A. Proofs

A.1. Analysis Of The Delayed Permuted Mirror Descent Algorithm

We will use throughout the proofs the well known Pythagorean Theorem for Bregman divergences, and the 'projection' lemma that considers the projection step in the algorithm.

Lemma 1. *Pythagorean Theorem for Bregman divergences*
Let v be the projection of w onto a convex set $\mathcal{W}$ w.r.t Bregman divergence $\triangle_{\psi}$: $v = argmin_{u\in\mathcal{W}}\triangle_{\psi}(u,w)$, then: $\triangle_{\psi}(u,w) \geq \triangle_{\psi}(u,v) + \triangle_{\psi}(v,w)$

Lemma 2. *Projection Lemma*
Let $\mathcal{W}$ be a closed convex set and let v be the projection of w onto $\mathcal{W}$, namely,
$v = \underset{x\in\mathcal{W}}{argmin}\|x-w\|^2$. *Then, for every $u \in \mathcal{W}$, $\|w-u\|^2 - \|v-u\|^2 \geq 0$*

The following lemma gives a bound on the distance between two consequent predictions when using the Euclidean mirror map:

Lemma 3. *Let $g \in \mathbb{R}^n$ s.t. $\|g\|_2 < G$, $\mathcal{W}$ a convex set, and $\eta > 0$ be fixed. Let $w \in \mathcal{W}$ and $w_2 = w - \eta\cdot g$. Then, for $w' = \underset{u\in\mathcal{W}}{argmin}\|w_2 - u\|_2^2$, we have that $\|w - w'\| \leq \eta\cdot G$*

Proof. From the projection lemma: $\|w_2 - w\|_2^2 \geq \|w' - w\|_2^2$ and so: $\|w_2 - w\|_2 \geq \|w' - w\|_2$. From definition: $\|w_2 - w\|_2 = \|\eta\cdot g\|_2 \leq \eta\cdot G$. and so we get: $\|w' - w\|_2 \leq \|w_2 - w\|_2 \leq \eta\cdot G$ □

We prove a modification of Lemma 2 given in (Menache et al., 2014) in order to bound the distance between two consequent predictions when using the negative entropy mirror map:

Lemma 4. *Let $g \in \mathbb{R}^n$ s.t. $\|g\|_1 \leq G$ for some $G > 0$ and let $\eta > 0$ be fixed, with $\eta < \frac{1}{\sqrt{2}\cdot G}$. For any distribution vector w in the $n - simplex$, if we define w' to be the new distribution vector*

$$\forall i \in \{1,...,n\}, w'_i = \frac{w_i\cdot\exp(-\eta\cdot g_i)}{\sum_{j=1}^n w_j\cdot\exp(-\eta\cdot g_j)}$$

Then $\|w - w'\|_1 \leq 3\eta G$

Proof. Since $\|g\|_\infty < G$ and $\eta < \frac{1}{\sqrt{2}\cdot G}$ we get that $\forall i : |\eta\cdot g_i| < 1$. We have that:

$$\|w - w'\|_1 = \sum_{i=1}^n |w_i - w'_i| = \sum_{i=1}^n \left| w_i \cdot \left(1 - \frac{\exp(-\eta\cdot g_i)}{\sum_{j=1}^n w_j\cdot\exp(-\eta\cdot g_j)}\right)\right|$$

Since $\|w\|_1 = 1$, we can apply Holder's inequality, and upper bound the above by

$$\underset{i}{max}\left|1 - \frac{\exp(-\eta\cdot g_i)}{\sum_{j=1}^n w_j\cdot\exp(-\eta\cdot g_j)}\right|$$

Using the inequality $1 - x \leq \exp(-x) \leq \frac{1}{1+x}$ for all $|x| \leq 1$, we know that

$$1 - \eta\cdot g_i \leq \exp(-\eta\cdot g_i) \leq \frac{1}{1+\eta\cdot g_i}$$

and since $-\eta G \leq \eta\cdot g_i \leq \eta G$ we have that

$$1 - \eta\cdot g_i \leq \exp(-\eta\cdot g_i) \leq \frac{1}{1+\eta\cdot g_i} \leq \frac{1}{1-\eta G}$$

and so we get:

$$1 - \frac{\frac{1}{1+\eta g_i}}{1+\eta G} \leq 1 - \frac{\exp(-\eta\cdot g_i)}{\sum_{j=1}^n w_j\cdot\exp(-\eta\cdot g_j)} \leq 1 - \frac{1-\eta\cdot g_i}{\frac{1}{1-\eta G}}$$

Using again the fact that $-\eta G \leq \eta \cdot g_i \leq \eta G$, we have

$$1-\frac{\frac{1}{1-\eta G}}{1+\eta G} \leq 1-\frac{\exp\left(-\eta \cdot g_i\right)}{\sum_{j=1}^{n} w_j \cdot \exp\left(-\eta \cdot g_j\right)} \leq 1-\frac{1-\eta G}{\frac{1}{1-\eta G}}$$

$$\implies \frac{-\eta^2 G^2}{1-\eta^2 G^2} = 1-\frac{1}{1-\eta^2 G^2} \leq 1-\frac{\exp\left(-\eta \cdot g_i\right)}{\sum_{j=1}^{n} w_j \cdot \exp\left(-\eta \cdot g_j\right)} \leq 1-\left(1-\eta G\right)^2 = 2\eta G+\eta^2 G^2$$

Now, since $\eta G < 1$, we get that:

$$\frac{-\eta^2 G^2}{1-\eta^2 G^2} \leq 1-\frac{\exp\left(-\eta \cdot g_i\right)}{\sum_{j=1}^{n} w_j \cdot \exp\left(-\eta \cdot g_j\right)} \leq 2\eta G+\eta G = 3\eta G$$

and so we can conclude that

$$\underset{i}{max}\left|1-\frac{\exp\left(-\eta \cdot g_i\right)}{\sum_{j=1}^{n} w_j \cdot \exp\left(-\eta \cdot g_j\right)}\right| \leq \underset{i}{max}\left(\left|\frac{-\eta^2 G^2}{1-\eta^2 G^2}\right|, \left|3\eta G\right|\right) \leq \underset{i}{max}\left(\frac{\eta G}{1-\eta^2 G^2}, 3\eta G\right)$$

Since $\eta < \frac{1}{\sqrt{2}G}$, we get $\left(\eta \cdot G\right)^2 < \frac{1}{2}$. Thus we get:

$$\underset{i}{max}\left|1-\frac{\exp\left(-\eta \cdot g_i\right)}{\sum_{j=1}^{n} w_j \cdot \exp\left(-\eta \cdot g_j\right)}\right| \leq \underset{i}{max}\left(2\eta G, 3\eta G\right) \leq 3\eta G$$

which gives us our desired bound. □

With the above two lemmas in hand, we bound the distance between consequent predictors by $c\eta G$, where c is a different constant in each mirror map: $c = 1$ for the euclidean case, and $c = 3$ for the negative entropy mirror map.
Note that both mapping are 1-strongly convex with respect to their respective norms. For other mappings with a different strong convexity constant, one would need to scale the step sizes according to the strong convexity parameter in order to get the bound.

