3.5, we have $(\varphi(\widetilde{\mathcal{T}}), E) \in \text{CIFSS}(G')$, where $\mu_{\varphi(\widetilde{\mathcal{T}})(\varepsilon)}(xy^{-1}) = \max\{\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(u) : u \in G, \varphi(u) = xy^{-1}\} \cup \{0\}\}$. Since φ is surjective, we have:

$$\max\left\{\left\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(u) : u \in G, \varphi(u) = xy^{-1}\right\} \cup \{0\}\right\} = \max\left\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(u) : u \in G, \varphi(u) = xy^{-1}\right\}.$$

Moreover, since φ is also a homomorphism, we have:

$$\max\left\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(u) : u \in G, \varphi(u) = xy^{-1}\right\}$$

$$\geqslant \max\Big\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}\left(pq^{-1}\right) : p,q \in G, \varphi(p)$$

$$= x, \varphi(q) = y\Big\}.$$

Since $(\widetilde{\mathcal{T}}, E) \in \mathrm{CIFSG}(G)$, $\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(pq^{-1}) \geqslant \min\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(p), \mu_{\widetilde{\mathcal{T}}(\varepsilon)}(q)\}$ for all $p, q \in G$. This implies that:

$$\max\Big\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}\left(pq^{-1}\right) : p,q \in G, \varphi(p) = x, \varphi(q) = y\Big\}$$

$$\geqslant \max\Big\{\min\Big\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(p), \mu_{\widetilde{\mathcal{T}}(\varepsilon)}(q)\Big\} : p, q$$

$$\in G, \varphi(p) = x, \varphi(q) = y\Big\}$$

$$\geqslant \min\left\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{T}}(\varepsilon)}(y)\right\}.$$

Similarly, for the non-membership function, we have the following:

$$\nu_{\varphi(\widetilde{\mathcal{T}})(\varepsilon)}(xy^{-1})$$

$$= \min\Big\{\Big\{\nu_{\widetilde{\mathcal{T}}(\varepsilon)}(u) : u \in G, \varphi(u) = xy^{-1}\Big\} \cup \{0\}\Big\}$$

$$= \min\left\{\nu_{\widetilde{\mathcal{T}}(\varepsilon)}(u) : u \in G, \varphi(u) = xy^{-1}\right\}$$

$$\leqslant \min\Big\{\nu_{\widetilde{\mathcal{T}}(\varepsilon)}(pq^{-1}) : p, q \in G, \varphi(p) = x, \varphi(q) = y\Big\}$$

$$\leqslant \min\Big\{\max\left\{\nu_{\widetilde{\mathcal{T}}(\varepsilon)}(p), \nu_{\widetilde{\mathcal{T}}(\varepsilon)}(q)\right\} : p, q$$

$$\in G, \varphi(p) = x, \varphi(q) = y\Big\}$$

$$\leqslant \max\left\{\nu_{\widetilde{\mathcal{T}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{T}}(\varepsilon)}(y)\right\}.$$

(ii) Let $x, y \in G$ and $s \in B$. Then, $(\varphi^{-1}(\widetilde{\mathcal{F}}), B) \in \mathrm{CIFSS}(G)$ by Definition 3.5. As $(\widetilde{\mathcal{F}}, B) \in \mathrm{CIFSG}(G')$, by applying Proposition 3.2, alongside with Definition 3.5, we obtain the following:

$$\mu_{\varphi^{-1}(\widetilde{\mathcal{F}})(s)}(xy^{-1}) = \mu_{\widetilde{\mathcal{F}}(s)}\left(\varphi\left(xy^{-1}\right)\right)$$

$$= \mu_{\widetilde{\mathcal{F}}(s)}\left(\varphi(x)(\varphi(y))^{-1}\right)$$

$$\geqslant \min\left\{\mu_{\widetilde{\mathcal{F}}(s)}(\varphi(x)), \mu_{\widetilde{\mathcal{F}}(s)}(\varphi(y))\right\}$$

$$= \min\left\{\mu_{\varphi^{-1}(\widetilde{\mathcal{F}})(s)}(x), \mu_{\varphi^{-1}(\widetilde{\mathcal{F}})(s)}(y)\right\}$$

$$\nu_{\varphi^{-1}(\widetilde{\mathcal{F}})(s)}(xy^{-1}) = \nu_{\widetilde{\mathcal{F}}(s)}\left(\varphi\left(xy^{-1}\right)\right)$$

$$= \nu_{\widetilde{\mathcal{F}}(s)}\left(\varphi(x)(\varphi(y))^{-1}\right)$$

$$\leqslant \max\left\{\nu_{\widetilde{\mathcal{F}}(s)}(\varphi(x)), \nu_{\widetilde{\mathcal{F}}(s)}(\varphi(y))\right\}$$

$$= \max\left\{\nu_{\varphi^{-1}(\widetilde{\mathcal{F}})(s)}(x), \nu_{\varphi^{-1}(\widetilde{\mathcal{F}})(s)}(y)\right\}.$$

This completes the proof. □

4. Normal complex intuitionistic fuzzy soft groups

In this section, the notion of CIFS-groups is extended by adding the normality condition to the existing conditions. Aygunoglu & Aygun [20] introduced the conditions for normality in the context of fuzzy soft sets. Here, these conditions are generalized for normality to be compatible with the CIFSS model; subsequently, these conditions are used to define the notion of normal CIFS-groups.

