UNIVERSIDADE DE SÃO PAULO

PUBLICAÇÕES

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01452-990 SÃO PAULO - SP
BRASIL

IFUSP/P-1136

QUASI-PERIODIC OSCILLATIONS ON A NONLINEAR TRANSIENT

H. Franco
Instituto de Física, Universidade de São Paulo

R.M. Pauletti
Escola Politécnica, Universidade de São Paulo

Janeiro/1995

Quasi-periodic oscillations on a nonlinear transient

H. Franco and R. M. Pauletti[+]

Instituto de Fisica, Universidade de São Paulo,
Caixa Postal 20515, Sao Paulo, 01498, Brazil

[+] Escola Politécnica, Universidade de São Paulo,

Abstract

The time-changing spectrum of a nonlinear transient is analyzed through the Gabor transform technique. Clusters of evenly spaced lines appear on spectrograms of the numerically computed oscillations. They can be explained by means of a simple model describing the dynamic of the energy exchanges between the external oscillating force and the nonlinear system. The resulting amplitude and frequency modulations are shown to produce the spectral line structures. Frequencies incommensurable with other present oscillations can be generated by the nonlinear system.

1. Introduction

Most literature on nonlinear oscillations [1,2,3,4,5] describes the *steady state* motion in detail, while the *transients* receive a qualitative description, typically an outline of the trajectories in the phase space and their topology. The time-domain information about the motion usually discards transients and covers the steady state, since the Fourier transform of a nonstationary oscillation has little physical sense.

Nevertheless, nonlinear transients are present on a variety of real-word problems. In the field of mechanic engineering, they can arise in the interaction of motors and structures. Acoustics has multiple problems that present nonlinear transients, such as transducers' dynamic, musical instruments' analysis, as well as sound perception and bioacoustics. Transients of electronic circuits, which can be nonlinear, have a potential interest on areas such as signal transmission. It should also be remembered that, in a semiclassical approach, most problems of interaction of radiation and matter deal with nonlinear oscillator's transients. These and many other phenomena cannot be fully understood by a steady state approach.

In the present work, however, we focus the attention precisely on the *transient* region. The *Gabor transform* technique was chosen to examine the spectrum evolution of the a nonlinear system thorough the transient. This technique has been previously used by the authors to analyze transients of a double potential well free oscillator [6]. In this article, we describe the "transient spectum" or spectrogram of a *forced* oscillator.

The spectrograms of the driven oscillator shows temporary *quasi-periodic* oscillations. Although they are transient phenomena, they bear a close resemblance with *steady-state* quasiperiodic oscillations. There are also important differences that we shall point later. Before we proceed, let us briefly recall some basic definitions of steady state oscillations.

Sinusoidally driven nonlinear oscillators can, in certain cases, exhibit *quasi-periodic* behavior[5]. In this regime, the spectrum has a discrete set of peaks, and all present frequencies can be expressed as *sums and differences* of a countable set of *basis frequencies* f_j and their harmonics. For a *p-periodic* oscillation, the allowed frequencies are:

$$f_i = \sum_{j=1}^{p} k_j \, f_j \quad , \tag{1}$$

where the k_j are integers. In other words, the frequencies obeys a sort of Ridberg-Ritz "combination principle".

In the case of *two* oscillating external forces, the basis frequencies are the same present in the driving force and their subharmonics. The power spectrum contains then the so called *combination tones.* In a system with an autonomous *limit cycle*, such as the van der Pol oscillator, the external frequency and the *limit cycle* frequency constitute the basis of the quasiperiodic oscillations. These are examples of *two-periodic* motions, and the resulting frequencies are the harmonics of the basis frequencies, as well as their sums and differences.

In this article, we show that the spectrogram of the transient region of a forced nonlinear oscillator shows series of evenly spaced peaks as a quasiperiodic oscillator. Each "time slice" of the spectrogram is equivalent to a *three-periodic* motion. As expected, one of the basis frequencies is the external force frequency. The 1/3 subharmonic of the driving force is also present in the basis. However, the third basis frequency is *not equal nor commensurable* to any characteristic system frequency. It is a function of the external force frequency *and magnitude*, and also as on the form of the system potential well. As the system approaches the steady state, these lines decrease in magnitude at different rates, so that each line has its own "*lifetime*".

