A. Proof of Equation (2)

First, for the first term in equation (2), we can derive:

$$
\begin{aligned}
&E\Big[Y_{t+k}(\bar{A}_{t-1}, a, A_{t+1}^{a_t=a}, \dots, A_{t+k-1}^{a_t=a})|H_t(\bar{A}_{t-1})\Big] \\
&= E\Big[Y_{t+k}(\bar{A}_{t-1}, a, A_{t+1}^{a_t=a}, \dots, A_{t+k-1}^{a_t=a})|H_t(\bar{A}_{t-1}), A_t = a\Big] \\
&= E\Big[Y_{t+k}(\bar{A}_{t-1}, A_t, A_{t+1}^{a_t=a}, \dots, A_{t+k-1}^{a_t=a})|H_t(\bar{A}_{t-1}), \\
&A_t = a\Big] \\
&= E\Big[E\Big\{Y_{t+k}(\bar{A}_{t-1}, A_t, A_{t+1}^{a_t=a}, \dots, A_{t+k-1}^{a_t=a})|H_t(\bar{A}_{t-1}), \\
&A_t = a, A_{t+1} = A_{t+1}^{a_t=a}, \dots, A_{t+k-1} = A_{t+k-1}^{a_t=a}\Big\}|H_t(\bar{A}_{t-1}), \\
&A_t = a\Big] \\
&= E\Big[E\Big\{Y_{t+k}|H_t(\bar{A}_{t-1}), A_t = a, A_{t+1} = A_{t+1}^{a_t=a}, \dots, \\
&A_{t+k-1} = A_{t+k-1}^{a_t=a}\Big\}|H_t(\bar{A}_{t-1}), A_t = a\Big] \\
&= E\Big[Y_{t+k}|H_t(\bar{A}_{t-1}), A_t = a\Big] \\
&= E\Big[Y_{t+k}|H_t, A_t = a\Big],
\end{aligned}
$$

where the first equation is based on the sequential ignorability assumption; The second, the third and the fifth equations are based on the property of the conditional expectation; The fourth and the last equations are based on the consistency assumption. Equation (2) can thus be proved.

B. Proof of Theorem 1

We need to show that under assumptions (3) and (7) :

$$
E\Bigg[\Big\{d(A_t, S_t) - E(d(A_t, S_t)|S_t)\Big\}\times \Big\{U_{t+k} - E(U_{t+k}|S_t)\Big\}\Bigg] = 0.
$$

By the property of conditional expectation, it is trivial to obtain that:

$$
\begin{aligned}
&E\Big[\Big\{d(A_t, S_t) - E(d(A_t, S_t)|S_t)\Big\} \times E(U_{t+k}|S_t)\Big] \\
&= E\Big(E\Big[\Big\{d(A_t, S_t) - E(d(A_t, S_t)|S_t)\Big\} \times E(U_{t+k}|S_t)\Big|S_t\Big]\Big) \\
&= E\Big[E\Big\{d(A_t, S_t)|S_t\Big\} - E\Big\{d(A_t, S_t)|S_t)\Big\} \times E(U_{t+k}|S_t)\Big] \\
&= 0.
\end{aligned}
$$

Thus Equation (8) is equivalent to: $E\Big[\Big\{d(A_t, S_t) - E(d(A_t, S_t)|S_t)\Big\} \times U_{t+k}\Big] = 0$. By the property of conditional expectation, it is sufficient to show that:

$$
E\Big[\Big\{d(A_t, S_t) - E(d(A_t, S_t)|S_t)\Big\} \times U_{t+k}\Big|S_t\Big] = 0,
$$

which is equivalent to:

$$
E\Big[d(A_t, S_t)U_{t+k}|S_t\Big] = E\Big[d(A_t, S_t)|S_t\Big]E\Big[U_{t+k}|S_t\Big]. \tag{B.1}
$$

From the definition of U_{t+k}, we can obtain that:

$$
U_{t+k} = Y_{t+k} - \tau_k(A_t, a_0, S_t).
$$

With consistency assumption, $Y_{t+k} = Y_{t+k}(\bar{A}_{t-1}, a = A_t, A_{t+1}^{a=A_t}, \dots, A_{t+k-1}^{a=A_t})$. Thus, $U_{t+k} = Y_{t+k}(\bar{A}_{t-1}, a_t = A_t, A_{t+1}^{a_t=A_t}, \dots, A_{t+k-1}^{a_t=A_t}) - \tau_k(A_t, a_0, S_t)$. By the consistency assumption , it is trivial to prove that $S_t(\bar{A}_{t-1}) = S_t$.

$$
\begin{aligned}
&E(U_{t+k}|S_t, A_t) = E\Big[Y_{t+k}(\bar{A}_{t-1}, a_t = A_t, A_{t+1}^{a_t=A_t}, \dots, \\
&A_{t+k-1}^{a_t=A_t}) - \tau_k(A_t, a_0, S_t)|S_t(\bar{A}_{t-1}), A_t\Big] \\
&= E\Big[Y_{t+k}(\bar{A}_{t-1}, a_t = A_t, A_{t+1}^{a_t=A_t}, \dots, A_{t+k-1}^{a_t=A_t}) - \\
&E\Big\{Y_{t+k}(\bar{A}_{t-1}, a_t = A_t, A_{t+1}^{a_t=A_t}, \dots, A_{t+k-1}^{a_t=A_t}) - \\
&Y_{t+k}(\bar{A}_{t-1}, a_t = a_0, A_{t+1}^{a_t=a_0}, \dots, A_{t+k-1}^{a_t=a_0})\Big|H_t(\bar{A}_{t-1}), \\
&A_t\Big\}\Big|S_t(\bar{A}_{t-1}), A_t\Big].
\end{aligned}
$$

We first take the conditional expectation with respect to $H_t(\bar{A}_{t-1}), A_t$. Then the first term and the second term are both $E\{Y_{t+k}(\bar{A}_{t-1}, a_t = A_t, A_{t+1}^{a_t=A_t}, \dots, A_{t+k-1}^{a_t=A_t})|H_t(\bar{A}_{t-1}), A_t\}$ and can be canceled. Thus the right side of the above equation is equal to:

$$
\begin{aligned}
&E\Big[E\{Y_{t+k}(\bar{A}_{t-1}, a_t = a_0, A_{t+1}^{a_t=a_0}, \dots, A_{t+k-1}^{a_t=a_0})|H_t(\bar{A}_{t-1}), \\
&A_t\}|S_t(\bar{A}_{t-1}), A_t\Big] = \\
&E\Big[Y_{t+k}(\bar{A}_{t-1}, a_t = a_0, A_{t+1}^{a_t=a_0}, \dots, A_{t+k-1}^{a_t=a_0})|S_t(\bar{A}_{t-1}), A_t\Big] \\
&= E\Big[Y_{t+k}(\bar{A}_{t-1}, a_t = a_0, A_{t+1}^{a_t=a_0}, \dots, A_{t+k-1}^{a_t=a_0})|S_t(\bar{A}_{t-1})\Big],
\end{aligned}
$$

where the last equation is based on assumption (7). Therefore,

$$
\begin{aligned}
&E(U_{t+k}|S_t, A_t) = \\
&E\Big[Y_{t+k}(\bar{A}_{t-1}, a_t = a_0, A_{t+1}^{a_t=a_0}, \dots, A_{t+k-1}^{a_t=a_0})|S_t(\bar{A}_{t-1})\Big]
\end{aligned}
$$

