UNIVERSIDADE DE SÃO PAULO

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CAIXA POSTAL 20516
01498 - SÃO PAULO - SP
BRASIL

PUBLICAÇÕES

IFUSP/P-896

MAGNETIC SURFACES IN NON-SYMMETRIC PLASMAS

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Fevereiro/1991

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Abstract: An averaging method is developed in order to determine analytically the magnetic structure of a system with symmetry broken by small perturbations. Particularly, Poincaré maps are obtained for toroidal helical fields using the typical parameters of the Brazilian Tokamak TBR–1. The technique is fairly general as the small parameter used in the construction of this theory is the original symmetry breaking perturbation parameter; it is applicable in analyses of fields in more compact devices as well. Unless the unperturbed system is doubly symmetric (circular cylinder) a single helical perturbation mode (m, n) can excite many other modes. The coupling between any two modes can be explained in a very natural way.

**Supported by FAPESP.*

1. INTRODUCTION

In symmetric systems magnetic field lines lie on surfaces (e.g. Edenstrasser (1980)). The escape of the lines from a limited region due to the instability of a magnetc surface against resonance perturbations represents one of the most serious problems in the confinement of plasmas. If the perturbation of the symmetry is not strong enough, part of the lines still forms magnetic surfaces.

Experiments in tokamaks have shown that instabilities can be controlled by external helical currents. Perturbations just bellow a critical value inhibit the Mirnov oscillations. Explanation of this stabilising effect is suggested by the Pulsator team (1985):fixed helical islands structure within the plasma would hinder a rotation of the MHD modes. Increasing the helical field, minor disruptions occur in the resonant surface and neighbouring rational surfaces until the confinement is totally lost; these disruptions can be explained in terms of randomization of the field lines. In spite of magnetic surfaces destruction it is still possible to find approximate solutions. Whatever is the method used we have to deal with some difficulties related to the resonance phenomena. A magnetic surface is very sensitive to any weak perturbation because it is shaped by infinite revolutions of the field line.

To our knowledge an averaging method was used by Morozov and Solov'ev (1966) and Solov'ev and Shafranov (1970) in the context of determination of an oveall structure of magnetic systems.

Our approach is similar in the sense that new coordinates are sought in order to find a first integral invariant of the system.

The perturbing field is characterized by a small parameter ϵ. The difference between these new coordinates and the original ones are not necessarily small terms; they are mostly dependent upon the original equilibrium field. Therefore, we do not deal with these new coordinates as representing approximate positions. We keep the same notations $\overline{\chi^i}$ as in Solov'ev and Shafranov because they result from an averaging procedure. Approximate magnetic surfaces are determined by this method; the wandering of the actual field lines in the vicinity of these surfaces is estimated.

In section 2 the method is developed for most general symmetric systems with a small perturbation.

In section 3 a toroidal helical system is analysed. Attention is paid to the mode coupling phenomena; it is shown how a sole helical perturbation mode (m, n) causes the formation of chains of islands at every resonance surface in a very simple way. A preliminary work on this subject had been presented at the Joint Varenna – Lausanne International Workshop on Theory of Fusion Plasmas (1990). The approaches are not quite the same. The present analysis is more comprehensive; the agreement with the numerical results is perfect.

2. GENERAL EQUATIONS

2.1. THE AVERAGING METHOD

The magnetic field is written as the symmetric equilibrium field $\vec{B}_0$ with a small perturbation $\vec{b}$:

$$\vec{B} = \vec{B}_0 + \vec{b} \tag{1}$$

The field line equation $\vec{B} \times d\vec{r} = 0$ is written explicitly as a 2–dimensional system of equations in terms of curvilinear coordinates χ^1, χ^2 and χ^3:

$$\frac{d\chi^1}{d\chi^3} = \frac{B^1}{B^3} \text{ and } \frac{d\chi^2}{d\chi^3} = \frac{B^2}{B^3} \tag{2}$$

where B^i are the contravariant components.

It is assumed that all the physical quantities are periodic functions of χ^3 with periodicity L.

An analogy to dynamic system equations can be established if χ^3 assumes the role of the time. The averaging method, as originally presented by Bogolyubov and Mitropol'skii (1961) is applicable when the motion is characterized by two different timescales; averaging over the fast time is performed and the average evolution is considered.

In the application of any averaging method, the choice of the coordinates is fundamental. We take for χ^1 a magnetic surface quantity of the unperturbed field with the meaning of radial variable; we call χ^3 the other relevant coordinate of this system; χ^2 will be an ignorable coordinate. The symmetry of the system is destroyed by the dependence of $\vec{b}$ upon χ^2. We, further, assume that B^3 is always different from zero.

With this choice:

$$B_0^1 = 0 \tag{3}$$

$$\frac{\partial}{\partial\chi^2} B_0^i = 0 \text{ and } \frac{\partial}{\partial\chi^2} g^{ij} = 0 \tag{4}$$

where g^{ij} are the contravariant metric elements.

The equation for the magnetic surfaces of the unperturbed plasma is regarded as known in terms of the magnetic flux $\Psi_0(\chi^1)$ of $\vec{B}_0$ through a constant χ^3 surface.

B_0^3 can be written as (Edenstrasser (1980)):

$$\sqrt{g}\, B_0^3 = \Psi_0'(\chi^1) \tag{5}$$

The prime denotes derivative with respect to χ^1.

If the perturbation is weak, (2) can be written in the form:

$$\frac{d\chi^1}{d\chi^3} = \epsilon f^1 (\chi^1, \chi^2, \chi^3)$$

and

$$\frac{d\chi^2}{d\chi^3} = F^2(\chi^1, \chi^3) + \epsilon f^2(\chi^1, \chi^2, \chi^3) \tag{6}$$

where ϵ is a dimensionless perturbation parameter of the order of b^3/B_0^3;

$$\epsilon f^1 = \frac{b^1}{B_0^3}\left(1 + \frac{b^3}{B_0^3}\right)^{-1};$$

$$F^2 = \frac{B_0^2}{B_0^3} = \frac{1}{\Psi_0'}\sqrt{g}\,B_0^2$$

and

$$\epsilon f^2 \simeq \frac{b^2}{B_0^3} - \frac{B_0^2}{B_0^3}\,\frac{b^3}{B_0^3}.$$

The method developed here can be applied in the resolution of any system of equations in the form (6).

We use the functions F^2 and f^i whenever the considerations are general and not restricted to magnetic field functions.

Terms of the first two orders of magnitude in the perturbation are retained. This is essential in most of expressions because in the integration processes the errors can accumulate and overwhelm the actual values; Poincaré plots obtained by numerial tracing of field lines clearly confirm this fact.

