

Nonlinear Data-Enabled Prediction and Control

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Abstract

The Willems’ fundamental lemma, which characterizes linear dynamics with measured trajectories, has found successful applications in controller design and signal processing, which has driven a broad research interest in its extension to nonlinear systems. In this work, we propose to apply the fundamental lemma to a reproducing kernel Hilbert space in order to extend its application to a class of nonlinear systems and we show its application in prediction and in predictive control.

Keywords: Willems’ fundamental lemma, Data-driven method, Reproducing Kernel Hilbert Space

1. Introduction

In the field of data-driven control, the characterization of system dynamics based on measured data serves as the driving force enabling controller design. Beyond running through a parametric modeling procedure (Ljung, 1999; Lanzetti et al., 2019; Chiuso and Pillonetto, 2019; Limanond and Si, 1998), non-parametric methods distinguish themselves by directly representing system dynamics with data (Kocijan, 2016; Hou, 1994; Guardabassi and Savaresi, 2000; Levine, 2018). In particular, there is a recent spark of interest in behavioral theory (Willems and Polderman, 1997) where system dynamics are characterized by trajectories. This viewpoint concludes a simple and closed representation for linear time invariant systems (Willems et al., 2005), coined Willems’ fundamental lemma, and has been successfully applied to predictive control (Coulson et al., 2019), named data-enabled predictive control. Its extension to nonlinear systems has received broad research attention, such as (Rueda-Escobedo and Schiffer, 2020; Guo et al., 2020; Bisoffi et al., 2020; Berberich and Allgöwer, 2020). However, most works assume an explicit knowledge of the model structure with fully measured states, which in general negates the necessity of a data-driven method.

Meanwhile, as control theory is well-developed for linear systems, several works have tried to map a nonlinear control problem into a linear one. Among those trials, Koopman operator based methods establish a special viewpoint by looking into the function evolution governed by system dynamics, where the Koopman operator is a linear composite operator (Koopman and Neumann, 1932) under an autonomous system. Even though this operator is only well-defined in forward-complete systems (Bittracher et al., 2015), it still serves as a successful heuristic motivating applications in system identification and controller design (Lian and Jones, 2019; Korda and Mezić, 2018).

This work is inspired by both the behavioral theory and the function space viewpoints of Koopman operator theory. In particular, the linear dynamics of a linear functional is studied under the framework of the Willems’ fundamental lemma, which ends up with a kernelized characterization of system responses. The contributions of this work are summarized in the following:

- Propose a kernelized characterization of linear dynamics evolving in the dual space of a reproducing kernel Hilbert space and explore its application in prediction and controller design.
- Discuss the existence and some relevant mathematical properties of the proposed dynamics. We further elaborate the special considerations with respect to numerical implementation.

An extended version of this work with more details is available on <https://arxiv.org/abs/2102.06553>.

1.1. Notation

$\{\cdot\}_{i=1}^T$ denotes a set of size T indexed by i , $\text{colspan}(A)$ is the column space of matrix A .

2. Background

2.1. Reproducing Kernel Hilbert Space

Definition 1 (*Saitoh and Sawano, 2016*) A reproducing kernel Hilbert space (**RKHS**) over a set X is a Hilbert space of functions from X to \mathbb{R} such that for each $x \in X$, the evaluation functional $E_x g := g(x)$ is bounded.

Given an RKHS H , the Riesz-representation theorem (*Pedersen, 2012*) guarantees that each $x \in X$ corresponds to an unique $k_x \in H$ such that $\langle g, k_x \rangle_H = g(x)$, where $\langle \cdot, \cdot \rangle_H$ denotes the inner product defined in H . The kernel function defined on H is given by $K(x, y) = \langle k_x, k_y \rangle_H$, and it is positive-semidefinite. In particular, the dual space H^* is the space of linear functionals over H . More background knowledge of RKHS theory is available in (*Berlinet and Thomas-Agnan, 2011*; *Steinwart and Christmann, 2008*; *Saitoh and Sawano, 2016*).

