Supplementary Material

"Robust Submodular Maximization: A Non-Uniform Partitioning Approach" (ICML 2017)

Ilija Bogunovic, Slobodan Mitrović, Jonathan Scarlett, and Volkan Cevher

A. Proof of Proposition 4.1

We have

$$\begin{aligned} |S_0| &= \sum_{i=0}^{\lceil \log \tau \rceil} \lceil \tau / 2^i \rceil 2^i \eta \\ &\leq \sum_{i=0}^{\lceil \log \tau \rceil} \left(\frac{\tau}{2^i} + 1 \right) 2^i \eta \\ &\leq \eta(\lceil \log \tau \rceil + 1)(\tau + 2^{\lceil \log \tau \rceil}) \\ &\leq 3\eta\tau(\lceil \log \tau \rceil + 1) \\ &\leq 3\eta\tau(\log k + 2). \end{aligned}$$

B. Proof of Proposition 4.4

Recalling that $\mathcal{A}_j(T)$ denotes a set constructed by the algorithm after j iterations, we have

$$\begin{aligned} f(\mathcal{A}_j(T)) - f(\mathcal{A}_{j-1}(T)) &\geq \frac{1}{\beta} \max_{e \in T} f(e|\mathcal{A}_{j-1}(T)) \\ &\geq \frac{1}{\beta} \max_{e \in T} f(e|\mathcal{A}_k(T)) \\ &\geq \frac{1}{\beta} \max_{e \in T \setminus \mathcal{A}_k(T)} f(e|\mathcal{A}_k(T)), \end{aligned} \tag{13}$$

where the first inequality follows from the β-iterative property (6), and the second inequality follows from $\mathcal{A}_{j-1}(S) \subseteq \mathcal{A}_k(S)$ and the submodularity of f.

Continuing, we have

$$\begin{aligned} f(\mathcal{A}_k(T)) &= \sum_{j=1}^{k} f(\mathcal{A}_j(T)) - f(\mathcal{A}_{j-1}(T)) \\ &\geq \frac{k}{\beta} \max_{e \in T \setminus \mathcal{A}_k(T)} f(e|\mathcal{A}_k(T)), \end{aligned}$$

where the last inequality follows from (13).

By rearranging, we have for any $e \in T \setminus \mathcal{A}_k(T)$ that

$$f(e|\mathcal{A}_k(T)) \leq \beta \frac{f(\mathcal{A}_k(T))}{k}.$$

C. Proof of Lemma 4.3

Recalling that $A_j(T)$ denotes the set constructed after j iterations when applied to T, we have

$$\begin{aligned} \max_{e \in T \setminus A_{j-1}(T)} f(e|A_{j-1}(T)) &\geq \frac{1}{k} \sum_{e \in \mathrm{OPT}(k,T) \setminus A_{j-1}(T)} f(e|A_{j-1}(T)) \\ &\geq \frac{1}{k} f(\mathrm{OPT}(k,T)|A_{j-1}(T)) \\ &\geq \frac{1}{k} \big(f(\mathrm{OPT}(k,T)) - f(A_{j-1}(T)) \big), \end{aligned} \tag{14}$$

where the first line holds since the maximum is lower bounded by the average, the line uses submodularity, and the last line uses monotonicity.

By combining the β-iterative property with (14), we obtain

$$f(\mathcal{A}_j(T)) - f(\mathcal{A}_{j-1}(T)) \geq \frac{1}{\beta} \max_{e \in T \setminus A_{j-1}(T)} f(e|A_{j-1}(T)) \\ \geq \frac{1}{k\beta}\big(f(\mathrm{OPT}(k,T)) - f(A_{j-1}(T))\big).$$

By rearranging, we obtain

$$f(\mathrm{OPT}(k,T)) - f(A_{j-1}(T)) \leq \beta k\big(f(\mathcal{A}_j(T)) - f(\mathcal{A}_{j-1}(T))\big). \tag{15}$$

We proceed by following the steps from the proof of Theorem 1.5 in (Krause & Golovin, 2012). Defining $\delta_j := f(\mathrm{OPT}(k,T)) - f(A_j(T))$, we can rewrite (15) as $\delta_{j-1} \leq \beta k(\delta_{j-1} - \delta_j)$. By rearranging, we obtain

$$\delta_j \leq \left(1 - \frac{1}{\beta k}\right)\delta_{j-1}.$$

Applying this recursively, we obtain $\delta_l \leq \big(1 - \frac{1}{\beta k}\big)^l \delta_0$, where $\delta_0 = f(\mathrm{OPT}(k,T))$ since f is normalized (i.e., $f(\emptyset) = 0$). Finally, applying $1 - x \leq e^{-x}$ and rearranging, we obtain

$$f(\mathcal{A}_l(T)) \geq \left(1 - e^{-\frac{l}{\beta k}}\right) f(\mathrm{OPT}(k,T)).$$

D. Proof of Theorem 4.5

D.1. Technical Lemmas

We first provide several technical lemmas that will be used throughout the proof. We begin with a simple property of submodular functions.