The fast timescale here is the periodicity in χ^3. Instead of dealing with actual field line coordinates new variables $\overline{\chi^1}(\chi^3)$ and $\overline{\chi^2}(\chi^3)$ are introduced in order to get integrable differential equations. We use the same notations as in Solov'ev and Shafranov (1970):

$$\bar{f}(\overline{\chi^1}, \overline{\chi^2}) = \frac{1}{L}\int_0^L f(\overline{\chi^1}, \overline{\chi^2}, \chi^3)\, d\chi^3$$

$$\tilde{f}(\overline{\chi^1}, \overline{\chi^2}, \chi^3) = f - \bar{f}$$

$$\hat{f}(\overline{\chi^1}, \overline{\chi^2}, \chi^3) = \widetilde{\int_0^{\chi^3} \tilde{f}\, d\chi^3} \qquad (7)$$

where the integrations are carried out with fixed $\overline{\chi^1}$ and $\overline{\chi^2}$.

The choice of $\overline{\chi^1}$ is not unique. We write:

$$\chi^1 \equiv \overline{\chi^1} + \delta\chi^1(\overline{\chi^1}, \overline{\chi^2}, \chi^3)$$

and

$$\chi^2 \equiv \overline{\chi^2} + \widetilde{\chi^2}(\overline{\chi^1}, \overline{\chi^2}, \chi^3) + \delta\chi^2(\overline{\chi^1}, \overline{\chi^2}, \chi^3) \qquad (8)$$

In principle, $\overline{\chi^1}$ can be the solutions of any autonomous system of two first order differential equations; $\widetilde{\chi^2}$ can be chosen arbitrarily. Because of (6) we expect $\delta\chi^1$ to be a small quantity ($O(\epsilon)$); we choose $\delta\chi^2$ to be of the same order of magnitude. Our idea is to build a peturbation method where the first order solution ($\delta\chi^i=0$) contains most of the main features of the magnetic structure; $\delta\chi^i$ will be regarded as second order corrections. In dealing with a resonance phenomena it is desirable that the resonance effects are not left to $\delta\chi^i$. We call zeroth order solution the unperturbed solution.

The equations (6) can be written:

$$\frac{d\overline{\chi^1}}{d\chi^3} + \frac{d}{d\chi^3}\delta\chi^1 = \epsilon f^1(\overline{\chi^1}, \overline{\chi^2} + \widetilde{\chi^2}, \chi^3) + \epsilon\left(\frac{\partial f^1}{\partial\overline{\chi^1}}\delta\chi^1 + \frac{\partial f^1}{\partial\overline{\chi^2}}\delta\chi^2\right) \qquad (9)$$

$$\frac{d\overline{\chi^2}}{d\chi^3} + \frac{d\widetilde{\chi^2}}{d\chi^3} + \frac{d}{d\chi^3}\delta\chi^2 = F^2(\overline{\chi^1}, \chi^3) + \frac{\partial F^2}{\partial\overline{\chi^1}}\delta\chi^1 + \epsilon f^2((\overline{\chi^1}, \overline{\chi^2} + \widetilde{\chi^2}, \chi^3) \qquad (9')$$

Consistently, terms of first two orders of magnitude are taken into account.

2.2. APPROXIMATE MAGNETIC SURFACES

As criteria for the choice of $\overline{\chi^1}$ we take:

(i) $$\frac{d\overline{\chi^1}}{d\chi^3} = \epsilon\overline{f^1(\overline{\chi^1},\overline{\chi^2} + \widetilde{\chi^2}, \chi^3)} + O(\epsilon^2)$$

$$\frac{d\overline{\chi^2}}{d\chi^3} = \overline{F^2(\overline{\chi^1},\chi^3)} + O(\epsilon) \qquad (10)$$

(ii) $\overline{\chi^1}$ define an average magnetic surface equation:

$$\Psi_H(\overline{\chi^1}, \overline{\chi^2}) = \text{constant} \qquad (11)$$

So, $\Psi_H(\chi^1 - \delta\chi^1, \chi^2 - \widetilde{\chi^2} - \delta\chi^2)$ would be an invariant of the system.

We call 1st order solution the magnetic surface:

$$\Psi_H(\chi^1, \chi^2 - \widetilde{\chi^2}) = \text{constant} \qquad (12)$$

just because it is found first. The second order solution takes into account $\delta\chi^i$:

$$\Psi_H(\chi^1 - \delta\chi^1, \chi^2 - \widetilde{\chi^2} - \delta\chi^2) = \text{constant} \qquad (13)$$

(iii) $\delta\chi^i = 0$ and the 1st order solution is the exact magnetic surface if, either:

$$\text{a) } f^i = 0 \quad \text{or} \quad \text{b) } \widetilde{F^2} = 0 \text{ and } \widetilde{f^i} = 0 \qquad (14)$$

An adequate choice of $\overline{\chi^1}$, $\widetilde{\chi^2}$, and Ψ_H is:

$$\widetilde{\chi^2} = \widehat{F^2}(\overline{\chi^1}, \chi^3) \qquad (15)$$

$$\frac{d\overline{\chi^1}}{d\chi^3} = \frac{\overline{b^1}}{B_0^3}\left(1 + \frac{\overline{b^3}}{B_0^3}\right)^{-1} \qquad (16)$$

$$\Psi_H(\overline{\chi^1}, \overline{\chi^2}) = \Psi_H^0(\overline{\chi^1}) - \int_c^{\overline{\chi^2} + \widehat{F^2}} \overline{\sqrt{g}\; b^1(\overline{\chi^1}, \chi^2, \chi^3)\; d\chi^2} \qquad (17)$$

where $\Psi_H^0(\overline{\chi^1}) = \int_0^{\overline{\chi^1}} \overline{\sqrt{g}\; B_0^2(\overline{\chi^1}, \chi^3)}\, d\overline{\chi^1}$

and c is a constant satisfying $\overline{\sqrt{g}\; b^2(\overline{\chi^1}, c, \chi^3)} = 0$

The functions in (16) are considered at $(\overline{\chi^1}, \overline{\chi^2} + \widehat{F^2}, \chi^3)$. We keep the form $\widehat{F^2}$ instead of writing explicitly as $\widehat{B_0^2/B_0^3}$ or $\widehat{\sqrt{g}\, B_0^2/\Psi_0'}$ because it has the dimension of χ^2.

In both cases in (14), L $\Psi_H(\overline{\chi^1}, \overline{\chi^2})$ is the exact magnetic flux through a constant $\overline{\chi^2}$ surface bounded by the principal magnetic axis and a magnegic surface $\overline{\chi^1}$.