A.1.1. PROOF OF THEOREM 1

We provide an upper bound on the regret of the algorithm, by competing against the best fixed action in each one of the sets of iterations- the first τ iterations and the last $M-\tau$ iterations in each block. This is an upper bound on competing against the best fixed predictor in hindsight for the entire sequence. Formally, we bound:

$$\begin{aligned}R(T) &= \mathbb{E}\left[\sum_{t=1}^{T} f_t\left(w_t\right)-\sum_{t=1}^{T} f_t\left(w^*\right)\right]\\ &\leq \mathbb{E}\left[\sum_{i=0}^{\frac{T}{M}-1}\left(\sum_{t=M\cdot i+1}^{M\cdot i+\tau} f_t\left(w_t\right)-f_t\left(w_f^*\right)+\sum_{t=M\cdot i+\tau+1}^{M\cdot(i+1)} f_t\left(w_t\right)-f_t\left(w_s^*\right)\right)\right]\end{aligned}$$

where

$$w_f^* = \underset{w\in\mathcal{W}}{\text{argmin}} \sum_{i=0}^{\frac{T}{M}-1}\sum_{t=M\cdot i+1}^{M\cdot i+\tau} f_t\left(w\right) \text{ and } w_s^* = \underset{w\in\mathcal{W}}{\text{argmin}} \sum_{i=0}^{\frac{T}{M}-1}\sum_{t=M\cdot i+\tau+1}^{M\cdot(i+1)} f_t\left(w\right)$$

where expectation is taken over the randomness of the algorithm.

The diameter of the domain $\mathcal{W}$ is bounded by B^2, and so $\triangle_\psi\left(w_f^*, w_0^f\right) \leq B^2$ and $\triangle_\psi\left(w_s^*, w_0^s\right) \leq B^2$. We start with a general derivation that will apply both for w^s and for w^f simultaneously. For the following derivation we use the notation w_j, w_{j+1} omitting the f, s superscript, for denoting subsequent updates of the predictor vector, whether it is w^s or w^f.

Denote by g_j the gradient used to update w_j, i.e., $\nabla\psi\left(w_{j+\frac{1}{2}}\right) = \nabla\psi\left(w_j\right) - \eta\cdot g_j$, and $w_{j+1} = \underset{w\in\mathcal{W}}{argmin}\triangle_\psi\left(w, w_{j+\frac{1}{2}}\right)$.
Looking at the update step in the algorithm, we have that $g_j = \frac{1}{\eta}\cdot\left(\nabla\psi\left(w_j\right) - \nabla\psi\left(w_{j+\frac{1}{2}}\right)\right)$ and thus:

$$\begin{aligned}\left\langle w_j - w^*, g_j\right\rangle &= \frac{1}{\eta}\cdot\left\langle w_j - w^*, \left(\nabla\psi\left(w_j\right) - \nabla\psi\left(w_{j+\frac{1}{2}}\right)\right)\right\rangle \\ &= \frac{1}{\eta}\cdot\left(\triangle_\psi\left(w^*, w_j\right) + \triangle_\psi\left(w_j, w_{j+\frac{1}{2}}\right) - \triangle_\psi\left(w^*, w_{j+\frac{1}{2}}\right)\right)\end{aligned}$$

We now use the Pythagorean Theorem to get:

$$\leq \frac{1}{\eta}\cdot\left(\triangle_\psi\left(w^*, w_j\right) + \triangle_\psi\left(w_j, w_{j+\frac{1}{2}}\right) - \triangle_\psi\left(w^*, w_{j+1}\right) - \triangle_\psi\left(w_{j+1}, w_{j+\frac{1}{2}}\right)\right)$$

When we sum terms for all updates of the predictor, w^f or w^s respectively, the terms $\triangle_\psi\left(w^*, w_j\right) - \triangle_\psi\left(w^*, w_{j+1}\right)$ will result in a telescopic sum, canceling all terms expect the first and last. Thus we now concentrate on bounding the term: $\triangle_\psi\left(w_j, w_{j+\frac{1}{2}}\right) - \triangle_\psi\left(w_{j+1}, w_{j+\frac{1}{2}}\right)$.

$$\begin{aligned}&\triangle_\psi\left(w_j, w_{j+\frac{1}{2}}\right) - \triangle_\psi\left(w_{j+1}, w_{j+\frac{1}{2}}\right) = \psi\left(w_j\right) - \psi\left(w_{j+1}\right) - \left\langle w_j - w_{j+1}, \nabla\psi\left(w_{j+\frac{1}{2}}\right)\right\rangle \\ &\underset{\psi\text{ 1-strong convex}}{\leq} \left\langle w_j - w_{j+1}, \nabla\psi\left(w_j\right) - \nabla\psi\left(w_{j+\frac{1}{2}}\right)\right\rangle - \frac{1}{2}\cdot\left\|w_j - w_{j+1}\right\|^2 \\ &= \left\langle w_j - w_{j+1}, \eta\cdot g_j\right\rangle - \frac{1}{2}\cdot\left\|w_j - w_{j+1}\right\|^2 \\ &\leq \eta\cdot G\cdot\left\|w_j - w_{j+1}\right\| - \frac{1}{2}\cdot\left\|w_j - w_{j+1}\right\|^2 \\ &\leq \frac{\left(\eta\cdot G\right)^2}{2}\end{aligned}$$

where the last inequality stems from the fact that $\left(\left\|w_j - w_{j+1}\right\|\cdot\frac{\sqrt{1}}{\sqrt{2}} - \frac{\eta\cdot G}{\sqrt{2}}\right)^2 \geq 0$