Lemma D.1 *For any submodular function f on a ground set V, and any sets $A, B, R \subseteq V$, we have*

$$f(A \cup B) - f(A \cup (B \setminus R)) \leq f(R \mid A).$$

Proof. Define $R_2 := A \cap R$, and $R_1 := R \setminus A = R \setminus R_2$. We have

$$\begin{aligned} f(A \cup B) - f(A \cup (B \setminus R)) &= f(A \cup B) - f((A \cup B) \setminus R_1) \\ &= f(R_1 \mid (A \cup B) \setminus R_1) \\ &\leq f(R_1 \mid (A \setminus R_1)) && (16) \\ &= f(R_1 \mid A) && (17) \\ &= f(R_1 \cup R_2 \mid A) && (18) \\ &= f(R \mid A), \end{aligned}$$

where (16) follows from the submodularity of f, (17) follows since A and R_1 are disjoint, and (18) follows since $R_2 \subseteq A$. □

The next lemma provides a simple lower bound on the maximum of two quantities; it is stated formally since it will be used on multiple occasions.

Lemma D.2 *For any set function f, sets A, B, and constant $\alpha > 0$, we have*

$$\max\{f(A), f(B) - \alpha f(A)\} \geq \left(\frac{1}{1+\alpha}\right) f(B), \tag{19}$$

and

$$\max\{\alpha f(A), f(B) - f(A)\} \geq \left(\frac{\alpha}{1+\alpha}\right) f(B). \tag{20}$$

Proof. Starting with (19), we observe that one term is increasing in $f(A)$ and the other is decreasing in $f(A)$. Hence, the maximum over all possible $f(A)$ is achieved when the two terms are equal, i.e., $f(A) = \frac{1}{1+\alpha} f(B)$. We obtain (20) via the same argument. □

The following lemma relates the function values associated with two buckets formed by PRO, denoted by X and Y. It is stated with respect to an arbitrary set E_Y, but when we apply the lemma, this will correspond to the elements of Y that are removed by the adversary.

Lemma D.3 *Under the setup of Theorem 4.5, let X and Y be buckets of* PRO *such that Y is constructed at a later time than X. For any set $E_Y \subseteq Y$, we have*

$$f(X \cup (Y \setminus E_Y)) \geq \frac{1}{1+\alpha} f(Y),$$

and

$$f(E_Y \mid X) \leq \alpha f(X), \tag{21}$$

where $\alpha = \beta \frac{|E_Y|}{|X|}$.

Proof. Inequality (21) follows from the β-iterative property of $\mathcal{A}$; specifically, we have from (8) that

$$f(e|X) \leq \beta \frac{f(X)}{|X|},$$

where e is any element of the ground set that is neither in X nor any bucket constructed before X. Hence, we can write

$$f(E_Y \mid X) \leq \sum_{e \in E_Y} f(e|X) \leq \beta \frac{|E_Y|}{|X|} f(X) = \alpha f(X),$$

where the first inequality is by submodularity. This proves (21).

Next, we write

$$f(Y) - f(X \cup (Y \setminus E_Y)) \leq f(X \cup Y) - f(X \cup (Y \setminus E_Y)) \tag{22}$$

$$\leq f(E_Y \mid X), \tag{23}$$

where (22) is by monotonicity, and (23) is by Lemma D.1 with $A = X$, $B = Y$, and $R = E_Y$.

Combining (21) and (23), together with the fact that $f(X \cup (Y \setminus E_Y)) \geq f(X)$ (by monotonicity), we have

$$f(X \cup (Y \setminus E_Y)) \geq \max\{f(X), f(Y) - \alpha f(X)\}$$

$$\geq \frac{1}{1+\alpha} f(Y), \tag{24}$$

where (24) follows from (19). □

Finally, we provide a lemma that will later be used to take two bounds that are known regarding the previously-constructed buckets, and use them to infer bounds regarding the next bucket.

Lemma D.4 *Under the setup of Theorem 4.5, let Y and Z be buckets of* PRO *such that Z is constructed at a later time than Y, and let $E_Y \subseteq Y$ and $E_Z \subseteq Z$ be arbitrary sets. Moreover, let X be a set (not necessarily a bucket) such that*

$$f((Y \setminus E_Y) \cup X) \geq \frac{1}{1+\alpha} f(Y), \tag{25}$$

and

$$f(E_Y \mid X) \leq \alpha f(X). \tag{26}$$

Then, we have

$$f(E_Z \mid (Y \setminus E_Y) \cup X) \leq \alpha_{\text{next}} f((Y \setminus E_Y) \cup X), \tag{27}$$

and

$$f((Z \setminus E_Z) \cup (Y \setminus E_Y) \cup X) \geq \frac{1}{1+\alpha_{\text{next}}} f(Z), \tag{28}$$

where

$$\alpha_{\text{next}} = \beta \frac{|E_Z|}{|Y|}(1+\alpha) + \alpha. \tag{29}$$

Proof. We first prove (27):