All the conditions established so far on the coordinates are not sufficient to fix χ^2. The system of equations (10) may admit as solutions, fixed points on $(\overline{\chi^1}, \overline{\chi^2})$ plane; a stable solution would be a magnetic axis and an unstable solution could be a hyperbolic line.On these fixed points the expressions in the righthand sides of (10) are zero. So, if the aim is to analyse the singular features of the magnetic structure a suitable choice of χ^2 must yield $\overline{F^2(\overline{\chi^1}, \chi^3)} = 0$.

Collecting all the remaining terms in (9) we are left with (Appendix A):

$$\frac{d}{d\chi^3}\delta\chi^1 \simeq \frac{1}{\Psi_0'}\widetilde{\sqrt{g}\, b^1} + \delta\chi^1 \frac{\partial}{\partial\overline{\chi^1}}\frac{\overline{b^1}}{B_0^3} + \delta\chi^2 \frac{\partial}{\partial\overline{\chi^2}}\frac{\overline{b^1}}{B_0^3} - \frac{\widetilde{b^1}}{B_0^3}\frac{\overline{b^3}}{B_0^3} - \frac{\overline{b^1}}{B_0^3}\frac{\widetilde{b^3}}{B_0^3} \qquad (18)$$

$$\frac{d}{d\chi^3}\delta\chi^2 \simeq -\overline{F^2}\;\overline{\frac{\partial F^2}{\partial\overline{\chi^1}}\;\frac{\partial}{\partial\overline{\chi^2}}\widehat{\delta\chi^1}} + \widetilde{\frac{\partial F^2}{\partial\overline{\chi^1}}\delta\chi^1} +$$

$$+\frac{1}{\Psi_0'}\left[\widetilde{\sqrt{g}\,b^2}-\widetilde{F^2\sqrt{g}\,b^3}-\overline{\frac{\partial\widehat{F^2}}{\partial\overline{\chi^1}}\sqrt{g}\;b^1}\right] \qquad (18')$$

The expressions in the righthand sides are evaluated at $(\overline{\chi^1}, \overline{\chi^2}+\widehat{F^2}, \chi^3)$.

In this derivation we do not take any assumption as to the nature of $\vec{b}$ except that it is solenoidal.

$\delta\chi^1$ is nearly oscillatory function of χ^3; $\delta\chi^2$ can be represented in the form:

$$\delta\chi^2 = \overline{\delta\chi^2}\,(\overline{\chi^1}, \overline{\chi^2}) + \widetilde{\delta\chi^2}\,(\overline{\chi^1}, \overline{\chi^2}, \chi^3) \qquad (19)$$

where $\overline{\delta\chi^2}$ is a correction to the average evolution of $\overline{\chi^2}$ and $\widetilde{\delta\chi^2}$ is the oscillatory part.

Using $\frac{d}{d\chi^3} \simeq \overline{F^2}\,\frac{\partial}{\partial\overline{\chi^2}} + \frac{\partial}{\partial\chi^3}$ in (18) we get:

$$\overline{\delta\chi^2} \cong \overline{\frac{\partial\widehat{F^2}}{\partial\overline{\chi^1}}\;\delta\chi^1} \qquad (20)$$

and

$$\frac{d}{d\chi^3}\,\widetilde{\delta\chi^2} \cong \widetilde{\frac{\partial F^2}{\partial\overline{\chi^1}}}\,\delta\chi^1 + \frac{1}{\Psi_0'}\left(\widetilde{\sqrt{g}\,b^2} - \widetilde{F^2\sqrt{g}\,b^3} - \overline{\frac{\partial\widehat{F^2}}{\partial\overline{\chi^1}}\sqrt{g}\;b^1}\right) \qquad (20')$$

The second order solution must satisfy (13), (18) and (20).

3. TOROIDAL HELICAL SYSTEMS

In order to describe toroidal systems, toroidal polar coordinates $(\rho_t, \theta_t, \varphi)$ have been introduced in terms of local polar coordinates (ρ, θ, φ) (Kucinski et al (1990)):

$$\rho_t = \rho\,(1 - \frac{\rho}{R_0}\cos\theta + (\frac{\rho}{2R_0})^2)^{1/2}$$

$$\sin\theta_t = \sin\theta\,(1 - \frac{\rho}{R_0}\cos\theta + (\frac{\rho}{2R_0})^2)^{-1/2} \qquad (21)$$

R_0 is the major axis of the system; ρ_t and θ_t have the meaning of radial and poloidal coordinates, respectively.

3.1. THE UNPERTURBED SYSTEM

We consider a self–consistent equilibrium solution for toroidal plasma with nearly circular cross–section written in terms of a cylindrical plasma (Kucinski et al (1990)).

The average value of ρ_t on a magnetic surface is taken here as the surface coordinate χ^1 $(\chi^1 \equiv \rho_0)$. ρ_0 must not be confused with the average coordinate $\overline{\chi^1}$ introduced in the previous section.

The average poloidal field is written in terms of a cylindrical magnetic flux function (Ψ_c) as:

$$\overline{B}_{\theta_t} = \rho_0\,\overline{\vec{B}_0\,\cdot\,\nabla\theta_t} \cong \Psi_c'(\rho_0)/R_0 \qquad (22)$$

A local safety factor is defined as:

$$q = \frac{d\varphi}{d\theta_t} = \frac{\vec{B}_0\,\cdot\,\nabla\varphi}{\vec{B}_0\,\cdot\,\nabla\theta_t} \qquad (23)$$

and all the relevant quantities are written in terms of q and Ψ_c'. The expression for q have the form (Appendix B):

$$q = \overline{q}\,(\rho_0) + \widetilde{q}_{max}\,(\rho_0)\,\cos\theta_t \qquad (24)$$

3.2. THE HELICAL PERTURBATION

We are especially concerned with the resonance phenomena; far from resonance a small peturbation does not significantly affect the magnetic surfaces. Here, the component of the helical field normal to an equilibrium plasma surface is written as:

$$\sqrt{g}\,b^1 = (\epsilon\Psi_c'\,\rho_0)_{\rho\,m,n} \quad b_0(\rho_0)\;\sin(m\theta_t - n\varphi) \qquad (25)$$

where the subscript indicates that the expression is taken at $\rho^{m,n}$; $\rho^{m,n}$ is the value of ρ_0 at a presumed resonance surface.

$b_0(\rho_0)$ is normalized at $\rho^{m,n}$ $(b_0(\rho^{m,n}) = \pm 1)$.