2.2. The Willems' Fundamental Lemma

Definition 2 A Hankel matrix of depth L associated with a vector signal $\{s_i\}_{i=1}^T$, $s_i \in \mathbb{R}^{n_s}$ is

$$\mathfrak{H}_L(s) := \begin{bmatrix} s_1 & s_2 & \dots & s_{T-L+1} \\ s_2 & s_3 & \dots & s_{T-L+2} \\ \vdots & \vdots & & \vdots \\ s_L & s_{L+1} & \dots & s_T \end{bmatrix}$$

Regarding a Hankel matrix $\mathfrak{H}_L(s)$, the signal sequence $\{s_i\}_{i=1}^T$ is persistently exciting of order L if $\mathfrak{H}_L(s)$ is full row rank. The Willems' fundamental lemma utilizes the Hankel matrices to characterize the response of the following deterministic linear time invariant (LTI) system, dubbed $\mathfrak{B}(A, B, C, D)$ with $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$,

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \quad (1)$$

whose order is denoted by $\mathfrak{D}(\mathfrak{B}(A, B, C, D))$ and all the L -step trajectories generated by this system is collected by $\mathfrak{B}_L(A, B, C, D)$. Given a sequence of input and output measurements $\{u_i\}_{i=1}^T$, $\{y_i\}_{i=1}^T$, we define the following stacked Hankel matrix

$$\mathfrak{H}_L(u, y) := \begin{bmatrix} \mathfrak{H}_L(u) \\ \mathfrak{H}_L(y) \end{bmatrix},$$

with which one has the following **Willems' Fundamental Lemma**.

Lemma 3 (*Willems et al., 2005, Theorem 1*) *Consider a controllable linear system and assume u is persistently excited of order $N \geq \mathfrak{D}(\mathfrak{B}(A, B, C, D)) + L$. Then $\text{colspan}(\mathfrak{H}_L(u, y)) = \mathfrak{B}_L(A, B, C, D)$.*

3. Nonlinear Data-enabled Methods

We consider a linear dynamical system $\mathfrak{B}(A, B, C, D)$ evolving in RKHS as

$$\begin{aligned} f_{i+1} &= \mathcal{A} f_i + \mathcal{B} E_{u_i} \\ E_{y_i} &= \mathcal{C} f_i + \mathcal{D} E_{u_i}, \end{aligned} \quad (2)$$

where f_i is a real-valued linear functional in RKHS H_x^* , similarly E_{u_i} and E_{y_i} are evaluation functionals in RKHS H_u^* and H_y^* . The kernels of H_u and H_y are $k_u(\cdot, \cdot)$ and $k_y(\cdot, \cdot)$ respectively. Meanwhile, the dynamics are modeled by bounded linear operators $\mathcal{A} : H_x^* \rightarrow H_x^*$, $\mathcal{B} : H_u^* \rightarrow H_x^*$, $\mathcal{C} : H_x^* \rightarrow H_y^*$ and $\mathcal{D} : H_u^* \rightarrow H_y^*$. It is noteworthy to point out that these dynamics are not necessarily infinite dimensional; more discussion is allocated to this point in Section 4.2.

Given a sequence of measurements $\{u_i\}_{i=1}^T$, $\{y_i\}_{i=1}^T$, we have two sequences of evaluation functionals as $\{E_{u_i}\}_{i=1}^T$, $\{E_{y_i}\}_{i=1}^T$. The corresponding n -column Hankel matrices are:

$$\mathfrak{H}_L(E_u) := \begin{bmatrix} E_{u_1} & E_{u_2} & \cdots & E_{u_{T-L+1}} \\ E_{u_2} & E_{u_3} & \cdots & E_{u_{T-L+2}} \\ \vdots & \vdots & & \vdots \\ E_{u_L} & E_{u_{L+1}} & \cdots & E_{u_T} \end{bmatrix}, \quad \mathfrak{H}_L(E_y) := \begin{bmatrix} E_{y_1} & E_{y_2} & \cdots & E_{y_{T-L+1}} \\ E_{y_2} & E_{y_3} & \cdots & E_{y_{T-L+2}} \\ \vdots & \vdots & & \vdots \\ E_{y_L} & E_{y_{L+1}} & \cdots & E_{y_T} \end{bmatrix}. \quad (3)$$