Take expectation with respect to S_t for both sides, we obtain that:

$$
\begin{aligned}
&E(U_{t+k}|S_t) = \\
&E\Big[Y_{t+k}(\bar{A}_{t-1}, a_t = a_0, A_{t+1}^{a_t=a_0}, \dots, A_{t+k-1}^{a_t=a_0})|S_t(\bar{A}_{t-1})\Big] \\
&= E(U_{t+k}|S_t, A_t).
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&E\Big[d(A_t,S_t)U_{t+k}|S_t\Big]\\
&=E\Big[E\{d(A_t,S_t)U_{t+k}|S_t,A_t\}\Big|S_t\Big]\\
&=E\Big[d(A_t,S_t)E\{U_{t+k}|S_t,A_t\}\Big|S_t\Big]\\
&=E\Big[d(A_t,S_t)E\{U_{t+k}|S_t\}\Big|S_t\Big]\\
&=E\Big\{d(A_t,S_t)|S_t\Big\}E\Big\{U_{t+k}|S_t\Big\}.
\end{aligned}$$

Thus Equation (B.1) can be proved. Therefore, Theorem 1 is proved.

C. Proof of Theorem 2

First of all,

$$\begin{aligned}
&\sqrt{n}(\hat{\phi}_k-\phi_k^*)\\
=&\Big[\mathbb{P}_n L_1(H;\hat{B},\hat{C})\Big]^{-1}\Big[\sqrt{n}\mathbb{P}_n\Big\{L_2(H;\hat{B},\hat{C},\hat{D})-\\
&L_1(H;\hat{B},\hat{C})\phi_k^*\Big\}\Big].
\end{aligned}$$

The second part on the right side can be written as:

$$\begin{aligned}
&\sqrt{n}\mathbb{P}_n\Big\{L_2(H;\hat{B},\hat{C},\hat{D})-L_1(H;\hat{B},\hat{C})\phi_k^*\Big\}\\
=&\sqrt{n}\Big[\mathbb{P}_n\{L_2(H;\hat{B},\hat{C},\hat{D})-L_1(H;\hat{B},\hat{C})\phi_k^*\}\\
&-\mathbb{P}_n\{L_2(H;B,C,D)-L_1(H;B,C)\phi_k^*\}\Big]\\
&+\sqrt{n}\Big[\mathbb{P}_n\{L_2(H;B,C,D)-L_1(H;B,C)\phi_k^*\}\\
&-E\{L_2(H;B,C,D)-L_1(H;B,C)\phi_k^*\}\Big].
\end{aligned}$$

Therefore, to prove Theorem 2, it is enough to show the following three equations:

$$\mathbb{P}_n L_1(H;\hat{B},\hat{C})\xrightarrow{p}E[L_1(H;B,C)], \tag{C.1}$$

$$\begin{aligned}
&\sqrt{n}\Big[\mathbb{P}_n\{L_2(H;\hat{B},\hat{C},\hat{D})-L_1(H;\hat{B},\hat{C})\phi_k^*\}-\\
&\mathbb{P}_n\{L_2(H;B,C,D)-L_1(H;B,C)\phi_k^*\}\Big]=o_p(1)
\end{aligned} \tag{C.2}$$

$$\begin{aligned}
&\sqrt{n}\Big[\mathbb{P}_n\{L_1(H;B,C)\phi_k^*-L_2(H;B,C,D)\}\\
&-E\{L_1(H;B,C)\phi_k^*-L_2(H;B,C,D)\}\Big]\\
&\xrightarrow{d}N\Big\{0,\Sigma(\phi_k^*;B,C,D)\Big\}.
\end{aligned} \tag{C.3}$$

Then with Slutsky's theorem, we can obtain that $\sqrt{n}(\hat{\phi}_k-\phi_k^*)$ converges in distribution to a mean zero normal random vector with variance-covariance matrix given by:

$$E^{-1}\Big\{L_1(H;B,C)\Big\}\Sigma(H;\phi_k^*,B,C,D)E^{-1}\Big\{L_1(H;B,C)\Big\}$$

C.1. Proof of Equation (C.1)

First, we can obtain:

$$\begin{aligned}
&\mathbb{P}_n L_1(H;\hat{B},\hat{C})-E[L_1(H;B,C)]\\
&=\{\mathbb{P}_n L_1(H;\hat{B},\hat{C})-\mathbb{P}_n L_1(H;B,C)\}\\
&+\{\mathbb{P}_n L_1(H;B,C)-E[L_1(H;B,C)]\}
\end{aligned}$$

The second part on the right is $o_p(1)$ by the law of large numbers. Therefore, we just need to prove that the first part is $o_p(1)$. With Taylor expansion and the mean value theorem, we can obtain:

$$\begin{aligned}
&\Big|\mathbb{P}_n L_1(H;\hat{B},\hat{C})-\mathbb{P}_n L_1(H;B,C)\Big|\\
=&\Big|\mathbb{P}_n\Big\{\frac{\partial L_1(H;B,C)}{\partial B}\Big|_{B'}(\hat{B}-B)+\\
&\frac{\partial L_1(H;B,C)}{\partial C}\Big|_{C'}(\hat{C}-C)\Big\}\Big|\\
\le&\Big|\mathbb{P}_n\Big\{\frac{\partial L_1(H;B,C)}{\partial B}\Big|_{B'}(\hat{B}-B)\Big\}\Big|+\\
&\Big|\mathbb{P}_n\Big\{\frac{\partial L_1(H;B,C)}{\partial C}\Big|_{C'}(\hat{C}-C)\Big\}\Big|
\end{aligned}$$

for some B' between $\hat{B}$ and B, and C' between $\hat{C}$ and C.

$$\begin{aligned}
&\Big|\mathbb{P}_n\Big\{\frac{\partial L_1(H;B,C)}{\partial B}\Big|_{B'}(\hat{B}-B)\Big\}\Big|\le\\
&\mathbb{P}_n\Big|\frac{\partial L_1(H;B,C)}{\partial B}\Big|_{B'}\Big|\sup_{s\in\mathcal{S}}\Big|\hat{B}-B\Big|
\end{aligned} \tag{C.4}$$

