Exponential Integration for Hamiltonian Monte Carlo Supplementary Material

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1. Proof of Proposition 1

Here, we prove the proposition stated in §3.1 of the main text. Since we propose only replacing the time integration step of HMC, to show the desired result we must show that our integrator is symplectic and time-reversible.

The one-step formulation in (7) defines a discrete flow $\Phi_h : (\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i) \mapsto (\boldsymbol{r}_{i+1}, \dot{\boldsymbol{r}}_{i+1})$ which is by definition time-reversible if $\Phi_h \circ \Phi_{-h} = \boldsymbol{I}$. Consequently, the integration $(\boldsymbol{r}_{i+1}, \dot{\boldsymbol{r}}_{i+1}) = \Phi_h(\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i)$ is time-reversible iff exchanging $i \leftrightarrow i+1$ and $h \leftrightarrow -h$ leaves it unaltered. By straightforward calculation, this is fulfilled for the choice $\boldsymbol{\psi}(\cdot) = \mathrm{sinc}(\cdot)\boldsymbol{\psi}_1(\cdot)$ and $\boldsymbol{\psi}_0(\cdot) = \cos(\cdot)\boldsymbol{\psi}_1(\cdot)$.

Moreover, Φ_h is by definition symplectic if it preserves the surface areas resulting from the projections onto planes of momenta and positions, i.e. iff for all $\boldsymbol{\xi}_0, \boldsymbol{\xi}_1 \in \mathbb{R}^{2n}$ the spanning function $\omega(\boldsymbol{\xi}, \boldsymbol{\mu}) := \boldsymbol{\xi}^{\mathrm{T}} \boldsymbol{J} \boldsymbol{\mu}$ satisfies $\omega(\Phi_h(\cdot,\cdot)\boldsymbol{\xi}_0, \Phi_h(\cdot,\cdot)\boldsymbol{\xi}_1) = \omega(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1)$. Because a differentiable flow can be approximated locally by a linear map, Φ_h is symplectic iff the mapping given by its Jacobian $\boldsymbol{J} := \Phi'_h$ is symplectic, i.e. $\boldsymbol{J}^{-1}\mathrm{adiag}(\boldsymbol{I}, -\boldsymbol{I})\boldsymbol{J} = \mathrm{adiag}(\boldsymbol{I}, -\boldsymbol{I})$ (Hairer et al., 2006). This is equivalent to $h^2/2\cos(h\boldsymbol{\Omega})\boldsymbol{\psi}(h\boldsymbol{\Omega})\boldsymbol{\phi}(h\boldsymbol{\Omega})\partial_{\boldsymbol{r}_i}\boldsymbol{F}[\boldsymbol{\phi}(h\boldsymbol{\Omega})\boldsymbol{r}_i] = h/2\,\boldsymbol{\Omega}^{-1}\sin(h\boldsymbol{\Omega})\boldsymbol{\psi}_0(h\boldsymbol{\Omega})\boldsymbol{\phi}(h\boldsymbol{\Omega})\partial_{\boldsymbol{r}_i}\boldsymbol{F}[\boldsymbol{\phi}(h\boldsymbol{\Omega})\boldsymbol{r}_i]$. Equating coefficients leads to $\boldsymbol{\psi}(\cdot) = \mathrm{sinc}(\cdot)\,\boldsymbol{\phi}(\cdot)$.

Hence the presented exponential integrators under the proposed restrictions on the filter functions are time-reversible and symplectic. The linear transformation in position space, $\boldsymbol{q} \mapsto \boldsymbol{r} = \boldsymbol{M}^{1/2}\boldsymbol{q}$, does not influence symplecticity since $\boldsymbol{p} \mapsto \dot{\boldsymbol{r}} = \boldsymbol{M}^{-1/2}\boldsymbol{p}$ in momentum space is transformed in a similar way.

2. Preconditioning for Leapfrog HMC

In §5.3 of the main text, we mentioned preconditioning for reducing the stiffness in executing leapfrogHMC.

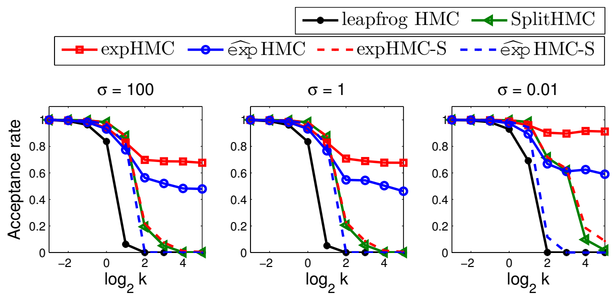
Table 1. Results of preconfitioning on the Pima Indian dataset. ESS ranges are shown as (min, median, max).

METHOD	TIME(S)	ESS	S/MIN ESS	RS	AR
$\sigma = 100$					
leapfrogHMC	6.7	(3606, 4021, 4303)	0.0018	1	0.88
expHMC (h, L)	7.9	(3627, 4166, 4520)	0.0022	0.85	0.96
expHMC (2h, L/2)	4.1	(2041, 2844, 3281)	0.0020	0.91	0.89
expHMC (4h, L/4)	2.2	(1940, 2555, 2996)	0.0011	**1.62**	0.87
$\sigma = 1$					
leapfrogHMC	6.7	(3690, 3984, 4295)	0.0018	1	0.87
expHMC (h, L)	7.9	(3631, 4298, 4597)	0.0022	0.83	0.96
expHMC (2h, L/2)	4.1	(2158, 2873, 3365)	0.0019	0.96	0.89
expHMC (4h, L/4)	2.2	(2024, 2609, 3043)	0.0011	**1.64**	0.88
$\sigma = 0.01$					
leapfrogHMC	5.9	(3751, 4109, 4403)	0.0016	1	0.88
expHMC (h, L)	7.0	(4550, 4880, 4987)	0.0015	1.02	0.99
expHMC (2h, L/2)	3.6	(4229, 4611, 4874)	0.0009	1.85	0.97
expHMC (4h, L/4)	1.9	(4178, 4610, 4852)	0.0005	**3.48**	0.97

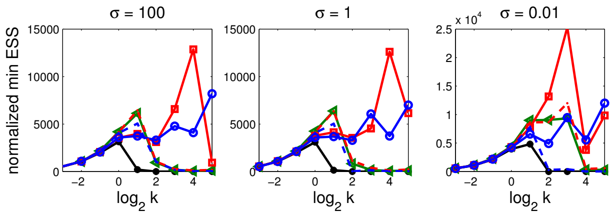
leapfrogHMC and our methods solve exactly the same ODE in (2), yet via different integrators. Hence for fair comparison, we precondition both methods, with preconditioned leapfrogHMC denoted as PLHMC. We conduct the experiments on Bayesian logistic regression using the Pima Indian dataset. We set $\boldsymbol{M}$ as the Hessian of $-\log p(\boldsymbol{\theta})$ at MAP, i.e., $\boldsymbol{\Sigma}^{-1}$ by Laplace approximation. We use $L = 100$ as in §5.3, but manually set h so that PLHMC has acceptance in $[0.6, 0.9]$. The results are summarized in Tables 1, in which expHMC has higher acceptance rate (AR) than PLHMC under the same step size and maintains AR > 0.87 for larger steps. More importantly, as σ shrinks (i.e., the ODE becomes stiffer), the relative speed (RS) of expHMC increases, demonstrating the advantage of our methods in dealing with stiff ODEs, even with preconditioning.

