Analysis and Optimization of Graph Decompositions by Lifted Multicuts (Supplement)

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Appendix

A. Multicuts

Proof of Lemma 2 First, we show that for any $\Pi \in D_G$, the image $\phi_G(\Pi)$ is a multicut of G. Assume the contrary, i.e. there exists a cycle C of G such that $|C \cap \phi_G(\Pi)| = 1$. Let $\{u, v\} = e \in C \cap \phi_G(\Pi)$, then for all $U \in \Pi$ it holds that $u \notin U$ or $v \notin U$. However, $C \setminus \{e\}$ is a sequence of edges $\{w_1, w_2\}, \ldots, \{w_{k-1}, w_k\}$ such that $u = w_1, v = w_k$ and $\{w_i, w_{i+1}\} \notin \phi_G(\Pi)$ for all $1 \leq i \leq k-1$. Consequently, since Π is a partition of V, there exists some $U \in \Pi$ such that

$$w_1 \in U \wedge w_2 \in U \wedge \ldots \wedge w_{k-1} \in U \wedge w_k \in U.$$

This contradicts $w_1 = u \notin U$ or $w_k = v \notin U$.

To show injectivity of ϕ_G, let $\Pi = \{U_1, \ldots, U_k\}$, $\Pi' = \{U'_1, \ldots, U'_\ell\}$ be two decompositions of G. Suppose $\Pi \neq \Pi'$. Then (w.l.o.g.) there exist some $u, v \in V$ with $\{u, v\} \in E$ and some $U_i \in \Pi$ such that $u, v \in U_i$ and for all $U'_j \in \Pi'$ it holds that $u \notin U'_j$ or $v \notin U'_j$. Thus, $\{u, v\} \in \phi_G(\Pi')$ but $\{u, v\} \notin \phi_G(\Pi)$, which means $\phi_G(\Pi) \neq \phi_G(\Pi')$.

For surjectivity, take some multicut $M \subseteq E$ of G. Let $\Pi = \{U_1, \ldots, U_k\}$ collect the node sets of the connected components of the graph $(V, E \setminus M)$. Apparently, Π defines a decomposition of G. We have $\{u, v\} \in \phi_G(\Pi)$ if and only if for all $U \in \Pi$ it holds that $v \notin U$ or $u \notin U$. The latter holds true if and only if $\{u, v\}$ is not contained in any connected component of $(V, E \setminus M)$, which is equivalent to $\{v, w\} \in M$. Hence, $\phi_G(\Pi) = M$.

Proof of Lemma 4 First, we show that for any $M \in M_{K_V}$ the image $\psi(M)$ is an equivalence relation on V. Since K_V is simple, we trivially have $\{v, v\} \notin M$ for any $v \in V$. Therefore, $(v, v) \in \psi(M)$, which means $\psi(M)$ is reflexive. Symmetry of $\psi(M)$ follows from $\{u, v\} = \{v, u\}$ for all $u, v \in V$. Now, suppose $(u, v), (v, w) \in \psi(M)$. Then $\{u, v, \}, \{v, w\} \notin M$ and thus $\{u, w\} \notin M$ (otherwise $C = \{u, v, w\}$ would be a cycle contradicting the definition of a multicut). Hence, $(u, w) \in \psi(M)$, which gives transitivity of $\psi(M)$.

Let M, M' be two multicuts of K_V with $\psi(M) = \psi(M')$. Then

$$\begin{aligned} \{u, v\} \in M &\iff (u, v) \notin \psi(M) \\ &\iff (u, v) \notin \psi(M') \\ &\iff \{u, v\} \in M'. \end{aligned}$$

Hence $M = M'$, so ψ is injective.

Let R be an equivalence relation on V and define M by

$$\{u, v\} \in M \iff (u, v) \notin R.$$

Transitivity of R implies that M is a multicut of K_V. Moreover, by definition, it holds that $\psi(M) = R$. Hence, ψ is also surjective.

B. Lifted Multicuts

Proof of Lemma 5 Let $x \in \{0,1\}^{E'}$ be such that $M' = x^{-1}(1)$ is a multicut of G' lifted from G. Every cycle in G is a cycle in G'. Moreover, for any path $vw = f \in F_{GG'}$ and any vw-path P in G, it holds that $P \cup \{f\}$ is a cycle in G'. Therefore, x satisfies all inequalities (4) and (5). Assume x violates some inequality of (6). Then there is an edge $vw \in F_{GG'}$ and some vw-cut C in G such that $x_{vw} = 0$ and for all $e \in C$ we have $x_e = 1$. Let Π be the partition of V corresponding to M' according to Lemma 2. There exists some $U \in \Pi$ with $v \in U$ and $w \in U$. However, for any $uu' = e \in C$ it holds that $u \notin U$ or $u' \notin U$. This means the subgraph $(U, E \cap \binom{U}{2})$ is not connected, as C is a vw-cut. Hence, Π is not a decomposition of G, which is a contradiction, because G is connected.

Now, suppose $x \in E'$ satisfies all inequalities (4)–(6). We show first that $M' = x^{-1}(1)$ is a multicut of G'. Assume the contrary, then there is a cycle C' in G' and some edge e' such that $C' \cap M' = \{e'\}$. For every $vw = f \in F_{GG'} \cap C' \setminus \{e'\}$ there exists a vw-path P in G such that $x_e = 0$ for all $e \in P$. Otherwise there would be some vw-cut in G

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Proceedings of the 34^{th} International Conference on Machine Learning, Sydney, Australia, PMLR 70, 2017. Copyright 2017 by the author(s).

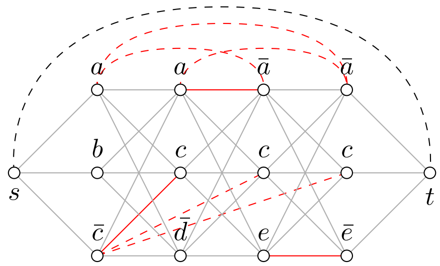

Figure 1. To show that the consistency problem is NP-hard, we reduce 3-SAT to this problem. Shown above is the instance of the consistency problem constructed for the instance of 3-SAT given by the form $(a \vee b \vee \bar{c}) \wedge (a \vee c \vee \bar{d}) \wedge (\bar{a} \vee c \vee e) \wedge (\bar{a} \vee c \vee \bar{e})$. Solid and dashed lines depict edges in E and $E' \setminus E$, respectively. Black means $\tilde{x}_e = 0$. Red means $\tilde{x}_e = 1$. Grey means $e \notin \operatorname{dom} \tilde{x}$.

violating (6), as G is connected. If we replace every such f with its associated path P in G, then the resulting cycle violates either (4) (if $e' \in E$) or (5) (if $e' \in F_{GG'}$). Thus, M' is a multicut of G'. By connectivity of G, the partition $\phi_{G'}^{-1}(M')$ is a decomposition of both G' and G. Therefore, $M = \lambda_{GG'}^{-1}(M') = \phi_G(\phi_{G'}^{-1}(M'))$ is a multicut of G and hence $M' = x^{-1}(1)$ is indeed lifted from G.

C. Partial Lifted Multicuts

Proof of Theorem 1 Firstly, we show that the consistency problem is in NP. For that, we show that verifying, for any given $x \in \{0,1\}^{E'}$, that x is a completion of $\tilde{x}$ and a characteristic function of a multicut of G' lifted from G is a problem of polynomial time complexity. To verify that x is a completion of $\tilde{x}$, we verify for every $e \in \operatorname{dom} \tilde{x}$ that $x_e = \tilde{x}_e$. This takes time $O(|E|)$. To verify that $x^{-1}(1)$ is a multicut of G' lifted from G, we employ a disjoint set data structure initialized with singleton sets V. For any $\{v, w\} \in x^{-1}(0)$, we call $\operatorname{union}(v, w)$. Then, we verify for every $\{v, w\} \in x^{-1}(1)$ that $\operatorname{find}(v) \neq \operatorname{find}(w)$. This takes time $O(|E| + |V| \log |V|)$.

To show that the consistency problem is NP-hard, we reduce 3-SAT to this problem. For that, we consider any instance of 3-SAT defined by a propositional logic formula A in 3-SAT form. An example is shown in Fig. 1. Let m be the number of variables and n the number of clauses in A.

In order to define an instance of the consistency problem w.r.t. this instance of 3-SAT, we construct in polynomial time a connected graph $G = (V, E)$, a graph $G' = (V, E')$ with $E \subseteq E'$, and a partial characteristic function $\tilde{x} \in \{0, 1, *\}^{E'}$ as described below. An example of this construction is shown in Fig. 1.

- There are $3n + 2$ nodes in V. Two nodes are denoted by s and t. Additional nodes are organized in n layers. For $j \in \{1, \ldots, n\}$, the j-th layer corresponds to the j-th clause in A, containing one node for each of the three literals[1] in the clause. Every node is labeled with its corresponding literal. Layer 0 contains only the node s. Layer $n + 1$ contains only the node t.
- Any two consecutive layers are connected such that their nodes together induce a complete bipartite subgraph of G. Additionally, any nodes v and w labeled with conflicting literals, a and $\bar{a}$, that are not already connected in G are connected in G' by an edge $\{v, w\} \in E' \setminus E$.
- For any edge $\{v, w\} \in E'$ whose nodes v and w are labeled with conflicting literals, we set $\tilde{x}_{vw} = 1$. In addition, we introduce the edge $\{s, t\} \in E' \setminus E$ and define $\tilde{x}_{st} = 0$. No other edges are in the domain of $\tilde{x}$.

