A. Proofs for Section 3 (Decomposing A)

Proof of Lemma 1. Writing each term in the decomposition (8) explicitly and summing these together we find a match between all entries to those of A.

As a representative example let us consider the last entry $A_{n,n} = P(n \mid n)$. The first term gives

$$(C_\beta \bar{A} B)_{[n,n]} = (1-\beta) P(\bar{n} \mid n).$$

The second term gives

$$\begin{aligned}(C_\beta \boldsymbol{\delta}^{\text{out}} \boldsymbol{c}_\beta^\intercal)_{[n,n]} &= -\beta(1-\beta)\delta_{n-1}^{\text{out}} \\ &= -\beta(1-\beta)(P(\bar{n} \mid n-1) - P(\bar{n} \mid n)).\end{aligned}$$

The third term,

$$\begin{aligned}(\boldsymbol{b}(\boldsymbol{\delta}^{\text{in}})^\intercal B)_{[n,n]} &= -\delta_{n-1}^{\text{in}} \\ &= -\beta(\alpha_{n-1}-\beta)P(\bar{n} \mid n-1) \\ &\quad -(1-\beta)(\alpha_n-\beta)P(\bar{n} \mid n).\end{aligned}$$

And lastly, the fourth term gives

$$\begin{aligned}(\kappa \boldsymbol{b} \boldsymbol{c}_\beta^\intercal)_{[n,n]} &= \beta(\alpha_{n-1}-\beta)P(\bar{n} \mid n-1) \\ &\quad -\beta(\alpha_n-\beta)P(\bar{n} \mid n).\end{aligned}$$

Putting $P(\bar{n} \mid n) = P(n-1 \mid n) + P(n \mid n)$ and $P(\bar{n} \mid n-1) = P(n-1 \mid n-1) + P(n \mid n-1)$, and summing all these four terms we obtain $P(n \mid n)$ as needed. The other entries of A are obtained similarly. □

B. Proofs for Section 4.1 (Minimality)

Let $H = (A, \boldsymbol{\theta}, \boldsymbol{\pi}^0)$ be a 2A-HMM. For any $k \geq 1$ the distribution $P_{H,k} \in \mathcal{P}_H$ can be cast in an explicit matrix form. Let $o \in \mathcal{Y}$. The *observable operator* $T_{\boldsymbol{\theta}}(o) \in \mathbb{R}^{n \times n}$ is defined by

$$T_{\boldsymbol{\theta}}(o) = \operatorname{diag}(f_{\theta_1}(o), f_{\theta_2}(o), \ldots, f_{\theta_n}(o)).$$

Let $y = (y_0, y_1, \ldots, y_{k-1}) \in \mathcal{Y}^k$ be a sequence of $k \geq 1$ initial consecutive observations. Then the distribution $P_{H,k}(y)$ is given by (Jaeger, 2000),

$$P_{H,k}(y) = \mathbf{1}_n^\intercal T_{\boldsymbol{\theta}}(y_{k-1}) A \ldots A\, T_{\boldsymbol{\theta}}(y_1) A\, T_{\boldsymbol{\theta}}(y_0) \boldsymbol{\pi}^0. \quad (34)$$

Proof of Theorem 1. Let us first show that $\boldsymbol{\delta}^{\text{out}} \neq 0$ is necessary for minimality, namely if $\boldsymbol{\delta}^{\text{out}} = 0$ then H is not minimal, regardless of the initial distribution $\boldsymbol{\pi}^0$. The non-minimality will be shown by explicitly constructing a $n-1$ state HMM equivalent to H. Let us denote the lifting of the merged transition matrix by

$$\tilde{A} = C_\beta \bar{A} B \in \mathbb{R}^{n \times n}.$$

Assume that $\boldsymbol{\delta}^{\text{out}} = 0$. We will shortly see that for any $\boldsymbol{\pi}^0$ and for any $k \geq 2$ consecutive observations $y = (y_0, y_1, \ldots, y_{k-1}) \in \mathcal{Y}^k$ we have that

$$\begin{aligned}&T_{\boldsymbol{\theta}}(y_{k-1}) A \ldots T_{\boldsymbol{\theta}}(y_1) A T_{\boldsymbol{\theta}}(y_0) \boldsymbol{\pi}^0 \quad (35) \\ - \;&T_{\boldsymbol{\theta}}(y_{k-1}) \tilde{A} \ldots T_{\boldsymbol{\theta}}(y_1) \tilde{A} T_{\boldsymbol{\theta}}(y_0) \boldsymbol{\pi}^0 \propto \boldsymbol{b}.\end{aligned}$$

Combining (35) with (34), and the fact that $\mathbf{1}_n^\intercal \boldsymbol{b} = 0$, we have that $\mathcal{P}_{(A,\boldsymbol{\theta},\boldsymbol{\pi}^0)} = \mathcal{P}_{(\tilde{A},\boldsymbol{\theta},\boldsymbol{\pi}^0)}$. Since $\tilde{A}$ has identical $(n-1)$-th and n-th columns, and $f_{\theta_{n-1}} = f_{\theta_n}$ we have that $\mathcal{P}_{(\tilde{A},\boldsymbol{\theta},\boldsymbol{\pi}^0)} = \mathcal{P}_{(\bar{A},\bar{\boldsymbol{\theta}},\bar{\boldsymbol{\pi}}^0)}$. Thus $H' = (\bar{A}, \bar{\boldsymbol{\theta}}, \bar{\boldsymbol{\pi}}^0)$ is an equivalent $(n-1)$-state HMM and H is not minimal, proving the claim. We prove (35) by induction on the sequence length $k \geq 2$. First note that since $\boldsymbol{\delta}^{\text{out}} = 0$, by Lemma 1 we have that

$$A = \tilde{A} + \boldsymbol{b}((\boldsymbol{\delta}^{\text{in}})^\intercal B + \kappa \boldsymbol{c}_\beta^\intercal).$$

Since for any $y \in \mathcal{Y}$, $T_{\boldsymbol{\theta}}(y)\boldsymbol{b} = f_{\theta_{\bar{n}}}(y)\boldsymbol{b}$, we have that

$$T_{\boldsymbol{\theta}}(y)A - T_{\boldsymbol{\theta}}(y)\tilde{A} = f_{\theta_{\bar{n}}}(y)\boldsymbol{b}((\boldsymbol{\delta}^{\text{in}})^\intercal B + \kappa \boldsymbol{c}_\beta^\intercal) \propto \boldsymbol{b}.$$

This proves the case $k = 2$. Next, assume (35) holds for all sequences of length at least 2 and smaller than k, namely, for some $a \in \mathbb{R}$

$$\begin{aligned}&T_{\boldsymbol{\theta}}(y_{k-2}) A \ldots T_{\boldsymbol{\theta}}(y_1) A T_{\boldsymbol{\theta}}(y_0) \boldsymbol{\pi}^0 \\ = \;&a\boldsymbol{b} + T_{\boldsymbol{\theta}}(y_{k-2}) \tilde{A} \ldots T_{\boldsymbol{\theta}}(y_1) \tilde{A} T_{\boldsymbol{\theta}}(y_0) \boldsymbol{\pi}^0.\end{aligned}$$

Using the fact that $B\boldsymbol{b} = 0$ we have $T_{\boldsymbol{\theta}}(y_{k-1})\tilde{A}\boldsymbol{b} = 0$. Inserting the expansion of A in the l.h.s of (35) we get

$$\begin{aligned}&f_{\theta_{\bar{n}}}(y_{k-1})\boldsymbol{b}\left((\boldsymbol{\delta}^{\text{in}})^\intercal B + \kappa \boldsymbol{c}_\beta^\intercal\right) \\ \times \;&\left(a\boldsymbol{b} + \tilde{A} T_{\boldsymbol{\theta}}(y_{k-2}) \ldots T_{\boldsymbol{\theta}}(y_1) \tilde{A} T_{\boldsymbol{\theta}}(y_0) \boldsymbol{\pi}^0\right).\end{aligned}$$

Since this last expression is proportional to $\boldsymbol{b}$ we are done.

(ii) The case $\pi_{\bar{n}}^0 = 0$ or $\beta^0 = \beta$. As we just saw, having $\boldsymbol{\delta}^{\text{out}} = 0$ implies that the HMM is not minimal. We now show that if $\boldsymbol{\delta}^{\text{in}} = 0$ then H is not minimal either. By contraposition this will prove the first direction of (ii).