$$\begin{aligned}
f(E_Z \mid (Y \setminus E_Y) \cup X) &= f((Y \setminus E_Y) \cup X \cup E_Z) - f((Y \setminus E_Y) \cup X) \\
&\leq f(X \cup Y \cup E_Z) - f((Y \setminus E_Y) \cup X) && (30)\\
&= f(E_Z \mid X \cup Y) + f(X \cup Y) - f((Y \setminus E_Y) \cup X) \\
&\leq f(E_Z \mid Y) + f(X \cup Y) - f((Y \setminus E_Y) \cup X) && (31)\\
&\leq \beta \frac{|E_Z|}{|Y|} f(Y) + f(X \cup Y) - f((Y \setminus E_Y) \cup X) && (32)\\
&\leq \beta \frac{|E_Z|}{|Y|}(1+\alpha) f((Y \setminus E_Y) \cup X) + f(X \cup Y) - f((Y \setminus E_Y) \cup X) && (33)\\
&\leq \beta \frac{|E_Z|}{|Y|}(1+\alpha) f((Y \setminus E_Y) \cup X) + f(E_Y \mid (Y \setminus E_Y) \cup X) && (34)\\
&\leq \beta \frac{|E_Z|}{|Y|}(1+\alpha) f((Y \setminus E_Y) \cup X) + f(E_Y \mid X) && (35)\\
&\leq \beta \frac{|E_Z|}{|Y|}(1+\alpha) f((Y \setminus E_Y) \cup X) + \alpha f(X) && (36)\\
&\leq \beta \frac{|E_Z|}{|Y|}(1+\alpha) f((Y \setminus E_Y) \cup X) + \alpha f((Y \setminus E_Y) \cup X) && (37)\\
&= \left(\beta \frac{|E_Z|}{|Y|}(1+\alpha) + \alpha\right) f((Y \setminus E_Y) \cup X)., && (38)
\end{aligned}$$

where: (30) and (31) follow by monotonicity and submodularity, respectively; (32) follows from the second part of Lemma D.3; (33) follows from (25); (34) is obtained by applying Lemma D.1 for $A = X$, $B = Y$, and $R = E_Y$; (35) follows by submodularity; (36) follows from (26); (37) follows by monotonicity. Finally, by defining $\alpha_{\text{next}} := \beta \frac{|E_Z|}{|Y|}(1+\alpha) + \alpha$ in (38) we establish the bound in (27).

In the rest of the proof, we show that (28) holds as well. First, we have

$$f((Z \setminus E_Z) \cup (Y \setminus E_Y) \cup X) \geq f(Z) - f(E_Z \mid (Y \setminus E_Y) \cup X) \tag{39}$$

by Lemma D.1 with $B = Z$, $R = E_Z$ and $A = (Y \setminus E_Y) \cup X$. Now we can use the derived bounds (38) and (39) to obtain

$$\begin{aligned}
f((Z \setminus E_Z) \cup (Y \setminus E_Y) \cup X) &\geq f(Z) - f(E_Z \mid (Y \setminus E_Y) \cup X) \\
&\geq f(Z) - \left(\beta \frac{|E_Z|}{|Y|}(1+\alpha) + \alpha\right) f((Y \setminus E_Y) \cup X).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
f((Z \setminus E_Z) \cup (Y \setminus E_Y) \cup X) &\geq \max\left\{ f((Y \setminus E_Y) \cup X), f(Z) - \left(\beta \frac{|E_Z|}{|Y|}(1+\alpha) + \alpha\right) f((Y \setminus E_Y) \cup X)\right\} \\
&\geq \frac{1}{1+\alpha_{\text{next}}} f(Z),
\end{aligned}$$

where the last inequality follows from Lemma D.1. □

Observe that the results we obtain on $f(E_Z \mid (Y \setminus E_Y) \cup X)$ and on $f((Z \setminus E_Z) \cup (Y \setminus E_Y) \cup X)$ in Lemma D.4 are of the same form as the pre-conditions of the lemma. This will allow us to apply the lemma recursively.

D.2. Characterizing the Adversary

Let E denote a set of elements removed by an adversary, where $|E| \le \tau$. Within S_0, PRo constructs $\lceil \log \tau \rceil + 1$ partitions. Each partition $i \in \{0, \dots, \lceil \log \tau \rceil\}$ consists of $\lceil \tau/2^i \rceil$ buckets, each of size $2^i \eta$, where $\eta \in \mathbb{N}$ will be specified later. We let B denote a generic bucket, and define E_B to be all the elements removed from this bucket, i.e. $E_B = B \cap E$.

The following lemma identifies a bucket in each partition for which not too many elements are removed.