Notice that:

$$\begin{aligned}
&E\Big|\frac{\partial L_1(H;B,C)}{\partial B_t}\Big|_{B'}\Big|\\
&=E\begin{pmatrix}2|B_t'-A_t^2| & |C_t-A_t|f_k(S_t)\\ |C_t-A_t|f_k(S_t) & 0\end{pmatrix}
\end{aligned}$$

By assumption 3, we can obtain that $E\{|C_t-A_t|f_k(S_t)\}<\infty$. Furthermore, $E|B_t'-A_t^2|\le E|B_t'-B_t|+E|B_t-A_t^2|\le E|\hat{B}_t-B_t|+E|B_t-A_t^2|$. By assumption 3, we can obtain that $E\{B_t-A_t^2\}<\infty$. Thus, If we can prove that:

$$\sup_s|\hat{B}_t(s)-B_t(s)|=o_p(1), \tag{C.5}$$

then $E\Big|\frac{\partial L_1(H;B,C)}{\partial B}|_{B'}\Big|<\infty$. Since

$$\mathbb{P}_n\Big|\frac{\partial L_1(H;B,C)}{\partial B}\Big|_{B'}\Big|\xrightarrow{p}E\Big|\frac{\partial L_1(H;B,C)}{\partial B_t}\Big|_{B'}\Big|,$$

we obtain that:

$$\mathbb{P}_n\Big|\frac{\partial L_1(H;B,C)}{\partial B}\Big|_{B'}\Big|=O_p(1).$$

Together with Equation (C.5), we obtain that the right side of Equation (C.4) is $o_p(1)$. Similarly, if we can prove that:

$$\sup_s |\hat{C}_t(s) - C_t(s)| = o_p(1), \tag{C.6}$$

and we can obtain:

$$\mathbb{P}_n \left| \frac{\partial L_1(H; B, C)}{\partial C} \Big|_{C'} (\hat{C} - C) \right| = o_p(1).$$

Then we can obtain that:

$$\left| \mathbb{P}_n L_1(H; \hat{B}, \hat{C}) - \mathbb{P}_n L_1(H; B, C) \right| = o_p(1).$$

Together with $\mathbb{P}_n L_1(H; B, C) - E[L_1(H; B, C)] \xrightarrow{p} 0$ by the law of large numbers, we can finish the proof for equation (C.1).

Below, we prove equation (C.6). Proof of equation (C.5) can be derived similarly. First, let the density of S_t be p_{S_t} and $\hat{p}_{S_t}(s) = \{\sum_{i=1}^n K_\lambda(s - S_t^i)\}/n$. Write $\hat{C}_t(s)$ as: $\hat{C}_t(s) = \hat{C}_{t,1}(s)/\hat{p}_{S_t}(s)$, where $\hat{C}_{t,1}(s) = \{\sum_{i=1}^n A_t^i K_\lambda(s - S_t^i)\}/n$. Also let $C_{t,1}(s) = C_t(s) p_{S_t}(s)$, then:

$$\sup_s |\hat{C}_t(s) - C(s)| = \sup_s |\frac{\hat{C}_{t,1}(s)}{\hat{p}_{S_t}(s)} - \frac{C_{t,1}(s)}{p_{S_t}(s)}| =$$
$$\sup_s \Big| \frac{\{\hat{C}_{t,1}(s) - C_{t,1}(s)\} p_{S_t}(s) - C_{t,1}(s)\{\hat{p}_{S_t}(s) - p_{S_t}(s)\}}{\hat{p}_{S_t}(s) p_{S_t}(s)} \Big|$$
$$\leq \sup_s \Big| \frac{\hat{C}_{t,1}(s) - C_{t,1}(s)}{\hat{p}_{S_t}(s)} \Big| + \sup_s \Big| \frac{C_{t,1}(s)\{\hat{p}_{S_t}(s) - p_{S_t}(s)\}}{\hat{p}_{S_t}(s) p_{S_t}(s)} \Big|.$$

Under the boundedness of $C_{t,1}(s)$ and the assumption that $p_{S_t}(s)$ is uniformly bounded away from 0, it suffices to show that:

$$\sup_s |\hat{C}_{t,1}(s) - C_{t,1}(s)| \to 0 \tag{C.7}$$

$$\sup_s |\hat{p}_{S_t}(s) - p_{S_t}(s)| \to 0 \tag{C.8}$$

We demonstrate the proof for equation (C.7). Equation (C.8) can be proved similarly. First notice:

$$\sup_s |\hat{C}_{t,1}(s) - C_{t,1}(s)| \leq$$
$$\sup_s |\hat{C}_{t,1}(s) - E\{\hat{C}_{t,1}(s)\}| + \sup_s |E\{C_{t,1}(s)\} - C_{t,1}(s)| \tag{C.9}$$

We prove the uniform convergence of the two parts on the right separately. First, we obtain:

$$E\{\hat{C}_{t,1}(s)\} = E\{A_t K_\Lambda(s - S_t)\}$$
$$= \int_{a_t} \int_{s_t} a_t K_\Lambda(s - s_t) p_{A_t|S_t}(a_t|s_t) p_{S_t}(s_t) ds_t da_t$$
$$= \int_{s_t} C_t(s) |\Lambda^{-1/2}| K\big(\Lambda^{-1/2}(s - s_t)\big) p_{S_t}(s_t) ds_t.$$

Let $v = \Lambda^{-1/2}(s - s_t)$, then $s_t = s - \Lambda^{1/2} v$. Let $\mathcal{V}_s = \{v : s - \Lambda^{-1/2} v \in \mathcal{S}\}$, then the above equation is equal to:

$$\int_{\mathcal{V}_s} C_t(s - \Lambda^{1/2} v) K(v) p_{S_t}(s - \Lambda^{1/2} v) dv$$
$$= \int_{\mathcal{V}_s} \Big\{ C_t(s) - v^T \Lambda^{1/2} \dot{C}_t(s') \Big\} K(v) \Big\{ p_{S_t}(s) -$$
$$v^T \Lambda^{1/2} \dot{p}_{S_t}(s'') \Big\} dv$$
$$= C_t(s) p_{S_t}(s) \Big\{ \int_{\mathcal{V}_s} K(v) dv \Big\} -$$
$$\Big\{ C_t(s) \dot{p}_{S_t}(s'')^T + p_{S_t} \dot{C}_t(s')^T \Big\} \Lambda^{\frac{1}{2}} \Big\{ \int_{\mathcal{V}_s} v K(v) dv \Big\},$$

where the first equation above is obtained by Taylor expansion; s' and s'' are vectors on the segment connecting s and S_t; for any function $g(s)$, $\dot{g}(s) = \partial g(s)/\partial s$. From the assumptions, $\Lambda \to 0$ as $n \to \infty$, thus $\inf_s \mathcal{V}_s \to \mathcal{S}$. $\inf_s \{\int_{\mathcal{V}_s} K(v) dv\} = 1 - O(\Lambda^{1/2})$. $\int_{\mathcal{V}_s} v K(v) dv \leq \int_{\mathcal{S}} v K(v) = O(1)$. Thus $E\{\hat{C}_{t,1}(s)\} = C_{t,1}(s) + O(\Lambda^{1/2})$. Next, we prove the uniform convergence of the first part of equation (C.9).

