A. Support Vector Technique

We describe Algorithm 2 in detail. Algorithm 2 takes as input the sample set S, the query sequence $\mathcal{F}$, the sensitivity of query Δ, the threshold τ, and the stop parameter s. Algorithm 2 outputs the result of each comparison with the threshold. In Algorithm 2, each noisy query output is compred with a noisy threshold at line 4 and outputs the result of comparison. Let $\top$ mean that $f_k(S) > \tau$. Algorithm 2 is terminated if outputs $\top$ s times.

Algorithm 2 Sparse Vector Technique (Dwork & Roth, 2014).

Require: Sample set S, query sequence $\mathcal{F}$, sensitivity of query Δ, threshold τ, stop parameter s

```
1:  ρ = Lap(2Δs/ε)
2:  count = 0
3:  for each f_i ∈ F do
4:      if f_i(S) + Lap(4Δs/ε) ≥ τ + ρ then
5:          Output ⊤
6:          ρ = Lap(2Δs/ε_1)
7:          count = count +1
8:          if count ≥ s then
9:              Abort
10:         end if
11:     else
12:         Output ⊥
13:     end if
14: end for
```

B. The proof of Theorem 2

Proof. By definition, we have

$$
\begin{aligned}
& \Pr[\mathcal{M}(S,\hat{\tau_\alpha}) = \mathsf{acc} | H_1 \text{ is true}] \\
= \; & \sup_{P\in\mathcal{P}} \Pr_{S\sim P}[\mathcal{M}(S,\hat{\tau_\alpha}) = \mathsf{acc}] \\
= \; & \sup_{P\in\mathcal{P}} \Big\{ \Pr_{S\sim P}[\mathcal{M}(S,\hat{\tau_\alpha}) = \mathsf{acc} | \chi^2(S) > \hat{\tau}_\alpha + \gamma] \Pr_{S\sim P}[\chi^2(S) > \hat{\tau}_\alpha + \gamma] \\
+ \; & \Pr_{S\sim P}[\mathcal{M}(S,\hat{\tau_\alpha}) = \mathsf{acc} | \chi^2(S) \le \hat{\tau}_\alpha + \gamma] \Pr_{S\sim P}[\chi^2(S) \le \hat{\tau}_\alpha + \gamma] \Big\} \\
\le \; & \sup_{P\in\mathcal{P}} \Big\{ \Pr_{S\sim P}[\mathcal{M}(S,\hat{\tau_\alpha}) = \mathsf{acc} | \chi^2(S) > \hat{\tau}_\alpha + \gamma] + \Pr_{S\sim P}[\chi^2(S) \le \hat{\tau}_\alpha + \gamma] \Big\}.
\end{aligned}
$$

For any $P \in \mathcal{P}$, $\Pr_{S\sim P}[\chi^2(S) \le \hat{\tau}_\alpha + \gamma] \le \beta_{\hat{\tau}_\alpha + \gamma}$ by definition. Thus, we have

$$
\Pr[\mathcal{M}(S,\hat{\tau}_\alpha) = \mathsf{acc} | H_1 \text{ is true}] \le \sup_{P\in\mathcal{P}} \Big\{ \Pr_{S\sim P}[\mathcal{M}(S,\hat{\tau}_\alpha) = \mathsf{acc} | \chi^2(S) > \hat{\tau}_\alpha + \gamma] + \beta_{\hat{\tau}_\alpha + \gamma} \Big\},
$$