3. Empirical Comparison with Splitting and Filter Selection

In §6 of the main text, we compared to splitting with a simple toy example (Shahbaba et al., 2014). In this section, we present more extensive empirical results comparing



(a) Acceptance rates under different $(hk, L/k)$



(b) Normalized min ESS under different $(hk, L/k)$

Figure 1. Sampling performance of each method under different $(hk, L/k)$ and σ (the variance of the prior in BLR) on the German credit dataset, including (a) acceptance rate and (b) normalized min ESS $= k \times$min ESS. expHMC-S and $\widehat{\text{exp}}$HMC-S with simple filters perform well at small k, so does splitHMC. expHMC and $\widehat{\text{exp}}$HMC with mollified filters perform the best at large k.

to splitHMC. We also examine the choice of filters (cf. §3.1 in main text) for our methods. We use expHMC-S, $\widehat{\text{exp}}$HMC-S, and rmExpHMC-S to denote the use of simple filters, and expHMC, $\widehat{\text{exp}}$HMC, and rmExpHMC to denote the use of mollified filters.

We take the same (h, L) as in §5.3 for each pair of dataset and σ (variance of the prior in BLR) but vary $(kh, L/k)$ for $k = \{2^{-3}, \ldots, 2^5\}$. Acceptance rates for different k on the German credit dataset are in Figure 1a, with time-normalized min ESS in Figure 1b. The normalized min ESS is computed by multiplying min ESS by k, since k^{-1} roughly estimates relative time consumption in sampling. rmExpHMC and rmExpHMC-S are omitted for clarity.

The acceptance and normalized min ESS of leapfrogHMC degrade drastically when $k > 1$ (larger step). splitHMC, expHMC-S, and $\widehat{\text{exp}}$HMC-S perform similarly when $k \leq 2$, with high acceptance rates and normalized min ESS. As k increases, however, all suffer a sharp drop in performance. expHMC and $\widehat{\text{exp}}$HMC maintain stability in this case, with sufficient acceptance rates and hence the highest normalized min ESS under large k. expHMC and $\widehat{\text{exp}}$HMC perform slightly worse than expHMC-S and $\widehat{\text{exp}}$HMC-S under small step sizes, mainly because the filters in use sacrifice too much accuracy for stability. Results of rmExpHMC, rmExpHMC-S and details on other datasets are included in Tables 3-7 and discussed later in §4.2 of this document.

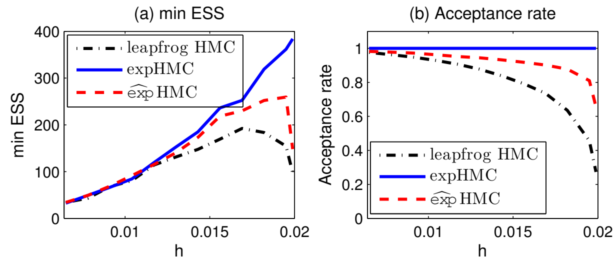


Figure 2. Sampling performance on a 100D Gaussian under different h, including (a) minimum effective sample size and (b) acceptance rate. Both expHMC and $\widehat{\text{exp}}$HMC outperform leapfrogHMC, especially under large h.

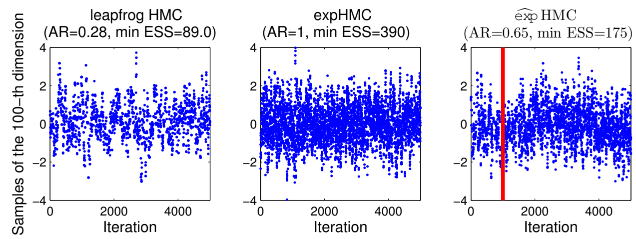


Figure 3. The 5000 samples of the 100th dimension by each method (with $h = 0.02$). As shown, samples of $\widehat{\text{exp}}$HMC becomes denser (right of the red line) as the estimated mean and covariance get updated and more accurate.

4. Extended Experimental Results

We conduct additional empirical studies following §5 of the main text, under the same setups and evaluation criteria.

4.1. Synthetic Examples

In this subsection, we present more synthetic examples and compare our expHMC and $\widehat{\text{exp}}$HMC to leapfrogHMC. For expHMC, we use the parameters of Gaussian distributions directly or average mixture components to obtain a Gaussian approximation.

100D Gaussian We sample from a 100D Gaussian with variance $(0.01k)^2$ in the kth dimension. Like Neal (2011), we uniformly sample $L \in \{1, \ldots, 50\}$ in each iteration, paired with fixed $h \in [0.0065, 0.02]$. We collect 5000 samples after 2000 iterations of burn-in, with $(N_1, N_2) = (1000, 500)$ for $\widehat{\text{exp}}$HMC.

We show the minimum effective sample size (min ESS) over 100 dimensions and acceptance rate (AR) from 10 trials in Figure 2. Both expHMC and $\widehat{\text{exp}}$HMC outperform HMC for large h. We show the 5000 samples of the 100th dimension when $h = 0.02$ in Figure 3. The samples by $\widehat{\text{exp}}$HMC become denser (better exploration) with a correspondingly increasing acceptance as the estimated mean and covariance become more accurate.

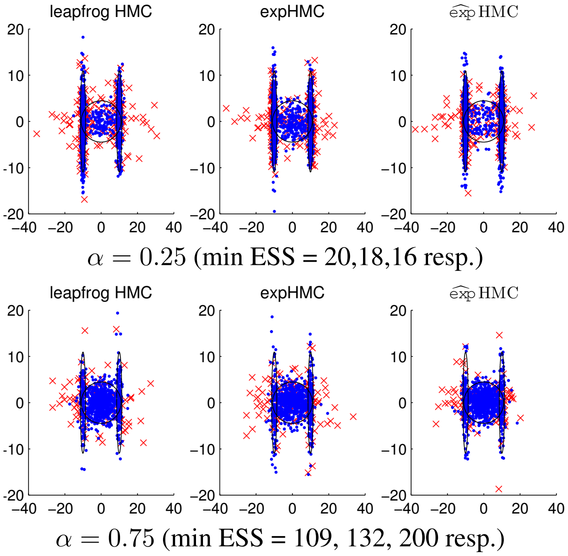


Figure 4. 1000 samples from two mixtures of Gaussians, with minimum ESS. For large α, the distribution is nearly Gaussian and exponential methods are more successful. All experiments in each row have similar acceptance rates.

Mixture of Gaussians Figure 4 shows samples from a 2D mixture of Gaussians of the form:

$$p(x) \propto \alpha p_0(x) + (1-\alpha)\left(\frac{1}{2}p_1(x) + \frac{1}{2}p_2(x)\right),$$

where $\alpha \in [0,1]$ trades off between the horizontal component p_0 and the vertical components p_1 and p_2.

Experiments are shown for $\alpha \in \{0.25, 0.75\}$; acceptance rates of all methods are roughly the same in both cases. For large α, the mixture is nearly Gaussian, and $\widehat{\text{exp}}$HMC has the highest min ESS. As α decreases, this approximation begins to fail and leapfrog performs comparably. This test illustrates when it is favorable to use exponential integration. When the distribution is well-approximated by a Gaussian, exponential integration improves sampling. When the distribution is strongly multi-modal, a single Gaussian does not suffice and the exponential method has less of an advantage.

4.2. Bayesian Logistic Regression

In this subsection, we provide detailed experimental results on sampling $\boldsymbol{\theta}$ from the posterior distribution of Bayesian logistic regression (BLR). The problem formulation, datasets in use, evaluation criteria, and experimental settings are described in §5.3 of the main text.

We compare the proposed expHMC (by Laplace approximation), $\widehat{\text{exp}}$HMC, and rmExpHMC with leapfrogHMC, splitHMC (Shahbaba et al., 2014), RMHMC (Girolami & Calderhead, 2011), and e-RMLMC (Lan et al., 2012). We also compare to expHMC-S, $\widehat{\text{exp}}$HMC-S, rmExpHMC-S using simple filters (cf. §3.1 of the main text) and to expHMC-B, which uses the Bayesian prior for Gaussian approximation. That is, since we apply the homogeneous Gaussian prior in BLR, the prior itself can be used for Gaussian approximation; the corresponding nonlinear component is the negative Fisher score. We consider the same parameters (h, L) as in the main text and further examine $(2h, L/2)$ and $(4h, L/4)$ for all variants of our methods except expHMC-B, which degrades drastically as step sizes increased, similar to leapfrogHMC, RMHMC, and e-RMLMC, The parameters of RMHMC and e-RMLMC are chosen based on the corresponding references.

