A. Derivations of the Optimal Measure

The problem of finding the optimal measure is as follows:

$$\min_{\mathbb{Q}} \Big[\mathbb{E}_{\mathbb{Q}}[S(\boldsymbol{x})] + \gamma \mathbb{D}_{KL}(\mathbb{Q}||\mathbb{P})\Big], \text{ s.t. } \int \mathrm{d}\mathbb{Q} = 1 \tag{18}$$

The minimum in (18) is attained at optimal measure $\mathbb{Q}^*$ given by:

$$\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} = \frac{\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))}{\mathbb{E}_{\mathbb{P}}[\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))]} \tag{19}$$

Next, we show the derivations of (19), which contain two parts. First, we will show the following inequality:

$$\gamma \log\bigg(\mathbb{E}_{\mathbb{P}}\Big[\exp\big(-\frac{1}{\gamma}S(\boldsymbol{x})\big)\Big]\bigg) \leqslant \Big[\mathbb{E}_{\mathbb{Q}}[S(\boldsymbol{x})] + \gamma \mathbb{D}_{KL}(\mathbb{Q}||\mathbb{P})\Big] \tag{20}$$

The second part is to show the minimum of the above inequality is reached at (19).

To prove the first part, we first express $\mathbb{E}_{\mathbb{P}}$ in the left-hand-side of (20) as a function of the expectation $\mathbb{E}_{\mathbb{Q}}$. More specifically, we have:

$$\log\bigg(\mathbb{E}_{\mathbb{P}}\Big[\exp\big(-\frac{1}{\gamma}S(\boldsymbol{x})\big)\Big]\bigg) = \log\bigg(\int \exp\big(-\frac{1}{\gamma}S(\boldsymbol{x})\big)\mathrm{d}\mathbb{P}\bigg) \tag{21}$$

$$= \log\bigg(\int \exp\big(-\frac{1}{\gamma}S(\boldsymbol{x})\big)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\mathrm{d}\mathbb{Q}\bigg) \tag{22}$$

$$\geqslant \int \log\bigg(\exp\big(-\frac{1}{\gamma}S(\boldsymbol{x})\big)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\bigg)\mathrm{d}\mathbb{Q} \tag{23}$$

where (23) is due to the Jensen's inequality that puts the $\log$ operator inside the integral. The measure $\mathbb{P}$ is absolute continuous with respect to $\mathbb{Q}$, hence the derivative $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}$ exists.

Moreover, using the property that $\log(ab) = \log a + \log b$ and $\log(1/a) = -\log a$, the right-hand-side of the above inequality can be written as:

$$\begin{aligned}\int \log\bigg(\exp\big(-\frac{1}{\gamma}S(\boldsymbol{x})\big)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\bigg)\mathrm{d}\mathbb{Q} &= \int \bigg(-\frac{1}{\gamma}S(\boldsymbol{x}) + \log\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\bigg)\mathrm{d}\mathbb{Q} \\ &= \int -\frac{1}{\gamma}S(\boldsymbol{x})\mathrm{d}\mathbb{Q} + \int \log\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\mathrm{d}\mathbb{Q} \\ &= \int -\frac{1}{\gamma}S(\boldsymbol{x})\mathrm{d}\mathbb{Q} - \int \log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\mathrm{d}\mathbb{Q} \\ &= -\frac{1}{\gamma}\mathbb{E}_{\mathbb{Q}}[S(\boldsymbol{x})] - \mathbb{D}_{\mathbb{KL}}(\mathbb{Q}||\mathbb{P})\end{aligned} \tag{24}$$

Hence, combining (23) and (24), we have:

$$\log\bigg(\mathbb{E}_{\mathbb{P}}\Big[\exp\big(-\frac{1}{\gamma}S(\boldsymbol{x})\big)\Big]\bigg) \geqslant -\frac{1}{\gamma}\mathbb{E}_{\mathbb{Q}}[S(\boldsymbol{x})] - \mathbb{D}_{\mathbb{KL}}(\mathbb{Q}||\mathbb{P}) \tag{25}$$

Finally, since $\gamma > 0$, multiply both sides of (25) by $-\gamma$ yields:

$$-\gamma \log\bigg(\mathbb{E}_{\mathbb{P}}\Big[\exp\big(-\frac{1}{\gamma}S(\boldsymbol{x})\big)\Big]\bigg) \leqslant \mathbb{E}_{\mathbb{Q}}[S(\boldsymbol{x})] + \gamma\mathbb{D}_{\mathbb{KL}}(\mathbb{Q}||\mathbb{P}) \tag{26}$$

This finishes the proof of (20), the first part of the theorem. Next, we will show the minimum is reached at $\mathbb{Q}^*$ given by (19).