$$\sup_s |\hat{C}_{t,1}(s) - E\{\hat{C}_{t,1}(s)\}|$$
$$= \sup_s |\frac{1}{n} \{\sum_{i=1}^n A_t^i K_\Lambda(s - S_t^i)\} - E\{A_t^i K_\Lambda(s - S_t^i)\}|$$
$$= \sup_s \Big| \int_{s_t} C_t(s_t) |\Lambda^{-\frac{1}{2}}| K\Big(\Lambda^{-\frac{1}{2}}(s - s_t)\Big) d\Big\{ F_n(s_t) -$$
$$F(s_t) \Big\} \Big|,$$

where $F_n(s_t)$ and $F(s_t)$ denote the empirical cumulative distribution and the cumulative distribution of S_t. Then with integration by part, the above equation is less or equal to:

$$|\Lambda^{-\frac{1}{2}}| \sup_{s, s_t} \Big| C_t(s_t) K\big(\Lambda^{-\frac{1}{2}}(s - s_t)\big) \Big\{ F_n(s_t) - F(s_t) \Big\} \Big|$$
$$+ \sup_s \Big| \int_{\mathcal{S}} \Big[\big\{ F_n(s_t) - F(s_t) \big\} d C_t(s_t) K\big(\Lambda^{-\frac{1}{2}}(s - s_t)\big) \Big] \Big|$$
$$\leq \xi_1 |\Lambda^{-\frac{1}{2}}| \sup_{s_t} |F_n(s_t) - F(s_t)|,$$

where ξ is a constant and the last inequality can be derived by the assumption for the boundedness of $C_t(s_t)$ and $K(\cdot)$. By lemma 2.1 of Schuster (1969), we obtain that: $P_{S_t}\{\sup_{s_t} |F_n(s_t) - F(s_t)| > \epsilon\} \leq \xi_2 \exp(-2n\epsilon^2)$.

Then:

$$P(\sup_s |\hat{C}_{t,1}(s) - E\{\hat{C}_{t,1}(s)\}| > \epsilon)$$
$$\leq P(\xi_1 |\Lambda^{-\frac{1}{2}}| \sup_{s_t} |F_n(s_t) - F(s_t)| > \epsilon)$$
$$= P(\sup_{s_t} |F_n(s_t) - F(s_t)| > \frac{\epsilon |\Lambda^{\frac{1}{2}}|}{\xi_1})$$
$$\leq \xi_2 \exp(-\frac{2n\epsilon^2 |\Lambda|}{\xi_1^2})$$

Thus, if $2n|\Lambda| \to \infty$ as $n \to \infty$, then the first part of equation (C.9) converges to 0. Equation (C.7) is then proved. With similar proof for equation (C.8), we can obtain formula (C.6). This ends the proof for formula (C.1).

C.2. Proof of Equation (C.2)

First we write the left side of the equation as:

$$\sqrt{n}\mathbb{P}_n\Big[\Big\{L_2(H;\hat{B},\hat{C},\hat{D}) - L_1(H;\hat{B},\hat{C},\hat{D})\phi_k^*\Big\} - \Big\{L_2(H;B,C,D) - L_1(H;B,C,D)\phi_k^*\Big\}\Big]$$
$$= \sqrt{n}\mathbb{P}_n\Big[\sum_{t=1}^{T-k+1}\{\hat{M}_{t,1}\hat{M}_{t,2} - M_{t,1}M_{t,2}\}\Big]$$
$$= \sum_{t=1}^{T-k+1}\sqrt{n}\mathbb{P}_n\Big[\{M_{t,1}(\hat{M}_{t,2} - M_{t,2}) + M_{t,2}(\hat{M}_{t,1} - M_{t,1}) + (\hat{M}_{t,1} - M_{t,1})(\hat{M}_{t,2} - M_{t,2})\}\Big],$$

where,

$$\hat{M}_{t,1} = \begin{pmatrix} A_t^2 - \hat{B}_t(S_t) \\ \{A_t - \hat{C}_t(S_t)\}f_k(S_t) \end{pmatrix}$$
$$\hat{M}_{t,2} = Y_{t+k} - \hat{D}_t(S_t) - \begin{pmatrix} A_t^2 - \hat{B}_t(S_t) \\ \{A_t - \hat{C}_t(S_t)\}f_k(S_t) \end{pmatrix}^T \phi_k^*$$
$$M_{t,1} = \begin{pmatrix} A_t^2 - B_t(S_t) \\ \{A_t - C_t(S_t)\}f_k(S_t) \end{pmatrix}$$
$$M_{t,2} = Y_{t+k} - D_t(S_t) - \begin{pmatrix} A_t^2 - B_t(S_t) \\ \{A_t - C_t(S_t)\}f_k(S_t) \end{pmatrix}^T \phi_k^*.$$

Thus, it is sufficient to show that:

$$\sqrt{n}\mathbb{P}_n M_{t,1}(\hat{M}_{t,2} - M_{t,2}) = o_p(1) \quad \text{(C.10)}$$
$$\sqrt{n}\mathbb{P}_n M_{t,2}(\hat{M}_{t,1} - M_{t,1}) = o_p(1) \quad \text{(C.11)}$$
$$\sqrt{n}\mathbb{P}_n (\hat{M}_{t,1} - M_{t,1})(\hat{M}_{t,2} - M_{t,2}) = o_p(1) \quad \text{(C.12)}$$

We first prove equation (C.10). Let $G_{t,1} = A_t^2 - B_t(S_t)$, $G_{t,2} = \{A_t - C_t(S_t)\}f_k(S_t)$, $G_{t,3} = Y_{t+k} - D_t(S_t)$ and $\hat{G}_{t,1} = A_t^2 - \hat{B}_t(S_t)$, $\hat{G}_{t,2} = \{A_t - \hat{C}_t(S_t)\}f_k(S_t)$, $\hat{G}_{t,3} = Y_{t+k} - \hat{D}_t(S_t)$. Then equation (C.10) can be written as:

$$\sqrt{n}\mathbb{P}_n \begin{pmatrix} G_{t,1} \\ G_{t,2} \end{pmatrix} \Big\{\hat{G}_{t,3} - G_{t,3} + \begin{pmatrix} \hat{G}_{t,1} - G_{t,1} \\ \hat{G}_{t,2} - G_{t,2} \end{pmatrix}^T \phi_k^*\Big\} = o_p(1)$$