Observe that $\tilde{x}$ is consistent iff there exists an st-path P in G such that no edge or chord $\{v, w\}$ of P is such that $\tilde{x}_{vw} = 1$. Any such path is called feasible. All other st-paths in G are called infeasible.

Now, we show firstly that the existence of a feasible path implies the existence of a solution to the given instance of 3-SAT. Secondly, we show that the existence of a solution to the given instance of 3-SAT implies the existence of a feasible path. That suffices.

1. Let P be a feasible path and let V_P its node set. An assignment χ to the variables of the instance of 3-SAT is constructed as follows: For any node $v \in V_P$ whose label is a variable a, we define $\chi(a) :=$ true. For any node $v \in V_P$ whose label is a negated variable $\bar{a}$, we define $\chi(a) :=$ false. All remaining variables are assigned arbitrary truth values. By the properties of P, χ is well-defined and $A[\chi]$ is true.
2. Let χ be a solution to the given instance of 3-SAT. As every clause of A contains one literal that is true, and by construction of G, we can choose an st-path in G along which all nodes are labeled with literals that are true for the assignment χ. By virtue of χ being a solution to the instance of 3-SAT, any pair of literals that are both true are non-conflicting. Thus, P has no edge or chord $\{v, w\}$ such that $\tilde{x}_{vw} = 1$.

Proof of Lemma 6 Firstly, suppose that $E \subseteq \operatorname{dom} \tilde{x}$. In this case, it is clear that $\tilde{x}$ is consistent iff $\tilde{x}$ satisfies all cycle inequalities (4) w.r.t. the graph $(V, E \cap \operatorname{dom} \tilde{x})$. This can be checked in time $O(|V| + |E'|)$ as follows: Label the maximal components of the subgraph $G_{\tilde{x}}$ of G induced by the edge set $\{e \in E : \tilde{x}_e = 0\}$. Then, for every $\{v, w\} \in E'$ with $\tilde{x}_{vw} = 1$, check if v and w are in distinct maximal

[1] A literal is either a variable a or a negated variable $\bar{a}$.

components of $G_{\tilde{x}}$. If so, $\tilde{x}$ is consistent, otherwise $\tilde{x}$ is inconsistent.

Now, suppose $\tilde{x} \in \{0,1,*\}^{E'}$ satisfies (10). We show that, similar to the first case, $\tilde{x}$ is consistent iff all inequalities (4) and (5) are satisfied w.r.t. the graph $(V, E' \cap \operatorname{dom} \tilde{x})$. This can be checked analogously to the first case.

Necessity of this condition is clear. To show sufficiency, assume this condition holds true. We construct some $x \in X_{GG'}[\tilde{x}]$ as follows. For all $e \in \operatorname{dom} \tilde{x}$, set $x_e := \tilde{x}_e$. For all $\{v,w\} = f \in E' \setminus E$ such that $f \notin \operatorname{dom} \tilde{x}$ and such that there is a vw-path P in G with $\tilde{x}_e = 0$ for all $e \in P$, set $x_f := 0$. For all remaining edges e, set $x_e := 1$. By construction, x satisfies (4), (5) and (6).

Proof of Theorem 2 To show that the maximal specificity problem is NP-hard, we reduce 3-SAT to this problem: For any given instance of 3-SAT we construct in polynomial time a connected graph $G = (V, E)$, a graph $G' = (V, E')$ with $E \subseteq E'$, and a partial characteristic function $\tilde{x} \in \{0,1,*\}^{E'}$ as in the proof of Thm. 1, except that now, we let $st \notin \operatorname{dom} \tilde{x}$.

We know that $\tilde{x}$ is consistent because $\mathbb{1} \in X_{GG'}[\tilde{x}]$. We show that $\tilde{x}$ is maximally specific iff the given instance of 3-SAT has a solution:

Firstly, every $e \in E' \setminus (\operatorname{dom} \tilde{x} \cup \{st\})$ is undecided, by the following argument: (i) There exists an $x \in X_{GG'}[\tilde{x}]$ with $x_e = 1$, namely $\mathbb{1}$. (ii) There exists an $x \in X_{GG'}[\tilde{x}]$ with $x_e = 0$, namely the $x \in \{0,1\}^{E'}$ with $x_e = 0$ and $\forall f \in E' \setminus \{e\} : x_f = 1$. To see that $x \in X_{GG'}[\tilde{x}]$, observe that $e \in E$ and $\tilde{x}^{-1}(0) = \emptyset$. Thus, st is the only edge in $E' \setminus \operatorname{dom} \tilde{x}$ that is possibly decided. That is:

$$E'[\tilde{x}] \subseteq \{st\} \cup \operatorname{dom} \tilde{x} \tag{42}$$

Thus, $\tilde{x}$ is maximally specific iff $\tilde{x}$ is undecided. More specifically, $\tilde{x}$ is maximally specific iff there exists an $x \in X_{GG'}[\tilde{x}]$ with $x_{st} = 0$, as we know of the existence of $\mathbb{1} \in X_{GG'}[\tilde{x}]$. Thus, $\tilde{x}$ is maximally specific iff the given instance of 3-SAT has a solution, by the arguments made in the proof of Thm. 1.

Proof of Lemma 7 Observe that $\tilde{x}$ is maximally specific iff $\operatorname{cl}_{GG'} \tilde{x} = \tilde{x}$. Thus, Lemma 7 follows from Lemma 11.

Proof of Lemma 8 Reflexivity is obvious. Antisymmetry: $(\tilde{x} \leq \tilde{x}' \wedge \tilde{x}' \leq \tilde{x}) \Rightarrow (\operatorname{dom} \tilde{x} = \operatorname{dom} \tilde{x}' \wedge \forall e \in \operatorname{dom} \tilde{x} : \tilde{x}_e = \tilde{x}'_e)$. Transitivity: Let $\tilde{x} \leq \tilde{x}' \leq \tilde{x}''$. Then $\operatorname{dom} \tilde{x} \subseteq \operatorname{dom} \tilde{x}' \subseteq \operatorname{dom} \tilde{x}''$ and $\forall e \in \operatorname{dom} \tilde{x} : \tilde{x}_e = \tilde{x}'_e = \tilde{x}''_e$.

Proof of Lemma 9 We show first that $\tilde{x}'$ is maximal w.r.t. $\leq$ in $\tilde{X}_{GG'}[\tilde{x}]$ iff it is maximally specific. This implies existence and uniqueness of the maximum of $\tilde{X}_{GG'}[\tilde{x}]$ by construction via $\operatorname{dom} \tilde{x}' = E'[\tilde{x}]$.

Let $\tilde{x}' \in \tilde{X}_{GG'}[\tilde{x}]$ be maximally specific and suppose $\tilde{x}' \leq \tilde{x}''$ for some $\tilde{x}'' \in \tilde{X}_{GG'}[\tilde{x}]$. Then $\operatorname{dom} \tilde{x}'' = \operatorname{dom} \tilde{x}'$, since $X_{GG'}[\tilde{x}'] \neq X_{GG'}[\tilde{x}'']$ if $\operatorname{dom} \tilde{x}'' \setminus E'[\tilde{x}] \neq \emptyset$. Thus, $\tilde{x}' = \tilde{x}''$, which means $\tilde{x}'$ is maximal w.r.t. $\leq$ in $\tilde{X}_{GG'}[\tilde{x}]$.

Conversely, any maximal element $\tilde{x}'$ of $\tilde{X}_{GG'}[\tilde{x}]$ w.r.t. $\leq$ must satisfy $E'[\tilde{x}] \subseteq \operatorname{dom} \tilde{x}'$, which means it is maximally specific.

Hence, the unique maximum $\tilde{x}' \in \tilde{X}_{GG'}[\tilde{x}]$ is obtained as follows. For an arbitrary $x \in X_{GG'}[\tilde{x}]$ define $\tilde{x}'$ via $\tilde{x}'_e := x_e$ for all decided edges $e \in E'[\tilde{x}]$.

Proof of Theorem 3 Let us have $\tilde{x}, \tilde{x}' \in \tilde{X}_{GG'}$.

- The implication $X_{GG'}[\tilde{x}] = X_{GG'}[\tilde{x}'] \Rightarrow \tilde{X}_{GG'}[\tilde{x}] = \tilde{X}_{GG'}[\tilde{x}']$: follows from the definition of $\tilde{X}_{GG'}[\tilde{x}]$ in Lemma 9.
- The implication $\tilde{X}_{GG'}[\tilde{x}] = \tilde{X}_{GG'}[\tilde{x}'] \Rightarrow \operatorname{cl}_{GG'} \tilde{x} = \operatorname{cl}_{GG'} \tilde{x}'$ follows from the definition of the closure of $\tilde{x}$ as the maximum of $\tilde{X}_{GG'}[\tilde{x}]$.
- The implication $\operatorname{cl}_{GG'} \tilde{x} = \operatorname{cl}_{GG'} \tilde{x}' \Rightarrow X_{GG'}[\tilde{x}] = X_{GG'}[\tilde{x}']$ follows from $\operatorname{cl}_{GG'} \tilde{x} = \operatorname{cl}_{GG'} \tilde{x}' \in \tilde{X}_{GG'}[\tilde{x}]$.