Lemma D.5 *Under the setup of Theorem 4.5, suppose that an adversary removes a set E of size at most τ from the set S constructed by* PRo*. Then for each partition i, there exists a bucket B_i such that $|E_{B_i}| \le 2^i$, i.e., at most 2^i elements are removed from this bucket.*

Proof. Towards contradiction, assume that this is not the case, i.e., assume $|E_{B_i}| > 2^i$ for every bucket of the i-th partition. As the number of buckets in partition i is $\lceil \tau/2^i \rceil$, this implies that the adversary has to spend a budget of

$$|E| \ge 2^i |E_{B_i}| > 2^i \lceil \tau/2^i \rceil = \tau,$$

which is in contradiction with $|E| \le \tau$. □

We consider $B_0, \dots, B_{\lceil \log \tau \rceil}$ as above, and show that even in the worst case, $f\left(\bigcup_{i=0}^{\lceil \log \tau \rceil} (B_i \setminus E_{B_i})\right)$ is almost as large as $f\left(B_{\lceil \log \tau \rceil}\right)$ for appropriately set η. To achieve this, we apply Lemma D.4 multiple times, as illustrated in the following lemma. We henceforth write $\eta_h := \eta/2$ for brevity.

Lemma D.6 *Under the setup of Theorem 4.5, suppose that an adversary removes a set E of size at most τ from the set S constructed by* PRo*, and let $B_0, \cdots, B_{\lceil \log \tau \rceil}$ be buckets such that $|E_{B_i}| \le 2^i$ for each $i \in \{1, \cdots \lceil \log \tau \rceil\}$ (*cf.*, Lemma D.5). Then,*

$$f\left(\bigcup_{i=0}^{\lceil \log \tau \rceil} (B_i \setminus E_{B_i})\right) \ge \left(1 - \frac{1}{1 + \frac{1}{\alpha}}\right) f\left(B_{\lceil \log \tau \rceil}\right) = \frac{1}{1+\alpha} f\left(B_{\lceil \log \tau \rceil}\right), \tag{40}$$

and

$$f\left(E_{B_{\lceil \log \tau \rceil}} \;\Big|\; \bigcup_{i=0}^{\lceil \log \tau \rceil - 1} (B_i \setminus E_{B_i})\right) \le \alpha f\left(\bigcup_{i=0}^{\lceil \log \tau \rceil - 1} (B_i \setminus E_{B_i})\right), \tag{41}$$

for some

$$\alpha \le \beta^2 \frac{(1+\eta_h)^{\lceil \log \tau \rceil} - \eta_h^{\lceil \log \tau \rceil}}{\eta_h^{\lceil \log \tau \rceil}}. \tag{42}$$

Proof. In what follows, we focus on the case where there exists some bucket B_0 in partition $i = 0$ such that $B_0 \setminus E_{B_0} = B_0$. If this is not true, then E must be contained entirely within this partition, since it contains τ buckets. As a result, (i) we trivially obtain (40) even when α is replaced by zero, since the union on the left-hand side contains $B_{\lceil \log \tau \rceil}$; (ii) (41) becomes trivial since the left-hand side is zero is a result of $E_{B_{\lceil \log \tau \rceil}} = \emptyset$.

We proceed by induction. Namely, we show that

$$f\left(\bigcup_{i=0}^{j} (B_i \setminus E_{B_i})\right) \ge \left(1 - \frac{1}{1 + \frac{1}{\alpha_j}}\right) f(B_j) = \frac{1}{1+\alpha_j} f(B_j), \tag{43}$$

and

$$f\left(E_{B_j} \;\Big|\; \bigcup_{i=0}^{j-1} (B_i \setminus E_{B_i})\right) \le \alpha_j f\left(\bigcup_{i=0}^{j-1} (B_i \setminus E_{B_i})\right), \tag{44}$$

for every $j \ge 1$, where

$$\alpha_j \le \beta^2 \frac{(1+\eta_h)^j - \eta_h^j}{\eta_h^j}. \tag{45}$$

Upon showing this, the lemma is concluded by setting $j = \lceil \log \tau \rceil$.

Base case $j = 1$. In the case that $j = 1$, taking into account that $E_{B_0} = \emptyset$, we observe from (43) that our goal is to bound $f(B_0 \cup (B_1 \setminus E_{B_1}))$. Applying Lemma D.3 with $X = B_0$, $Y = B_1$, and $E_Y = E_{B_1}$, we obtain

$$f(B_0 \cup (B_1 \setminus E_{B_1})) \geq \frac{1}{1+\alpha_1} f(B_1),$$

and

$$f(E_{B_1} \mid B_0) \leq \alpha_1 f(B_0),$$

where $\alpha_1 = \beta \frac{|E_{B_1}|}{|B_0|}$. We have $|B_0| = \eta$, while $|E_{B_1}| \leq 2$ by assumption. Hence, we can upper bound α_1 and rewrite as

$$\alpha_1 \leq \beta \frac{2}{\eta} = \beta \frac{1}{\eta_h} = \beta \frac{(1+\eta_h) - \eta_h}{\eta_h} \leq \beta^2 \frac{(1+\eta_h) - \eta_h}{\eta_h},$$

where the last inequality follows since $\beta \geq 1$ by definition.

Inductive step. Fix $j \geq 2$. Assuming that the inductive hypothesis holds for $j - 1$, we want to show that it holds for j as well.