For simplicity, we further define $v(\{u_i\}_{i=1}^L, \{y_i\}_{i=1}^L) := [E_{u_1}, \dots, E_{u_L}, E_{y_1}, \dots, E_{y_L}]^\top$. The Gram matrix of the stacked Hankel matrix $\mathfrak{H}_L(E_u, E_y) := [v_1, \dots, v_n]$ is then defined by

$$\begin{aligned} K_{i,j} &:= k(v(u_i, y_i), v(u_j, y_j)) \\ &= \sum_{k=0}^{L-1} \langle E_{u_{i+k}}, E_{u_{j+k}} \rangle_{H_u^*} + \sum_{k=0}^{L-1} \langle E_{y_{i+k}}, E_{y_{j+k}} \rangle_{H_y^*} \\ &\stackrel{(a)}{=} \sum_{k=0}^{L-1} k_u(u_{i+k}, u_{j+k}) + \sum_{k=0}^{L-1} k_y(y_{i+k}, y_{j+k}), \end{aligned} \quad (4)$$

where (a) holds by the fact that the Hilbert space is self-dual (Pedersen, 2012). The corresponding RKHS generated by $k(v(u_i, y_i), v(u_j, y_j))$ is constructed by the following product topology (Berlinet and Thomas-Agnan, 2011, Chapter 1.4),

$$H^* := \underbrace{H_u^* \otimes \cdots \otimes H_u^*}_{L \text{ times}} \otimes \underbrace{H_y^* \otimes \cdots \otimes H_y^*}_{L \text{ times}}.$$

With the Fundamental Lemma 3, we can state the following theorem

Theorem 4 Consider a controllable linear system $\mathfrak{B}(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ and assume E_u is persistently excited of order $N \geq \mathfrak{D}(\mathfrak{B}(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})) + Ln_u$. A trajectory of length L $\{\tilde{u}_i\}_{i=1}^L$ and $\{\tilde{y}_i\}_{i=1}^L$ is an element of $\mathfrak{B}_L(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ if and only if there exists $g \in \mathbb{R}^n$ such that

$$g^T K g + k(\tilde{v}, \tilde{v}) - 2 \sum_{i=1}^n g_i k(\tilde{v}, v_i) = 0, \quad (5)$$

where $\tilde{v} := v(\{\tilde{u}_i\}_{i=1}^L, \{\tilde{y}_i\}_{i=1}^L)$.

Proof The inputs and outputs sequence containing the evaluation functionals is an element of $\mathfrak{B}_L(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. Hence, by Fundamental Lemma 3, $\tilde{v} \in \text{colspan}(\mathfrak{H}_L(E_u, E_y))$ and there exists $g = [g_1, \dots, g_n]^T \in \mathbb{R}^n$ such that

$$\begin{aligned} \sum_{i=1}^n g_i v_i = \tilde{v} &\stackrel{(a)}{\iff} \left\| \sum_{i=1}^n g_i v_i - \tilde{v} \right\| = 0 \\ &\stackrel{(b)}{\iff} \sum_{i,j=1}^n g_i \langle v_i, v_j \rangle - 2 \sum_{i=1}^n g_i \langle v_i, \tilde{v} \rangle + \langle \tilde{v}, \tilde{v} \rangle = 0 \\ &\iff g^T K g + k(\tilde{v}, \tilde{v}) - 2 \sum_{i=1}^n g_i k(\tilde{v}, v_i) = 0. \end{aligned}$$

(a) holds by the uniqueness of the zero-element in a Hilbert space and (b) follows equation (4). ■

Remark 5 We intentionally do not state that $\text{colspan}(\mathfrak{H}_L(E_u, E_y)) = \mathfrak{B}_L(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ in this theorem. One reason is that working with an arbitrary linear functional is not necessarily trivial in most RKHS. The most important reason is that the evaluation functional is not necessarily dense in the RKHS, while dynamics (2) only considers evaluation functionals. This concern can be fixed by referring to the quotient space $\mathfrak{B}_L(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})/\mathbb{R}$; more details can be found in the appendix of the extended version.

Remark 6 One may be concerned whether it is valid to apply the fundamental lemma in the dual space. Besides the original proof in (Willems et al., 2005), Lemma 3 can be proven by showing that all trajectories span the range of a linear operator. In a similar way, the fundamental lemma holds in the dual space by considering the corresponding linear operator in the dual space.