3.3. MAGNETIC ISLANDS

The structure near the resonance is analysed choosing:

$$\chi^1 \equiv \rho_0\,,\quad \chi^2 \equiv m\theta_t - n\varphi \quad\text{and}\quad \chi^3 \equiv \theta_t \tag{26}$$

Thus, the averages are taken along helical lines. $2\pi\Psi_H^0$ has the meaning of magnetic flux of $\vec{B}_0$ through a helical ribbon.

The expressions for F^2, $\widetilde{F^2}$ and $\widehat{F^2}$ become:

$$F^2 = m - n\bar{q} - n\tilde{q}_{max}\cos\theta_t;$$
$$\widetilde{F^2} = -n\tilde{q}_{max}\cos\theta_t$$

and

$$\widehat{F^2} = -n\tilde{q}_{max}\sin\theta_t \tag{27}$$

The unperturbed stream function $\Psi_H^0\,(\chi^1)$ has a maximum value at the rational surface $\bar{q}\,(\rho^{m,n}) = m/n$ as can be quickly inferred from:

$$\Psi_H^{0\,\prime} = \overline{\sqrt{g}\;B_0^2} = \Psi_c'\overline{F^2} = \Psi_c'\,(m - n\bar{q}) \tag{28}$$

Near this surface:

$$\Psi_H^0 \cong -\frac{1}{2}\,(n\bar{q}'\,\Psi_c')_{\rho^{m,n}}\,(\Delta\overline{\rho}_0)^2 + \text{an irrelevant constant}$$

where

$$\Delta\overline{\rho_0} \equiv \overline{\rho_0} - \rho^{m,n}$$

For the perturbed system, expression (17) yields:

$$\Psi_H = \Psi_H^0 + (\epsilon\Psi_c'\,\rho_0)_{\rho^{m,n}}\;b_0(\overline{\rho_0})\;\overline{\cos\,(\chi^2 - n\tilde{q}_{max}\sin\theta_t)}$$

Making use of the well known expression:

$$e^{i\chi\sin\theta} = \sum_{\ell=-\infty}^{+\infty} J_\ell\,(\chi)\,e^{i\ell\theta}\ ,$$

where $J_\ell(\chi)$ are cylindrical Bessel functions, and taking the average we have:

$$\Psi_H \cong -\frac{1}{2}\,(n\bar{q}'\,\Psi_c')_{\rho^{m,n}}\,(\Delta\rho_0)^2 + \epsilon\,\delta\Psi_H\cos\overline{\chi^2} \tag{29}$$

where

$$\delta\Psi_H \equiv (\Psi_c'\,\rho_0\,J_0)_{\rho^{m,n}} \quad\text{and}\quad J_0 \equiv J_0\,(n\tilde{q}_{max})$$

This expression can be compared with the universal Hamiltonian of a non linear pendulun (Sagdeev et al (1988)). The constancy of Ψ_H determines the relation:

$$\frac{\Delta\rho_0}{\rho^{m,n}} \cong \pm\frac{W}{K}\,(1 - K^2\sin^2\frac{\overline{\chi^2}}{2})^{1/2} \tag{30}$$

where K is a real positive number and

$$W \equiv \epsilon^{1/2}\left[\frac{4\,|\,\delta\Psi_H|}{\Psi_c'\;n\bar{q}'\;\rho_0^2}\right]^{1/2}_{\rho^{m,n}} = \left[\frac{4\epsilon J_0}{n\bar{q}'\;\rho_0}\right]^{1/2}_{\rho^{m,n}}$$

if $\delta\Psi_H \geq 0$.

In the case of $\delta\Psi_H < 0$ we have:

$$\frac{\Delta\overline{\rho}_0}{\rho^{m,n}} \cong \pm\frac{W}{K}\,(1 - K^2\cos^2\frac{\overline{\chi^2}}{2})^{1/2} \tag{30'}$$

In what follows we shall assume $\Psi_c' > 0$ and $b_0 > 0$ and consequently $\delta\Psi_H > 0$.

$\overline{\chi^2}\,(\theta_t)$ and $\Delta\rho_0\,(\theta_t)$ can be written in terms of Jacobian elliptic functions.

(30) is clearly a resonant solution; the departure of $\overline{\rho_0}$ from the resonance value $\rho^{m,n}$ is of the order of $\epsilon^{1/2}$. The diagram $(\overline{\chi^2}, \overline{\rho_0})$ is similar to the phase diagram of a pendular motion.

$K = 1$ corresponds to the separatrix, $K > 1$ to the surfaces inner to the separatrix and $K < 1$ to the outer surfaces.

The 1st order solution that defines approximate magnetic surfaces comes from (13). It is enough to replace the variables in (30) by:

$$\bar{\rho}_0 = \rho_0 \text{ and } \overline{\chi^2} = \chi^2 - \widehat{F^2} = m\theta_t - n\varphi + n\tilde{q}_{max} \sin\theta \qquad (31)$$

In order to draw a Poincaré map we consider a plane cutting accross the torus ($\varphi = 0$, for instance); every time the field line crosses this plane, the position is marked. If we put $\varphi = \ell.2\pi$, ℓ an integer number, in (31) and use (30) we must have an approximate representantion of the Poincaré map. The separatrix equation (at $\varphi = 0$) becomes:

$$\frac{\rho_0 - \rho^{m,n}}{\rho^{m,n}} \cong \pm W \cos((m\theta_t + nq_{max} \sin\theta_t)/2) \qquad (32)$$

($\rho^{m,n}$ W) has the meaning of halfwidth of the islands. This depends mostly on the value of q'.

3.4. THE SPURIOUS BEHAVIOUR OF THE LINES

The second order corrections are derived from (18), (19) and (20) using explicit expressions for $\sqrt{g}b^i$.

From (25):

$$\sqrt{g}\, b^1 (\bar{\rho}_0, \overline{\chi^2} - n\tilde{q}_{max} \sin\theta_t) = (\epsilon\, \Psi_c' \rho_0)_{\rho^{m,n}}\, b_0(\bar{\rho}_0) \sum_{\ell \neq 0} J_{-\ell} \sin(\overline{\chi^2} + \ell\theta_t) \qquad (33)$$

If $\delta\rho_0$ is sought in the form of Fourier series we find:

$$\delta\rho_0 \cong \sum_{\ell \neq 0} \left[\frac{\epsilon \;\; \rho_0 J_{-\ell}}{m + \ell - n\bar{q}}\right]_{\rho^{m,n}} \cos(\overline{\chi^2} + \ell\theta_t) \qquad (34)$$

This expression evidently shows resonance denominators at every rational surfaces $\bar{q} = (m + \ell)/n$ except at $\rho^{m,n}$. The resonant effect around $\rho^{m,n}$ is removed and $\delta\rho_0$ represents a dispersion of the order of ϵ.