We write

$$f\left(\bigcup_{i=0}^{j} (B_i \setminus E_{B_i})\right) = f\left(\left(\bigcup_{i=0}^{j-1} (B_i \setminus E_{B_i})\right) \cup (B_j \setminus E_{B_j})\right),$$

and apply Lemma D.4 with $X = \bigcup_{i=0}^{j-2} (B_i \setminus E_{B_i})$, $Y = B_{j-1}$, $E_Y = E_{B_{j-1}}$, $Z = B_j$, and $E_Z = E_{B_j}$. Note that the conditions (25) and (26) of Lemma D.4 are satisfied by the inductive hypothesis. Hence, we conclude that (43) and (44) hold with

$$\alpha_j = \beta \frac{|E_{B_j}|}{|B_{j-1}|}(1 + \alpha_{j-1}) + \alpha_{j-1}.$$

It remains to show that (45) holds for α_j, assuming it holds for α_{j-1}. We have

$$\begin{aligned}
\alpha_j &= \beta \frac{|E_{B_j}|}{|B_{j-1}|}(1 + \alpha_{j-1}) + \alpha_{j-1} \\
&\leq \beta \frac{1}{\eta_h}\left(1 + \beta \frac{(1+\eta_h)^{j-1} - \eta_h^{j-1}}{\eta_h^{j-1}}\right) + \beta \frac{(1+\eta_h)^{j-1} - \eta_h^{j-1}}{\eta_h^{j-1}} && (46) \\
&\leq \beta^2 \left(\frac{1}{\eta_h}\left(1 + \frac{(1+\eta_h)^{j-1} - \eta_h^{j-1}}{\eta_h^{j-1}}\right) + \frac{(1+\eta_h)^{j-1} - \eta_h^{j-1}}{\eta_h^{j-1}}\right) && (47) \\
&= \beta^2 \left(\frac{1}{\eta_h} \frac{(1+\eta_h)^{j-1}}{\eta_h^{j-1}} + \frac{(1+\eta_h)^{j-1} - \eta_h^{j-1}}{\eta_h^{j-1}}\right) \\
&= \beta^2 \left(\frac{(1+\eta_h)^{j-1}}{\eta_h^{j}} + \frac{\eta_h(1+\eta_h)^{j-1} - \eta_h^{j}}{\eta_h^{j}}\right) \\
&= \beta^2 \frac{(1+\eta_h)^{j} - \eta_h^{j}}{\eta_h^{j}},
\end{aligned}$$

where (46) follows from (45) and the fact that

$$\beta \frac{|E_{B_j}|}{|B_{j-1}|} \leq \beta \frac{2^j}{2^{j-1}\eta} = \beta \frac{2}{\eta} = \beta \frac{1}{\eta_h},$$

by $|E_{B_j}| \leq 2^j$ and $|B_{j-1}| = 2^{j-1}\eta$; and (47) follows since $\beta \geq 1$. □

Inequality (45) provides an upper bound on α_j, but it is not immediately clear how the bound varies with j. The following lemma provides a more compact form.

Lemma D.7 *Under the setup of Lemma D.6, we have for $2\lceil \log \tau \rceil \le \eta_h$ that*

$$\alpha_j \le 3\beta^2 \frac{j}{\eta} \tag{48}$$

Proof. We unfold the right-hand side of (45) in order to express it in a simpler way. First, consider $j = 1$. From (45) we obtain $\alpha_1 \le 2\beta^2 \frac{1}{\eta}$, as required. For $j \ge 2$, we obtain the following:

$$\begin{aligned}
\beta^{-2}\alpha_j &\le \frac{(1+\eta_h)^j - \eta_h^j}{\eta_h^j} \\
&= \sum_{i=0}^{j-1} \binom{j}{i} \frac{\eta_h^i}{\eta_h^j} && (49) \\
&= \frac{j}{\eta_h} + \sum_{i=0}^{j-2} \binom{j}{i} \frac{\eta_h^i}{\eta_h^j} && (50) \\
&= \frac{j}{\eta_h} + \sum_{i=0}^{j-2} \left(\frac{\prod_{t=1}^{j-i}(j-t+1)}{\prod_{t=1}^{j-i} t} \frac{\eta_h^i}{\eta_h^j} \right) \\
&\le \frac{j}{\eta_h} + \frac{1}{2} \sum_{i=0}^{j-2} j^{j-i} \frac{\eta_h^i}{\eta_h^j} && (51) \\
&= \frac{j}{\eta_h} + \frac{1}{2} \sum_{i=0}^{j-2} \left(\frac{j}{\eta_h} \right)^{j-i} \\
&= \frac{j}{\eta_h} + \frac{1}{2} \left(-1 - \frac{j}{\eta_h} + \sum_{i=0}^{j} \left(\frac{j}{\eta_h} \right)^{j-i} \right),
\end{aligned}$$

where (49) is a standard summation identity, and (51) follows from $\prod_{t=1}^{j-i}(j-t+1) \le j^{j-i}$ and $\prod_{t=1}^{j-i} t \ge 2$ for $j - i \ge 2$. Next, explicitly evaluating the summation of the last equality, we obtain