Proof of Lemma 10 Let $x \in X_G$ and define $y = \operatorname{cl}_{GG'} x$. Since $\operatorname{dom} x = E$, it holds that $E'[x] = E'$, i.e. all edges are decided. Therefore, $y^{-1}(1)$ is a multicut of G' and for all $\{v,w\} = f \in E' \setminus E$ it holds that $y_f = 0$ iff there is a vw-path P in G such that $x_e = 0$ for all $e \in P$. By Lemma 5, this implies $y^{-1}(1) = \lambda_{GG'}(x^{-1}(1))$.

Proof of Theorem 4 Computing closures is at least as hard as deciding maximal specificity: To decide maximal specificity of $\tilde{x} \in \tilde{X}_{GG'}$, compute its closure $\operatorname{cl}_{GG'} \tilde{x}$. Then $\tilde{x}$ is maximally specific iff $\operatorname{dom} \tilde{x} = \operatorname{dom} \operatorname{cl}_{GG'} \tilde{x}$, i.e., if $\tilde{x} = \operatorname{cl}_{GG'} \tilde{x}$. By Theorem 2, this means computing closures is NP-hard.

Proof of Lemma 11 Let $\tilde{x} \in \tilde{X}_{GG'}$ and $\tilde{y} = \operatorname{cl}_{GG'} \tilde{x}$.

Suppose first that $E = E'$. We describe how to compute $\tilde{y}$ efficiently. Obviously, we must set $\tilde{y}_e = \tilde{x}_e$ for all $e \in \operatorname{dom} \tilde{x}$. Furthermore, we must set $\tilde{y}_{vw} = 0$ for all $\{v,w\} \in E \setminus \operatorname{dom} \tilde{x}$ such that there is a vw-path P in G with $\tilde{x}_e = 0$ for all $e \in P$. Moreover, we must set $\tilde{y}_{vw} = 1$ for all $\{v,w\} \in E \setminus \operatorname{dom} \tilde{x}$ that satisfy

$$\begin{aligned} &\exists P \in vw\text{-paths}(G)\ \exists! e \in P : \\ &\qquad \tilde{x}_e = 1\ \wedge\ \forall e' \in P \setminus \{e\} : \tilde{x}_{e'} = 0\ . \end{aligned} \tag{43}$$

Therefore, initialize a disjoint-set data structure with singleton sets V. Apply the union operation on all edges $e \in \operatorname{dom} \tilde{x}$ where $\tilde{x}_e = 0$, i.e. *contract* all 0-labeled edges. Then, set $\tilde{y}_e = 0$ for all edges that connect nodes of the same component. If there is an edge e' between two components such that $\tilde{x}_{e'} = 1$, then for all edges e between those components set $\tilde{y}_e = 1$. The remaining edges are undecided by $\tilde{x}$. In case we only want to decide maximal specificity, we can stop upon finding the first edge $e \in \operatorname{dom} \tilde{y} \setminus \operatorname{dom} \tilde{x}$.

Now suppose that $E \subseteq \operatorname{dom} \tilde{x}$. In this case, all edges are decided, because $\tilde{x}|_E \in X_G$. According to Lemma 10, the closure $\tilde{y}$ corresponds to the lifting of $\tilde{x}|_E$ to G'. Therefore, to obtain $\tilde{y}$, compute the decomposition of G associated to $\tilde{x}|_E$ using, e.g., a disjoint-set data structure. Set $\tilde{y}_e = 0$ if e is an edge within a component. Set $\tilde{y}_e = 1$ if e is an edge between components.

D. Metrics

Proof of Theorem 5 Symmetry and non-negativity follow directly from the definition, and so does $d^\mu_{E'}(x,x) = 0$ for all $x \in X_{GG''}$. For any $e \in E'$, the form d^1_e on $E' \times E'$ is a Hamming metric on words of length 1 from the alphabet $\{0,1\}$. Therefore, it satisfies the triangle inequality. Hence, for any $x, y, z \in X_{GG''}$:

$$\begin{aligned} d^\mu_{E'}(x,z) &= \sum_{e \in E'} \mu_e d^1_e(x,z) && (44)\\ &\le \sum_{e\in E'} \mu_e (d^1_e(x,y) + d^1_e(y,z)) && (45)\\ &= \sum_{e\in E'} \mu_e d^1_e(x,y) + \sum_{e\in E'} \mu_e d^1_e(y,z) && (46)\\ &= d^\mu_{E'}(x,y) + d^\mu_{E'}(y,z), && (47)\end{aligned}$$

Thus, $d^\mu_{E'}$ is a pseudo-metric on $X_{GG''}$.

If $E \subseteq E'$, then $G' = G''$ and thus, $X_{GG''} = X_{GG'} \subseteq X_{G'}$. For any two $x, x' \in X_{GG''} \subseteq X_{G'}$, it holds that $d^\mu_{E'}(x,x') = 0$ iff $d^1_e(x,x') = 0$ for all $e \in E'$, i.e. iff $x = x'$. Conversely, suppose there exists some $e \in E \setminus E'$. Define $x, x' \in X_{GG''}$ via $x_{e'} = x'_{e'} = 1$ for all $e' \in E'' \setminus \{e\}$ and $x_e = 1$, $x'_e = 0$. It holds that $x \neq x'$ but $d^\mu_{E'}(x,x') = 0$.

Proof of Theorem 6 We first prove that $\tilde{d}^\theta_{E'}$ is a metric on $\hat{X}_{GG'}$. For any $\tilde{x} \in \hat{X}_{GG'}$, we have $\operatorname{cl}_{GG'} \tilde{x} = \tilde{x}$. Thus, for all $\tilde{x}, \tilde{x}' \in \hat{X}_{GG'}$, we have $\tilde{d}^\theta_{E'}(\tilde{x},\tilde{x}') = d^\theta_{E'}(\tilde{x},\tilde{x}')$. Therefore, positive definiteness and symmetry are obvious from the definition of $d^\theta_{E'}(\tilde{x},\tilde{x}')$. To establish the triangle inequality for $d^\theta_{E'}$, we prove it for θd^1_e and any edge $e \in E'$. Let $\tilde{x}, \tilde{y}, \tilde{z} \in \hat{X}_{GG'}$ and consider the inequality

$$\theta d^1_e(\tilde{x},\tilde{z}) \le \theta d^1_e(\tilde{x},\tilde{y}) + \theta d^1_e(\tilde{y},\tilde{z}). \qquad (48)$$

Table 1. The left- and right-hand side of the inequality $\theta d^1_e(\tilde{x},\tilde{z}) \le \theta d^1_e(\tilde{x},\tilde{y}) + \theta d^1_e(\tilde{y},\tilde{z})$ for all possible combinations of values $\tilde{x}_e, \tilde{y}_e, \tilde{z}_e$ where $\tilde{x}, \tilde{y}, \tilde{z} \in \hat{X}_{GG'}$. The right-hand side is always greater or equal the left-hand side iff $0.5 \le \theta$.

$\tilde{x}_e$	$\tilde{y}_e$	$\tilde{z}_e$	lhs	rhs
0	0	0	0	0
1	1	1	0	0
*	*	*	0	0
0	*	1	1	2θ
0	1/0	1	1	1
0	0/*	*	θ	θ
1	1/*	*	θ	θ

In Tab. 1, the left-hand side and right-hand side of (48) are evaluated for all possible assignments of values to $\tilde{x}_e, \tilde{y}_e, \tilde{z}_e$. It is apparent form this table that (48) holds iff $\theta \ge 0.5$.

We now show that $\tilde{d}^\theta_{E'}$ is a pseudo-metric on $\tilde{X}_{GG'}$. Symmetry and non-negativity are obvious from the definition. For all $\tilde{x} \in \tilde{X}_{GG'}$, we have $\tilde{d}^\theta_{E'}(\tilde{x},\tilde{x}) = 0$. Since $\tilde{d}^\theta_{E'}(\tilde{x},\tilde{x}') = \tilde{d}^\theta_{E'}(\operatorname{cl}_{GG'}\tilde{x}, \operatorname{cl}_{GG'}\tilde{x}')$ and $\operatorname{cl}_{GG'}\tilde{x} \in \hat{X}_{GG'}$ for any $\tilde{x} \in \tilde{X}_{GG'}$, the triangle inequality follows from the fact that $\tilde{d}^\theta_{E'}$ is a metric on $\hat{X}_{GG'}$.

Finally, it holds that $\tilde{d}^\theta_{E'}(\tilde{x},\tilde{x}') = 0$ iff $\operatorname{cl}_{GG'}\tilde{x} = \operatorname{cl}_{GG'}\tilde{x}'$, which in turn is equivalent to $\tilde{X}_{GG'}[\tilde{x}] = \tilde{X}_{GG'}[\tilde{x}']$, by Theorem 3. This proves property (24).

E. Polyhedral Optimization

Proof of Theorem 7 The all-one vector $\mathbb{1} \in \{0,1\}^{E'}$ is such that $\mathbb{1} \in X_{GG'}$.