Therefore, it is equivalent to show all the following equations :

$$\sqrt{n}\mathbb{P}_n G_{t,1}\{\hat{G}_{t,3} - G_{t,3}\} = o_p(1)$$
$$\sqrt{n}\mathbb{P}_n G_{t,2}\{\hat{G}_{t,3} - G_{t,3}\} = o_p(1)$$
$$\sqrt{n}\mathbb{P}_n G_{t,1}\{\hat{G}_{t,1} - G_{t,1}\} = o_p(1)$$
$$\sqrt{n}\mathbb{P}_n G_{t,2}\{\hat{G}_{t,2} - G_{t,2}\} = o_p(1)$$
$$\sqrt{n}\mathbb{P}_n G_{t,1}\{\hat{G}_{t,2} - G_{t,2}\} = o_p(1)$$
$$\sqrt{n}\mathbb{P}_n G_{t,2}\{\hat{G}_{t,1} - G_{t,1}\} = o_p(1) \quad \text{(C.13)}$$

We show the proof of the last equation above. The rest of the equations can be proved similarly. First write it as:

$$\sqrt{n}\mathbb{P}_n G_{t,2}\{\hat{G}_{t,1} - G_{t,1}\}$$
$$= -\sqrt{n}\mathbb{P}_n\{A_t - C_t(S_t)\}\{\hat{B}_t(S_t) - B_t(S_t)\}$$

Let $\hat{B}_{t,1}(s) = \sum_{j=1}^n {A_t^j}^2 K_\Lambda(S_t^j - s)/n$, $B_{t,1}(s) = B_t(s)p_{S_t}(s)$. Then $\hat{B}_t(s) = \hat{B}_{t,1}(s)/\hat{p}_{S_t}(s)$. If we can obtain that

$$\lim_{n\to\infty} \text{Var}\Big\{\sqrt{n|\Lambda^{1/2}|}\Big(\hat{B}_{t,1}(s) - B_t(s)\hat{p}_{S_t}(s)\Big)\Big\} < \infty, \quad \text{(C.14)}$$

then from appendix B.1 of Zhu et al. (2020), we obtain that: under the assumptions: $\sqrt{n|\Lambda^{1/2}|}\Big(\hat{B}_{t,1}(s) - B_t(s)\hat{p}_{S_t}(s)\Big)$ converge in distribution to a mean 0 normal distribution. Together with equation C.8 and the assumption that $p_{S_t}(s)$ is bounded away from 0, we can obtain that,

$$\sqrt{n|\Lambda|^{\frac{1}{2}}}\{\hat{B}_t(S_t) - B_t(S_t)\}$$
$$= \sqrt{n|\Lambda|^{\frac{1}{2}}}\Big\{\frac{\hat{B}_{t,1}(S_t) - B_t(S_t)\hat{p}_{S_t}(S_t)}{p_{S_t}(S_t)}\Big\} + o_p(1)$$
$$= \sqrt{n|\Lambda|^{\frac{1}{2}}}\Big\{\frac{1}{n}\sum_{j=1}^n \tilde{B}_t^j(S_t)K_\Lambda(S_t^j - S_t)\Big\} + o_p(1)$$

where $\tilde{B}_t^j(s) = \{A_t^2 - E(A_t^2|S_t = s)\}/p_{S_t}(s)$.

Then:

$$\sqrt{n}\mathbb{P}_n G_{t,2}\{\hat{G}_{t,1} - G_{t,1}\}$$
$$= -\sqrt{n}\mathbb{P}_n\{A_t - C_t(S_t)\}\{\hat{B}_t(S_t) - B_t(S_t)\}$$
$$= -\frac{1}{n\sqrt{|\Lambda^{\frac{1}{2}}|}}\sum_{i=1}^n\Big\{A_t^i - C_t(S_t^i)\Big\}\Big[\sqrt{n|\Lambda^{\frac{1}{2}}|}\Big\{\frac{1}{n}\sum_{j=1}^n \tilde{B}_t^j(S_t^i)K_\Lambda(S_t^j - S_t^i)\Big\} + o_p(1)\Big]$$
$$= -\frac{\sqrt{n}}{n^2}\sum_{i=1}^n\Big\{A_t^i - C_t(S_t^i)\Big\}\Big\{\sum_{j=1}^n \tilde{B}_t^j(S_t^i)K_\Lambda(S_t^j - S_t^i)\Big\} + o_p(1)$$
$$= \frac{\sqrt{n}}{n^2}\sum_{i=1}^n\sum_{j=1}^n\Big\{A_t^i - C_t(S_t^i)\Big\}\tilde{B}_t^j(S_t^i)K_\Lambda(S_t^j - S_t^i) + o_p(1)$$

The third equation above is based on $\sqrt{n|\Lambda^{\frac{1}{2}}|} \to \infty$ and $\frac{\sqrt{n}}{n}\sum_{i=1}^{n}(A_t^i - C_t(S_t^i)) \xrightarrow{d} N(0, E\{\text{Var}(A_t|S_t\})$. Thus we just need to prove that the first term is $o_p(1)$. The first term above is a $\sqrt{n}$ times a U-statistic plus an $o_p(1)$ term when written as:

$$\frac{\sqrt{n}}{n^2}\sum_{j=1}^{n}\sum_{i<j}\Big[\tilde{B}_t^j(S_t^i)K_\Lambda(S_t^j - S_t^i)\{A_t^i - C_t(S_t^i)\}$$
$$+ \tilde{B}_t^i(S_t^j)K_\lambda(S_t^j - S_t^i)\Big\{A_t^j - C_t(S_t^j)\Big\}\Big]$$
$$+ \frac{1}{n}\Big[\frac{\sqrt{n}}{n}\sum_{i=1}^{n}\tilde{B}_t^i(S_t^i)\Big\{A_t^i - C_t(S_t^i)\Big\}\Big].$$