We now continue with the analysis referring to w^f and w^s separately. Summing over $j = \tau+1$ to $\left(\frac{T}{M}+1\right)\cdot\tau$ for w^f (these are the $\frac{T}{M}\tau$ iterations in which the first sub-algorithm is in use), and from $j = 1$ to $\frac{T}{M}\cdot\left(M-\tau\right)$ for w^s (these are the $\frac{T}{M}\left(M-\tau\right)$ iterations in which the second sub-algorithm is in use) we get:

For w^f:

$$\begin{aligned}&\sum_{j=\tau+1}^{\left(\frac{T}{M}+1\right)\cdot\tau}\left\langle w_j^f - w_f^*, g_j\right\rangle \\ &= \sum_{j=\tau+1}^{\left(\frac{T}{M}+1\right)\cdot\tau}\left\langle w_j^f - w_f^*, \nabla f_{T_1(j-\tau)}\left(w_{j-\tau}^f\right)\right\rangle \\ &= \sum_{j=\tau+1}^{\left(\frac{T}{M}+1\right)\cdot\tau}\frac{1}{\eta}\cdot\left\langle w_j^f - w_f^*, \left(\nabla\psi\left(w_j^f\right) - \nabla\psi\left(w_{j+\frac{1}{2}}^s\right)\right)\right\rangle \\ &= \sum_{j=\tau+1}^{\left(\frac{T}{M}+1\right)\cdot\tau}\frac{1}{\eta}\cdot\left(\triangle_\psi\left(w_f^*, w_j^f\right) + \triangle_\psi\left(w_j^f, w_{j+\frac{1}{2}}^f\right) - \triangle_\psi\left(w_f^*, w_{j+\frac{1}{2}}^f\right)\right) \\ &\leq \sum_{j=\tau+1}^{\left(\frac{T}{M}+1\right)\cdot\tau}\frac{1}{\eta}\cdot\left(\triangle_\psi\left(w_f^*, w_j^f\right) + \triangle_\psi\left(w_j^f, w_{j+\frac{1}{2}}^f\right) - \triangle_\psi\left(w_f^*, w_{j+1}^f\right) - \triangle_\psi\left(w_{j+1}^f, w_{j+\frac{1}{2}}^f\right)\right) \\ &\leq \frac{1}{\eta}\cdot\sum_{j=\tau+1}^{\left(\frac{T}{M}+1\right)\cdot\tau}\triangle_\psi\left(w_f^*, w_j^f\right) - \triangle_\psi\left(w_f^*, w_{j+1}^f\right) + \frac{1}{\eta}\cdot\sum_{j=\tau+1}^{\left(\frac{T}{M}+1\right)\cdot\tau}\triangle_\psi\left(w_j^f, w_{j+\frac{1}{2}}^f\right) - \triangle_\psi\left(w_{j+1}^f, w_{j+\frac{1}{2}}^f\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\eta} \cdot \triangle_\psi \left(w_f^*, w_{\tau+1}^f \right) - \triangle_\psi \left(w_f^*, w^f_{\left(\frac{T}{M}+1\right)\cdot\tau} \right) + \frac{1}{\eta} \cdot \sum_{j=\tau+1}^{\left(\frac{T}{M}+1\right)\cdot\tau} \triangle_\psi \left(w_j^f, w_{j+\frac{1}{2}}^f \right) - \triangle_\psi \left(w_{j+1}^f, w_{j+\frac{1}{2}}^f \right) \\
&\le \frac{1}{\eta_f} \cdot \triangle_\psi \left(w_f^*, w_{\tau+1}^f \right) + \frac{1}{\eta_f} \cdot \frac{T}{M} \cdot \tau \cdot \frac{\left(\eta_f \cdot G\right)^2}{2} \\
&\le \frac{1}{\eta_f} \cdot B^2 + \frac{T}{M} \cdot \tau \cdot \frac{\eta_f \cdot G^2}{2}
\end{aligned}
$$

For w^s:

$$
\begin{aligned}
&\sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)} \left\langle w_j^s - w_s^*, g_j \right\rangle \\
&= \sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)} \left\langle w_j^s - w_s^*, \nabla f_{T_2(j)}\left(w_j^s\right) \right\rangle \\
&= \sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)} \frac{1}{\eta} \cdot \left\langle w_j^s - w_s^*, \left(\nabla\psi\left(w_j^s\right) - \nabla\psi\left(w_{j+\frac{1}{2}}^s\right) \right) \right\rangle \\
&= \sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)} \frac{1}{\eta} \cdot \left(\triangle_\psi\left(w_s^*, w_j^s\right) + \triangle_\psi\left(w_j^s, w_{j+\frac{1}{2}}^s\right) - \triangle_\psi\left(w_s^*, w_{j+\frac{1}{2}}^s\right) \right) \\
&\le \sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)} \frac{1}{\eta} \cdot \left(\triangle_\psi\left(w_s^*, w_j^s\right) + \triangle_\psi\left(w_j^s, w_{j+\frac{1}{2}}^s\right) - \triangle_\psi\left(w_s^*, w_{j+1}^s\right) - \triangle_\psi\left(w_{j+1}^s, w_{j+\frac{1}{2}}^s\right) \right) \\
&\le \frac{1}{\eta} \cdot \sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)} \triangle_\psi\left(w_s^*, w_j^s\right) - \triangle_\psi\left(w_s^*, w_{j+1}^s\right) + \frac{1}{\eta} \cdot \triangle_\psi\left(w_j^s, w_{j+\frac{1}{2}}^s\right) - \triangle_\psi\left(w_{j+1}^s, w_{j+\frac{1}{2}}^s\right) \\
&= \frac{1}{\eta} \cdot \triangle_\psi\left(w_s^*, w_1^s\right) - \triangle_\psi\left(w_s^*, w^s_{\left(\frac{T}{M}+1\right)\cdot\tau}\right) + \frac{1}{\eta} \cdot \sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)} \triangle_\psi\left(w_j^s, w_{j+\frac{1}{2}}^s\right) - \triangle_\psi\left(w_{j+1}^s, w_{j+\frac{1}{2}}^s\right) \\
&\le \frac{1}{\eta_s} \cdot \triangle_\psi\left(w_s^*, w_1^s\right) + \frac{1}{\eta_s} \cdot \frac{T}{M} \cdot (M-\tau) \cdot \frac{\left(\eta_s \cdot G\right)^2}{2} \\
&\le \frac{1}{\eta_s} \cdot B^2 + \frac{T}{M} \cdot (M-\tau) \cdot \frac{\eta_s \cdot G^2}{2}
\end{aligned}
$$