3.1. Nonlinear Data-enabled Prediction

Given an input-output sequence of length T_m , $\tilde{u}_m := \{\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_{T_m}\}$ and $\tilde{y}_m := \{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{T_m}\}$, an open-loop prediction of length T_p is to predict $\tilde{y}_p := \{\tilde{y}_{T_m+1}, \tilde{y}_{T_m+2}, \dots, \tilde{y}_{T_m+T_p}\}$ if a sequence of inputs $\tilde{u}_p := \{\tilde{u}_{T_m+1}, \tilde{u}_{T_m+2}, \dots, \tilde{u}_{T_m+T_p}\}$ is applied from $T_m + 1$ to $T_m + T_p$. Theorem 4 indicates that the prediction problem is equivalent to the following optimization problem

$$\min_{y_p, g} g^T K g + k(\tilde{v}, \tilde{v}) - 2 \sum_{i=1}^n g_i k(\tilde{v}, v_i), \quad (6)$$

where K is computed from the Hankel matrix $\mathfrak{H}_{T_m+T_p}(E_u, E_y)$ and $\tilde{v} := v(\{\tilde{u}_i\}_{i=1}^{T_m+T_p}, \{\tilde{y}_i\}_{i=1}^{T_m+T_p})$. Note that the prediction is achieved by an optimization problem instead of by solving the nonlinear equation (5) in order to better accommodate the numerical solvability of nonlinear equation (5), the presence of measurement noise, the model mismatch and the infinite dimensionality. At the same time, it is noteworthy to point out that a solution to equation (5) is a global minimizer of problem (6).

Remark 7 A kernel heuristic is mentioned in (Berberich and Allgöwer, 2020, Section V), which relies heavily on their presumed Hammerstein and/or Wiener system structure. In particular, their algorithm decouples the reconstruction of the predicted trajectory and the selection of the weight by assuming an inverse map from RKHS to the states, which is not valid for most RKHS. As discussed in Remark 5, our approach looks into a quotient space resulting in an unified prediction structure. We will show that Hammerstein and Wiener systems are special cases of the proposed framework in Section 4.1

3.2. Nonlinear Data-enabled Predictive Control

To convert the method in Section 3.1 into a predictive control scheme, the input sequence is optimized so that the corresponding output sequence is most desirable. Under a receding-horizon scheme, it leads to an optimistic bi-level problem (Dempe, 2002, Chapter 2) as follows:

$$\begin{aligned} \min_{u_p, y_p, g} \quad & \sum_{i=0}^{T_p} l(u_{T_m+i}, y_{T_m+1+i}) \\ \text{s.t.} \quad & u_{T_m+i} \in \mathcal{U}, \quad y_{T_m+1+i} \in \mathcal{Y} \\ & y_p \in \operatorname{argmin}_{\tilde{y}_p} g^T K g + k(\tilde{v}, \tilde{v}) - 2 \sum_{i=1}^n g_i k(\tilde{v}, v_i), \end{aligned} \quad (7a)$$

where $l(\cdot, \cdot)$ is the stage cost and \mathcal{U}, \mathcal{Y} are constraints for control input and outputs.

4. Discussion and Practical Issues

In Section 3, the theory and the applications have been built, more theoretical details and practical issues are elaborated in this section.

4.1. On Existence of the Proposed Model

One obvious question is that whether the proposed model (2) makes any practical sense. Above all, the proposed model (2) includes standard linear systems (1) as a special instance.

Lemma 8 If H_x, H_u, H_y are RKHS whose kernel are $k(x, y) = x^T y$, then model (2) is equivalent to a standard linear system (1).

Proof By Riesz representation theorem, for each $f_i \in H_x^*$ there exists a unique $x_i \in H_x$ such that $\forall \tilde{x} \in H_x, f_i(\tilde{x}) = \langle x_i^T, \tilde{x} \rangle = x_i^T \tilde{x}$. The following proof finds the x_{i+1} such that $\langle x_{i+1}, \cdot \rangle = f_{i+1}(\cdot)$. For any $\tilde{x} \in H_x$, equation (2) gives

$$\begin{aligned} f_{i+1}(\tilde{x}) &= \mathcal{A} f_i(\tilde{x}) + \mathcal{B} E_{u_i}(\tilde{x}) \stackrel{(1)}{=} \langle \mathcal{A} x_i, \tilde{x} \rangle + \langle \mathcal{B} u_i, \tilde{x} \rangle \\ &= \langle \mathcal{A} x_i + \mathcal{B} u_i, \tilde{x} \rangle \implies f_{i+1}(\cdot) = \langle \mathcal{A} x_i + \mathcal{B} u_i, \cdot \rangle, \end{aligned}$$

where (1) follows the definition of kernel function. In a similar way, $E_{y_i} = \mathcal{C} f_i + \mathcal{D} E_{u_i}$ can be reformulated. Hence, we conclude a standard linear system (1). \blacksquare

Remark 9 Lemma 8 can be generalized to Hammerstein systems and Wiener systems. Without loss of generality, we consider a Hammerstein system, whose input nonlinearity enters the dynamics through an N_ϕ dimensional map $\phi(\cdot)$ as follows,

$$\begin{aligned} x_{i+1} &= Ax_i + B\phi(u_i) \\ y_i &= Cx_i + D\phi(u_i) . \end{aligned}$$

If H_u is generated by the kernel $k(x, y) = \langle \phi(x), \phi(y) \rangle_{\mathbb{R}^{N_\phi}}$ with N_ϕ the dimension of $\phi(\cdot)$ and H_x , H_y are generated by a linear kernel, then the resulting dynamics of (2) is a Hammerstein system following a similar proof of Lemma 8.