Sampling performance is summarized in Tables 3-7. The relative speed (RS) represents the relative min ESS/sec of each method compared to that of leapfrogHMC. As presented, expHMC, $\widehat{\text{exp}}$HMC, and rmExpHMC always achieve higher acceptance rates (AR) against leapfrogHMC with fixed (h, L); the improvement of relative speed becomes more obvious as σ decreases (i.e., the ODE becomes stiff). RMHMC and e-RMLMC, though with the highest min ESS, require more computational time, leading to poor relative speed. The use of exponential integration allow expHMC, $\widehat{\text{exp}}$HMC, and rmExpHMC to take larger step sizes (i.e., fewer steps) with limited decrease in acceptance rates and min ESS, especially under small σ, leading to a significant gains in relative speed.

When compared to expHMC-B, expHMC outperforms in acceptance rate and has higher relative speed in most cases. In our experiments, expHMC-B exhibits a quick drop in both measures when doubling the step size, suggesting that a better Gaussian approximation (like Laplace approximation or more sophisticated methods) has a significant effect on performance.

splitHMC, introduced in the main text, also analytically integrates a Gaussian approximation. With small step sizes, splitHMC outperforms expHMC, $\widehat{\text{exp}}$HMC, and rmExpHMC in acceptance rate and relative speed, indicating that mollified filters (cf. §3.1) sacrifice too much accuracy for stability. In this case, expHMC-S, $\widehat{\text{exp}}$HMC-S, and rmExpHMC-S with simple filters perform equally well to splitHMC. With large steps, however, all methods adopting either simple filters or simple explicit integrators drastically degrade (see experiments marked with (4h)). In contrast, expHMC, $\widehat{\text{exp}}$HMC, and rmExpHMC still exhibit fairly high acceptance rates thanks to stable filtering.

In most cases, $\widehat{\text{exp}}$HMC performs comparably to or better than expHMC, particularly under large σ. This demonstrates the effectiveness of using empirical means and covariances for the Gaussian approximation in exponential integration. For large σ (less stiff ODE, less peaked distribution), this observation also suggests that empirical statistics provide a better Gaussian approximation than the Laplace

Table 2. MEG datasets ($d = 5$, $N = 17730$ and 25 sample dimensions). "AR" means acceptance rate, and "RS" denotes relative speed; ESS ranges are shown as (min, median, max).

METHOD	TIME(S)	ESS	S/MIN ESS	RS	AR
$\sigma = 100$					
leapfrogHMC (h, L)	196	(2680, 3889, 4811)	0.073	1	0.85
leapfrogHMC (2h, L/2)	109	(498, 688, 952)	0.220	0.33	0.26
$\widehat{\text{exp}}$HMC (h, L)	207	(3256, 4813, 5000)	0.063	1.15	0.95
$\widehat{\text{exp}}$HMC (2h, L/2)	109	(2579, 3330, 4233)	0.042	**1.73**	0.82
$\widehat{\text{exp}}$HMC (4h, L/4)	62	(1143, 1731, 2424)	0.054	1.34	0.64
$\widehat{\text{exp}}$HMC-S (h, L)	207	(3377, 4933, 5000)	0.061	1.19	0.98
$\widehat{\text{exp}}$HMC-S (2h, L/2)	108	(2915, 4242, 4978)	0.037	**1.98**	0.89
$\widehat{\text{exp}}$HMC-S (4h, L/4)	54	(4, 10, 28)	13.25	0.006	0.003

method, which is aware of only local structure around modes.

rmExpHMC, similar to $\widehat{\text{exp}}$HMC, updates the mean and covariance periodically. rmExpHMC generally outperforms $\widehat{\text{exp}}$HMC in the BLR experiments, validating the use of manifold structures for Gaussian approximation.

4.3. Independent Component Analysis

We include in Tables 2 the detailed results on ICA, following §5.4 of the main text. Similar to the observations in the BLR cases, $\widehat{\text{exp}}$HMC-S performs better than $\widehat{\text{exp}}$HMC under small step sizes. The highest relative speed is achieved by $\widehat{\text{exp}}$HMC-S at $2h$. However, by further increasing the step (i.e. to $4h$), both acceptance and relative speed drop drastically for $\widehat{\text{exp}}$HMC-S. On the other hand, expHMC suffers less in such case, demonstrating its stability under large step sizes.

5. Computational Complexity

The computational complexity of our methods is approximately the same as that of leapfrogHMC. This is because a large number of operations can be precomputed. Concretely, the computations in each iteration of our methods are shown in (7) and inside the "for" loop of Figure 2 in the main text. Since the matrices $\boldsymbol{S}$, $\boldsymbol{C}$, $\boldsymbol{\Omega}$, $\boldsymbol{\Omega}^{-1}$, and those of the filters (i.e., $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, $\boldsymbol{\psi}_0$, $\boldsymbol{\psi}_1$) can be precomputed, the required operations are simply matrix-vector products and the computation of the distribution (included in the function $\boldsymbol{F}(\cdot)$), which is also needed in leapfrogHMC.

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Table 3. Australian credit dataset ($d = 14$, $N = 690$ and 15 regression coefficients). "AR" means acceptance rate, and "RS" denotes relative speed; ESS ranges are shown as (min, median, max).