We are after bounding the regret, which in itself is upper bounded by the sum of the regret accumulated by each sub-algorithm, considering iterations in the first τ and last $M-\tau$ per block separately, as mentioned above. Using the convexity of f_t for all t, we bound these terms:

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=0}^{\frac{T}{M}-1} \sum_{t=M\cdot i+1}^{M\cdot i+\tau} f_t\left(w_t\right) - f_t\left(w_f^*\right) + \sum_{i=0}^{\frac{T}{M}-1} \sum_{t=M\cdot i+\tau+1}^{M\cdot(i+1)} f_t\left(w_t\right) - f_t\left(w_s^*\right) \right] \\
&\le \mathbb{E}\left[\sum_{i=0}^{\frac{T}{M}-1} \sum_{t=M\cdot i+1}^{M\cdot i+\tau} \left\langle w_t - w_f^*, \nabla f_t\left(w_t\right) \right\rangle + \sum_{i=0}^{\frac{T}{M}-1} \sum_{t=M\cdot i+\tau+1}^{M\cdot(i+1)} \left\langle w_t - w_s^*, \nabla f_t\left(w_t\right) \right\rangle \right] \\
&= \mathbb{E}\left[\sum_{j=1}^{\frac{T}{M}\cdot\tau} \left\langle w_j^f - w_f^*, \nabla f_{T_1(j)}\left(w_j^f\right) \right\rangle + \sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)} \left\langle w_j^s - w_s^*, \nabla f_{T_2(j)+\tau}\left(w_j^s\right) \right\rangle \right]
\end{aligned}
$$

In the last equality of the above derivation, we simply replace notations, writing the gradient $\nabla f_t\left(w_t\right)$ in notation of T_1

and T_2. T_1 contains all time points in the first τ iterations of each block, and T_2 contains all time points in the first $M-\tau$ iterations of each block.

Note that what we have bounded so far is $\sum_{j=\tau+1}^{(\frac{T}{M}+1)\cdot\tau}\langle w_j^f - w_f^*, \nabla f_{T_1(j-\tau)}(w_{j-\tau}^f)\rangle$ for w^f and $\sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)}\left\langle w_j^s - w_s^*, \nabla f_{T_2(j)}\left(w_j^s\right)\right\rangle$ for w^s, which are not the terms we need to bound in order to get a regret bound since they use the *delayed* gradient, and so we need to take a few more steps in order to be able to bound the regret.

We begin with w^f:

$$
\begin{aligned}
\sum_{i=0}^{\frac{T}{M}-1}\sum_{t=M\cdot i+1}^{M\cdot i+\tau}\left\langle w_t - w_f^*, \nabla f_t\left(w_t\right)\right\rangle &= \sum_{j=1}^{\frac{T}{M}\cdot\tau}\left\langle w_j^f - w_f^*, \nabla f_{T_1(j)}\left(w_j^f\right)\right\rangle \\
&= \sum_{j=1}^{\frac{T}{M}\cdot\tau}\left\langle w_{j+\tau}^f - w_f^*, \nabla f_{T_1(j)}\left(w_j^f\right)\right\rangle + \left\langle w_j^f - w_{j+\tau}^f, \nabla f_{T_1(j)}\left(w_j^f\right)\right\rangle \\
&= \sum_{j=\tau+1}^{(\frac{T}{M}+1)\cdot\tau}\left\langle w_j^f - w_f^*, \nabla f_{T_1(j-\tau)}\left(w_{j-\tau}^f\right)\right\rangle + \left\langle w_{j-\tau}^f - w_j^f, \nabla f_{T_1(j-\tau)}\left(w_{j-\tau}^f\right)\right\rangle \\
&\leq \frac{1}{\eta_f}\cdot B^2 + \frac{T}{M}\cdot\tau\cdot\frac{\eta_f\cdot G^2}{2} + \sum_{j=\tau+1}^{(\frac{T}{M}+1)\cdot\tau}\left\langle w_{j-\tau}^f - w_j^f, \nabla f_{T_1(j-\tau)}\left(w_{j-\tau}^f\right)\right\rangle \\
&\leq \frac{1}{\eta_f}\cdot B^2 + \frac{T}{M}\cdot\tau\cdot\frac{\eta_f\cdot G^2}{2} + \sum_{j=\tau+1}^{(\frac{T}{M}+1)\cdot\tau}\|w_{j-\tau}^f - w_j^f\|\cdot\|\nabla f_{T_1(j-\tau)}\left(w_{j-\tau}^f\right)\| \\
&\leq \frac{1}{\eta_f}\cdot B^2 + \frac{T}{M}\cdot\tau\cdot\frac{\eta_f\cdot G^2}{2} + \sum_{j=\tau+1}^{(\frac{T}{M}+1)\cdot\tau}\sum_{i=1}^{\tau}\|w_{j-i}^f - w_{j-i+1}^f\|\cdot G
\end{aligned}
$$

The last term in the above derivation, is the sum of differences between consecutive predictors. This difference, is determined by the mirror map in use, the step size η_f, and the bound over the norm of the gradient used in the update stage of the algorithm, G. This is because every consecutive predictor is received by taking a gradient step from the previous predictor, in the dual space, with a step size η_f, and projecting back to the primal space by use of the bregman divergence with the specific mirror map in use. We denote the bound on this difference by $\Psi_{(\eta_f,G)}$, i.e., $\forall j, j+1 : \|w_j^f - w_{j+1}^f\| \leq \Psi_{(\eta_f,G)}$. Continuing our derivation, we have:

$$
\begin{aligned}
&\leq \frac{1}{\eta_f}\cdot B^2 + \frac{T}{M}\cdot\tau\cdot\frac{\eta_f\cdot G^2}{2} + \sum_{j=\tau+1}^{(\frac{T}{M}+1)\cdot\tau}\sum_{i=1}^{\tau}\Psi_{(\eta_f,G)}\cdot G \\
&\leq \frac{1}{\eta_f}\cdot B^2 + \frac{T}{M}\cdot\tau\cdot\frac{\eta_f\cdot G^2}{2} + \frac{T}{M}\cdot\tau^2\cdot\Psi_{(\eta_f,G)}\cdot G
\end{aligned}
$$

Since this upper bound does not depend on the permutation,and holds for every sequence, it holds also in expectation, i.e.