For any $e \in E$, $x^e \in \{0,1\}^{E'}$ such that $x^e_e = 0$ and $x^e_{E\setminus\{e\}} = 1$ and $x^e_{F_{GG'}} = 1$ holds $x^e \in X_{GG'}$.

For any $f \in F_{GG'}$, any f-feasible $x^f \in \{0,1\}^{E'}$ is such that $x^f \in X_{GG'}$. Moreover, x^f can be chosen such that one shortest path connecting the two nodes in f is the only component containing more than one node.

For any $e \in E$, let $y^e \in \mathbb{R}^{E'}$ such that

$$y^e = \mathbb{1} - x^e \ . \qquad (49)$$

For any $f \in F_1$, choose an f-feasible x^f and let $y^f \in \mathbb{R}^{E'}$ such that

$$y^f = \mathbb{1} - x^f - \sum_{\{e \in E | x^f_e = 0\}} y^e \ . \qquad (50)$$

For any $n \in \mathbb{N}$ such that $n > 1$ and any $f \in F_n$, choose an f-feasible x^f and let $y^f \in \mathbb{R}^{E'}$ such that

$$y^f = \mathbb{1} - x^f - \sum_{\{f' \in F_{GG'} | f' \neq f \wedge x^f_{f'} = 0\}} y^{f'} - \sum_{\{e \in E | x^f_e = 0\}} y^e \ . \qquad (51)$$

Here, $\ell(f') < \ell(f) \le n$, by definition of f-feasibility. Thus, all $y^{f'}$ are well-defined by induction (over n).

Observe that $\{y^e \mid e \in E'\}$ is the unit basis in $\mathbb{R}^{E'}$. Moreover, each of its elements is a linear combination of $\{\mathbb{1} - x^e \mid e \in E'\}$ which is therefore linearly independent.

Thus, $\{\mathbb{1}\} \cup \{x^e \mid e \in E'\}$ is affine independent. It is also a subset of $X_{GG'}$ and, therefore, a subset of $\Xi_{GG'}$. Thus, $\dim \Xi_{GG'} = |E'|$.

Proof of Lemma 12 Let $\{v, w\} = f \in F_{GG'}$ and let $d(v, w)$ the length of a shortest vw-path in G. Then, $d(v, w) > 1$ because $F_{GG'} \cap E = \emptyset$.

If $d(v, w) = 2$, there exists a $u \in V$ such that $\{v, u\} \in E$ and $\{u, w\} \in E$. Moreover, $\{v, u\} \notin F_{GG'}$ and $\{u, w\} \notin F_{GG'}$, as $F_{GG'} \cap E = \emptyset$. Thus, $f \in F_1$.

If $d(v, w) = m$ with $m > 2$, consider any shortest vw-path P in G. Moreover, let $F' \subseteq F_{GG'}$ such that, for any $\{v', w'\} = f' \in F_{GG'}$, $f' \in F'$ iff $v' \in P$ and $w' \in P$ and $f' \neq f$. If $F' = \emptyset$ then $f \in F_1$. Otherwise:

$$\forall \{v', w'\} \in F' : \quad d(v', w') < m \tag{52}$$

and thus:

$$\forall f' \in F' \, \exists n_{f'} \in \mathbb{N} : \quad f' \in F_{n_{f'}} \tag{53}$$

by induction (over m). Let

$$n = \max_{f' \in F'} n_{f'} \quad . \tag{54}$$

Then, $f \in F_{n+1}$.

Proof of Lemma 13 For any $\{v, w\} = f \in F_{GG'}$, let P be a shortest vw-path in G and let

$$F'_{GG'} := \{\{v', w'\} \in F_{GG'} \mid v' \in P \wedge w' \in P\} \tag{55}$$

$$F''_{GG'} := F_{GG'} \setminus F'_{GG'} \quad . \tag{56}$$

Moreover, let $x \in \{0, 1\}^{E'}$ with $x_P = 0$ and $x_{E \setminus P} = 1$ and $x_{F'_{GG'}} = 0$ and $x_{F''_{GG'}} = 1$. P has no chord in E, because it is a shortest path. Thus, $x \in X_{GG'}$.

Proof of Theorem 8 Let $S = \{x \in X_{GG'} \mid x_e = 1\}$ and put $\Sigma = \operatorname{conv} S$.

To show necessity, suppose there is some $vw = f \in F_{GG'}$ such that e connects a pair of v-w-cut-vertices. Then, for any vw-path P in G, either $e \in P$ or e is a chord of P. We claim that we have $x_f = 1$ for any $x \in S$. This gives $\dim \Sigma \le |E'| - 2$, so the inequality $x_e \le 1$ cannot define a facet of $\Xi_{GG'}$. If there are no vw-paths that have e as a chord, then $\{e\}$ is a vw-cut and the claim follows from the corresponding inequality of (6). Otherwise, every vw-path P that has e as a chord contains a subpath P' such that $P' \cup \{e\}$ is a cycle. Thus, for any $x \in S$, the inequalities (4) or (5) (for $e \in E$ or $e \in F_{GG'}$, respectively) imply the existence of some $e_{P'} \in P'$ such that $x_{e_{P'}} = 1$. Let $\mathcal{P}$ denote the set of all such paths P'. Apparently, the collection $\bigcup_{P' \in \mathcal{P}} \{e_{P'}\} \cup \{e\}$ is a v-w-separating set of edges. Therefore, it contains some subset C that is a vw-cut. This gives $x_f = 1$ via the inequality of (6) corresponding to C.

We turn to the proof of sufficiency. Assume there is no $vw = f \in F_{GG'}$ such that e connects a pair of v-w-cut-vertices in G. The construction of an affine independent $|E'|$-element-subset of $S \subset X_{GG'}$ is analogous to the proof of Theorem 7. The assumption guarantees for any $f \in F_{GG'}$ with $f \neq e$ the existence of an f-feasible $x \in S$ such that there is a vw-path P with $x_P = 0$. In particular, the hierarchy on $F_{GG'}$ defined by the level function ℓ remains unchanged (if $e \in F_{GG'}$, then $\ell(e) \ge \ell(f)$ for all $f \in F_{GG'}$). Hence, $\dim \Sigma = |E'| - 1$, which means Σ is a facet of $\Xi_{GG'}$.

Proof of Theorem 9 Let $S = \{x \in X_{GG'} \mid x_e = 0\}$ and put $\Sigma = \operatorname{conv} S$.

Consider the case that $e \in E$. Let $G_{[e]}$ and $G'_{[e]}$ be the graphs obtained from G and G', respectively, by contracting the edge e. The lifted multicuts $x^{-1}(1)$ for $x \in S$ correspond bijectively to the multicuts of $G'_{[e]}$ lifted from $G_{[e]}$. This implies $\dim \Sigma = \dim \Xi_{G_{[e]} G'_{[e]}}$. The claim follows from Theorem 7 and the fact that $G'_{[e]}$ has $|E'| - 1$ many edges if and only if e is not contained in any triangle in G'.

Now, suppose $uv = e \in F_{GG'}$. We show necessity of Conditions (a)-(c) by proving that if any of them is violated, then all $x \in S$ satisfy some additional, orthogonal equality and thus, $\dim \Sigma \le |E'| - 2$.

First, assume that (a) is violated. Hence, there are edges $e', e'' \in E'$ such that $T = \{e, e', e''\}$ is a triangle in G'. Every $x \in S$ satisfies the cycle inequalities

$$x_{e'} \le x_e + x_{e''} \tag{57}$$

$$x_{e''} \le x_e + x_{e'} \tag{58}$$

by Lemma 3 applied to the multicut $x^{-1}(1)$ of G'. Every $x \in S$ satisfies $x_{e'} = x_{e''}$, by (57) and (58) and $x_e = 0$.

Next, assume that (b) is violated. Consider a violating pair $\{u', v'\} \neq \{u, v\}$, $u' \neq v'$ of u-v-cut-vertices. For every $x \in S$, there exists a uv-path P in G with $x_P = 0$, as $x_e = 0$. Any such path P has a sub-path P' from u' to v' because u' and v' are u-v-cut-vertices.

- If the distance of u' and v' in G' is 1, then $u'v' \in E'$. If $u'v' \in P$, then $x_{u'v'} = 0$ because $x_P = 0$. Otherwise,

$x_{u'v'} = 0$ by $x_{P'} = 0$ and the cycle/path inequality

$$x_{u'v'} \leq \sum_{\hat{e} \in P'} x_{\hat{e}} \quad . \tag{59}$$

Thus $x_{u'v'} = 0$ for all $x \in S$.