$$\begin{aligned}
\beta^{-2}\alpha_j &\le \frac{j}{\eta_h} + \frac{1}{2} \left(-1 - \frac{j}{\eta_h} + \frac{1 - \left(\frac{j}{\eta_h}\right)^{j+1}}{1 - \frac{j}{\eta_h}} \right) \\
&\le \frac{j}{\eta_h} + \frac{1}{2} \left(-1 - \frac{j}{\eta_h} + \frac{1}{1 - \frac{j}{\eta_h}} \right) \\
&= \frac{j}{\eta_h} + \frac{1}{2} \left(\frac{\left(\frac{j}{\eta_h}\right)^2}{1 - \frac{j}{\eta_h}} \right) && (52) \\
&= \frac{j}{\eta_h} + \frac{j}{2\eta_h} \left(\frac{\frac{j}{\eta_h}}{1 - \frac{j}{\eta_h}} \right), && (53)
\end{aligned}$$

where (52) follows from $(-a-1)(-a+1) = a^2 - 1$ with $a = j/\eta_h$.

Next, observe that if $j/\eta_h \le 1/2$, or equivalently

$$2j \le \eta_h, \tag{54}$$

then we can weaken (53) to

$$\beta^{-2}\alpha_j \le \frac{j}{\eta_h} + \frac{j}{2\eta_h} = \frac{3}{2}\frac{j}{\eta_h} = 3\frac{j}{\eta}, \tag{55}$$

which yields (48).

□

D.3. Completing the Proof of Theorem 4.5

We now prove Theorem 4.5 in several steps. Throughout, we define μ to be a constant such that $f(E_1 \mid (S \setminus E)) = \mu f(S_1)$ holds, and we write $E_0 := E_S^* \cap S_0$, $E_1 := E_S^* \cap S_1$, and $E_{B_i} := E_S^* \cap B_i$, where E_S^* is defined in (9). We also make use of the following lemma characterizing the optimal adversary. The proof is straightforward, and can be found in Lemma 2 of (Orlin et al., 2016)

Lemma D.8 (Orlin et al., 2016) *Under the setup of Theorem 4.5, we have for all $X \subset V$ with $|X| \le \tau$ that*

$$f(\mathrm{OPT}(k, V, \tau) \setminus E^*_{\mathrm{OPT}(k,V,\tau)}) \le f(\mathrm{OPT}(k - \tau, V \setminus X)).$$

Initial lower bounds: We start by providing three lower bounds on $f(S \setminus E_S^*)$. First, we observe that $f(S \setminus E_S^*) \ge f(S_0 \setminus E_0)$ and $f(S \setminus E_S^*) \ge f\left(\bigcup_{i=0}^{\lceil \log \tau \rceil} (B_i \setminus E_{B_i})\right)$. We also have

$$\begin{aligned}
f(S \setminus E) &= f(S) - f(S) + f(S \setminus E) \\
&= f(S_0 \cup S_1) + f(S \setminus E_0) - f(S \setminus E_0) - f(S) + f(S \setminus E) && (56)\\
&= f(S_1) + f(S_0 \mid S_1) + f(S \setminus E_0) - f(S) - f(S \setminus E_0) + f(S \setminus E) \\
&= f(S_1) + f(S_0 \mid (S \setminus S_0)) + f(S \setminus E_0) - f(E_0 \cup (S \setminus E_0)) - f(S \setminus E_0) + f(S \setminus E) && (57)\\
&= f(S_1) + f(S_0 \mid (S \setminus S_0)) - f(E_0 \mid (S \setminus E_0)) - f(S \setminus E_0) + f(S \setminus E) \\
&= f(S_1) + f(S_0 \mid (S \setminus S_0)) - f(E_0 \mid (S \setminus E_0)) - f(E_1 \cup (S \setminus E)) + f(S \setminus E) && (58)\\
&= f(S_1) + f(S_0 \mid (S \setminus S_0)) - f(E_0 \mid (S \setminus E_0)) - f(E_1 \mid S \setminus E) \\
&= f(S_1) - f(E_1 \mid S \setminus E) + f(S_0 \mid (S \setminus S_0)) - f(E_0 \mid (S \setminus E_0)) \\
&\ge (1 - \mu) f(S_1), && (59)
\end{aligned}$$

where (56) and (57) follow from $S = S_0 \cup S_1$, (58) follows from $E_S^* = E_0 \cup E_1$, and (59) follows from $f(S_0 \mid (S \setminus S_0)) - f(E_0 \mid (S \setminus E_0)) \ge 0$ (due to $E_0 \subseteq S_0$ and $S \setminus S_0 \subseteq S \setminus E_0$), along with the definition of μ.

By combining the above three bounds on $f(S \setminus E_S^*)$, we obtain

$$f(S \setminus E_S^*) \ge \max \left\{ f(S_0 \setminus E_0), (1 - \mu) f(S_1), f\left(\bigcup_{i=0}^{\lceil \log \tau \rceil} (B_i \setminus E_{B_i}) \right) \right\}. \quad (60)$$

We proceed by further bounding these terms.