METHOD	TIME(S)	ESS	S/MIN ESS	RS	AR
$\sigma = 100$					
leapfrogHMC	8.5	(3152, 3584, 3962)	0.0027	1	0.82
RMHMC	58.2	(4894, 5000, 5000)	0.0119	0.23	0.92
e-RMLMC	44.4	(4898, 5000, 5000)	0.0091	0.30	0.92
splitHMC (h)	11.1	(3917, 4516, 4942)	0.0028	0.95	0.96
splitHMC (2h)	5.9	(539, 830, 1059)	0.011	0.25	0.36
splitHMC (4h)	3.3	(53, 130, 232)	0.062	0.04	0.11
expHMC-B (h)	10.5	(3130, 3550, 3967)	0.0034	0.79	0.82
expHMC (h)	10.3	(2944, 3585, 4044)	0.0035	0.78	0.89
expHMC (2h)	5.3	(1291, 1983, 2491)	0.0041	0.66	0.73
expHMC (4h)	2.8	(846, 1523, 1940)	0.0034	0.81	0.67
$\widehat{\text{exp}}$HMC (h)	10.8	(3435, 4056, 4550)	0.0031	0.86	0.93
$\widehat{\text{exp}}$HMC (2h)	5.6	(2131, 2678, 3135)	0.0026	1.04	0.79
$\widehat{\text{exp}}$HMC (4h)	3.0	(1657, 2236, 2650)	0.0018	1.50	0.72
rmExpHMC (h)	10.4	(3543, 4216, 4635)	0.0029	0.92	0.94
rmExpHMC (2h)	5.5	(2430, 3118, 3523)	0.0023	1.20	0.84
rmExpHMC (4h)	2.9	(2080, 2799, 3297)	0.0014	1.94	0.80
expHMC-S (h)	10.5	(3944, 4437, 4911)	0.0027	1.01	0.96
expHMC-S (2h)	5.6	(564, 847, 1060)	0.01	0.27	0.37
expHMC-S (4h)	2.9	(86, 164, 296)	0.034	0.08	0.13
$\widehat{\text{exp}}$HMC-S (h)	11.4	(3955, 4531, 4886)	0.0029	0.94	0.96
$\widehat{\text{exp}}$HMC-S (2h)	6.3	(567, 784, 979)	0.011	0.24	0.34
$\widehat{\text{exp}}$HMC-S (4h)	3.1	(10, 21, 79)	0.3	0.009	0.001
rmExpHMC-S (h)	10.1	(4176, 4653, 4972)	0.0024	1.12	0.97
rmExpHMC-S (2h)	5.3	(699, 982, 1261)	0.0076	0.36	0.40
rmExpHMC-S (4h)	2.8	(135, 236, 354)	0.021	0.13	0.15
$\sigma = 1$					
leapfrogHMC	8.8	(3169, 3556, 3898)	0.0028	1	0.81
RMHMC	58.1	(4836, 5000, 5000)	0.012	0.23	0.92
e-RMLMC	44.7	(4562, 5000, 5000)	0.0098	0.29	0.92
splitHMC (h)	11.2	(4042, 4596, 4927)	0.0028	1.0	0.96
splitHMC (2h)	6.0	(461, 708, 945)	0.013	0.21	0.32
splitHMC (4h)	3.3	(73, 139, 251)	0.044	0.06	0.12
expHMC-B (h)	10.6	(3072, 3529, 3815)	0.0034	0.79	0.81
expHMC (h)	10.2	(3017, 3628, 4117)	0.0034	0.82	0.89
expHMC (2h)	5.3	(1540, 2142, 2615)	0.034	0.81	0.73
expHMC (4h)	2.8	(1005, 1585, 2035)	0.028	1.00	0.68
$\widehat{\text{exp}}$HMC (h)	10.6	(3550, 4019, 4455)	0.0030	0.93	0.92
$\widehat{\text{exp}}$HMC (2h)	5.5	(2016, 2608, 3025)	0.0027	1.02	0.78
$\widehat{\text{exp}}$HMC (4h)	2.9	(1512, 2141, 2549)	0.0019	1.4	0.71
rmExpHMC (h)	10.4	(3797, 4245, 4584)	0.0027	1.02	0.94
rmExpHMC (2h)	5.5	(2419, 3038, 3484)	0.0023	1.23	0.83
rmExpHMC (4h)	2.9	(2139, 2822, 3300)	0.0014	2.06	0.81
expHMC-S (h)	10.4	(4027, 4498, 4864)	0.0026	1.08	0.96
expHMC-S (2h)	5.4	(506, 721, 928)	0.011	0.26	0.33
expHMC-S (4h)	2.9	(96, 186, 325)	0.03	0.09	0.14
$\widehat{\text{exp}}$HMC-S (h)	11.2	(4148, 4556, 4839)	0.0027	1.02	0.96
$\widehat{\text{exp}}$HMC-S (2h)	6.3	(468, 649, 828)	0.013	0.21	0.30
$\widehat{\text{exp}}$HMC-S (4h)	3.0	(16, 32, 111)	0.19	0.015	0.001
rmExpHMC-S (h)	10.1	(4097, 4608, 4926)	0.0025	1.13	0.97
rmExpHMC-S (2h)	5.3	(581, 842, 1045)	0.0092	0.30	0.35
rmExpHMC-S (4h)	2.9	(186, 294, 450)	0.015	0.18	0.17
$\sigma = 0.01$					
leapfrogHMC	8.1	(3421, 3876, 4179)	0.0024	1	0.85
RMHMC	57.9	(4953, 5000, 5000)	0.0117	0.20	0.93
e-RMLMC	44.5	(4998, 5000, 5000)	0.0089	0.27	0.93
splitHMC (h)	10.5	(4506, 4930, 5000)	0.0023	1.01	0.99
splitHMC (2h)	5.6	(3570, 4065, 4398)	0.0016	1.51	0.90
splitHMC (4h)	3.1	(2389, 2852, 3239)	0.0013	1.81	0.80
expHMC-B (h)	9.9	(3302, 3807, 4173)	0.003	0.80	0.85
expHMC (h)	9.7	(4098, 4613, 4915)	0.0024	1.01	0.97
expHMC (2h)	5.0	(3286, 4003, 4407)	0.0015	1.55	0.91
expHMC (4h)	2.6	(2817, 3775, 4229)	0.0009	2.56	0.89
$\widehat{\text{exp}}$HMC (h)	10.2	(3985, 4442, 4815)	0.0026	0.93	0.95
$\widehat{\text{exp}}$HMC (2h)	5.2	(2746, 3369, 3771)	0.0019	1.25	0.85
$\widehat{\text{exp}}$HMC (4h)	2.8	(2382, 2941, 3353)	0.0012	2.02	0.80
rmExpHMC (h)	9.9	(4231, 4718, 4980)	0.0023	1.03	0.98
rmExpHMC (2h)	5.2	(3639, 4218, 4636)	0.0014	1.69	0.93
rmExpHMC (4h)	2.7	(3290, 3783, 4155)	0.0008	2.88	0.88
expHMC-S (h)	9.8	(4402, 4844, 4994)	0.0022	1.06	0.99
expHMC-S (2h)	5.0	(3532, 4047, 4388)	0.014	1.69	0.90
expHMC-S (4h)	2.7	(2396, 3015, 3444)	0.011	2.14	0.80
$\widehat{\text{exp}}$HMC-S (h)	10.5	(4268, 4765, 4988)	0.0025	0.97	0.98
$\widehat{\text{exp}}$HMC-S (2h)	5.8	(2957, 3501, 3893)	0.002	1.22	0.85
$\widehat{\text{exp}}$HMC-S (4h)	2.9	(8, 17, 97)	0.38	0.006	0.001
rmExpHMC-S (h)	9.6	(4428, 4821, 5000)	0.0022	1.11	0.99
rmExpHMC-S (2h)	5.0	(3545, 4089, 4381)	0.0014	1.70	0.90
rmExpHMC-S (4h)	2.7	(2833, 3378, 3774)	0.0010	2.49	0.83

Table 4. German credit dataset ($d = 24$, $N = 1000$ and 25 regression coefficients). "AR" means acceptance rate, and "RS" denotes relative speed; ESS ranges are shown as (min, median, max).