The dispersion grows near other rational surfaces; when a line of force moves far away from the average position the average magnetic surface loses the meaning. This would represent the beginning of overlap of two neighbouring resonances

Once $\delta\rho_0$ is known the determination of $\delta\chi^2$ is a matter of elementary calculus.

$$\overline{\delta\chi^2} \cong \left[\epsilon \frac{\overline{F^2}}{1 - \overline{F^2}^2} n\tilde{q}'_{max} \rho_0 J_1\right]_{\rho^{m,n}} \sin \overline{\chi^2} \qquad (35)$$

$$\cong O(\epsilon^{3/2})$$

If terms of the order of $\epsilon^{3/2}$ are to be taken into account, all the terms in (18) must be considered in order to be coherent.

For the calculation of $\delta\chi^2$ we have to know $\sqrt{g}\, b^2$ and $\sqrt{g}\, b^3$. If these are of the same order of magnitude, all the conclusions drawn so far are not changed.

The analytic expression for the separatrix with the corrections $\delta\chi^i$ is:

$$\frac{\rho_0 - \delta\rho_0 - \rho^{m,n}}{\rho^{m,n}} = \pm W \cos((m\theta_t + n\tilde{q}_{max} \sin\theta_t + \delta\theta_t)/2) \qquad (36)$$

Finding $\overline{\chi^2}$ as function of θ_t, the relation:

$$\overline{\chi^2}(\theta_t) + \widetilde{\chi^2}(\theta_t) + \delta\chi^2(\theta_t) = m\theta_t - n\varphi \qquad (37)$$

can be used in order to determine the value of θ_t after each toroidal revolution ($\varphi = \ell\, 2\pi$).

The departure of ρ_t from the first order value is evaluated from (36).

The Poincaré map can be obtained without solving directly a system of standard mapping equations.

3.5. SATELLITE ISLANDS

As the second order corrections $\delta\chi^i$ exhibit resonance denominators the expressions obtained so far are not very accurate near other rational surfaces; near each rational surface $\bar{q} = (m + \Delta m)/n$ similar procedure can be followed using for the coordinate χ^2 the expression $\chi^2 \equiv (m + \Delta m)\theta_t - n\varphi$.

A chain of satellite islands is found in the form (Appendix C):

$$\frac{\Delta \rho_0}{\rho^{m+\Delta m,n}} \cong \pm W \cos(((m + \Delta m)\ \theta_t + n\tilde{q}_{max} \sin\theta_t)/2) \qquad (38)$$

where the width is now given by:

$$W = \left[\frac{4\epsilon}{n\bar{q}'\rho_0} \left| J_{\Delta m}(n\tilde{q}_{max}) \right| \right]^{1/2}_{\rho^{m+\Delta m,n}}$$

if $J_{-\Delta m}(n\tilde{q}_{max}) > 0$. If $J_{-\Delta m} < 0$ we have a sine function instead of a cosine.

The coupling between the modes occurs through $\widehat{F^2}$. The width of the satellite islands are fundamentally dependent on the asymmetry of q around the magnetic axis ($\tilde{q}_{max}$); indirectly, they are determined by the value of $\bar{q}$, $\bar{q}'$, the aspect ratio and the poloidal beta (β_p).

For $n\tilde{q}_{max} \lesssim 1$ the correction to $\bar{\rho}_0$ is proportional to J_0 ($n\tilde{q}_{max}$) whereas, near the main islands the dominant terms in $\delta\rho_0$ are proportional to J_1 ($n\tilde{q}_{max}$). So, we may expect a higher scattering of the field line around the hyperbolic lines in neighbouring rational surfaces. However, the main features of the magnetic structure must depend upon local values of q, $\bar{q}'$(shear) and $\vec{b}$ rather than on the way the mode has been excited.

In reallity, it is unlikely to have a single perturbing mode. A more general expression for the perturbation could be:

$$\sqrt{g}\, b^1 = (\epsilon\, \Psi_c' \rho_0)_{\rho^{m,n}} \sum_{m_p} b_{m_p} (\rho_0) \cos(m_p\ \theta_t - n\varphi).$$

In the vicinity of the rational surface (m,n) the function $\delta\Psi_H$ in (29) must be the sum:

$$\delta\Psi_H = (\Psi_c' \rho_0 \sum_{m_p} b_{m_p} J_{m_p - m})_{\rho^{m,n}}$$

instead of a single term.

The positions of the hyperbolic lines depend upon the resultant sign of $\delta\Psi_H$.

3.6. NUMERICAL CALCULATIONS

For the sake of numerical calculations we assumed that $\vec{b}$ is due to pairs of currents $\pm I_H$ wound on the tokamak vessel at $\rho_t = a_v$. So, inside the tokamak $\vec{b}$ is derived from a scalar potential and $b^2 \cong mb^3$ with an accuracy of the order of $(\rho_t/R_0)^2$ (Appendix D). With only one perturbing mode (m,n) they take the form:

$$\sqrt{g}\, b^3 = (\epsilon \Psi_c' \rho_0)_{\rho^{m,n}} \frac{b_0'}{m} \cos(m\theta_t - n\varphi)$$

$$\sqrt{g}\, b^2 = m \sqrt{g}\, b^3$$

$$b_0 = \left[\frac{\rho_0}{\rho^{m,n}} \right]^m$$

ϵ is the amplitude of variation of $b^3/B_0^3 = b_{\theta_t}/B_{\theta_t}$ at $\rho^{m,n}$ and the value is given by:

$$\epsilon = \frac{mn}{\pi a_v} \mu_0 I_H \left[\frac{\rho^{m,n}}{a_v} \right]^{m-1} \frac{R_0}{\Psi_c'}$$

The unperturbed system function Ψ_c' is written in terms of the plasma current density

$$\Psi_c'(\rho) = \frac{R_0}{\rho} \int_0^\rho \mu_0 J_\varphi(\chi)\, \chi d\chi$$

with $J_\varphi(\rho)\ \alpha\ (1 - \rho^2/a^2)^\gamma$.

Typical values in TBR–1 experiments are used (Vannucci et al (1988)): plasma current (I_p) = 10 kA; R_0 = 0,30m; radius of the limiter a = 0,08 m; B_φ = 0,5 T; coefficient of asymmetry at the plasma surface (Λ) = 0,28; radius of the vessel (a_v) = 0,11m and $\bar{q}$ ranging from 1 on the axis to $\simeq$ 4 (or $\gamma = 3$).

The field line is numerically traced for 700 toroidal circuits in the vicinity of each rational surface and plotted in $\varphi = 0$ poloidal plane.