$$
\mathbb{E}\left[\sum_{i=0}^{\frac{T}{M}-1}\sum_{t=M\cdot i+1}^{M\cdot i+\tau} f_t\left(w_t\right) - f_t\left(w_f^*\right)\right] \leq \frac{1}{\eta_f}\cdot B^2 + \frac{T}{M}\cdot\tau\cdot\frac{\eta_f\cdot G^2}{2} + \frac{T}{M}\cdot\tau^2\cdot\Psi_{(\eta_f,G)}\cdot G
$$

We now turn to w^s

$$
\begin{aligned}
&\sum_{i=0}^{\frac{T}{M}-1}\sum_{t=M\cdot i+\tau+1}^{M\cdot(i+1)} f_t\left(w_t\right) - f_t\left(w_s^*\right) \\
&\leq \sum_{i=0}^{\frac{T}{M}-1}\sum_{t=M\cdot i+\tau+1}^{M\cdot(i+1)}\left\langle w_t - w_s^*, \nabla f_t\left(w_t\right)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)} \left\langle w_j^s - w_s^*, \nabla f_{T_2(j)+\tau}\left(w_j^s\right)\right\rangle \\
&= \sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)} \left\langle w_j^s - w_s^*, \nabla f_{T_2(j)}\left(w_j^s\right)\right\rangle + \sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)} \left\langle w_j^s - w_s^*, \nabla f_{T_2(j)+\tau}\left(w_j^s\right) - \nabla f_{T_2(j)}\left(w_j^s\right)\right\rangle \\
&\leq \frac{1}{\eta_s}\cdot B^2 + \frac{T}{M}\cdot(M-\tau)\cdot\frac{\eta_s\cdot G^2}{2} + \sum_{j=1}^{\frac{T}{M}\cdot(M-\tau)} \left\langle w_j^s - w_s^*, \nabla f_{T_2(j)+\tau}\left(w_j^s\right) - \nabla f_{T_2(j)}\left(w_j^s\right)\right\rangle
\end{aligned}
$$

We now look at the expression $\left\langle w_j^s - w_s^*, \nabla f_{T_2(j)+\tau}\left(w_j^s\right) - \nabla f_{T_2(j)}\left(w_j^s\right)\right\rangle$ for any j.

We first notice that for any j, w_j^s only depends on gradients of time points: $T_2(1), T_2(2), ..., T_2(j-1)$.
We also notice that given the functions received at these time points, i.e, given $f_{T_2(1)}, f_{T_2(2)}, ..., f_{T_2(j-1)}$, w_j^s is no longer a random variable.
We have that for all j, $T_2(j)$ and $T_2(j)+\tau$ are both time points that are part of the same M-sized block. Suppose we have observed n functions of the block to which $T_2(j)$ and $T_2(j)+\tau$ belong. All of these n functions are further in the past than both $T_2(j)$ and $T_2(j)+\tau$, because of the delay of size τ. We have $M-n$ functions in the block that have not been observed yet, and since we performed a random permutation within each block, all remaining functions in the block have the same expected value. Formally, given w_j^s, the expected value of the current and delayed gradient are the same, since we have: $\mathbb{E}[\nabla f_{T_2(j)+\tau}\left(w_j^s\right)|w_j^s] = \frac{1}{M-n}\cdot\sum_{i=1}^{M-n}\nabla f_{T_2(j)+i}\left(w_j^s\right) = \mathbb{E}[\nabla f_{T_2(j)}\left(w_j^s\right)|w_j^s]$. As mentioned above, this stems from the random permutation we performed within the block - all $M-n$ remaining functions (that were not observed yet in this block) have an equal (uniform) probability of being in each location, and thus the expected value of the gradients is equal. From the law of total expectation we have that

$$
\mathbb{E}[\nabla f_{T_2(j)+\tau}\left(w_j^s\right)] = \mathbb{E}[\mathbb{E}[\nabla f_{T_2(j)+\tau}\left(w_j^s\right)|w_j^s]] = \mathbb{E}[\mathbb{E}[\nabla f_{T_2(j)}\left(w_j^s\right)|w_j^s]] = \mathbb{E}[\nabla f_{T_2(j)}\left(w_j^s\right)]
$$

ans thus $\mathbb{E}[\nabla f_{T_2(j)+\tau}\left(w_j^s\right) - \nabla f_{T_2(j)}\left(w_j^s\right)] = 0$.

We get that $\mathbb{E}[\left\langle w_j^s - w_s^*, \nabla f_{T_2(j)+\tau}\left(w_j^s\right) - \nabla f_{T_2(j)}\left(w_j^s\right)\right\rangle] = 0$

So we have that the upper bound on the expected regret of the time point in which we predict with w^s is:

$$
\mathbb{E}\left[\sum_{i=0}^{\frac{T}{M}-1}\sum_{t=M\cdot i+\tau+1}^{M\cdot(i+1)} f_t(w_t) - f_t(w_s^*)\right] \leq \frac{1}{\eta_s}\cdot B^2 + \frac{T}{M}\cdot(M-\tau)\cdot\frac{\eta_s\cdot G^2}{2}
$$

Summing up the regret of the two sub-algorithms, we get:

$$
\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^{T} f_t(w_t) - f_t(w^*)\right] &\leq \mathbb{E}\left[\sum_{i=0}^{\frac{T}{M}-1}\sum_{t=M\cdot i+1}^{M\cdot i+\tau} f_t(w_t) - f_t\left(w_f^*\right) + \sum_{i=0}^{\frac{T}{M}-1}\sum_{t=M\cdot i+\tau+1}^{M\cdot(i+1)} f_t(w_t) - f_t(w_s^*)\right] \\
&\leq \frac{B^2}{\eta_f} + \eta_f\cdot\frac{T\tau}{M}\cdot\frac{G^2}{2} + \frac{T\tau^2}{M}\cdot G\cdot\Psi_{(\eta_f,G)} + \frac{B^2}{\eta_s} + \eta_s\cdot\frac{T\cdot(M-\tau)}{M}\cdot\frac{G^2}{2}
\end{aligned}
$$

which gives us the bound.
For $\Psi_{(\eta_f,G)} \leq c\cdot\eta_f\cdot G$ where c is some constant, choosing the step sizes, η_f, η_s optimally:

$$
\eta_f = \frac{B\cdot\sqrt{M}}{G\cdot\sqrt{T\cdot\tau\cdot\left(\frac{1}{2}+c\cdot\tau\right)}}, \eta_s = \frac{B\cdot\sqrt{2M}}{G\cdot\sqrt{T\cdot(M-\tau)}}
$$

we get the bound:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=1}^{T} f_t\left(w_t\right)-f_t\left(w^*\right)\right] \\
&=\sqrt{\frac{T\cdot\tau}{M}}\cdot B\cdot G\cdot\sqrt{\frac{1}{2}+c\cdot\tau}+\sqrt{\frac{T\cdot\tau}{M}}\cdot B\cdot G\cdot\frac{1}{\sqrt{\frac{1}{2}+c\tau}}+\sqrt{\frac{T\cdot\tau}{M}}\cdot B\cdot G\cdot\frac{c\tau}{\sqrt{\frac{1}{2}+c\tau}} \\
&+\sqrt{\frac{2\cdot T\cdot(M-\tau)}{M}}\cdot B\cdot G \\
&\leq c\cdot\sqrt{\frac{T\cdot\tau}{M}}\cdot B\cdot G\cdot\sqrt{\frac{1}{2}+c\cdot\tau}+\sqrt{\frac{2\cdot T\cdot(M-\tau)}{M}}\cdot B\cdot G \\
&=\mathcal{O}\left(\sqrt{\frac{T\cdot\tau^2}{M}}+\sqrt{\frac{T\cdot(M-\tau)}{M}}\right)=\mathcal{O}\left(\sqrt{T}\cdot\left(\sqrt{\frac{\tau^2}{M}}+1\right)\right)
\end{aligned}$$

A.2. Lower Bound For Algorithms With No Permutation Power

Theorem 3. *For every (possible randomized) algorithm A, there exists a choice of linear, 1-Lipschitz functions over $[-1,1]\subset\mathbb{R}$, with τ a fixed size delay of feedback, such that the expected regret of A after T rounds (with respect to the algorithm's randomness), is*

$$\mathbb{E}\left[R_A\left(T\right)\right]=\mathbb{E}\left[\sum_{t=1}^{T} f_t\left(w_t\right)-\sum_{t=1}^{T} f_t\left(w^*\right)\right]=\Omega\left(\sqrt{\tau T}\right), \textit{ where } w^*=\underset{w\in\mathcal{W}}{\textit{argmin}}\sum_{t=1}^{T} f_t\left(w\right)$$

Proof. First, we note that in order to show that for every algorithm, there exists a choice of loss functions by an oblivious adversary, such that the expected regret of the algorithm is bounded from below, it is enough to show that there exists a distribution over loss function sequences such that for any algorithm, the expected regret is bounded from below, where now expectation is taken over both the randomness of the algorithm and the randomness of the adversary. This is because if there exists such a distribution over loss function sequences, then for any algorithm, there exists some sequence of loss functions that can lead to a regret at least as high. To put it formally, if we mark $\underset{alg}{\mathbb{E}}$ the expectation over the randomness of the algorithm, and $\underset{f_1,\ldots,f_T}{\mathbb{E}}$ the expectation over the randomness of the adversary, then:

$$\exists \text{ a (randomized) adversary s.t. } \forall \text{ algorithm A, } \underset{f_1,\ldots,f_T}{\mathbb{E}}\ \underset{alg}{\mathbb{E}}\left[R_A\left(T\right)\right]>\Omega\left(\sqrt{\tau T}\right)\rightarrow$$
$$\forall \text{ algorithm A, } \exists f_1,\ldots,f_T \text{ s.t. } \underset{alg}{\mathbb{E}}\left[R_A\left(T\right)\right]>\Omega\left(\sqrt{\tau T}\right)$$

Thus, we prove the first statement above, that immediately gives us the second statement which gives the lower bound.

We consider the setting where $\mathcal{W}=[-1,1]$, and $\forall t\in[1,T]: f_t\left(w_t\right)=\alpha_t\cdot w_t$ where $\alpha_t\in\{1,-1\}$. We divide the T rounds to blocks of size τ. α_t is chosen in the following way: if α_t is the first α in the block, it is randomly picked, i.e, $\Pr\left(\alpha=\pm 1\right)=\frac{1}{2}$. Following this random selection, the next $\tau-1$ α's of the block will be identical to the first α in it, so that we now have a block of τ consecutive functions in which α is identical. We wish to lower bound the expected regret of any algorithm in this setting.

Consider a sequence of predictions by the algorithm $w_1,w_2,\ldots,w_T$. Denote by $\alpha_{i,j}$ the j'th α in the i'th block, and similarly for $w_{i,j},f_{i,j}$. We denote the entire sequence of α's by $\bar{\alpha}_{(1\to T)}$, and the sequence of α's until time point j in block i by $\bar{\alpha}_{(1\to i,j)}$. Notice that $w_{i,j}$ is a function of the α's that arrive up until time point $i\cdot\tau+j-\tau-1$. We denote these α's as $\bar{\alpha}_{(1\to i,j-\tau-1)}$.

Then the expected sum of losses is:

$$\mathbb{E}\left[\sum_{t=1}^{T} f_t\left(w_t\right)\right]=\mathbb{E}\left[\sum_{i=1}^{\frac{T}{\tau}}\sum_{j=1}^{\tau} f_{i,j}\left(w_{i,j}\right)\right]$$

$$= \sum_{i=1}^{\frac{T}{\tau}} \sum_{j=1}^{\tau} \mathbb{E}\left[f_{i,j}\left(w_{i,j}\right)\right]$$

$$= \sum_{i=1}^{\frac{T}{\tau}} \sum_{j=1}^{\tau} \mathbb{E}_{\bar{\alpha}_{(1\to T)}}\left[\alpha_{i,j} \cdot w_{i,j}\right]$$

$$= \sum_{i=1}^{\frac{T}{\tau}} \sum_{j=1}^{\tau} \mathbb{E}_{\bar{\alpha}_{(1\to i,j-\tau-1)}}\left[\mathbb{E}_{\bar{\alpha}_{(i,j-\tau\to T)}}\left[\alpha_{i,j} \cdot w_{i,j} | \bar{\alpha}_{(1\to i,j-\tau-1)}\right]\right]$$

$$= \sum_{i=1}^{\frac{T}{\tau}} \sum_{j=1}^{\tau} \mathbb{E}_{\bar{\alpha}_{(1\to i,j-\tau-1)}}\left[w_{i,j} \cdot \mathbb{E}_{\bar{\alpha}_{(i,j-\tau\to T)}}\left[\alpha_{i,j} | \bar{\alpha}_{(1\to i,j-\tau-1)}\right]\right]$$