Beyond the standard linear system, the heuristic generalized linear systems considered in Koopman operator based control (Korda and Mezić, 2018) is also a subset of the proposed model. In particular, this generalized linear model has a form of

$$\begin{aligned} \phi(x_{i+1}) &= A\phi(x_i) + Bu \\ y_i &= C\phi(x_i) , \end{aligned}$$

where $\phi(\cdot)$ are some chosen/learned basis functions. If there is some RKHS, such that $\phi(\cdot) \in H_x$, this model can be shown to be a special case of the proposed model in a way similar to Lemma 8. In this case, H_y will be a subspace of H_x (Berlinet and Thomas-Agnan, 2011, Chapter 1). Note that if $\phi(\cdot)$ is a polynomial, then H_x will be the space equipped with the exponential kernel $k(x, y) = e^{x^T y}$.

4.2. On Persistent Excitation

The assumption of persistent excitation of E_u in Theorem 4 is defined according to the rank of its Hankel matrix $\mathfrak{H}_L(E_u)$. Checking rank of a matrix defined by functionals is not trivial, instead, the following procedure can simplify the rank calculation.

$$\text{rank}(\mathfrak{H}_L(E_u)) = \text{rank}(\mathfrak{H}_L(E_u)^T \mathfrak{H}_L(E_u)) =: K_u , (K_u)_{i,j} = \sum_{k=0}^{L-1} k_u(u_{i+k}, u_{j+k}) .$$

Hence, the condition of persistent excitation is determined by the rank of the corresponding Gram matrix. However, the condition of persistent excitation is only well defined for finite order dynamics, where H_x , H_u , H_y are also finite dimensional, such as the RKHS corresponding to a linear kernel or a polynomial kernel. If, instead, the RKHS is infinite dimensional, persistent excitation is no longer guaranteed, and Theorem 4 only serves as a heuristic. In this case, more informative data is more preferable if we still consider the condition of persistent excitation. This claim follows the fact that $\text{rank}(K_u) \geq \text{trace}(K_u)$ when E_{u_i} is within a unit ball of H_u^* ¹. Therefore, the nuclear norm is the convex envelop of the rank function (Fazel et al., 2001). Given the fact the $k_u(\cdot, \cdot)$ is strictly positive definite (Steinwart and Christmann, 2008, Lemma 4.55), i.e. $k_u(u_i, u_i) > 0 \forall i \in \mathbb{Z}$. Therefore, the rank of K_u is non-decreasing regarding the amount of data. In conclusion, a relaxed condition of persistent excitation is required for an infinite dimensional system, and therefore needs more in-depth exploration.

1. This condition holds for many stationary kernels, such as the RBF kernel.

4.3. On Choice of the Kernel

Even though the proposed method is non-parametric, the choice of kernel still determines the final performance. Above all, due to the unique correspondence between the RKHS and the kernel function by Moore-Aronszajn theorem (Berlinet and Thomas-Agnan, 2011, Theorem 3), the choice of kernel function reflects our a-priori knowledge. In particular, linear and polynomial kernels imply symmetric dynamics around 0. Exponential kernel $k(x, y) = e^{x^T y}$ is used when H_x, H_u, H_y is spanned by polynomials, because polynomials are dense in the corresponding RKHS (Steinwart and Christmann, 2008, Chapter 4). Moreover, if the trajectories that are close to each other in the state space also show similar behavior, the RBF kernel $k(x, y) = e^{-\frac{\|x-y\|^2}{2}}$ can be used.

Beyond the a-priori knowledge about the system, the choice of kernel function also affects the solvability of Problem (6) and Problem (7). We observe that exponential kernel has relatively low numerical stability as two distant trajectories result in a large gradient. We also observed that the RBF kernel leads to pervasive local minima, which causes poor performance when the Problem (7) and Problem (6) are solved by gradient based algorithms. Based on our experiments, the kernel function $k(x, y) = e^{-\frac{\|x-y\|^2}{2}} e^{x^T y}$ in general has the best performance for examples tried so far.