METHOD	TIME(S)	ESS	S/MIN ESS	RS	AR
$\sigma = 100$					
leapfrogHMC	11.9	(3176, 3688, 4067)	0.0037	1	0.84
RMHMC	117.4	(4725, 5000, 5000)	0.025	0.15	0.90
e-RMLMC	104.0	(4644, 5000, 5000)	0.022	0.17	0.91
splitHMC (h)	16.5	(4226, 4836, 5000)	0.0039	0.95	0.98
splitHMC (2h)	8.8	(3219, 3790, 4247)	0.0027	1.36	0.88
splitHMC (4h)	4.9	(232, 417, 607)	0.021	0.18	0.20
expHMC-B (h)	14.1	(3166, 3628, 4005)	0.0045	0.82	0.84
expHMC (h)	13.7	(3543, 4257, 4704)	0.0039	0.96	0.94
expHMC (2h)	7.2	(1865, 2971, 3534)	0.0038	0.97	0.83
expHMC (4h)	3.8	(969, 1893, 2403)	0.0039	0.95	0.70
$\widehat{\text{exp}}$HMC (h)	14.1	(3559, 4166, 4584)	0.0040	0.94	0.93
$\widehat{\text{exp}}$HMC (2h)	7.3	(1903, 2560, 3015)	0.0038	0.97	0.77
$\widehat{\text{exp}}$HMC (4h)	3.9	(755, 1241, 1598)	0.0051	0.73	0.55
rmExpHMC (h)	14.0	(3708, 4455, 4859)	0.0038	0.98	0.96
rmExpHMC (2h)	7.3	(2752, 3605, 4074)	0.0026	1.40	0.88
rmExpHMC (4h)	3.9	(1800, 2823, 3357)	0.0022	1.71	0.79
expHMC-S (h)	13.9	(4121, 4716, 4998)	0.0034	1.1	0.98
expHMC-S (2h)	7.2	(3039, 2971, 4170)	0.0024	1.58	0.88
expHMC-S (4h)	3.8	(258, 478, 649)	0.015	0.25	0.22
$\widehat{\text{exp}}$HMC-S (h)	15.2	(3969, 4645, 4987)	0.0038	0.98	0.97
$\widehat{\text{exp}}$HMC-S (2h)	8.1	(2717, 3230, 3602)	0.0030	1.25	0.81
$\widehat{\text{exp}}$HMC-S (4h)	3.9	(9, 18, 61)	0.43	0.009	0.001
rmExpHMC-S (h)	13.7	(4272, 4785, 5000)	0.0032	1.15	0.98
rmExpHMC-S (2h)	7.2	(3290, 3859, 4208)	0.0022	1.70	0.88
rmExpHMC-S (4h)	3.9	(309, 528, 719)	0.013	0.29	0.24
$\sigma = 1$					
leapfrogHMC	12.1	(3205, 3667, 4028)	0.0038	1	0.84
RMHMC	117.7	(4829, 5000, 5000)	0.024	0.16	0.90
e-RMLMC	103.7	(4762, 5000, 5000)	0.022	0.17	0.91
splitHMC (h)	16.6	(4209, 4795, 5000)	0.004	0.96	0.98
splitHMC (2h)	8.9	(3219, 3803, 4245)	0.0028	1.38	0.88
splitHMC (4h)	4.9	(254, 442, 636)	0.019	0.20	0.21
expHMC-B (h)	14.2	(3079, 3625, 4030)	0.0046	0.83	0.84
expHMC (h)	13.7	(3520, 4268, 4692)	0.0039	0.97	0.94
expHMC (2h)	7.1	(1931, 3010, 3555)	0.0037	1.03	0.83
expHMC (4h)	3.8	(856, 1865, 2405)	0.0044	0.86	0.70
$\widehat{\text{exp}}$HMC (h)	14.0	(3611, 4167, 4594)	0.0039	0.97	0.93
$\widehat{\text{exp}}$HMC (2h)	7.3	(1860, 2509, 2967)	0.0039	0.96	0.77
$\widehat{\text{exp}}$HMC (4h)	3.8	(799, 1243, 1664)	0.0048	0.79	0.55
rmExpHMC (h)	13.9	(3866, 4464, 4832)	0.0036	1.05	0.96
rmExpHMC (2h)	7.3	(2471, 3602,4176)	0.0029	1.29	0.88
rmExpHMC (4h)	3.9	(1352, 2760, 3351)	0.0029	1.31	0.79
expHMC-S (h)	13.7	(4146, 4720, 5000)	0.0033	1.14	0.98
expHMC-S (2h)	7.2	(3135, 3729, 155)	0.0023	1.66	0.88
expHMC-S (4h)	3.9	(296, 509, 715)	0.013	0.29	0.23
$\widehat{\text{exp}}$HMC-S (h)	15.0	(4041, 4603, 4964)	0.0037	1.02	0.97
$\widehat{\text{exp}}$HMC-S (2h)	8.2	(2590, 3111, 3615)	0.0032	1.19	0.81
$\widehat{\text{exp}}$HMC-S (4h)	4.1	(8, 14, 61)	0.52	0.007	0.001
rmExpHMC-S (h)	13.7	(4195, 4734, 5000)	0.0033	1.16	0.98
rmExpHMC-S (2h)	7.1	(3319, 3857, 4202)	0.0021	1.77	0.88
rmExpHMC-S (4h)	3.9	(372, 562, 739)	0.01	0.37	0.25
$\sigma = 0.01$					
leapfrogHMC	11.2	(3870, 4490, 4895)	0.0035	1	0.93
RMHMC	118.1	(4814, 5000, 5000)	0.025	0.12	0.91
e-RMLMC	103.7	(4783, 5000, 5000)	0.022	0.16	0.92
splitHMC (h)	15.7	(4451, 4995, 5000)	0.0035	0.81	0.99
splitHMC (2h)	8.4	(4318, 4966, 5000)	0.0019	1.48	0.98
splitHMC (4h)	4.7	(2158, 2632, 3033)	0.0022	1.32	0.71
expHMC-B (h)	13.6	(3917, 4384, 4746)	0.0035	1.00	0.93
expHMC (h)	13.0	(4283, 4847, 5000)	0.0030	0.94	0.99
expHMC (2h)	6.7	(3938, 4520, 4944)	0.0017	1.67	0.96
expHMC (4h)	3.6	(3426, 3966, 4525)	0.0010	2.74	0.91
$\widehat{\text{exp}}$HMC (h)	13.4	(4172, 4700, 4990)	0.0032	0.89	0.97
$\widehat{\text{exp}}$HMC (2h)	7.0	(3185, 3830, 4289)	0.0022	1.31	0.89
$\widehat{\text{exp}}$HMC (4h)	3.7	(1369, 1958, 2397)	0.0027	1.07	0.68
rmExpHMC (h)	13.2	(4386, 4846, 5000)	0.0030	1.16	0.99
rmExpHMC (2h)	6.9	(4099, 4660, 4977)	0.0017	2.08	0.97
rmExpHMC (4h)	3.7	(3485, 4042, 4552)	0.0011	3.27	0.90
expHMC-S (h)	13.1	(4381, 4935, 5000)	0.0030	0.96	0.99
expHMC-S (2h)	6.7	(4251, 4858, 5000)	0.0016	1.82	0.98
expHMC-S (4h)	3.7	(2188, 2695, 3100)	0.017	1.71	0.72
$\widehat{\text{exp}}$HMC-S (h)	14.0	(4305, 4847, 5000)	0.0032	0.88	0.99
$\widehat{\text{exp}}$HMC-S (2h)	7.7	(3893, 4433, 4863)	0.002	1.44	0.94
$\widehat{\text{exp}}$HMC-S (4h)	4.0	(57, 107, 175)	0.07	0.04	0.10
rmExpHMC-S (h)	12.9	(4359, 4942, 5000)	0.0030	1.18	0.99
rmExpHMC-S (2h)	6.7	(4351, 4909, 5000)	0.0016	2.26	0.98
rmExpHMC-S (4h)	3.6	(2245, 2712, 3099)	0.0016	2.15	0.72

Table 5. Heart dataset ($d = 13, N = 270$ and 14 regression coefficients). "AR" means acceptance rate, and "RS" denotes relative speed; ESS ranges are shown as (min, median, max).