To prove the second part, we will substitute (19) to the right-hand-side of (25) to show that the infimum is reached with this $\mathbb{Q}^*$. More specifically,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}^*}[S(\boldsymbol{x})] + \gamma \mathbb{D}_{\mathbb{KL}}(\mathbb{Q}^* || \mathbb{P}) &= \mathbb{E}_{\mathbb{Q}^*}[S(\boldsymbol{x})] + \gamma \int \log \frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} \mathrm{d}\mathbb{Q}^* \\
&= \mathbb{E}_{\mathbb{Q}^*}[S(\boldsymbol{x})] + \gamma \int \log \frac{\exp(-\frac{1}{\gamma} S(\boldsymbol{x}))}{\mathbb{E}_{\mathbb{P}}[\exp(-\frac{1}{\gamma} S(\boldsymbol{x}))]} \mathrm{d}\mathbb{Q}^* \\
&= \mathbb{E}_{\mathbb{Q}^*}[S(\boldsymbol{x})] + \gamma \int -\frac{1}{\gamma} S(\boldsymbol{x}) \mathrm{d}\mathbb{Q}^* - \gamma \int \log \left(\mathbb{E}_{\mathbb{P}} \Big[\exp\big(-\frac{1}{\gamma} S(\boldsymbol{x}) \big) \Big] \right) \mathrm{d}\mathbb{Q}^* \qquad (27) \\
&= \mathbb{E}_{\mathbb{Q}^*}[S(\boldsymbol{x})] - \int S(\boldsymbol{x}) \mathrm{d}\mathbb{Q}^* - \gamma \log \left(\mathbb{E}_{\mathbb{P}} \Big[\exp\big(-\frac{1}{\gamma} S(\boldsymbol{x}) \big) \Big] \right) \int \mathrm{d}\mathbb{Q}^* \\
&= \mathbb{E}_{\mathbb{Q}^*}[S(\boldsymbol{x})] - \mathbb{E}_{\mathbb{Q}^*}[S(\boldsymbol{x})] - \gamma \log \left(\mathbb{E}_{\mathbb{P}} \Big[\exp\big(-\frac{1}{\gamma} S(\boldsymbol{x}) \big) \Big] \right) \qquad (28) \\
&= -\gamma \log \left(\mathbb{E}_{\mathbb{P}} \Big[\exp\big(-\frac{1}{\gamma} S(\boldsymbol{x}) \big) \Big] \right)
\end{aligned}
$$

where (27) is due to the property $\log(a/b) = \log a - \log b$ and (28) is because $\mathbb{Q}^*$ is a probability measure hence $\int \mathrm{d}\mathbb{Q}^* = 1$. Hence the infimum is reached and this finishes the proof of the second part.

B. Proof of Theorem 3

Theorem 3. *For the intensity control problem in (4), we have:* $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}(\boldsymbol{u})} = \exp\big(\mathcal{D}(\boldsymbol{u})\big)$*, where* $\mathcal{D}(\boldsymbol{u})$ *is expressed as*

$$\sum_{i=1}^{M} \int_0^T \big(u_i(s) - 1\big)\lambda_i(s)\mathrm{d}s - \int_0^T \log\big(u_i(s)\big)\mathrm{d}N_i(s)$$

Proof. Intuitively, the derivative $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}(\boldsymbol{u})$ means the relative density of probability distribution $\mathbb{P}$ with respect to $\mathbb{Q}$. The change of probability measure happens because the intensity of the point process that drives the SDE in (2) is changed from $\boldsymbol{\lambda}(t)$ to $\boldsymbol{\lambda}(\boldsymbol{u}, t)$ in (4). Hence $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}(\boldsymbol{u})$ describes the change of probability measure for point processes and is the *likelihood ratio* between the uncontrolled and controlled point process (Brémaud, 1981):