- If the distance of u' and v' in G' is 2, there is a $u'v'$-path in G' consisting of two distinct edges $e', e'' \in E'$. We show that all $x \in S$ satisfy $x_{e'} = x_{e''}$:
  - If $e' \in P$ and $e'' \in P$ then $x_{e'} = x_{e''} = 0$ because $x_P = 0$.
  - If $e' \in P$ and $e'' \notin P$ then $x_{e'} = x_{e''} = 0$ by $x_P = 0$ and the cycle/path inequality

$$x_{e''} \leq \sum_{\hat{e} \in P' \setminus \{e'\}} x_{\hat{e}} \quad . \tag{60}$$

  - If $e' \notin P$ and $e'' \notin P$ then $x_{e'} = x_{e''}$ by $x_P = 0$ and the cycle/path inequalities

$$x_{e''} \leq x_{e'} + \sum_{\hat{e} \in P'} x_{\hat{e}} \tag{61}$$

$$x_{e'} \leq x_{e''} + \sum_{\hat{e} \in P'} x_{\hat{e}} \quad . \tag{62}$$

Now, assume that (c) is violated. Hence, there exists a u-v-cut-vertex t and a u-v-separating set of vertices $\{s, s'\}$ such that $\{ts, ts', ss'\}$ is a triangle in G'. We have that all $x \in S$ satisfy $x_{ss'} = x_{ts} + x_{ts'}$ as follows. At most one of x_{ts} and $x_{ts'}$ is 1, because t is a u-v-cut-vertex and $\{s, s'\}$ is u-v-separating as well. Moreover, $x_{ts} + x_{ts'} = 0$ if and only if $x_{ss'} = 0$.

Proof of Theorem 10 Note that both C and $P \cup \{f\}$ are cycles in G'. We show that, for any chordal cycle C' of G' and any $e \in C'$, the inequality

$$x_e \leq \sum_{e' \in C' \setminus \{e\}} x_{e'} \tag{63}$$

is not facet-defining for $\Xi_{G'}$. This implies that (63) cannot be facet-defining for $\Xi_{GG'}$ either, as $\Xi_{GG'} \subseteq \Xi_{G'}$ and $\dim \Xi_{GG'} = \dim \Xi_{G'}$. Hence, for facet-definingness of (4) and (5), it is necessary that C and $P \cup \{f\}$ be chordless in G'.

For this purpose, consider some cycle C' of G' with a chord $uv = e' \in E'$. We may write $C' = P_1 \cup P_2$ where P_1 and P_2 are edge-disjoint uv-paths such that $C_1 = P_1 \cup \{e'\}$ and $C_2 = P_2 \cup \{e'\}$ are cycles in G'. Let $e \in C'$, then either $e \in P_1$ or $e \in P_2$. W.l.o.g. we may assume $e \in P_1$. The inequalities

$$x_e \leq \sum_{e'' \in C_1 \setminus \{e\}} x_{e''}, \tag{64}$$

$$x_{e'} \leq \sum_{e'' \in C_2 \setminus \{e'\}} x_{e''} \tag{65}$$

are both valid for $\Xi_{G'}$. Moreover, since $e' \in C_1$, (64) and (65) imply (63) via

$$\begin{aligned} x_e &\leq \sum_{e'' \in C_1 \setminus \{e\}} x_{e''} = \sum_{e'' \in C_1 \setminus \{e, e'\}} x_{e''} + x_{e'} \\ &\leq \sum_{e'' \in C_1 \setminus \{e, e'\}} x_{e''} + \sum_{e'' \in C_2 \setminus \{e'\}} x_{e''} \\ &= \sum_{e'' \in C' \setminus \{e\}} x_{e''} . \end{aligned} \tag{66}$$

Thus, (63) is not facet-defining for $\Xi_{G'}$.

For the proof of sufficiency, suppose the cycle C of G is chordless in G' and let $e \in C$. Let Σ be a facet of $\Xi_{GG'}$ such that $\Sigma_{GG'}(e, C) \subseteq \Sigma$ and suppose it is induced by the inequality

$$\sum_{e' \in E'} a_{e'} x_{e'} \leq \alpha \tag{67}$$

with $a \in \mathbb{R}^{E'}$ and $\alpha \in \mathbb{R}$, i.e., $\Sigma = \operatorname{conv} S$, where

$$S := \left\{ x \in X_{GG'} \;\middle|\; \sum_{e' \in E'} a_{e'} x_{e'} = \alpha \right\} . \tag{68}$$

For convenience, we also define the linear space

$$L := \left\{ x \in \mathbb{R}^{E'} \;\middle|\; \sum_{e' \in E'} a_{e'} x_{e'} = \alpha \right\} . \tag{69}$$

As $0 \in S_{GG'}(e, C) \subseteq S$, we have $\alpha = 0$. We show that (67) is a scalar multiple of (4) and thus $\Sigma_{GG'}(e, C) = \Sigma$.

Let $y \in \{0, 1\}^{E'}$ be defined by

$$y_C = 0, \quad y_{E' \setminus C} = 1, \tag{70}$$

i.e. all edges except C are cut. Then $y \in S_{GG'}(e, C) \subseteq S$, since C is chordless.

For any $e' \in C \setminus \{e\}$, the vector $x \in \{0, 1\}^{E'}$ with

$$x_{C \setminus \{e, e'\}} = 0, \quad x_{E' \setminus C \cup \{e, e'\}} = 1 \tag{71}$$

holds $x \in S_{GG'}(e, C) \subseteq S$. Therefore, $y - x \in L$. Thus,

$$\forall e' \in C \setminus \{e\} : \quad a_{e'} = -a_e \quad . \tag{72}$$

It remains to show that $a_{e'} = 0$ for all edges $e' \in E' \setminus C$. We proceed by considering edges from E and $F_{GG'}$ separately. We consider the nodes $u, v \in V$ such that $uv = e'$. W.l.o.g., we assume that v does not belong to C. This is possible because C does not have a chord in G'.

Firstly, consider $e' \in E$ and distinguish the following cases:

(i) If e' connects two nodes not contained in C or it is the only edge connecting some node in C to v, then for $x \in \{0,1\}^{E'}$, defined by

$$x_C = 0, \quad x_{e'} = 0, \quad x_{E' \setminus (C \cup \{e'\})} = 1, \tag{73}$$

it holds that $x \in S_{GG'}(e, C) \subseteq S$. Therefore, $y - x \in L$, which evaluates to $a_{e'} = 0$.

(ii) Otherwise, let $E'_{C,v} := \{\{u', v\} \in E' \mid u' \text{ belongs to } C\}$ denote the set of edges in E' that connect v to some node in C. By assumption, we have that $|E'_{C,v}| \geq 2$. Now, pick some direction on C and traverse C from one endpoint of e to the other endpoint of e. We may order the edges $E'_{C,v} = \{e_1, \ldots, e_k\}$ such that the endpoint of e_i appears before the endpoint of e_{i+1} in the traversal of C. We show that $a_{e_i} = 0$ for all $1 \leq i \leq k$:

For the vector $x \in \{0,1\}^{E'}$ defined by

$$x_{e''} = \begin{cases} 0 & \text{if } e'' \in C \\ 0 & \text{if } e'' \in E'_{C,v} \\ 1 & \text{else,} \end{cases} \tag{74}$$

it holds that $x \in S_{GG'}(e, C) \subseteq S$. Therefore, $y - x \in L$. Thus:

$$\sum_{1 \leq i \leq k} a_{e_i} = 0. \tag{75}$$

Consider the $m \in \{1, \ldots, k\}$ such that $e' = e_m$. For any i with $1 \leq i \leq m-1$, consider the following construction that is illustrated also in Fig. 2: Let $e'' \in C$ be some edge between the endpoints of e_i and e_{i+1}. If $e_i \in E$, define $x \in \{0,1\}^{E'}$ via

$$x_e = x_{e''} = 1 \tag{76}$$
$$x_{C \setminus \{e, e''\}} = 0 \tag{77}$$
$$\forall j \leq i: \quad x_{e_j} = 0 \tag{78}$$
$$\forall j > i: \quad x_{e_j} = 1 \tag{79}$$

If $e_i \in F_{GG'}$, define $x \in \{0,1\}^{E'}$ via

$$x_e = x_{e''} = 1 \tag{80}$$
$$x_{C \setminus \{e, e''\}} = 0 \tag{81}$$
$$\forall j \leq i: \quad x_{e_j} = 1 \tag{82}$$
$$\forall j > i: \quad x_{e_j} = 0 \tag{83}$$

Either way, it holds that $x \in S_{GG'}(e, C) \subseteq S$ and thus, $y - x \in L$. If $e_i \in E$, this yields

$$0 = \sum_{1 \leq j \leq i} a_{e_j} - a_e - a_{e''} = \sum_{1 \leq j \leq i} a_{e_j} \tag{84}$$

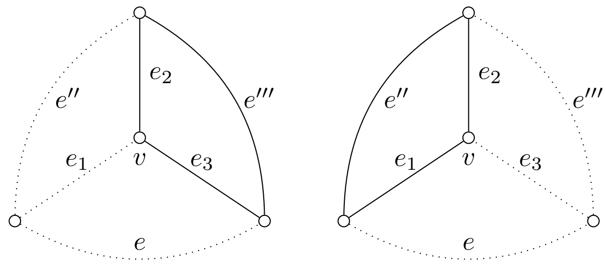

Figure 2. The figure illustrates the argument from case (ii) in the proof of Theorem 10 for the cycle $C = \{e, e'', e'''\}$. In this example, $e_3 = e'$, $e_1 \in F_{GG'}$ and $e_2 \in E$. The left multicut is chosen for $i = 1$ and the right one for $i = 2$.

by (72). If $e_i \in F_{GG'}$, we similarly obtain

$$0 = \sum_{i+1 \leq j \leq k} a_{e_j} - a_e - a_{e''} = \sum_{i+1 \leq j \leq k} a_{e_j}. \tag{85}$$

Together with (75), this yields $\sum_{1 \leq j \leq i} a_{e_j} = 0$ as well. Applying this argument repeatedly from $i = 1$ to $i = m-1$, we conclude that $a_{e_1} = \ldots = a_{e_{m-1}} = 0$. By reversing the order of the edges in $E'_{C,v}$, it can be shown analogously that $a_{e_k} = a_{e_{k-1}} = \ldots = a_{e_{m+1}} = 0$. Thus, by (75), $a_{e'} = a_{e_m} = 0$.