$$= \sum_{i=1}^{\frac{T}{\tau}} \sum_{j=1}^{\tau} \mathbb{E}_{\bar{\alpha}_{(1\to i,j-\tau-1)}}\left[w_{i,j} \cdot \mathbb{E}_{\bar{\alpha}_{(i,1\to i,j)}}\left[\alpha_{i,j} | \bar{\alpha}_{(1\to i,j-\tau-1)}\right]\right]$$

$$= \sum_{i=1}^{\frac{T}{\tau}} \sum_{j=1}^{\tau} \mathbb{E}_{\bar{\alpha}_{(1\to i,j-\tau-1)}}\left[w_{i,j} \cdot \mathbb{E}_{\alpha_{i,1}}\left[\alpha_{i,1}\right]\right]$$

$$= \sum_{i=1}^{\frac{T}{\tau}} \sum_{j=1}^{\tau} \mathbb{E}_{\bar{\alpha}_{(1\to i,j-\tau-1)}}\left[w_{i,j} \cdot \left(\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1)\right)\right] = 0$$

The last equality is true because every first α in any block has probability $\frac{1}{2}$ to be either $+1$ or -1.

We now continue to the expected sum of losses for the optimal choice of $w^* = argmin_{w\in\mathcal{W}}\left(\sum_{t=1}^{T} f_t(w)\right)$. Note that in this setting, $w^* \in \{+1, -1\}$ and is with opposite sign to the majority of α's in the sequence.

$$\mathbb{E}\left[\sum_{t=1}^{T} f_t(w^*)\right] = \mathbb{E}\left[\sum_{i=1}^{\frac{T}{\tau}} \sum_{j=1}^{\tau} f_{i,j}(w^*)\right] = \mathbb{E}\left[\sum_{i=1}^{\frac{T}{\tau}} \sum_{j=1}^{\tau} \alpha_{i,j} \cdot w^*\right]$$
$$= \mathbb{E}\left[\sum_{i=1}^{\frac{T}{\tau}} \tau \cdot \alpha_{i,1} \cdot w^*\right] = \tau \cdot \mathbb{E}\left[\sum_{i=1}^{\frac{T}{\tau}} \alpha_{i,1} \cdot w^*\right]$$
$$= -\tau \cdot \mathbb{E}\left[|\sum_{i=1}^{\frac{T}{\tau}} \alpha_{i,1}|\right]$$

Using Khintchine inequality we have that:

$$-\tau \cdot \mathbb{E}\left[|\sum_{i=1}^{\frac{T}{\tau}} \alpha_{i,1} \cdot 1|\right] \le -\tau \cdot C \cdot \sqrt{\left(\sum_{i=1}^{\frac{T}{\tau}} 1^2\right)} = -\tau \cdot C \cdot \sqrt{\frac{T}{\tau}} = -\Omega\left(\sqrt{\tau \cdot T}\right)$$

where C is some constant.

Thus we get that for a sequence of length T the expected regret is:

$$\mathbb{E}\left[\sum_{t=1}^{T} f_t(w_t)\right] - \mathbb{E}\left[\sum_{t=1}^{T} f_t(w^*)\right] = \Omega\left(\sqrt{\tau \cdot T}\right)$$

$\square$

A.3. Proof of Theorem 2

Proof. First, we note that to show that for every algorithm, there exists a choice of loss functions by an oblivious adversary, such that the expected regret of the algorithm is bounded from below, it is enough to show that there exists a distribution over loss function sequences such that for any algorithm, the expected regret is bounded from below, where now expectation is taken over both the randomness of the algorithm and the randomness of the adversary. This is because if there exists such a distribution over loss function sequences, then for any algorithm, there exists some sequence of loss functions that can lead to a regret at least as high. To put it formally, if we mark $\mathop{\mathbb{E}}_{alg}$ the expectation over the randomness of the algorithm, and $\mathop{\mathbb{E}}_{f_1,\ldots,f_T}$ the expectation over the randomness of the adversary, then:

$$\exists \text{ a (randomized) adversary s.t. } \forall \text{ algorithm A, } \mathop{\mathbb{E}}_{f_1,\ldots,f_T} \mathop{\mathbb{E}}_{alg}\left[R_A\left(T\right)\right] > \Omega\left(\sqrt{\tau T}\right) \rightarrow$$
$$\forall \text{ algorithm A, } \exists f_1,...,f_T \text{ s.t. } \mathop{\mathbb{E}}_{alg}\left[R_A\left(T\right)\right] > \Omega\left(\sqrt{\tau T}\right)$$

Thus, we prove the first statement above, that immediately gives us the second statement which is indeed our lower bound.

We consider the setting where $\mathcal{W} = [-1, 1]$, and $\forall t \in [1, T] : f_t(w_t) = \alpha_t \cdot w_t$ where $\alpha_t \in \{1, -1\}$. We start by constructing our sequence of α's. We divide the T iterations to blocks of size $\frac{\tau}{3}$. In each block, all α's are identical, and are chosen to be $+1$ or -1 w.p. $\frac{1}{2}$. This choice gives us blocks of $\frac{\tau}{3}$ consecutive functions in which α is identical within each block. Let M be a permutation window of size smaller than $\frac{\tau}{3}$. We notice first that since $M < \frac{\tau}{3}$ and the sequence of α's is organized in blocks of size $\frac{\tau}{3}$, then even after permutation, the time difference between the first and last time we encounter an α is $\leq \tau$, which means we will not get the feedback from the first time we encountered this α before encountering the next one, and we will not be able to use it for correctly predicting α's of this (original) block that arrive later. This is the main idea that stands in the basis of this lower bound.

Formally, consider a sequence of $w_1, w_2, ..., w_T$ chosen by the algorithm. Denote by $\alpha_{i,j}$ the j'th α in the i'th block, and similarly for $w_{i,j}, f_{i,j}$. We denote the entire sequence of α's by $\bar{\alpha}_{(1\to T)}$, and the sequence of α's until time point j in block i by $\bar{\alpha}_{(1\to i,j)}$. For simplicity we will denote β_t as the α that was presented at time t, after permutation, i.e. $\beta_t := \alpha_{\sigma^{-1}(()t)}$. Notice that $w_{i,j}$ is a function of the β's that arrive up until time point $i \cdot \left(\frac{\tau}{3}\right) + j - \tau - 1$. We denote these β's as $\bar{\beta}_{(1\to i,j-\tau-1)}$. I.e $w_{i,j} = g\left(\bar{\beta}_{(1\to i,j-\tau-1)}\right)$ where g is some function.