The maps are compared to the 1^{st} order solution $\Psi_H\ (\chi^1,\ \chi^2 - \tilde{\chi}^2) =$ constant surfaces in the figures 1 to 4. Whenever there is no overlapping of islands the separatrix is sharply defined and the agreement between both maps is perfect. In the case of $I_H = 80A$ and $(m,n) = (2,1)$, following the line around the principal resonance for another 700 revolutions did not show any significant change in the picture. In all the cases considered, overlapping of (3,1) and (4,1) islands had already began; the values of ϵ are $\leq 1\%$.

CONCLUSIONS

1. In toroidal plasmas a single perturbing helical mode (m,n) causes the formation of islands at every rational surface. The plasma is all tied up by chains of islands. It seems vey likely that this structure would hinder rotations of MHD modes, thus stabilizing the plasma. The widths of the satellite islands are fundamentally dependent on the asymmetry of the pitch angle of $\vec{B}_0$ around the principal magnetic axis.

2. The hyperbolic lines are destroyed because of toroidality, irrespective of overlap of neighbouring islands. Minor disruptions can be expected before islands overlapping.

The technique developed here is fairly general; it can be applied to any symmetric system under the influence of small perturbations in order to get a bird's eye view of the magnetic structure, as well as to investigate the scattering of the field lines.

To us, in the numerical tracing of a field line near a separatrix the analytic representantion of the map had proved to be very helpful to find the starting point of integration.

Above all, the application of the method is very simple.

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APPENDIX A

From the definition of Ψ_H (17) we derive:

$$\frac{\partial\Psi_H}{\partial\overline{\chi^2}} = -\overline{\sqrt{g}\ b^1\ (\overline{\chi^1},\ \overline{\chi^2} + \widehat{F^2},\ \chi^3)} \qquad (A.1)$$

and

$$\frac{\partial\Psi_H}{\partial\overline{\chi^1}} = \overline{\sqrt{g}\ B_0^2\ (\overline{\chi^1},\ \chi^3)} - \overline{\frac{\partial F^2}{\partial\overline{\chi^1}}\ \sqrt{g}\ b^1} - \overline{\int_C^{\overline{\chi^2} + \widehat{F^2}} \frac{\partial}{\partial\overline{\chi^1}}\ \sqrt{g}\ b^1\ d\overline{\chi^2}} \qquad (A.2)$$

Using the divergence – free condition for $\vec{b}$ in the form:

$$\int_C^{\overline{\chi^2}+\widehat{F^2}} \frac{\partial}{\partial\overline{\chi^i}}\sqrt{g}\, b^i\, d\overline{\chi^2} = 0 \tag{A.3}$$

the last term in the expression (A.2) becomes:

$$-\overline{\int_C^{\overline{\chi^2}+\widehat{F^2}} \frac{\partial}{\partial\overline{\chi^1}}\sqrt{g}\,b^1\,d\overline{\chi^2}} = \overline{\sqrt{g}\,b^2\,(\overline{\chi^1},\ \overline{\chi^2}+\widehat{F^2},\ \chi^3)} - \overline{\widetilde{F^2}\sqrt{g}\,b^3} \tag{A.4}$$

if $\overline{\sqrt{g}\,b^2\,(\overline{\chi^1},\ C,\ \chi^3)} = 0$. Consequently:

$$\frac{\partial\Psi_H}{\partial\overline{\chi^1}} = \overline{\sqrt{g}\,B_0^2} + \overline{\sqrt{g}\,b^2} - \overline{\frac{\partial\widehat{F^2}}{\partial\overline{\chi^1}}\sqrt{g}\,b^1} - \overline{\widetilde{F^2}\sqrt{g}\,b^3} \tag{A.5}$$

where $\sqrt{g}\,b^i$ are taken at $(\overline{\chi^1}, \overline{\chi^2}+\widehat{F^2}, \chi^3)$.

The condition (16) becomes:

$$\frac{d\overline{\chi^1}}{d\chi^3} = -\frac{1}{\Psi_0'}\frac{\partial\Psi_H}{\partial\overline{\chi^2}}\left(1+\frac{\overline{b^3}}{B_0^3}\right)^{-1} \tag{A.6}$$

If $\frac{d\Psi_H}{d\chi^3}$ is to be zero we must have:

$$\frac{d\overline{\chi^2}}{d\chi^3} = \frac{1}{\Psi_0'}\frac{\partial\Psi_H}{\partial\overline{\chi^1}}\left(1+\frac{\overline{b^3}}{B_0^3}\right)^{-1} \tag{A.7}$$

Using (A.1), (A.5), (A.6) and (A.7) together with

$$\frac{d\widetilde{\chi^2}}{d\chi^3} = \frac{d\widehat{F^2}}{d\chi^3} = \frac{d\overline{\chi^1}}{d\chi^3}\frac{\partial\widehat{F^2}}{\partial\overline{\chi^1}} + \frac{\partial\widehat{F^2}}{\partial\chi^3}$$

in the expression (9) the differential equations (18) are derived for $\delta\chi^i$.

APPENDIX B.

When a cylindrical plasma with circular cross section and poloidal flux $2\pi\Psi_c(\rho)$ is bent into a torus, the poloidal flux $2\pi\Psi_0$ of the self–consistent magnetic field of the resultant system is (Kucinski et al (1990)):

$$\Psi_0\,(\rho_t,\,\theta_t) \cong \Psi_c(\rho_t) + \cos\theta_t\,\Psi_c'(\rho_t)\int_{\rho_t}^{a}\frac{\chi}{R_0}\Lambda\,(\chi)d\chi \tag{B.1}$$

where ρ_t, θ_t, φ are the toroidal polar coordinates; $\Lambda(\rho_t) = -1 + \beta_p + \frac{l_i\,(\rho_t)}{2}$ is the coefficient of asymmetry of the poloidal field; β_p is the poloidal beta;

$$\frac{l_i}{2} = \frac{1}{\rho_t^2\,\Psi_c'^2}\int_0^{\rho_t}\rho_t\,\Psi_c'^2 d\rho_t$$

and $\rho_t = a$ is the outermost plasma surface.