So assume that $\boldsymbol{\delta}^{\text{in}} = 0$. Lemma 1 implies

$$A = \tilde{A} + (C_\beta(\boldsymbol{\delta}^{\text{out}}) + \kappa \boldsymbol{b})\boldsymbol{c}_\beta^\intercal. \quad (36)$$

Now note that for all $y \in \mathcal{Y}$, $\boldsymbol{c}_\beta^\intercal T_{\boldsymbol{\theta}}(y) = f_{\theta_{n-1}}(y)\boldsymbol{c}_\beta^\intercal$ and since either $\pi_{\bar{n}}^0 = 0$ or $\beta^0 = \beta$ we have that $\boldsymbol{c}_\beta^\intercal \boldsymbol{\pi}^0 = 0$. Thus $\boldsymbol{c}_\beta T_{\boldsymbol{\theta}}(y)\boldsymbol{\pi}^0 = 0$ and we find that

$$\begin{aligned}&T_{\boldsymbol{\theta}}(y_{k-1}) A \ldots A T_{\boldsymbol{\theta}}(y_1) A T_{\boldsymbol{\theta}}(y_0) \boldsymbol{\pi}^0 \quad (37) \\ &= T_{\boldsymbol{\theta}}(y_{k-1}) A \ldots A T_{\boldsymbol{\theta}}(y_1) \tilde{A} T_{\boldsymbol{\theta}}(y_0) \boldsymbol{\pi}^0. \quad (38)\end{aligned}$$

Now since $\boldsymbol{c}_\beta^\intercal C_\beta = 0$ we have that for any $y \in \mathcal{Y}$, $\boldsymbol{c}_\beta^\intercal T_{\boldsymbol{\theta}}(y)\tilde{A} = 0$ and thus expanding A by (36) we find that for any $y \in \mathcal{Y}$,

$$A T_{\boldsymbol{\theta}}(y)\tilde{A} = \left(\tilde{A} + (C_\beta(\boldsymbol{\delta}^{\text{out}}) + \kappa \boldsymbol{b})\boldsymbol{c}_\beta^\intercal\right) T_{\boldsymbol{\theta}}(y)\tilde{A} = \tilde{A} T_{\boldsymbol{\theta}}(y)\tilde{A}.$$

Thus each A in the right hand side of (37) can be replaced by $\tilde{A}$ and we conclude that $\mathcal{P}_{(A,\boldsymbol{\theta},\boldsymbol{\pi}^0)} = \mathcal{P}_{(\tilde{A},\boldsymbol{\theta}_n,\boldsymbol{\pi}^0)}$. Similarly to the case $\boldsymbol{\delta}^{\text{out}} = 0$ we have that $H' = (\bar{A}, \bar{\boldsymbol{\theta}}, \bar{\boldsymbol{\pi}}^0)$ is an equivalent $(n\text{–}1)$-state HMM and thus H is not minimal.

In order to prove the other direction we will show that if H is not minimal then either $\boldsymbol{\delta}^{\text{out}} = 0$ or $\boldsymbol{\delta}^{\text{in}} = 0$. This is equivalent to the condition $\boldsymbol{\delta}^{\text{out}}\boldsymbol{\delta}^{\text{in}^\intercal} = 0$.

Assuming H is not minimal, there exists an HMM H' with $n' < n$ states such that $\mathcal{P}_{H'} = \mathcal{P}_H$. Assumptions (A1-A3) readily imply that H' must have $n' = n-1$ states and that the unique $n-1$ output components are identical for H and H'. Since $\mathcal{P}_{H'}$ is invariant to permutations, we may assume that $\bar{\boldsymbol{\theta}}' = \bar{\boldsymbol{\theta}}$ and consequently the kernel matrices in (1) for both H and H' are equal $\bar{K} = \bar{K}'$.

Let $A' \in \mathbb{R}^{(n\text{–}1)\times(n\text{–}1)}$ be the transition matrix of H' and define $H'' = (A'', \boldsymbol{\theta}'', \pi'')$ as the equivalent n-state HMM to H' by setting $\beta'' = \beta$, $A'' = C_\beta A' B$, $\boldsymbol{\theta}'' = \bar{\boldsymbol{\theta}}' B$ and $\boldsymbol{\pi}'' = C_\beta \bar{\boldsymbol{\pi}}'$. Note that for H'', by construction we have $\boldsymbol{\delta}^{\text{out}''}(\boldsymbol{\delta}^{\text{in}''})^\intercal = 0$.

Now, by the equivalence of the two models H and H'', we have that the second order moments $\mathcal{M}^{(2)}$ given in (12) are the same for both. By the fact that $\bar{K}'' = \bar{K}$, $\bar{\boldsymbol{\pi}}'' = \bar{\boldsymbol{\pi}}$ and by (22) in Lemma 3 we must have that $\boldsymbol{\delta}^{\text{out}}(\boldsymbol{\delta}^{\text{in}})^\intercal = \boldsymbol{\delta}^{\text{out}''}(\boldsymbol{\delta}^{\text{in}''})^\intercal$. Thus $\boldsymbol{\delta}^{\text{out}}(\boldsymbol{\delta}^{\text{in}})^\intercal = 0$ and the claim is proved.

(i) The case $\pi^0_{\bar{n}} \neq 0$ and $\beta^0 \neq \beta$. We saw above that if H is minimal then $\boldsymbol{\delta}^{\text{out}} \neq 0$. Thus, in order to prove the claim we are left to show that if H is not minimal then $\boldsymbol{\delta}^{\text{out}} = 0$.

So assume H is not minimal and let H'' be constructed as above. By way of contradiction assume $\boldsymbol{\delta}^{\text{out}} \neq 0$. As we just saw, since H is not minimal then $\boldsymbol{\delta}^{\text{out}}(\boldsymbol{\delta}^{\text{in}})^\intercal = 0$. Thus by the assumption $\boldsymbol{\delta}^{\text{out}} \neq 0$ we must have $\boldsymbol{\delta}^{\text{in}} = 0$. This implies that A is in the form (36). Since $\mathcal{P}_H = \mathcal{P}_{H''}$ we have $P_{H,2} = P_{H'',2}$ where:

$$
\begin{aligned}
P_{H,2} &= \mathbf{1}_n^\intercal T_{\boldsymbol{\theta}}(y_2) A T_{\boldsymbol{\theta}}(y_1)\boldsymbol{\pi}^0 \\
&= \mathbf{1}_n^\intercal T_{\boldsymbol{\theta}}(y_2)\Big(\tilde{A} + (C_\beta \boldsymbol{\delta}^{\text{out}} + \kappa \boldsymbol{b})\boldsymbol{c}_\beta^\intercal\Big) T_{\boldsymbol{\theta}}(y_1)\boldsymbol{\pi}^0 \\
P_{H'',2} &= \mathbf{1}_n^\intercal T_{\boldsymbol{\theta}}(y_2) A'' T_{\boldsymbol{\theta}}(y_1)\boldsymbol{\pi}^0.
\end{aligned}
$$

In addition, by the fact that $\bar{K}'' = \bar{K}$, $\bar{\boldsymbol{\pi}}'' = \bar{\boldsymbol{\pi}}$ we must have that $M^{(1)} = M''^{(1)}$, where $M^{(1)}$ is defined in (16) and $M''^{(1)}$ is defined similarly with the parameters of H'' instead of H. By (21) in Lemma 3 we thus have

$$M''^{(1)} = A' = \bar{A} = M^{(1)}.$$

Hence $A'' = \tilde{A}$ and $P_{H,2} = P_{H'',2}$ is equivalent to

$$\mathbf{1}_n^\intercal T_{\boldsymbol{\theta}}(y_2)\Big(C_\beta \boldsymbol{\delta}^{\text{out}} + \kappa \boldsymbol{b}\Big)\boldsymbol{c}_\beta^\intercal T_{\boldsymbol{\theta}}(y_1)\boldsymbol{\pi}^0 = 0. \tag{39}$$

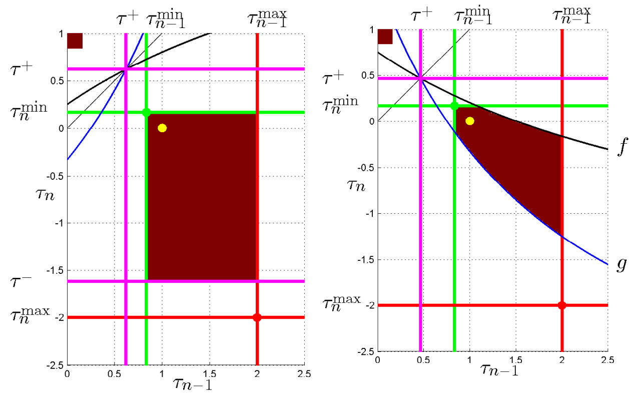


Figure 4. The feasible region Γ_H (shaded) in the $(\tau_{n\text{–}1}, \tau_n)$ plane. Any $(\tau_{n\text{–}1}, \tau_n) \in \Gamma_H$ induces an HMM equivalent to H via Lemma 2. The pair $(\tau_{n\text{–}1}, \tau_n) = (1, 0)$, corresponding to the original transition matrix A, is indicated by a yellow point. Left: $\alpha_{n\text{–}1} \geq \alpha_n$ and Right: $\alpha_{n\text{–}1} < \alpha_n$.

Now, note that $\forall y_1, y_2 \in \mathcal{Y}$ we have

$$
\begin{aligned}
\mathbf{1}_n^\intercal T_{\boldsymbol{\theta}}(y_2)\boldsymbol{b} &= 0 \\
\boldsymbol{c}_\beta^\intercal T_{\boldsymbol{\theta}}(y_1)\boldsymbol{\pi}^0 &= (\beta^0 - \beta)\pi^0_{\bar{n}} f_{\theta_{\bar{n}}}(y_1) \\
\mathbf{1}_n^\intercal T_{\boldsymbol{\theta}}(y_2) C_\beta &= (f_{\theta_1}(y_2), \ldots, f_{\theta_{n\text{–}1}}(y_2)).
\end{aligned}
$$

Thus, (39) is given by

$$(\beta^0 - \beta)\pi^0_{\bar{n}} f_{\theta_{n\text{–}1}}(y_1)\Big(f_{\theta_1}(y_2), \ldots, f_{\theta_{n\text{–}1}}(y_2)\Big) \cdot \boldsymbol{\delta}^{\text{out}} = 0.$$

Since by assumption $(\beta^0 - \beta)\pi^0_{\bar{n}} \neq 0$ we have $\forall y_1, y_2 \in \mathcal{Y}$

$$f_{\theta_{n\text{–}1}}(y_1)\Big(f_{\theta_1}(y_2), \ldots, f_{\theta_{n\text{–}1}}(y_2)\Big) \cdot \boldsymbol{\delta}^{\text{out}} = 0.$$

For each $i \in [n\text{–}1]$, multiplying by $f_{\theta_i}(y_2)$ and integrating over $y_1, y_2 \in \mathcal{Y}$ we get

$$\bar{K}\boldsymbol{\delta}^{\text{out}} = 0.$$

Since $\bar{K}$ is full rank we must have $\boldsymbol{\delta}^{\text{out}} = 0$ in contradiction to the assumption $\boldsymbol{\delta}^{\text{out}} \neq 0$. This concludes the proof of the Theorem. □

C. Proofs for Section 4.2 (Identifiability)

C.1. Proof of Theorem 2

Before characterizing Γ_H let us first give some intuition on the role of $(\tau_{n\text{–}1}, \tau_n)$. Consider the $n-1$ dimensional columns $\{\bar{\boldsymbol{a}}_i \,|\, i \in [n]\}$ of the matrix BA. These can be plotted on the $n-1$ dimensional simplex, as shown in Fig.5 (top), for $n = 4$ and aliased states $\{3, 4\}$. Recall that

$$A_H(\tau_{n\text{–}1}, \tau_n) = S(\tau_{n\text{–}1}, \tau_n)^{-1} A S(\tau_{n\text{–}1}, \tau_n)$$

and let $\{\bar{\boldsymbol{a}}_{H,i} \,|\, i \in [n]\}$ be the columns of the matrix $BA_H \in \mathbb{R}^{n-1\times n}$.