The second term above is $o_p(1)$ because of the law of large numbers. The expectation of the U-statistics is equal to :

$$\frac{n-1}{n}E\Big[\tilde{B}_t^j(S_t^i)K_\Lambda(S_t^j - S_t^i)\Big\{A_t^i - C_t(S_t^i)\Big\}\Big]$$
$$= \frac{n-1}{n}E\Big[\frac{{A_t^j}^2 - E(A_t^2|S_t = S_t^i)}{p_{S_t}(S_t^i)}K_\Lambda(S_t^j - S_t^i)$$
$$\Big\{A_t^i - E(A_t|S_t = S_t^i)\Big\}\Big]$$
$$= \frac{n-1}{n}E\Bigg(E\Big[\Big\{A_t^i - E(A_t|S_t = S_t^i)\Big\}\Big|S_t^i, A_t^j, S_t^j\Big]$$
$$\frac{{A_t^j}^2 - E(A_t^2|S_t = S_t^i)}{p_{S_t}(S_t^i)}K_\lambda(S_t^j - S_t^i)\Bigg)$$
$$= 0.$$

By the properties of U-statistics, the variance of $\sqrt{n}$ times the U-statistics converge to:

$$\text{Var}\Big\{E\Big[\tilde{B}_t^j(S_t^i)K_\Lambda(S_t^i - S_t^j)\{A_t^i - C_t(S_t^i)\}$$
$$+ \tilde{B}_t^i(S_t^j)K_\Lambda(S_t^j - S_t^i)\{A_t^j - C_t(S_t^i)\}\Big|S_t^i, A_t^i\Big]\Big\}$$

We can obtain:

$$E\Big[\tilde{B}_t^j(S_t^i)K_\Lambda(S_t^j - S_t^i)\{A_t^i - C_t(S_t^i)\}+$$
$$\tilde{B}_t^i(S_t^j)K_\lambda(S_t^i - S_t^j)\{A_t^j - C_t(S_t^j)\}\Big|S_t^i, A_t^i\Big]$$
$$= E\Big[\tilde{B}_t^j(S_t^i)K_\Lambda(S_t^j - S_t^i)\{A_t^i - C_t(S_t^i)\}|S_t^i, A_t^i\Big]$$
$$= E\Big[\{{A_t^j}^2 - B_t(S_t^i)\}K_\Lambda(S_t^j - S_t^i)|S_t^i, A_t^i\Big]\frac{A_t^i - C_t(S_t^i)}{p_{S_t}(S_t^i)}$$

From calculation in section C.1, we can obtain that:

$$\sup_s |E\{{A_t^j}^2 K_\Lambda(S_t^j - s)\} - B_t(s)p_{S_t}(s)| = O(|\Lambda^{\frac{1}{2}}|)$$
$$\sup_s |E\{K_\Lambda(S_t^j - s)\} - p_{S_t}(s)| = O(|\Lambda^{\frac{1}{2}}|).$$

Thus,

$$E\Big[\{{A_t^j}^2 - B_t(S_t^i)\}K_\Lambda(S_t^j - S_t^i)|S_t^i, A_t^i\Big] \le \xi_3|\Lambda^{\frac{1}{2}}$$

for some constant ξ_3. Thus,

$$\text{Var}\Big\{E\Big[\tilde{B}_t^j(S_t^i)K_\Lambda(S_t^i - S_t^j)\{A_t^i - C_t(S_t^i)\}$$
$$+ \tilde{B}_t^i(S_t^j)K_\Lambda(S_t^j - S_t^i)\{A_t^j - C_t(S_t^i)\}\Big|S_t^i, A_t^i\Big]\Big\}$$
$$\le \xi^2|\Lambda|\text{Var}\{\frac{A_t^i - C_t(S_t^i)}{p_{S_t}(S_t^i)}\}$$

Then as long as $\text{Var}\{(A_t^i - C_t(S_t^i))/p_{S_t}(S_t^i)\} < \infty$, the variance of the U-statistics converges to 0. Since we assumed that $p_{S_t}(s)$ is bounded away from 0 and $E(A_t^2|S_t = S_t^i) < \infty$, this conditional can be satisfied. Thus, both the expectation and the variance of the $\sqrt{n}$ times the U-statistics converge to 0, so $\sqrt{n}$ times the U-statistic converges in probability to 0. Thus equation (C.13) can be proved. With similar proof for the other equations above equation (C.13), we can obtain equation (C.10). Equation (C.11) can be proved similarly. Equation (C.12) can be proved with similar calculation. We omit the details here due to the length of the proof. This completes the proof for equation (C.2).

C.3. Proof for equation C.3

Equation C.3 can be simply obtained with the law of large numbers. Thus the proof for theorem is completed.

D. Details of the Simulation Setting

D.1. Form for lag k effect under the simulation setting

The true value for the lag 2 effect is:

$$E(Y_{t+2}|A_t = a, S_t) - E(Y_{t+2}|A_t = 0, S_t) =$$
$$-(\tau_1\eta_2 + \tau_2 - \beta_1\eta_2)(\tau_2 + \tau_1\eta_2)a^2 +$$
$$\Big\{\theta_1\eta_2 + \beta_0(\tau_1\eta_2 + \tau_2)\Big\}a$$
$$+ \Big\{(\tau_1\eta_2 + \tau_2)(-2\tau_1\eta_1 + \beta_1\eta_1) + \beta_1\tau_1\eta_1\eta_2\Big\}aX_t.$$

For $k \ge 3$: If we have:

$$E(Y_{t+k-1}|A_t = a, S_t) = \alpha_{k-1,1}X_t + \alpha_{k-1,2}X_t^2 +$$
$$\alpha_{k-1,3}A_t^2 + \alpha_{k-1,4}A_t + \alpha_{k-1,5}A_tX_t.$$

Then

$$E(Y_{t+k}|A_t = a, S_t) = \alpha_{k-1,1}X_{t+1} + \alpha_{k-1,2}X_{t+1}^2 +$$
$$\alpha_{k-1,3}A_{t+1}^2 + \alpha_{k-1,4}A_{t+1} + \alpha_{k-1,5}A_{t+1}X_{t+1}$$
$$= \alpha_{k-1,1}(\eta_1X_t + \eta_2A_t) + \alpha_{k-1,2}(\eta_1X_t + \eta_2A_t)^2 +$$
$$\alpha_{k-1,3}\Big\{\tau_1\eta_1X_t + (\tau_1\eta_2 + \tau_2)A_t\Big\}^2$$
$$+ \alpha_{k-1,4}\Big\{\tau_1\eta_1X_t + (\tau_1\eta_2 + \tau_2)A_t\Big\} + \alpha_{k-1,5}\Big\{\tau_1\eta_1X_t$$
$$+ (\tau_1\eta_2 + \tau_2)A_t\Big\}(\eta_1X_t + \eta_2A_t)$$
$$= \alpha_{k,1}X_t + \alpha_{k,2}X_t^2 + \alpha_{k,3}A_t^2 + \alpha_{k,4}A_t + \alpha_{k,5}A_tX_t,$$