The conditions for intuitionistic fuzzy soft normality are described in Lemma 4.1, whereas the notion of a normal CIFS-group is proposed in Definition 4.1.

Lemma 4.1. *Let $(\widetilde{\mathcal{F}}, E) \in \mathrm{CIFSG}(G)$ and $\varepsilon \in E$. Then, the following statements are equivalent:*

(i) $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}) \geqslant \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)$ *and* $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}) \leqslant \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)$, *for all* $x, y \in G$;

(ii) $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)$ *and* $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)$, *for all* $x, y \in G$;

(iii) $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(yx)$ *and* $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(yx)$, *for all* $x, y \in G$.

Proof.

(i)⇒(ii) Let $x, y \in G$. Then, $y = x^{-1}(xyx^{-1})(x^{-1})^{-1}$, and both $x^{-1}, xyx^{-1} \in G$. As a result, we have:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x^{-1}(xyx^{-1})(x^{-1})^{-1})$$

$$\geqslant \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}),$$

and:

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x^{-1}(xyx^{-1})(x^{-1})^{-1})$$

$$\leqslant \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}).$$

Hence, statement (ii) now follows;

(ii)⇒(iii) Let $x, y \in G$. Then, $xy = x(yx)x^{-1}$ and $yx \in G$. As a result, we now have:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x(yx)x^{-1}) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(yx),$$

and:

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x(yx)x^{-1}) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(yx).$$

Hence, statement (iii) now follows;

(iii)⇒(i) Let $x, y \in G$. Then $(x^{-1})(xy) = y$, where $xy, x^{-1} \in G$. Thus, we obtain:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}((xy)x^{-1}) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(x^{-1}(xy)) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y),$$

and:

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}((xy)x^{-1}) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(x^{-1}(xy)) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y).$$

Hence, statement (i) now follows. □

Definition 4.1. Let $(\widetilde{\mathcal{F}}, E) \in \mathrm{CIFSG}(G)$. Then $(\widetilde{\mathcal{F}}, E)$ is said to be a *Normal Complex Intuitionistic Fuzzy Soft Group* on G ($(\widetilde{\mathcal{F}}, E) \in \mathrm{NCIFSG}(G)$), if the conditions:

$$\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}) \geqslant \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y),$$

and:

$$\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}) \leqslant \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y),$$

are satisfied for all $\varepsilon \in E$ and $x, y \in G$.

Example 4.1. Consider the example described in Example 3.1. We define an $(\widetilde{\mathcal{H}}, E) \in \mathrm{CIFSS}(G)$ as $(\widetilde{\mathcal{H}}, E) = \{\widetilde{\mathcal{H}}(a), \widetilde{\mathcal{H}}(b)\}$, where $\widetilde{\mathcal{H}}(a) = \widetilde{\mathcal{F}}(a)$, and $\widetilde{\mathcal{F}}(a)$ is as defined in Example 3.1. and:

$$\widetilde{\mathcal{H}}(b) = \left\{ \begin{array}{c} (1, \mu_3, \nu_3), ((12), \mu_1, \nu_1), ((13), \mu_1, \nu_1), \\ ((23), \mu_1, \nu_1), ((123), \mu_1, \nu_1), \\ ((132), \mu_1, \nu_1) \end{array} \right\}.$$

Then, $(\widetilde{\mathcal{H}}, E) \in \mathrm{CIFSG}(G)$ and it also satisfies the conditions for normality described in Definition 4.1. Hence, $(\widetilde{\mathcal{H}}, E) \in \mathrm{NCIFSG}(G)$.

Theorem 4.1. *Let $(\widetilde{\mathcal{F}}, E) \in \mathrm{CIFSG}(G)$. Then, the following statements are equivalent:*

(i) $(\widetilde{\mathcal{F}}, E) \in \mathrm{NCIFSG}(G)$;

(ii) $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}) \geqslant \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)$ *and* $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}) \leqslant \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)$, *for all* $\varepsilon \in E$ *and* $x, y \in G$;

(iii) $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)$ *and* $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)$, *for all* $\varepsilon \in E$ *and* $x, y \in G$;

(iv) $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy) = \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(yx)$ *and* $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xy) = \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(yx)$, *for all* $\varepsilon \in E$ *and* $x, y \in G$.