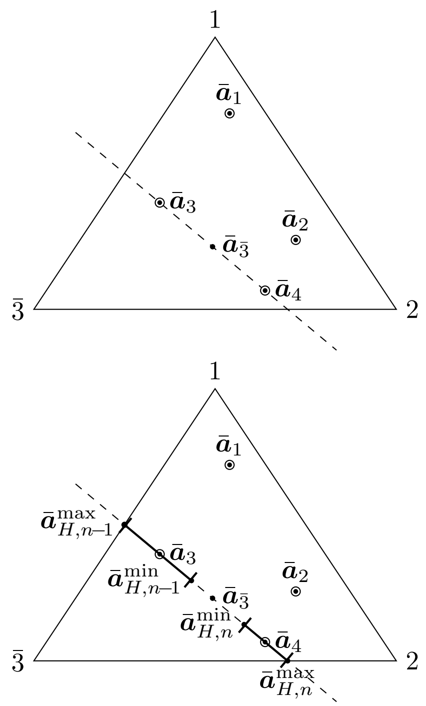


Figure 5. **Top**: Plotting the columns of BA on the simplex for a 2A-HMM with aliased states $\{3,4\}$. Here, $\bar{\boldsymbol{a}}_{\bar{3}} = \beta\bar{\boldsymbol{a}}_3 + (1-\beta)\bar{\boldsymbol{a}}_4$. **Bottom**: Any vectors $\bar{\boldsymbol{a}}_{H,n-1}, \bar{\boldsymbol{a}}_{H,n}$ within the depicted bars results in a matrix A_H with all entries non-negative, except in possibly the 2×2 aliased block.

Since $BS(\tau_{n-1},\tau_n)^{-1} = B$ we have $BA_H = BAS(\tau_{n-1},\tau_n)$. So the non-aliased columns of BA_H are unaltered from these of BA, i.e. for all $i \in [n-2]$, $\bar{\boldsymbol{a}}_i = \bar{\boldsymbol{a}}_{H,i}$. The new aliased columns of BA_H are

$$\begin{aligned}\bar{\boldsymbol{a}}_{H,n-1} &= \bar{\boldsymbol{a}}_n + \tau_{n-1}\boldsymbol{\delta}^{\text{out}}\\ \bar{\boldsymbol{a}}_{H,n} &= \bar{\boldsymbol{a}}_n + \tau_n\boldsymbol{\delta}^{\text{out}}.\end{aligned}$$

Thus τ_{n-1} (τ_n) determines the position of the vector $\bar{\boldsymbol{a}}_{H,n-1}$ ($\bar{\boldsymbol{a}}_{H,n}$) along the ray passing through $\bar{\boldsymbol{a}}_{n-1}$ and $\bar{\boldsymbol{a}}_n$ (dashed line in Fig.5).

Hence a necessary condition for A_H to be a valid transition matrix is that $\bar{\boldsymbol{a}}_{H,n-1} \geq 0$ and $\bar{\boldsymbol{a}}_{H,n} \geq 0$, and one cannot take τ_{n-1} and τ_n arbitrarily. In particular, there are $\tau^{\max}_{n-1}$ and $\tau^{\max}_n$ such that $\bar{\boldsymbol{a}}_{H,n-1}$ and $\bar{\boldsymbol{a}}_{H,n}$ are as "far" apart as possible by putting them on the opposite sides of the ray connecting them, such that both sit on the simplex boundary. This is achieved by taking

$$\begin{aligned}\bar{\boldsymbol{a}}^{\max}_{H,n-1} &= \bar{\boldsymbol{a}}_n + \tau^{\max}_{n-1}\boldsymbol{\delta}^{\text{out}}\\ \bar{\boldsymbol{a}}^{\max}_{H,n} &= \bar{\boldsymbol{a}}_n + \tau^{\max}_n\boldsymbol{\delta}^{\text{out}},\end{aligned}$$

where

$$\begin{aligned}\tau^{\max}_{n-1} &= \min_{j\in\mathcal{X}\setminus\{n-1,n\}} \frac{\frac{1}{2}(1+\text{sign}(\delta^{\text{out}}_j))-(\bar{\boldsymbol{a}}_n)_j}{\delta^{\text{out}}_j} &\geq 0\\ \tau^{\max}_n &= \max_{j\in\mathcal{X}\setminus\{n-1,n\}} \frac{\frac{1}{2}(1-\text{sign}(\delta^{\text{out}}_j))-(\bar{\boldsymbol{a}}_n)_j}{\delta^{\text{out}}_j} &\leq 0.\end{aligned}$$

(see Fig.5, bottom). Since we assumed as a convention that $\tau_{n-1} > \tau_n$ we have that any $\tau_{n-1} \leq \tau^{\max}_{n-1}$ and $\tau_n \geq \tau^{\max}_n$ results in a non negative matrix BA_H. Note that $BA_H \geq 0$ implies $A_{H[1:n-2,1:n]} \geq 0$.

Next, consider the new relative probabilities $\alpha_{H,i}$ as defined by (4) with A_H replacing A. One can verify that these satisfy

$$\alpha_{H,i} = \frac{\alpha_i - \tau_n}{\tau_{n-1}-\tau_n}, \quad i \in \text{supp}_{\text{in}}\setminus\{n-1,n\}.$$

Obviously, a necessary condition for A_H to be a valid transition matrix is that

$$0 \leq \alpha_{H,i} = \frac{\alpha_i - \tau_n}{\tau_{n-1}-\tau_n} \leq 1, \quad i \in \text{supp}_{\text{in}}\setminus\{n-1,n\}. \tag{40}$$

Define the minimal and maximal relative probabilities of the non-aliased states by

$$\begin{aligned}\alpha^{\min} &= \min\{\alpha_i \,|\, i \in \text{supp}_{\text{in}}\setminus\{n-1,n\}\}\\ \alpha^{\max} &= \max\{\alpha_i \,|\, i \in \text{supp}_{\text{in}}\setminus\{n-1,n\}\}.\end{aligned}$$

Let $\alpha^{\min}_H$ and $\alpha^{\max}_H$ be defined similarly. Taking

$$\begin{aligned}\tau^{\min}_{n-1} &= \alpha^{\max}\\ \tau^{\min}_n &= \alpha^{\min},\end{aligned}$$

we have $\alpha^{\min}_H = 0$ and $\alpha^{\max}_H = 1$. Hence, for any $\tau_{n-1} \geq \tau^{\min}_{n-1}$ and $\tau_n \leq \tau^{\min}_n$ the constraint (40) holds and consequently $A_{H[1:n,1:n-2]}$ is non-negative. The corresponding columns $\bar{\boldsymbol{a}}^{\min}_{H,n-1} = \bar{\boldsymbol{a}}_n + \tau^{\min}_{n-1}\boldsymbol{\delta}^{\text{out}}$ and $\bar{\boldsymbol{a}}^{\min}_{H,n} = \bar{\boldsymbol{a}}_n + \tau^{\min}_n\boldsymbol{\delta}^{\text{out}}$ are depicted in Fig.5 (bottom).

Combining the above constraints we have that the four parameters $\tau^{\min}_{n-1}, \tau^{\min}_n, \tau^{\max}_{n-1}, \tau^{\max}_n$ define the rectangle

$$\Gamma_1 = [\tau^{\min}_{n-1}, \tau^{\max}_{n-1}] \times [\tau^{\max}_n, \tau^{\min}_n], \tag{41}$$

which characterize the equivalent matrices A_H having all entries non-negative except of possibly in the 2×2 aliased block (see Fig.4). Thus we must have $\Gamma_H \subset \Gamma_1$.