2. Transient analysis

The system chosen to the transient analysis is the *Duffing oscillator*, defined by the following DE :

$$m\ddot{x} + c\dot{x} - k_1 x - k_2 x^2 - k_3 x^3 = F_0 \cos(\omega t) \tag{2}$$

The oscillator's parameters are m=1, $k_1 = 4\ \pi^2$, $k_2 = 0$, $k_3 = 4\ \pi^2$ and c = 0.025. The potential curve has the form of a single symmetric potential well, with a small oscillation frequency 1 Hz. The external force has a strength F_O=4 and a frequency 1.5 Hz. The damping constant c is chosen so that the relative energy dissipation per cycle is small. The initial position is x_O=0 and the velocity v_O=120 m/s. We have employed in all numerical simulations the DGEAR routine, which implements the Adams-Moulton algorithm with polynomials up to 12th order. The transient obtained using the above parameters is shown in fig 1.

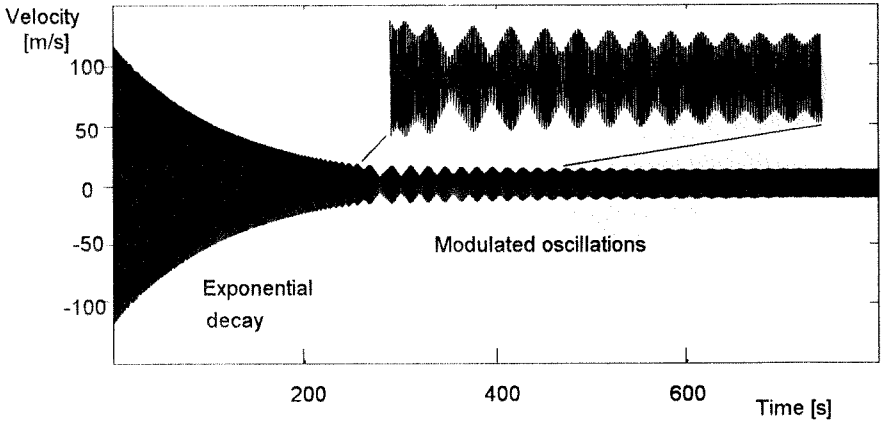

Figure 1. Transient of the forced Duffing oscillator.

The picture actually shows the *envelope*, since individual cycles are too small in the graphic scale. After a steady energy decay, the system undergoes transient amplitude modulations (AM) which eventually vanish. We analyze here this second phase of the transient.

Gabor transform [7,8] is implemented through a dedicated code written on MATLAB® language and using direct calculation from definition and using *Morlet wavelets* [9], which has the form of Gaussian wave packets, as basis elements. The Gabor transform $G_s(b,\omega)$ of a function $s(T_j)$, a *sampled* version of a continuous time function $s(t)$, reads then:

$$G_s(b,\omega) = \left\langle s(t),\, g_{(b,\omega)}(t) \right\rangle = \sum_{k=1}^{N} s(T_k)\, g^*_{(b,\omega)}(T_k) \quad , \tag{3}$$

where :

$$g_{(b,\omega)}(t) = e^{i\omega(t-b)}\ e^{\left(\frac{t-b}{2\sigma_t}\right)^2} \tag{4}$$

In words, the Gabor transform coefficients are obtained through the hilbertian scalar product between the basis function and the analyzed function. The sampling rate should, of course, respect the limits imposed by the Nyquist theorem [10].

Both numerical simulation and Gabor analysis are performed using floating point double precision numbers with 16 digit mantissa and 3 digit exponent. The Gabor transform maps the original function s(t) onto the *time-frequency* phase space. The time mean square deviation of the Morlet wavelets of equation (4) is $\sigma_t = 100/\sqrt{2\pi} = 39.86$ s . This peculiar value comes from the choice of the *frequency resolution*, linked with σ_t through the uncertainty relations for Gaussian wave packets. We use only the square modulus of the transform, $|\, G(b,\omega)\,|^2$, which can be interpreted as an *energy density* [11] and is also called Gabor *spectrogram*.

We show in figure 2 a typical spectrogram of the nonlinear transient. It can be read as a chart of the power spectral density in the time-frequency domain. We use a gray scale with a density proportional to the *logarithm* of the spectral power. The main line at 1.5 Hz corresponds, of course, to the external frequency. Surrounding this line, there are *sidebands*, constituted by evenly spaced lines. If the system has simple AM oscillations, there would be just a single pair of extra lines. The oscillations can however be identified as *simultaneous frequency and amplitude modulation* (AM-FM). The *modulation frequency* is identical to the third basis element of the quasi-periodic oscillations mentioned above.