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}(\boldsymbol{u})} = \frac{\exp\big(\mathcal{L}(\boldsymbol{\lambda})\big)}{\exp\big(\mathcal{L}(\boldsymbol{\lambda}(\boldsymbol{u}))\big)} = \exp\big(\mathcal{D}(\boldsymbol{u})\big),$$

where $\mathcal{L}$ is the *log-likelihood* for the multi-dimension point process with $\mathcal{L}(\boldsymbol{\lambda}) = \sum_{i=1}^{M} \mathcal{L}(\lambda_i)$. It is defined as the summation of log-likelihood $\mathcal{L}(\lambda_i)$ of each dimension i, where $\mathcal{L}(\lambda_i)$ is defined as follows (Aalen et al., 2008):

$$\mathcal{L}(\lambda_i(t)) = \int_0^T \log(\lambda_i(t))\mathrm{d}N_i(t) - \int_0^T \lambda_i(t)\mathrm{d}t \tag{29}$$

where the operation $\int f(t)\mathrm{d}N(t)$ is defined as the summation of the value of function f at each event time: $\int f(t)\mathrm{d}N(t) := \sum_i f(t_i)$.

Hence, $\mathcal{D}(\boldsymbol{u})$ denotes the difference of the log-likelihood between these two point processes:

$$\begin{aligned}
&\mathcal{D}(\boldsymbol{u}) = \mathcal{L}(\boldsymbol{\lambda}(t)) - \mathcal{L}(\tilde{\boldsymbol{\lambda}}(\boldsymbol{u}(t), t)) \\
&= \sum_{i=1}^{M} \bigg(\int_0^T \Big(\tilde{\lambda}_i(u_i(s), s) - \lambda_i(s)\Big)\mathrm{d}s - \int_0^T \log\Big(\frac{\tilde{\lambda}_i(u_i(s), s)}{\lambda_i(s)}\Big)\mathrm{d}N_i(s) \bigg) \\
&= \sum_{i=1}^{M} \bigg(\int_0^T \Big(u_i(s)\lambda_i(s) - \lambda_i(s)\Big)\mathrm{d}s - \int_0^T \log\Big(u_i(s)\Big)\mathrm{d}N_i(s) \bigg) \\
&= \sum_{i=1}^{M} \bigg(\int_0^T \Big(u_i(s) - 1\Big)\lambda_i(s)\mathrm{d}s - \int_0^T \log\Big(u_i(s)\Big)\mathrm{d}N_i(s) \bigg)
\end{aligned} \tag{30}$$

where M is the dimension of point process. (30) comes from the form of control in (4). $\lambda_i(t), N_i(t), u_i(t)$ denote the i-th dimension of $\boldsymbol{\lambda}(t), \boldsymbol{N}(t), \boldsymbol{u}(t)$.

□

C. Derivations of the Optimal Control Policy in (14)

We will formulate our objective function based on the form of optimal measure $\mathbb{Q}^*$ in (10). More specifically, we find a control $\boldsymbol{u}$ which pushes the controlled measure $\mathbb{Q}(\boldsymbol{u})$, as close to the optimal measure as possible. This leads to minimizing the Kullback-Leibler (KL) distance:

$$\boldsymbol{u}^* = \operatorname*{argmin}_{\boldsymbol{u}>0} \mathbb{D}_{KL}(\mathbb{Q}^*||\mathbb{Q}(\boldsymbol{u})) \tag{31}$$

This objective function is in sharp contrast to traditional methods that solve the optimal control problem by computing the solution the HJB PDE, which have severe limitations in scalability and feasibility to nonlinear jump diffusion SDEs.

Next we simplify the objective function. According to the definition of KL divergence and chain rule of derivatives, we have:

$$\mathbb{D}_{KL}(\mathbb{Q}^*||\mathbb{Q}(\boldsymbol{u})) = \mathbb{E}_{\mathbb{Q}^*}\left[\log\left(\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{Q}(\boldsymbol{u})}\right)\right] = \mathbb{E}_{\mathbb{Q}^*}\left[\log\left(\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}(\boldsymbol{u})}\right)\right] \tag{32}$$

The derivative $\mathrm{d}\mathbb{Q}^*/\mathrm{d}\mathbb{P}$ is given in (19) and $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}(\boldsymbol{u})$ is given in Theorem 3. Hence, we then substitute $\mathrm{d}\mathbb{Q}^*/\mathrm{d}\mathbb{P}$ and $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}(\boldsymbol{u})$ to (32). After removing terms which are independent of $\boldsymbol{u}$, the objective function (31) is simplified as:

$$\boldsymbol{u}^* = \operatorname*{argmin}_{\boldsymbol{u}>0} \mathbb{E}_{\mathbb{Q}^*}[\mathcal{D}(\boldsymbol{u})]$$