We are left to find the conditions under which the 2×2 aliased block is non-negative. Writing A_H explicitly we have that these conditions are

$$\begin{aligned} A_{H,n,n\text{-}1} &= \tau_{n\text{-}1}(\tau_{n\text{-}1}-\alpha_{n\text{-}1})P(\bar{n}\,|\,n\text{-}1) \\ &\quad + (1-\tau_{n\text{-}1})(\tau_{n\text{-}1}-\alpha_n)P(\bar{n}\,|\,n) \geq 0 \end{aligned} \tag{42}$$

$$\begin{aligned} A_{H,n\text{-}1,n} &= \tau_n(\alpha_{n\text{-}1}-\tau_n)P(\bar{n}\,|\,n\text{-}1) \\ &\quad + (1-\tau_n)(\alpha_n-\tau_n)P(\bar{n}\,|\,n) \geq 0 \end{aligned} \tag{43}$$

$$\begin{aligned} A_{H,n\text{-}1,n\text{-}1} &= \tau_{n\text{-}1}(\alpha_{n\text{-}1}-\tau_n)P(\bar{n}\,|\,n\text{-}1) \\ &\quad + (1-\tau_{n\text{-}1})(\alpha_n-\tau_n)P(\bar{n}\,|\,n) \geq 0 \end{aligned} \tag{44}$$

$$\begin{aligned} A_{H,n,n} &= \tau_n(\tau_{n\text{-}1}-\alpha_{n\text{-}1})P(\bar{n}\,|\,n\text{-}1) \\ &\quad + (1-\tau_n)(\tau_{n\text{-}1}-\alpha_n)P(\bar{n}\,|\,n) \geq 0. \end{aligned} \tag{45}$$

As the case $P(\bar{n}\,|\,n\text{-}1) = P(\bar{n}\,|\,n) = 0$ is trivial, we assume that at least one of $P(\bar{n}\,|\,n\text{-}1), P(\bar{n}\,|\,n)$ is nonzero (and since by convention $P(\bar{n}\,|\,n\text{-}1) \geq P(\bar{n}\,|\,n)$, this is equivalent to $P(\bar{n}\,|\,n\text{-}1) > 0$).

Recall that by definition $\delta^{\text{out}}_{n\text{-}1} = P(\bar{n}\,|\,n\text{-}1) - P(\bar{n}\,|\,n)$ (see (5)). We now consider the cases $\delta^{\text{out}}_{n\text{-}1} = 0$ and $\delta^{\text{out}}_{n\text{-}1} > 0$ separately.

The case $\delta^{\text{out}}_{n\text{-}1} = 0$. Consider first the off-diagonal constraint (43) for $A_{H,n\text{-}1,n} \geq 0$, taking the form

$$\tau_n(1-(\alpha_{n\text{-}1}-\alpha_n))) \leq \alpha_n.$$

Denote

$$\tau^0 = \alpha_n/(1-(\alpha_{n\text{-}1}-\alpha_n)).$$

Since $\alpha_{n\text{-}1} - \alpha_n \leq 1$ we need $\tau_n \leq \tau^0$. Similarly, (42) is satisfied if and only if $\tau_{n\text{-}1} \geq \tau^0$. Thus in order for the off-diagonal entries $A_{H,n,n\text{-}1}, A_{H,n\text{-}1,n}$ to be non-negative we need $(\tau_{n\text{-}1},\tau_n) \in \Gamma^0_2$ where

$$\Gamma^0_2 = [\tau^0,\infty] \times [-\infty,\tau^0]. \tag{46}$$

Next, the on-diagonal constraint (44) for $A_{H,n\text{-}1,n\text{-}1} \geq 0$ is equivalent to

$$\tau_n \leq \alpha_n + \tau_{n\text{-}1}(\alpha_{n\text{-}1}-\alpha_n). \tag{47}$$

Similarly, the on-diagonal constrain (45) for $A_{H,n,n} \geq 0$ is

$$\tau_n(\alpha_{n\text{-}1}-\alpha_n) \leq \tau_{n\text{-}1} - \alpha_n. \tag{48}$$

Define the two linear functions $g^0, f^0 : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} g^0(\tau_{n\text{-}1}) &= \alpha_n + \tau_{n\text{-}1}(\alpha_{n\text{-}1}-\alpha_n) \\ f^0(\tau_{n\text{-}1}) &= \frac{\tau_{n\text{-}1}-\alpha_n}{\alpha_{n\text{-}1}-\alpha_n}. \end{aligned}$$

Note that τ^0 is a fixed point of both g^0 and f^0,

$$\tau^0 = g^0(\tau^0) = f^0(\tau^0).$$

Note also that for $\alpha_{n\text{-}1} \geq \alpha_n$ the functions g^0 and f^0 are increasing, while for $\alpha_{n\text{-}1} < \alpha_n$ they are decreasing. Thus, if $\alpha_{n\text{-}1} \geq \alpha_n$ the constrains (47,48) are automatically satisfied for $(\tau_{n\text{-}1},\tau_n) \in \Gamma^0_2$, so in this case $A_{H,n\text{-}1,n\text{-}1}, A_{H,n,n}$ are also guaranteed to be non-negative.

If $\alpha_{n\text{-}1} < \alpha_n$ then with $\tau_{n\text{-}1} \geq \tau_n$ (as we assume here) we have $f^0(\tau_{n\text{-}1}) \leq g^0(\tau_{n\text{-}1}) \leq \tau_0$ and the constraints (47,48) take the form $f^0(\tau_{n\text{-}1}) \leq \tau_n \leq g^0(\tau_{n\text{-}1})$. Thus, in order for the on-diagonal entries $A_{H,n\text{-}1,n\text{-}1}$ and $A_{H,n,n}$ to be non-negative we must have $(\tau_{n\text{-}1},\tau_n) \in \Gamma^0_3$, where

$$\Gamma^0_3 = \{(\tau_{n\text{-}1},\tau_n) \in \Gamma_1 \mid f^0(\tau_{n\text{-}1}) \leq \tau_n \leq g^0(\tau_{n\text{-}1})\}. \tag{49}$$

We are left to ensure that for $\alpha_{n\text{-}1} < \alpha_n$ the off diagonal entries are also non-negative. Indeed, since $\tau_n \leq \tau_{n\text{-}1}$, τ^0 is a fixed point and $g(\tau_{n\text{-}1}), f(\tau_{n\text{-}1})$ are decreasing, for any $(\tau_{n\text{-}1},\tau_n) \in \Gamma^0_3$ we automatically have that $\tau_0 \leq \tau_{n\text{-}1}$ and $\tau_n \leq \tau_0$, so $(\tau_{n\text{-}1},\tau_n) \in \Gamma^0_3$ implies $(\tau_{n\text{-}1},\tau_n) \in \Gamma^0_2$. Thus all entries of the aliasing block are guaranteed to be non-negative.

To conclude, we have shown that for $\delta^{\text{out}}_{n\text{-}1} = 0$ the feasible region (10) is given by

$$\Gamma^0_H = \begin{cases} \Gamma_1 \cap \Gamma^0_2 & \alpha_{n\text{-}1} \geq \alpha_n \\ \Gamma_1 \cap \Gamma^0_3 & \alpha_{n\text{-}1} < \alpha_n. \end{cases}$$

The case $\delta^{\text{out}}_{n\text{-}1} > 0$. This case has the same characteristics as for the $\delta^{\text{out}}_{n\text{-}1} = 0$ case, but it is a bit more complex to analyze. Define $\tau^{\pm}$ (as the analogues of τ^0) by

$$\begin{aligned} \tau^{\pm} &= \frac{1}{2\delta^{\text{out}}_{n\text{-}1}}\Big(\alpha_{n\text{-}1}P(\bar{n}\,|\,n\text{-}1) \\ &\qquad -(1+\alpha_n)P(\bar{n}\,|\,n) \pm \sqrt{\Delta}\Big), \end{aligned} \tag{50}$$

where

$$\begin{aligned} \Delta &= \Big(\alpha_{n\text{-}1}P(\bar{n}\,|\,n\text{-}1) - (1+\alpha_n)P(\bar{n}\,|\,n)\Big)^2 \\ &\qquad + 4\alpha_n P(\bar{n}\,|\,n)\delta^{\text{out}}_{n\text{-}1} \geq 0. \end{aligned}$$

And define the regions

$$\Gamma_2 = [\tau^+,\infty] \times [\tau^-,\tau^+] \tag{51}$$

$$\Gamma_3 = \{(\tau_{n\text{-}1},\tau_n) \in \Gamma_1 \mid f(\tau_{n\text{-}1}) \leq \tau_n \leq g(\tau_{n\text{-}1})\} \tag{52}$$

where the functions $g, f : \mathbb{R} \to \mathbb{R}$ are given by

$$\begin{aligned} g(\tau_{n\text{-}1}) &= \left(\frac{\alpha_{n\text{-}1}P(\bar{n}\,|\,n\text{-}1) - \alpha_n P(\bar{n}\,|\,n)}{\delta^{\text{out}}_{n\text{-}1}}\right) \\ &\quad - \frac{P(\bar{n}\,|\,n\text{-}1)P(\bar{n}\,|\,n)(\alpha_{n\text{-}1}-\alpha_n)}{(\delta^{\text{out}}_{n\text{-}1})^2} \\ &\quad \times \left(\tau_{n\text{-}1} - \left(\frac{-P(\bar{n}\,|\,n)}{\delta^{\text{out}}_{n\text{-}1}}\right)\right)^{-1} \end{aligned} \tag{53}$$

and

$$f(\tau_{n-1}) = \left(\frac{-P(\bar{n}\,|\,n)}{\delta^{\text{out}}_{n-1}}\right) - \frac{P(\bar{n}\,|\,n-1)P(\bar{n}\,|\,n)(\alpha_{n-1}-\alpha_n)}{(\delta^{\text{out}}_{n-1})^2} \times \left(\tau_{n-1} - \left(\frac{\alpha_{n-1}P(\bar{n}\,|\,n-1) - \alpha_n P(\bar{n}\,|\,n)}{\delta^{\text{out}}_{n-1}}\right)\right)^{-1}. \quad (54)$$

Lemma 8. *Let Γ_1 be defined in (41) and let Γ_2 and Γ_3 be defined according to whether $\delta^{\text{out}}_{n-1} = 0$ (46,49) or not (51,52). Then the feasible region Γ_H satisfies*

$$\Gamma_H = \begin{cases} \Gamma_1 \cap \Gamma_2 & \alpha_{n-1} \geq \alpha_n \\ \Gamma_1 \cap \Gamma_3 & \alpha_{n-1} < \alpha_n. \end{cases}$$