Finally, even with all points discussed above, the choice of the kernel function is still non-trivial in general. If a dictionary of kernel functions $\{k(\cdot, \cdot)\}_{i=1}^{N_k}$ is available, the choice of the kernel can be optimized over the positive-weighted sum $k(\cdot, \cdot) = \sum_{i=1}^{N_k} \alpha_i k(\cdot, \cdot)$ via minimization of the prediction error in the training data set.

4.4. On Stochastic Model with Measurement Noise

When the data is contaminated by measurement noise, a kernel mean embedding (Muandet et al., 2016) can be used, which evaluates the kernel function with respect to its distribution as

$$\mathbb{E}_{\tilde{X}} k(\cdot, x), \tag{8}$$

where $x \sim \tilde{X}$ relating to the distribution of the measurement noise. If the distribution of measurement noise is known, such as Gaussian distribution, the Equation (8) has closed explicit form. If noise is unknown, then (8) can be evaluated by the empirical distribution.

Remark 10 Notice that the measurement noise in each column of the Hankel matrix is not i.i.d. Hence, the empirical distribution has slower convergence rate than $O(\frac{1}{N})$.

5. Numerical Results

In this section, the proposed prediction and control scheme is validated by a nonlinearly damped pendulum and a bilinear DC motor. The prediction problem is solved with CASADI (Andersson et al., 2019) in MATLAB and the bi-level predictive control optimization is solved by FMINCON interfacing CASADI. It is noteworthy that the bi-level optimization cannot be solved by reformulating the lower problem with its KKT conditions, because the lower problem is non-convex. The experiments are carried out with an Intel i7-4500U(1.8 GHz) and a 1333MHz 8 GB memory.

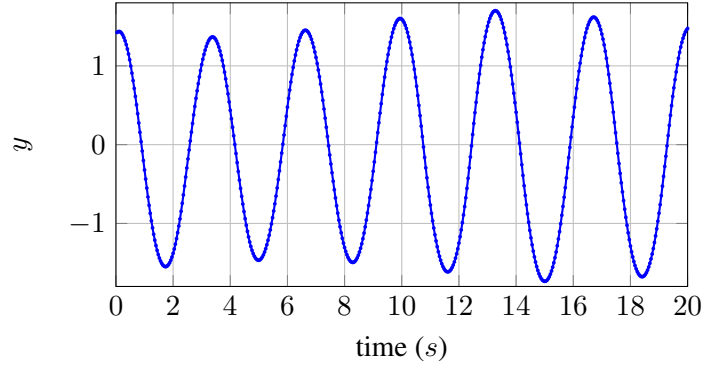


Figure 1: Snapshot of training data use in damped pendulum model

5.1. Damped Pendulum

We consider a force acting on the tip of the damped pendulum. The dynamics are

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{2g}{l} \sin(x_1) - \mu x_2^3 + \frac{1}{l} |\cos(x_1)| u$$

with $g = 9.8N/kg$, $l = 0.5m$ and $\mu = 0.1$ denoting a nonlinear friction factor. Only x_1 is observed as $y = x_1$. The output training data is generated by random input of ranging from -1 to 1, the sequence is measured with a sampling time of 0.04 seconds. 500 data points are used to defined the Problem (6) and Problem (7). Part of the output training data is shown in Figure 1, in which one can see the nonlinear modulation effect of the damped pendulum model. An open-loop prediction of $T_p = 60$ step is carried out with $T_m = 10$ previous step measured and the result is shown in Figure 2, where the kernel for inputs is $k_u(x, y) = 0.2e^{-\frac{\|x-y\|^2}{6}} + e^{x^T y} + 0.01e^{-\frac{\|x-y\|^2}{6}} e^{x^T y}$ and the one for outputs is $k_y(x, y) = 0.2e^{-\frac{\|x-y\|^2}{6}} + e^{x^T y} + 0.01e^{-\frac{\|x-y\|^2}{6}} e^{x^T y} + (1 + x^T y)^2$.