Next we parameterize $\boldsymbol{u}(t)$ as a piecewise constant function on $[0, T]$ as follows.

$$\boldsymbol{u}(t) = \begin{cases} \vdots \\ \boldsymbol{u}^k \quad \text{for } t \in [k\Delta t, (k+1)\Delta t) \\ \vdots \end{cases}$$

More specifically, the k-th piece is defined on $[k\Delta t, (k+1)\Delta t)$ as $\boldsymbol{u}^k$, where $k = 0, \cdots, K-1$, $t_k = k\Delta t$ and $T = t_K$. Then we have:

$$\mathbb{E}_{\mathbb{Q}^*}[\mathcal{D}(\boldsymbol{u})] = \sum_{i=1}^{M}\sum_{k=1}^{K}\left(\mathbb{E}_{\mathbb{Q}^*}\Big[\int_{t_k}^{t_{k+1}}(u_i^k - 1)\lambda_i(s)\mathrm{d}s\Big] - \mathbb{E}_{\mathbb{Q}^*}\Big[\int_{t_k}^{t_{k+1}}\log(u_i^k)\mathrm{d}N_i(s)\Big]\right) \tag{33}$$

where u_i^k is the i-th dimension of $\boldsymbol{u}^k$. To compute u_i^k, we can neglect the two summation terms in (33) and only focus on the parts that involves u_i^k. Then we move u_i^k outside of the expectation and discard any constant terms. This yields the function that only involves u_i^k:

$$f(u_i^k) = u_i^k \mathbb{E}_{\mathbb{Q}^*}\Big[\int_{t_k}^{t_{k+1}}\lambda_i(s)\mathrm{d}s\Big] - \log(u_i^k)\mathbb{E}_{\mathbb{Q}^*}\Big[\int_{t_k}^{t_{k+1}}\mathrm{d}N_i(s)\Big] \tag{34}$$

We can then show $f(u_i^k)$ is convex in u_i^k. More specifically, it is in the form of $f(x) = ax - \log(x)b$ with $a > 0, b > 0$ and $f''(x) > 0$. Finally, setting $f'(u_i^k) = 0$ yields u_i^{k*}:

$$u_i^{k*} = \frac{\mathbb{E}_{\mathbb{Q}^*}\big[\int_{t_k}^{t_{k+1}}\mathrm{d}N_i(s)\big]}{\mathbb{E}_{\mathbb{Q}^*}\big[\int_{t_k}^{t_{k+1}}\lambda_i(s)\mathrm{d}s\big]} \tag{35}$$

However, u_i^{k*} is still not computable since the expectation is taken under the optimal probability measure $\mathbb{Q}^*$. Since we only known the SDE of the uncontrolled dynamics and can only compute the expectation under $\mathbb{P}$, we need to change the expectation from $\mathbb{E}_{\mathbb{Q}^*}$ to $\mathbb{E}_{\mathbb{P}}$ to compute u_i^{k*}.

To do this, we will use the following lemma.

Lemma 4. *Let the probability measure $\mathbb{Q}^*$ be defined as $\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} = \frac{\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))}{\mathbb{E}_{\mathbb{P}}[\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))]}$ in (10), and $g(\boldsymbol{x}) : \Omega \to \Re$ be any measurable function. Then we have:*

$$\mathbb{E}_{\mathbb{Q}^*}[g(\boldsymbol{x})] = \frac{\mathbb{E}_{\mathbb{P}}\Big[\exp\big(-\frac{1}{\gamma}S(\boldsymbol{x})\big)g(\boldsymbol{x})\Big]}{\mathbb{E}_{\mathbb{P}}[\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))]}$$

Proof.

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^*}[g(\boldsymbol{x})] &= \int g(\boldsymbol{x})\mathrm{d}\mathbb{Q}^* \\
&= \int g(\boldsymbol{x})\frac{\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))\mathrm{d}\mathbb{P}}{\mathbb{E}_{\mathbb{P}}[\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))]} \\
&= \frac{\int \Big(g(\boldsymbol{x})\exp\big(-\frac{1}{\gamma}S(\boldsymbol{x})\big)\Big)\mathrm{d}\mathbb{P}}{\mathbb{E}_{\mathbb{P}}[\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))]} \\
&= \frac{\mathbb{E}_{\mathbb{P}}\Big[\exp\big(-\frac{1}{\gamma}S(\boldsymbol{x})\big)g(\boldsymbol{x})\Big]}{\mathbb{E}_{\mathbb{P}}[\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))]}
\end{aligned}$$