where

$$\alpha_{k,1} = \Big\{\alpha_{k-1,1} + \alpha_{k-1,4}\tau_1\Big\}\eta_1,$$

$$\alpha_{k,2} = \Big\{\alpha_{k-1,2} + \alpha_{k-1,3}\tau_1^2 + \alpha_{k-1,5}\tau_1\Big\}\eta_1^2,$$

$$\alpha_{k,3} = \Big\{\alpha_{k-1,2}\eta_2^2 + \alpha_{k-1,3}(\tau_1\eta_2 + \tau_2)^2 + \alpha_{k-1,5}(\tau_1\eta_2 + \tau_2)\eta_2\Big\},$$

$$\alpha_{k,4} = \Big\{\alpha_{k-1,1}\eta_2 + \alpha_{k-1,4}(\tau_1\eta_2 + \tau_2)\Big\},$$

$$\alpha_{k,5} = \Big\{2\eta_1\eta_2\alpha_{k-1,2} + \alpha_{k-1,3}2\tau_1\eta_1(\tau_1\eta_2 + \tau_2) + \alpha_{k-1,5}\big[\tau_1\eta_1\eta_2 + \eta_1(\tau_1\eta_2 + \tau_2)\big]\Big\}.$$

Then lag k effect is:

$$\alpha_{k,3}A_t^2 + \alpha_{k,4}A_t + \alpha_{k,5}A_tX_t.$$

D.2. Proof of Assumption (7) under the Simulation Setting

According to the data generation model for our simulation setting,

$$Y_{t+1}(\bar{A}_{t-1}, a_t = a) = \theta_1 X_t + \theta_2 A_{t-1} - a(a - \beta_0 - \beta_1 X_t) + \epsilon_{t+1}.$$

$$A_t \sim \text{Normal}(\tau_1 X_t + \tau_2 A_{t-1}).$$

When $\theta_2 = 0$,

$$Y_{t+1}(\bar{A}_{t-1}, a_t = a) = \theta_1 X_t - a(a - \beta_0 - \beta_1 X_t) + \epsilon_{t+1}.$$

is independent of A_t given $S_t = X_t$. Thus assumption (7) is satisfied for $k = 1$. However, when $\theta_2 \neq 0$, this assumption is not satisfied for $k = 1$.

For $k = 2$, first since $X_{t+1}(\bar{A}_{t-1}, a_t = 2) \sim \text{Normal}(\eta_1 X_t + \eta_2 A_t, \sigma^2)$, it is trivial to see that:

$$X_{t+1}(\bar{A}_{t-1}, a_t = a) \perp A_t | X_t. \tag{D.1}$$

Since

$$A_{t+1}(\bar{A}_{t-1}, a_t = a) \sim \text{Normal}\Big(\tau_1 X_{t+1}(\bar{A}_{t-1}, a_t = a) + \tau_2 a_t, \sigma^2\Big),$$

we can obtain that

$$A_{t+1}(\bar{A}_{t-1}, a_t = a) \perp A_t | X_t. \tag{D.2}$$

Therefore,

$$Y_{t+2}(\bar{A}_{t-1}, a_t = a, A_{t+1}^{a_t=a}) = \theta_1 X_{t+1}(\bar{A}_{t-1}, a_t = a) + \theta a - A_{t+1}(\bar{A}_{t-1}, a_t = a)\Big\{A_{t+1}(\bar{A}_{t-1}, a_t = a) - \beta_0 - \beta_1 X_{t+1}(\bar{A}_{t-1}, a_t = a)\Big\} + \epsilon_{t+2}$$

is independent of A_t given X_t based on Equation (D.1) and (D.2). This is true even when $\theta_2 \neq 0$.

Using induction, we can also prove that for any $k \geq 3$, $Y_{t+k}(\bar{A}_{t-1}, a_t = a, A_{t+1}, \ldots, A_{t+k-1}) \perp A_t | X_t$.

D.3. Additional Simulation Results

The true parameters for $k = 2, 3$ when $\theta_2 = -0.1$ are : $(\alpha_2, \beta_{2,0}, \beta_{2,1}) = (-0.21, 0.06, -0.08)$; $(\alpha_3, \beta_{3,0}, \beta_{3,1}) = (-0.0125, -0.05, -0.03)$. Table 1 presents the result for the estimated parameters when $S_t = X_t$. As shown in the table, the estimated parameters appeared to be unbiased.

Table 1. Simulation results from 200 replicates when $\theta_2 = -0.1$.

		α_k				$\beta_{k,0}$				$\beta_{k,1}$			
k	n	Bias[1]	SD[1]	SE[1]	CP	Bias[1]	SD[1]	SE[1]	CP	Bias[1]	SD[1]	SE[1]	CP
2	100	1.9	31.4	29.2	91.5	-1.2	23.4	22.4	93.5	-4.7	79.1	68.3	93.0
	200	-1.0	23.8	20.9	91.5	-0.1	16.6	15.8	93.0	3.6	56.3	47.9	90.5
	400	-1.1	14.8	14.8	95.5	1.0	11.9	11.1	93.5	1.0	33.6	33.2	95.0
3	100	2.0	32.2	26.8	88.5	4.2	22.2	21.1	94.0	2.9	75.1	67.1	90.5
	200	-3.1	19.5	18.8	93.0	0.6	15.6	14.7	91.5	5.3	50.8	45.7	92.0
	400	1.1	15.7	13.3	89.5	0.7	10.8	10.3	94.0	-2.5	36.7	31.5	92.0

[1] Note: These columns are in 10^{-3} scale
[2] Note: SD refers to the standard deviation of the estimated parameters from 200 replicates, SE refers to the mean of the estimated standard errors calculated by our covariance function, CP refers to the coverage probability of the 95% confidence intervals calculated using the estimated standard errors.
[3] Note: The worst case Monte Carlo standard error for proportions is 2.3%.

Table 2. Simulation results from 200 replicates when $\theta_2 = -0.1$.

k	n	Parameter	Bias[1]	SD[1]	SE[1]	CP
1	100	α_k	18.3	23.4	21.8	83.5
		$\beta_{k,0}$	11.7	15.9	14.8	82.0
		$\beta_{k,1}$	-20.5	59.3	54.5	91.5
		$\beta_{k,2}$	12.8	32.9	31.2	90.5
	200	α_k	9.2	13.8	15.0	92.5
		$\beta_{k,0}$	6.6	12.2	10.3	85.5
		$\beta_{k,1}$	-5.2	35.5	37.5	97.0
		$\beta_{k,2}$	4.6	21.1	21.3	95.5
	400	α_k	5.3	11.0	10.5	92.0
		$\beta_{k,0}$	3.4	8.1	7.3	88.0
		$\beta_{k,1}$	-2.6	28.4	26.1	93.0
		$\beta_{k,2}$	1.7	14.5	14.9	96.0

[1] Note: These columns are in 10^{-3} scale
[2] Note: SD refers to the standard deviation of the estimated parameters from 200 replicates, SE refers to the mean of the estimated standard errors calculated by our covariance function, CP refers to the coverage probability of the 95% confidence intervals calculated using the estimated standard errors.
[3] Note: The worst case Monte Carlo standard error for proportions is 2.3%.