METHOD	TIME(S)	ESS	S/MIN ESS	RS	AR
$\sigma = 100$					
leapfrogHMC	4.7	(3271, 3670, 4022)	0.0014	1	0.85
RMHMC	39.6	(4917, 5000, 5000)	0.0081	0.18	0.91
e-RMLMC	30.5	(4817, 5000, 5000)	0.0063	0.22	0.92
splitHMC (h)	6.3	(3734, 4319, 4653)	0.0017	0.86	0.95
splitHMC (2h)	3.3	(699, 1103, 1440)	0.0048	0.30	0.48
splitHMC (4h)	1.9	(42, 94, 184)	0.045	0.03	0.09
expHMC-B (h)	6.7	(3223, 3650, 4029)	0.0021	0.67	0.85
expHMC (h)	6.5	(2638, 3156, 3582)	0.0025	0.58	0.87
expHMC (2h)	3.4	(969, 1448, 1879)	0.0035	0.41	0.68
expHMC (4h)	1.9	(537, 862, 1240)	0.0035	0.42	0.58
$\widehat{\text{exp}}$HMC (h)	6.4	(3259, 3810, 4173)	0.0020	0.73	0.92
$\widehat{\text{exp}}$HMC (2h)	3.3	(1724, 2294, 2716)	0.0019	0.75	0.76
$\widehat{\text{exp}}$HMC (4h)	1.8	(1181, 1747, 2182)	0.0015	0.95	0.67
rmExpHMC (h)	6.5	(3362, 3883, 4379)	0.0019	0.72	0.93
rmExpHMC (2h)	3.4	(1938, 2465, 2889)	0.0017	0.80	0.80
rmExpHMC (4h)	1.9	(1746, 2288, 2792)	0.0011	1.32	0.76
expHMC-S (h)	6.6	(3799, 4242, 4593)	0.0017	0.83	0.96
expHMC-S (2h)	3.3	(815, 1149, 1471)	0.0041	0.35	0.49
expHMC-S (4h)	1.8	(69, 131, 228)	0.026	0.05	0.10
$\widehat{\text{exp}}$HMC-S (h)	7.6	(3951, 4426, 4709)	0.0019	0.75	0.96
$\widehat{\text{exp}}$HMC-S (2h)	4.1	(900, 1176, 1404)	0.0045	0.32	0.45
$\widehat{\text{exp}}$HMC-S (4h)	2.0	(9, 21, 75)	0.23	0.006	0.01
rmExpHMC-S (h)	6.3	(4048, 4410, 4723)	0.0015	0.90	0.96
rmExpHMC-S (2h)	3.2	(1012, 1266, 1478)	0.0032	0.44	0.48
rmExpHMC-S (4h)	1.8	(163, 267, 372)	0.01	0.13	0.14
$\sigma = 1$					
leapfrogHMC	4.8	(3300, 3697, 4025)	0.0014	1	0.83
RMHMC	39.8	(4943, 5000, 5000)	0.0081	0.18	0.91
e-RMLMC	30.3	(4832, 5000, 5000)	0.0063	0.22	0.92
splitHMC (h)	6.3	(3930, 4483, 4819)	0.0016	0.90	0.95
splitHMC (2h)	3.3	(624, 930, 1175)	0.0054	0.27	0.41
splitHMC (4h)	1.8	(52, 106, 205)	0.035	0.04	0.10
expHMC-B (h)	6.7	(3254, 3584, 3930)	0.0021	0.67	0.83
expHMC (h)	6.4	(2751, 3342, 3764)	0.0023	0.62	0.88
expHMC (2h)	3.3	(960, 1458, 1923)	0.0035	0.42	0.68
expHMC (4h)	1.8	(611, 1035, 1468)	0.0029	0.50	0.61
$\widehat{\text{exp}}$HMC (h)	6.4	(3425, 3952, 4332)	0.0019	0.77	0.92
$\widehat{\text{exp}}$HMC (2h)	3.3	(1903, 2411, 2835)	0.0018	0.82	0.77
$\widehat{\text{exp}}$HMC (4h)	1.79	(1606, 2204, 2611)	0.0011	1.30	0.72
rmExpHMC (h)	6.6	(3439, 4061, 4477)	0.0019	0.73	0.93
rmExpHMC (2h)	3.4	(1775, 2561, 3156)	0.0019	0.73	0.81
rmExpHMC (4h)	1.8	(1989, 2541, 3117)	0.0009	1.51	0.80
expHMC-S (h)	6.5	(3969, 4435, 4780)	0.0016	0.89	0.95
expHMC-S (2h)	3.5	(669, 979, 1227)	0.0052	0.28	0.43
expHMC-S (4h)	1.8	(51, 116, 213)	0.036	0.04	0.11
$\widehat{\text{exp}}$HMC-S (h)	7.7	(4042, 4563, 4870)	0.0019	0.76	0.96
$\widehat{\text{exp}}$HMC-S (2h)	4.2	(704, 986, 1170)	0.006	0.24	0.40
$\widehat{\text{exp}}$HMC-S (4h)	2.1	(18, 27, 127)	0.11	0.013	0.01
rmExpHMC-S (h)	6.3	(4102, 44589, 4908)	0.0015	0.91	0.96
rmExpHMC-S (2h)	3.3	(811, 1069, 1353)	0.0041	0.35	0.43
rmExpHMC-S (4h)	1.8	(147, 291, 409)	0.012	0.12	0.18
$\sigma = 0.01$					
leapfrogHMC	4.6	(3002, 3386, 3729)	0.0015	1	0.79
RMHMC	39.7	(4980, 5000, 5000)	0.0080	0.19	0.94
e-RMLMC	30.4	(4987, 5000, 5000)	0.0061	0.25	0.94
splitHMC (h)	5.9	(4434, 4921, 5000)	0.0013	1.16	0.99
splitHMC (2h)	3.2	(1871, 2340, 2674)	0.0017	0.91	0.71
splitHMC (4h)	1.8	(1903, 2306, 2624)	0.0009	1.66	0.71
expHMC-B (h)	6.3	(3014, 3532, 3868)	0.0021	0.71	0.83
expHMC (h)	6.1	(4253, 4706, 4965)	0.0014	1.07	0.98
expHMC (2h)	3.2	(3949, 4405, 4755)	0.0008	1.92	0.95
expHMC (4h)	1.7	(4118, 4510, 4860)	0.0004	3.77	0.95
$\widehat{\text{exp}}$HMC (h)	6.1	(3849, 4305, 4630)	0.0016	0.98	0.94
$\widehat{\text{exp}}$HMC (2h)	3.1	(2814, 3310, 3686)	0.0011	1.38	0.84
$\widehat{\text{exp}}$HMC (4h)	1.7	(2701, 3318, 3756)	0.0006	2.47	0.84
rmExpHMC (h)	6.2	(4209, 4744, 4979)	0.0015	1.01	0.98
rmExpHMC (2h)	3.2	(3780, 4316, 4614)	0.0008	1.77	0.94
rmExpHMC (4h)	1.7	(3412, 3850, 4243)	0.0005	2.97	0.90
expHMC-S (h)	6.0	(4401, 4843, 4998)	0.0014	1.13	0.99
expHMC-S (2h)	3.2	(1966, 2336, 2736)	0.0016	0.95	0.71
expHMC-S (4h)	1.8	(2130, 2525, 2895)	0.0008	1.85	0.72
$\widehat{\text{exp}}$HMC-S (h)	6.9	(4122, 4611, 4898)	0.0019	0.91	0.97
$\widehat{\text{exp}}$HMC-S (2h)	3.8	(1604, 2058, 2437)	0.0045	0.64	0.67
$\widehat{\text{exp}}$HMC-S (4h)	1.9	(405, 423, 531)	0.005	0.32	0.01
rmExpHMC-S (h)	5.9	(4400, 4858, 5000)	0.0014	1.11	0.99
rmExpHMC-S (2h)	3.1	(1876, 2347, 2716)	0.0017	0.90	0.70
rmExpHMC-S (4h)	1.7	(1572, 1857, 2163)	0.0011	1.38	0.62

Table 6. Pima Indian dataset ($d = 7, N = 532$ and 8 regression coefficients). "AR" means acceptance rate, and "RS" denotes relative speed; ESS ranges are shown as (min, median, max).