5.2. Bilinear Motor

We consider a bilinear motor (Daniel-Berhe and Unbehauen, 1998) whose dynamics is

$$\dot{x}_1 = -\frac{R_a}{L_a} x_1 + \frac{k_m}{L_a} x_2 u + \frac{u_a}{L_a}, \quad \dot{x}_2 = -\frac{B}{J} x_2 + \frac{k_m}{J} x_1 u - \frac{\tau}{J}$$

where x_1 is the rotor current, x_2 is the angular velocity and the control input u is the stator current. Only the stator current is measured as $y = x_1$. The parameters are $L_a = 0.314$, $R_a = 12.345$, $k_m = 0.253$, $J = 0.00441$, $B = 0.00732$, $\tau = 1.47$, and $u_a = 60$. Due to the synthetic effect of the bilinear term and the bias term, the responses of the system at different operating points are wildly different, hence the data is generated by a sequence of $\mathcal{N}(\mu_n, \sigma_n^2)$, where the mean of the random signal μ_n is time varying in order to excite more modes around different operating points. In particular, the mean μ_n ranges from -0.5 to 0.5, and the variance $\sigma_n = 1$. 700 datapoints are used to defined the prediction and the optimal control problem. The sampling time of the generated sequence is 0.01 seconds. An open-loop prediction of 30 is carried out with 40 previous step measured, the corresponding result is shown in Figure 3,

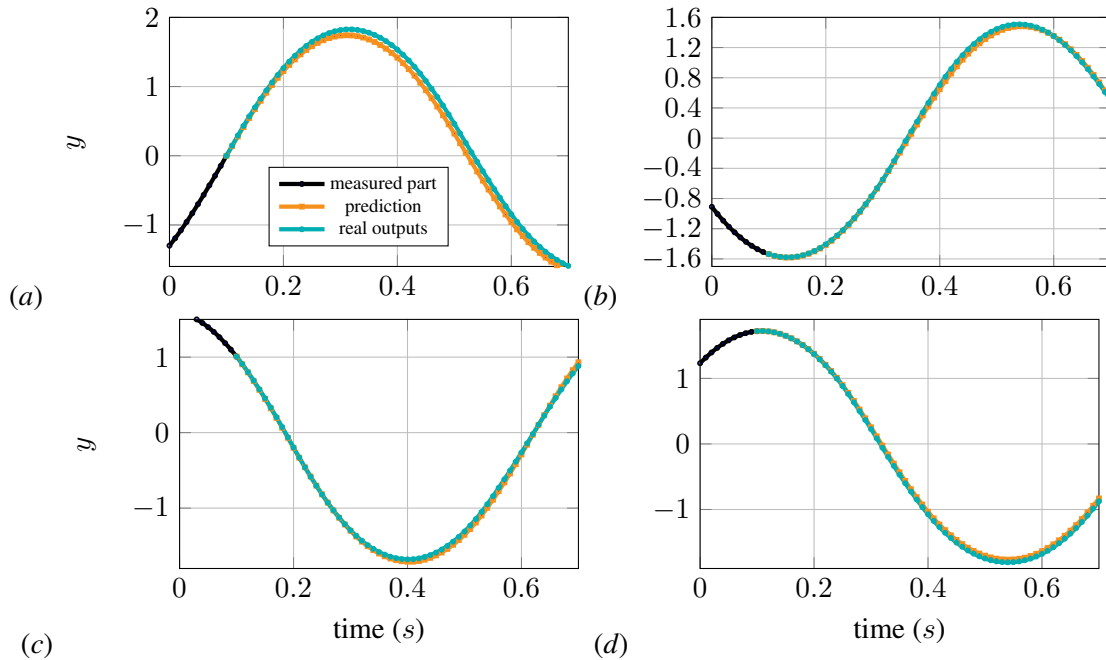


Figure 2: Open-loop prediction of damped pendulum. An asymmetric oscillation is observed and is learnt by the proposed method in subfigure (b). Each subplot is an open-loop prediction evaluated on different data. By referring to Section 3.1, the black curves are \tilde{y}_m and orange curves are the open-loop prediction solved by problem (6)

where the kernel for inputs is $k_u(x, y) = 0.1e^{-\frac{\|x-y\|^2}{4}} + e^{-\frac{\|x-y\|^2}{4}}e^{x^T y}$ and the one for outputs is $k_y(x, y) = 0.1e^{-\frac{\|x-y\|^2}{4}} + e^{-\frac{\|x-y\|^2}{4}}e^{x^T y}$.