The average value of ρ_t on a magnetic surface (ρ_0) is also the value of ρ_t at $\cos\theta_t = 0$ on this surface:

$$\Psi_0\,(\rho_t,\,\theta_t) = \Psi_0\,(\rho_0,\,\theta_t = \frac{\pi}{2}) = \Psi_c\,(\rho_0) \tag{B.2}$$

From (B.1) and (B.2) it follows:

$$\rho_t \simeq \rho_0 - \int_{\rho_0}^{a}\frac{\chi}{R_0}\Lambda\,(\chi)\,d\chi\,\cos\theta_t$$

Using: $\quad R^2\,\vec{B}_0\cdot\nabla\varphi = \text{constant}$

$$\vec{B}_0\cdot\nabla\theta_t = \frac{1}{R_0\,\rho_t}\frac{\partial\Psi_0}{\partial\rho_t}$$

$$R^2 \simeq \left(1 - \frac{2\rho_t}{R_0}\cos\theta_t\right)R_0^2$$

an approximate expression for the local safety factor $q(\rho_0, \theta_t) = \dfrac{\vec{B}_0 \cdot \nabla\varphi}{\vec{B}_0 \cdot \nabla\theta_t}$ can be derived

as:

$$q = \bar{q}(\rho_0) + \tilde{q}_{max}(\rho_0)\cos\theta_t$$

$$\bar{q} = \overline{\frac{\rho_0 \; B\varphi}{R_0 B_{\theta_t}}}$$

$$\tilde{q}_{max} = \bar{q}\,\frac{a}{R_0}\left(\frac{\rho_0}{a}(2+\Lambda) - \frac{1}{\rho_0}\int_{\rho_0}^{a} \frac{\chi}{a}\Lambda(\chi)\,d\chi\right)$$

APPENDIX C

$$\sqrt{g}\, b^1 = (\epsilon \Psi_c' \rho_0)_{\rho\, m,n}\, b_0(\rho_0)\sin(m\theta_t - n\varphi)$$

$$= (\epsilon \Psi_c' \rho_0)_{\rho\, m,n}\, b_0(\rho_0)\sin(\chi^2 - \Delta m\theta_t)$$

where $\chi^2 \equiv (m + \Delta m)\theta_t - n\varphi$.

If $b_0(\rho_0)$ is renormalized in order to be ± 1 at $\rho^{m+\Delta m,n}$:

$$\Psi_H = \Psi_H^0 + (\epsilon \Psi_c' \rho_0)_{\rho\, m + \Delta m,n} \quad b_0(\rho_0)\,\overline{\cos(\chi^2 - n\tilde{q}_{max}\sin\theta_t - \Delta m\theta_t)}$$

$$\overline{\cos(\chi^2 - n\tilde{q}_{max}\sin\theta_t - \Delta m\theta_t)} = \sum_{\ell=-\infty}^{+\infty} J_\ell(n\tilde{q}_{max})\,\overline{\cos(\chi^2 - \Delta m\theta_t - \ell\theta_t)}$$

$$= J_{-\Delta m}(n\tilde{q}_{max})\cos\overline{\chi^2}$$

Then,

$$\Psi_H = \Psi_H^0 + \delta\Psi_H \cos\overline{\chi^2}$$

with $\delta\Psi_H = (\epsilon \Psi_c' \rho_0 J_{-\Delta m})_{\rho\, m + \Delta m,n}$

If $J_{-\Delta m} > 0$, $\delta\Psi_H > 0$ and we have the expression (38) for the separatrix.

APPENDIX D

$\vec{b} = \nabla\phi(\chi^1, \chi^2, \chi^3)$; $\chi^2 \equiv m\theta_t - n\varphi$; $\chi^3 \equiv \theta_t$.

$$b^2 = \frac{\partial\phi}{\partial\chi^i} g^{i2} \text{ and } b^3 = \frac{\partial\phi}{\partial\chi^i} g^{i3}$$

$$g^{12} = \nabla\chi^1 \cdot (m\nabla\theta_t - n\nabla\varphi) = m\nabla\chi^1 \cdot \nabla\theta_t$$

$$g^{22} = (m\nabla\theta_t - n\nabla\varphi) = m^2\nabla\theta_t^2 + n^2\nabla\varphi^2 \simeq m^2\nabla\theta_t^2$$

$$g^{32} = \nabla\theta_t \cdot (m\nabla\theta_t - n\nabla\varphi) = m\,\nabla\theta_t^2$$

Therefore $g^{i2} \simeq m g^{i3}$ and $b^2 \simeq m\, b^3$; $\nabla \cdot \vec{b} = 0$ is written $\dfrac{\partial}{\partial\chi^i}\sqrt{g}\, b^i = 0$.

$$\frac{\partial}{\partial\chi^1}\sqrt{g}\, b^1 + \left(m\frac{\partial}{\partial\chi^2} + \frac{\partial}{\partial\chi^3}\right)\sqrt{g}\, b^3 = 0.$$

Taking $\sqrt{g}\, b^1 = (\epsilon \Psi_c' \rho_0)_{\rho\, m,n}\, b_0(\rho_0)\sin\chi^2$ we derive:

$$\sqrt{g}\, b^3 = \Psi_c'\, b^3 / B_0^3 = (\epsilon \Psi_c' \rho_0)_{\rho\, m,n}\, \frac{b_0'}{m}\cos\chi^2.$$

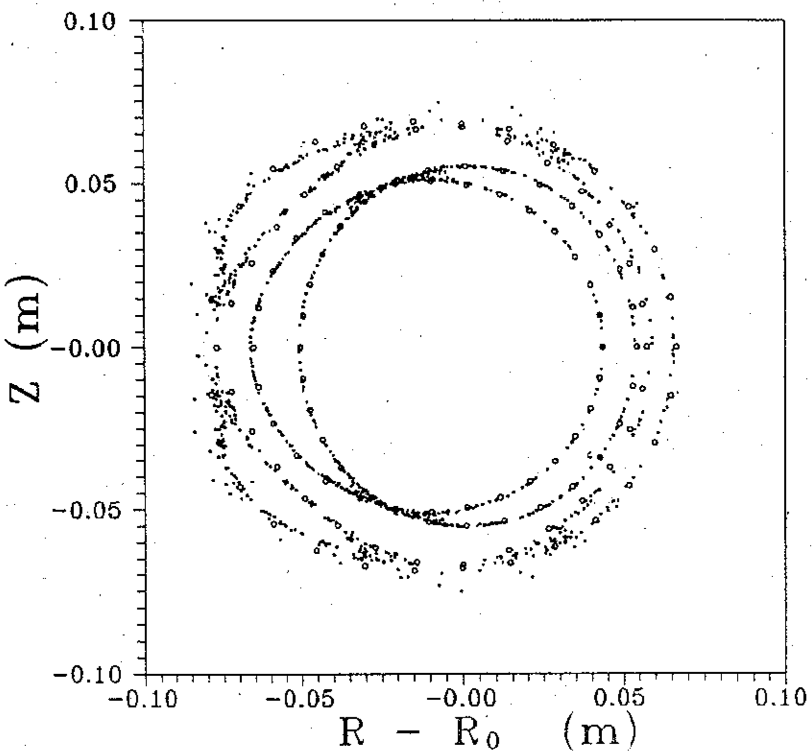

Fig. 1 - The Poincaré map obtained by numerical integration is represented by dots; 700 toroidal circuits are considered in the vicinity of each resonance surface. Small circles represent the 1^{st} order analytic solution for the corresponding map. In parentesis is the perturbing field mode (m,n). I_H=80A; (2,1).