METHOD	TIME(S)	ESS	S/MIN ESS	RS	AR
$\sigma = 100$					
leapfrogHMC	6.3	(3213, 3640, 3916)	0.0019	1	0.82
RMHMC	28.3	(4983, 5000, 5000)	0.0057	0.34	0.95
e-RMLMC	19.8	(4948, 5000, 5000)	0.0040	0.48	0.95
splitHMC (h)	7.8	(4356, 4736, 4982)	0.0018	1.08	0.98
splitHMC (2h)	4.2	(885, 1231, 1459)	0.0047	0.41	0.48
splitHMC (4h)	2.3	(499, 756, 963)	0.0047	0.42	0.39
expHMC-B (h)	8.0	(3307, 3593, 3857)	0.0024	0.79	0.82
expHMC (h)	7.8	(3758, 4231, 4572)	0.0021	0.94	0.95
expHMC (2h)	4.1	(2694, 3299, 3729)	0.0015	1.29	0.88
expHMC (4h)	2.2	(2555, 3350, 3780)	0.0008	2.30	0.88
$\widehat{\text{exp}}$HMC (h)	7.7	(3876, 4370, 4695)	0.0020	0.98	0.95
$\widehat{\text{exp}}$HMC (2h)	4.0	(3025, 3524, 3904)	0.0013	1.47	0.89
$\widehat{\text{exp}}$HMC (4h)	2.1	(2845, 3349, 3861)	0.0008	2.58	0.85
rmExpHMC (h)	7.8	(4143, 4515, 4820)	0.0019	1.00	0.97
rmExpHMC (2h)	4.1	(3229, 3751, 4153)	0.0013	1.49	0.91
rmExpHMC (4h)	2.2	(3232, 3807, 4218)	0.0007	2.78	0.90
expHMC-S (h)	7.9	(4314, 4667, 4933)	0.0018	1.06	0.98
expHMC-S (2h)	4.1	(1081, 1344, 1583)	0.0038	0.51	0.49
expHMC-S (4h)	2.2	(627, 894, 1124)	0.0036	0.55	0.41
$\widehat{\text{exp}}$HMC-S (h)	8.8	(4290, 4644, 4943)	0.0021	0.95	0.97
$\widehat{\text{exp}}$HMC-S (2h)	4.8	(878, 1070, 1280)	0.0055	0.35	0.44
$\widehat{\text{exp}}$HMC-S (4h)	2.3	(10, 23, 58)	0.24	0.008	0.01
rmExpHMC-S (h)	7.6	(4339, 4722, 4941)	0.0018	1.08	0.98
rmExpHMC-S (2h)	4.0	(1095, 1331, 1582)	0.0037	0.52	0.48
rmExpHMC-S (4h)	2.2	(402, 621, 789)	0.0054	0.35	0.34
$\sigma = 1$					
leapfrogHMC	6.3	(3219, 3574, 3862)	0.0020	1	0.82
RMHMC	28.6	(4984, 5000, 5000)	0.0057	0.34	0.95
e-RMLMC	19.8	(5000, 5000, 5000)	0.0040	0.5	0.96
splitHMC (h)	7.9	(4393, 4740, 4954)	0.0018	1.09	0.98
splitHMC (2h)	4.2	(1048, 1296, 1566)	0.0040	0.49	0.51
splitHMC (4h)	2.3	(531, 780, 1060)	0.0044	0.45	0.43
expHMC-B (h)	8.0	(3227, 3548, 3781)	0.0025	0.80	0.81
expHMC (h)	8.1	(3759, 4185, 4567)	0.0022	0.90	0.95
expHMC (2h)	4.2	(2788, 3334, 3717)	0.0015	1.29	0.89
expHMC (4h)	2.3	(2265, 3063, 3638)	0.0010	1.95	0.87
$\widehat{\text{exp}}$HMC (h)	7.7	(3993, 4307, 4674)	0.0019	1.02	0.96
$\widehat{\text{exp}}$HMC (2h)	4.0	(3098,3691, 4067)	0.0013	1.52	0.89
$\widehat{\text{exp}}$HMC (4h)	2.1	(2743, 3312, 3782)	0.0008	2.52	0.86
rmExpHMC (h)	7.8	(4126, 4469, 4720)	0.0019	1.05	0.97
rmExpHMC (2h)	4.1	(3232, 3814, 4174)	0.0013	1.58	0.91
rmExpHMC (4h)	2.2	(3261, 3919, 4342)	0.0007	2.96	0.91
expHMC-S (h)	7.7	(4255, 4642, 4877)	0.0018	1.09	0.98
expHMC-S (2h)	4.1	(977, 1301, 1608)	0.0042	0.47	0.52
expHMC-S (4h)	2.2	(553, 835, 1081)	0.0039	0.50	0.43
$\widehat{\text{exp}}$HMC-S (h)	8.7	(4234, 4598, 4850)	0.0021	0.95	0.97
$\widehat{\text{exp}}$HMC-S (2h)	5.0	(789, 1040, 1304)	0.0063	0.31	0.45
$\widehat{\text{exp}}$HMC-S (4h)	2.3	(8, 16, 101)	0.28	0.007	0.01
rmExpHMC-S (h)	7.6	(4301, 4790, 4954)	0.0018	1.14	0.98
rmExpHMC-S (2h)	4.0	(1039, 1274, 1529)	0.0038	0.52	0.50
rmExpHMC-S (4h)	2.2	(342, 503, 699)	0.0063	0.32	0.31
$\sigma = 0.01$					
leapfrogHMC	6.0	(3865, 4216, 4475)	0.0015	1	0.89
RMHMC	28.3	(4987, 5000, 5000)	0.0057	0.27	0.95
e-RMLMC	20.2	(4999, 5000, 5000)	0.0040	0.38	0.95
splitHMC (h)	7.1	(4569, 4920, 4998)	0.0016	1.09	0.99
splitHMC (2h)	3.8	(4237, 4653, 4914)	0.0009	1.70	0.97
splitHMC (4h)	2.1	(3644, 4034, 4300)	0.0006	2.64	0.93
expHMC-B (h)	7.4	(3536, 3940, 4212)	0.0021	0.71	0.89
expHMC (h)	7.3	(4239, 4706, 4957)	0.0017	0.89	0.99
expHMC (2h)	3.8	(4164, 4559, 4880)	0.0009	1.69	0.97
expHMC (4h)	2.0	(4226, 4668, 4953)	0.0005	3.21	0.97
$\widehat{\text{exp}}$HMC (h)	7.1	(4141, 4622, 4920)	0.0017	0.90	0.98
$\widehat{\text{exp}}$HMC (2h)	3.7	(3771, 4159, 4483)	0.0010	1.57	0.93
$\widehat{\text{exp}}$HMC (4h)	2.0	(3540, 3941, 4305)	0.0006	2.79	0.90
rmExpHMC (h)	7.3	(4316, 4807, 4981)	0.0017	0.89	0.99
rmExpHMC (2h)	3.8	(4284, 4607, 4851)	0.0009	1.68	0.97
rmExpHMC (4h)	2.0	(3758, 4181, 4550)	0.0005	2.80	0.93
expHMC-S (h)	7.2	(4402, 4837, 4981)	0.0016	0.95	0.99
expHMC-S (2h)	3.7	(4264, 4669, 4938)	0.0009	1.79	0.97
expHMC-S (4h)	2.0	(3838, 4292, 4635)	0.0005	2.9	0.94
$\widehat{\text{exp}}$HMC-S (h)	8.0	(4416, 4802, 4992)	0.0018	0.85	0.99
$\widehat{\text{exp}}$HMC-S (2h)	4.3	(3788, 4282, 4600)	0.0011	1.35	0.94
$\widehat{\text{exp}}$HMC-S (4h)	2.5	(11 ,20, 55)	0.22	0.007	0.01
rmExpHMC-S (h)	7.0	(4445, 4857, 5000)	0.0016	0.95	0.99
rmExpHMC-S (2h)	3.7	(4093, 4591, 4871)	0.0009	1.67	0.97
rmExpHMC-S (4h)	2.0	(3860, 4299, 4667)	0.0005	2.89	0.93

Table 7. Ripley dataset ($d = 2, N = 250$ and 7 regression coefficients). "AR" means acceptance rate, and "RS" denotes relative speed; ESS ranges are shown as (min, median, max).