Proof. As the case $\delta^{\text{out}}_{n-1} = 0$ was treated above, we consider the case $\delta^{\text{out}}_{n-1} > 0$. Consider first the off diagonal constraint (43). Multiplying by (-1), we need to solve the following inequality for $\tau \in \mathbb{R}$,

$$\tau^2\delta^{\text{out}}_{n-1} - \tau(\alpha_{n-1}P(\bar{n}\,|\,n-1) -(1+\alpha_n)P(\bar{n}\,|\,n)) - \alpha_n P(\bar{n}\,|\,n) \leq 0. \quad (55)$$

We first solve with equality to find the solutions τ^-, τ^+ given in (50). Thus, since $\Delta \geq 0$ we have that any feasible τ_n must satisfy $\tau^- \leq \tau_n \leq \tau^+$. Note that the constraint (42) for τ_{n-1} is the complement of (43), and by assumption $\tau_{n-1} \geq \tau_n$, so (42) is satisfied iff $\tau^+ \leq \tau_{n-1}$. Thus, the region Γ_2 given in (51) indeed characterize the non-negativity of both $A_{n-1,n}$ and $A_{n,n-1}$. With some algebra, τ^+ and τ^- can be shown to satisfy the following useful relations:

- If $\alpha_{n-1} \geq \alpha_n$ then

$$-\frac{P(\bar{n}\,|\,n-1)}{\delta^{\text{out}}_{n-1}} \leq \tau^- \leq 0 \quad (56)$$

and

$$0 \leq \tau^+ \leq \frac{\alpha_{n-1}P(\bar{n}\,|\,n-1) - \alpha_n P(\bar{n}\,|\,n)}{\delta^{\text{out}}_{n-1}}. \quad (57)$$

- If $\alpha_{n-1} < \alpha_n$ then

$$\tau^- \leq -\frac{P(\bar{n}\,|\,n)}{\delta^{\text{out}}_{n-1}} \leq \frac{\alpha_{n-1}P(\bar{n}\,|\,n-1) - \alpha_n P(\bar{n}\,|\,n)}{\delta^{\text{out}}_{n-1}} \leq \tau^+. \quad (58)$$

We proceed to handle the constraints (44) and (45) corresponding to the region Γ_3. We begin by solving the inequality (44):

$$-\tau_n(P(\bar{n}\,|\,n) + \tau_{n-1}\delta^{\text{out}}_{n-1}) + \alpha_n P(\bar{n}\,|\,n) + \tau_{n-1}(\alpha_{n-1}P(\bar{n}\,|\,n-1) - \alpha_n P(\bar{n}\,|\,n)) \geq 0.$$

Note that for $(\tau_{n-1}, \tau_n) \in \Gamma_1$ we have $(P(\bar{n}\,|\,n) + \tau_{n-1}\delta^{\text{out}}_{n-1}) \geq 0$. Rearranging we get that in order for $A_{H,n-1,n-1}$ to be non-negative we must have that

$$\text{if} \quad \tau_{n-1} \geq -\frac{P(\bar{n}\,|\,n)}{\delta^{\text{out}}_{n-1}} \quad \text{then} \quad \tau_n \leq g(\tau_{n-1}), \quad (59)$$

where g is the function given in (53). Similarly, consider the condition (45),

$$\tau_n\Big(\alpha_n P(\bar{n}\,|\,n) - \alpha_{n-1}P(\bar{n}\,|\,n-1) + \tau_{n-1}\delta^{\text{out}}_{n-1}\Big) + P(\bar{n}\,|\,n)(\alpha_n - \tau_{n-1}) \geq 0.$$

Rearranging we find that in order for $A_{H,n,n} \geq 0$ we must have

$$\begin{cases} \tau_n \leq f(\tau_{n-1}) & \tau_{n-1} \leq \frac{\alpha_{n-1}P(\bar{n}\,|\,n-1) - \alpha_n P(\bar{n}\,|\,n)}{P(\bar{n}\,|\,n-1) - P(\bar{n}\,|\,n)} \\ \tau_n \geq f(\tau_{n-1}) & \text{otherwise}, \end{cases} \quad (60)$$

where the function f is given in (54). Note that g (res. f) defines the boundary where (44) (res. (45)) changes sign, namely any pair $(\tau_{n-1}, \tau_n) = (\tau_{n-1}, g(\tau_{n-1}))$ is on the curve making Equation (44) equal zero, and similarly $f(\tau_{n-1})$ is such that $(\tau_{n-1}, \tau_n) = (\tau_{n-1}, f(\tau_{n-1}))$ is on the curve making (45) equal zero. Having the boundaries g, f in our disposal let us first consider the case $\alpha_{n-1} \geq \alpha_n$.

The sub-case $\alpha_{n-1} \geq \alpha_n$. We show that in this case, having $(\tau_{n-1}, \tau_n) \in \Gamma_1 \cap \Gamma_2$ already ensures that conditions (59) and (60) are trivially met, which in turn implies the non-negativity of both $A_{H,n-1,n-1}$ and $A_{H,n,n}$. This is done by showing that for any $(\tau_{n-1}, \tau_n) \in \Gamma_1 \cap \Gamma_2$ the curve $(\tau_{n-1}, g(\tau_{n-1}))$ is above (τ_{n-1}, τ^+). Similarly, for $\tau_{n-1} < (\alpha_{n-1}P(\bar{n}\,|\,n-1) - \alpha_n P(\bar{n}\,|\,n))/\delta^{\text{out}}_{n-1}$ the curve $(\tau_{n-1}, f(\tau_{n-1}))$ is above (τ_{n-1}, τ^+) and for $\tau_{n-1} > (\alpha_{n-1}P(\bar{n}\,|\,n-1) - \alpha_n P(\bar{n}\,|\,n))/\delta^{\text{out}}_{n-1}$ the curve $(\tau_{n-1}, f(\tau_{n-1}))$ is below (τ_{n-1}, τ^-), thus making conditions (59) and (60) true. Toward this end consider the equality $g(\tau) = f(\tau)$ given by

$$0 = \Big(\alpha_{n-1}P(\bar{n}\,|\,n-1) + (1-\alpha_n)P(\bar{n}\,|\,n)\Big) \times \Big(\tau^2\delta^{\text{out}}_{n-1} + \tau((1+\alpha_n)P(\bar{n}\,|\,n) -\alpha_{n-1}P(\bar{n}\,|\,n-1)) - \alpha_n P(\bar{n}\,|\,n)\Big).$$

Thus if $(\alpha_{n-1}P(\bar{n}\,|\,n-1) + (1-\alpha_n)P(\bar{n}\,|\,n)) = 0$ we have that $g = f$ identically. Otherwise we need to solve again (55) so the solutions are τ^+, τ^- with $g(\tau^+) = f(\tau^+)$ and $g(\tau^-) = f(\tau^-)$. In addition one can show that τ^+ and τ^- are in fact fixed points of both g and f, so together we have

$$\tau^+ = g(\tau^+) = f(\tau^+)$$
$$\tau^- = g(\tau^-) = f(\tau^-).$$

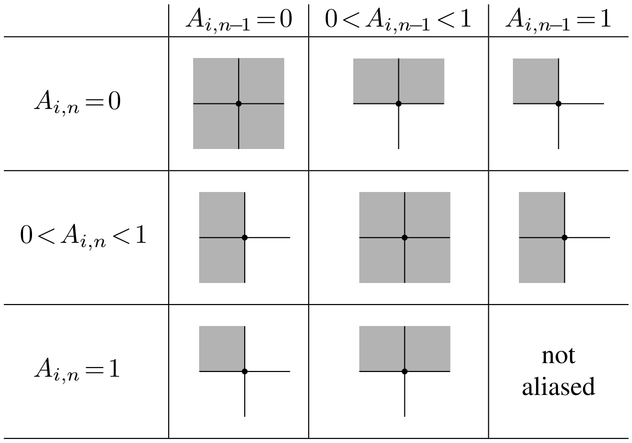


Table 1. For any state $i \in \mathcal{X} \setminus \{n-1, n\}$ with $A_{i,n} \in \{0,1\}$ or $A_{i,n-1} \in \{0,1\}$, pick the relevant diagram.

Inspecting $g(\tau_{n-1})$ one can see that for $\tau_{n-1} \geq -P(\bar{n} \,|\, n)/\delta^{\text{out}}_{n-1}$, $g(\tau_{n-1})$ is monotonic increasing and concave. Since by (57) we have $\tau^+ \geq -P(\bar{n} \,|\, n)/\delta^{\text{out}}_{n-1}$ we get that for $\tau_{n-1} \geq \tau^+$ we must have $g(\tau_{n-1}) \geq \tau^+$ as needed. Similarly, for $\tau^+ \leq \tau_{n-1} < (\alpha_{n-1} P(\bar{n} \,|\, n-1) - \alpha_n P(\bar{n} \,|\, n))/\delta^{\text{out}}_{n-1}$ the function $f(\tau_{n-1})$ is increasing and convex and thus above τ^+, while for $(\alpha_{n-1} P(\bar{n} \,|\, n-1) - \alpha_n P(\bar{n} \,|\, n))/\delta^{\text{out}}_{n-1} < \tau_{n-1}$ it is increasing but always below τ^-. Thus, for $\alpha_{n-1} \geq \alpha_n$ we have that $\Gamma_1 \cap \Gamma_2$ also characterize the non-negativity of $A_{H,n-1,n-1}$ and $A_{H,n,n}$ as claimed.