Next, consider $e' \in F_{GG'}$ and distinguish the following additional cases:

(iii) Suppose there is a uv-path P' in G that does not contain any node from C. Define $x \in \{0,1\}^{E'}$ via

$$x_{e''} = \begin{cases} 0 & \text{if } e'' \in C \\ 0 & \text{if } e'' = e' \\ 0 & \text{if } e'' \in P' \text{ or } e'' \text{ is a chord of } P' \\ 1 & \text{else.} \end{cases} \tag{86}$$

Then $x \in S_{GG'}(e, C) \subseteq S$ and thus $y - x \in L$. This gives

$$a_{e'} + \sum_{e'' \in P'} a_{e''} + \sum_{\substack{e'' \text{ chord} \\ \text{of } P'}} a_{e''} = 0. \tag{87}$$

We argue that all terms except $a_{e'}$ vanish by induction over the level function $\ell(e')$. If $\ell(e') = 1$, then P' does not have any chords from $F_{GG'}$, thus $a_{e'} = 0$, because $a_{e''} = 0$ for all $e'' \in E$ as shown previously in the cases (i) and (ii). If $\ell(e') > 1$, then for any chord $e'' \in F_{GG'}$ of P' it holds that $\ell(e'') < \ell(e')$. The induction hypothesis provides $a_{e''} = 0$ and hence we conclude $a_{e'} = 0$.

(iv) Suppose u is contained in C. Pick a shortest uv-path P' in G. We argue inductively over the length of P', which we denote by $d(P')$. If $d(P') = 1$, then P' consists of only one edge from E. This situation is in fact already covered by case (ii). If $d(P') > 1$, then we employ an argument similar to (ii) as follows. Let $F_{C,v} := \{\{u', v\} \in F_{GG'} \mid u' \text{ belongs to } C\} = \{f_1, \ldots, f_k\}$ be the set of edges $f_i \in F_{GG'}$ that connect v to some node in C. Again, assume they are ordered such that the endpoint of f_i appears before the endpoint of f_{i+1} on C in a traversal from e to itself. For the vector $x \in \{0,1\}^{E'}$ defined by

$$x_{e''} = \begin{cases} 0 & \text{if } e'' \in C \\ 0 & \text{if } e'' \in P' \text{ or } e'' \text{ is a chord of } P' \\ 0 & \text{if } e'' = u'v' \text{ where } u' \text{ belongs to } C, \\ & \qquad v' \neq v \text{ belongs to } P' \\ 0 & \text{if } e'' \in F_{C,v} \\ 1 & \text{else,} \end{cases} \tag{88}$$

it holds that $x \in S_{GG'}(e, C) \subseteq S$ and thus $y - x \in L$. This yields

$$\sum_{e'' \in P'} a_{e''} + \sum_{\substack{e'' \text{ chord} \\ \text{of } P'}} a_{e''} + \sum_{\substack{e''=u'v': \\ u' \text{ belongs to } C, \\ v' \neq v \text{ belongs to } P'}} a_{e''} + \sum_{e'' \in F_{C,v}} a_{e''} = 0 \tag{89}$$

and thus

$$\sum_{1 \leq i \leq k} a_{f_i} = \sum_{e'' \in F_{C,v}} a_{e''} = 0, \tag{90}$$

as all other terms vanish (apply the induction hypothesis to all $u'v' \in F_{GG'}$ where u' belongs to C and $v' \neq v$ belongs to P'). Let m be the highest index such that the endpoint of f_m appears before the endpoint of P' on C. Now, for any i with $1 \leq i \leq m$, pick an edge $e'' \in C$ between the endpoint of f_i and the endpoint of f_{i+1} and before the endpoint of P' on C. Define $x \in \{0,1\}^{E'}$ by

$$x_g = \begin{cases} 0 & \text{if } g \in C \setminus \{e, e''\} \\ 0 & \text{if } g \in P' \text{ or } g \text{ is a chord of } P' \\ 0 & \text{if } g = u'v' \text{ where} \\ & u' \text{ appears before endpoint of } P' \text{ on } C, \\ & v' \neq v \text{ belongs to } P' \\ 0 & \text{if } g = f_j \ \forall j > i \\ 1 & \text{else.} \end{cases} \tag{91}$$

Then, it holds that $x \in S_{GG'}(e, C) \subseteq S$ and thus $y - x \in L$. This yields, after removing all zero terms (apply the induction hypothesis once more),

$$\sum_{i+1 \leq j \leq k} a_{f_j} = 0. \tag{92}$$

Together with (90), we obtain

$$\sum_{1 \leq j \leq i} a_{f_i} = 0. \tag{93}$$

Applying this argument repeatedly for $i = 1$ to $i = m$, we conclude $a_{f_1} = \ldots = a_{f_m} = 0$. Similarly, we obtain $a_{f_k} = a_{f_{k-1}} = \ldots = a_{f_m} = 0$, by reversing the direction of traversal of C and employing the same reasoning.

(v) Finally, suppose neither u nor v belong to the cycle C, but every uv-path in G shares at least one node with C. Let P' be such a uv-path. Define the vector $x \in \{0,1\}^{E'}$ by

$$x_{e''} = \begin{cases} 0 & \text{if } e'' \in C \\ 0 & \text{if } e'' = e' \\ 0 & \text{if } e'' \in P' \text{ or } e'' \text{ is a chord of } P' \\ 0 & \text{if } e'' = u'v' \text{ where } u' \text{ belongs to } C, \\ & \qquad v' \text{ belongs to } P' \\ 1 & \text{else.} \end{cases} \tag{94}$$

It holds that $x \in S_{GG'}(e, C) \subseteq S$ and thus $y - x \in L$. This gives

$$a_{e'} + \sum_{e'' \in P'} a_{e''} + \sum_{\substack{e'' \text{ chord} \\ \text{of } P'}} a_{e''} + \sum_{\substack{e''=u'v': \\ u' \text{ belongs to } C, \\ v' \text{ belongs to } P'}} a_{e''} = 0. \tag{95}$$

We argue inductively over the level function $\ell(e')$. If $\ell(e') = 1$, then P' does not have any chords and our consideration in cases (i)–(iv) yield that all terms except $a_{e'}$ vanish. If $\ell(e') > 1$, then we additionally employ the induction hypothesis to achieve the same result. Hence, it holds that $a_{e'} = 0$ as well.

The proof of sufficiency in the second assertion is completely analogous (replace C by $P \cup \{f\}$ and e by f). The chosen multicuts remain valid, because $e = f$ is the only edge in the cycle that is not contained in E.

Proposition 1 *For every connected graph $G = (V, E)$, every graph $G' = (V, E')$ with $E \subseteq E'$, every $vw \in F_{GG'}$ and every $C \in vw$-cuts(G), the following holds:*

(a) Every $x \in S_{GG'}(vw, C)$ defines a decomposition of G into (vw, C)-connected components. That is, every maximal component of the graph $(V, \{e \in E | x_e = 0\})$ is (vw, C)-connected. At most one of these is properly (vw, C)-connected. It exists iff $x_{vw} = 0$.

(b) For every (vw, C)-connected component (V^*, E^*) of G, the $x \in \{0,1\}^{E'}$ such that $\forall rs \in E'(x_{rs} = 0 \Leftrightarrow r \in V^* \wedge s \in V^*)$ is such that $x \in S_{GG'}(vw, C)$.

Proof of Proposition 1 a) Let $x \in S_{GG'}(vw, C)$ arbitrary. Let $E_0 := \{e \in E | x_e = 0\}$ and let $G_0 := (V, E_0)$.

If $x_{vw} = 1$ then $\forall e \in C : x_e = 1$, by (35). Thus, every component of G_0 is improperly (vw, C)-connected.

If $x_{vw} = 0$ then

$$\exists e \in C(x_e = 0 \wedge \forall e' \in C \setminus \{e\}(x_{e'} = 1)) \tag{96}$$

by (35). Let (V^*, E^*) the maximal component of G_0 with

$$e \in E^* \quad . \tag{97}$$

Clearly:

$$\forall e' \in C \setminus \{e\} : \ e' \notin E^* \tag{98}$$

by (96) and definition of G_0. There does not exist a $C' \in vw$-cuts(G) with $x_{C'} = 1$, because this would imply $x_{vw} = 1$, by (6). Thus, there exists a $P \in vw$-paths(G) with $x_P = 0$, as G is connected. Any such path P has $e \in P$, as $P \cap C \neq \emptyset$ and $C \cap E_0 = \{e\}$ and $P \subseteq E_0$. Thus:

$$v \in V^* \wedge w \in V^* \tag{99}$$

by (97). (V^*, E^*) is properly (vw, C)-connected, by (97), (98) and (99). Any other component of G_0 does not cross the cut, by (96), (97) and definition of G_0, and is thus improperly (vw, C)-connected.

b) We have

$$\forall st \in E : \ x_{st} = 0 \Leftrightarrow st \in E^* \tag{100}$$

by the following argument:

- If $st \in E^*$, then $s \in V^* \wedge t \in V^*$, as (V^*, E^*) is a graph. Thus, $x_{st} = 0$, by definition of x.
- If $st \notin E^*$ then $s \notin V^* \vee t \notin V^*$, as (V^*, E^*) is a component of G. Thus, $x_{st} = 1$, by definition of x.