Going back to our main idea of the construction, we can put it in this new terminology- since the delay is τ and the permutation window is $M < \frac{\tau}{3}$, for any i, j, the first time we encountered $\alpha_{\sigma^{-1}(i,j)}$ is less than τ iterations ago, and thus, $\beta_{i,j}$ is independent of $\bar{\beta}_{(1\to i,j-\tau-1)}$, while $w_{i,j}$ is a function of it: $w_{i,j} = g\left(\bar{\beta}_{(1\to i,j-\tau-1)}\right)$.

With this in hand, we look at the sum of losses of the predictions of the algorithm, $w_1, w_2, ..., w_T$:

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^{T} f_t\left(w_t\right)\right] &= \mathbb{E}\left[\sum_{i=1}^{T/\frac{\tau}{3}}\sum_{j=1}^{\frac{\tau}{3}} f_{i,j}\left(w_{i,j}\right)\right] \\
&= \sum_{i=1}^{T/\frac{\tau}{3}}\sum_{j=1}^{\frac{\tau}{3}} \mathbb{E}\left[f_{i,j}\left(w_{i,j}\right)\right] \\
&= \sum_{i=1}^{T/\frac{\tau}{3}}\sum_{j=1}^{\frac{\tau}{3}} \mathbb{E}_{\bar{\beta}_{(1\to T)}}\left[\beta_{i,j}\cdot w_{i,j}\right] \\
&= \sum_{i=1}^{T/\frac{\tau}{3}}\sum_{j=1}^{\frac{\tau}{3}} \mathbb{E}_{\bar{\beta}_{(1\to i,j-\tau-1)}}\left[\mathbb{E}_{\bar{\beta}_{(i,j-\tau\to T)}}\left[\beta_{i,j}\cdot w_{i,j}|\bar{\beta}_{(1\to i,j-\tau-1)}\right]\right] \\
&= \sum_{i=1}^{T/\frac{\tau}{3}}\sum_{j=1}^{\frac{\tau}{3}} \mathbb{E}_{\bar{\beta}_{(1\to i,j-\tau-1)}}\left[w_{i,j}\cdot \mathbb{E}_{\bar{\beta}_{(i,j-\tau\to T)}}\left[\beta_{i,j}|\bar{\beta}_{(1\to i,j-\tau-1)}\right]\right] \\
&= \sum_{i=1}^{T/\frac{\tau}{3}}\sum_{j=1}^{\frac{\tau}{3}} \mathbb{E}_{\bar{\beta}_{(1\to i,j-\tau-1)}}\left[w_{i,j}\cdot \mathbb{E}_{\bar{\beta}_{(i,j-\tau\to T)}}\left[\alpha_{\sigma^{-1}(i,j)}\right]\right]
\end{aligned}$$

$$= \sum_{i=1}^{T/\frac{\tau}{3}} \sum_{j=1}^{\frac{\tau}{3}} \mathbb{E}_{\bar{\beta}_{(1\to i,j-\tau-1)}} \left[w_{i,j} \cdot \left(\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) \right) \right] = 0$$

where the last equality stems from the fact that $\beta_{i,j} = \alpha_{\sigma^{-1}(i,j)}$ is equal to the expected value of the first time we encountered the α that corresponds to $\alpha_{\sigma^{-1}(i,j)}$, i.e, the first α that came from the same block of $\alpha_{\sigma^{-1}(i,j)}$. This expectation is 0 since we choose $\alpha = 1$ or $\alpha = -1$ with probability $\frac{1}{2}$ for each block.

We now continue to the expected sum of losses for the optimal choice of $w^* = argmin_{w \in \mathcal{W}} \left(\sum_{t=1}^{T} f_t(w) \right)$. Note that after permutation, the expected sum of losses of the optimal w remains the same since it is best predictor over the entire sequence, and so for simplicity we look at the sequence of α's as it is chosen initially. Also, in this setting, $w^* \in \{+1, -1\}$ and is with opposite sign to the majority of α's in the sequence.

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^{T} f_t(w^*)\right] &= \mathbb{E}\left[\sum_{i=1}^{T/\frac{\tau}{3}} \sum_{j=1}^{\frac{\tau}{3}} f_{i,j}(w^*)\right] = \mathbb{E}\left[\sum_{i=1}^{T/\frac{\tau}{3}} \sum_{j=1}^{\frac{\tau}{3}} \alpha_{i,j} \cdot w^*\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{T/\frac{\tau}{3}} \frac{\tau}{3} \cdot \alpha_{i,1} \cdot w^*\right] = \frac{\tau}{3} \cdot \mathbb{E}\left[\sum_{i=1}^{T/\frac{\tau}{3}} \alpha_{i,1} \cdot w^*\right] \\
&= -\frac{\tau}{3} \cdot \mathbb{E}\left[\left| \sum_{i=1}^{T/\frac{\tau}{3}} \alpha_{i,1} \right| \right]
\end{aligned}$$

Using Khintchine inequality we have that:

$$\begin{aligned}
-\frac{\tau}{3} \cdot \mathbb{E}\left[\left| \sum_{i=1}^{T/\frac{\tau}{3}} \alpha_{i,1} \cdot 1 \right| \right] &\leq -\frac{\tau}{3} \cdot C \cdot \sqrt{\left(\sum_{i=1}^{T/\frac{\tau}{3}} 1^2 \right)} = -\frac{\tau}{3} \cdot C \cdot \sqrt{\frac{T}{\frac{\tau}{3}}} \\
&= -\Omega\left(\sqrt{\frac{\tau}{3} \cdot T} \right) = -\Omega\left(\sqrt{\tau \cdot T} \right)
\end{aligned}$$

where C is some constant.

Thus we get that overall expected regret for any algorithm with permutation power $M < \frac{\tau}{3}$ is:

$$\mathbb{E}\left[\sum_{t=1}^{T} f_t(w_t)\right] - \mathbb{E}\left[\sum_{t=1}^{T} f_t(w^*)\right] = \Omega\left(\sqrt{\tau \cdot T} \right)$$

as in the adversarial case.

□