$\square$

Finally, applying Lemma 4 to (35) yields the following expression for the optimal policy:

$$u_i^{k*} = \frac{\mathbb{E}_{\mathbb{Q}^*}\big[\int_{t_k}^{t_{k+1}}\mathrm{d}N_i(s)\big]}{\mathbb{E}_{\mathbb{Q}^*}\big[\int_{t_k}^{t_{k+1}}\lambda_i(s)\mathrm{d}s\big]} = \frac{\frac{\mathbb{E}_{\mathbb{P}}\big[\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))\int_{t_k}^{t_{k+1}}\mathrm{d}N_i(s)\big]}{\mathbb{E}_{\mathbb{P}}\big[\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))\big]}}{\frac{\mathbb{E}_{\mathbb{P}}\big[\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))\int_{t_k}^{t_{k+1}}\lambda_i(s)\mathrm{d}s\big]}{\mathbb{E}_{\mathbb{P}}\big[\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))\big]}} = \frac{\mathbb{E}_{\mathbb{P}}\big[\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))\int_{t_k}^{t_{k+1}}\mathrm{d}N_i(s)\big]}{\mathbb{E}_{\mathbb{P}}\big[\exp(-\frac{1}{\gamma}S(\boldsymbol{x}))\int_{t_k}^{t_{k+1}}\lambda_i(s)\mathrm{d}s\big]} \tag{36}$$

The derivation of the optimal policy is now complete.

D. Derivations of the Control Cost

We will derive the control cost in (9), which comes naturally from the dynamics. According to the definition of the KL divergence, we have:

$$\mathbb{D}_{KL}(\mathbb{Q}||\mathbb{P}) := \mathbb{E}_{\mathbb{Q}}[\log(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}})] = \mathbb{E}_{\mathbb{Q}}[C(\boldsymbol{u})] \tag{37}$$

Hence, the next step is to compute the derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$. This derivative means the relative density of probability distribution $\mathbb{Q}$ with respect to $\mathbb{P}$. According to (Brémaud, 1981), we have:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\Big(\sum_i \int_0^T \log\Big(\frac{\tilde{\lambda}_i(u_i(t),t)}{\lambda_i(t)}\Big)\mathrm{d}N_i(u_i(t),t) - \int_0^T (\tilde{\lambda}_i(u_i(t),t) - \lambda_i(t))\mathrm{d}t\Big), \tag{38}$$

Using the relationship that $\lambda_i(u_i(t),t) = \lambda_i(t)u_i(t)$, we have:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\log(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}})] &= \mathbb{E}_{\mathbb{Q}}\Big[\sum_i \int_0^T \log\Big(\frac{\tilde{\lambda}_i(u_i(t),t)}{\lambda_i(t)}\Big)\mathrm{d}\tilde{N}_i(u_i(t),t) - \int_0^T (\tilde{\lambda}_i(u_i(t),t) - \lambda_i(t))\mathrm{d}t\Big] &(39)\\
&= \mathbb{E}_{\mathbb{Q}}\Big[\sum_i \int_0^T \log\Big(u_i(t)\Big)\mathrm{d}\tilde{N}_i(u_i(t),t) - \int_0^T \Big(1 - \frac{1}{u_i(t)}\Big)\tilde{\lambda}_i(u_i(t),t)\mathrm{d}t\Big] &(40)\\
&= \mathbb{E}_{\mathbb{Q}}\Big[\sum_i \int_0^T \log\Big(u_i(t)\Big)\tilde{\lambda}_i(u_i(t),t)\mathrm{d}t + \int_0^T \Big(1 - \frac{1}{u_i(t)}\Big)\tilde{\lambda}_i(u_i(t),t)\mathrm{d}t\Big] &(41)
\end{aligned}$$

Note that (40) to (41) follows from the Campbell theorem (Daley & Vere-Jones, 2007). Therefore, the control cost is:

$$\begin{aligned}
C(\boldsymbol{u}) &= \int_0^T \sum\nolimits_i \Big(\log(u_i(t)) + \frac{1}{u_i(t)} - 1\Big)\tilde{\lambda}_i(u_i(t),t)\mathrm{d}t \\
&= \int_0^T \sum\nolimits_i \Big(\log(u_i(t)) + \frac{1}{u_i(t)} - 1\Big)u_i(t)\lambda_i(t)\mathrm{d}t
\end{aligned}$$