$\square$

C. The proof of Theorem 3

Proof. Fix the sample S. Then, the conditional distribution $\Pr[\mathcal{M}_\Delta(S,\hat{\tau}_\alpha)=\mathsf{acc}|S]$ is obtained as

$$\begin{aligned}\Pr[\mathcal{M}_\Delta(S,\hat{\tau}_\alpha)=\mathsf{acc}|S] &= \Pr\left[\chi^2(S)+Lap\left(\frac{\Delta}{\epsilon}\right)\le\hat{\tau}_\alpha\right]\\ &= \Pr\left[Lap\left(\frac{\Delta}{\epsilon}\right)\le\hat{\tau}_\alpha-\chi^2(S)\right]\\ &= \frac{\epsilon}{2\Delta}\int_{-\infty}^{\hat{\tau}_\alpha-\chi^2(S)}\exp\left(\frac{x\epsilon}{\Delta}\right)dx\\ &= \frac{1}{2}\exp\left(\frac{(\hat{\tau}_\alpha-\chi^2(S))\epsilon}{\Delta}\right).\end{aligned}$$

Under the condition $\chi^2(S)>\hat{\tau}_\alpha+\gamma$, we have

$$\begin{aligned}\Pr[\mathcal{M}_\Delta(S,\hat{\tau}_\alpha)=\mathsf{acc}|S] =&\Pr\left[\chi^2(S)+Lap\left(\frac{\Delta}{\epsilon}\right)\le\hat{\tau}_\alpha\right]\\ \le&\frac{1}{2}\exp\left(\frac{-\gamma\epsilon}{\Delta}\right).\end{aligned}\tag{8}$$

The gamma error is rearranged as

$$E(\hat{\tau}_\alpha,\gamma,\mathcal{M}_\Delta) = \sup_{P\in\mathcal{P}}\mathop{\mathrm{E}}_{S\sim P}[\Pr[\mathcal{M}_\Delta(S,\hat{\tau}_\alpha)=\mathsf{acc}|S]|\chi^2(S)>\hat{\tau}_\alpha+\gamma].\tag{9}$$

Substituting Eq. 8 into Eq. 9 gives the claim. □

D. The proof of Lemma 1

Proof. Let $\chi^2(c_{11},c_{10})=\tau_\alpha$. Eq. 4 is rearranged as

$$Ac_{11}^2+Bc_{10}^2 + 2Cc_{11}c_{10}+D(c_{11}+c_{10})=0,\tag{10}$$

where $A=(N_0^2N+\tau_\alpha N_1N_0)$, $B=(N_1^2N+\tau_\alpha N_1N_0)$, $C=N_1N_0(\tau_\alpha-N)$, and $D=-\tau_\alpha N_1N_0N$. Eq. 10 is a quadratic form, and it is an ellipse if and only if $AB-C^2>0$. For any $N_1>0$, $N_0>0$, $N>0$, and $\tau_\alpha>0$, we have

$$\begin{aligned}AB-C^2 &= (N_0^2N+\tau_\alpha N_1N_0)(N_1^2N+\tau_\alpha N_1N_0)-\{N_1N_0(\tau_\alpha-N)\}^2\\ &= \tau_\alpha NN_1N_0(N_1+N_0)^2>0.\end{aligned}$$

Thus, we get the claim. □

E. The affine transformation V

The affine transformation V that transforms the ellipse derived in Eq. 10 to the unit circle is defined as follows:

$$V((c_{11},c_{10})^t)=\\ \begin{pmatrix}\sqrt{\frac{\lambda_1}{R}} & 0\\ 0 & \sqrt{\frac{\lambda_2}{R}}\end{pmatrix}\left(\begin{pmatrix}\frac{C}{\sqrt{C^2+(\lambda_1-A)^2}} & \frac{(\lambda_1-A)}{\sqrt{C^2+(\lambda_1-A)^2}}\\ \frac{-(\lambda_1-A)}{\sqrt{C^2+(\lambda_1-A)^2}} & \frac{C}{\sqrt{C^2+(\lambda_1-A)^2}}\end{pmatrix}\begin{pmatrix}c_{11}\\ c_{10}\end{pmatrix}+\frac{D}{2\sqrt{C^2+(\lambda_1-A)^2}}\begin{pmatrix}\frac{C+\lambda_1-A}{\lambda_1}\\ \frac{C+\lambda_2-B}{\lambda_2}\end{pmatrix}\right).$$

where λ_1 and λ_2 are the eigenvalues of matrix $\begin{pmatrix}A & C\\ C & B\end{pmatrix}$ and

$$R=\frac{D^2\left(\lambda_2(C+\lambda_1-A)^2+\lambda_1(C+\lambda_2-B)^2\right)}{4\lambda_1\lambda_2\left(C^2+(\lambda_1-A)^2\right)}.\tag{11}$$