Bounding the first term in (60): Defining $S_0' := \mathrm{OPT}(k - \tau, V \setminus E_0) \cap (S_0 \setminus E_0)$ and $X := \mathrm{OPT}(k - \tau, V \setminus E_0) \setminus S_0'$, we have

$$\begin{aligned}
f(S_0 \setminus E_0) + f(\mathrm{OPT}(k - \tau, V \setminus S_0)) &\ge f(S_0') + f(X) && (61)\\
&\ge f(\mathrm{OPT}(k - \tau, V \setminus E_0)) && (62)\\
&\ge f(\mathrm{OPT}(k, V, \tau) \setminus E^*_{\mathrm{OPT}(k,V,\tau)}), && (63)
\end{aligned}$$

where (61) follows from monotonicity, i.e. $(S_0 \setminus E_0) \subseteq S_0'$ and $(V \setminus S_0) \subseteq (V \setminus E_0)$, (62) follows from the fact that $\mathrm{OPT}(k - \tau, V \setminus E_0) = S_0' \cup X$ and submodularity,[2] and (63) follows from Lemma D.8 and $|E_0| \le \tau$. We rewrite (63) as

$$f(S_0 \setminus E_0) \ge f(\mathrm{OPT}(k, V, \tau) \setminus E^*_{\mathrm{OPT}(k,V,\tau)}) - f(\mathrm{OPT}(k - \tau, V \setminus S_0)). \quad (64)$$

Bounding the second term in (60): Note that S_1 is obtained by using $\mathcal{A}$ that satisfies the β-iterative property on the set $V \setminus S_0$, and its size is $|S_1| = k - |S_0|$. Hence, from Lemma 4.3 with $k - \tau$ in place of k, we have

$$f(S_1) \ge \left(1 - e^{-\frac{k - |S_0|}{\beta(k - \tau)}}\right) f(\mathrm{OPT}(k - \tau, V \setminus S_0)). \quad (65)$$

[2]The submodularity property can equivalently be written as $f(A) + f(B) \ge f(A \cup B) + f(A \cap B)$.

Bounding the third term in (60)**:** We can view S_1 as a large bucket created by our algorithm after creating the buckets in S_0. Therefore, we can apply Lemma D.4 with $X = \bigcup_{i=0}^{\lceil \log \tau \rceil - 1} (B_i \setminus E_{B_i})$, $Y = B_{\lceil \log \tau \rceil}$, $Z = S_1$, $E_Y = E_S^* \cap Y$, and $E_Z = E_1$. Conditions (25) and (26) needed to apply Lemma D.4 are provided by Lemma D.6. From Lemma D.4, we obtain the following with α as in (42):

$$f\left(E_1 \;\middle|\; \left(\bigcup_{i=0}^{\lceil \log \tau \rceil} (B_i \setminus E_{B_i})\right) \cup (S_1 \setminus E_1)\right) \leq \left(\beta \frac{|E_1|}{|B_{\lceil \log \tau \rceil}|}(1+\alpha) + \alpha\right) f\left(\bigcup_{i=0}^{\lceil \log \tau \rceil} (B_i \setminus E_{B_i})\right). \tag{66}$$

Furthermore, noting that the assumption $\eta \geq 4(\log k + 1)$ implies $2\lceil \log \tau \rceil \leq \eta_h$, we can upper-bound α as in Lemma D.7 by (48) for $j = \lceil \log \tau \rceil$. Also, we have $\beta \frac{|E_1|}{|B_{\lceil \log \tau \rceil}|} \leq \beta \frac{\tau}{2^{\lceil \log \tau \rceil} \eta} \leq \frac{\beta}{\eta}$. Putting these together, we upper bound (66) as follows:

$$\begin{aligned}
f\left(E_1 \;\middle|\; \left(\bigcup_{i=0}^{\lceil \log \tau \rceil} (B_i \setminus E_{B_i})\right) \cup (S_1 \setminus E_1)\right) &\leq \left(\frac{\beta}{\eta}\left(1 + \frac{3\beta^2 \lceil \log \tau \rceil}{\eta}\right) + \frac{3\beta^2 \lceil \log \tau \rceil}{\eta}\right) f\left(\bigcup_{i=0}^{\lceil \log \tau \rceil} (B_i \setminus E_{B_i})\right) \\
&\leq \frac{5\beta^3 \lceil \log \tau \rceil}{\eta} f\left(\bigcup_{i=0}^{\lceil \log \tau \rceil} (B_i \setminus E_{B_i})\right),
\end{aligned}$$

where we have used $\eta \geq 1$ and $\lceil \log \tau \rceil \geq 1$ (since $\tau \geq 2$ by assumption). We rewrite the previous equation as

$$\begin{aligned}
f\left(\bigcup_{i=0}^{\lceil \log \tau \rceil} (B_i \setminus E_{B_i})\right) &\geq \frac{\eta}{5\beta^3 \lceil \log \tau \rceil} f\left(E_1 \;\middle|\; \left(\bigcup_{i=0}^{\lceil \log \tau \rceil} (B_i \setminus E_{B_i})\right) \cup (S_1 \setminus E_1)\right) \\
&\geq \frac{\eta}{5\beta^3 \lceil \log \tau \rceil} f(E_1 \mid (S \setminus E)) \qquad (67) \\
&= \frac{\eta}{5\beta^3 \lceil \log \tau \rceil} \mu f(S_1), \qquad (68)
\end{aligned}$$

where (67) follows from submodularity, and (68) follows from the definition of μ.