The sub-case $\alpha_{n-1} < \alpha_n$. Note that by (58) we have $\tau^+ \geq (\alpha_{n-1} P(\bar{n} \,|\, n-1) - \alpha_n P(\bar{n} \,|\, n))/\delta^{\text{out}}_{n-1}$. Thus for $\tau^+ \leq \tau_{n-1}$ both g and f are decreasing and convex and $f(\tau_{n-1}) \leq g(\tau_{n-1})$. Thus in order to ensure (59, 60) we need $f(\tau_{n-1}) \leq \tau_n \leq g(\tau_{n-1})$. Thus, Γ_3 as defined in (52) characterize the non-negativity of $A_{H,n,n}, A_{H,n-1,n-1}$. Finally we need to show that having $(\tau_{n-1}, \tau_n) \in \Gamma_3$ also ensures the non-negativity of $A_{H,n,n-1}$ and $A_{H,n-1,n}$. But by (58) we have that for $\tau_{n-1} \geq \tau^+$ both $g(\tau_{n-1}), f(\tau_{n-1}) \geq \tau^-$ and thus $\Gamma_3 \subset \Gamma_2$. Hence we have shown that Γ_H is characterized as claimed. □

Lemma 9. *The set Γ_H is connected.*

Proof. If $\alpha_{n-1} \geq \alpha_n$ then $\Gamma_H = \Gamma_1 \cap \Gamma_2$ is a rectangle and thus connected. In the case $\alpha_{n-1} < \alpha_n$ we have that $f(\tau_{n-1}) \leq g(\tau_{n-1})$ and are both decreasing and convex thus the region Γ_3 with intersection with a rectangle is a connected set. □

C.2. Conditions for $|\Gamma_H| = 1$

Let us first write

$$(\tau_{n-1}, \tau_n) = (1, 0) + (\Delta\tau_{n-1}, \Delta\tau_n).$$

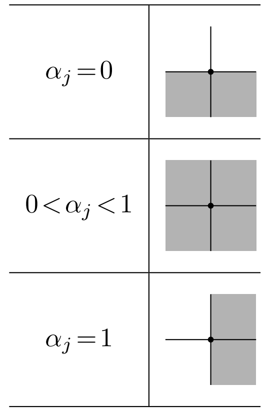


Table 2. For any state $j \in \text{supp}_{\text{in}} \setminus \{n-1, n\}$ with $\alpha_j \in \{0,1\}$ pick the corresponding diagram.

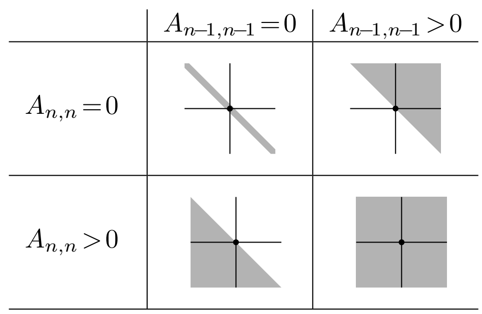


Table 4. If $\alpha_{n-1} < \alpha_n$ pick the relevant diagram from here.

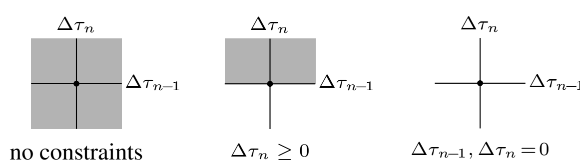


Figure 6. The effective feasible region for various constraints, ensuring $A_H(1 + \Delta\tau_{n-1}, \Delta\tau_n) \geq 0$ for $|\Delta\tau_{n-1}|, |\Delta\tau_n| << 1$.

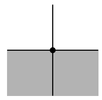
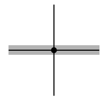
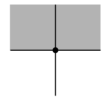
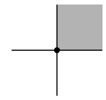
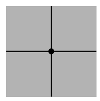
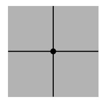
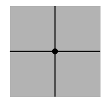
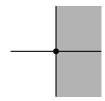
	$A_{n-1,n-1} < A_{n,n}$	$A_{n-1,n-1} = A_{n,n}$	$A_{n,n} < A_{n-1,n-1} < A_{\bar{n},n-1}$	$A_{n-1,n-1} = A_{\bar{n},n-1}$
$A_{n-1,n} = 0$				
$A_{n-1,n} > 0$				

Table 3. If $\alpha_{n-1} \geq \alpha_n$ pick the relevant diagram from here.

We characterize the conditions for $|\Gamma_H| = 1$ by determining the geometrical constraints the entries of the transition matrix A pose on $(\Delta\tau_{n-1}, \Delta\tau_n)$ in order to ensure $A_H(1 + \Delta\tau_{n-1}, \Delta\tau_n) \geq 0$. Note that $|\Gamma_H| = 1$ iff these constraints imply that $(\Delta\tau_{n-1}, \Delta\tau_n) = (0, 0)$ is the *unique* feasible pair.

As a first example, consider a 2A-HMM H having a transition matrix with all entries being strictly positive, $A \geq \epsilon > 0$. Since the mapping (9) is continuous in $\Delta\tau_{n-1}, \Delta\tau_n$, there exists a neighborhood $N \subset \mathbb{R}^2$ of $(\Delta\tau_{n-1}, \Delta\tau_n) = (0, 0)$, such that for *any* $(\Delta\tau_{n-1}, \Delta\tau_n) \in N$ the matrix $A_H(1 + \Delta\tau_{n-1}, \Delta\tau_n)$ is non-negative, and thus $N \subset \Gamma_H$. This condition can be represented in the $(\Delta\tau_{n-1}, \Delta\tau_n)$ plane (i.e. $\mathbb{R}^2$) as the "full" diagram Fig.6. On the other hand, the condition that $(\Delta\tau_{n-1}, \Delta\tau_n) = (0, 0)$ is the unique feasible pair can be represented by a point like diagram as in Fig.6.

In general, the entries of the transition matrix A put constraints on the feasible $(\Delta\tau_{n-1}, \Delta\tau_n)$ only when $(\tau_{n-1}, \tau_n) = (1, 0)$ is on the boundary of Γ_H. These constraints can be explicitly determined in terms of A's entries by considering the exact characterization of Γ_H given in Theorem 2. Note however that by the fact that Γ_H is connected, and as far as the condition $|\Gamma_H| = 1$ is concerned, we only need to consider the shape of these constraints in a small neighborhood of $(\tau_{n-1}, \tau_n) = (1, 0)$, i.e for $|\Delta\tau_{n-1}|, |\Delta\tau_n| << 1$. Any such neighborhood can be represented on the $\mathbb{R}^2$ plane (as in Fig.6). The shape of this neighborhood for a given H is called the *effective feasible region* of Γ_H.

Now, as the example with $A \geq \epsilon > 0$ shows, a non-trivial constraint on the (effective) feasible region must results from A having some zeros entries. Each such a zero entry, as determined by its position in A, put a boundary constraint on $(\Delta\tau_{n-1}, \Delta\tau_n)$. These in turn corresponds to a suitable diagram in $\mathbb{R}^2$ (as the diagram for $\Delta\tau_n \geq 0$ in Fig.6). The effective feasible region of A is obtained by taking the intersection of all these diagrams. The exact correspondence between A's entries and the corresponding diagrams is given in Tables 1,2,3,4. The procedure for determining the effective feasible region of a 2A-HMM is given in Algorithm 1. The correctness of the algorithm is demonstrated in the proof of Lemma 8.

Algorithm 1 determining the effective feasible region for minimal 2A-HMM H

permute aliased states so that $P(\bar{n} \,|\, n-1) \geq P(\bar{n} \,|\, n)$

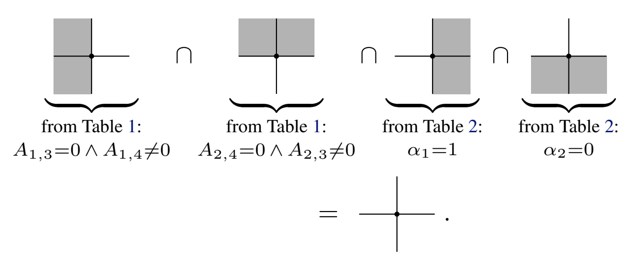
collect the following diagrams:

- $\forall i \in [n-2]$ with $A_{i,n} \in \{0, 1\}$ or $A_{i,n-1} \in \{0, 1\}$ pick the relevant diagram in Table 1
- $\forall j \in \text{supp}_{\text{in}} \setminus \{n-1, n\}$ with $\alpha_j \in \{0, 1\}$ pick corresponding diagram in Table 2
- if $\alpha_{n-1} \geq \alpha_n$ pick relevant diagram in table 3 and if $\alpha_{n-1} < \alpha_n$ pick relevant diagram in Table 4

Return the intersection of all the regions obtained in previous step

C.3. Examples.

We demonstrate our Algorithm 1 for determining the identifiability of 2A-HMMs on the 2A-HMM given in Section 6, shown in Fig 2 (left). Going through the steps of Algorithm 1 we get the following diagrams for the effective feasible region:



Since their intersection results in a point like diagram, this 2A-HMM is identifiable.

More generally, for a minimal stationary 2A-HMM satisfying Assumptions (A1-A2) with aliased states n and $n-1$, a sufficient condition for uniqueness is the following constraints on the allowed transitions between the hidden states: $\exists i_{n-1}, j_{n-1}, i_n, j_n \in [n-2]$ such that

✓	$i_{n-1} \to n-1 \to j_{n-1}$	✗	$i_{n-1} \to n \to *$
✓	$i_n \to n \to j_n$	✗	$i_n \to n-1 \to *$
		✗	$* \to n-1 \to j_n$
		✗	$* \to n \to j_{n-1}.$

One can check that these conditions give the same set of diagrams as above.