F. The proof of Theorem 5

Proof. We can rewrite Eq. 10 as

$$(c_{11}, c_{10}) \begin{pmatrix} A & C \\ C & B \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{10} \end{pmatrix} + D\,(1,1) \begin{pmatrix} c_{11} \\ c_{10} \end{pmatrix} = 0, \tag{12}$$

where A, B, C and D are defined in Appendix D. By eigendecomposition of the matrix $\begin{pmatrix} A & C \\ C & B \end{pmatrix}$, we obtain as

$$\begin{pmatrix} A & C \\ C & B \end{pmatrix} = P^T \mathrm{diag} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P,$$

where $\lambda_1, \lambda_2 = \frac{(A+B) \pm \sqrt{(A+B)^2 - 4AB + 4C^2}}{2}$ and

$$P = \begin{pmatrix} \frac{C}{\sqrt{C^2+(\lambda_1-A)^2}} & \frac{-(\lambda_1-A)}{\sqrt{C^2+(\lambda_1-A)^2}} \\ \frac{\lambda_1-A}{\sqrt{C^2+(\lambda_1-A)^2}} & \frac{C}{\sqrt{C^2+(\lambda_2-A)^2}} \end{pmatrix}.$$

Let $\begin{pmatrix} \hat{c}_{11} \\ \hat{c}_{10} \end{pmatrix} = P^T \begin{pmatrix} c_{11} \\ c_{10} \end{pmatrix}$. Then, we can rewrite Eq. 12 as

$$\begin{aligned} &\lambda_1 \left(\hat{c}_{11} + \frac{D}{2\lambda_1 \sqrt{C^2 + (\lambda_1 - A)^2}} (C + \lambda_1 - A) \right)^2 + \lambda_2 \left(\hat{c}_{10} + \frac{D}{2\lambda_2 \sqrt{C^2 + (\lambda_1 - A)^2}} (C + \lambda_2 - B) \right)^2 \\ = \quad & \frac{D^2 \left(\lambda_2 (C + \lambda_1 - A)^2 + \lambda_1 (C + \lambda_2 - B)^2 \right)}{4\lambda_1 \lambda_2 \left(C^2 + (\lambda_1 - A)^2 \right)}. \end{aligned} \tag{13}$$

The right hand side is equivalent to R. Since $A > 0$ and $B > 0$, we have $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. Thus, by definition $R \geq 0$. Dividing the right hand side and left hand side of Eq. 13 by R gives $\dot{c}_{11}^2 + \dot{c}_{10}^2 = 1$ where

$$\begin{aligned} \dot{c}_{11} =& \sqrt{\frac{\lambda_1}{R}} \left(\tilde{c}_{11} + \frac{D}{2\lambda_1 \sqrt{C^2 + (\lambda_1 - A)^2}} (C + \lambda_1 - A) \right), \\ \dot{c}_{10} =& \sqrt{\frac{\lambda_2}{R}} \left(\hat{c}_{10} + \frac{D}{2\lambda_2 \sqrt{c^2 + (\lambda_1 - A)^2}} (C + \lambda_2 - B) \right). \end{aligned}$$

Consequently, $\chi^2(c_{11}, c_{10}) = \tau_\alpha$ if and only if the vector $(\dot{c}_{11}, \dot{c}_{10})$ is on the boundary of the unit circle.