METHOD	TIME(S)	ESS	S/MIN ESS	RS	AR
$\sigma = 100$					
leapfrogHMC	4.4	(3239, 3710, 4087)	0.0013	1	0.87
RMHMC	11.0	(4349, 4780, 4931)	0.0025	0.53	0.86
e-RMLMC	8.2	(3985, 4578, 4847)	0.0021	0.62	0.88
splitHMC (h)	5.5	(3480, 4043, 4395)	0.0016	0.85	0.93
splitHMC (2h)	2.9	(692, 875, 1093)	0.0042	0.32	0.40
splitHMC (4h)	1.6	(79, 130, 199)	0.021	0.066	0.10
expHMC-B (h)	5.8	(3325, 3729, 4122)	0.0018	0.72	0.88
expHMC (h)	5.8	(2849, 3611, 4037)	0.0020	0.66	0.91
expHMC (2h)	3.0	(919, 1442, 1753)	0.0033	0.41	0.66
expHMC (4h)	1.6	(546, 851, 1135)	0.0029	0.46	0.55
$\widehat{\text{exp}}$HMC (h)	5.8	(3498, 3991, 4478)	0.0017	0.82	0.93
$\widehat{\text{exp}}$HMC (2h)	3.0	(1608, 1926, 2218)	0.0019	0.72	0.69
$\widehat{\text{exp}}$HMC (4h)	1.6	(942, 1357, 1740)	0.0017	0.79	0.63
rmExpHMC (h)	5.9	(3044, 3961, 4432)	0.0019	0.67	0.93
rmExpHMC (2h)	3.1	(1393, 1857, 2479)	0.0022	0.59	0.74
rmExpHMC (4h)	1.6	(722, 1209, 1873)	0.0023	0.57	0.71
expHMC-S (h)	6.1	(3566, 4041, 4384)	0.0017	0.79	0.93
expHMC-S (2h)	3.1	(672, 863, 1040)	0.0045	0.30	0.41
expHMC-S (4h)	1.6	(82, 128, 207)	0.020	0.07	0.12
$\widehat{\text{exp}}$HMC-S (h)	6.5	(3589, 4126, 4535)	0.0018	0.75	0.93
$\widehat{\text{exp}}$HMC-S (2h)	4.0	(743, 882, 1066)	0.0054	0.25	0.40
$\widehat{\text{exp}}$HMC-S (4h)	1.9	(31, 46, 74)	0.06	0.02	0.03
rmExpHMC-S (h)	5.6	(3607, 4077, 4527)	0.0016	0.83	0.93
rmExpHMC-S (2h)	3.0	(767, 1002, 1166)	0.0038	0.34	0.43
rmExpHMC-S (4h)	1.6	(132, 168, 256)	0.012	0.11	0.14
$\sigma = 1$					
leapfrogHMC	4.4	(2749, 3120, 3425)	0.0016	1	0.81
RMHMC	11.0	(4939, 5000, 5000)	0.0022	0.73	0.93
e-RMLMC	8.2	(4805, 5000, 5000)	0.0017	0.94	0.93
splitHMC (h)	5.5	(3056, 3500, 3839)	0.0018	0.90	0.89
splitHMC (2h)	2.9	(373, 515, 652)	0.0078	0.21	0.24
splitHMC (4h)	1.6	(52, 84, 158)	0.031	0.05	0.07
expHMC-B (h)	5.8	(2740, 3190, 3490)	0.0021	0.76	0.81
expHMC (h)	6.1	(2511, 3288, 3666)	0.0024	0.66	0.88
expHMC (2h)	3.2	(816, 1331, 1640)	0.0040	0.41	0.62
expHMC (4h)	1.7	(554, 979, 1345)	0.0031	0.52	0.59
$\widehat{\text{exp}}$HMC (h)	5.7	(3075, 3694, 3988)	0.0019	0.87	0.91
$\widehat{\text{exp}}$HMC (2h)	3.0	(1208, 1704, 2000)	0.0024	0.66	0.65
$\widehat{\text{exp}}$HMC (4h)	1.6	(1253, 1842, 2327)	0.0013	1.27	0.70
rmExpHMC (h)	5.8	(2582, 3628, 4045)	0.0023	0.71	0.91
rmExpHMC (2h)	3.0	(1248, 2098, 2501)	0.0024	0.66	0.75
rmExpHMC (4h)	1.6	(1181, 2045, 2488)	0.0014	1.16	0.75
expHMC-S (h)	5.7	(3007, 3550, 3861)	0.0019	0.86	0.89
expHMC-S (2h)	3.0	(337, 483, 607)	0.009	0.18	0.25
expHMC-S (4h)	1.6	(82, 147, 223)	0.019	0.08	0.10
$\widehat{\text{exp}}$HMC-S (h)	7.0	(3088, 3570, 3864)	0.0023	0.72	0.89
$\widehat{\text{exp}}$HMC-S (2h)	4.0	(393, 494, 602)	0.01	0.16	0.24
$\widehat{\text{exp}}$HMC-S (4h)	2.0	(19, 34, 52)	0.11	0.015	0.02
rmExpHMC-S (h)	5.6	(3144, 3618, 3932)	0.0018	0.90	0.89
rmExpHMC-S (2h)	2.9	(453, 590, 675)	0.0065	0.25	0.28
rmExpHMC-S (4h)	1.6	(119, 170, 248)	0.01	0.012	0.10
$\sigma = 0.01$					
leapfrogHMC	3.9	(3483, 3817, 4072)	0.0011	1	0.84
RMHMC	10.6	(4970, 5000, 5000)	0.0021	0.53	0.96
e-RMLMC	8.3	(5000, 5000, 5000)	0.0017	0.65	0.95
splitHMC (h)	5.0	(4472, 4778, 4955)	0.0011	1.0	0.98
splitHMC (2h)	2.7	(469, 657, 816)	0.0059	0.19	0.30
splitHMC (4h)	1.5	(351, 589, 707)	0.043	0.26	0.29
expHMC-B (h)	5.4	(3084, 3494, 3844)	0.0018	0.61	0.85
expHMC (h)	5.4	(4083, 4502, 4800)	0.0013	0.85	0.97
expHMC (2h)	2.8	(3286, 3866, 4194)	0.0009	1.32	0.90
expHMC (4h)	1.5	(3556, 4279, 4745)	0.0004	2.67	0.94
$\widehat{\text{exp}}$HMC (h)	5.3	(4033, 4332, 4771)	0.0013	0.86	0.96
$\widehat{\text{exp}}$HMC (2h)	2.7	(3279, 3700, 4052)	0.0008	1.35	0.89
$\widehat{\text{exp}}$HMC (4h)	1.5	(3478, 3944, 4346)	0.0004	2.67	0.91
rmExpHMC (h)	5.4	(4220, 4683, 4916)	0.0013	0.86	0.97
rmExpHMC (2h)	2.8	(3269, 3858, 4215)	0.0009	1.29	0.90
rmExpHMC (4h)	1.5	(3858, 4275, 4588)	0.0004	2.81	0.93
expHMC-S (h)	5.3	(4444, 4762, 4960)	0.0012	0.94	0.98
expHMC-S (2h)	2.8	(542, 724, 881)	0.0052	0.22	0.32
expHMC-S (4h)	1.5	(424, 699, 865)	0.0035	0.32	0.31
$\widehat{\text{exp}}$HMC-S (h)	6.2	(4358, 4679, 4929)	0.0014	0.79	0.97
$\widehat{\text{exp}}$HMC-S (2h)	3.4	(447, 623, 776)	0.0076	0.15	0.31
$\widehat{\text{exp}}$HMC-S (4h)	1.8	(11, 19, 72)	0.16	0.007	0.01
rmExpHMC-S (h)	5.2	(4310, 4716, 4959)	0.0012	0.91	0.98
rmExpHMC-S (2h)	2.7	(539, 776, 935)	0.0050	0.22	0.33
rmExpHMC-S (4h)	1.5	(258, 456, 666)	0.0057	0.19	0.26