Consider the decomposition of G into (V^*, E^*) and singleton components. $E_1 := \{e \in E | x_e = 1\}$ is the set of edges that straddle distinct components of this decomposition, by (100). Therefore, E_1 is a multicut of G, by Lemma 2. Thus, (4) holds, by Lemma 3.

For any $st \in F_{GG'}$ and any $P \in st$-paths(G), distinguish two cases:

- If $P \subseteq E^*$, then $s \in V^* \wedge t \in V^*$, as (V^*, E^*) is a graph. Thus, $x_{st} = 0$, by definition of x. Moreover, $x_P = 0$, by (100). Hence, (5) evaluates to $0 = 0$.
- Otherwise, there exists an $e \in P$ such that $e \notin E^*$. Therefore, $x_e = 1$, by (100). Thus, (5) holds, as the r.h.s. is at least 1.

For any $st \in F_{GG'}$ and any $C' \in st$-cuts(G), distinguish two cases:

- If $C' \cap E^* = \emptyset$ then $s \notin V^* \vee t \notin V^*$. Therefore, $x_{st} = 1$, by definition of x. Moreover, $x_{C'} = 1$, by (100). Thus, (6) evaluates to $0 = 0$.
- Otherwise, there exists an $e \in C'$ such that $e \in E^*$. Therefore, $x_e = 0$, by (100). Thus, (6) holds, as the r.h.s. is at least 1.

Proof of Theorem 11 Assume that C1 does not hold (as in Fig. 4a). Then, there exists an $e \in C$ such that no (vw, C)-connected component of G contains e. Thus, for all $x \in S_{GG'}(vw, C)$:

$$x_e = 1 \tag{101}$$

by Proposition 1. Now, $\dim \Sigma_{GG'}(vw, C) \leq |E'| - 2$, by (35) and (101). Thus, $\Sigma_{GG'}(vw, C)$ is not a facet of $\Xi_{GG'}$, by Theorem 7.

Assume that C2 does not hold. Then, for any $e \in C$ there exists some number m such that for all (vw, C)-connected components (V^*, E^*) with $e \in E^*$ it holds that $|F \cap F_{V^*}| = m$. Thus, we can write

$$C = \bigcup_{m=0}^{|F|} C(F, m), \tag{102}$$

where $C(F, m) := \big\{e \in C \mid |F \cap F_{V^*}| = m \ \forall \ (vw, C)$-connected (V^*, E^*) with $e \in E^*\big\}$. It follows that for all $x \in S_{GG'}(vw, C)$ we have the equality

$$\sum_{m=0}^{|F|} m \sum_{e \in C(F,m)} (1 - x_e) = \sum_{f' \in F} (1 - x_{f'}) \tag{103}$$

by the following argument:

- If $x_e = 1$ for all $e \in C$, then $x_{f'} = 1$ for all $v'w' = f' \in F$, since C is also a $v'w'$-cut. Thus, (103) evaluates to $0 = 0$.
- Otherwise there exists precisely one edge $e \in C$ such that $x_e = 0$. Let m be such that $e \in C(F, m)$. By definition of $C(F, m)$, there are exactly m edges $f' \in F$ with $x_{f'} = 0$. Thus, (103) evaluates to $m = m$.

Assume that condition C3 does not hold. Then there exists an $f' \in F_{GG'}(vw, C)$, a set $\emptyset \neq F \subseteq F_{GG'}(vw, C)$ and

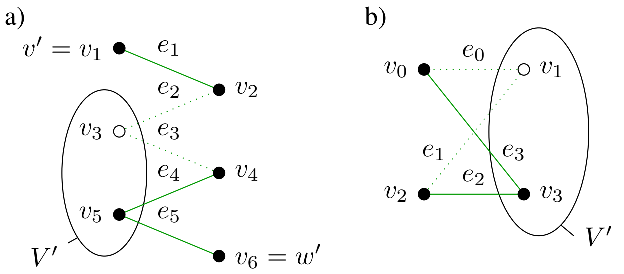

Figure 3. Depicted are the nodes (in black) and edges (in green) on a path **(a)** and on a cycle **(b)**, respectively. Nodes in the set V' are are either in V^* (filled circle) or not in V^* (open circle). Consequently, pairs of consecutive edges are either cut (dotted lines) or not cut (solid lines).

some $k \in \mathbb{N}$ such that for all (vw, C) connected components (V^*, E^*) and (V^{**}, E^{**}) with $f' \in F_{V^*}$ and $f' \notin F_{V^{**}}$ it holds that

$$|F \cap F_{V^*}| = k \text{ and } |F \cap F_{V^{**}}| = 0. \tag{104}$$

In other words, for all $x \in S_{GG'}(vw, C)$ it holds that $x_{f'} = 0$ iff there are exactly k edges $f'' \in F$ such that $x_{f''} = 0$. Similarly, it holds that $x_{f'} = 1$ iff for all $f'' \in F$ we have $x_{f''} = 1$. Therefore, all $x \in S_{GG'}(vw, C)$ satisfy the additional equality

$$k(1 - x_{f'}) = \sum_{f'' \in F} 1 - x_{f''}. \tag{105}$$

Assume that C4 does not hold. Then, there exist $v' \in V(v, C)$ and $w' \in V(w, C)$ and a $v'w'$-path $P = (V_P, E_P)$ in $G'(vw, C)$ such that every properly (vw, C)-connected component (V^*, E^*) of G holds:

$$(v' \in V^* \;\wedge\; V(w, C) \cap V_P \subseteq V^*) \tag{106}$$
$$\vee \quad (w' \in V^* \;\wedge\; V(v, C) \cap V_P \subseteq V^*) \;. \tag{107}$$

Let $v_1 < \cdots < v_{|V_P|}$ the linear order of the nodes V_P and let $e_1 < \cdots < e_{|E_P|}$ the linear order of the edges E_P in the $v'w'$-path P. Now, for all $x \in S_{GG'}(vw, C)$:

$$x_{vw} = \sum_{j=1}^{|E_P|} (-1)^{j+1} x_{e_j} \tag{108}$$

by the following argument: $|E_P|$ is odd, as the path P alternates between the set $V(v, C)$ where it begins and the set $V(w, C)$ where it ends. Thus,

$$\sum_{j=1}^{|E_P|} (-1)^{j+1} x_{e_j} = x_{e_1} - \sum_{j=1}^{(|E_P|-1)/2} (x_{e_{2j}} - x_{e_{2j+1}}) \;. \tag{109}$$

Distinguish two cases:

- If $x_{vw} = 1$, then $x_{E_P} = 1$, by (35) and (6). Thus, (108) evaluates to $1 = 1$, by (109).

- If $x_{vw} = 0$, the decomposition of G defined by x contains precisely one properly (vw, C)-connected component (V^*, E^*) of G, by Proposition 1. Without loss of generality, (106) holds. Otherwise, that is, if (107) holds, exchange v and w.

  Consider the nodes V_P as depicted in Fig. 3a: $v_1 = v' \in V^*$, by (106). For every even j, $v_j \in V(w, C)$, by definition of P. Thus:

  $$\forall j \in \{1, \ldots, (|E_P| + 1)/2\} : \; v_{2j} \in V^* \tag{110}$$

  by (106).

  Consider the edges E_P as depicted in Fig. 3a: $e_1 = v_1 v_2 \in E^*$, as $v_1 \in V^*$ and $v_2 \in V^*$ and as (V^*, E^*) is a component of G. Thus,

  $$x_{e_1} = 0 \tag{111}$$

  by Proposition 1. For every $j \in \{1, \ldots, (|E_P|-1)/2\}$, distinguish two cases:

  - If $v_{2j+1} \in V^*$, then $e_{2j} = v_{2j} v_{2j+1} \in E^*$ and $e_{2j+1} = v_{2j+1} v_{2j+2} \in E^*$, because $v_{2j} \in V^*$ and $v_{2j+2} \in V^*$, by (110), and because (V^*, E^*) is a component of G. Thus:

    $$x_{e_{2j}} = 0 \;\wedge\; x_{e_{2j+1}} = 0 \;. \tag{112}$$

  - If $v_{2j+1} \notin V^*$, then $e_{2j} = v_{2j} v_{2j+1}$ and $e_{2j+1} = v_{2j+1} v_{2j+2}$ straddle distinct components of the decomposition of G defined by x, because $v_{2j} \in V^*$ and $v_{2j+2} \in V^*$, by (110). Thus:

    $$x_{e_{2j}} = 1 \;\wedge\; x_{e_{2j+1}} = 1 \;. \tag{113}$$

  In any case:

  $$\forall j \in \{1, \ldots, (|E_P| - 1)/2\} : \; x_{e_{2j}} - x_{e_{2j+1}} = 0 \;. \tag{114}$$

  Thus, (108) evaluates to $0 = 0$, by (109), (111), (114).