Furthermore, a predictive control scheme is tested, where the 15 previous steps are used for a prediction horizon of 8 steps. The stage cost is $l(u, y) = (y - y_{\text{ref}})^T (y - y_{\text{ref}}) + 0.01u^T u$. The proposed predictive control scheme is compared to the nonlinear model predictive control, which has explicit knowledge of the system dynamics and full access of the state measurement. A step-like reference signal is tracked with outcome shown in Figure 4 without considering output constraints², it is observed that the proposed method shows competitive performance against the model based control law, however, it fails to converge to the upper reference with a subtle bias and it has slight overshoot with respect to both set points.

Remark 11 *The bi-level problem is currently hard to solve in real-time, it takes 15 minutes to solve the predictive control problem. Hence, a four step closed-loop control simulation takes one hour.*

6. Conclusion

This paper presents a novel data-driven method by extending the applications of the Willems' fundamental lemma in a RKHS. The resulting kernelized characterization of system dynamics is stud-

². The consideration of output constraints makes the bi-level highly unsolvable.

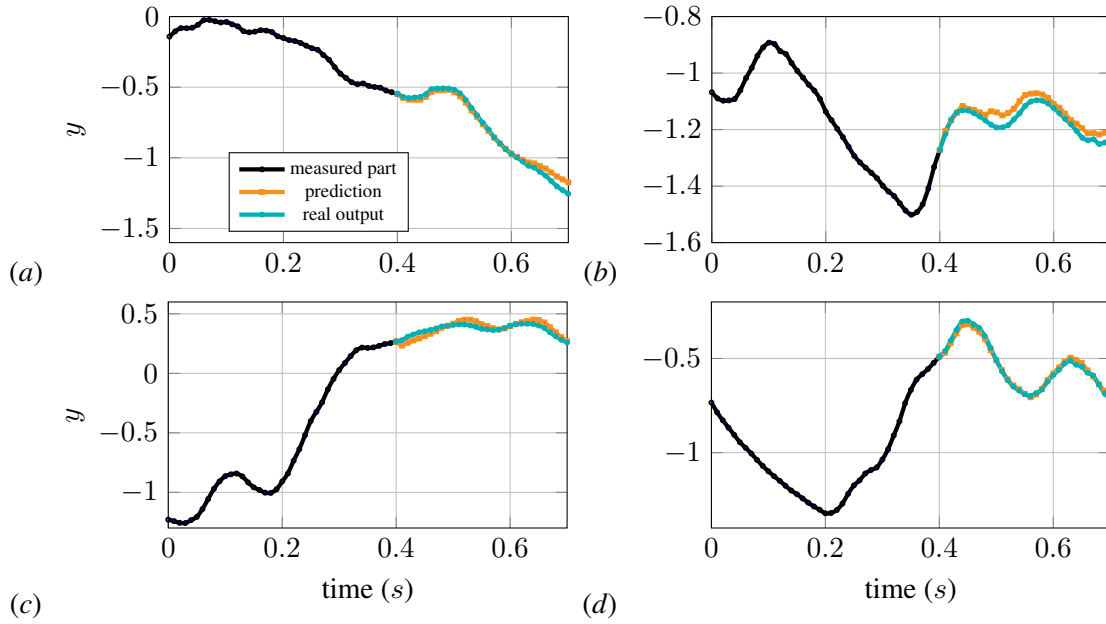


Figure 3: Open-loop output prediction with random biased input sequence of bilinear motor model. Interpretation of each subplot is similar to Figure 2

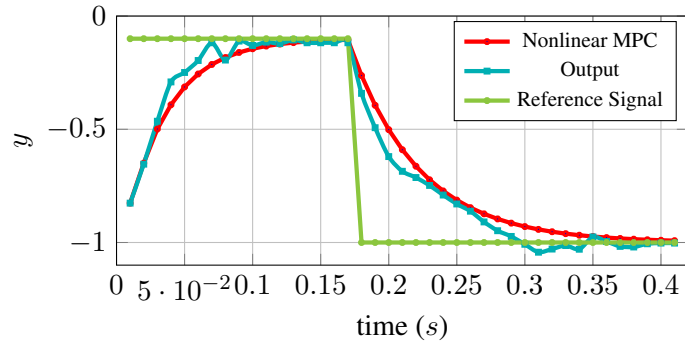


Figure 4: Closed-loop MPC control output

ied and applied in prediction and predictive control with numerical validations. In conclusion, the proposed method generalizes the behavioral theory to some nonlinear dynamics, its early-stage theoretical result is presented in this work.

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