The relationship between (c_{11}, c_{10}) and $(\dot{c}_{11}, \dot{c}_{10})$ is obtained as

$$\begin{pmatrix} \dot{c}_{11} \\ \dot{c}_{10} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\lambda_1}{R}} & 0 \\ 0 & \sqrt{\frac{\lambda_2}{R}} \end{pmatrix} \left(\begin{pmatrix} \frac{C}{\sqrt{C^2+(\lambda_1-A)^2}} & \frac{(\lambda_1-A)}{\sqrt{C^2+(\lambda_1-A)^2}} \\ \frac{-(\lambda_1-A)}{\sqrt{C^2+(\lambda_1-A)^2}} & \frac{C}{\sqrt{C^2+(\lambda_1-A)^2}} \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{10} \end{pmatrix} + \frac{D}{2\sqrt{C^2 + (\lambda_1 - A)^2}} \begin{pmatrix} \frac{C+\lambda_1-A}{\lambda_1} \\ \frac{C+\lambda_2-B}{\lambda_2} \end{pmatrix} \right).$$

Thus, by the definition of V, we have $(\dot{c}_{11}, \dot{c}_{10})^t = V((c_{11}, c_{10})^t)$. Since $R \geq 0$, $\chi^2(c_{11}, c_{10}) > \tau_\alpha$ if and only if $1 < \|(\dot{c}_{11}, \dot{c}_{10})\|_2 = \|V((c_{11}, c_{10})^t)\|_2$. □

G. The proof of Lemma 2

Proof. Let S and S' be databases such that $d(S, S') = 1$. Let $\mathbf{c} = (c_{11}, c_{10})^t$ and $\mathbf{c}' = (c'_{11}, c'_{10})^t$ be the elements of the contingency table derived from S and S', respectively. Then, we have

$$\begin{aligned} |\|V(\mathbf{c})\|_2 - \|V(\mathbf{c}')\|_2| =& \sqrt{\|V(\mathbf{c})\|_2^2 + \|V(\mathbf{c}')\|_2^2 - 2\|V(\mathbf{c})\|_2 \|V(\mathbf{c}')\|_2} \\ =& \sqrt{\|V(\mathbf{c}) - V(\mathbf{c}')\|_2^2 + 2(V(\mathbf{c}))^t V(\mathbf{c}') - 2\|V(\mathbf{c})\|_2 \|V(\mathbf{c}')\|_2}. \end{aligned}$$

From CauchySchwarz inequality, we have

$$|\|V(\mathbf{c})\|_2 - \|V(\mathbf{c}')\|_2| \leq \|V(\mathbf{c}) - V(\mathbf{c}')\|_2.$$

From the definition of V, we have

$$V(\mathbf{c}) - V(\mathbf{c}') = \begin{pmatrix} \sqrt{\frac{\lambda_1}{R}} & 0 \\ 0 & \sqrt{\frac{\lambda_2}{R}} \end{pmatrix} \begin{pmatrix} \frac{C}{\sqrt{C^2+(\lambda_1-A)^2}} & \frac{(\lambda_1-A)}{\sqrt{C^2+(\lambda_1-A)^2}} \\ \frac{-(\lambda_1-A)}{\sqrt{C^2+(\lambda_1-A)^2}} & \frac{C}{\sqrt{C^2+(\lambda_1-A)^2}} \end{pmatrix} (\mathbf{c} - \mathbf{c}').$$

Since $d(S, S') = 1$, an element of $\mathbf{c} - \mathbf{c}'$ is either of 1 or -1 and the other is 0. Consequently, we have

$$\begin{aligned} \|V(\mathbf{c}) - V(\mathbf{c}')\|_2 &= \sqrt{\frac{1}{C^2 + (\lambda_1 - A)^2}\left(\left(C\sqrt{\frac{\lambda_1}{R}}\right)^2 + \left(-(\lambda_1 - A)\sqrt{\frac{\lambda_2}{R}}\right)^2\right)} \\ &\leq \sqrt{\left(\sqrt{\frac{\lambda_1}{R}}\right)^2 + \left(\sqrt{\frac{\lambda_2}{R}}\right)^2} = \sqrt{\frac{\lambda_1 + \lambda_2}{R}}. \end{aligned}$$