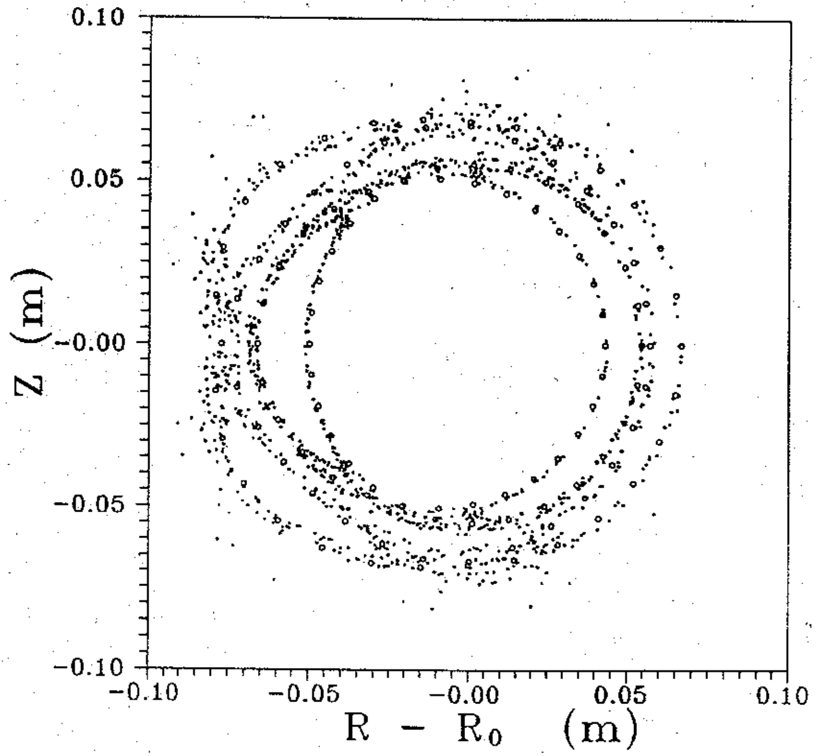

Fig. 2 - The same as in the Fig. 1. I_H=90A; (2,1).

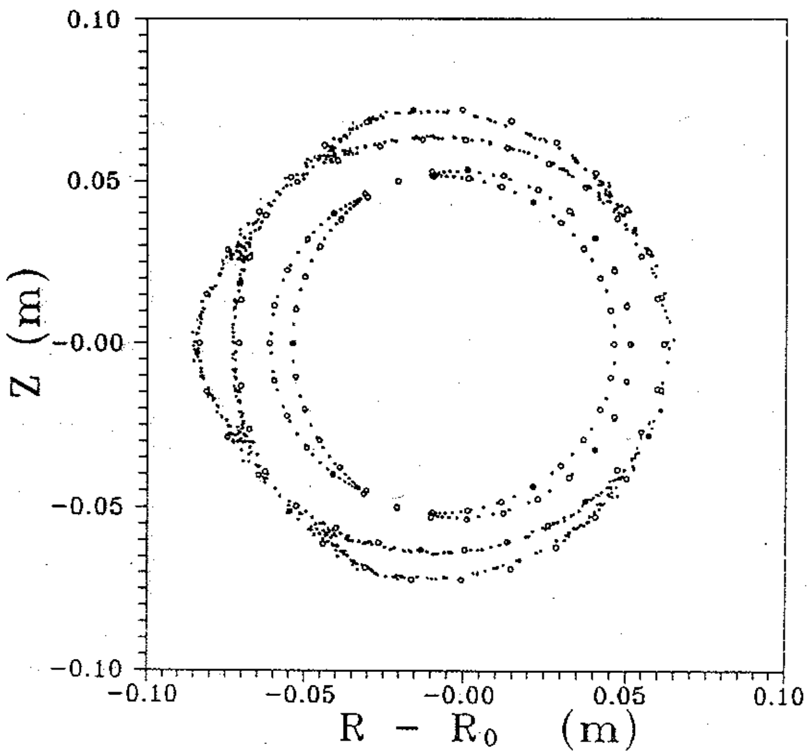

Fig. 3 - The same as in the other figures.
I_H=60A; (3,1)

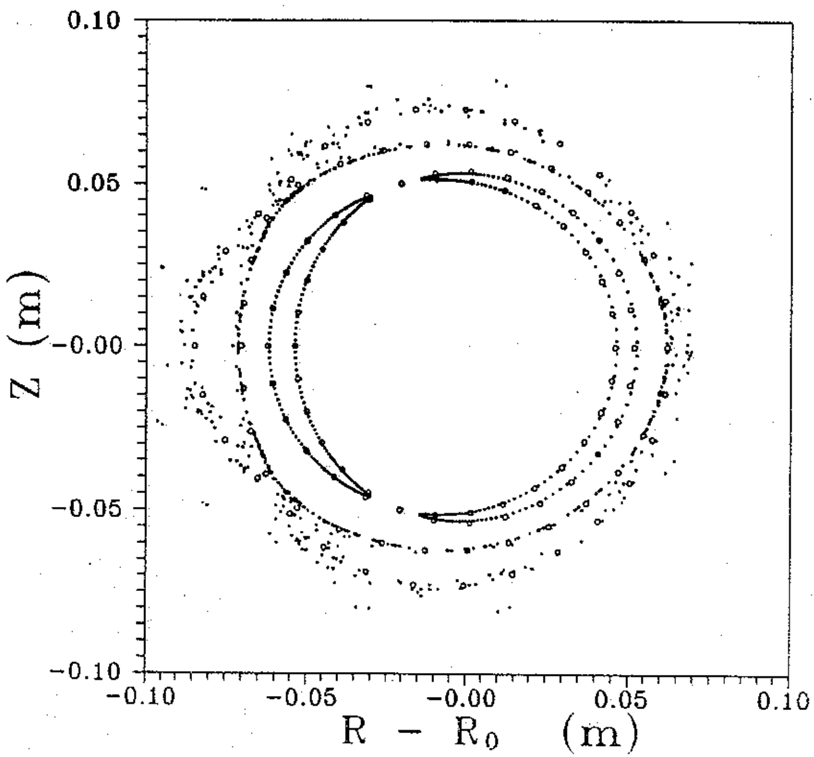

Fig. 4 - The same as in the other figures.
I_H= 80A; (3,1)