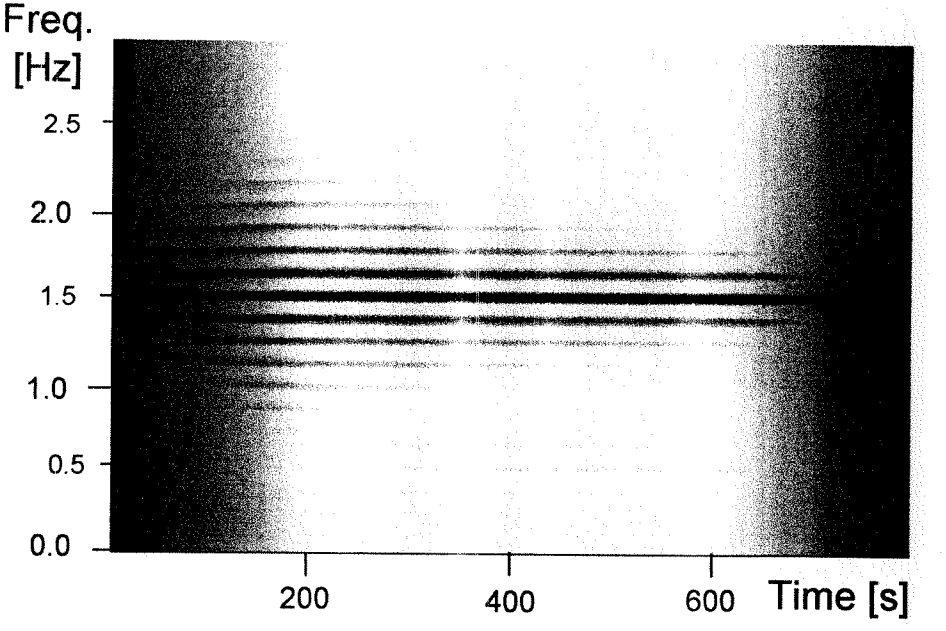

Figure 1. Narrowband Gabor spectrogram of the nonlinear transient

3. A model for the sidebands structure.

The modulated oscillations can be interpreted as follows: The nonlinear oscillator and the external force exchange energy back and forth, so that the system energy oscillates during the transient. The damping term ensures the approach to an equilibrium state, while energy oscillations decrease. The frequency of these oscillations can be estimated through a simple model involving energy balance considerations. Only the main hypotheses of the derivation are sketched here.

We approximate the system motion to a sinusoidal function of time, so that only the coupling between the *first harmonic* of the motion and the external force is taken on account. Moreover, we assume *slowly varying* energy and frequency. The energy transfer rate between the oscillator and the external force is then estimated using a "power factor" similar to the one used in electric circuits. We assume also a small phase departure from steady state condition, and obtain the following estimation of the modulation frequency :

$$f_m = \omega_m/2\pi \cong \sqrt{\frac{1}{2}\frac{d\omega(E)}{dE}\bigg|_{\omega_0} F_0 \; \mathrm{v}_{\max}(\omega_0)} \qquad (5)$$

The function $\omega(E)$ is the angular frequency of the *free oscillator*, as a function of the total energy E. The derivative of this frequency is evaluated for the free oscillator at the frequency ω_0 of the external force. F_0 is the magnitude of the external sinusoidal force, as in equation (2), while $\mathrm{v}_{\max}(f_0)$ is the maximum velocity of the free oscillator when it oscillates at the same frequency f_0 as the external force. These functions depend on the detailed form of the potential well, and require a separate numerical evaluation.

Due to the oscillating energy exchange, the system motion is both amplitude and frequency modulated. The system oscillations can be expanded over a set of sinusoidal oscillations, yielding the spectrum of an AM-FM *wideband* modulated oscillation [12]:

$$x(t) = A_c \sum_{n=-\infty}^{\infty} \left\{ J_n(\beta) + \frac{m_a}{2}[\, J_{n-1}(\beta) + J_{n+1}(\beta)] \right\} \times \exp\,[\mathrm{i}\,(\omega_c + n\,\omega_m t)] \qquad (6)$$

where J_n are Bessel functions of first kind and ω_c , the "carrier frequency" is identified with ω_0 , the external force frequency. This expression shows that the AM-FM oscillations appear in the frequency domain as a cluster of evenly spaced sharp peaks, with magnitudes decreasing as n increases. The parameter β indicates the *modulation depth*. In our problem, this parameter decreases exponentially in time, while the system approaches the steady state. The energy oscillation model above fits well the line spacing observed on the spectrograms with the external force F_0 ranging from 0.5 to 10.

The spectrogram shown on figure 2 expands the first harmonic region only, covering the range from 0 to 3 Hz. There are, however, similar "multiplet-like" structures at higher frequencies. We show in figure 3 a "time slice" of the Gabor transform of the oscillations, which is roughly a "local spectrum". There are clusters of evenly spaced lines surrounding the frequencies of the *odd harmonics* of the sinusoidal external force. The potential symmetry "forbids" the even harmonics that would be present if the quadratic term of equation (2) were not zero.