For $k = 1$, we can correct the bias by estimating the parameters with $S_t = (X_t, A_{t-1})$. The model for the lag 1 treatment effect is thus: $\tau_{t,1} = \alpha_1 a^2 + (\beta_{1,0} a + \beta_{1,1} X_t + \beta_{1,2} A_{t-1}) a$. The true parameters are: $(\alpha_1, \beta_{1,0}, \beta_{1,1}, \beta_{1,2}) = (-1, 0, 2, 0)$. Since the dimension of S_t has increased, we use the bandwidth $\lambda_j = n^{-1/4} \mathrm{sd}(S_{t,j})$. The estimated parameters are presented in Table 2. From the results we see that the estimated parameters appeared to be unbiased. However, the estimated standard deviation was smaller than the actual standard deviation, leading to lower coverage probability when sample size was small. This implies that when the dimension of covariates increases, the estimated standard error converges slower to the actual standard deviation.

E. Additional Results for Ohio Type 1 Diabetes Dataset

We applied the proposed method for the Ohio Type 1 Diabetes Dataset. The dataset consists of data from 6 patients. The result for patient 6 has been presented in the article. In Table 3, we present the additional results from all the patients. It is likely that the decision process for insulin dosage is different for each patient. Thus using the same set of S_t for all patients might not be the optimal choice.

Table 3. Estimated variables with the Ohio type 1 diabetes dataset

Patient 1					
k	1	2	3	4	Weighted
α_k	17(7.6)	21.8(10.6)	16.6(10.6)	10.9(12.7)	16(9.1)
$\beta_{k,0}$	-172.3(73.1)	-92.3(103.9)	-11(112.4)	-50(116.4)	-63.2(86.3)
$\beta_{k,1}$	1.4(0.7)	3.1(1.4)	3.6(1.6)	3.3(1.7)	3.1(1.3)
$\beta_{k,2}$	-0.8(3.1)	2.1(4.7)	5.9(4.8)	3.9(5.9)	2.9(3.6)
$\beta_{k,3}$	-9.1(51.5)	-104.4(80.4)	-93.7(88.3)	34.9(66.7)	-59.6(56.9)
$\beta_{k,4}$	12.9(8.9)	2.6(19.2)	-11.6(23)	-170(35.2)	-110.4(33.6)
Patient 2					
k	1	2	3	4	Weighted
α_k	0.2(0.7)	0.5(0.8)	0.6(1)	-0.8(1.1)	0.1(0.8)
$\beta_{k,0}$	-8.8(11.1)	14.1(18)	18(22.6)	14.7(24.2)	10.7(17.8)
$\beta_{k,1}$	0.6(0.3)	-0.2(0.2)	0(0.2)	-0.1(0.5)	0.1(0.2)
$\beta_{k,2}$	0.4(0.1)	0.2(0.3)	0.1(0.3)	0.3(0.3)	0.3(0.2)
$\beta_{k,3}$	-2.6(6.8)	-19.1(16.1)	-22.2(22.5)	-5.3(23.8)	-13.1(16.9)
$\beta_{k,4}$	-2.1(2.2)	-3(4)	-4.6(4.5)	-6.3(5.2)	-4.2(4)
Patient 3					
k	1	2	3	4	Weighted
α_k	0.6(1)	-2.2(1.8)	-5(2.6)	-4.6(2.9)	-2.9(1.9)
$\beta_{k,0}$	-14(39.7)	69.2(57.3)	159.8(81.9)	173.5(88.3)	98.8(63.9)
$\beta_{k,1}$	0.1(0.1)	0.4(0.3)	0.6(0.4)	0.7(0.4)	0.4(0.3)
$\beta_{k,2}$	0.1(0.1)	0.2(0.2)	0.4(0.3)	0.3(0.3)	0.3(0.2)
$\beta_{k,3}$	-11.2(46.6)	-78.1(67.3)	-148.4(93.1)	-166(98.1)	-103.3(74.8)
$\beta_{k,4}$	2.3(2.5)	3.4(2.9)	5.9(3.3)	7.5(3.2)	4.8(2.7)
Patient 4					
k	1	2	3	4	Weighted
α_k	-1.2(5.1)	-7.4(5.9)	-6.5(4.8)	-5.8(4.6)	-6.2(4.4)
$\beta_{k,0}$	-129(138.1)	162.5(95.5)	154.2(124.4)	167.4(104.2)	97.8(72.4)
$\beta_{k,1}$	0.2(0.4)	0.1(0.5)	-0.8(0.7)	-0.4(1)	-0.1(0.5)
$\beta_{k,2}$	0.4(0.5)	1.3(0.9)	1.2(0.8)	1.2(0.8)	0.7(0.6)
$\beta_{k,3}$	130.6(169.5)	-173.4(130.2)	-146.4(162.6)	-182.3(143.9)	-96(97)
$\beta_{k,4}$	1.2(3.5)	-8.2(10.1)	-13.9(12.3)	-21.2(9.7)	-10.3(7.7)
Patient 5					
k	1	2	3	4	Weighted
α_k	2.7(2.9)	6.7(4.4)	6.6(4.9)	2.7(5.2)	3.9(3.7)
$\beta_{k,0}$	-271(110.8)	-514.8(139.1)	-321(160.8)	-110.6(151.9)	-303.4(127.8)
$\beta_{k,1}$	0.4(0.3)	1.2(0.7)	0.1(0.7)	-0.6(0.8)	0.3(0.6)
$\beta_{k,2}$	0.6(0.3)	0.5(0.6)	0.5(0.7)	0.8(0.7)	0.6(0.5)
$\beta_{k,3}$	173.1(87.1)	324.2(119.8)	201.9(134.1)	78.7(125.6)	198.1(103.7)
$\beta_{k,4}$	28.8(13)	12.4(10.8)	-5.1(6)	-1.6(13.2)	8.4(7.8)

[1] Note: These columns are in 10^{-2} scale .
[2] Note: The numbers in the parenthesis are the estimated standard errors calculated by the covariance formula.
[3] Note: The last column presents the estimated parameters for the lag 4 weighted advantage with $w_1 = w_2 = w_3 = w_4 = 1/4$.

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