D. Proofs for Section 5 (Learning)

D.1. Proof of Lemma 3

The claim in (21) that $M^{(1)} = BAC_\beta = \bar{A}$ follows directly from (14) and (16). Next, we have

$$\begin{aligned} R^{(2)} &= BAAC_\beta - BAC_\beta BAC_\beta \\ &= BA(I_n - C_\beta B)AC_\beta \\ &= BA\boldsymbol{b}\boldsymbol{c}_\beta^\intercal AC_\beta, \end{aligned}$$

where the last equality is by the fact that $(I_n - C_\beta B) = \boldsymbol{b}\boldsymbol{c}_\beta^\intercal$. Since $BA\boldsymbol{b} = \boldsymbol{\delta}^{\text{out}}$ and $\boldsymbol{c}_\beta^\intercal AC_\beta = (\boldsymbol{\delta}^{\text{in}})^\intercal$ we have $R^{(2)} = \boldsymbol{\delta}^{\text{out}}(\boldsymbol{\delta}^{\text{in}})^\intercal$ as claimed in (22).

As for (23) we have,

$$\begin{aligned} R^{(3)} &= BAAAC_\beta - BAAC_\beta BAC_\beta \\ &\quad - BAC_\beta BAAC_\beta + BAC_\beta BAC_\beta BAC_\beta \\ &= BA(I_n - C_\beta B)A(I_n - C_\beta B)AC_\beta \\ &= \boldsymbol{\delta}^{\text{out}}\boldsymbol{c}_\beta^\intercal A\boldsymbol{b}(\boldsymbol{\delta}^{\text{in}})^\intercal. \end{aligned}$$

Since by definition $\kappa = \boldsymbol{c}_\beta^\intercal A\boldsymbol{b}$ we have $R^{(3)} = \kappa R^{(2)}$ and the claim in (23) is proved.

Finally,

$$F^{(c)} = BA\Big(\operatorname{diag}(\mathcal{K}_{[\cdot,c]}) - C_\beta \operatorname{diag}(\bar{K}_{[\cdot,c]})B\Big)AC_\beta.$$

Since

$$\operatorname{diag}(\mathcal{K}_{[\cdot,c]}) - C_\beta \operatorname{diag}(\bar{K}_{[\cdot,c]})B = \bar{K}_{n-1,c}\boldsymbol{b}(\boldsymbol{c}_\beta)^\intercal$$

we have that $F^{(c)} = \bar{K}_{n-1,c}R^{(2)}$ as claimed in (24).

D.2. Proof of Lemma 4

Assumption (A2) combined with the fact that the HMM has a finite number of states imply that the HMM is *geometrically* ergodic: there exist parameters $G < \infty$ and $\psi \in [0,1)$ such that from any initial distribution $\boldsymbol{\pi}^0$,

$$\left\|A^t\boldsymbol{\pi}^0 - \boldsymbol{\pi}\right\|_1 \le 2G\psi^t, \quad \forall t \in \mathbb{N}. \tag{61}$$

Thus, we may apply the following concentration bound, given in Kontorovich & Weiss (2014):

Theorem 4. *Let $Y = Y_0, \ldots, Y_{T-1} \in \mathcal{Y}^T$ be the output of a HMM with transition matrix A and output parameters $\boldsymbol{\theta}$. Assume that A is geometrically ergodic with constants G, ψ. Let $F : (Y_0, \ldots, Y_{T-1}) \mapsto \mathbb{R}$ be any function that is Lipschitz wit constatnt l with respect to the Hamming metric on $\mathcal{Y}^T$. Then, for all $\epsilon > 0$,*

$$\Pr(|F(Y) - \mathbf{E}F| > \epsilon T) \le 2\exp\left(-\frac{T(1-\psi)^2\epsilon^2}{2l^2G^2}\right). \tag{62}$$

In order to apply the theorem note that $\forall t \in \{1,2,3\}$, $\mathbf{E}[\hat{\mathcal{M}}_{ij}^{(t)}] = \mathcal{M}_{ij}^{(t)}$ for any $i,j \in [n-1]$. In addition, following Assumption (A3), $(T-t)\hat{\mathcal{M}}_{ij}^{(t)}$ is $(t+1)L^2$-Lipschitz with respect to the Hamming metric on $\mathcal{Y}^T$. Thus, taking $\epsilon \approx T^{-\frac{1}{2}}$ in Theorem 4 and applying a union bound on i,j readily gives

$$\hat{\mathcal{M}}^{(t)} = \mathcal{M}^{(t)} + O_P\left(T^{-\frac{1}{2}}\right).$$

The kernel-free moments $\hat{M}^{(t)}$ given in (25) incur additional error which results in a factor of at most $1/(\sigma_{\min}(\bar{K})^2 \min_i \pi_i)$ hidden in the O_P notation. Since $R^{(t)}$ are (low order) polynomials of $M^{(t)}$, the asymptotics $O_P\left(T^{-\frac{1}{2}}\right)$ carry on to the error in $\hat{R}^{(t)}$. A similar argument yields the claim for $F^{(c)}$.

D.3. Proof of Lemma 6

Let σ_1 and $\hat{\sigma}_1$ be the largest singular values of $R^{(2)}$ and $\hat{R}^{(2)}$, respectively. Combining Weyl's Theorem (Stewart & Sun, 1990) with Lemma 4 gives

$$|\sigma_1 - \hat{\sigma}_1| \le \left\|R^{(2)} - \hat{R}^{(2)}\right\|_{\text{F}} = O_P(T^{-\frac{1}{2}}),$$

Recall that under the null hypothesis $\mathcal{H}_0$, we have $\sigma_1 = 0$. Thus, with high probability $\hat{\sigma}_1 < \xi_0 T^{-\frac{1}{2}}$, for some $\xi_0 > 0$. In contrast, under $\mathcal{H}_1$ we have $\sigma_1 = \sigma > 0$, thus for some $\xi_1 > 0$, $\hat{\sigma}_1 > \sigma - \xi_1 T^{-\frac{1}{2}}$. Hence, taking T sufficiently large, we have that for any $c_h > 0$ and $0 < \epsilon < \frac{1}{2}$, with $h_T = c_h T^{-\frac{1}{2}+\epsilon}$,

$$\begin{aligned} &\text{in case } \mathcal{H}_0: && \hat{\sigma}_1 < h_T \\ &\text{in case } \mathcal{H}_1: && \hat{\sigma}_1 > h_T, \end{aligned}$$

with high probability. Thus, the correct detection of aliasing is with high probability.

D.4. Proof of Lemma 7

Let us define the following score function for any $i \in [n-1]$,

$$\text{score}(i) = \sum_{j \in [n-1]} \left\|\hat{F}^{(j)} - \bar{K}_{i,j}\hat{R}^{(2)}\right\|_{\text{F}}^2.$$

According to Eq. (30) the chosen aliased component is the index with minimal score. Hence, in order to prove the Lemma we need to show that

$$\lim_{T\to\infty} \Pr(\exists i \neq n-1 : \text{score}(i) < \text{score}(n-1)) = 0.$$

By Lemma 4 and (20) we have

$$\begin{aligned}\hat{F}^{(j)} &= F^{(j)} + \frac{\xi_F^{(j)}}{\sqrt{T}} = \bar{K}_{n-1,j} R^{(2)} + \frac{\xi_F^{(j)}}{\sqrt{T}} \\ \hat{R}^{(2)} &= R^{(2)} + \frac{\xi_R}{\sqrt{T}},\end{aligned}$$

for some $\xi_R, \xi_F^{(j)} \in \mathbb{R}^{(n-1)\times(n-1)}$ with $O_P(\xi_R) = 1$ and $O_P(\xi_F^{(j)}) = 1$. Thus,

$$\text{score}(n-1) = \frac{1}{\sqrt{T}} \sum_{j\in[n-1]} \left\| \xi_F^{(j)} - \bar{K}_{j,n-1}\xi_R \right\|_{\text{F}}^2 \xrightarrow{P} 0.$$

In contrast, for any $i \neq n-1$ we may write $\text{score}(i)$ as

$$\sum_{j\in[n-1]} \left\| (\bar{K}_{j,i} - \bar{K}_{j,n-1})R^{(2)} + \frac{1}{\sqrt{T}}(\xi_F^{(j)} - \bar{K}_{j,n-1}\xi_R) \right\|_{\text{F}}^2 .$$

Applying the (inverse) triangle inequality we have

$$\text{score}(i) \geq \sigma^2 \left\| \bar{K}_{[\cdot,n-1]} - \bar{K}_{[\cdot,i]} \right\|^2 - O_P(T^{-\frac{1}{2}}).$$

Since $\bar{K}$ is full rank, $\sigma^2 \left\| \bar{K}_{[\cdot,n-1]} - \bar{K}_{[\cdot,i]} \right\|^2 > 0$. Thus, for any $i \neq n-1$ as $T \to \infty$, w.h.p $\text{score}(i) > \text{score}(n-1)$. Taking a union bound over i yields the claim.