Combining the bounds: Returning to (60), we have

$$\begin{aligned}
f(S \setminus E_S^*) &\geq \max\left\{ f(S_0 \setminus E_0), (1-\mu) f(S_1), f\left(\bigcup_{i=0}^{\lceil \log \tau \rceil} (B_i \setminus E_{B_i})\right)\right\} \\
&\geq \max\left\{ f(S_0 \setminus E_0), (1-\mu) f(S_1), \frac{\eta}{5\beta^3 \lceil \log \tau \rceil} \mu f(S_1)\right\} \qquad (69) \\
&\geq \max\{ f(\mathrm{OPT}(k, V, \tau) \setminus E^*_{\mathrm{OPT}(k,V,\tau)}) - f(\mathrm{OPT}(k-\tau, V \setminus S_0)), \\
&\qquad (1-\mu)\left(1 - e^{-\frac{k-|S_0|}{\beta(k-\tau)}}\right) f(\mathrm{OPT}(k-\tau, V \setminus S_0)), \\
&\qquad \frac{\eta}{5\beta^3 \lceil \log \tau \rceil} \mu \left(1 - e^{-\frac{k-|S_0|}{\beta(k-\tau)}}\right) f(\mathrm{OPT}(k-\tau, V \setminus S_0))\} \qquad (70) \\
&\geq \max\{ f(\mathrm{OPT}(k, V, \tau) \setminus E^*_{\mathrm{OPT}(k,V,\tau)}) - f(\mathrm{OPT}(k-\tau, V \setminus S_0)), \\
&\qquad \frac{\frac{\eta}{5\beta^3 \lceil \log \tau \rceil}}{1 + \frac{\eta}{5\beta^3 \lceil \log \tau \rceil}} \left(1 - e^{-\frac{k-|S_0|}{\beta(k-\tau)}}\right) f(\mathrm{OPT}(k-\tau, V \setminus S_0))\} \qquad (71) \\
&= \max\{ f(\mathrm{OPT}(k, V, \tau) \setminus E^*_{\mathrm{OPT}(k,V,\tau)}) - f(\mathrm{OPT}(k-\tau, V \setminus S_0)), \\
&\qquad \frac{\eta}{5\beta^3 \lceil \log \tau \rceil + \eta} \left(1 - e^{-\frac{k-|S_0|}{\beta(k-\tau)}}\right) f(\mathrm{OPT}(k-\tau, V \setminus S_0))\} \\
&\geq \frac{\frac{\eta}{5\beta^3 \lceil \log \tau \rceil + \eta}\left(1 - e^{-\frac{k-|S_0|}{\beta(k-\tau)}}\right)}{1 + \frac{\eta}{5\beta^3 \lceil \log \tau \rceil + \eta}\left(1 - e^{-\frac{k-|S_0|}{\beta(k-\tau)}}\right)} f(\mathrm{OPT}(k, V, \tau) \setminus E^*_{\mathrm{OPT}(k,V,\tau)}), \qquad (72)
\end{aligned}$$

where (69) follows from (68), (70) follows from (64) and (65), (71) follows since $\max\{1-\mu, c\mu\} \geq \frac{c}{1+c}$ analogously to (19), and (72) follows from (20). Hence, we have established (72).

Turning to the permitted values of τ, we have from Proposition 4.1 that

$$|S_0| \ \leq \ 3\eta\tau(\log k + 2).$$

For the choice of τ to yield valid set sizes, we only require $|S_0| \leq k$; hence, it suffices that

$$\tau \leq \frac{k}{3\eta(\log k + 2)}. \tag{73}$$

Finally, we consider the second claim of the lemma. For $\tau \in o\big(\frac{k}{\eta(\log k)}\big)$ we have $|S_0| \in o(k)$. Furthermore, by setting $\eta \geq \log^2 k$ (which satisfies the assumption $\eta \geq 4(\log k + 1)$ for large k), we get $\frac{k-|S_0|}{\beta(k-\tau)} \to \beta^{-1}$ and $\frac{\eta}{5\beta^3\lceil\log\tau\rceil+\eta} \to 1$ as $k \to \infty$. Hence, the constant factor converges to $\frac{1-e^{-1/\beta}}{2-e^{-1/\beta}}$, yielding (11). In the case that GREEDY is used as the subroutine, we have $\beta = 1$, and hence the constant factor converges t $\frac{1-e^{-1}}{2-e^{-1}} \geq 0.387$. If THRESHOLDING-GREEDY is used, we have $\beta = \frac{1}{1-\epsilon}$, and hence the constant factor converges to $\frac{1-e^{\epsilon-1}}{2-e^{\epsilon-1}} \geq (1-\epsilon)\frac{1-e^{-1}}{2-e^{-1}} \geq (1-\epsilon)0.387$.