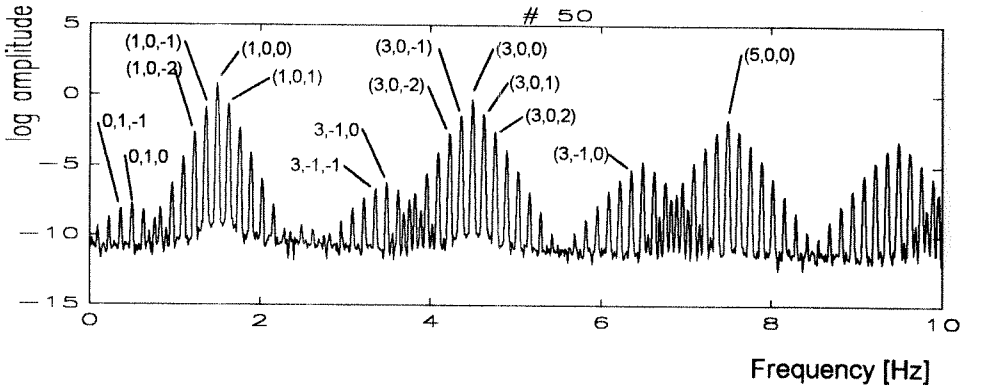

Figure 3. Spectrum of a section of the simulation, calculated through the Gabor transform. Logarithm of the Gabor transform magnitude is used in the vertical scale

Clusters of evenly spaced peaks are also visible around the frequency $f_0/3$, a *subharmonic* of the driving force (visible on the bottom of figure 2). Moreover, they are present around the frequencies (n–1/3) F_0 , so that this subharmonic is also a basis of the quasiperiodic movement. Some of the lines are labeled with the integer numbers k_j of equation (1). The first integer is the coefficient of the frequency f_0, the second, of the subharmonic $f_0/3$ and the third multiplies the modulating frequency f_m.

4. Conclusions

Gabor transform can map the spectrum of transient nonlinear vibrations. We examine the synchronization process of a forced oscillator with a cubic nonlinearity. The spectrograms generated by the Gabor transform shows clusters of evenly spaced lines, a signature of *quasi-periodic* oscillations. The structure of the line clusters correspond s to a *3-periodic* movement. One of the basis frequency present is incommensurable with the external force frequency. These spectra are identified with simultaneous amplitude and frequency modulation of the system. A simple model, based on energy transfer considerations, can explain the AM-FM behavior and corectly calculate the modulating frquency.

REFERENCES

[1] J.J. Stoker, *"Nonlinear Vibrations in Mechanical and Electrical Systems"*, Interscience Publishers, New York, 1950.

[2] N. Minorsky, *Nonlinear Oscillations*, D. Van Nostrand Co. INC., Princeton,1962.

[3] A.J. Lichtenberg and M.A. Lieberman ; *"Regular and Stochastic Motion"*, Springer-Verlag, New York, 1983.

[4] J.M.T. Thompson and H.B. Steward, *"Nonlinear Dynamics and Chaos"*, John Willey & Sons, Chichester, 1987

[5] T. S. Parker and L.O. Chua *"Practical Numerical Algorithms for Chaotic Systems"*, Springer-Verlag, New York, 1989.

[6] H. Franco and R.M.O Pauleti *"Analysis of Nonlinear Ocillation by Gabor Spectrograms"* Accepted for publication in *Nonlinear Dynamics.*

[7] D. Gabor , "Theory of Communication"Journal of the Institution of Electrical Engineers (part III) 63 429-441 (1946).

[8] I. Daubechies, *"The Wavelet transform, Time-frequency localization and Signal Analysis", IEEE* Transations on Information Theory **IT-36** 961-1005 (1990).

[9] J. Morlet and A. Grossman, *"Wave Propagation and Smpling Theory - Part II: Sampling Theory and Complex Waves"* Geophysiscs 47 222-236 (1982).

[10] A.V. Oppenheim and R.W. Schafer, *Discrete Time Signal Processing*; Prentice Hall,Inc. ed. , N.J. 1989.

[11] N. Delprat *et al., Assymptotic Wavelet and Gabor Analysis: Extraction of Instantaneous Frequency , IEEE Transaction on Information Theory v. 38, 644-664 (1992).*

[12] P.F. Panter, *"Modulation, Noise and Spectral Analysis"*, McGraw Hill Co., New York 1965.