Assume that C5 does not hold. Then, there exists a cycle $Y = (V_Y, E_Y)$ in $G'(vw, C)$ such that every properly (vw, C)-connected component (V^*, E^*) of G holds:

$$V_Y \cap V(v, C) \subseteq V^* \tag{115}$$
$$\vee \quad V_Y \cap V(w, C) \subseteq V^* \;. \tag{116}$$

Let $v_0 < \cdots < v_{|V_Y|-1}$ an order on V_Y such that $v_0 \in V(v, C)$ and, for all $j \in \{0, \ldots, |E_Y| - 1\}$:

$$e_j := \{v_j, v_{j+1 \bmod |E_Y|}\} \in E_Y \;. \tag{117}$$

Now, for all $x \in S_{GG'}(vw, C)$:

$$0 = \sum_{j=0}^{|E_Y|-1} (-1)^j x_{e_j} \tag{118}$$

by the following argument: $|E_Y|$ is even, as the cycle Y alternates between the sets $V(v, C)$ and $V(w, C)$. Thus,

$$\sum_{j=0}^{|E_Y|-1} (-1)^j x_{e_j} = \sum_{j=0}^{(|E_Y|-2)/2} (x_{e_{2j}} - x_{e_{2j+1}}) \ . \tag{119}$$

Distinguish two cases:

- If $x_{vw} = 1$, then $x_{E_Y} = 1$, by (35) and (6). Thus, (118) evaluates to $0 = 0$, by (119).
- If $x_{vw} = 0$, the decomposition of G defined by x contains precisely one properly (vw, C)-connected component (V^*, E^*) of G, by Proposition 1. Without loss of generality, (115) holds. Otherwise, that is, if (116) holds, exchange v and w.

  Consider the nodes V_Y as depicted in Fig. 3b: For every even j, $v_j \in V(v, C)$, by definition of Y and the order. Thus:

  $$\forall j \in \{0, \ldots, (|E_Y| - 2)/2\} : \ v_{2j} \in V^* \tag{120}$$

  by (115).

  Consider the edges E_Y as depicted in Fig. 3b: For every $j \in \{0, \ldots, (|E_Y| - 2)/2\}$, distinguish two cases:

  - If $v_{2j+1} \in V^*$, then $e_{2j} = v_{2j}v_{2j+1} \in E^*$ and $e_{2j+1} = v_{2j+1}v_{2j+2 \bmod |E_Y|} \in E^*$, because $v_{2j} \in V^*$ and $v_{2j+2 \bmod |E_Y|} \in V^*$, by (120), and because (V^*, E^*) is a component of G. Thus:

    $$x_{e_{2j}} = 0 \ \wedge \ x_{e_{2j+1}} = 0 \ . \tag{121}$$

  - If $v_{2j+1} \notin V^*$, then $e_{2j} = v_{2j}v_{2j+1}$ and $e_{2j+1} = v_{2j+1}v_{2j+2 \bmod |E_Y|}$ straddle distinct components of the decomposition of G defined by x, because $v_{2j} \in V^*$ and $v_{2j+2 \bmod |E_Y|} \in V^*$, by (120). Thus:

    $$x_{e_{2j}} = 1 \ \wedge \ x_{e_{2j+1}} = 1 \ . \tag{122}$$

  In any case:

  $$\forall j \in \{0, \ldots, (|E_Y| - 2)/2\} : \ x_{e_{2j}} - x_{e_{2j+1}} = 0 \ . \tag{123}$$

  Thus, (118) evaluates to $0 = 0$, by (119) and (123).

Acknowledgements

The examples depicted in Fig. 4h and 4j were proposed by Banafsheh Grochulla and Ashkan Mokarian, respectively.

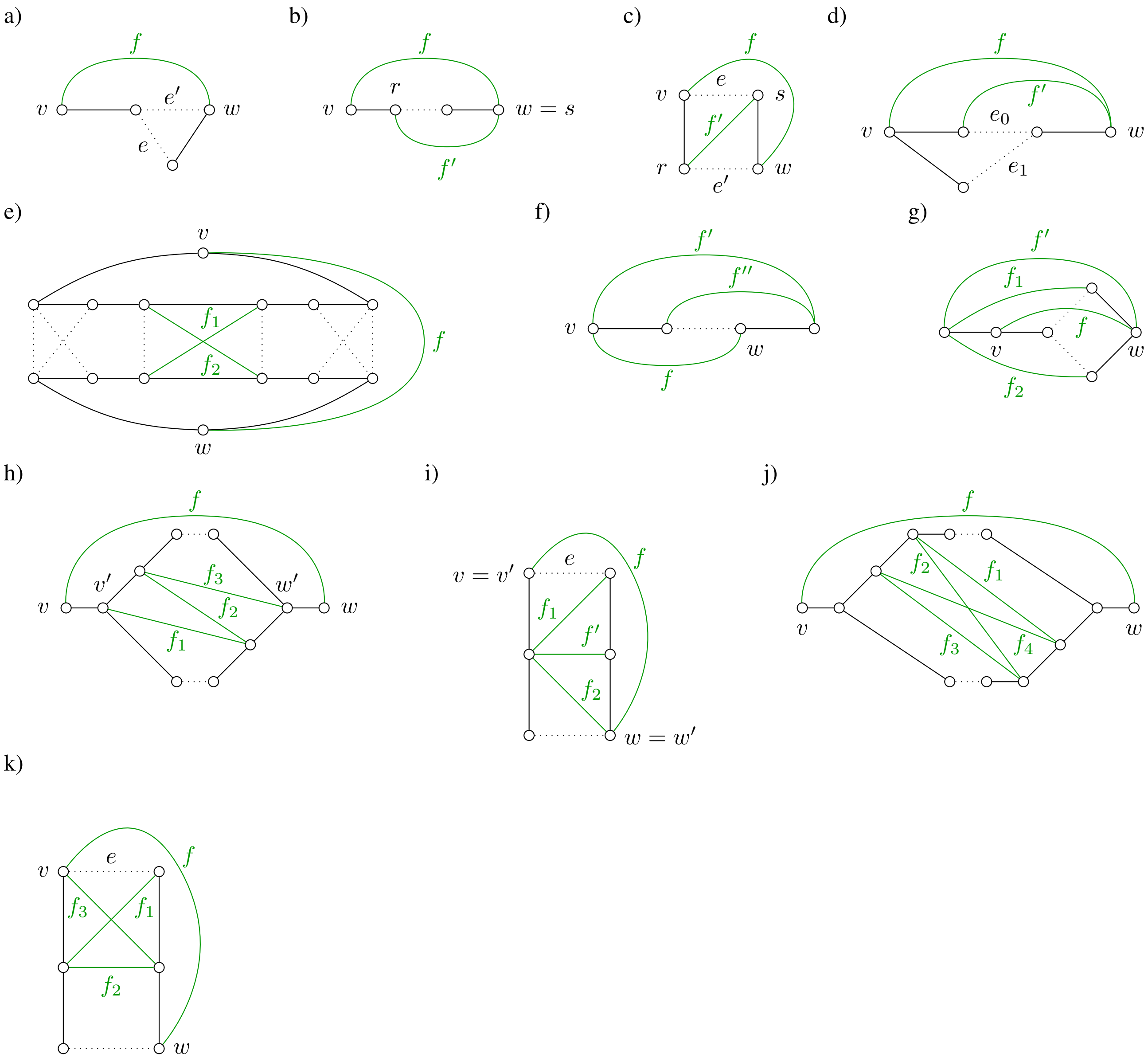

Figure 4. Depicted above are graphs $G = (V, E)$ (in black) and $G' = (V, E')$ with $E \subseteq E'$ (E' in green), distinct nodes $v, w \in V$ and a vw-cut C of G (as dotted lines). In any of the above examples, one condition of Theorem 11 is violated and thus, $\Sigma_{GG'}(vw, C)$ is not a facet of the lifted multicut polytope $\Xi_{GG'}$. **a)** Condition C1 is violated for e. **b)** Condition C2 is violated as r and s are connected in any (vw, C)-connected component. **c)** Condition C2 is violated as r and s are not connected in any (vw, C)-connected component. **d)** Condition C2 is violated. Specifically, $C(\{f'\}, 1) = \{e_0\}$ and $C(\{f'\}, 0) = \{e_1\}$ in the proof of Theorem 11. **e)** Condition C2 is violated for $F = \{f_1, f_2\}$. **f)** Condition C3 is violated. **g)** Condition C3 is violated for $F = \{f_1, f_2\}$ and $k = 1$. **h)** Condition C4 is violated for the $v'w'$-path $f_1 f_2 f_3$. **i)** Condition C4 is violated for the $v'w'$-path $e f_1 f_2$. **j)** Condition C5 is violated for the cycle $f_1 f_2 f_3 f_4$. **k)** Condition C5 is violated for the cycle $e f_1 f_2 f_3$.