D.5. Estimating γ and β

We now show how to estimate γ and β. As discussed in Section 5.4, this is done by searching for γ', β' ensuring the non-negativity of (33), namely, $A'_H(\gamma', \beta') \geq 0$, where

$$A'_H(\gamma', \beta') \equiv C_{\beta'} \bar{A} B + \gamma' C_{\beta'} \boldsymbol{u}\boldsymbol{c}_{\beta'}{}^{\mathsf{T}} + \frac{\sigma}{\gamma'} \boldsymbol{b}\boldsymbol{v}^{\mathsf{T}} B + \kappa\, \boldsymbol{b}\boldsymbol{c}_{\beta'}{}^{\mathsf{T}}.$$

We pose this as a non-linear two dimensional optimization problem. For any $\gamma' \geq 0$ and $0 \leq \beta' \leq 1$ define the objective function $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$h(\gamma', \beta') = \min_{i,j\in[n]} \{\gamma' A'_H(\gamma', \beta')_{ij}\}.$$

Note that $h(\gamma', \beta') \geq 0$ iff $A'_H(\gamma', \beta')$ does not have negative entries. Recall that by the identifiability of H, if we constrain $\gamma' \geq 0$ then the constraint $A'_H(\gamma', \beta') \geq 0$ has the unique solution (γ, β) (this is the equivalent to the convention $\tau_{n-1} \geq \tau_n$ made in Section 4.2). Namely, any $(\gamma', \beta') \neq (\gamma, \beta)$ results in at least one negative entry in $A'_H(\gamma', \beta')$. Hence, $h(\gamma', \beta')$ has a unique maximum, obtained at the true (γ, β). In addition, since $\|\boldsymbol{u}\|_2 = \|\boldsymbol{v}\|_2 = 1$, a feasible solution must have $\gamma' \leq 2/\sigma$. So our optimization problem is:

$$(\hat{\gamma}, \hat{\beta}) = \operatorname*{argmax}_{(\gamma',\beta')\in[0,\frac{2}{\sigma}]\times[0,1]} h(\gamma', \beta') \tag{63}$$

This two dimensional optimization problem can be solved by either brute force or any non-linear problem solver.

In practice, we solve the optimization problem (63) with the empirical estimates plugged in, that is

$$\hat{A}'_H(\gamma', \beta') = C_{\hat{\beta}} \hat{\bar{A}} B + \gamma' C_{\beta'} \hat{\boldsymbol{u}}_1 \boldsymbol{c}_{\beta'}^{\mathsf{T}} + \frac{\hat{\sigma}_1}{\gamma'} \boldsymbol{b}\hat{\boldsymbol{v}}_1^{\mathsf{T}} B + \hat{\kappa}\, \boldsymbol{b}\boldsymbol{c}_{\beta'}^{\mathsf{T}}.$$

The empirical objective function $\hat{h}(\gamma', \beta')$ is defined similarly. Such a perturbation may results in a problem with many feasible solutions, or worse, with no feasible solutions at all. Nevertheless, as shown in the proof of Theorem 3, this method is consistent. Namely, as $T \to \infty$, the above method will return an arbitrarily close solution (in $\|\cdot\|_{\text{F}}$) to the true transition matrix A, with high probability.

D.6. Proof of Theorem 3

Recall the definitions of $A'_H(\gamma', \beta')$ and its empirical version $\hat{A}'_H(\gamma', \beta')$, given in the previous Section D.5. To prove the theorem we show that

$$\left\| \hat{A}'_H(\hat{\gamma}, \hat{\beta}) - A'_H(\gamma, \beta) \right\|_{\text{F}} \xrightarrow{P} 0.$$

Toward this goal we bound the l.h.s by

$$\left\| \hat{A}'_H(\hat{\gamma}, \hat{\beta}) - A'_H(\hat{\gamma}, \hat{\beta}) \right\|_{\text{F}} + \left\| A'_H(\hat{\gamma}, \hat{\beta}) - A'_H(\gamma, \beta) \right\|_{\text{F}}, \tag{64}$$

and show that each term converges to 0 in probability.

We shall need the following lemma, establishing the *point-wise* convergence in probability of $\hat{A}_H$ to A_H:

Lemma 10. *For any* $0 < \gamma'$ *and* $0 \leq \beta' \leq 1$,

$$\left\| \hat{A}_H(\gamma', \beta') - A_H(\gamma', \beta') \right\|_{\text{F}} = o_P(1).$$

Proof. By (29), $\hat{\bar{A}} \xrightarrow{P} \bar{A}$. In addition, in Section D.3 we saw $\hat{\sigma}_1 \xrightarrow{P} \sigma$ and one can easily show that $\hat{\kappa} \xrightarrow{P} \kappa$. Thus, in order to prove the claim it suffices to show that $\hat{\boldsymbol{u}}_1 \xrightarrow{P} \boldsymbol{u}$ and $\hat{\boldsymbol{v}}_1 \xrightarrow{P} \boldsymbol{v}$. By Wedin's Theorem (Stewart & Sun, 1990):

$$\|\hat{\boldsymbol{u}}_1 - \boldsymbol{u}\|_2 \leq C \frac{\left\| \hat{R}^{(2)} - R^{(2)} \right\|_2}{\sigma},$$

for some $C > 0$. Combining this with Lemma 4 gives that $\|\hat{\boldsymbol{u}}_1 - \boldsymbol{u}\|_2 = O_P(T^{-\frac{1}{2}})$. The same argument goes for $\|\hat{\boldsymbol{v}}_1 - \boldsymbol{v}\|_2$. □

We begin with the second term in (64). The first step is showing that the estimated parameters $\hat{\gamma}, \hat{\beta}$ in (63) converge with probability to the true parameters γ, β. We first need to following lemma, establishing the convergence of $\hat{h}$ to h uniformly in probability:

Lemma 11. *For any $\epsilon > 0$,*

$$\Pr\left(\sup_{(\gamma',\beta')\in[0,2]\times[0,1]} \left|\hat{h}(\gamma',\beta') - h(\gamma',\beta')\right| > \epsilon\right) = o(1).$$

Proof. Note that $\hat{h}(\gamma',\beta')$ is the value of the minimal entry of a matrix with all entries being polynomials of γ', β' with bounded coefficients. Thus $\hat{h}$ is Lipschitz. In addition $[0,2]\times[0,1]$ is compact and, similarly to Lemma 10, $\hat{h}(\gamma',\beta')$ converges in probability pointwise to $h(\gamma',\beta')$. Hence, the claim follows by Newey (1991, Corollary 2.2). □

Lemma 12. $(\hat{\gamma},\hat{\beta}) \xrightarrow{P} (\gamma,\beta)$.

Proof. Recall that $(\hat{\gamma},\hat{\beta})$ are the maximizers of $\hat{h}(\gamma',\beta')$ and (γ,β) are the maximizers of $h(\gamma',\beta')$, over $(\gamma',\beta') \in [0,2]\times[0,1]$. To prove the claim we need to show that for any $\delta > 0$,

$$\Pr\left(\left\|(\hat{\gamma},\hat{\beta}) - (\gamma,\beta)\right\| > \delta\right) = o(1).$$

Toward this end define

$$\epsilon(\delta) \equiv h(\gamma,\beta) - \max_{\|(\gamma',\beta')-(\gamma,\beta)\|>\delta} h(\gamma',\beta').$$

Note that $\epsilon(\delta) > 0$ since $h(\gamma',\beta')$ has the *unique* maximum (γ,β).

Now,by Lemma 11, we have that

$$\Pr\left(\sup_{\gamma',\beta'} \left|\hat{h}(\gamma',\beta') - h(\gamma',\beta')\right| > \epsilon(\delta)/4\right) = o(1). \quad (65)$$

Thus, if we show that $\sup\left|\hat{h} - h\right| \leq \epsilon(\delta)/4$ implies $\left\|(\hat{\gamma},\hat{\beta}) - (\gamma,\beta)\right\| \leq \delta$ then the claim is proved. So assume

$$\sup_{\gamma',\beta'} \left|\hat{h}(\gamma',\beta') - h(\gamma',\beta')\right| \leq \epsilon(\delta)/4.$$

Toward getting a contradiction let us assume that $\left\|(\hat{\gamma},\hat{\beta}) - (\gamma,\beta)\right\| > \delta$. Then the following relations hold:

$$\begin{aligned} h(\hat{\gamma},\hat{\beta}) &\leq h(\gamma,\beta) - \epsilon(\delta) \\ \hat{h}(\hat{\gamma},\hat{\beta}) &\leq h(\hat{\gamma},\hat{\beta}) + \epsilon(\delta)/4 \\ \hat{h}(\gamma,\beta) &\geq h(\gamma,\beta) - \epsilon(\delta)/4. \end{aligned}$$

Thus,

$$\hat{h}(\hat{\gamma},\hat{\beta}) \leq \hat{h}(\gamma,\beta) - \epsilon(\delta)/2,$$

in contradiction to the optimality of $(\hat{\gamma},\hat{\beta})$. □

By Lemma 12, $(\hat{\gamma},\hat{\beta}) \xrightarrow{P} (\gamma,\beta)$. Since H is minimal, Theorem 1 implies $\gamma > 0$ and thus $\hat{\gamma} \geq_P \gamma/2$. In addition, A_H is continuous in the compact set $[\gamma/2, 2]\times[0,1]$. Thus, by the continuous mapping theorem we have

$$\left\|A_H(\hat{\gamma},\hat{\beta}) - A_H(\gamma,\beta)\right\|_{\mathrm{F}} \xrightarrow{P} 0.$$

This proves the case for the right term of (64).

The convergence in probability of the left term of (64) to zero is a direct consequence of the following uniform convergence lemma:

Lemma 13.

$$\sup_{(\gamma',\beta')\in[\frac{\gamma}{2},2]\times[0,1]} \left\|\hat{A}_H(\gamma',\beta') - A_H(\gamma',\beta')\right\|_{\mathrm{F}} = o_P(1).$$

Proof. Since $\gamma' \geq \gamma/2$ we have that for any $i,j \in [n]$, $\hat{A}_H(\gamma',\beta')_{ij}$ is Lipschitz. In addition, by Lemma 10, for any $(\gamma',\beta') \in [\frac{\gamma}{2},2]\times[0,1]$, each entry $\hat{A}_H(\gamma',\beta')_{ij}$ converge pointwise in probability to $A_H(\gamma',\beta')_{ij}$. Finally, $[\frac{\gamma}{2},2]\times[0,1]$ is compact. Thus, the claim follows from Newey (1991, Corollary 2.2) with an application of a union bound over $i,j \in [n]$. □