Hence,

$$\Delta = \max_{S,S':d(S,S')=1} |\|V(\mathbf{c})\|_2 - \|V(\mathbf{c}')\|_2| \leq \sqrt{\frac{\lambda_1 + \lambda_2}{R}}.$$

By using Eq. 11, we get

$$\begin{aligned} \Delta &= \sqrt{\frac{\lambda_1 + \lambda_2}{R}} \\ &= 2\sqrt{\frac{(N_0^2 + N_1^2)N + 2\tau_\alpha N_0 N_1}{\tau_\alpha N_0 N_1 N^2}}. \end{aligned}$$

□

H. The proof of Theorem 7

Proof. As the same manner of the proof of Theorem 3, we have

$$\Pr[\mathcal{M}_{\Delta_V}(S, \hat{\tau}_\alpha) = \mathsf{acc}|S] = \frac{1}{2}\exp\left(\frac{(1 - ||V((c_{11}, c_{10})^t)||_2)\epsilon}{\Delta_V}\right)$$

Define $g(c_{11}, c_{10}) = N_1 N_0 (c_{11}^2 + c_{10}^2) - N_1 N_0 N (c_{11} + c_{10}) + 2N_1 N_0 c_{11} c_{10}$. Under the condition $\chi^2(S) > \hat{\tau}_\alpha + \gamma$, we have

$$\|V((c_{11}, c_{10})^t)\|_2^2 + \frac{\gamma g(c_{11}, c_{10})}{R} \geq 1.$$

Hence,

$$\|V((c_{11}, c_{10})^t)\|_2 \geq \sqrt{1 - \frac{\gamma g(c_{11}, c_{10})}{R}}.$$

By the definition of R and g, $\frac{\gamma g(c_{11}, c_{10})}{R} = -4\gamma \frac{N_1 N_0}{\hat{\tau_\alpha} N^2}$. Hence,

$$\Pr[\mathcal{M}_{\Delta_V}(S, \hat{\tau}_\alpha) = \mathsf{acc}|S] \leq \frac{1}{2}\exp\left(\frac{\epsilon N}{2}\left(1 - \sqrt{1 + \frac{4\gamma M_1 M_0}{\hat{\tau_\alpha} N^2}}\right)\sqrt{\frac{\hat{\tau_\alpha} N_1 N_0}{(N_1^2 + N_0^2)N + 2\hat{\tau_\alpha} N_1 N_0}}\right).$$

Thus, we get the claim by Eq. 9. □

I. Algorithm of Unit Circle Mechanism + SVT

We describe Algorithm 3 in detail. Algorithm 3 takes as input sample sets $S^1, \cdots, S^K$, the significance level α, the privacy budget ϵ, and two stop parameters $s_1 \leq s_2$. Algorithm 3 is terminated if (1) it rejects at most s_1 null hypothesis, or (2) it outputs s_2 test results. In Algorithm 3, the outer for-loop (line 3 - 26) is the main loop of SVT. The test statistic for S^k is evaluated at line 5 and is compared with a noisy threshold at line 12. To keep the type-I error as, at most, α per test, we want that $\hat{d}^k < 1 + \rho$ holds with the probability of at least $1 - \alpha$, where ρ is the noise that SVT requires to add a threshold. To attain this, Algorithm 3 generates a sample distribution of the randomized test statistics by Monte Carlo sampling at the inner for-loop (line 6 - 10). What differs from the normal SVT framework are s_2 and Monte Carlo sampling to find a new threshold to control FWER. For Algorithm 3, if marginals $N_0^k, N_1^k, M_0^k, M_1^k$ are public, then the computation of threshold τ^k does not consume an additional privacy budget Therefore, Algorithm 3 requires the same privacy budget as SVT does.