Proof.

(i)⇒(ii) This follows directly from Definition 4.1;

(ii) ⇒ (iii) and (iii) ⇒ (iv) These follow directly from Lemma 4.1;

(iv)⇒(i) Statement (ii) follows directly from Definition 4.1 and, therefore, statement (i) follows by Lemma 4.1. □

Proposition 4.1. *Let $(\widetilde{\mathcal{F}}, E) \in \mathrm{NCIFSG}(G)$ and $\emptyset \subset D \subseteq E$. Then $(\widetilde{\mathcal{F}}, D) \in \mathrm{NCIFSG}(G)$ as well.*

Proof. Let $x, y \in G$. By Theorem 4.1, it follows that $\mu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}) \geqslant \mu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)$ and $\nu_{\widetilde{\mathcal{F}}(\varepsilon)}(xyx^{-1}) \leqslant \nu_{\widetilde{\mathcal{F}}(\varepsilon)}(y)$, for all $\varepsilon \in E$. Since $\emptyset \subset D \subseteq E$, such statement holds for all $\varepsilon \in D$ too. This completes the proof. □

Theorem 4.2. *Let:*

$(\widetilde{\mathcal{F}}_1, E_1), (\widetilde{\mathcal{F}}_2, E_2) \in \mathrm{NCIFSG}(G)$.

Then:

$(\widetilde{\mathcal{F}}_1, E_1) \widetilde{\cap} (\widetilde{\mathcal{F}}_2, E_2) \in \mathrm{NCIFSG}(G)$.

Proof. The proof is straightforward by Definitions 2.14 and 4.1. □

Remark. Similar to Theorem 3.3, this property also holds for the restricted intersection operation between CIFSSs.

Theorem 4.3. *Let $\varphi : G \to G'$ be a surjective group homomorphism. Let $(\widetilde{\mathcal{T}}, E) \in \mathrm{NCIFSG}(G)$ and $(\widetilde{\mathcal{F}}, B) \in \mathrm{NCIFSG}(G')$. Then:*

(i) $(\varphi(\widetilde{\mathcal{T}}), E) \in \mathrm{NCIFSG}(G')$ *provided that:*

$$\max\left\{\min\left\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(p), \mu_{\widetilde{\mathcal{T}}(\varepsilon)}(q)\right\} : \begin{array}{c} p, q \in G \\ \varphi(p) = x, \varphi(q) = y \end{array}\right\}$$

$$\geqslant \min\left\{\mu_{\widetilde{\mathcal{T}}(\varepsilon)}(x), \mu_{\widetilde{\mathcal{T}}(\varepsilon)}(y)\right\},$$

and:

$$\min\left\{\max\left\{\nu_{\widetilde{\mathcal{T}}(\varepsilon)}(p), \nu_{\widetilde{\mathcal{T}}(\varepsilon)}(q)\right\} : \begin{array}{c} p, q \in G \\ \varphi(p) = x, \varphi(q) = y \end{array}\right\}$$

$$\leqslant \max\left\{\nu_{\widetilde{\mathcal{T}}(\varepsilon)}(x), \nu_{\widetilde{\mathcal{T}}(\varepsilon)}(y)\right\}$$

for all $x, y \in G'$,

(ii) $(\varphi^{-1}(\widetilde{\mathcal{F}}), B) \in \mathrm{NCIFSG}(G)$.

Proof. The proof can be derived from Theorem 3.4, Lemma 4.1, and Definition 4.1. □

5. Conclusion

This paper presented the initial theory of complex fuzzy algebra. We defined and developed the algebraic structures pertaining to groups and subgroups for the Complex Intuitionistic Fuzzy Soft Set (CIFSS) model. The notions of CIF-subgroups, CIFS-groups, and normal CIFS-groups were introduced. The fundamental properties and structural characteristics of these algebraic structures were then examined and verified. All of these were accomplished by carefully defining some important concepts pertaining to the structure of the CIFSS model and also carefully generalizing some of the well-known operations and relations that exist between intuitionistic fuzzy soft sets to be made compatible with the structure of the CIFSS model, in which the membership and non-membership functions are defined in terms of complex numbers. Furthermore, in this paper, we contextualized the phase term by using it to represent the different cycles of alternating groups, thereby proposing a new way of interpreting the phase term.

6. Further direction of this work

Our research in this area is still ongoing. We are currently in the midst of extending the CIFSG structure introduced in this paper to introduce more advanced algebraic structures, such as CIFS cyclic groups, abelian groups, dihedral groups, symmetric groups, and alternating groups, using the concepts and theory developed in this paper. The work presented in this paper can also be used as a basis to develop other algebraic theories of complex fuzzy based models.

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