Algorithm 3 Unit Circle Mechanism + SVT

Require: Sample set $S^1, \cdots, S^K$, significance level α, privacy budget ϵ, stop parameters $s_1 \leq s_2$,

1: $count_1 = 0, \;\; count_2 = 0$
2: $\rho = Lap(\frac{2s_1 \Delta_{V, \frac{\alpha}{s_2}}(N_0, N_1)}{\epsilon})$
3: **for** each sample set S^k **do**
4:     Evaluate contingency table from S^k
5:     $\hat{d}^k(S^k) = ||V((c_{11}^k, c_{10}^k)^t)||_2 + Lap(\frac{4s_1 \Delta_{V, \frac{\alpha}{s_2}}(N_0, N_1)}{\epsilon})$
6:     **for** $j = 1$ to m **do**
7:         $S^{k,j} \sim mult(\frac{N_1 M_1^k}{N^2}, \frac{N_0 M_1^k}{N^2}, \frac{N_1 M_0^k}{N^2}, \frac{N_0 M_0^k}{N^2})$
8:         Evaluate contingency table from $S^{k,j}$
9:         $\hat{d}^{k,j}(S^{k,j}) = ||V((c_{11}^{k,j}, c_{10}^{k,j})^t)||_2 + Lap(\frac{4s_1 \Delta_{V, \frac{\alpha}{s_2}}(N_0^{kj}, N_1^{kj})}{\epsilon}) - \rho$
10:     **end for**
11:     Let $\hat{\tau}^k$ be the $\lceil (m+1)(1-\alpha) \rceil$th largest value in $\{\hat{d}^{k,j}\}_{j=1,\cdots,m}$
12:     **if** $\hat{d}^k(S^k) > \hat{\tau}^k + \rho$ **then**
13:         Return rej
14:         $\rho = Lap(\frac{2s_1 \Delta_{V, \frac{\alpha}{s_2}}(N_0, N_1)}{\epsilon})$
15:         $count_1 = count_1 + 1$
16:         **if** $count_1 \geq s_1$ **then**
17:             Abort
18:         **end if**
19:     **else**
20:         Output acc
21:     **end if**
22:     $count_2 = count_2 + 1$
23:     **if** $count_2 \geq s_2$ **then**
24:         Abort
25:     **end if**
26: **end for**

J. Algorithm of Unit Circle Mechanism + EM

We describe Algorithm 4 in detail. Algorithm 4 takes the sample sets $S^1, \cdots, S^K$, the significance level α, the privacy budget ϵ, the stop parameter s_1. Algorithm 4 outputs s_1 test results. Let $\mathcal{E}_q^\epsilon$ be the exponential mechanism with privacy budget ϵ and score function q. Algorithm 4 first calculates the score function defined by Jhonson et al. (Johnson & Shmatikov, 2013) (line 1) and chooses the sample sets associated with the top s_1 significant random variable pairs by the exponential mechanism (line 5). Then, the mechanism gets the results of the test by using the unit circle mechanism (line 9). In Algorithm 4, we spend privacy budget $\frac{\epsilon}{2s_1}$ for the exponential mechanism and $\frac{\epsilon}{2s_1}$ for the unit circle mechanism s_1 times, respectively.

Algorithm 4 Unit Circle Mechanism + EM

Require: Sample sets $S^1, \cdots, S^K$, significance level α, privacy budget ϵ, stop parameter s_1,

1: Calculate score function $q(S^k)$ for each sample set S^k
2: $I = \emptyset$
3: **for** $j = 1$ to s_1 **do**
4:     **repeat**
5:         $\hat{S} \leftarrow \mathcal{E}_q^{\frac{\epsilon}{2s_1}}$
6:
7:     **until** $\hat{S} \notin I$
8:     $I \leftarrow I \cup \{\hat{S}\}$
9:     Run Algorithm 1 with Sample set $\hat{S}$ and significance Level $\frac{\alpha}{K}$ and privacy budget $\frac{\epsilon}{2s_1}$
10: **end for**